Mathematics AA HL Internal Assessment

Exploration: The mathematics behind locating a lost phone.

1 Introduction

Within an area an electromagnetic waves emitting device, such as a mobile phone, is situated at an unknown location. However, there are 3 base stations, placed at a known distance away from each other. These base station receive the EM radiation from the phone. The role of these base stations, is to measure the intensity of the EM wave received. This mathematical exploration aims to find the position of the phone relative to one of the base stations, using the distances between each base station, and the ratio of the intensity of the EM wave the base stations receive.

Let the EM waves transmitting device be denoted as P at position (P_x, P_y). Let the three base stations be denoted as B_0, B_1 and, B_2. The intensities received from them are I_0, I_1 and, I_2 respectively. If B_0 is located at position $0, 0$, then B_1 is located at $(0, L_1)$ and B_2 at (x_2, y_2), such that, $x_2\mathbb{R} - 0, y_2 \in \mathbb{R}$ and B_2 is distance L_2 from B_0. They are set up in a way, such that B_0, B_1 and B_2 are noncollinear. Using the known ratio of intensities $k_1 = I_0 : I_1$ and $k_2 = I_0 : I_2$, this exploration aims to determine (P_x, P_y), where $k_1, k_2 > 0$

1.1 Background Knowledge

Intensity of light: The EM wave transmitting device transmits EM waves uniformally in all direction following the inverse square law, which states that the strength of the wave is proportional to $\frac{1}{d^2}$, where d is the distance from transmitter. The formula for intensity of light, which can be measured by the base stations is, $I = \frac{P_w}{4\pi d^2}$, where P_w is the power of the EM wave transmitted from the GPS device, and d is the distance between point of measurement of intensity and source (phone) [1]

For example, if the Power of the EM wave is given to be $P_w = 1600W$ and B_0 measures an intensity of $I_0 = 100W/m^2$, then the distance between B_0 and P is

$$d = \sqrt{\frac{1600W}{100\frac{W}{m^2}4\pi}} = \frac{2}{\sqrt{\pi}} = 1.12837\ldots m \approx 1.13m$$

Hence, in this example, the device is about $1.13m$ away from the base station. Since, d is a scalar not a vector, the direction of d is unknown. With a fixed value $1.13m$, P lies on a circle with the radius $1.13m$ and centered at B_0.

Therefore, it is also important to know the conditions under which of a 2 variable quadratic equation takes form of a circle.

Properties of quadratic equations with 2 variables: If an equation is in the form of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and $B^2 - 4AC < 0$ [2], the function is a circle or an ellipse. Additionally, this equation is a function of a circle under the conditions that $A > 0, B = 0, C = A$ For example, the is quadratic equation: $0.7x^2 0xy + 0.7y^2 + 6.8x - 1.1y + 7.3 = 0$, is in the general form of a 2 variable quadratic equation, where:

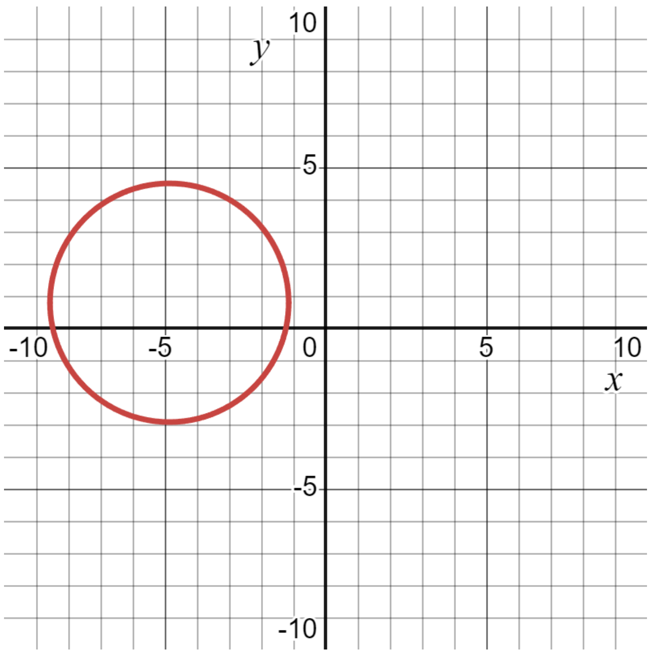
$$A = 0.7,\ \ B = 0$$

$$C = 0.7,\ \ D = 6.8$$

$$E = -1.1,\ \ F = 7.3$$

Here, $B^2 - 4AC = -1.96 < 0$, and $A = 0.7 > 0, B = 0, C = A$. Hence, the plot of this function (*see Figure 1*) is a circle or ellipse:

Figure 1: The graph of the 2 variable quadratic from the example



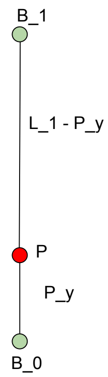
Let I_0 and

2 Solution

2.1 Solution on 1 Dimensional Case

Firstly, a 1 dimensional case is considered where, only the base stations B_0 and B_1 with the distance between them being L_1, are considered. The phone P, located at P_y, has can move between the two base stations.

Figure 2: Experimental setup for 2 sources with GPS device moving only vertically



I_0 and I_1 are the intensities of EM waves received at B_0 and B_1 respectively. Let, k_1 be the ratio of $I_0 : I_1$. The EM wave at source has a power of $P_w[W]$.
Using the equation of Intensity, $I_0 = \frac{P_w}{4\pi P_y^2}$ and $I_1 = \frac{P_w}{4\pi(L_1 - P - y)^2}$. Hence,

$$k_1 = \frac{I_0}{I_1} = \frac{P_w}{4\pi P_y^2} \cdot \frac{4\pi(L_1 - P - y)^2}{P_w}$$

$$k_1 = \left(\frac{L_1 - P_y}{P_y^2}\right)^2 = \left(\frac{L_1}{P_y} - 1\right)^2$$

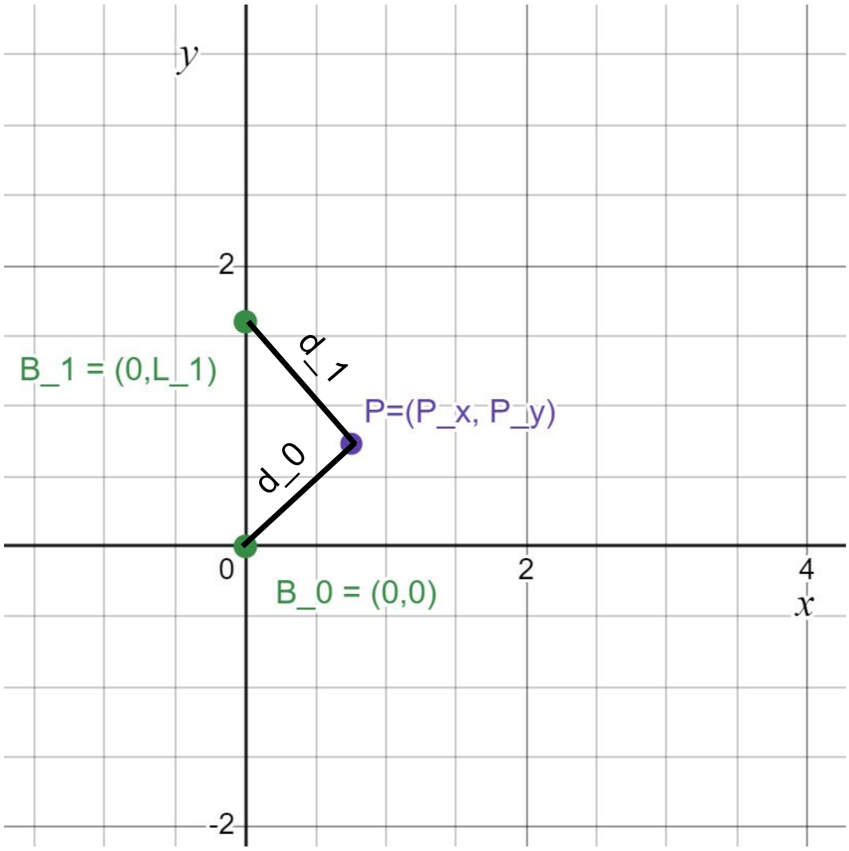
$$P_y = \frac{L_1}{\sqrt{k_1} + 1} \tag{1}$$

Hence, knowing the ratio of intensities received between the two sources, on a 1 dimensional line such that B_0,B_1 and P are collinear, the location of P is $P_y = \frac{L_1}{\sqrt{k_1}+1}$.

2.2 Solution on a 2D area

In the 1 Dimensional case, the location of P was defined by P_y, as there was only one degree of freedom. In a 2 dimensional case, the location of P is defined by $(P_x, andP_y)$.

Figure 3: Experimental setup for 2 base stations with the Phone having 2 degrees of freedom (x-axis and y-axis)



The point or phone's location, P is unknown. However, the distance between a base station and P can be calculated using the Pythagoras theorem. The distance between B_0 and P is $d_0 = \sqrt{P_y^2 + P_x^2}$ and with B_1, $d_1 = \sqrt{(L_1 - P_y)^2 + P_x^2}$.

Trying to find the equation of k_1, with d_0 and d_1, gives:

$$k_1 = \frac{I_0}{I_1} = \frac{P_w}{4\pi d_0^2} \cdot \frac{4\pi d_1^2}{P_w}$$

$$k_1 = \frac{I_0}{I_1} = \frac{P_w}{4\pi \sqrt{P_y^2 + P_x}^2} \cdot \frac{4\pi \sqrt{(L_1 - P_y)^2 + P_x^2}^2}{P_w}$$

Simplifying this equation gives:

$$k_1 = \frac{(L_1 - P_y)^2 + P_x^2}{P_x^2 + P_y^2}$$

$$k_1(P_x^2 + P_y^2) = (L_1 - P_y)^2 + P_x^2$$

$$P_x^2(k_1 - 1) = (L_1 - P_y)^2 - k_1 P_y^2 = L_1^2 + 2L_1 P_y^2 - P_y^2(k - 1)$$

This equation can be rearranged to:

$$(k_1 - 1)P_x^2 + (k_1 - 1)P_y^2 + 2L_1 P_y - L_1^2 = 0 \tag{2}$$

The *equation 2*, can be arranged in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where

$$A = (k_1 - 1),\ B = 0$$

$$C = (k_1 - 1),\ D = 0$$

$$E = 2L_1,\ F = -L_1^2$$

This gives 3 cases, depending on the value of k_1. **Case 1:** $k_1 > 1$

In this case, the conditions for the equation forming a circle are satisfied as $A > 0, B = 0, C = A$. Hence, this equation can be converted to a standard form equation of a circle, to find its center and radius.

$$(k_1-1)P_x^2+(k_1-1)P_y^2+2L_1P_y=L_1^2$$

$$(k_1-1)\left(P_x^2+P_y^2+\frac{2L}{k_1}P_y\right)=L_1^2$$

$$P_x^2+\left(P_y+\frac{L_1}{k_1-1}\right)^2-\left(\frac{L_1}{k_1-1}\right)^2=\frac{L_1^2}{k_1-1}$$

$$P_x^2+\left(P_y-\frac{-L_1}{k_1-1}\right)^2=\frac{L_1^2}{(k_1-1)}+\frac{L_1^2}{(k_1-1)^2}$$

$$P_x^2+\left(P_y-\frac{-L_1}{k_1-1}\right)^2=\frac{L_1^2\cdot(k_1-1)}{(k_1-1)\cdot(k_1-1)}+\frac{L_1^2}{(k_1-1)^2}$$

$$P_x^2+\left(P_y-\frac{-L_1}{k_1-1}\right)^2=\frac{L_1^2(k_1)}{(k_1-1)^2}$$

$$C_1: P_x^2+\left(P_y-\frac{-L_1}{k_1-1}\right)^2=\left(\frac{L_1}{k_1-1}\sqrt{k_1}\right)^2 \tag{3}$$

This shows that, knowing the ratio of intensities from 2 base stations the possible coordinate of P for the circle C_1 seen in *equation 3*, which is centered at $(0,\frac{-L_1}{k_1-1})$ and has a radius $R_0=\frac{L_1}{k_1-1}\sqrt{k_1}$ and is a distance $S_{c1}=\frac{-L_1}{k_1-1}$ units away from B_0. It should also be noted, that the center of C_1 lies on the same line $M_1: x=0$ that passes through B_0 and B_1.

Case 2: $k_1=1$

In such a case, $A=0, B=0, C=0, D=0$. Hence, the *equation 2*, simplifies to

$$0x^2+0xy+0y^2+0x+2L_1y-L_1^2=0$$
$$y=\frac{L_1^2}{2L_1}=\frac{L_1}{2}$$

As this equation is an equation of a horizontal line T_1, the infinitely many possible positions of P are $(P_x,P_y)=(x,\frac{L_1}{2})$, where $x, L_1\in\mathbb{R}$.

Case 3: $0 < k_1 < 1$

In such cases, instead of defining k_1 as the ratio of $I_0 : I_1$ it can be defined as $I_1 : I_0$, which would lead to the first case, where the possible positions of P form a circle with *equation 3*. Hence, this case can be disregarded as it has the same effect as **Case 1**.

Taking into account the 3rd base station, this procedure can be repeated using the ratio $k_2 = I_0 : I_1$. Here, B_1 is disregarded and instead B_2 is used. The *Figure 4*, shows the base stations taken into account when calculating C_1, and *Figure 5*, shows the base stations this procedure is extended to.

Figure 4

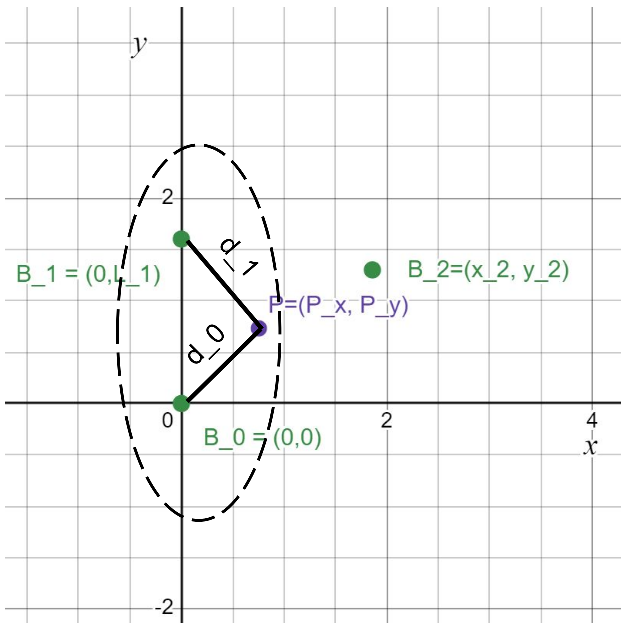
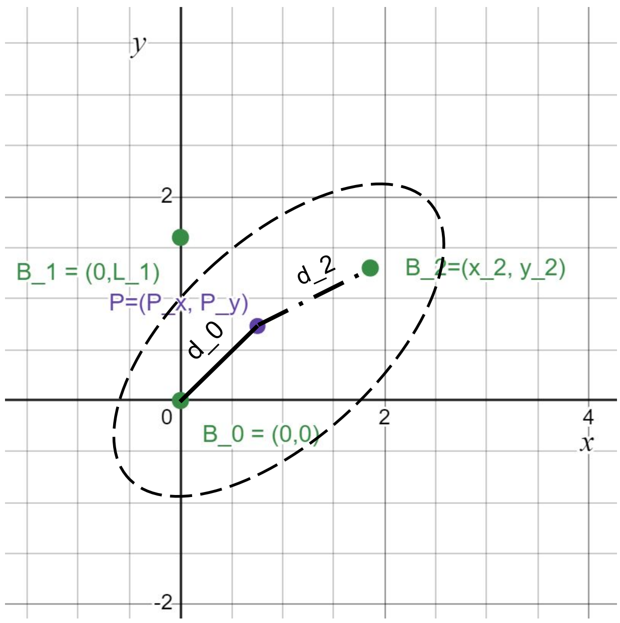


Figure 5



Again here two cases must be considered:

Case 1: $k_2 > 1$

When the equation for C_1 was found, it was noted that the circle is distance $S_{c1} = \frac{-L_1}{k_1-1}$ units from B_0. Let C_2 be the circle made up of positions that P could be at, considering only the ratio k_2. Since, the same procedure as the one used to find C_1 is used, we know that the the center of C_2 is $S_{c2} = \frac{-L_2}{k_2-2}$, where L_2 is the distance between B_0 and B_1. This applies to the radius also: $R_2 = \left(\frac{L_2}{k_1-1}\sqrt{k_1}\right)$ The center of C_2 also lies on the line M_2 that passes through the points B_0 and B_2. Since the position of B_2 is known to be (x_2, y_2), and B_0 is at $(0,0)$:

$$M_2 : y = \frac{y_2}{x_2}x$$

Let the center of C_2 be denoted as (x_{c2}, y_{c2}). Since this point lies on the line M_2, we know that $y_{c2} = \frac{y_2}{x_2}x_{c2}$. Additionally, using Pythagoras theorem, we have:

$$\begin{aligned}
x_{c2}^2 + y_{c2}^2 &= S_{c2}^2 \\
y_{c2} &= \sqrt{S_{c2}^2 - x_{c2}^2} \\
y_{c2} &= \sqrt{\left(\frac{-L_2}{k_2 - 2}\right) - x_{c2}^2}
\end{aligned}$$

This gives a system of equations with two variables:

$$\begin{cases} y_{c2} = \frac{y_2}{x_2}x_{c2} \\ y_{c2} = \sqrt{\left(\frac{-L_2}{k_2-2}\right) - x_{c2}^2} \end{cases} \tag{4}$$

The *equations 4*, can be solved to find the center of C_2.

$$\left(\frac{y_2}{x_2}\right)^2 x_{c2} = \sqrt{\left(\frac{-L_2}{k_2 - 1}\right)^2 - x_{c2}^2}$$

$$\left(\frac{y_2}{x_2}\right)^2 x_{c2}^2 = \left(\frac{-L_2}{k_2 - 1}\right)^2 - x_{c2}^2$$

$$x_{c2}^2 \left(\left(\frac{y_2}{x_2}\right)^2 + 1\right) = \left(\frac{-L_2}{k_2 - 1}\right)^2$$

$$x_{c2} = \frac{-L_1}{k_1 - 1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2 + 1}}$$

Hence,

$$y_{c2} = \frac{y_2}{x_2} \cdot x_{c2}$$
$$y_{c2} = \frac{y_2}{x_2} \cdot \frac{-L_1}{k_1 - 1} \sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2 + 1}}$$

This gives the final question for the circle:

$$C_2 : \left(x - \frac{-L_1}{k_1 - 1} \sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2 + 1}} \right)^2 + \left(y - \frac{y_2}{x_2} \cdot \frac{-L_1}{k_1 - 1} \sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2 + 1}} \right)^2 = \left(\frac{L_2}{k_1 - 1} \sqrt{k_1} \right)^2$$

Case 2: $k_2 = 1$

Just like **Case 2:** $k_1 = 1$, the possible positions of P will also be on a line: T_2. This line will be perpendicular to M_2 and pass through the midpoint between B_0 and B_1.

$$T_2 : \left(y - \frac{y_2}{2}\right) = -\frac{x_2}{y_2}\left(x - \frac{x_2}{2}\right)$$
$$T_2 : y = -\frac{x_2}{y_2} x + \frac{x_2^2 + y_2^2}{2y_2}$$
$$T_2 : y = -\frac{x_2}{y_2} x + \frac{L_2^2}{2y_2}$$

2.3 Finding the exact location of P

Since, there are there are two cases for both k_1 and k_2, a total of 4 cases must be considered.

Case 1: $k_1 > 1$ **and** $k_2 > 1$

In this case, there are two circles formed C_1 and C_2. Since, the device P lies on both circles, it must be one of the intersections between the two circles. The intersection(s) could be found by system of equations of the two circles. But, this leads to the system of equation would be a quadratic in some cases as there would be two intersection points and would make find the solutions of this quadratic relatively difficult.

A simpler solution is using the cosine rule. Let's consider $\vec{R_{1p}}$, $\vec{R_{2p}}$ and $\vec{R_D}$

- $\vec{R_{1p}}$ is the vector of magnitude R_1, and direction from the center of C_1 to P.
- $\vec{R_{2p}}$ is the vector of magnitude R_2, and direction from the center of C_2 to P.

- $\vec{R_D}$ is the vector of magnitude $\sqrt{(0-x_c)^2 + \left(\frac{-L_1}{k_1-1} - y_c\right)^2}$, and direction from the center of C_1 to C_2

$$\vec{R_D} = \begin{pmatrix} -x_c \\ \\ \frac{-L_1}{k_1-1} - y_c \end{pmatrix}$$

The three vectors form a triangle. Let α be the angle between $\vec{R_D}$ and $\vec{R_{1p}}$. Let θ be the angle between $\vec{R_{1p}}$ and the x-axis. Therefore, $\alpha + \theta = \pi + \arctan\left(\frac{\frac{-L_1}{k_1-1} - y_c}{-x_c}\right)$.

Then using the cosine rule, α can be calculated.

$$R_2^2 = R_1^2 + |\vec{R_D}|^2 - 2 \cdot R_1^2 \cdot |\vec{R_D}|^2 \cos(\alpha)$$

$$\alpha = \arccos\left(\frac{R_2^2 - R_1^2 - |\vec{R_D}|^2}{-2 \cdot R_1^2 \cdot |\vec{R_D}|^2}\right)$$

Then θ is calculated using:

$$\theta = \pi + \arctan\left(\frac{\frac{-L_1}{k_1-1} - y_c}{-x_c}\right) - \alpha$$

Since, θ is the angle of the vector $\vec{R_1}$, and has a starting position at $\left(0, \frac{-L_1}{k_1-1}\right)$, the coordinates of P are:

$$P\ (P_x, P_y) = \left(R_1\cos(\theta), R_1\sin(\theta) + \frac{-L_1}{k_1-1}\right) \tag{5}$$

Case 2: $k_1 = 1$ **and** $k_2 > 1$

In this case, and intesection between a line and a circle is used. Since $k_1 = 1$ and $k_2 > 1$ the phone P lies on the line T_1 and on the circle C_2. Since p This gives a system of equations:

$$\begin{cases} P_y = \frac{L_1}{2} \\ \\ (P_x - x_{c2})^2 + (P_y - y_{c2})^2 = R_2^2 \end{cases} \tag{6}$$

Which can be simplified into:

$$\begin{cases} P_y = \frac{L_1}{2} \\ P_x = \sqrt{R_2^2 - (P_y - y_{c2})^2} + x_{c2} \end{cases}$$

And solving this equation gives the value for $P_x = \sqrt{R_2^2 - \left(\frac{L_1}{2} - y_{c2}\right)^2} + x_{c2}$

$$P\ (P_x, P_y) = \left(\sqrt{R_2^2 - \left(\frac{L_1}{2} - y_{c2}\right)^2} + x_{c2},\ \frac{L_1}{2}\right) \tag{7}$$

Case 3: $k_1 > 1$ **and** $k_2 = 1$

In this case, also an intersection between the line T_2 and the circle C_1 is to be found.

$$\begin{cases} P_y = -\frac{x_2}{y_2} P_x + \frac{L_2^2}{2y_2} \\ P_x^2 + \left(P_y - \frac{-L_1}{k_1 - 1}\right) R_1^2 \end{cases}$$

Substituing P_y in the equation of C_1 with $-\frac{x_2}{y_2} P_x + \frac{L_2^2}{2y_2}$, gives:

$$P_x^2 + \left(\frac{-x_2}{y_2} P_x + \frac{L_2^2}{2y_2} - \frac{L_1}{k_1 - 1}\right)^2 = R_1^2$$

Let: $a = \frac{-x_2}{y_2}$, $b = \frac{L_2^2}{2y_2}$ and $c = \frac{L_1}{k_1 - 1}$

$$\begin{aligned} &P_x^2 + (aP_x + b - c)^2 = R_1^2 \\ &P_x = \frac{-2a(b-c) \pm \sqrt{(2ab - 2ac)^2 - 4(1 + a^2)((b-c)^2 - R_1^2)}}{2(1 + a^2)} \end{aligned}$$

Hence, solving this system gives the position of P to be:

$$P\ (P_x, P_y) = \left(P_x, \frac{x_2}{y_2} P_x + \frac{L_2^2}{2y_2}\right) \tag{8}$$

The whole equation isn't displayed due to it being lengthy and unable to fit into one page.

Case 4: $k_1 = 1$ **and** $k_2 = 1$

In this case, it is an intersection between T_1 and T_2:

$$\begin{cases} P_y = \frac{L_1}{2} \\ P_y = -\frac{x_2}{y_2} P_x + \frac{L_2^2}{2y_2} \end{cases}$$

Solving this system of equations gives:

$$\frac{L_1}{2} = -\frac{x_2}{y_2} P_x + \frac{L_2^2}{2y_2}$$

$$P_x = \left(\frac{L_1}{2} - \frac{L_2^2}{2y_2} \right) \cdot \frac{-y_2}{x_2}$$

$$P\ (P_x, P_y) = \left(\left(\frac{L_1}{2} - \frac{L_2^2}{2y_2} \right) \cdot \frac{-y_2}{x_2},\ \frac{L_1}{2} \right) \tag{9}$$

3 Generalized Formula for Location of P

Using the given knowns:

- k_1: Ratio of Intensities received from B_0 and B_1
- k_2: Ratio of Intensities received from B_0 and B_2
- B_0: Location of Receiver 0 at $(0, 0)$
- B_1: Location of Receiver 1 at $(0, L)$
- B_2: Location of Receiver 2 at (x_2, y_2) for $x_2, y_2 \in \mathbb{R}$
- L_1: Distance between B_0 and B_1
- L_2: Distance between B_0 and B_2
- x_{c2}: $\frac{-L_1}{k_1-1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2+1}}$
- y_{c2}: $\frac{y_2}{x_2} \cdot \frac{-L_1}{k_1-1}\sqrt{\frac{1}{\left(\frac{y_2}{x_2}\right)^2+1}}$

P is located at (P_x, P_y)

$$P_x = \begin{cases} P_x\ from\ equation\ 5,\ if\ k_1 > 1\ and\ k_2 > 1 \\ P_x\ from\ equation\ 7,\ if\ k_1 >= 1 and\ k_2 > 1 \\ P_x\ from\ equation\ 8,\ if\ k_1 > 1\ and\ k_2 = 1 \\ P_x\ from\ equation\ 9,\ if\ k_1 =\ \ and\ k_2 = 1 \end{cases}$$

$$P_y = \begin{cases} P_y\ from\ equation\ 5,\ if\ k_1 > 1\ and\ k_2 > 1 \\ P_y\ from\ equation\ 7,\ if\ k_1 >= 1 and\ k_2 > 1 \\ P_y\ from\ equation\ 8,\ if\ k_1 > 1\ and\ k_2 = 1 \\ P_y\ from\ equation\ 9,\ if\ k_1 =\ \ and\ k_2 = 1 \end{cases}$$

4 Conclusion

Using the location of 3 Base stations, and the ratio of intensities received by them by a lost electromagnetic wave emmiting device can be found using an equation for different conditions of the intensities. The location of this EM device is relative to the the base station B_0 and is found using intersection of 2 circles, a circle and line or 2 lines, depending on the conditions of the ratio of intensities.

4.1 Evaluation

The model created works only in a 2 dimensional system. Such systems could be implemented for cities or even small countries that don't have drastic altitude changes. Althought this model find the exact location of P, in real life, certain limitations such as measurement uncertainty which propagates as one intensity is divided by the other (such as $\frac{I_0}{I_1}$).

This model could be expanded into a 3 dimensional system, however, during my research for this exploration it was found that GPS devices apply a much simpler and different approach. A GPS system consists of 4 satellites placed in 3D space, that send a signal to the lost device, with a time stamp. The distance between the device and each of the satellites is determined using the equation $c = \frac{D}{\Delta t}$, where Δt is the time taken for the signal to reach the device, $c \approx 3 \cdot 10^8 m/s$ is the speed of electromagnetic waves and D is the distance. After finding the 4 Distances, they are used to draw spheres centered at each satellites and of radius with their corresponding distance. The intersection of the 4 spheres is where the lost device is located.

Albeit, this positioning system not working on a global scale, this could work on local scales. Devices such as apple air tags, that use nearby apple devices to transmit their location could be utilizing this methodology of positioning.

References

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[2] Varsity tutors. *Conic Sections and Standard Forms of Equations.* Accessed: 2024/April/04. URL: https://www.varsitytutors.com/hotmath/hotmath_help/topics/conic-sections-and-standard-forms-of-equations.