tendency to grow from zero population up to some carrying capacity K.

Originally a much stricter interpretation was proposed, and the model was argued to be a universal law of growth (Pearl 1927). The logistic equation was tested in laboratory experiments in which colonies of bacteria, yeast, or other simple organisms were grown in conditions of constant climate, food supply, and absence of predators. For a good review of this literature, see Krebs (1972, pp. 190–200). These experiments often yielded sigmoid growth curves, in some cases with an impressive match to the logistic predictions.

On the other hand, the agreement was much worse for fruit flies, flour beetles, and other organisms that have complex life cycles, involving eggs, larvae, pupae, and adults. In these organisms, the predicted asymptotic approach to a steady carrying capacity was never observed—instead the populations exhibited large, persistent fluctuations after an initial period of logistic growth. See Krebs (1972) for a discussion of the possible causes of these fluctuations, including age structure and time-delayed effects of overcrowding in the population.

For further reading on population biology, see Pielou (1969) or May (1981). Edelstein–Keshet (1988) and Murray (1989) are excellent textbooks on mathematical biology in general.

2.4 Linear Stability Analysis

So far we have relied on graphical methods to determine the stability of fixed points. Frequently one would like to have a more quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by *linearizing* about a fixed point, as we now explain.

Let x^* be a fixed point, and let $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for η. Differentiation yields

$$\dot{\eta} = \tfrac{d}{dt}(x - x^*) = \dot{x},$$

since x^* is constant. Thus $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$. Now using Taylor's expansion we obtain

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2),$$

where $O(\eta^2)$ denotes quadratically small terms in η. Finally, note that $f(x^*) = 0$ since x^* is a fixed point. Hence

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2).$$

Now if $f'(x^*) \neq 0$, the $O(\eta^2)$ terms are negligible and we may write the approximation

$$\dot{\eta} \approx \eta f'(x^*).$$

This is a linear equation in η, and is called the ***linearization about*** x^*. It shows that *the perturbation* $\eta(t)$ *grows exponentially if* $f'(x^*) > 0$ *and decays if* $f'(x^*) < 0$. If $f'(x^*) = 0$, the $O(\eta^2)$ terms are not negligible and a nonlinear analysis is needed to determine stability, as discussed in Example 2.4.3 below.

The upshot is that the slope $f'(x^*)$ at the fixed point determines its stability. If you look back at the earlier examples, you'll see that the slope was always negative at a stable fixed point. The importance of the *sign* of $f'(x^*)$ was clear from our graphical approach; the new feature is that now we have a measure of *how* stable a fixed point is—that's determined by the *magnitude* of $f'(x^*)$. This magnitude plays the role of an exponential growth or decay rate. Its reciprocal $1/|f'(x^*)|$ is a ***characteristic time scale***; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

EXAMPLE 2.4.1:

Using linear stability analysis, determine the stability of the fixed points for $\dot{x} = \sin x$.

Solution: The fixed points occur where $f(x) = \sin x = 0$. Thus $x^* = k\pi$, where k is an integer. Then

$$f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd}. \end{cases}$$

Hence x^* is unstable if k is even and stable if k is odd. This agrees with the results shown in Figure 2.1.1. ■

EXAMPLE 2.4.2:

Classify the fixed points of the logistic equation, using linear stability analysis, and find the characteristic time scale in each case.

Solution: Here $f(N) = rN\left(1 - \frac{N}{K}\right)$, with fixed points $N^* = 0$ and $N^* = K$. Then $f'(N) = r - \frac{2rN}{K}$ and so $f'(0) = r$ and $f'(K) = -r$. Hence $N^* = 0$ is unstable and $N^* = K$ is stable, as found earlier by graphical arguments. In either case, the characteristic time scale is $1/|f'(N^*)| = 1/r$. ■

EXAMPLE 2.4.3:

What can be said about the stability of a fixed point when $f'(x^*) = 0$?

Solution: Nothing can be said in general. The stability is best determined on a case-by-case basis, using graphical methods. Consider the following examples:

$$\text{(a) } \dot{x} = -x^3 \qquad \text{(b) } \dot{x} = x^3 \qquad \text{(c) } \dot{x} = x^2 \qquad \text{(d) } \dot{x} = 0$$

Each of these systems has a fixed point $x^* = 0$ with $f'(x^*) = 0$. However the stability is different in each case. Figure 2.4.1 shows that (a) is stable and (b) is unstable. Case (c) is a hybrid case we'll call ***half-stable***, since the fixed point is attracting from the left and repelling from the right. We therefore indicate this type of fixed point by a half-filled circle. Case (d) is a whole line of fixed points; perturbations neither grow nor decay.

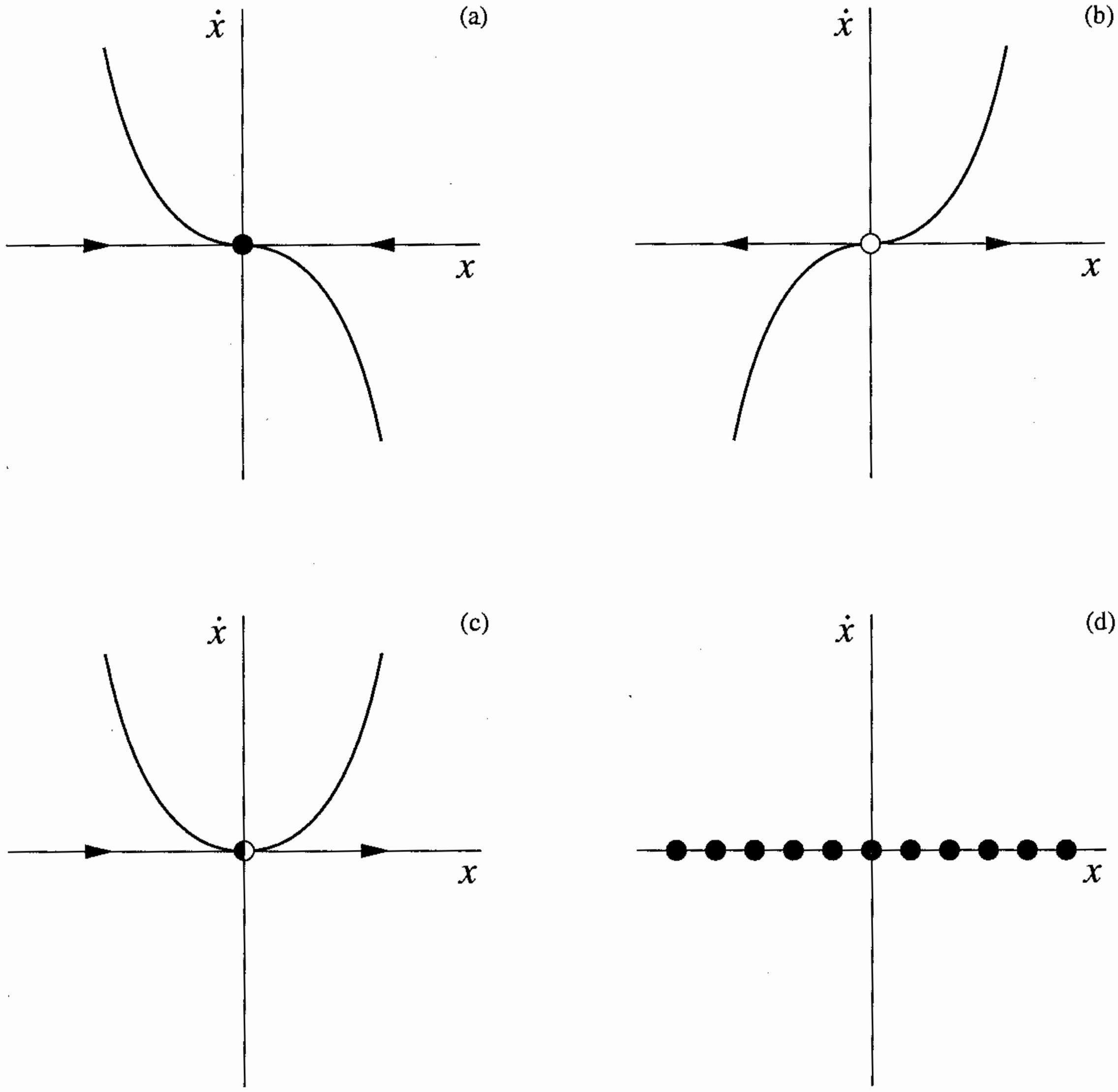

Figure 2.4.1

These examples may seem artificial, but we will see that they arise naturally in the context of *bifurcations*—more about that later. ■

2.5 Existence and Uniqueness

Our treatment of vector fields has been very informal. In particular, we have taken a cavalier attitude toward questions of existence and uniqueness of solutions to the system $\dot{x} = f(x)$. That's in keeping with the "applied" spirit of this book. Nevertheless, we should be aware of what can go wrong in pathological cases.

EXAMPLE 2.5.1:

Show that the solution to $\dot{x} = x^{1/3}$ starting from $x_0 = 0$ is *not* unique.

Solution: The point $x = 0$ is a fixed point, so one obvious solution is $x(t) = 0$ for all t. The surprising fact is that there is *another* solution. To find it we separate variables and integrate:

$$\int x^{-1/3} dx = \int dt$$

so $\frac{3}{2} x^{2/3} = t + C$. Imposing the initial condition $x(0) = 0$ yields $C = 0$. Hence $x(t) = \left(\frac{2}{3} t\right)^{3/2}$ is also a solution! ■

When uniqueness fails, our geometric approach collapses because the phase point doesn't know how to move; if a phase point were started at the origin, would it stay there or would it move according to $x(t) = \left(\frac{2}{3} t\right)^{3/2}$? (Or as my friends in elementary school used to say when discussing the problem of the irresistible force and the immovable object, perhaps the phase point would explode!)

Actually, the situation in Example 2.5.1 is even worse than we've let on—there are *infinitely* many solutions starting from the same initial condition (Exercise 2.5.4).

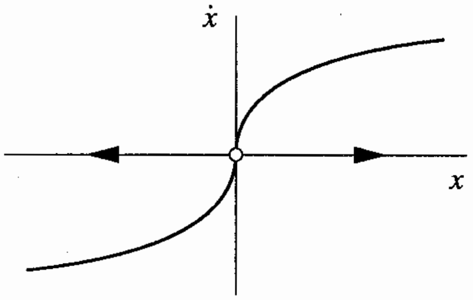

Figure 2.5.1

What's the source of the non-uniqueness? A hint comes from looking at the vector field (Figure 2.5.1). We see that the fixed point $x^* = 0$ is *very* unstable—the slope $f'(0)$ is infinite.

Chastened by this example, we state a theorem that provides sufficient conditions for existence and uniqueness of solutions to $\dot{x} = f(x)$.

Existence and Uniqueness Theorem: Consider the initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_0.$$

Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis, and suppose that x_0 is a point in R. Then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

For proofs of the existence and uniqueness theorem, see Borrelli and Coleman (1987), Lin and Segel (1988), or virtually any text on ordinary differential equations.

This theorem says that *if $f(x)$ is smooth enough,* then solutions exist and are unique. Even so, there's no guarantee that solutions exist forever, as shown by the next example.

EXAMPLE 2.5.2:

Discuss the existence and uniqueness of solutions to the initial value problem $\dot{x} = 1 + x^2$, $x(0) = x_0$. Do solutions exist for all time?

Solution: Here $f(x) = 1 + x^2$. This function is continuous and has a continuous derivative for all x. Hence the theorem tells us that solutions exist and are unique for any initial condition x_0. But *the theorem does not say that the solutions exist for all time;* they are only guaranteed to exist in a (possibly very short) time interval around $t = 0$.

For example, consider the case where $x(0) = 0$. Then the problem can be solved analytically by separation of variables:

$$\int \frac{dx}{1+x^2} = \int dt,$$

which yields

$$\tan^{-1} x = t + C$$

The initial condition $x(0) = 0$ implies $C = 0$. Hence $x(t) = \tan t$ is the solution. But notice that this solution exists only for $-\pi/2 < t < \pi/2$, because $x(t) \to \pm\infty$ as $t \to \pm\pi/2$. Outside of that time interval, there is no solution to the initial value problem for $x_0 = 0$. ■

The amazing thing about Example 2.5.2 is that the system has solutions that reach infinity *in finite time*. This phenomenon is called ***blow-up***. As the name suggests, it is of physical relevance in models of combustion and other runaway processes.

There are various ways to extend the existence and uniqueness theorem. One can allow f to depend on time t, or on several variables $x_1, \ldots, x_n$. One of the most useful generalizations will be discussed later in Section 6.2.

From now on, we will not worry about issues of existence and uniqueness—our vector fields will typically be smooth enough to avoid trouble. If we happen to come across a more dangerous example, we'll deal with it then.

2.6 Impossibility of Oscillations

Fixed points dominate the dynamics of first-order systems. In all our examples so far, all trajectories either approached a fixed point, or diverged to $\pm\infty$. In fact, those are the *only* things that can happen for a vector field on the real line. The reason is that trajectories are forced to increase or decrease monotonically, or remain constant (Figure 2.6.1). To put it more geometrically, the phase point never reverses direction.

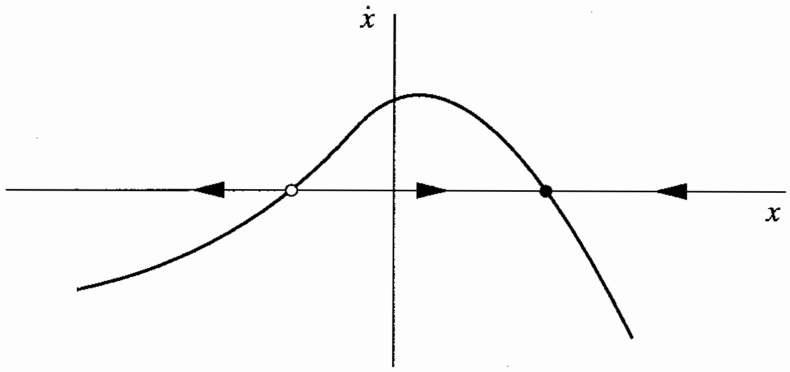

Figure 2.6.1

Thus, if a fixed point is regarded as an equilibrium solution, the approach to equilibrium is always *monotonic*—overshoot and damped oscillations can never occur in a first-order system. For the same reason, undamped oscillations are impossible. Hence *there are no periodic solutions to* $\dot{x} = f(x)$.

These general results are fundamentally topological in origin. They reflect the fact that $\dot{x} = f(x)$ corresponds to flow on a *line*. If you flow monotonically on a line, you'll never come back to your starting place—that's why periodic solutions are impossible. (Of course, if we were dealing with a *circle* rather than a line, we *could* eventually return to our starting place. Thus vector fields on the circle can exhibit periodic solutions, as we discuss in Chapter 4.)

Mechanical Analog: Overdamped Systems

It may seem surprising that solutions to $\dot{x} = f(x)$ can't oscillate. But this result becomes obvious if we think in terms of a mechanical analog. We regard $\dot{x} = f(x)$ as a limiting case of Newton's law, in the limit where the "inertia term" $m\ddot{x}$ is negligible.

For example, suppose a mass m is attached to a nonlinear spring whose restoring force is $F(x)$, where x is the displacement from the origin. Furthermore, suppose that the mass is immersed in a vat of very viscous fluid, like honey or motor oil (Figure 2.6.2), so that it is subject to a damping force $b\dot{x}$. Then Newton's law is

$$m\ddot{x} + b\dot{x} = F(x).$$

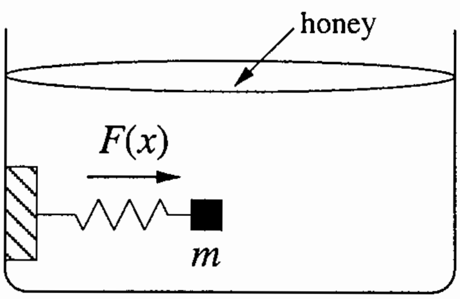

Figure 2.6.2

If the viscous damping is strong compared to the inertia term ($b\dot{x} >> m\ddot{x}$), the system should behave like $b\dot{x} = F(x)$, or equivalently $\dot{x} = f(x)$, where $f(x) = b^{-1}F(x)$. In this ***overdamped*** limit, the behavior of the mechanical system is clear. The mass prefers to sit at a stable equilibrium, where $f(x) = 0$ and $f'(x) < 0$. If displaced a bit, the mass is slowly dragged back to equilibrium by the restoring force. No overshoot can occur, because the damping is enormous. And undamped oscillations are out of the question! These conclusions agree with those obtained earlier by geometric reasoning.

Actually, we should confess that this argument contains a slight swindle. The neglect of the inertia term $m\ddot{x}$ is valid, but only after a rapid initial transient during which the inertia and damping terms are of comparable size. An honest discussion of this point requires more machinery than we have available. We'll return to this matter in Section 3.5.

2.7 Potentials

There's another way to visualize the dynamics of the first-order system $\dot{x} = f(x)$, based on the physical idea of potential energy. We picture a particle sliding down the walls of a potential well, where the ***potential*** $V(x)$ is defined by

$$f(x) = -\frac{dV}{dx}.$$

As before, you should imagine that the particle is heavily damped—its inertia is completely negligible compared to the damping force and the force due to the potential. For example, suppose that the particle has to slog through a thick layer of goo that covers the walls of the potential (Figure 2.7.1).

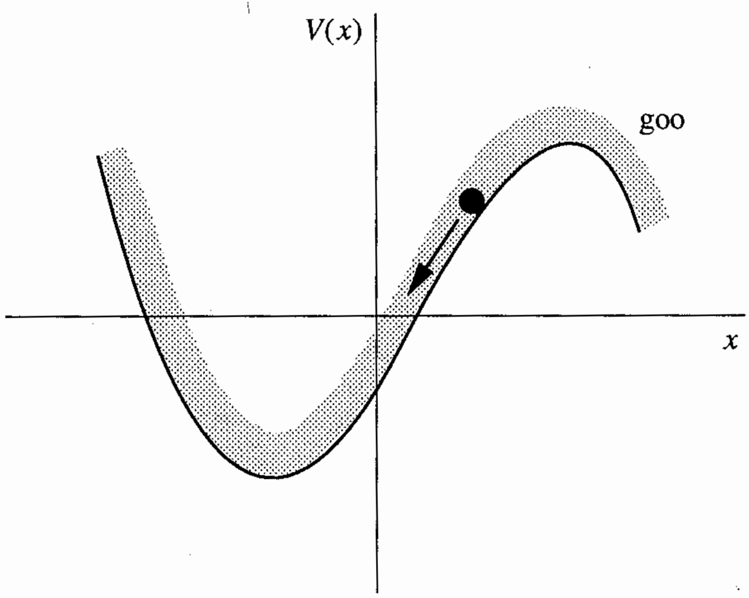

Figure 2.7.1

The negative sign in the definition of V follows the standard convention in physics; it implies that the particle always moves "downhill" as the motion proceeds. To see this, we think of x as a function of t, and then calculate the time-derivative of $V(x(t))$. Using the chain rule, we obtain

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}.$$

Now for a first-order system,

$$\frac{dx}{dt} = -\frac{dV}{dx},$$

since $\dot{x} = f(x) = -dV/dx$, by the definition of the potential. Hence,

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0.$$

Thus $V(t)$ *decreases along trajectories,* and so the particle always moves toward lower potential. Of course, if the particle happens to be at an ***equilibrium*** point where $dV/dx = 0$, then V remains constant. This is to be expected, since $dV/dx = 0$ implies $\dot{x} = 0$; equilibria occur at the fixed points of the vector field. Note that local minima of $V(x)$ correspond to *stable* fixed points, as we'd expect intuitively, and local maxima correspond to unstable fixed points.

EXAMPLE 2.7.1:

Graph the potential for the system $\dot{x} = -x$, and identify all the equilibrium points.

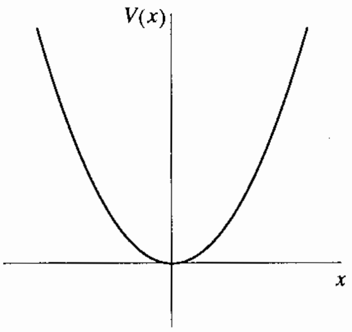

Figure 2.7.2

Solution: We need to find $V(x)$ such that $-dV/dx = -x$. The general solution is $V(x) = \frac{1}{2}x^2 + C$, where C is an arbitrary constant. (It always happens that the potential is only defined up to an additive constant. For convenience, we usually choose $C = 0$.) The graph of $V(x)$ is shown in Figure 2.7.2. The only equilibrium point occurs at $x = 0$, and it's stable. ■

EXAMPLE 2.7.2:

Graph the potential for the system $\dot{x} = x - x^3$, and identify all equilibrium points.

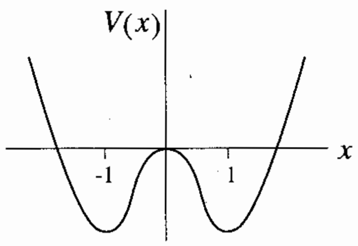

Figure 2.7.3

Solution: Solving $-dV/dx = x - x^3$ yields $V = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$. Once again we set $C = 0$. Figure 2.7.3 shows the graph of V. The local minima at $x = \pm 1$ correspond to stable equilibria, and the local maximum at $x = 0$ corresponds to an unstable equilibrium. The potential shown in Figure 2.7.3 is often called a ***double-well potential***, and the system is said to be ***bistable***, since it has two stable equilibria. ■