

# Learning Dynamics: Assignement 1

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## 1 Hawk-Dove game

### 1.1 Expected payoff for player 1

Given player 2's mixed strategy  $\alpha_2$ , player 1 expected payoff for the pure strategy  $Hawk(p)$  is:

$$E_1(H, \alpha_2) = q\left(\frac{1}{2}(V - D)\right) + (1 - q)V = V - \frac{q}{2}(V - D) \quad (1)$$

Given player 2's mixed strategy  $\alpha_2$ , player 1 expected payoff for the pure strategy  $Dove(1 - p)$  is:

	Hawk	Dove
Hawk	$\frac{1}{2}(V - D), \quad \frac{1}{2}(V - D)$	$V, \quad 0$
Dove	$0, \quad V$	$\frac{V}{2} - T, \quad \frac{V}{2} - T$

Figure 1: Hawk-Dove game

$$E_1(D, \alpha_2) = q \cdot 0 + (1 - q)\left(\frac{V}{2} - T\right) = (1 - q)\left(\frac{V}{2} - T\right) \quad (2)$$

## 1.2 Expected payoff for player 2

Given player 1's mixed strategy  $\alpha_1$ , player 2 expected payoff for the pure strategy  $Hawk(q)$  is:

$$E_2(H, \alpha_1) = p\left(\frac{1}{2}(V - D)\right) + (1 - p)V = V - \frac{p}{2}(V - D) \quad (3)$$

Given player 1's mixed strategy  $\alpha_1$ , player 2 expected payoff for the pure strategy  $Dove(1 - q)$  is:

$$E_2(D, \alpha_1) = p \cdot 0 + (1 - p)\left(\frac{V}{2} - T\right) = (1 - p)\left(\frac{V}{2} - T\right) \quad (4)$$

## 1.3 Player 1's mixed strategies

Since this is a game with two-players two actions *The linearity implies 3 possible outcomes:*

player 1's unique best response is the pure strategy  $Hawk(p)$  when:

$$\begin{aligned} \iff E_1(H, \alpha_2) &> E_1(D, \alpha_2) \\ \iff V - \frac{q}{2}(V - D) &> (1 - q)\left(\frac{V}{2} - T\right) \\ \iff V - \frac{V}{2}q + \frac{D}{2}q &> \frac{V}{2} - T - \frac{V}{2}q + T \\ \iff q &> -\frac{V}{D} \end{aligned}$$

player 1's unique best response is the pure strategy  $Dove(1 - p)$  when:

$$\begin{aligned} \iff E_1(H, \alpha_2) &< E_1(D, \alpha_2) \\ \iff q &< -\frac{V}{D} \end{aligned}$$

all player 1's mixed strategies are all best responses when:

$$\begin{aligned} \iff E_1(H, \alpha_2) &= E_1(D, \alpha_2) \\ \iff q &= -\frac{V}{D} \end{aligned}$$

#### 1.4 Player 2's mixed strategies

This game is symmetric, the payoff matrix of player 1 and 2 are the same ( $A = B^T$ ). The strategies of player 2 are the same than the player 1, except that it depends on  $p$  and not on  $q$ .

Player 2's unique best response is pure strategy *Hawk*( $q$ ) when  $p > -\frac{V}{D}$ , is pure strategy *Dove*( $1 - q$ ) when  $p < -\frac{V}{D}$  and all mixed strategies are all best response when  $p = -\frac{V}{D}$ .

#### 1.5 Discussions

### 2 Social Dilemma

Figure 2 represents the expected payoff of  $A$  based on her beliefs. For example the first column indicates the payoff of  $A$  when she believes that  $B$  will defect in all games type.

Since  $A$  is sure that each game is equally likely, there is a probability of  $1/3$  that she is in Prisoner's dilemma, probability of  $1/3$  in Stag-Hunt game and probability of  $1/3$  in Snowdrift game.

Figure 3 represents the best response of player  $B$  in all three games, she doesn't need to form any beliefs since she knows in which game she is playing.

	DDD	CCC	CCD	CDC	CDD	DCC	DCD	DDC
C	1/3	3	<b>8/3</b>	4/3	1	7/3	<b>2</b>	2/3
D	<b>2/3</b>	<b>4</b>	7/3	<b>11/3</b>	<b>2</b>	<b>8/3</b>	1	<b>7/3</b>

Figure 2: Expected payoff of player  $A$  for all possible combinations of player  $B$ 's types

	C	D	C	D	C	D
C	2	<b>5</b>	<b>5</b>	2	2	<b>5</b>
D	0	<b>1</b>	0	<b>1</b>	<b>1</b>	0

Figure 3: Best response of player  $B$  against player  $A$  in all games (in the following order): Prisoner's dilemma, Stag-hunt game, Snowdrift game

## 2.1 Payoff function

Let's associate Prisoner's dilemma, Stag-hunt game and Snowdrift game with the indexes  $I = \{1, 2, 3\}$ , respectively. The actions cooperate and defect be the set  $A = \{C, D\}$ . Let  $b_i \forall i \in I$ , the beliefs of  $A$ , with  $b_i \in A$  an action. Let  $o_i(x, y) \forall i \in I \wedge \forall x, y \in A$  the outcomes of  $A$  for game  $i$  and action  $x$  for player  $A$  and action  $y$  for  $B$ . The payoff function is  $p(a, b_1, b_2, b_3) \forall a \in A$ :

$$p(a, b_1, b_2, b_3) = \frac{1}{3}(o_1(a, b_1) + o_2(a, b_2) + o_3(a, b_3)) \quad (5)$$

For example the payoff for  $A$  when she plays cooperate (C) and believes  $b_1 = D, b_2 = D, b_3 = D$ , corresponding the first row and column of Figure 2, is:

$$p(C, D, D, D) = \frac{1}{3}(0 + 0 + 1) = 1/3 \quad (6)$$

Other examples:

$$p(C, C, C, D) = \frac{1}{3}(2 + 5 + 1) = 8/3 \quad (7)$$

$$p(D, C, C, C) = \frac{1}{3}(5 + 2 + 5) = 4 \quad (8)$$

## 2.2 Best response and equilibrium

The best response given the action of all games type for player  $A$  are the bolded values in Figure 2. The best response for all games for player  $B$  are the bolded values in Figure 3. To find the pure Nash Equilibria, we need to match the best response of  $A$  with the best response of  $B$  for each beliefs. There are two pure Nash Equilibria, shown in red in Figure 2,  $\{C, (DCD)\}$  and  $\{D, (DDC)\}$ . Indeed, for  $\{C, (DCD)\}$ , when  $A$  plays action  $C$ , the best response of  $B$  in Prisoner's dilemma is  $D$ , in Stag-hunt game it is  $C$  and in Snowdrift game it is  $D$ , so it is a pure Nash Equilibria. The same goes for  $\{D, (DDC)\}$ .

## 3 Sequential truel

The diagram representing the subgames are drawn as trees in Figure 4, 5 and 6. When we will refer to  $T_1$ ,  $T_2$  and  $T_3$  it means that we are referring to, respectively, Figure 4, 5 and 6. In the subgames, the action  $t(i)$  where  $i \in \{A, B, C\}$ , means that the current player is targeting player  $i$ . The current player can of course not target himself.

The main game is represented by  $T_1$ . As mentioned in the assignment specifications, the subgames when  $A$  fails to hit is intended target are the same, it is  $T_2$ . When in subgame  $T_2$ , we also found that when  $B$  misses his intended target, the subgames are the same, it is  $T_3$ .

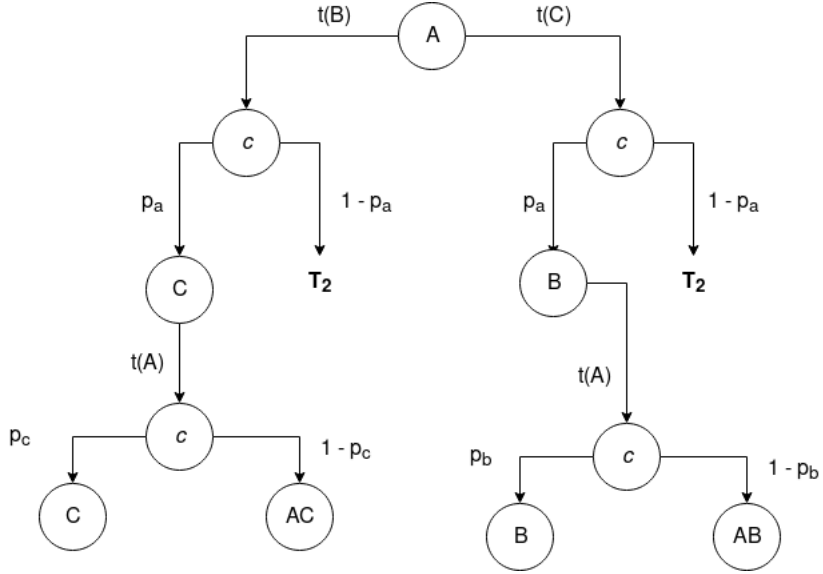


Figure 4:  $T_1$ , main subgame

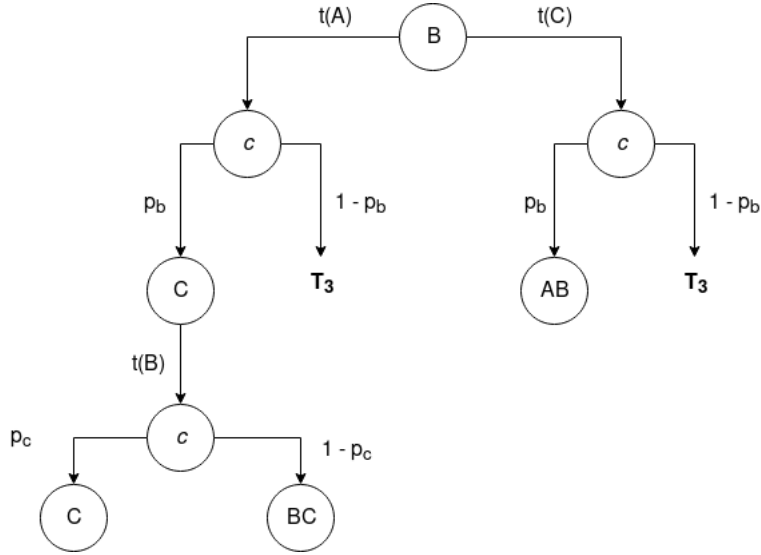


Figure 5:  $T_2$ , subgame when  $A$  misses her intended target

### 3.1 Subgame perfect equilibria

#### 3.1.1 C equilibrium in $T_3$

To find the SPE, we have to use the backward induction, so we need to find the SPE of  $T_3$  first. Player  $C$  always stay alive whatever target she is choosing.

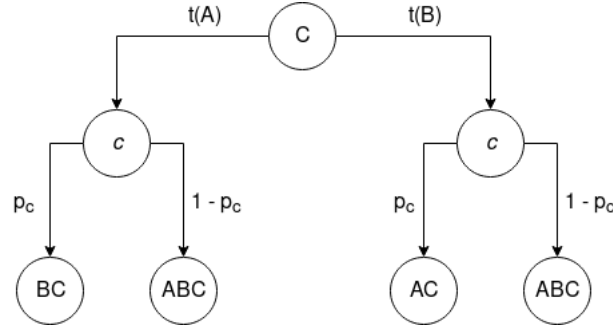


Figure 6:  $T_3$ , subgame when  $A$  and  $B$  miss their intended targets

The SPE is  $t(A)$  if  $p_a > p_b$ , and  $t(B)$  if  $p_b > p_a$ .

### 3.1.2 B equilibrium in $T_2$

Let's consider subgame  $T_2$ . Player  $B$  needs either to target  $A$  or target  $C$ . If  $B$  targets  $A$ , she has less chance to survive because, since next turn  $C$  will play and will still be alive whatever the result is of this shoot, so  $B$  will always target  $C$  in  $T_3$ .

Formally, if  $B$  misses, her outcome will be the same because it will have the SPE of subgame  $T_3$ , which is unique. If  $B$  targets  $A$  and hits her, she has a probability of  $1 - p_c$  to stay alive. If  $B$  targets  $C$  and hits her, she has a probability of 1 to stay alive. So  $B$  will always choose to target  $C$ .

### 3.1.3 A equilibrium in $T_1$

Let's consider the subgame  $T_1$  (the full game). Intuitively, it is best for  $A$  to target the player with the biggest probability, because she will have more chance to stay alive if she manages to eliminate the strongest opponent.

Formally, if  $A$  misses her intended target, the outcome will be the same since  $T_2$  has an unique SPE. If  $A$  targets  $B$  and hits her, she has a probability of  $1 - p_c$  to stay alive, because  $C$  is the remaining shooter and has a probability of  $1 - p_c$  to fail. If  $A$  targets  $C$  and hits her, she has a probability of  $1 - p_b$  to stay alive, because  $B$  is the remaining shooter and has a probability of  $1 - p_b$  to fail. So  $A$  targets  $B$  if  $(1 - p_c) > (1 - p_b)$ , which can be simplified as  $p_c < p_b$ , and  $A$  targets  $C$  if  $(1 - p_b) > (1 - p_c)$ , which can be simplified as  $p_b < p_c$ .

### 3.1.4 Weakness is strength

In the previous paragraph, we explained that if  $p_c > p_b$   $A$  will target  $C$ . If  $C$  is the target, her probability of survival is the probability that  $A$  misses and  $B$

misses, formally:

$$(1 - p_a)(1 - p_b) = 1 - p_b - p_a + p_a p_b = 1 - p_a - p_b(1 - p_a) \quad (9)$$

If  $p_b > p_c$   $A$  will target  $B$ , then  $C$ 's chance of survival is if  $A$  hits her target and  $C$  will be the last player with a bullet or  $A$  misses her target and  $B$  misses also her target. Formally:

$$p_a + (1 - p_a)(1 - p_b) = p_a + 1 - p_b - p_a + p_a p_b = 1 - p_b(1 - p_a) \quad (10)$$

The difference between the probability (9) and (10), is that (9) is smaller than (10) because of the decrease of  $p_a$ , so (9) will always be smaller than (10). The probability of survival of  $C$  when  $p_c > p_b$  will always be smaller than the probability of survival of  $C$  when  $p_b > p_c$ .  $C$  is always better off when  $p_b > p_c$ .