Foundations of Force and Torque for Classical Many-Particle Systems

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1 Force on a System of Particles

To describe any process in nature, we need a system of reference, a coordinate system where we can track the position, velocity, and acceleration of objects. I will pick an inertial coordinate system, which means that an object without any external forces will move with a constant velocity (that velocity might be zero). This means we will use a coordinate system where Newton's 1st Law holds.

Imagine all objects in the Universe are made up of N point particles (particles with no size, so to a very good approximation, they exist at a single point). Perhaps these particles are the atoms that make up objects, or the protons, neutrons, and electrons, or whatever. I will ask that the mass of each point particle remains constant. The i-th particle could potentially feel a force from each of the other particles, but we can define the word "object" so nothing else but these N particles can cause a force. The force of particle j on particle i is written like $\vec{F}_{ji}$, and generally takes on some magnitude and direction for all different values of $j = 1, 2, \ldots, N$, with the condition that if $j = i$, $\vec{F}_{ii} = 0$ (a particle cannot exert a force on itself).

Newton's 2nd Law for the i-th particle with mass, m_i (I recommend referring to Appendix 4.1 periodically, along with other Appendices, for notation and other fundamental ideas):

$$m_i\ddot{\vec{r_i}} = \sum_{j=1}^{N} \vec{F}_{ji} \tag{1}$$

If we focus on some subset of these particles, $i = 1, 2, \ldots, n$, this will become our system, with the other $N - n$ particles in the environment. Since Newton's 2nd Law applies to all particles, individually, certainly we could sum the left side of Eq 1 for each i in our n-particle system, together and the right side of Eq 1 for each i in our n-particle system, and the equation remains valid.

$$\sum_{i=1}^{n} m_i\ddot{\vec{r_i}} = \sum_{i=1}^{n} \dot{\vec{p_i}} = \sum_{i=1}^{n}\sum_{j=1}^{N} \vec{F}_{ji} \tag{2}$$

The second part of the equality is because the mass stays constant with time meaning we can pull it out of the derivative (because the time derivative of mass is zero when we apply the product rule):

$$\frac{d}{dt}\vec{p_i} = \frac{d}{dt}(m_i\dot{\vec{r}}_i) = \dot{\vec{r}}_i\frac{d}{dt}m_i + m_i\frac{d}{dt}\dot{\vec{r}}_i = m_i\frac{d}{dt}\dot{\vec{r}}_i = m_i\ddot{\vec{r}}_i$$

I am doing this for all i terms in the sum. Recall that $\vec{F}_{ji} = 0$ when $j = i$, but when $j \neq i$, the force's magnitude and direction depends on both the source particle j, causing the force, and the system particle, i, feeling the force. This says that if you sum up all the forces on all the partices in your system, due to all particles in the Universe, you can relate that to the mass and acceleration of all objects in your system.

This is not that useful until you multiply the left side by M/M where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system, which stays constant with time.

$$\begin{aligned} M\left(\frac{\sum_{i=1}^{n} m_i\ddot{\vec{r_i}}}{M}\right) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \vec{F}_{ji} \\ M\ddot{\vec{r}}_{cm} = \dot{\vec{p}}_{cm} &= \sum_{i=1}^{n}\sum_{j=1}^{N} \vec{F}_{ji} \end{aligned} \tag{3}$$

The second term in the last set of equations works because M and all m_i's are constant in time, and because the derivative is a linear operator so we can distribute it to all terms in the summation:

$$\begin{aligned} \frac{d}{dt}\vec{p}_{cm} &= \frac{d}{dt}\left[M\left(\frac{\sum_{i=1}^{n} m_i\dot{\vec{r_i}}}{M}\right)\right] = M\left(\frac{\frac{d}{dt}\left[\sum_{i=1}^{n} m_i\dot{\vec{r_i}}\right]}{M}\right) = M\left(\frac{\sum_{i=1}^{n}\frac{d}{dt}[m_i\dot{\vec{r_i}}]}{M}\right) \\ &= M\left(\frac{\sum_{i=1}^{n} m_i\frac{d}{dt}\dot{\vec{r_i}}}{M}\right) = M\left(\frac{\sum_{i=1}^{n} m_i\ddot{\vec{r_i}}}{M}\right) \end{aligned}$$

1.1 Only External Forces Matter

The right-hand side of Eq 3 can be simplified with Newton's 3rd law, relating forces between two particles. The inner sum can be split into those values of j inside the system ($j = 1, 2, \ldots, n$) and those j values outside the system, ($j = n + 1, n + 2, \ldots, N$) – the outer summation distributes:

$$\dot{\vec{p}}_{cm} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ji}\right) + \left(\sum_{i=1}^{n}\sum_{j=n+1}^{N} \vec{F}_{ji}\right)$$

The first term represents the sum of all internal forces and the second represents the sum of all external forces. For every pair of particles inside the system there are two forces added in the sum of internal forces, for example: for the 3rd and 5th particle, there is $j = 3$, $i = 5$ and $j = 5$, $i = 3$.

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ji}\right) = \vec{F}_{12} + \vec{F}_{13} + \ldots + \vec{F}_{35} + \ldots + \vec{F}_{53} + \ldots + \vec{F}_{n(n-1)}$$

These two force terms $\vec{F}_{35}$ and $\vec{F}_{53}$ are equal and opposite so the sum of them will cancel. Newton's 3rd Law says this is true for every pair of particles. The entire first term cancels meaning the final form of Newton's 2nd Law for the center of mass of a system of particles becomes:

$$\dot{\vec{p}}_{cm} = \left(\sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{F}_{ji} \right) = \vec{F}_{total,external} \tag{4}$$

This means we can disregard any internal forces, and only count the external forces that act on our system; this is enough to track the change in momentum of the center of mass of our system of n particles, relative to our inertial coordinate system.

1.2 A Tip of the Hat to Energy

Though this document is not about energy, the concept of energy is clearly extremely important. One other reason I am pursuing this temporary diversion is because I have previously seen published work arguing that Newton's 2nd Law for the center of mass of a system of particles, which will yield the Work-Energy Theorem, does not relate to the same kind of energy as with the 1st Law of Thermodynamics. This paper was published for physics teachers and compared the Work-Energy theorem to the 1st Law of Thermodynamics, which explicitly involes internal energy systems, and focused on problems with kinetic friction.

It is true that, when starting from Newton's 2nd Law for the center of mass of particles, you cannot discuss work done when internal parts of the system are displaced by some force acting on them. But the center of mass form of Newton's 2nd Law is not the most fundamental form. I wish to show that you can recover a term for the kinetic energy of particles relative to the center of mass, when you start from the more general form of Newton's 2nd Law for individual particles then work your way up to a system of particles. I will not pursue this endeavor fully, since this is not the point of this document. I will simply show that work due to external forces on different particles in the system does contribute to the kinetic energy of the system as a whole, though not to motion of its center of mass. Let's look at the acceleration of particle i, in an inertial reference frame:

$$\begin{aligned}
m_i \ddot{\vec{r}}_i &= \sum_{j=1}^{N} \vec{F}_{ji} \\
\ddot{\vec{r}}_i &= \frac{d}{dt} \dot{\vec{r}}_i \\
&= \frac{d}{dt} \left(\dot{x}_i \hat{x} + \dot{y}_i \hat{y} + \dot{z}_i \hat{z} \right) \\
&= \frac{d\dot{x}_i}{dt} \hat{x} + \frac{d\dot{y}_i}{dt} \hat{y} + \frac{d\dot{z}_i}{dt} \hat{z} \\
&= \dot{x}_i \frac{d\dot{x}_i}{dx_i} \hat{x} + \dot{y}_i \frac{d\dot{y}_i}{dy_i} \hat{y} + \dot{z}_i \frac{d\dot{z}_i}{dz_i} \hat{z} \\
&= \frac{1}{2} \frac{d(\dot{x}_i^2)}{dx_i} \hat{x} + \frac{1}{2} \frac{d(\dot{y}_i^2)}{dy_i} \hat{y} + \frac{1}{2} \frac{d(\dot{z}_i^2)}{dz_i} \hat{z} \\
\ddot{\vec{r}}_i \cdot d\vec{r}_i &= \frac{1}{2} \frac{d(\dot{x}_i^2)}{dx_i} dx_i + \frac{1}{2} \frac{d(\dot{y}_i^2)}{dy_i} dy_i + \frac{1}{2} \frac{d(\dot{z}_i^2)}{dz_i} dz_i \\
&= \frac{1}{2} \left(d(\dot{x}_i^2) + d(\dot{y}_i^2) + d(\dot{z}_i^2) \right) = \frac{1}{2} d(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) = \frac{1}{2} d(\dot{\vec{r}}_i^{\,2})
\end{aligned}$$

This demonstration shows that we can take the dot product of Newton's 2nd law for a single particle with its own infinitesimal displacement and get something that we will see is useful.

$$
\begin{aligned}
m_i\ddot{\vec{r}}_i \cdot d\vec{r}_i &= \sum_{j=1}^{N} \vec{F}_{ji} \cdot d\vec{r}_i \\
\frac{1}{2}m_i d(\dot{\vec{r}}_i^2) &= \sum_{j=1}^{N} \vec{F}_{ji} \cdot d\vec{r}_i \\
\sum_{i=1}^{n} \frac{1}{2}m_i d(\dot{\vec{r}}_i^2) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \vec{F}_{ji} \cdot d\vec{r}_i \\
\sum_{i=1}^{n} \int_0^t \frac{1}{2}m_i d(\dot{\vec{r}}_i^2) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_i \\
\sum_{i=1}^{n} \frac{1}{2}m_i \Delta(\dot{\vec{r}}_i^2) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_i \qquad (5)
\end{aligned}
$$

On the third line, I am summing the equation from the second line for all n particles in the system. In the last line, the Δ operator yields the difference between the quantity it operates on in the final state and that in the initial state. This is the Work-Energy theorem for an explicit system of particles. It differs from the aforementioned work by considering the motion of all particles in the system, rather than just the motion of the center of mass. I can relate the two by shifting the coordinate system of Equation 5 so it is relative to the center of mass:

$$
\begin{aligned}
\vec{r}_i &= \vec{r}_{cm} + \vec{r}_{i\,cm} \\
\dot{\vec{r}}_i &= \dot{\vec{r}}_{cm} + \dot{\vec{r}}_{i\,cm} \\
d\vec{r}_i &= d\vec{r}_{cm} + d\vec{r}_{i\,cm} \\
\sum_{i=1}^{n} \frac{1}{2}m_i \Delta((\dot{\vec{r}}_{cm} + \dot{\vec{r}}_{i\,cm})^2) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{cm} + \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{i\,cm} \\
\sum_{i=1}^{n} \frac{1}{2}m_i \Delta(\dot{\vec{r}}_{cm}^2 + \dot{\vec{r}}_{i\,cm}^2 + 2\dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{i\,cm}) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{cm} + \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{i\,cm} \\
\sum_{i=1}^{n} \frac{1}{2}m_i \Delta(\dot{\vec{r}}_{cm}^2) + \sum_{i=1}^{n} \frac{1}{2}m_i \Delta(\dot{\vec{r}}_{i\,cm}^2) + \sum_{i=1}^{n} \frac{1}{2}m_i \Delta(2\dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{i\,cm}) &= \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{cm} + \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{i\,cm} \qquad (6) \\
\sum_{i=1}^{n} \frac{1}{2}m_i \Delta(2\dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{i\,cm}) &= \Delta\left[\dot{\vec{r}}_{cm} \cdot \left(\sum_{i=1}^{n} m_i \dot{\vec{r}}_{i\,cm}\right)\right] = 0
\end{aligned}
$$

The dot-product term in the last line can be shuffled, as shown, because of the linear nature of the summation, the dot product, and the Δ. It cancels because the term in the inner-most parenthese is a constant times the momentum of the center of mass, relative to the center of mass, which is zero. See the Appendix, Equation 99 for details. Now, the process of deriving the Work-Energy theorem can be repeated in the same fashion starting from Newton's 2nd Law for the center of mass of a system. This yields the following:

$$
\frac{1}{2}M\Delta(\dot{\vec{r}}_{cm}^2) = \sum_{i=1}^{n}\sum_{j=1}^{N} \vec{F}_{ji} \cdot d\vec{r}_{cm} \qquad (7)
$$

Here, M is the total mass of the system of n particles. The left and right side of this equation are precisely the first terms on the left and right side of Equation 6. This means these terms are identically equal and can cancel from this equation, yielding:

$$
\sum_{i=1}^{n} \frac{1}{2}m_i \Delta(\dot{\vec{r}}_{i\,cm}^2) = \sum_{i=1}^{n}\sum_{j=1}^{N} \int_0^t \vec{F}_{ji} \cdot d\vec{r}_{i\,cm} \qquad (8)
$$

This term on the left side does not cancel like other summations of position and velocity vectors with respect to the center of mass, because this velocity vector is squared. Cancelation occurred before because for every position vector on one side of the center of mass, there was another on the other side that canceled during a weighted average, as the center of mass equation does. Likewise, no part of the right side of this equation cancels either. Before, for internal forces to cancel, we needed just that the forces be equal and opposite according to Newton's 3rd law. In this representation, we need both the forces and their infinitesimal displacements to cancel. Pairs of forces due to the same interaction due cancel, and they do interact for the same period of time but their infinitesimal displacements are not the same during this time. Internal forces and their associated work can affect the kinetic energy of the individual particles comprising the system.

Because the left side of the equation only involves motion relative to the center of mass, internal forces and their associated work cannot alter the kinetic energy of the center of mass of the system. This fits our observations. An explosion inside a system can drive the different parts of the system apart, but will not result in motion of the center of mass unless there is an external force. The relevant example might be a bullet firing inside of a gun. The bullet and gas gains kinetic energy, the gun recoils back gaining kinetic energy, but the center of mass stays put if this is the only work done. A cleaner example is a rocket being fired in space. The fuel goes one way, the rocket another, but they can separately gain kinetic energy from work due to the other part of the system. This also says friction from internal forces can increase the kinetic energy of the particles, something we would call internal energy or thermal energy, but they cannot produce an increase in kinetic energy of the system's center of mass.

Realize that in demonstrating this, I have just shown that there is a contribution of work in the more general Work-Energy theorem for an explicit system of particles that contributes to a change in kinetic energy in the individual particles relative to the center of mass. I have not assumed any particular type of object or motion such as a the rigid rotating objects I will get to later. This is a general term that is true regardless of the type of motion. It even includes thermal motion of particles. From here, I could try to specify forces and show that the work from these forces contribute to different kinds of energy, including electric potential energy holding charges together, or the Lennard-Jones potential energy which comes from a combination of an electric dipole attraction and repulsion from the Pauli-exclusion principle, responsible for the stability of matter. I will not pursue this further but know that it can be done. The Work-Energy theorem for an explicit system of particles and the corresponding Newton's 2nd Law are consistent with internal energy and the 1st Law of Thermodynamics.

2 Torque on a System of Particles

Let's now start over with Eq 1 for Newton's 2nd law for a single particle with mass m_i. Take the cross product of that particle's position vector with each side of the equation to get a starting point for Newton's 2nd law for rotation:

$$\vec{r}_i \times m_i \ddot{\vec{r}}_i = \vec{r}_i \times \dot{\vec{p}}_i = \vec{r}_i \times \sum_{j=1}^{N} \vec{F}_{ji} \tag{9}$$

The cross product is a linear operation, meaning we can distribute it over the sum of the forces to define the torque due to a single force, relative to our coordinate system.

$$\vec{r}_i \times \dot{\vec{p}}_i = \sum_{j=1}^{N} \vec{r}_i \times \vec{F}_{ji} = \sum_{j=1}^{N} \vec{\tau}_{ji}$$

Note that the time derivative of the angular momentum of particle i is equal to the left-hand side because $\dot{\vec{r}}_i$ is colinear with $\vec{p}_i$ so their cross product is zero:

$$\dot{\vec{L}}_i = \frac{d}{dt}\vec{L}_i = \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \dot{\vec{r}}_i \times \vec{p}_i + \vec{r}_i \times \dot{\vec{p}}_i = \vec{r}_i \times \dot{\vec{p}}_i$$

Making this substitution, we get Newton's 2nd law for rotation for a point particle i, relative to our coordinate system (of course):

$$\dot{\vec{L}}_i = \sum_{j=1}^{N} \vec{\tau}_{ji} \tag{10}$$

Like before, we can sum the left-hand side and right-hand side for all particles $i = 1, 2, \ldots, n$ in the system.

$$\sum_{i=1}^{n} \dot{\vec{L}}_i = \sum_{i=1}^{n} \vec{r}_i \times m_i \ddot{\vec{r}}_i = \sum_{i=1}^{n} \sum_{j=1}^{N} \vec{\tau}_{ji} \tag{11}$$

2.1 Only External Torques Matter if Forces are Colinear

Like with forces, we can expand the inner sum over the $j = 1, 2, \ldots, N$ source particles, exerting torques on our subset of these, the particles $i = 1, 2, \ldots, n$ in our system, into those inside and outside of our system:

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\vec{\tau}_{ji} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\vec{\tau}_{ji} + \sum_{j=n+1}^{N}\vec{\tau}_{ji}\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\vec{r}_i \times \vec{F}_{ji} + \sum_{j=n+1}^{N}\vec{r}_i \times \vec{F}_{ji}\right)$$

It's true, like before, that Newton's 3rd Law says that $\vec{F}_{ji} = -\vec{F}_{ij}$, but the position vector for particles i and j are different, so the overall cross product terms will not cancel in the internal summation as easily as before. We have to use the added condition that the forces, not only are equal and opposite, but point along the line separating particles i and j. This means we are considering forces that are either mutually attractive or repulsive, they do not push or pull at right angles to the line connecting the two particles. It is not hard to verify that the magnetic part of the Lorentz force does not obey this property, in general (it does under special configurations and velocities). This may argue for an expansion of our definition of a particle. We assumed N point particles in the Universe, with n in our system. If we say the remaining $N - n$ can contain, somehow, other force generators like electric and magnetic fields, then these fields can carry momentum and energy, which they do. We could discretize the fields, which obey Maxwell's equations, into little chunks. Stepping into more exotic, fundamental physics terminology for a moment, perhaps the force on each chunk is a generalized force acquired by the partial derivatives the Hamiltonian with respect to the generalized coordinates. There's still a couple of problems, though. First, these discretized chunks of electric and magnetic field do not carry mass, even though they do carry energy. If the field-particles are only outside of the system, perhaps this is okay. It may be possible to reconcile this somewhat strained view. Certainly, the correspondence principle formulated by Neils Bohr and the logical motivation behind it do argue that more accurate, more fundamental models of reality need to conform to classical physics under the appropriate limits (like large quantum numbers). Classical physics is an emergent property of quantum physics, though the emergence is not always as simple as how the motions of atoms can be averaged to get that of continuous matter. To my knowledge, though, force exerted on a field is not generally discussed, which is probably one reason modern physics or even the physics of fields avoid forces in favor of conserved quantities that relate to symmetries/invariants of the action and the equations of motion like momentum and energy. This is a possible point of divergence from the classical realm. We knew it had to happen some time, because of the existence of non-classical regimes like quantum mechanics, and general relativity. I'll proceed with the assumption of colinear, Newton's 3rd Law pairs of forces.

There are two vectors we could use for this line separating the particles, but I will pick the one that points from i to j, which I will represent with $\vec{r}_{ji} = \vec{r}_j - \vec{r}_i$, similar in notation to the center of mass momenta and angular momenta. Consistent with past notation, it can be read, "the position of particle j, relative to particle i". Recall we are considering forces that are colinear with $\vec{r}_{ji}$. I will expand just the internal summation and focus on a single pair of particles, $i = 3$ and $j = 5$, like before. Then for just one of the pair, I will utilize this relative position vector and make use of the colinearity of these forces.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\vec{\tau}_{ji} &= \vec{\tau}_{12} + \vec{\tau}_{13} + \ldots + \vec{\tau}_{35} + \ldots + \vec{\tau}_{53} + \ldots + \vec{\tau}_{n(n-1)} \\
&= \vec{\tau}_{12} + \vec{\tau}_{13} + \ldots + \left(\vec{r}_5 \times \vec{F}_{35}\right) + \ldots + \left(\vec{r}_3 \times \vec{F}_{53}\right) + \ldots + \vec{\tau}_{n(n-1)} \\
&= \vec{\tau}_{12} + \vec{\tau}_{13} + \ldots + \left((\vec{r}_3 - \vec{r}_{35}) \times \vec{F}_{35}\right) + \ldots + \left(\vec{r}_3 \times \vec{F}_{53}\right) + \ldots + \vec{\tau}_{n(n-1)} \\
&= \vec{\tau}_{12} + \vec{\tau}_{13} + \ldots + \left(\vec{r}_3 \times \vec{F}_{35} - \vec{r}_{35} \times \vec{F}_{35}\right) + \ldots + \left(\vec{r}_3 \times \vec{F}_{53}\right) + \ldots + \vec{\tau}_{n(n-1)} \\
&= \vec{\tau}_{12} + \vec{\tau}_{13} + \ldots + \left(\vec{r}_3 \times \vec{F}_{35}\right) + \ldots + \left(\vec{r}_3 \times \vec{F}_{53}\right) + \ldots + \vec{\tau}_{n(n-1)}
\end{aligned}$$

Finally, these two highlighted pairs of torques cancel due to Newton's 3rd law. Non-colinear, Newton's 3rd Law pairs of forces do not generally produce torques that cancel. But with our assumed forces, all internal torques have a twin torque that cancels it. Only external torques matter, when forces are colinear:

$$\sum_{i=1}^{n}\dot{\vec{L}}_i = \sum_{i=1}^{n}\sum_{j=n}^{N}\vec{\tau}_{ji} = \vec{\tau}_{total,external} \tag{12}$$

2.2 The Center of Mass Reference Frame is Special

To relate this to the center of mass, I will write each particle's relative to the center of mass vector, with this new vector relative to the center of mass as $\vec{r}_{i\,cm}$ so that $\vec{r}_i = \vec{r}_{cm} + \vec{r}_{i\,cm}$. Think of this subscript notation for motion vectors as saying, "of i, relative to the center of mass". The previous notation for motion vectors, with just a single item, can then consistently be thought of as saying, "of i, relative to the origin" or "of the center of mass, relative to the origin". In order to take the derivative of these vectors, I have to assume that the coordinate system for $\vec{r}_{i\,cm}$ is not rotating. Clearly it will be translating, but it cannot rotate. With this, because of the linearity of the derivative, we have $\dot{\vec{r}}_i = \dot{\vec{r}}_{cm} + \dot{\vec{r}}_{i\,cm}$, and $\ddot{\vec{r}}_i = \ddot{\vec{r}}_{cm} + \ddot{\vec{r}}_{i\,cm}$. If I had used a rotating coordinate system, such as one that is fixed to the object moving and rotating in space, there would need to be extra terms for the first and second time derivatives (see the Appendix on NonInertial Reference Frames).

$$\sum_{i=1}^{n}(\vec{r}_{cm} + \vec{r}_{i\,cm}) \times m_i(\ddot{\vec{r}}_{cm} + \ddot{\vec{r}}_{i\,cm}) = \sum_{i=1}^{n}\sum_{j=n+1}^{N}(\vec{r}_{cm} + \vec{r}_{i\,cm}) \times \vec{F}_{ji}$$

I'll distribute this cross product on the left-hand side and look at each of terms, individually:

$$\begin{aligned}\sum_{i=1}^{n}(\vec{r}_{cm} + \vec{r}_{i\,cm}) \times m_i(\ddot{\vec{r}}_{cm} + \ddot{\vec{r}}_{i\,cm}) =& \sum_{i=1}^{n}\vec{r}_{cm} \times m_i\ddot{\vec{r}}_{cm} + \sum_{i=1}^{n}\vec{r}_{cm} \times m_i\ddot{\vec{r}}_{i\,cm} + \\ & \sum_{i=1}^{n}\vec{r}_{i\,cm} \times m_i\ddot{\vec{r}}_{cm} + \sum_{i=1}^{n}\vec{r}_{i\,cm} \times m_i\ddot{\vec{r}}_{i\,cm} \\ =& \vec{A} + \vec{B} + \vec{C} + \vec{D}\end{aligned} \tag{13}$$

I'll start by looking at the middle two terms, $\vec{B}$ and $\vec{C}$, which cancel with relative ease:

$$\vec{B} = \sum_{i=1}^{n}\vec{r}_{cm} \times m_i\ddot{\vec{r}}_{i\,cm} = \vec{r}_{cm} \times \sum_{i=1}^{n} m_i\ddot{\vec{r}}_{i\,cm} = 0$$

I pulled the center of mass vector out of the sum because of the distributive property of the cross product, and the fact that it is not explicitly dependent on i in the summation. The last step, setting $\vec{B}$ to zero, because, from Eq 99, the total system momentum relative to the center of mass is zero at all times, so the change in the total momentum must be zero, too.

$$\vec{C} = \sum_{i=1}^{n}\vec{r}_{i\,cm} \times m_i\ddot{\vec{r}}_{cm} = -\sum_{i=1}^{n}\ddot{\vec{r}}_{cm} \times m_i\vec{r}_{i\,cm} = -\ddot{\vec{r}}_{cm} \times \sum_{i=1}^{n} m_i\vec{r}_{i\,cm} = 0$$

The first manipulation follows from two things, that the scalar multiplication of the mass can be as associated with either vector in the cross product, and because the cross product is anti-commutative. The term is zero because, after pulling out the cross product with the acceleration of the center of mass, in the inertial coordinate system, we get the position of the center of mass in the center of mass coordinate system, which is zero, see Eq 98.

Eq 13 simplifies to $\vec{A}$, $\vec{D}$, and the right hand side of the equation:

$$\begin{aligned}\sum_{i=1}^{n}\vec{r}_{cm} \times m_i\ddot{\vec{r}}_{cm} + \sum_{i=1}^{n}\vec{r}_{i\,cm} \times m_i\ddot{\vec{r}}_{i\,cm} &= \sum_{i=1}^{n}\sum_{j=n+1}^{N}\vec{r}_{cm} \times \vec{F}_{ji} + \sum_{i=1}^{n}\sum_{j=n+1}^{N}\vec{r}_{i\,cm} \times \vec{F}_{ji} \\ \vec{r}_{cm} \times \sum_{i=1}^{n} m_i\ddot{\vec{r}}_{cm} + \sum_{i=1}^{n}\vec{r}_{i\,cm} \times m_i\ddot{\vec{r}}_{i\,cm} &= \vec{r}_{cm} \times \sum_{i=1}^{n}\sum_{j=n+1}^{N}\vec{F}_{ji} + \sum_{i=1}^{n}\sum_{j=n+1}^{N}\vec{r}_{i\,cm} \times \vec{F}_{ji}\end{aligned}$$

The second line follows because $\vec{r}_{cm}$ does not explicitly depend on i or j and the cross product is distributive over addition of vectors. Notice the first term on the right hand side is equal to the first term on the left hand side, which follows from taking the cross product of $\vec{r}_{cm}$ with both sides of the basic form of Newton's 2nd law for a system of particles, Eq 2. This means they cancel from both sides and we're left with just:

$$\sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i \ddot{\vec{r}}_{i\,cm} = \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{r}_{i\,cm} \times \vec{F}_{ji}$$

$$\sum_{i=1}^{n} \dot{\vec{L}}_{i\,cm} = \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{\tau}_{ji\,cm} \tag{14}$$

But the left-hand side is the definition of the change of the total angular momenta of a system of particles, now in a coordinate system with its origin (rotation point) at the center of mass, and the right-hand side is sum of all the torques on all n particles in the system, but now in a coordinate system with its origin (rotation point) at the center of mass, see Eq 11 and Eq 9.

Let's review what was done to understand the significance of this result. I started by assuming some inertial coordinate system, one where Newton's 1st law holds, then shifted to a coordinate system whose origin is at the center of mass of the system of particles, but one that is not rotating in space. The center of mass is not necessarily inertial, but this shift shows that picking this as the origin leaves this Newton's 2nd Law for rotation looking the same as before, just shifted. Other noninertial origins to the coordinate system not at the center of mass, even if the basis vectors are not rotating, would in general produce extra terms in the equation for Newton's 2nd Law for rotation, when shifted like we did. You can see this because vectors $\vec{B}$ and $\vec{C}$ would not be zero, and vector $\vec{A}$ would not cancel with the first term in the expanded torque equation.

The conclusion is that shifting the origin to the center of mass, but with non-rotating basis vectors, does not produce fictitious forces and torque about that point cleanly determines the angular momentum at that point. The center of mass is a special origin but you still need nonrotating basis vectors for Newton's 2nd Law for rotation to come out nicely.

2.3 Torque from a Constant Gravitational Force

If the force on each particle looks like the approximate force of gravity near the surface of the Earth or other massive body, $m\vec{g}$, then the total external force in Newton's 2nd Law for translation, Eq 4, becomes:

$$\vec{F}_{total,external} = \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{F}_{ji} = \sum_{i=1}^{n} \vec{F}_{i,total} = \sum_{i=1}^{n} m_i \vec{g} = \dot{\vec{p}}_{cm} = M \ddot{\vec{r}}_{cm}$$

Newton's 2nd Law for rotation, Eq 12, becomes:

$$\begin{aligned}
\vec{\tau}_{total,external} &= \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{r}_i \times \vec{F}_{ji} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{i,total} \\
\vec{\tau}_{total,external} &= \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{g} = -\vec{g} \times \sum_{i=1}^{n} m_i \vec{r}_i = -M\vec{g} \times \left(\frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \right) = -M\vec{g} \times \vec{r}_{cm} \\
&= \vec{r}_{cm} \times (M\vec{g})
\end{aligned}$$

If I make the origin (rotation point) of our coordinate system at the center of mass and pick nonrotating basis vectors, then Newton's 2nd law for rotation applies only with a straight-forward shift in the position vectors:

$$\vec{\tau}_{total,external} = \vec{r}_{cm\,cm} \times (M\vec{g}) = 0 = \sum_{i=1}^{n} \dot{\vec{L}}_{i\,cm} \tag{15}$$

This says that a constant force of gravity exerts no torque about the center of mass and will not change the system's angular momentum when measured in a nonrotating reference frame relative to the object's center of mass. If all other forces are at the center of mass or otherwise produce no torque, the system will balance at its center of mass.

3 Motion of Rigid Objects

When a system of particles is rigid, the spacing between all particles is constant with time. If the object is rigid and rotating and translating through space, we can define two coordinate systems: one unprimed, inertial system and possibly outside the body of the object, and another primed system with its origin and basis vectors fixed to the body at the center of mass and therefore noninertial. See the Appendix on NonInertial Reference Frames. I will use the vector quantities, $\vec{r}_i$ is the position of the i-th particle relative to the inertial frame, $\vec{r}_{cm}$ is the position of the noninertial frame with respect to the inertial frame, and $\vec{r}_{i\,cm}$ is the position of the particle with respect to the noninertial frame, so that:

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_{i\,cm} \tag{16}$$

Euler's rotation theorem says that, in three dimensions, any displacement of a rigid body so that a point on the rigid body remains fixed is equivalent to a single rotation about some axis that runs through the fixed point. It also means that any composition of two rotations is also a rotation. We can represent the velocity of a particle in the system in the inertial frame, $\dot{\vec{r}}_i$, as the sum of the velocity of the origin and the velocity due to the angular velocity, $\vec{\omega}$ of the noninertial reference frame.

$$\dot{\vec{r}}_i = \dot{\vec{r}}_{cm} + \vec{\omega} \times \vec{r}_{i\,cm} \tag{17}$$

Had the basis vectors not been fixed to the body, or if the object was not rigid, an extra term would be added, $\dot{\vec{r}}'_{i\,cm}$ showing the change in the particle's position with respect to the body-fixed coordinates. I included the prime for emphasis that the time derivative must be computed in a noninertial reference frame. However, since I picked the center of mass as the origin of this coordinate system, even if the basis vectors were not fixed to the system, this term would still vanish if summed over, since the total momentum of the particles relative to the center of mass is zero, so its change must be zero, too.

The acceleration of a particle in this rigid object and coordinates fixed to the center of mass can be represented as follows:

$$\begin{aligned}\ddot{\vec{r}}_i &= \ddot{\vec{r}}_{cm} + \ddot{\vec{r}}'_{i\,cm} + \dot{\vec{\omega}} \times \vec{r}_{i\,cm} + 2\vec{\omega} \times \dot{\vec{r}}'_{i\,cm} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm}) \\ &= \ddot{\vec{r}}_{cm} + \dot{\vec{\omega}} \times \vec{r}_{i\,cm} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm})\end{aligned} \tag{18}$$

Newton's second law for rotation Eq 12 can be expanded using this new relationship:

$$\begin{aligned}\sum_{i=1}^{n} \vec{r}_i \times m_i \ddot{\vec{r}}_i &= \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{r}_i \times \vec{F}_{ji} \\ \sum_{i=1}^{n} (\vec{r}_{cm} + \vec{r}_{i\,cm}) \times m_i \left(\ddot{\vec{r}}_{cm} + \dot{\vec{\omega}} \times \vec{r}_{i\,cm} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm})\right) &= \sum_{i=1}^{n} \sum_{j=n+1}^{N} (\vec{r}_{cm} + \vec{r}_{i\,cm}) \times \vec{F}_{ji}\end{aligned} \tag{19}$$

The left-hand side can be expanded into the following:

$$\begin{aligned}LHS =& \sum_{i=1}^{n} \vec{r}_{cm} \times m_i \ddot{\vec{r}}_{cm} + \sum_{i=1}^{n} \vec{r}_{cm} \times m_i (\dot{\vec{\omega}} \times \vec{r}_{i\,cm}) + \sum_{i=1}^{n} \vec{r}_{cm} \times m_i (\vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm})) + \\ & \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i \ddot{\vec{r}}_{cm} + \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i (\dot{\vec{\omega}} \times \vec{r}_{i\,cm}) + \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i \left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm})\right) \\ =& \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E} + \vec{F}\end{aligned} \tag{20}$$

The right-hand side can be expanded into the following, much simpler equation:

$$\begin{aligned}RHS =& \vec{r}_{cm} \times \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{F}_{ji} + \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{r}_{i\,cm} \times \vec{F}_{ji} \\ =& \vec{G} + \vec{H}\end{aligned} \tag{21}$$

Term $\vec{A}$ can simply to the following by noticing $\vec{r}_{cm}$ does not depend on i or j:

$$\vec{A} = \vec{r}_{cm} \times \sum_{i=1}^{n} m_i \ddot{\vec{r}}_{cm} = \vec{G}$$

This is equal to $\vec{G}$ because they are the cross product of $\vec{r}_{cm}$ with both sides of the simplest form of Newton's 2nd Law for a system of particles. Since the left-hand side equals the right-hand side, $\vec{A}$ cancels with $\vec{G}$.

Term $\vec{D}$ cancels as well:

$$\vec{D} = \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i \ddot{\vec{r}}_{cm} = -\ddot{\vec{r}}_{cm} \times \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} = 0$$

The summation in the second to last part of the equality is the position of the center of mass in the center of mass coordinate system, which is zero.

Terms $\vec{B}$ and $\vec{E}$ can be expanded with a triple product rule:

$$\vec{B} = \sum_{i=1}^{n} \vec{r}_{cm} \times m_i(\dot{\vec{\omega}} \times \vec{r}_{i\,cm}) = \sum_{i=1}^{n} \Big(m_i(\vec{r}_{cm} \cdot \vec{r}_{i\,cm})\dot{\vec{\omega}} - m_i(\vec{r}_{cm} \cdot \dot{\vec{\omega}})\vec{r}_{i\,cm} \Big)$$
$$= \dot{\vec{\omega}} \left(\vec{r}_{cm} \cdot \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \right) - (\vec{r}_{cm} \cdot \dot{\vec{\omega}}) \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} = 0$$
$$\vec{E} = \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i(\dot{\vec{\omega}} \times \vec{r}_{i\,cm}) = \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\dot{\vec{\omega}} - m_i \vec{r}_{i\,cm}(\vec{r}_{i\,cm} \cdot \dot{\vec{\omega}})$$
$$= \dot{\vec{\omega}} \sum_{i=1}^{n} m_i r_{i\,cm}^2 - \dot{\vec{\omega}} \cdot \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \vec{r}_{i\,cm} \tag{22}$$

Term $\vec{B} = 0$ because both summations are a constant factor of the center of mass, relative to the center of mass, which is always zero. Term $\vec{E}$ can be simplified by introducing tensor representation, or the tensor product and a matrix. The first part is aligned with an identity matrix, δ^{α}_{β}.

I get:

$$E^{\alpha} = \left(\sum_{i=1}^{n} m_i (r_{i\,cm}^2 \delta^{\alpha}_{\beta} - [r_{i\,cm}]_{\beta} [r_{i\,cm}]^{\alpha}) \right) \dot{\omega}^{\beta} \tag{23}$$

The term in parentheses is the rotational inertia tensor, or the moment of inertia. Using the tensor product $\otimes$ and the identity matrix $\mathbb{I}$, I get:

$$\overline{I} = \sum_{i=1}^{n} m_i (r_{i\,cm}^2 \mathbb{I} - \vec{r}_{i\,cm} \otimes \vec{r}_{i\,cm}) \tag{24}$$
$$\vec{L} = \overline{I}\vec{\omega}$$

For a rigid body and a reference frame, including basis vectors, fixed to the object, the rotational inertia is a constant. This can be seen because $\dot{\vec{r}}'_{i\,cm}$ is fixed. We will see the added factor needed when taking the derivative of $\vec{L}$ in a noninertial reference frame (like the one we are using fixed to the rigid body).

I can expand the terms $\vec{C}$ and $\vec{F}$ with a vector triple product and quadruple product, respectively.

$$\vec{C} = \sum_{i=1}^{n} \vec{r}_{cm} \times m_i(\vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm})) = \vec{r}_{cm} \times \sum_{i=1}^{n} m_i(\vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm}))$$
$$= \vec{r}_{cm} \times \sum_{i=1}^{n} m_i((\vec{\omega} \cdot \vec{r}_{i\,cm})\vec{\omega} - \omega^2 \vec{r}_{i\,cm})$$
$$= (\vec{r}_{cm} \times \vec{\omega}) \left(\vec{\omega} \cdot \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \right) - \omega^2 \vec{r}_{cm} \times \left(\sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \right) = 0$$
$$\vec{F} = \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i \left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm}) \right)$$
$$= \sum_{i=1}^{n} m_i \left(\vec{\omega}(\vec{r}_{i\,cm} \cdot (\vec{\omega} \times \vec{r}_{i\,cm})) - (\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) \right)$$
$$= -\sum_{i=1}^{n} m_i (\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) \tag{25}$$

$\vec{C}$ is equal to zero because the terms inside the summations are a constant factor of the center of mass of the object, relative to the center of mass, which is zero.

The first term in $\vec{F}$ is zero because of the cyclical nature of the scalar triple product, and that $\vec{r}_{i\,cm}$ is obviously colinear with itself, so its cross product with itself is zero. The second term in $\vec{F}$ can be confirmed, and has been, by performing a cross product with a vector triple product. It does not appear to cancel and should be the modification of Newton's 2nd law for rotation of a rigid body with noninertial coordinates. You can also look at just shifting $\vec{L}$ and defining the rotational inertia in an inertial coordinate system, which is probably simpler. The $\vec{F}$ term is the torque due to the fictional centrifugal force. It may cancel by symmetry but I'm not sure. There should be an extra term for it being noninertial.

We're left with the following:

$$\sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\dot{\vec{\omega}} - m_i(\vec{r}_{i\,cm} \cdot \dot{\vec{\omega}})\vec{r}_{i\,cm} - \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) = \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{r}_{i\,cm} \times \vec{F}_{ji} \tag{26}$$

With the definition of the rotational inertia, and angular momentum, I get the following:

$$\overline{I}\dot{\vec{\omega}} - \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) = \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{r}_{i\,cm} \times \vec{F}_{ji} \tag{27}$$

Suppose we take the time derivative of $\vec{L} = \overline{I}\vec{\omega}$, and relate this for the inertial (unprimed) and noninertial (primed - fixed to object) coordinate system:

$$\begin{aligned}\vec{\tau}_{total,external} = \dot{\vec{L}} &= \dot{\vec{L}}' + \vec{\omega} \times \vec{L} \\ &= \overline{I}\dot{\vec{\omega}} + \vec{\omega} \times (\overline{I}\vec{\omega}) \end{aligned} \tag{28}$$

Recall that the rotational inertia tensor is constant in the fixed-to-rigid-body coordinate system. The second term in this equation can be simplified by noticing the colinearity of $\vec{\omega}$ with itself:

$$\begin{aligned}\vec{\omega} \times (\overline{I}\vec{\omega}) &= \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\vec{\omega} \times \vec{\omega} - m_i(\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) \\ &= -m_i(\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) \end{aligned} \tag{29}$$

This is exactly the term we have from applying the shifted, noninertial position vectors directly to Newton's 2nd Law of rotational motion. They are consistent. See Chapter 13 Rigid Body Motion and Rotational Dynamics and Analytical Mechanics for lots more information. And, perhaps, Quaternion applications to rigid body motion for many benefits in actually computing these vectors, and with no down sides that I am aware of.

3.1 Rotational Inertia - Moment of Inertia

The moment of inertia tensor was given by the following with its origin at the center of mass:

$$I^{\alpha}_{\beta} = \sum_{i=1}^{n} m_i(r^2_{i\,cm}\delta^{\alpha}_{\beta} - [r_{i\,cm}]_{\beta}[r_{i\,cm}]^{\alpha}) \tag{30}$$

$$L^{\alpha} = I^{\alpha}_{\beta}\omega^{\beta} \tag{31}$$

$$\overline{I} = \sum_{i=1}^{n} m_i(r^2_{i\,cm}\mathbb{I} - \vec{r}_{i\,cm} \otimes \vec{r}_{i\,cm}) \tag{32}$$

$$\vec{L} = \overline{I}\vec{\omega} = \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\vec{\omega} - m_i\vec{r}_{i\,cm}(\vec{r}_{i\,cm} \cdot \vec{\omega}) \tag{33}$$

3.2 Rotational Inertia - Inertial Coordinates at Center of Mass

With an origin at the center of mass but nonrotating basis vectors, I'll see how this affects the equation for Newton's 2nd Law for rotation and the rotational inertia tensor.

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_{i\,cm} \tag{34}$$

But I already know from Eq 14 that, with the assumption of a nonrotating coordinate system, Newton's 2nd Law for rotation can be shifted to the center of mass frame without any extra terms:

$$\sum_{i=1}^{n} \dot{\vec{L}}_{i\,cm} = \sum_{i=1}^{n} \vec{r}_{i\,cm} \times m_i\ddot{\vec{r}}_{i\,cm} = \sum_{i=1}^{n} \sum_{j=n+1}^{N} \vec{\tau}_{ji\,cm}$$

One assumption we did not make at the time was that the system of particles was rigid. In our inertial coordinates, we can define the angular velocity vector $\vec{\omega}$ pointing normal to the axis of rotation. All particles share this same angular velocity vector, and we'll assume they rotate around the center of mass. See Appendix 4.3 for an explanation of the following relationship, which holds for our assumptions:

$$\dot{\vec{r}}_{i\,cm} = \vec{\omega} \times \vec{r}_{i\,cm} \tag{35}$$
$$\ddot{\vec{r}}_{i\,cm} = \dot{\vec{\omega}} \times \vec{r}_{i\,cm} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{i\,cm}) \tag{36}$$

If I insert this into Newton's 2nd Law above we get the exact same form as before, in Eq 27, with a noninertial coordinate system:

$$\sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\dot{\vec{\omega}} - m_i\vec{r}_{i\,cm}(\vec{r}_{i\,cm} \cdot \dot{\vec{\omega}}) - \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{\omega})(\vec{\omega} \times \vec{r}_{i\,cm}) = \sum_{i=1}^{n}\sum_{j=n+1}^{N} \vec{\tau}_{ji\,cm} \tag{37}$$

However, now we need to interpret $\vec{\omega}$ as the rotation of the particles in the system around the center of mass, not the rotation of the coordinate system itself. The centrifugal term, vector triple product, we interpreted as the noninertial term for the derivative of the angular momentum. And, before, the rotational inertia was constant, now it changes with time as the system rotates. Now, perhaps this centrifugal term is equal to the time derivative of the rotational inertia. If we can represent the angular momentum as the same $\vec{L} = \overline{I}\vec{\omega}$, and apply the time derivative to this in the inertial frame, we should get the second summation term for the angular momentum in the equation above.

$$\begin{aligned}\dot{\vec{L}} &= \frac{d}{dt}\left(\sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\vec{\omega} - m_i\vec{r}_{i\,cm}(\vec{r}_{i\,cm} \cdot \vec{\omega})\right) &(38)\\ &= \sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\dot{\vec{\omega}} - m_i\dot{\vec{r}}_{i\,cm}(\vec{r}_{i\,cm} \cdot \vec{\omega}) - m_i\vec{r}_{i\,cm}(\dot{\vec{r}}_{i\,cm} \cdot \vec{\omega}) - m_i\vec{r}_{i\,cm}(\vec{r}_{i\,cm} \cdot \dot{\vec{\omega}})\end{aligned}$$

Keep in mind, that while the vector $\vec{r}_{i\,cm}$ does change with time, that is because it is rotating. The magnitude of the vector does not change because it is a rigid body. This is the reason the second line doesn't have any terms for the derivative of $\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm}$.

$$\begin{aligned}\dot{\vec{L}} &= \left(\sum_{i=1}^{n} m_i(\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\dot{\vec{\omega}} - m_i\vec{r}_{i\,cm}(\vec{r}_{i\,cm} \cdot \dot{\vec{\omega}})\right) - \left(\sum_{i=1}^{n} m_i\dot{\vec{r}}_{i\,cm}(\vec{r}_{i\,cm} \cdot \vec{\omega}) + m_i\vec{r}_{i\,cm}(\dot{\vec{r}}_{i\,cm} \cdot \vec{\omega})\right)\\ &= \left(\sum_{i=1}^{n} m_i(r_{i\,cm}^2\mathbb{I} - \vec{r}_{i\,cm} \otimes \vec{r}_{i\,cm})\right)\dot{\vec{\omega}} - \left(\sum_{i=1}^{n} m_i\dot{\vec{r}}_{i\,cm}(\vec{r}_{i\,cm} \cdot \vec{\omega}) + m_i\vec{r}_{i\,cm}(\dot{\vec{r}}_{i\,cm} \cdot \vec{\omega})\right)\\ &= \overline{I}\dot{\vec{\omega}} - \left(\sum_{i=1}^{n} m_i\dot{\vec{r}}_{i\,cm}(\vec{r}_{i\,cm} \cdot \vec{\omega}) + m_i\vec{r}_{i\,cm}(\dot{\vec{r}}_{i\,cm} \cdot \vec{\omega})\right)\\ &= \overline{I}\dot{\vec{\omega}} - \left(\sum_{i=1}^{n} m_i(\vec{\omega} \times \vec{r}_{i\,cm})(\vec{r}_{i\,cm} \cdot \vec{\omega}) + m_i\vec{r}_{i\,cm}((\vec{\omega} \times \vec{r}_{i\,cm}) \cdot \vec{\omega})\right)\\ &= \overline{I}\dot{\vec{\omega}} - \sum_{i=1}^{n} m_i(\vec{\omega} \times \vec{r}_{i\,cm})(\vec{r}_{i\,cm} \cdot \vec{\omega}) &(39)\end{aligned}$$

The second to last line follows from Eq 35, the velocity of a particle in the rigid rotating system, with inertial coordinates. While the last line follows from the cyclical nature of the scalar triple product and the colinearity of $\vec{\omega}$ with itself. This shows that the time derivative of the rotational inertia tensor in an inertial, center of mass reference frame is equal to the added term in a noninertial, center of mass reference frame (cf. Eq 27).

3.3 Rotational Inertia, Explicitly

The rotational inertia tensor at the center of mass, again, is as follows:

$$I^{\alpha}_{\beta} = \sum_{i=1}^{n} m_i(r_{i\,cm}^2\delta^{\alpha}_{\beta} - [r_{i\,cm}]_{\beta}[r_{i\,cm}]^{\alpha})$$
$$\overline{I} = \sum_{i=1}^{n} m_i(r_{i\,cm}^2\mathbb{I} - \vec{r}_{i\,cm} \otimes \vec{r}_{i\,cm})$$

In explicit matrix form, with xyz aligned in any general way, fixed or unfixed to the body, but with the origin at the center of mass, the rotational inertia becomes the following (I will temporarily drop the center of mass subscript for readability):

$$\overline{I} = \sum_{i=1}^{n} m_i \begin{bmatrix} x_i^2 + y_i^2 + z_i^2 & 0 & 0 \\ 0 & x_i^2 + y_i^2 + z_i^2 & 0 \\ 0 & 0 & x_i^2 + y_i^2 + z_i^2 \end{bmatrix} - \sum_{i=1}^{n} m_i \begin{bmatrix} x_i^2 & x_i y_i & x_i z_i \\ y_i x_i & y_i^2 & y_i z_i \\ z_i x_i & z_i y_i & z_i^2 \end{bmatrix}$$

$$\overline{I} = \sum_{i=1}^{n} m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & x_i^2 + z_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & x_i^2 + y_i^2 \end{bmatrix}$$

Note that since the components all depend on i the simplified form of the matrix will depend on the problem. In many problems, you can assume a continuum of point particles making up the system's rigid body, in which case you get an integral, with $m_i \to dm = \rho dV$, where ρ can depend on the coordinates. Since the matrix is a sum of all point particles, you can compute the rotational inertia for two distinct pieces of your rigid object about the same point and the total rotational inertia will be the sum of the two of them. If you don't want to use the center of mass, the parallel axis theorem says how you can modify the rotational inertia by rotating about an axis parallel to the angular velocity vector through the center of mass.

3.4 Parallel Axis Theorem

Until now, I have used the angular momentum (and therefore, the rotational inertia) relative to the center of mass. Sometimes you want the angular momentum and rotational inertia relative to some other point, but with the same direction for $\vec{\omega}$. This is the parallel axis theorem. It's easiest to derive the general equation for this shift starting from the angular momentum at a general point first, then shift to the center of mass. Let's pick an inertial frame. If the coordinates are in an inertial frame, $\vec{\omega}$ can be considered the rotation around the origin, which is the same for all particles in the rigid object, so the following relationships are true:

$$\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i \tag{40}$$

Inserting this into the general equation for angular momentum, from Newton's 2nd law for rotation, I get:

$$\begin{aligned} \vec{L} &= \sum_{i=1}^{n} \vec{r} \times \vec{p}_i = \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{r}}_i \\ &= \sum_{i=1}^{n} \vec{r}_i \times m_i(\vec{\omega} \times \vec{r}_i) \\ &= \sum_{i=1}^{n} m_i(\vec{r}_i \cdot \vec{r}_i)\vec{\omega} - \vec{r}_i(\vec{r}_i \vec{\omega}) \\ &= \left(\sum_{i=1}^{n} m_i(r_i^2 \mathbb{I} - \vec{r}_i \otimes \vec{r}_i) \right) \vec{\omega} = \overline{I}\vec{\omega} \\ \overline{I} &= \sum_{i=1}^{n} m_i(r_i^2 \mathbb{I} - \vec{r}_i \otimes \vec{r}_i) \end{aligned} \tag{41}$$

If we now shift this to the center of mass, with the same $\vec{\omega}$ so that $\vec{r}_i = \vec{r}_{i\,cm} + \vec{d}$, and noting that scalar multiplication with a matrix and the tensor product are linear, I get:

$$
\begin{aligned}
\overline{I} =& \sum_{i=1}^{n} m_i \left[(\vec{r}_{i\,cm} + \vec{d}) \cdot (\vec{r}_{i\,cm} + \vec{d})\mathbb{I} - (\vec{r}_{i\,cm} + \vec{d}) \otimes (\vec{r}_{i\,cm} + \vec{d}) \right] \\
=& \sum_{i=1}^{n} m_i \left[\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm}\mathbb{I} + 2\vec{d} \cdot \vec{r}_{i\,cm}\mathbb{I} + \vec{d} \cdot \vec{d}\mathbb{I} - \vec{r}_{i\,cm} \otimes \vec{r}_{i\,cm} - \vec{d} \otimes \vec{r}_{i\,cm} - \vec{r}_{i\,cm} \otimes \vec{d} - \vec{d} \otimes \vec{d} \right] \\
=& \sum_{i=1}^{n} m_i \left((\vec{r}_{i\,cm} \cdot \vec{r}_{i\,cm})\mathbb{I} - \vec{r}_{i\,cm} \otimes \vec{r}_{i\,cm} \right) + \sum_{i=1}^{n} m_i (\vec{d} \cdot \vec{d})\mathbb{I} \\
&+ 2\sum_{i=1}^{n} m_i (\vec{d} \cdot \vec{r}_{i\,cm})\mathbb{I} - \sum_{i=1}^{n} m_i \vec{d} \otimes \vec{r}_{i\,cm} - \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \otimes \vec{d} - \sum_{i=1}^{n} m_i \vec{d} \otimes \vec{d} \\
=& \overline{I}_{cm} + (\vec{d} \cdot \vec{d})\mathbb{I} \sum_{i=1}^{n} m_i - (\vec{d} \otimes \vec{d}) \sum_{i=1}^{n} m_i \\
&+ 2\mathbb{I} \left(\vec{d} \cdot \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \right) - \vec{d} \otimes \sum_{i=1}^{n} m_i \vec{r}_{i\,cm} - \left(\sum_{i=1}^{n} m_i \vec{r}_{i\,cm} \right) \otimes \vec{d} \\
\overline{I} =& \overline{I}_{cm} + M \left((\vec{d} \cdot \vec{d})\mathbb{I} - \vec{d} \otimes \vec{d} \right)
\end{aligned}
\tag{42}
$$

Every term in the second line of the second to last equality cancels because the summations are a constant factor times the center of mass position with respect to the center of mass, which is zero. The result is the parallel axis theorem, where $\vec{d}$ points from the center of mass to the new point. We'll see in the next section that with the right choice of basis vectors, the off-diagonal terms become zero for both this shift and in the equation for the rotational inertia in the center of mass.

3.5 Rotational Inertia - Principal Axes

The equation for the rotational inertia yields a 3x3 real, symmetric matrix, as can be seen by the explicit form given above or by the fact that switching the α and β in the tensor version gives the same values for all α and β. All square, real, symmetric matrices can be diagonalized with positive eigenvalues along the diagonal and orthogonal eigenvectors. This is a form of Sylvester's law of inertia. These eigenvectors, for bodies with constant density, are the axes of rotational symmetry. I will not prove any of this, but the diagonal eigenvalues are called the principal moments of inertia, and the orthogonal eigenvectors are called the object's principal axes.

I'll do an example for a rectangular solid with sides $a \neq b \neq c$, and a uniform mass density, $\rho = M/(abc)$, rotating about its center of mass. I'll pick a coordinate basis fixed to the solid with origin at the center of mass, so that the terms in the rotational inertia are constant with time. If I align the coordinate basis vectors like you would naturally expect, $\hat{x}$ points parallel to the a side, $\hat{y}$ points parallel to the b side, and $\hat{z}$ points parallel to the c side, we can solve for the rotational inertia with an integral that simplifies greatly. The point mass is a tiny piece of the continuum $m_i = dm = \rho dxdydz$ and the limits of integration extend from $(-a/2, -b/2, -c/2) \to (a/2, b/2, c/2)$. I'll introduce the common shorthand notation:

$$
\equiv \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

First, I'll solve the off-diagonal terms, I_{xy}, I_{xz}, and I_{yz}. Starting with I_{xy}, whatever we get here, our result could depend on the mass and the dimensions a, b, and c. Whatever result we get here, if we simply send replace b with c and vice versa, this will take I_{xy} to I_{xz}. Likewise, replacing a with b and vice versa, this will take I_{xz} to I_{yz}. It turns out, it will not be this complicated.

$$
\begin{aligned}
I_{xy} &= \frac{M}{abc} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} xydxdydz = \frac{M}{abc} \int_{-a/2}^{a/2} xdx \int_{-b/2}^{b/2} ydy \int_{-c/2}^{c/2} dz \\
&= \frac{M}{abc} \left[\frac{x^2}{2} \right]_{-a/2}^{a/2} \left[\frac{y^2}{2} \right]_{-b/2}^{b/2} [z]_{-c/2}^{c/2} = \frac{M}{abc} \left[\frac{a^2}{4} - \frac{a^2}{4} \right] \left[\frac{b^2}{4} - \frac{b^2}{4} \right] \left[\frac{c}{2} - \frac{-c}{2} \right] = 0
\end{aligned}
\tag{43}
$$

Because of the symmetry between the three off-diagonal terms, performing the above mapping means all three off-diagonal terms are zero, in this example.

$$
I_{xy} = I_{xz} = I_{yz} = 0 \tag{44}
$$

Now, I'll solve the diagonal term, I_{xx} and be prepared to make a similar mapping to find the other two terms:

$$I_{xx} = \frac{M}{abc}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}(y^2+z^2)dxdydz = \frac{M}{abc}\int_{-a/2}^{a/2}dx\left(\int_{-b/2}^{b/2}dy\int_{-c/2}^{c/2}(y^2+z^2)dz\right)$$
$$= \frac{M}{abc}\int_{-a/2}^{a/2}dx\left(\int_{-b/2}^{b/2}dy\left[y^2z+\frac{z^3}{3}\right]_{-c/2}^{c/2}\right) = \frac{M}{abc}\int_{-a/2}^{a/2}dx\left(\int_{-b/2}^{b/2}dy\left[y^2c+\frac{c^3}{12}\right]\right)$$
$$= \frac{M}{ab}\int_{-a/2}^{a/2}dx\left(\int_{-b/2}^{b/2}\left[y^2+\frac{c^2}{12}\right]dy\right) = \frac{M}{ab}\int_{-a/2}^{a/2}dx\left[\frac{y^3}{3}+y\frac{c^2}{12}\right]_{-b/2}^{b/2} = \frac{M}{ab}a\left[\frac{b^3}{12}+b\frac{c^2}{12}\right]$$
$$I_{xx} = \frac{M}{12}(b^2+c^2) \tag{45}$$
$$I_{yy} = \frac{M}{abc}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}(x^2+z^2)dxdydz = \frac{M}{12}(a^2+c^2) \tag{46}$$
$$I_{zz} = \frac{M}{abc}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}(x^2+y^2)dxdydz = \frac{M}{12}(a^2+b^2) \tag{47}$$

By picking the coordinate bases along the symmetry axes of this uniform-density rectangular solid, we get the off-diagonal moments canceling:

$$\overline{I}_{cm} = \frac{M}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

Note that what matters is how spread out the matter is from the center of mass. With matter more spread, there is a larger rotational inertia about that axis.

If we want the same orientation for our coordinates, but shifting the origin from the center of mass to one corner, we can use the parallel axis theorem, with $\vec{d} = -(a,b,c)/2$. $|d|^2 = (a^2+b^2+c^2)/4$ and the rotational inertia matrix becomes:

$$\overline{I} = \overline{I}_{cm} + \frac{M}{4}\begin{bmatrix} a^2+b^2+c^2 & 0 & 0 \\ 0 & a^2+b^2+c^2 & 0 \\ 0 & 0 & a^2+b^2+c^2 \end{bmatrix} - \frac{M}{4}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

The final rotational inertia oriented along the symmetry axes with its origin at the corner is the following:

$$\overline{I} = M\begin{bmatrix} (b^2+c^2)/3 & -ab/4 & -ac/4 \\ -ab/4 & (a^2+c^2)/3 & -bc/4 \\ -ac/4 & -bc/4 & (a^2+b^2)/3 \end{bmatrix}$$

This can be confirmed by recomputing the rotational inertia with the different limits on the integrals.

Is it possible to have coordinates fixed at the corner of the rectangular solid and diagonalize the rotational inertia matrix? It must because the matrix is real and symmetric. But what principal axes do you get? By inputting the form of $\overline{I}$ for the rectangular solid, with origin at the corner, into Wolfram Alpha to find the eigenvalues and eigenvectors (which are the principal axes) I get a hideous set of equations that take up many pages. This matrix should be diagonalizable but does not appear to line up with the diagonal, and to solve it is a cubic equation. The center of mass is best. We can be more general. Luckily the parallel axis theorem shows that once we start with a diagonal rotational inertia matrix, shifting from the center of mass origin to a new origin directly along one of the principal axes still yields a diagonal rotational inertia matrix. If $\vec{d}$ is parallel to say the third principal axis, call it $\hat{3}$, then:

$$\overline{I} = \overline{I}_{cm} + M\left((\vec{d}\cdot\vec{d})\mathbb{I} - \vec{d}\otimes\vec{d}\right)$$
$$\vec{d} = d\hat{3} \tag{48}$$
$$\overline{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} + M\begin{bmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^2 \end{bmatrix} - M\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^2 \end{bmatrix}$$
$$\overline{I} = \begin{bmatrix} I_1+Md^2 & 0 & 0 \\ 0 & I_2+Md^2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{49}$$

Indeed, the rotational inertia matrix is still diagonal and the same principal axes are true for this point shifted along one of the principal axes for the center of mass, by a vector $\vec{d}$.

3.6 Newton's 2nd Law for Rotation - Euler Equations

It's customary to call the body-fixed, center of mass frame coordinates the $(1, 2, 3)$ frame. The corresponding diagonalized eigenvalues of the rotational inertia matrix are I_1, I_2, I_3. Recall that if the coordinates are not body-fixed, then the terms in the rotational inertia matrix will change with time as the object rotates, meaning it will not, in general, be diagonalized since only occasionally will the symmetry axes line up with the coordinate axes.

Suppose we have body-fixed, center of mass coordinates lined up with the symmetry axes of our rigid object so that the rotational inertia matrix is diagonalized like mentioned above. In that frame, Newton's 2nd law for rotation is:

$$\begin{aligned}
\vec{\tau}_{total,external} &= \dot{\vec{L}} + \vec{\omega} \times \vec{L} \\
&= \overline{I}\dot{\vec{\omega}} + \vec{\omega} \times (\overline{I}\vec{\omega}) \\
\overline{I}\vec{\omega} &= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix} \\
\vec{\omega} \times (\overline{I}\vec{\omega}) &= \begin{bmatrix} \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2 \\ \omega_3 I_1 \omega_1 - \omega_1 I_3 \omega_3 \\ \omega_1 I_2 \omega_2 - \omega_2 I_1 \omega_1 \end{bmatrix} = \begin{bmatrix} \omega_2\omega_3(I_3 - I_2) \\ \omega_3\omega_1(I_1 - I_3) \\ \omega_1\omega_2(I_2 - I_1) \end{bmatrix} \\
\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} &= \begin{bmatrix} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{bmatrix}
\end{aligned} \tag{50}$$

These are called Euler's equations. They are a nonlinear coupled system of equations. There is an analytical solution that involves Jacobi elliptic functions. I, however, want to make some approximations to explore this more simply, or at least in a way that's more comfortable to the less mathematically sophisticated. Also, while this is about as clean of equations as we could hope for with rotational motion, in real problems this equation is only good if we can relate the body-fixed coordinates to the inertial ones. Torque and angular velocity are clean and easy to analyze in nonrotating coordinates, but are difficult to view from coordinates that are fixed to a rotating object. Keep in mind that the angular velocity, the rotation rate of the body-fixed coordinate system relative to the inertial one, will be the same in both coordinate systems, though their components will be different. The rotational inertia matrix is easy to write down and analyze in body-fixed coordinates aligned with the symmetry axes of the object because it is constant in time, but here the torque and angular velocity are difficult to make sense because we don't rotate with the object.

3.6.1 Euler's Equations with a Torque-free Example

Before getting any more complicated, let's look at a torque-free problem. Euler's equations become:

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} -\omega_2\omega_3(I_3 - I_2)/I_1 \\ -\omega_3\omega_1(I_1 - I_3)/I_2 \\ -\omega_1\omega_2(I_2 - I_1)/I_3 \end{bmatrix} \tag{51}$$

Even more simply, let's look at one that starts out with angular velocity completely aligned with one axis of the body-fixed frame, say ω_3 (they're symmetric anyway), so that all others are zero, but may not remain this way. Euler's equations become:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{bmatrix}$$

This is pretty bland. If angular velocity is completely aligned with just one symmetry axis then it will always remain this way. Real life is not that simple though. We can't perfectly align the angular velocity, we can just minimize the other components. Let's take a look at what happens when there is a small wobble along the other directions; so that at $t = 0$, the angular velocity is $\vec{\omega} = \delta\omega\hat{\omega}_1 + \delta\omega\hat{\omega}_2 + \omega_3\hat{\omega}_3$ and $\delta\omega \ll \omega_3$, with $\delta\omega$ and ω_3 positive constants. Writing out Euler's equation at this initial time, we get:

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}_{t=0} = \begin{bmatrix} -\omega_3\delta\omega(I_3 - I_2)/I_1 \\ -\omega_3\delta\omega(I_1 - I_3)/I_2 \\ -\delta\omega^2(I_2 - I_1)/I_3 \end{bmatrix}_{t=0}$$

Notice, first of all, that $\dot{\omega}_3$ is equal to a very small number if $\delta\omega$ is small, meaning that ω_3 will not change rapidly and we can treat it effectively as a constant, which I'll just call ω_0 so it is not confused with being a variable. The other two equations are not negligible, to first order in $\delta\omega$, so for a future time, ω_2 and ω_3 must remain variables.

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} \approx \begin{bmatrix}-\omega_2\omega_0(I_3 - I_2)/I_1\\ -\omega_1\omega_0(I_1 - I_3)/I_2\\ 0\end{bmatrix} \tag{52}$$

This has now become a linear system of equations. This solution is linearized in the vicinity of the initial conditions and is therefore inadequate for long term behavior. I will press on, though. In general, we must consider how ω_1 and ω_2 behave in concert. The two appear as exponentials, either increasing or decreasing depending on the overall sign, which in turn depends on which of the three principal moments of inertia are bigger than the others. However, it is not that simple, because the equations are coupled. If the constant coefficients in each equation are positive, the two will increase as an exponential, so the rotation around those axes will become unstable and the object will tumble. If one is increasing and the other decreasing, they don't behave like real exponentials at all. We'll need another method to be sure of their behavior, which will, in turn, reassure us of our previous conclusions.

Now, we can analyze the different possible orientations of the three axes. For a rectangular solid, it matters along which axis the matter is more spread out from the center of mass. If matter is most spread out from the 3-axis, I_3 will be largest. If the matter is most condensed along the 3-axis, I_3 will be the smallest.

Suppose that I_3 is in the middle, so that $I_1 < I_3 < I_2$. ω_1 will be like an increasing exponential, as long as ω_2 is not a decreasing exponential. A glance at ω_2 show that it, too, will be an increasing exponential. This shows that spinning an object with three distinct principal axes, about its intermediate axis is unstable, and will cause the object to tumble. If you try it, you will see this is true.

Suppose that I_3 is in the middle, so that $I_2 < I_3 < I_1$. This, oddly enough, yields something like two decreasing exponential equations. Note that these components of angular velocity are not strictly positive. As the object rotates, tiny wobbles from air resistance will occur in both the positive and negative directions. For this orientation, positive wobbles will be damped but negative wobbles will amplify, meaning that both scenarios, this and the last orientation of the axes, will be unstable given long enough time rotating in the air.

Suppose that I_3 is the largest, so that $I_3 > I_2 > I_1$. We get conflicting results, at first glance, as previously mentioned. This same is true for I_3 being the smallest. We have a linear system of differential equations, and we should treat it this way: with matrix methods. The most simplified form of the vector equation, Eq 52, can be written like a case of matrix multiplication. Eliminating $\omega_3 \approx \omega_0$:

$$\dot{\vec{\omega}} = \overline{A}\vec{\omega} \tag{53}$$

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\end{bmatrix} = -\omega_0 \begin{bmatrix}0 & (I_3 - I_2)/I_1\\ (I_1 - I_3)/I_2 & 0\end{bmatrix} \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \tag{54}$$

Let's try to diagonalize this, because the eigenvalues and eigenvectors will give the solution to the system of equations, with the following:

$$\vec{\omega}(t) = C_1\vec{u}_1 e^{\lambda_1 t} + C_2\vec{u}_2 e^{\lambda_2 t} \tag{55}$$

Here, C_1 and C_2 are constants determined by the initial conditions, $\vec{u}_1$ and $\vec{u}_2$ are the eigenvectors of the matrix, and λ_1 and λ_2 are the corresponding eigenvalues. This can be shown to be the solution as follows, assuming the matrix $\overline{A}$ is diagonalizable (which it is in this case), and keeping in mind that the inverse of an orthogonal matrix is equal to its transpose:

$$\begin{aligned}
\dot{\vec{x}} &= \overline{A}\vec{x}\\
\overline{P}^T\dot{\vec{x}} &= \overline{P}^T\overline{A}(\overline{P}\overline{P}^T)\vec{x}\\
\overline{P}^T\dot{\vec{x}} &= (\overline{P}^T\overline{A}\overline{P})\overline{P}^T\vec{x}\\
\overline{P}^T\dot{\vec{x}} &= \overline{D}(\overline{P}^T\vec{x})\\
\dot{\tilde{x}} &= \overline{D}\tilde{x}\\
\begin{bmatrix}\vec{u}_1\cdot\dot{\vec{x}}\\ \vec{u}_2\cdot\dot{\vec{x}}\end{bmatrix} &= \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\vec{u}_1\cdot\vec{x}\\ \vec{u}_1\cdot\vec{x}\end{bmatrix} = \begin{bmatrix}\lambda_1\vec{u}_1\cdot\vec{x}\\ \lambda_2\vec{u}_2\cdot\vec{x}\end{bmatrix}
\end{aligned}$$

Here, $\overline{P}$ is the matrix of column-eigenvectors. To find out the general solution for our problem, I will find the eigenvalues and eigenvectors for $\overline{A}$ for this problem:

$$\begin{aligned}
\overline{A}\vec{u} &= \lambda\vec{u} \\
(\overline{A} - \lambda\mathbb{I})\vec{u} &= 0 \\
\det(\overline{A} - \lambda\mathbb{I}) &= 0 \\
\begin{vmatrix} -\lambda & -\omega_0(I_3 - I_2)/I_1 \\ -\omega_0(I_1 - I_3)/I_2 & -\lambda \end{vmatrix} &= 0 \\
\lambda_\pm &= \pm|\omega_0|\sqrt{\frac{(I_1 - I_3)(I_3 - I_2)}{I_1 I_2}} \qquad (56)
\end{aligned}$$

I had to be very careful with the sign because ω_0 could be negative. The term in the square root can also be negative. It is enough of a start to see that the eigenvalues for when I_3 is the intermediate principal axis are both real, yielding an exponential. When I_3 is not the intermediate axis, the eigenvalues are imaginary, yielding a solution with a complex exponential, meaning the wobble oscillates in time and does not become unstable. I will proceed with finding the eigenvectors to show that what will happen for the two real-valued eigenvalues, with I_3 being an intermediate axis. We do this by row-reducing the matrix above and solving for the eigenvectors to make this equation always zero.

$$(\overline{A} - \lambda_\pm\mathbb{I})\vec{u}_\pm = 0$$

$$\begin{aligned}
\begin{bmatrix} \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} & -\omega_0(I_3 - I_2)/I_1 \\ -\omega_0(I_1 - I_3)/I_2 & \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} \end{bmatrix} &= \begin{bmatrix} \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} & -\omega_0(I_3 - I_2)/I_1 \\ \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} & \frac{-I_2}{\omega_0(I_1-I_3)}\frac{\omega_0^2(I_1-I_3)(I_3-I_2)}{I_1 I_2} \end{bmatrix} \\
&= \begin{bmatrix} \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} & -\omega_0(I_3 - I_2)/I_1 \\ \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} & -\omega_0(I_3 - I_2)/I_1 \end{bmatrix} \\
&= \begin{bmatrix} \mp|\omega_0|\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} & -\omega_0(I_3 - I_2)/I_1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mp|\omega_0|\sqrt{\frac{(I_1 - I_3)(I_3 - I_2)}{I_1 I_2}}u_{\pm1} - \omega_0\frac{I_3 - I_2}{I_1}u_{\pm2} &= 0 \qquad \text{(pick } u_{\pm2} = 1\text{)} \\
\mp|\omega_0|\sqrt{\frac{(I_1 - I_3)(I_3 - I_2)}{I_1 I_2}}u_{\pm1} - \omega_0\frac{I_3 - I_2}{I_1}(1) &= 0 \\
u_{\pm1} &= \mp\frac{\omega_0}{|\omega_0|}\frac{(I_3 - I_2)}{I_1}\sqrt{\frac{I_1 I_2}{(I_1 - I_3)(I_3 - I_2)}} \qquad (57)
\end{aligned}$$

I cannot simplify this any further since the difference of moments could be negative and bringing it inside the square root would destroy that. Notice that, in row reducing, we had to assume that $I_1 \neq I_3 \neq I_2$ so we didn't divide by zero. So these solutions will not work for more symmetric objects. Such detailed solutions won't be as difficult, since cancelation will happen right at the beginning. I won't go into much detail on that, I will just give a quick statement about the most symmetric system (the sphere), but you can easily redo this with those and other symmetric assumptions, like a cylinder about different axes. We have the eigenvectors, so here's the general solution.

$$\vec{\omega}(t) = C_+\vec{u}_+e^{\lambda_+ t} + C_-\vec{u}_-e^{\lambda_- t} \qquad (58)$$

$$\lambda_\pm = \pm|\omega_0|\sqrt{\frac{(I_1 - I_3)(I_3 - I_2)}{I_1 I_2}} \qquad (59)$$

$$\begin{bmatrix}\omega_1(t) \\ \omega_2(t)\end{bmatrix} = C_+\begin{bmatrix} -\frac{\omega_0}{|\omega_0|}\frac{(I_3-I_2)}{I_1}\sqrt{\frac{I_1 I_2}{(I_1-I_3)(I_3-I_2)}} \\ 1 \end{bmatrix}e^{\lambda_+ t} + C_-\begin{bmatrix} +\frac{\omega_0}{|\omega_0|}\frac{(I_3-I_2)}{I_1}\sqrt{\frac{I_1 I_2}{(I_1-I_3)(I_3-I_2)}} \\ 1 \end{bmatrix}e^{\lambda_- t} \qquad (60)$$

You can easily take the derivative of each component of the angular velocity to see that it does satisfy the Euler's equations simplified for this problem. Note, again, that only when I_3 is an intermediate value do we get a real exponential, otherwise, ω_1 and ω_2 oscillate but are stable. The C_+ solution is the increasing exponential and C_- the decreasing exponential. We really want to see the conditions on C_+ and C_- to know when each of the two solutions dominates. It's clear that, the increasing exponential will dominate whenever it can, which is when $C_+ \neq 0$. These coefficients are the constants of integration set for when $t = 0$, so let's set $t = 0$ and require that the ω components are small values for the initial conditions. For generality, we don't want them to be the same, or to be strictly positive, so I'll call them $\delta\omega_1$ and $\delta\omega_2$, which can be positive or negative.

$$\begin{bmatrix}\omega_1(t=0)\\ \omega_2(t=0)\end{bmatrix} = \begin{bmatrix}\delta\omega_1\\ \delta\omega_2\end{bmatrix} = C_+ \begin{bmatrix}-\frac{\omega_0}{|\omega_0|}\frac{(I_3-I_2)}{I_1}\sqrt{\frac{I_1 I_2}{(I_1-I_3)(I_3-I_2)}}\\ 1\end{bmatrix} + C_- \begin{bmatrix}+\frac{\omega_0}{|\omega_0|}\frac{(I_3-I_2)}{I_1}\sqrt{\frac{I_1 I_2}{(I_1-I_3)(I_3-I_2)}}\\ 1\end{bmatrix}$$

$$\delta\omega_1 = \frac{\omega_0}{|\omega_0|}\frac{(I_3-I_2)}{I_1}\sqrt{\frac{I_1 I_2}{(I_1-I_3)(I_3-I_2)}}(C_- - C_+)$$

$$\delta\omega_2 = C_+ + C_-$$

$$C_+ = -\frac{1}{2}\left(\delta\omega_1 \frac{|\omega_0|}{\omega_0}\frac{I_1}{(I_3-I_2)}\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} - \delta\omega_2\right) \tag{61}$$

$$C_- = \frac{1}{2}\left(\delta\omega_1 \frac{|\omega_0|}{\omega_0}\frac{I_1}{(I_3-I_2)}\sqrt{\frac{(I_1-I_3)(I_3-I_2)}{I_1 I_2}} + \delta\omega_2\right) \tag{62}$$

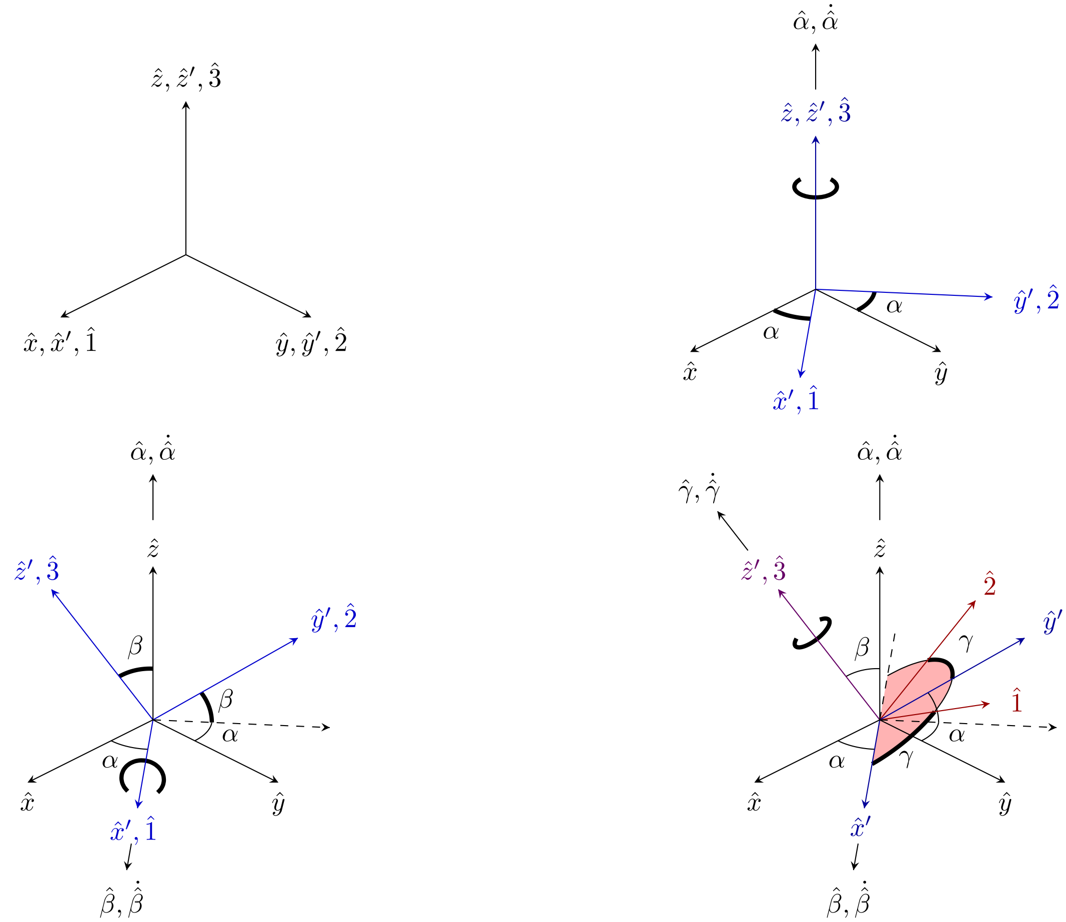
Only in the very rare instance that $C_+ = 0$ does the system not become unstable, which is not theoretically impossible under highly idealized conditions, but small variations from this that can easily occur from air resistance will make $C_+ \neq 0$ even if it temporarily does equal this value. Finally note that the system is unstable, regardless of whether I_1 is the greatest or least principal moment, because the eigenvalues, eigenvectors, and coefficients get two factors of a negative sign inside the square root, when I_1 switches with I_2 as the greatest and least values. For a demonstration of this, with perhaps some minor glitches (that don't change whether the system is stable or unstable), see my simulations at www.glowscript.org/#/user/owendix/. Recall that this solution is linearized in the vicinity of the initial conditions, so its long-term behavior is not to be trusted as physical. Thorough solutions will use constants of the motion, including angular momentum and energy, and involve Jacobi elliptic integrals, as previously mentioned.

If we have a sphere, then $I_1 = I_2 = I_3$ (or at least approximately for approximate spheres) and the rate of change of the three components of angular velocity are zero. For a cylinder, two of the principal moments are the same and the other could be larger or smaller than this, depending on if its a short-fat cylinder or tall-thin cylinder. I'll let the more interested reader try these simplere scenarios.

3.6.2 Euler's Angles

For Euler's angles useful for solving problems with torque, see Analytical Mechanics, Chapter 13 on Rotational Motion, or some other resource. Euler's angles or something comparable (but physicists generally use these) let you understand both precession and nutation. I am partial to use of the variables (α, β, γ) representing the ordered set of rotations since it is much easier to remember which comes first than (ϕ, θ, ψ) - or whatever order they are commonly represented in. Wikipedia uses the former. All that you need to remember, then is how they are used to tilt the different axes. Three sets of coordinates are used: a (noninertial) body-fixed set of coordinates $(1, 2, 3)$ which align with the principal axes of the system so the rotational inertia matrix is diagonal. Does this require putting the origin at the center of mass? I previously derived that you can shift the fixed point along one of the 3 principal axes, but keep the principal axes pointing in the same direction, and still have rotational inertia matrix be diagonal (though with different eigenvalues).

I will be using the 3-1-3 sequence of rotating the axes: first rotate around the 3-axis by an angle α, then around the (now displaced) 1-axis by an angle β, finally again around the new 3-axis by an angle γ. For help visualizing it, the 3-axis is generally picked to point along the primary angular velocity vector, say, for a rotating wheel. There is another component to this, though, as the system precesses, nutates, or tumbles. Then there is a noninertial set of primed coordinates (x', y', z') where z' aligns with the 3-axis and, since the system can precess and nutate, the coordinates are noninertial, but the x' and y' axes do not rotate with the body completely, so that γ gives the deviation of the 1- and 2-axes from x' and y'. Finally there is the inertial, unprimed (x, y, z) set of coordinates, that are easiest for us to observe. The Euler angles are applied by first starting with all three coordinate systems aligned in the order written. The angles are all applied in the right-handed sense. α rotates the primed and body-fixed coordinates around the z, z', and 3 axes which are still colinear.



The vector $\vec{\alpha}$ points along the z axis, even after rotating the other angles. Second, $\vec{\beta}$ rotates the z' and 3 axes away from the z axis, around the x' axis, and always points along x' axis even after the final rotation. Lastly, γ rotates the 1 and 2 axes within the plane of x' and y'. z' and 3 remain colinear so that $\vec{\gamma}$ points along these two axes. With these defined you can use the angles to relate the three coordinate systems and solve Euler's equations with torque. For compactness, I will abbreviate sin and cos of the angles as shown below.

$$
\begin{gathered}
c_\alpha = \cos\alpha \quad c_\beta = \cos\beta \quad c_\gamma = \cos\gamma \\
s_\alpha = \sin\alpha \quad s_\beta = \sin\beta \quad s_\gamma = \sin\gamma \\
\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ -s_\alpha c_\beta & c_\alpha c_\beta & s_\beta \\ s_\alpha s_\beta & -c_\alpha s_\beta & c_\beta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{1} \\ \hat{2} \\ \hat{3} \end{bmatrix} \\
\begin{bmatrix} \hat{1} \\ \hat{2} \\ \hat{3} \end{bmatrix} = \begin{bmatrix} (c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma) & (s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma) & s_\beta s_\gamma \\ -(c_\alpha s_\gamma + s_\alpha c_\beta c_\gamma) & -(s_\alpha s_\gamma - c_\alpha c_\beta c_\gamma) & s_\beta c_\gamma \\ s_\alpha s_\beta & -c_\alpha s_\beta & c_\beta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} c_\gamma & s_\gamma & 0 \\ -s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} \\
\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} c_\alpha & -s_\alpha c_\beta & s_\alpha s_\beta \\ s_\alpha & c_\alpha c_\beta & -c_\alpha s_\beta \\ 0 & s_\beta & c_\beta \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} (c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma) & -(c_\alpha s_\gamma + s_\alpha c_\beta c_\gamma) & s_\alpha s_\beta \\ (s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma) & -(s_\alpha s_\gamma - c_\alpha c_\beta c_\gamma) & -c_\alpha s_\beta \\ s_\beta s_\gamma & s_\beta c_\gamma & c_\beta \end{bmatrix} \begin{bmatrix} \hat{1} \\ \hat{2} \\ \hat{3} \end{bmatrix}
\end{gathered}
\tag{63}
$$

These transformations can be acquired by looking at the projections of the different unit vectors onto the other frames' unit vectors, then making some substitutions. They can also be acquired through a composition of multiple rotations about the 3, then 1, then 3 axes by the different respective angles. I've put these transformations in the clearer matrix form, which also lets you see that the inverse transformation matrices are simply the transpose of each other, since all three coordinate systems are orthogonal. Though I have cast it in terms of a vector of unit vectors, they can equally apply to the components of a vector in those coordinates. I will demonstrate this. Let $\overline{A}$ be the final matrix in Eq 63 so that $\vec{u}(xyz) = \overline{A}\vec{u}(123)$, where $\vec{u}$ is the vector of unit vectors in the coordinate system described. This relationship says the following is true for another vector, such as $\vec{\omega}$, which does not need to be modified for being in a rotating reference frame. Only its components change.

$$
\begin{aligned}
\hat{x} &= A_{x1}\hat{1} + A_{x2}\hat{2} + A_{x3}\hat{3} \\
\hat{y} &= A_{y1}\hat{1} + A_{y2}\hat{2} + A_{y3}\hat{3} \\
\hat{z} &= A_{z1}\hat{1} + A_{z2}\hat{2} + A_{z3}\hat{3} \\
\vec{\omega} &= \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z} \\
&= \omega_x\left(A_{x1}\hat{1} + A_{x2}\hat{2} + A_{x3}\hat{3}\right) + \omega_y\left(A_{y1}\hat{1} + A_{y2}\hat{2} + A_{y3}\hat{3}\right) + \omega_z\left(A_{z1}\hat{1} + A_{z2}\hat{2} + A_{z3}\hat{3}\right) \\
&= \left(\omega_x A_{x1} + \omega_y A_{y1} + \omega_z A_{z1}\right)\hat{1} + \left(\omega_x A_{x2} + \omega_y A_{y2} + \omega_z A_{z2}\right)\hat{2} + \left(\omega_x A_{x3} + \omega_y A_{y3} + \omega_z A_{z3}\right)\hat{3} \\
&= \omega_1\hat{1} + \omega_2\hat{2} + \omega_3\hat{3} \\
\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} &= \begin{bmatrix}\omega_x A_{x1} + \omega_y A_{y1} + \omega_z A_{z1}\\ \omega_x A_{x2} + \omega_y A_{y2} + \omega_z A_{z2}\\ \omega_x A_{x3} + \omega_y A_{y3} + \omega_z A_{z3}\end{bmatrix} = \begin{bmatrix}A_{x1} & A_{y1} & A_{z1}\\ A_{x2} & A_{y2} & A_{z2}\\ A_{x3} & A_{y3} & A_{z3}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \\
\vec{\omega}_{123} &= \overline{A}^T\vec{\omega}_{xyz} \\
\vec{\omega}_{xyz} &= \overline{A}\vec{\omega}_{123}
\end{aligned}
$$

We need to relate these angles to the object's angular velocity and the angular velocity primed coordinate system. Imagine an infinitesimal rotation about some general direction, $d\vec{\theta}$. The angle vectors may not add to $\vec{\theta}$, but because we are using infinitesimal vectors $(d\vec{\alpha}, d\vec{\beta}, d\vec{\gamma})$ will add to $d\vec{\theta}$. Dividing by the infinitesimal time interval, we get:

$$\vec{\omega} = \dot{\vec{\alpha}} + \dot{\vec{\beta}} + \dot{\vec{\gamma}} \tag{64}$$

$$\vec{\omega}' = \dot{\vec{\alpha}} + \dot{\vec{\beta}} \tag{65}$$

$\vec{\omega}'$ should be interpreted as the rate of rotation of the $\hat{z}'$ axis and therefore the $(\hat{x}', \hat{y}', \hat{z}')$ coordinate system relative to the inertial one. It is not the rate of rotation around the $\hat{z}'$ axis, since that is clearly $\dot{\vec{\gamma}}$. Correspondingly, $\vec{\omega}$ is the rotation of the $(\hat{1}, \hat{2}, \hat{3})$ coordinate system relative to the inertial coordinate system. These will be useful for solving rotation problems.

Usually, an energy equation is used to find the three nutational modes (it is also possible there will be no nutation, with just precession or with a sleeping top). There may be a way to show this nutation using Euler's equations, but since they are nonlinear, it may be difficult. Also note that, since torque from gravity will always keep the L_z and $L_{z'}$ components the same, nutation must be due to the fact that $\vec{\omega}$ doesn't point perfectly along the symmetry axis (which also occurs with zero torque - hence a poorly spiraled football) and because $\vec{\omega}$ doesn't point along $\vec{L}$, since the rotation matrix would have to be proportional to an identity matrix, not even just diagonal, for $\vec{\omega}$ to be colinear with $\vec{L}$.

3.6.3 Euler's Equations and Angles with a Gravitational Torque

We can use the angle relationships above to find Euler's equations for a rotating object with cylindrical symmetry, like a bike wheel or top, attached at a single fixed point which is not the center of mass. Luckily the parallel axis theorem shows that once we start with a diagonal rotational inertia matrix, shifting from the center of mass origin to a new origin directly along one of the principal axes still yields a diagonal rotational inertia matrix. This is exactly what we want since the ideal example is a bike wheel or top fixed at one end, which has a symmetry axis (one principal axis) along the axle and, because of its cylindrical symmetry along that axis, we can pick any two perpendicular vectors as the others, see Eq 49. The principal axes will still be aligned with the symmetry axes for the object in the center of mass frame. The components of the rotational inertia with the fixed point at the origin will use the following variables, true for both the (x', y', z') and $(1, 2, 3)$ reference frames again because of the object's cylindrical symmetry:

$$\overline{I} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_s \end{bmatrix}$$

Let's assume the object's total mass is M and the axle has a length of $2L$ so that the center of mass is halfway along the axle. Because the force of gravity points down, along the $\hat{z}$ axis, and $\hat{z}'$ and $\hat{3}$ point along the axle, the torque from gravity will be aligned with $\hat{x}'$ at all times:

$$
\begin{aligned}
\vec{\tau} &= MgL\sin\beta\hat{x}' \\
&= MgL\sin\beta(\cos\gamma\hat{1} - \sin\gamma\hat{2})
\end{aligned}
$$

Both the primed and body-fixed coordinates are noninertial so Euler's equations will involve the cross product with the angular velocity. Having already computed what this is for body-fixed coordinates, the $(1,2,3)$ frame's Euler's equations follow directly.

$$\vec{\omega} = (\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma)\hat{1} + (\dot{\alpha}\sin\beta\cos\gamma - \dot{\beta}\sin\gamma)\hat{2} + (\dot{\alpha}\cos\beta + \dot{\gamma})\hat{3}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix} = \begin{bmatrix}MgL\sin\beta\cos\gamma\\ -MgL\sin\beta\sin\gamma\\ 0\end{bmatrix} = \begin{bmatrix}I\dot{\omega}_1 + \omega_2\omega_3(I_s - I)\\ I\dot{\omega}_2 + \omega_3\omega_1(I - I_s)\\ I_s\dot{\omega}_3\end{bmatrix} =$$
$$\begin{bmatrix}I(\ddot{\alpha}\sin\beta\sin\gamma + \dot{\alpha}\dot{\beta}\cos\beta\sin\gamma + \dot{\alpha}\dot{\gamma}\sin\beta\cos\gamma + \ddot{\beta}\cos\gamma - \dot{\beta}\dot{\gamma}\sin\gamma) + (\dot{\alpha}\sin\beta\cos\gamma - \dot{\beta}\sin\gamma)(\dot{\alpha}\cos\beta + \dot{\gamma})(I_s - I)\\ I(\ddot{\alpha}\sin\beta\cos\gamma + \dot{\alpha}\dot{\beta}\cos\beta\cos\gamma - \dot{\alpha}\dot{\gamma}\sin\beta\sin\gamma - \ddot{\beta}\sin\gamma - \dot{\beta}\dot{\gamma}\cos\gamma) + (\dot{\alpha}\cos\beta + \dot{\gamma})(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma)(I - I_s)\\ I_s(\ddot{\alpha}\cos\beta - \dot{\alpha}\sin\beta + \ddot{\gamma})\end{bmatrix}$$

The latter relationships, Euler's equations in the $(1,2,3)$ frame, are pretty hideous. The added information tracks the rotation by the angle γ around the $\hat{3}$ axis, but the equations are complicated enough, as they without this information. I'll only pursue Euler's equations in the (x', y', z') frame. Note that, since we aren't including any torque due to friction of the wheel moving around the axle, the expected gradual decrease in $\dot{\gamma}$ will not be there.

Recall that $\vec{\omega}$ in Euler's equation (whose general form is repeated below) is the rotation of the coordinate system. It was derived using a body-fixed coordinate system, so that the rotation of the coordinate system told you the rotation of all particles in the object. Naively trying to transform Euler's equation in the finished form will lead to errors, since the first terms in Euler's equations involved $\vec{\omega}$ and the second terms involved both $\vec{\omega}'$ in the cross product, and $\vec{\omega}$ in the angular momentum. We can take advantage of how $\overline{I}$ is an identical diagonal matrix in both $(1,2,3)$ and (x', y', z'). The following shows what I mean and the correct way to do it, with all quantities in the (x', y', z') basis:

$$\vec{\tau} = \dot{\vec{L}} + \vec{\omega}' \times \vec{L}$$
$$\begin{bmatrix}L_{x'}\\ L_{y'}\\ L_{z'}\end{bmatrix} = \begin{bmatrix}I\omega_{x'}\\ I\omega_{y'}\\ I_s\omega_{z'}\end{bmatrix}$$
$$\begin{bmatrix}\tau_{x'}\\ \tau_{y'}\\ \tau_{z'}\end{bmatrix} = \begin{bmatrix}I\dot{\omega}_{x'} + \omega'_{y'}\omega_{z'}I_s - \omega'_{z'}\omega_{y'}I\\ I\dot{\omega}_{y'} + \omega'_{z'}\omega_{x'}I - \omega'_{x'}\omega_{z'}I_s\\ I_s\dot{\omega}_{z'} + \omega'_{x'}\omega_{y'}I - \omega'_{y'}\omega_{x'}I\end{bmatrix} \tag{66}$$

The angular velocity vectors, for tracking the rotation of the primed coordinate system and motion of the object, itself, are:

$$\vec{\omega}' = \dot{\beta}\hat{x}' + \dot{\alpha}\sin\beta\hat{y}' + \dot{\alpha}\cos\beta\hat{z}' \tag{67}$$
$$\vec{\omega} = \dot{\beta}\hat{x}' + \dot{\alpha}\sin\beta\hat{y}' + (\dot{\alpha}\cos\beta + \dot{\gamma})\hat{z}' \tag{68}$$

The angular momentum vector expanded in this basis is:

$$\begin{bmatrix}L_{x'}\\ L_{y'}\\ L_{z'}\end{bmatrix} = \begin{bmatrix}I\dot{\beta}\\ I\dot{\alpha}\sin\beta\\ I_s(\dot{\alpha}\cos\beta + \dot{\gamma})\end{bmatrix} = \begin{bmatrix}I\dot{\beta}\\ I\dot{\alpha}\sin\beta\\ I_sS\end{bmatrix} \tag{69}$$

I defined the term S, as the component of the object's angular velocity along the primary axis, $\hat{z}'$:

$$S = \omega_{z'} = \dot{\alpha}\cos\beta + \dot{\gamma} \tag{70}$$

This will be useful because, as we'll see, S is constant for this problem.

Noting that the only difference between $\vec{\omega}$ and $\vec{\omega}'$ is the z'-component, we can now put this together to get Euler's equations in the (x', y', z') frame.

$$\begin{bmatrix}\tau_{x'}\\ \tau_{y'}\\ \tau_{z'}\end{bmatrix} = \begin{bmatrix}MgL\sin\beta\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}I\dot{\omega}_{x'} + \omega'_{y'}\omega_{z'}I_s - \omega'_{z'}\omega_{y'}I\\ I\dot{\omega}_{y'} + \omega'_{z'}\omega_{x'}I - \omega'_{x'}\omega_{z'}I_s\\ I_s\dot{\omega}_{z'}\end{bmatrix} = \begin{bmatrix}I\ddot{\beta} + \dot{\alpha}\sin\beta SI_s - \dot{\alpha}\cos\beta\dot{\alpha}\sin\beta I\\ I(\ddot{\alpha}\sin\beta + \dot{\alpha}\dot{\beta}\cos\beta) + \dot{\alpha}\dot{\beta}\cos\beta I - \dot{\beta}SI_s\\ I_s\dot{S}\end{bmatrix}$$
$$= \begin{bmatrix}I\ddot{\beta} + I_sS\dot{\alpha}\sin\beta - I\dot{\alpha}^2\sin\beta\cos\beta\\ I\ddot{\alpha}\sin\beta + 2I\dot{\alpha}\dot{\beta}\cos\beta - I_sS\dot{\beta}\\ I_s\dot{S}\end{bmatrix} \tag{71}$$

The third term of Euler's equation, the z'-component is:

$$\begin{aligned} 0 = I_s \dot{S} = \dot{L}_{z'} \\ I_s S = constant \end{aligned}$$

This says that the component of angular momentum along the $\hat{z}'$-direction is constant, which is understandable since torque from gravity points perpendicular to $\hat{z}'$. It also points perpendicular to $\hat{z}$, so we should expect this angular momentum components is constant, too. To see if this is true, I'll need to transform into those coordinates and pull out the $\hat{z}$-component of $\vec{\omega}$ and $\vec{L}$.

$$\begin{aligned}
\vec{\omega} =& \dot{\beta}\hat{x}' + \dot{\alpha}\sin\beta\hat{y}' + (\dot{\alpha}\cos\beta + \dot{\gamma})\hat{z}' \\
=& \dot{\beta}(\cos\alpha\hat{x} + \sin\alpha\hat{y}) + \dot{\alpha}\sin\beta(-\sin\alpha\cos\beta\hat{x} + \cos\alpha\cos\beta\hat{y} + \sin\beta\hat{z}) \\
& + (\dot{\alpha}\cos\beta + \dot{\gamma})(\sin\alpha\sin\beta\hat{x} - \cos\alpha\sin\beta\hat{y} + \cos\beta\hat{z}) \\
=& \left(\dot{\beta}\cos\alpha - \dot{\alpha}\sin\beta\sin\alpha\cos\beta + (\dot{\alpha}\cos\beta + \dot{\gamma})\sin\alpha\sin\beta\right)\hat{x} \\
& + \left(\dot{\beta}\sin\alpha + \dot{\alpha}\sin\beta\cos\alpha\cos\beta - (\dot{\alpha}\cos\beta + \dot{\gamma})\cos\alpha\sin\beta\right)\hat{y} \\
& + (\dot{\alpha}\sin\beta\sin\beta + (\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta)\,\hat{z} \\
=& \left(\dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta\right)\hat{x} + \left(\dot{\beta}\sin\alpha - \dot{\gamma}\cos\alpha\sin\beta\right)\hat{y} + (\dot{\alpha} + \dot{\gamma}\cos\beta)\,\hat{z} \\
\omega_z =& \dot{\alpha} + \dot{\gamma}\cos\beta
\end{aligned}$$

This component of angular velocity makes sense from a geometric perspective, just by looking at the figures showing Euler's angles. Since $\vec{L} = \overline{I}\vec{\omega}$, I either have to transform $\overline{I}$, as well, or notice the following change of basis relationship:

$$\begin{aligned}
\vec{L} &= \overline{I}\vec{\omega} \\
\overline{P}\vec{L} &= \overline{PI}\vec{\omega} \\
\overline{P}\vec{L} &= \overline{PI}\left(\overline{P}^T\overline{P}\right)\vec{\omega} \\
\overline{P}\vec{L} &= \left(\overline{PIP}^T\right)\left(\overline{P}\vec{\omega}\right)
\end{aligned}$$

Here, $\overline{P}$ is the matrix transforming (x', y', z') into (x, y, z). The approach of transforming $\overline{I}$ to the new coordinate system comes from the last line, but our way around this step comes from the second line. We can just left-multiply $\overline{I}\vec{\omega}$ by the change of basis matrix, which is essentially what I did to $\vec{\omega}$ to get ω_z a moment ago. Since $\overline{I}$ is diagonal and we only need the L_z-component, one might naively think that left-multiplying $\overline{I}\vec{\omega}$ by $\overline{P}$ is equivalent to just multiplying ω_z by I_s:

$$L_z = \left(\overline{P}\left(\overline{I}\vec{\omega}\right) \cdot \hat{z}\right) \neq I_s(\dot{\alpha} + \dot{\gamma}\cos\beta)$$

The last relationship is not the correct form for L_z because different terms in $\overline{I}\vec{\omega}$ simplified to make our form for $\omega_z = \dot{\alpha} + \dot{\gamma}\cos\beta$. I'll compute the full form for $\vec{\omega}$ and $\overline{I}$ in the (x, y, z) basis. Note that $\overline{P}$ is the matrix taking the column vector (x', y', z') to (x, y, z) from our transformation matrices above, as you would intuitively expect, since our goal is to take vectors in the former basis to the latter basis. Starting from vectors and matrices in (x', y', z'):

$$\begin{aligned}
\vec{L} = \overline{I}\vec{\omega} &= \begin{bmatrix} I\omega_{x'} \\ I\omega_{y'} \\ I_s\omega_{z'} \end{bmatrix} = \begin{bmatrix} I\dot{\beta} \\ I\dot{\alpha}\sin\beta \\ I_s(\dot{\alpha}\cos\beta + \dot{\gamma}) \end{bmatrix} = \begin{bmatrix} I\dot{\beta} \\ I\dot{\alpha}\sin\beta \\ I_s S \end{bmatrix} \\
\overline{P} &= \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} && (72) \\
\overline{P}\vec{L} = \overline{P}(\overline{I}\vec{\omega}) &= \begin{bmatrix} I\dot{\beta}\cos\alpha - I\dot{\alpha}\sin\beta\sin\alpha\cos\beta + I_s(\dot{\alpha}\cos\beta + \dot{\gamma})\sin\alpha\sin\beta \\ I\dot{\beta}\sin\alpha + I\dot{\alpha}\sin\beta\cos\alpha\cos\beta - I_s(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\alpha\sin\beta \\ I\dot{\alpha}\sin\beta\sin\beta + I_s(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta \end{bmatrix} \\
\vec{L}_{(x,y,z)} &= \begin{bmatrix} I\dot{\beta}\cos\alpha + (I_s S - I\dot{\alpha}\cos\beta)\sin\alpha\sin\beta \\ I\dot{\beta}\sin\alpha + (I\dot{\alpha}\cos\beta - I_s S)\cos\alpha\sin\beta \\ I\dot{\alpha}\sin^2\beta + I_s S\cos\beta \end{bmatrix} && (73) \\
&= \begin{bmatrix} I\dot{\beta}\cos\alpha + (I_s - I)\dot{\alpha}\sin\alpha\sin\beta\cos\beta + I_s\dot{\gamma}\sin\alpha\sin\beta \\ I\dot{\beta}\sin\alpha + (I - I_s)\dot{\alpha}\cos\alpha\sin\beta\cos\beta - I_s\dot{\gamma}\cos\alpha\sin\beta \\ \dot{\alpha}(I\sin^2\beta + I_s\cos^2\beta) + I_s\dot{\gamma}\cos\beta \end{bmatrix} && (74) \\
\vec{\omega}_{(x,y,z)} &= \begin{bmatrix} \dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta \\ \dot{\beta}\sin\alpha - \dot{\gamma}\cos\alpha\sin\beta \\ \dot{\alpha} + \dot{\gamma}\cos\beta \end{bmatrix} && (75)
\end{aligned}$$

The result in the second to last line is $\vec{L}$ in the (x, y, z) basis. The last line is $\vec{\omega}$ in the (x, y, z) basis, which I included for easy future comparison. For sake of completeness, I will calculate $\overline{I}$ in the (x, y, z) basis:

$$\begin{aligned}
\overline{PIP}^T &= \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_s \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} I\cos\alpha & I\sin\alpha & 0 \\ -I\sin\alpha\cos\beta & I\cos\alpha\cos\beta & I\sin\beta \\ I_s\sin\alpha\sin\beta & -I_s\cos\alpha\sin\beta & I_s\cos\beta \end{bmatrix} \\
\overline{I}_{(x,y,z)} &= \begin{bmatrix} I\cos^2\alpha + \sin^2\alpha(I\cos^2\beta + I_s\sin^2\beta) & \sin\alpha\cos\alpha(I - I\cos^2\beta - I_s\sin^2\beta) & (I_s - I)\sin\alpha\sin\beta\cos\beta \\ \sin\alpha\cos\alpha(I - I\cos^2\beta - I_s\sin^2\beta) & I\sin^2\alpha + \cos^2\alpha(I\cos^2\beta + I_s\sin^2\beta) & (I - I_s)\cos\alpha\sin\beta\cos\beta \\ (I_s - I)\sin\alpha\sin\beta\cos\beta & (I - I_s)\cos\alpha\sin\beta\cos\beta & I\sin^2\beta + I_s\cos^2\beta \end{bmatrix}
\end{aligned} \tag{76}$$

Notice that $\overline{I}_{(x,y,z)}$ is still symmetric, as it should be. Our primary goal in doing this transformation was to see if the z-component of $\vec{L}$ is constant with Euler's equations for our rotating bike wheel under gravity scenario. This component and its derivative are:

$$\begin{aligned}
L_z &= \dot{\alpha}(I\sin^2\beta + I_s\cos^2\beta) + I_s\dot{\gamma}\cos\beta = I\dot{\alpha}\sin^2\beta + I_s\dot{\alpha}\cos^2\beta + I_s\dot{\gamma}\cos\beta \\
\dot{L}_z &= I\ddot{\alpha}\sin^2\beta + 2I\dot{\alpha}\dot{\beta}\sin\beta\cos\beta + I_s\ddot{\alpha}\cos^2\beta - 2I_s\dot{\alpha}\dot{\beta}\sin\beta\cos\beta + I_s\ddot{\gamma}\cos\beta - I_s\dot{\beta}\dot{\gamma}\sin\beta
\end{aligned}$$

Written in terms of the spin, S, these become:

$$\begin{aligned}
L_z &= I\dot{\alpha}\sin^2\beta + I_s S\cos\beta && (77) \\
\dot{L}_z &= I\ddot{\alpha}\sin^2\beta + 2I\dot{\alpha}\dot{\beta}\sin\beta\cos\beta + I_s\dot{S}\cos\beta - I_s S\dot{\beta}\sin\beta \\
&= I\ddot{\alpha}\sin^2\beta + 2I\dot{\alpha}\dot{\beta}\sin\beta\cos\beta - I_s S\dot{\beta}\sin\beta && (78)
\end{aligned}$$

The very last line follows from what we previously determined, that the angular momentum z'-component is constant, which means $\dot{S} = 0$. Let's compare this to the y'-component of Euler's equation:

$$\begin{aligned}
\tau_{y'} &= 0 = I\ddot{\alpha}\sin\beta + 2I\dot{\alpha}\dot{\beta}\cos\beta - I_s S\dot{\beta} \\
0\sin\beta &= 0 = I\ddot{\alpha}\sin^2\beta + 2I\dot{\alpha}\dot{\beta}\sin\beta\cos\beta - I_s S\dot{\beta}\sin\beta = \dot{L}_z
\end{aligned}$$

The z-component of angular momentum is, indeed, constant.

Looking back at $\vec{L}_{(x',y',z')}$ and $\vec{\omega}_{(x',y',z')}$, notice that they must point in different directions, due to I and I_s being different values. The angle between these two vector quantities, along with the angles between $\vec{\omega}$ and $\hat{z}'$ and between $\vec{L}$ and $\hat{z}'$ turn out to be quite hideous. The most useful way to approach this is with conservation of energy, but that is not what I wish this document to be about. The relevant sections of Analytical Mechanics by Fowles and Cassiday, 6th ed. include include section 9.8 and the previous sections to establish their notation.

I can assume an initial condition for the angular velocity and momentum. If the bike wheel is started at some nonzero angle β_o from the z-axis, with initial conditions:

$$\begin{aligned}
\dot{\alpha}_o &= 0 \\
\dot{\beta}_o &= 0 \\
\dot{\gamma}_o &> 0 \\
\Rightarrow S &= \dot{\alpha}\cos\beta + \dot{\gamma} = \dot{\gamma}_o
\end{aligned}$$

Since S was determined in our general analysis to be constant, then it will always remain this value, which is why I did not give it a subscript specifying this as its initial state. Euler's equations become:

$$\begin{aligned}
\begin{bmatrix} MgL\sin\beta \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} I\ddot{\beta} + I_sS\dot{\alpha}\sin\beta - I\dot{\alpha}^2\sin\beta\cos\beta \\ I\ddot{\alpha}\sin\beta + 2I\dot{\alpha}\dot{\beta}\cos\beta - I_sS\dot{\beta} \\ I_s\dot{S} \end{bmatrix} = \begin{bmatrix} I\ddot{\beta} + (I_s - I)\dot{\alpha}^2\sin\beta\cos\beta + I_s\dot{\alpha}\dot{\gamma}\sin\beta \\ I\ddot{\alpha}\sin\beta + (2I - I_s)\dot{\alpha}\dot{\beta}\cos\beta - I_s\dot{\beta}\dot{\gamma} \\ I_s\dot{S} \end{bmatrix} \\
\begin{bmatrix} MgL\sin\beta_o \\ 0 \\ 0 \end{bmatrix}_{t=0} &= \begin{bmatrix} I\ddot{\beta} \\ I\ddot{\alpha}\sin\beta \\ I_s\dot{S} \end{bmatrix}_{t=0} \qquad (79)
\end{aligned}$$

This seems to say that, the first moment the wheel is released from rest, the axle drops slightly (β accelerates in the positive direction). The initial angular velocity and angular momentum are:

$$\begin{aligned}
\begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix} &= \begin{bmatrix} \dot{\beta} \\ \dot{\alpha}\sin\beta \\ \dot{\alpha}\cos\beta + \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \dot{\beta} \\ \dot{\alpha}\sin\beta \\ S \end{bmatrix} \\
\begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix}_{t=0} &= \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma}_o \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 0 \\ S \end{bmatrix}_{t=0} \\
\begin{bmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{bmatrix} &= \begin{bmatrix} I\omega_{x'} \\ I\omega_{y'} \\ I_s\omega_{z'} \end{bmatrix} = \begin{bmatrix} I\dot{\beta} \\ I\dot{\alpha}\sin\beta \\ I_s(\dot{\alpha}\cos\beta + \dot{\gamma}) \end{bmatrix} = \begin{bmatrix} I\dot{\beta} \\ I\dot{\alpha}\sin\beta \\ I_sS \end{bmatrix} \\
\begin{bmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{bmatrix}_{t=0} &= \begin{bmatrix} 0 \\ 0 \\ I_s\dot{\gamma}_o \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 0 \\ I_sS \end{bmatrix}_{t=0}
\end{aligned}$$

Both vectors are fully aligned with the $\hat{z}'$ axis in the initial state. Let's look a moment later:

$$\begin{aligned}
\begin{bmatrix} MgL\sin\beta \\ 0 \\ 0 \end{bmatrix}_{t=\epsilon} &= \begin{bmatrix} I\ddot{\beta} \\ I\ddot{\alpha}\sin\beta - I_sS\dot{\beta} \\ I_s\dot{S} \end{bmatrix}_{t=\epsilon} \\
\begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix}_{t=\epsilon} &= \begin{bmatrix} \dot{\beta} \\ 0 \\ \dot{\gamma} \end{bmatrix}_{t=\epsilon} = \begin{bmatrix} \dot{\beta} \\ 0 \\ S \end{bmatrix}_{t=\epsilon} \\
\begin{bmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{bmatrix}_{t=\epsilon} &= \begin{bmatrix} I\dot{\beta} \\ 0 \\ I_s\dot{\gamma} \end{bmatrix}_{t=\epsilon} = \begin{bmatrix} I\dot{\beta} \\ 0 \\ I_sS \end{bmatrix}_{t=\epsilon}
\end{aligned}$$

β continues to increase, as does $\dot{\beta}$. Now also, α begins to acquire some positive velocity, because its value for acceleration is positive:

$$\ddot{\alpha} = \frac{I_sS\dot{\beta}}{I\sin\beta}$$

The angular velocity and angular momentum are no longer aligned with the $\hat{z}'$ axis. Both have some component in the positive $\hat{x}'$ direction, at this instant. Which one points farther from the $\hat{z}'$ direction depends on whether I is less than or greater than I_s. We'll see later that, for approximate values involved in a bike wheel demonstration, $\vec{L}$ always points farther from the $\hat{z}'$ axis than does $\vec{\omega}$.

Another moment later we have:

$$\begin{bmatrix} MgL\sin\beta \\ 0 \\ 0 \end{bmatrix}_{t=2\epsilon} = \begin{bmatrix} I\ddot{\beta} + I_s S\dot{\alpha}\sin\beta - I\dot{\alpha}^2\sin\beta\cos\beta \\ I\ddot{\alpha}\sin\beta + 2I\dot{\alpha}\dot{\beta}\cos\beta - I_s S\dot{\beta} \\ I_s\dot{S} \end{bmatrix}_{t=2\epsilon} = \begin{bmatrix} I\ddot{\beta} + (I_s - I)\dot{\alpha}^2\sin\beta\cos\beta + I_s\dot{\alpha}\dot{\gamma}\sin\beta \\ I\ddot{\alpha}\sin\beta + (2I - I_s)\dot{\alpha}\dot{\beta}\cos\beta - I_s\dot{\beta}\dot{\gamma} \\ I_s\dot{S} \end{bmatrix}_{t=2\epsilon}$$

$$\begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix}_{t=2\epsilon} = \begin{bmatrix} \dot{\beta} \\ \dot{\alpha}\sin\beta \\ \dot{\alpha}\cos\beta + \dot{\gamma} \end{bmatrix}_{t=2\epsilon} = \begin{bmatrix} \dot{\beta} \\ \dot{\alpha}\sin\beta \\ S \end{bmatrix}_{t=2\epsilon}$$

$$\begin{bmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{bmatrix}_{t=2\epsilon} = \begin{bmatrix} I\dot{\beta} \\ I\dot{\alpha}\sin\beta \\ I_s(\dot{\alpha}\cos\beta + \dot{\gamma}) \end{bmatrix}_{t=2\epsilon} = \begin{bmatrix} I\dot{\beta} \\ I\dot{\alpha}\sin\beta \\ I_s S \end{bmatrix}_{t=2\epsilon}$$

Now we have the full Euler's equation for our scenario. Recognizing that $\dot{\alpha} > 0$, $\dot{\beta} > 0$, and $\beta > 0$, the bike wheel's axis of symmetry will begin to move in a more complicated pattern. The angular velocity and momentum both have components in both the $\hat{x}'$ and $\hat{y}'$ directions. Since we know the z'-component of angular momentum is constant throughout this process, it may end up precessing around the $\hat{z}'$ axis, or not. It is not clear from these equations. We can find the angle that the angular velocity and angular momentum vectors make from the $\hat{x}'$ axis:

$$\begin{aligned} \frac{\omega_{y'}}{\omega_{x'}} &= \tan\Theta_{\omega,x'} = \frac{\dot{\alpha}\sin\beta}{\dot{\beta}} \\ \frac{L_{y'}}{L_{x'}} &= \tan\Theta_{L,x'} = \frac{I\dot{\alpha}\sin\beta}{I\dot{\beta}} = \frac{\dot{\alpha}\sin\beta}{\dot{\beta}} \end{aligned} \tag{80}$$

This is true for this problem, in general, not at specific times. The two vectors make the same angle from $\hat{x}'$, meaning that the extra vector deviation from the $\hat{z}'$, is always colinear for $\vec{\omega}$ and $\vec{L}$. This vector deviation and $\hat{z}'$ form a plane that contains the bike wheel's axis of symmetry, the angular velocity, and the angular momentum vectors.

Since $\dot{\alpha}$, β, and $\dot{\beta}$ can and often do change, this angle will change. Again, whether these vectors precess around the $\hat{z}'$ axis is not obvious. If $\dot{\alpha}$ and $\dot{\beta}$ oscillate and with the same frequency, as appears to happen with nutation, then it is possible that the angle made from the $\hat{x}'$ axis is constant if they also oscillate with the same phase. In order for the vector to precess around the $\hat{z}'$ symmetry axis, it seems the tangent of this angle would go to infinity, and indeed it would if the term were oscillating and $\dot{\beta}$ was going to zero. However, I should point out that because the $\hat{z}'$ axis is itself oscillating, $\vec{\omega}$ making a constant angle with the $\hat{x}'$ is not a sort of passive oscillation of $\vec{\omega}$ around the $\hat{z}'$ axis because $\omega_{x'}$ would change sign if this was to happen.

In one phase in nutational motion, it seems $\dot{\alpha}$ is roughly constant and $\dot{\beta}$ is oscillating. Another mode appears to have the two out of phase, possibly by 90 degrees. Yet another appears they may be in phase, but $\dot{\alpha}$ may actually have half the frequency of $\dot{\beta}$. This means, in all these nutational modes, it seems that the $\vec{\omega}$ and $\vec{L}$ vectors do precess around the $\hat{z}'$ symmetry axis.

If $\dot{\alpha}$ is constant and $\dot{\beta}$ is very small or zero, as with non-nutating precession, then Equations 80 approache infinity meaning that the angle from the $\hat{x}'$ axis approaches 90 degrees. This is positive if $\dot{\alpha}$ is positive and would mean the angular velocity and angular momentum vectors are pointing in the $\hat{z}$ direction, upward, and not precessing around the $\hat{z}'$ symmetry axis.

We can look at the deviation of the angular velocity and angular momentum vectors from the $\hat{z}'$ axis by recognizing that the x' and y' components of each vector can add to a single vector perpendicular to $\hat{z}'$, and the angle that $\vec{\omega}$ and $\vec{L}$ make from the $\hat{z}'$ axis are given by:

$$\begin{aligned} \tan\Theta_{\omega,z'} &= \frac{|\vec{\omega}_{x'} + \vec{\omega}_{y'}|}{\omega_{z'}} = \frac{\sqrt{\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2}}{S} \\ \tan\Theta_{L,z'} &= \frac{|\vec{L}_{x'} + \vec{L}_{y'}|}{L_{z'}} = \frac{I}{I_s}\frac{\sqrt{\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2}}{S} \\ \frac{\tan\Theta_{L,z'}}{\tan\Theta_{\omega,z'}} &= \frac{I}{I_s} \end{aligned} \tag{81}$$

Again, this is true for this problem, in general, not just for a specific time. It is clear that the individual angles do change as $\dot{\alpha}$, β, and $\dot{\beta}$ change, but the ratio of these angles is always constant for this rigid body. We will see that for a typical bike wheel demonstration, the angular momentum points farther from the $\hat{z}'$ axis, the axis of symmetry, than the angular velocity does. Notice also that, since S is constant but $\dot{\alpha} > 0$ and $\beta > 0$, then $\dot{\gamma}$ must have decreased slightly, since:

$$\dot{\gamma} = S - \dot{\alpha}\cos\beta$$

I will stray away from forces and torques momentarily. Though it is not obvious, this makes some sense from an energy perspective, because the kinetic energy was entirely due to rotation around the axis, initially. However, as the entire bike wheel gains kinetic energy from rotating in the $\hat{\alpha}$ and $\hat{\beta}$ directions, those particles that were not moving before are now, and this kinetic energy has to come from somewhere. It could come from the very slight drop in height of those particles, and therefore, a drop in gravitational potential energy. A more thorough analysis of the energy, using work due to the force of gravity, should reveal more insight but I will not pursue this here.

At this point, it seems the future motion of α, and β depend more carefully on the particular values of I, I_s, and β. Some light could be shed on this by approximating a real bike wheel to see which moment of inertia term is greater. Recall the entire bike wheel apparatus has mass M and length $2L$ with a center of mass at L from the origin. If the wheel part is approximated as a thin, uniform disk with radius R, a small thickness d, and mass $m \lesssim M$, and the axle which does not spin with the wheel has length $2L \approx 2R$, mass $\mu \ll m \lesssim M$, and very small radius ρ, then we could either calculate or just look up the values for the rotational inertia terms for each part of the bike wheel. Recall that, since the rotational inertia involes a sum or integral, the total rotational inertia is the sum of the component rotational inertia terms.

$$\begin{aligned}
\overline{I}_{cm} &= \overline{I}_{wheel,cm} + \overline{I}_{axle,cm} \\
\overline{I}_{wheel,cm} &= \begin{bmatrix} \frac{m}{12}(3R^2+d^2) & 0 & 0 \\ 0 & \frac{m}{12}(3R^2+d^2) & 0 \\ 0 & 0 & \frac{m}{2}R^2 \end{bmatrix} \approx \begin{bmatrix} \frac{m}{4}R^2 & 0 & 0 \\ 0 & \frac{m}{4}R^2 & 0 \\ 0 & 0 & \frac{m}{2}R^2 \end{bmatrix} \\
\overline{I}_{axle,cm} &= \begin{bmatrix} \frac{\mu}{12}(3\rho^2+4L^2) & 0 & 0 \\ 0 & \frac{\mu}{12}(3\rho^2+4L^2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \approx \begin{bmatrix} \frac{\mu}{3}L^2 & 0 & 0 \\ 0 & \frac{\mu}{3}L^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\overline{I}_{cm} &\approx \begin{bmatrix} \frac{m}{4}R^2 + \frac{\mu}{3}L^2 & 0 & 0 \\ 0 & \frac{m}{4}R^2 + \frac{\mu}{3}L^2 & 0 \\ 0 & 0 & \frac{m}{2}R^2 \end{bmatrix} \approx \begin{bmatrix} \frac{M}{4}R^2 & 0 & 0 \\ 0 & \frac{M}{4}R^2 & 0 \\ 0 & 0 & \frac{M}{2}R^2 \end{bmatrix} \\
\overline{I} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_s \end{bmatrix} \approx \begin{bmatrix} \frac{M}{4}R^2 + ML^2 & 0 & 0 \\ 0 & \frac{M}{4}R^2 + ML^2 & 0 \\ 0 & 0 & \frac{M}{2}R^2 \end{bmatrix} \\
I &\approx \frac{M}{4}R^2 + ML^2 \approx \frac{5M}{4}R^2 \\
I_s &\approx \frac{M}{2}R^2 \approx \frac{2}{5}I
\end{aligned}$$

So for typical bike wheels for a demonstration like this, I is significantly greater than I_s, due to having to rotate the entire apparatus about its endpoint. This result supports the point that for a typical bike wheel precessing about the end point, the angle that the angular momentum makes from the $\hat{z}'$ axis, the axis of symmetry, is greater than that made by the angular velocity, though they always lie in the same plane containing themselves and the $\hat{z}'$ axis. Specifically, we found previously that:

$$\frac{\tan\Theta_{L,z'}}{\tan\Theta_{\omega,z'}} = \frac{I}{I_s} \approx \frac{5}{2}$$

These angles depend on $\dot{\alpha}$, β, and $\dot{\beta}$, as shown in Equations 81.

What we know so far is consistent with the notion that the bike wheel will precess in the positive $\hat{\alpha}$ direction, which matches simpler scenario of a bike wheel at $\beta = 90° = \pi/2$. We saw from Equation 80 that when this happens, the angular velocity and angular momentum vectors point vertically up (if $\dot{\alpha} > 0$). I know from energy considerations in other sources that giving the bike wheel a sharp push forward and backward ($\dot{\alpha} > 0, \dot{\alpha} < 0$, respectively) can yield two different nutational modes. Another mode occurs when $\dot{\alpha} = 0$ initially, when the bike wheel has the correct spin, and angle β. During nutation in general, we saw from Equations 80 that $\vec{\omega}$ and $\vec{L}$ can wobble in more complicated but interesting ways, as the $\dot{\alpha}$ and $\dot{\beta}$ angles oscillate. This includes precession of $\vec{\omega}$ and $\vec{L}$ around the $\hat{z}'$ symmetry axis. While this occurs, the fixed components of angular momentum ($L_{z'}$ and L_z) will not oscillate, despite the oscillation of the symmetry axis of the bike wheel, and the angular velocity.

I could pursue this further to find α, β, and γ with time. Some insight can be made by using the graph method for analyzing nonlinear differential equations, defining the rates of change of each Euler angle's velocity so it becomes a system of four equations and four variables, but each equation is only a first order autonomous differential equation. Then one could identify the stable points by setting the various rates to zero, perhaps linearizing, and looking at different regimes. This yields a slight bit of insight like the condition for stable precession when $\dot{\beta} = 0$ and $\ddot{\alpha} = 0$, but the real interesting behavior appears outside of these stable point regimes, during nutation. Instead, I prefer to refer the reader, to the relevant sections of Analytical Mechanics by Fowles and Cassiday, 6th ed. include include section 9.8 and the previous sections to establish their notation, or another analytical mechanics book.

Computer simulation would be a better tool for analyzing this rotational motion further. This is greatly helped by modifying Euler's equations for quaternions. It significantly simplifies problems for computation and clarity, with no down sides that I know of. See the introduction to quaternions with an applciation to rigid body dynamics. A particular unit-length quaternion can be used for rotation and its conditions are very simple which one can find with 5 minutes of Googling. There are relationships between quaternion multiplication (which is noncommutative, like matrix multiplication) and vector dot and cross products. Quaternion use for rotation has the benefits of being more numerically stable, less computationally expensive (because you need to compute fewer terms), is immune to gimbal lock which occurs with matrix multiplication, and finally its representation of rotation around some generic vector in three dimensions by a certain angle is much simpler than rotation matrices, which require either a change of basis to align the rotation axes with a generic vector or a hideous single 3×3 matrix.

3.7 Quaternion use for Euler's Equations Simulation

Quaternions were invented by William Rowan Hamilton in 1843 as a generalization of complex numbers. Where $i = \sqrt{-1}$ and $i^2 = -1$, and a complex number can be made of a real and imaginary part: $a + bi$, a quaternion has three different imaginary-like parts: $a + bi + cj + dk$. These imaginary numbers obey the following multiplication rules:

$$ijk = -1, \tag{82}$$

$$i^2 = j^2 = k^2 = -1. \tag{83}$$

From these, you can derive the following multiplication table and you discover that quaternions are non-commutative, as this multiplication table is neither symmetric nor anti-symmetric:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

While quaternion multiplication is not commutative, it is associative.

There is a simpler way to represent quaternions and quaternion multiplication:

$$q = (a, \vec{b}) \tag{84}$$

$$q_1 q_2 = (a_1 a_2 - \vec{b}_1 \cdot \vec{b}_2, a_1 \vec{b}_2 + a_2 \vec{b}_1 + \vec{b}_1 \times \vec{b}_2) \tag{85}$$

Quaternions can have a conjugate and inverse like so:

$$q^c = (a, -\vec{b}) \tag{86}$$

$$q^{-1} = \frac{(a, -\vec{b})}{qq^c} \tag{87}$$

The denominator can be thought of as the norm of a quaternion because the result is always real and the result is consistent with the idea of the norm of a scalar and a vector, the quaternion's constituents:

$$qq^c = q^c q = (a^2 + |b|^2, 0) = a^2 + |b|^2 \tag{88}$$

This means if you want to normalize a quaternion, you can apply the following:

$$q' = \frac{q}{\sqrt{qq^c}} \tag{89}$$

A particular unit-length quaternion can be used for rotation, as can the product of multiple similar quaternions. If an object rotates by θ around a unit-vector $\hat{n}$, the rotation quaternion would be:

$$q = (\cos(\theta/2), \sin(\theta/2)\hat{n}) \tag{90}$$

The inverse of this rotation quaternion undoes the rotation:

$$q^{-1} = q^c = (\cos(\theta/2), -\sin(\theta/2)\hat{n})$$

It would be applied to another vector $\vec{x}$ made into a pure-imaginary quaternion like so:

$$q(0, \vec{x})q^{-1} \tag{91}$$

The resultant imaginary part gives the rotated vector. Multiple rotations around different axes and with different angles, can be composed into a single quaternion, with the first rotation on the right. The Euler equations are simplest to solve in the 123-coordinate system, but to visualize, we need to convert it to the xyz system. To solve the Euler equations numerically, and do so while avoiding gimbal lock which happens when axes overlap and a degree of freedom is lost, I need to maintain the quaternion to rotate/transform the angular velocity components in the 123 system back to the xyz system.

Recall my particular Euler angles come from a 3-1-3 sequence. The transformation matrix from Eq 63 is equivalent to taking a product of rotations, in the 123 basis, by $-\gamma$ about the 3 axis, then $-\beta$ about the 1 axis, then $-\alpha$ about the 3 axis. Because these are orthogonal matrices and the transpose is the inverse, a negative transpose of a rotation matrix about a simple axis is the same as the original. This inspired the idea of taking the product of rotation quaternions, from right to left, by the positive angles: γ, β, α. For simplicity, I will adopt similar abbreviations as Eq 63.

$$\begin{aligned}
c_a &= \cos(\alpha/2) \quad c_b = \cos(\beta/2) \quad c_g = \cos(\gamma/2) \\
s_a &= \sin(\alpha/2) \quad s_b = \sin(\beta/2) \quad s_g = \sin(\gamma/2) \\
q &= (c_a, 0, 0, s_a)(c_b, s_b, 0, 0)(c_g, 0, 0, s_g) \qquad (92) \\
&= \begin{bmatrix} (c_a c_g - s_a s_g)c_b \\ (c_a c_g + s_a s_g)s_b \\ (s_a c_g - c_a s_g)s_b \\ (s_a c_g + c_a s_g)c_b \end{bmatrix} \qquad (93)
\end{aligned}$$

Notice how symmetric this quaternion's form is, compared to the transformation matrices of Eq 63.

To update the quaternion after a change in angle of $\vec{d\theta}_{123} = \vec{\omega}_{123}dt$, right multiply this quaternion by:

$$dq = (\cos(d\theta/2), \sin(d\theta/2)d\hat{\theta}) \tag{94}$$

Note that this is equivalent to left-multiplying by the quaternion in the xyz-coordinate system. It makes sense that we need to right-multiply by this dq to construct a quaternion that will take vectors in the 123 frame to the xyz frame. The quaternion in Eq 93 is a composite of rotations from right to left, in the opposite order in which they occured. This dq is the last rotation to occur so it must be the first to be undone when applied to vectors for converting bases. The $\vec{\omega}$ vector in the xyz-coordinate system is acquired by applying q from Eq 93 in the manner prescribed by Eq 91. You do the same thing to rotate the vectors defining the orientation of your object.

The last thing to note is how to get the 3-1-3 Euler angles from the quaternion in Eq 93. With the quaternion as written, the following relationship gives the set of angles. Use the atan2 function as described below to allow for the full domain for each angle. This relationship uses the form $q = (q_0, q_1, q_2, q_3)$ and note that if you take $q_2 \to -q_2$, this switches α and γ:

$$\begin{aligned}
\alpha &= \operatorname{atan2}\{2(q_1q_3 + q_0q_2), 2(q_0q_1 - q_2q_3)\} \\
\beta &= \operatorname{acos}\left\{(q_3^2 - q_2^2) + (q_0^2 - q_1^2)\right\} \\
\gamma &= \operatorname{atan2}\{2(q_1q_3 - q_0q_2), 2(q_0q_1 + q_2q_3)\} \qquad (95)
\end{aligned}$$

The products of 2 in the atan2 functions are superfluous, since they cancel, and can therefore be left out, but they demonstrate the use of the double angle formulas for verifying these relationships. Otherwise, terms have been associated for better numerical stability. I have implemented these quaternion methods in two simulations of rotational motion at www.glowscript.org/#/user/owendix/.

4 Appendix

4.1 Reference for the Basics

- All position, velocity, and acceleration values, as well as all quantities derived from them, assume some inertial coordinate system has been chosen. An inertial coordinate system is one where objects moving without external forces move with a constant velocity.

- Vectors are written like $\vec{a}$, while scalars, including the magnitudes of vectors, are written like a.
- Dot notation is used to mean the full time derivative of a quantity, which is distributive over addition and constant factors can be effectively factored out of a derivative, because the derivative of the factor is zero when expanded with the product rule (here m is constant):

$$\frac{d}{dt}(\vec{a}+m\vec{b}) = \dot{\vec{a}} + \frac{d}{dt}(m\vec{b}) = \dot{\vec{a}} + \dot{m}\vec{b} + m\dot{\vec{b}} = \dot{\vec{a}} + m\dot{\vec{b}}$$

- The position vector of some object is $\vec{r}$
- The object's velocity, $\vec{v} = d\vec{r}/dt$ is written as $\dot{\vec{r}}$
- The object's acceleration is $\vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2 = \ddot{\vec{r}}$
- The momentum of an object is $\vec{p} = m\vec{v} = m\dot{\vec{r}}$
- The angular momentum of an object is $\vec{L} = \vec{r} \times \vec{p}$, where $\times$ indicates the vector cross product.
- The cross product $\vec{a} = \vec{b} \times \vec{c}$ yields a vector $\vec{a}$ perpendicular to the plane of $\vec{b}$ and $\vec{c}$ in a right-handed sense (which is only a plane if they are not colinear) and having magnitude $a = bc\sin\alpha$, where α is the angle between the vectors $\vec{b}$ and $\vec{c}$, found by translating them so their tails are placed at the same point. Note if $\vec{b}$ and $\vec{c}$ are colinear then $\alpha = 0$ or π and $a = 0$. The components of $\vec{a} = (a_x, a_y, a_z)$ can be calculated with a 3×3 determinant with rows equal to $(\hat{x}, \hat{y}, \hat{z})$, $\vec{b}$, and $\vec{c}$, respectively. This yields:

$$\vec{a} = (a_x, a_y, a_z) = (b_y c_z - b_z c_y, b_z c_x - b_x c_z, b_x c_y - b_y c_x) \tag{96}$$

- You can verify from the above expression that scalar multiplication of vectors can be factored out of a cross product, so $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times b)$. Cross products are also distributive over addition of vectors $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$, and $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ and are anti-commutative so $\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$. The distributive property means that, if a vector is not explicitly dependent on the index of a summation, its cross product with other vectors can be pulled out of the sum.
- The torque due to some force $\vec{F}$ that acts on a particle at point $\vec{r}$ (relative to the origin of the coordinate system) is written as $\vec{\tau} = \vec{r} \times \vec{F}$.
- The center of mass of an object, which is a weighted average of the how the mass of the object is distributed, with particles $k = 1, 2, \ldots, K$ and total mass $M = \sum_{k=1}^{K} m_k$ is given by:

$$\vec{r}_{cm} = \frac{\sum_{k=1}^{K} m_k \vec{r}_k}{M} \tag{97}$$

- You can get two somewhat obvious but significant ideas by shifting the position vector $\vec{r}_i$, relative to some generic origin so a new one has its origin at the center of mass, itself: $\vec{r}_i = \vec{r}_{cm} + \vec{r}_{i\,cm}$. Read $\vec{r}_{i\,cm}$ as the position of the i-th particle relative to the center of mass. Substituting this:

$$\vec{r}_{cm} = \frac{\sum_{k=1}^{K} m_k(\vec{r}_{cm} + \vec{r}_{k\,cm})}{M} = \frac{\sum_{k=1}^{K} m_k\vec{r}_{cm} + \sum_{k=1}^{K} m_k\vec{r}_{k\,cm}}{M} = \vec{r}_{cm} + \frac{\sum_{k=1}^{K} m_k\vec{r}_{k\,cm}}{M}$$

$$\vec{r}_{cm} = \vec{r}_{cm} + \vec{r}_{cm\,cm}$$

$$0 = \vec{r}_{cm\,cm} = \frac{\sum_{k=1}^{K} m_k\vec{r}_{k\,cm}}{M} \tag{98}$$

$$0 = \dot{\vec{r}}_{cm\,cm} = M\dot{\vec{r}}_{cm\,cm} = \vec{p}_{cm\,cm} = \sum_{k=1}^{K} m_k\dot{\vec{r}}_{k\,cm} \tag{99}$$

$\vec{r}_{cm\,cm}$ should be read as the position of the center of mass, relative to the center of mass. The second to last line means that with the center of mass at the origin of the system, the center of mass of the system is at the origin (obviously). The first step in the last line of equations follows by taking the time derivative of all sides. This means that the velocity of the center of mass, relative to the center of mass is zero, and therefore, the total momentum of the system, relative to the center of mass, (and its rate of change) are zero too. These ideas are significant, and will come up later.

4.2 NonInertial Reference Frames

An inertial coordinate system is one where Newton's 1st law holds: where objects move in a straight line, as measured by that coordinate system. A noninertial coordinate system is one where this does not hold. If the origin of a coordinate system is accelerating through space, it is easy to see, that measurements of a particle under no external forces, will not match that for a particle moving in a straight line with a constant speed. Also if the basis vectors of a coordinate system are rotating, even with a constant speed and orientation, the measured position for a particle will not match that for a straight line, either. What will happen, precisely?

Any vector, $\vec{A}$, exists independent of its coordinate system but measurements of its direction and size need to be relative to something, and a coordinate system serves that purpose. The vector $\vec{A}$ can be expanded in terms of its degree of overlap with each basis vector in the coordinate system:

$$\vec{A} = \sum_\mu A_\mu \hat{e}_\mu = \sum_\mu A'_\mu \hat{e}'_\mu \tag{100}$$

Here the two sets of unit-length, basis vectors for the different primed and unprimed coordinate systems are given by $\hat{e}_\mu$ and $\hat{e}'_\mu$, respectively, and μ runs over the different coordinate bases. The degree of overlap of $\vec{A}$ with each coordinate system is given by the following, $A_\mu = \vec{A}\cdot\hat{e}_\mu$ and $A'_\mu = \vec{A}\cdot\hat{e}'_\mu$. In the language of differential geometry, the basis vectors (not unit length) for a coordinate system are given by differentials of different coordinates and partial derivatives with respect to coordinates (for contravariant and covariant vectors), and the dot product is computed using the metric. The basis vectors are like a directional derivative of your space with respect to each coordinate. I'll be using the simpler, less generalized unit vectors and the dot product will be computed in flat space (the metric is the identity tensor).

The differential geometric way does give us a quick way to find the unit vectors for a change of basis with a system of spatial coordinates: take the derivative with respect to the new coordinates of the position vector (written in terms of the new coordinates), then normalize. Here's an example of a change of basis from 2-dimensional cartesian coordinates $\{\hat{e}_x, \hat{e}_y\} = \{\hat{x}, \hat{y}\}$ to 2-dimensional polar coordinates $\{\hat{e}_r, \hat{e}_\theta\} = \{\hat{r}, \hat{\theta}\}$ (cylindrical coordinates in 3-dimensions but setting $z = 0$).

$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{r} = r\cos\theta\hat{x} + r\sin\theta\hat{y}$$

$$\hat{e}_r = \left|\frac{\partial \vec{r}}{\partial r}\right|^{-1}\frac{\partial \vec{r}}{\partial r} = \cos\theta\hat{x} + \sin\theta\hat{y} \tag{101}$$

$$\hat{e}_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right|^{-1}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \tag{102}$$

Let's consider the unprimed coordinate to be constant, fixed in space, and the primed coordinate system to generally be varying (though I won't assume how just yet).

$$\frac{d\vec{A}}{dt} = \dot{\vec{A}} = \sum_\mu \dot{A}_\mu \hat{e}_\mu \tag{103}$$

$$\dot{\vec{A}} = \sum_\mu \dot{A}'_\mu \hat{e}'_\mu + A'_\mu \dot{\hat{e}}'_\mu \tag{104}$$

In the previous example, the cartesian coordinates may be the unprimed system, fixed in space, and the polar coordinates, since their direction depends on their location in space, could be the primed coordinates. This is not the only way to have variable coordinates; we could define coordinates as being fixed to a body as it moves and rotates through space. The first term in the second line, for the primed system, is in correspondence to the unprimed system. The second term is clearly the difference. To know how the primed system's basis vectors change with time, we need to compare them to another coordinate system. It turns out most productive to expand this term in the primed basis vectors, since we ultimately want a relation between one side of an equation with one basis and the other side with the other basis.

$$\dot{\hat{e}}'_\mu = \sum_\nu \left(\dot{\hat{e}}'_\mu \cdot \hat{e}'_\nu\right)\hat{e}'_\nu \tag{105}$$

The term in parentheses is an antisymmetric tensor, it switches sign when μ and ν are exchanged, which can be seen by the following given that $\hat{e}'_\mu \cdot \hat{e}'_\nu = \delta'_{\mu\nu}$:

$$0 = \frac{d}{dt}\left(\hat{e}'_\mu \cdot \hat{e}'_\nu\right) = \dot{\hat{e}}'_\mu \cdot \hat{e}'_\nu + \hat{e}'_\mu \cdot \dot{\hat{e}}'_\nu$$

$$\dot{\hat{e}}'_\mu \cdot \hat{e}'_\nu = -\hat{e}'_\mu \cdot \dot{\hat{e}}'_\nu$$

Now, regardless of whether the primed system is rotating or translating or both, the dot product between two vectors requires they be associated with the same point. This means that no translation of the two coordinate systems can affect this dot product, whether that translation is inertial or not. In flat space, vectors can be parallel transported without changing the direction of the vector and in a way that is independent of the path taken to transport the vector. In curved space, this may pose a problem, but we are working in flat space, here. If you define:

$$\begin{aligned}
\left(\dot{\hat{e}}'_\mu \cdot \hat{e}'_\nu\right) &= \sum_\sigma \epsilon_{\mu\nu\sigma}\dot{\Omega}'_\sigma && (106)\\
\omega_\sigma &= \dot{\Omega}'_\sigma \\
A'_\mu \dot{\hat{e}}'_\mu &= A'_\mu \sum_\nu \sum_\sigma \epsilon_{\mu\nu\sigma}\omega_\sigma \hat{e}'_\nu \\
\dot{\vec{A}} = \sum_\mu \dot{A}'_\mu \hat{e}'_\mu + A'_\mu \dot{\hat{e}}'_\mu &= \sum_\mu \dot{A}'_\mu \hat{e}'_\mu + \sum_\mu A'_\mu \sum_\nu \sum_\sigma \epsilon_{\mu\nu\sigma}\omega_\sigma \hat{e}'_\nu && (107)
\end{aligned}$$

The second line, defining the angular velocity, can be seen by noting that, if rotation is all that affects the dot product of the change in the prime basis vectors with the basis vector itself, and these unit vectors do not change length, then rotation is all that will affect it. Defining the angular velocity proportional to their degree of rotation with this levi-cevita symbol is perfectly natural.

Let's explicitly right out the different components of the second term in $\dot{\vec{A}}$, where the only nonzero terms come from $\mu \neq \nu \neq \sigma$:

$$\begin{aligned}
A'_1(\epsilon_{123}\omega_3\hat{e}'_2 + \epsilon_{132}\omega_2\hat{e}'_3) &= A'_1(\omega_3\hat{e}'_2 - \omega_2\hat{e}'_3)\\
A'_2(\epsilon_{231}\omega_1\hat{e}'_3 + \epsilon_{213}\omega_3\hat{e}'_1) &= A'_2(\omega_1\hat{e}'_3 - \omega_3\hat{e}'_1)\\
A'_3(\epsilon_{312}\omega_2\hat{e}'_1 + \epsilon_{321}\omega_1\hat{e}'_2) &= A'_3(\omega_2\hat{e}'_1 - \omega_1\hat{e}'_2)
\end{aligned}$$

The vector is actually the sum of these and we wish to see the components that line up with the $\{\hat{e}'_j\}$ basis vectors, so lets add and combine these:

$$\begin{aligned}
\sum_\mu A'_\mu \dot{\hat{e}}'_\mu &= (\omega_2 A'_3 - \omega_3 A'_2)\hat{e}'_1 + (\omega_3 A'_1 - \omega_1 A'_3)\hat{e}'_2 + (\omega_1 A'_2 - \omega_2 A'_1)\hat{e}'_3\\
&= \vec{\omega} \times \vec{A}'\\
\dot{\vec{A}} &= \dot{\vec{A}}' + \vec{\omega} \times \vec{A} && (108)
\end{aligned}$$

The $\vec{A}$ in the cross product term can be represented in any coordinate system, as I will soon argue. This is true for any vector, comparing its change in an inertial or noninertial (primed) coordinate system. The primed coordinate system could be translating or rotating and the angular velocity may change with time. Note the angular velocity does not have a prime because, since it is the rotation of the primed coordinate system itself and not the rotation of something within it, it is represented in both coordinate systems the same. The following supports that:

$$\dot{\vec{\omega}} = \dot{\vec{\omega}}' + \vec{\omega} \times \vec{\omega} = \dot{\vec{\omega}}' \tag{109}$$

The cross product of $\vec{\omega}$ with itself will always be zero. Recall a vector is independent of its coordinate representation, just what numbers you have aligned with each basis depends on the coordinate choice. This means that the cross product term can be computed in either coordinate system, because the two vectors and the result are independent of coordinates. Only the time derivative needs to be computed in the primed basis. Of course, when you add the two terms together, you need a common basis.

It had bothered me, previously, that when adding angular momenta of the orbit and spin of an object, they could simply be added directly. In a fixed coordinate system, perhaps it shouldn't have, but the rotations are about different points. This result may help alleviate some of that unease but I think it doesn't really relate. The real help may be that, since we can mentally consider only part of the system, say the spinning part, its angular momenta will be conserved (in that it can only change from an angular impulse from a net torque acting for a period of time) and the same is true if we mentally expand our system to include all parts. Then again, perhaps it does help. I'll know when I apply these results to conservation of angular momentum.

Let's apply this to a position vector to see how velocity and acceleration change a changing coordinate basis. Then

I'll shift I'll use primes to represent quantities that use the noninertial system.

$$\dot{\vec{r}} = \dot{\vec{r}}' + \vec{\omega} \times \vec{r}$$
$$\dot{\vec{r}} = \left(\frac{d}{dt}\bigg|_{'} + \vec{\omega}\times \right) \vec{r} \tag{110}$$

Again, the $\vec{\omega}$ cross product can be computed in either coordinate system, but just convert the result to the primed system afterward to be able to add this vector to the other term. The second line writes the equation like a linear operator. Applying two operators:

$$\ddot{\vec{r}} = \left(\frac{d}{dt}\bigg|_{'} + \vec{\omega}\times \right) \left(\frac{d}{dt}\bigg|_{'} + \vec{\omega}\times \right) \vec{r}$$
$$\ddot{\vec{r}} = \ddot{\vec{r}}' + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{111}$$

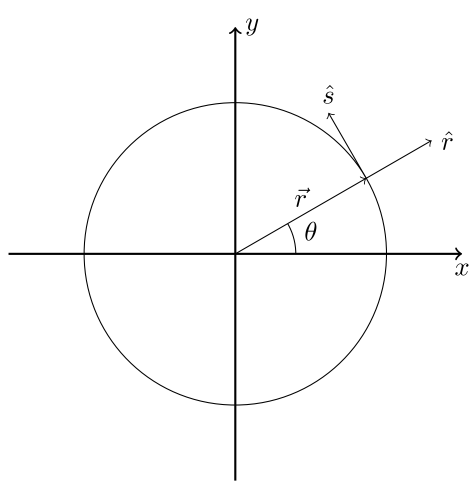
Now let's shift the vector, showing a translating origin. This could not be done for a general vector because the units of the length-shift do not match the units of the other vector. Angular momentum has already been done in the main section, for the center of mass frame with the result that fictitious torques do not appear from translation to the center of mass-origin. I'll use the shift $\vec{\rho} = \vec{R} + \vec{r}$ so that $\vec{\rho}$ is the position of the particle in an inertial frame, $\vec{R}$ is the position of the noninertial frame, relative to the inertial frame, and $\vec{r}$ is the position of the particle in the noninertial frame, making it the only noninertial position vector. This vector $\vec{r}$ will be expanded using the noninertial relationships derived above.

$$\dot{\vec{\rho}} = \dot{\vec{R}} + \dot{\vec{r}}' + \vec{\omega} \times \vec{r}$$
$$\ddot{\vec{\rho}} = \ddot{\vec{R}} + \ddot{\vec{r}}' + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{112}$$

We can find the fictitious forces by multiplying the second line by the total mass of the system, M:

$$M\ddot{\vec{\rho}} = M\ddot{\vec{R}} + M\ddot{\vec{r}}' + M\dot{\vec{\omega}} \times \vec{r} + 2M\vec{\omega} \times \dot{\vec{r}}' + M\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{113}$$
$$M\dot{\vec{\omega}} \times \vec{r} \qquad \text{(transverse “force”)}$$
$$2M\vec{\omega} \times \dot{\vec{r}}' \qquad \text{(coriolis “force”)}$$
$$M\vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad \text{(centrifugal “force”)}$$

4.3 Motion in a Circle



Refer to the picture above and start from the equation for arclength, with angle θ measured in radians (defining the radians), then take the derivative of both sides, assuming the particle is confined to a perfect circle:

$$\ell = r\theta \tag{114}$$
$$\vec{v} = r\omega\hat{s} \tag{115}$$

Taking another derivative, with constant r, we get:

$$\vec{a} = r\dot{\omega}\hat{s} + r\omega\dot{\hat{s}}$$
$$= r\alpha\hat{s} + r\omega\dot{\hat{s}} \tag{116}$$

The first term contains the magnitude of the angular acceleration. The second term is nonzero because the tangent vector $\hat{s}$ does change with time as the particle rotates. The change in $\hat{s}$ must be perpendicular to $\hat{s}$ because its length does not change, visually you can see that $\hat{s}$ will point in the negative $\hat{r}$ direction as θ increases. More explicitly, though, we can expand $\hat{s}$ and $\hat{r}$ in terms of the fixed $\hat{x}$ and $\hat{y}$, then take the derivative of $\hat{s}$:

$$\begin{aligned}
\hat{r} &= \cos\theta\hat{x} + \sin\theta\hat{y} \\
\hat{s} &= -\sin\theta\hat{x} + \cos\theta\hat{y} \\
\dot{\hat{s}} &= -\omega\cos\theta\hat{x} - \omega\sin\theta\hat{y} \\
&= -\omega(\cos\theta\hat{x} + \sin\theta\hat{y}) \\
&= -\omega\hat{r}
\end{aligned}$$

We end up with,

$$\vec{a} = r\alpha\hat{s} - \omega^2 r\hat{r} = r\alpha\hat{s} - \frac{v^2}{r}\hat{r} \tag{117}$$

We can do this a bit more rigorously, since we want rotational motion vectors to have direction and magnitude. If we define the vector $\vec{\theta}$ as out of the plane, it would be convenient because it would use a constant direction characterizing the plane of rotation, which actually helps relate to quaternion rotation, nicely. Doing this:

$$\hat{\theta} = \hat{r} \times \hat{s} = \hat{\omega} \tag{118}$$

where the final equality follows because the particle is confined to a fixed circle. Note what we did previously, setting the derivative of the angular speed to get the angular acceleration, is still valid since the direction of angular velocity does not change for motion in a fixed circle.

We can substitute for $\hat{s} = \hat{\omega} \times \hat{r}$, because:

$$\hat{\omega} \times \hat{r} = (\hat{r} \times \hat{s}) \times \hat{r} = -\hat{r} \times (\hat{r} \times \hat{s}) = \hat{s}(\hat{r}\cdot\hat{r}) - \hat{r}(\hat{r}\cdot\hat{s}) = \hat{s} \tag{119}$$

Then the velocity of any particle moving in a circle will be given by the following, with $\vec{r}$ relative to the center of the circle:

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{120}$$
$$= r\omega\hat{s} \tag{121}$$

A derivative of the Eq 120 will yield the acceleration of a particle moving in a circle, with $\vec{r}$ relative to the center:

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}$$
$$= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{122}$$
$$= \alpha r\hat{s} - \omega^2 r\hat{r} \tag{123}$$

This is consistent with our previous attempt using magnitudes.

Note that Eq 120 and Eq 122 are consistent with the results of explicitly relating an inertial and noninertial coordinate system, as it should be. Here, polar coordinates can be considered fixed to the particle as it moves through space, then the angular velocity of the particle is the angular velocity of the coordinate system since its orientation changes keeping $\hat{r}$ always pointed away from the origin of the cartesian coordinate system. I'll use primes to represent quantities that use the polar, noninertial system.

$$\dot{\vec{r}} = \dot{\vec{r}}' + \vec{\omega} \times \vec{r}$$
$$= \left(\frac{d}{dt}\bigg|' + \vec{\omega}\times\right)\vec{r} \tag{124}$$

The result of the cross product with $\vec{\omega}$ is independent of coordinate choice, since the $\vec{\omega}$ and the vector are both independent of coordinate choice. Applying two operators:

$$\ddot{\vec{r}} = \left(\frac{d}{dt}\bigg|' + \vec{\omega}\times \right) \left(\frac{d}{dt}\bigg|' + \vec{\omega}\times \right) \vec{r}$$
$$= \ddot{\vec{r}}' + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (125)$$

The vector $\vec{r}$ pointing from the origin of the fixed frame to the particle does not change in the polar coordinate system. It always points along $\hat{r}$ and has constant length. So,

$$\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r} \quad (126)$$
$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (127)$$

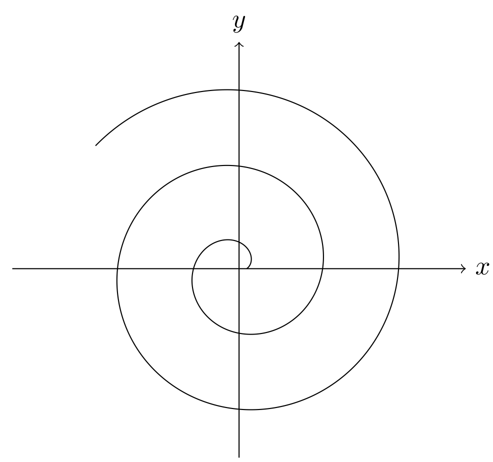
Which is, again, consistent with previous results.

4.4 The Spiral from a Rotating Sprinkler in Space

Imagine a sprinkler at the origin of a coordinate system rotates in the positive angular velocity direction at a constant speed, $\Omega_o = 2\pi/T$, where T is its period of rotation. As it rotates, it fires water (or any particle, really) radially outward at a constant rate so that these particles leave the origin at the same constant speed v_o for all ejected particles. The pattern of water made should be some sort of spiral; let's compare to two different possible spirals.

An archimedian spiral in polar coordinates and then cartesian coordinates, respectively, has the equations:

$$r = a + b\theta \quad (128)$$
$$x = r\cos\theta = (a + b\theta)\cos\theta \quad (129)$$
$$y = r\sin\theta = (a + b\theta)\sin\theta \quad (130)$$

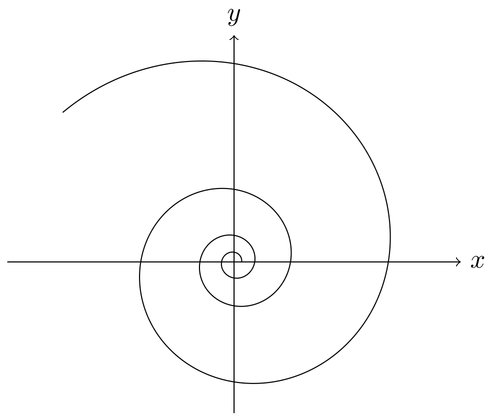


Archimedian Spiral $a = 0.1, b = 0.16$

This is the most general spiral where each branch is equally spaced with the next closer or farther branch at the same θ.

A logarithmic spiral has the equations:

$$r = ae^{b\theta} \quad (131)$$
$$x = ae^{b\theta}\cos\theta \quad (132)$$
$$y = ae^{b\theta}\sin\theta \quad (133)$$



Logarithmic Spiral $a = 0.1, b = 0.16$

This spiral has the feature that the radial change with θ at a given point (r, θ) is proportional the value of r at that point.

For our scenario, the radial distance from the origin will depend on the speed and time from when it was ejected, call it $t_e = t_e(\theta)$ which can be depend on the angle within the spiral we are looking at.

$$r = v_o[t - t_e] \tag{134}$$

I really want r to be a function of θ only, so I need to find $t_e(\theta)$. If the sprinkler starts $(t = 0)$ at $\theta = 0$, then t_e is given by:

$$t_e(\theta) = \theta/\Omega_o \tag{135}$$

This makes the radial distance:

$$r = v_o[t - \theta/\Omega_o] \tag{136}$$

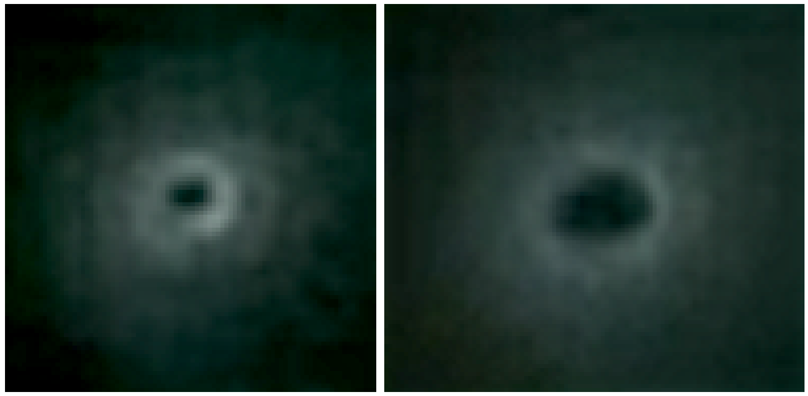
Now, when we view a spiral of sprinkler, we are seeing a series of snapshots with fixed t. So consider t to be some fixed parameter, and we can analyze the shape of this spiral at this fixed time. This makes the equation above that for an archimedian spiral (in polar coordinates), with parameters $a = v_o t$ and $b = -v_o\Omega_o$. The negative sign in b affects the direction of rotation of the spiral snapshot.

Compare the shape of an archimedian spiral generated by ejecting mass at a constant speed from an object rotating at a constant rotation rate to the Spiral over Norway (2009) which has several youtube videos.

The middle of the spiral is difficult to make out with how thick the spiral lines are, as you look farther from the center, it appears to match much closer to an archimedian spiral than a logarithmic one, since the spiral lines separate at a pretty constant spacing, the key signature of an archimedian spiral. The spiral appears to be rotating clockwise, making b a positive number. We can estimate the Ω_o at roughly 1 revolution every 2-3 seconds or $\Omega_o \approx 2.5 rad/s$, but not the v_o for the rate that matter leaves the rocket. This would depend on properties of the leak, like the hole size, the pressure in the tank, etc. which we cannot know.

It is not clear whether the fuel is ignited. Other evidence points to a rocket, spinning and ejecting fuel at a roughly constant rate, being a good explanation. There is a Wikipedia entry for this event. At the end, the blackness expands because the fuel stream stops and is not replenished by the previously constant stream.



Two factors may affect the accuracy of our assumptions of a rotating source of matter. We assumed constant rotation and that the matter traveled outward at a constant rate. The two factors in question, to the degree that they matter - which is possibly quite slight, seem to cancel so that an apparently archimedian spiral should still fit the shape of this exhaust. The rocket loses mass as it spins which would tend to increase its angular velocity with time, however if the mass is not lost very fast this will be minimal. This perhaps slight increase in its rotation rate with time would tend to condense the spirals closer to the rocket compared to those that are formed by matter released a little while earlier, making it appear a bit more logarithmic. Again, if this is a slow enough loss compared to its rotational inertia which it quite possibly is, it may not be visible over a time scale of the life of the matter in the spiral at any given time. The other factor is that the matter being ejected won't expand outward at a constant velocity, exactly, since there is atmosphere in the way and gravity pulling it down. However, just like how air in a room is relatively homogeneous over those length scales, and roughly behaves the ideal gas law (meaning interactions are minimal), this would have a minimal effect. It would condense the entire spiral and if the ejected matter was actually brought to a rough halt, may make the outer part of the spiral seem to condense the most. Air resistance on small particles like gas is roughly linear with velocity, not quadratic like for larger objects, so its effects would not be seen the most at the inside of the spiral. Again, though, these two effects seem to be quite small. So we should still expect an archimedian spiral. For a demonstration of this, see my simulations at www.glowscript.org/#/user/owendix/.