Quantum Information Science HW P3

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In [4]:
```
# Imports
import math
import matplotlib.pyplot as plt

# Imports from Qiskit
import qiskit
from qiskit import QuantumCircuit
from qiskit.circuit.library import GroverOperator, MCMT, ZGate
from qiskit.visualization import plot_distribution
from qiskit.transpiler.preset_passmanagers import generate_preset_pass_manager

# Imports from Qiskit Runtime
from qiskit_ibm_runtime import QiskitRuntimeService
from qiskit_ibm_runtime import SamplerV2 as Sampler
```

Question 1

Derivation of three Grover's iterations for N = 32:

Definitions

- $U_\omega = I - 2|\omega\rangle\langle\omega|$
- **Grover Diffusion Operator:** $U_{|\psi_0\rangle} = 2|\psi_0\rangle\langle\psi_0| - I$

Initialization

Apply Hadamard gates to achieve a uniform superposition of basis states:

$$|\psi_0\rangle = \frac{1}{\sqrt{32}} \sum_{x=0}^{31} |x\rangle$$

Iteration 1

$$U_\omega(|\psi_0\rangle) = (I - 2|\omega\rangle\langle\omega|)|\psi_0\rangle = |\psi_0\rangle - (2|\omega\rangle\langle\omega|\psi_0\rangle) = |\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_0\rangle}(|\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = (2|\psi_0\rangle\langle\psi_0| - I) \cdot (|\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle)$$

$$U_{|\psi_0\rangle}(|\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = (2|\psi_0\rangle\langle\psi_0|\psi_0\rangle - |\psi_0\rangle) - (\tfrac{4}{\sqrt{32}}|\psi_0\rangle\langle\psi_0|\omega\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle)$$

$$U_{|\psi_0\rangle}(|\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = 2|\psi_0\rangle - |\psi_0\rangle - \tfrac{4}{\sqrt{32}}|\psi_0\rangle\langle\psi_0|\omega\rangle + \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_0\rangle}(|\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = 2|\psi_0\rangle - |\psi_0\rangle - \tfrac{4}{32}|\psi_0\rangle + \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_0\rangle}(|\psi_0\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = \tfrac{7}{8}|\psi_0\rangle + \tfrac{1}{2\sqrt{2}}|\omega\rangle = |\psi_1\rangle$$

Iteration 2

$$U_\omega(|\psi_1\rangle) = (I - 2|\omega\rangle\langle\omega|)|\psi_1\rangle = |\psi_1\rangle - (2|\omega\rangle\langle\omega|\psi_1\rangle) = |\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_1\rangle}(|\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = (2|\psi_1\rangle\langle\psi_1| - I) \cdot (|\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle)$$

$$U_{|\psi_1\rangle}(|\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = (2|\psi_1\rangle\langle\psi_1|\psi_1\rangle - |\psi_1\rangle) - (\tfrac{4}{\sqrt{32}}|\psi_1\rangle\langle\psi_1|\omega\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle)$$

$$U_{|\psi_1\rangle}(|\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = 2|\psi_1\rangle - |\psi_1\rangle - \tfrac{4}{\sqrt{32}}|\psi_1\rangle\langle\psi_1|\omega\rangle + \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_1\rangle}(|\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = 2|\psi_1\rangle - |\psi_1\rangle - \tfrac{4}{32}|\psi_1\rangle + \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_1\rangle}(|\psi_1\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = \tfrac{7}{8}|\psi_1\rangle + \tfrac{1}{2\sqrt{2}}|\omega\rangle = |\psi_2\rangle$$

Iteration 3

$$U_\omega(|\psi_2\rangle) = (I - 2|\omega\rangle\langle\omega|)|\psi_2\rangle = |\psi_2\rangle - (2|\omega\rangle\langle\omega|\psi_2\rangle) = |\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_2\rangle}(|\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = (2|\psi_2\rangle\langle\psi_2| - I)\cdot(|\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle)$$

$$U_{|\psi_2\rangle}(|\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = (2|\psi_2\rangle\langle\psi_2|\psi_2\rangle - |\psi_2\rangle) - (\tfrac{4}{\sqrt{32}}|\psi_2\rangle\langle\psi_2|\omega\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle)$$

$$U_{|\psi_2\rangle}(|\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = 2|\psi_2\rangle - |\psi_2\rangle - \tfrac{4}{\sqrt{32}}|\psi_2\rangle\langle\psi_2|\omega\rangle + \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_2\rangle}(|\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = 2|\psi_2\rangle - |\psi_2\rangle - \tfrac{4}{32}|\psi_2\rangle + \tfrac{2}{\sqrt{32}}|\omega\rangle$$

$$U_{|\psi_2\rangle}(|\psi_2\rangle - \tfrac{2}{\sqrt{32}}|\omega\rangle) = \tfrac{7}{8}|\psi_2\rangle + \tfrac{1}{2\sqrt{2}}|\omega\rangle = |\psi_3\rangle$$

Simplification

Known states:

- $|\psi_0\rangle = \dfrac{1}{\sqrt{32}}\displaystyle\sum_{x=0}^{31}|x\rangle$
- $|\psi_1\rangle = \frac{7}{8}|\psi_0\rangle + \frac{1}{2\sqrt{2}}|\omega\rangle$
- $|\psi_2\rangle = \frac{7}{8}|\psi_1\rangle + \frac{1}{2\sqrt{2}}|\omega\rangle$
- $|\psi_3\rangle = \frac{7}{8}|\psi_2\rangle + \frac{1}{2\sqrt{2}}|\omega\rangle$

Simplifying:

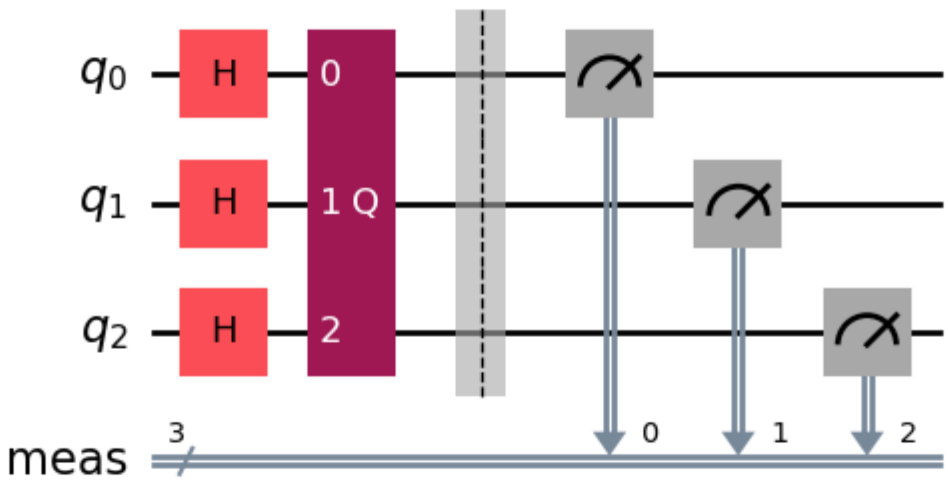
- $|\psi_3\rangle = \frac{7}{8}(\frac{7}{8}|\psi_1\rangle + \frac{1}{2\sqrt{2}}|\omega\rangle) + \frac{1}{2\sqrt{2}}|\omega\rangle$
- $|\psi_3\rangle = \frac{49}{64}|\psi_1\rangle + \frac{15\sqrt{2}}{32}|\omega\rangle$
- $|\psi_3\rangle = \frac{49}{64}(\frac{7}{8}|\psi_0\rangle + \frac{1}{2\sqrt{2}}|\omega\rangle) + \frac{15\sqrt{2}}{32}|\omega\rangle$
- $|\psi_3\rangle = \frac{343}{512}|\psi_0\rangle + \frac{169\sqrt{2}}{256}|\omega\rangle$
- $|\psi_3\rangle = \dfrac{343\sqrt{2}}{4096}\displaystyle\sum_{x=0}^{31}|x\rangle + \dfrac{169\sqrt{2}}{256}|\omega\rangle$

Question 2

In [5]:
```python
# Code source: https://learning.quantum.ibm.com/tutorial/grovers-algorithm

# Oracle Function
def grover_oracle(marked_states):
    """Build a Grover oracle for multiple marked states

    Here we assume all input marked states have the same number of bits

    Parameters:
        marked_states (str or list): Marked states of oracle

    Returns:
        QuantumCircuit: Quantum circuit representing Grover oracle
    """
    if not isinstance(marked_states, list):
        marked_states = [marked_states]
    # Compute the number of qubits in circuit
    num_qubits = len(marked_states[0])

    qc = QuantumCircuit(num_qubits)
    # Mark each target state in the input list
    for target in marked_states:
        # Flip target bit-string to match Qiskit bit-ordering
        rev_target = target[::-1]
        # Find the indices of all the '0' elements in bit-string
        zero_inds = [ind for ind in range(num_qubits) if rev_target.startswith("0", ind)]
        # Add a multi-controlled Z-gate with pre- and post-applied X-gates (open-controls)
        # where the target bit-string has a '0' entry
        qc.x(zero_inds)
        qc.compose(MCMT(ZGate(), num_qubits - 1, 1), inplace=True)
        qc.x(zero_inds)
    return qc

# Define the Grover Search Instance
marked_states = ["011", "100"] # States to search for with oracle
oracle = grover_oracle(marked_states)
oracle.draw(output="mpl", style="iqp")
plt.show()
```

```python
# Use builtin Grover Operator to amplify states marked by oracle
grover_op = GroverOperator(oracle)
grover_op.decompose().draw(output="mpl", style="iqp")

# Calculate the optimal number of Grover iterations to perform
optimal_num_iterations = math.floor(
    math.pi / (4 * math.asin(math.sqrt(len(marked_states) / 2**grover_op.num_qubits)))
)

# Create the full Grover Search circuit
qc = QuantumCircuit(grover_op.num_qubits)
qc.h(range(grover_op.num_qubits)) # Create even superposition of all basis states
qc.compose(grover_op.power(optimal_num_iterations), inplace=True) # Apply Grover operator the optimal number of times
qc.measure_all()
qc.draw(output="mpl", style="iqp")
```

Out[5]:



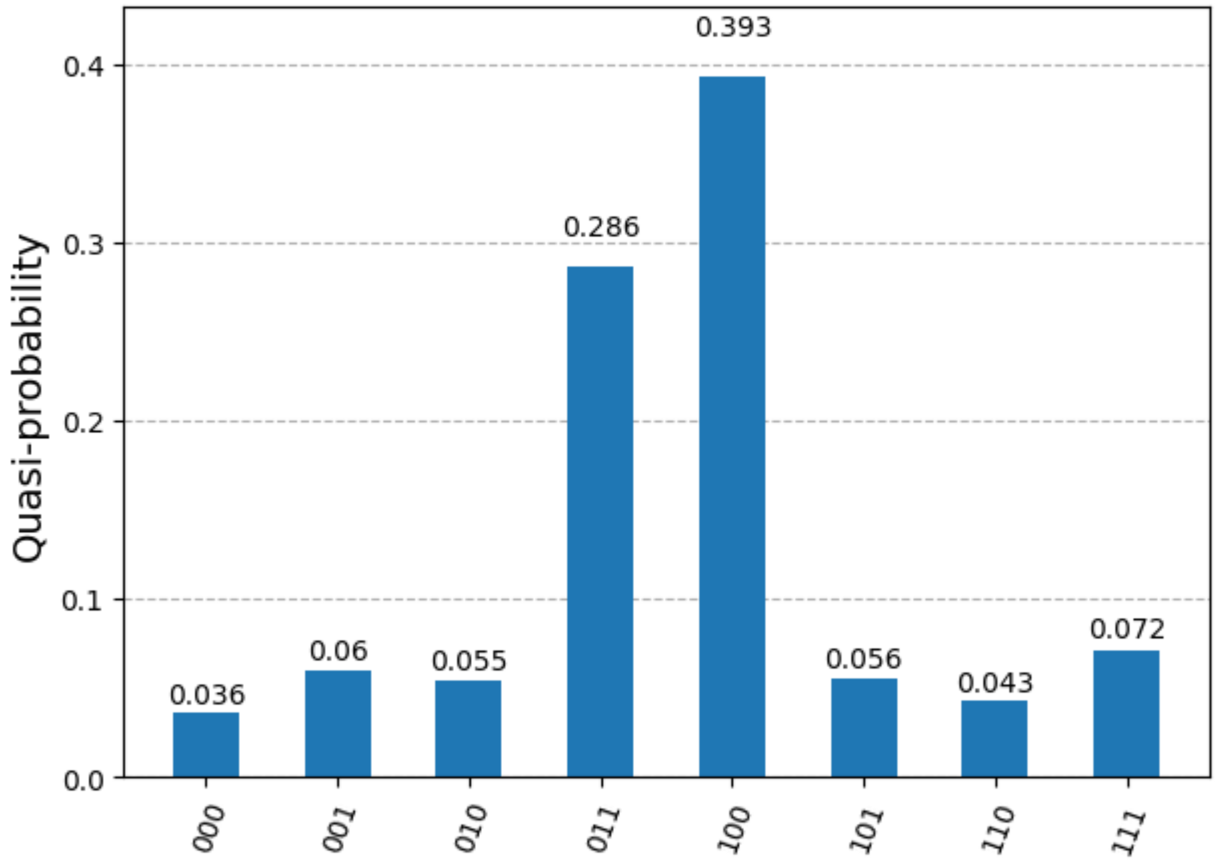
In [6]:

```python
# Select backend service with fewest jobs in queue
service = QiskitRuntimeService(channel="ibm_quantum")
backend = service.least_busy(operational=True, simulator=False)
print(backend.name)

# Transpile quantum circuit to run on quantum hardware
```

```python
target = backend.target
pm = generate_preset_pass_manager(target=target, optimization_level=3)
circuit_isa = pm.run(qc)

# Runs simulation
# To run on local simulator:
#   1. Use the SatetvectorSampler from qiskit.primitives instead
sampler = Sampler(mode=backend)
sampler.options.default_shots = 10_000
result = sampler.run([circuit_isa]).result()
dist = result[0].data.meas.get_counts()
plot_distribution(dist)
```

```
ibm_brisbane
```

Out[6]:



We can see that the two "marked" states (011 and 100) had their amplitudes amplified by the Grover Search circuit when compared against the uniform superposition of the other three-qubit states. This distribution, averaged over a statistically significant amount of runs on real quantum hardware, demonstrates the effectiveness of the Grover Search algorithm when the problem can be structured in the form of an oracle.