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PREPARATORY WORK FOR THE MASTER THESIS

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# Implementing data structures for Partially Ordered Set

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# 1 Introduction

Automata theory is used in various field in computer science and has shown to be an interesting way to resolve important problems, such as synthesis of computer systems or the universality problem. It has been proved that those problems are PSPACE-complete [?].

More efficient algorithms to resolve those problems have been implemented using antichain based-algorithms. Antichains are data structures that allow to represent a partially ordered set, in a more compact way.

The goal of this preparatory work is to motivate the interest that are made in the antichains data structure, and why we need an efficient implementation. It also defines the desired objectives and propose and state-of-the-art of the existing implementation.

**Motivation** An example that highlights the use of antichains in automata theory is the universality problem. It is the problem that for an automaton  $A$ , is the language of  $A$  equivalent to the language of all the words on the alphabet, that is, is  $L(A) = \Sigma^*$ . The language of  $A$  is universal if and only if the language of the complementary of  $A$  accepts no words, that is,  $L(\bar{A}) = \emptyset$  if and only if  $L(A) = \Sigma^*$ . The only way to compute the complementary of  $A$  is to first determine it, which is hard. Now if  $L(\bar{A})$  is the empty set, it means that there is no set of states that for which there exists a word such that this word is accepted by  $\bar{A}$ . A function can be defined to check the existence of such sets. Let the function  $EAcc^A(w) = X = \{P \subseteq Q, P' \subseteq Q, P' \cap F = \emptyset \mid \exists w \in \Sigma^* s.t. P \xrightarrow{w} P'\}$ . The goal is to show that if there exists no set of states  $P$  such that  $P \cap F = \emptyset$  i.e.  $X = \emptyset$ , then it is shown that  $L(\bar{A}) = \emptyset$ , therefore  $A$  is universal. The difficulty of the problem arise at the computation of  $X$ . Checking intersection between the accepting states set and all the subsets is costly. For a subset  $P \subseteq Q$ , for all subsets  $p$  such that  $p \subseteq P$ , it is known that, computing the work  $w$  starting from  $p$  or  $P$ , will lead to the same final state. Therefore it is more interesting to compute the maximal sets  $\lceil X \rceil$ , which correspond to an antichain of set of states that are  $\subseteq$ -uncomparable.

In the antichain algorithms paper [DWRLH06], the algorithm proposed follow an idea equivalent by using game theory. The universality problem is reduce to a two-player reachability game, which can be done in polynomial time. The objective of the game is for the protagonist to establish that the automaton  $A$  is not universal. To this end, the protagonist will provide a word, a letter at a time, and find a strategy that try to show that  $A$  ends in a rejecting state. The protagonist only has a strategy to win the game if and only if  $A$  is not universal.

**Objective** this thesis is to provide an efficient implementation of different data structures that allow to compactly represent partially ordered sets, specif-

In the paper, should I use We or I, or nothing ?

cite the original papers of those problems (w/ complexity)

Talk about the problems, complexity and alternative (Safra vs antichain)

We have to define complementary with det. computation to understand why we use set of state

Define  $F$  bar, final state of  $A$  bar

$EAcc$  function def is not correct, how to correctly formulate it ?

$EAcc$ , set or function ?

Costly vs expensive vs complexe vs hard etc..

Include that UP is a problem important

ically antichains and pseudo-antichains. The first step is to implement in Java, classes that will be provided to the Owl library [Sal16]. Owl is a LTL to deterministic automata translations tool-set written in Java. A second step will be to implement antichain-based algorithms using the new antichains implementation and study the performance.

### Related work

AaPAL is a generic library is a that was implemented in the frame of Aaron Bohy's PhD thesis [Boh14b] to provide an antichain library. It is implemented in C.

An implementation of antichains in Java have been done by De Causmaecker and De Wannemacker in [DCDW]. The algorithms to find the ninth Dedekind number uses antichains and they needed to implement a representation of antichains. To improve efficiency and performances, Hoedts in [Hoe] has extended [DCDW] antichains implementation by using bit sequence instead of tree representation.

Include difference between BDDs and Antichains

How are antichains implemented in Acacia+ ? Is it AaPAL or another impl. ?

**Structure of the work** This work is the introduction to next year thesis. Therefore the content concern only the preliminaries, the goal is to properly define the subject and existing implementation. In Section ??, we formally define antichains, and give examples of such data structures. In Section ??, we cite and summarize the work that has been done by the community for antichains implementation. In 4, we propose an overview of next year work and possibilities.

Discuss impl. specifics etc in chapter Implementation

## 2 Data Structures

In this section, we will provide formal definitions of the data structures that we will implement. We recall the notion of binary relations and important properties of such relations. We then define partially ordered set, totally order set and closed set. Finally we give a formal definition for antichains and pseudo-antichains.

What operations are implemented in those papers ?

The definitions and examples for this section are based on [Boh14b].

What domain (of sets) is used in those papers ?

### 2.1 Binary relations

A binary relation for an arbitrary set  $S$  is a set of pair  $R \subseteq S \times S$ . There are five important properties: reflexivity, transitivity, symmetry, antisymmetry and total.

Research other possible related works

A relation  $R$  on  $S$  is said to be:

- Reflexive: iff  $\forall s \in S$  it holds that  $(s, s) \in R$
- Transitive: iff  $\forall s_1, s_2, s_3 \in S$ , if  $(s_1, s_2) \in R$  and  $(s_2, s_3) \in R$  then it holds that  $(s_1, s_3) \in R$
- Symmetric: iff  $(s_1, s_2) \in R$  then  $(s_2, s_1) \in R$ .
- Antisymmetric: iff  $(s_1, s_2) \in R$  and  $(s_2, s_1) \in R$  then  $s_1 = s_2$
- Total: iff  $\forall s_1, s_2 \in S$  then  $(s_1, s_2) \in R$  or  $(s_2, s_1) \in R$

**Orders** A *partial order* is a binary relation that is *reflexive*, *transitive* and *antisymmetric*. We note a partial order relation by  $R$ . We note  $s_1 R s_2$  to show the belonging of a binary relation to a partial order, which is equivalent to  $(s_1, s_2) \in R$ . A *total order* is a partial order that is *total*.

**Example 2.1.1.** For example, the comparison of natural numbers is a partial order. Let  $\leq$  be a binary relation on  $\mathbb{N}$  such that  $\leq \subseteq \mathbb{N}^2$ . The binary relation is defined following the usual semantic of the symbol, i.e.  $n_1 \leq n_2$  if and only if  $n_1$  is smaller or equal to  $n_2$ . Based on this,  $\leq$  is a partial order. It is reflexive, transitive and antisymmetric. The binary relation  $\leq$  on natural numbers is actually a total order since all the natural numbers can be compared against each other.

## 2.2 Partially ordered set

An arbitrary set  $S$  associated with a partial order  $\preceq$  is called a *partially ordered set* or *poset*. It is denoted by the pair  $\langle S, \preceq \rangle$ .

**Comparable** Let  $s_1, s_2 \in S$  and  $\langle S, \preceq \rangle$  a poset. The two elements  $s_1$  and  $s_2$  are called *comparable* if either  $s_1 \preceq s_2$  or  $s_2 \preceq s_1$ . If neither of those two comparisons are correct, then  $s_1$  and  $s_2$  are called *uncomparable*.

**Bounds** Let  $\langle S, \preceq \rangle$  a partially ordered set. We denote the *greatest lower bound* of the two elements  $s_1, s_2 \in S$  by  $s_1 \sqcap s_2 \in S$ . The greatest lower bound is defined as follow:  $s_1 \sqcap s_2 \preceq s_1$ ,  $s_1 \sqcap s_2 \preceq s_2$  and for all  $s' \in S$  we have that if  $s' \preceq s_1$  and  $s' \preceq s_2$  then  $s' \preceq s_1 \sqcap s_2$ .

**Example 2.2.1.** A more interesting example is the set inclusion comparison. Let  $A$  be automaton defined in Example ???. Let  $\langle 2^{Q_A}, \subseteq \rangle$ , a poset.

**Lattices** A *lower semilattice* is a poset  $\langle S, \preceq \rangle$  where for all pair of elements  $s_1, s_2 \in S$ , we have that the greatest lower bound  $s_1 \sqcap s_2$  exists.

Include definition of least upper bound

Still need to define the automaton, here or in another section.

## 2.3 Antichains

### 2.3.1 Closed sets

A closed set is a set  $L \subseteq S$  of a lower semilattice  $\langle S, \preceq \rangle$  where  $\forall \ell \in L$  we have that  $\forall s \in S$  such that  $s \preceq \ell$ , then  $s \in L$ .

Note that for two closed sets  $L_1, L_2 \subseteq S$ , we have that  $L_1 \cup L_2$  and  $L_1 \cap L_2$  are also closed sets, but  $L_1 \setminus L_2$  does not result necessarily to a closed set.

**Maximal/minimal elements** We denote by  $\lceil L \rceil$  the set of maximal elements of a closed set  $L$  which correspond to  $\lceil L \rceil = \{\ell \in L \mid \forall \ell' \in L : \ell \preceq \ell' \Rightarrow \ell = \ell'\}$ . Alternatively, to represent the set of minimal elements, the notation  $\lfloor L \rfloor$  is used which has the following semantic  $\lfloor L \rfloor = \{\ell \in L \mid \forall \ell' \in L : \ell' \preceq \ell \Rightarrow \ell' = \ell\}$ .

Meaning of  $\lceil \rceil$  vs  $\lfloor \rfloor$  : in set definition ?

**Closure** A lower closure of a set  $L$  on  $S$  noted  $\downarrow L$  is the set of all elements of  $S$  that are smaller or equal to an element of  $L$  i.e.  $\downarrow L = \{s \in S \mid \exists \ell \in L : s \preceq \ell\}$ . Note that for a closed set  $L$  we have that  $\downarrow L = L$ .

**Antichain** An antichain of a poset  $\langle S, \preceq \rangle$  is a set  $\alpha \subseteq S$  where all element of  $\alpha$  are uncomparable with respect to the partial order  $\preceq$ . Antichains allow to represent closed set in a more compact way. For a closed set  $L \subseteq S$  we can retrieve all elements of  $L$  by using the antichain  $\alpha = \lceil L \rceil$ . With respect to the definition of the lower closure we have that  $\downarrow \alpha = L$ .

## 2.4 Operations on antichains

**Proposition 2.4.1.** Let  $\alpha_1, \alpha_2 \subseteq S$  two antichains and  $s \in S$ :

- $s \in \downarrow \alpha_1$  iff  $\exists a \in \alpha_1$  such that  $s \preceq a$
- $\downarrow \alpha_1 \subseteq \downarrow \alpha_2$  iff  $\forall a_1 \in \alpha_1, \exists a_2 \in \alpha_2$  such that  $a_1 \preceq a_2$
- $\downarrow \alpha_1 \cup \downarrow \alpha_2 = \downarrow [\alpha_1 \cup \alpha_2]$
- $\downarrow \alpha_1 \cap \downarrow \alpha_2 = \downarrow [\alpha_1 \cap \alpha_2]$

Cite original paper, FJR11 from Bohy' phd

## 3 Implementation

Java already provide built-in implementation for Set.

In this thesis we are more interested in partially ordered sets as totally ordered sets are already implemented in Java built-in SortedSet.

Give complete definition for intersection

Includes limitation of Java built-in and different possible solution for antichains found on stack overflow and others

### 3.1 Summary of objectives

The main focus of the thesis is to be able to provide an efficient implementation of antichains and pseudo-antichains in Java. The first step is to provide an interface for the different operations that can be applied to antichains. We then give a description of the implementation. Antichains provide a way to represent in a compact way partially ordered set that are closed. Pseudo-antichains are an extension of antichains and provide a compact way to represent partially ordered sets. Pseudo-antichains does not specifically require closed set.

Our goal is to find a way to not keep all the closure all element of the antichain in memory, but be able to retrieve those elements or check the belonging of a closure element from the uncomparable elements of the antichain.

### 3.2 Existing implementation

#### 3.2.1 AaPAL

Bohy's **Antichain and Pseudo-Antichain Library** [Boh14a] is an open-source C library for the manipulation of antichains and pseudo-antichains data structures.

**Antichain representation** An antichain is represented by a struct, containing as attributes the size of the antichain, and the uncomparable elements of the antichains, as a list. The list is manipulated using the `GSList` object from the `glib` library. To allow modularity, the type of the elements is `void`.

**Operations** The operations implemented in AaPAL are described in this section. An interesting remark is that most of the complexity is given as a parameter to the functions. For example the function to compare two elements in an antichain is given as a parameter. It means that the complexity to define the domain of the antichain, must be implemented in the compare function.

Operations such as creating an antichain, adding an element to an antichain, checking emptiness or cloning an antichain are implemented in AaPAL. In addition to that

- create add elem clone
- interesection Union
- compare 2 antichains
- containing element closure size

#### 3.2.2 ABD

#### 3.2.3 Hoedt's ABD modifications

Referring to [Hoe]

Is actually [Hoe] what we what to do, and if not, what will be the differences ?



### 3.3 Difficulties

One of the difficulties is to define the domain of the elements that the antichains should work on. In AaPAL the all complexity is implemented in the `compare_elements` that must be given to the library functions. In that case, all the complexity of the must be implemented by the user to define the domain of the antichains.

What is the difference between domain, univers and set ?

#### 3.3.1 Tree vs Bitarray representations

### 3.4 Possible solutions

In the case of the domain specifications, a good approach could be to provide an interface/abstract class to the users, to let them provide their own implementation. In addition to that, usual domain such a natural numbers or others well known domain used, could be directly implemented in the library.

### 3.5 New implementation

## 4 Next year overview

### 4.1 Interface and implementation

The interface proposed in ?? gives a first overview of the desired implementation.

### 4.2 Possible extensions

As we mainly focus on efficiency, it could be interesting to use a C implementation such as AaPAL, and provide bindings to Java. We could try this method as an alternative to a pure Java implementation and compare performances.

Fill-in bib correctly!

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