1. Determine whether the following functions are inner products.

   (a) Let $V = \mathbb{R}^2$ and define $\langle (a, b), (c, d) \rangle = ac - bd$.

   (b) Let $V = \mathbb{C}^n$, for $v = (w_1, \dots, w_n)$, $u = (z_1, \dots, z_n)$ and define $\langle v, u \rangle = a_1 w_1 \overline{z_1} + \cdots + a_n w_n \overline{z_n}$, where $a_i \in \mathbb{C}$.

   (c) Let $V = M_2(\mathbb{R})$ and define $\langle A, B \rangle = \operatorname{tr}(A + B)$.

   (d) Let $V = \operatorname{Fun}([-1, 1], \mathbb{R})$ and define $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx$.

(e) (Optional) Let $V = \mathcal{P}(\mathbb{R})$ and define $\langle p(x), q(x) \rangle = \int_{-1}^{1} p'(x)q(x)\,dx$.

2. (Don't look at the textbook yet). The goal of this question is to prove:

   Theorem: Cauchy-Schwarz Inequality Suppose $u, v \in V$. Then
   $$|\langle u, v\rangle| \leq ||u||\,||v||.$$

   Hint: Apply the Pythagorean Theorem to an orthogonal decomposition $u = \lambda v + w$.

3. Given vectors $u, v \in V$ and an inner product on V, recall the *orthogonal decomposition of* u *and* v is an equation of the form $u = \lambda v + w$ where $\lambda \in F$ and $\langle v, w \rangle = 0$.

   Using inner product from 1.d) and using the vectors $u(x) = 5x^3$ and $v(x) = x$ find the orthogonal decomposition of u and v.