A CENTRAL LIMIT THEOREM FOR THE KONTSEVICH-ZORICH COCYCLE

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Abstract. In this note, we show that a central limit theorem holds for the top exterior power of the Kontsevich-Zorich cocycle. In particular, we show that a central limit theorem holds for the lift of the (leafwise) hyperbolic Brownian motion to the Hodge bundle, and then show that a (possibly degenerate) central limit theorem holds for the lift of the Teichmüller geodesic flow to the same bundle. We show that the variance of the random cocycle is positive if the second top Lyapunov exponent of the cocycle is positive.

1. Introduction

Following the potential-theoretic approach to studying Lyapunov exponents in Teichmüller dynamics (pioneered by Kontsevich-Zorich [**Kon97**] and further developed by Forni [**For02**]), and inspired by the work of Le Jan [**LJ94**], we study the trajectories of the (foliated) hyperbolic Brownian motion, prove a central limit theorem for the lift of these trajectories to the Hodge bundle, and deduce a central limit theorem for the top exterior power of the Kontsevich-Zorich cocycle.

More precisely, let $\pi : \mathbb{P}(\mathbf{H}) \to X$ be the projectivized absolute (real) Hodge bundle over an $\mathrm{SL}_2(\mathbb{R})$ orbit closure X, whose fiber over each point in X is $H^1(S, \mathbb{R})$, and with ν an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on X. For $g \in \mathrm{SL}_2(\mathbb{R})$, the Kontsevich-Zorich cocycle $(g)_*$ is the lift of the action of g to $\mathbb{P}(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. Furthermore, $(g)_*$ acts symplectically since it preserves the intersection form on $H^1(S, \mathbb{R})$.

For our purposes, we will in fact be concerned with the top exterior power of the Hodge bundle that is symplectic orthogonal to the tautological subbundle (spanned by $[\mathrm{Re}\ \omega]$ and $[\mathrm{Im}\ \omega]$), and we continue to call this subbundle $\mathbb{P}(\mathbf{H})$. This bundle supports an $\mathrm{SO}_2(\mathbb{R})$-invariant probability measure $\hat{\nu}$ such that, for ν-a.e ω, the conditional measure on $\mathbb{P}(\mathbf{H}_\omega)$ is the Haar measure. An Euclidean structure is in fact given by the Hodge norm (see 2.5 for the definition), and which we use in the sequel.

Therefore, we fix the $\mathrm{SO}_2(\mathbb{R})$-invariant Hodge norm $\|\cdot\|_{\pi(\cdot)}$ on $\mathbf{H}$. Define $\sigma : \mathrm{SL}_2(\mathbb{R}) \times \mathbb{P}(\mathbf{H}) \to \mathbb{R}$ by

$$\sigma(g, \mathbf{v}) = \log \frac{\|(g)_*\mathbf{v}\|_{g\pi(\mathbf{v})}}{\|\mathbf{v}\|_{\pi(\mathbf{v})}}$$

For $\omega \in X$, let $\mathbf{v}_\omega$ in $\mathbb{P}(\mathbf{H}_\omega)$ be the projectivization of any Lagrangian subspace (of dimension $g-1$) in $\mathbb{P}(\mathbf{H})$. For $\hat{\nu}$-a.e. $\mathbf{v} = (\omega, \mathbf{v}_\omega)$, it is a consequence of the multiplicative ergodic theorem that

$$\lim_{T\to\infty} \frac{\sigma(g_T, \mathbf{v})}{T} = \sum_{i=2}^{g} \lambda_i$$

where, together with $\lambda_1 = 1$, λ_i are the top g Lyapunov exponents of the Kontsevich-Zorich cocycle, and where $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The top g exponents determine the entire Lyapunov spectrum by symplecticity. Note that $\lambda_2 = 0$ if and only if $\sum_{i=2}^{g} \lambda_i = 0$, since the exponents are ordered so that $\lambda_2 \geq \cdots \geq \lambda_g \geq 0$.

Our main result is the following:

Theorem 1.1. *There exists* $\Phi_{g_\infty} \geq 0$ *such that*

$$\lim_{T\to\infty} \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}) \ : \ a \leq \frac{1}{\sqrt{T}}(\sigma(g_T, \mathbf{v}) - T\lambda) \leq b\right\}\right) = \frac{1}{\sqrt{2\pi\Phi_{g_\infty}}} \int_a^b \exp(-x^2/\Phi_{g_\infty})dx.$$

Remark 1.2. The statement also holds for when $\Phi_{g_\infty} = 0$, in that the resulting distribution is a delta distribution.

To prove this, we will first work with the hyperbolic Brownian motion, which is the diffusion process generated by the foliated hyperbolic Laplacian. Let ρ be a (foliated) hyperbolic Brownian motion trajectory starting at a (generic) basepoint $\omega \in X$, defined almost everywhere with respect to a probability measure $\mathbb{P}_\omega$ on the space of such trajectories W_ω. This process is in fact defined on $X^* = \mathrm{SO}_2(\mathbb{R})\backslash X$. Moreover, ρ can be lifted to $\mathrm{SL}_2(\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ by abuse of notation. Additionally, the space X gives rise to a product space $X^W := X \otimes W$ whose fiber over each point ω in X is W_ω, and which also supports a measure $\nu_{\mathbb{P}} := \nu \otimes \mathbb{P}$, whose conditional measure over a point ω is $\mathbb{P}_\omega$. We can thus similarly define the product W-Hodge bundle $\mathbb{P}^W(\mathbf{H})$, whose fiber over each point (ω, ρ) in X^W is $\mathbf{H}_\omega$. A pair $(\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H})$ is thus defined to be the lift of the path ρ (starting at ω) to $\mathbb{P}^W(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}} := \hat{\nu} \otimes \mathbb{P}$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}_\omega$. We therefore also have

Theorem 1.3. *There exists $\Phi_{\rho_\infty} \geq 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}_{\mathbb{P}} \left(\left\{ (\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H}) \ : \ a \leq \frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\lambda) \leq b \right\} \right) = \frac{1}{\sqrt{2\pi\Phi_{\rho_\infty}}} \int_a^b \exp(-x^2/\Phi_{\rho_\infty}) dx.$$

Moreover, if $\lambda_2 > 0$, then $\Phi_{\rho_\infty} > 0$.

Remark 1.4. Observe that for $g = 2$, our two results reduce to ones that concern the second Lyapunov exponent λ_2.

Some ingredients of our proof include

- results of Avila-Gouëzel-Yoccoz [**AGY06**] and Avila-Gouëzel [**AG13**] on the spectral gap of the leafwise hyperbolic Laplacian to show existence of a solution of the leafwise Poisson's equation (see Appendix A),
- elementary stochastic calculus to extract and control the necessary oscillations,
- and an asymptotic estimate due to Ancona [**Anc90**] to relate the geodesic flow with the Brownian motion.

1.1. **Related results.** The paper of Daniels-Deroin [**DD19**] adapts the Teichmüller dynamics methodology to more general compact Kahler manifolds, and one in which the methods in this note are applicable, provided that we can prove existence of a solution to Poisson's equation for the corresponding Laplacian. In [**DFV17**], Dolgopyat-Fayad-Vinogradov prove a central limit theorem for the time integral of sufficiently regular zero-average observables of the pushforward of a small horocyclic arc by the geodesic flow, as the basepoint varies generically with respect to an ergodic P-invariant measure, where P is the upper triangular subgroup of $\mathrm{SL}_2(\mathbb{R})$ (their results are in fact much more general, cf. [**DFV17**, Theorem 7.1], but we present their theorem in the $\mathrm{SL}_2(\mathbb{R})$ setting for simplicity) - it would be interesting to prove exponential mixing for the g_t-action on the Hodge bundle (more precisely for the g_t-action on a ν-strongly-irreducible $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle of the Hodge bundle, and where $\hat{\nu}$ in this case would be a P-invariant measure), and then apply their result to the Kontsevich-Zorich cocycle. In [**AS21**], a mechanism to produce oscillations for the Kontsevich-Zorich cocycle is presented, where the basepoint is a fixed surface – we hope that the results presented here can be brought to bear on the scope of the result in [**AS21**], and on the limiting measures of Teichmüller horocyclic arcs as in the recent work of Chaika-Khalil-Smillie [**CKS21**].

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2. Preliminaries

2.1. **Translation surfaces.** Let S be a Riemann surface of genus $g \geq 2$, and ω a holomorphic 1-form on S. The pair (S, ω) is said to be a translation surface, since ω gives a (degenerate) flat metric on S, and ω is invariant under translations when it is written in local coordinates. The zero set of ω characterizes the singularity set of the conical metric. The area of a translation surface is given by $\int_S \omega \wedge \overline{\omega}$. We will refer to the pair (S, ω) as just ω.

2.2. **Moduli Space.** Let $\mathcal{TH}_g$ be the Teichmüller space of unit-area translation surfaces of genus $g \geq 2$, and let $\mathcal{H}_g = \mathcal{TH}_g / \mathrm{Mod}_g$ be the corresponding moduli space, where Mod_g is the mapping class group. The space $\mathcal{H}_g$ is partitioned into strata $\mathcal{H}(\kappa) = \mathcal{H}(\kappa_1, \ldots, \kappa_n)$, which consist of unit-area translation surfaces whose singularities have cone angle $2\pi(1 + \kappa_i)$, and $\sum \kappa_i = 2g - 2$. One can also define local period coordinates on a stratum, where all changes of coordinates are given by affine maps.

2.3. $\mathrm{SL}_2(\mathbb{R})$ **action.** There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on translation surfaces and on their moduli. It is shown in [**EM18, EMM15**] that for any $\omega \in \mathcal{H}(\kappa)$, the closure X of $\mathrm{SL}_2(\mathbb{R}) \cdot \omega$ is an affine invariant submanifold, and supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability Lebesgue measure ν.

2.4. **Kontsevich-Zorich cocycle.** Let $\widehat{\mathbf{H}} = \mathcal{TH}_g \times H^1(S, \mathbb{R})$, and define the trivial cocycle $\widehat{(g_T)_*} : \widehat{\mathbf{H}} \to \widehat{\mathbf{H}}$ with $\widehat{(g_T)_*}(\omega, c) = (g_T \omega, c)$ for $\omega \in \mathcal{TH}_g$ and $c \in H^1(S, \mathbb{R})$. The absolute (real) Hodge bundle is given by $\mathbf{H} = \widehat{\mathbf{H}} / \mathrm{Mod}_g$ and the Kontsevich-Zorich cocycle $(g_T)_*$ is the projection of $\widehat{(g_T)_*}$ to $\mathbf{H}$.

2.5. **Hodge inner product and the second fundamental form.** Given two holomorphic 1-forms ω_1, ω_2 in $\Omega(S)$, where $\Omega(S)$ is the vector space of holomorphic 1-forms on S, the Hodge inner product is defined to be

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_S \omega_1 \wedge \overline{\omega_2}$$

Moreover, the Hodge representation theorem implies that for any given cohomology class $c \in H^1(S, \mathbb{R})$, there is a unique holomorphic 1-form $h(c) \in \Omega(S)$, such that $c = [\mathrm{Re}\ h(c)]$ (cf. [**FMZ12**]). We define the Hodge inner product for two real cohomology classes $c_1, c_2 \in H^1(S, \mathbb{R})$ as

$$A_\omega(c_1, c_2) := \langle h(c_1), h(c_2) \rangle$$

The second fundamental form B_ω is defined as

$$B_\omega(c_1, c_2) := \frac{i}{2} \int_S \frac{h(c_1)h(c_2)}{\omega^2} \omega \wedge \overline{\omega}$$

It is known that B_ω does not vanish identically in the symplectic orthogonal of the tautological subbundle on all but two orbit closures [**Aul16, EKZ14, Möl11, AN20**]. These two orbit closures are referred to in the literature as *Eierlegende Wollmilchsau* and *Ornithorynque*, and have many special properties.

For any Lagrangian subspace $\mathbf{c}_\omega$ in $\mathbf{H}_\omega$, it also follows from the work of Forni [**For02**] (see also [**FMZ12**, Corollary 2.2]) that

$$\left| \frac{d}{dt} \sigma(g_t, \mathbf{c}_\omega) \right| \leq g - 1 \tag{2.5.1}$$

Let $\{c_1, c_2, \ldots, c_g\}$ be a Hodge-orthonormal basis of $\mathbf{c}$ in $H^1(S, \mathbb{R})$, and let A^g_ω (resp., B^g_ω) be the corresponding representation matrix of the Hodge inner product A_ω (resp., of B_ω). The eigenvalues of B^g_ω are denoted by $\Lambda_i(\omega)$, where $1 = |\Lambda_1(\omega)| > |\Lambda_2| \geq \cdots \geq |\Lambda_g| \geq 0$. Moreover, the norm squared of these eigenvalues, $|\Lambda_i(\omega)|^2$, are continuous, bounded functions on $\mathcal{H}_g$ (cf. [**FMZ12**], Lemma 2.3).

2.6. **Foliated Hyperbolic Laplacian.** The space $\mathcal{H}_g$, is foliated by the orbits of the $\mathrm{SL}_2(\mathbb{R})$-action, whose leaves are isometric to the unit cotangent bundle of the Poincaré disk $\mathbb{D}$. For $\omega \in \mathcal{H}_g$, the Teichmüller disk $L_\omega := \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cdot \omega$ is isometric to $\mathbb{D}$, and so is endowed with the (foliated) hyperbolic gradient ∇_{L_ω} and hyperbolic Laplacian Δ_{L_ω}.

Remark 2.1. Observe that for $\omega \in X$, the Teichmüller disk L_ω is identified with $\mathbb{D}$ via the map $(t, \theta) \mapsto \mathrm{SO}_2(\mathbb{R}) \cdot g_t r_\theta \omega$.

Now suppose that $f : X \to \mathbb{R}$ is an $\mathrm{SO}_2(\mathbb{R})$-invariant C^∞-function in the direction of the leaf. For $\omega \in X$ and for L_ω the Teichmüller disk passing through ω, we define $\Delta f(\omega) := \Delta_{L_\omega} f|_{L_\omega}(\omega)$, where $f|_{L_\omega}$ is the restriction of f to L_ω. We also define the leafwise gradient similarly.

Observe that the Hodge inner product $A_\omega(\cdot,\cdot)$ is invariant under the action of $\mathrm{SO}_2(\mathbb{R})$, and so defines a real-analytic function on the Teichmüller disk. In the sequel, we will only work in a given Teichmüller disk, so the norm will read $(\cdot,\cdot)_z$ for a complex parameter $z \in \mathbb{D}$. For any Lagrangian $(g-1)$-plane $\mathbf{v} = (\omega, \mathbf{v}_\omega)$ in the symplectic orthogonal of the tautological subbundle (with the origin $z = 0$ corresponding to ω as in 2.1), define

$$\sigma(z,\mathbf{v}) := \log|\det A_z^{(g-1)}|^{1/2},$$

where $A_{z,ij}^{(g-1)} = A_z(\mathbf{v}_i, \mathbf{v}_j)$ and $\{\mathbf{v}_i\}$ is an ordered basis of $\mathbf{v}$.

Remark 2.2. In fact, this is an abuse of notation since we originally lifted elements of $\mathrm{SL}_2(\mathbb{R})$ to the Hodge bundle. This is not an issue since the Hodge norm is $\mathrm{SO}_2(\mathbb{R})$-invariant.

We recall the following fundamental fact

Theorem 2.3. **[For02, FMZ12]** *Let $\mathbf{v}$ be any Lagrangian subspace in the symplectic orthogonal of the tautological subbundle. We have the following equalities*

$$\Delta_{L_\omega}\sigma(z,\mathbf{v}) = 2\sum_{i=2}^{g} |\Lambda_i(z)|^2$$

$$|\nabla_{L_\omega}\sigma(z,\mathbf{v})|^2 = \left|\sum_{i=2}^{g} \Lambda_i(z)\right|^2$$

In particular, the Laplacian is independent of the choice of a Lagrangian subspace. Moreover, we have that for ν a.e. ω,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \Delta_{L_\omega}\sigma(g_t,\mathbf{v})\,dt = \int_X 2\sum_{i=2}^{g}|\Lambda_i(\omega)|^2 d\nu = 2\sum_{i=2}^{g}\lambda_i$$

Remark 2.4. Observe that in genus $g = 2$, we have

$$\Delta_{L_\omega}\sigma(z,\mathbf{v}_z) = 2|\nabla_{L_\omega}\sigma(z,\mathbf{v}_z)|^2 = 2|\Lambda_2(z)|^2$$

but in general, for any Lagrangian subspace, the norm squared of the gradient $|\nabla_{L_\omega}\sigma(z,\mathbf{v}_z)|^2$ is not a function on moduli space the same way that the Laplacian is.

2.7. **Harmonic measures.** We say that a probability measure μ on $\mathrm{SO}_2(\mathbb{R})\backslash X$ is harmonic if for all bounded functions $f : \mathrm{SO}_2(\mathbb{R})\backslash X \to \mathbb{R}$ of class C^∞ in the leaf direction,

$$\int_{\mathrm{SO}_2(\mathbb{R})\backslash X} \Delta f(\omega)\,d\mu = \int_{\mathrm{SO}_2(\mathbb{R})\backslash X} \Delta_{L_\omega} f|_{L_\omega}(\omega)\,d\mu = 0.$$

Such a measure is also ergodic if $\mathrm{SO}_2(\mathbb{R})\backslash X$ cannot be partitioned into two union of leaves, each of which having positive μ measure. We refer the reader to the interesting paper of Lucy Garnett [**Gar83**] for details and for an ergodic theorem for such measures. It is also a fact, due to Bakhtin-Martinez [**BM08**], that harmonic measures on $\mathrm{SO}_2(\mathbb{R})\backslash X$ are in one-to-one correspondence with P-invariant measures on X. This is closely related to a classical fact due to Furstenberg [**Fur63a, Fur63b**] that P-invariant measures are in one-to-one correspondence with (admissible) stationary measures, and that harmonic measures are stationary. In the case of $\mathrm{SL}_2(\mathbb{R})$, these three notions are therefore closely related.

2.8. **Hyperbolic Brownian Motion.** Following the normalization used in [**For02**] (which is a standard normalization, see also [**Hel00**]), for $z = re^{i\theta}$ with $\theta \in [0, 2\pi]$, write

$$t := \frac{1}{2}\log\frac{1+r}{1-r}. \tag{2.8.1}$$

Since the Hodge norm is $\mathrm{SO}_2(\mathbb{R})$-invariant, it suffices to study the diffusion process generated by $\frac{1}{2}\Delta_{L_\omega}$, where the leafwise hyperbolic Laplacian in geodesic polar coordinates is given by

$$\Delta_{L_\omega} = \frac{\partial^2}{\partial t^2} + 2\coth(2t)\frac{\partial}{\partial t} + \frac{4}{\sinh^2(2t)}\frac{\partial^2}{\partial\theta^2}. \tag{2.8.2}$$

Moreover, let $(W_\omega^{(i)}, \mathbb{P}_\omega^{(i)})$, $i = 1, 2$, be two copies of the space of Brownian trajectories $C(\mathbb{R}^+, \mathbb{R})$ starting at the origin (with the origin corresponding to a random point ω), together with the standard Wiener measure, and such that $W_\omega^{(1)}$ and $W_\omega^{(2)}$ are independent. Set $W_\omega = W_\omega^{(1)} \times W_\omega^{(2)}$ and $\mathbb{P}_\omega = \mathbb{P}_\omega^{(1)} \times \mathbb{P}_\omega^{(2)}$. The hyperbolic Brownian motion is the diffusion process $\rho_s = (t_s, \theta_s)$ generated by the (leafwise) hyperbolic Laplacian. It follows by Ito's formula [**FLJ12**, Theorem VI.5.6] that the generator determines the trajectories of the diffusion process ρ_t which are solutions of the following stochastic differential equations

$$dt_s = dW_s^{(1)} + \coth(2t_s)ds \tag{2.8.3}$$

$$d\theta_s = \frac{2}{\sinh(2t_s)}dW_s^{(2)} \tag{2.8.4}$$

with $\rho_0 = 0$.

In addition, for an $\mathrm{SO}_2(\mathbb{R})$-invariant function $f : X \to \mathbb{R}$, where f is of class C^2 along $\mathrm{SL}_2(\mathbb{R})$ orbits, Ito's formula gives

$$f(\rho_T) - f(\rho_0) = \int_0^T \left(\frac{\partial}{\partial t} f(\rho_s), \frac{2}{\sinh(2t_s)} \frac{\partial}{\partial \theta} f(\rho_s)\right) \cdot \left(dW_s^{(1)}, dW_s^{(2)}\right) \quad (2.8.5)$$

$$+ \int_0^T \left(\frac{1}{2}\frac{\partial^2}{\partial t^2} f(\rho_s) + \frac{1}{2} 2\coth(2t_s) \frac{\partial}{\partial t} f(\rho_s) + \frac{1}{2}\frac{4}{\sinh^2(2t_s)} \frac{\partial^2}{\partial \theta^2} f(\rho_s)\right) ds \quad (2.8.6)$$

$$= \int_0^T \nabla_{L_\omega} f(\rho_s) \cdot (dW_s^{(1)}, dW_s^{(2)}) + \frac{1}{2}\int_0^T \Delta_{L_\omega} f(\rho_s) ds \quad (2.8.7)$$

Finally, we note that the foliated heat semigroup D_t is given as follows

$$D_s f(x) := \int_X \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty f(z) p_\omega(t,s) \sinh(t) dt\, d\theta\, d\nu \quad (2.8.8)$$

where $p_\omega(t,s)$ is the (foliated) hyperbolic heat kernel at time s; in other words, for $x, y \in L_\omega$, this is the transition probability kernel $p_\omega(x,y;s)$, with $d_{\mathbb{D}}(x,y) = t$.

3. Proofs of Main Theorems

3.1. **Distributional Convergence in Theorem 1.3.** Recall that ρ_s is the diffusion process generated by the foliated hyperbolic Laplacian. We are interested in studying the term

$$\frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\sum_{i=2}^{g} \lambda_i). \quad (3.1.1)$$

Set $\lambda = \sum_{i=2}^g \lambda_i$. By applying Ito's formula, we obtain,

$$\frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\lambda) = \frac{\sigma(\rho_0, \mathbf{v})}{\sqrt{T}} + \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega}\sigma(\rho_s, \mathbf{v}) \cdot (dW_s^{(1)}, dW_s^{(2)}) \quad (3.1.2)$$

$$+ \frac{1}{2\sqrt{T}}\int_0^T (\Delta_{L_\omega}\sigma(\rho_s, \mathbf{v}) - 2\lambda) ds \quad (3.1.3)$$

It then follows then by Corollary A.1 that the Poisson equation $\Delta_{L_\omega} u(z) = \Delta_{L_\omega}\sigma(z, \mathbf{v}) - 2\lambda$ has an L^2 solution $u(z)$ (of class C^∞ along $\mathrm{SL}_2(\mathbb{R})$ orbits), so that, by applying Ito's formula on $u(z)$, we get

$$\frac{1}{\sqrt{T}}(u(\rho_T)-u(0)) = \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega} u(\rho_s)\cdot(dW_s^{(1)},dW_s^{(2)}) + \frac{1}{2\sqrt{T}}\int_0^T \Delta u(\rho_s)ds \tag{3.1.4}$$

$$= \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega} u(\rho_s)\cdot(dW_s^{(1)},dW_s^{(2)}) \tag{3.1.5}$$

$$+ \frac{1}{2\sqrt{T}}\int_0^T (\Delta_{L_\omega}\sigma(\rho_s,\mathbf{v})-2\lambda)ds \tag{3.1.6}$$

So we have that

$$\frac{1}{2\sqrt{T}}\left(\int_0^T (\Delta_{L_\omega}\sigma(\rho_s,\mathbf{v})-\lambda)ds\right) = \frac{1}{\sqrt{T}}(u(\rho_T)-u(0)) \tag{3.1.7}$$

$$- \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega} u(\rho_s)\cdot(dW_s^{(1)},dW_s^{(2)}) \tag{3.1.8}$$

Define

$$M_T = \int_0^T \nabla_{L_\omega}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))\cdot(dW_s^{(1)},dW_s^{(2)}) \tag{3.1.9}$$

We then have

$$\frac{1}{\sqrt{T}}(\sigma(\rho_T,\mathbf{v})-T\sum_{i=2}^{g}\lambda_i) = \frac{1}{\sqrt{T}}(u(\rho_T)-u(0)+\sigma(\rho_0,\mathbf{v})) + \frac{1}{\sqrt{T}}M_T \tag{3.1.10}$$

Next, we study the quadratic variation $\langle M_T, M_T\rangle_{\hat{\nu}_{\mathbb{P}}}$. Recalling that the covariance of two Ito integrals with respect to independent Brownian motions is zero, we have:

$$\langle M_T, M_T\rangle_{\hat{\nu}_{\mathbb{P}}} = \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\left(\int_0^T \nabla_{L_\omega}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))\cdot(dW_s^{(1)},dW_s^{(2)})\right)^2\right] \tag{3.1.11}$$

$$= \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\left(\int_0^T \frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))dW_s^{(1)}\right)^2\right] \tag{3.1.12}$$

$$+ \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\left(\int_0^T \frac{2}{\sinh(2t_s)}\frac{\partial}{\partial\theta}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))dW_s^{(2)}\right)^2\right] \tag{3.1.13}$$

Applying Ito's isometry [**FLJ12**, Lemma VI.4.3] on the expectation of the square of the Ito integrals on the RHS yields

$$\langle M_T, M_T\rangle_{\hat{\nu}_{\mathbb{P}}} = \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T \left(\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))\right)^2 ds\right] \tag{3.1.14}$$

$$+ \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T \left(\frac{2}{\sinh(2t_s)}\frac{\partial}{\partial\theta}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))\right)^2 ds\right] \tag{3.1.15}$$

$$= \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T |\nabla_{L_\omega}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))|^2 ds\right]. \tag{3.1.16}$$

Observe that $|\nabla u|^2 \in L^1(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ by Corollary A.1. Therefore, by Oseledec's theorem, Fubini's theorem, and the dominated convergence theorem, we have convergence with respect to the measure $\hat{\nu}$ on $\mathbb{P}(\mathbf{H})$:

$$\Phi_{\rho_\infty} := \lim_{T\to\infty}\frac{1}{T}\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T |\nabla_{L_\omega}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))|^2 ds\right] \tag{3.1.17}$$

$$= \int_{\mathbb{P}(\mathbf{H})} |B_\omega([\mathbf{E}^+(\omega)],[\mathbf{E}^+(\omega)]) - \nabla_{L_\omega} u(\omega))|^2 d\hat{\nu} \tag{3.1.18}$$

$$= \int_X |B_\omega([\mathbf{E}^+(\omega)],[\mathbf{E}^+(\omega)]) - \nabla_{L_\omega} u(\omega))|^2 d\nu \tag{3.1.19}$$

where $\mathbf{E}^+(\omega)$ denotes the top unstable Lyapunov subspace of the A-action (see also [**For02**, Corollary 5.5]). This shows that the random cocycle converges in distribution. □

3.2. **Distributional Convergence in Theorem 1.1.** Observe that $t_s = d_{\mathbb{D}}(0,\rho_s)$, and that it is rotationally invariant. We will need the following useful lemma:

Lemma 3.1. [**FLJ12**, Lemma VII.7.2.1] *For all $\omega \in X$, there exists an $\mathbb{P}_\omega$-almost everywhere converging process η_s such that $t_s = W_s^{(1)} + s + \eta_s$.*

Proof. It is a classical fact that $t_s \to \infty$ $\mathbb{P}_\omega$-almost everywhere. This implies that $\lim_{s\to\infty}\coth(2t_s) = 1$ almost everywhere. Setting $\eta_s := t_s - W_s^{(1)} - s$, so that, together with 2.8.3, we get

$$\eta_s = \int_0^s (\coth(2t_s) - 1)ds = \int_0^s \frac{2ds}{e^{4t_s} - 1},$$

which converges almost everywhere, as desired. □

Next, it will be crucial to stop the radial process before it exits the region bounded by a circle of geodesic radius T, and so for each T, we define the stopping time τ_T as follows

$$\tau_T := \inf\{s > 0 \ : \ T = d_{\mathbb{D}}(0, \rho_s)\} \tag{3.2.1}$$

$$= \inf\{s > 0 \ : \ T = W_s^{(1)} + s + \eta_s\} \tag{3.2.2}$$

where the second equality follows by Lemma 3.1. Next, we will need the following lemma:

Lemma 3.2. *For all $\omega \in X$, we have $\lim_{T\to\infty} \tau_T/T = 1$ $\mathbb{P}_\omega$-almost everywhere. Moreover, we have that as $T \to \infty$, $\tau_t \to \infty$ $\mathbb{P}_\omega$-almost everywhere.*

Proof. Observe that we have $\tau_T = T - W_{\tau_T}^{(1)} - \eta_{\tau_T}$. The lemma then follows immediately from the definition of the stopping time and the law of the iterated logarithm. □

See also [**EFLJ01**, Lemma 4.2] for related and interesting results on this stopping time.

Recall that $\mathbb{P}_\omega$ is the Wiener measure on the space of all Brownian trajectories W_ω starting at the origin (corresponding to the random point ω). Let $\mathbb{P}_\omega^\theta$ be the Wiener measure on the space W_ω^θ corresponding to all paths starting at the origin and conditioned to exit at the point $e^{i\theta}$ in $\partial\mathbb{D}^2$. To relate the conditioned process ρ_s^θ to the unconditioned process ρ_s, we will need the following lemma:

Lemma 3.3.

$$\mathbb{P}_\omega = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}_\omega^\theta \, d\theta \tag{3.2.3}$$

Proof. Recall that W_ω is the space of all hyperbolic Brownian motion trajectories starting at the origin, with $\mathbb{P}_\omega$ the corresponding Wiener measure. There exists a map $\Theta : W_\omega \to \partial\mathbb{D}^2$, defined $\mathbb{P}_\omega$-almost everywhere, such that $\Theta(\rho) = \rho_\infty$, where ρ_∞ is the limit point of ρ on $\partial\mathbb{D}^2$. It is a classical fact that the pushforward measure $\Theta_*(\mathbb{P}_\omega)$ equals Leb, where Leb is the normalized Lebesgue measure on $[0, 2\pi]$. We also recall that the foliated process is in fact defined on $\mathrm{SO}_2(\mathbb{R})\backslash X$ and that $\hat{\nu}$ is $\mathrm{SO}_2(\mathbb{R})$-invariant, and so our disintegration claim follows. □

Remark 3.4. See also [**Fra05**, Lemma 8] for a short potential theoretic proof (using Doob's h-process) of this fact. The approach to proving the CLT in [**Fra05**], with the aid of a stopping time, is what we will essentially follow in the sequel, though in our case the proof here is simpler, in view of the Lipschitz property of the Kontsevich-Zorich cocycle and Ancona's estimate.

Remark 3.5. It is worth repeating and adapting what is written in the introduction in view of the application of the conditioned process in the sequel. The conditioned process is in fact defined on $X^* = \mathrm{SO}_2(\mathbb{R})\backslash X$. Moreover, ρ^θ

can be lifted to $\mathrm{SL}_2(\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ^θ by abuse of notation. Additionally, the space X gives rise to a product space $X^{W^\theta} := X \otimes W^\theta$ whose fiber over each point ω in X is W^θ_ω, and which also supports a measure $\nu_{\mathbb{P}^\theta} := \nu \otimes \mathbb{P}^\theta$, whose conditional measure over a point ω is $\mathbb{P}^\theta_\omega$. We can thus similarly define the product W^θ-Hodge bundle $\mathbb{P}^{W^\theta}(\mathbf{H})$, whose fiber over each point (ω, ρ^θ) in X^{W^θ} is $\mathbf{H}_\omega$. A pair $(\rho^\theta, \mathbf{v}) \in \mathbb{P}^{W^\theta}(\mathbf{H})$ is thus defined to be the lift of the path ρ^θ (starting at ω) to $\mathbb{P}^{W^\theta}(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}^\theta} := \hat{\nu} \otimes \mathbb{P}^\theta$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}^\theta_\omega$.

We recall the following fundamental result due to Ancona [**Anc90**] (see also [**Gru98**, Lemma 4.1]):

Theorem 3.6. [**Anc90**, Théorème 7.3] *For all $\omega \in X$, and $\mathbb{P}_\omega$-almost all paths ρ starting at ω, we have that $d_{\mathbb{D}}(\rho_0\rho_\infty, \rho_T) = O(\log T)$ as $T \to \infty$, where $\rho_0\rho_\infty$ is the geodesic ray with $\rho_0 \in \mathbb{D}$ and $\rho_\infty \in \partial\mathbb{D}$.*

Now observe that our aim is to study

$$\Sigma^g(T, [a,b]) := \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}) \;:\; a \leq \frac{1}{\sqrt{T}}(\sigma(g_T, \mathbf{v}) - T\lambda) \leq b\right\}\right) \tag{3.2.4}$$

as $T \to \infty$.
Let

$$\Sigma^\rho(T, [a,b]) := \hat{\nu}_{\mathbb{P}}\left(\left\{(\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H}) \;:\; a \leq \frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T}, \mathbf{v}) - T\lambda) \leq b\right\}\right) \tag{3.2.5}$$

Lemma 3.7. *The quantity*

$$|\Sigma^g(T, [a,b]) - \Sigma^\rho(T, [a,b])| \to 0 \tag{3.2.6}$$

as $T \to \infty$, $\mathbb{P}_\omega$-almost everywhere and for all $\omega \in X$.

Proof. By applying the disintegration in Lemma 3.3, 3.2.5 is also equal to

$$\Sigma^{\rho}(T,[a,b]) = \mathrm{Leb}\otimes\hat{\nu}_{\mathbb{P}^{\theta}}\Big(\Big\{(\theta,\rho^{\theta},\mathbf{v})\in[0,2\pi]\otimes\mathbb{P}^{W^{\theta}}(\mathbf{H})\ : \tag{3.2.7}$$

$$a \leq \frac{1}{\sqrt{T}}(\sigma(\rho^{\theta}_{\tau_T},\mathbf{v}) - T\lambda) \leq b\Big\}\Big) \tag{3.2.8}$$

$$= \mathrm{Leb}\otimes\hat{\nu}_{\mathbb{P}^{\theta}}\Big(\Big\{(\theta,\rho^{\theta},\mathbf{v})\in[0,2\pi]\otimes\mathbb{P}^{W^{\theta}}(\mathbf{H})\ : \tag{3.2.9}$$

$$a \leq \frac{1}{\sqrt{T}}(\sigma(g_T r_\theta,\mathbf{v}) - T\lambda) + \frac{1}{\sqrt{T}}(\sigma(\rho^{\theta}_{\tau_T},\mathbf{v}) - \sigma(g_T r_\theta,\mathbf{v})) \leq b\Big\}\Big) \tag{3.2.10}$$

Theorem 3.6 applied on τ_T gives that, for all $\omega\in X$, $d_{\mathbb{D}}(g_T r_\theta\cdot 0,\rho^{\theta}_{\tau_T}) = O(\log\tau_T)$ $\mathbb{P}^{\theta}_{\omega}$-almost everywhere as $T\to\infty$. Together with Lemma 3.2, the lemma now follows by the Lipschitz property of the Kontsevich-Zorich cocycle (by the derivative bound in 2.5.1). □

Therefore, it suffices to study the limiting distribution of the quantity

$$\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T},\mathbf{v}) - T\lambda).$$

Observe that we have that for all $\omega\in X$, and $\mathbb{P}_{\omega}$-almost everywhere, $\tau_t\to\infty$ as $T\to\infty$. By applying the stopping time identity $T=\tau_T+W^{(1)}_{\tau_T}+\eta_{\tau_T}$, a straightforward calculation shows the following equality:

$$\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T},\mathbf{v}) - T\lambda) = -\frac{1}{\sqrt{T}}\eta_{\tau_T}\lambda \tag{3.2.11}$$

$$-\frac{1}{\sqrt{T}}W^{(1)}_{\tau_T}\lambda \tag{3.2.12}$$

$$+\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T},\mathbf{v}) - \tau_T\lambda) \tag{3.2.13}$$

So this reduces the proof of the theorem to controlling three terms on the RHS of the previous equality. First, we can observe that 3.2.11 clearly converges to zero $\mathbb{P}_{\omega}$-almost everywhere by Lemma 3.1. Next, it follows by Lemma 3.2 that

$$\lim_{T\to\infty}\frac{\lambda}{\sqrt{T}}W^{(1)}_{\tau_T}\xrightarrow{d}W^{(1)}_{\lambda^2}$$

and in particular the asymptotic variance of 3.2.12 is λ^2. The asymptotic variance of 3.2.13 converges to Φ_{ρ_∞} by Theorem 1.3 and Lemma 3.2, together with the simple observation that

$$\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T},\mathbf{v}) - \tau_T\lambda) = \frac{\sqrt{\tau_T}}{\sqrt{T}}\frac{1}{\sqrt{\tau_T}}(\sigma(\rho_{\tau_T},\mathbf{v}) - \tau_T\lambda) = \frac{\sqrt{\tau_T}}{\sqrt{T}}\frac{1}{\sqrt{\tau_T}}M_{\tau_T}.$$

Remark 3.8. In fact, since M_{τ_T} is an Ito process (see Eq. 3.1.9), it follows also that $\frac{1}{\sqrt{\tau_T}} M_{\tau_T}$ converges in distribution to a centered Gaussian random variable with variance Φ_{ρ_∞}.

The following lemma concerns the covariance of the terms 3.2.12 and 3.2.13, and shows that it converges almost everywhere:

Lemma 3.9. $Cov_{\hat{\nu}_{\mathbb{P}}}\left(\frac{1}{\sqrt{T}} M_{\tau_T}, -\frac{\lambda}{\sqrt{T}} W^{(1)}_{\tau_T}\right) \to -\lambda^2$

Proof. We will first need the following fact, which follows by [**For02**, Lemma 3.1], together with the observation that $\Delta_{L_\omega}(\sigma - u) = 2\lambda$, we have,

$$\frac{\partial}{\partial t}\frac{1}{2\pi}\int_0^{2\pi}(\sigma(z,\mathbf{v}) - u(z))d\theta = \frac{1}{\sinh(2t)}\int_0^t \frac{1}{2\pi}\int_0^{2\pi}\Delta_{L_\omega}(\sigma - u)d\theta \sinh(2r)dr \tag{3.2.14}$$

$$= \lambda\frac{\cosh(2t) - 1}{\sinh(2t)} \tag{3.2.15}$$

$$= \lambda \tanh(t) \tag{3.2.16}$$

We are now ready to calculate the covariance. We have

$$\mathrm{Cov}_{\hat{\nu}_{\mathbb{P}}}\left(\frac{1}{\sqrt{T}} M_{\tau_T}, -\frac{\lambda}{\sqrt{T}} W^{(1)}_{\tau_T}\right) \tag{3.2.17}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{\tau_T}\nabla_{L_\omega}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))\cdot(dW^{(1)}_s, dW^{(2)}_s)\int_0^{\tau_T} dW^{(1)}_s\right] \tag{3.2.18}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{\tau_T}\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))dW^{(1)}_s\int_0^{\tau_T} dW^{(1)}_s\right] \tag{3.2.19}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{\tau_T}\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))ds\right] \tag{3.2.20}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{T}\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v}) - u(\rho_s))ds\right] + o(1) \tag{3.2.21}$$

$$= -\frac{\lambda^2}{T}\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T \tanh(t_s)ds\right] + o(1) \tag{3.2.22}$$

$$\to -\lambda^2 \tag{3.2.23}$$

where 3.2.19 follows by the independence of $W^{(1)}_s$ and $W^{(2)}_s$, and where 3.2.20 follows by an application of Ito's inner product (a more general case of Ito's isometry, which follows by applying the polarization identity), which also holds for our stopping time – in fact, Ito's isometry holds for stochastic integrals with infinite time horizon, and so it also follows for our defined stopping time (see also [**FLJ12**, Lemma VI.4.3]). We also note that 3.2.21 holds thanks to

Lemma 3.2 and the identity within its proof. Finally, 3.2.22 holds thanks to 3.2.16, together with the rotational invariance of the hyperbolic heat kernel. ∎

To conclude the proof of Theorem 1.1, we observe that since the Brownian motion has normally distributed independent increments, a linear combination of Brownian motion terms is also normally distributed. This, together with convergence of the asymptotic covariance in Lemma 3.9, completes the proof, and in particular we have that the asymptotic variance Φ_{g_∞} is

$$\Phi_{g_\infty} = \Phi_{\rho_\infty} + \lambda^2 + 2 \lim_{T\to\infty} \mathrm{Cov}\left(\frac{2}{\sqrt{T}} M_{\tau_T}, -\frac{\lambda\sqrt{2}}{\sqrt{T}} W^{(1)}_{\tau_T} \right) \tag{3.2.24}$$

$$= \Phi_{\rho_\infty} + \lambda^2 - 2\lambda^2 = \Phi_{\rho_\infty} - \lambda^2 \tag{3.2.25}$$

∎

4. Positivity of the variance

4.1. **Random cocycle.** Recall that 3.2.25 says that $\Phi_{g_\infty} = \Phi_{\rho_\infty} - \lambda^2$, and so we also have the following important corollary:

Corollary 4.1. *If $\lambda_2 > 0$, then $\Phi_{\rho_\infty} > 0$*

Proof. Since, by construction, $\Phi_{g_\infty} \geq 0$, and we have that $\Phi_{\rho_\infty} \geq \lambda^2 > 0$, and it is clear that, since $\lambda = \sum \lambda_i$, we have $\lambda^2 \geq \lambda_2^2$. ∎

4.2. **Deterministic cocycle.**

Remark 4.2. It is not clear to us how our formulas can be leveraged to deduce positivity of the variance for the deterministic cocycle. Our claim therefore is that the deterministic cocycle converges in distribution with a $\sqrt{T}$ normalization, and we hope that this result could be useful to specifically address the question of positivity of the variance via different methods.

Appendix A. Solving Poisson's equation

The purpose of this section is to prove a straightforward corollary of the spectral gap of the foliated Laplacian due to Avila-Gouëzel, Avila-Gouëzel-Yoccoz, which will be key to the proofs of our main theorems.

Corollary A.1. [**AGY06, AG13**] *For $f \in L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ with $\int_X f d\mu = 0$, we can find $u \in L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ with $\Delta u = f$. As a consequence, $|\nabla u|^2$ is also in $L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$.*

We follow closely the notation in Avila-Gouëzel, [**AG13**, Section 3.4], and we refer to their paper for more details and references. In particular, following their notation, and for ξ varying in the space Ξ of all unitary irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, let H_ξ be a family of representations. For us, we will be concerned with the following spherical decomposition

$$L^2(\mathrm{SO}_2(\mathbb{R})\backslash X,\nu) \simeq \int_\Xi H_\xi^{\mathrm{SO}_2(\mathbb{R})} dm(\xi) \tag{A.0.1}$$

where $H_\xi^{\mathrm{SO}_2(\mathbb{R})}$ is the set of $\mathrm{SO}_2(\mathbb{R})$-invariant vectors contained in H_ξ, and m a measure on Ξ. It is a fact that the spectrum of the Laplacian Δ on $L^2(\mathrm{SO}_2(\mathbb{R})\backslash X,\nu)$ is equal to the set $\{(1-s(\xi)^2)/4\}$, where ξ is a spherical representation in supp m. Finally, taking into account the direct integral in A.0.1, we also have that the L^2 norm of a function $f : X \to \mathbb{R}$ is given as

$$\|f\|^2 = \int \|f_\xi\|^2_{H_\xi} dm(\xi). \tag{A.0.2}$$

Proof of Corollary A.1. Let

$$u := \int_\Xi \frac{4}{(1-s(\xi)^2)} f_\xi dm(\xi) \tag{A.0.3}$$

Since the spherical unitary irreducible representations are the principal (for which s is purely imaginary) and the complementary ones, we also have

$$\|u\|^2 = \int_\Xi \left\| \frac{4}{(1-s(\xi)^2)} f_\xi \right\|^2_{H_\xi} dm(\xi) \tag{A.0.4}$$

$$= \int_{\Xi_{\mathrm{comp}}} \left\| \frac{4}{(1-s(\xi)^2)} f_\xi \right\|^2_{H_\xi} dm(\xi) + \int_{\Xi_{\mathrm{princ}}\subset[0,\infty]} \left\| \frac{4}{(1+y^2)} f_y \right\|^2_{H_y} dy \tag{A.0.5}$$

We also have that the spectra $\sigma(\Delta)\cap(0,1/4) = \sigma(\Omega)\cap(0,1/4)$, and that the spectral measures coincide, which follows since, in the interval $(0,1/4)$, the complementary series representations are all spherical. In particular, it follows from the work of Avila-Gouëzel and Avila-Gouëzel-Yoccoz that there are finitely many representations in the complementary series (where $s(\xi)$ lies in $(0,1)$) that appear in the decomposition of $L^2(\mathrm{SO}_2(\mathbb{R})\backslash X,\nu)$ into irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, and they have finite multiplicity. This immediately gives

$$\int_{\Xi_{\mathrm{comp}}} \left\| \frac{4}{(1-s(\xi)^2)} f_\xi \right\|^2_{H_\xi} dm(\xi) < \infty \tag{A.0.6}$$

Moreover, and recalling again that for irreducible unitary representations of the principal series, $s(\xi)$ is equal to iy. As a consequence, we also have

$$\int_{\Xi_{\mathrm{princ}} \subset [0,\infty]} \left\| \frac{4}{(1+y^2)} f_y \right\|_{H_y}^2 dy \leq 16 \int_{\Xi_{\mathrm{princ}} \subset [0,\infty]} \|f_y\|_{H_y}^2 \, dy < \infty \tag{A.0.7}$$

giving us that the solution u is in L^2.

To show that $|\nabla u|^2$ is also in L^2, we recall that for X, Y the generators of the geodesic flow (X) and the orthogonal geodesic flow (Y), respectively, we have that $\Delta u = -(X^2 + Y^2)u$ since u is an $\mathrm{SO}_2(\mathbb{R})$-invariant function. We claim that $\langle \Delta u, u \rangle_{L^2} = \|\nabla u\|_{L^2}^2$. Since X and Y are volume-preserving, and therefore also skew-adjoint, we have that

$$\langle \Delta u, u \rangle_{L^2} = \langle Xu, Xu \rangle_{L^2} + \langle Yu, Yu \rangle_{L^2} \tag{A.0.8}$$

$$= \|\nabla u\|_{L^2}^2 \tag{A.0.9}$$

as desired. □

Remark A.2. In fact, more can be said of the radial and angular derivatives of u via the representation theory of $\mathrm{SL}_2(\mathbb{R})$, and we refer to Flaminio-Forni **[FF03]** for details in that direction.

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