Homework 1

1. Find the smallest positive number $u > 0$, written as a negative power of 10, $u = 10^{-m}$, where $m \in \mathbb{N}$, which satisfies the property:

$$1 +_c u \neq 1$$

In the above relation, $+_c$ denotes the computer-implemented addition operation. The number u is known as ***machine precision***.

2. Operation $+_c$ is ***non-associative***: consider the numbers $x=1.0$, $y = u/10$, $z = u/10$, where u is the above-computed machine precision (the value that satisfies the relations $1 +_c u \neq 1$ and $1 +_c u/10 = 1$). Verify that the computer addition operation is non-associative.

$$(x +_c y) +_c z \neq x +_c (y +_c z).$$

Find an example that shows the computer multiplication operation is also non-associative.

3. **Rational function approximations for *tan*[1]**

Consider the rational functions from Figure 1. Toate funcţiile din această figură sunt aproximări ale funcției ***tangentă*** pentru $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\tan(a) \approx T(i,a) \quad , \quad \forall a \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Implement functions *T(i,a), i=4, 5, 6, 7, 8, 9.*

Generate 10.000 random numbers in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Compute the approximate values of the ***tan*** function using the 6 above-mentioned rational functions. Consider that the value of the ***tan*** function computed using the mathematical library of the program you are working with (usually math.tan) is the exact value of the ***tan*** function.

$$v_{exact} = \tan(a) = math.\tan(a).$$

For each of the 10.000 generated numbers, establish which are the three functions that provided the best approximations (those approximations that provided the smallest error).

$$error_i\,(a) = |\, T(i,a) - v_{exact}\,|.$$

[1] https://mae.ufl.edu/~uhk/IEEETrigpaper8.pdf

Use these results to compute an hierarchy of the six rational functions.

$$T(1,a) = a$$

$$T(2,a) = \frac{3a}{3-a^2}$$

$$T(3,a) = \frac{15a-a^3}{15-6a^2}$$

$$T(4,a) = \frac{105a-10a^3}{105-45a^2+a^4}$$

$$T(5,a) = \frac{945a-105a^3+a^5}{945-420a^2+15a^4}$$

10395 a

$$T(6,a) = \frac{10395-1260a^3+21a^5}{10395-4725a^2+210a^4-a^6}$$

$$T(7,a) = \frac{135135a-17325a^3+378a^5-a^7}{135135-62370a^2+3150a^4-28a^6}$$

$$T(8,a) = \frac{2027025a-270270a^3+6930a^5-36a^7}{2027025-945945a^2+51975a^4-630a^6+a^8}$$

$$T(9,a) = \frac{34459425a-4729725a^3+135135a^5-990a^7+a^9}{34459425-16216200a^2+945945a^4-13860a^6+45a^8}$$

Figure 1 – Rational function approximations for the *tan* function

Errata:

$$\boldsymbol{T(6,a) = \frac{10395a-1260a^3+21a^5}{10395-4725a^2+210a^4-a^6}}$$

Bonus 10pt: Approximate the values of functions *sin* and *cos* using the formulae (6) and (7) from the cited reference.