Dynamic Programming

by

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Dynamic Programming

1) Solution: Result of a sequence of decisions.

2) Why Dynamic Programming,

3) Principle of optimality,

4) Difference between DP & Greedy Method.

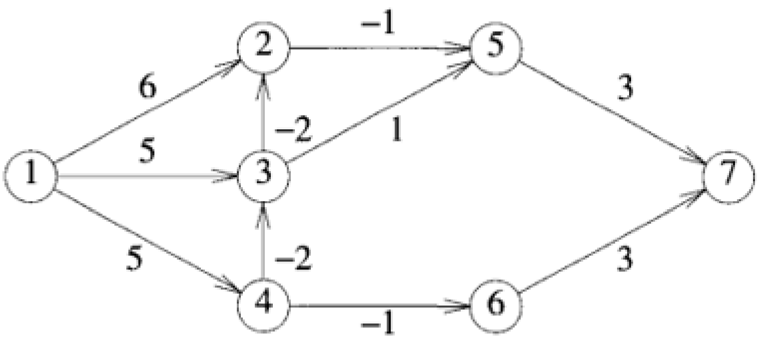
5) Similar to divide and conquer, DP solves a problem by combining solutions of subproblems

- Subproblems are independent for D&C so more works as it solves common subproblems
- In contrast the subproblems are not independent rather they are overlapped in DP.

  So, DP solves every subproblem just once and saves its answer in a table, thereby avoiding

  re-computation of the answer every time the subproblem is encountered.

Single Source Shortest Paths



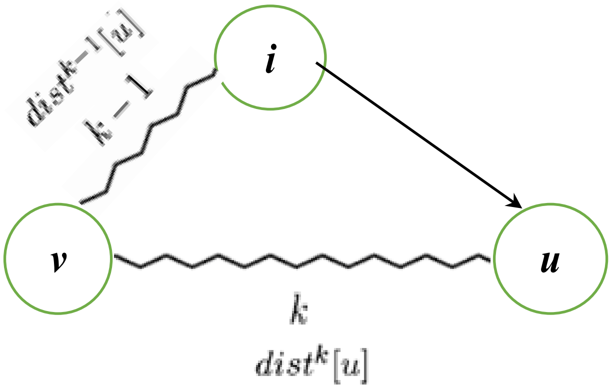
(a) A directed graph

Note: If there is no cycle of negative length, any shortest path has at most $n - 1$ edges for a graph of nodes n.

$\text{dist}^k[u]$: Length of the shortest path from source v to u having at most k edges.

$\therefore \text{dist}^1[u] = \text{cost}(v, u)$

We have to find out $\text{dist}^{n-1}[u]\ \forall\ u$.



1. If the shortest path from v to u with at most k, $k > 1$, edges has no more than $k-1$ edges, then $dist^k[u] = dist^{k-1}[u]$.

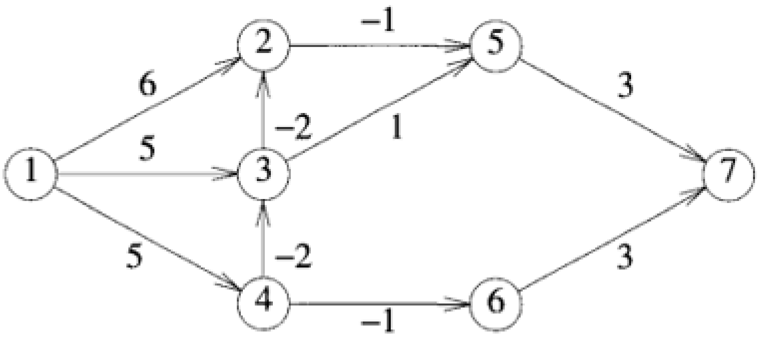
2. If it has exactly *k* edges,

then $dist^k[u] = dist^{k-1}[i] + cost[i, u]$

$$dist^k[u] = \min \{dist^{k-1}[u], \min_i \{dist^{k-1}[i] + cost[i, u]\}\}$$

This recurrence can be used to compute $dist^k$ from $dist^{k-1}$, for $k = 2, 3, \ldots, n-1$.

An Illustration:



(a) A directed graph

k	$dist^k[1..7]$						
	1	2	3	4	5	6	7
1	0	6	5	5	∞	∞	∞
2	0	3	3	5	5	4	∞
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	3

(b) $dist^k$

```
Algorithm BellmanFord(v, cost, dist, n)
// Single-source/all-destinations shortest
// paths with negative edge costs
{
    for i := 1 to n do // Initialize dist.
        dist[i] := cost[v, i];
    for k := 2 to n − 1 do
        for each u such that u ≠ v and u has
                at least one incoming edge do
A           for each ⟨i, u⟩ in the graph do
                if dist[u] > dist[i] + cost[i, u] then
                    dist[u] := dist[i] + cost[i, u];
}
```

A $\longrightarrow$ requires $O(n^2)$ for adjacency matrix (or) $O(e)$ for adjacency list.

$\therefore$ Time complexity: $O(n^3)$ for adjacency matrix, $O(ne)$ for adjacency list.

All Pairs Shortest Path

(Floyd Warshall Algorithm)

Statement: To determine a matrix A such that A(*i*, *j*) is the length of the shortest path from *i* to *j*.

Assumption: No negative cycles.

Let $A^k(i,j)$ be the shortest path going through no vertex higher than k.

Then, $A^0(i, j) = \text{cost}\ (i, j)$

If the shortest path goes through *k*

$$A^k(i,j) = A^{k-1}(i,k) + A^{k-1}(k,j) \longrightarrow \quad (1)$$

If not

$$A^k(i,j) = A^{k-1}(i,j) \longrightarrow \quad (2)$$

Combining (1) and (2), we get

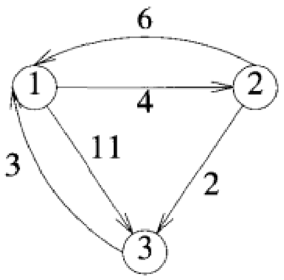
$$A^k(i,j) = \min\ \{A^{k-1}(i,j),\ A^{k-1}(i,k) + A^{k-1}(k,j)\}$$

```
Algorithm AllPaths(cost, A, n)
// cost[1 : n, 1 : n] is the cost adjacency matrix of a graph with
// n vertices; A[i, j] is the cost of a shortest path from vertex
// i to vertex j. cost[i, i] = 0.0, for 1 ≤ i ≤ n.
{
    for i := 1 to n do
        for j := 1 to n do
            A[i, j] := cost[i, j]; // Copy cost into A.
    for k := 1 to n do
        for i := 1 to n do
            for j := 1 to n do
                A[i, j] := min(A[i, j], A[i, k] + A[k, j]);
}
```

Time Complexity: $O(n^3)$

An Illustration:

Let A = $\begin{matrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & \infty & 0 \end{matrix}$



A^0	1	2	3
1	0	4	11
2	6	0	2
3	3	∞	0

(b) A^0

A^1	1	2	3
1	0	4	11
2	6	0	2
3	3	7	0

(c) A^1

A^2	1	2	3
1	0	4	6
2	6	0	2
3	3	7	0

(d) A^2

A^3	1	2	3
1	0	4	6
2	5	0	2
3	3	7	0

(e) A^3

Longest Common Subsequence Problem

Subsequence: Z = (B, C, D, B) is a subsequence of X =(A, B, A, C, D,A,D,A,B)

Common subsequence: (B, C, A) is a common subsequence of

X = (A, B, C, B, D, A, B) and Y = (B, D, C, A, B, D)

Longest common subsequence: The common subsequence of two sequences with largest length.

(B, C, A, B) is a LCS of X and Y

(B, C, B, D) is another LCS of X and Y.

NOTE – Thus LCS of two sequence is not unique.

Longest-common-subsequence problem: Given two sequences X= (x_1, x_2,…, x_m) and Y = (y_1, y_2,….., y_n) and wish to find a maximum length common subsequence of X and Y.

Brute Force Approach:

Step1: Enumerate all subsequence of X.

Step2: check each subsequence to see if it is also subsequence of Y.

Step3: Find the largest one of them in step 2.

Time Complexity:

1) $\exists\ 2^m$ subsequences of X

2) So, it has exponential time complexity

Dynamic Programming Approach:

NOTE –LCS problem has optimal substructure property as noted by the following theorem

$X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_n)$

$X = (x_1, x_2, \ldots, x_i)$, $i = 0, 1, \ldots, m$ is defined as the i^{th} prefix of X.

e.g. X = (A, B, C, B, D) then X_4 = (A, B, C, B), X_0 is the empty sequence.

$Z = (z_1, z_2, \ldots, z_k)$ be any LCS X and Y.

1) If $x_m = y_n$ then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.

2) If $x_m \neq y_n$ then $z_k \neq x_m$ implies Z is an LCS of X_{m-1} and Y.

3) If $x_m \neq y_n$ then $z_k \neq y_n$ implies Z is an LCS of X and Y_{n-1}.

A recursive solution to problem:

Theorem implies that there are either one or two subproblems when finding an LCS of X and Y.

1)If $x_m = y_n$ then we are to find an LCS of X_{m-1} and Y_{n-1}.

2)If $x_m \neq y_n$ then we are to solve two subproblems

2.1) we have to find LCS of X_{m-1} and Y.

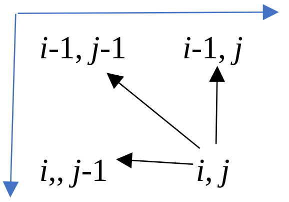
2.2) we have to find LCS of X and Y_{n-1}.

Whichever is longer will be LCS of X and Y.

Therefore, if c[i, j] denotes the length of LCS of X_i and Y_j. then:

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

NOTE – if either i=0 or j=0, one of the sequence has length 0, so the LCS has length 0.



Computing the length of the LCS

```
LCS-LENGTH(X, Y)
m = X.length
n = Y.length
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i − 1, j − 1] + 1
            b[i, j] = "↖"
        elseif c[i − 1, j] ≥ c[i, j − 1]
            c[i, j] = c[i − 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j − 1]
            b[i, j] = "←"
return c and b
```

Time complexity: entries are captured in row-major order from left to right, and each entry take O(1) time to execute. Since there are ‘m’ rows and ‘n’ columns it require O(m*n) time to execute.

An Illustration:

X = (A,B,C,B,D,A,B) and Y = (B,D,C,A,B,A)

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

We simply begin at b[m,n] and trace through the table by following the arrows. Whenever we encounter a "↖" in entry b[i,j] it implies that $x_m = y_n$ is an element of the LCS .

NOTE : the element of the LCS are encountered in reverse order using this method.

Thus following the "↖" we got BCBA as the LCS.

Following procedure runs it

```
PRINT-LCS(b, X, i, j)
if i == 0 or j == 0
        return
if b[i, j] == "↖"
        PRINT-LCS(b, X, i − 1, j − 1)
        print x_i
elseif b[i, j] == "↑"
        PRINT-LCS(b, X, i − 1, j)
else PRINT-LCS(b, X, i, j − 1)
```

Time Complexity: Since at least one of i and j is decremented at each stage in the recursion, it requires O(m+n).

Overall Time Complexity: O(m*n)

Overall Space Complexity: O(m*n)

Tutorial: Determine an LCS of <1,0,0,1,0,1,0,1> and <0,1,0,1,1,0,1,1,0>
Ans : 100110

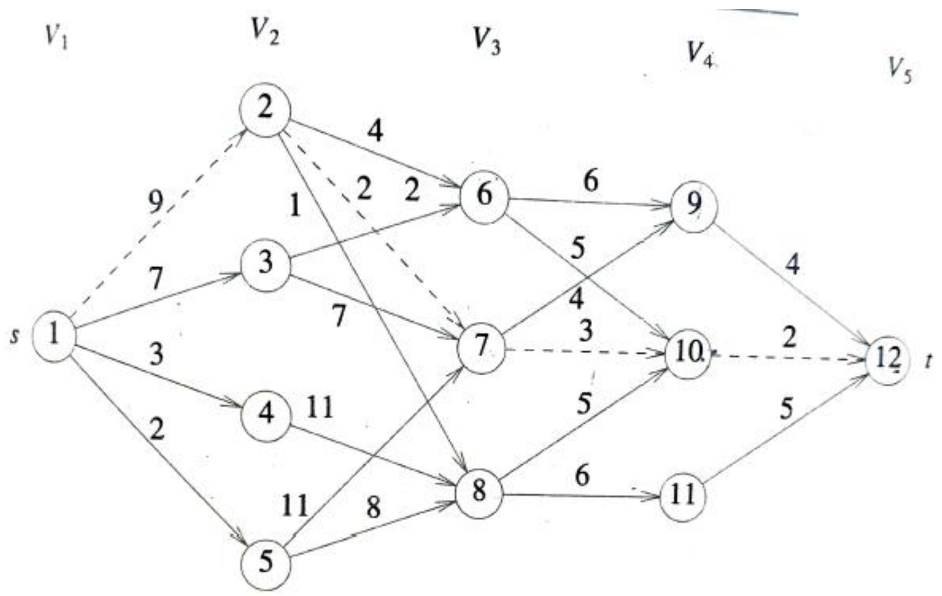
Multistage graph problem

A multistage graph: A directed graph in which vertices are partitioned into k >= 2 disjoint sets V_1, V_2, …, V_k such that

1) $|V_1| = |V_k| = 1$

2) If there exists an edge <u, v> then if u $\in V_i$ then v must belong to V_{i+1}.

V_1: source (s) and V_k: sink (t)

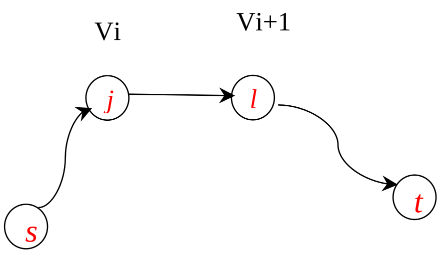


Problem Statement: To determine a minimum-cost path from s to t.

Let c[i, j] be the cost of the edge <i, j>

Let P(i, j) be the minimum cost path from vertex j in V_i to t and

Let cost (i, j) be its cost.

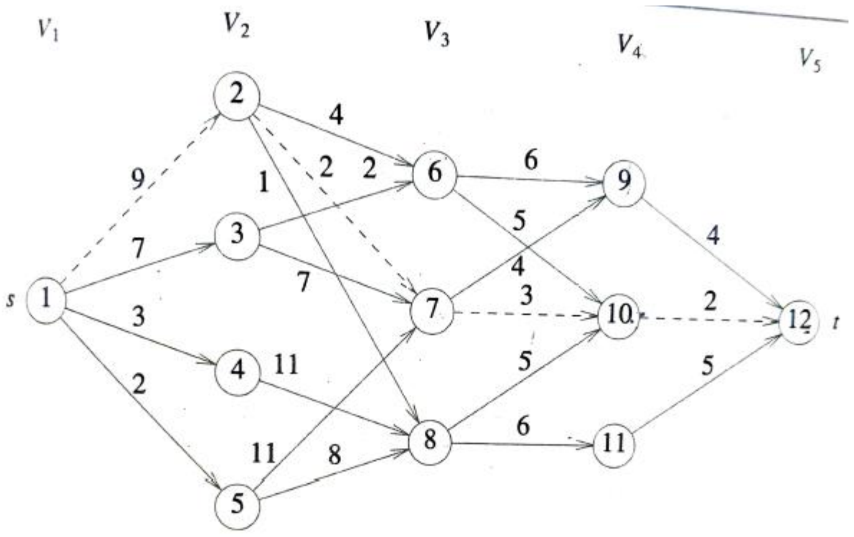


Initialization: cost (k-1, j) = c[j, t]

Solution: We have to find the Cost(1, s).

Intermediate Calculations:

$$cost(i, j) = \min_{\substack{l \in V_{i+1} \\ \langle j,l \rangle \in E}} \{c(j, l) + cost(i+1, l)\}$$



$$cost(i,j) = \min_{\substack{l \in V_{i+1} \\ (j,l) \in E}} \{c(j,l) + cost(i+1,l)\}$$

Let d(*i*, *j*) be the value of *l* that minimizes Cost(*i*+1, *l*) + c(*j*, *l*).

Then d(3, 6) = 10, d(3, 7) = 10, d(3, 8) = 10;

d(2, 2) = 7, d(2, 3) = 6, d(2, 4) = 8, d(2, 5) = 8;

d(1, 1) = 2

Then minimum cost path is 1, v1, v2, …, vk-1, t.

Here, v2 = d(1, 1) = 2;

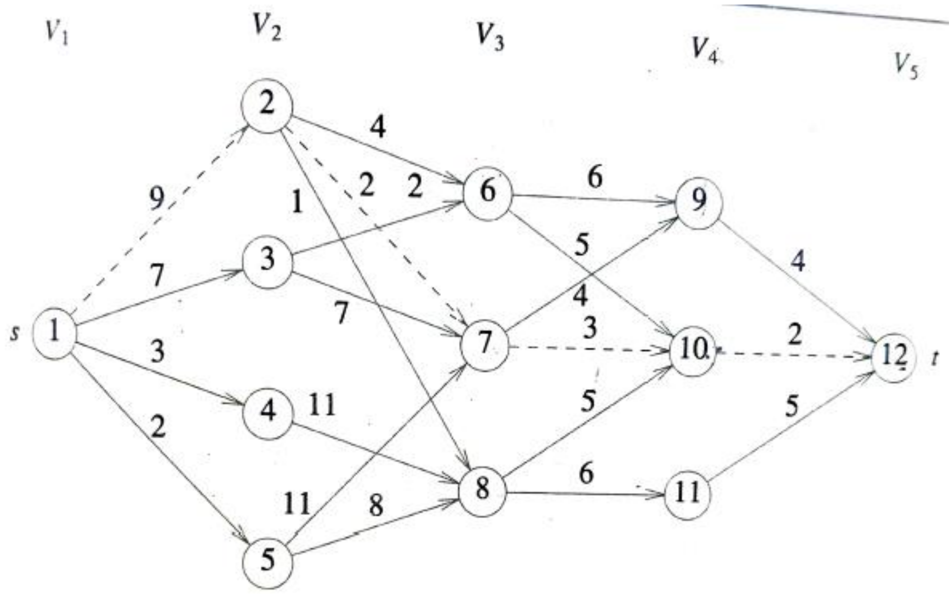
v3 = d(2, d(1, 1)) = d(2, 2) = 7

v4 = d(3, d(2, d(1, 1))) = d(3, 7) = 10

Therefore, the minimum cost path is 1, 2, 7, 10, 12

Algorithm using Forward Approach:

$$cost(i,j) = \min_{\substack{l \in V_{i+1} \\ \langle j,l \rangle \in E}} \{c(j,l) + cost(i+1,l)\}$$




```
Algorithm FGraph(G, k, n, p)
// The input is a k-stage graph G = (V, E) with n vertices
// indexed in order of stages. E is a set of edges and c[i, j]
// is the cost of ⟨i, j⟩. p[1 : k] is a minimum-cost path.
{
    cost[n] := 0.0;
    for j := n - 1 to 1  step -1 do
    { // Compute cost[j].
        Let r be a vertex such that ⟨j, r⟩ is an edge
        of G and c[j, r] + cost[r] is minimum;
        cost[j] := c[j, r] + cost[r];
        d[j] := r;
    }
    // Find a minimum-cost path.
    p[1] := 1; p[k] := n;
    for j := 2 to k - 1 do p[j] := d[p[j - 1]];
}
```

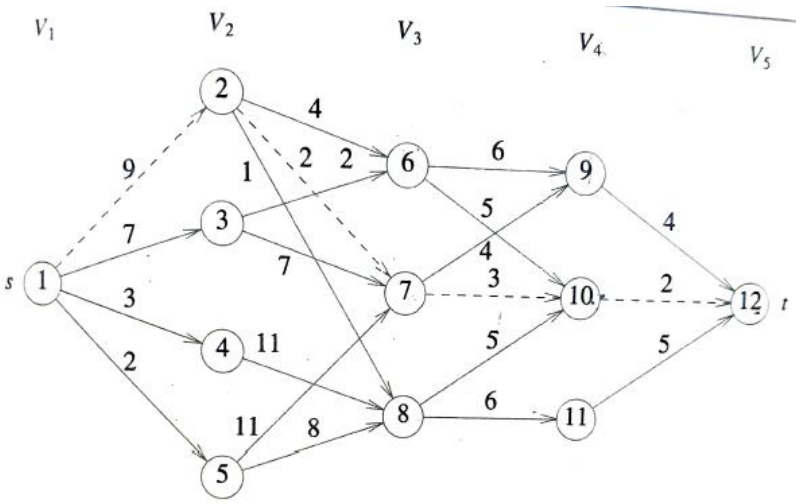
Time Complexity: If the graph is represented by **adjacency list**, then it is:

$$\Theta(|V| + |E|)$$

Solution Using Backward Approach:

The multistage graph problem can also be solved using the backward approach. Let $bp(i,j)$ be a minimum-cost path from vertex s to a vertex j in V_i. Let $bcost(i,j)$ be the cost of $bp(i,j)$. From the backward approach we obtain

$$bcost(i,j) = \min_{\substack{l \in V_{i-1} \\ \langle l,j\rangle \in E}} \{bcost(i-1,l) + c(l,j)\}$$



$$\begin{aligned}
bcost(3,6) &= \min\ \{bcost(2,2) + c(2,6), bcost(2,3) + c(3,6)\} \\
&= \min\ \{9+4, 7+2\} \\
&= 9 \\
bcost(3,7) &= 11 \\
bcost(3,8) &= 10 \\
bcost(4,9) &= 15
\end{aligned}$$

$$\begin{aligned}
bcost(4,10) &= 14 \\
bcost(4,11) &= 16 \\
bcost(5,12) &= 16
\end{aligned}$$

```
Algorithm BGraph(G, k, n, p)
// Same function as FGraph
{
    bcost[1] := 0.0;
    for j := 2 to n do
    { // Compute bcost[j].
        Let r be such that ⟨r, j⟩ is an edge of
        G and bcost[r] + c[r, j] is minimum;
        bcost[j] := bcost[r] + c[r, j];
        d[j] := r;
    }
    // Find a minimum-cost path.
    p[1] := 1; p[k] := n;
    for j := k − 1 to 2 do p[j] := d[p[j + 1]];
}
```