$$\max_{w \in \mathbb{R}^p} \frac{w^T A w}{w^T B w}$$

$\tilde{w} = B^{1/2} w \iff w = B^{-1/2} \tilde{w}$

$$\max_{\tilde{w} \in \mathbb{R}^n} \left( \frac{\tilde{w}^T B^{-1/2} A B^{-1/2} \tilde{w}}{\tilde{w}^T \tilde{w}} \right)$$

$\Updownarrow$

PCA pb applied to matrix $\underbrace{B^{-1/2} A B^{-1/2}}_{= C \in \mathbb{R}^{p \times p}}$.

Transform PCA into SVD problem.

$M_1 = \mathrm{sqrt}\left(B^{-1/2} A B^{-1/2}\right)$

$C^{1/2} = B^{-1/2} A^{1/2}$

$C^{\frac{1}{2}} C^{\frac{1}{2}T} = B^{-1/2} A^{1/2} A^{1/2} B^{-1/2}$

$= B^{-1/2} A B^{-1/2}$

SVD of $M_2 = B^{-1/2} A^{1/2}$

$M_2 = U D V^T$

$M_2 M_2^T = U D V^T V D^T U^T$

$= \underline{\underline{U D D^T U^T}}$

$U = \mathrm{PCA}$ of $M_2 M_2^T$

$= B^{-1/2} A B^{-1/2}$ ✓

$$\max_{u,v} \; u^T M_2 v \qquad \|u\|_2^2 = 1 \quad P_1(u) \leq c_1$$
$$\qquad\qquad\qquad \|v\|_2^2 = 1 \quad P_2(v) \leq c_2$$

Here, only interested in penalising $u$:

BUT we are interested in $\underline{B^{-1/2} u} = w$

$$\max_{u,v} \; u^T M_2 v \qquad \|u\|_2^2 = 1, \quad P_1(B^{-1/2} u) \leq c_1$$

$$\max_{w,v} \; w^T \underbrace{B^{1/2} M_2}_{=} v, \quad \|B^{1/2} w\|_2^2 = 1, \quad P_1(w) \leq c_1$$

$A^{1/2}$ $\qquad w^T B w = 1$

Solution in $v$ is trivial $\qquad \max_v \; v^T A^{1/2} w \quad \|v\|_2^2 = 1$

$$v^* = \frac{A^{1/2} w}{\sqrt{w^T A w}}$$

What about $w$ step? $\qquad \max_w \; w^T a \quad$ s.t. $\begin{cases} w^T B w = 1 \\ \|w\|_1 \leq c \end{cases}$ solve numerically. LASSO.

Group lasso penalty? $\quad$ we can replace $P_1(w)$ w. group lasso

Goal: solve $\min \; -u^T M v + \lambda \|u\|_1$

$\text{s.t.} \quad v^T v \leq 1$

$u^T B u \leq 1$

Algo $(M, B, \lambda)$

1. $v^{it1} \leftarrow \dfrac{M u^{it1}}{\sqrt{u^{it1\,T} M^T M \, u^{it1}}}$

2. solve $\begin{cases} \min & -u^T M v^{it1} + \lambda \|u\|_1 \\ \text{s.t.} & u^T B u \leq 1 \end{cases}$ routine

$u^{it1} \leftarrow u^*$

Routine $(v, B, \lambda)$

$\min \; -u^T M v + \lambda \|u\|_1$

$\text{s.t.} \quad u^T B u \leq 1$

1. Initialize $u^0$
2. $d_{t+1} \leftarrow -Mv + \lambda \, \text{sgn}(u_t)$
3. $s_{t+1} \leftarrow \dfrac{B^{-1} d_{t+1}}{\sqrt{d_{t+1}^T B^{-2} d_{t+1}}}$
4. $u_{t+1} \leftarrow u_t + \frac{2}{t+2}(s_{t+1} - u_t)$

until convergence