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Utility and mechanism design in multi-agent systems: An overview

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ABSTRACT

Future cities promise to be more autonomous than ever, largely owing to our ability of coordinating complex systems in real time: fleets of self-driving cars will offer on-demand transportation services, delivery drones will fly parcels in our skies, power plants will provide renewable energy reliably. In many of these systems, there is no single decision-maker with full information and authority. Instead, the system performance greatly depends on the decisions made by interacting entities with local information and limited communication capabilities. Game theory, intended as the study of multi-agent decision-making, is a fitting paradigm to tackle many of the associated challenges. Moving from this observation, in this paper we review how tools and ideas from game theory can be brought to bear on the coordination of multi-agent systems. At the heart of the proposed approach is the design and influence of agents' preferences so that their local optimization induces a desirable system behavior. Its applicability spans a variety of settings irrespective of whether the decision makers are strategic (e.g., drivers in a road network), or not (e.g., delivery drones). Along the way, we also discuss future research directions and connections with related research areas including algorithmic game theory, incentive and mechanism design, economics, computational complexity, and approximation algorithms.

1. Introduction

Future cities promise to be wonders of seamless cyber–human interaction: app-based services will bring self-driving taxis at the click of a button (Greenemeier, 2018; Verger, 2018), autonomous drones will deliver goods over the ground or by air (Kellermann et al., 2020; Pogue, 2016), and many routine tasks will be automated, freeing up more time for social and human-centric activities (Bloomfield, 1995; Manyika & Sneader, 2018). In this context, crucial infrastructures such as transportation and energy distribution will play an increasingly central role, fostering the development of smart, green, and reliable cities, enhancing productivity and driving economic growth.

Amongst the many technological advancements responsible for the realization of such futuristic vision, distributed decision-making is playing a prominent role and is establishing itself as the hidden technology underpinning many of these systems' operations (Jones, 2014; NSF, 2017). It is no coincidence that, in very recent years, the field has seen a surge of interest. Contrary to traditional architectures, in distributed decision-making there is no single entity with full information and authority. Rather, decisions are made by an often large collection of interacting agents, each with local information and limited communication capabilities. In many cases, decision-making agents are human-users, e.g., drivers on a road-network. We refer to this first class of systems as *sociotechnical systems*. In other cases, decision-making agents correspond to fully programmable machines, e.g., delivery drones. We refer to this second class of systems as *engineered multi-agent systems*. In both cases, the common challenge is to coordinate the agents to induce a desirable system behavior.

Sociotechnical systems

While the design and operation of large-scale infrastructures, such as transportation and energy distribution systems, is typically grounded on purely engineering principles, their performance is increasingly dependent on the interaction between human agents and the underlying technological infrastructure (Jones, 2014; Manyika & Sneader, 2018; NSF, 2017). The rapid penetration of smartphones and the decrease in mobile communication costs has resulted in an even tighter integration between these two components, further blurring the boundaries between engineered systems and the social fabric. As a consequence, the operation of such systems – often referred to as sociotechnical systems – requires interdisciplinary considerations at the interface between economics, engineering, and social sciences.

In this context, when the agents' individual objectives are not aligned to the "greater good", the corresponding system performance

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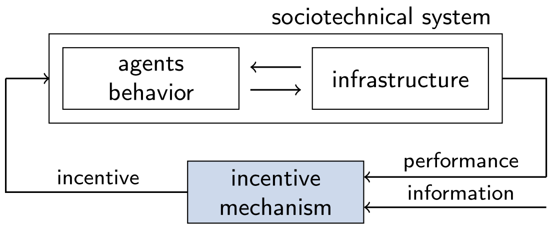

Fig. 1. Architectural overview on the design of behavior-influencing mechanism for sociotechnical systems. At the top, sociotechnical systems are captured by the interaction of agents behavior and the underlying technological infrastructure. An incentive mechanism (blue block) measures the current performance and leverages any additional information so as to improve the resulting performance, by affecting the agents' behavior through incentives. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

often suffers significant degradation (Anderson & Moore, 2006; Macfarlane, 2019). Within the environmental sciences this phenomenon has been extensively studied, and, in its extreme form, is referred to as the "tragedy of the commons" (Hardin, 1968). Within the realm of sociotechnical systems, a prime example is provided by road-traffic routing: when drivers choose routes minimizing their own travel time, the aggregate congestion could be much higher compared to that of a centrally-imposed routing (Roughgarden & Tardos, 2002).

While improved performances could be attained if a central operator could dictate the choices of each user, imposing such control is often impossible, with traffic routing providing an immediate illustration. Hence, indirect interventions have been proposed to influence the resulting system behavior (Abnett & Twidale, 2021; Graham-Rowe et al., 2011; Jones et al., 2019). Such a control approach, whereby the agents' behavior is indirectly influenced, necessitates modeling the emergent behavior. The latter is often characterized through the notion of Nash equilibrium, a configuration in which rational agents maximize their individual utilities, given the choice of the others. Within such settings, many of the existing approaches (including coordination mechanisms (Christodoulou et al., 2009), taxation mechanisms (Beckmann et al., 1956; Pigou, 1920), information provision (Castiglioni et al., 2021; Wu et al., 2021), cost sharing rules (Gkatzelis et al., 2016; Marden & Wierman, 2013), etc.) involve the introduction of incentives that modify the agents' behavior by indirectly influencing their *utilities* (see Fig. 1). For example, monetary taxes (e.g., congestion pricing) are commonly utilized to incentivize congestion-aware traffic-routing. Such monetary taxes are factored-in through the agents' utilities, and thus influence the emergent behavior. Following this approach prompts a crucial question:

How should incentives be designed to influence agents' utilities and induce a desirable system behavior?

The field of utility design and that of mechanism design aim at answering this question relative to relevant classes of problems (two of which are discussed next), employing different models of emergent behavior, and imposing various constraints on the utilities' structure and their properties. This direction bears significant value for the design of control architectures whenever strategic agents influence the resulting performances.

Example 1 (*Incentive Design for Traffic-Routing*)**.** A set of agents needs to travel on a common road-traffic network, each departing and arriving from a chosen origin/destination. Towards this goal, agents are free to travel on a desired route. Once they have chosen their route, they experience a corresponding travel time and increase the congestion on the chosen path. The system operator is interested in minimizing a global metric describing the congestion on the network, such as the sum of all agents' travel times. To incentivize congestion-aware routing, the operator imposes (possibly congestion-dependent) tolls on the network, thus influencing the agents' perceived utilities which factor in both the travel time and the monetary tolls. How should tolls be designed?

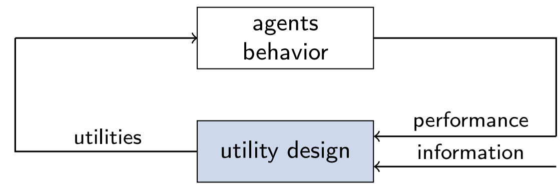

Fig. 2. Architectural overview on the design of utilities for engineered multiagent systems. A utility design mechanism measures the current performance and leverages any additional information so as to design the agents' utilities and improve the resulting performance. In this context, both the utilities and the agents behavior are subject to our design. Nevertheless, given the wide availability of algorithms ensuring convergence to commonly employed equilibrium notions, this manuscript focuses on the design of agents utilities (blue block). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 2 (*Mechanism Design in Auctions*)**.** A set of resources needs to be allocated across different agents (e.g., intersections slots in autonomous driving, user impressions on a well-visited webpage), each with a different private valuation for the resources. The system operator is interested in allocating the resources in order to maximize the social welfare, measured for example by the sum of the agents' valuations over the resources they have been assigned. Agents submit bids for the resources. The operator decides which agent receives which resources (i.e., decides on an outcome), and at what price. Both outcome and prices are functions of the received bids and directly impact the agents' utilities, often modeled as the difference between the valuation of the accrued resources and the payed price. How should the outcome and price functions be designed?

Engineered multi-agent systems

Many autonomous systems of interest, such as self-driving taxis or delivery drones, draw their strength from the ability to coordinate large fleets of vehicles over spatially distant regions. When agents are fully programmable, as considered in this section, achieving this goal often involves the definition of a system-level objective and its decomposition into individual agent's tasks. Such problems are typically posed as optimization problems, where physics and operational limitations enters as constraints on the decision variables (Kelly, 1997; Lin et al., 2006; Srikant, 2004).

Whilst commonly employed approaches for designing local algorithms leverage distributed reformulation of existing centralized schemes (e.g., distributed gradient ascent, primal–dual and Newton's method (Ozdaglar & Srikant, 2007)), a different approach entails formulating the original coordination problem through the language of game theory (Marden & Shamma, 2018b), see Fig. 2. Pursuing this direction requires identifying (i) the agents or players in the game; (ii) for each agent, a set of admissible decisions; (iii) for each agent a utility or cost function modeling their preferences over the collective agents' decisions. In addition, designing distributed algorithms through a game-theoretic lens requires identifying a solution concept for the game (e.g., a Nash equilibrium), and a class of algorithms ensuring convergence to the chosen solution concept.

Steps (i), and (ii) are often immediate. In addition, the field of learning in games provides readily available algorithms ensuring convergence to commonly employed equilibrium notions (Hart & Mas-Colell, 2003; Marden & Shamma, 2012; Marden et al., 2014; Papadimitriou & Roughgarden, 2008; Young, 2004). Therefore, the crux of the approach resides in the design of agents' utilities ensuring that the resulting system behavior is desirable, or close to being so.

Example 3 (*Sensor Coverage*)**.** A collection of possibly heterogeneous mobile sensors (the agents) need to allocate themselves over a previously unknown region with the goal of surveilling the area as best possible. Each agent has limited communication capabilities, prompting the need for distributed algorithms. Towards this goal, agents are assigned local utility functions so that their individual maximization results in good overall surveillance. How should the local utilities be chosen?

Contrary to the approach taken for sociotechnical systems, where game theory was utilized as a *descriptive tool* and the design component limited to influencing the agents' behavior, the design of distributed algorithms for engineered systems prompts the use of game theory as a *prescriptive tool*. In spite of these differences, central to both problems is the design of utilities producing desirable system behavior.

An overarching question

The previous sections highlighted how seemingly different problems, including incentive design in sociotechnical systems and distributed control in engineered multi-agent systems, give rise to strikingly similar questions when analyzed through a common game-theoretic lens. In this section we describe such game-theoretic framework and the corresponding research questions abstracting from these two classes of problems.

We consider a system composed of multiple decision-making agents $i \in \{1, \ldots, n\}$, each equipped with a set of admissible actions $a_i \in \mathcal{A}_i$. The quality of a joint-action, also referred to as an allocation, $a = (a_1, \ldots, a_n)$ is measured by the system-cost function $C(a)$, which the designer wishes to minimize. Towards this goal, each agent i is equipped with a cost function $C_i(a)$ describing the cost experienced by the agent in allocation a.[1] Here, the agents' cost functions are subject of our design. Finally, the emergent behavior is modeled through the notion of Nash equilibrium, or generalizations thereof. We are interested in the following question.

Overarching question: *How should agents' cost functions be designed so that the emergent behavior is desirable?*

In this manuscript, we review general methodologies to address this question and showcase corresponding results on thoroughly studied classes of problems, such as those introduced earlier in Examples 1 and 2.

We conclude remarking that the notion of "desirable system behavior" has been left purposely vague thus far, as different performance metrics can be utilized. For example, if multiple emergent behaviors are possible, a cautious system designer might want to ensure that the *worst* emergent behavior achieves a low system-cost. A more optimistic designer might instead be satisfied as soon as the *best* emergent behavior achieves low system-cost.

1.1. Structure of the manuscript

We conclude the introduction by providing an overview of the most relevant research threads connected to the agenda previously described (Section 1.2). Section 2 presents background notions from game-theory, including equilibrium definitions and performance metrics employed throughout. In Section 3 we recall how the smoothness framework can be leveraged to quantify the equilibrium performance. We also provide insight on the origin of the smoothness inequality.

Armed with all these tools, Sections 4 and 5 present the most central results for the design of optimal incentives in congestion games, and for the design of mechanisms in auctions. We conclude the manuscript highlighting recent and future directions in Section 6.

[1] Equivalently, we can formulate this as a maximization problem, in which case we refer to agents' utilities as opposed to agents' costs.

1.2. Related works

The overarching question considered in this manuscript connects with multiple research threads at the interface between Automatic Control, Game Theory, Economics and Computer Science. In the ensuing paragraphs we provide an overview (by no means exhaustive) of the most relevant literature pertaining to these fields.

Efficiency quantification. The performance of the emergent behavior is typically measured through two metrics known as the *price of anarchy* and the *price of stability*.

Price of anarchy. Introduced by Koutsoupias and Papadimitriou (1999), the price of anarchy measures the quality of the worst-performing equilibrium relative to an optimal allocation. Since its inception, there has been significant interest in evaluating this metric for broad classes of problems including transportation networks (Aland et al., 2011; Awerbuch et al., 2013; Bhawalkar et al., 2014; Christodoulou & Koutsoupias, 2005b; Correa et al., 2008; Suri et al., 2007), auctions (Caragiannis, Kaklamanis et al., 2011; Christodoulou et al., 2016; Eden et al., 2020; Leme & Tardos, 2010; Lucier & Borodin, 2010), supply-chain (Perakis & Roels, 2007), covering problems (Gairing, 2009), network games (Demaine et al., 2012). Many of these studies have been fostered by the development of the so-called *smoothness* framework, consisting of a general recipe for computing price of anarchy bounds, and including a black-box extension allowing to port results originally derived for pure Nash equilibria to more permissive equilibrium notions (e.g., coarse correlated equilibria) and approximate equilibria (Roughgarden, 2015a). Motivated by the design of agents' utilities optimizing the price of anarchy, Chandan et al. (2019) recently introduced the notion of generalized smoothness and showed how the approach yields provably tighter values compared to the original smoothness condition.

While providing a self-contained environment to reason about the price of anarchy, the smoothness framework moves the problem of estimating the price of anarchy to that of determining parameters (λ, μ) satisfying a collection of inequalities (one per each allocation). In many cases determining such parameters requires exploiting the specific structure of the problem at hand, although a number of black-box computational approaches have recently appeared. Most notably, Bilò (2018) and Chandan et al. (2019) provide tractable programs for the characterization of the price of anarchy. Their approach leverages a linear programming reformulation of the smoothness condition, and a concise game parametrization to obtain polynomially-sized programs.

Price of stability. The price of stability measures the quality of the best-performing equilibrium relative to an optimal allocation (Anshelevich et al., 2008). Although this metric has also been extensively studied, much fewer results are available in comparison to the price of anarchy. Notable exceptions includes congestion games with polynomial cost functions (Caragiannis, Flammini et al., 2011; Christodoulou & Gairing, 2015; Christodoulou & Koutsoupias, 2005a) and network design problems (Anshelevich et al., 2008). Quantifying the price of stability is particularly challenging as its analysis requires an equilibrium selection argument whereby the performance of the best equilibrium is considered, as opposed to the performance of any equilibrium. Inspired by the approach of Christodoulou and Gairing (2015), Chandan et al. (2021a) recently propose a modified notion of smoothness that applies to and beyond congestion games yielding improved bounds on the price of stability.

Interventions and Incentive Design. The study and design of interventions encompasses multiple research areas spanning across different communities including Transportation, Economics, Computer Science, and Public Health. As such, it is impossible to provide a concise overview here.

Rather, we focus on the design of interventions for efficient road-traffic routing (Example 1). Two main directions are typically pursued. A first approach is based on upgrading the existing infrastructure through the development of new roads or by increasing the capacity of existing ones. From a mathematical standpoint, this formulation results in a bi-level program, whose solution is particularly challenging (Farahani et al., 2013). A second and more common approach consists in influencing the drivers' behavior through monetary tolls (Beckmann, 1967; Beckmann et al., 1956; Pigou, 1920), artificial currencies (Censi et al., 2019; Gorokh et al., 2021; Salazar et al., 2021), information design (Castiglioni et al., 2021; Wu et al., 2021). The design of monetary incentives has received significant attention, partially owing to its relatively simple implementation, within two of the most-commonly employed settings: atomic congestion games and their continuous flow counterpart.

Relative to atomic congestion games, Caragiannis et al. (2010) first and Bilò and Vinci (2019) later propose tolling mechanisms with associated performance guarantees in the form of price of anarchy bounds. More recently, Paccagnan et al. (2021) design local tolling mechanism (i.e., tolling mechanisms utilizing solely local information) that optimize the price of anarchy. Beyond the local information setting, the question of designing optimal tolls in congestion games has been settled very recently by Paccagnan and Gairing (2021). In this work, the authors provide tight hardness of approximation results, and design tractable tolls yielding matching performances. For example, when cost functions are polynomials of maximum degree d, the best-achievable approximation equals the $(d+1)$st Bell number.

Finally, we observe that the design of optimal tolls is far better understood when considering the continuous flow approximation of the original atomic congestion game model. In these settings, owing to the uniqueness of Nash equilibria, marginal cost tolls are known to incentivize optimal behavior, that is, their price of anarchy and price of stability is equal to one. Nevertheless, Paccagnan et al. (2021) show that, when utilized on the original atomic setting, marginal cost tolls do not improve – and, instead, can significantly deteriorate – the resulting performance.

Mechanism Design. While the study of mechanism design was initiated in Economics,[2] the discipline is fundamentally guided by a social engineering question: "How should we design agents' interactions in order to achieve a desirable objective?" As such, mechanism design theory has since then been widely adopted, especially within Computer Science, Operations Research and, naturally, Engineering. While providing an overview of the related literature is beyond the scope of this manuscript, we refer the reader to Börgers and Krahmer (2015) and Nisan et al. (2007) for treatments of mechanism design from Economics and Computer Science perspectives, respectively.

The classical approach to mechanism design focuses on the design of truthful mechanisms (i.e. mechanisms ensuring that reporting the truthful bid is always in the agents' interests) and on the efficiency of the truth-telling equilibria (Nisan & Ronen, 2001). Given that truthfulness is typically a very demanding property, an alternative and more recent approach consists in designing mechanism whose equilibrium performance is desirable (e.g., through performance metrics such as the price of anarchy and price of stability), regardless of whether or not they induce truthful bidding (Hartline et al., 2014; Roughgarden et al., 2017). In this manuscript, we focus on this latter approach, given its relevance to the overarching question presented. In this context, Christodoulou et al. (2016), Roughgarden (2015b) and Syrgkanis and Tardos (2013) were the first to show that the smoothness framework can be applied to provide equilibrium efficiency bounds, and that such bounds extend to more general notions of equilibrium. The majority of the results in strategic mechanism design consist of bounds on the price of anarchy for commonly studied mechanisms. For example, Roughgarden et al. (2017) and Syrgkanis and Tardos (2013) show that the first-price mechanism has price of anarchy of $1-1/e$ in simultaneous, multiple item auctions with additive valuations. Notably, smoothness bounds apply to equilibrium efficiency even within the Bayesian setting, where agents possess common priors on the system state. We refer to Hartline et al. (2014) and Roughgarden et al. (2017) for more details on the strategic approach to mechanism design.

Game-theoretic approaches to distributed control. Preliminary work highlighting how tools and ideas from game theory can be brought to bear on the development of distributed control algorithms initiated with (Arslan et al., 2007; Marden et al., 2009a). Since then, a host of works pursued similar directions, the most relevant of which are discussed in a recent review article (Marden & Shamma, 2018b) and book chapter (Marden & Shamma, 2018a). Most notably, Gopalakrishnan et al. (2011) and Marden and Shamma (2018a) introduce an architectural overview of the approach, decomposing the problem into a *utility design* and an *learning design* component. The first component is concerned with the design of utility functions ensuring that the resulting emergent behavior (e.g., Nash equilibrium) is desirable from the standpoint of the system designer. The second component focuses on the development of distributed algorithms coordinating the agents towards one such equilibrium configuration.

Utility design. The question of designing agents' utilities to ensure desirable emergent behavior was initially formulated in (Marden & Wierman, 2013), whereby the authors focus on separable resource allocation problems (e.g., sensor coverage) and model the emergent behavior through the notion of pure Nash equilibrium. Drawing inspiration from the literature of cost-sharing rules, they identify several utility design methodologies and discuss their properties with particular attention to the existence of equilibria, informational requirements, and tractability of computation.

Whilst many works have embraced this approach, the majority of the existing results focus on characterizing the performance (price of anarchy) of given design methodologies as opposed to their systematic optimization, with only a few exceptions (Bilò & Vinci, 2019; Caragiannis et al., 2010; Gairing, 2009; Gkatzelis et al., 2016; Marden & Wierman, 2013). Nevertheless, a series of works (Chandan et al., 2019; Paccagnan et al., 2019; Paccagnan & Marden, 2022) recently tackled and settled the utility design question for the case of separable resource allocation problems. Specifically, the authors provide a systematic methodology to design utilities that optimize the price of anarchy through the solution of tractable linear programs. Much less is known beyond the resource additive case, with recent results providing initial answers, for example, for interesting classes of submodular maximization problems (Sessa et al., 2019).

Finally, we observe that, in contrast to the design of utilities optimizing the price of anarchy, utilities optimizing the price of stability (best-case performance) are available for general classes of problems, as discussed in Section 4.

Learning design. The field of learning in games is concerned with how (if at all) rational agents converge to an equilibrium allocation, such as that of Nash, through a dynamic process whereby their choices are revised over time (see Marden and Shamma (2018a, 2018b)). Different agents' dynamics have been investigated in the years, and, by now, we have a comprehensive understanding of when learning happens (i.e., what assumptions and natural dynamics guarantee equilibrium convergence), and when it does not, see for example the books by Fudenberg et al. (1998) and Young (2004).

Interestingly, Hart and Mas-Colell (2003) establish that there exists no "natural" dynamics that converge to Nash equilibria in *all* games (here "natural" excludes dynamics relying on centralized coordination such as exhaustive search). Nevertheless, the answer is more reassuring when considering classes of games with specific structure, e.g., potential games. Within this class, equilibrium convergence is guaranteed for a variety of dynamics, most notably best-response dynamics (Rosenthal,

[2] The 2007 Nobel Memorial Prize in Economic Sciences was awarded to Hurwicz, Maskin and Myerson "for having laid the foundations of mechanism design theory".

1973), as well as fictitious play (whereby agents use the empirical frequency of other agents' past play to guide their decision) and corresponding variations where the observational requirements are further reduced (Berger, 2007; Marden et al., 2009b, 2009).

When the equilibrium notion is weakened and coarse correlated equilibria are considered, convergence can be guaranteed in general settings employing classical algorithms such as no-regret dynamics (Cesa-Bianchi & Lugosi, 2006; Hart & Mas-Colell, 2000; Roughgarden, 2015a) or the celebrated "Ellipsoid against Hope" (Jiang & Leyton-Brown, 2015; Papadimitriou & Roughgarden, 2008). From a computational perspective, while pure and mixed Nash equilibria are intractable to compute in the worst-case (formally, PLS and CLS complete (Daskalakis & Papadimitriou, 2011; Fabrikant et al., 2004)), coarse correlated equilibria can be computed in polynomial time in the size of the problem input (Jiang & Leyton-Brown, 2015; Papadimitriou & Roughgarden, 2008).

Complexity barriers to optimal utilities. From a system-level perspective, the ideal allocation corresponds to a solution of $\min_{a\in\mathcal{A}} C(a)$. Nevertheless, many of the system-level optimization problems considered are intractable to solve, even when assuming the presence of a centralized decision-maker with complete information and control over the agents. For example, minimizing the congestion in affine congestion games (a prototypical model used to describe incentive design in traffic routing as in Example 1) is NP-hard to approximate within a factor of 2 (see Paccagnan and Gairing (2021)). As a result, hardness of approximation imposes an *inherent barrier* on what can be achieved through the design of utilities whenever the equilibrium notion considered can be computed efficiently. For example, in affine congestion games, we should not expect being able to design tractable incentives yielding a price of anarchy below 2 for coarse correlated equilibria. If this were the case, we would have obtained a tractable approach (determine incentives and compute an equilibrium of the incentivized game) breaching the hardness barrier — a contradiction.

Connection to non-oblivious local search

The design of agents' utilities ensuring desirable system behavior is intimately connected with a recently developed approach in combinatorial optimization, referred to as *non-oblivious local search* (Filmus & Ward, 2014; Ward, 2012). Non-oblivious local search is concerned with the design of approximation algorithms for combinatorial optimization problems of the form $\min_{a\in\mathcal{A}} C(a)$, whereby algorithms are guided by an auxiliary function distinct from the problem's objective (thus the name "non-oblivious"). Whenever the underlying optimization problems includes the presence of multiple decision makers, i.e., the system-level cost takes the form $C(a_1,\ldots,a_n)$, this approach is equivalent to the design of agents' utilities studied here.

To see this, note that non-oblivious local search focuses on the design of a modified potential function $\Phi : \mathcal{A} \to \mathbb{R}$ so that a locally optimum allocation with respect to Φ, i.e., an allocation a^* such that $\Phi(a^*) \le \Phi(a_i, a^*_{-i})$ for all a_i and i, has best possible approximation with respect to the original system-level cost C (Ward, 2012). One can then interpret Φ as the agents' common utility in a so-called "common interest" game, and observe that the notion of "locally optimum allocation" is identical to that of pure Nash equilibrium in potential games, as in this setting $\Phi(a^*)-\Phi(a_i,a^*_{-i}) \le 0$ for all a_i and i is equivalent to the definition of Nash equilibrium for a^*. Therefore, designing the potential $\Phi(\cdot)$ to attain the best possible approximation (non-oblivious local search) is equivalent to designing utilities in a potential game so as to optimize the price of anarchy (utility design). A concrete example is given by the results from Gairing (2009) and Filmus and Ward (2012) which are, mutatis mutandis, equivalent.

2. Equilibria, and performance metrics

In this section, we recall fundamental game-theoretic concepts, revise commonly employed notions describing the emergent behavior, and formally introduce the performance metrics used throughout.

We begin by recalling the notion of cost minimization game. In a cost minimization game, we are given a set of agents $N = \{1,\ldots,n\}$, whereby each agent $i \in N$ is given an action set $\mathcal{A}_i$, and a corresponding cost function $C_i : \mathcal{A} \to \mathbb{R}_{\ge 0}$, where $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$. Furthermore, we assume the presence of a system level cost function $C : \mathcal{A} \to \mathbb{R}_{\ge 0}$. The most commonly studied system-level cost function is that of social welfare, $C(a) = \sum_{i\in N} C_i(a)$; however, in this manuscript, we consider arbitrary system-level cost functions. We refer to one such game as $G = \{N, \{\mathcal{A}_i\}_{i\in N}, \{C_i\}_{i\in N}, C\}$.

Equilibrium notions. A variety of solution concepts is commonly employed to describe the emergent behavior arising from the presence of rational decision-makers. The most widely known is that of pure Nash equilibrium.

Definition 1. Given a strategic game G, an action profile $a^{\text{ne}} \in \mathcal{A}$ is a pure Nash equilibrium if for all agents $i \in N$

$$C_i(a^{\text{ne}}) \le C_i(a_i, a^{\text{ne}}_{-i}) \quad \forall a_i \in \mathcal{A}_i,$$

where $a_{-i} = (a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$ denotes the actions of all agents but i.

Informally, a Nash equilibrium is an action profile where no agent can decrease their cost through unilateral deviations. While pure Nash equilibria are commonly employed, their existence is not always guaranteed. Even when this is the case, they suffer from their computational complexity (Daskalakis et al., 2009). Moving beyond, mixed Nash equilibria are a natural extension of pure Nash equilibria to scenarios where the agents can select their actions probabilistically. Despite guaranteed existence, mixed Nash equilibria also suffer from similar computational issues (Daskalakis & Papadimitriou, 2011). Nevertheless, coarse correlated equilibria (a weaker notion of pure and mixed Nash equilibria) are guaranteed to exist and can be computed in polynomial time (Hart & Mas-Colell, 2000; Jiang & Leyton-Brown, 2015; Papadimitriou & Roughgarden, 2008). Let $\Delta(\mathcal{A})$ denote the simplex over the finite set $\mathcal{A}$. A coarse correlated equilibrium is characterized by a joint distribution $\sigma \in \Delta(\mathcal{A})$ that satisfies the following condition.

Definition 2. Given a strategic game G, a joint distribution $\sigma \in \Delta(\mathcal{A})$ is a coarse correlated equilibrium if for all $i \in N$

$$\mathbb{E}_{a\sim\sigma}[C_i(a)] \le \mathbb{E}_{a\sim\sigma}[C_i(a'_i, a_{-i})] \quad \forall a'_i \in \mathcal{A}_i.$$

In other words, a joint distribution is a coarse correlated equilibrium if no agent can decrease the expected cost by unilaterally deviating to a pure action. In addition to its guaranteed existence, coarse correlated equilibria can be efficiently computed through simple learning algorithms. We refer the interested reader to Jiang and Leyton-Brown (2015), Papadimitriou and Roughgarden (2008), Roughgarden (2015a) and references therein.

Example 4 (*Equilibrium Notions*)**.** Consider a game with two agents. Each agent can select the action Stop or Go. Given an action profile, i.e., a choice of actions for both agents, the row agent (resp. column agent) incurs a cost equal to the first entry (resp. second entry) in the following matrix.

	Stop	Go
Stop	$(1,1)$	$(1,0)$
Go	$(0,1)$	$(2,2)$

The game has two pure Nash equilibria. Indeed, (Stop, Go) is a pure Nash equilibrium since the row player cannot decrease the cost (currently 1) by moving to the second row where it would equal 2. Analogously, for the column player when moving to the first column. Following an identical reasoning, we note that (Go, Stop) is also a pure Nash equilibrium. Finally, observe that the game has a continuum of coarse correlated equilibria. Indeed, any distribution placing zero mass on (Stop, Stop), (Go, Go) while placing mass σ_1, σ_2 on actions (Stop, Go) and (Go, Stop) is a coarse correlated equilibrium. To see this, note that the row agent's expected cost equals $1 \cdot \sigma_1 + 0 \cdot \sigma_2$. At the same time, if she was to deviate to a deterministic action corresponding to the first row (resp. second row), she would incur a cost equal to 1 (resp. $2 \cdot \sigma_1 + 0 \cdot \sigma_2$). The latter is no lower than $1 \cdot \sigma_1 + 0 \cdot \sigma_2$. Similarly for the column agent.

Performance metrics. We are now concerned with substantiating the notion of "desirable system behavior" introduced earlier. Specifically, we will investigate two commonly employed performance metrics often referred to as the *price of anarchy* and the *price of stability*. While these metrics are relative to the equilibrium notion considered (e.g., pure or mixed Nash equilibrium, coarse correlated equilibrium, etc.), we begin by focusing on pure Nash equilibria for simplicity.

The first metric of interest, referred to as the price of anarchy, measures the *worst-case* ratio between the system-level cost at a Nash equilibrium and an optimal allocation. For a given game G,

$$\mathrm{PoA}(G) = \sup_{a^{\mathrm{ne}} \in \mathrm{NE}(G)} \frac{C(a^{\mathrm{ne}})}{\mathrm{Opt}(G)} \geq 1, \tag{1}$$

where $\mathrm{NE}(G)$ denotes the set of pure Nash equilibria of G and $\mathrm{Opt}(G) = \min_{a \in \mathcal{A}} C(a)$ the minimum system-level cost. To ensure that $\mathrm{PoA}(G)$ is well-defined, we will assume that $\mathrm{Opt}(G)$ is strictly positive. Observe that, by definition, $\mathrm{PoA}(G) \geq 1$, and the smaller the price of anarchy, the better performance guarantees we can provide for *all* pure Nash equilibria. Finally, we observe that the price of anarchy provides a multiplicative bound on the cost incurred at any Nash equilibrium allocation with respect to the minimum cost, i.e., $C(a^{\mathrm{ne}}) \leq \mathrm{PoA}(G) \cdot C(a^{\mathrm{opt}})$, similarly to the notion of approximation ratio in combinatorial optimization.

The second metric we consider is a more optimistic variant referred to as the price of stability. It measures the *best-case* ratio between the system-level cost at a Nash equilibrium, and an optimal allocation. For a given game G,

$$\mathrm{PoS}(G) = \inf_{a^{\mathrm{ne}} \in \mathrm{NE}(G)} \frac{C(a^{\mathrm{ne}})}{\mathrm{Opt}(G)} \geq 1.$$

Observe that, by definition, $\mathrm{PoS}(G) \leq \mathrm{PoA}(G)$. Contrary to the price of anarchy, the price of stability provides performance certificates limited to the best-performing equilibrium. In addition to its more optimistic perspective on the emergent equilibrium allocation, the price of stability is relevant whenever simple dynamics guarantee convergence to specific types of equilibrium, e.g., the best Nash equilibrium, see Marden and Shamma (2012).

Finally, when we are interested in providing performance guarantees against a family of games $\mathcal{G}$, we extend the definitions of price of anarchy and price of stability (with slight abuse of notation) as follows

$$\mathrm{PoA}(\mathcal{G}) = \sup_{G \in G} \mathrm{PoA}(G), \quad \mathrm{PoS}(\mathcal{G}) = \sup_{G \in \mathcal{G}} \mathrm{PoS}(G). \tag{2}$$

While it is possible to introduce the above metrics with respect to all equilibrium class mentioned, we do not pursue this direction for two reasons. First, this allows to simplify the exposition. Second, in many cases, the price of anarchy (resp. price of stability) over pure, mixed Nash and coarse correlated equilibria will coincide, as we clarify later.

3. Smoothness: a recipe for price of anarchy and price of stability bounds

In this section we introduce a methodology, often referred to as the *smoothness* framework, for providing bounds on the price of anarchy and stability. Its relevance to our agenda stems from the fact that designing optimal utilities crucially hinges on the ability to characterize such performance metrics for *given* choices of utilities. For this purpose, we first consider the problem of *quantifying* the price of anarchy and price of stability.

Price of anarchy. Since the notions of price of anarchy and price of stability were introduced in 1999 (Koutsoupias & Papadimitriou, 1999) and in 2008 (Anshelevich et al., 2008), respectively, there has been considerable interest in quantifying these performance metric for various classes of games. While most of the approaches had initially leveraged domain specific analysis, a successful attempt in providing a unifying framework for characterizing the price of anarchy came with the introduction of the *smoothness* approach in 2009 (Roughgarden, 2015a). Since then, the approach has been adapted to address also the price of stability (Bilò, 2018; Chandan et al., 2021a).

Definition 3 (*Smoothness*)**.** A cost minimization game G is (λ, μ)-smooth if the following conditions hold:

C1. $C(a) \leq \sum_{i=1}^n C_i(a)$ for all $a \in \mathcal{A}$;
C2. There exist $\lambda > 0$ and $\mu < 1$ such that

$$\sum_{i=1}^n C_i(a_i', a_{-i}) \leq \lambda C(a') + \mu C(a) \quad \forall a, a' \in \mathcal{A}. \tag{3}$$

Informally, a game is (λ, μ)-smooth if, in addition to C1, it is possible to provide on upper-bound on the term $\sum_{i=1}^n C_i(a_i', a_{-i})$ as a linear combination of $C(a')$ and $C(a)$, for all allocations a and a'. When a game G is (λ, μ)-smooth, its price of anarchy is readily upper bounded by (see Roughgarden (2015a))

$$\mathrm{PoA}(G) \leq \frac{\lambda}{1 - \mu}. \tag{4}$$

This result naturally extends to a class of games $\mathcal{G}$ when the smoothness conditions holds for each and every game in $\mathcal{G}$.[3] The proof (see Footnote 3) highlights how the desired bound on the price of anarchy follows almost directly from the very definition of (λ, μ)-smoothness, thus suggesting that determining parameters (λ, μ) satisfying condition C2 might not be much simpler than computing the price of anarchy from the ground up. Naturally, one can optimize the resulting upper bound by searching for the best-possible parameters. For a given class of games $\mathcal{G}$, such best-achievable bound is referred to as the Robust Price of Anarchy (Roughgarden, 2015a),

$$\mathrm{RPoA}(\mathcal{G}) := \inf_{\lambda > 0,\ \mu < 1} \left\{ \frac{\lambda}{1 - \mu} \text{ s.t. Eq. (3) holds } \forall G \in \mathcal{G} \right\}. \tag{5}$$

Observe that $\mathrm{RPoA}(\mathcal{G})$ is only an upper bound on $\mathrm{PoA}(\mathcal{G})$, and that, in general, such bound could be not tight (i.e., $\mathrm{PoA}(\mathcal{G}) < \mathrm{RPoA}(\mathcal{G})$ may hold).

Whilst the smoothness framework has underpinned many exciting results, it suffers from a number of drawbacks: (i) the requirement in C1 limits the applicability of the approach; (ii) the bounds obtained are not tight in general, i.e., $\mathrm{PoA} \neq \mathrm{RPoA}$; and (iii) as a consequence of the previous item, smoothness often provides inaccurate indications regarding the design of optimal utilities (i.e., utilities designed to optimize RPoA often do not optimize the true PoA). Motivated by these issue, Chandan et al. (2019) introduced the notion of *generalized*

[3] Proving the statement in (4) is educational, and can be carried out in one line for pure Nash equilibria: $C(a^{\mathrm{ne}}) \leq \sum_{i=1}^n C_i(a^{\mathrm{ne}}) \leq \sum_{i=1}^n C_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) \leq \lambda C(a^{\mathrm{opt}}) + \mu C(a^{\mathrm{ne}})$ where the chain of inequalities holds thanks to C1, the definition of Nash equilibrium, and C2, respectively.

smoothness, which we report next maintaining the (λ,μ) notation for consistency.

Definition 4 (*Generalized Smoothness*). A cost minimization game G is (λ,μ)-generalized smooth if there exist $\lambda>0$ and $\mu<1$ such that

$$\sum_{i=1}^{n} C_i(a_i',a_{-i})-\sum_{i=1}^{n} C_i(a)+C(a)\le \lambda C(a')+\mu C(a)\quad \forall a,a'\in\mathcal{A}. \tag{6}$$

Notably, generalized smoothness and smoothness coincide when $C(a)=\sum_{i=1}^{n} C_i(a)$. Additionally, following the same line of argument in Footnote 3, the price of anarchy of a (λ,μ)-generalized smooth game is upper bounded by $\lambda/(1-\mu)$. In the same spirit as above, we introduce $\mathrm{GPoA}(\mathcal{G})$ as the best attainable bound on the price of anarchy using the generalized smoothness approach. The latter is defined as in (5) whereby Eq. (3) is replaced with (6).

At first, generalized smoothness appears a simple modification of the original definition. Nevertheless, such modification is sufficient to address the points highlighted above. Most notably, generalized smoothness always yields provably tighter bounds, i.e., $\mathrm{GPoA}(\mathcal{G})\le\mathrm{RPoA}(\mathcal{G})$, see Chandan et al. (2019). Finally, similarly to smoothness, price of anarchy bounds obtained through generalized smoothness extend from pure Nash to mixed Nash and coarse correlated equilibria. The next theorem summarizes these results.

Theorem 1 (*Smoothness for Price of Anarchy*). *Let G be a (λ,μ)-generalized smooth game. Then*

(i) $PoA(G)\le\lambda/(1-\mu)$;
(ii) The bound in (i) extends to mixed Nash and coarse correlated equilibria.

A bird's eye view on the origin of smoothness

While many of the original attempts at deriving price of anarchy bounds seemed to utilize ad-hoc machineries, Roughgarden (2015a) realized that such techniques can be unified through a common framework, which he put forward and called *smoothness*. In this context, the smoothness conditions in (3) and (4) might seem difficult to grasp, and their origin might appear related to an "abstraction exercise" rather than anchored on a theoretical foundation. Here, we show that this is *not* the case. Instead, the smoothness conditions can be reinterpreted as the constraints of a natural dual program connected to the very definition of the price of anarchy. Complimentary to this, we will also clarify why price of anarchy bounds derived thorough the smoothness framework extend to coarse correlated equilibria.

Towards this goal, we fix a game G, and consider its price of anarchy over coarse correlated equilibria, i.e.,

$$\mathrm{PoA}(G)=\sup_{\sigma\in\mathrm{CCE}(G)}\frac{\sum_{a\in\mathcal{A}}\sigma_a C(a)}{C(a^{\mathrm{opt}})},$$

where $\mathrm{CCE}(G)$ denotes the set of coarse correlated equilibria of G, and σ_a is the probability associated to action $a\in\mathcal{A}$. It is immediate to observe that the following set of distributions $\sigma\in\Delta(\mathcal{A})$ is a superset of $\mathrm{CCE}(G)$

$$\mathrm{CCE}^{+}(G)=\left\{\sigma\in\Delta(\mathcal{A})\ \text{s.t.}\sum_{a\in\mathcal{A}}\sum_{i}\sigma_a[C(a)-C(a_i^{\mathrm{opt}},a_{-i})]\le 0\right\},$$

since we summed the coarse correlated equilibrium conditions over the players, and verify only deviations $a_i'=a_i^{\mathrm{opt}}$. Therefore,

$$\mathrm{PoA}(G)\le\sup_{\sigma\in\mathrm{CCE}^{+}(G)}\frac{\sum_{a\in\mathcal{A}}\sigma_a C(a)}{C(a^{\mathrm{opt}})}. \tag{7}$$

Note that the right hand side in (7) defines a linear program in the unknown σ, since the set CCE^{+} is represented as a polytope and the cost function is linear in σ. We are interested in taking the dual of this program. Before doing so, we equivalently divide the equilibrium inequality appearing in the definition of CCE^{+} by $C(a^{\mathrm{opt}})>0$. By strong duality (Bertsekas et al., 2003), the price of anarchy is then upper bounded by the value of the following program

$$\begin{aligned}&\inf_{\rho\in\mathbb{R},\nu\ge 0}\ \rho\\ &\text{s.t.}\quad C(a^{\mathrm{opt}})-\rho C(a)+\nu\sum_{i=1}^{n}[C_i(a)-C_i(a_i^{\mathrm{opt}},a_{-i})]\ge 0\\ &\qquad\qquad\forall a\in\mathcal{A},\end{aligned}$$

where we associated the Lagrange multiplier ρ to the constraint $\sum_{a\in\mathcal{A}}\sigma_a=1$, and ν to the equilibrium condition. It is now immediate to observe that the definition of $\mathrm{GPoA}(G)$ coincides with the above program, whereby the inequality is now required to hold for all allocations a' and not only for the optimal one a^{opt}.[a] To recover the variables (λ,μ), simply use the following change of coordinates $\nu=1/\lambda$ and $\rho=(1-\mu)/\lambda$, see also the forthcoming (10).

[a] Strengthening the inequality this way is natural as the optimal allocation is typically not known or difficult to compute.

Price of stability. Moving beyond the price of anarchy, smoothness-like approaches have also been employed to bound the price of stability. Whilst the price of stability is a more optimistic variant of the price of anarchy (we have seen that $\mathrm{PoS}(\mathcal{G})\le\mathrm{PoA}(\mathcal{G})$), its characterization can be *significantly* more challenging. Indeed, determining the price of stability of given utilities requires performing a worst-case analysis of the *best-performing* equilibrium. As such, the analysis calls for an "equilibrium selection" argument allowing to single out a good equilibrium (ideally the best) over each instance G before taking the worst-case over the entire class $\mathcal{G}$. This is a nontrivial task in general.

Nevertheless, for specialized classes of games, such as potential games,[4] it is possible to provide a general approach. In this context, it is common to utilize the potential function to single out a specific equilibrium. Indeed, since the potential minimizer is guaranteed to be a Nash equilibrium (Rosenthal, 1973), the worst-case over best-performing equilibria is replaced by the worst-case over potential minimizers. Building upon this idea allows to develop the following smoothness-like argument (Chandan et al., 2021a).

Theorem 2 (*Smoothness for Price of Stability (Chandan et al., 2021a)*). *Let $\mathcal{G}$ be a set of potential games with n agents. Suppose there exist $\zeta>0$, $\lambda>0$ and $\mu<1$ such that, for every game $G\in\mathcal{G}$,*

$$\begin{aligned}C(a)+\sum_{i=1}^{n}[C_i(a_i',a_{-i})-C_i(a)]+\zeta[\Phi(a')-\Phi(a)]\\ \le\lambda C(a')+\mu C(a)\end{aligned} \tag{8}$$

for all $a,a'\in\mathcal{A}$. Then, $PoS(\mathcal{G})\le\lambda/(1-\mu)$.

The idea behind the latter condition consists in strengthening the inequality appearing in the definition of generalized smoothness (used to bound the price of anarchy in Section 3) by adding the contribution $\Phi(a')-\Phi(a)$. A different perspective clarifying why this term is added to the inequality can be obtained following a similar procedure to that in the previous panel whereby the search is restricted to equilibria whose

[4] A game G is potential if there exists a single real-valued function $\Phi:\mathcal{A}\to\mathbb{R}$ such that $J_i(a_i,a_{-i})-J_i(a_i',a_{-i})=\Phi(a_i,a_{-i})-\Phi(a_i',a_{-i})$ for all pairs a,a' and all agents i.

potential is lower than that at the optimum. The corresponding dual program will feature the constraint appearing in (8).

Once again, the best upper bound on the price of stability obtained through this approach is given by

$$\mathrm{GPoS}(\mathcal{G}) := \inf_{\zeta>0,\ \lambda>0,\ \mu<1} \left\{ \frac{\lambda}{1-\mu} \text{ s.t. Eq. (8) holds } \forall G \in \mathcal{G} \right\}. \tag{9}$$

Having replaced best-performing equilibria with potential-minimizing equilibria, one would expect GPoS to be an inexact bound on the price of stability, i.e., $\mathrm{PoS}(\mathcal{G}) \leq \mathrm{GPoS}(\mathcal{G})$, perhaps even strictly. Nevertheless for many important classes of problems (e.g., unincentivized congestion games), $\mathrm{PoS}(\mathcal{G}) = \mathrm{GPoS}(\mathcal{G})$, see, e.g., Chandan et al. (2021a) and Christodoulou and Gairing (2015).

Analytical and computational perspective to smoothness. While smoothness and generalized smoothness provide solid ground to bound the price of anarchy, their concrete utilization brings forward a number of challenges, most notably in deriving good estimates of the parameters (λ, μ). Towards this goal, two approaches are typically pursued.

A first approach, followed by many precursors of smoothness (Aland et al., 2011; Awerbuch et al., 2013; Christodoulou & Koutsoupias, 2005b; Gairing, 2009; Suri et al., 2007), consists in leveraging the specific structure of the agents' cost functions to derive analytical values on admissible pairs (λ, μ), thus obtaining upper bounds on the price of anarchy. Lower bounds, are then derived by constructing a "bad" game instance. Following this approach, often requires refined arguments to ensure matching upper and lower bounds. One class of problems where this technique is fruitfully employed is that of auctions, presented in Section 5.

A second approach is based on the use of computational techniques to determine the best possible parameters (λ, μ). Still, solving for (5) is often computationally demanding. Even when the class $\mathcal{G}$ contains a single instance, (5) corresponds to a linear program with exponentially many constraints (one per allocation pair). The situation worsens when $\mathcal{G}$ contains infinitely many games as often the case, or when we are interested in designing optimal utilities. In the latter case, the number of decision variables (the utilities) is also exponential. To see how RPoA (equivalently GPoA) corresponds to a linear program of exponential size even for a single instance G, note that the definition of RPoA can be rewritten as follows

$$\begin{aligned} \rho^{\mathrm{opt}} = &\sup_{\nu\geq 0,\ \rho\in\mathbb{R}} \quad \rho \\ \text{s.t.}\quad & C(a') - \rho C(a) + \nu\left[\sum_{i=1}^{n} C_i(a) - \sum_{i=1}^{n} C_i(a_i', a_{-i})\right] \geq 0 \\ & \qquad\qquad \forall a, a' \in \mathcal{A}, \end{aligned} \tag{10}$$

where we made use of the following change of variables $\nu = 1/\lambda$ and $\rho = (1-\mu)/\lambda$. Fortunately, when games can be *concisely parametrized* such curse of dimensionality can be avoided. Incentive design in static traffic routing can be studied employing this framework as presented in Section 4.

4. Resource allocation with congestion

In this section, we introduce a commonly employed class of resource allocation problems and aim at designing utilities optimizing the price of anarchy or the price of stability.

In the model considered, a set of agents needs to select subsets of a set of resources, whereby each resource is associated to a cost depending on the number of agents concurrently choosing it (informally, the load). A prototypical example is that of road-traffic routing where resources corresponds to edges in the road network, and resource costs describe the edges' travel times. A different example, pertaining to welfare maximization problems, is given by weighted set covering problems (Gairing, 2009). These classes of problems can be described through the well-known congestion game model (Rosenthal, 1973).

In a congestion game we are given a set of agents N, and a set of resources $\mathcal{R}$. Each agent must choose a subset of the set of resources $a_i \in \mathcal{A}_i \subseteq 2^{\mathcal{R}}$. The cost for using each resource $r \in \mathcal{R}$ depends only on the total number of agents concurrently selecting that resource, and is denoted with $f_r : \mathbb{N} \to \mathbb{R}_{\geq 0}$, We refer to f_r as to the *agents' resource cost*. Once all agents have made a choice $a_i \in \mathcal{A}_i$, each of them incurs a cost obtained by summing the costs of all resources she selected

$$C_i(a) = \sum_{r\in a_i} f_r(|a|_r), \tag{11}$$

where $|a|_r$ denotes the number of players selecting resource r in allocation $a = (a_1, \dots, a_n)$. Similarly, the system-level cost also depends on the number of agents currently selecting the resources through the function $c_r : \mathbb{N} \to \mathbb{R}_{\geq 0}$, referred to as *system resource cost*,

$$C(a) = \sum_{r\in a} c_r(|a|_r). \tag{12}$$

Example 5 (*Traffic Routing*). Within the context of traffic routing, $\mathcal{R}$ describes the set of edges representing the road network. Each agent $i \in N$ selects a path connecting her origin to her destination node. Here $\mathcal{A}_i$ contains all feasible paths for agent i. The travel time incurred when transiting on edge $r \in \mathcal{R}$ is captured by the function ℓ_r, and depends solely on the number of users traveling through that very edge. The system cost describes the time spent on the network by *all* agents, i.e., $C(a) = \sum_{i\in N} \sum_{r\in a_i} \ell_r(|a|_r)$, which can be equivalently written as $C(a) = \sum_{r\in a} |a|_r \ell_r(|a|_r)$, thus corresponding to the choice of $c_r(x) = x\ell_r(x)$. When no incentives are employed, the agents' costs capture their individual travel time, thus taking the form in Eq. (11) with $f_r(x) = \ell_r(x)$. Similarly, when tolls are imposed on the network, i.e., each edge is associated to a congestion-dependent tolling function τ_r, the agents' cost accounts for both their travel time and the tolls levied along their route, thus corresponding to $f_r(x) = \ell_r(x) + \tau_r(x)$.

In congestion games, the system cost $C(a)$ depends only on the total load on the resources, while $C_i(a)$ additionally depends on the resources chosen by agent i. These properties allow congestion games to be *concisely parametrized*. Specifically, these properties are key to (i) tame the curse of dimensionality appearing in (10) by allowing a transformation of the optimization problem defining GPoA to a linear program of *polynomial* size, and to (ii) show that GPoA = PoA. The result is a simple recipe for determining the price of anarchy as the solution of a tractable linear program.

Before presenting this result, we introduce the notation used in the remainder of this section. We denote with $\mathcal{I}(n)$ the set of integer triplets (x, y, z) on the boundary of the set $\{(x, y, z) \in \mathbb{R}^3_{\geq 0} \text{ s.t. } 1 \leq x + y + z \leq n\}$, see Fig. 3. Additionally, we let $\mathcal{G}$ be the class of congestion games with n agents, where the agents and the system resource costs are obtained by non-negative combination of given basis functions. That is, where $c_r(x) = \sum_{j=1}^{m} \alpha_r^j c^j(x)$, $\alpha_r^j \geq 0$, and $f_r(x) = \sum_{j=1}^{m} \alpha_r^j f^j(x)$, given $c^1, \dots, c^m$ and $f^1, \dots, f^m$. This structural assumption is classical and adopted in the vast majority of the literature. For example, polynomial congestion games correspond to employing polynomial basis functions, see Paccagnan et al. (2021) for an extended discussion. We are now ready to state the first main result.

Theorem 3 (*PoA as a Tractable LP (Chandan et al., 2019)*). *$\mathrm{PoA}(\mathcal{G}) = \mathrm{GPoA}(\mathcal{G}) = 1/\rho^{\mathrm{opt}}$, where ρ^{opt} is the value of the following linear program*

$$\begin{aligned} &\underset{\nu\geq 0,\ \rho\in\mathbb{R}}{\mathit{maximize}} \quad \rho \\ &\mathit{subject\ to:} \\ &c^j(x+z) - \rho c^j(x+y) + \nu[f^j(x+y)y - f^j(x+y+1)z] \geq 0, \\ &\qquad\qquad \forall j = 1, \dots, m, \quad \forall (x,y,z) \in \mathcal{I}(n), \end{aligned} \tag{13}$$

and we set $c^j(0) = f^j(0) = f^j(n+1) = 0$.

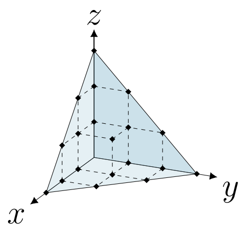

Fig. 3. The diamonds identify all points belonging to the set $\mathcal{I}(n)$. In this figure, $n = 3$.

The previous result is interesting for at least two reasons. First, it provides a mechanistic view on computing the price of anarchy: given agents' and system resource costs, simply solve a linear program of polynomial size. For this purpose, Matlab and Python code formulating and solving these programs are available in Chandan et al. (2013). In a similar spirit, we observe that Theorem 3 recovers and unifies a variety of existing results on the price of anarchy (see Chandan et al. (2019) for a discussion), including classical results on the exact price of anarchy in polynomial congestion games (Aland et al., 2011; Awerbuch et al., 2013; Christodoulou & Koutsoupias, 2005b). Second, Theorem 3 serves as a building block to resolve the utility design question too, both for the case where resource costs need to be designed only using local information and for the case where they can be designed using global information. Intuitively, optimizing GPoA is equivalent to optimizing PoA since GPoA = PoA. The remainder of this section addresses this topic.

Optimal utilities based on local information. In congestion games, given system resource costs $\{c_r\}_{r\in\mathcal{R}}$, the utility design problem reduces to that of designing agents' resource costs $\{f_r\}_{r\in\mathcal{R}}$ optimizing the price of anarchy or stability.

In this section, we focus on the design of agents' resource costs using *local information* only. That is, we constrain the design of the agent resource cost f_r to using information regarding the corresponding system resource cost c_r only. No other information about the instance G at hand can be employed. This informational constraint is motivated by real-world scenarios where privacy, scalability, and robustness are paramount. Indeed, designing agents' resource costs employing global information (such as knowledge of all feasible sets $\mathcal{A}_i$, or all resource costs) is impractical, as it requires the system designer to access private information and is often associated to limited robustness (e.g., within the context of traffic routing, a design based on global information needs to be updated even when a single agent changes destination).

When concerned with optimizing the price of anarchy, one can exploit the program in Theorem 3, allowing agents' resource costs to become decision variables. This observation, jointly with a result on the structure of optimal agents' resource costs, allows to settle the question completely.

Theorem 4 (*Local Utilities Optimizing PoA* (*Paccagnan et al., 2021*)). *Agents' resource costs (based on local information) minimizing the price of anarchy over congestion games with n agents and system resource costs $c_r(x) = \sum_{j=1}^m \alpha_r^j c^j(x)$, are given by $f_r^{\text{opt}}(x) = \sum_{j=1}^m \alpha_r^j f^j(x)$ where each pair (f^j, ρ^j) solves the following linear program of polynomial size:*

$$\begin{aligned}
&\underset{f\in\mathbb{R}^n,\rho\in\mathbb{R}}{\textit{maximize}} \quad \rho \\
&\textit{subject to:} \\
&c^j(x+z) - \rho c^j(x+y) + f(x+y)y - f(x+y+1)z \geq 0, \\
&\qquad\qquad\qquad\qquad \forall (x,y,z) \in \mathcal{I}(n).
\end{aligned} \tag{14}$$

The corresponding optimal price of anarchy is

$$PoA^{\text{opt}}(\mathcal{G}) = \max_{j\in\{1,\dots,m\}} \frac{1}{\rho^j}.$$

The previous statement contains two important results. First, it shows that the optimal resource cost f_r is obtained by taking the linear combination of f^j with the *same* coefficients defining the corresponding system resource cost c_r (structural property). Second, it provides a concrete approach to compute f^j by solving a tractable linear program.

We now turn the attention to the price of stability, whose computation is in general particularly challenging as discussed in Section 3. However, a simple design based on local information – referred to as marginal cost mechanism – achieves the best-possible price of stability.

Theorem 5 (*Utilities Optimizing PoS*)**.** *Agents' resource cost minimizing the price of stability over congestion games with n agents and system resource costs $c_r(x) = \sum_{j=1}^m \alpha_r^j c^j(x)$, are given by $f_r^{\text{opt}}(x) = \sum_{j=1}^m \alpha_r^j f^j(x)$ where*

$$f^j(x) = c^j(x) - c^j(x-1), \quad \forall j \in \{1,\dots,m\}, \forall x \in \mathbb{N},$$

and $f^j(0) = c^j(0) = 0$. The resulting price of stability is 1.

The above result is well-known (see e.g., (Chandan et al., 2021a)) and shows that a simple design approach yields the best-possible price of stability. In other words, this design choice ensures that an optimal allocation a^{opt} is always an equilibrium of the game. We note that such result extends to games with very general structure. Indeed, the marginal cost mechanisms always guarantees a price of stability of 1. However, as shown in Paccagnan et al. (2021), the price of anarchy guarantees of the marginal cost mechanism can be much worse than when no incentive is used.

Optimal utilities based on global information. In this section we are interested in a tractable design methodology, although we do not impose any limitation on what piece of information is utilized in such process. Therefore, at least in principle, agents' resource cost can be designed utilizing complete knowledge of the instance G at hand. We focus solely on the price of anarchy, as the design in Theorem 5 cannot be improved even when using global information since it already attains $\text{PoS}(\mathcal{G}) = 1$.

In this context, the problem of designing optimal utilities minimizing the price of anarchy was recently solved in Paccagnan and Gairing (2021). Their approach is based on defining a parametrized class of agents' resource costs, whose set of parameters is determined through the solution of a convex and tractable optimization problem. Specifically, let $p^j(v) = \mathbb{E}_{Y\sim\text{Poi}(v)}[c^j(Y)]$ and consider agents' resource cost of the form $f_r(x,v) = \sum_{j=1}^m \alpha_r^j f^j(x,v)$, where

$$f^j(x,v) = \frac{(x-1)!}{v^x} \sum_{i=0}^{x-1} \frac{p^j(v) - c^j(i)}{i!} v^i,$$

and parameters $\{v_r\}_{r\in\mathcal{R}}$ are computed solving the continuous relaxation of $\min_{a\in\mathcal{A}} \tilde{C}(a)$ where $\tilde{C}$ is defined as C upon replacing $c^j(x)$ with $p^j(x)$. The following holds.

Theorem 6 (*Global Utilities Optimizing PoA* (*Paccagnan & Gairing, 2021*))**.** *The agents' resource costs defined above achieve a price of anarchy equal to $\max_j PoA_j$, where*

$$PoA_j = \sup_{x\in\mathbb{N}} \frac{\mathbb{E}_{P\sim\text{Poi}(x)}[c^j(P)]}{c^j(x)}.$$

No polynomial time algorithm can provide a better approximation, unless P = NP.

The theorem contains two statements. The first one provides a tractable methodology to determine agents' resource costs. The second shows that the design proposed is optimal in a strong sense. First, no other agents' resource cost derived efficiently (in polynomial time) can provide better price of anarchy values. Further, and more fundamental, no other tractable approach regardless of whether is it based on the utility design framework studied here, or any other principle, can improve upon this result.

5. Mechanism design in auctions

In this section, we focus on mechanism design in auctions and review a number of results pertaining to the performance attained by classical designs, e.g., first-price, second-price, and all-pay auctions. Throughout, we highlight the connections between mechanism design and utility design, and illustrate how the smoothness framework provides, once again, valuable insight.

Auctions with additive valuations. An auction with additive valuations consists of a set of n agents (or bidders) $N = \{1, \dots, n\}$ and a set of m indivisible items. Each agent $i \in N$ has private values $v_{i,j} \geq 0$ for every item j and a set of possible actions $\mathcal{A}_i$, where each action $a_i \in \mathcal{A}_i$ contains bids $a_{i,j}$ for every item j. We let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ denote the joint action space. A *mechanism* $\mathcal{M} = (x, p)$ consists of an allocation function x and a payment function p that map joint actions $a = (a_1, \dots, a_n) \in \mathcal{A}$ to allocations of the items and payments, respectively. Specifically, $x_i(a)$ denotes the set of items allocated to agent i while $p_i(a)$ the corresponding price payed. When mechanism $\mathcal{M}$ is employed, agents receive utilities *summing* the value of received items minus the price payed , i.e.,

$$U_i(a) = \sum_{j \in x_i(a)} v_{i,j} - p_i(a), \quad \forall i \in N. \tag{15}$$

Similarly, the system designer (or auctioneer) earns a revenue equal to the sum of the payed prices $\mathrm{Rev}(a) = \sum_{i \in N} p_i(a)$. Examples of commonly studied mechanisms are the well-known first-price, second-price and all-pay auctions, reviewed next.

Example 6 (*First-Price, Second-Price and All-Pay Auctions*)**.** In many of the commonly studied mechanisms, each item j is allocated to the highest bidder, i.e., we fix $x_i(a) = \{j \text{ s.t. } i \in \arg\max_{k \in N} a_{k,j}\}$ where ties are broken arbitrarily. What should the accompanying payment function be? In a first price auction, each agent is charged their bid for every item they are allocated, i.e., $p_i(a) = \sum_{j \in x_i(a)} a_{i,j}$. In a second price auction, each agent is charged the second highest bid on every item they are allocated, i.e., $p_i(a) = \sum_{j \in x_i(a)} \max_{k \neq i} a_{k,j}$. In an all-pay auction, each agent is charged the sum of her bids, i.e., $p_i(a) = \sum_j a_{i,j}$.

As discussed in Section 1.2, we focus here on the *strategic approach* to mechanism design, whose goal is that of devising mechanisms optimizing the equilibrium performance, regardless of whether they induce truthful bidding. In this context, mechanism design and utility design bear important similarities: both fields are concerned with the design of utility functions ensuring efficient outcomes through careful crafting of the agents' utilities. However, within the realm of auctions, the design of the agents' utilities is further constrained by their very structure, as in (15), where only the allocation and payment rule can be designed. This introduces an additional challenge, so that most of the existing literature focuses on characterizing the performance of well-known mechanisms, as opposed to its systematic optimization.

In this section we employ the notion of Nash equilibrium to model the emergent behavior, and remark that, since auctions typically result in maximization problems, the definitions of pure Nash equilibrium and price of anarchy need to be suitably adjusted.[5] We measure the price of anarchy with respect to the commonly studied "total welfare" corresponding to the sum of the bidders and auctioneer utilities[6]

$$W(a) = \sum_{i \in N} \sum_{j \in x_i(a)} v_{i,j}.$$

Finally, we denote with $\mathrm{Opt}(v)$ the maximum total welfare , given the valuation profile v.

Once again, the smoothness framework provides a systematic approach to bounding the equilibrium performance. However, there are important nuances on how this approach is applied to mechanism design.[7] For this reason, we begin with a classical result and include its proof for pedagogical reasons. While we state this result for *single item* auctions, the argument generalizes to multiple-items auctions.

Theorem 7 (*PoA in First-Price, Single Item Auctions (Syrgkanis & Tardos, 2013)*)**.** *The price of anarchy in any first-price, single item auction is at least* $1 - 1/e \approx 0.632$.

Proof. As in (Syrgkanis & Tardos, 2013) we first show that the price of anarchy is at least 0.5. We then provide details on how this argument can be optimized to retrieve the $1 - 1/e$ bound. Note that for the case of single item auctions, the valuation v_i and actions $a_i \in \mathcal{A}_i$ simply represent agent i's valuation and bids on the item, respectively. While the proof style follows a smoothness-like approach, its main difference resides in imposing such smoothness conditions only for carefully constructed unilateral deviations from the equilibrium, contrary to the more demanding original definition presented in (3).

Towards this goal, we observe that we can bound the utility of each agent $i \in N$ for bidding *half* her true valuation. That is, fix $a_i' = v_i/2$, it holds that $U_i(a_i', a_{-i}) \geq v_i/2 - p(a)$ for *any* joint action $a \in \mathcal{A}$. This follows as either the agent wins the auction so that

$$U_i(a_i', a_{-i}) = v_i - a_i' = \frac{1}{2} v_i \geq \frac{1}{2} v_i - p(a),$$

or she loses the auction and thus

$$U_i(a_i', a_{-i}) = 0 \geq \frac{1}{2} v_i - p(a),$$

where the inequality holds since $p(a) \geq v_i/2$ as she lost the auction and thus the first-price must be no-lower than her bid. It follows that

$$U_i(a_i', a_{-i}) \geq \left(\frac{1}{2} v_i - p(a)\right) \cdot x_i^{\mathrm{opt}}(v)$$

also holds since $U_i(a_i', a_{-i}) \geq 0$. With slight abuse of notation, in the above we let $x_i^{\mathrm{opt}}(v) = 1$ if the item is given to agent i in the optimal allocation under the valuation profile v, and zero else. Summing up these inequalities, one concludes. Indeed, for any pure Nash equilibrium a^{ne}, it is

$$\sum_{i \in N} U_i(a^{\mathrm{ne}}) \geq \sum_{i \in N} U_i(a_i', a_{-i}^{\mathrm{ne}}) \tag{16}$$

$$\geq \sum_{i \in N} \left(\frac{1}{2} v_i - p(a^{\mathrm{ne}})\right) \cdot x_i^{\mathrm{opt}}(v) \tag{17}$$

$$= \frac{1}{2} \mathrm{Opt}(v) - p(a^{\mathrm{ne}}). \tag{18}$$

Using the definition of $W(a)$, and rearranging gives

$$W(a^{\mathrm{ne}}) = \sum_{i \in N} U_i(a^{\mathrm{ne}}) + p(a^{\mathrm{ne}}) \geq \frac{1}{2} \mathrm{Opt}(v).$$

In order to show the sharper $1 - 1/e$ bound, one can follow the same set of arguments as above and replace the deterministic bid $a_i' = v_i/2$ with a randomized uniform bid with support $[0, (1 - 1/e)v_i]$. The

[5] A pure Nash equilibrium is as in Definition 1 whereby agents costs are substituted by agents' utilities, and the inequality is reversed. The price of anarchy is defined as in (1) whereby the system-cost is replaced by the system welfare and the supremum by the infimum. Contrary to the case of cost minimization problems, the price of anarchy is upper bounded by one, and the higher the price of anarchy, the better the performance guarantees. Later in this section we discuss extensions to Bayesian Nash equilibria, whereby a common prior distribution on the valuations is known.

[6] Another well-studied welfare function is the total revenue achieved by the auctioneer, $W(a) = \sum_{i \in N} p_i(a)$. However, we do not delve into resulting pertaining to revenue maximization here for brevity.

[7] This connection has been studied extensively by, e.g., Hartline et al. (2014), Roughgarden et al. (2017) and Syrgkanis and Tardos (2013).

resulting efficiency lower bound applies to all mixed Nash equilibria and, thus, applies to the efficiency of all pure Nash equilibria as well. □

A similar approach can be used to prove the following result on all-pay, single item auctions.

Theorem 8 (*Price of Anarchy in Additive All-Pay Auctions*). *The price of anarchy of mixed Nash equilibria in any all-pay, single item auction is at least* $1/2$.[8]

The proof of the above result follows traditional smoothness arguments, and, thus, the efficiency guarantees hold for more general notions of equilibrium (e.g., mixed Nash, coarse correlated). Roughgarden et al. (2017), Syrgkanis and Tardos (2013) show that the application of the smoothness argument in this setting extends beyond deterministic valuations to the common Bayesian setting where agents' valuations are drawn from a commonly known prior distribution and where the focus is on Bayesian Nash equilibrium.

The use of smoothness arguments in the context of mechanism design has also been pursued for other auction formats (e.g., second-price, fixed-price auctions (Feldman et al., 2014; Syrgkanis & Tardos, 2013)), for multiple item and combinatorial auctions (Bhawalkar & Roughgarden, 2011; Christodoulou et al., 2016; Feldman et al., 2013), and for various system-level objectives (e.g., total revenue, fairness (Filos-Ratsikas et al., 2019; Hartline et al., 2014)). As in the proof of Theorem 7, in many of these works, the notion of smoothness has had to be adapted in order to provide meaningful price of anarchy bounds (Feldman et al., 2013; Hartline et al., 2014; Syrgkanis & Tardos, 2013).

6. Recent and future directions

The results presented in the previous sections provide a solid foundation regarding utility design in congestion games and auctions. Yet, many exciting research questions are being explored at the time of writing. Here, we highlight some of these recent developments.

6.1. Local vs. global information

As we have seen, the design of agents' utility functions heavily relies on the information available to the system designer. It is therefore natural to study how the achievable performance depends on the degree of informational availability. While this is a broad research area , in the following we focus on a well studied class of problems for concreteness.

Specifically, we consider congestion games and compare the design of incentives based on local and global information. As discussed in Section 4, we refer to local incentives when each agents resource cost is designed solely using local properties of that resource. On the contrary, when all information pertaining to the instance at hand can be used, we refer to the design as global. Naturally, local incentives utilize significantly less amount of information, and thus are often preferred in the applications. Within this context, utilities optimizing the price of anarchy based either on local or global information are known (see Section 4).

Relative to the thoroughly studied class of polynomial congestion games, Table 1 compares the performance (price of anarchy) of optimal utilities designed with local and global information. Immediately, we observe that the use of incentives significantly improves the performance (compare the first column with the second and third in Table 1).

Table 1
Price of anarchy values for congestion games with resource costs of degree at most d. All results are tight for pure Nash and also hold for coarse correlated equilibria. Respectively, columns 2–4 feature the price of anarchy with no incentives (Aland et al., 2011; Awerbuch et al., 2013; Christodoulou & Koutsoupias, 2005b), with optimal incentives using global information (matching the hardness of approximation) (Bilò & Vinci, 2019; Caragiannis et al., 2010; Paccagnan & Gairing, 2021), and with optimal incentives using local information (Paccagnan et al., 2021). Interestingly, optimal incentives relying only on local information perform closely to optimal incentives designed using global information, with a difference in performance below 1% for $d = 1$.

d	No incentive	Global design	Local design
1	2.50	2	2.012
2	9.58	5	5.101
3	41.54	15	15.551
4	267.64	52	55.452
5	1513.57	203	220.401

However, while the performance achievable by incentives utilizing global information matches the hardness factor from Theorem 6, local incentives achieve an almost identical performance while requiring much less information. This is remarkable as it provides the system designer with an easy-to-implement recipe with almost optimal performance. Whether there are other classes of problems for which this is the case, is an interesting direction for future work.

6.2. Performance tradeoffs

While the majority of the existing literature focuses on optimizing either one of the price of anarchy or price of stability, in a series of recent works, Chandan et al. (2019), Filos-Ratsikas et al. (2019) and Ramaswamy Pillai et al. (2021) investigate whether there exists an inherent performance tradeoff between these two quantities. Specifically, Ramaswamy Pillai et al. (2021) consider utility design in set covering problems, Filos-Ratsikas et al. (2019) study mechanism design for machine scheduling, while Chandan et al. (2021a) focus on the classical setting of incentive design in congestion games. Interestingly, the authors show that in *all* these settings there is a fundamental tradeoff between the price of anarchy and the price of stability. More concretely, the authors characterize such tradeoff for each of these problems by providing corresponding Pareto frontiers, and proving that incentives/mechanisms minimizing the price of anarchy have *matching* price of stability. This observation alone is interesting, as it suggests that improving the price of anarchy necessarily comes at the expense of the price of stability. In Fig. 4, we plot the performance tradeoff in congestion games with affine resource costs and local incentives from Chandan et al. (2021a).

6.3. Matching the hardness frontier

Many of the problems considered thus far, e.g., system-cost minimization in congestion games, are intractable to solve exactly (formally NP-hard) even when the designer has perfect control over all the agents' decisions. In this setting, polynomial time approximations are known for many of these problems. Can the utility design approach presented here match these centralized polynomial time approximations or is there an inherent gap on the achievable performance? Interestingly, a number of recent works have shown that judiciously designed utilities can be computed in polynomial time, and their corresponding equilibrium efficiency matches the performance achievable by the best centralized polynomial time algorithm. Hence, for many problem settings, there is no performance degradation in employing a utility-design approach.

For example, Paccagnan and Gairing (2021) show that, in congestion games, tractably designed incentives have a price of anarchy matching the best-achievable approximation factor. For the case of polynomial system resource costs of degree d, this factor reduces to

[8] Interestingly, all-pay auctions do not possess pure Nash equilibria, and thus the price of anarchy is not well-defined for this class of equilibria. Thus, the extension theorems proving that smoothness bounds apply to more general notions of equilibrium are especially important in this setting.

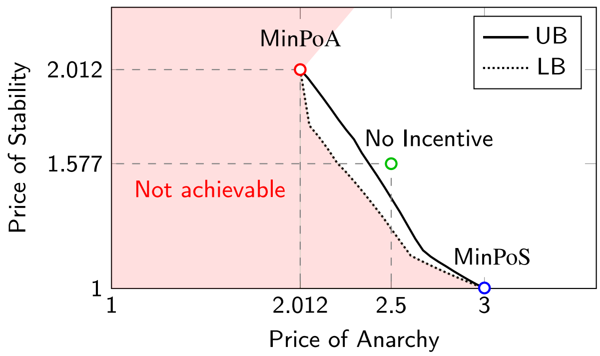

Fig. 4. *The Pareto frontier between the price of anarchy and price of stability in congestion games with affine resource costs and local incentives.* The exact Pareto frontier lies within the region below the upper bound (UB, solid) and above the lower bound curves (LB, dotted). All points above the upper bound are suboptimal. All points in the red region are unachievable owing to the lower bound, and the fact that PoA ≥ PoS by definition. The joint price of anarchy and stability values for the mechanism that minimizes the price of anarchy (in red), no incentive (in green), and the mechanism that minimizes the price of stability (in blue) are plotted as circles. Observe that the joint performance guarantees provided by no incentive fall above the upper bound, and are thus Pareto suboptimal. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the $(d+1)$st Bell Number. For weighted set covering problems, Gairing (2009) shows a similar result, whereby the approximation factor is $1-1/e$. For more general submodular maximization problems with curvature c, Chandan et al. (2021b) and Filmus and Ward (2014) provide a $1-c/e$ approximation through utility design, again matching the hardness frontier. Note, however, that this is not always the case. One notable exception is that of machine scheduling. Within this setting, the best achievable price of anarchy equals n (the number of machines) (Filos-Ratsikas et al., 2019), while the best-known polynomial time algorithm yields a 2-approximation (Lenstra et al., 1990).

7. Conclusion

Whether we are interested in distributed control of multi-agent systems, or we are concerned with the design of incentives ensuring desirable system behavior, this manuscript highlighted the relevance of Game Theory – and in particular Utility Design – to these agendas. Building atop this observation, we presented a cohesive set of tools to quantify the performance of given utility functions, and to design such utilities optimally. Central to this approach is the smoothness framework and corresponding modifications, which we have shown to provide a general recipe to tackle these problems. En route, we have also highlighted recent developments and important connections between the utility design approach and related approaches, most notably in the field of algorithmic game theory, approximation theory, and computational complexity. While we presented (in our view) many exciting results, we expect more to come in the near future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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