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PRICING CREDIT DEFAULT SWAPS UNDER LÉVY MODELS

By

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ABSTRACT

In this thesis, we discuss the traditional structural model and a new structural model. Under both models, we assume the asset price is driven by a Variance Gamma(VG) process. We price the CDS under the traditional structural model in which default is triggered by the crossing of a preset barrier. Monte Carlo method and a partial integro-differential equation(PIDE) method are used under this model. We introduce the new model in which the time of default is specified as the first jump of the log-returns of a firm's stock price below a level. We price the CDS under the new structural model with a constant default level in particular. The sensitivity of the parameters of the traditional structural model and the new structural model are presented.

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Some Definitions and Notations

Definition 0.0.1. (Characteristic Exponent) *For a random variable* Θ*, define its characteristic function* $\phi(u)$ *as,*

$$\phi(u) = \mathbb{E}(e^{iu\Theta})$$

Then the characteristic exponent $\Psi(u)$ *is*

$$\Psi(u) := -log\phi(u)$$

Definition 0.0.2. (Poisson Distribution) *A random variable* N *is following Poisson Distribution with parameter* λ *if it satisfies*

$$\mathbb{P}(N = k) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2....$$

Proposition 0.0.3. *The mean of a Poisson random variable is* λ

Proposition 0.0.4. *The characteristic function of a Poisson random variable is* $\phi(u) = e^{-\lambda(1-e^{iu})}$

Definition 0.0.5. (Poisson Process) *A Poisson process* $(N_t)_{t\geq 0}$ *with intensity* λ *is a counting process adapted to* $(\mathcal{F}_t)_{t\geq 0}$ *with stationary and independent increments, i.e.*
(i)For s,t ≥ 0*,* $N_{t+s} - N_t$ *is equal in distribution to* N_s*.*
(ii)For $0 \leq s \leq t$*,* $N_t - N_s$ *is independent of* $N_u, u \leq s$*.*
(iii)N_t *is a Poisson distribution with parameter* λt*.*

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CHAPTER 1

INTRODUCTION

Generally, there are two classes of models of default risk. The first class, called structural model, is linking credit risk with the equity market. The original idea was presented by Merton [1]; he observed that both equities and debts can be viewed as options on the value of a firm's asset. In Merton's model, when the value of equity drops to zero, shareholders walk away and the firm defaults on its debt. Another approach among this class is the so-called First-Passage approach [2]. The debt holders have the right to force the firm into default and obtain the firm's remaining assets if the firm value falls to a specified level. In effect, the equity holders then have a barrier option on the firm's assets which gets knocked out if the value of the firm reaches the barrier. A major drawback of models where the firm's assets follow a diffusion process was recognized by Zhou (1997). In [3], Zhou then proposed a model where the dynamics of the firm value is described by a continuous diffusion component and a discontinuous jump component. Jessica Cariboni and Wim Schoutens [4] proposed a different model under which the asset price process is described by an exponential of a Lévy process and is completely driven by jumps. The Variance Gamma process is used in their model.

The second class, the reduced-form models, however, does not directly relate to capital structure of the firm. It specifies the default time as the first jump of a counting process with a stochastic intensity. Jarrow and Turnbull [5] and Lando [6] introduce one reduced form approach which models default as the first arrival of a Poisson process. The reduced form models are popular for their easy implementation and use but it loses the direct intuition.

In this paper, we discuss two kinds of structural models. One of them, we call the old structural model, in which the time of default is given by the first passage time of the stochastic process below a predetermined barrier. The other model, we call the new structural model, which is invented by Pierre Garreau and Alec Kercheval [7], specifies the default time to be the time of the first jump of log-return of a firm's stock price. Under both of them, the underlying asset is described by an exponential Lévy Process. The reason for using Lévy process instead of a diffusion process is that the first passage time of a diffusion is predictable. However, computing the distribution of the first passage time of a Levy process is a difficult problem in general. Monte Carlo simulations and partial-differential equation approaches are possible ways. Before the presence of Pierre Garreau's work [7], however, there were no consistent ways to solve the partial integro-differential equation (PIDE) associated with the first passage time problem of Levy processes in an efficient and general way. Pierre Garreau[7] presented a new method to compute default probabilities in the framework of structural models when the underlying asset follows a general Levy Process. The method he introduced was shown to be suited for any choice of Levy process and faster than both Monte Carlo and finite difference methods. But the survival probability is still not tractable. Apart from this reason, the old structural model still has many drawbacks as stated in [7]. First, the default is defined as the time when the value of the firm or the value of its stock prices crosses a lower threshold. However, this is not realistic because, in the real world, bankruptcy is not immediate. Secondly, in the old model, it is not possible to use the PIDE formulation to value credit derivatives written on a basket of defaultable securities . The approach of solving PIDE is not easily applied to higher dimensional problems. What's more, the calibration of models and valuing of the default probabilities become more difficult in higher dimension case. Thirdly, the true value of a firm is not observable for market participants [8]. The assumption in the old structural model is that the lower threshold is constant and can be deduced from the quarterly reports of the firm. In reality, its true value is a function of several forms of debts and financial instruments, which are unknown to outside investors. In essence, it is stochastic and can only be guessed. All of these motivate to the new structural model. This new structural model is based on the traditional structural models and borrows mathematics from the theory of reduced form models. In this new approach, the daily log-return of a security rather than the stock price is used. The new model combines the advantages from both the structural approach and reduced form approach.

This paper is structured as follows: chapter 2 introduces the basic definitions and properties of Lévy Processes. Chapter 3 introduces the formula of pricing a CDS and Variance Gamma model. Pricing a CDS under the old structural model is also presented in this chapter. Procedure of valuing credit default swaptions by Monte-Carlo Method and PIDE methods are presented. Chapter 4 introduces the new structural model. Only constant default level is considered in this paper. A comparison regarding the sensitivity of model parameters between the old and the new structural models is presented.

CHAPTER 2

AN INTRODUCTION TO LÉVY PROCESSES

Empirical evidence suggests that the underlying (log) normal distribution is not accurately describing the true behavior of the asset. The underlying distribution of stock returns is skewed and has a positive excess kurtosis. Moreover, default events are triggered by sudden shocks in the asset price which cannot be captured by models with a continuous asset path value. Instead, Lévy processes are used. Lévy based models have already shown their capabilities in equity models([9]) and fixed income models ([10]). In addition, the underlying distributions in the Lévy models are very flexible and can take into account asymmetry and fat-tail behaviour. Another advantage of this approach is that the presence of jumps in the underlying process allows for instantaneous default. Hence, there is no need to build this in artificially by making the default barrier stochastic.
Before we investigate the structure of Lévy Processes, we must first define them.

2.1 Lévy Processes and Infinite Divisibility

Definition 2.1.1. (Lévy Process) *A process $X = X_t : t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a levy processs if it possesses the following properties:*
(i) The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.
(ii)$P(X_0 = 0)$=1.
(iii)For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.
(iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $X_u : u \leq s$.

From(i), we can see that the Levy Process is a càdlàg path. (iii) is referred to as the stationary property and (iv) is the independent increment property. From the definition above, it's easy to see that the Brownian Motion and Poisson Process are cases of Levy Processes. There are a variety of Lévy Processes. Any infinitely divisible distribution can be used to construct a Lévy Process.

Definition 2.1.2. (Infinite divisibility) *A random variable X has an infinitely divisible distribution if for each n=1,2,... there exist a sequence of i.i.d random variables $X_{1,n}, X_{2,n}, ..., X_{n,n}$ such that*

$$X \stackrel{d}{=} X_{1,n} + X_{2,n} + ... + X_{n,n}$$

As we know, a random variable can be completely described by its characteristic function or by its characteristic exponent Ψ. Thus we can tell whether a random variable is infinitely divisible via its characteristic exponent.

Proposition 2.1.3. *If $\Psi(u)$ is the characteristic exponent of random variable X, then X has infinitely divisible distribution if for all $n \geq 1$ there exists a characteristic exponent of a probability distribution, say Ψ_n, such that $\Psi(u) = n\Psi_n(u)$ for all $u \in R$.*

Proposition 2.1.4. *For a Lévy process X_t, $t > 0$, the distribution of X_t is infinitely divisible.*

Actually, $\forall$ t and each $n = 1, 2, ...$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + (X_{3t/n} - X_{2t/n}) + ... + (X_t - X_{(n-1)t/n}) \tag{2.1}$$

By the definition of Lévy process, X has stationary independent increments.

Denote the characteristic function of X_t as ϕ_t, denote the characteristic exponent as Φ_t. So from above 2.1, then

$$\phi_t = (\phi_{t/n})^n$$

The characteristic exponent thus to have

$$\Psi_t = n\Psi_{t/n}$$

Theorem 2.1.5. *The characteristic exponent Φ_t of a Lévy process X_t satisfies the following equation:*

$$\Psi_t = t\Psi_1 \tag{2.2}$$

Proof. when t is a rational number, $t = \frac{m}{n}$, as

$$X_m = X_{m/n} + (X_{2m/n} - X_{m/n}) + (X_{3m/n} - X_{2m/n})... + (X_m - X_{(n-1)m/n})$$

we have

$$\Psi_m = n\Psi_{m/n}$$

let $m = n$,Then

$$\Psi_m = m\Psi_1$$

Then

$$\Psi_{m/n} = \frac{1}{n}\Psi_m = \frac{m}{n}\Psi_1$$

The statement is true when t is a rational number. When t is a irrational number, we could find a series of rational numbers which approach t from the right hand side, i.e.$lim_{n\to\infty} t_n = t$, t_n are rational numbers. Then

$$\Psi_t = lim_{n\to\infty}\Psi_{t_n} = lim_{n\to\infty} t_n \Psi_1 = t\Psi_1$$

This is based on the fact that X_t is right continuous, so is Ψ_t. □

So we have the following equation

$$E(e^{i\theta X_t}) = e^{-t\Psi(\theta)} \tag{2.3}$$

where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of X_1, which has an infinitely divisible distribution. So for a lévy process X, for all t, the characteristic function of X_t can be represented by the characteristic function of X_1.

As we know, if we can give out an explicit expression of the characteristic exponent of random variables, then we have characterised the infinitely divisible distributions. The Levy-Khintchine formula gives out such expression.

For simplicity in notation and proof in Levy-Ito decomposition, we now consider lévy process defined on one dimensional R rather than R^d, $d > 1$. The results for a high dimensional version will be given in the next, but the idea and theory is quite similar with those of one dimension.

Theorem 2.1.6. (Levy-Khintchine formula) *A probability law μ of a real valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_R e^{i\theta x}\mu(dx) = e^{-\Psi(\theta)} for \quad \theta \in R \tag{2.4}$$

if and only if there exists a triple (a, σ, Π), where $a \in R$, $\sigma \geq 0$ and Π is a measure concentrated on $R \setminus \{0\}$ satisfying $\int_R (1 \wedge x^2)\Pi(dx) < \infty$, such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{(|x|<1)})\Pi(dx) \tag{2.5}$$

for every $\theta \in R$.

So we can see that a Lévy process can be fully characterised by $\Psi(\theta)$. While $\Psi(\theta)$ has a particular expression as described in $Levy - Khintchine\ formula$, we call $\Psi(\theta)$ the characteristic exponent of the Levy process.
Each Levy process can be associated with an infinitely divisible distribution. Actually, any infinitely divisible distribution can be used to construct a levy process X, such that X_1 has that distribution.

Theorem 2.1.7. (Levy-Khintchine formula for Levy process) *Suppose that $a \in R, \sigma \geq 0$ and Π is a measure concentrated on $R \setminus \{0\}$ such that $\int_R (1 \wedge x^2)\Pi(dx) < \infty$. From this triple define for each $\theta \in R$,*

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{(|x|<1)})\Pi(dx) \tag{2.6}$$

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ on which a Levy process is defined having characteristic exponent Ψ.

Write $\Psi(\theta)$ as sum of three parts,

$$\Psi(\theta) = \Psi^{(1)}(\theta) + \Psi^{(2)}(\theta) + \Psi^{(3)}(\theta) \tag{2.7}$$

$$\begin{aligned} \Psi^{(1)} &= ia\theta + \frac{1}{2}\sigma^2\theta^2 \\ \Psi^{(2)} &= \Pi(R \setminus (-1,1)) \int_{|x|\geq 1} (1 - e^{i\theta x}) \frac{\Pi(dx)}{\Pi(R \setminus (-1,1))} \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
\Psi^{(3)} &= \int_{0<|x|<1} (1 - e^{i\theta x} + i\theta x 1_{(|x|<1)})\Pi(dx) \\
&= \sum_{n\geq 0} \{\lambda_n \int_{2^{-(n+1)}\leq |x| < 2^{-n}} (1 - e^{i\theta x}) F_n(dx) \\
&+ i\theta\lambda_n \int_{2^{-(n+1)}\leq |x| < 2^{-n}} x F_n(dx)\}
\end{aligned}
\tag{2.9}
$$

where $\lambda_n = \Pi\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\}$ and $F_n(dx) = \Pi(dx)/\lambda_n$.

The idea of Levy-Ito decomposition is trying to find three processes in which each of them corresponds to one characteristic function. $\Psi^{(1)}$ obviously corresponds to the Brownian motion with drift. $\Psi^{(2)}$ corresponds to the Compound Poisson Process, $\Psi^{(3)}$ corresponds to a process which is a limit of a compound poisson process with drift.

The Lévy-Ito Decompositon Theorem says that a Lévy Process is a sum of three independent Lévy Process. That implies that the characteristic exponent of a lévy process can be written as the sum of each characteristic exponent. So if we can find three such independent Lévy Processes, more specifically, one is the Brownian Motion, one is the Compound Poisson Process and one is the square integrable martingale, whose exponent character is $\Psi^{(3)}$. And if the three processes are also Lévy Processes, then the theorem is proved, as the independent sum of the Lévy Processes is still a Lévy Process. Actually ,Lévy-Ito Decomposition Theorem gives out the structure of a lévy process, a diffusion process plus the jump component with different jump size. To see those, we must introduce the new concept of Poisson Random Measure.It is the core of the Lévy-Ito Decomposition Theorem.

2.2 Poisson Random Measure

Defining a random measure often involves two measurable spaces. Generally speaking, it is a random variable in one space but a measure in the other. In the space where it is defined as a measure, it is natural for us to define integrals with respect to this measure. The integral is still a random variable in the other space. The Poisson random measure is one kind of random measure. The integral defined by the Poisson random measure is the essence of Lévy-Ito decomposition theorem.

Definition 2.2.1. (Random measure) *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $(S, \mathcal{S})$ *a measurable space. The function M:* $\Omega \times \mathcal{S} \to [0, \infty]$ *is called a random measure if*
(i) $\forall A \in \mathcal{S}$*,M(.,A) is a random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$*.*
(ii) $\forall \omega \in \Omega$ *,* $M(\omega, .)$ *is a measure on* $(S, \mathcal{S})$

Definition 2.2.2. (Poisson random measure) *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $(S, \mathcal{S}, \eta)$ *a measurable space. A random measure X is a Poisson random measure on* $\mathcal{S}$ *with intensity* η *if:*

(i) $\forall A \in \mathcal{S}$, *X(A) is a Poisson random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with parameter* $\eta(A)$.
(ii)$\forall \{A_i\}_{i=1}^n \in \mathcal{S}$ *such that* $A_i \bigcap A_j = \emptyset$ *if* $i \neq j$, *the r.v.* $X(A_i)$ *and* $X(A_j)$ *are independent.*

If we think of $(S, \mathcal{S}, \eta)$ as a product of two measurable spaces, $([0,\infty), \mathcal{B}[0,\infty), Leb^1) \times (E, \mathcal{E}, \pi)$, one measurable space takes the time component and the other takes the space component. Then, $S = [0,\infty) \times E$. Define the σ-algebra on S as the one which is generated by all measurable rectangles[2].Define the measure η as the product measure $Leb \times \pi$ in general way[3].Then we can define the Temporal Poisson random measure. Temporal Poisson random measure is at the center of the Lévy Process. TPRM is nothing more than a poisson random measure defined on the specific space.
We know when defining a stochastic process $X_t(\omega)$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is often constructed by a filtration $(\mathcal{F}_t)_{t\geqslant 0}$. $X_t(\omega)$ is required to be adapted to the filtration. Similar to the stochastic process, we need a Poisson random measure to be adapted to $(\mathcal{F}_t)_{t\geqslant 0}$.

Definition 2.2.3. (Adaptedness of a counting measure) *Let* $(E, \mathcal{E})$ *be a measure space. A random counting measure X on* $([0,\infty) \times E, \mathcal{B}[0,\infty) \otimes \mathcal{E})$ *is said to be adapted to the filtration* $(\mathcal{F}_t)_{t\geqslant 0}$ *if*

$$\forall B \in \mathcal{B}[0,\infty) \otimes \mathcal{E}, X(B) \in \mathcal{F}_t.$$

Definition 2.2.4. (Temporal Poisson random measure-TPRM) *Let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ *be a filtered probability space,* $(E, \mathcal{E}, \pi)$ *a measure space, and Leb denote Lebesgue measure on* $[0,\infty)$. *A temporal Poisson random measure (TPRM) X on E with intensity* π *is an* $\mathcal{F}$*-adapted Poisson random measure on* $([0,\infty) \times E, \mathcal{B}[0,\infty) \otimes \mathcal{E})$ *with intensity* $Leb \times \pi$. *For* $t \geq 0, A \in \mathcal{E}$ *we write* $X_t(A) = X([0,t] \times A)$.

With this TPRM, we can define the integral with this measure. For $t \geq 0, A \in \mathcal{E}$, we can write the measure in a integral form

$$X(\omega, [0,t] \times A) = \int_0^t \int_A X(\omega, ds \times dx) \tag{2.10}$$

We have known that a Poisson random measure is counting the events falling in the set, so the integral above gives out the number of events falling in the set A between time 0 and time t. Further, we consider integral with measurable function x

$$\int_0^t \int_A xX(\omega, ds \times dx) \tag{2.11}$$

This integral is the key for understanding Levy-Ito decomposition. Actually, this integral is equivalent to the compound Poisson process if $\pi(A) < \infty$.
Roughly speaking, X contains the information about how frequently the jumps happen. The x that appears in $\int_0^t \int_A xX(\omega, ds \times dx)$ represents the jump size, so the integral represents the position at time t after having jumps through time 0 to time t. We can see how this describes the jump structure by the following examples.

[1] Lebesgue measure on $[0,\infty)$
[2] the sets with the form $A \times B$, $A \in \mathcal{B}[0,\infty)$, $B \in \mathcal{E}$
[3] if C is a measurable rectangle,i.e. $C = A \times B$. $A \in \mathcal{B}[0,\infty)$, $B \in \mathcal{E}$, then $\eta(C) = Leb(A) \times \pi(B)$

When given the space $([0,\infty) \times R, \mathcal{B}[0,\infty) \times \mathcal{B}(R), dt \times \pi(dx))$, whether or not there exits a TPRM with intensity π? This can be showed in the next theorem.

Theorem 2.2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(S, \mathcal{S}, \eta)$ a measurable space. There exists a Poisson random measure on $(S, \mathcal{S}, \eta)$ with intensity η*

Proof. The proof is trying to construct a random measure on $(S, \mathcal{S}, \eta)$ from a Poisson random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.
First, suppose that S is a space with finite measure. i.e. $\eta(S) < \infty$. There exists a standard construction of infinite product space,say $(\Omega, \mathcal{F}, \mathbb{P})$, on which the independent random variables N and $\{v_1, v_2, ...\}$ are collectively defined such that N has a poisson distribution with parameter $\eta(S)$ and each of variables v_i have distribution $\eta(dx)/\eta(S)$ on S. Define a function X on $\mathcal{S}$, where $X : A \to \{0, 1, 2...\} \bigcup \{\infty\}$. For each $A \in \mathcal{S}$,

$$X(A) = \Sigma_{i=1}^{N} 1_{v_i \in A}$$

then $X(S) = N$. And X(A) is obviously measurable for each $A \in \mathcal{S}$.
A poisson process $M_t : t \geq 0$,the law of $T_1, T_2, ...T_n$ (jump arrival time) is conditional on the event that $M_t = n$ is the same as the law of an ordered independent sample size of n from uniform distribution on $[0, t]$. So the random variable $X(A)$, conditioned on the event $\{N = n\}$, is a binomial random variable with probability of success given by $\eta(A)/\eta(S)$.
For any disjoint $A_1, ...A_k$ in S.

$$P(X(A_1) = n_1, ...X(A_k) = n_k | N = n) = \frac{n!}{(n - \sum_{i=1}^{k} n_i)! n_1! ... n_k!} \prod_{i=0}^{k} (\frac{\eta(A_i)}{\eta(S)})^{n_i} \tag{2.12}$$

Integrating out the condition on N it follows that

$$\begin{aligned} P(X(A_1) = n_1, ...X(A_k) = n_k) &= \sum_{n \geq \sum_{i=1}^{k} n_i} e^{-\eta(S)} \frac{(\eta(S))^n}{n!} \frac{n!}{(n - \sum_{i=1}^{k} n_i)! n_1! ... n_k!} \prod_{i=0}^{k} (\frac{\eta(A_i)}{\eta(S)})^{n_i} \\ &= \sum_{n \geq \sum_{i=1}^{k} n_i} e^{-\eta(S)} \frac{(\eta(S))^{n - \sum_{i=1}^{k} n_i}}{(n - \sum_{i=1}^{k} n_i)!} (\prod_{i=1}^{k} e^{-\eta(A_i)} \frac{(\eta(A_i))^{n_i}}{n_i!}) \\ &= \prod_{i=1}^{k} e^{-\eta(A_i)} \frac{(\eta(A_i))^{n_i}}{n_i!} \end{aligned} \tag{2.13}$$

So $X(A_1)$,$X(A_2)$,...$X(A_k)$ are independent and Poisson distributed. It's easy to conclude X is a Poisson random measure with intensity η. When we proved the result for the space with finite measure, it's easy to get the same result for σ-finite space. □

Example 2.2.6 (Compound Poisson process)**.** *Let $E = R\backslash\{0\}$, $\mathcal{E} = \mathcal{B}(E)$. Suppose $X = X_t : t \geq 0$ is a compound Poisson process taking the form*

$$X_t = \sum_{i=1}^{N_t} \xi_i, t \geq 0 \tag{2.14}$$

$\xi_i : i \geq 1$ is independent and identically distributed random variables with common distribution F on $(E, \mathcal{E})$. Further let $T_i : i \geq 0$ be the times of arrival of the Poisson process $N = N_t : t \geq 0$ with $\lambda > 0$ independent of $\{\xi_i\}_{i \geq 0}$.

Define the random measure X on $[0, \infty) \times E$: $\forall B \in \mathcal{B}[0, \infty) \times \mathcal{E}$,$\omega \in \Omega$

$$X(\omega, B) = \sharp\{i \geq 0 : (T_i(\omega), \xi_i(\omega)) \in B\} = \sum_{i=1}^{\infty} 1_{((T_i, \xi_i) \in B)} \tag{2.15}$$

For $t \geq 0$, $B \in \mathcal{B}$, we get

$$X(\omega, [0, t] \times B) = \int_0^t \int_B X(\omega, ds \times dx) = \sum_{i=1}^{N_t} I(Y_i(\omega), B) \tag{2.16}$$

where $I(Y_i(\omega), B) = 1$ if $Y_i(\omega) \in B$, otherwise $I(Y_i(\omega), B) = 0$.
So X counts the number of those jumps of this compound poisson process that have jump size falls in set B between time 0 to time $t.X(\omega, [0, t] \times B)$ is a Poisson random variable with parameter $\lambda t F(B)$ X also satisfies the independent conditions for a TPRM(the proof is same in lemma 1). So X is a TPRM on E with intensity $\pi(dx) = \lambda F(dx)$.

From this example, we can see that TPRM is the tool to count the number of jumps of compound poisson process. It has been shown that given a compound poisson process, there is a TPRM associated with it. Then, if we are given a TPRM on $E = R^d \setminus \{0\}$ with intensity π, can we construct a compound poisson process with it? The answer is true.In the following section, we will see the integral $\int_0^t \int_A x X(\omega, ds \times dx)$ could be used to represent the compound poisson process itself. Further, we could use TPRM to construct a compound poisson process with drift which is a square integrable martingale. These two processes are two components of Levy Process, as a Lévy process is roughly a linear Brownian motion plus a compound poisson process plus a square integrable martingale is the limit of a series of compound poisson processes with drift.
In general, with different conditions of the intensity π, we can construct processes with different properties. We will present this in the next chapter.

2.3 Lévy-Ito Decomposition Theorem

The lemma listed in the following are the essence of Levy-Ito decomposition Theorem.

Lemma 2.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.Take $S = [0, \infty)$ and $\mathcal{S}$ to be the Borel σ-algebra on S. Theorem*
Suppose that N is a Poisson random measure on $(S, \mathcal{S}, \eta)$. Let $f : S \to R$ be a measurable function.
(i) Then

$$X = \int f(x)N(dx) \tag{2.17}$$

is almost surely absolutely convergent if and only if

$$\int_S (1 \wedge |f(x)|)\eta(dx) < \infty \tag{2.18}$$

(ii)When condition $\int_S (1 \wedge |f(x)|)\eta(dx) < \infty$ holds, then

$$E(e^{i\beta X}) = exp\{-\int_S (1 - e^{i\beta f(x)})\eta(dx)\} \tag{2.19}$$

for any $\beta \in R$.
(iii)Further

$$E(X) = \int_S f(x)\eta(dx) when \int_S |f(x)|\eta(dx) < \infty \tag{2.20}$$

and

$$E(X^2) = \int_S f(x)^2\eta(dx) + (\int_S f(x)\eta(dx))^2 when \int_S f(x)^2\eta(dx) < \infty \tag{2.21}$$

Proof. (i) Like the mechanism which we learned in measure theory, we can prove this theorem by first proving the simple functions case.

$$f(x) = \sum_{i=1}^{n} f_i 1_{A_i}(x),$$

where f_i is constant and $A_i : i = 1, ..., n$ are disjoint sets in $\mathcal{S}$ and further $\eta(A_1 \bigcup ... \bigcup A_n) < \infty$. For such functions we have

$$X = \sum_{i=1}^{n} f_i N(A_i)$$

which is clearly finite with probability one. N is a poisson random measure so $N(A_i)$ has Poisson distribution with parameter $\eta(A_i) < \infty$ and $N(A_i)$ are independent. The characteristic function of a Poisson distribution with parameter $\lambda > 0$ is $exp\{-\lambda(1 - e^{-\theta})\}$),Then

$$\begin{aligned} E(e^{-\theta X}) &= \Pi_{i=1}^{n} E(e^{-\theta f_i N(A_i)}) \\ &= \Pi_{i=1}^{n} exp\{-(1 - e^{-\theta f_i})\eta(A_i)\} \\ &= exp\{-\Sigma_{i=1}^{n}(1 - e^{-\theta f_i})\eta(A_i)\} \end{aligned} \tag{2.22}$$

So

$$E(e^{-\theta X}) = exp\{-\int_S (1 - e^{-\theta f(x)})\eta(dx)\}$$

Next we think about a positive measurable function f. There exists a pointwise increasing sequence of positive simple function $\{f_n : n \geq 0\}$ such that $lim_{n\uparrow\infty} f_n = f$. We use the monotone convergence

theorem ,bounded convergence and dominated convergence theorem to get the same result. Last, we consider a general measurable function f which can be written as $f = f^+ - f^-$.
To prove (ii) and (iii), we first prove positive simple function ,then positive measurable function, in the last, measurable function. $\square$

Lemma 2.3.2. *Suppose that N is a Poisson random measure on* $([0,\infty)\times R, B[0,\infty)\times\mathcal{B}(R), dt\times\Pi(dx))$ *where* Π *is a measure concentrated on* $R\backslash\{0\}$ *and* $B\in\mathcal{B}(R)$ *such that* $0<\Pi(B)<\infty$. *Then*

$$X_t \equiv \int_{[0,t]}\int_B xN(ds\times dx), t\geq 0 \tag{2.23}$$

is a compound Poisson process with arrival rate $\Pi(B)$ *and jump distribution* $\Pi(B)^{-1}\Pi(dx)|_B$.

Proof. Since $\Pi(B)<\infty$, Next note that for all $0\leq s<t<\infty$,

$$X_t - X_s = \int_{(s,t]}\int_B xN(ds\times dx) \tag{2.24}$$

which is independent of $X_u : u\leq s$ as N gives independent counts over disjoint regions. As from Lemma 1.3.1(ii)

$$\begin{aligned} E(e^{i\theta X_t}) &= exp\{-t\int_B(1-e^{i\theta x})\Pi(dx)\} \\ &= exp\{-t\Pi(B)\int_B(1-e^{i\theta x})\frac{\Pi(dx)}{\Pi(B)}\} \end{aligned} \tag{2.25}$$

So

$$\begin{aligned} E(e^{i\theta(X_t-X_s)}) &= \frac{E(e^{i\theta X_t})}{E(e^{i\theta X_s})} \\ &= exp\{-(t-s)\int_B(1-e^{i\theta x})\Pi(dx)\} \\ &= E(e^{i\theta X_{t-s}}) \end{aligned}$$

so increments are stationary. From characteristic function of X, we can see X_t is a compound Poisson process with jump distribution and arrival rate given by $\Pi(B)^{-1}\Pi(dx)|_B$ and $\Pi(B)$. $\square$

Lemma 2.3.3. *Suppose that N is a Poisson random measure on* $([0,\infty)\times R, B[0,\infty)\times\mathcal{B}(R), dt\times\Pi(dx))$ *where* Π *is a measure concentrated on* $R\backslash\{0\}$ *with the additional assumption that* $\int_B|x|\Pi(dx)<\infty$.
(i)The compound Poisson process with drift

$$M_t := \int_{[0,t]}\int_B xN(ds\times dx) - t\int_B x\Pi(dx), t\geq 0 \tag{2.26}$$

is a P-martingale with respect to the filtration

$$\mathcal{F}_t = \sigma(N(A) : A\in\mathcal{B}[0,t]\times\mathcal{B}(R)), t>0$$

(ii) If further, $\int_B x^2\Pi(dx)<\infty$ *then it is a square integrable martingale.*

Proof. (i)First note that the process $M = M_t : t > 0$ is adapted to the filtration $\mathcal{F}_t : t \geq 0$. Next note that for each $t > 0$,

$$E(|M_t|) \leq E(\int_{[0,t]} \int_B |x| N(ds \times dx) + t \int_B |x| \Pi(dx))$$

is finite because $\int_B |x|\Pi(dx)$ is. Next use the fact that M has stationary independent increments to deduce that for $0 \leq s \leq t < \infty$,

$$\begin{aligned} E(M_t - M_s|\mathcal{F}_s) &= E(M_{t-s}) \\ &= E(\int_{[s,t]} \int_B xN(ds \times dx)) - (t-s) \int_B x\Pi(dx) \\ &= 0 \end{aligned} \tag{2.27}$$

(ii)From Lemma 1.3.1(iii), as $\int_B |x|\Pi(dx) < \infty$,together with the assumption that $\int_B x^2\Pi(dx) < \infty$, we have

$$\begin{aligned} E(\{M_t + t\int_B x\Pi(dx)\}^2) &= E(\{\int_{[0,t]} \int_B xN(ds \times dx)\}^2) \\ &= \int_{[0,t]} \int_B x^2 dt \times \Pi(dx) + (\int_{[0,t]} \int_B xdt \times \Pi(dx))^2 \\ &= t\int_B x^2\Pi(dx) + t^2(\int_B x\Pi(dx))^2 \end{aligned} \tag{2.28}$$

Also,

$$E(\{M_t + t\int_B x\Pi(dx)\}^2) = E(M_t^2) + 2tE(M_t \int_B x\Pi(dx)) + t^2 E((\int_B x\Pi(dx))^2) \tag{2.29}$$

and with the martingale property that $E(M_t) = 0$

$$E(\{M_t + t\int_B x\Pi(dx)\}^2) = E(M_t^2) + t^2(\int_B x\Pi(dx))^2 \tag{2.30}$$

It follows that

$$E(M_t^2) = t\int_B x^2\Pi(dx) < \infty \tag{2.31}$$

as required. □

In conclusion, suppose that N is a Poisson random measure on $([0,\infty) \times R, \mathcal{B}[0,\infty) \times \mathcal{B}(R), dt \times \pi(dx))$ where π is a measure concentrated on $R\backslash\{0\}$ and $B \in R$.Then we have:

(1).$\int_B 1 \wedge |x|\pi(dx) < \infty$ if and only if $\int_{[0,t]} \int_B xN(ds \times dx)$ is almost surely absolutely convergent.

(2).If $\int_B 1 \wedge |x|\pi(dx) < \infty$, denote $X_t = \int_{[0,t]} \int_B xN(ds \times dx), t \geq 0$,then

$$E(e^{i\beta X_t}) = exp\{-\int_B (1 - e^{i\beta x})\pi(dx)\} \tag{2.32}$$

(3).If $0 < \pi(B) < \infty$. Then

$$\int_{[0,t]}\int_B xN(ds \times dx), t \geq 0 \tag{2.33}$$

is a compound Poisson process with arrival rate $\pi(B)$ and jump distribution $\pi(B)^{-1}\pi(dx)|_B$.
(4).if $\int_B |x|\pi(dx) < \infty$.
The compound Poisson process with drift

$$M_t := \int_{[0,t]}\int_B xN(ds \times dx) - t\int_B x\pi(dx), t \geq 0 \tag{2.34}$$

is a P-martingale with respect to the filtration

$$\mathcal{F}_t = \sigma(N(A) : A \in \mathcal{B}[0,t] \times \mathcal{R}), t > 0 \tag{2.35}$$

(5). If $\int_B |x|\pi(dx) < \infty$, $\int_B x^2\pi(dx) < \infty$ then

$$M_t := \int_{[0,t]}\int_B xN(ds \times dx) - t\int_B x\pi(dx), t \geq 0 \tag{2.36}$$

a square integrable martingale.
If $\int_B 1 \wedge |x|^2\pi(dx) < \infty$, then from the above lemmas, we get

$$\int_{[0,t]}\int_{x\geq 1} xN(ds \times dx) \tag{2.37}$$

is a compound poisson process. In addition, for each $1 > \epsilon > 0$,

$$\int_{[0,t]}\int_{\epsilon\leq|x|<1} xN(ds \times dx) - t\int_{\epsilon\leq|x|<1} x\pi(dx) \tag{2.38}$$

is a compound poisson process with drift which is square integrable martingale.It can be shown in [11] that there exists a Levy process, which is also a square integrable martingale, to which the above process converges uniformly on $[0,T]$ along an appropriate deterministic subsequence in ϵ.

Theorem 2.3.4. [11] *Suppose that N is a Poisson random measure on $([0,\infty) \times R, B[0,\infty) \times \mathcal{B}(R), dt \times \Pi(dx))$ where Π is a measure concentrated on $R\backslash\{0\}$ and $\int_{(-1,1)} x^2\pi(dx) < \infty$. For each $\epsilon \in (0,1)$ define the martingale*

$$M_t^\epsilon = \int_{[0,t]}\int_{B_\epsilon} xN(ds \times dx) - t\int_{B_\epsilon} x\pi(dx), t \geq 0 \tag{2.39}$$

and let $\mathcal{F}_t^*$ be equal to the completion of $\bigcap_{s>t}\mathcal{F}_s$ by the null sets of $\mathbb{P}$ where

$$\mathcal{F}_t = \sigma(N(A) : A \in \mathcal{B}[0,t] \times \mathcal{R}), t > 0$$

Then there exists a martingale $M = M_t : t \geq 0$ with the following properties,
(i)for each $T > 0$, there exists a deterministic subsequence $\{\epsilon_n^T : n = 1, 2, ...\}$ with $\epsilon_n^T \downarrow 0$ along which

$$\mathbb{P}(lim_{n\uparrow\infty} sup_{0\leq s\leq T}(M_s^{\epsilon_n^T} - M_s)^2 = 0) = 1 \tag{2.40}$$

(ii)it is adapted to the filtration $\{\mathcal{F}_t^* : t \geq 0\}$,
(iii)it has right continuous paths with left limits almost surely, (iv)it has, at most, a countable number of discontinuities on $[0, T]$ almost surely and
(v)it has stationary and independent increments.
In short, there exists a Lévy process, which is also a martingale with a countable number of jumps to which, for any fixed $T > 0$, the sequence of martingales $\{M_t^\epsilon : t \leq T\}$ converges uniformly on $[0, T]$ with probability one along a subsequence in ϵ which may depend on T.

Theorem 2.3.5. (Lévy-Ito decomposition) *Given any $a \in R, \sigma \geq 0$ and measure Π concentrated on $R \setminus \{0\}$ satisfying*

$$\int_R (1 \wedge x^2)\Pi(dx) < \infty. \tag{2.41}$$

there exists a probability space on which three independent Levy processes exist, $X^{(1)}, X^{(2)}$ and $X^{(3)}$ where $X^{(1)}$ is a linear Brownian motion with drift given by

$$X_t^{(1)} = \sigma B_t - at, t \geq 0 \tag{2.42}$$

$X^{(2)}$ is a compound Poisson process given by

$$X_t^{(2)} = \sum_{t=1}^{N_t} \xi_i, t \geq 0 \tag{2.43}$$

$N_t : t \geq 0$ is a Poisson process with rate $\Pi(R \backslash (-1, 1))$, and $\{\xi_i : i \geq 1\}$ are independently identically distributed with distribution $\Pi(dx)/\Pi(R \setminus (-1, 1))$ concentrated on $x : |x| \geq 1$. $X^{(3)}$ is a square integrable martingale with an almost surely countable number of jumps on each finite time interval which are of magnitude less than unity and with characteristic exponent given by $\Psi^{(3)}$. By taking $X = X^{(1)} + X^{(2)} + X^{(3)}$ we see that the conclusion of Theorem above holds, that there exists a probability space on which a Levy process is defined with characteristic exponent

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{(|x|<1)})\Pi(dx) \tag{2.44}$$

for each $\theta \in R$.
(ii)Conversely, if X is a Lévy process, there exists $a \in R$, $\sigma \geq 0$, and a TPRM N with intensity π, as above, so that $X = X^{(1)} + X^{(2)} + X^{(3)}$.
The measure π is called the Lévy measure of X.

Now we could start to develop a main idea in the Levy-Ito decomposition. We can take $X^{(1)}$ to be the linear Brownian motion defined on some probability space $(\Omega^1, \mathcal{F}^1, P^1)$, and from the theorem we prove in the previous section, there exists $(\Omega^2, \mathcal{F}^2, P^2)$, on which one may construct a Poisson random measure N on $([0, \infty) \times R, B[0, \infty) \times \mathcal{B}(R), dt \times \Pi(dx))$. Now define

$$X_t^{(2)} = \int_{[0,t]} \int_{|x| \geq 1} xN(ds \times dx), t \geq 0 \tag{2.45}$$

and as $\Pi(R \backslash (-1, 1)) < \infty$, it is a compound Poisson process with rate $\Pi(R \setminus (-1, 1))$ and jump distribution $\Pi(R \setminus (-1, 1))^{-1}\Pi(dx)|_{R \backslash (-1,1)}$. (We may assume without loss of generality that $\Pi(R \setminus$

$(-1,1)) > 0$ as otherwise we may simply take the process $X^{(2)}$ as the process which is identically zero.)
Next we construct a Levy Process having only small jumps. For each $1 > \epsilon > 0$ define similarly the compound Poisson process with drift

$$X_t^{3,\epsilon} = \int_{[0,t]} \int_{\epsilon \le |x| < 1} xN(ds \times dx) - t \int_{\epsilon \le |x| < 1} x\Pi(dx), t \ge 0 \tag{2.46}$$

(As in the definition of $X^{(2)}$) we shall assume without loss of generality $\Pi(x : |x| < 1) > 0$ otherwise the process $X^{(3)}$ may be taken as the process which is identically zero). We can compute its characteristic exponent,

$$\Psi^{(3,\epsilon)}(\theta) := \int_{\epsilon \le |x| < 1} (1 - e^{i\theta x} + i\theta x)\Pi(dx). \tag{2.47}$$

According to theorem 2.3.4,as we have $\int_{(-1,1)} x^2\Pi(dx) < \infty$, there exists a Levy process, which is also a square integrable martingale, defined on $(\Omega^2, \mathcal{F}^2, P^2)$, to which $X^{(3,\epsilon)}$ converges uniformly on $[0, T]$ along an appropriate deterministic subsequence in ϵ.The characteristic exponent of this latter Levy process is equal to

$$\Psi^{(3)}(\theta) := \int_{|x| < 1} (1 - e^{i\theta x} + i\theta x)\Pi(dx). \tag{2.48}$$

And we can also show they are three independent processes.
To conclude,define the process

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, t \ge 0$$

This process is defined on the product space

$$(\Omega, \mathcal{F}, P) = ((\Omega^1, \mathcal{F}^1, P^1)) \times (\Omega^2, \mathcal{F}^2, P^2) \tag{2.49}$$

has stationary independent increments, has paths that are right continuous with left limits and has characteristic exponent

$$\Psi(\theta) = \Psi^{(1)}(\theta) + \Psi^{(2)}(\theta) + \Psi^{(3)}(\theta) \tag{2.50}$$

$$= ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{|x|<1})\Pi(dx) \tag{2.51}$$

In the end, we give out the Ito-Lévy decomposition for higher dimensional case.

Theorem 2.3.6. (Ito-Lévy decomposition) *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$ be a filtered probability space.*
(i) Given a vector $a \in \mathbb{R}^d$, a $d \times d$ positive semi-definite matrix Σ, a d-dimensional Brownian motion $(W_t)_{t\ge 0}$ and, independent of it, a TPRM X on $R = \mathbb{R}^d \setminus \{0\}$ of intensity π, with π a measure on E such that $\pi[|x|^2 \wedge 1] < \infty$.
Let $Y^{(1)}$,$Y^{(2)}$,$Y^{(3)}$ be, respectively, a linear Brownian motion with drift, a compound Poisson process, and an $\mathcal{F}_t$-martingale defined as follows:

$$Y_t^{(1)} = -at + \Sigma W_t \tag{2.52}$$

$$Y_t^{(2)} = \int_0^t \int_{|x|\geq 1} xX(ds \times dx) \tag{2.53}$$

$$Y_t^{(3)} = lim_{\epsilon\downarrow 0} \int_0^t \int_{\epsilon<|x|<1} x\{X(dx \times dx) - ds\pi(dx)\} \tag{2.54}$$

where the limit converges pointwise almost surely on Ω and uniformly in t on bounded intervals. Then the stochastic process

$$Y = Y^{(1)} + Y^{(2)} + Y^{(3)} \tag{2.55}$$

is a Lévy process on $\mathbb{R}^d$.

(ii)Conversely, if Y is a Lévy process on $\mathbb{R}^d$, there exists a, a unique Σ, and a TPRM X with intensity π, as above, so that $Y = Y^{(1)} + Y^{(2)} + Y^{(3)}$.

The measure π is called the Lévy measure of Y.

CHAPTER 3

CREDIT DEFAULT SWAP

A credit default swap (CDS) is a financial swap agreement that the seller of the CDS will compensate the buyer in the event of a loan default or other credit event. It is the most common credit derivative. The buyer of a CDS is trying to transfer the credit risk, which is given by the reference entity, by making a series of payments to the seller on predetermined dates. Those payments are called premium payments. The buyer would stop paying the premium payments if the reference entity defaults. Meanwhile, in exchange, the seller of a CDS pays a protection payment to the buyer to protect them from this default event.

On a standard CDS, the maturity date, annual spread and settlement in case of default are determined. There are two different settlements: physical settlement and cash settlement. If the credit event occurs, the buyer has the right to sell a particular bond (or loan) issued by the company for its par value (physical settlement) or receive a cash settlement based on the difference between the defaulted bond par value and its market price.The market value of the reference obligation after a default per unit of face value is commonly referred to as the recovery rate. The recovery rate in this paper is set to be a constant R. In this project, we will take cash settlement. By assuming cash settlement, and a constant recovery rate R, the protection payment can then be denoted as (1-R) per notional amount insured.

To value a CDS, it is necessary to get the present value of the stream of those premium payments and protection payments. From the side of the buyer of a CDS, he needs to make premium payments if there is no default. On the contrary, the seller of the contract has to make protection payments if there is a default. Those claims can be valued by the risk-neutral method.

3.1 Pricing a CDS

We assume that we work on the filtered probability space $(F, P, (\mathcal{F}_t)t \geq 0)$ and represent the default time τ with a non-negative random variable. Under no arbitrage opportunities, there is a measure Q which is equivalent to the real probability measure P which makes the discounted price of an asset a martingale.We write Q for a risk neutral probability and E_t^Q the expectation under Q. We further assume the interest rate to be a constant r. Assume a contingent claim which has the payoff at time t:

$$V_t = \begin{cases} 1, & \text{if } \tau > \text{t}; \\ 0, & \text{if } \tau \leq \text{t}. \end{cases}$$

Then the initial value of the this claim which is denoted by V_0 satisfies:

$$V_0 = e^{-rt} E^Q[1_{\tau > t}] \tag{3.1}$$

where

$$1_{\tau>t} = \begin{cases} 1, & \text{if } \tau > \text{t;} \\ 0, & \text{if } \tau \leq \text{t.} \end{cases}$$

Define the risk-neutral survival probability as the probability under the risk-neutral measure of the survival of an entity through time t. Denote this probability as:

$$P(t) \equiv P(\tau > t) \tag{3.2}$$

Then

$$E_t^Q[1_{\tau>t}] = P(t) \cdot 1 + (1 - P(t)) \cdot 0 \tag{3.3}$$

Then the initial value of this contingent claim then can be expressed by the survival probability:

$$V_0 = e^{-rt} P(t) \tag{3.4}$$

Having recall the risk-neutral valuation method, now we we can start to get the present value of premium payments and protection payments.

Premium leg
Consider a CDS initiated and starting at time $t = 0$ maturing at time T with an annual premium denoted c. The premium, also called the spread, is quoted as a percentage of the par value of debt insured through the contract, typically named the notional amount. For the contract buyer, they need to make the payment on specific dates. For the continuous case, the payment should be paid continuously until the default appears or the contract expires. One way to figure out the formula is to divide the period into n parts, then take the limit of the discrete case. Divide the period of contract into n parts. Assume the buyer of the contract pays on payment dates $T_0 = 0$,T_2..$T_n = T$,$T_i = iT/n$
Denote $\delta t = T_i - T_{i-1} = T/n$ as the time between payment dates. For a contract with a notional amount equal to 1, the premium payment at time $T_0, T_1, ...T_{i-1}$ is:

$$\delta t c \tag{3.5}$$

conditioned on no default before time T_i.

If default appears during the time interval $(\frac{iT}{n}, \frac{(i+1)T}{n}]$, then the buyer should pay at time $\frac{jT}{n}, j = 0, 1, 2, ..i$ for $c\frac{T}{n}$. This contingent claim could be valued by the risk-neutral method. The present value of premium is then

$$c\frac{T}{n}(1 + e^{-r\frac{T}{n}} + e^{-r\frac{2T}{n}} + ... + e^{-r\frac{iT}{n}})$$

multiplied by

$$P(\frac{iT}{n}) - P(\frac{(i+1)T}{n})$$

If there is no default during the whole life of contract [0,T], the buyer pays

$$c\frac{T}{n}(1 + e^{-r\frac{T}{n}} + e^{-r\frac{2T}{n}} + ... + e^{-r\frac{iT}{n}} + e^{-rT})$$

with probability P(T). So the present value of the claim in this case is

$$c\frac{T}{n}(1+e^{-r\frac{T}{n}}+e^{-r\frac{2T}{n}}+...+e^{-r\frac{iT}{n}}+e^{-rT})\times P(T). \tag{3.6}$$

Now, we get all possible payments, if we denote the value of the premium leg $\Pi^{Prem}(T,c)$ (the present value with maturity time T and spread c). Then it is expressed as:

$$\begin{aligned}\Pi^{Prem}(T,c) = lim_{n\longrightarrow n-1}c\frac{T}{n}(1+e^{-r\frac{T}{n}}+e^{-r\frac{2T}{n}}+...+e^{-r\frac{iT}{n}})(P(\frac{iT}{n})-P(\frac{(i+1)T}{n}))\\ +c\frac{T}{n}(1+e^{-r\frac{T}{n}}+e^{-r\frac{2T}{n}}+...+e^{-r\frac{iT}{n}}+e^{-rT})P(T)\end{aligned} \tag{3.7}$$

As n goes to infinity, then the present value of premium in continuous case is

$$\begin{aligned}\Pi^{Prem}(T) &= lim_{n\longrightarrow\infty}(\sum_{i=1}^{n-1}c\frac{T}{n}(1+e^{-r\frac{T}{n}}+e^{-r\frac{2T}{n}}+...+e^{-r\frac{iT}{n}})(P(\frac{iT}{n})-P(\frac{(i+1)T}{n}))\\ &+c\frac{T}{n}(1+e^{-r\frac{T}{n}}+e^{-r\frac{2T}{n}}+...+e^{-r\frac{iT}{n}}+e^{-rT})P(T))\\ &= lim_{n\longrightarrow\infty}c\frac{T}{n}\sum_{i=1}^{n-1}e^{-r\frac{iT}{n}}P(\frac{iT}{n})\\ &= c\int_0^T e^{-rs}P(s)ds\end{aligned} \tag{3.8}$$

Protection leg
The above formula is from the side of the buyer of the CDS, for computing the present value of the payment from the seller. Risk-neutral valuation methods can be applied again. We call this payment the protection leg. But there is some difference here. The protection payment is only paid after a default. It can only be paid once but the premium is paid in series. Generally, the premium is paid annually, but the protection payment could be paid at any time as a default could occur at any time during the year. One way to deal with this problem is dividing the life of the contract into n parts equally. We assume the protection payment to only be paid on those n dates. To be more clear, between $t=0$ and $t=T$, T is the maturity time. The protection payment could only be paid at $T_i=\frac{iT}{n}, i=1,2,...n$. If default occurs between two payment dates, say $t+\frac{i-1}{n}, t+\frac{i}{n}$, then the seller pays the protection payment to the buyer on $t+\frac{i}{n}$ with money (1-R).

So similarly, with the risk-neutral valuation, the present value of the protection leg, denoted $\Pi^{Prot}(T)$, can be written as:

$$\Pi^{Prot}(T)=(1-R)\Sigma_{i=1}^{n}e^{-rT_i}[P(T_{i-1})-P(T_i)] \tag{3.9}$$

In summary, if default happens between $[\frac{iT}{n},\frac{(i+1)T}{n}]$, the seller pays $(1-R)$ at $\frac{(i+1)T}{n}$. We have the formula for this discrete case, then for the continuous case, just take the limit. We get

$$\Pi^{Prot}(T)=lim_{n\longrightarrow\infty}\sum_{i=1}^{n}e^{-r\frac{iT}{n}}[-P\frac{iT}{n}+P(\frac{(i-1)T}{n})]=-\int_0^T e^{-rs}dP(s) \tag{3.10}$$

Once we have the premium leg and protection leg, we could start to value CDS and the par spread.Let $V(T, c)$ denote the value of a CDS to the protection buyer at time $t = 0$, maturing at time T with an annual spread c. As the value of the CDS is what he claims minus what he pays, an expression of its value can be found by

$$V^{CDS} = (1 - R)(-\int_0^T exp(-rs)dP(s)) - c\int_0^T exp(-rs)P(s)ds \tag{3.11}$$

where R is the recovery rate and r is the default-free discount rate.
It usually does not cost anything when entering a CDS. By setting $V(T_N, c)$ equal to zero and solving for the fair spread, which makes the contract fairly priced, we get the par spread c^* that makes the CDS price equal to zero:

$$\begin{aligned} c^* &= \frac{(1-R)(-\int_0^T exp(-rs)dP(s))}{\int_0^T exp(-rs)P(s)ds} \\ &= \frac{(1-R)(1 - exp(-rT)P(T) - r\int_0^T exp(-rs)P(s)ds)}{\int_0^T exp(-rs)P(s)ds} \end{aligned} \tag{3.12}$$

We can see the spread is determined by survival probability up to maturity time T.

3.2 Credit Risk Modeling

In this chapter, the old structural model is presented and the new model will be presented in the next chapter.

We describe the asset value of the firm by a exponential Lévy process $S = S_t, t \geq 0$.

$$S_t = S_0 exp(X_t), S_0 > 0$$

where $X = X_t, t \geq 0$ is a Lévy process.
L is the preset level which is used to define default, R is the recovery rate for a CDS, S_0 is the initial value of the asset price. Denote the risk-neutral probability of no-default between 0 and t as P(t). Then default is defined to occur when

$$S_t = S_0 exp(X_t) \leq L$$

Note the survival probability under risk neutral probability P(t):

$$\begin{aligned} P(t) &= P_Q(X_s > log(L/S_0), forall \quad 0 \leq s \leq t); \\ &= P_Q(min_{0 \leq s \leq t} X_S > log(L/S_0)); \\ &= E_Q[1(min_{0 \leq s \leq t} X_s > log(L/S_0))]; \\ &= E_Q[1(min_{0 \leq s \leq t} S_s > L)] \end{aligned}$$

The subindex Q refers to the fact that we are working in a risk-neutral setting.
Let us denote by

$$BDOB(t, L) = exp(-rt)E_Q[1(min_{0 \leq s \leq t} S_s > L)] \tag{3.13}$$

the initial price of a binary down-and-out barrier option with maturity t and barrier level L; this option pays out a unit currency if the asset price S remains above the barrier during the lifetime of the option and zero otherwise. Similarly, we denote down-and-in barrier option

$$BDIB(t, L) = exp(-rt)E_Q[1(min_{0\leq s\leq t}S_s \leq L)] \tag{3.14}$$

Note

$$BDIB(t, L) + BDOB(t, L) = exp(-rt) \tag{3.15}$$

As

$$BDOB(t, L) = exp(-rt)P(t) \tag{3.16}$$

We can rewrite the par spread c^* in terms of the binary barrier prices as

$$c^* = \frac{(1-R)(1-BDOB(T,L)) - r\int_0^T BDOB(s,L)ds)}{\int_0^T BDOB(s,L)ds} \tag{3.17}$$

Under this model, two numerical methods are considered. Both of them are trying to transform the problem to computing a binary down-and-in barrier options. First is the Monte Carlo method. The second numerical method is trying to solve a PIDE (partial integral differential equation) by finite difference method.To estimate the price of a binary down-and-out barrier option we first calculate the price of a binary down-and-in barrier option with barrier L and time to maturity T.

3.3 The Variance Gamma Model

Variance Gamma Process is one kind of Levy process. We assume the underlying asset follows exponential variance gamma process. We price a one dimensional CDS under this VG model.

Definition 3.3.1. (Gamma distribution) *We say a random variable follows a gamma-(α, β) distribution if the density function of this r.v. is*

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\alpha x}dx \tag{3.18}$$

which is concentrated on $(0, \infty)$.

Proposition 3.3.2. *The Levy-Khintchine decomposition of a gamma distribution is*

$$\sigma = 0, \Pi(dx) = \beta x^{-1}e^{-\alpha x}1_{x>0}dx, a = \frac{\beta}{\alpha}(e^{-1} - 1) \tag{3.19}$$

Proof. We have

$$\int_{(0,\infty)} e^{i\theta x}\mu_{\alpha,\beta}(dx) = \frac{1}{(1-i\theta/\alpha)^\beta} = [\frac{1}{(1-i\theta/\alpha)^{\beta/n}}]^n \tag{3.20}$$

So the distribution is infinitely invisible.

$$\frac{1}{(1-i\theta/\alpha)^\beta} = e^{-\int_0^\infty(1-e^{i\theta x})\beta x^{-1}e^{-\alpha x}dx} \tag{3.21}$$

And the characteristic exponent is

$$\int_0^\infty (1 - e^{i\theta x})\beta x^{-1} e^{-\alpha x} dx \tag{3.22}$$

So

$$\begin{aligned}
\sigma &= 0 \\
\Pi(dx) &= \beta x^{-1} e^{-\alpha x} 1_{x>0} dx \\
a &= -\int_0^1 x\Pi(dx) = -\int_0^1 \beta e^{-\alpha x} dx = \frac{\beta}{\alpha}(e^{-1} - 1)
\end{aligned} \tag{3.23}$$

So the characteristic exponent is

$$\Psi(\theta) = \beta \int_0^\infty (1 - e^{i\theta x}) \frac{1}{x} e^{-\alpha x} dx = \beta log(1 - i\theta/\alpha) \tag{3.24}$$

□

Proposition 3.3.3. *Gamma distribution has expectation μ*

$$\mu = \frac{\beta}{\alpha} \tag{3.25}$$

and variance ν

$$\nu = \frac{\beta}{\alpha^2} \tag{3.26}$$

So

$$\begin{aligned}
\alpha &= \frac{\mu}{\nu} \\
\beta &= \frac{\mu^2}{\nu}
\end{aligned} \tag{3.27}$$

So a gamma distribution can be represented by μ and ν as follows:

Proposition 3.3.4. *For a gamma distribution, if the mean is μ and the variance is ν, then the density of this gamma distribution is*

$$\mu_{\alpha,\beta}(dx) = \frac{\frac{\mu}{\nu}^{\frac{\mu^2}{\nu}}}{\Gamma(\frac{\mu^2}{\nu})} x^{\frac{\mu^2}{\nu}-1} e^{-\frac{\mu}{\nu}x} dx \tag{3.28}$$

which is concentrated on $(0, \infty)$. And the Lévy-Khintchine decomposition of a gamma distribution is

$$a = \mu(e^{-1} - 1), \sigma = 0, \Pi(dx) = \frac{\mu^2}{\nu} x^{-1} e^{-\frac{\mu}{\nu}x} 1_{x>0} dx \tag{3.29}$$

Definition 3.3.5. (Gamma Process) *The Gamma process $\gamma(t; \mu, \nu)$ is a Lévy process with Lévy triplet $[a, \sigma, \Pi(dx)]$, where $a = \mu(e^{-1} - 1), \sigma = 0, \Pi(dx) = \frac{\mu^2}{\nu} x^{-1} e^{-\frac{\mu}{\nu}x} 1_{x>0} dx$, $\mu \in R, \nu \in R$.*

Proposition 3.3.6. *The gamma process $\gamma(t)$ is the process of independent gamma increments over non-overlapping intervals of time $(t, t+h)$. The density, $f_h(g)$, of the increments $g = \gamma(t+h;\mu,v) - \gamma(t;\mu,v)$ is given by the gamma density function with mean μh and variance vh. Specifically,*

$$f_h(g) = (\frac{\mu}{v})^{\frac{\mu^2 h}{v}} \frac{g^{\frac{\mu^2 h}{v}-1} exp(-\frac{\mu}{v}g)}{\Gamma(\frac{\mu^2 h}{v})}, g > 0 \tag{3.30}$$

where $\Gamma(x)$ is the gamma function. We denote $\gamma(t;\mu,v)$ as a gamma process with mean rate μ and variance ν.

Also, for $\gamma(t;\mu,\nu)$, we can equivalently represented it by parameters α and β as $\gamma(t;\alpha,\beta)$, where

$$\alpha = \frac{\mu}{\nu}$$

$$\beta = \frac{\mu^2}{\nu}$$

Definition 3.3.7. (Variance Gamma Process) *Let*

$$b(t;\theta,\sigma) = \theta t + \sigma W(t)$$

where $W(t)$ is a standard Brownian motion. The process $b(t;\theta,\sigma)$ is a Brownian motion with drift θ and volatility σ, and the gamma process has a unit mean rate, $\gamma(t;1,v)$ Then a VG process $X(t;\sigma,v,\theta)$, is defined as

$$X(t;\sigma,v,\theta) = b(\gamma(t;1,v);\theta,\sigma)$$

The VG process is obtained on evaluating Brownian motion at a time given by the gamma process.

Proposition 3.3.8. [12] *The VG process can be expressed as the difference of two independent increasing gamma processes, specifically*

$$X(t;\sigma,v,\theta) = \gamma_p(t;\mu_p,v_p) - \gamma_n(t;\mu_n,v_n) \tag{3.31}$$

$$\begin{aligned}
\mu_p &= \sqrt{(\frac{\theta^2}{4} + \frac{\sigma^2}{2v})} + \frac{\theta}{2} \\
v_p &= v(\sqrt{(\frac{\theta^2}{4} + \frac{\sigma^2}{2v})} + \frac{\theta}{2})^2 \\
\mu_n &= \sqrt{(\frac{\theta^2}{4} + \frac{\sigma^2}{2v})} - \frac{\theta}{2} \\
v_n &= v(\sqrt{(\frac{\theta^2}{4} + \frac{\sigma^2}{2v})} - \frac{\theta}{2})^2
\end{aligned} \tag{3.32}$$

To simplify the notation, we define C,M,G as follows

$$\begin{aligned} \frac{1}{v} &= C \\ \frac{1}{v\mu_p} &= M \\ \frac{1}{v\mu_n} &= G \end{aligned} \tag{3.33}$$

From proposition 3.3.8, we know VG process is a Lévy process.

Proposition 3.3.9. *The Lévy triplet of a VG-process is given by $[a, 0, \Pi(dx)]$, where*

$$a = \frac{-C(G(exp(-M)-1) - M(exp(-G)-1))}{MG} \tag{3.34}$$

$$\Pi(dx) = (Cx^{-1}e^{-Mx}1_{x>0} + C|x|^{-1}e^{Gx}1_{x<0})(dx) \tag{3.35}$$

Proof. We have

$$\mu_p = \frac{C}{M}, \mu_n = \frac{C}{G}, \nu_p = \frac{C}{M^2}, \nu_n = \frac{C}{G^2}$$

Forγ_p

$$\Pi_1(dx) = \frac{\mu_p^2}{v_p}x^{-1}e^{-\frac{\mu_p}{v_p}x}1_{x>0}dx = \frac{1}{v}x^{-1}e^{-\frac{1}{v\mu_p}x}1_{x>0}dx$$

Forγ_n

$$\Pi_2(dx) = \frac{\mu_n^2}{v_n}x^{-1}e^{-\frac{\mu_n}{v_n}x}1_{x>0}dx = \frac{1}{v}x^{-1}e^{-\frac{1}{v\mu_n}x}1_{x>0}dx$$

So

$$\Pi(dx) = Cx^{-1}e^{-Mx}1_{x>0} + C|x|^{-1}e^{Gx}1_{x<0}$$

Forγ_p

$$a_1 = -\int_0^1 x\Pi(dx) = -\int_0^1 Ce^{-Mx}dx = -C\frac{e^{-Mx}}{-M}|_0^1 = \frac{-Ce^{-M}}{-M} - \frac{C}{M} = \frac{Ce^{-M} - C}{M}$$

Forγ_n

$$a_2 = -\int_{-1}^0 x\Pi(dx) = \int_{-1}^0 Ce^{Gx}dx = C\frac{e^{Gx}}{G}|_{-1}^0 = \frac{C}{G} - \frac{Ce^{-G}}{G} = \frac{-C(e^{-G}-1)}{G}$$

So

$$\begin{aligned} a = a_1 + a_2 &= \frac{-CMe^{-G} + CM + CGe^{-M} - CG}{GM} = \frac{C(M(-e^{-G}+1)) + G(e^{-M}-1)}{GM} \\ &= \frac{-C(M(e^{-G}-1)) - G(e^{-M}-1)}{GM} \end{aligned}$$

□

3.3.1 Monte Carlo Method

We simulate a VG process $X^{VG} = X_t^{(VG)}, t \geq 0$ with parameters $C, G, M > 0$ by simulating the difference of two independent gamma process,$\gamma^{(1)}$,$\gamma^{(2)}$. A VG process can be decomposed as $X_t^{(VG)} = \gamma_t^{(1)} - \gamma_t^{(2)}$, where $\gamma_t^{(1)}, t \geq 0$ is a Gamma process with parameters $\beta^1 = C$ and $\alpha^1 = M$ and $\gamma_t^{(2)}, t \geq 0$ is a Gamma process with parameters $\beta^2 = C$ and $\alpha^2 = G$.

Algorithm

(i)Simulate at time points $n\Delta t, n = 0, 1, ...$

(ii)Set $\gamma_0^{(1)} = 0$,$\gamma_0^{(2)} = 0$.

(iii)Generate independent Gamma$(\beta^1 \Delta t, \alpha^1)$ random numbers $g_n^1, n \geq 1$, generate independent Gamma$(\beta^2 \Delta t, \alpha^2)$ random numbers $g_n^2, n \geq 1$by using a gamma random numbers generator.

(iv)Set

$$\gamma_{n\Delta t}^1 = \gamma_{(n-1)\Delta t}^1 + g_n^1, n \geq 1$$

$$\gamma_{n\Delta t}^2 = \gamma_{(n-1)\Delta t}^2 + g_n^2, n \geq 1$$

Then

$$X_n^{VG} = \gamma_n^1 - \gamma_n^2$$

One sees a path of a VG process with parameters C=20, G=40 and M=50.

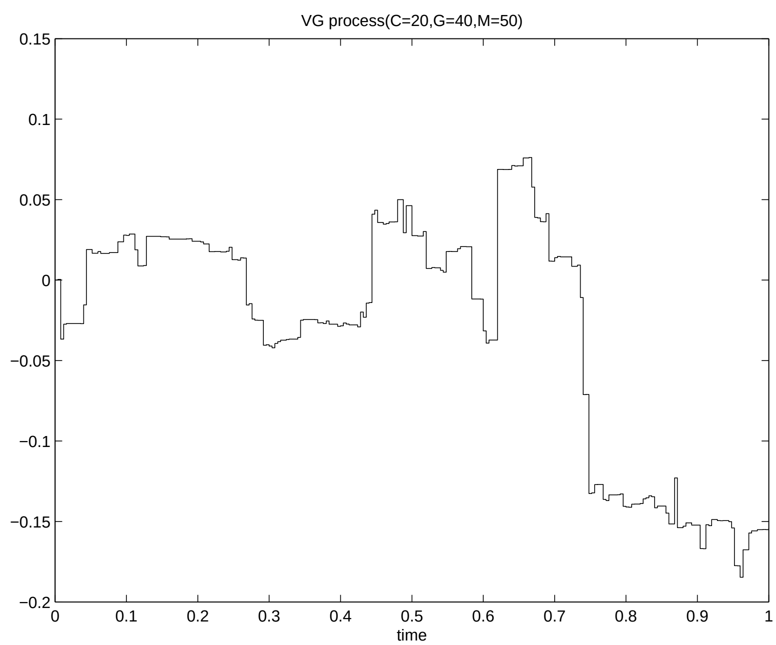

Figure 3.1: A sample path of a VG process with parameter C=20, G=40 and M=50, dividing with 250 time steps.

3.3.2 PIDE Pricing Method

Assume r is the constant continuously compounded interest rate, and a risky asset with price process $S = S_t, t \geq 0$. We assume the asset pays out a continuous dividend yield of q. The risk-neutral dynamics for the asset price are given by

$$S_t = S_0 exp((r-q)t + X_t + \omega t) \tag{3.36}$$

where $X = X_t, t \geq 0$ is a VG process. The risk-neutral drift rate for the asset is $r - q$ and thus to have $E(S_t) = S_0 exp((r-q)t)$, we set

$$\omega = \nu^{-1} log(1 - \frac{1}{2}\sigma^2\nu - \theta\nu). \tag{3.37}$$

Consider a binary down-and-in option with barrier L and time to maturity T. This claim pays out 1 if the asset price S goes below the barrier during the lifetime of the option and zero otherwise. Then the price of the claim at time $t < T$ is given by

$$V(t) = exp(-r(T-t))E_Q[1(min_{0\leq s\leq T} S_s \leq L)|\mathcal{F}_t] \tag{3.38}$$

where $\mathcal{F}_t$ is the filtration by the process S_t. Assume $S_0 > L$, and denote $\upsilon(S_t, t)$ as the price at time t of the down-and-in barrier option under the assumption that the stock price has not crossed the barrier L prior to time t.

$$\upsilon(S_t, t) = exp(-r(T-t))E_Q[1(min_{0\leq s\leq T} S_s \leq L)|\mathcal{F}_t] \tag{3.39}$$

If we define ρ to be the first time at which the asset price reaches the barrier L. In other words, ρ is chosen in a path-dependent way so that $S(t) > L$ for $0 \leq t \leq \rho$ and $S(\rho) = L$. Hence, $e^{-rt}\upsilon(S_t, t)$ is a martingale up to time ρ, hence the infinitesimal generator $\mathcal{L}$, of the underlying Lévy process (Markov process) applied to $e^{-rt}\upsilon(S_t, t)$ yields zero:

$$\mathcal{L}(exp(-rt)\upsilon(S_t, t)) = 0 \tag{3.40}$$

Definition 3.3.10. *Denote C_0 as the space of all the continuous functions. Let (X_t) be real valued Lévy process, a function $f \in C_0$, is said to belong to the domain $\mathcal{D}_{\mathcal{L}}$ of the infinitesimal generator of X_t if the limit*

$$\mathcal{L}f(X_t) := lim_{h\downarrow 0} \frac{E(f(X_{t+h}) - f(X_t)|\mathcal{F}_t)}{h} \tag{3.41}$$

exists in C_0. The operator $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \to C_0$ thus defined is called the infinitesimal generator of the process (X_t).

Let $\mathcal{A}$ be the space of infinitely differential functions on the real line s.t. $lim_{|x|\to\infty} f^k(x)P(x) = 0$ for any polynomial P(x),$\forall k \in \mathbb{Z}^+$.Fourier transform is one-to-one on $\mathcal{A}$ to itself. $\forall f, g \in \mathcal{A}$, define the inner product $< f, g >:= \int f(x)g(x)dx$.

Theorem 3.3.11. *Let (X_t) be real valued Lévy process, and assume the Lévy-Kintchin formula for this process is $\Psi(\mu) = i\beta\mu + \frac{\sigma^2\mu^2}{2} + \int(1 - e^{i\mu y} + i\mu y)\nu(dy)$, then $\mathcal{A} \subset \mathcal{D}_{\mathcal{L}}$, and $\forall f \in \mathcal{A}$,the infinitesimal generator $\mathcal{L}$*

$$\mathcal{L}(f(x)) = \beta f'(x) + \frac{\sigma^2}{2}f''(x) + \int(f(x+y) - f(x) - yf'(x))\nu(dy) \tag{3.42}$$

Proof. Use μ_t to denote the density function of X_t, $\hat{\mu}_t$ to denote the Fourier transform for X_t, then

$$\hat{\mu}_t(\eta) = \int e^{ix\eta}\mu_t dx = e^{t\psi(\eta)}$$

$$\psi(\eta) = -i\beta\eta - \frac{\sigma^2\eta^2}{2} + \int (e^{i\eta y} - 1 - i\eta y)\nu(dy)$$

For any $f \in \mathcal{A}$, there exists unique $g \in \mathcal{A}$, s.t. $f(x) = \int e^{i\eta x}g(\eta)d\eta$. We denote $e^{i\eta x}g(\eta) = g_x(\eta)$. Then $f(x) = \int g_x(\eta)d\eta =< 1, g_x >$, and then

$$\begin{aligned} < \hat{\mu}_t, g_x > &= \int (e^{ix\eta}g(\eta))(\int e^{iy\eta}\mu_t(dy))d\eta \\ &= \int \mu_t(dy) \int e^{i(x+y)\eta}g(\eta)d\eta \\ &= \int f(x+y)\mu_t(dy) \\ &= T_t f(x) \end{aligned}$$

So use Taylor expansion,

$$\begin{aligned} T_t f(x) =< \hat{\mu}_t, g_x > = \int e^{t\psi(\eta)}g_x(\eta)d\eta \\ =< 1, g_x > +t < \psi, g_x > + \frac{t^2}{2}H(t,x) \end{aligned}$$

where $|H(x,t)| \leq sup_{0\leq s\leq t}| < \psi^2 e^{s\psi}, g_x > | \leq < |\psi|^2, |g_x| >$. We know $< |\psi|, |g_x| > and < |\psi|^2, |g_x| >$ are finite, the bounds are independent with x, hence when $t \downarrow 0$,

$$\frac{1}{t}(T_t f(x) - f(x)) \to < \psi, g_x >$$

uniformly on x. So

$$\mathcal{L}f(x) =< \psi, g_x >=< i\beta\mu - \frac{\sigma^2\mu^2}{2} + \int (e^{i\mu y} - 1 - i\mu y)\nu(dy), g_x(\mu) >$$

and observe

$$f'(x) = i\int \mu g_x(\eta)d\eta =< i\eta, g_x(\eta) >$$

$$f''(x) = i^2 \int \eta^2 g_x(\mu)d\mu =< -\eta^2, g_x(\eta) >$$

$$\begin{aligned} < \int (e^{i\eta y} - 1 - i\eta y)\nu(dy), g_x(\eta) > &= \int \nu(dy)(\int (e^{i\eta y} - 1 - i\eta y)g_x(\eta)d\eta) \\ &= \int (f(x+y) - f(x) - yf'(x))\nu(dy) \end{aligned}$$

□

Proposition 3.3.12. *Let (X_t) be real valued Lévy process, and assume the Lévy-Kintchin formula for this process is $\Psi(\mu) = i\beta\mu + \frac{\sigma^2\mu^2}{2} + \int(1 - e^{i\mu y} + i\mu y)\nu(dy)$, then $\mathcal{A} \subset \mathcal{D}_{\mathcal{L}}$, and if f is a function with two parameters $f(X_t, t)$ and f is infinitely differentiable with respect to X_t and t, then the infinitesimal generator $\mathcal{L}$ of f is*

$$\mathcal{L}(f(x,t)) = \frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int (f(x+y) - f(x) - y\frac{\partial f}{\partial x})\nu(dy) \tag{3.43}$$

Proof.

$$\begin{aligned}
\mathcal{L}f := & lim_{h\downarrow 0}\frac{E(f(X_{t+h}, t+h) - f(X_t, t)|\mathcal{F}_t)}{h} \\
&= lim_{h\downarrow 0}\frac{E(f(X_{t+h}, t) + h\frac{\partial f}{\partial t} - f(X_t, t)|\mathcal{F}_t)}{h} \\
&= lim_{h\downarrow 0}\frac{E(f(X_{t+h}, t) - f(X_t, t)|\mathcal{F}_t)}{h} + \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int (f(x+y) - f(x) - y\frac{\partial f}{\partial x})\nu(dy)
\end{aligned}$$

□

We now develop the specific form of the PIDE

$$\upsilon_t + (r - q + \omega)S\upsilon_S + \int_{-\infty}^{+\infty}[\upsilon(Se^y, t) - \upsilon(s,t)]k(y)dy = r\upsilon \tag{3.44}$$

where

$$\begin{aligned}
k(y)dy = 1_{y>0}\frac{e^{-\lambda_p y}}{\nu y} + 1_{y<0}\frac{e^{-\lambda_n |y|}}{\nu|y|} \\
\lambda_p = -\frac{1}{\upsilon\mu_p} \\
\lambda_n = -\frac{1}{\upsilon\mu_n}
\end{aligned} \tag{3.45}$$

Because $(S(t), t)$ can reach any point in the rectangle $\{(x,t); x > L, 0 \le t < T\}$, so the PIDE is satisfied in $\{(x,t); x > L, 0 \le t < T\}$. And $\upsilon(S(t), t)$ satisfies the following boundary conditions:

$$\upsilon(L, t) = exp(-r(T-t))E_Q[1(min_{S_T \le L}|\mathcal{F}_t] \tag{3.46}$$

and

$$\upsilon(S(T), T) = 0 \tag{3.47}$$

where $L < S(T)$. For the value of European binary option $V_{European}(S(t), t) = exp(-r(T-t))E_Q[1(min_{S_T \le L}|\mathcal{F}_t]$, we can derive the same PIDE formula on the entire region and with the boundary condition.

$$V_{European}(S(t), T) = 1(min_{S_T \le L}) \tag{3.48}$$

Make the change of variable $x = log(S)$, and denote $W(x,t) := \upsilon(S,t)$. Then

$$\begin{aligned} W(x+y,t) &= \upsilon(Se^y,t) \\ W_x(x,t) &= S\upsilon_S(S,t) \\ W_t(x,t) &= \upsilon_t(S,t) \end{aligned} \tag{3.49}$$

Now substituting these values in the above PIDE, we get

$$W_t + (r-q+\omega)W_t + \int_{-\infty}^{+\infty}[W(x+y,t) - W(x,t)]k(y)dy = rW \tag{3.50}$$

with the following condition [13]

$$W(x,T) = 1_{\{L>e^x\}} \tag{3.51}$$

where L is the barrier level we set, and with boundary conditions

$$\begin{aligned} W(-\infty,t) &= e^{-rt} \\ W(\infty,0) &= 0 \end{aligned} \tag{3.52}$$

and domain

$$[0,T] \times [ln(S_{min}, ln(S_{max}))] \tag{3.53}$$

Discretize the domain with M by N. With $N+1$ mesh points in x direction and $M+1$ mesh points in t direction. Approximate the derivatives by:

$$W_t \simeq \frac{W_{i,j+1} - W_{i,j}}{\triangle t}$$

$$W_x \simeq \frac{W_{i+1,j} - W_{i-1,j}}{2\triangle x}$$

Then we have

$$\begin{aligned} &\frac{W_{i,j+1} - W_{i,j}}{\Delta t} + (r-q+\omega)\frac{W_{i+1,j} - W_{i-1,j}}{2\Delta x} \\ &\qquad\qquad + \int_{-\infty}^{+\infty}[W(x_i+y,t_{j+1}) - W_{i,j+1}]k(y)dy = rW_{i,j} \end{aligned}$$

multiply by Δt, and rearrange the above finite difference equation, we have

$$\begin{aligned} &W(x_{i-1},t_j) + (1+r\Delta t)W(x_i,t_j) - \frac{a\Delta t}{2\Delta x}W(x_{i+1},t_j) \\ &\qquad = W(x_i,t_{j+1}) + \Delta t\int_{-\infty}^{+\infty}[W(x_i+y,t_{j+1}) - W(x_i,t_{j+1})]k(y)dy \end{aligned} \tag{3.54}$$

We divided the right hand side integral into six different integrals.

$$\int_{-\infty}^{+\infty}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy = \tag{3.55}$$

$$\int_{-\infty}^{x_0-x_i}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy+ \tag{3.56}$$

$$\int_{x_0-x_i}^{-\Delta x}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy+ \tag{3.57}$$

$$\int_{-\Delta x}^{0}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy+ \tag{3.58}$$

$$\int_{0}^{\Delta x}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy+ \tag{3.59}$$

$$\int_{\Delta x}^{(N-i)\Delta x}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy+ \tag{3.60}$$

$$\int_{(N-i)\Delta x}^{+\infty}[W(x_i+y,t_{j+1})-W(x_i,t_{j+1})]k(y)dy \tag{3.61}$$

For each of the six integrals, we use the first order Taylor expansion to approximate the integrand [14]. For $x_i - y < S_{min}$, we take

$$W(x_i-y,t_{j+1}) = e^{-r(T-t_{j+1})} \tag{3.62}$$

For $x_i + y > S_{max}$, we take

$$W(x_i+y,t_{j+1}) = 0 \tag{3.63}$$

For $x_{i+k+1} > x_i + y > x_{i+k}$, we use the following approximation:

$$W(x_i+y,t_{j+1}) - W(x_{i+k},t_{j+1}) \simeq \frac{W(x_{i+k+1},t_{j+1})-W(x_{i+k},t_{j+1})}{\Delta x}(y-k\Delta x) \tag{3.64}$$

After we write out the approximation formula for the right hand side integral, we can now solve the equation 3.54.

So when we implement this algorithm, we can proceed as done for the European binary option and at each time step, after computing the new values $W(x_i, x_{j+1})$, we impose:

$$W_{x_i,t_{j+1}} = exp(-r(M-(j+1))\triangle t)$$

The price of barrier down-and-out option will be derived from the price of barrier down-and-in option.

3.3.3 Comparison of Two numerical methods

Method	M(stock)	N(time)	c^*(in bp)	BDIB	cpu(in sec)
PIDE	100	100	137	0.0262	0.01
PIDE	150	150	134	0.0256	0.04
PIDE	200	200	133	0.0253	0.1
PIDE	250	250	132	0.0252	0.19
PIDE	500	250	132	0.0252	0.71
PIDE	250	500	132	0.0253	0.36
Method	iterations	N	c^*(in bp)	BDIB	cpu(in sec)
MC	10000	80	131	0.0257	70
MC	10000	250	133	0.0255	231
MC	100000	250	135	0.0260	3794
MC	500000	250	133	0.0255	18999
MC	1000000	250	132	0.0254	35093

Table 3.1: spread and BDIB prices for PIDE method and Monte Carlo method under the VG model. The parameters are:$\sigma = 0.20722; \upsilon = 0.50215; \theta = -0.22898; r = 0.0421; q = 0; L = 50; S_0 = 100; R = 0.5; T = 1$ The domain: S_{min}=33.25, S_{max}=210

In PIDE method, M denotes the number of points we take on the log-stock direction, N denotes the time steps we take. In the Monte Carlo method, VG process is simulated as a difference of independent gamma processes. From the table, we see that it's time consuming to compute the survival probability by Monte Carlo method. PIDE method has shown a great improvement in time. It gets a good result when M and N are both greater or equal to 200. However, the PIDE method is very sensitive to the stock domain we choose. S_{min} and S_{max} should be selected very carefully to get an accurate result.

CHAPTER 4

A NEW STRUCTURAL MODEL

The new structural model in [7] preserves the framework linking the credit risk with the financial securities of the firm. Meanwhile, it is a tractable method which shares the same mathematics theory with the intensity models, or the reduced form models. In the reduced form models, the default time τ, is modeled by a stochastic process. This process is called the intensity of default. Assume $(\Omega, \mathcal{G}, \mathbb{P})$ is a probability space, and that the random time of default τ is a non negative random variable on $(\Omega, \mathcal{G}, \mathbb{P})$ such that $\mathbb{P}(\tau = 0) = 0$, and $\mathbb{P}(\tau > t) > 0$ for any $t \geq 0$. Assume we have a model to compute the survival probability $G(t) = \mathbb{P}(\tau > t)$. G is the probability that the firm does not default between time 0 and time t. Assume $G(t)$ is differentiable and then we may define the function γ:

$$\gamma(t) = -\frac{G'(t)}{G(t)} \tag{4.1}$$

Then

$$G(t) = e^{-\int_0^t \gamma(u)du} \tag{4.2}$$

The function γ is called the intensity function of the random time of default τ. The representation of γ is the subject of the reduced form. It can be showed that the new structural model can be put in the framework of an intensity based model.

4.1 One dimensional structural model

Recall for any one dimensional Lévy process Y on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{P}, (\mathcal{G}_t)_{t\geq 0})$, there exists two constants $\gamma \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$, a $\mathbb{P}$ Brownian motion W and a TPRM X on $E = \mathbb{R} \setminus \{0\}$ adapted to $\mathcal{G}$ with intensity λ such that $\forall\ \omega \in \Omega$, $t \geq 0$,

$$\begin{aligned} Y_t(\omega) = -\gamma t + \sigma W_t + \int_0^t \int_{|x|\geq 1} xX(\omega, ds \times dx) \\ + \lim_{\epsilon \downarrow 0} \int_0^t \int_{\epsilon<|x|<1} x\{X(\omega, ds \times dx) - ds \times \lambda(dx)\} \end{aligned} \tag{4.3}$$

Equivalently, X is a Poisson random measure on $R_+ \times E$ with intensity $Leb \times \lambda$. λ is the Lévy measure of Y, it satisfies $\lambda[|x|^2 \wedge 1] < \infty$. The jump measure of the process Y which counts the number of jumps of Y in a given set of $\mathcal{E} = \mathcal{B}(E)$ is precisely X. In particular we have $\forall \omega \in \Omega, A \in \mathcal{E}, t \geq 0$

$$X_t(\omega, A) = X(\omega, [0, t] \times A) = \sharp\{s \geq 0 : (s, \triangle Y_s(\omega)) \in [0, t] \times A\} \tag{4.4}$$

4.1.1 Constant threshold

Assume that the interest rate process $(r_t)_{t\geq 0}$ is constant and equal to r, and we model the movements of S with the following stochastic differential equation, for $t \geq 0$:

$$dS_t = S_{t^-}(rdt + \sigma dW_t + \int_{\mathbb{R}} (e^x - 1)\{X(dt \times dx) - dt \times \lambda(dx)\}), S_0 = s > 0 \tag{4.5}$$

4.5 is satisfied by the process $S_t = sexp\{rt + Y_t - \psi(-i)\}$, where Y is the Lévy process in and ψ is the characteristic exponent of Y. With this specification, the discounted value of the stock price $\tilde{S}_t = S_t/e^{-rt}$ is an $\mathcal{G}_t-$martingale under $\mathbb{P}$.

Definition 4.1.1. (Default time, constant level) *Let S be the stock price of the firm. We define the time of default as the first time the log-returns of S records an amplitude greater than $a < 0$.*

$$\tau = inf\{t > 0 : logS_t/S_{t^-} \leq a\} \tag{4.6}$$

We call the parameter a the default level of the firm.

Proposition 4.1.2. (Default probability, constant level) *Let S be the exponential Lévy process and define the default time τ of a firm with stock S as in Def. 4.1.1. Then the survival probability of the firm up to time $t > 0$ is given by*

$$P(\tau > t) = e^{-t\Lambda(a)}$$

where $\Lambda(x) = \int_{\mathcal{I}(x)} \lambda(d\omega)$ is the tail integral of the process Y.

Proof. If S satisfies the stochastic differential equation, i.e. $S_t = sexp\{rt + Y_t - \psi(-i)\}$ with Y as in 4.5. and ψ is the characteristic exponent of Y. Therefore,

$$\forall \omega \in \Omega, t \geq 0, logS_t/S_t^- = \triangle Y_t \tag{4.7}$$

The jump measure of Y is the Poisson random measure X on $E = \mathbb{R} \setminus \{0\}$ with intensity λ. The default time τ is then

$$\tau = inf\{t > 0 : \triangle Y_t \leq a\} = inf\{t > 0 : X_t(-\infty, a] > 0\} \tag{4.8}$$

The process N such that for $t \in \mathbb{R}_+$, $N_t = X_t(-\infty, a]$ is a Poisson process of intensity $\Lambda(a)$, where Λ is the tail integral of Y. With these remarks, the probability of default in this model is then

$$\mathbb{P}(\tau \leq t) = 1 - \mathbb{P}(X_t(-\infty, a] = 0) = 1 - e^{-t\Lambda(a)} \tag{4.9}$$

□

Proposition 4.1.3. (Local default rate, constant level). *Let S be the exponential Lévy process and define the default time τ of a firm with stock S as in Def. 3.1. The local default rate LDR_t is given by*

$$LDR_t \equiv lim_{h\downarrow 0} \frac{\mathbb{P}(\tau \leq t + h | \tau > t)}{h} = \Lambda(a) \tag{4.10}$$

Proposition 4.1.4. (spread, constant level) *Assume the stock price follows exponential VG process. Then CDS spread at par under this new model is*

$$c^* = \Lambda(a)(1-R) \tag{4.11}$$

Proof.

$$\begin{aligned}
c^* &= \frac{(1-R)(1-exp(-rT)P(T) - r\int_0^T exp(-rs)P(s)ds)}{\int_0^T exp(-rs)P(s)ds} \\
&= \frac{(1-R)(1-exp(-rT)exp(-T\Lambda(a)) - r\int_0^T exp(-rs)exp(-s\Lambda(a))ds)}{\int_0^T exp(-rs)exp(-s\Lambda(a))ds} \\
&= \frac{(1-R)(1-e^{-T(r+\Lambda(a))} - r\int_0^T e^{-s(r+\Lambda(a))}ds)}{\int_0^T e^{-s(r+\Lambda(a))}ds} \\
&= \frac{(1-R)(1-e^{-T(r+\Lambda(a))} - r(\frac{e^{-(r+\Lambda(a))s}}{-(r+\Lambda(a))}\mid_0^T))}{\frac{e^{-(r+\Lambda(a))s}}{-(r+\Lambda(a))}\mid_0^T))} \\
&= \frac{(1-R)[1-e^{-T(r+\Lambda(a))} + \frac{r}{r+\Lambda(a)}(e^{-(r+\Lambda(a))T}-1)]}{\frac{e^{-(r+\Lambda(a))T}-1}{-(r+\Lambda(a))}} \\
&= \Lambda(a)(1-R)
\end{aligned}$$

□

The constant default level fails to explain the term structure of the CDS. Hence, the stochastic default level are considered in [15]. Due to the length of this thesis, we will not discuss stochastic default level here.

4.2 Comparison of Two Models

We investigate the sensitivity of the parameters (σ, θ, v) from the VG model under two different models. The initial stock price is set to be 100, the barrier level is set to be 50. Interest rate r is set to be 0.0421.

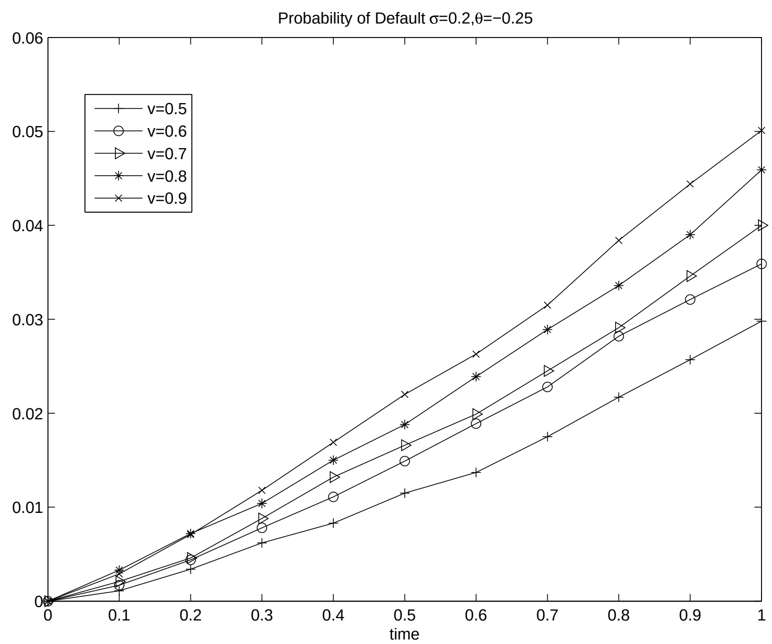

Figure 4.1: Kurtosis sensitivity: Plot of Default Probability $\mathbb{P}(\tau \leq t)$ versus time $t = 0.1, 0.2, ...1$ for old structural model. The parameters of VG model is $\sigma = 0.2, \theta = -0.25$, $v = 0.5, 0.6, 0.7, 0.8, 0.9$;

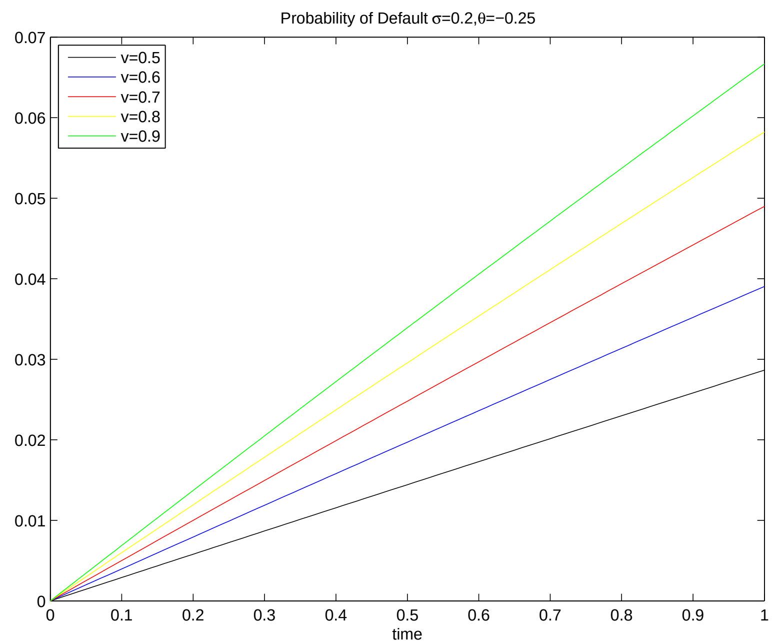

Figure 4.2: Kurtosis sensitivity: Plot of Default Probability $\mathbb{P}(\tau \leq t)$ versus time $t \in [0, 1]$ for new structural model. The parameters of VG model is $\sigma = 0.2, \theta = -0.25$, $v = 0.5, 0.6, 0.7, 0.8, 0.9$;

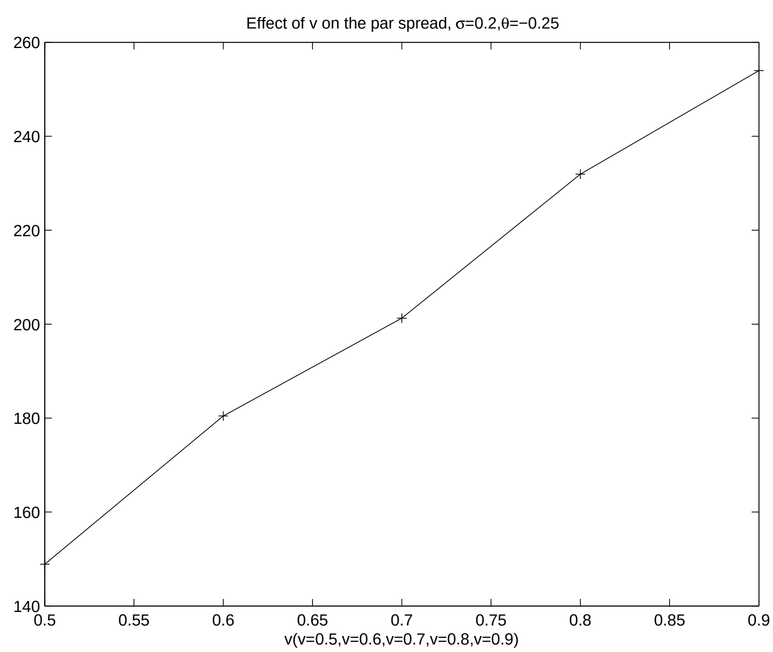

Figure 4.3: Kurtosis sensitivity: Plot of CDS spread c versus time v for old structural model. The parameters of VG model is $\sigma = 0.2, \theta = -0.25$, and choose $v = 0.5, 0.6, 0.7, 0.8, 0.9$;

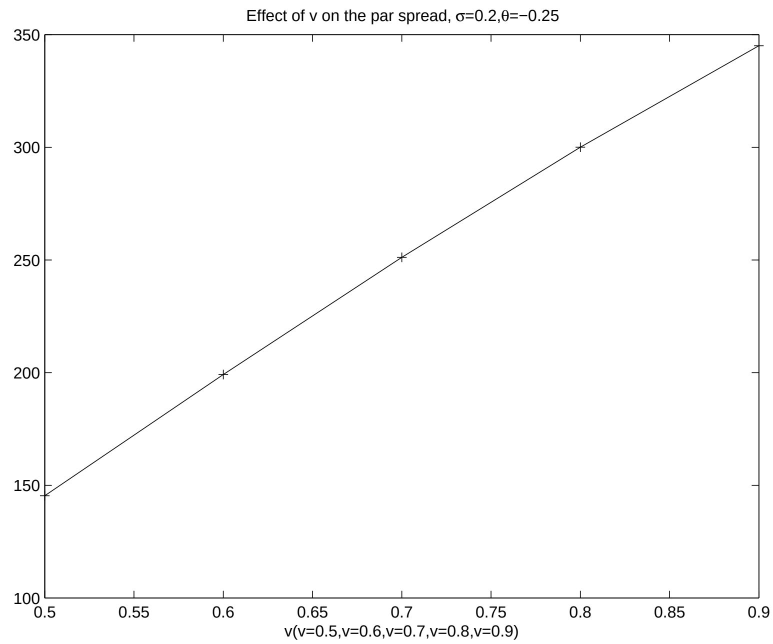

Figure 4.4: Kurtosis sensitivity: Plot of CDS spread c versus time v for new structural model. The parameters of VG model is $\sigma = 0.2, \theta = -0.25$, and $v = 0.5, 0.6, 0.7, 0.8, 0.9$;

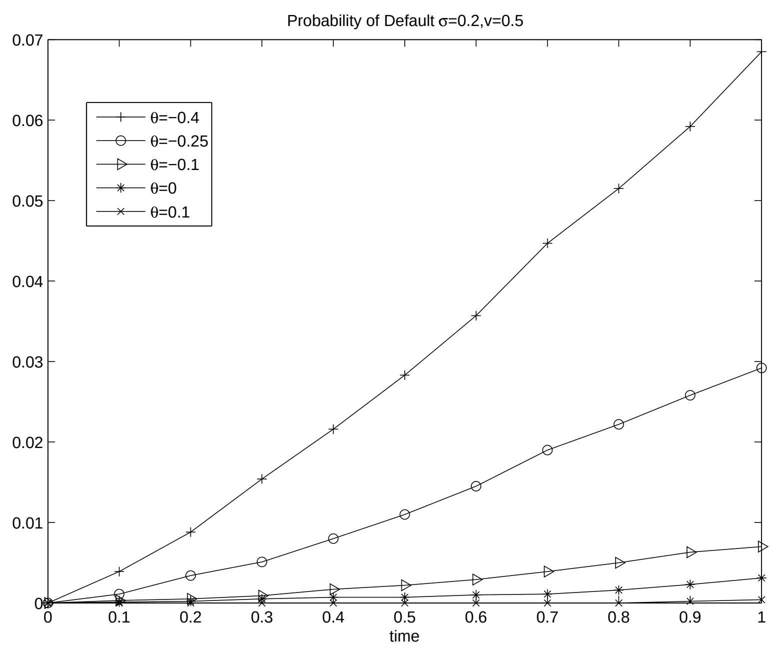

Figure 4.5: Skewness sensitivity: Plot of Default Probability $\mathbb{P}(\tau \leq t)$ versus time $t = 0.1, 0.2, ...1$ for old structural model. The parameters of VG model is $\sigma = 0.2, v = 0.5$, and $\theta = -0.4, -0.25, -0.1, 0, 0.1$;

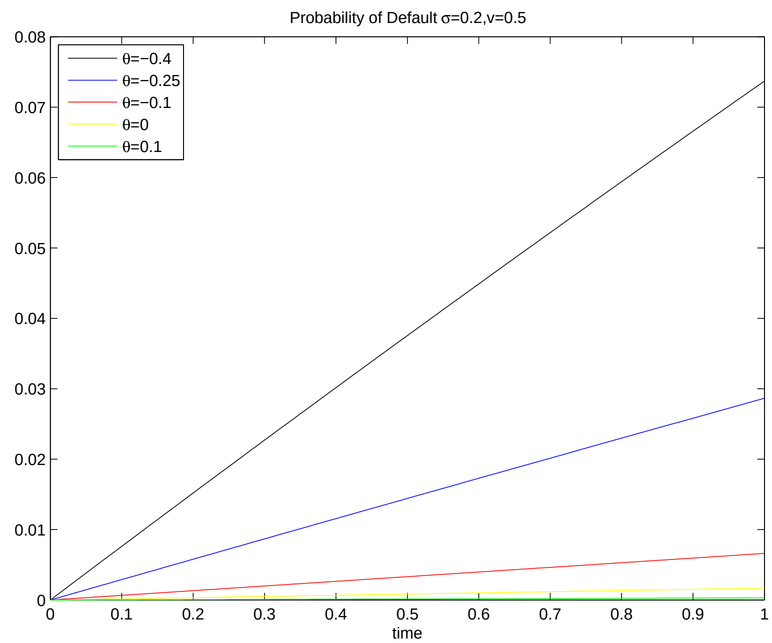

Figure 4.6: Skewness sensitivity: Plot of Default Probability $\mathbb{P}(\tau \leq t)$ versus time $t \in [0, 1]$ for new structural model. The parameters of VG model is $\sigma = 0.2, v = 0.5$, and $\theta = -0.4, -0.25, -0.1, 0, 0.1$;

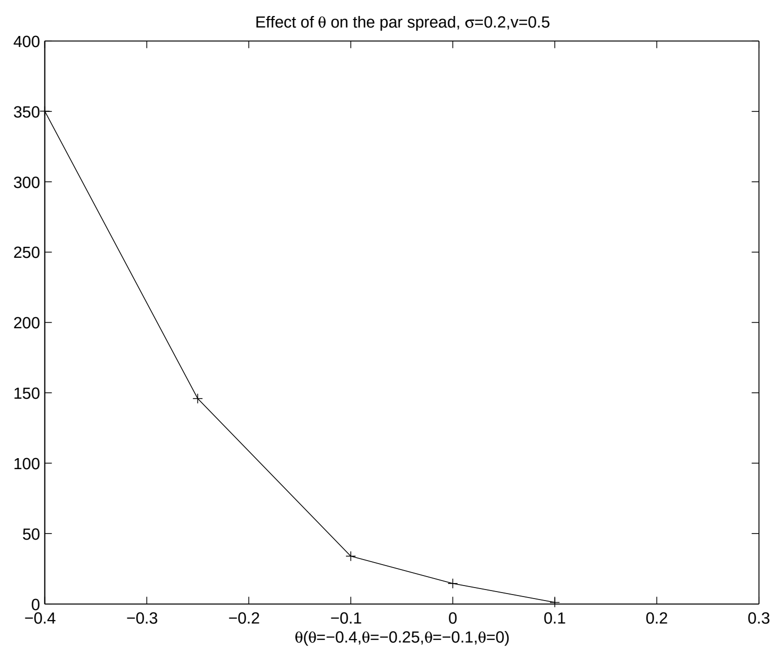

Figure 4.7: Skewness sensitivity: Plot of CDS spread c versus time θ for old structural model. The parameters of VG model is $\sigma = 0.2, v = 0.5$, and $\theta = -0.4, -0.25, -0.1, 0, 0.1$;

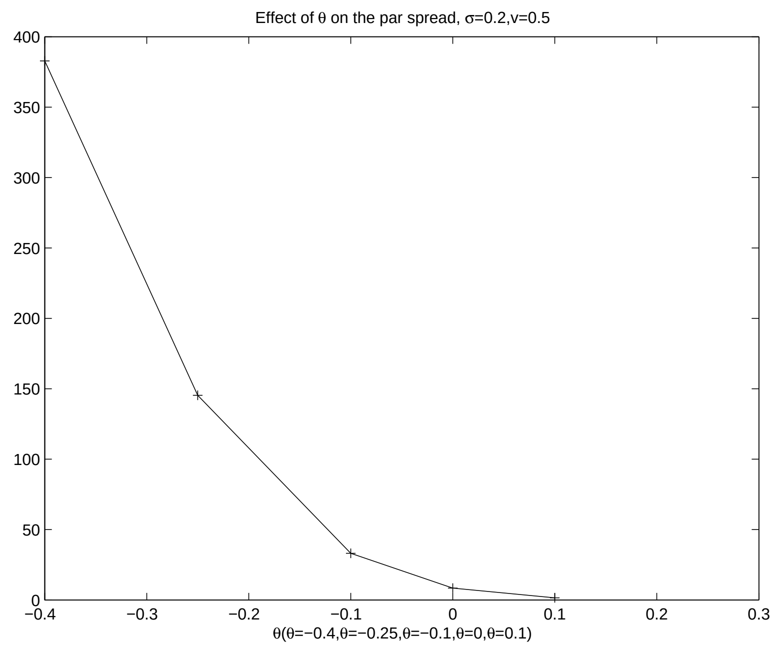

Figure 4.8: Skewness sensitivity: Plot of CDS spread c versus time θ for new structural model. The parameters of VG model is $\sigma = 0.2, v = 0.5$, and $\theta = -0.4, -0.25, -0.1, 0, 0.1$;

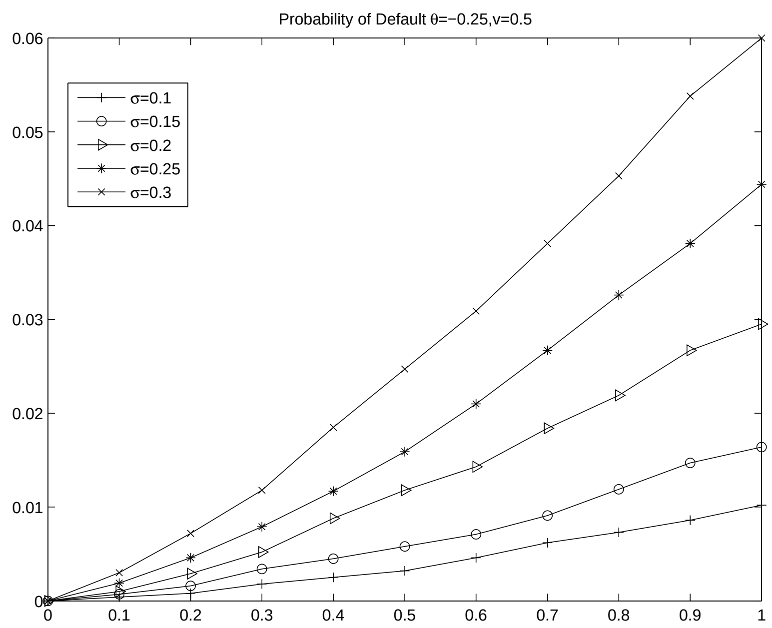

Figure 4.9: Variance sensitivity: Plot of Default Probability $\mathbb{P}(\tau \leq t)$ versus time $t = 0.1, 0.2, ...1$ for old structural model. The parameters of VG model is $\theta = -0.25, v = 0.5$, and $\sigma = 0.1, 0.15, 0.2, 0.25, 0.3$;

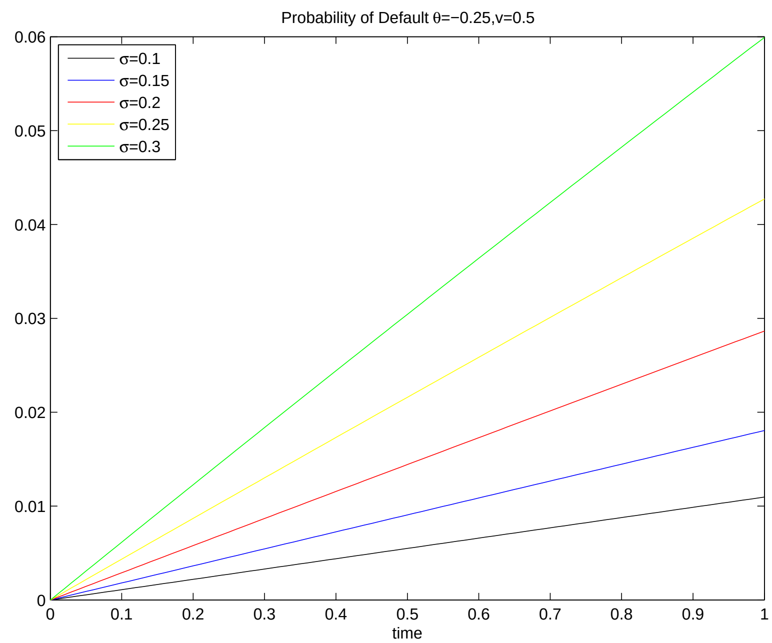

Figure 4.10: Variance sensitivity: Plot of Default Probability $\mathbb{P}(\tau \leq t)$ versus time $t \in [0, 1]$ for new structural model. The parameters of VG model is $\theta = -0.25, v = 0.5$, and $\sigma = 0.1, 0.15, 0.2, 0.25, 0.3$;

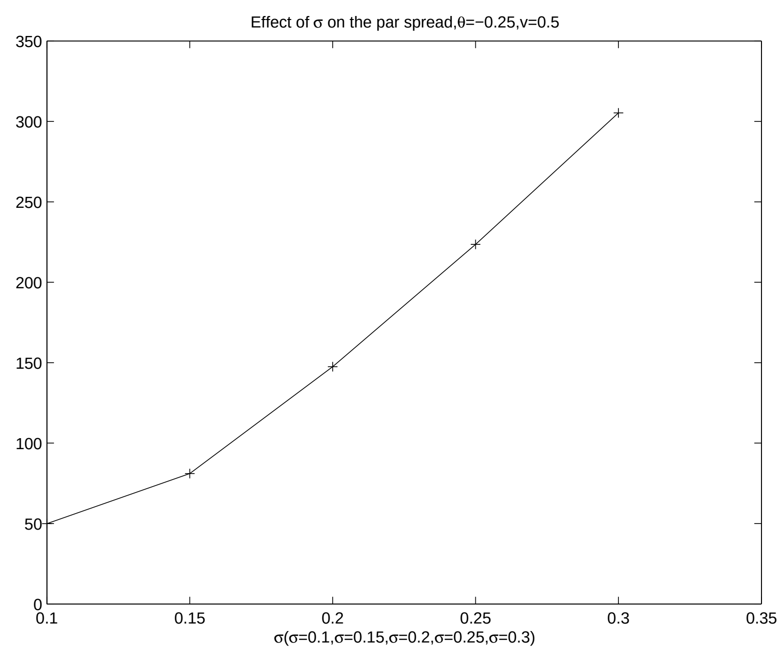

Figure 4.11: Variance sensitivity: Plot of CDS spread c versus time σ for new structural model. The parameters of VG model is $\theta = -0.25, v = 0.5$, and $\theta = 0.1, 0.15, 0.2, 0.25, 0.3$;

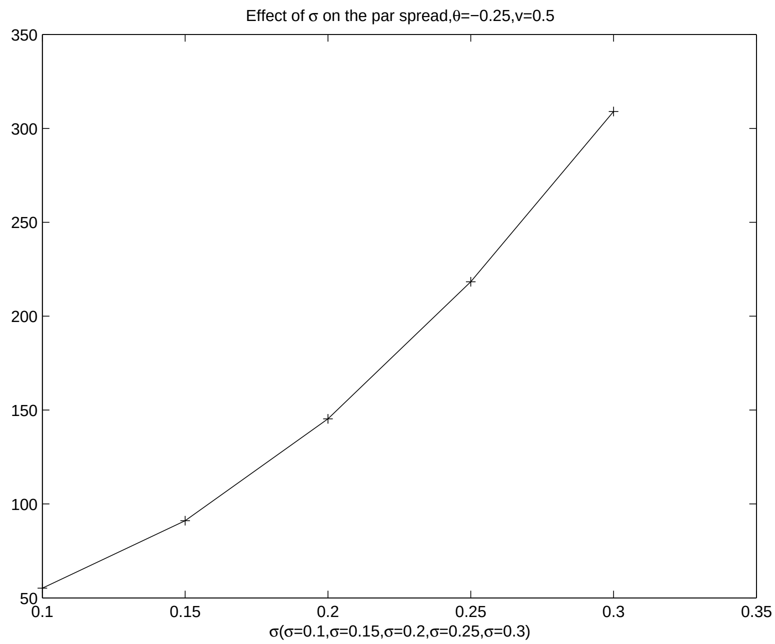

Figure 4.12: Variance sensitivity: Plot of CDS spread c versus time σ for new structural model. The parameters of VG model is $\theta = -0.25, v = 0.5$, and $\theta = 0.1, 0.15, 0.2, 0.25, 0.3$;

4.3 Conclusion

To investigate the sensitivity of one parameter of the models, we vary the parameter with the other two parameters fixed. We first vary the kurtosis parameter ν keeping all other parameters fixed. Next, we vary the skewness parameter θ and keep all other parameters fixed. At last, we vary the variance parameter σ and keep all other parameters fixed. For the new structural model, we assume the default level is constant. The constant level we choose is the one makes the CDS price accurate under the new structural model.

From the figure 4.1 to figure 4.4, we can see for both models higher kurtosis ν give rise to higher default probabilities and higher par spreads. From figure 4.5 to figure 4.8, more negative skewness θ also give rise to a higher default probabilities and higher par spreads. And from figure 4.9 to figure 4.12, higher variance σ give rise to higher default probabilities and higher par spreads. From the figure 4.1 to figure 4.4, we see the new model is more sensitive to the kurtosis parameter ν than the old structural model. They don't have significant difference in the sensitivity of the other two parameters.

The new structural model outperforms the old structural model in its simplicity of computing default probability and the CDS spreads. No PIDE or Monte Carlo methods are involved in the new model. We only need to compute a integral with respect to a lévy measure. Thus, the higher dimensional case are possible under this model. However, the old structural model is more intuitive and less sensitive to parameters. Selection of the default level is another important question in the new model.

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