1 Black-Scholes Dynamics

This section will introduce the Black-Scholes model for stock price dynamics modified to account for stock dividends. Further, the idea of a risk-free savings account will be developed as it is necessary to complete the derivation of an analytic option pricing formula.

1.1 Stochastic Differential Equation

The dynamics of a stock price at time T are given by the following equation known as the Black-Scholes SDE:

$$dS_T = rS_t dt + \sigma S_t dW_t$$

This equation can be modified to the following form to account for a stock's dividend yield, if any:

$$\boldsymbol{dS_T = (r - q)S_t dt + \sigma S_t dW_t} \tag{1}$$

Using Ito's Lemma, the SDE can be solved for the stock price as given by the following equation:

$$\boldsymbol{S_T = S_t \exp\left\{(r - q - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t)\right\}} \tag{2}$$

The stochastic process S is the geometric Brownian motion and is therefore log-normally distributed at all times t with the Wiener increment W. This denotes the stochastic volatility term as the product σW_{T-t}. q is the annually quoted dividend yield of the stock S. r represents the continuously compounded interest rate.

1.2 Risk-free Savings Account

Assuming r is constant, a risk-neutral savings account can be denoted as:

$$\boldsymbol{\beta_T = \beta_t e^{r(T-t)}} \tag{3}$$

The importance of defining a risk-neutral measure is that we will use this to change to an equivalent probability measure in the pricing technique used in the following section.

1.3 Historical Volatility

Historical volatility **(HV)** is a measure of the previously realized volatility of an asset over a given period of time. For this model, this measure of volatility refers to the underlying asset. Being that this is an observed value, it is straight forward to calculate the historical volatility of the underlying asset. The **HV** is just the standard deviation of the log returns series given by the asset. This can be calculated over any interval of time, but it is important to note that this measure can be sensitive to extreme values that occur during abnormal market conditions. Furthermore, extending the time interval too far into the past may lead to issues of data relevancy since the older the price information, the less relevant it may be to the future. The last thing to note on **HV** is that for the purposes of pricing options, this is not the value to be used as the σ parameter in the Black-Scholes formula. In practice, the market provides prices and all other parameters except for σ. Therefore, another measure must be estimated using what parameters are available given the market data.

1.4 Implied Volatility

Implied volatility **(IV)** is not a given feature of the option contract nor is it directly observable. In the market, the price and other parameters will be available and the **IV** must be numerically estimated. This can be implemented in various ways, but in `European_Option.py` a root search using `SciPy.optimize.root_scalar` is performed. This uses the van Wijngaarden-Dekker-Brent method as the minimization algorithm by taking in all factors of the option, including a zeroed-out pricing function and the market price. The altered pricing function is the difference between the Black-Scholes pricing formula and the observed price. Minimizing this difference is done by altering σ to wherever the function evaluates to zero. Getting as close to zero as possible will approximate the solution for **IV**.

As the implied volatility considers the option's realized price in the market, it offers unique information about the option. **IV** is often called forward-looking because it captures the market expectations about future moves. When markets are bullish, the implied volatility is typically lower. It is a forecast of how prices will probabilistically behave in the future base on supply and demand.

2 Vanilla European Options

2.1 Payoff Function

Now we will express the payoff of European Call and Put Options as the difference between the stock price at maturity and the strike price. Given that there is a right not to exercise, the payoff is maximum of this difference and zero. For a Call Option:

$$\boldsymbol{C(t,S) = \max[S_T - K, 0]} \tag{4}$$

Similarly, for a Put Option:

$$\boldsymbol{P(t,S) = \max\left[K - S_T, 0\right]} \tag{5}$$

Equation (4) can be solved for a closed-form option pricing solution using the stock price dynamics defined for the asset S. There are many techniques to obtain this solution including Gaussian Shift Theorem or Fourier Transform, but here we take the Martingale approach to employ the Change of Measure technique.

2.2 Pricing Solution

First, we take the expected value of the discounted payoff under the risk-neutral probability measure β (the risk-free savings account) as:

$$C(t,S) = \mathbb{E}_\beta\left[e^{-r(T-t)}\max[S_T - K, 0]|\mathcal{F}_t\right]$$

$$= e^{-r(T-t)}\left(\mathbb{E}_\beta\left[S_T\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right] - K\mathbb{E}_\beta\left[\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right]\right)$$

$$= e^{-r(T-t)}\left(\mathbb{E}_\beta\left[S_T\mathbb{I}_{\{S_T>K\}}|\mathcal{F}_t\right] - KQ_\beta\{\ln(S_T) > \ln(K)\}\right)$$

The second term can be evaluated directly by taking the difference of the stock price dynamics given by $\ln(S_T)$ and the log-strike, $\ln(K)$, under the probability measure Q_β. Where Φ denotes the Normal distribution:

$$\ln(S_T) \overset{Q_\beta}{\sim} \Phi\left(\ln(S_t) + (r - q - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)\right)$$

Note that in the above equation, the diffusion term, $\sigma(W_T - W_t)$, has been evaluated to $\sigma^2(T-t)$ by Ito's Isometry.

$$KQ_\beta\{\ln(S_T) > \ln(K)\} = K\Phi\left(\frac{\ln(\frac{S_t}{K}) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}\right)$$

The first term requires changing to an equivalent Martingale measure to cancel the S_T in the expectation. This further requires Girsanov's Theorem and the use of an appropriate Radon/Nikodym derivative such that the product of the Radon/Nikodym derivative and an appropriate correction term is the exponential Martingale. Thereby, the new Wiener increment given by applying Girsanov's theorem will be a standard Brownian motion under the new measure. First, note that the following expression does not give the exponential Martingale:

$$\frac{S_T\beta_t}{S_t\beta_T} = e^{-r(T-t)}\exp\left\{\sigma(W_T - W_t) + r(T-t) - q(T-t) - \frac{1}{2}\sigma^2(T-t))\right\}$$

To obtain the exponential Martingale, a correction term is needed:

$$\frac{d\hat{Q}}{dQ_\beta} = \underbrace{\overbrace{e^{q(T-t)}}^{\substack{\textbf{Correction}\\ \textbf{Term}}}\ \frac{S_T\beta_t}{S_t\beta_T}}_{\substack{\textbf{Radon/Nikodym}\\ \textbf{Derivative}}} = \underbrace{\exp\left\{\sigma(W_T - W_t) - \frac{1}{2}\sigma^2(T-t))\right\}}_{\textbf{The Exponential Martingale}}$$

Applying Girsanov's Theorem:

$$d\hat{W}_t = dW_t - \sigma dt$$

The above equation changes the stock prices dynamics of S by using the substitution $dW_t = d\hat{W}_t + \sigma dt$:

$$S_T = S_t((r - q + \sigma^2)dt + \sigma d\hat{W}_t)$$

Likewise,

$$\mathbf{S_T} = \mathbf{S_t}\exp\left\{(\mathbf{r} - \mathbf{q} + \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}})(\mathbf{T} - \mathbf{t}) + \sigma(\mathbf{d\hat{W}_T} - \mathbf{d\hat{W}_t})\right\} \tag{6}$$

Another way of describing the Radon/Nikodym Derivative is that it is the factor by which the probability measure dQ_β is equal to $d\hat{Q}$. By substituting out dQ_β, we can change the measure of the expectation into something that can be evaluated. Finally we can apply the measure change to the first term as:

$$\mathbb{E}_\beta \left[e^{-r(T-t)} S_T \mathbb{I}_{\{S_T>K\}} | \mathcal{F}_t\right] = \int_\Omega e^{-r(T-t)} S_T \mathbb{I}_{\{S_T>K\}} dQ_\beta$$

Given that:

$$dQ_\beta = e^{-q(T-t)} \frac{S_t \beta_T}{S_T \beta_t} d\hat{Q}$$

We can substitute out dQ_β to arrive at:

$$= \int_\Omega e^{-r(T-t)} S_T \mathbb{I}_{\{S_T>K\}} e^{-q(T-t)} \frac{S_t \beta_T}{S_T \beta_t} d\hat{Q}$$

$$= e^{q(T-t)} S_t \hat{\mathbb{E}} \left[\mathbb{I}_{\{S_T>K\}}\right] = e^{-q(T-t)} S_t \hat{Q}\{S_T > K\}$$

Using equation (5) we can evaluate this as follows:

$$\ln(S_T) \overset{\hat{Q}}{\sim} \Phi\left(\ln(S_t) + (r - q + \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)\right)$$

$$\hat{Q}\{S_T > K\} = \Phi\left(\frac{\ln(\frac{S_t}{K}) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}\right)$$

Collecting terms, the price of the European Call Option is:

$$\mathrm{d1} = \frac{\ln(\frac{S_t}{K}) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

$$\mathrm{d2} = \frac{\ln(\frac{S_t}{K}) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}$$

$$\boxed{\mathbf{C(t,S) = e^{-q(T-t)} S_t \Phi(d1) - e^{-r(T-t)} K \Phi(d2)}} \tag{7}$$

By exploiting the symmetry of the payoff equation, the price of a European Put Option can be calculated similarly as:

$$\boxed{\mathbf{P(t,S) = e^{-r(T-t)} K \Phi(-d2) - e^{-q(T-t)} S_t \Phi(-d1)}} \tag{8}$$

3 Deriving the Traditional Greeks

Option Greeks are a group of several sensitivity measures derived from the pricing equation. These are used to calculate and manage risk exposure by hedging portfolios against a given Greek. For instance, Delta-hedging is performed by taking an offsetting position at discrete time points to approximately neutralize any risk generated by Delta. In the following section, we will derive the traditional option Greeks as well as several higher order Greeks.

3.1 Delta

The Delta of an option measures the sensitivity of the price to changes in the underlying. Its interpretation is the change in price relative to a 1 dollar change in the price of the underlying. This is calculated by taking the partial derivative of the Call pricing function with respect to S_t.

$$\frac{\partial C}{\partial S} = \frac{\partial}{\partial S}(e^{-q(T-t)}S_t\Phi(d1) - e^{-r(T-t)}K\Phi(d2))$$

$$= e^{-q(T-t)}\Phi(\mathrm{d1}) + S_te^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial S} - Ke^{-r(T-t)}\Phi'(\mathrm{d2})\frac{\partial(\mathrm{d2})}{\partial S}$$

Now, the goal is to change the third term to prove that it actually cancels out the second term. Taking advantage of the fact that we are differentiating with respect to S_t, observe that:

$$\frac{\partial(\mathrm{d1})}{\partial S} = \frac{\partial(\mathrm{d2})}{\partial S}$$

So that we can get:

$$= e^{-q(T-t)}\Phi(\mathrm{d1}) + S_te^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial S} - Ke^{-r(T-t)}\Phi'(\mathrm{d2})\frac{\partial(\mathrm{d1})}{\partial S}$$

Next, we can further alter the third term by exploiting the following identity:

$$\mathrm{d2} = \mathrm{d1} - \sqrt{\sigma^2(T-t)}$$

Where:

$$\Phi'(\mathrm{d2}) = \Phi'\left(\mathrm{d1} - \sqrt{\sigma^2(T-t)}\right)$$

is the Probability Density Function (PDF) of the Normal Distribution:

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\mathrm{d1}-\sqrt{\sigma^2(T-t)}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d1^2-\frac{1}{2}\sigma^2(T-t)+\mathrm{d1}\sqrt{\sigma^2(T-t)}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d1^2} e^{-\frac{1}{2}\sigma^2(T-t)+\mathrm{d1}\sqrt{\sigma^2(T-t)}}$$

$$= \Phi'(\mathrm{d1}) e^{-\frac{1}{2}\sigma^2(T-t)+\mathrm{d1}\sqrt{\sigma^2(T-t)}}$$

Substitute in the identity for d1:

$$= \Phi'(\mathrm{d1}) e^{-\frac{1}{2}\sigma^2(T-t)+\left(\frac{\ln(\frac{S_t}{K})+(r-q+\frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}\right)\sqrt{\sigma^2(T-t)}}$$

$$= \Phi'(\mathrm{d1}) \left(\frac{S_t}{K}\right) e^{(r-q)(T-t)}$$

Finally, we can plug the above term into the third term of the original equation to give:

$$= e^{-q(T-t)}\Phi(\mathrm{d1}) + S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial S} - K e^{-r(T-t)}\Phi'(\mathrm{d1})\left(\frac{S_t}{K}\right) e^{(r-q)(T-t)}\frac{\partial(\mathrm{d1})}{\partial S}$$

$$= e^{-q(T-t)}\Phi(\mathrm{d1}) + S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial S} - S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial S}$$

Finally, the second and third terms cancel out to give the solution for the Delta of a European Call as:

$$\boxed{\mathbf{\Delta}_{\text{Call}} = \mathbf{e^{-q(T-t)}\Phi}(\text{d1})} \tag{9}$$

The solution for a the Delta of a Put Option is taken in the exact same way as the partial derivative of the Put pricing function with respect to S_t:

$$\frac{\partial P}{\partial S} = -\left[= e^{-q(T-t)}\Phi(\text{d1}) + S_t e^{-q(T-t)}\Phi'(\text{d1})\frac{\partial(\text{d1})}{\partial S} - Ke^{-r(T-t)}\Phi'(\text{d2})\frac{\partial(\text{d2})}{\partial S}\right]$$

$$\boxed{\mathbf{\Delta}_{\text{Put}} = -\mathbf{e^{-q(T-t)}\Phi}(\text{-d1})} \tag{10}$$

3.2 Gamma

Gamma of an option measures the sensitivity of Delta to changes in the price of the underlying asset. This mathematically means that Gamma is a second order measurement of the option pricing function differentiated again with respect to the underlying asset price. Given that we already have the first partial, we can just differentiate the formula for Delta with respect to S_t. Note that the solutions for Gamma will be the same for both Calls and Puts.

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\Delta_{\text{Call}}$$

$$= e^{-q(T-t)}\frac{\partial}{\partial S}\Phi(\text{d1}) = e^{-q(T-t)}\Phi'(\text{d1})\frac{\partial}{\partial S}(\text{d1})$$

Finally, evaluating the partial of d1 with respect to S_t gives the solution for Gamma of both European Call and Put Options:

$$\boxed{\mathbf{\Gamma} = \frac{\mathbf{e^{-q(T-t)}}}{\mathbf{S_t}\sqrt{\sigma^{\mathbf{2}}\mathbf{(T-t)}}}\mathbf{\Phi'}(\text{d1})} \tag{11}$$

3.3 Vega

An option's Vega is interpreted as the sensitivity of the the option's price to changes in volatility. The change in price and sigma are positively correlated meaning that they increase and decrease together. Vega is the same for both Calls and Puts and will always be positive for any option. It is calculated by taking the partial derivative of the pricing function with respect to σ.

$$\frac{\partial C}{\partial \sigma} = e^{-q(T-t)} S_t \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) - K e^{-r(T-t)} \Phi'(\text{d2}) \frac{\partial}{\partial \sigma}(\text{d2})$$

By using the same techniques we used when calculating Delta to alter terms with d2 into terms with d1 we have:

$$= e^{-q(T-t)} S_t \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) - K e^{-r(T-t)} \Phi'(\text{d1}) \left(\frac{S_t}{K}\right) e^{(r-q)(T-t)} \frac{\partial}{\partial \sigma}\left(\text{d1} - \sqrt{\sigma^2 (T-t)}\right)$$

$$= e^{-q(T-t)} S_t \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) - S_t e^{-q(T-t)} \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}\left(\text{d1} - \sqrt{\sigma^2 (T-t)}\right)$$

$$= e^{-q(T-t)} S_t \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) - S_t e^{-q(T-t)} \Phi'(\text{d1}) \left(\frac{\partial}{\partial \sigma}(\text{d1}) - \frac{\partial}{\partial \sigma}\sqrt{\sigma^2 (T-t)}\right)$$

$$= e^{-q(T-t)} S_t \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) - S_t e^{-q(T-t)} \Phi'(\text{d1}) \left(\frac{\partial}{\partial \sigma}(\text{d1}) - \sqrt{(T-t)}\right)$$

$$= e^{-q(T-t)} S_t \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) - S_t e^{-q(T-t)} \Phi'(\text{d1}) \frac{\partial}{\partial \sigma}(\text{d1}) + S_t e^{-q(T-t)} \sqrt{(T-t)} \Phi'(\text{d1})$$

The first two terms of the above equation will cancel to give the solution for Vega of European Call and Put Options:

$$\boxed{\mathcal{V} = \mathbf{S_t e^{-q(T-t)}} \sqrt{\mathbf{(T-t)}} \mathbf{\Phi'}(\text{d1})} \tag{12}$$

3.4 Theta

Theta is known as an option's time decay and measures the sensitivity of the option's price to its time to maturity. Theta will have a different function for Calls and Puts. It is calculated by taking the partial derivative of each pricing function with respect to the time until maturity, T.

$$\frac{\partial C}{\partial T} = S_t \frac{\partial}{\partial T}\left(e^{-q(T-t)}\Phi(\mathrm{d1})\right) - K\frac{\partial}{\partial T}\left(e^{-r(T-t)}\Phi(\mathrm{d2})\right)$$

The derivation for Theta will be the most complex so far as the equations can get quite long. Upon distributing the partial derivative operator and evaluating with product rule we obtain:

$$= S_t\left(e^{-q(T-t)}\frac{\partial}{\partial T}\Phi(\mathrm{d1}) + \frac{\partial}{\partial T}e^{-q(T-t)}\Phi(\mathrm{d1})\right)$$
$$-K\left(e^{-r(T-t)}\frac{\partial}{\partial T}\Phi(\mathrm{d2}) + \frac{\partial}{\partial T}e^{-r(T-t)}\Phi(\mathrm{d2})\right)$$

We can once again use the identities from calculating Delta in order to alter the strike term and simplify the expression by canceling like terms. First, substitute in the identities used in the Delta derivation. Doing so we arrive at:

$$= S_t e^{-q(T-t)}\frac{\partial(\mathrm{d1})}{\partial T}\Phi'(\mathrm{d1}) + S_t q e^{-q(T-t)}\Phi(\mathrm{d1})$$
$$-Ke^{-r(T-t)}\Phi'\left(\mathrm{d1} - \sqrt{\sigma^2(T-t)}\right)\left(\frac{\partial \mathrm{d1}}{\partial T} - \frac{\partial\sqrt{\sigma^2(T-t)}}{\partial T}\right)$$
$$-Kre^{-r(T-t)}\Phi(\mathrm{d1} - \sqrt{\sigma^2(T-t)})$$

Which further simplifies to:

$$= S_t e^{-q(T-t)}\frac{\partial(\mathrm{d1})}{\partial T}\Phi'(\mathrm{d1}) + S_t q e^{-q(T-t)}\Phi(\mathrm{d1})$$
$$-S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial T} - S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\sigma}{2\sqrt{T-t}}$$
$$-Kre^{-r(T-t)}\Phi(\mathrm{d2})$$

Finally, after canceling like terms we have Theta for the European Call Option. Note that the final term leaves the prime notation to indicate the normal PDF.

$$\boxed{\mathbf{\Theta_{Call} = S_t q e^{-q(T-t)}\Phi(\text{d1}) - K r e^{-r(T-t)}\Phi(\text{d2}) - S_t e^{-q(T-t)}\Phi'(\text{d1})\frac{\sigma}{2\sqrt{T-t}}}} \tag{13}$$

For the European Put Option, Theta can be worked out the same way, however, the results are symmetric. Take the partial of the Put price funtion with respect to T.

$$\frac{\partial P}{\partial T} = -S_t\frac{\partial}{\partial T}\left(e^{-q(T-t)}\Phi(\text{-d1})\right) + K\frac{\partial}{\partial T}\left(e^{-r(T-t)}\Phi(\text{-d2})\right)$$

Evaluating in the same manner leads to:

$$= S_t e^{-q(T-t)}\frac{\partial(\text{d1})}{\partial T}\Phi'(\text{d1}) - S_t q e^{-q(T-t)}\Phi(\text{-d1})$$

$$-S_t e^{-q(T-t)}\Phi'(\text{-d1})\frac{\partial(\text{-d1})}{\partial T} - S_t e^{-q(T-t)}\Phi'(\text{-d1})\frac{\sigma}{2\sqrt{T-t}}$$

$$+K r e^{-r(T-t)}\Phi(\text{-d2})$$

By canceling terms, we have the solution for Theta of a European Put Option:

$$\boxed{\mathbf{\Theta_{Put} = -S_t q e^{-q(T-t)}\Phi(\text{-d1}) + K r e^{-r(T-t)}\Phi(\text{-d2}) - S_t e^{-q(T-t)}\Phi'(\text{d1})\frac{\sigma}{2\sqrt{T-t}}}} \tag{14}$$

3.5 Rho

Rho measures the sensitivity of the option price to changes in the interest rate. There are different formulas for both Put and Call options, but each are symmetric. To derive Rho, take the partial derivative of the pricing function with respect to the interest rate, r.

$$\frac{\partial C}{\partial r} = S_t e^{-q(T-t)}\Phi'(\text{d1})\frac{\partial(\text{d1})}{\partial r} - \left(K e^{-r(T-t)}\Phi'(\text{d2})\frac{\partial(d2)}{\partial r} - K(T-t)e^{-r(T-t)}\Phi(\text{d2})e^{-r(T-t)}\right)$$

$$= S_t e^{-q(T-t)}\Phi'(\text{d1})\frac{\partial(\text{d1})}{\partial r} - K e^{-r(T-t)}\Phi'(\text{d2})\frac{\partial(\text{d2})}{\partial r} + K(T-t)e^{-r(T-t)}\Phi(\text{d2})$$

Again, use the identities from the Delta derivation to simplify:

$$= S_t e^{-q(T-t)}\Phi'(\text{d1})\frac{\partial(\text{d1})}{\partial r} - Ke^{-r(T-t)}\Phi'(\text{d1})\left(\frac{S_t}{K}\right)e^{(r-q)(T-t)}\frac{\partial(\text{d1})}{\partial r}$$

$$+K(T-t)e^{-r(T-t)}\Phi(\text{d2})$$

Canceling terms gives the solution for Rho of a European Call:

$$\boxed{\rho_{\mathbf{Call}} = \mathbf{K}(\mathbf{T}-\mathbf{t})\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{\Phi}(\text{d2})} \tag{15}$$

Put Options have a symmetrical Rho that can be derived similarly.

$$\frac{\partial P}{\partial T} = -S_t e^{-q(T-t)}\frac{\partial}{\partial T}\left(\Phi(-\text{d1})\right) + K\frac{\partial}{\partial T}\left(e^{-r(T-t)}\Phi(\text{-d2})\right)$$

Simplifying the same way for the Call Option gives Rho as:

$$\boxed{\rho_{\mathbf{Put}} = -\mathbf{K}(\mathbf{T}-\mathbf{t})\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{\Phi}(\text{-d2})} \tag{16}$$

3.6 Lambda

Lambda is often referred to as the elasticity or leverage of the option. It measures the percentage change in price per the percentage change in the price of the underlying. Interestingly, Lambda for both Calls and Puts can be calculated as the respective Delta times the ratio of the underlying stock price to the value of the option. Note further that the product of lambda and the volatility of the underlying yields the volatility of the option. In terms of performing a derivation, we can use the previously derived Delta multiplied by the ratio of the price of the asset and the price of the option. Lambda for Calls and Puts are respectively given as:

$$\boxed{\lambda_{\mathbf{Call}} = \frac{\mathbf{e}^{-\mathbf{q}(\mathbf{T}-\mathbf{t})}\mathbf{S_t}\mathbf{\Phi}(\mathbf{d1})}{(\mathbf{e}^{-\mathbf{q}(\mathbf{T}-\mathbf{t})}\mathbf{S_t}\mathbf{\Phi}(\mathbf{d1}) - \mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{K}\mathbf{\Phi}(\mathbf{d2}))}} \tag{17}$$

$$\boxed{\lambda_{\mathbf{Put}} = \frac{-\mathbf{e}^{-\mathbf{q}(\mathbf{T}-\mathbf{t})}\mathbf{S_t}\mathbf{\Phi}(-\mathbf{d1})}{(-\mathbf{e}^{-\mathbf{q}(\mathbf{T}-\mathbf{t})}\mathbf{S_t}\mathbf{\Phi}(-\mathbf{d1}) + \mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{K}\mathbf{\Phi}(-\mathbf{d2}))}} \tag{18}$$

3.7 Epsilon

Epsilon measures the sensitivity of price to changes in the dividend yield. Note that the formula for row, as well as the derivation, will be very similar to Rho as the terms of r (for Rho) and q (for Epsilon) each take similar positions in the pricing formula. Like Rho, there will be two respective formulas for Epsilon of Call and Put options.

$$\frac{\partial C}{\partial q} = S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial q} - S_t(T-t)e^{-q(T-t)}\Phi(\mathrm{d1}) - Ke^{-r(T-t)}\Phi'(\mathrm{d2})\frac{\partial(\mathrm{d2})}{\partial q}$$

Once again, use the same techniques as Delta; that, in the final term, the partial $\frac{\partial(\mathrm{d2})}{\partial q}$ is equal to $\frac{\partial(\mathrm{d1})}{\partial q}$ and that we can simplify the PDF of d2 into terms of d1. Doing so gives:

$$Ke^{-r(T-t)}\Phi'(\mathrm{d2})\frac{\partial(\mathrm{d2})}{\partial q} = S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial q}$$

Plugging this into the original formula gives the equation into like terms that can then be canceled:

$$S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial q} - S_t(T-t)e^{-q(T-t)}\Phi(\mathrm{d1}) - S_t e^{-q(T-t)}\Phi'(\mathrm{d1})\frac{\partial(\mathrm{d1})}{\partial q}$$

Canceling terms we have the solution for Epsilon of a European Call option:

$$\boxed{\epsilon_{\mathbf{Call}} = -\mathbf{S_t}(\mathbf{T}-\mathbf{t})\mathbf{e^{-q(T-t)}}\boldsymbol{\Phi}(\mathrm{d1})} \tag{19}$$

Epsilon for European Put options are derived in the same way, exhibiting a similar symmetry to the other Greeks:

$$\boxed{\epsilon_{\mathbf{Put}} = \mathbf{S_t}(\mathbf{T}-\mathbf{t})\mathbf{e^{-q(T-t)}}\boldsymbol{\Phi}(\text{-d1})} \tag{20}$$

4 Non-traditional Greeks

Non-traditional Greeks include many higher-order sensitivity measurements that are newer and gaining popularity. There are many second-order and third-order Greeks that will be presented. In fact, the previously mentioned Gamma is among the second-order Greeks, but is already popular for the purposes of hedging. The lesser known second-order derivatives are Vanna, Charm, Vomma, Veta, and Vera. Furthermore,the third-order Greeks to be presented are Speed, Zomma, Color, and Ultima.

4.1 Charm

Where Theta is sometimes called option price decay, Charm is referred to as Delta decay. Recall that Gamma measures sensitivity of Delta with respect to changes in the Spot price. Charm measures the sensitivity of Delta to changes in the time to maturity. It can also be thought of as measuring the sensitivity of Theta to changes in the spot price. Likewise, it can be calculated by either taking the partial derivative of Theta with respect to the spot price or by taking the partial derivative of Delta with respect to the time to maturity. Charm is given as:

$$\boxed{\text{Charm} = \left(\frac{\mathbf{2r(T-t)} - \mathrm{d2}\sqrt{\sigma^{\mathbf{2}}\mathbf{(T-t)}}}{\mathbf{2(T-t)}\sqrt{\sigma^{\mathbf{2}}\mathbf{(T-t)}}}\right)\mathbf{\Phi'}(\mathrm{d1})} \tag{21}$$

4.2 Vanna

Vanna is another measure of Delta as it relates to Vega and volatility. Like Charm, it can be calculated in two different ways as the partial derivative of Vega with respect to the spot price or as the partial derivative of Delta with respect to the volatility. Vanna is often used as a check for the performance of Delta- or Vega-hedged portfolios as the volatility changes over time. Vanna is given by the following fromula for both European Put and Call options.

$$\boxed{\text{Vanna} = \left(\frac{-\mathbf{e}^{-\mathbf{q(T-t)}}\mathrm{d2}}{\sigma}\right)\mathbf{\Phi'}(\mathrm{d1})} \tag{22}$$

4.3 Veta

Veta, also known as Vega decay, is another measure of Vega that is taken with respect to time to maturity. It is another second-order sensitivity that is calculated as either the derivative of Vega with respect to the time to maturity or as the derivative of Theta with respect to volatility. Its worth noting that the derivation of Veta requires the use of a three term product rule as the variable for time to maturity is contained within all three of the multiplied terms in the Vega formula. The formula for Veta of European Calls and Puts is given by the following.

$$\boxed{\text{Veta} = -\mathbf{S_t e^{-q(T-t)}}\sqrt{\mathbf{(T-t)}}\boldsymbol{\Phi}(\text{d1})\left(\mathbf{q} + \frac{(\mathbf{r}-\mathbf{q})\text{d1}}{\sqrt{\sigma^{\mathbf{2}}(\mathbf{T}-\mathbf{t})}} - \frac{\mathbf{1}+\text{d1d2}}{\mathbf{2}(\mathbf{T}-\mathbf{t})}\right)} \tag{23}$$

4.4 Vomma

Vomma is the second-order derivative of the pricing function taken with respect to volatility. It can be thought of as the sensitivity of Vega to changes in volatility and is therefore calculated as the partial derivative of Vega with respect to volatility. Vomma for vanilla European Call and Put options is given by the following formula.

$$\boxed{\text{Vomma} = \left(\frac{\mathbf{S_t e^{-q(T-t)}}\sqrt{\mathbf{(T-t)}}\text{d1d2}}{\sigma}\right)\boldsymbol{\Phi}'(\text{d1})} \tag{24}$$

4.5 Speed

Speed is a third-order sensitivity that measures how Gamma changes with the changes in the price of the underlying asset. Specifically, it measures this sensitivity with respect to only one underlying asset. This distinction is important given the possibility for the option to be written on multiple different assets. For the case of the vanilla European option that is written on one asset only, the derivation is straightforward. Speed is calculated as the partial derivative of Gamma with respect to the underlying stock price.

$$\boxed{\text{Speed} = -\left(\frac{\text{d1} + \sqrt{\sigma^{\mathbf{2}}(\mathbf{T}-\mathbf{t})}}{\mathbf{S_t}}\right)\boldsymbol{\Gamma}} \tag{25}$$

4.6 Color

Color is another measure of Gamma, capturing the sensitivity to the time to maturity. Therefore, it is another third-order partial derivative of the pricing function. It can be calculated as the derivative of Gamma with respect to time to maturity and is given for both Calls and Puts by the following equation.

$$\boxed{\text{Color} = -\left(\frac{\sigma + \left(\ln(\frac{\mathbf{K}}{\mathbf{S_t}}) + \left(\frac{(\mathbf{r}+\sigma^{\mathbf{2}})(\mathbf{T}-\mathbf{t})}{\mathbf{2}}\right)\right)\mathrm{d1}}{\mathbf{2}\sigma^{\mathbf{2}}(\mathbf{T}-\mathbf{t})^{\mathbf{2}}}\mathbf{S_t}\right)\mathbf{\Phi}'(\mathrm{d1})} \tag{26}$$

4.7 Zomma

Zomma is the third-order sensitivity measure of Gamma to volatility. It can be calculated by taking the partial derivative of Gamma with respect to volatility. The formula for Zomma is given below.

$$\boxed{\text{Zomma} = \left(\frac{\mathrm{d1d2}-\mathbf{1}}{\sigma}\right)\mathbf{\Gamma}} \tag{27}$$

4.8 Ultima

Ultima is the third order sensitivity to volatility and can be calculated by taking the appropriately ordered partial derivative of the price, Vega, or Vomma with respect to volatility. The formula for Ultima is given as follows.

$$\boxed{\text{Ultima} = \frac{-\mathcal{V}}{\sigma^{\mathbf{2}}}\left(\mathrm{d1d2}(\mathbf{1}-\mathrm{d1d2}) + \mathrm{d1}^{\mathbf{2}} + \mathrm{d2}^{\mathbf{2}}\right)} \tag{28}$$