A New Hybrid Model for Default Risk

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Default Risk Models

There are general two major approaches to model default risk: the structural (firm value based) model and the reduced form model. Structural models assume that the information available is held by the firm's manager, while reduced form models assume that it is the information observable to the market. Default risk can be captured by credit default swap(CDS) spread. CDS is the most popular and liquid credit derivative. To price a CDS or a basket CDS, the reduced form models are the preferred approach because of the information assumptions and its tractability. While credit risk modeling has two key challenges: incomplete information and multidimensional default risk. We proposed a hybrid model which takes advantages of structural model and reduced form model to resolve the above two challenges.

In the pricing problem, the key task is to find out the survival probability. Denote the default time by τ, the risk-neutral survival probability $P(t)$ is defined as

$$P(t) \equiv P(\tau > t)$$

Model Framework

- We use stock price as a proxy for a firm's value and model the price via an exponential Levy process.
  Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space and assume that the d stock prices $S = (S_t^1, S_t^2, ..., S_t^d)$ of d firms respectively can be written in terms of a d dimensional Lévy process $Y = (Y_t^1, Y_t^2, ..., Y_t^d)$ on $\mathbb{R}^d$.
  For Y, there exists a vector $\vec{b} \in \mathbb{R}^d$, a unique Σ, and a Temporal Poisson random measure(TPRM) X on $E = \mathbb{R}^d \setminus \{0\}$ with intensity π such that
  $$Y_t(\omega) = -\vec{b}t + \Sigma W_t + \int_0^t \int_{|x|\geq 1} xX(ds \times dx)$$
  $$+lim_{\epsilon\downarrow 0} \int_0^t \int_{\epsilon<|x|<1} x\{X(dx \times dx) - ds\pi(dx)\}$$
  and
  $$S_t^j = S_0^j exp\{rt + Y_t^j + t\psi^j(-i)\}$$
  where ψ^j is the characteristic exponent of Y_1^j, $j = 1, 2, ..., d$.
- (Default time) Let S_t be the stock price of the firm. We define the time of default as the first time the log-return of S_t jumps below a level $a(t) < 0$.
  $$\tau = inf\{t > 0 : logS_t/S_{t^-} \leq a(t)\}$$
  We call $a(t)$ the default level of the firm, $a(t)$ could be stochastic.
- The default level $a(t)$ allows the model incorporate the observable or latent risk factors such as volatility, interest rate.

Tail Integrals

- (signed tail integral) Let Y be a R^d-valued Lévy process with Levy measure π. The signed tail integral of Y is the function $\overline{\Lambda} : (R \setminus \{0\})^d \to R$ defined by
  $$\overline{\Lambda}(x_1, ..., x_d) := \Pi_{i=1}^d sgn(x_i)\pi(\Pi_{j=1}^d \mathcal{I}(x_j))$$
- (I-marginal (signed)tail integral) Let Y be a R^d-valued Lévy process and let $I \subset \{1, ..., d\}$ non-empty. The I-marginal (signed)tail integral Λ_I of Y is the (signed)tail integral of the process $Y^I := (Y^i)_{i\in I}$.

Default Probability For One Firm

$$P(\tau > t) = E(e^{-\int_0^t \Lambda(a(s))ds})$$

where $\Lambda(x)$ is the tail integral of the Lévy process Y, $a(s)$ is the default level.

Jump Dependence

- The information of jump component of stock prices are contained in tail integrals and marginal tail integrals.
- Lévy copulas can be used to link marginal tail integrals with tail integrals.
- Default dependency structure or jump dependency are specified by a Lévy copula.
- The joint default probability could be represented only via marginal tail integrals.

Default Probability For N Firms

- For simplicity, we only show results for N=3. Assume (τ^1, τ^2, τ^3) are the survival times of three firms at level $\vec{a} = (a_t^1, a_t^2, a_t^3)$
  $$\mathbb{P}(\tau^1 > x_1, \tau^2 > x_2, \tau^3 > x_3)$$
  $$= E[e^{-\int_0^{x_1} \Lambda_1(a^1(s))ds} e^{-\int_0^{x_2} \Lambda_2(a^2(s))ds} e^{-\int_0^{x_3} \Lambda_3(a^3(s))ds}$$
  $$e^{\int_0^{min(x_1,x_2)} \Lambda_{12}(s)ds} e^{\int_0^{min(x_1,x_3)} \Lambda_{13}(s)ds} e^{\int_0^{min(x_2,x_3)} \Lambda_{23}(s)ds}$$
  $$e^{-\int_0^{min(x_1,x_2,x_3)} \Lambda_{123}(s)ds}]$$
- There exists a Lévy copula F to represent tail integrals via marginal tail integrals. Thus the joint default probability could be represented as
  $$\mathbb{P}(\tau^1 > x_1, \tau^2 > x_2, \tau^3 > x_3)$$
  $$= E[e^{-\int_0^{x_1} \Lambda_1(a^1(s))ds} e^{-\int_0^{x_2} \Lambda_2(a^2(s))ds} e^{-\int_0^{x_3} \Lambda_3(a^3(s))ds}$$
  $$e^{\int_0^{min(x_1,x_2)} F^{12}(\overline{\Lambda}(a^1(s)),\overline{\Lambda}(a^2(s)))ds} e^{\int_0^{min(x_1,x_3)} F^{13}(\overline{\Lambda}(a^1(s)),\overline{\Lambda}(a^3(s)))ds} e^{\int_0^{min(x_2,x_3)} F^{23}(\overline{\Lambda}(a^2(s)),\overline{\Lambda}(a^3(s)))ds}$$
  $$e^{-\int_0^{min(x_1,x_2,x_3)} -F^{123}(\overline{\Lambda}(a^1(s)),\overline{\Lambda}(a^2(s)),\overline{\Lambda}(a^3(s)))ds}]$$

References

[1] Pierre Garreau and Alec Kercheval.
A structural jump threshold framework for credit risk.
2014.

[2] Peter Tankov.
Simulation and option pricing in lévy copula models.
Mathematical Modelling of Financial Derivatives, IMA Volumes in Mathematics and Applications, Springer, 2006.

Simulation of Two Dimensional Default

The following graphs are simulations of the default times τ_1 and τ_2 of the two firms (N and M) for default levels x, y fixed to $a = x = y$ ranging from -1 to -5. In the case of complete dependence, see Fig 3, firm M always defaults first. In the case of a larger value of default level a, the chance of both firm defaulting together is higher. When the jumps in two firms' stocks are independent however, τ_1 and τ_2 are uniformly distributed in the square, see Fig 1: the default times are independent.

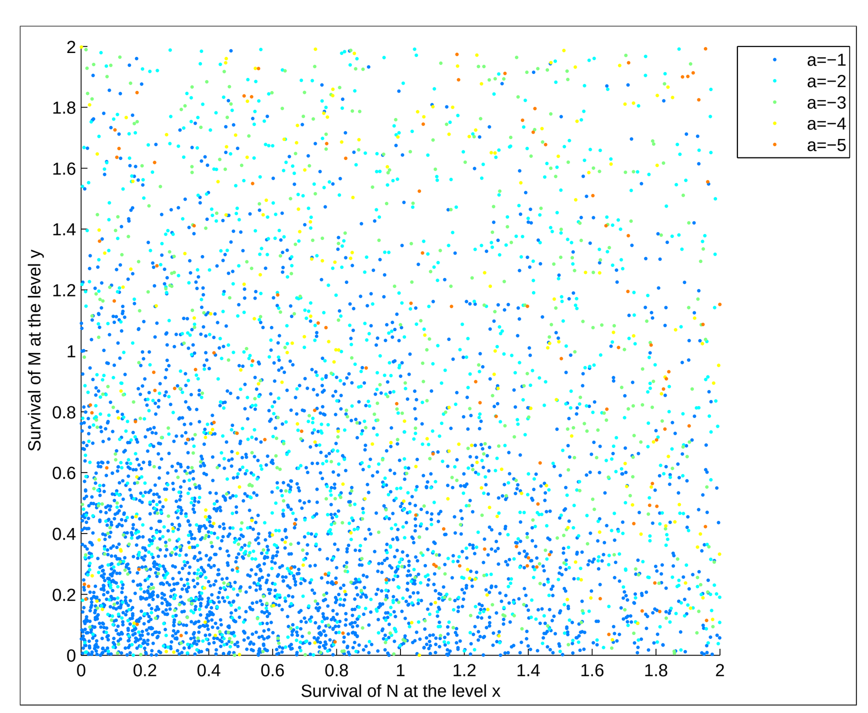

Figure 1: Default intensities are independent

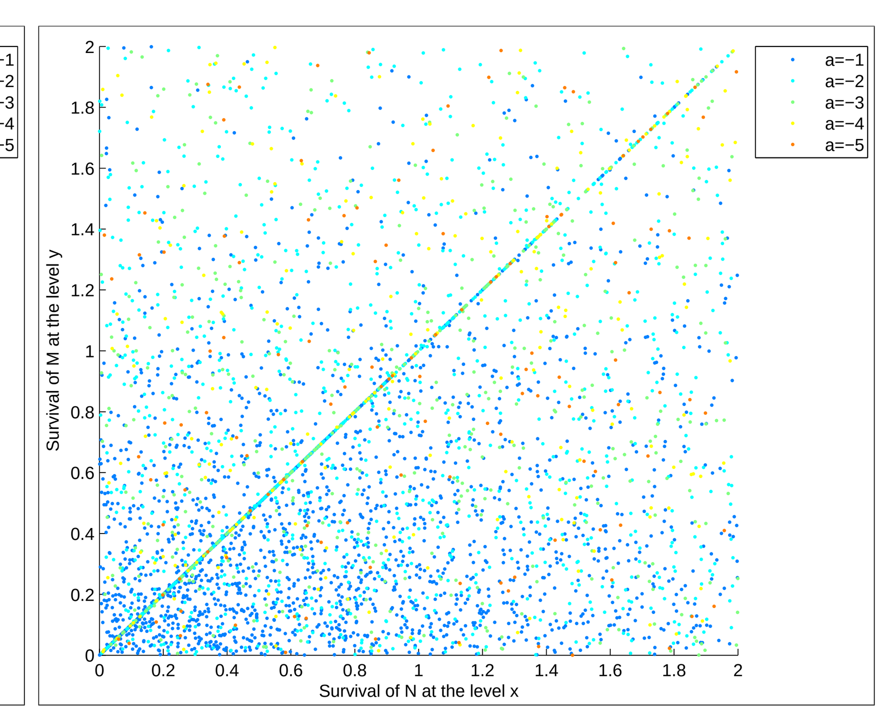

Figure 2: Default intensities are partially dependent

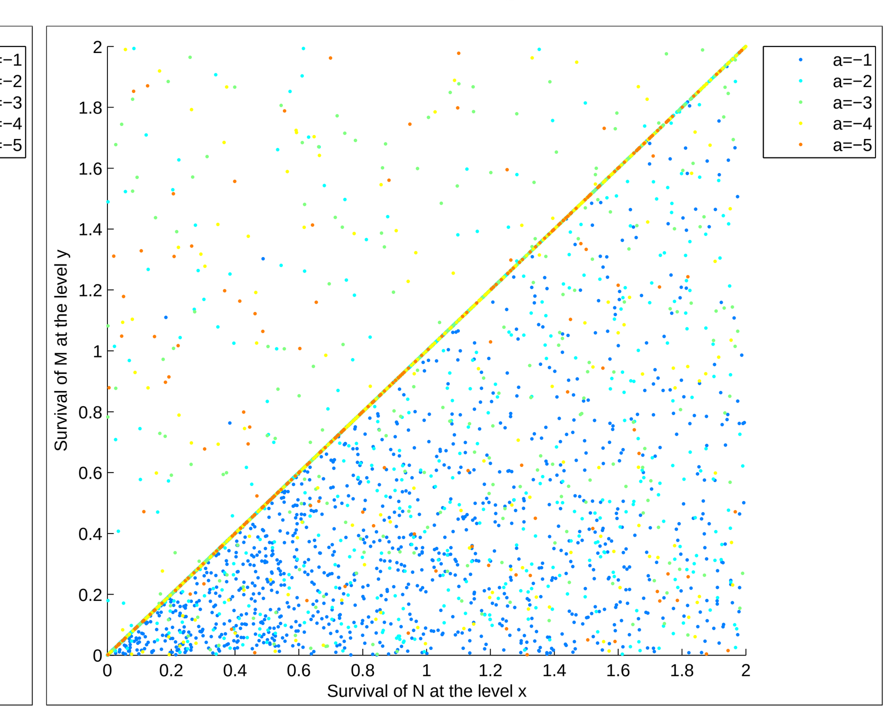

Figure 3: Default intensities are highly dependent