Contents

1 Copies of some equations and their major sources

Stix 2.45 (sign as for Whistler mode):

$$\mu^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e \cos\theta)} \tag{1}$$

Doppler resonance condition:

$$\omega - k_\parallel v_\parallel = \frac{n\Omega_e}{\gamma} \tag{2}$$

2 Resonant frequencies in high-density approximation

Start from Stix Equation 2.45 for the Right hand mode, noting that Ω_c contains the *sign* of the particle charge:

$$\mu^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e \cos\theta)} \tag{3}$$

along with the resonance condition

$$\omega - k_\parallel v_\parallel = \frac{n\Omega_e}{\gamma} \tag{4}$$

The latter is rewritten as

$$k_\parallel = k\cos\theta = \frac{1}{v_\parallel}\left(\omega - \frac{n\Omega_e}{\gamma}\right) \tag{5}$$

and so as $\mu^2 = c^2k^2/\omega^2 = ((ck_\parallel)/(\omega\cos\theta))^2$ we get

$$\mu = \frac{c(\gamma\omega - n\Omega_e)}{\gamma\omega v_\parallel \cos\theta}. \tag{6}$$

Now we equate 3 and the square of 6:

$$\frac{\omega(\omega + \Omega_e\cos\theta) - \omega_{pe}^2}{\omega(\omega + \Omega_e\cos\theta)} = \frac{c^2}{\gamma^2\omega^2 v_\parallel^2 \cos^2\theta}(\gamma\omega - n\Omega_e)^2 \tag{7}$$

which rearranges to

$$(\gamma^2\omega^2 v_\parallel^2\cos^2\theta)(\omega(\omega + \Omega_e\cos\theta) - \omega_{pe}^2) = c^2\omega(\omega + \Omega_e\cos\theta)(\gamma\omega - n\Omega_e)^2. \tag{8}$$

Cancel one ω from each side:

$$(\gamma^2\omega v_\parallel^2\cos^2\theta)(\omega(\omega + \Omega_e\cos\theta) - \omega_{pe}^2) = c^2(\omega + \Omega_e\cos\theta)(\gamma\omega - n\Omega_e)^2 \tag{9}$$

and multiply out:

$$(\gamma^2\omega v_\parallel^2\cos^2\theta)(\omega(\omega + \Omega_e\cos\theta) - \omega_{pe}^2) = c^2(\omega + \Omega_e\cos\theta)(\gamma^2\omega^2 - 2n\Omega_e\gamma\omega + n^2\Omega_e^2) \tag{10}$$

and again

$$\begin{aligned} &\gamma^2\omega^3 v_\parallel^2\cos^2\theta + \gamma^2\omega^2 v_\parallel^2\cos^3\theta\Omega_e - \gamma^2\omega v_\parallel^2\cos^2\theta\omega_{pe}^2 = \\ &c^2\gamma^2\omega^3 + c^2\Omega_e\cos\theta\gamma^2\omega^2 - 2c^2\omega^2 n\Omega_e\gamma - 2c^2n\Omega_e^2\gamma\omega\cos\theta + c^2\omega n^2\Omega_e^2 + c^2n^2\Omega_e^3\cos\theta \end{aligned} \tag{11}$$

Write $v_\parallel/c = \tilde{v}$ and divide through by c^2:

$$\begin{aligned} &\gamma^2\omega^3\tilde{v}^2\cos^2\theta + \gamma^2\omega^2\tilde{v}^2\cos^3\theta\Omega_e - \gamma^2\omega\tilde{v}^2\cos^2\theta\omega_{pe}^2 = \\ &\gamma^2\omega^3 + \Omega_e\cos\theta\gamma^2\omega^2 - 2\omega^2 n\Omega_e\gamma - 2n\Omega_e^2\gamma\omega\cos\theta + \omega n^2\Omega_e^2 + n^2\Omega_e^3\cos\theta \end{aligned} \tag{12}$$

Now collect up coefficients of each power:

$$\begin{aligned} &\left(\gamma^2\tilde{v}^2\cos^2\theta - \gamma^2\right)\omega^3 \\ &+ \left(\gamma^2\tilde{v}^2\cos^3\theta\Omega_e - \Omega_e\cos\theta\gamma^2 + 2n\Omega_e\gamma\right)\omega^2 \\ &+ \left(-\gamma^2\tilde{v}^2\cos^2\theta\omega_{pe}^2 + 2n\Omega_e^2\gamma\cos\theta - n^2\Omega_e^2\right)\omega \\ &+ \left(-n^2\Omega_e^3\cos\theta\right) \end{aligned} \tag{13}$$

Or as $\omega, \omega_{pe}, |\Omega_e| \gg 1$ we can make the substitution $\omega_0 = \omega/\omega_{ref}$ to minimise the differences in size between the coefficients, giving:

$$\begin{aligned}
&\left(\gamma^2 \tilde{v}^2 \cos^2\theta - \gamma^2\right)(\omega_{ref}\omega_0)^3 \\
&+ \left(\gamma^2 \tilde{v}^2 \cos^3\theta \Omega_e - \Omega_e \cos\theta \gamma^2 + 2n\Omega_e\gamma\right)(\omega_{ref}\omega_0)^2 \\
&+ \left(-\gamma^2 \tilde{v}^2 \cos^2\theta \omega_{pe}^2 + 2n\Omega_e^2 \gamma \cos\theta - n^2\Omega_e^2\right)\ \omega_{ref}\omega_0 \\
&+ \left(-n^2 \Omega_e^3 \cos\theta\right)
\end{aligned} \tag{14}$$

or dividing through by $(\omega_{ref})^3$:

$$\begin{aligned}
&\left(\gamma^2 \tilde{v}^2 \cos^2\theta - \gamma^2\right)(\omega_0)^3 \\
&+ \left(\gamma^2 \tilde{v}^2 \cos^3\theta (\Omega_e/\omega_{ref}) - (\Omega_e/\omega_{ref}) \cos\theta \gamma^2 + 2n(\Omega_e/\omega_{ref})\gamma\right)(\omega_0)^2 \\
&+ \left(-\gamma^2 \tilde{v}^2 \cos^2\theta (\omega_{pe}/\omega_{ref})^2 + 2n(\Omega_e/\omega_{ref})^2 \gamma \cos\theta - n^2(\Omega_e/\omega_{ref})^2\right)\ \omega_0 \\
&+ \left(-n^2 (\Omega_e/\omega_{ref})^3 \cos\theta\right)
\end{aligned} \tag{15}$$

which are expected to be order 1 for $n \sim 1$ and $\tilde{v}, \cos\theta \sim 1$.

3 Normalisation of angular distribution, $N(\omega)$

We need to calculate the factor $N(\omega)$ in Lyons 1974 J Plasma Phys, 12, 417, Equation A3 or A5 for arbitrary μ. I.e.

$$N(\omega) = \frac{1}{(2\pi)^2} \int_0^\infty g_\omega(\tan^{-1} x) \left| J\left(\frac{k_\perp, k_\parallel}{\omega, x}\right)\right| k_\perp dx \tag{16}$$

Using Eq A6:

$$J\left(\frac{k_\perp, k_\parallel}{\omega, x}\right) = -\left.\frac{\partial k_\perp}{\partial x}\right|_\omega \cdot \left.\frac{\partial k_\parallel}{\partial \omega}\right|_{k_\perp}, \tag{17}$$

Using $k_\perp = x k_\parallel$ and $\mu = ck/\omega$ we get

$$\frac{\partial k_\perp}{\partial x} = \frac{\partial}{\partial x}\left.\left(x \cos(\mathrm{atan} x) \frac{\mu(x)\omega}{c}\right)\right|_\omega. \tag{18}$$

Using the chain rule:

$$\frac{\partial k_\perp}{\partial x} = \frac{\partial}{\partial x}\left(x\cos(\mathrm{atan}x)\right)\frac{\mu(x)\omega}{c} + x\cos(\mathrm{atan}x)\frac{\omega}{c}\frac{\partial \mu(x)}{\partial x}. \tag{19}$$

Note that

$$\cos(\mathrm{atan}x) = \frac{1}{\sqrt{(x^2+1)}}, \sin(\mathrm{atan}x) = \frac{x}{\sqrt{(x^2+1)}} \tag{20}$$

then we get

$$\frac{\partial k_\perp}{\partial x} = \frac{1}{(x^2+1)^{\frac{3}{2}}}\frac{\mu(x)\omega}{c} + \frac{x}{\sqrt{(x^2+1)}}\frac{\omega}{c}\frac{\partial \mu(x)}{\partial x}. \tag{21}$$

However, we need to evaluate this at a fixed ω which means that $ck/\mu = \text{Const}$ and as k is a free variable the $\partial\mu/\partial x$ must be 0. This then becomes

$$\left.\frac{\partial k_\perp}{\partial x}\right|_\omega = \frac{\omega}{c\sqrt{(x^2+1)}}\frac{\mu(x)}{(1+x^2)}. \tag{22}$$

The other term we need is

$$\left.\frac{\partial k_\parallel}{\partial \omega}\right|_{k_\perp}, \tag{23}$$

We can implicitly differentiate $k^2 = k_\parallel^2 + k_\perp^2$ as $2kdk = 2k_\perp dk_\perp + 2k_\parallel dk_\parallel$ which for $k_\perp$ held constant means

$$\frac{\partial k}{\partial \omega} = \frac{\partial k_\parallel}{\partial \omega}\frac{k_\parallel}{k}. \tag{24}$$

Now

$$\frac{ck}{\omega} = \mu(\omega) \tag{25}$$

$$\frac{\partial \mu}{\partial \omega} = -\frac{ck}{\omega^2} + \frac{c}{\omega}\frac{\partial k}{\partial \omega} \tag{26}$$

So

$$\begin{aligned}\frac{\partial k_\parallel}{\partial \omega} &= \frac{k}{k_\parallel}\frac{\partial k}{\partial \omega} \\ &= \frac{k}{k_\parallel}\left(\frac{\partial \mu}{\partial \omega} + \frac{ck}{\omega^2}\right)\frac{\omega}{c}.\end{aligned} \tag{27}$$

Alternately we write $k_\parallel = (k^2 - k_\perp^2)^{1/2}$ and differentiate this by ω to get the same result. We note that this term is to be evaluated at constant $k_\perp$, or equivalently constant $x\mu/\sqrt{x^2+1}$, and that $k/k_\parallel = \sqrt{x^2+1}$
Using these we get:

$$N(\omega) = \frac{2\omega^2}{(2\pi)^2 c^2}\int_0^\infty g_\omega(\tan^{-1} x)\left|\frac{\mu(x)}{(1+x^2)}\right|\left|\frac{\partial \mu}{\partial \omega} + \frac{ck}{\omega^2}\right| k_\perp dx. \tag{28}$$

The final step is to eliminate $k_\perp$ in favour of $\omega, x, \mu(\omega, x)$, using $k_\perp = \mu\omega\sin(\text{atan}x)/c$ to get

$$N(\omega) = \frac{2\omega^3}{(2\pi)^2 c^3}\int_0^\infty g_\omega(\text{atan}x)\frac{x\mu^2(x,\omega)}{(x^2+1)^{3/2}}\left|\frac{\partial \mu}{\partial \omega} + \frac{\mu}{\omega}\right| dx. \tag{29}$$

4 Obtaining Lyons A7

Substitute Lyons Eq (12) for Whistler mode (rewritten in terms of $x = \tan\theta$):

$$\mu^2 = \frac{\omega_{pe}^2}{\Omega_c^2}\frac{1+M}{M}\Psi^{-1} =: A\Psi^{-1} \tag{30}$$

$$\Psi = 1 - \frac{\omega^2}{\Omega_p\Omega_e} - \frac{x^2}{2(x^2+1)} + \left[\frac{x^4}{4(x^2+1)^2} + \left(\frac{\omega}{\Omega_p}\right)^2 (1-M)^2 \frac{1}{x^2+1}\right]^{1/2} \tag{31}$$

into 29 with $M = m_e/m_p$ to get

$$N(\omega) = \frac{\omega^3}{(2\pi)^2 c^3} \int_{-\infty}^{\infty} g_\omega(\mathrm{atan}x) \frac{x}{(1+x^2)^{3/2}} A\Psi^{-1} \left(A^{1/2} \frac{\partial \Psi^{-1/2}}{\partial \omega} + \frac{A^{1/2}\Psi^{-1/2}}{\omega} \right) dx. \tag{32}$$

Then

$$\frac{\partial \Psi^{-1/2}}{\partial \omega} = -\frac{1}{2} \Psi^{-3/2} \frac{\partial \Psi}{\partial \omega} \tag{33}$$

$$\frac{\partial \Psi}{\partial \omega} = -\frac{2\omega}{\Omega_p \Omega_e} + \left[\frac{x^4}{4(x^2+1)^2} + \left(\frac{\omega}{\Omega_p}\right)^2 (1-M)^2 \frac{1}{x^2+1} \right]^{-1/2} \frac{\omega}{\Omega_p^2} (1-M)^2 \frac{1}{x^2+1} \tag{34}$$

The part in square brackets can be re-written as

$$\begin{aligned} \left[\frac{x^4}{4(x^2+1)^2} + \left(\frac{\omega}{\Omega_p}\right)^2 (1-M)^2 \frac{1}{x^2+1} \right]^{-1/2} &= \left(\Psi - 1 + \frac{\omega^2}{\Omega_p \Omega_e} + \frac{x^2}{2(x^2+1)} \right)^{-1} \\ &= (1+x^2) \left[(1+x^2) \left(\Psi - 1 + \frac{\omega^2}{\Omega_p \Omega_e} \right) + \frac{x^2}{2} \right]^{-1} \end{aligned} \tag{35}$$

So

$$\left(\frac{\partial \Psi^{-1/2}}{\partial \omega} + \frac{\Psi^{-1/2}}{\omega} \right) = \frac{\Psi^{-1/2}}{\omega} \left(1 + \Psi^{-1} \left[\frac{\omega^2}{\Omega_p \Omega_e} - \frac{1}{2} \left[(1+x^2) \left(\Psi - 1 + \frac{\omega^2}{\Omega_p \Omega_e} \right) + \frac{x^2}{2} \right]^{-1} \frac{\omega^2}{\Omega_p^2} (1-M)^2 \right] \right) \tag{36}$$

And finally

$$\begin{aligned} N(\omega) = A^{3/2} \frac{\omega^2}{(2\pi)^2 c^3} \int_{-\infty}^{\infty} g_\omega(x) \frac{x}{(1+x^2)^{3/2}} (\Psi)^{-\frac{3}{2}} \times \\ \left(1 + \Psi^{-1} \left[\frac{\omega^2}{\Omega_p \Omega_e} - \frac{1}{2} \left[(1+x^2) \left(\Psi - 1 + \frac{\omega^2}{\Omega_p \Omega_e} \right) + \frac{x^2}{2} \right]^{-1} \frac{\omega^2}{\Omega_p^2} (1-M)^2 \right] \right) dx. \end{aligned} \tag{37}$$

Note that the second factor is defined as $I(\omega)$,

$$I(\omega) := \left(1 + \Psi^{-1} \left[\frac{\omega^2}{\Omega_p \Omega_e} - \frac{1}{2} \left[(1+x^2) \left(\Psi - 1 + \frac{\omega^2}{\Omega_p \Omega_e} \right) + \frac{x^2}{2} \right]^{-1} \frac{\omega^2}{\Omega_p^2} (1-M)^2 \right] \right) \tag{38}$$

5 G2 transformation to tan axis

G2 in Albert [1] is integral over theta, but in our case we have a $\tan(\theta)$ axis so we transform

$$x = \tan(\theta) \tag{39}$$

so

$$\frac{\mathrm{d}\tan(\theta)}{\mathrm{d}\theta} = \sec^2(\theta) = \frac{1}{\cos^2(\theta)} \tag{40}$$

i.e.

$$\mathrm{d}\theta = \cos^2(\theta)\mathrm{d}\tan\theta \tag{41}$$

providing $\cos\theta \neq 0$. Similarly, the $\sin\theta$ term is

$$\sin\theta = \tan\theta\cos\theta \tag{42}$$

so using $\tan^2\theta + 1 = \sec^2\theta$ we get

$$\sin\theta\mathrm{d}\theta = \tan\theta\cos^3\theta\mathrm{d}\tan\theta = \frac{\tan\theta}{(\tan^2\theta+1)^{3/2}}\mathrm{d}\tan\theta = \frac{x}{(x^2+1)^{3/2}}\mathrm{d}x \tag{43}$$

Then Albert's [1] Eq 3 for G2 becomes

$$G_2 = g_\omega(\theta)/\left[\int g_\omega(\mathrm{atan}(\theta'))\mu^2\left|\mu+\omega\frac{\partial\mu}{\partial\omega}\right|\frac{x}{(x^2+1)^{3/2}}\mathrm{d}x\right] \tag{44}$$

6 Relation of Alberts G1 and G2 to Lyons equations

Consider Albert [1] Eq 2 (Al2) which is

$$D^{nx}_{\alpha\alpha} = \frac{B^2_{wave}}{B^2_0}(mv)^2\frac{\pi}{2}\frac{\Omega^2_c}{\Delta\omega}\frac{\cos^2\theta}{|v_\parallel/c|^3}\Phi^2_n\frac{(-\sin^2\alpha+\omega_n/\omega)^2}{|1-(\partial\omega/\partial k_\parallel)_x/v_\parallel|}G_1G_2 \tag{45}$$

$$G_1(\omega) = \frac{(\Delta\omega)B^2(\omega)}{\int B^2(\omega')\mathrm{d}\omega'} \tag{46}$$

$$G_2(\omega,\theta) = \frac{g_\omega(\theta)}{\int g_\omega(\theta')\mu^2|\mu+\omega\partial\mu/\partial\omega|\sin\theta'\mathrm{d}\theta'} \tag{47}$$

noting that we have restored the normalisation factors from

$$D_{xx} = \Omega_c(B^2_{wave}/B^2_0)(mv)^2D^{norm}_{xx} \tag{48}$$

and that

$$B^2_{wave} = \int B^2(\omega')\mathrm{d}\omega' \tag{49}$$

Φ^2_n is as in Lyons, Eq 9 (Ly9) and contains some Bessel functions etc etc
Ly2 should match Al2. Take Ly2 and subs for Θ as given before 9, and get $|B|$ from the appendix:

$$D^{nk_\perp}_{\alpha\alpha} = \lim_{V\to\infty}\frac{\pi q^2}{(2\pi)^2Vm^2v_\parallel}\left[\frac{-\sin^2\alpha+n\Omega/\omega_k}{\cos\alpha}\right]^2\frac{|B_k|^2|\Phi_{n,k}|^2}{\mu^2|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|} \tag{50}$$

$$|B_k|^2 = \frac{V}{N(\omega)}B^2(\omega)g_\omega(\theta) \tag{51}$$

where q, m are the (per species) charge and mass respectively. Derivation of $N(\omega)$ following Lyons was done above.

They look fairly consistent, so let's rearrange the Lyons one to match the Albert:

$$D^{nk_\perp}_{\alpha\alpha} = \lim_{V\to\infty}\frac{\pi q^2}{(2\pi)^2Vm^2v_\parallel}\left[\frac{-\sin^2\alpha+n\Omega/\omega_k}{\cos\alpha}\right]^2\frac{|\Phi_{n,k}|^2}{\mu^2|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}\frac{V}{N(\omega)}B^2(\omega)g_\omega(\theta) \tag{52}$$

Cancel the V and subs the prefactors of N from Eq 29, i.e. define

$$N(\omega) =: N'(\omega)\frac{2\omega^3}{(2\pi)^2c^3} \tag{53}$$

$$D^{nk_\perp}_{\alpha\alpha} = \frac{\pi q^2}{2m^2 v_\parallel}\left[\frac{-\sin^2\alpha + n\Omega/\omega_k}{\cos\alpha}\right]^2 \frac{|\Phi_{n,k}|^2}{\mu^2|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}\frac{1}{N'(\omega)}B^2(\omega)g_\omega(\theta)\frac{c^3}{\omega^3} \tag{54}$$

$\cos^2\alpha$ can be replaced by $(v_\parallel/v)^2$ to get

$$D^{nk_\perp}_{\alpha\alpha} = \frac{\pi q^2 v^2}{2m^2(v_\parallel/c)^3\omega^3}\frac{1}{\mu^2}|\Phi_{n,k}|^2\frac{(-\sin^2\alpha + n\Omega/\omega_k)^2}{|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}\frac{1}{N'(\omega)}B^2(\omega)g_\omega(\theta) \tag{55}$$

Tackle G1 and G2 next. Starting from Eq 29 and Eq 53

$$N'(\omega) = \int_{-\infty}^{\infty} g_\omega(\mathrm{atan}x)\frac{\mu(x)^2 x}{(1+x^2)^{3/2}}\left|\frac{\partial\mu}{\partial\omega} + \frac{\mu}{\omega}\right| dx \tag{56}$$

and Eq 47 with Eq 43 which gives a denominator of

$$G2_{denom} = \int g_\omega(\theta')\mu^2\left|\mu + \omega\frac{\partial\mu}{\partial\omega}\right|\sin\theta' \mathrm{d}\theta' \tag{57}$$

$$= \int g_\omega(\mathrm{atan}(x'))\mu^2\left|\mu + \omega\frac{\partial\mu}{\partial\omega}\right|\frac{x'}{((x')^2+1)^{3/2}}\mathrm{d}x' \tag{58}$$

i.e.

$$G2_{denom} = \omega N'(\omega) \tag{59}$$

For G1 we just absorb the B^2_{wave} and $\Delta\omega$ and so Eq 55 becomes

$$D^{nk_\perp}_{\alpha\alpha} = \frac{\pi q^2 v^2}{2m^2(v_\parallel/c)^3\omega^3}\frac{1}{\mu^2}|\Phi_{n,k}|^2\frac{(-\sin^2\alpha + n\Omega/\omega_k)^2}{|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}\frac{G_1}{\Delta\omega}B^2_{wave}\omega G_2 \tag{60}$$

Recalling

$$\frac{ck}{\omega} = \mu(\omega) \tag{61}$$

we get

$$D^{nk_\perp}_{\alpha\alpha} = \frac{\pi q^2 v^2}{2m^2(v_\parallel/c)^3}\frac{1}{c^2k^2}|\Phi_{n,k}|^2\frac{(-\sin^2\alpha + n\Omega/\omega_k)^2}{|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}\frac{G_1}{\Delta\omega}B^2_{wave}G_2 \tag{62}$$

Then using $\Omega_c = qB_0/m_e c$ we find

$$D^{nk_\perp}_{\alpha\alpha} = \frac{B^2_{wave}}{B^2_0}\frac{\pi v^2\Omega_c^2}{2\Delta\omega(v_\parallel/c)^3}\frac{1}{k^2}|\Phi_{n,k}|^2\frac{(-\sin^2\alpha + n\Omega/\omega_k)^2}{|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}G_1G_2 \tag{63}$$

However, this is the expression for $D(k_\perp)$ while Albert has $D(x)$, so comparing Ly1 to Alb5 implies

$$D_{\alpha\alpha} = \int xD(x)\mathrm{d}x = \int k_\perp D(k_\perp)\mathrm{d}k_\perp \tag{64}$$

and we must multiply the Lyons result by $k_\perp/xdk_\perp/dx$. We have $k_\perp = k\sin\theta = kx/\sqrt{(}x^2+1)$ so $dk_\perp/dx = k(x^2+1)^{-3/2}$ and

$$\frac{k_\perp}{x}\frac{\mathrm{d}k_\perp}{\mathrm{d}x} = \frac{k^2}{(x^2+1)} = k^2\cos^4\theta$$

so we end up with, after reordering terms

$$D^{nx}_{\alpha\alpha} = \frac{B^2_{wave}}{B^2_0}\frac{\pi}{2}v^2\frac{\Omega^2_c}{\Delta\omega}\frac{\cos^4\theta}{(v_\parallel/c)^3}|\Phi_{n,k}|^2\frac{(-\sin^2\alpha + n\Omega/\omega_k)^2}{|1-(\partial\omega_k/\partial k_\parallel)_x/v_\parallel|}G_1G_2 \tag{65}$$

NB NB NB we have some erroneous m^2 and a cos^4 instead of cos^2, compared to the Albert result which is

$$D^{nx}_{\alpha\alpha} = \frac{B^2_{wave}}{B^2_0}(mv)^2\frac{\pi}{2}\frac{\Omega^2_c}{\Delta\omega}\frac{\cos^2\theta}{|v_\parallel/c|^3}\Phi^2_n\frac{(-\sin^2\alpha + \omega_n/\omega)^2}{|1-(\partial\omega/\partial k_\parallel)_x/v_\parallel|}G_1G_2 \tag{66}$$

FOR THE m^2 note that the duplicate of Lyons in Glauert/Horne doesn't have the $1/m$ factor at all...

NB NB I think this is because they use $g(\theta)$ vs $g(\tan(\theta))$ respectively!!

7 Evaluation of resonant frequencies

```
a = (vel_cos - 1.0) * gamma_sq;
b = (vel_cos*cos_th*gamma_sq + 2.0*gamma_particle*n - gamma_sq*cos_th)*om_ce/om_ce_ref;
c = ((2.0 * gamma_particle * n * cos_th - n * n)* std::pow(om_ce/om_ce_ref, 2) -
      std::pow(om_pe_loc/om_ce_ref, 2)*vel_cos*gamma_sq);
d = -n*n*std::pow(om_ce/om_ce_ref, 3)*cos_th;
```

8 Full evaluation of resonant frequencies

The previous section showed the cubic polynomial to be solved to find the resonant frequency in the high-density limit. Here we show the end result for the full, 10th order, version.

8.1 Coefficients in polynomial for omega

From Clare's notes we have the following

$$A\mu^4 - B\mu^2 + C = 0$$

and the contributions from each term broken down by powers of omega are:

$$
\begin{aligned}
\omega^{10}\text{coeff}:& A_{n6}\mu_{n4}^4 & -B_{n6}\mu_{n2}^2 + C_{n8} \\
\omega^{9}\text{coeff}:& A_{n6}\mu_{n3}^4 & -B_{n6}\mu_{n1}^2 \\
\omega^{8}\text{coeff}:& A_{n6}\mu_{n2}^4 + A_{n4}\mu_{n4}^4 & -B_{n6}\mu_{n0}^2 - B_{n4}\mu_{n2}^2 + C_{n6} \\
\omega^{7}\text{coeff}:& A_{n6}\mu_{n1}^4 + A_{n4}\mu_{n3}^4 & -B_{n4}\mu_{n1}^2 \\
\omega^{6}\text{coeff}:& A_{n6}\mu_{n0}^4 + A_{n4}\mu_{n2}^4 + A_{n2}\mu_{n4}^4 & -B_{n4}\mu_{n0}^2 - B_{n2}\mu_{n2}^2 + C_{n4} \\
\omega^{5}\text{coeff}:& A_{n4}\mu_{n1}^4 + A_{n2}\mu_{n3}^4 & -B_{n2}\mu_{n1}^2 \\
\omega^{4}\text{coeff}:& A_{n4}\mu_{n0}^4 + A_{n2}\mu_{n2}^4 + A_{n0}\mu_{n4}^4 & -B_{n2}\mu_{n0}^2 - B_{n0}\mu_{n2}^2 + C_{n2} \\
\omega^{3}\text{coeff}:& A_{n2}\mu_{n1}^4 + A_{n0}\mu_{n2}^4 & -B_{n0}\mu_{n1}^2 \\
\omega^{2}\text{coeff}:& A_{n2}\mu_{n0}^4 + A_{n0}\mu_{n2}^4 & -B_{n0}\mu_{n0}^2 \\
\omega^{1}\text{coeff}:& A_{n0}\mu_{n1}^4 + A_{n0}\mu_{n1}^4 & \\
\omega^{0}\text{coeff}:& A_{n0}\mu_{n0}^4 &
\end{aligned}
$$

with

$$
\begin{aligned}
A_{n6} &= 1 \\
A_{n4} &= -\omega_p^2 - (\Omega_e^2 + \Omega_i^2) \\
A_{n2} &= \Omega_e^2\Omega_i^2 + \omega_p^2(\Omega_e^2 + \Omega_i^2)\cos^2\psi - \sin^2\psi(\omega_{pe}^2\Omega_i^2 + \omega_{pi}^2\Omega_e^2) \\
A_{n0} &= -\omega_p^2\Omega_e^2\Omega_i^2\cos^2\psi
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
B_{n6} &= 2 \\
B_{n4} &= -2(\Omega_e^2 + \Omega_i^2) - 4\omega_p^2 \\
B_{n2} &= 2\Omega_e^2\Omega_i^2 + (2 + \sin^2\psi)(\omega_{pe}^2\Omega_i^2 + \omega_{pi}^2\Omega_e^2) + 2(\omega_{pe}^4 + \omega_{pi}^4) + 4\omega_{pe}^2\omega_{pi}^2 + \omega_p^2(\Omega_e^2 + \Omega_i^2)(1 + \cos^2\psi) \\
B_{n0} &= -2\omega_{pe}^4\Omega_i^2 - 2\omega_{pi}^4\Omega_e^2 - \omega_p^2\Omega_e^2\Omega_i^2(1 + \cos^2\psi) + \omega_{pe}^2\omega_{pi}^2((1 + \cos^2\psi)(\Omega_e - \Omega_i)^2 - 4\Omega_e\Omega_i)
\end{aligned} \tag{68}
$$

and

$$
\begin{aligned}
C_{n8} &= 1 \\
C_{n6} &= -\Omega_e^2 - \Omega_i^2 - 3\omega_p^2 \\
C_{n4} &= 3\omega_p^4 + \omega_p^2(\Omega_e - \Omega_i)^2 + \Omega_e^2\Omega_i^2 \\
C_{n2} &= -\omega_p^2(\Omega_e\Omega_i - \omega_p^2)^2
\end{aligned} \tag{69}
$$

where $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$. For μ we can expand out the terms using

$$L = \frac{c^2}{\gamma^2 v_\parallel^2 \cos^2\psi}$$

as

$$
\begin{aligned}
\mu_{n4}^4 &= L^2\gamma^4 \\
\mu_{n3}^4 &= -4L^2 n\gamma^3\Omega_e \\
\mu_{n2}^4 &= 6L^2 n^2\gamma^2\Omega_e^2 \\
\mu_{n1}^4 &= -4L^2 n^3\gamma\Omega_e^3 \\
\mu_{n0}^4 &= L^2 n^4\Omega_e^4
\end{aligned} \tag{70}
$$

and

$$
\begin{aligned}
\mu_{n2}^2 &= L\gamma^2 \\
\mu_{n1}^2 &= -2Ln\gamma\Omega_e \\
\mu_{n0}^2 &= Ln^2\Omega_e^2
\end{aligned} \tag{71}
$$

8.2 Coefficients in ion-negligible form

When the ion frequencies are negligible with respect to the electron frequencies, we can simplify the A, B, C expressions significantly. Assuming $\omega, \omega_{pe}, \Omega_e \gg \omega_{pi}, \Omega_i$ gives

$$
\begin{aligned}
A_{n6} &= 1 \\
A_{n4} &= -\omega_{pe}^2 - \Omega_e^2 \\
A_{n2} &= \omega_{pe}^2\Omega_e^2\cos^2\psi \\
A_{n0} &= 0
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
B_{n6} &= 2 \\
B_{n4} &= -2\Omega_e^2 - 4\omega_{pe}^2 \\
B_{n2} &= 2\omega_{pe}^4 + \omega_{pe}^2\Omega_e^2(1+\cos^2\psi) \\
B_{n0} &= 0
\end{aligned} \tag{73}
$$

and

$$
\begin{aligned}
C_{n8} &= 1 \\
C_{n6} &= -\Omega_e^2 - 3\omega_{pe}^2 \\
C_{n4} &= 3\omega_{pe}^4 + \omega_{pe}^2\Omega_e^2 \\
C_{n2} &= -\omega_{pe}^6
\end{aligned} \tag{74}
$$

8.3 Rescaling to keep terms order 1

Since this is a 10th order polynomial in a large number, the high powers involved are troublesome for numerical solutions. So we prefer to solve an equation where all terms are order 1. In particular, we want to solve for ω/ω_{ref} using the cyclotron and plasma frequencies similarly reduced. Therefore we: multiply the coefficient for ω^m by ω_{ref}^m; divide the entire equation by ω_{ref}^l where l is chosen to get the terms to be generally amenable; rewrite in terms of the normalised frequencies $\tilde{\omega}_{pe} = \omega_{pe}/\omega_{ref}$ etc. Taking just $l = 10$ we just replace all the frequencies in μ and in A, B, C with their normalised versions.

9 Calculation of D

9.1 Breakdown of terms

Since D is evaluated at each p, α and to do so requires integrating over wave angle and totalling over both n and the resonant omegas we have 4 or 5 nested loops unavoidably. Therefore we break the calculation down and hoist everything as high as possible. Taking

$$D_{\alpha\alpha}^{nx} = \frac{B_{wave}^2}{B_0^2}(mv)^2\frac{\pi}{2}\frac{\Omega_c^2}{\Delta\omega}\frac{\cos^2\theta}{|v_\parallel/c|^3}\Phi_n^2\frac{(-\sin^2\alpha + \omega_n/\omega)^2}{|1-(\partial\omega/\partial k_\parallel)_x/v_\parallel|}G_1G_2 \tag{75}$$

we break the terms down as follows. Note that ω is the resonant solution in the innermost loop

- Constant factors,
$$\frac{\pi\Omega_c^2c^3m^2}{2B_0^2}$$
This m^2 I can't match with Lyons though, see above....
- Particle parallel momentum $v_\parallel$,
$$\frac{1}{v_\parallel^3}$$
- Particle pitch angle α, which gives the
$$v^2 = \frac{v_\parallel^2}{\cos^2\alpha}$$
- Wave normal angle, which we keep as $x = \tan\theta$ and integrate over and includes $x\cos^2\mathrm{atan}(x)$
- Resonance number, n
- Resonant omega solution, ω in the above equation and dictating anything containing μ also, i.e.
$$\Phi_n^2\frac{(-\sin^2\alpha + \omega_n/\omega)^2}{|1-(\partial\omega/\partial k_\parallel)_x/v_\parallel|}(G_1B_{wave}^2/\Delta\omega)G_2$$

It's also useful to have some estimates of the sizes of the various terms:

- $$\frac{\pi\Omega_c^2c^3m^2}{2B_0^2} = 0.5\pi q^2c^3 \simeq 1e-12$$

- $B_{wave}^2 \sim (100\text{pT})^2 \sim 1e-20T$
- Alternately we combine as $0.5\pi\Omega_c^2 c^3 m^2 \simeq 1e-26$ and $B_{wave}^2/B_0^2 \simeq 1e-6$ (for 100pT waves and 100nT background)
- $$|1 - (\partial\omega/\partial k_\parallel)_x/v_\parallel| \simeq 1$$
- $\Phi \sim 1$
- $G_1 \sim 1$
- G_2

9.2 Limiting n

Since we're working from simulation data we have fairly easy to define limits on $k_\parallel$, and once we have a max for that we can limit n for each $v_\parallel$ and α. From the resonance condition, we know that

$$n_{max} = \frac{(\omega - k_{\parallel,max} v_\parallel)\gamma}{s\Omega_c}$$

and that we're considering ω from 0 to Ω_c

10 Really important subtleties that often get glossed over

10.1 What D is actually a function of, and the various coordinate transforms

In both Lyons and Albert D is given without arguments specified, so one probably has to infer based on Lyons first equation and the subscripts, as well as the listed transforms to D_{EE} that it's v, α although they helpfully use $v_\parallel$ on the RHS not $v\cos\alpha$.

10.2 Arguments of Bessel functions in Phi

In Lyons (between Eq 3 and 4) we have that the Bessel function args Big-Theta or Phi are

$$k_\perp v_\perp/\Omega_l$$

Relativistically, this becomes

$$\frac{k_\perp p_\perp}{m\Omega_l}$$

and note that $p = \gamma m v$ Note Ω_l is signed!. Albert converts these to

$$nx\tan\alpha(\omega - \omega_n)/\omega_n$$

which DOES NOT WORK for $n = 0$. The conversion is done using the resonance condition to get $k_\parallel v_\parallel = \omega - \omega_n$ along with $k_\perp = k_\parallel x$ and $v_\perp = v_\parallel \tan\alpha$ and finally replacing $\Omega_l = s\Omega_c = \omega_n\gamma/n$. So we can simply cancel the n.

However we note there might be some subtlety for Landau $n = 0$ resonance.

10.3 Sign of cyclotron frequency

Lyons considers a signed Ω_l but an unsigned Ω_e. Albert has only unsigned Ω_c.

10.4 Sign of resonant omega and multiple root counting

The previous subsection and the fact that negative wave frequency ω is mostly ignored means I don't know which solutions should be considered. In the resonance condition

$$\omega - k_\| v_\| = -\frac{n|\Omega_c|}{\gamma}$$

we get plus and minus signs for the same omega value for pairwise sign swaps of $k_\|$, $v_\|$ and n. It's not clear in Albert whether these things should be signed or whether sign is absorbed into angle, but in Lyons with the $\perp\|$ breakdown both must be signed.

11 Bounce averaging of D

The equations for bounce averaging the various D components are:

$$\langle D_{\alpha\alpha}\rangle = \frac{1}{\tau_B}\int_0^{\tau_B} D_{\alpha\alpha}(\alpha)\left(\frac{\partial\alpha_{eq}}{\partial\alpha}\right)^2 \mathrm{d}t \tag{76}$$

$$\langle D_{\alpha p}\rangle = \frac{1}{\tau_B}\int_0^{\tau_B} D_{\alpha p}(\alpha)\left(\frac{\partial\alpha_{eq}}{\partial\alpha}\right) \mathrm{d}t \tag{77}$$

$$\langle D_{pp}\rangle = \frac{1}{\tau_B}\int_0^{\tau_B} D_{pp}(\alpha)\mathrm{d}t \tag{78}$$

with τ_B the bounce period of the particle, $\tau_B = 4R_0 S(\alpha_{eq})/v_0$ and $S(\alpha_{eq}) \simeq 1.3 - 0.56\sin\alpha_{eq}$ Using a dipole field, we have the field strength and particle pitch angles given by

$$B = B_{eq} f(\lambda) \tag{79}$$

$$f(\lambda) = \frac{(1+3\sin^2\lambda)^{1/2}}{\cos^6\lambda} \tag{80}$$

$$\sin^2\alpha = f(\lambda)\sin^2\alpha_{eq} \tag{81}$$

Then the time integrals can be converted to latitude integrals using

$$v_0\cos\alpha\mathrm{d}t = R_0\cos\lambda(1+3\sin^2\lambda)^{1/2}\mathrm{d}\lambda \tag{82}$$

although the $\cos\alpha$ factor makes them potentially degenerate at the mirror point. The equations are then

$$\langle D_{\alpha\alpha}\rangle = \frac{1}{S(\alpha_{eq})}\int_0^{\lambda_m} D_{\alpha\alpha}(\alpha)\frac{\cos\alpha\cos^7\lambda}{\cos^2\alpha_{eq}}\mathrm{d}\lambda \tag{83}$$

$$\langle D_{\alpha p}\rangle = \frac{1}{S(\alpha_{eq})}\int_0^{\lambda_m} D_{\alpha p}(\alpha)\frac{\sin\alpha\cos^7\lambda}{\sin\alpha_{eq}\cos\alpha_{eq}}\mathrm{d}\lambda \tag{84}$$

$$\langle D_{pp}\rangle = \frac{1}{S(\alpha_{eq})}\int_0^{\lambda_m} D_{pp}(\alpha)\frac{\sin^2\alpha\cos^7\lambda}{\sin^2\alpha_{eq}\cos\alpha}\mathrm{d}\lambda \tag{85}$$

NB the third equation has a $\cos\alpha$ in the denominator and is thus ill defined at the mirror point λ_m Using 80 and 80 we can rewrite these as

$$\langle D_{\alpha\alpha}\rangle = \frac{1}{S(\alpha_{eq})}\int_0^{\lambda_m} D_{\alpha\alpha}(\alpha)\frac{\cos\alpha\cos^7\lambda}{\cos^2\alpha_{eq}}\mathrm{d}\lambda \tag{86}$$

$$\langle D_{\alpha p}\rangle = \frac{1}{S(\alpha_{eq})}\int_0^{\lambda_m} D_{\alpha p}(\alpha)\frac{\cos^4\lambda(1+3\sin^2\lambda)^{1/4}}{\cos\alpha_{eq}}\mathrm{d}\lambda \tag{87}$$

$$\langle D_{pp}\rangle = \frac{1}{S(\alpha_{eq})}\int_0^{\lambda_m} D_{pp}(\alpha)\frac{\cos\lambda(1+3\sin^2\lambda)^{1/2}}{\cos\alpha}\mathrm{d}\lambda \tag{88}$$

11.1 Analytic integrals for bounce-average testing

We can substitute a suitable form of D to allow the analytic evaluation of the bounce averaging. While we can always consume the entire α dependence that way, it's more useful to evaluate slightly different things in each case. For the $\alpha\alpha$ coefficient we use the original Eq 83:

$$D_{\alpha\alpha} = \frac{1}{\cos\alpha} \tag{89}$$

$$\langle D_{\alpha\alpha}\rangle = \frac{1}{S(\alpha_{eq})}\frac{1}{\cos^2\alpha_{eq}}\left[1225\sin\lambda + 245\sin 3\lambda + 49\sin 5\lambda + 5\sin 7\lambda\right]/2240 \tag{90}$$

similarly the mixed coefficient:

$$D_{\alpha p} = \frac{1}{\sin\alpha} \tag{91}$$

$$\langle D_{\alpha p}\rangle = \frac{1}{S(\alpha_{eq})}\frac{1}{\cos\alpha_{eq}\sin\alpha_{eq}}\left[1225\sin\lambda + 245\sin 3\lambda + 49\sin 5\lambda + 5\sin 7\lambda\right]/2240 \tag{92}$$

For the pp we use the rewritten component Eq 88:

$$D_{pp} = \cos\alpha \tag{93}$$

$$\langle D_{pp}\rangle = \frac{1}{S(\alpha_{eq})}\frac{1}{6}\left[3\sin\lambda\sqrt{3\sin^2\lambda+1} + \sqrt{3}\sinh^{-1}(\sqrt{3}\sin\lambda)\right] \tag{94}$$

A Failing attempt to derive Lyons N factor using IBP

Since we have a term $k_\perp \frac{\partial k_\perp}{\partial x} = \frac{\partial (k_\perp^2)}{2\partial x}$ it suggests we can use integration by parts on this. We need

$$N(\omega) = \frac{2}{(2\pi)^2} \int_0^\infty g_\omega(\tan^{-1} x) \left| \frac{\partial k_\perp}{\partial x} k_\perp \right|_\omega \left| \frac{\partial k_\parallel}{\partial \omega} \right|_{k_\perp} dx \tag{95}$$

The other term is as in 3.

Using the standard expression for 3-term IBP, i.e.

$$\int_a^b uvdw = [uvw]_a^b - \int_a^b uwdv - \int_a^b vwdu \tag{96}$$

we put

$$u = \sqrt{x^2+1} \left(\frac{\partial \mu}{\partial \omega} + \frac{\mu}{\omega} \right) \bigg|_{k_\perp} \tag{97}$$

$$v = g(x) \tag{98}$$

$$dw = \frac{\partial}{\partial x} \left(\frac{x^2 \mu^2}{x^2+1} \right) \tag{99}$$

Then we need $\frac{\partial u}{\partial x}$. But we know that in u $x\mu/\sqrt{x^2+1}$ is constant. So

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left[\frac{x^2+1}{x} \left(\frac{\partial}{\partial \omega} \frac{x\mu}{\sqrt{x^2+1}} + \frac{x\mu}{\omega\sqrt{x^2+1}} \right) \right] \tag{100}$$

$$= \frac{\partial}{\partial x} \left(\frac{x^2+1}{x} \right) \left(\frac{\partial}{\partial \omega} \frac{x\mu}{\sqrt{x^2+1}} + \frac{x\mu}{\omega\sqrt{x^2+1}} \right) \tag{101}$$

And w is trivially integrated. Now taking terms one by one in 96 and assuming well behaved $\mu, g(x)$ and $g(x) \to 0, x \to \infty$ and symmetry of g, hence antisymmetry of dg/dx:

$$[uvw] = 0 \tag{102}$$

$$\int uwdv = \int f(x^2) \frac{dg}{dx}$$

$$= 0 \; \left(f(x^2) \text{ symmetric by definition } \frac{dg}{dx} \text{ antisymmetric} \right) \tag{103}$$

$$\int vwdu = \int g(x) \frac{\mu^2(x^2-1)x}{(x^2+1)^{3/2}} \left(\frac{\partial \mu}{\partial \omega} + \frac{\mu}{\omega} \right) dx \tag{104}$$

Then the full expression becomes

$$N(\omega) = \frac{\omega^3}{(2\pi)^2 c^3} \int_{-\infty}^\infty g_\omega(\text{atan}x) \frac{\mu(x)^2 x(x^2-1)}{(1+x^2)^{3/2}} \left| \frac{\partial \mu}{\partial \omega} + \frac{\mu}{\omega} \right| dx. \tag{105}$$

******* NB NB NB NB We expected instead this:

$$N(\omega) = \frac{\omega^3}{(2\pi)^2 c^3} \int_{-\infty}^\infty g_\omega(\text{atan}x) \frac{\mu(x)^2 x}{(1+x^2)^{3/2}} \left| \frac{\partial \mu}{\partial \omega} + \frac{\mu}{\omega} \right| dx. \tag{106}$$

and the other derivation appears to get this correctly

References

[1] J. M. Albert. Evaluation of quasi-linear diffusion coefficients for whistler mode waves in a plasma with arbitrary density ratio. *JGR (Space Physics)*, 110:A03218, March 2005.