Faculty of Engineering, Mathematics and Science

School of Computer Science & Statistics

M.Sc. Computer Science **Hilary Term 2016**
(Integrated Computer Science,
Interactive Entertainment Technology)

Semester I Annual Examination

Numerical Methods and Advanced Mathematical Modelling I

12th Jan 2016 **Exam Hall** **14:00–16:00 hrs**

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Instructions to Candidates:

Answer all **four** questions. All questions carry equal marks.

You may not start this examination until you are instructed to do so by the Invigilator.

Clarification:

The term "program fragment" is used to mean some program code that is not necessarily a complete program nor entirely syntactically correct. It should have sufficient detail to describe the essential elements of the solution.

Materials permitted for this examination:

Non-programmable calculators are permitted for this examination—please indicate make and model on your answer book.

Mathematical formulae and tables booklet—available from the invigilators.

Graph paper—available from the invigilators.

Question 1. The Fourier transform of $g(t)$, a complex-valued function of an independent variable t, is:

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{+\infty} g(t) \exp[-\mathrm{j}\, 2\pi\, f_T\, t]\, \mathrm{d}t.$$

(i) Explain the role of Euler's formula in interpreting the expression.

[2 marks]

(ii) Explain why complex numbers are used in the expression.

[3 marks]

(iii) Explain the meaning of the *power spectrum* that can be calculated from the output of the transform. Suggest a practical application that undertakes an analysis of the spectrum.

[6 marks]

(iv) Specify the *discrete* equivalent of the *inverse* Fourier transform.

[2 marks]

(v) Write a SciPy program fragment that implements methods for one-dimensional *rectangle, sinc, signum,* and *triangle* functions and calculates their power spectra using a discrete Fourier transform method provided.

[6 marks]

(vi) Sketch the above functions and their power spectra. Comment on their appearance.

[6 marks]

Question 2. The *convolution* of two functions is equivalent to the inverse Fourier transform of the *product* of their transforms, e.g. in the two-dimensional case:

$$\iint_{-\infty}^{+\infty} g(\xi, \eta) h(x - \xi, y - \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta = \mathcal{F}^{-1}\{G(f_X, f_Y) H(f_X, f_Y)\}.$$

(i) Explain why convolution (or correlation) is usually computed as the product of Fourier transforms.

[6 marks]

(ii) Give examples of two significantly different-looking two-dimensional convolution filter kernels and describe their effect on an image. Suggest a practical application of each kernel.

[6 marks]

(iii) Explain the role of the Fourier transform in an approach to quantify the *similarity* of two images.

[4 marks]

(iv) Write a SciPy program fragment that reduces the number of numbers required to encode a two-dimensional image to about 10% of the original, using a discrete Fourier transform method provided. Try to use information from the power spectrum to maintain the best possible image quality.

[9 marks]

Question 3. A complex-valued phasor $U(\boldsymbol{p})$, a function of position $\boldsymbol{p}$, wave amplitude $A(\boldsymbol{p})$, and wave phase $\phi(\boldsymbol{p})$ can be used to model the propagation of a light wave through space:

$$U(\boldsymbol{p}) = A(\boldsymbol{p}) \exp[+\mathrm{j}\, \phi(\boldsymbol{p})].$$

(i) Making reference to the Helmholtz Equation and its solution, describe in detail how $U(\boldsymbol{p})$ and related functions are used to model light wave propagation.

[13 marks]

(ii) Write a SciPy program fragment that shows an animation of one-dimensional light wave propagation, modeled as you have described above, over a range of times and/or positions.

[12 marks]

Question 4. The Rayleigh-Sommerfeld diffraction formula can be used to calculate the complex-valued phasor $U(\boldsymbol{p}_0)$ at position $\boldsymbol{p}_0$ located on the opposite side of a plane aperture with extent Σ to a point source of light with wave amplitude $A(\boldsymbol{p}_2)$ at position $\boldsymbol{p}_2$:

$$U(\boldsymbol{p}_0) = \frac{A(\boldsymbol{p}_2)}{\mathrm{j}\,\lambda} \iint_{\Sigma} \frac{\exp[\mathrm{j}\, k\, (r_{21} + r_{01})]}{r_{21}\, r_{01}} \cos(\boldsymbol{n}, \boldsymbol{r}_{21})\, \mathrm{d}s$$

where λ is wavelength, $k = \frac{2\pi}{\lambda}$, $\boldsymbol{n}$ is a vector normal to the aperture, $\boldsymbol{r}_{21}$ is a vector from $\boldsymbol{p}_2$ to the aperture and $\boldsymbol{r}_{01}$ is a vector from $\boldsymbol{p}_0$ to the aperture with magnitudes r_{21} and r_{01} respectively.

(i) Explain in detail your understanding of this formula and how it was derived. Use appropriate drawings.

[13 marks]

(ii) Write a SciPy program fragment to calculate images that approximate the Fraunhofer diffraction patterns of a rectangular aperture and a circular aperture.

[12 marks]