极坐标解平面问题

极坐标与直角坐标的转换关系

坐标	直角坐标(x,y)	极坐标(r,θ)
坐标关系	$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$	$\begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$
一次偏导的关系	$\begin{cases} \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \\ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \end{cases}$;	$\begin{cases} \frac{\partial\theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \\ \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r} \end{cases}$
	$\begin{cases} \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \end{cases}$ $[\beta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ 且$[\beta]^{-1} = [\beta]^T$ 变换：$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{Bmatrix} = [\beta]\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$ 逆变换：$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [\beta]^{-1}\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{Bmatrix}$	
二次偏导的关系	$\begin{cases} \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\ \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} = \sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\ \frac{\partial^2}{\partial x\partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\sin 2\theta}{2}\frac{\partial^2}{\partial r^2} + \frac{\cos 2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin 2\theta}{2r}\frac{\partial}{\partial r} - \frac{\cos 2\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\sin 2\theta}{2r^2}\frac{\partial^2}{\partial\theta^2} \end{cases}$	
调和算子	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$	$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$
双调和算子	$\nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\frac{\partial^4}{\partial x^2\partial y^2}\right)$	$\nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$

基本量间关系

	直角坐标(x,y)	极坐标(r,θ)	平面轴对称问题(r)
位移	$\boldsymbol{u}(u,v)$ $\begin{Bmatrix} u \\ v \end{Bmatrix} = [\beta]^{-1}\begin{Bmatrix} u_r \\ u_\theta \end{Bmatrix}$	$\boldsymbol{u}(u_r,u_\theta)$ $\begin{Bmatrix} u_r \\ u_\theta \end{Bmatrix} = [\beta]\begin{Bmatrix} u \\ v \end{Bmatrix}$	$\boldsymbol{u}(u_r,0)$
应力	$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$	$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_r & \tau_{r\theta} \\ \tau_{\theta r} & \sigma_\theta \end{bmatrix}$	$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_r & 0 \\ 0 & \sigma_\theta \end{bmatrix}$
	$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = [\beta]\begin{bmatrix} \sigma_r & \tau_{r\theta} \\ \tau_{\theta r} & \sigma_\theta \end{bmatrix}[\beta]^T$ $\begin{cases} \sigma_x = \dfrac{\sigma_r+\sigma_\theta}{2} + \dfrac{\sigma_r-\sigma_\theta}{2}cos2\theta - \tau_{r\theta}sin2\theta \\ \sigma_y = \dfrac{\sigma_r+\sigma_\theta}{2} - \dfrac{\sigma_r-\sigma_\theta}{2}cos2\theta + \tau_{r\theta}sin2\theta \\ \tau_{xy} = \dfrac{\sigma_r-\sigma_\theta}{2}sin2\theta + \tau_{r\theta}cos2\theta \end{cases}$	$\begin{bmatrix} \sigma_r & \tau_{r\theta} \\ \tau_{\theta r} & \sigma_\theta \end{bmatrix} = [\beta]^T\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}[\beta]$ $\begin{cases} \sigma_r = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}cos2\theta - \tau_{xy}sin2\theta \\ \sigma_\theta = \dfrac{\sigma_x+\sigma_y}{2} - \dfrac{\sigma_x-\sigma_y}{2}cos2\theta + \tau_{xy}sin2\theta \\ \tau_{r\theta} = \dfrac{\sigma_x-\sigma_y}{2}sin2\theta + \tau_{xy}cos2\theta \end{cases}$	
应变	同上	同上	$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon_\theta \end{bmatrix}$
应力函数	$\phi(x,y)$	$\phi(r,\theta)$	$\phi(r)$

基本方程和边界条件间的关系

	直角坐标(x,y)	极坐标(r,θ)	平面轴对称问题(r)
平衡方程	$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + F_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial xy} + \dfrac{\partial \sigma_y}{\partial y} + F_y = 0 \end{cases}$	$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \\ \dfrac{\partial \tau_{\theta r}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + 2\dfrac{\tau_{r\theta}}{r} + F_\theta = 0 \end{cases}$	$\dfrac{\partial \sigma_r}{\partial r} + \dfrac{\sigma_r - \sigma_\theta}{r} + F_r = 0$
几何方程	$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \varepsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \end{cases}$	$\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}$	$\begin{cases} \varepsilon_r = \dfrac{du_r}{dr} \\ \varepsilon_\theta = \dfrac{u_r}{r} \\ \gamma_{r\theta} = 0 \end{cases}$
本构方程	4.7-4.8	$\varepsilon_r = \frac{1}{E^*}(\sigma_r - v^*\sigma_\theta)$ $\sigma_r = \frac{2G^*}{1-v^*}(\varepsilon_r + v^*\varepsilon_\theta)$ $\varepsilon_\theta = \frac{1}{E^*}(\sigma_\theta - v^*\sigma_r)$ 或 $\sigma_\theta = \frac{2G^*}{1-v^*}(\varepsilon_\theta + v^*\varepsilon_r)$ $\gamma_{r\theta} = \frac{1}{G^*}\tau_{r\theta}$ $\tau_{r\theta} = G^*\gamma_{r\theta}$ 轴向分量： 平面应力：$\begin{cases} \varepsilon_z = -\frac{v}{E}(\sigma_r + \sigma_\theta) \\ \sigma_z = 0 \end{cases}$ 平面应变：$\begin{cases} \varepsilon_z = 0 \\ \sigma_z = v(\sigma_r + \sigma_\theta) \end{cases}$	$\varepsilon_r = \frac{1}{E^*}(\sigma_r - v^*\sigma_\theta)$ $\sigma_r = \frac{2G^*}{1-v^*}(\varepsilon_r + v^*\varepsilon_\theta)$ $\varepsilon_\theta = \frac{1}{E^*}(\sigma_\theta - v^*\sigma_r)$ 或 $\sigma_\theta = \frac{2G^*}{1-v^*}(\varepsilon_\theta + v^*\varepsilon_r)$ $\gamma_{r\theta} = \tau_{r\theta} = 0$ $\gamma_{r\theta} = \tau_{r\theta} = 0$ 轴向分量： 平面应力：$\begin{cases} \varepsilon_z = -\frac{v}{E}(\sigma_r + \sigma_\theta) \\ \sigma_z = 0 \end{cases}$ 平面应变：$\begin{cases} \varepsilon_z = 0 \\ \sigma_z = v(\sigma_r + \sigma_\theta) \end{cases}$
协调方程	4.10（平面应变）	$\dfrac{\partial^2 \varepsilon_\theta}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial \varepsilon_r}{\partial \theta^2} - \dfrac{1}{r}\dfrac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \dfrac{2}{r}\dfrac{\partial \varepsilon_\theta}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial \varepsilon_r}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial \gamma_{r\theta}}{\partial \theta} = 0$	$\dfrac{\partial^2 \varepsilon_\theta}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial \varepsilon_\theta}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial \varepsilon_r}{\partial r} = 0$

<table>
<tr>
<td>双调和方程</td>
<td>$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+2\frac{\partial^4}{\partial x^2\partial y^2}\right)\phi = 0$</td>
<td>$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0$</td>
<td>$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d^2}{d\theta^2}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d^2}{d\theta^2}\right)\phi = \frac{d^4\phi}{dr^4}+\frac{2}{r}\frac{d^3\phi}{dr^3}-\frac{1}{r^2}\frac{d^2\phi}{dr^2}+\frac{1}{r^3}\frac{d\phi}{dr} = 0$</td>
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<tr>
<td rowspan="2">应力公式</td>
<td>$\begin{cases}\sigma_x = \frac{\partial^2\phi}{\partial y^2}\\ \sigma_y = \frac{\partial^2\phi}{\partial x^2}\\ \tau_x = -\frac{\partial^2\phi}{\partial x\partial y}\end{cases}$</td>
<td>$\begin{cases}\sigma_r = \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\\ \sigma_\theta = \frac{\partial^2\phi}{\partial r^2}\\ \tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\end{cases}$</td>
<td>$\begin{cases}\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r}\\ \sigma_\theta = \frac{\partial^2\phi}{\partial r^2}\\ \tau_{r\theta} = 0\end{cases}$
$\phi = Alnr + Br^2lnr + Cr^2 + D$</td>
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<tr>
<td colspan="3">应力第一不变量与坐标系无关：$I_1 = \sigma_x + \sigma_y = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = \nabla^2\phi = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = \sigma_r + \sigma_\theta$</td>
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<tr>
<td>边界条件</td>
<td>• 在Γ_u上：$\begin{cases}u_x = \bar{u}_y\\ u_y = \bar{u}_y\end{cases}$
• 在Γ_σ上：$\begin{cases}\sigma_x\cos(\boldsymbol{v},\boldsymbol{e_x}) + \tau_{r\theta}\cos(\boldsymbol{v},\boldsymbol{e_y}) = \overline{T}_x\\ \tau_{xy}\cos(\boldsymbol{v},\boldsymbol{e_x}) + \sigma_y\cos(\boldsymbol{v},\boldsymbol{e_y}) = \overline{T}_y\end{cases}$</td>
<td>• 在Γ_u上：$\begin{cases}u_r = \bar{u}_r\\ u_\theta = \bar{u}_\theta\end{cases}$
• 在Γ_σ上$\begin{cases}\sigma_r\cos(\boldsymbol{v},\boldsymbol{e_r}) + \tau_{r\theta}\cos(\boldsymbol{v},\boldsymbol{e_\theta}) = \overline{T}_r\\ \tau_{r\theta}\cos(\boldsymbol{v},\boldsymbol{e_r}) + \sigma_\theta\cos(\boldsymbol{v},\boldsymbol{e_\theta}) = \overline{T}_\theta\end{cases}$</td>
<td>• 在Γ_u上：$u_r = \bar{u}_r$
• 在Γ_σ上：$\sigma_r = \overline{T}_r$</td>
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平面轴对称问题解的具体形式

<table>
<tr><th>解的形式</th><th>系数</th><th colspan="2">简化解的形式</th></tr>
<tr>
<td>$$\begin{cases} \sigma_r = \dfrac{A}{r^2} + B(1 + 2lnr) + 2C \\ \sigma_\theta = -\dfrac{A}{r^2} + B(3 + 2lnr) + 2C \\ \tau_{r\theta} = \dfrac{\sigma_x - \sigma_y}{2} sin2\theta + \tau_{xy} cos2\theta \end{cases}$$</td>
<td>A、B、C</td>
<td rowspan="2">环向闭合圆/环域，位移单值要求：B = 0；

对于实心圆域，还要求 A = 0；</td>
<td>$$\begin{cases} \sigma_r = \dfrac{A}{r^2} + 2C \\ \sigma_\theta = -\dfrac{A}{r^2} + 2C \\ \tau_{r\theta} = 0 \end{cases}$$</td>
</tr>
<tr>
<td>$$\begin{cases} \varepsilon_r = \dfrac{1}{E^*}\left[\dfrac{(1 + v^*)A}{r^2} + (1 - 3v^*)B + 2(1 - v^*)Blnr + 2(1 - v)\right] C \\ \varepsilon_\theta = \dfrac{1}{E^*}\left[-\dfrac{(1 + v^*)A}{r^2} + (3 - v^*)B + 2(1 - v^*)Blnr + 2(1 - v)C\right] \\ \gamma_{r\theta} = 0 \end{cases}$$</td>
<td>A、B、C</td>
<td>$$\begin{cases} \varepsilon_r = C_1 - \dfrac{C_2}{r^2} \\ \varepsilon_\theta = C_1 + \dfrac{C_2}{r^2} \\ \gamma_{r\theta} = 0 \end{cases}$$</td>
</tr>
<tr>
<td>$$\begin{cases} u_r = \dfrac{1}{E^*}\left[\dfrac{-A(1 + v^*)}{r} + 2(1 - v^*)Brlnr - (1 + v^*)Br + 2(1 - v^*)Cr\right. \\ u_\theta = \dfrac{4Br\theta}{E^*} + Hr + Kcos\theta - Isin\theta \end{cases}$$</td>
<td>A、B、C、H、K、I，并与θ相关</td>
<td>限制原点刚体位移：I = K = 0；限制刚体转动：H = 0；</td>
<td>$$\begin{cases} u_r = C_1 r + \dfrac{C_2}{r} \\ u_\theta = 0 \end{cases}$$
$C_1 = 2C\frac{1-v}{E}$； $C_2 = -A\frac{1+v}{E}$</td>
</tr>
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