Example Theorem Lemma[section] Definition[section] Fact[section] Corollary[section]

An Introduction to Controllability of Nonlinear Systems

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Outline

Introduction

Reachability, accessibility and controllability

Manifolds, vector fields and flows

Operations on vector fields

The Jacobi-Lie bracket

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- Introduction
- Reachability, accessibility and controllability
- Manifolds, vector fields and flows
- Operations on vector fields
- The Jacobi-Lie bracket

Controllability of linear systems

$$\dot{x} = Ax + Bu \quad x(t) \in \mathbb{R}^n$$

Given a fixed time T and starting from the zero state at time $t = 0$, can we reach any arbitrary state x_f in time T, using control $\mathrm{u}(\cdot) : [0, T] \to \mathbb{R}$ belongs to a class of piecewise continuous functions ?

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A standard test for linear time-invariant systems

$$\text{Rank} \quad [B, AB, A^2B, \ldots, A^{n-1}B] = n?$$

Yes $\Rightarrow$ the system is controllable.

No $\Rightarrow$ the system is NOT controllable. other notions - stabilizable, uncontrollable subspace, etc.

The class of control systems

Affine in the control nonlinear system

$$\dot{x} = f_0(x) + \sum_{i=1}^{p} f_i(x)u_i \qquad\qquad x(t) \in V \subset \mathbb{R}^n, u(t) \in U$$

- $f_0(\cdot)$ termed the drift vector field and $\{f_1(\cdot), f_2(\cdot), , \ldots, f_p(\cdot), \}$ (control vector fields) are assumed to be smooth in their argument x. They are termed $\mathbf{C}^\infty$ vector fields.
- A control system is denoted by - $\Sigma = (M, \mathfrak{F} = \{f_0, f_1, \ldots, f_p\}, U)$ where U denotes the set where the control function $u(t)$ takes values in.

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Controllability and notation

Where can I go from x_0 ?

- What are the points in the state-space that I can reach in a finite time T ? (reachable set.)
- Does the reachable set contain a neighbourhood ? (accessible.)
- Does there exist a small neighbourhood containing my present state in which I can reach every point ? (locally controllable.)
- Can I reach all points in the state-space ? (global controllability.)

Reachable sets

$$\mathfrak{R}_\Sigma(x_0, T) = \{x(T) | x(s) \quad \text{is a trajectory generated by} \quad u : [0, T] \to U\}$$

$$\mathfrak{R}_\Sigma(x_0, \leq T) = \bigcup_{t \in [0,T]} \mathfrak{R}_\Sigma(x_0, t) \qquad \mathfrak{R}_\Sigma(x_0) = \bigcup_{t \geq 0} \mathfrak{R}_\Sigma(x_0, t)$$

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Reachable sets

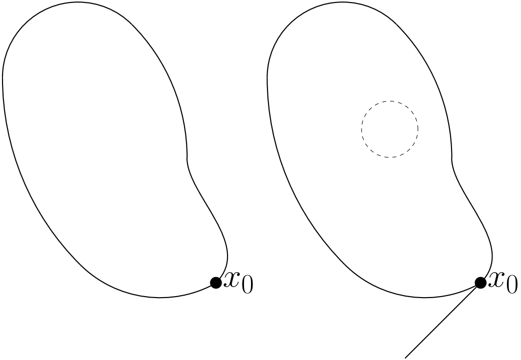


Figure: $\mathfrak{R}_\Sigma(x_0, \leq T)$

Accessibility

Accessible

Σ is accessible from x_0 if $\mathrm{int}(\mathfrak{R}_\Sigma(x_0)) \neq \emptyset$

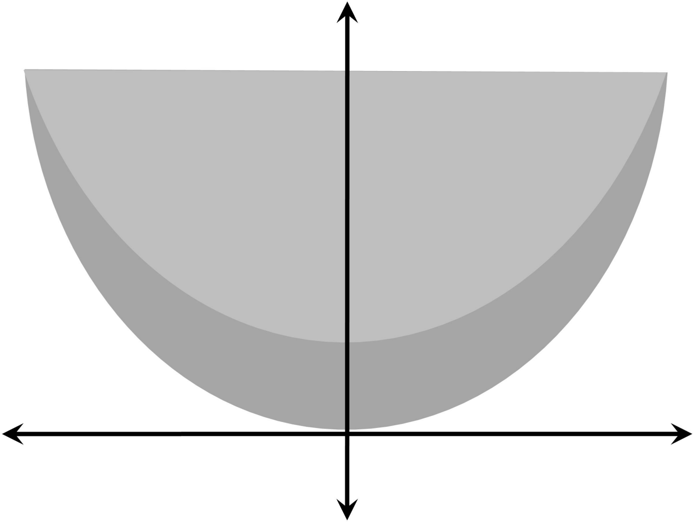
Example 1

$\Sigma = (\mathbb{R} \times \mathbb{R}, \mathfrak{F} = \{f_0, f_1\}, [-1, 1])$

$$\dot{x}_1 = u \quad \dot{x}_2 = x_1^2$$

Take $U = [-1, 1]$. Examine accessibility at $(0, 0)$.

Accessibility



Strong accessibility

Strongly accessible

Σ is strongly accessible from x_0 if $\mathrm{int}(\mathfrak{R}_\Sigma(x_0, T)) \neq \emptyset$ for each $T > 0$.

Strong accessibility

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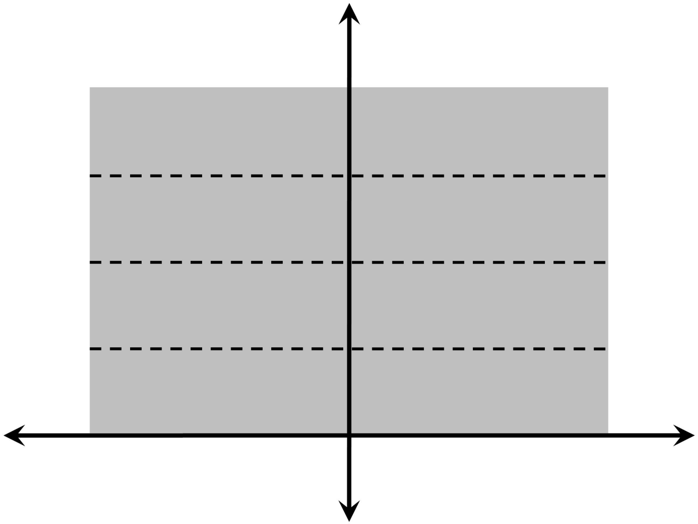
Example 2

$\Sigma = (\mathbb{R} \times \mathbb{R}, \mathfrak{F} = \{f_0, f_1\}, \mathbb{R})$

$$\dot{x}_1 = u \quad \dot{x}_2 = 1$$

Examine strong accessibility from $(0, 0)$ in the previous example.

Strong accessibility



Local controllability

Locally controllable

Σ is locally controllable from x_0 if $x_0 \in \mathrm{int}(\mathfrak{R}_\Sigma(x_0))$

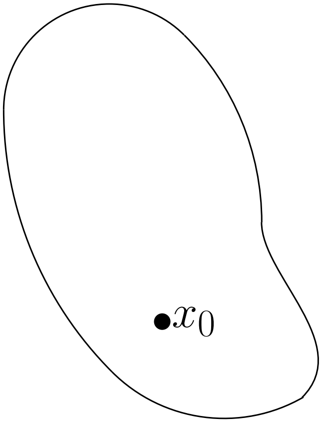


FIGURE: $x_0 \in \operatorname{int}(\mathfrak{R}_\Sigma(x_0))$

Small-time local controllability

STLC

Σ is small-time locally controllable from x_0 if there exists $T > 0$ such that $x_0 \in \operatorname{int}(\mathfrak{R}_\Sigma(x_0, \leq t))$ for each $t \in (0, T]$.

Small-time local controllability

STLC

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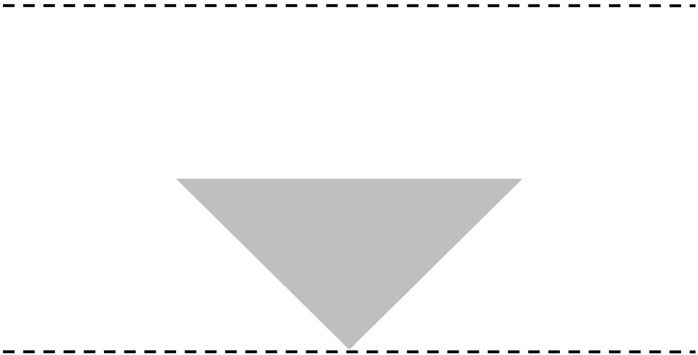
Example 3

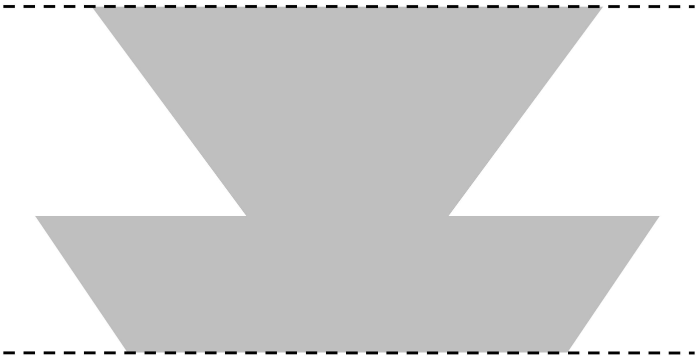
$\Sigma = (\mathbb{R} \times \mathbb{S}^1, \mathfrak{F} = \{f_0, f_1\}, [-1, 1])$

$$\dot{x}_1 = u \quad \dot{x}_2 = 1$$

Small time controllability



Small time controllability



Global controllability

Global controllability

Σ is globally controllable from x_0 if $(\mathfrak{R}_\Sigma(x_0)) = M$

[1] A. Lewis: *A brief on controllability of nonlinear systems*, URL: http://www.mast.queensu.ca/ andrew/, Dated: 08/07/2002

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The examples in the last few slides were taken from Lewis. [1]

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High school physics

The cross product

- Vector space $\mathbb{R}^3$ and the cross-product operation $\times$.
  - $a \times (b \times c) = (a \times b) \times c$ - associative.
  - $(\alpha_1 a_1 + \alpha_2 a_2) \times b = \alpha_1 (a_1 \times b_1) + \alpha_2 (a_2 \times b_2)$ - linearity. (*holds in the second argument as well.*)
  - $a \times b = -b \times a$ - skew-commutative.
  - $a \times (b \times c) + c \times (a \times b) + b \times (c \times a) = 0$ - the Jacobi-Lie identity.

Comment: the cross-product of two vectors in $\mathbb{R}^3$ gave a new *direction*.

High school physics

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An alternate notation

$$a \times b \leftrightarrow \hat{a}b \qquad \hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Smooth manifolds and differential geometry

- The study of differential geometry in our context is motivated by the need to study dynamical systems that evolve on spaces other than the usual Euclidean space.
- Single pendulum, double pendulum

- Definition

  A manifold is a topological space M with the following property. For any $x \in M$, there exists a neighbourhood B of x which is homeomorphic to R^n (for some fixed $n \geq 0$). (We shall need more - "smooth" manifolds)

Vector fields

Vector fields on smooth manifolds

- A **vector field** on a smooth manifold M is a map $\mathbf{X}$ that assigns to each $p \in M$ a tangent vector $\mathbf{X}(p)$ in $T_p(M)$.
- If this assignment is smooth, the vector field is called **smooth** or $\mathbf{C}^\infty$.

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The algebraic structure

- The collection of all C^∞ vector fields on a manifold M denoted by $\mathcal{X}(M)$ is endowed with an algebraic structure as follows: Let $\mathbf{V}, \mathbf{W} \in \mathcal{X}(X)$, $a \in R$ and $f \in C^\infty(M)$
  - $\mathbf{V} + \mathbf{W}(p) = \mathbf{V}(p) + \mathbf{W}(p)$ $(\in \mathcal{X}(M))$
  - $a(\mathbf{V})(p) = a\mathbf{V}(p)$ $(\in \mathcal{X}(M))$
  - $(f\mathbf{V})(p) = f(p)\mathbf{V}(p)$ $(\in \mathcal{X}(M))$

Flow of a vector field

Flow of $X(x)$

The flow of the vector field $X(x)$, denoted by $\Phi(t, x_0)$, is a mapping from $(-a, a) \times U \to \mathbb{R}^n$ (where $a(> 0) \in \mathbb{R}$ and U is an open region in the state-space) and satisfies the differential equation

$$\frac{d\Phi(t, x_0)}{dt} = X(\Phi(t, x_0)) \quad \forall t \in (-a, a), x(0) = x_0 \in U.$$

over the interval $(-a, a)$ and with initial conditions starting in the region U.

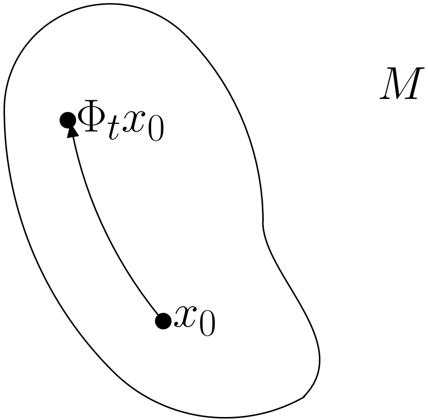

FIGURE: Flow of a vector field

Properties of flows

The group structure

Denote

$$\Phi_t(x_0) = \Phi(t, x_0)$$

The set of transformations $\{\Phi_t\} : U \to \mathbb{R}^n$ satisfies the following properties.

- $\Phi_{t+s} x_0 = \Phi_t \circ \Phi_s x_0 \quad \forall t, s, t+s \in (-a, a)$ - the group binary operation.)
- $\Phi_0 x_0 = x_0$ (the group identity.)
- For a fixed $t \in (-a, a)$ we have $\Phi_t \Phi_{-t} x_0 = x_0 \Rightarrow [\Phi_t]^{-1} = \Phi_{-t}$ (existence of an inverse.)

The flow of a linear system

The group property

Remark

The three properties mentioned above impart a group structure to the set $\{\Phi_t\}$*. This set is called a one-parameter (time) group of diffeomorphisms (*Φ_t *and its inverse are smooth mappings).*

The flow of a linear system

The group property

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Linear flow

Remark

For a linear system described by

$$\dot{x} = Ax \qquad A \in \mathbb{R}^{n \times n}$$

the flow $\Phi_t x_0 = e^{At} x_0$ where $\{e^{At} : t \in (-\infty, \infty)\}$ constitutes the one-parameter group of diffeomorphisms.

Push-forward of vector fields

- Suppose $f : M \to N$ is a diffeomorphism, then the *push-forward* of a vector field X on M by f is the vector field $f_* X$ on N defined by

$$(f_* X)(f(x)) = T_x f(X(x)) \quad \forall x \in M$$

- In coordinates

$$y = f(x) \qquad (f_* X)(y) = Df(x).X(x) = \frac{dy}{dx} \cdot X(x)$$

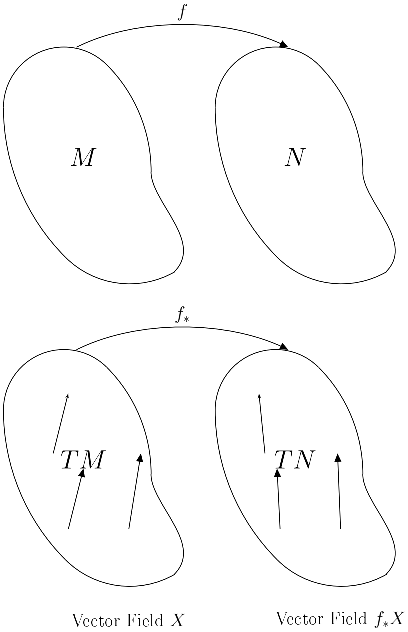

FIGURE: Push-forward

Pull-back of vector fields

The pull-back

- Suppose $f : M \to N$ is a diffeomorphism, then the *pull-back* of a vector field Y on N by f is the vector field f^*Y on N defined by

$$f^*Y = (f^{-1})_*Y = Tf^{-1} \circ Y \circ f$$

- In coordinates

$$y = f(x) \qquad (f_*X)(y) = Df(x).X(x) = \frac{dy}{dx} \cdot X(x)$$

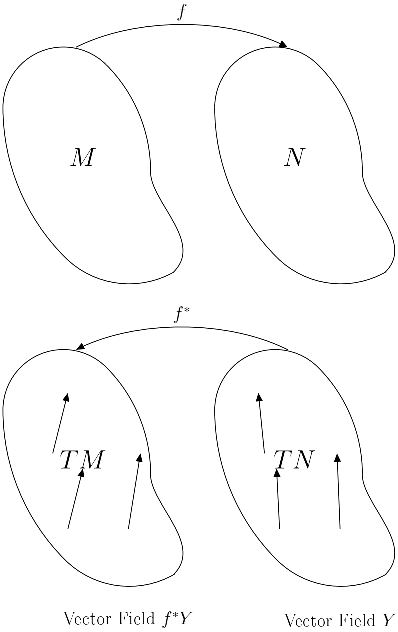

Figure: Pull-back

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Operations on vector fields

The Gradient

Consider a smooth function $g(\cdot) : U \to \mathbb{R}$. The gradient of such a function, denoted by ∇g, is defined as

$$\nabla g(x) = \left[\begin{array}{ccc} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{array} \right]$$

alternate notation: $\text{grad}(g)$.

The Lie derivative of a function

The Lie derivative

The Lie derivative of a function f along X is

$$(\mathcal{L}_X f)(x) = \frac{d}{dt}|_{t=0} f \circ \Phi_t(x) = \frac{d}{dt}|_{t=0} (\Phi_t^* f)(x)$$

In coordinates we have

$$(\mathcal{L}_X f)(x) = \left[\begin{array}{ccc} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{array} \right] X(x)$$

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Alternate notation

$$(Xf)(x) = \frac{d}{dt}|_{t=0} f \circ \Phi_t(x) = \lim_{t \to 0} \frac{f(\Phi_t x) - f(x)}{t}$$

The Lie derivative of a vector field

The pull-back

The Lie derivative of Y along X is

$$\mathcal{L}_X Y \triangleq \frac{d}{dt}|_{t=0} \Phi_t^* Y$$

where Φ is the flow of X.

Explicitly

$$(\mathcal{L}_X Y)(x) = \frac{d}{dt}|_{t=0} (D\Phi_t(z))^{-1} \cdot Y(\Phi_t(x))$$

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Operation on vector fields

The Jacobi-Lie bracket

The Jacobi-Lie bracket of two vector fields is an operation between two vector fields that yields another vector field. For two vector fields X and Y, both defined from U to $\mathbb{R}^n$, it is defined as

$$[X, Y] = (\mathcal{L}_X Y) = (DY) \cdot X - (DX) \cdot Y$$

and satisfies the following properties (for any three vector fields X, Y, Z)

- $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ - linearity in the first argument (also hold for the second argument)
- $[X, Y] = -[Y, X]$ - skew-commutative.
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [X, Z]] = 0$ - the Jacobi-Lie identity

More properties

The Jacobi-Lie bracket

Let X generate $\{\Phi_t\}$ and Y generate $\{\Psi_t\}$. Then $[X, Y] = 0$ if and only if $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for all $, t \in \mathbb{R}$.

Orbits

A complete vector field

A vector field X is said to be *complete* if the flow $\Phi_t x_0$ exists for all $x_0 \in M$ and all $t \in \mathbb{R}$.

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Orbits

An orbit of the family of complete vector fields $\mathcal{X} = \{X_i : i = 1, \ldots, p\}$ through the point $x \in M$ is the set

$$\mathcal{O}_x = \{(\Phi_{t_k})_{X_k} \circ \cdots (\Phi_{t_1})_{X_1}(x) : (t_1, \ldots, t_k) \in \mathbb{R}^k, \{X_1, \ldots, X_k\} \in \mathcal{X}\}$$

where k is an arbitrary integer.

Reachable sets and orbits

Reachable set in time T

A reachable set of the family of complete vector fields $\mathcal{X} = \{X_i : i = 1, \ldots, p\}$ from x in exactly time T is

$$\mathfrak{R}_{\mathcal{X}}(x, T) = \{(\Phi_{t_k})_{X_k} \circ \cdots (\Phi_{t_1})_{X_1}(x) : (t_1, \ldots, t_k) \in \mathbb{R}^k_{\geq 0}, t_1 + \ldots + t_k = T,$$

$$\{X_1, \ldots, X_k\} \in \mathcal{X}\}$$

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Reachable sets and orbits

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where k is an arbitrary integer.

The Orbit theorem

The Orbit theorem

The orbit of the family of vector field $\mathcal{X}$ through each point x of M is a connected submanifold of M.

The Lie algebra

The Lie algebra

For any family of vector fields $\mathcal{X}$ defined on M, $\mathrm{Lie}(\mathcal{X})$ is the smallest vector subspace S of $\mathbf{C}^\infty(M)$ that also satisfies

$$[X, S] \subset S \quad \forall X \in \mathcal{X}$$

The Lie algebra

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$$[X, S] \subset S \quad \forall X \in \mathcal{X}$$

An example

Let Ax be a linear vector field defined on $\mathbb{R}^n$ ($x \in \mathbb{R}^n$) and $B \in \mathbb{R}^n$ be a constant vector field on $\mathbb{R}^n$.

$$\mathrm{Lie}(\{Ax, B\}) = \mathrm{span}\{Ax, B, AB, A^2B, \ldots, A^{n-1}B\}$$

At $x = 0$,

$$\mathrm{Lie}_0(\{Ax, B\}) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{n-1}B\}$$

Frobenius' theorem

Involutivity

A family $\mathcal{X}$ of vector fields is said to be involutive if for any vector fields X and Y in $\mathcal{X}$,

$$[X, Y](x) \subset \text{span}\{\mathcal{X}(x)\} \quad \forall x \in M.$$

Frobenius' theorem

Let $\mathcal{X}$ be an involutive family of smooth vector fields for which the dimension of the linear span of $\mathcal{X}(x)$ is constant for all $x \in M$. Then the tangent space at a point x of an orbit of $\mathcal{X}$ is equal to the linear span of $\mathcal{X}(x)$.

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