UNIVERSITÉ LIBRE DE BRUXELLES

DÉPARTMENT D'INFORMATIQUE

MEMO-F-403
PREPARATORY WORK FOR THE MASTER THESIS

Data structures for Partially Ordered Sets

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Contents

Todo list

1 Introduction

Theoritical context Data structures play an important role in algorithms complexity. With the objective to improve standard algorithms in automata theory, researchers at ULB have implemented new algorithms to resolve important problems in the field. Those implementations uses antichains, which are data structures that allow to represent elements in a partially ordered set, in a more compact way. The goal of this preparatory work is to motivate the interest that are made in data structures for partially ordered set, especially antichains, and why we need efficient implementation. It also defines the desired objectives and propose a state of the art of the various existing implementations. In this work, we will mainly focus on the usage of partially ordered sets and antichains in automata theory-related problems.

Reference researchers ?

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Extend this paragraph by talking about diffent problems;algorihtms where antichains can be used

Checking universality with antichains An example that highlights the use of antichains in automata theory is the universality problem. It is the problem that for a finite automaton, we want to check if the language of this automaton equivalent to the language of all the words on the alphabet. The universality problem is a classical theoritical problem, and many verifications-related problems can be reduced in polynomial time to it.

Let $A = \langle Q_A, \Sigma, q_0, \delta, F_A \rangle$ be a finite automaton, we want to check whether or not the language of A is universal, that is, $L(A) = \Sigma^*$. The language of A is universal if and only if the language of the complementary of A accepts no words, that is, $L(A) = \Sigma^*$ if and only if $L(\bar{A}) = \emptyset$. Therefore, the goal is to find a computation path of the automaton on a word such that the path start in the initial state, and the final target is a non-accepting state.

Let's first define an interesting proprety of non-deterministic automaton before explaining the use of antichains for the universality problem. Let q and f two subsets of states such that $q, f \subseteq Q_A$. When computing a letter σ on the deterministic equivalent A_d, that is $q \xrightarrow{\sigma} f$, it is known that for all set of states $q' \subseteq q$, the resulting subset f' when computed on A_d such that $q' \xrightarrow{\sigma} f'$, it holds that $f' \subseteq f$.

The algorithms proposed in [DWRLH06], doesn't build the deterministic automaton, but rather do it implicitly. The algorithms procedures is to build iteratively a set of subsets of states that have as target the non-accepting states.At each iteration, only maximal set of states, based on the inclusion operation are kept in memory for the next iteration. Also, the idea is based on adding in the targets, all the predecessors.

Concrete example Let's take the example from [DWRLH06] shown in Figure 2. This example is an non-deterministic finite automata. The goal is to check if wheter or not this automata is universal. The standard algorithm for this problem is to first build the deterministic equivalent automata. Figure 1 shows the equivalent deterministic finite automata. Instead of building the deterministic

talk about subset construction

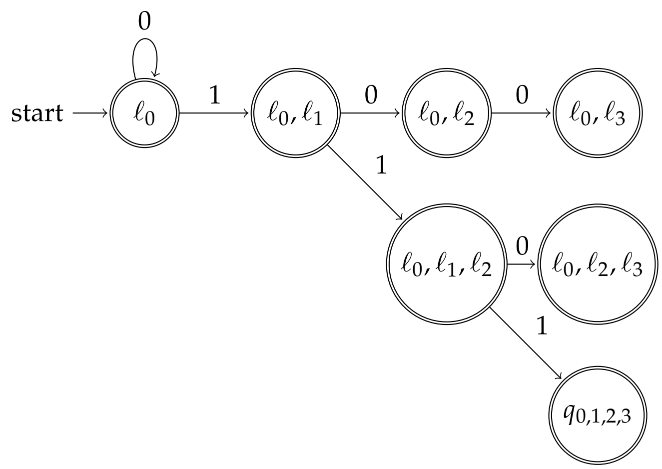

Figure 1:

Example 1.0.1.

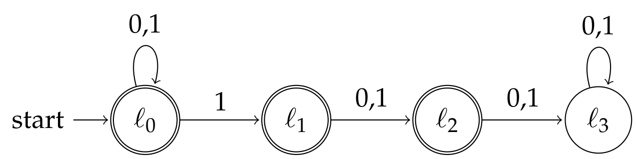

Figure 2:

automaton, Raskin et al. [DWRLH06] have proposed an algorithm that will focus on maximal sets because of specifics of proprety in automata theory.

In the specific example of 1. Instead on checking each state of the deterministic finite automata, it is better to check only the maximal path. Indeed, the focus is made on checking if the final state of a word computation is non-accepting i.e. $q_f \cap F_A = \emptyset$. Because of the proprety of the inclusion computation, checking only the maximal set of states is sufficient. Computing the workd 1^k, will lead to a maximal path for this automaton. For any word $w < 1^k$ of size k, the path of computation will be included in the path of 1^k. Since the objective is to find the final state as a non-accepting, if the final state is non-accepting for 1^k, that is $f_{1^k} \cap F = \emptyset$ then for the word w, since $f_w \subseteq f_{1^k}$, it holds that $f_w \cap F = \emptyset$.

In [DWRLH06], the algorithm proposed follow an equivalent idea by using game theory. The universality problem is reduce to a two-player reachability game, which can be done in polynomial time. The objective of the game is for the protagonist to establish that the automaton A is not universal. To this end, the protagonist will provide a word, a letter at a time, and find a strategy that try to show that A ends in a rejecting state. The protagonist only has a strategy to win the game if and only if A is not universal.

Objective The objective of the final work is to provide an efficient implementation of different data structures that allow to compactly represent partially ordered sets, specifically antichains. The first step is to implement in Java, classes that will be provided to the `Owl` library [Sal16]. `Owl` is a LTL to deterministic automata translations tool-set written in `Java`. A second step will be to implement antichain-based algorithms using the new antichains implementation and study the performance.

Include examples from Guillermo correction

Related work There are two interesting implementations that were found. The first one is `AaPAL` (Antichain and Pseudo-Antichain Library), a generic library that was implemented in the frame of Aaron Bohy's PhD thesis [Boh14b] to provide an antichain library. It is implemented in `C`. The other implementation of antichains have been done by De Causmaecker and De Wannemacker in [DCDW]. The algorithms to find the ninth Dedekind number uses antichains and they needed to implement a representation of antichains. Their implementation is using `Java`. To improve efficiency and performances, Hoedt in [Hoe] has extended [DCDW] antichains implementation by using bit sequence instead of tree reprensentation.

Structure of the preparatory work This paper is the introduction to next year thesis, therefore the content concern only the preliminaries. The goal is to properly define the subject, existing implementations and the desired objectives. In Section 2, we formally define antichains, and give examples of such data structures. In Section **??**, we summarize the work that has been done by others for antichains implementation. In 4, we propose an overview of next year work and possibilities.

2 Data Structures

In this section, we will provide formal definitions of the data structures that we will implement. We recall the notion of binary relations and important propreties of such relations. We then define partially ordered set, totally order set and closed set. Finally we give a formal definition for antichains.

The definitions and examples for this section are based on [Boh14b] and [Maq].

2.1 Binary relations

A binary relation for an arbitrary set S is a set of pair $R \subseteq S \times S$. There are five important properties: reflexitivity, transitivity, symmetry, antisymmetry and total.

A relation R on S is said to be:

- Reflexive: iff $\forall s \in S$ it holds that $(s, s) \in R$
- Transitive: iff $\forall s_1, s_2, s_3 \in S$, if $(s_1, s_2) \in R$ and $(s_2, s_3) \in R$ then it holds that $(s_1, s_3) \in R$
- Symmetric: iff $(s_1, s_2) \in R$ then $(s_2, s_1) \in R$.
- Antisymmetric: iff $(s_1, s_2) \in R$ and $(s_2, s_1) \in R$ then $s_1 = s_2$
- Total: iff $\forall s_1, s_2 \in S$ then $(s_1, s_2) \in R$ or $(s_2, s_1) \in R$

Orders A *partial order* is a binary relation that is *reflexive*, *transitive* and *antisymmetric*. We note a partial order relation by R. We note $s_1 R s_2$ to show the belonging of a binary relation to a partial order, which is equivalent to $(s_1, s_2) \in R$. A *total order* is a partial order that is *total*.

Example 2.1.1. For example, the comparison of natural numbers is a partial order. Let $\leq$ be a binary relation on $\mathbb{N}$ such that $\leq \subseteq \mathbb{N}^2$. The binary relation is defined following the usual semantic of the symbol, i.e. $n_1 \leq n_2$ if and only if n_1 is smaller or equal to n_2. Based on this, $\leq$ is a partial order. It is reflexive, transitive and antisymmetric. The binary relation $\leq$ on natural numbers is actually a total order since all the natural numbers can be compared against each other.

2.2 Partially ordered set

An arbitrary set S associated with a partial order $\preceq$ is called a *partially ordered set* or *poset*. It is denoted by the pair $\langle S, \preceq \rangle$.

Comparable Let $s_1, s_2 \in S$ and $\langle S, \preceq \rangle$ a poset. The two elementes s_1 and s_2 are said to be *comparable* if either $s_1 \preceq s_2$ or $s_2 \preceq s_1$. If neither of those two comparisons are correct, then s_1 and s_2 are said to be *incomparable*.

Bounds Let $\langle S, \preceq \rangle$ a partially ordered set. A *lower bound* of $P \subseteq S$ is an element $s \in S$ such that for all $p \in P$, it holds that $s \preceq p$. The *greatest lower bound* of elements of a set $P \subseteq S$ is an element $s \in S$ defineds as follow: for all $p \in P$ it holds that $s \preceq p$, and for all $s' \in S$ we have that if $s' \preceq p$ then $s' \preceq s$.

The greatest lower bound is unique. It means that if two elements $s_1, s_2 \in S$ are $\preceq$-incomparable and for a subset $P \subseteq S$, for all $p \in P$, if it holds that $s_1 \preceq p$ and $s_2 \preceq p$ and if all others elements $s' \in S$ with s' a lower-bound for P such that $s' \preceq s_1$ and $s' \preceq s_2$; the greatest lower bound is said to be undefined.

For a set of two elements $P = \{p_1, p_2\}$, we denote by $p_1 \sqcap p_2$ the greatest lower bound.

Lattices A *lower semilattice* is a poset $\langle S, \preceq \rangle$ where for all pair of elements $s_1, s_2 \in S$, we have that the greatest lower bound $s_1 \sqcap s_2$ exists.

Include definition of least upper bound, if necessary

Example 2.2.1. Let $\langle 2^{Q_A}, \subseteq \rangle$, a poset with Q_A a set of 3 elements. Figure 3 shows a graph with the incomparable elements of such poset.

Fix figure formulation as we havent spoke about antichains at this section yet

2.3 Antichains

Closed sets A closed set is a set $L \subseteq S$ of a lower semilattice $\langle S, \preceq \rangle$ where $\forall \ell \in L$ we have that $\forall s \in S$ such that $s \preceq \ell$, then $s \in L$. Note that for two closed sets $L_1, L_2 \subseteq S$, we have that $L_1 \cup L_2$ and $L_1 \cap L_2$ are also closed sets, but $L_1 \setminus L_2$ does not result necessarily to a closed set.

Maximal/minimal elements We denote by $\lceil L \rceil$ the set of maximal elements of a closed set L which correspond to $\lceil L \rceil = \{\ell \in L | \forall \ell' \in L : \ell \preceq \ell' \Rightarrow \ell = \ell'\}$. Alternatively, to represent the set of minimal elements, the noation $\lfloor L \rfloor$ is used which has the following semantic $\lfloor L \rfloor = \{\ell \in L | \forall \ell' \in L : \ell' \preceq \ell \Rightarrow \ell = \ell'\}$.

Meaning of — vs . vs : in set definition ?

Closure A *lower closure* of a set L on S noted $\downarrow L$ is the set of all elements of S that are *smaller or equal* to an element of L i.e. $\downarrow L = \{s \in S \mid \exists \ell \in L \cdot s \preceq \ell\}$. Note that for a closed set L we have that $\downarrow L = L$.

Antichain An *antichain* of a poset $\langle S, \preceq \rangle$ is a set $\alpha \subseteq S$ where all element of α are incomparable with respect to the partial order $\preceq$. Otherwise, if all elements are comparable the set is called a *chain*. Antichains allow to represent closed set in a more compact way. For a closed set $L \subseteq S$ we can retrieve all elements of L by using the antichain $\alpha = \lceil L \rceil$. With respect to the definition of the lower closure we have that $\downarrow \alpha = L$.

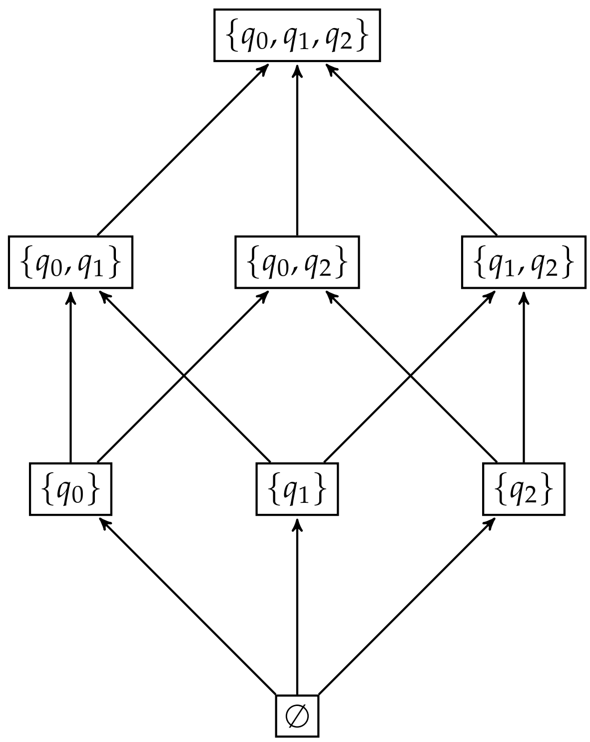

Figure 3: Example of antichains using the poset $\langle 2^{Q_A}, \subseteq \rangle$, with Q_A the set of states of an automaton A, which correspond to $Q_A = \{q_0, q_1, q_2\}$. Each directed edge of the graph correspond to a valid comparison using the set inclusion $\subseteq$. For example $\{q_0\} \rightarrow \{q_0, q_1\}$ corresponds to the inclusion $\{q_0\} \subseteq \{q_0, q_1\}$. Two elements of the graph with no connection, means that the elements are incomparable. For example, $\alpha = \{\{q_0\}, \{q_1\}, \{q_2\}\}$ is a set of incomparable elements and α is called an antichain.

2.4 Operations on antichains

This section list the classical propreties of antichains. All the examples are based on the poset defined in Example 2.2.1.

Proposition 2.4.1. Let $\alpha_1, \alpha_2 \subseteq S$ two antichains and $s \in S$:

- $s \in {\downarrow}\alpha_1$ iff $\exists a \in \alpha_1$ such that $s \preceq a$

  Example 2.4.1. Let $\alpha_1 = \{\{q_0, q_1\}\}$, $\{q_0\}$ belongs to the lower closure α_1 because, $\{q_0\} \subseteq \{q_0, q_1\}$.

- ${\downarrow}\alpha_1 \subseteq {\downarrow}\alpha_2$ iff $\forall a_1 \in \alpha_1, \exists a_2 \in \alpha_2$ such that $a_1 \preceq a_2$

  Example 2.4.2.

- ${\downarrow}\alpha_1 \cup {\downarrow}\alpha_2 = {\downarrow}\lceil \alpha_1 \cup \alpha_2 \rceil$

  Example 2.4.3.

- ${\downarrow}\alpha_1 \cap {\downarrow}\alpha_2 = {\downarrow}\lceil \alpha_1 \sqcap \alpha_2 \rceil$ where $\alpha_1 \sqcap \alpha_2$ is defined as $\alpha_1 \sqcap \alpha_2 = \{a_1 \sqcap a_2 | a_1 \in \alpha_1, a_2 \in \alpha_2\}$

  Example 2.4.4.

Give complete definition for interesection

3 Existing implementations

Research other possible related works

3.1 AaPAL

Talk about AaPAL in Acacia+ ?

Bohy's **A**ntichain **a**nd **P**seudo-**A**ntichain **L**ibrary [Boh14a] is an open-source generic library for the manipulation of antichains and pseudo-antichains data structures, implemented in `C`. In this section we will mainly focus on the implementation of antichains.

Pseudo-antichains might be interesting for next year maybe ?

Antichain representation An antichain is represented by a `struct`, containing as attributes the size of the antichain, and the incomparable elements of the antichains, as a list. The list is manipulated using the `GSList` object from the `glib` library. To allow modularity, the type of the elements is `void`.

Cite glib library

Operations The operations implemented in `AaPAL` are the union, intersection and appartenance defined in Proposition 2.4.1. An interesting remark is that most of the complexity is given as a paramater to the functions. For example the function to compare two elements in an antichain is given as a parameter. It means that the complexity to define the domain of the antichain, must be implemented in the compare function. Same pattern goes for the intersection operation, the function to compute the intersection must be provided by the user. Also basic operations such as creating an antichain, adding an element to an antichain, checking emptiness or cloning an antichain are implemented. in `AaPAL`.

Find usage of AaPAL (see Bohy's PHD)

3.2 Antichains for Dedekind's problem

Dedekind's problem The Dedekind's problem correspond to a problem introduced by Richard Dedekind in the 19th century. He defined a rapidly growing sequence of natural numbers. The n_{th} Dedekind number is the count of antichains in the powerset of set with n elements.

De Causemaecker and De Wannemacker [DCDW] improved and implemented a multithreaded algorithm to find the n_{th} Dedekind number that allow to compute the n_{th} Dedekind number from the powerset of a set with $(n-2)$-elements, that is re-use the number of antichains for $(n-2)$-set to compute the n_{th} Dedekind number.

De Causemaecker and De Wannemacker only provide the executable of their algorithm in their paper for the Dedekind algorithm [DCDW]. Hoedt proposed in [Hoe] an algorithm to find the ninth Dedekind number, which requires a representation of antichains. The representation used is an extension of the implementation proposed by De Causmaecker and De Wannemacker. The source of his implementation was found in is personal GitHub [Hoe15]. We will therefore only focus on the implementation of Hoedt, which is an extension of the implementation proposed by De Causmaecker and De Wannemacker, but by representing an antichains using bitarray-like methods.

Antichains representation According to Hoedt in [Hoe], the first proposed representation in [DCDW] was done by using a `TreeSet`. With the objective to improve the performance of the important operations, for example the intersection, Hoedt used another way to represent antichains, by using a bitarray-like representation. The goal of the bitarray representation is to represent set of natural numbers by a binary representation. For example the subset $\{1,2\}$ would be represented by the binary number 11_2 and the subset $\{1,3\}$ by 101_2 corresping respectively to 3_{10} and 5_{10}. Then an antichain is represented by enabling the bit of the corresponding number of an element of the antichain. Therefore, an antichain $\alpha = \{\{1,2\},\{1,3\}\}$ is represented by the bitarray where the bit at indices 3 and 5 are enabled, that is 10100.

Tree vs Bitarray representations

4 Next year overview

4.1 Requirements

Different requirements were defined regarding the desired implementation. The first requirement is that the library will have to be used in `Owl`, therefore the language of reference is `Java 10`.

Talk about Owl

Java profiling

4.2 Framework and implementation

The goal will be to try to find a good way to implement antichains representation and operations. One of the possible way to do so is to provide a framework that will allow to implement operations functions depending on the universe of the antichains. In `AaPAL` all the complexity is implemented in the `compare_elements` that must be given to the library functions. It that case, all the complexity must be implemented by the user to define the universe of the antichains. An example of specific implementation for natural numbers is the one proposed by Hoedt, presented in section 3.2, using bitarray. A first step is to proposed an framework for users to implement the specifics of the universe necessary for their algorithms. Next year thesis objective will be to propose a generic and abstract API, and implement specifics for known universe and test them on state-of-the-art algorithms, especially in automata theory.

Include algorithms to test i.e. Boolean function from Guillermo

Write a paragraph about why Java and not C

Fill-in bib correctly!

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