Mathematics AA Internal Assessment

RQ: How to find the relative of device using the intensity of electromagnatic wave recieved by 3 recievers.

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1 Introduction

A GPS is lost within a relatively flat town. The GPS transmits an electromagnetic wave in all directions. The GPS device is located at point $G = (x_1, y_1)$, where $x_g, y_g \in \mathbb{R}$. The town consists of 3 devices placed that can receive EM waves and measure the intensity of those waves. The 3 receivers are placed in such a way, such that they form a triangle. The distance between each source is known, and the intensity of the EM waves received are also known.

The aim is of this internal assessment is to find the location of the GPS device using the known variables.

Firstly, a solution will be found for a 1 dimensional town, where only 2 sources are located and, the GPS can move in only on one axis. Then the GPS device is allowed to move on both the x and y-axis, and all possible points the GPS device could be at, by knowing the EM wave intensity from receivers is found. A 3rd receiver is then introduced, and knowing the intensity of EM wave at the 1st and 3rd receiver is found. The GPS device should be located both at the place where the function of the device location from 1st and 2nd source and 1st and 3rd source intersect.

1.1 Background Knowledge

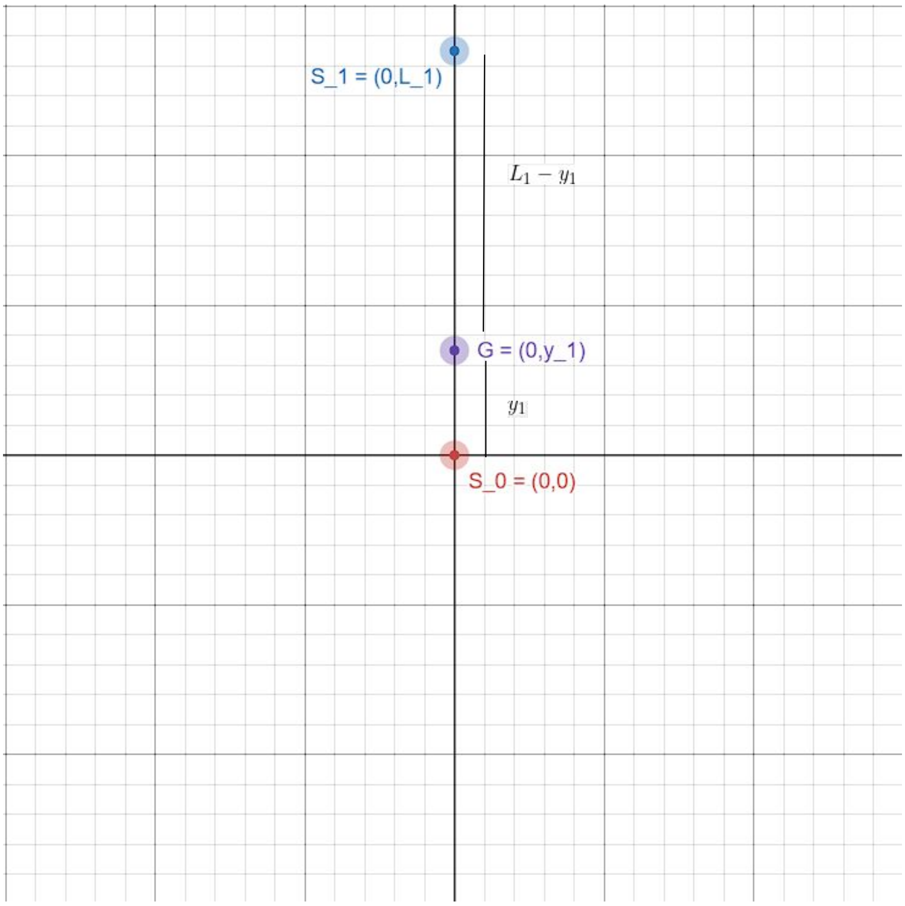
Intensity of light: The GPS device transmits EM waves in all directions, which follows the inverse square law. The formula for intensity of light, which can be measured by the receiver is, $I = \frac{P}{4\pi d^2}$, where P is the power of the EM wave transmitted from the GPS device, and d is the distance between point of measurement of intensity and source [1]

Properties of quadratic equations with 2 variables: If an equation is in the form of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and $B^2 - 4AC < 0$ [2], the function is a circle or an ellipse. This condition helps recognize *equation 2* as a circle, and hence attempt to convert it to standard form to find the center and radius.

2 Solution

Initially, 2 receivers (S_0 and S_1) are placed at a distance L_1 from each other. S_0 and S_1 are placed on a coordinate system at points such that $S_0 = (0, 0)$ and $S_1 = (0, L_1)$. The GPS device G is located at $G = (0, y_1)$, where $y_g \in (0, L_1)$. This setup is shown in *figure 1.*

Figure 1: Experimental setup for 2 sources with GPS device moving only vertically



Let I_0 and I_1 be the intensities of EM waves received at S_0 and S_1 respectively. Let, k_1 be the ratio of $I_0 : I_1$, where $k_1 > 1$.

Using *equation...*, $I_0 = \frac{P_g}{4\pi y_1^2}$ and $I_0 = \frac{P_g}{4\pi(L_1 - y_1)^2}$. Hence,

$$k_1 = \frac{I_0}{I_1} = \frac{P_g}{4\pi y_1^2} \times \frac{4\pi(L_1 - y_1)^2}{P_g}$$

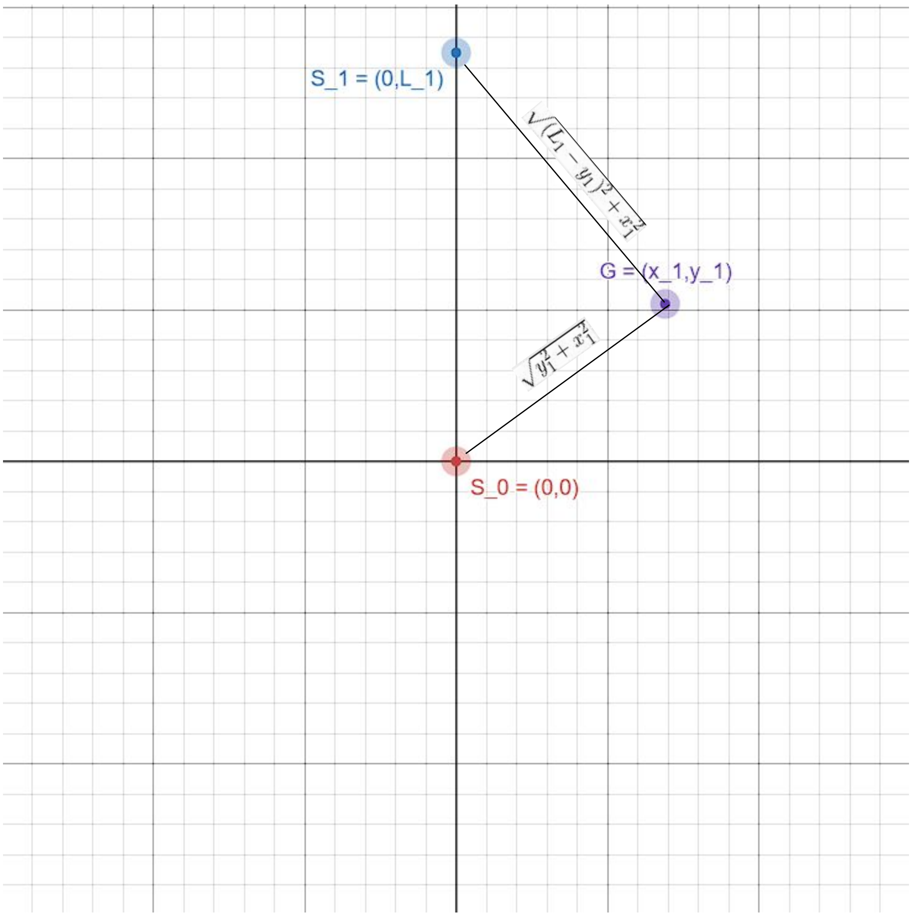
$$k_1 = \left(\frac{L_1 - y_1}{y_1}\right)^2 = \left(\frac{L_1}{y_1} - 1\right)^2 \tag{1}$$

$$y_1 = \frac{L}{\sqrt{k_1} + 1}$$

Hence, knowing the ratio of intensities between the two sources, and G being able to move only vertically between the two sources, the location of $G = (0, \frac{L}{\sqrt{k_1}+1})$.

Allowing G to move both vertically and horizontally gives the setup seen in *figure 2*

Figure 2: Experimental setup for 2 sources with GPS device moving in all directions



Due to GPS having another degree of freedom, k_1, turns into a function, and the GPS device, G, lies on this function.

$$k_1 = \frac{I_0}{I_1} = \frac{P_g}{4\pi\sqrt{y_1^2 + x_1 2}^2} \times \frac{4\pi\sqrt{(L_1 - y_1)^2 + x_1^2}^2}{P_g}$$

Simplifying this equation gives:

$$
\begin{aligned}
&k_1 = \frac{(L_1 - y_1)^2 + x_1^2}{x_1^2 + y_1^2} \\
&k_1(x_1^2 + y_1^2) = (L_1 - y_1)^2 + x_1^2 \\
&x_1^2(k_1 - 1) = (L_1 - y_1)^2 - k_1 y_1^2 = L_1^2 + 2L_1 y_1^2 - y_1^2(k - 1)
\end{aligned}
$$

This equation can be rearranged to:

$$
(k_1 - 1)x_1^2 + (k_1 - 1)y_1^2 + 2L_1 y_1 - L_1^2 = 0 \tag{2}
$$

This equation, satisfies the conditions for the function to be an ellipse or circle. Hence, this equation can be converted to a standard form equation of a circle, to find its centre.

$$
(k_1 - 1)x_1^2 + (k_1 - 1)y_1^2 + 2L_1 y_1 = L_1^2
$$

$$
(k_1 - 1)\left(x_1^2 + y_1^2 + \frac{2L}{k_1} y_1\right) = L_1^2
$$

$$
x_1^2 + \left(y_1 + \frac{L_1}{k_1 - 1}\right)^2 - \left(\frac{L_1}{k_1 - 1}\right)^2 = \frac{L_1^2}{k_1 - 1}
$$

$$
x_1^2 + \left(y_1 - \frac{-L_1}{k_1 - 1}\right)^2 = \frac{L_1^2}{(k_1 - 1)} + \frac{L_1^2}{(k_1 - 1)^2} \tag{3}
$$

$$
x_1^2 + \left(y_1 - \frac{-L_1}{k_1 - 1}\right)^2 = \frac{L_1^2 \times (k_1 - 1)}{(k_1 - 1) \times (k_1 - 1)} + \frac{L_1^2}{(k_1 - 1)^2}
$$

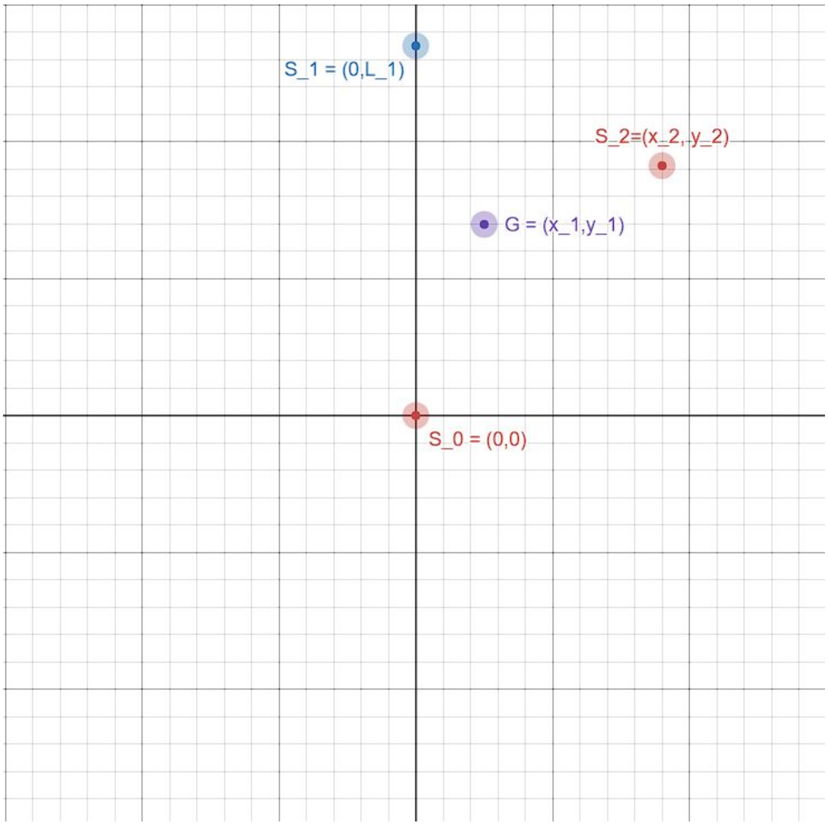
$$
x_1^2 + \left(y_1 - \frac{-L_1}{k_1 - 1}\right)^2 = \frac{L_1^2(k_1)}{(k_1 - 1)^2}
$$

$$
x_1^2 + \left(y_1 - \frac{-L_1}{k_1 - 1}\right)^2 = \left(\frac{L_1}{k_1 - 1}\sqrt{k_1}\right)^2
$$

The point (x_1, y_1) lies on the circle with the equation $C_1 = x^2 + \left(y - \frac{-L_1}{k_1 - 1}\right)^2 = \left(\frac{L_1}{k_1 - 1}\sqrt{k_1}\right)^2$, center at $(0, \frac{-L_1}{k_1 - 1})$ and radius $R_1 = \frac{L_1}{k_1 - 1}\sqrt{k_1}$. It is also important to know that the center of the circle is the $\frac{-L_1}{k_1 - 1}$ units away from S_0

This process can be repeated with a 3rd receiver, located at $S_2 = (x_2, y_2)$, where $x_2, y_2 \in \mathbb{R}$ and is distance L_2 units away from S_0.

Figure 3: Experimental setup for 3 sources with GPS device moving in all directions



The next step is to find the circle where G lies on, based on the received intensities I_0 and I_2, at S_0 and S_2, respectively and their ratio $k_2 = I_0 : I_2$.
Since the circle found using *equation 3*, the centre is $\frac{-L_1}{k_1 - 1}$ units away from S_0, the circle C_2 of possible points that G lies on, has the equation $(x - x_c)^2 + (y - y_c)^2 = R_2^2$, where (x_c, C_y) is the center, which need to be calculated, and it has the radius: $R_2 = \frac{L_2}{k_2 - 1}\sqrt{k_2}$. The radius of the second circle is in the same form as of the first circle is because if the previous coordinate system was transformed in such a way that S_0 and S_2 will lie on the same x-axis, and the steps in *equation 2*, with k_2 and L_2, the radius would be

$R_2 = \frac{L_2}{k_2-1}\sqrt{k_2}$.

The point (x_c, y_c), lies on a line that connects the points S_0 and S_2. Since, the location of $S_2 = (x_2, y_2)$ is known and the line passes through the origin, the equation of that line N is $y = \frac{y_2}{x_2}x$ Substituting $x = x_c$ and $y = y_c$ into line N, gives: $y_c = \frac{y_2}{x_2}x_c$.

Additionally, the distance of the center of the circle from the center is known. Using Pythagoras' theorem, $x_c^2 + y_c^2 = \left(\frac{-L_1}{k_1-1}\right)^2$. Solving for y_c gives: $y_c = \sqrt{\left(\frac{-L_1}{k_1-1}\right)^2 - x_c^2}$

This gives the following system of equation:

$$\begin{cases} y_c = \frac{y_2}{x_2}x_c \\ y_c = \sqrt{\left(\frac{-L_1}{k_1-1}\right)^2 - x_c^2} \end{cases}$$

$$\left(\frac{y_2}{x_2}\right)^2 x_c = \sqrt{\left(\frac{-L_1}{k_1-1}\right)^2 - x_c^2}$$

$$\left(\frac{y_2}{x_2}\right)^2 x_c^2 = \left(\frac{-L_1}{k_1-1}\right)^2 - x_c^2$$

$$x_c^2\left(\left(\frac{y_2}{x_2}\right)^2 + 1\right) = \left(\frac{-L_1}{k_1-1}\right)^2 \tag{4}$$

$$x_c = \frac{-L_1}{k_1-1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2 + 1}}$$

Hence,

$$y_c = \frac{y_2}{x_2} \times \frac{-L_1}{k_1-1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2 + 1}}$$

Knowing, that the point G must lie on both the circles, means the GPS devices is located at the intersection of both circles calculated.

$$\begin{cases} (x - x_c)^2 + (y - y_c)^2 = R_2^2 \\ x^2 + \left(y - \frac{-L_1}{k_1 - 1}\right)^2 = R_1^2 \end{cases} \tag{5}$$

Solving this system of equation gives:

$$(x - x_c)^2 + (y - y_c)^2 - R_2^2 = x^2 + \left(y - \frac{-L_1}{k_1 - 1}\right)^2 - R_1^2$$

$$x^2 - 2x_c x + x_c^2 + y^2 - 2y_c y + y_c^2 - R_2^2 = x^2 + y^2 - 2\frac{-L_1}{k_1 - 1}y - R_1^2$$

$$-2x_c x - 2y_c y + 2\frac{-L_1}{k_1 - 1}y - \left(\frac{-L_1}{k_1 - 1}\right)^2 + x_c^2 + y_c^2 - R_2^2 + R_1^2 = 0$$

$$2(-x_c)x - 2(y_c + \frac{-L_1}{k_1 - 1})y - \left(\frac{-L_1}{k_1 - 1}\right)^2 + x_c^2 + y_c^2 - R_2^2 + R_1^2 = 0$$

This equation is the equation of a line, solving for y gives the line:

$$M : y = \frac{2(-x_c)}{2\left(y_c - \frac{-L_1}{k_1 - 1}\right)}x + \frac{x_c^2 + y_c^2 - \left(\frac{-L_1}{k_1 - 1}\right)^2 - R_2^2 + R_1^2}{2\left(y_c - \frac{-L_1}{k_1 - 1}\right)} \tag{6}$$

The next step is to find the intersection between the line M and any one of the circles. Unfortunately, due to the high complexity of the equations, another method of intersection between two circles must be used. This method is called triangulation as it utilizes angles to find the GPS device.

2.1 Finding general equation for GPS device

The point G is R_1 distance from Circle 1, and R_2 from Circle 2. The distance the centers of 2 circles is:

$$R_D = \sqrt{(0 - x_c)^2 + \left(\frac{-L_1}{k_1 - 1} - y_c\right)^2}$$

Let $\vec{R_D}$, $\vec{R_2}$ and $\vec{R_1}$ be vectors, with magnitudes R_D, R_2 and R_1, respectively. $\vec{R_D}$, $\vec{R_2}$ and $\vec{R_1}$ form a triangle, with angle α between $\vec{R_D}$ and $\vec{R_1}$ and angle θ between the x-axis and $\vec{R_1}$. Additionally,

$$\vec{R_D} = \begin{pmatrix} -x_c \\ \\ \frac{-L_1}{k_1-1} - y_c \end{pmatrix} \tag{7}$$

With this, we know that $\alpha + \theta = \pi + \arctan\left(\frac{\frac{-L_1}{k_1-1} - y_c}{-x_c}\right)$. To calculate α, the cosine rule can be used:

$$R_2^2 = R_1^2 + R_D^2 - 2R_1^2 R_D^2 \cos(\alpha)$$

$$\alpha = \arccos\left(\frac{R_2^2 - R_1^2 - R_D^2}{-2R_1^2 R_D^2}\right)$$

To calculate θ, the following formula is used:

$$\theta = \pi + \arctan\left(\frac{\frac{-L_1}{k_1-1} - y_c}{-x_c}\right) - \alpha$$

θ is the angle of a line drawn from origin to the point G. The point G is at

$$G = \left(R_1 \cos(\theta), R_1 \sin(\theta) + \frac{-L_1}{k_1 - 1}\right) \tag{8}$$

The y coordinate is shifted down by $\frac{-L_1}{k_1-1}$, because R_1 is the distance from the center of circle 1 to the point G. Not adding this constant would result in the point G being higher than the intersection. Thus, the point must be shifted the same amount as the location of the center of circle 1.

3 Generalized Formula for Location of G

Using the given knowns:

- k_1: Ratio of Intensities received from S_0 and S_1
- k_2: Ratio of Intensities received from S_0 and S_2
- S_0: Location of Receiver 0 at $(0, 0)$

- S_1: Location of Receiver 1 at $(0, L)$
- S_2: Location of Receiver 2 at (x_2, y_2) for $x_2, y_2 \in \mathbb{R}$
- L_1: Distance between S_0 and S_1
- L_2: Distance between S_0 and S_2
- x_c: $\frac{-L_1}{k_1-1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2+1}}$
- y_c: $\frac{y_2}{x_2} \times \frac{-L_1}{k_1-1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2+1}}$

There are 2 cases to condsider:

Case 1: $k_1, k_2 \neq 1$

A general formula to find the location of the point G where the GPS devices is situated, for this point:

$$G = \left(R_1 \cos(\theta), R_1 \sin(\theta) + \frac{-L_1}{k_1 - 1}\right)$$

$$G = \left(\frac{L_1\sqrt{k_1}}{k_1 - 1}\cos\left(\pi + \arctan\left(\frac{\frac{-L_1}{k_1-1} - y_c}{-x_c}\right) - \alpha\right), \frac{L_1\sqrt{k_1}}{k_1 - 1}\left(\pi + \arctan\left(\frac{\frac{-L_1}{k_1-1} - y_c}{-x_c}\right) - \alpha\right)\right) \tag{9}$$

Case 2: $k_1, k_2 = 1$

In this case, the GPS device would be located at the center point of all 3 Recievers.

4 Conclusion

The general equation for the location of the GPS device was found using minimal information such as location of the 3 receivers and the ratio of intensities received by them. This exploration initially attempted to solve the system of equations *see equation 5*, but due to increasing complexity, a vector approach was applied. Using the cosine rule, coordinates of the point G were found in the polar form, and then using vector components, it was converted to Cartesian form, to get the final coordinates as $G = (R_1 \cos(\theta), R_1 \sin(\theta))$ *see expanded version in equation 9*

5 Evaluation

The model created works only in a 2 dimensional system. Such systems could be implemented for cities or even small countries that don't have drastic altitude changes. This model not only pinpoints the exact location of the GPS, but also if the ratio of intensities creates 2 circles that don't intersect, this model predicts a point where the GPS is most likely to be.

This model must be expanded into a 3 dimensional system. For a 1 dimensional system, 2 receivers were required. 2 dimensional system required 3 receivers. Similarly, in a 3 dimensional system, 4 receivers will be required. The 2 dimensional system could be easily expanded to 3 dimensions, and using a similar approach, the GPS would be located at a point where 3 Spheres intersect. This is where the problem arrises, due to the added dimension, solving the system of equations of 3 intersecting spheres might prove to be extremely complicated. Although a vector approach could be attempted, it isn't guaranteed to work.

GPS systems in today's world, use GPS satellites in space that apply this 3 dimensional approach. This shows that it is possible to calculate this intersection point of 3 spheres, placed at a known location.

Albeit the impracticality, this exploration serves as an important to into the understanding of GPS systems and the complexity of mathematics behind it.

References

[1] University of Hawai OER. *Energy in Waves: Intensity.* Accessed: 2024/April/04. URL: https://pressbooks-dev.oer.hawaii.edu/collegephysics/chapter/16-11-energy-in-waves-intensity/.

[2] Varsity tutors. *Conic Sections and Standard Forms of Equations*. Accessed: 2024/April/04. URL: https://www.varsitytutors.com/hotmath/hotmath_help/topics/conic-sections-and-standard-forms-of-equations.