Computational Topology, Homework 2

[[**XXX TODO:** your name here]]

due: 17 February 2022

This homework assignment should be submitted as a single PDF file to D2L.

General homework expectations:

- Homework should be typeset using LaTex.
- Answers should be in complete sentences, and make sense without seeing the question.
- You will not plagiarize, nor will you share your written solutions with classmates. (But, discussing the questions is highly encouraged).
- List collaborators at the start of each question using the `collab` command.
- Put your answers where the `todo` command currently is (and remove the `todo`, but not the word `Answer`).
- If you are asked to come up with an algorithm, you are expected to give an algorithm that beats the brute force (and, if possible, of optimal time complexity). With your algorithm, please provide the following:
  - *What*: A prose explanation of the problem and the algorithm, including a description of the input/output.
  - *How*: Describe how the algorithm works, including giving psuedocode for it. Be sure to reference the pseudocode from within the prose explanation.
  - *How Fast*: Runtime, along with justification. (Or, in the extreme, a proof of termination).
  - *Why*: Justify why this algorithm works. At a minimum, I expect a statement of the loop invariant for each loop, or recursion invariant for each recursive function.

CSCI 535 Problem 2-1

Collaborators on this problem: *[[**XXX TODO:** list your collaborators here]]*

Function Space Let (X, d) be a metric space. That is, X is a set and $d\colon X \times X \to \mathbb{R}$ is a distance metric on X. A *metric ball* at $x \in X$ with radius $r \geq 0$ is the (open) set: $B_r(x) := \{x' \in X \mid d(x, x') < r\}$. We can use the set of all metric balls to generate a topology on X. When we do so, we call this a *metric* topology on X.

One of my favorite types of topological spaces are where the points represent functions. For example, let $(X, \mathcal{T}_X)$ be a topological space and let $(Y, \mathcal{T}_Y)$ be a metric space (corresponding to the distance function $d_Y\colon Y \times Y \to \mathbb{R}$). Let $C(X, Y)$ denote the set of all continuous functions from X to Y. We can topologize $C(X, Y)$ using the L_∞-metric; that is, we define a distance metric $\ell_\infty\colon C(X, Y) \times C(X, Y) \to \mathbb{R}$ by

$$\ell_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

(a) Prove that this is a metric.

Answer A function l_∞ : C(X,Y) x C(X,Y) $\to R$ is said to be metric on C(X,Y) if:

- $l_\infty(f, g) \geq 0$ for all f, g $\in C(X, Y)$: $l_\infty(f, g)$ is the supremum of the distance $d_Y(f(x), g(x))$ for all $x \in X$. And since the distance is never zero, the supremum which is the smallest upperbound of it has to be ≥ 0
- $l_\infty(f, g) = 0$ if $f = g$: If both f and g are the same point, the distance between the two points is zero.
- $l_\infty(f, g) = l_\infty(g, f)$ for all f, g $\in C(X, Y)$: The distance between two points in the topological space remains the same no matter which direction it is measured from i.e $f(x) \to g(x)$ or $g(x) \to f(x)$
- $l_\infty(f, g) \leq l_\infty(f, h) + l_\infty(h, g)$ for all f, g, h $\in C(X, Y)$: If a third point h is introduced, considering the triangle rule, $d_Y(f(x), g(x))$ will always be greater than $d_Y(f(x), h(x)) + d_Y(h(x), g(x))$ since C(X,Y) are continuous functions from X to Y.

(b) Suppose the topology on X is the indiscrete topology; that is, $\mathcal{T}_X = \{\emptyset, X\}$. Describe the topological space whose set is $C(X, \mathbb{R})$ and whose topology is generated by metric balls in ℓ_∞. (Note: when I use $\mathbb{R}$ without explicitly stating the topology, please assume that we are using the standard topology).

Answer Since $\mathcal{T}_X = \{\emptyset, X\}$ is an indiscrete(trivial) topology, then $C(X, \mathbb{R})$ is continuous for any topology in $\mathbb{R}$ because $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = \mathbb{R}$, both of which are always open in any topology on $\mathbb{R}$.

CSCI 535 Problem 2-2

Collaborators on this problem: *[[**XXX TODO:** list your collaborators here]]*

We can also define an L_p-norm in Euclidean space: $d_p \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$d_p(x, y) = \left(\sum_{i=1,2,\dots d} |x_i - y_i|^p \right)^{1/p} .$$

In fact, you should be quite familiar with this metric, as d_2 is the Euclidean distance. You might also know d_1 as the Manhattan distance. We define $d_\infty(x, y) := \lim_{p \to \infty} d_p(x, y) = \max_{i=1,2,\dots d} |x_i - y_i|$.

Let $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ be two topological spaces. We say that two $\mathcal{T}_1$ and $\mathcal{T}_2$ are *equivalent topologies* if for each open set $A \in \mathcal{T}_1$, there exists $A', A'' \in T_2$, where neither A' nor A'' are $\emptyset$ or X itself and $A' \subseteq A \subseteq A''$. (And, symmetrically for $B \in T_2$). If you have a basis for your topology, you just need to prove that this property holds on the basis elements.

Prove the following:

Theorem 1 (Euclidean Space)**.** *Let $d \in \mathbb{N}$. Let $(\mathbb{R}^d, \mathcal{T}_1)$ be the metric topology induced from the metric d_1, and let $(\mathbb{R}^d, \mathcal{T}_\infty)$ be the metric topology induced from the metric d_∞. Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent topologies.*

Proof. $\square$

CSCI 535 Problem 2-3

Collaborators on this problem: *[[**XXX TODO:** list your collaborators here]]*

Planar Graph Coloring Planar graph coloring (two credits). Recall that every planar graph has a vertex of degree at most five. We can use this fact to show that every planar graph has a vertex 6-coloring, that is, a coloring of each vertex with one of six colors such that any two adjacent vertices have different colors. Indeed, after removing a vertex with fewer than six neighbors we use induction to 6-color the remaining graph and when we put the vertex back we choose a color that differs from the colors of its neighbors. Refine the argument to prove that every planar graph has a vertex 5-coloring.

Answer HE-CT Part I, Question 7 (Planar Graph Coloring).

Answer [[**XXX TODO:** answer here]]

CSCI 535 Problem 2-4

Collaborators on this problem: *[[**XXX TODO:** list your collaborators here]]*

Two-Coloring 2-coloring (two credits). Let K be a triangulation of an orientable 2- manifold without boundary. Construct L by decomposing each edge into two and each triangle into six. To do this, we add a new vertex in the interior of each edge. Similarly, we add a new vertex in the interior of each triangle, connecting it to the six vertices in the boundary of the triangle. The resulting structure is the same as the barycentric subdivision of K, which we will define in Chapter III. (i) Show that the vertices of L can be 3-colored such that no two neighboring vertices receive the same color. (ii) Prove that the triangles of L can be 2-colored such that no two triangles sharing an edge receive the same color.

Answer HE-CT Part II, Question 2 (2-*Coloring*).

Answer [[**XXX TODO:** answer here]]

CSCI 535 Problem 2-5

Collaborators on this problem: *[[**XXX TODO:** list your collaborators here]]*

SoCG The Symposium for Computational Geometry (SoCG), pronounced "sausage", is the main conference in computational geometry and computational topology. Choose a paper published in SoCG (any year). If you were a reviewer of that paper, then you would expect to provide a brief (1 paragraph) summary of the paper, highlighting the main contributions of the paper. Write a maximum of one page explaining the main result of the paper that you chose. In future assignments, we will continue to work on how to write a review, so keep the opinions limited in this assignment.

If you need some inspiration on papers, here are some recommendations:

- Adamaszek, Adams, Gasparovic, Gommel, Purvine, Sazdanovic, Wang, Wang, and Ziegelmeier. Vietoris-Rips and Cech Complexes of Metric Gluings. SoCG 2018.
- Amezquita, Quigley, Ophelders, Munch, and Chitwood. Quantifying Barley Morphology using Euler Characteristic Curves. SoCG 2020.
- Chambers and Wang. Measuring Similarity Between Curves on 2-Manifolds via Homotopy Area. SoCG 2013.
- Driemel, Phillips, and Psarros. On the VC Dimension of Metric Balls under Fréchet and Hausdorff Distances. SoCG 2019
- Edelsbrunner and Osang. The Multi-cover Persistence of Euclidean Balls. SoCG 2018.
- Sheehy. The Persistent Homology of Distance Functions under Random Projection. SoCG 2014.

Answer [[**XXX TODO:** answer here]]