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ON THE DETERMINATION OF THE ROOTS OF DISPERSION EQUATIONS BY USE OF WINDING NUMBER INTEGRALS

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1. INTRODUCTION

The determination of roots of dispersion equations is a problem that manifests itself in almost every branch of acoustics and structural vibration and only rarely are the roots readily available. To quote a simple example, the dispersion equation of a fluid-loaded flat plate (see reference [1]) is quintic in square wavenumber and consequently the roots cannot be expressed in closed form. When roots of dispersion equations are not immediately available, the usual way to determine them invariably involves using Newton–Raphson iteration to refine their positions from approximate initial values. Newton–Raphson iteration may work well enough in some circumstances, but problems often arise in an unpredictable way. The problem of slow convergence, often exacerbated by unpredictability of which root is converged upon, occurs all too frequently, especially where the equation concerned contains transcendental functions.

This note is concerned with an entirely different method based on the winding number integral, being a contour integral around a closed path in the complex plane to determine the presence of zeros and poles in the enclosed area. The use of the winding number integral is not new—it was used with great effect by Doolittle *et al.* [2] to confirm that areas of the complex plane are free of roots in their search for creeping ray solutions to acoustic scattering from cylinders. However, where roots were found in search areas these authors reverted to Newton–Raphson iteration to determine their positions. It is on this point that we propose a different procedure: having determined that a closed area in the complex plane contains roots, using the winding number integral, the location of those roots may be deduced, *with a high degree of accuracy*, by using higher moments of the winding integral. For a single root, the first moment is sufficient, for two roots the first two moments are required and so on. In section 2 we describe how the method works and in section 3, it is applied to the well known thick plate dispersion equation for the Rayleigh–Lamb modes as an example of its efficacy.

2. THE WINDING NUMBER INTEGRAL METHOD

Typically, a dispersion equation, $D(k, \omega)$, say, is analytic in the wavenumber, k, and the frequency, ω, except where branch cuts occur. These may arise because of acoustic interaction with a structure. For example the dispersion equation for a fluid-loaded flat plate (see reference [1]),

$$D(k, \omega) = (Bk^4 - \sigma\omega^2)\gamma + \rho\omega^2 = 0, \tag{1}$$

with $\gamma = \sqrt{k^2 - (\omega/c)^2}$, c the speed of sound, B the plate bending rigidity, σ the area density of plate and ρ the fluid density, is analytic in k except at the two branch points at $\pm\omega/c$ and along the associated cuts. We shall now explore how, for a fixed frequency ω, the values of k may be sought out. With k and z, and f and D, taken as equivalents, the task is to solve $f(z) = 0$. Equivalently, instead of searching for the zeros at $f(z)$ one

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may equally seek out the simple poles of $F(z)=1/f(z)$ and, for reasons to be explained later, this we shall do; the winding number integral method is quite insensitive as to whether zeros of a function or poles of its reciprocal are sought out. Let us now consider the function $F(z)$ to be analytic over a region in the complex z-plane except for n_p simple poles. For completeness we assume that the function also contains n_z zeros. Then the integral

$$\frac{1}{2\pi \mathrm{i}}\oint \frac{F'}{F}\,\mathrm{d}z = n_z - n_p \tag{2}$$

defines the winding number as the difference between the number of zeros and poles, $n_z - n_p$, of the function F, where c is the anticlockwise path enclosing the region. This result is also called the principle of the argument (see section 6.2 of the book by Copson [3]). From a computational point of view, we note that equation (2) is essentially an analytical continuation of log F around the closed path c. Therefore it is a simple matter to divide the path c into a series of chords and examine whether each chord takes log F across the negative real axis (where conventionally the cut of the logarithmic function is taken); if it crosses from above, then the winding number is increased by 1, and if from below, reduced by 1.

Now dispersion functions can usually be arranged so as not to contain poles, which would give rise to zeros of the function F. When such an arrangement has been made, the winding number integral (2) then gives the number of roots of the dispersion equation as n_p. However, even if it has not been possible to remove poles from the dispersion function, all is not lost, because the fact that, say, a region contains a zero and a pole, for example, may be detected by taking higher moments of the winding number integral. That is to say, we weight the integrand in equation (2) by z^n. It is easy to demonstrate that

$$I_n = \frac{1}{2\pi \mathrm{i}}\oint z^n \frac{F'}{F}\,\mathrm{d}z = \sum_i (z_z^i)^n - \sum_j (z_p^j)^n, \tag{3}$$

being a simple corollary of the principle of the argument. Here z_z^i is the location of the ith zero and z_p^j is the location of the jth pole. Thus the higher moments of the winding integral generate algebraic equations in the zero and pole positions inside the contour c, the solution of which will yield the pole and zero positions. For example, in the instance of a pole and zero existing inside the region, although the winding number is zero, the first moment integral is *not*. It is

$$I_1 = z_z - z_p: \tag{4}$$

i.e., the difference between the two locations. By taking the next moment

$$I_2 = (z_z)^2 - (z_p)^2, \tag{5}$$

the pole and zero positions may easily be established as

$$z_z = \tfrac{1}{2}\{(I_2/I_1) + I_1\}, \qquad z_p = \tfrac{1}{2}\{(I_2/I_1) - I_1\}. \tag{6}$$

However, let us return to the situation where it is known that the dispersion function contains only zeros, hence F contains only poles. Then the pole locations will be given as follows: (a) for the winding number equal to 1, the single pole is at

$$z_p = I_1; \tag{7}$$

(b) when the winding number is equal to 2, two poles are determined by the roots of the quadratic

$$z^2 - I_1 z + \tfrac{1}{2}(I_1^2 - I_2) = 0; \tag{8}$$

(c) when the winding number is equal to 3, three poles are determined by the roots of the cubic

$$z^3 - I_1 z^2 + \tfrac{1}{2}(I_1^2 - I_2)z + (I_1 I_2/2) - (I_1^3/6) - (I_3/3) = 0. \quad (9)$$

Although it appears that our procedure is to establish the winding number first, followed by evaluation of the necessary moments integrals, in fact we establish in parallel the first three moments in anticipation of finding up to three poles. Also, as with the winding number itself not being calculated from (2) directly, but by an analytic continuation of $\log F$ around the path, the moment I_n is calculated via

$$I_n = -\frac{1}{2\pi i}\left\{[z^n \log F] - n \oint_c z^{n-1} \log F \,\mathrm{d}z\right\}, \quad (10)$$

where $\log F$ is again analytically continued around the path c. This is a superior form of equation (3) because the requirement to determine the derivative of F has been obviated.

The question arises as to the procedure if there are four or more roots. Clearly we could establish higher moments and, say, for four roots, by solving the appropriate quartic with coefficients derived from I_1, I_2, I_3, I_4 establish their positions. However, we note that $n_p = 4$ is the largest number of poles in the search region that can be handled before encountering the fundamental theorem that quintic, and higher order equations do not yield solutions in closed form. The resolution must be to split a region into smaller regions such that the occurrence of more than three poles in a region becomes impossible. Fortunately, it is usually possible by inspection of dispersion equations to establish appropriate sizes to search areas in the complex plane.

We return to the earlier point that the preferred function for the winding integral was $1/D$ and thus the roots of the dispersion equation would correspond to poles. Firstly, it is now clear why the winding number method is completely insensitive to taking the reciprocal: with equation (10) taken as the definitive form, for the winding number or its higher moments, only $\log F$ is required and reciprocation simply results in a sign change of I_n. Secondly, physical problems giving rise to dispersion equations are often forcing problems in which the system response is calculated from an integral transform in which the reciprocal of the dispersion appears as a factor. For our example of the fluid-loaded flat plate again, in plane symmetry, the plate response to a time-periodic normal line force is [1]

$$\xi(x) = \frac{F_0}{2\pi}\int_c \frac{\gamma\, e^{ikx}}{D(k,\omega)}\,\mathrm{d}k, \quad (11)$$

where D is as given by equation (1) and the contour, c, is the real axis indented under any poles arising from zeros of $D(k, \omega)$ in $\mathrm{Re}\,(k) > 0$ and over any poles in $\mathrm{Re}\,(k) < 0$. The fact that solution of equation (11) may require determination of the residues of $1/D$ (a point to which we shall return shortly) makes the reciprocal function the natural one to work with.

Up to now, we have said nothing about the shape of the search areas for the dispersion equation roots of the numerical integration schemes. In all our applications of the winding number integral to date, we have used two: firstly, the rectangular search area makes it simple to cover large areas of the complex plane with perhaps small overlap at the boundaries of adjacent search areas; secondly, a circular search area around individual poles found from the rectangular searches to reduce any small remnant error. For the rectangular search areas, each side is divided into equal (only possible if the ratio of sides of the rectangle is rational) or near-equal lengths. Integration may then proceed by application of trapezium rule along these lengths.

The numerical evaluation of winding number integrals around the circular path proceeds by replacing the circle by a regular polygon of n sides and again applying the trapezium rule around the sides of the polygon. That is to say, the integration takes *chord*wise steps around the circle. It turns out that, if this scheme is applied to a "model" function of a simple pole, $R/(z-z_p)$ (as we shrink the circle radius, all functions will tend asymptotically to this form), the chordwise scheme is *exact* for refining z_p. However, while pole locations are being refined via the circular integrations, their residues may also be calculated by using the standard Cauchy formula

$$R=\frac{1}{2\pi \mathrm{i}}\oint_c F\,\mathrm{d}z. \tag{12}$$

Now for the "model" function of a simple pole $R/(z-z_p)$ it turns out that the chordwise scheme is not the optimum: by defining $\mathrm{d}z=\mathrm{i}r\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$, where r is the circular path radius, numerical integration then takes place by application of trapezium rule at the end points of successive *arcs*. It turns out that this *arc*-wise scheme is *exact* for the residue calculation when applied to the "model" $R/(z-z_p)$. Thus we have a curious dichotomy of integration schemes arising from the fact that each is exact in its particular application when applied to the model single pole function of a complex variable. A bonus arising from the exactness of each scheme applied to the model function is that the number of steps for the circular integration path may be quite small. Typically, we have found ten steps to be adequate for all applications to date.

3. THE DISPERSION CURVES OF A THICK PLATE

To illustrate the ability of the method of winding number integrals for detection of the presence of roots of a dispersion equation *and* determination of their locations, we sought out an equation that is sufficiently well known that the solutions have already been worked out. Yet the determination of those solutions is to be sufficiently non-trivial as to give the method a fair test.

We have chosen, as our example, the classical problem of the propagating modes supported by a thick plate. The dispersion equation was worked out some time ago by

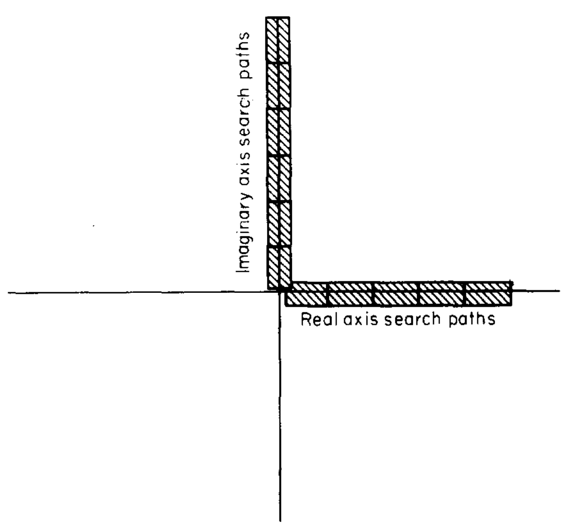

Figure 1. The search areas for roots of the Rayleigh–Lamb equation.

Rayleigh [4] and Lamb [5], for these modes which now bear their joint names. It is

$$D(k, \omega) = \{(\tan \beta b)/(\tan \alpha b)\} + \{4\alpha\beta k^2/(k^2 - \beta^2)^2\}^{\pm 1} = 0, \tag{13}$$

with the +1 for symmetric modes and the −1 for antisymmetric modes. Here $2b$ is the plate thickness, $\alpha^2 = (\omega/c_l)^2 - k^2$, $\beta^2 = (\omega/c_t) - k^2$ and c_l, c_t are respectively the longitudinal and transverse wave speeds in the solid. z with normalized $\bar{k} = bK$, $\bar{\alpha} = b\alpha$, $\bar{\beta} = b\beta$ and $\Omega = b\omega/c_l$, we may rearrange equation (13) to give the two dispersion equations for antisymmetric and symmetric modes as (a) symmetric,

$$(\bar{k}^2 - \bar{\beta}^2)^2 (\sin \bar{\beta}/\bar{\beta}) \cos \bar{\alpha} + 4\bar{\alpha}\bar{k}^2 \cos \bar{\beta} \sin \bar{\alpha} = 0, \tag{14}$$

and (b) antisymmetric,

$$4\bar{\beta} \sin \bar{\beta} \cos \alpha + (\bar{k}^2 - \bar{\beta}^2)^2 \cos \bar{\beta} (\sin \bar{\alpha}/\alpha) = 0, \tag{15}$$

where $\bar{\beta}^2 = \Omega^2 - \bar{k}^2$, $\bar{\alpha}^2 = c_r^2\Omega^2 - k^2$ and $c_r = c_t/c_l$. We note that equation (14) and (15) have been put in a form such that the branch points and associated cuts implied by the

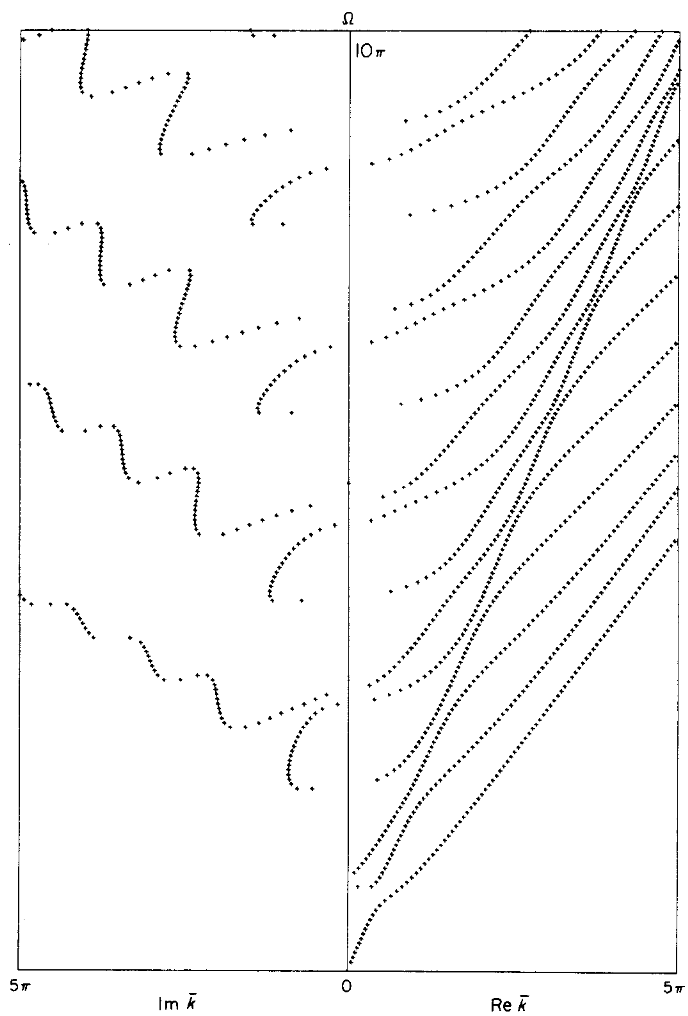

Figure 2. Wavenumber roots of the symmetric mode solutions of the Rayleigh–Lamb equation for non-dimensional frequency intervals of 0·1π.

definition of $\bar{\alpha}$ and $\bar{\beta}$ do not appear in the dispersion equations. It is not always possible to do this and, where not, the winding integrals must have paths *that do not cross such cuts*. This means that, where a cut goes through a region, the region must be split into two search areas—one each side of the cut.

We have chosen a range of $0 < \Omega < 10\pi$ and $0 < \bar{k} < 5\pi$ to display the dispersion curves in order that direct comparison may be made with Mindlin's frequency spectrum of the Rayleigh–Lamb equation [6], as reproduced by Graff [7, p. 453], for example. (Note that Mindlin's non-dimensional scheme puts his quantities at $2/\pi \times$ those here.) We stepped the frequency range in increments of $0{\cdot}1\pi$. At each frequency step, the real axis and the positive imaginary axis was searched for poles by rectangular search areas, as shown in Figure 1. The search areas measure $1 \times \frac{1}{2}$. The wavenumber roots of the symmetric and antisymmetric equations respectively are shown in Figures 2 and 3. Note that the left-hand sides of the diagrams have been used to display those roots lying on the imaginary axis. The correspondence between Mindlin's curves and those here is apparent.

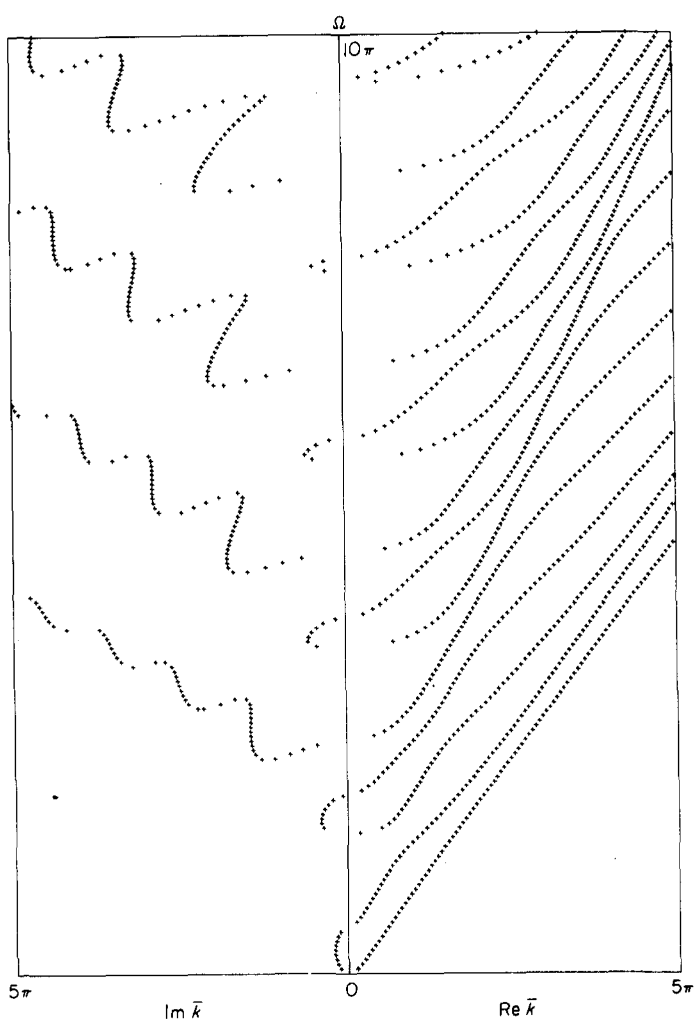

Figure 3. Wavenumber roots of the antisymmetric mode solutions of the Rayleigh–Lamb equation for non-dimensional frequency intervals of $0{\cdot}1\pi$.

4. CLOSING REMARKS

We have presented here a method which we intended specifically for deducing the roots of dispersion equations, but it is of course applicable to solution for the zeros of any analytic function. It seems to us unlikely that this method is new because of its simplicity and efficacy, but it seems not to have enjoyed wide dissemination and, as such, we feel that it is worthy of further press—perhaps to the extent that it could appear as a numerical method in standard texts on computational methods.

We close by mentioning that, although the method fulfilled the role as an elegant way of deducing the Rayleigh-Lamb modes, we had a more pressing problem at hand. In 1985, Hodges *et al.* [8] worked out the dispersion curves of a periodically ribbed cylinder in air in its angular order components, n_c (i.e., decoupled motions having variations with rotation about the cylinder axis of the form cos $n_c\theta$). These were worked out by solving a matrix eigenvalue problem to give a set of frequencies for a given axial wavenumber. The addition of fluid-loading precludes solution by such means. Over the years we have

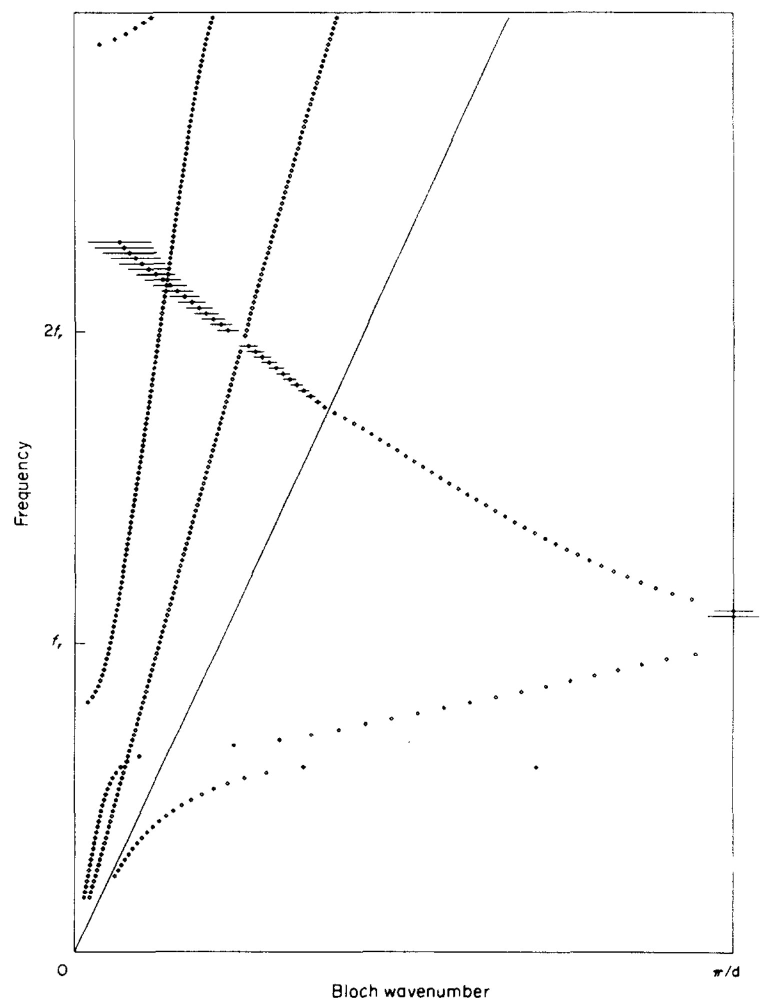

Figure 4. The angular order $n_c = 0$ dispersion curves for the ribbed cylinder described in reference [8] with the cylinder immersed in water. The wavenumber range is from 0 to the Nyquist wavenumber of rib separation and frequency is given in terms of the cylinder ring frequency.

sought to include fluid-loading on the outside of the cylinder and solve for a set of *complex* wavenumbers for a given *real* frequency, this being the natural way if one wishes to solve for a forced fluid-loaded ribbed cylinder. The alternative, which is easier to do, of solving for complex frequencies for a real wavenumber faced a number of as yet unresolved conceptual problems. It is therefore to solve the complex wavenumber dispersion of the fluid-loaded ribbed cylinder that we opted for the method described here. We hope to describe this work more fully at some time in the future, but for now, present the "fluid-loaded" equivalent of the Hodges *et al.* Figure 5 for $n_c = 0$, as Figure 4. The horizontal bars visible represent the imaginary part of wavenumber and the diagonal line on the diagram is the axial "sonic" line separating wavenumbers that are acoustically fast and those that are acoustically slow.

We believe that the prospect of being able to deduce zeros of a function analytic over a region by contour integrals around the boundary is a thoroughly useful one. It has enabled us to solve for the fluid-coupled modes of a ribbed cylinder with relative ease, and could be useful for the solution of similar problems.

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