Optimal Growth I: The Stochastic Optimal Growth Model

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Contents

Introduction

- The Stochastic Optimal Growth Model involves the study of how economic agents allocate consumption and capital each period when production is hit with random shocks.
- This model is useful because it captures how productivity shocks impact an agent's decision about consumption in the future.
- In recent times, the extensions of the stochastic growth model have been featured in representing economic problems in business cycles and asset pricing.

Introduction

We are going to

- Discuss the model outline which entails how an agent makes their consumption decisions based on their resources available.
- Review topics in mathematics that we will need in order to solve the model.
- Explain how to solve the model.

Model Outline

- Consider a single sector infinite horizon economy populated with a single agent.
- An agent born in period t is endowed with $y_t \geq 0$ units of a consumption good (and none at other times).
- After receiving the endowment, the agent makes a decision of how much to consume and invest.
- The amount invested is directly transformed into capital and will be used to produce output which is available to the agent at the beginning of period $t + 1$.

Model Outline

- Denote consumption and capital at time t as c_t and k_{t+1}, respectively. We write the agent's resource constraint:

$$c_t + k_{t+1} \leq y_t \quad \forall t \in \{0, 1, 2, ...\}. \tag{1}$$

Model Outline

- Consider a production function $f : \mathbb{R}_+ \to \mathbb{R}_+$,
- where f is increasing and continuous on $\mathbb{R}_+$.

$$y_{t+1} = f(k_{t+1}, \epsilon_{t+1}) \tag{2}$$

- The production process works as follows: the agent chooses what to consume and what to invest in production in period t, then before the end of the period, productivity is hit with a stochastic shock.
- Denote this productivity shock at time t as ϵ_{t+1}.
- $\{\epsilon_t\}_{t=0}^{\infty}$ is $i.i.d$ and each ϵ_t follows a common distribution denoted as ϕ.

Model Outline

- Preferences are represented by a utility function $u : \mathbb{R}_+ \to \mathbb{R}$.
- Where $u(\cdot)$ is bounded, continuous, and strictly increasing on $\mathbb{R}_+$.
- The agent's goal is to choose a consumption profile $\{c_t\}_{t=0}^{\infty}$ so that they maximize their lifetime utility.

Model Outline

- Given y_0, the agent solves the following problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{3}$$

subject to

$$c_t + k_{t+1} \leq y_t \tag{4}$$

$$y_{t+1} = f(k_{t+1}, \epsilon_{t+1}) \tag{5}$$

- $\beta \in (0,1)$ represents the discount factor by which utility is discounted.

Mathematics Review

- Mappings and Operators
  - A **mapping**, also known as a function, is a relation that pairs an element from its domain with an element from its range.
  - An **operator**, on the other hand, is mapping of a function to a function.

A Simple Example

Suppose that f is a mapping with domain and range $\mathbb{R}$. We write $f : \mathbb{R} \rightarrow \mathbb{R}$. The mapping f can be represented by a function, say, $f(x) = x^2$.

Next, we define an operator F such that $F : \mathbb{R} \rightarrow \mathbb{R}$. For simplicity, suppose that $F(f(x)) = x$. That is, the operator F maps the function x^2 to some simple linear function x.

Mathematics Review

- The Supremum of a Set
    - The **supremum** of a set, denoted as *sup*, is the least upper bound of a set.

Example #1

Let $A = \{1, 2, 3\}$. Then the supremum of the set A, $\sup A$, is 3.

Example #2

Let $B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then $\sup B = 1$.

Mathematics Review

- The Expectation of a Continuous Random Variable
  - Let X be a continuous random variable with probability density function $f(x)$. We have the following properties:

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx \tag{6}$$

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx \tag{7}$$

Example

Let $X \sim N(0,1)$ and suppose that $g(X) = X^2$. Then we have

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx = 0$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)dx = 1$$

The Value Function Approach

- We assume that it is possible to to calculate the value of the discounted value of utility that the agent receives when solving the optimization problem showed in the model outline.
- The variable y_t is considered a **state variable** since it describes the state of the model economy in period t.
- We only choose c_t as the **control variable** because the agent chooses how much to consume in each period and after we solve for c_t we can pin down the value of k_{t+1} by using the agent's resource constraint.

The Value Function Approach

- We are looking for an optimal policy function, σ, that maps y_t to a decision c_t. That is,

$$\sigma : \mathbb{R} \rightarrow \mathbb{R} \tag{8}$$

  - $\sigma(y_t) = c_t$
  - $\sigma(\cdot)$ is called a **Markov policy**.
  - $0 \leq \sigma(y_t) \leq y_t \implies$ such a policy is feasible.

- The goal is to find the optimal σ that maximizes the agent's lifetime utility.

The Value Function Approach

- Using the fact that $\sigma(y_t) = c_t$ we write the agent's lifetime utilty in the following way:

$$V_\sigma(y_0) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(\sigma(y_t))\right] \tag{9}$$

- Next, we write the **Value Function**:

$$V^*(y_0) := \sup_{\sigma \in \Sigma} V_\sigma(y_0) \tag{10}$$

  - Note 1: Σ denotes the set of all feasible consumption policies.
  - Note 2: we are looking for a σ^* that optimizes equation (9).

The Value Function Approach

- Using the definition of the supremum of a set (as mentioned on slide 11) we re-write equation (10)

$$V(y_t) = \max_{\{c_{t+i}\}_{i=0}^{\infty}} \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i u(c_{t+i}) \right] \tag{11}$$

- To reduce the dimensionality of the problem, we use recursion to simplify equation (11)

$$V(y_t) = \max_{c_t} \left[u(c_t) + \beta \mathbb{E}_t V(y_{t+1}) \right] \tag{12}$$

The Value Function Approach

- Substituting the following into equation (12)
  - $c_t = \sigma(y_t)$
  - $y_{t+1} = f(k_{t+1}, \epsilon_{t+1}) = f(y_t - \sigma(y_t), \epsilon_{t+1})$
- and using the property of expected value shown in equation (7) gives us the **Bellman Equation**.

$$V(y_t) = \max_{0 \leq \sigma(y_t) \leq y_t} \left[u(\sigma(y_t)) + \beta \int V(f(y_t - \sigma(y_t), z))\phi(z)dz \right] \tag{13}$$

- Note that $\phi(z)$ denotes the probability density function of ϵ_{t+1}

The Fitted Value Algorithm

- Next, we want to solve the bellman equation. This will be done using the **Fitted Value Iteration Algorithm**.
- We will closely follow the algorithm provided by Sargent and Stachurski (2016).
- Step 1: Consider a finite dimensional vector whose elements are those associated with some function evaluated at points $\{y_1, \ldots, y_n\}$. Set $(g_1, \ldots, g_n) = (g(y_1), \ldots, g(y_n))$ and set a tolerance limit on the random shock ϵ_{t+1} (Sargent and Stachurski, 2016).

The Fitted Value Algorithm

- Step 2: Using the data generated, construct an approximation of the value function, $\tilde{V}$, by using linear interpolation (Sargent and Stachurski, 2016).

A Simple Example on Linear Interpolation

- Suppose we have two data points: $(x_0, y_0) = (5, 3)$, and $(x_1, y_1) = (2, 6)$. Then we use the following formula to estimate a function between these two points:

$$y = y_0 + (y_1 - y_0)\frac{x - x_0}{x_1 - x_0} \tag{14}$$

- Using the points above we get that the estimated function between these points is:

$$y = 8 - x \tag{15}$$

The Fitted Value Algorithm

- Step 3: Define the Bellman equation by first defining an operator, F, that takes the value function approximated in step 2 and maps it to the Bellman equation shown in equation (13). This "Bellman operator" returns the Bellman equation (Sargent and Stachurski, 2016).
- Step 4: Repeatedly solve the Bellman equation from step 3 for values $\{y_1, \ldots, y_n\}$ and store the results into an array $(F(\tilde{V}(y_1)), \ldots, F(\tilde{V}(y_n)))$ (Sargent and Stachurski, 2016).
- Step 5: If the tolerance limit is reached, then stop and return the optimal policy from the array in step 4; otherwise, set $(g_1, \ldots, g_n) = (F(\tilde{V}(y_1)), \ldots, F(\tilde{V}(y_n)))$ and repeat steps 2 through 5 (Sargent and Stachurski, 2016).

The Fitted Value Algorithm

Intuition

- From steps 1 and 2 we get our estimated value function $\tilde{V}$.
- In step 3 we define the bellman operator F
- Then solve for values of $(y_1, \ldots, y_n)$ and store into an array $(F(\tilde{V}(y_1), \ldots, F(\tilde{V}(y_n))$.
- If this process is repeated we get the following sequence: $\{\tilde{V}, F\tilde{V}, F^2\tilde{V}, ...\}$. Then, we use the fact that this sequence converges to the optimal value V^*.
- This is known as the fitted value iteration technique.

Example & Output from Python

Example on page 396 from Sargent and Stachurski (2016)

- $y_t = k_t^\alpha$
- $u(c_t) = \ln(c_t)$
- ϕ represents the distribution of $e^{\mu + s\epsilon}$, where $\epsilon \sim N(0,1)$.
- The solution is

$$V^*(y) = \frac{\ln(1-\beta\alpha)}{1-\beta} + \frac{\mu + \alpha\ln(\alpha\beta)}{1-\alpha} + \frac{1}{1-\alpha\beta}\ln(y) \tag{16}$$

- The optimal consumption policy is found to be

$$\sigma^*(y) = (1-\alpha\beta)y \tag{17}$$

Example & Output from Python

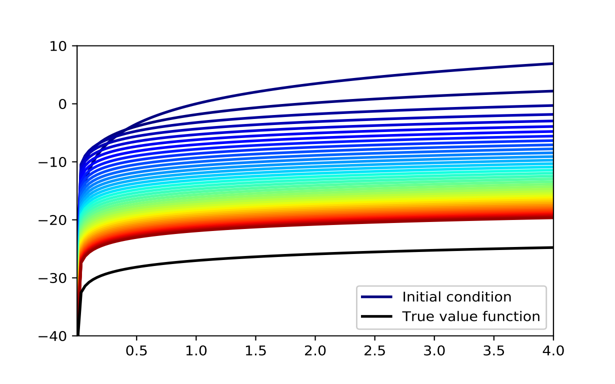
Example (continued)

- Assume that $\alpha = 0.6$, $\beta = 0.96$, $\mu = 0$, $s = 0.1$, and the shock size is 250.
- To start off, we consider the initial condition to be $V(y) = 5\ln(y)$. First we want to check that when we repeatedly apply the Bellman Operator we converge closely to the true value function $V^*(y)$.

Example & Output from Python

Example (continued)

- After running the code we get the following output:

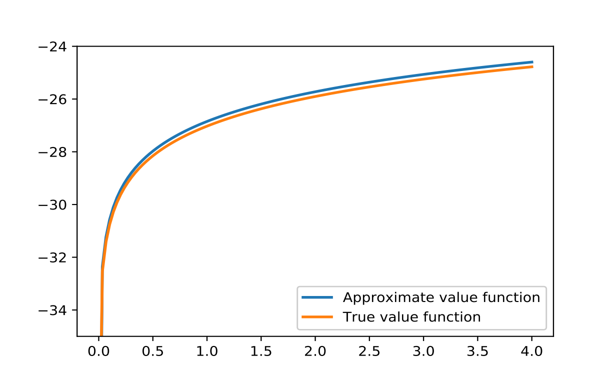


- Clearly we are converging closer and closer to the true value of $V^*(y)$ (the black line).

Example & Output from Python

Example (continued)

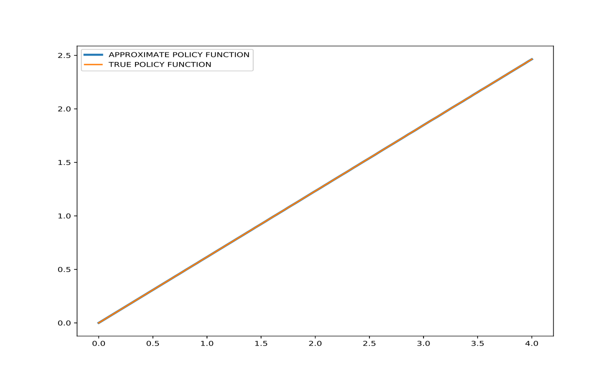
- Next we write a function that iterates until a certain tolerance limit is reached. Our estimation is very close to the true value function (as seen in the figure below).



Example & Output from Python

Example (continued)

- Now we proceed to estimate the optimal consumption policy. Graphically it is seen that our estimation of $\sigma^*(y)$ is identical to the true value of $\sigma^*(y)$.



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