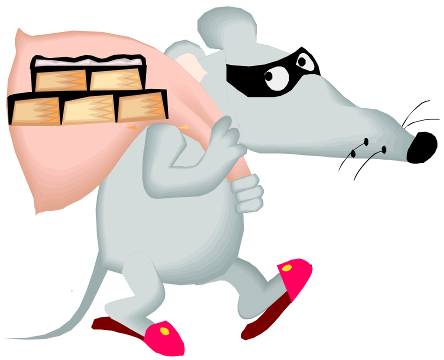
Algorithms:
Dynamic Programming

0-1 Knapsack Problem

The Knapsack Problem (Review)

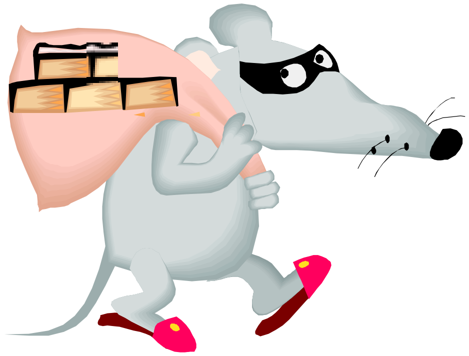
There are two versions of the problem:

(1) "0-1 knapsack problem"



Items are indivisible: you either take an item or not. Solved with *dynamic programming*.

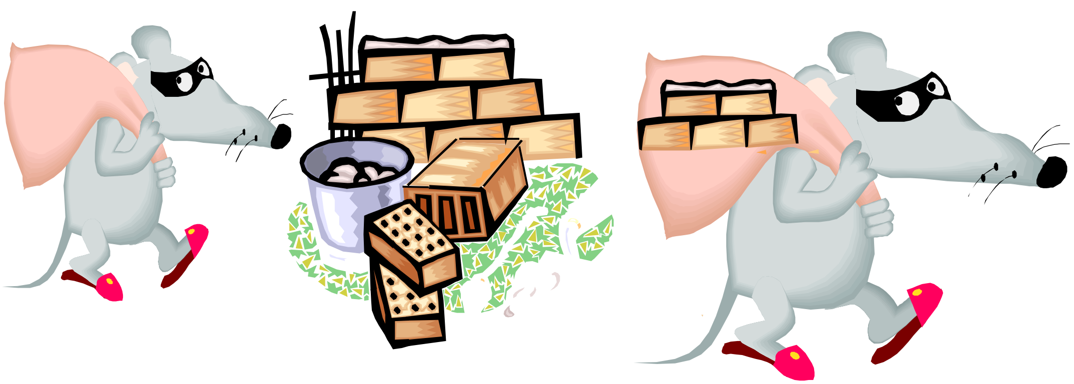
(2) "Fractional knapsack problem"



Items are divisible: you can take any fraction of an item. Solved with a *greedy algorithm*.

Optimal Substructure Property

- Both problems exhibit the optimal substructure property.
- To show this for both the problems, consider the most valuable load weighing at most W pounds
  - Q: If we remove item j from the load, what do we know about the remaining load?
  - A: The remaining load must be the most valuable load weighing at most $W - w_j$ that the thief could take from the n-1 original items excluding item j.



0-1 Knapsack Problem

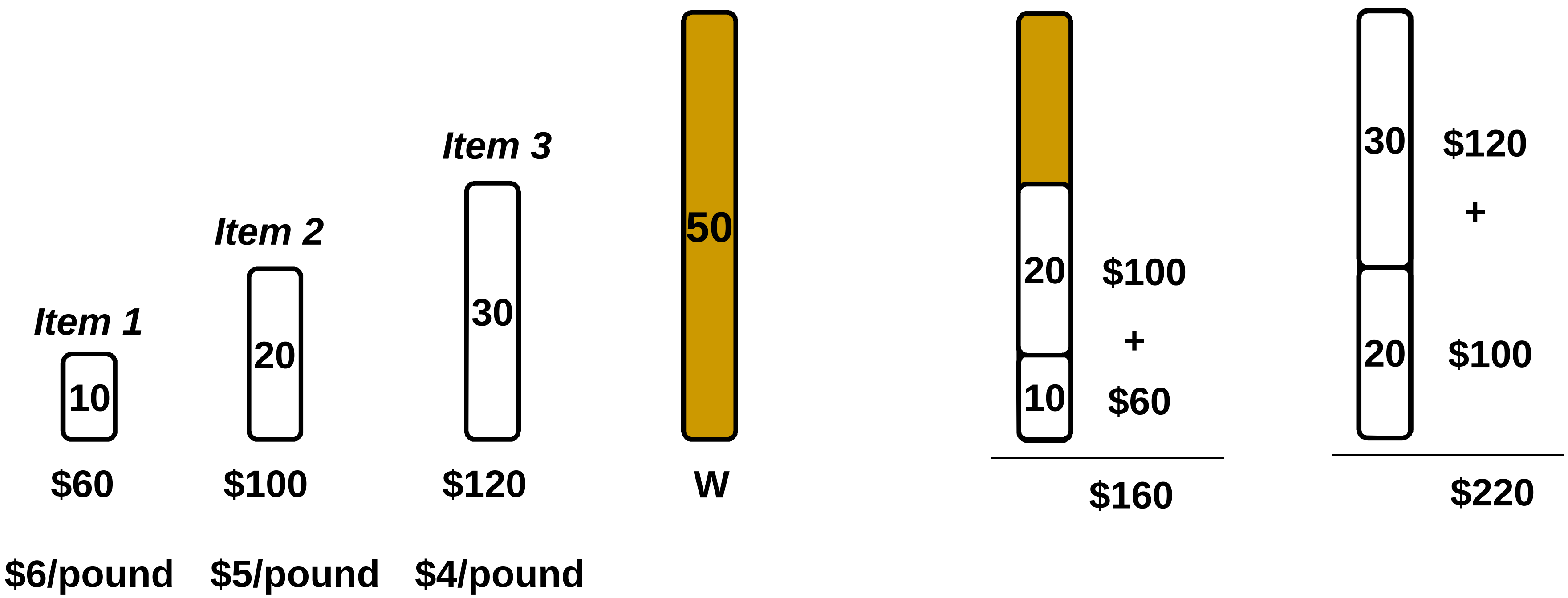
- Thief has a knapsack with maximum capacity *W*, and a set *S* consisting of *n* items
- Each item *i* has some weight w_i and benefit value v_i (all w_i , v_i and *W* are integer values)
- Problem: How to pack the knapsack to achieve maximum total value of packed items?
- Goal:

find x_i such that for all $x_i \in \{0, 1\}$, $i = 1, 2, .., n$

$\sum w_i x_i \leq W$ and

$\sum x_i v_i$ is maximum

0-1 Knapsack - Greedy Strategy Fails



0-1 Knapsack: Brute-Force Approach

- Since there are n items, there are 2^n possible combinations of items.
- We go through all combinations and find the one with the most total value and with total weight less or equal to W.
- Running time will be $O(2^n)$.

0-1 Knapsack - Dynamic Programming

- $P(i, w)$ – the maximum profit that can be obtained from items 1 to i, if the knapsack has size w
- Case 1: thief takes item i

$$P(i, w) = v_i + P(i - 1, w - w_i)$$

- Case 2: thief does not take item i

$$P(i, w) = P(i - 1, w)$$

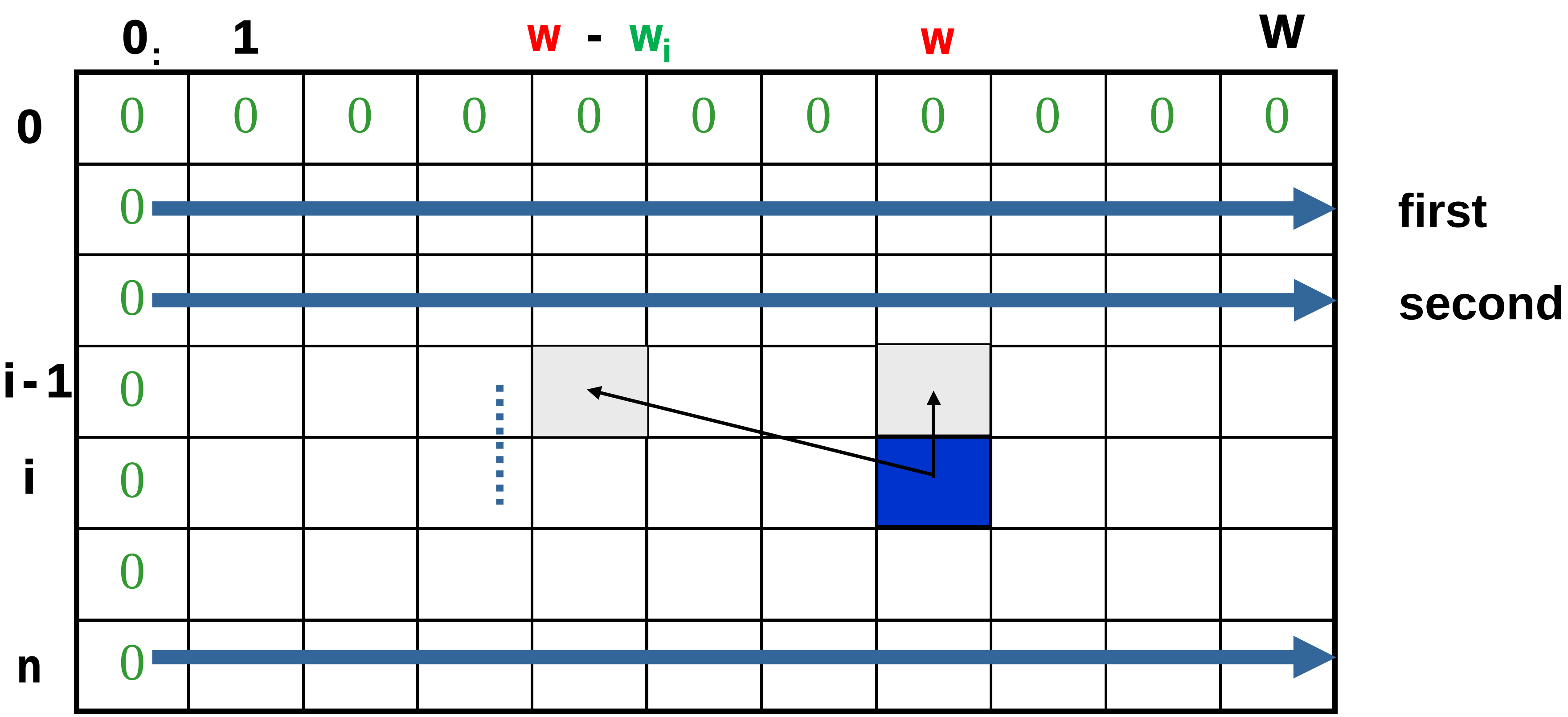
Recursive Formula

$$P[i,w] = \begin{cases} P[i-1,w] & \text{if } w < w_i \\ \max\{v_i + P[i-1, w-w_i], P[i-1,w]\} & \text{else} \end{cases}$$

- The best subset that has the total weight *w*, either contains item *i* or not.
- First case: $w < w_i$. Item *i* can't be part of the solution, since if it was, the total weight would be > *w*, which is unacceptable.
- Second case: $w >= w_i$. Then the item *i* <u>can</u> be in the solution, and we choose the case with greater value.

Dependencies among Subproblems

$$P(i, w) = \max \{\overbrace{v_i + P(i - 1, w - w_i)}^{\textbf{Item } i \textbf{ was taken}},\ \overbrace{P(i - 1, w)}^{\textbf{Item } i \textbf{ was not taken}}\}$$



Overlapping Subproblems

$P(i, w) = \max \{v_i + P(i - 1, w\text{-}w_i), P(i - 1, w)\}$

	0	1				w					W
0	0	0	0	0	0	0	0	0	0	0	0
	0										
	0										
i-1	0										
i	0										
	0										
n	0										

***E.g.*: All the subproblems shown in grey may depend on P(i-1, w)**

W = 5

Item	Weight	Value
1	2	12
2	1	10
3	3	20
4	2	15

Example:

$$P(i, w) = \max\{v_i + P(i-1, w-w_i), P(i-1, w)\}$$

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

P(1, 1) = P(0, 1) = 0

P(1, 2) = max{12+0, 0} = 12

P(1, 3) = max{12+0, 0} = 12

P(1, 4) = max{12+0, 0} = 12

P(1, 5) = max{12+0, 0} = 12

P(2, 1)= max{10+0, 0} = 10

P(2, 2)= max{10+0, 12} = 12

P(2, 3)= max{10+12, 12} = 22

P(2, 4)= max{10+12, 12} = 22

P(2, 5)= max{10+12, 12} = 22

P(3, 1)= P(2,1) = 10

P(3, 2)= P(2,2) = 12

P(3, 3)= max{20+0, 22}=22

P(3, 4)= max{20+10,22}=30

P(4, 5)= max{20+12,22}=32

P(4, 1)= P(3,1) = 10

P(4, 2)= max{15+0, 12} = 15

P(4, 3)= max{15+10, 22}=25

P(4, 4)= max{15+12, 30}=30

P(4, 5)= max{15+22, 32}=37

Reconstructing the Optimal Solution

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

- Item 4
- Item 2
- Item 1

- Start at P(n, W)
- When you go left-up ⇒ item i has been taken
- When you go straight up ⇒ item i has not been taken

0-1 Knapsack Algorithm (DP)

```
for w = 0 to W
    P[0, w] = 0
for i = 0 to n
    P[i, 0] = 0
    for w = 0 to W
        if w_i <= w // item i can be part of the solution
            if (v_i + P[i-1, w-w_i] > P[i-1, w])
                P[i, w] = v_i + P[i-1, w- w_i]
            else
                P[i, w] = P[i-1, w]
        else P[i, w] = P[i-1, w]  // w_i > w
```

Running time: $O(n*W)$