Learning Dynamics: Assignement 1

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Contents

1 Hawk-Dove game

The game payoff matrix is shown in Figure 1.

1.1 Expected payoff for player 1

Given player 2's mixed strategy α_2, player 1 expected payoff for the pure strategy $Hawk(p)$ is:

$$E_1(H, \alpha_2) = q(\frac{1}{2}(V - D)) + (1 - q)V = V - \frac{q}{2}(V + D) \tag{1}$$

	Hawk	Dove
Hawk	$\frac{1}{2}(V-D), \quad \frac{1}{2}(V-D)$	$V, \quad 0$
Dove	$0, \quad V$	$\frac{V}{2}-T, \quad \frac{V}{2}-T$

Figure 1: Hawk-Dove game

Given player 2's mixed strategy α_2, player 1 expected payoff for the pure strategy $Dove(1-p)$ is:

$$E_1(D,\alpha_2) = q\cdot 0 + (1-q)(\frac{V}{2}-T) = (1-q)(\frac{V}{2}-T) \tag{2}$$

1.2 Expected payoff for player 2

Given player 1's mixed strategy α_1, player 2 expected payoff for the pure strategy $Hawk(q)$ is:

$$E_2(H,\alpha_1) = p(\frac{1}{2}(V-D)) + (1-p)V = V - \frac{p}{2}(V+D) \tag{3}$$

Given player 1's mixed strategy α_1, player 2 expected payoff for the pure strategy $Dove(1-q)$ is:

$$E_2(D,\alpha_1) = p\cdot 0 + (1-p)(\frac{V}{2}-T) = (1-p)(\frac{V}{2}-T) \tag{4}$$

1.3 Player 1's mixed strategies

Since this is a game with two-players two actions *The linearity implies 3 possible outcomes*:

player 1's unique best response is the pure strategy $Hawk(p)$ when:

$$\begin{aligned}
\iff & \quad E_1(H,\alpha_2) > E_1(D,\alpha_2) \\
\iff & V - \frac{q}{2}(V+D) > (1-q)(\frac{V}{2}-T) \\
\iff & V - \frac{V}{2}q - \frac{D}{2}q > \frac{V}{2} - T - \frac{V}{2}q + Tq \\
\iff & \quad \frac{V}{2} + T > \frac{D}{2}q + Tq \\
\iff & \quad \frac{\frac{V}{2}+T}{\frac{D}{2}+T} > q
\end{aligned}$$

player 1's unique best response is the pure strategy $Dove(1-p)$ when:

$$\iff E_1(H,\alpha_2) < E_1(D,\alpha_2)$$
$$\iff \quad \frac{\frac{V}{2}+T}{\frac{D}{2}+T} < q$$

all player 1's mixed strategies are all best responses when:

$$\iff E_1(H,\alpha_2) = E_1(D,\alpha_2)$$
$$\iff \quad \frac{\frac{V}{2}+T}{\frac{D}{2}+T} = q$$

1.4 Player 2's mixed strategies

This game is symmetric, the payoff matrix of player 1 and 2 are the same ($A = B^T$). The strategies of player 2 are the same than the player 1, except that it depends on p and not on q.

Player 2's unique best response is pure strategy $Hawk(q)$ when $p < \frac{\frac{V}{2}+T}{\frac{D}{2}+T}$, is pure strategy $Dove(1-q)$ when $p > \frac{\frac{V}{2}+T}{\frac{D}{2}+T}$ and all mixed strategies are all best response when $p = \frac{\frac{V}{2}+T}{\frac{D}{2}+T}$.

1.5 Result based on conditions

Note When we mention *the result* in the next paragraphs, we refer to the fraction $\frac{\frac{V}{2}+T}{\frac{D}{2}+T}$.

Whatever values T has, it will not change the result, since it is present in denominator and numerator of it. The choices depend on V and D.

If $V = D$, the result will be equal to 1, it means that p and q are equal to 1 when $V = D$, then all mixed strategies are best response for both players.

If $V > D$, it means that the the numerator will always be bigger than the denominator, so always bigger than 1. Since p and q are probablities they cannot go beyond 1, so p and q will always be smaller than the result. So if $V > D$, player 1 and 2 will play the pure strategy $Hawk(p)$ and $Hawk(q)$.

If $V < D$, it means that the the numerator will always be smaller than the denominator, so always smaller than 1. So if $V < D$, player 1 and 2 will play the pure strategy $Dove(1-p)$ and $Dove(1-q)$. So displaying become more beneficial than escalating when $V < D$.

2 Social Dilemma

Figure 2 represents the expected payoff of A based on her beliefs. For example the first column indicates the payoff of A when she believes that B will defect in all games type.

Since A is sure that each game is equally likely, there is a probability of 1/3 that she is in Prisonner's dilemma, probability of 1/3 in Stag-Hunt game and probability of 1/3 in Snowdrift game.

Figure 3 shows the outcomes for the three games and highlights the best response of player B, since she doesn't need to form any beliefs because she knows in which game she is playing.

	DDD	CCC	CCD	CDC	CDD	DCC	DCD	DDC
C	1/3	3	**8/3**	4/3	1	7/3	**2**	2/3
D	**2/3**	**4**	7/3	**11/3**	**2**	**8/3**	1	**7/3**

Figure 2: Expected payoff of player A for all possible combinations of player B's types

	C	D	C	D	C	D
C	2	**5**	**5**	2	2	**5**
D	0	**1**	0	**1**	**1**	0

Figure 3: Best response of player B against player A in all games (in the following order): Prisonner's dilemma, Stag-hunt game, Snowdrift game

2.1 Payoff function

Let's associate Prisonner's dilemma, Stag-hunt game and Snowdrift game with the indexes $I = \{1, 2, 3\}$, respectively. The actions cooperate and defect be the set $A = \{C, D\}$. Let $b_i \ \forall i \in I$, the beliefs of A, with $b_i \in A$ an action. Let $o_i(x, y) \ \forall i \in I \wedge \forall x, y \in A$ the outcomes of A for game i and action x for player A and action y for B. The payoff function is $p(a, b_1, b_2, b_3) \ \forall a \in A$:

$$p(a, b_1, b_2, b_3) = \frac{1}{3}(o_1(a, b_1) + o_2(a, b_2) + o_3(a, b_3)) \tag{5}$$

For example the payoff for A whe she plays cooperate (C) and believes $b_1 = D, b_2 = D, b_3 = D$, corresponding to the first row and column of Figure 2, is:

$$p(C, D, D, D) = \frac{1}{3}(0 + 0 + 1) = 1/3 \tag{6}$$

Other examples:

$$p(C, C, C, D) = \frac{1}{3}(2 + 5 + 1) = 8/3 \tag{7}$$

$$p(D, C, C, C) = \frac{1}{3}(5 + 2 + 5) = 4 \tag{8}$$

All the payoff function results are shown in Figure 2.

2.2 Best response and equilibrium

The best response given the action of all games type for player A are the bolded values in Figure 2. The best response for all games for player B are the bolded values in Figure 3. To find the pure Nash Equilibria, we need to match the best response of A with the best response of B for each beliefs. There are two pure Nash Equilibria, shown in red in Figure 2, $\{C, (DCD)\}$ and $\{D, (DDC)\}$. Indeed, for $\{C, (DCD)\}$, when A plays action C, the best response of B in Prisonner's dilemma is D, in Stag-hunt game it is C and in Snowdrift game it is D, so it is a pure Nash Equilibria. The same goes for $\{D, (DDC)\}$.

3 Sequential truel

The diagram representing the subgames are drawn as trees in Figure 4, 5 and 6. When we will refer to T_1, T_2 and T_3 it means that we are refering to, respectively, Figure 4, 5 and 6. In the subgames, the action $t(i)$ where $i \in \{A, B, C\}$, means that the current player is targeting player i. The current player can of course not target himself.

The main game is represented by T_1. As mentioned in the assignement specifications, the subgames when A fails to hit is intended target are the same, it is T_2. When in subgame T_2, we also found that when B misses his intended target, the subgames are the same, it is T_3.

3.1 Subgame perfect equilibria

3.1.1 C equilibrium in T_3

To find the SPE, we have to use the backward induction, so we need to find the SPE of T_3 first. Player C always stay alive whatever target she is choosing. The SPE is $t(A)$ if $p_a > p_b$, and $t(B)$ if $p_b > p_a$.

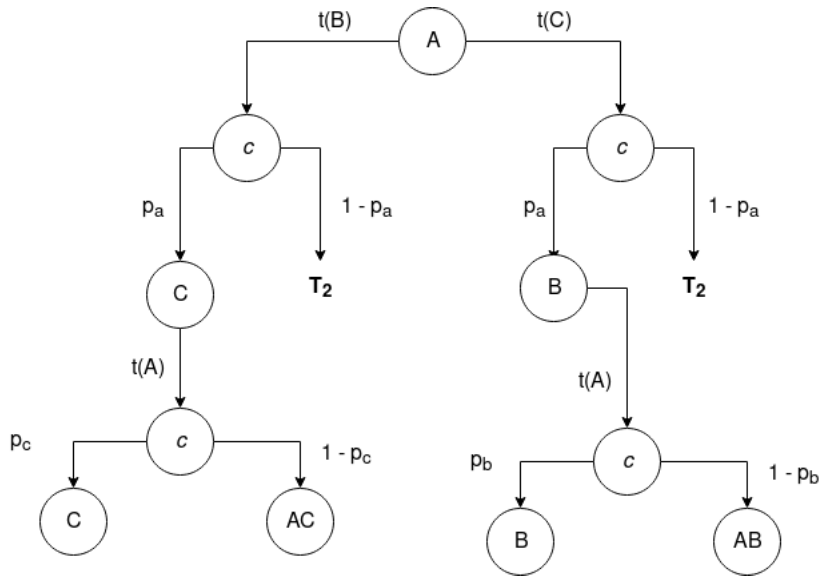

Figure 4: T_1, main subgame

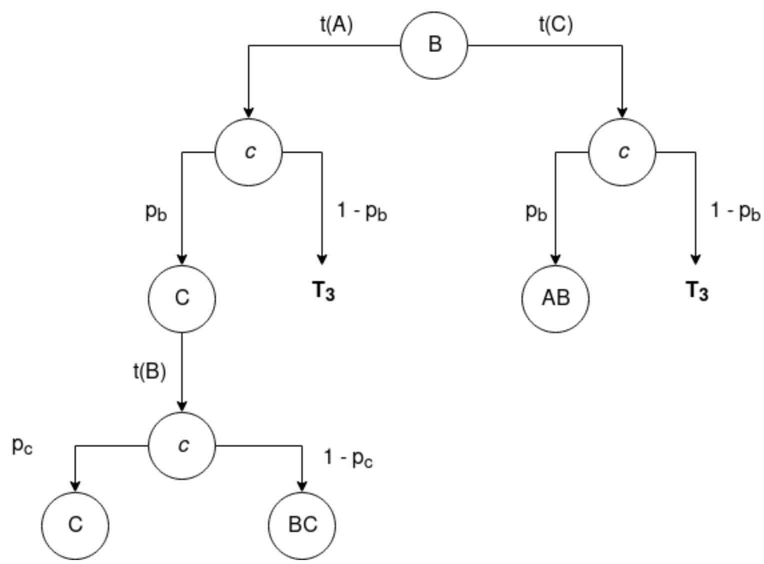

Figure 5: T_2, subgame when A misses her intended target

3.1.2 B equilibrium in T_2

Let's consider subgame T_2. Player B needs either to target A or target C. If B targets A, she has less chance to survive because next turn C will play and

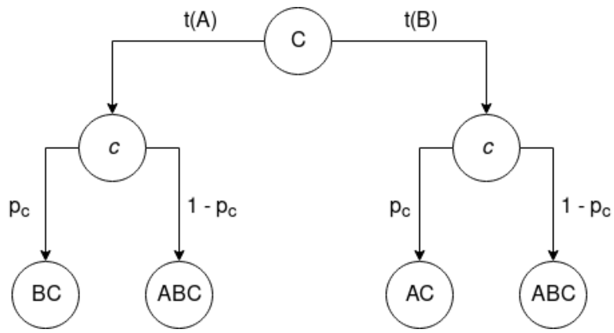

Figure 6: T_3, subgame when A and B miss their intended targets

will still be alive whatever the result of this shoot is. So B will always target C in T_3.

Formally, if B misses, her outcome will be the same because it will have the SPE of subgame T_3, which is unique. If B targets A and hits her, she has a probabilty of $1 - p_c$ to stay alive. If B targets C and hits her, she has a probablity of 1 to stay alive. So B will always choose to target C.

3.1.3 A equilibrium in T_1

Let's consider the subgame T_1 (the full game). Intuitively, it is best for A to target the player with the biggest probability, because she will have more chance to stay alive if she manages to eliminate the strongest opponent.

Formally, if A misses her intended target, the outcome will be the same since T_2 has an unique SPE. If A targets B and hits her, she has a probability of $1 - p_c$ to stay alive, because C is the remaining shooter and has a probability of $1 - p_c$ to fail. If A targets C and hits her, she has a probability of $1 - p_b$ to stay alive, because B is the remaining shooter and has a probability of $1 - p_b$ to fail. So A targets B if $(1 - p_c) > (1 - p_b)$, which can be simplified as $p_c < p_b$, and A targets C if $(1 - p_b) > (1 - p_c)$, which can be simplified as $p_b < p_c$.

3.2 Weakness is strength

In section 3.1.3, we explained that if $p_c > p_b$ A will target C. If C is the target, her probablity of survival is the probability that A misses and B misses, formally:

$$(1 - p_a)(1 - p_b) = 1 - p_b - p_a + p_a p_b = 1 - p_a - p_b(1 - p_a) \tag{9}$$

If $p_b > p_c$ A will target B, then C's chance of survival is: A hits her target and C will be the last player with a bullet or A misses her target and B misses also her target. Formally:

$$p_a + (1 - p_a)(1 - p_b) = p_a + 1 - p_b - p_a + p_a p_b = 1 - p_b(1 - p_a) \tag{10}$$

The difference between the probablity (9) and (10), is that (9) is smaller than (10) because of the decrease of p_a, so (9) will always be smaller than (10). The probablity of survival of C when $p_c > p_b$ will always be smaller than the probability of survival of C when $p_b > p_c$. C is always better off when $p_b > p_c$.