Algorithms:
Greedy Method

Shortest Path Problems

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Shortest-Path

- Given a graph (directed or undirected) $G = (V, E)$ with weight function $w: E \to \mathbf{R}$ and a vertex $s \in V$, find for all vertices $v \in V$ the minimum possible weight for path from s to v.

- The weight of path $p = v_1 \to v_2 \to \ldots \to v_k$ is

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$

- Shortest path = a path of the minimum weight

- Algorithm will compute a shortest-path tree.

Shortest-Path Problems

- Shortest-Path problems
  - **Single-Source (Single-Destination):** Find a shortest path from a given source (vertex *s*) to each of the vertices.
  - **Single-Pair:** Given two vertices, find a shortest path between them. Solution to single-source problem solves this problem efficiently, too.
  - **All-Pairs:** Find shortest-paths for every pair of vertices. Dynamic programming algorithm.

Single-Source Shortest Path

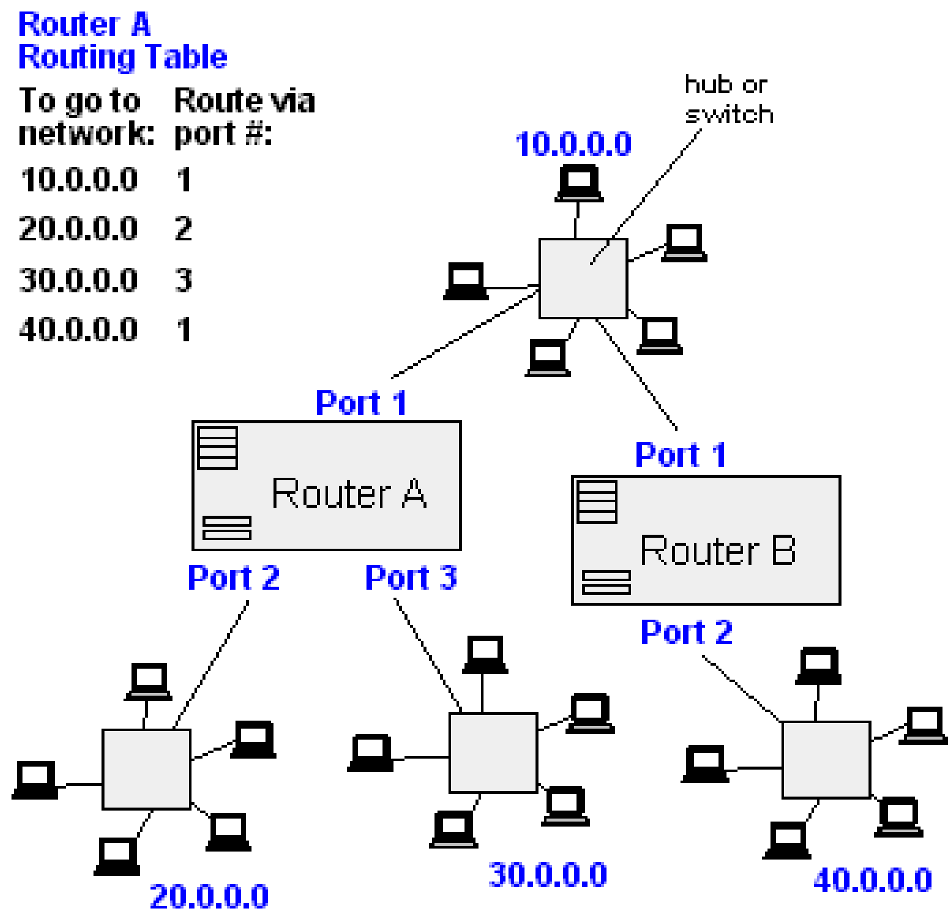
- Given a graph (directed or undirected) $G = (V, E)$ with weight function $w: E \rightarrow \mathbf{R}$ and a vertex $s \in V$, find for all vertices $v \in V$ the minimum possible weight for path from s to v.

- We will discuss two general case algorithms:
  - **Dijkstra's Algorithm** (positive edge weights only)
  - **Bellman-Ford Algorithm** (positive and negative edge weights)

- If all edge weights are equal (let's say 1), the problem is solved by BFS in $\Theta(V + E)$ time.

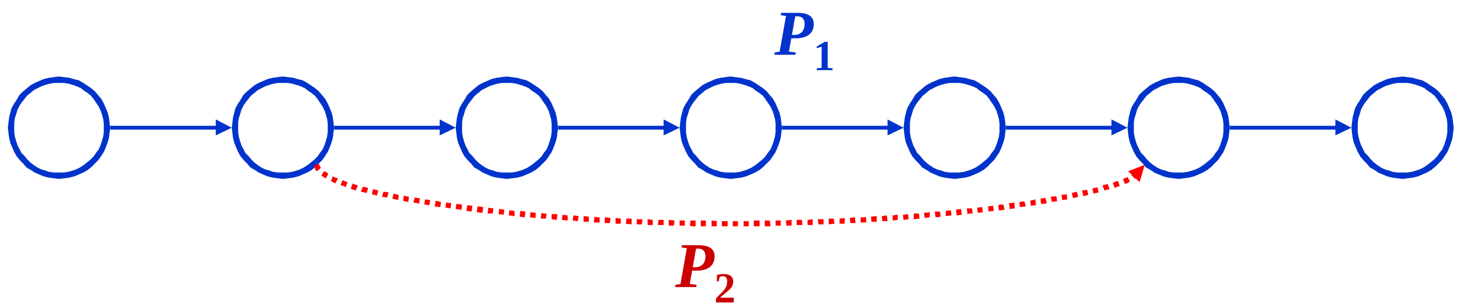
Single-Source Shortest Path

- Traffic Information Systems are most prominent use
- Mapping (Map Quest, Google Maps)
- Routing Systems



Shortest Path Properties

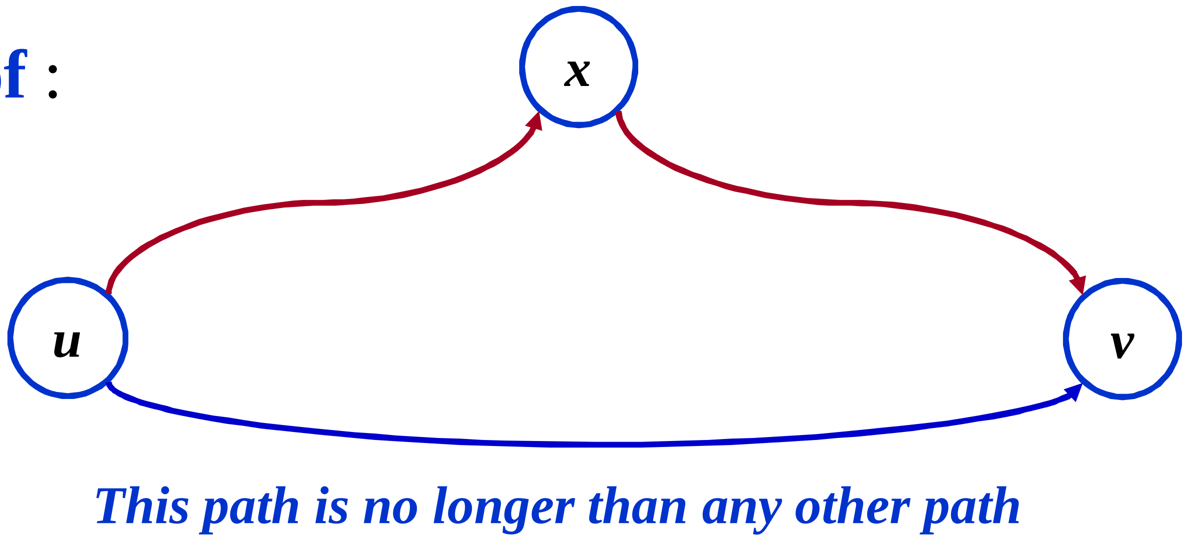
- The shortest path problem satisfies the *optimal substructure property*:
  - Subpaths of shortest paths are shortest paths.



  - **Proof**: suppose some subpath P_1 is not a shortest path
    - There must then exist a shorter subpath P_2
    - Could substitute the subpath P_1 by the shorter path P_2
    - But then overall path is not the shortest path. Contradiction

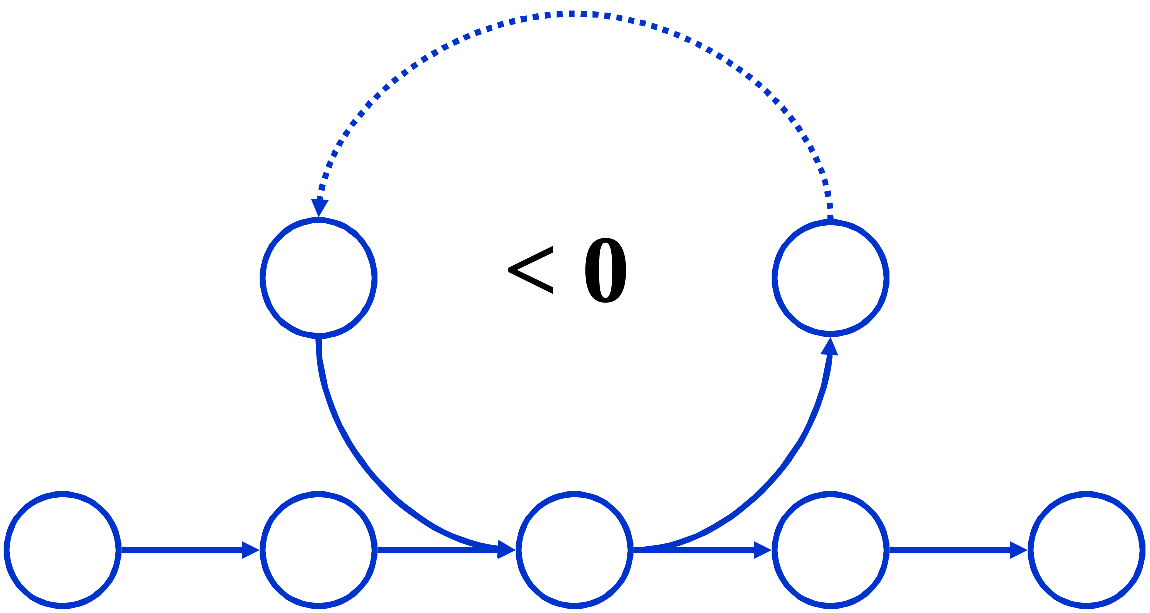
Shortest Path Properties

- Define $\delta(u, v)$ to be the weight of the shortest path from u to v
- Shortest paths satisfy the *triangle inequality*: $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$
- **Proof** :



Shortest Path Properties

- In graphs with negative weight cycles, some shortest paths will not exist (*Why* ?):



Relaxation

- A key technique in shortest path algorithms is *relaxation*
  - Idea: for all *v*, maintain upper bound d[*v*] on δ(s, v)

```
Relax(u,v,w) {
    if (d[v] > d[u]+w(u,v))
       then d[v] = d[u]+w(u,v);
}
```

Bellman-Ford Algorithm

```
BellmanFord()
   for each v ∈ V
      d[v] = ∞;
   d[s] = 0;

   for i=1 to |V|-1
      for each edge (u,v) ∈ E
         Relax(u,v,w);

   for each edge (u,v) ∈ E
      if (d[v] > d[u] + w(u,v))
           return "no solution";
```

Initialize d[]*, which will converge to shortest-path value* δ

Relaxation: Make |V|-1 passes, relaxing each edge

Test for solution
Under what condition do we get a solution?

```
Relax(u,v,w): if (d[v] > d[u]+w(u,v))
                 then d[v]=d[u]+w(u,v)
```

Bellman-Ford Algorithm

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BellmanFord()
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      d[v] = ∞;
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   for i=1 to |V|-1
      for each edge (u,v) ∈ E
         Relax(u,v,w);

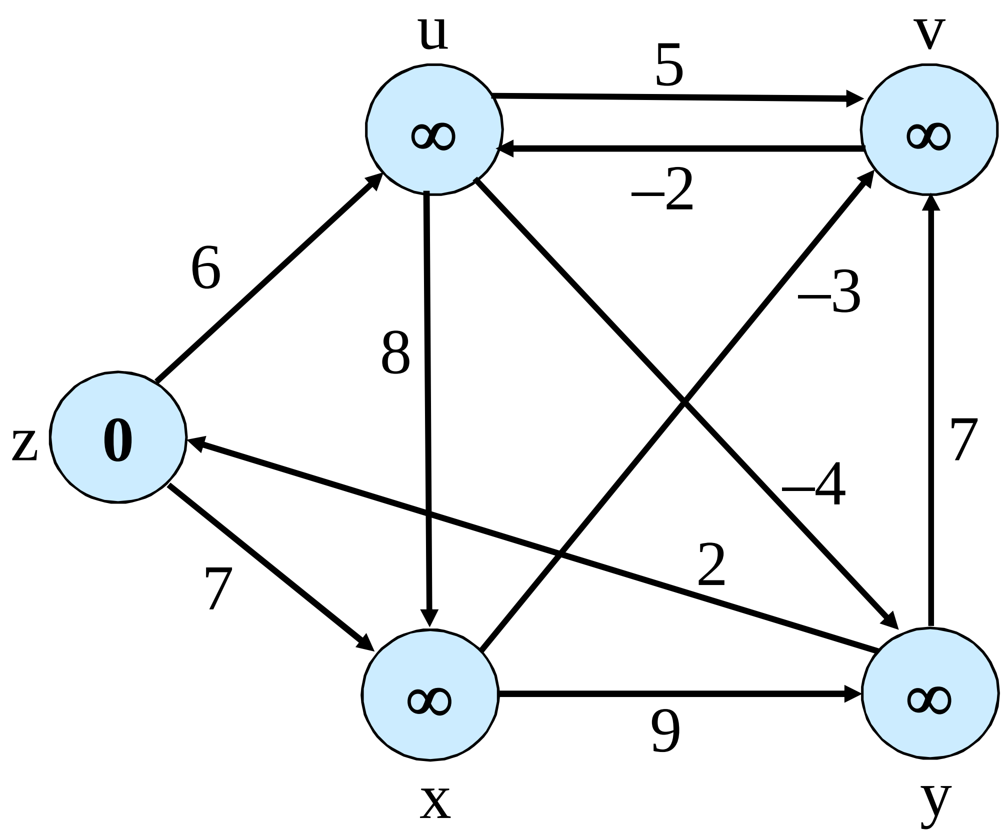
   for each edge (u,v) ∈ E
      if (d[v] > d[u] + w(u,v))
           return "no solution";
```

Q: What will be the running time?

A: O(V E)

```
Relax(u,v,w): if (d[v] > d[u]+w(u,v))
                then d[v]=d[u]+w(u,v)
```

Bellman-Ford Algorithm: Example



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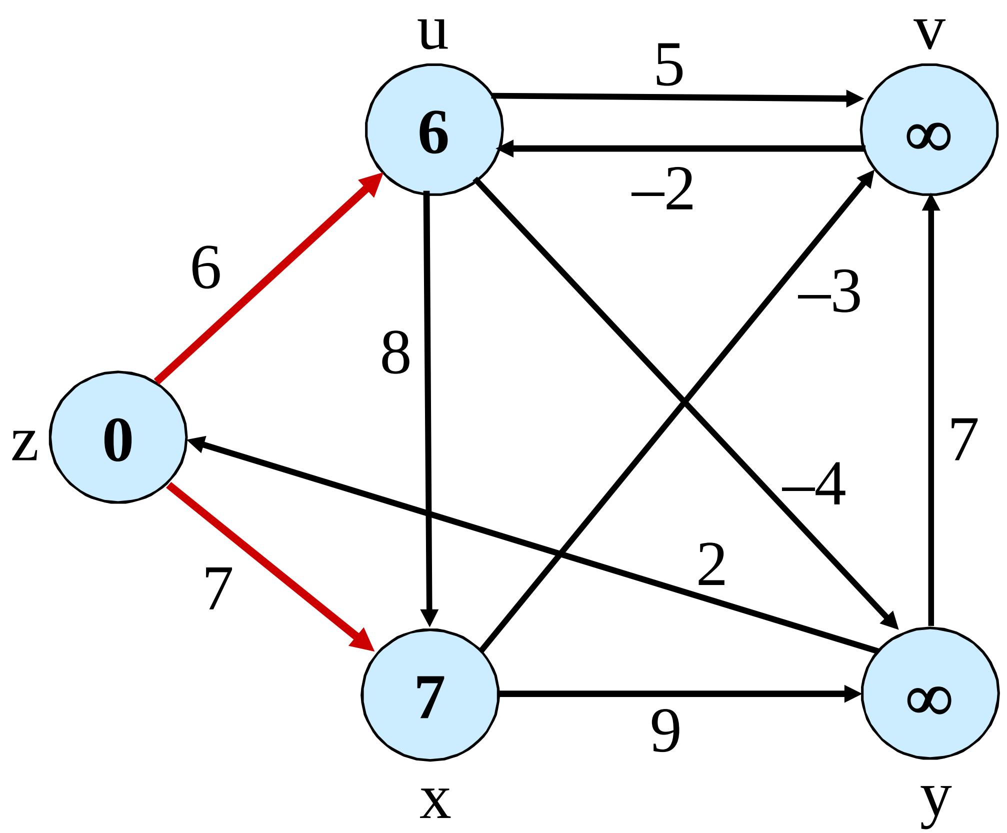

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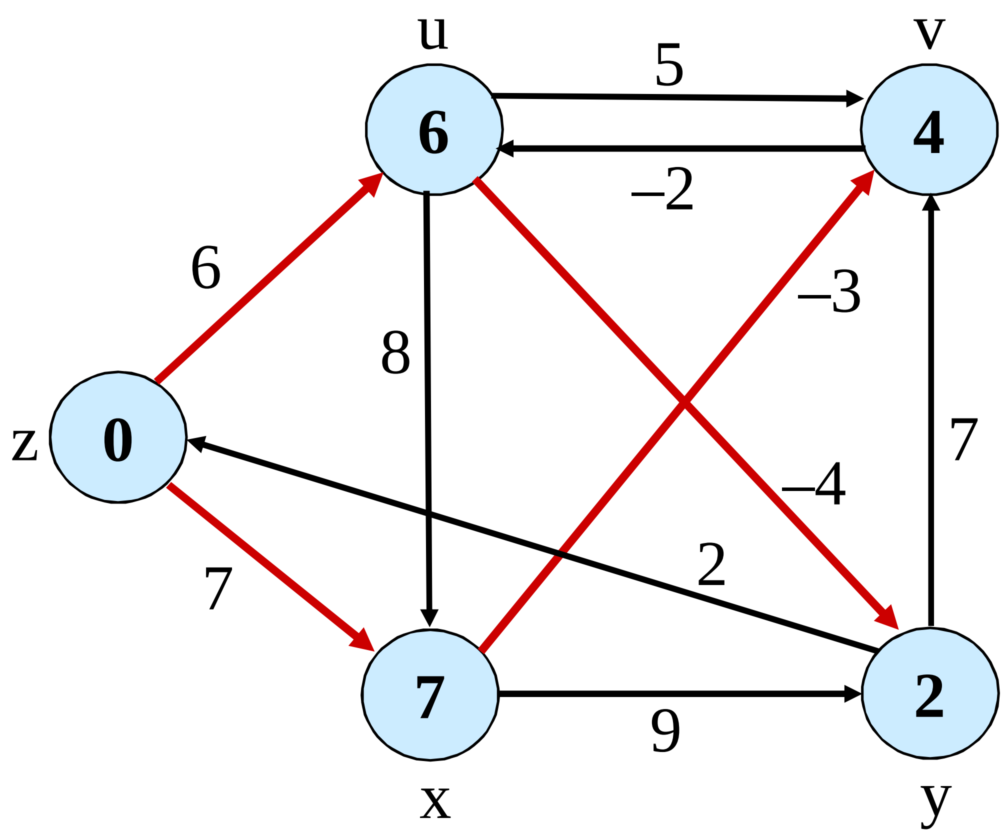

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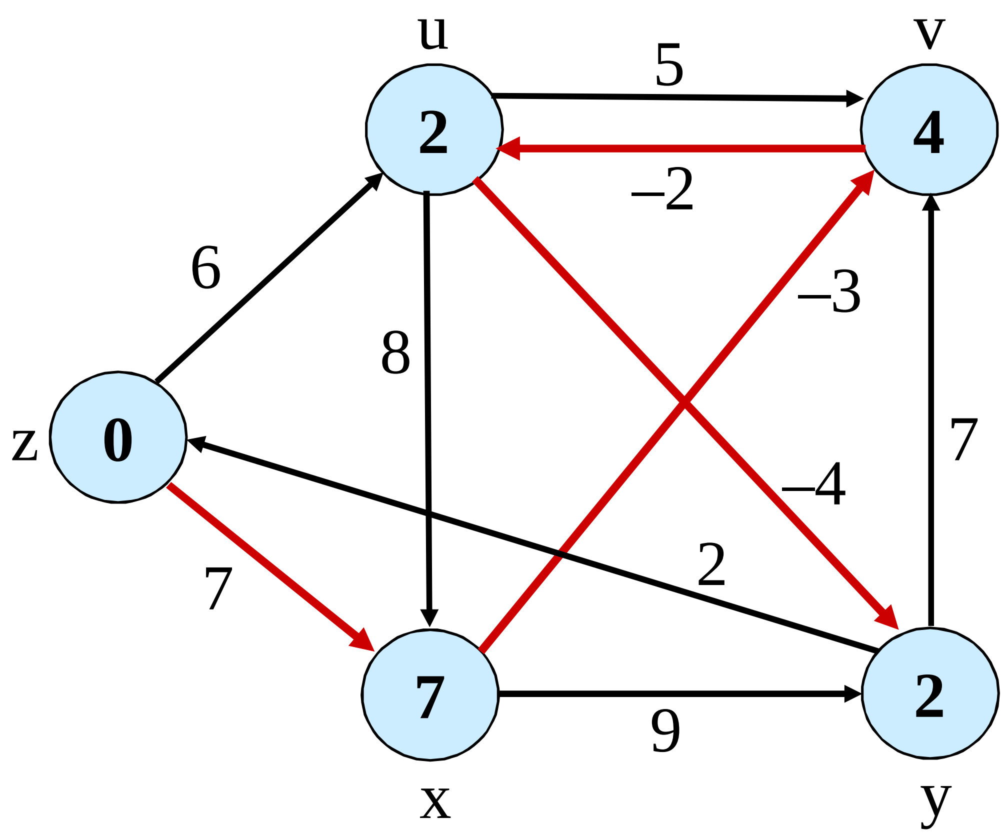

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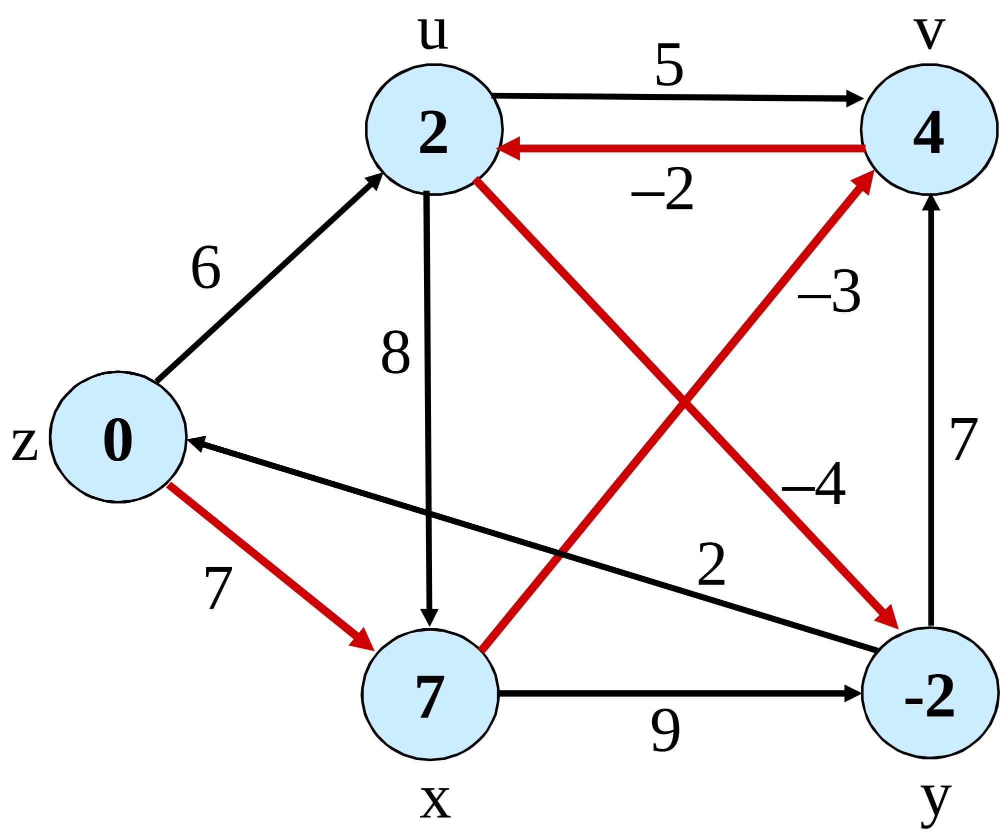

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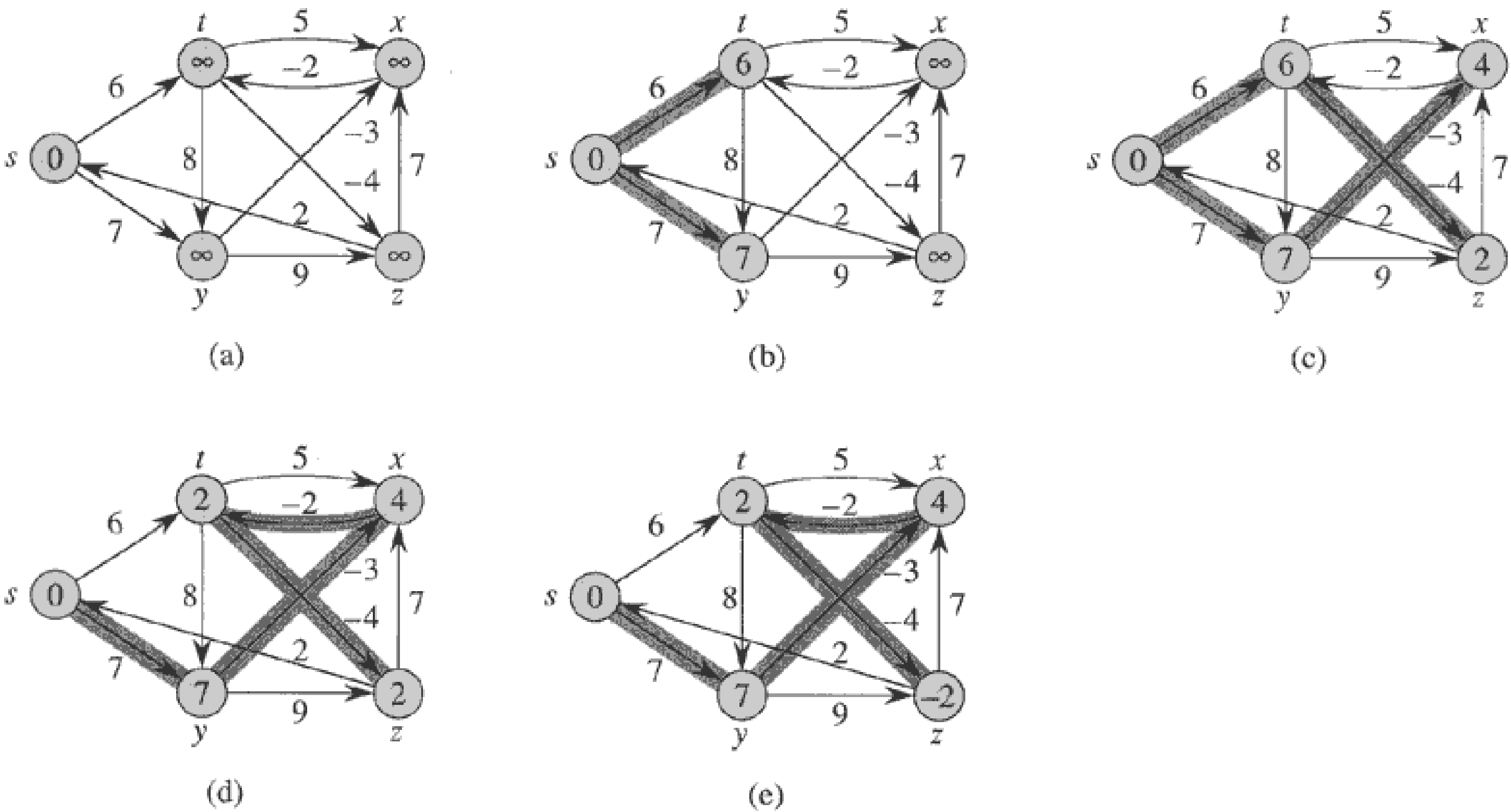


Dijkstra's Algorithm

- If no negative edge weights, we can beat Bellman-Ford Algorithm
- Similar to breadth-first search
  - Grow a tree gradually, advancing from vertices taken from a queue
- Also similar to Prim's algorithm for MST
  - Use a priority queue keyed on d[v]

Dijkstra's Algorithm

```
Dijkstra(G)
   for each v ∈ V
      d[v] = ∞;
   d[s] = 0; S = ∅; Q = V;

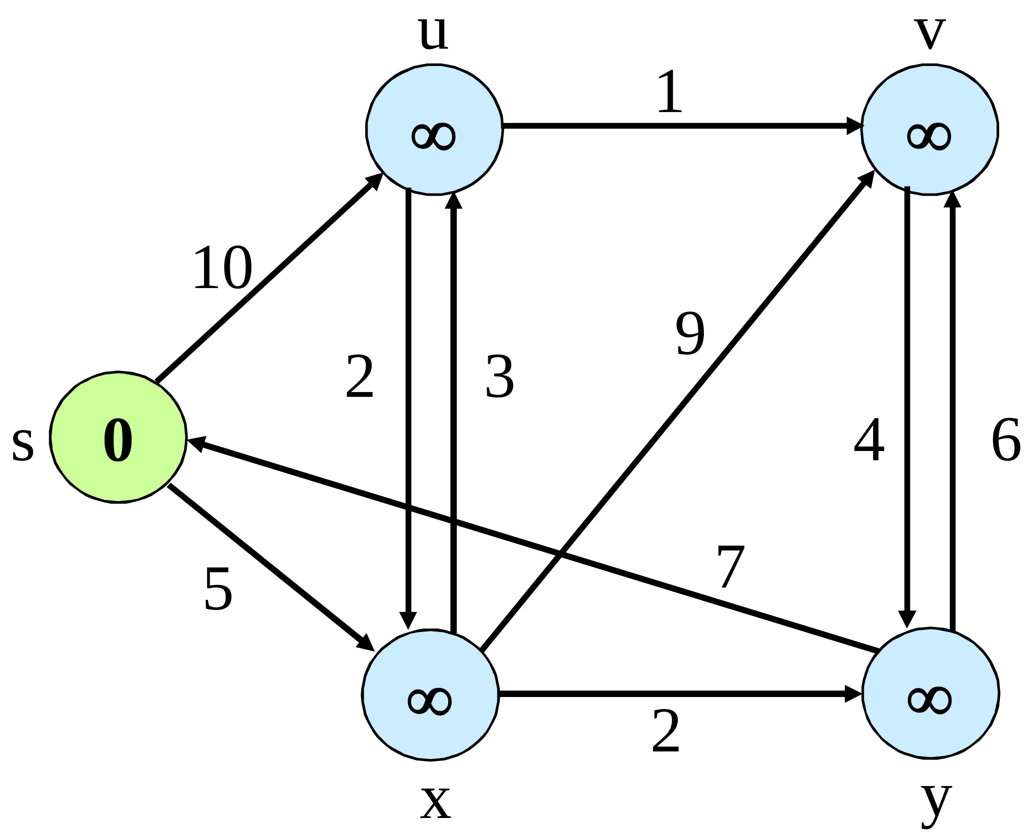
   while (Q ≠ ∅)
      u = ExtractMin(Q);
      S = S U {u};

      for each v ∈ u->Adj[]
         if (d[v] > d[u]+w(u,v))
            d[v] = d[u]+w(u,v);
```

Relaxation Step

Note: this is really a call to Q->DecreaseKey()

Dijkstra’s Algorithm: Example



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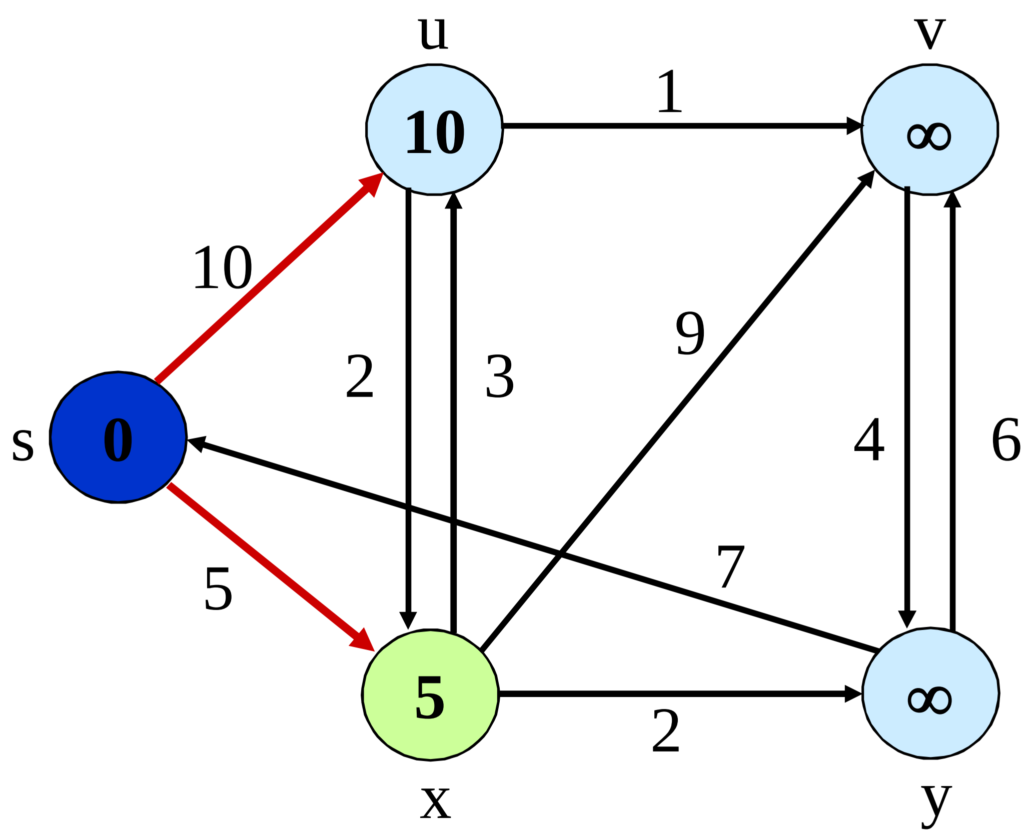

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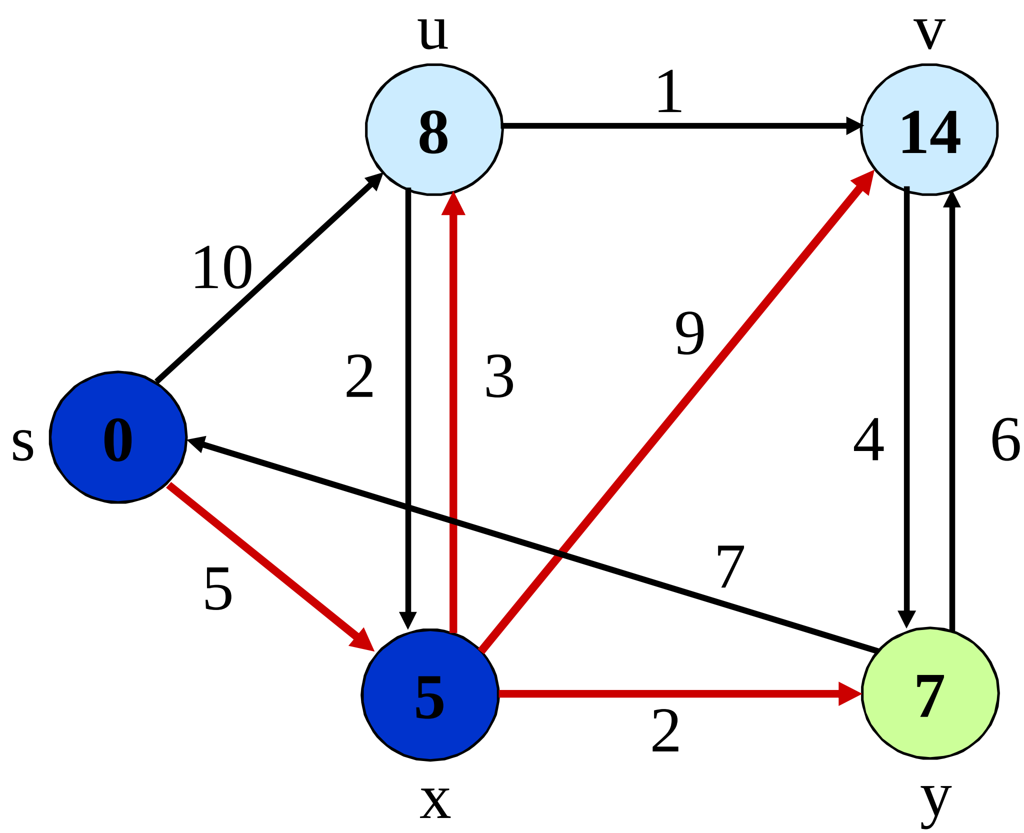

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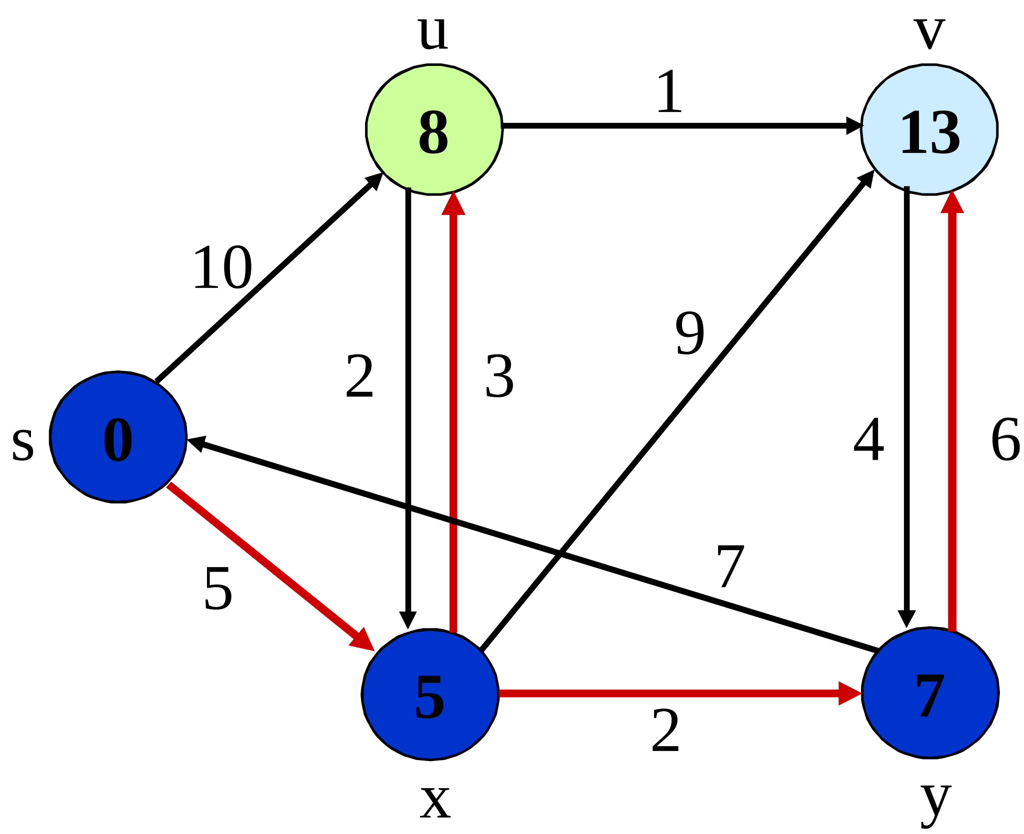

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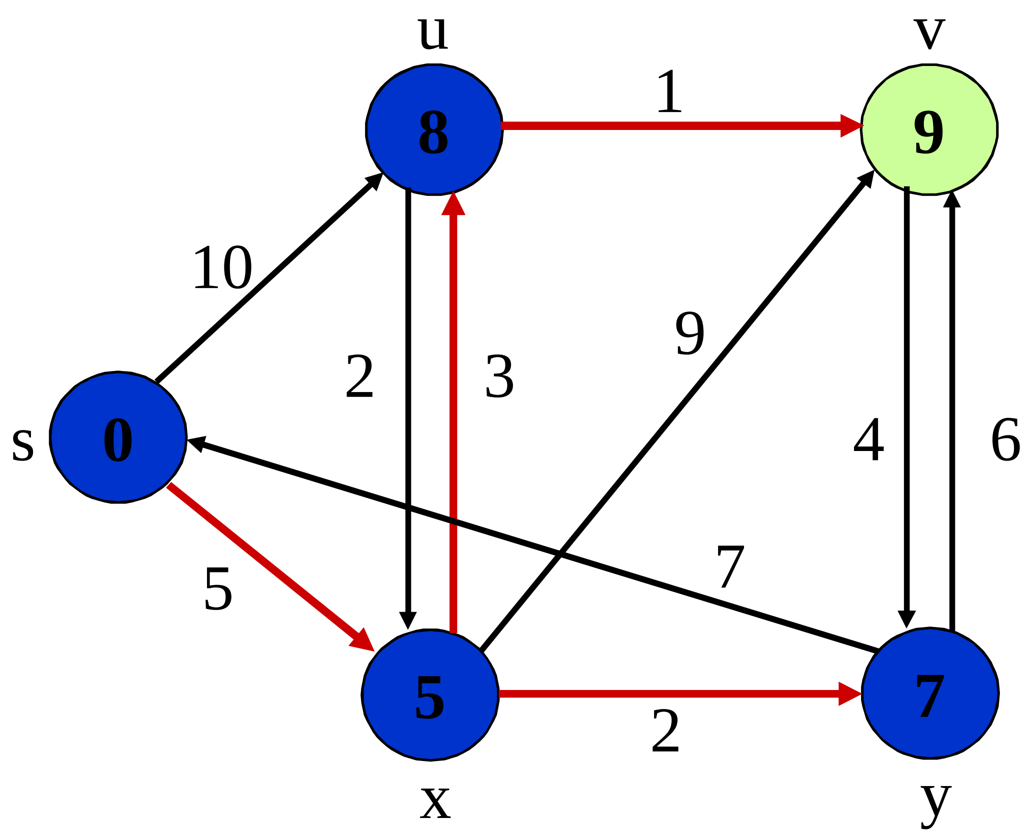

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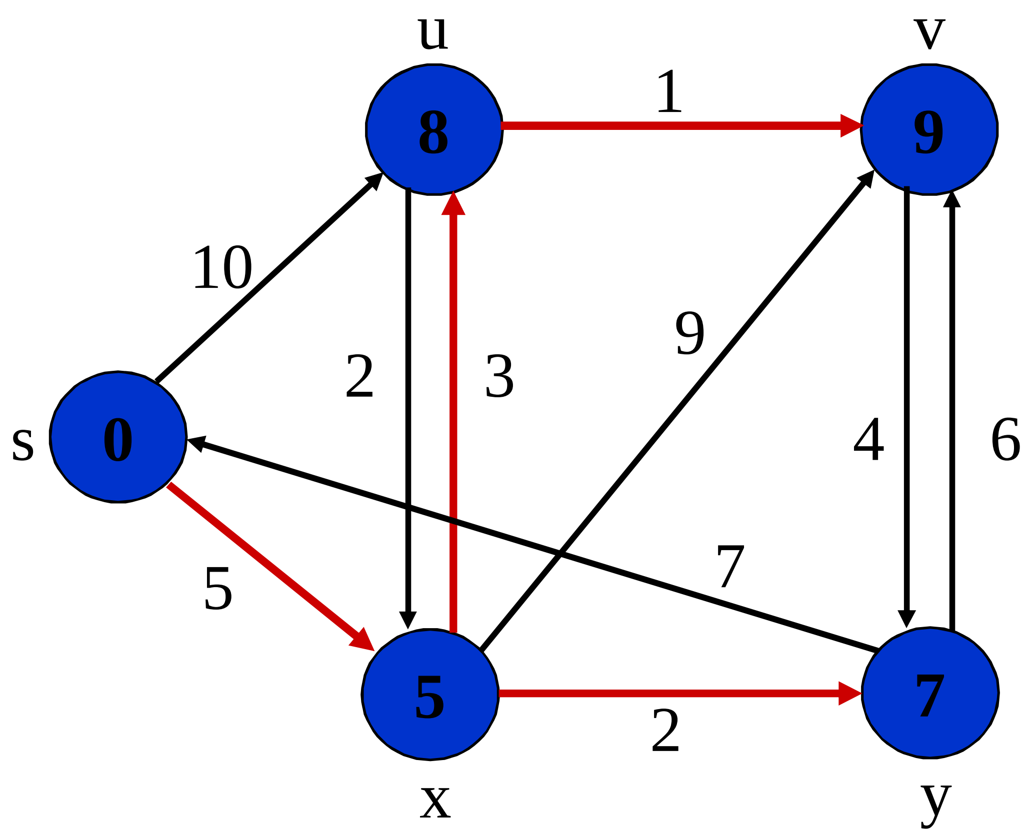

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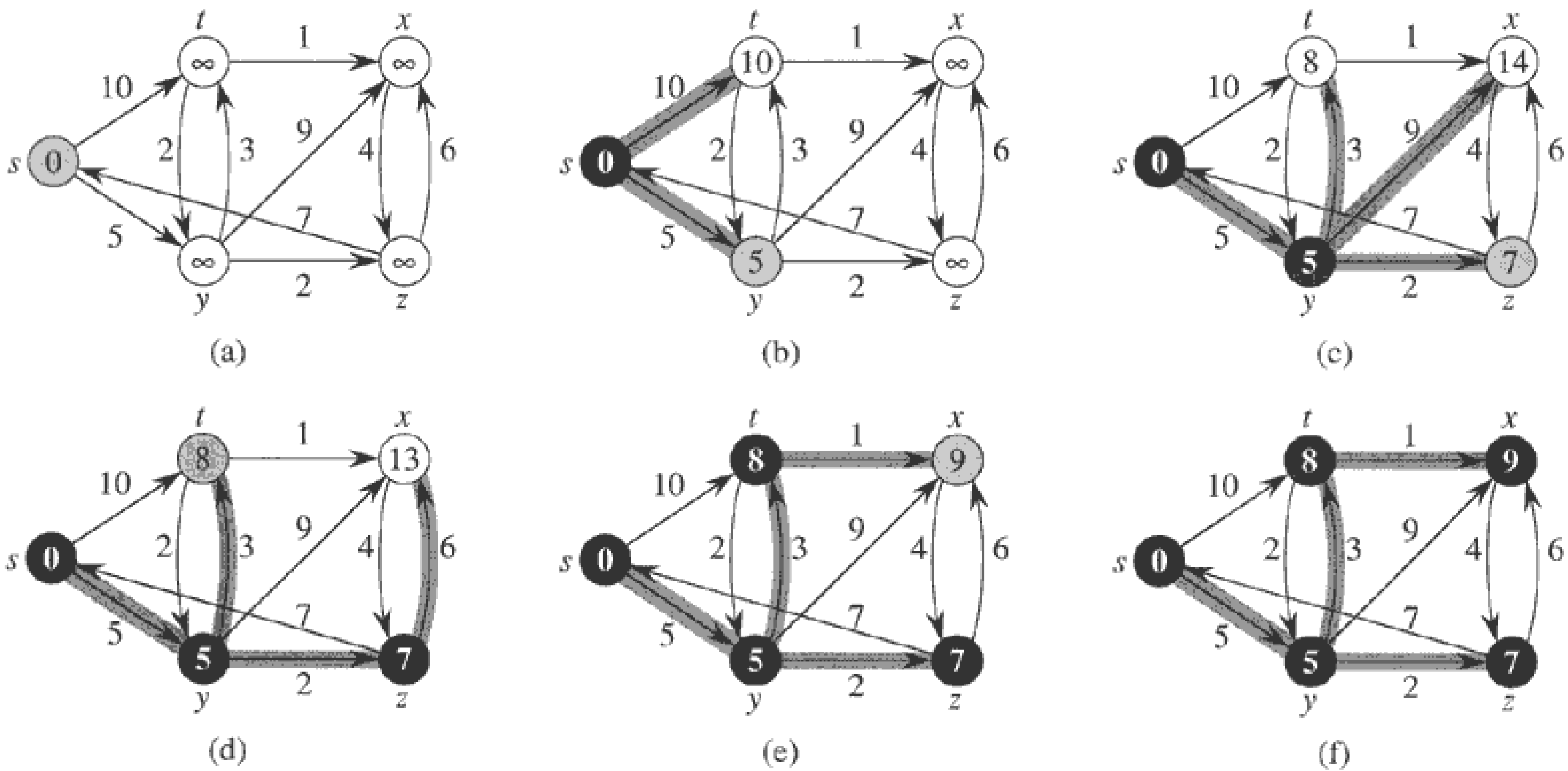

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   while (Q ≠ ∅)
      u = ExtractMin(Q);
      S = S U {u};
      for each v ∈ u->Adj[]
         if (d[v] > d[u]+w(u,v))
            d[v] = d[u]+w(u,v);
```

How many times is ExtractMin() called?

How many times is DecraseKey() called?

What will be the total running time?

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```

A: O(E lg V) using binary heap for Q
Can achieve O(V lg V + E) with Fibonacci heaps

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Correctness: we must show that when u is removed from Q, it has already converged