Integrated Science 2/3: **Electronics** **S. Adam & J. Gruber**
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Lecture Notes: Charging a capacitor

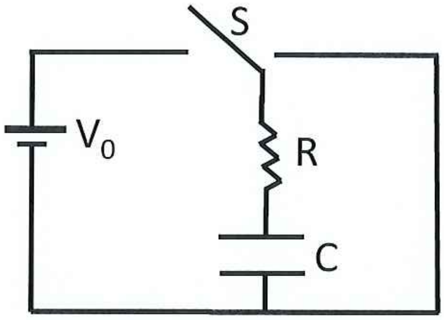


Figure 1: RC Circuit

Consider the circuit shown in Fig. 1. In these notes, we will examine this problem first using the traditional analytic approach, and then using the Strogatz non-linear dynamics graphical approach.

Part 1: Setting up the ODE

When the switch is in the left position, using Kirchoff's loop law, we know that at any moment in time, the voltage drop across the resistor and capacitor must equal the potential of the battery. Or mathematically, $V_R + V_C = V_0$. We also know that $V_R = IR$, $Q = CV_C$, where R is the total resistance in the circuit, and C is the total capacitance of the circuit. Here $Q(t)$ is the charge on the capacitor and

$$I(t) = \frac{d}{dt}Q(t),$$

is the current in the circuit. Simple algebra gives the following differential equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V_0. \tag{1}$$

We now want to cast the equation in a dimensionless form. To do this, we define a state variable X, a dimensionless time $\tau = t/RC$, and a boundary condition $X_f = CV_0$. Let us define our initial and final boundary conditions in terms of the new variables (since this will be useful later). When $t = 0$, we have $Q(t) = 0$. This initial condition corresponds to $\tau = 0$, and $X = 0$. At final time $t \to \infty$, we have $Q = cV_0$, or in the new variables, when $\tau \to \infty$, $X \to X_f$. Finally, for a given time t, we have $\tau = t/RC$ and $dt = RCd\tau$. Plugging this into the equation above, we get

$$\frac{R}{RC}\frac{dX}{d\tau} + \frac{X}{C} = \frac{X_f}{C},$$

and finally multiplying both sides by C, and bringing X to the rhs we get the form we want

$$\boxed{\frac{dX}{d\tau} = X_f - X.}$$

Part 2: Solving the ODE

Rearranging our ODE again, we can get it in the following form (where I have used primes to denote the integration variables

$$\begin{aligned} \frac{dX'}{X_f - X'} &= d\tau', \\ \int \frac{dX'}{X_f - X'} &= \int d\tau'. \end{aligned}$$

Determining the limits of the integral is a bit tricky. In the previous section, we noted that when $\tau = 0$, $X = 0$, and for given time τ, we want $X(\tau)$. Using this information, we can write

$$\int_0^{X(\tau)} \frac{dX'}{X_f - X'} = \int_0^{\tau} d\tau' = \tau, \tag{2}$$

where the integral over $d\tau'$ is trivial. The integral over dX' needs a bit more massaging, we can introduce $y = X_f - X'$, and $dy = -dX'$. And when $X' = 0, y = X_f$. When $X' = X(\tau)$, $y = X_f - X$. Putting this together, we have

$$\int_0^{X} \frac{dX'}{X_f - X'} = \int_{X_f}^{X_f - X} \frac{-dy}{y} = -\ln\left(\frac{X_f - X}{X_f}\right) = \tau. \tag{3}$$

Rearranging terms, this gives us

$$\boxed{X(\tau) = X_f\left[1 - e^{-\tau}\right].}$$

And now putting back the dimensions (i.e. using $\tau = t/RC$, $X_f = CV_0$, and $Q(t) = X$, we get

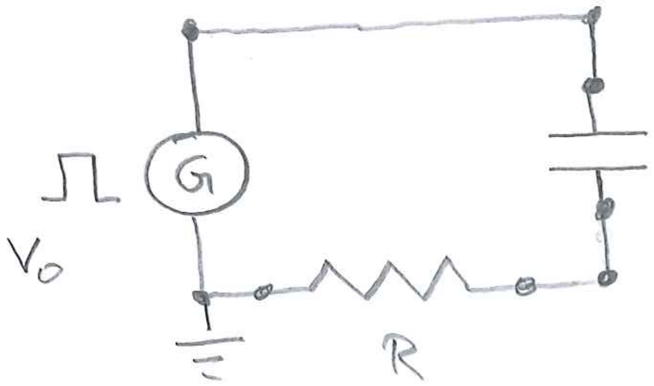
$$\boxed{Q(t) = CV_0\left[1 - \exp\left(\frac{-t}{RC}\right)\right]}.$$

One final question we can ask is at what time will the capacitor be half charged or, find $t_{1/2}$ such that $Q(t_{1/2}) = CV_0/2$. Plugging this in to the equaiton, we get

$$\begin{aligned} Q(t_{1/2}) = \frac{CV_0}{2} &= CV_0\left[1 - \exp\left(\frac{-t_{1/2}}{RC}\right)\right]. \\ 1 &= 2 - 2\exp\left(\frac{-t_{1/2}}{RC}\right), \\ \exp\left(\frac{-t_{1/2}}{RC}\right) &= \frac{1}{2}, \\ \ln\left(\exp\left(\frac{-t_{1/2}}{RC}\right)\right) &= \ln\frac{1}{2}, \\ \frac{t_{1/2}}{RC} &= \ln 2, \\ & \boxed{t_{1/2} = (\ln 2)RC}. \end{aligned}$$

Part 3: Geometric Analysis (Flow on a line)

$$I = \frac{d}{dt} Q$$



$$u_C = \frac{Q}{C}$$

$$V_0 = u_R + u_C$$

$$V_0 = R \cdot \dot{Q} + \frac{Q}{C}$$

$$u_R = R \cdot I$$
$$= R \cdot \dot{Q}$$

$\rightsquigarrow$ ODE:

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{R \cdot C}$$

Not trivial to solve
↳ let's think first!

Build it, measure!

V_0
u_C
u_R

$R = 1\Omega$

$C = 100\,\mu F$

$V_0 = 10V$

"Dynamical system"

1st order ODE system

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2 \ldots x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) \end{aligned}\right\}$$

simplest case: 1D system, 1st order

$$\dot{x} = f(x) \quad , \; x = x(t)$$

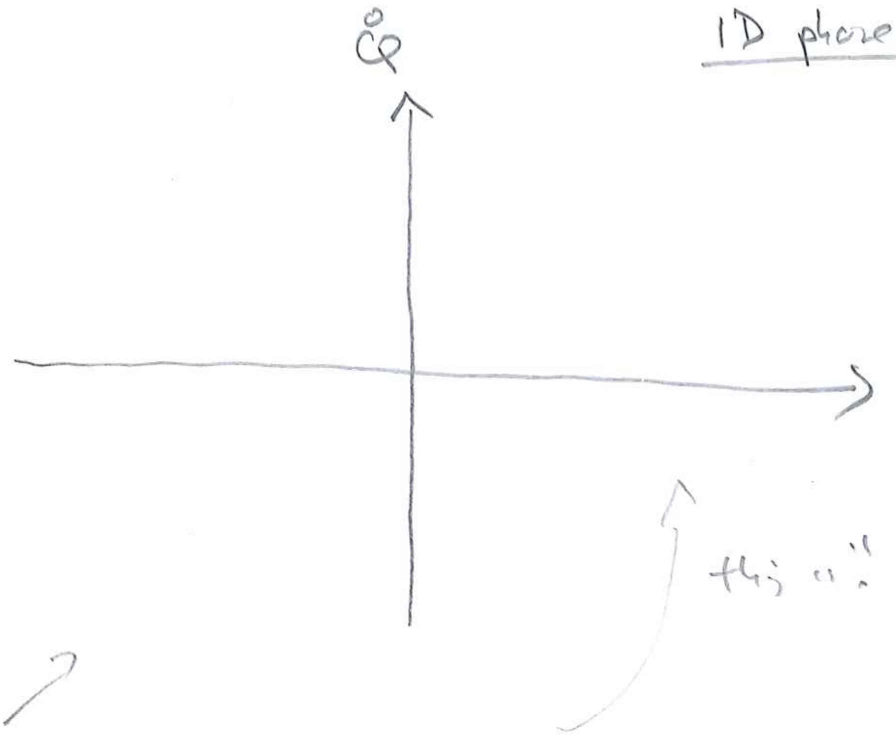
⇒ ~~"Interpret ODE's as vector field"~~

e.g.: $\dot{Q} = f(Q) = -\frac{Q}{R \cdot C} + \frac{U_0}{R}$ ← straight line

(2) $\dot{\phi} = mQ + b$

Idea of "phase fluid" moves on line

1D phase space

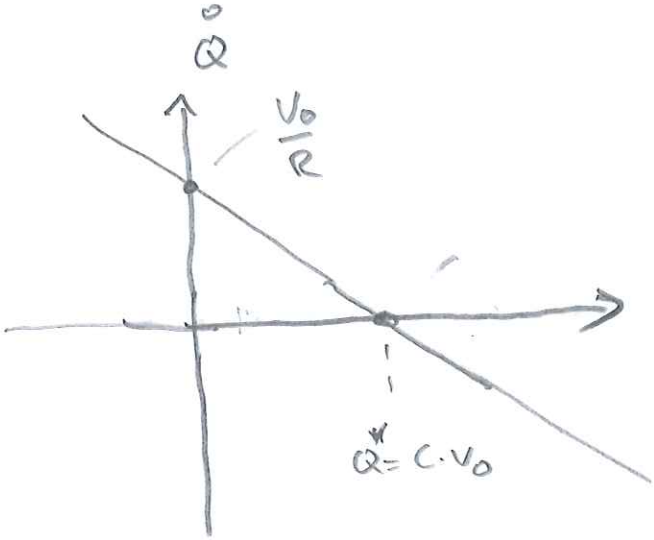


this is !!

$\dot{Q} = 0$

this is not phase space

(1)



1D phase space

Straight line: $y = mx + b$

$$\dot{Q} = \underbrace{-\frac{1}{CR}}_{m} Q + \underbrace{\frac{V_0}{R}}_{b}$$

$Q = 0 \leadsto \dot{Q} = \frac{V_0}{R}$

$\dot{Q} = 0 \iff -\frac{1}{CR} Q + \frac{V_0}{R} = 0$

$\iff Q - C V_0 = 0$

$\iff Q = C \cdot V_0$

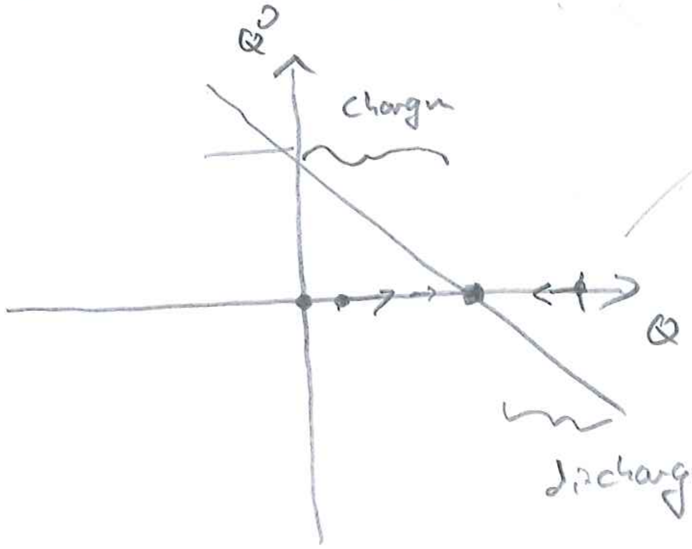
meaning of this point?

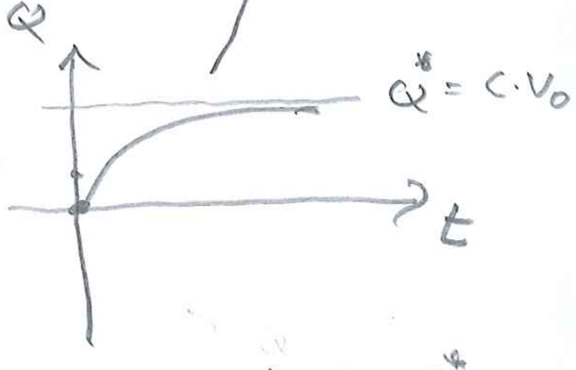
$I = \frac{V_0}{R}$ initial current

$Q = C \cdot V_0$ ~ recall $C = \frac{Q}{V_0}$

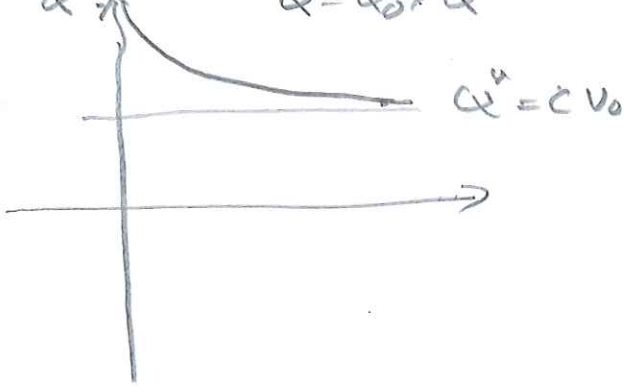
Idea of "phase fluid" – the state of the system

e.g. start @ $Q = Q_0 < Q^*$





$Q = Q_0 > Q^*$



$\leadsto$ Q^* is a "<u>fixed point</u>" or "<u>attractor</u>"

3) But is it exponential

How to know?

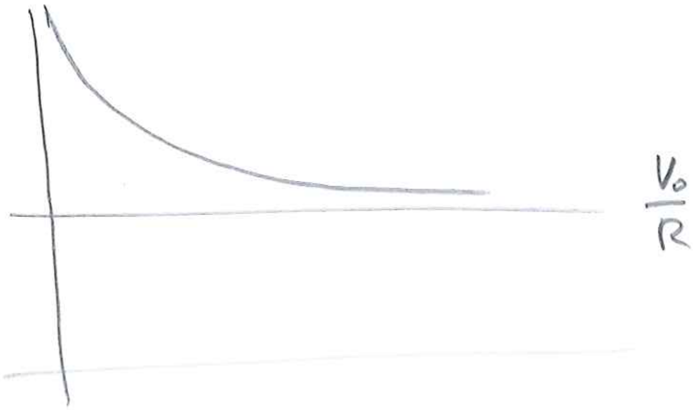
i) Integrating Factor ~ no maths!

ii) Ansatz

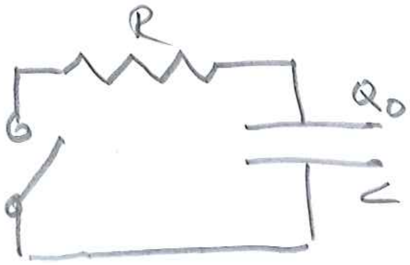
iii) Think:

Does this shape depend on V?

↪ Potential!



Probably not! ~ how about $V_0 = 0$



← Do you remember this?

$$\leadsto \quad \dot{Q} = -\frac{1}{CR} Q + \cancel{\frac{V_0}{R}}$$

$$\leadsto \quad \dot{Q} = -\frac{1}{CR} Q$$

↪ this is an exponential function (you may remember)

If not:

$$\frac{dQ}{dt} = -\frac{1}{CR} Q$$

$$\leadsto \quad \frac{dQ}{Q} = -\frac{1}{CR} dt$$

③

$$\ln Q = -\frac{1}{CR}\cdot t + K$$

$$\frac{d}{dQ}\ln Q = \frac{1}{Q}$$

$$\frac{d}{dt}\left[\frac{1}{CR}t + K\right] = -\frac{1}{CR}$$

$$\rightsquigarrow e^{\ln Q} = e^{-\frac{1}{CR}t + K}$$

$$Q(t) = e^{K}\cdot e^{-\frac{1}{CR}t} \quad (*)$$

Boundary condition:

$$Q(0) = Q_0 \overset{*}{\rightsquigarrow} Q(0) = e^{K} e^{-\frac{1}{CR}0}$$

$$Q(0) = e^{K}$$

$$\rightsquigarrow K = \ln Q(0)$$

$\Rightarrow$ Solution:

$$\boxed{Q(t) = Q(0)\, e^{-\frac{1}{CR}t}}$$

think about what that means?!?

✓ yap!

Now:

i) Integrate ✗

ii) Ansatz

iii) Think ✓

iv) Trick

$\rightarrow$ Stop! ?

What trick

$$V_0 = R \cdot \dot{Q} + \frac{Q}{C}$$

„coordinate transformation"

↪ describe in terms of I

$$\frac{d}{dt} V_0 = R \frac{d}{dt} \dot{Q} + \frac{d}{dt} \frac{Q}{C}$$

$$\frac{d}{dt} Q = \dot{Q} = I$$

0

$$R \cdot \dot{I} + \frac{I}{C} = 0$$

$$\rightsquigarrow \dot{I} = \frac{-1}{RC} \cdot I$$

↝ As before!

$$\boxed{I(t) = I_0 \cdot e^{-\frac{1}{RC} t}}$$

where $I_0 = \frac{V_0}{R}$!

(homework!)