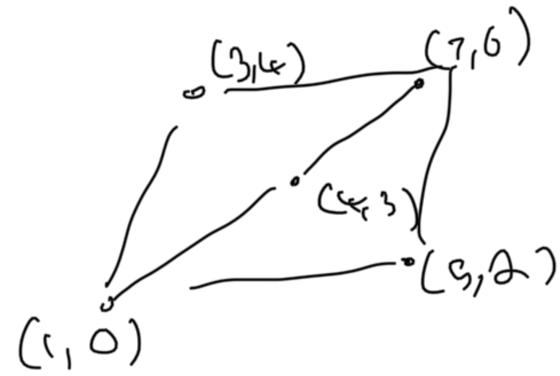
Homework 3

1) (a), (c), (e) center symmetric

2) (a) ϕ is a parallelogram over the $(\hat{x}_1, \hat{x}_2)$ plane with vertices $(3,4), (7,6), (5,2), (1,0)$



Proving the property:

$$N(x_1, x_2) = \left(\text{Relu}(4\varepsilon_1 + 5\varepsilon_2 + 10),\ \text{Relu}(-\varepsilon_1 + \varepsilon_2 - 1) \right)$$

Notice $4\varepsilon_1 + 5\varepsilon_2 + 10 \geq 1$ & $-3 \leq -\varepsilon_1 + \varepsilon_2 - 1 \leq 1$

$\Rightarrow \text{Relu}(4\varepsilon_1 + 5\varepsilon_2 + 10) = 4\varepsilon_1 + 5\varepsilon_2 + 10 \geq 1$ $\Rightarrow \text{Relu}(-\varepsilon_1 + \varepsilon_2 - 1) \leq 1$

$x_5 \geq 1$ & $x_6 \leq 1$ $\therefore \underline{x_5 \geq x_6}$

(b) For the affine case, we use the usual transform (sum coefficients, scalar multiply.)

$$\therefore \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} \varepsilon_1 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} \varepsilon_2$$

For Relu using the 1-D Relu transformer in the NN verif book.

$$\text{relu}^a(\langle c_1, c_2, c_3 \rangle) = \begin{cases} \langle c_1, c_2, c_3 \rangle & \text{if } l \geq 0 \\ \langle 0, 0, 0 \rangle & \text{if } u \leq 0 \\ \langle \lambda c_1, \lambda c_2, \lambda c_3, 0 \rangle + \langle \eta, 0, 0, \eta \rangle & \text{else} \end{cases}$$

where

- $\langle c_1, c_2, c_3 \rangle = c_1 + c_2\varepsilon_1 + c_3\varepsilon_2$
- l is the lower bound, u is the upper bound

• x is ...

• $\lambda = \frac{u}{u-l}$

• $\eta = \frac{u(1-\lambda)}{2}$

So, $x_5 = \text{Relu}(\underbrace{x_1 + 2x_2}) \rightarrow 10 + 4\varepsilon_1 + 5\varepsilon_2 \geq 1$

$\Rightarrow \text{relu}^a(\langle 10, 4, 5 \rangle) = \underline{\underline{\langle 10, 4, 5 \rangle}}$

$x_6 = \text{Relu}(\underbrace{x_2 - x_1}) \rightarrow -1 - \varepsilon_1 + \varepsilon_2 \in [-3, 1]$

$\Rightarrow \text{relu}^a(\langle -1, -1, 1 \rangle) = \langle -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, 0 \rangle + \langle \frac{3}{8}, 0, 0, \frac{3}{8} \rangle$

$= \underline{\underline{\langle \frac{1}{8}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{8} \rangle}}$

$\therefore$ After the abstraction, we get

$10 + 4\varepsilon_1 + 5\varepsilon_2 + 0\varepsilon_3$

$\frac{1}{8} - \frac{1}{4}\varepsilon_1 + \frac{1}{4}\varepsilon_2 + \frac{3}{8}\varepsilon_3$

Now, solve $\left\{ \begin{array}{l} \min \quad x - y \\ x = 10 + 4\varepsilon_1 + 5\varepsilon_2 + 0\varepsilon_3 \\ y = \frac{1}{4} - \frac{1}{4}\varepsilon_1 + \frac{1}{4}\varepsilon_2 + \frac{3}{4}\varepsilon_3 \\ -1 \leq \varepsilon_i \leq 1 \end{array} \right\}$

Here as well, we have $x \geq 1$ & $y \leq 1$ so yes we can prove the prt using the above transformer.

3) $\underline{t^{\#}}$:

$+^{\#}(\top, a) = \top$ for any $a \in \Pi$

$$+^{\#}(+, a) = \begin{cases} + & \text{if } a = +, 0 \\ \top & \text{if } a = -, \top \end{cases}$$

$$+^{\#}(-, a) = \begin{cases} - & \text{if } a = -, 0 \\ \top & \text{if } a = +, \top \end{cases}$$

$$+^{\#}(0, a) = \begin{cases} 0 & \text{if } a = 0 \\ - & \text{if } a = - \\ + & \text{if } a = + \\ \top & \text{if } a = \top \end{cases}$$

$\underline{\cdot^{\#}}$

Let $$-a := \begin{cases} - & \text{if } a = + \\ + & \text{if } a = - \\ 0 & \text{if } a = 0 \\ \top & \text{if } a = \top \end{cases}$$

$$\cdot^{\#}(\lambda, a) = \begin{cases} a & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -a & \text{if } \lambda < 0 \end{cases}$$

$\underline{\text{Relu}^{\#}}$

$$\text{Relu}^{\#}(a) = \begin{cases} a & \text{if } a = + \\ 0 & \text{if } a = 0, - \\ \top & \text{if } a = \top \end{cases}$$

note: we need $\text{Relu}^{\#}(\top) = \top$ because $x \geq 0$ in the only thing covering A is $\top$ ($+$ is $x > 0$ not $x \geq 0$)

(b) $x \geq 5 \rightsquigarrow +$

$\cdot^{\#}(3, +) \rightarrow +$

note: -1 is just $-$

$+^{\#}(+, -1) \to T$

$Relu^{\#}(T) \to T$

$+^{\#}(T, 1) \to T$

in A.
Similarly 1 is +.
($x < 0 \Rightarrow -$
$x > 0 \Rightarrow +$
$x = 0 \Rightarrow 0$).

cannot prove

$\forall x \in \mathbb{R}.\ x \geq 5 \Rightarrow N(x) > 0$ using A

(c) $x \geq 5 \to [5, \infty)$

$3^{\#}[5, \infty) = [15, \infty)$

$[15, \infty) - [1, 1] = [14, \infty)$

$Relu^{\#}([14, \infty)) = [Relu(14), \infty)$

(we define $Relu^{\#}([a, \infty)) = [Relu(a), \infty)$
$Relu^{\#}((-\infty, a]) = [0, Relu(a)]$)

$[14, \infty) + 1 = [15, \infty)$

$\therefore$ yes, we can prove

$\forall x \in \mathbb{R}.\ x \geq 5 \Rightarrow N(x) > 0$

Let z1,z2 be the original zonotopes and z the union

For the zonotope union, I did the following (see code):

-Created variables for the coefficients of z

-Created variables for \epsilon_i_vs for each vertex v (we want different \epsilon_is for different vertices).

- For each vertex v, added constraint that says -1<=\epsilon_i_v <=1 for each i and that

(C0 + c1\epsilon_1_v + c2\epsilon_2_v + c3\epsilon_3_v = x_coordinate_of_v

and similarly for (d0,d1,d2,d3) and y_coordinate_of_v)

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checking
[d1 = 1/2,
 d3 = 8,
 e23 = -24/41,
 d2 = -7,
 e24 = -20/41,
 e11 = 0,
 c3 = 12,
 e12 = 0,
 e34 = -111/164,
 e25 = 8/41,
 e32 = -1/2,
 c1 = -1/2,
 e22 = 0,
 e31 = 0,
 e13 = 0,
 e21 = 0,
 e14 = 0,
 e15 = 0,
 e33 = -21/41,
 e35 = -27/82,
 c2 = -1/4,
 d0 = 2,
 c0 = 6]
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