

Learning Dynamics: Assignement 1

Hakim Boulahya
hboulahy@ulb.ac.be

Université Libre de Bruxelles

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1 Hawk-Dove game

The game payoff matrix is shown in Figure 1.

1.1 Expected payoff for player 1

Given player 2's mixed strategy α_2 , player 1 expected payoff for the pure strategy $Hawk(p)$ is:

$$E_1(H, \alpha_2) = q\left(\frac{1}{2}(V - D)\right) + (1 - q)V = V - \frac{q}{2}(V + D) \quad (1)$$

	Hawk	Dove
Hawk	$\frac{1}{2}(V - D), \quad \frac{1}{2}(V - D)$	$V, \quad 0$
Dove	$0, \quad V$	$\frac{V}{2} - T, \quad \frac{V}{2} - T$

Figure 1: Hawk-Dove game

Given player 2's mixed strategy α_2 , player 1 expected payoff for the pure strategy *Dove*($1 - p$) is:

$$E_1(D, \alpha_2) = q \cdot 0 + (1 - q)\left(\frac{V}{2} - T\right) = (1 - q)\left(\frac{V}{2} - T\right) \quad (2)$$

1.2 Expected payoff for player 2

Given player 1's mixed strategy α_1 , player 2 expected payoff for the pure strategy *Hawk*(q) is:

$$E_2(H, \alpha_1) = p\left(\frac{1}{2}(V - D)\right) + (1 - p)V = V - \frac{p}{2}(V + D) \quad (3)$$

Given player 1's mixed strategy α_1 , player 2 expected payoff for the pure strategy *Dove*($1 - q$) is:

$$E_2(D, \alpha_1) = p \cdot 0 + (1 - p)\left(\frac{V}{2} - T\right) = (1 - p)\left(\frac{V}{2} - T\right) \quad (4)$$

1.3 Player 1's mixed strategies

Since this is a game with two-players two actions *The linearity implies 3 possible outcomes:*

player 1's unique best response is the pure strategy *Hawk*(p) when:

$$\begin{aligned}
&\iff E_1(H, \alpha_2) > E_1(D, \alpha_2) \\
&\iff V - \frac{q}{2}(V + D) > (1 - q)\left(\frac{V}{2} - T\right) \\
&\iff V - \frac{V}{2}q - \frac{D}{2}q > \frac{V}{2} - T - \frac{V}{2}q + Tq \\
&\iff \frac{V}{2} + T > \frac{D}{2}q + Tq \\
&\iff \frac{\frac{V}{2} + T}{\frac{D}{2} + T} > q
\end{aligned}$$

player 1's unique best response is the pure strategy $Dove(1 - p)$ when:

$$\begin{aligned} &\iff E_1(H, \alpha_2) < E_1(D, \alpha_2) \\ &\iff \frac{\frac{V}{2} + T}{\frac{D}{2} + T} < q \end{aligned}$$

all player 1's mixed strategies are all best responses when:

$$\begin{aligned} &\iff E_1(H, \alpha_2) = E_1(D, \alpha_2) \\ &\iff \frac{\frac{V}{2} + T}{\frac{D}{2} + T} = q \end{aligned}$$

1.4 Player 2's mixed strategies

This game is symmetric, the payoff matrix of player 1 and 2 are the same ($A = B^T$). The strategies of player 2 are the same than the player 1, except that it depends on p and not on q .

Player 2's unique best response is pure strategy $Hawk(q)$ when $p < \frac{\frac{V}{2} + T}{\frac{D}{2} + T}$, is pure strategy $Dove(1 - q)$ when $p > \frac{\frac{V}{2} + T}{\frac{D}{2} + T}$ and all mixed strategies are all best response when $p = \frac{\frac{V}{2} + T}{\frac{D}{2} + T}$.

1.5 Result based on conditions

Note When we mention *the result* in the next paragraphs, we refer to the fraction $\frac{\frac{V}{2} + T}{\frac{D}{2} + T}$.

Whatever values T has, it will not change the result, since it is present in denominator and numerator of it. The choices depend on V and D .

If $V = D$, the result will be equal to 1, it means that p and q are equal to 1 when $V = D$, then all mixed strategies are best response for both players.

If $V > D$, it means that the the numerator will always be bigger than the denominator, so always bigger than 1. Since p and q are probabilities they cannot go beyond 1, so p and q will always be smaller than the result. So if $V > D$, player 1 and 2 will play the pure strategy $Hawk(p)$ and $Hawk(q)$.

If $V < D$, it means that the the numerator will always be smaller than the denominator, so always smaller than 1. So if $V < D$, player 1 and 2 will play the pure strategy $Dove(1 - p)$ and $Dove(1 - q)$. So displaying become more beneficial than escalating when $V < D$.

2 Social Dilemma

Figure 2 represents the expected payoff of A based on her beliefs. For example the first column indicates the payoff of A when she believes that B will defect in all games type.

Since A is sure that each game is equally likely, there is a probability of $1/3$ that she is in Prisonner's dilemma, probability of $1/3$ in Stag-Hunt game and probability of $1/3$ in Snowdrift game.

Figure 3 shows the outcomes for the three games and highlights the best response of player B , since she doesn't need to form any beliefs because she knows in which game she is playing.

	DDD	CCC	CCD	CDC	CDD	DCC	DCD	DDC
C	1/3	3	8/3	4/3	1	7/3	2	2/3
D	2/3	4	7/3	11/3	2	8/3	1	7/3

Figure 2: Expected payoff of player A for all possible combinations of player B 's types

	C	D	C	D	C	D
C	2	5	5	2	2	5
D	0	1	0	1	1	0

Figure 3: Best response of player B against player A in all games (in the following order): Prisonner's dilemma, Stag-hunt game, Snowdrift game

2.1 Payoff function

Let's associate Prisonner's dilemma, Stag-hunt game and Snowdrift game with the indexes $I = \{1, 2, 3\}$, respectively. The actions cooperate and defect be the set $A = \{C, D\}$. Let $b_i \forall i \in I$, the beliefs of A , with $b_i \in A$ an action. Let $o_i(x, y) \forall i \in I \wedge \forall x, y \in A$ the outcomes of A for game i and action x for player A and action y for B . The payoff function is $p(a, b_1, b_2, b_3) \forall a \in A$:

$$p(a, b_1, b_2, b_3) = \frac{1}{3}(o_1(a, b_1) + o_2(a, b_2) + o_3(a, b_3)) \quad (5)$$

For example the payoff for A when she plays cooperate (C) and believes $b_1 = D, b_2 = D, b_3 = D$, corresponding to the first row and column of Figure 2, is:

$$p(C, D, D, D) = \frac{1}{3}(0 + 0 + 1) = 1/3 \quad (6)$$

Other examples:

$$p(C, C, C, D) = \frac{1}{3}(2 + 5 + 1) = 8/3 \quad (7)$$

$$p(D, C, C, C) = \frac{1}{3}(5 + 2 + 5) = 4 \quad (8)$$

All the payoff function results are shown in Figure 2.

2.2 Best response and equilibrium

The best response given the action of all games type for player A are the bolded values in Figure 2. The best response for all games for player B are the bolded values in Figure 3. To find the pure Nash Equilibria, we need to match the best response of A with the best response of B for each beliefs. There are two pure Nash Equilibria, shown in red in Figure 2, $\{C, (DCD)\}$ and $\{D, (DDC)\}$. Indeed, for $\{C, (DCD)\}$, when A plays action C , the best response of B in Prisoner's dilemma is D , in Stag-hunt game it is C and in Snowdrift game it is D , so it is a pure Nash Equilibria. The same goes for $\{D, (DDC)\}$.

3 Sequential truel

The diagram representing the subgames are drawn as trees in Figure 4, 5 and 6. When we will refer to T_1 , T_2 and T_3 it means that we are referring to, respectively, Figure 4, 5 and 6. In the subgames, the action $t(i)$ where $i \in \{A, B, C\}$, means that the current player is targeting player i . The current player can of course not target himself.

The main game is represented by T_1 . As mentioned in the assignment specifications, the subgames when A fails to hit is intended target are the same, it is T_2 . When in subgame T_2 , we also found that when B misses his intended target, the subgames are the same, it is T_3 .

3.1 Subgame perfect equilibria

3.1.1 C equilibrium in T_3

To find the SPE, we have to use the backward induction, so we need to find the SPE of T_3 first. Player C always stay alive whatever target she is choosing. The SPE is $t(A)$ if $p_a > p_b$, and $t(B)$ if $p_b > p_a$.

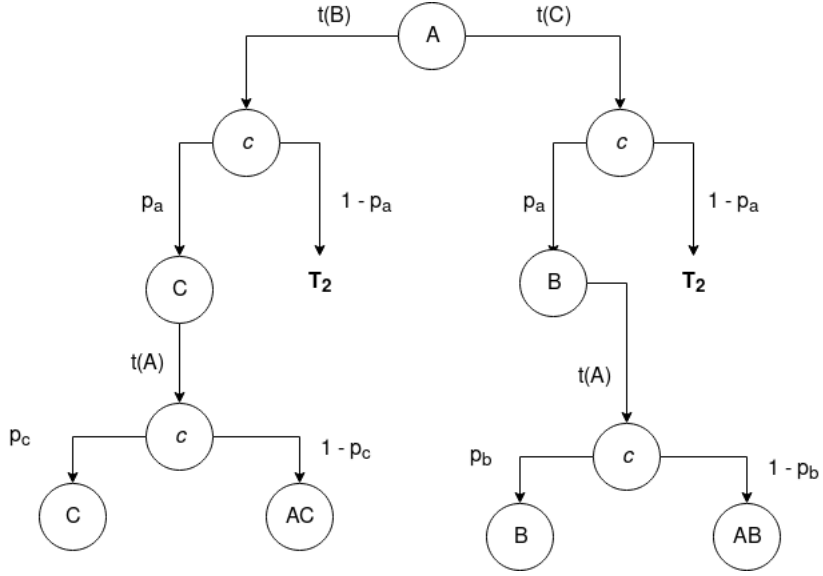


Figure 4: T_1 , main subgame

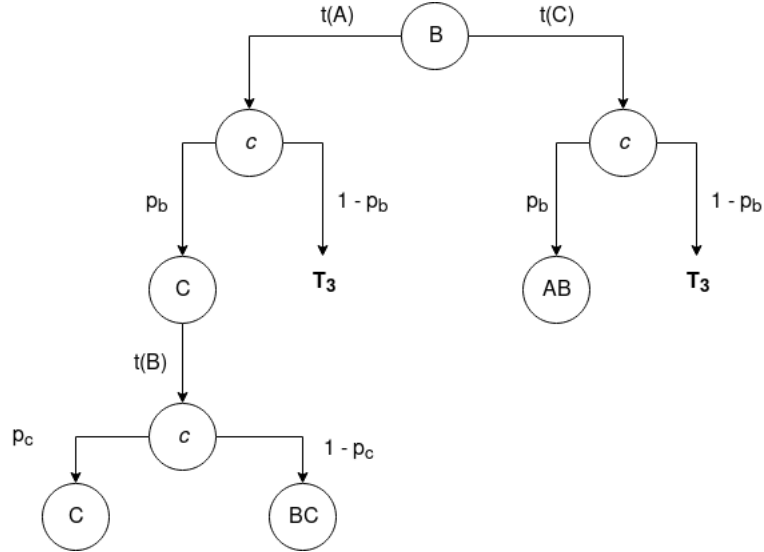


Figure 5: T_2 , subgame when A misses her intended target

3.1.2 B equilibrium in T_2

Let's consider subgame T_2 . Player B needs either to target A or target C . If B targets A , she has less chance to survive because next turn C will play and

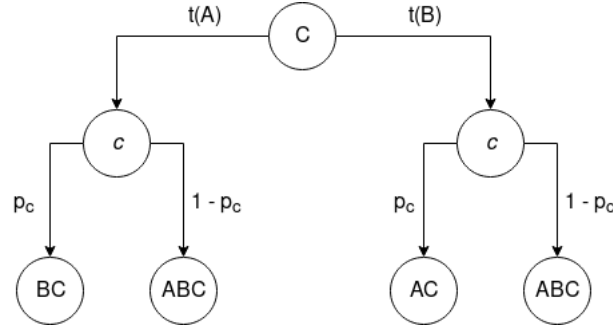


Figure 6: T_3 , subgame when A and B miss their intended targets

will still be alive whatever the result of this shoot is. So B will always target C in T_3 .

Formally, if B misses, her outcome will be the same because it will have the SPE of subgame T_3 , which is unique. If B targets A and hits her, she has a probability of $1 - p_c$ to stay alive. If B targets C and hits her, she has a probability of 1 to stay alive. So B will always choose to target C .

3.1.3 A equilibrium in T_1

Let's consider the subgame T_1 (the full game). Intuitively, it is best for A to target the player with the biggest probability, because she will have more chance to stay alive if she manages to eliminate the strongest opponent.

Formally, if A misses her intended target, the outcome will be the same since T_2 has an unique SPE. If A targets B and hits her, she has a probability of $1 - p_c$ to stay alive, because C is the remaining shooter and has a probability of $1 - p_c$ to fail. If A targets C and hits her, she has a probability of $1 - p_b$ to stay alive, because B is the remaining shooter and has a probability of $1 - p_b$ to fail. So A targets B if $(1 - p_c) > (1 - p_b)$, which can be simplified as $p_c < p_b$, and A targets C if $(1 - p_b) > (1 - p_c)$, which can be simplified as $p_b < p_c$.

3.2 Weakness is strength

In section 3.1.3, we explained that if $p_c > p_b$ A will target C . If C is the target, her probability of survival is the probability that A misses and B misses, formally:

$$(1 - p_a)(1 - p_b) = 1 - p_b - p_a + p_a p_b = 1 - p_a - p_b(1 - p_a) \quad (9)$$

If $p_b > p_c$ A will target B , then C 's chance of survival is: A hits her target and C will be the last player with a bullet or A misses her target and B misses also her target. Formally:

$$p_a + (1 - p_a)(1 - p_b) = p_a + 1 - p_b - p_a + p_a p_b = 1 - p_b(1 - p_a) \quad (10)$$

The difference between the probability (9) and (10), is that (9) is smaller than (10) because of the decrease of p_a , so (9) will always be smaller than (10). The probability of survival of C when $p_c > p_b$ will always be smaller than the probability of survival of C when $p_b > p_c$. C is always better off when $p_b > p_c$.