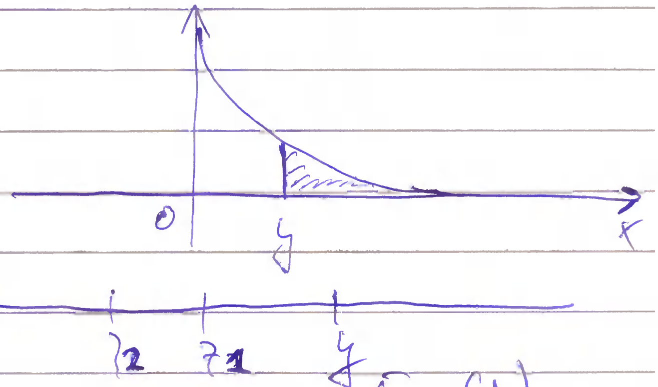
Консултация за DU

2011г. юли 1 зад.

$\xi \in Exp(\lambda)$, $\lambda > 0$

$f(x) = \frac{1}{\lambda} e^{-1/\lambda x}$, $x \geq 0$

$\rightarrow EX = \lambda$



a) $\eta = \min(\xi_1, \xi_2)$, където $\xi_1 \perp\!\!\!\perp \xi_2$ и $\xi_1, \xi_2 \in Exp(\lambda)$

не е диф. ф-я

$F_\eta(y)$ – ф-я на разпр. $F_\eta(y) = P(\eta \leq y)$

↳ плътността е $F'_\eta(y)$

$P(\eta \leq y) = P(\min(\xi_1, \xi_2) \leq y) =$

$= 1 - P(\min(\xi_1, \xi_2) > y) = 1 - P(\xi_1 > y, \xi_2 > y) =$

$= 1 - P(\xi_1 > y) \cdot P(\xi_2 > y) = 1 - \underbrace{[P(\xi > y)]}_{\text{дясна опашка на р-ето}}^2 = 1 - [\bar{F}_\xi(y)]^2$

$\bar{F}_\xi(x)$ – дясна опашка
$1 - F_\xi(x) = P(\xi > x)$

$P(\xi > y) = \int_y^\infty \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} dx =$

$= -\int_y^\infty d e^{-\frac{1}{\lambda}x} dx = -e^{-\frac{1}{\lambda}x} \Big|_{x=y}^{\infty} = e^{-\frac{1}{\lambda}y} - \underbrace{\lim e^{-\frac{1}{\lambda}x}}_{0} = e^{-\frac{1}{\lambda}y}$

Отг.: $\underline{1 - e^{-\frac{2}{\lambda}y}}$ $\Rightarrow \eta \in Exp\left(\frac{\lambda}{2}\right)$

б) $\xi_1, \xi_2, \ldots, \xi_n$ -> нез. набл. над $\xi \in Exp(\lambda)$

(Извадка: съвкупност от нез. еднакво разпр. сл. вел. с разпр. като характеристиката на изследваната на попул.)

Д(оказателство), че $\bar{\xi}_n = \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n}$ е еф. оценка за пар. λ

$t(\xi_1, \xi_2, \ldots, \xi_n)$ => статистика

<u>Деф.</u> Статистика, която е иззета за стойности на неизв. параметър, се нарича оценка (точкова оценка)

Св-ва на оценките:

- неизместеност (unbiased): $Et(\xi_1, \ldots, \xi_n) = \lambda$ ($\overset{\text{непр. ф-я}}{\tau}(\theta)$, където θ е неизв. парам.)

- състоятелност (consistency): $P(|t(\xi_1, \ldots, \xi_n) - \tau(\theta)| > \varepsilon) \xrightarrow[n \to \infty]{} 0 \quad \forall \varepsilon > 0$

- ефективна: неизместена оценка с минимална дисперсия.

Метод на макс. правдоподобие:

$L(\xi_1, \xi_2, \ldots, \xi_n | \underset{(\theta)}{\lambda})$ -> ф-я на правдоподобие

$$= f_\xi(\xi_1 | \lambda) \cdot f_\xi(\xi_2 | \lambda) \cdot f \ldots \ldots f_\xi(\xi_n | \lambda)$$

Ако $\frac{\partial \ln(L(\xi_1, \ldots, \xi_n | \lambda))}{\partial \lambda} = G(n, \lambda)(t(\xi_1, \ldots, \xi_n) - \lambda)$

=> $t(\xi_1, \ldots, \xi_n)$ е с минимална дисперсия.

$$L(z_1,\dots,z_n|\lambda)=\prod_{i=1}^{n} f_z(z_i|\lambda)=\prod_{i=1}^{n}\frac{1}{\lambda}e^{-\frac{1}{\lambda}z_i}=$$

$$=\frac{1}{\lambda}e^{-\frac{1}{\lambda}z_1}\cdot\frac{1}{\lambda}e^{-\frac{1}{\lambda}z_2}\dots\frac{1}{\lambda}e^{-\frac{1}{\lambda}z_n}=$$

$$=\left(\frac{1}{\lambda}\right)^n e^{-\frac{1}{\lambda}(z_1+\dots+z_n)}=\left(\frac{1}{\lambda}\right)^n e^{-\frac{1}{\lambda}\sum_{i=1}^{n}z_i}$$

$$\ln L(z_1,\dots,z_n|\lambda)=n\ln\frac{1}{\lambda}-\frac{1}{\lambda}\sum_{i=1}^{n}z_i=-n\ln\lambda-\frac{1}{\lambda}\sum_{i=1}^{n}z_i$$

(*) $$\frac{\partial\ln(L(z_1,\dots,z_n|\lambda))}{\partial\lambda}=-\frac{n}{\lambda}+\frac{1}{\lambda^2}\cdot\sum_{i=1}^{n}z_i=0\quad/\cdot\lambda^2$$

$$\rightarrow -n\lambda+\sum_{i=1}^{n}z_i=0\quad\Rightarrow\hat{\lambda}=\frac{z_1+\dots+z_n}{n}$$

$$(*)=-\frac{n}{\lambda}+\frac{1}{\lambda^2}\sum_{i=1}^{n}z_i=\underbrace{\frac{n}{\lambda^2}}_{G(n,\lambda)}\underbrace{\left(\frac{\sum_{i=1}^{n}z_i}{n}-\lambda\right)}_{(t(z_1\dots z_n)-\lambda)}$$

where $t(z_1\dots z_n)=\hat{\lambda}$.

$\Rightarrow\hat{\lambda}$ е с мин. дисперсия

$$E\left(\frac{z_1+\dots+z_n}{n}\right)=\frac{1}{n}\left(E(z_1+\dots+z_n)\right)=\frac{1}{n}\left(\underbrace{Ez_1}_{Ez}+\dots+\underbrace{Ez_n}_{Ez}\right)=$$

$$=\frac{1}{\cancel{n}}\cdot\cancel{n}\,Ez=Ez=\lambda\Rightarrow\hat{\lambda}\text{ е неизместена оценка за }\lambda$$

⊕ мин.дис. ⇒ $\hat{\lambda}$ е ефективна оценка б) $H_0: \lambda = 2$ $\qquad \alpha = 5\% = 0.05$

$H_1: \lambda = 3$ $\qquad n = 10$

Нека $L_0(\xi_1 \dots \xi_n | \lambda)$ е ф. правдоподобие при вярна H_0 и
$L_1(\xi_1 \dots \xi_n | \lambda)$ е ф. правдоподобие при вярна H_1.

! Лема на Нейман - Пирсън:

(**) $\alpha = P\left(L_0(\xi_1 \dots \xi_n | \lambda) \le k L_1(\xi_1, \dots, \xi_n | \lambda) \middle| H_0\right)$

$$L_0(\xi_1 \dots \xi_n | \lambda) = \left(\frac{1}{\lambda}\right)^n e^{-\frac{1}{\lambda}\sum_{i=1}^{n}\xi_i}\Bigg|_{\lambda=2} = \left(\frac{1}{2}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\xi_i}$$

$$L_1(\xi_1 \dots \xi_n | \lambda) = \left(\frac{1}{3}\right)^n e^{-\frac{1}{3}\sum_{i=1}^{n}\xi_i}$$

(**) $$= P\left(\left(\frac{1}{2}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\xi_i} \le k \left(\frac{1}{3}\right)^n e^{-\frac{1}{3}\sum_{i=1}^{n}\xi_i} \middle| H_0\right) = \qquad \ln k_1$$

$$= P\left(e^{-\frac{1}{2}\sum_{i=1}^{n}\xi_i} \le k_1 e^{-\frac{1}{3}\sum_{i=1}^{n}\xi_i} \middle| H_0\right) \overset{\ln}{=} P\left(-\frac{1}{2}\sum_{i=1}^{n}\xi_i \le -\frac{1}{3}\sum_{i=1}^{n}\xi_i + k_2 \middle| H_0\right)$$

$$P\left(\frac{1}{6}\sum_{i=1}^{n}\xi_i \ge \overset{-k_2}{k_3} \middle| H_0\right) \overset{k_4 = 6k_3}{=} P\left(\sum_{i=1}^{n}\xi_i \ge k_4 \middle| H_0\right) =$$

$$= P\Bigg(\underbrace{\frac{\sum_{i=1}^{n}\xi_i - E\sum_{i=1}^{n}\xi_i}{\sqrt{D\sum\xi_i}}}_{\text{ЦГТ}\ \eta \to N(0,1)} \ge \frac{k_4 - E\sum\xi_i}{\sqrt{D\xi_i}} \Bigg| H_0\Bigg) = \qquad \Bigg| \sum_{i=1}^{n}\xi_i \in Gama(2, n)$$

(от таблицата $N(\mu, \sigma^2)$)

$$= P\left(\eta \ge \frac{k_4 - 2n}{\sqrt{n \cdot 2^2}}\right) = P(\eta \ge k_5) \Rightarrow k_5 = 1,65$$

$$\frac{k_4 - 2n}{2\sqrt{n}} = 1,65 \to k_4 = 1,65 \cdot 2\sqrt{n} + 2n = 1,65 \cdot 2 \cdot \sqrt{10} + 2 \cdot 10$$
$$= 3,3 \cdot \sqrt{10} + 20 =$$
$$= 30,428$$

$$\Rightarrow W = \left\{\sum_{i=1}^{10}\xi_i \ge 30,42\right\}$$

оптимална критична област (ОКО)

2016 септ. 1 зад. а)

Нека $X \in U(0,a)$ $EX = a/2$ $DX = \frac{a^2}{12}$

$Y \in U(0,b)$ $EY = b/2$ $DY = b^2/12$

$\zeta \in U(c,d)$

$E\zeta = \frac{d+c}{2}$

$D\zeta = \frac{(d-c)^2}{12}$

$f_\zeta(x) = \frac{1}{d-c}, x \in [c,d]$; $0, x \notin [c,d]$

$F_\zeta(x) = \begin{cases} \frac{x-c}{d-c}, & x \in [c,d] \\ 0, & x \leq c \\ 1, & x > d \end{cases}$

Ако $X \perp\!\!\!\perp Y$, $E(2X-3Y) = ?$

$D(2X-3Y) = ?$

$E(2X-3Y) = 2EX - 3EY = a - \frac{3b}{2}$

$D(2X-3Y) = D(2X) + D(-3Y) = 4D(X) + (-3)^2 DY =$

$= 4 \cdot \frac{a^2}{12} + 9\frac{b^2}{12} = \frac{a^2}{3} + \frac{3b^2}{4}$

б) $\rho(X,Y) = 0{,}8$

$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{DX} \cdot \sqrt{DY}}$

$cov(a\zeta, \eta) = E(a\zeta\eta) - Ea\zeta E\eta = aE\zeta\eta - aE\zeta E\eta = a \cdot cov(\zeta,\eta)$

$D(\zeta+\eta) = E(\zeta+\eta - E(\zeta+\eta))^2 = E(\zeta+\eta)^2 - (E(\zeta+\eta))^2 =$

$= E(\zeta^2 + 2\zeta\eta + \eta^2) - (E\zeta)^2 - 2E\zeta E\eta - (E\eta)^2 =$

$= E\zeta^2 - (E\zeta)^2 + E\eta^2 - (E\eta)^2 + \underbrace{2(E\zeta\eta - E\zeta E\eta)}_{2 \cdot cov(\zeta,\eta)}$

$D(\zeta+\eta) = D\zeta + D\eta + 2\,cov(\zeta,\eta)$

$E(2X-3Y) = a - \frac{3b}{2}$ (не се влияе от независимостта)

$D(2X-3Y) = D(2X) + D(-3Y) + 2cov(2X, -3Y) =$

$= \frac{a^2}{3} + \frac{3b^2}{4} + 2 \cdot 2 \cdot (-3) cov(X,Y) =$

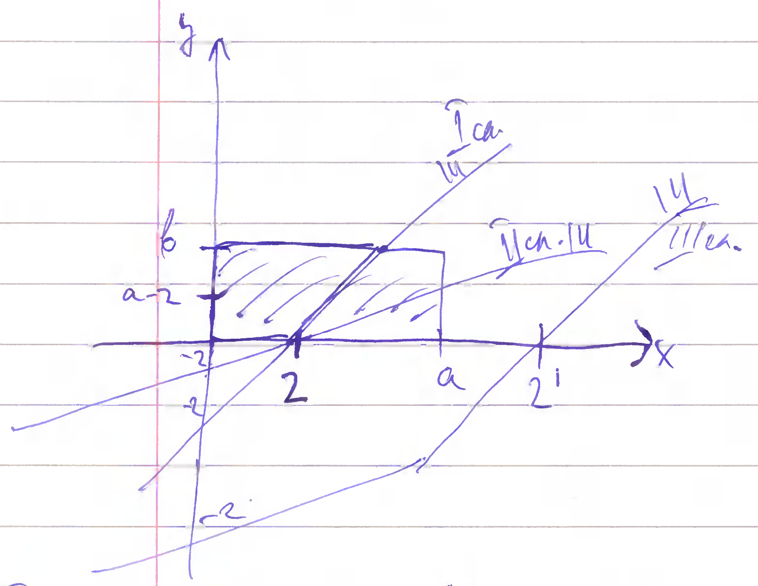
$$= \frac{a^2}{3} + \frac{3b^2}{4} - 12\,cov(X,Y) = \frac{a^2}{3} + \frac{3b^2}{4} - 12 \cdot 0{,}8\sqrt{DX}\sqrt{DY} =$$

$$= \frac{a^2}{3} + \frac{3b^2}{4} - \cancel{12} \cdot 0{,}8 \cdot \frac{a}{\cancel{\sqrt{12}}} \cdot \frac{b}{\cancel{\sqrt{12}}} = \frac{a^2}{3} + \frac{3b^2}{4} - \frac{4ab}{5} \to$$

$$\to \frac{20a^2 + 45b^2 - 48ab}{60} = \frac{16a^2 + 39b^2 + (4a-6b)^2}{60} > 0$$

в) X и Y - незав. $\to f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{a}\cdot\frac{1}{b}, & x\in(0,a),\ y\in(0,b) \\ 0, & \text{вир. сл.} \end{cases}$

$P(X-Y)=2$



I сл. $a > 2$ и $2+b < a$, тогава $P(X-Y<2) = \int_0^b \left(\int_0^{2+y} \frac{1}{ab}\,dx \right) dy =$

$$= \frac{1}{ab} \cdot \frac{2+2+b}{2} \cdot b = \frac{4+b}{2a}$$

II сл. Ако $a > 2$ и $a-2 \le b$

$P(X-Y<2) = 1 - \int_2^a \left(\int_0^{x-2} \frac{1}{ab}\,dy \right) dx = 1 - (a-2)^2 \cdot \frac{1}{2} \cdot \frac{1}{ab} =$

$$= 1 - \frac{(a-2)^2}{2ab}$$

III сп. Ако $a \leq 2$
$\cancel{P(X-Y)\geq 2}$ $P(X-Y<2)=1$

2016 тома 2 зад.

$X \sim Po(\lambda)$

$P(X=x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x=0,1,2,\dots$

а) $EX=?$
$DX=?$

$$EX = \sum_{x=0}^{\infty} x \cdot P(X=x) = \sum_{x=0}^{\infty} x e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{n=0}^{\infty} n \frac{e^{-\lambda}\lambda^n}{\cancel{n}\cdot(n-1)!} =$$

$$= \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{(n-1)!} = e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{(n-1)!} = e^{-\lambda}\lambda \cdot \sum_{n=1}^{\infty} \lambda^{n-1}/(n-1)! =$$

$$= e^{-\lambda}\sum_{\substack{k=0 \\ (k=n-1)}}^{\infty} \lambda^k / k! = e^{-\lambda}\cdot\lambda e^{\lambda} = \lambda$$

$$DX = EX^2 - (EX)^2 \Rightarrow EX^2 = \sum_{n=0}^{\infty} n^2 P(X=n) = \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda}\lambda^n}{n!} =$$

$$= \sum_{n=1}^{\infty} \frac{n e^{-\lambda}\lambda^n}{(n-1)!} = \sum_{n=1}^{\infty} \frac{(n-1)e^{-\lambda}\lambda^n}{(n-1)!} + \underbrace{\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{(n-1)!}}_{\lambda} =$$

$$= \sum_{n=2}^{\infty} \frac{e^{-\lambda}\lambda^n}{(n-2)!} + \lambda = e^{-\lambda}\lambda^2 \sum_{n=2}^{\infty}\frac{\lambda^{n-2}}{(n-2)!} + \lambda =$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \lambda = \lambda^2 + \lambda \rightarrow DX = \lambda^2 + \lambda - \lambda^2 = \lambda$$

б) $X_1 \sim Po(\lambda_1), X_2 \sim Po(\lambda_2), \quad \lambda_1, \lambda_2 > 0, \quad X_1 \perp\!\!\!\perp X_2$
$S_2 = X_1 + X_2$?

$\xi \in Po(\lambda)$

$\varphi_\xi(t) \to$ пораждаща ф-я на ξ

$$\varphi_\xi(t) = \sum_{k=0}^{\infty} t^k . P(\xi = k) = \sum \frac{t^k e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} =$$

$$= e^{-\lambda} e^{t\lambda} = e^{-(1-t)\lambda}$$

$\varphi_{X_1}(t) = e^{-(1-t)\lambda_1}$

$\varphi_{X_2}(t) = e^{-(1-t)\lambda_2}$

От $X_1 \perp\!\!\!\perp X_2 \to \varphi_{X_1+X_2}(t) = \varphi_{X_1}(t)\,\varphi_{X_2}(t)$

$\varphi(S_2) = \varphi_{X_1+X_2}(t) = e^{-(1-t)(\lambda_1+\lambda_2)} \to S_2 \in Po(\lambda_1+\lambda_2)$

в) $X_1 \ldots X_n$ — незав. наблюд. над $X \sim Po(\lambda),\ \lambda > 0$

$S_n = X_1 + \ldots + X_n$

$H_0: \lambda = 3 \qquad n = 4$

$H_1: \lambda = 6 \qquad S_n = 15 = X_1 + X_2 + X_3 + X_4$

$$L_0(X_1, \ldots, X_n \mid \lambda = 3) = \prod_{i=1}^{n} P(\xi = X_i) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{X_i}}{(X_i)!}\Bigg|_{\lambda=3} =$$

$$= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} X_i}}{\prod_{i=1}^{n} (X_i)!}\Bigg|_{\lambda=3} = \frac{e^{-3n} 3^{\sum X_i}}{\prod_{i=1}^{n} (X_i)!}$$

$$L_1(X_1 \ldots X_n \mid \lambda = 6) = \prod_{i=1}^{n} P(\xi = X_i) = \ldots = \frac{e^{-6n} 6^{\sum X_i}}{\prod_{i=1}^{n} (X_i)!}$$

$$\alpha = P\left(L_0(X_1 \ldots X_n \mid \lambda = 3) \leq k . L_1(X_1 \ldots X_n \mid \lambda = 6) \mid H_0\right) =$$

$$= P\left(\frac{e^{-3n} 3^{\sum X_i}}{\cancel{\prod_{i=1}^{n} (X_i)!}} \leq k_1 \frac{e^{-6n} 6^{\sum X_i}}{\cancel{\prod_{i=1}^{n} (X_i)!}} \Bigg| H_0\right) = P\left(3^{\sum X_i} \leq k\, 6^{\sum X_i} \mid H_0\right)$$

$= P\left(\left(\frac{3}{6}\right)^{\sum X_i} \leq k_1 \mid H_0\right) = P\left(\left(\frac{1}{2}\right)^{\sum X_i} \leq k_1 \mid H_0\right) \stackrel{\ln}{=}$

$= P\left(\ln\left(\frac{1}{2}\right) \cdot \sum X_i \leq \underbrace{\ln k_1}_{k_2} \mid H_0\right) = P\left(\underbrace{\sum_{i=1}^{n} X_i}_{S_n} \geq k_2 \mid H_0\right)$

$S_n \sim Po(n\lambda)$

При H_0 $S_n \sim Po(3n) \to S_4 \sim Po(12)$

$W_\alpha^* = \{S_n \geq k_2\}$

За частния случай $n=4$ и $S_n = 15$

$p\text{-value} = P(S_n \geq 15)$

От таблицата $\Rightarrow P(S_n < 15) = P(S_n \leq 14) =$
$= 0,772$

$\Rightarrow P(S_n \geq 15) = 1 - 0,772 = \underline{0,228}$

2014г. юли 2 зад. а) $p \in (0,1)$ вер-за успех в 1 опит
X – брой на неуспехите до първа поява на успех
$X \sim Ge_0(p)$

X	0	1	2	3	k ...
p_X	p	$p(1-p)$	$p(1-p)^2$	$p(1-p)^3$	$p(1-p)^k$ ...

б) Пораждаща ф-я на X: $\varphi_X(t) = \sum_{k=0}^{\infty} t^k P(X=k) = \sum_{k=0}^{\infty} t^k p(1-p)^k =$
$= p\sum_{k=0}^{\infty} [t(1-p)]^k$. За $|t|<1 \to p\frac{1}{1-t(1-p)} = \frac{p}{1-t(1-p)} = \frac{p}{1-tq}$

9

б) $EX = ?$

$DX = ?$

$$\varphi'_X(t) = \sum k t^{k-1} \cdot P(X=k)$$

$$\varphi'_X(1) = \sum_{k=1}^{\infty} k\, P(X=k) = EX$$

$$\varphi''_X(t) = \sum_{k=2}^{\infty} k(k-1)\, t^{k-2} P(X=k) \Rightarrow \varphi''_X(1) = E[X(X-1)] = EX^2 - EX$$

$$DX = EX^2 - (EX)^2 = \varphi''_X(1) + \varphi'_X(1) - (\varphi'_X(1))^2 =$$

$$\varphi'_X(t) = \left[p\,(1-qt)^{-1}\right]' = p(-1)(1-qt)^{-2} \cdot (-q) =$$

$$= \frac{p \cdot q}{(1-qt)^2}$$

$$\varphi'_X(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$\varphi''_X(t) = \left[pq\,(1-qt)^{-2}\right]' = (-2)qp\,(1-qt)^{-3}(-q) =$$

$$= \frac{2pq^2}{(1-qt)^3}$$

$$\varphi''_X(1) = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

$$DX = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q^2+qp}{p^2} = \frac{q(q+p)}{p^2} =$$

$$= \frac{q}{p^2}$$

г) $P(X>k),\ k \geq 0 \Rightarrow \sum_{n=k+1}^{\infty} P(X=n) = \sum_{n=k+1}^{\infty} p(1-p)^n =$

$$= p\sum_{n=k+1}^{\infty} (1-p)^n = p\left((1-p)^{k+1} + \ldots +\right) = p(1-p)^{k+1}\left[\underbrace{1 + (1-p) + (1-p)^2 + \ldots}_{``1/p"}\right] =$$

$= \underline{(1-p)^{k+1}}$

2014г. сеш. 2 зад.

$\tilde{X} \to$ брой на опитите до първи успех

$\tilde{X} \in Ge_1(p)$

$P(X=k) = p(1-p)^{k-1}, k=1,2,\dots$

$\tilde{X} = X+1$

$E\tilde{X} = EX+1 = \frac{1}{p} - 1 + 1 = \frac{1}{p}$; $D\tilde{X} = D(X+1) = DX = \frac{q}{p^2}$

$\varphi_{\tilde{X}}(t) = \frac{pt}{(1-qt)}$ $\qquad P(\tilde{X}>k) = (1-p)^k$, $k \geq 0$

Задача. Нека $X_1, X_2, \dots, X_n$ са незав. наб. над $X \in N(\mu, \sigma^2)$

μ – незав., σ^2 – известно

Намерете опт. кр. област за проверка на:

$H_0: \mu = \mu_0$

$H_1: \mu = \mu_1$

При ниво на значимост α.

Решение: $L_0(X_1 \dots X_n) = \prod_{i=1}^{n} f_X(X_i \mid \mu = \mu_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu_0)^2}{2\sigma^2}} =$

$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(X_i-\mu_0)^2}{2\sigma^2}}$

$L_1(X_1 \dots X_n) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(X_i-\mu_1)^2}{2\sigma^2}}$

От Лемата на Н-П:

$$\alpha = P\left(L_0(\dots) \le k L_1(\dots) \mid H_0\right) = P\left(\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(X_i-\mu_0)^2}{2\sigma^2}} \le k \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(X_i-\mu_1)^2}{2\sigma^2}}\right) =$$

$$= P\left(e^{-\frac{\sum(X_i-\mu_0)^2 - \sum(X_i-\mu_1)^2}{2\sigma^2}} \le k \mid H_0\right) \overset{\ln}{=}$$

$$= P\left(-\sum\frac{(X_i-\mu_0)^2}{2\sigma^2} + \sum\frac{(X_i-\mu_1)^2}{2\sigma^2} \le k_1 \mid H_0\right) =$$

$$= P\left(\sum_{i=1}^n\left[(X_i-\mu_0)^2 - (X_i-\mu_1)^2\right] \le k_2 \mid H_0\right) =$$

$$= P\left(\sum_{i=1}^n\left[\cancel{X_i^2} - 2\mu_0 X_i + \mu_0^2 - \cancel{X_i^2} + 2X_i\mu_1 - \mu_1^2\right] \le k_2 \mid H_0\right) =$$

$$= P\left(\left[\sum_{i=1}^n 2X_i(\mu_1-\mu_0)\right] + n\mu_0^2 - n\mu_1^2 \le k_2 \mid H_0\right) =$$

$$= P\left(2(\mu_1-\mu_0)\sum_{i=1}^n X_i \le k_3 \mid H_0\right) \begin{cases} \xrightarrow{\mu_0>\mu_1} P\left(\sum X_i \le k_5 \mid H_0\right) \ (1) \\ \xrightarrow{\mu_0<\mu_1} P\left(\sum X_i \ge k_5 \mid H_0\right) \ (2) \end{cases}$$

k_4

(1) $\Rightarrow W^* = \{\sum X_i \le \text{crit. value}\}$

(2) $\rightarrow W^* = \{\sum X_i \ge \text{crit. value}\}$

$\sum_{i=1}^n X_i \in N(n\mu, n\sigma^2)$

При H_0: $\sum_{i=1}^n X_i \in N(n.\mu_0, n\sigma^2)$

$$\frac{\sum X_i - n\mu_0}{\sqrt{n\sigma^2}} \sim N(0,1)$$

+ за упр. : двусмислено H_0 и H_1

2017г. септ. 2 зад.

б) $X_1 \in Exp(\lambda_1)$

$$P(X_1 \le E[X_1]) = P(X_1 \le \lambda_1^{-1}) = F_{X_1}(\lambda_1^{-1}) = 1 - e^{-\frac{1}{\lambda_1}\lambda_1} = 1 - e^{-1}$$

г) $H_0: \lambda = 2 \qquad \alpha = 0.025$
$H_1: \lambda = 4 \qquad n = 7$

$$P\left(e^{-\frac{1}{2}\sum X_i} < k e^{-\frac{1}{4}\sum X_i} \mid H_0\right) = P\left(-\frac{1}{2}\sum X_i < k_1 - \frac{1}{4}\sum X_i \mid H_0\right) =$$

$$= P\left(\frac{1}{4}\sum X_i \ge k_3 \mid H_0\right) = P\left(\underbrace{\sum X_i}_{\sim \text{Гамма}(2,n)} \ge k_4 \mid H_0\right) =$$

$$= P\left(\frac{\sum X_i - E\sum X_i}{\sqrt{D\sum X_i}} \ge \frac{k_4 - E\sum X_i}{\sqrt{D\sum X_i}} \mid H_0\right) =$$

$$= P\left(Y \ge \frac{k_4 - 2n}{\sqrt{2^2 \cdot n}}\right) = P(Y \ge k_5)$$

$$k_4 = 1,96 + 2\sqrt{7} + 14 = \underline{\underline{24,37}} \qquad \Rightarrow W^* = \left\{\sum_{i=1}^{7} X_i \ge 24,37\right\}$$

в) $X \in Exp(\lambda)$; $Y = X^2 + 3X$

$$EY = E(X^2 + 3X) = EX^2 + 3EX = \frac{2}{\lambda^2} + \frac{3}{\lambda}$$

$f_Y(y) = ?$ $\quad P(Y \le y) = 1 - P(Y > y)$
$\quad\quad 1 - e^{-\lambda y}$

(на другата страница)

$$DX = EX^2 - (EX)^2 \qquad E[X^n] = \frac{n!}{\lambda^n} \text{ за } Exp(\lambda)$$

$$\frac{1}{\lambda^2} \qquad \frac{2}{\lambda^2} \qquad \frac{1}{\lambda^2}$$

а) $X_1 \in Exp(\lambda_1)$, $X_2 \in Exp(\lambda_2)$
$P(X_2 < X_1) = ?$

Съвместната плътност на X_1 и X_2:
$\lambda_1 \lambda_2 . e^{-\lambda_1 X_1} . e^{-\lambda_2 X_2}$

$$P(X_2 < X_1) = \int_0^\infty \int_0^{X_1} \lambda_1 e^{-\lambda_1 X_1} . \lambda_2 e^{-\lambda_2 X_2} \, dX_2 \, dX_1 =$$

$$= \int_0^\infty \lambda_1 (e^{\lambda_2 X_1} - 1) e^{-X(\lambda_1+\lambda_2)} dX_1 = \ldots \frac{\lambda_1}{\lambda_1+\lambda_2}$$

в) (решено от Ники Николов)

$y = X^2 + 3X$ $\quad X_{1,2} = \frac{-3 \pm \sqrt{9+4y}}{2}$ $\quad X_{1,2}$ трябва да е ≥ 0

$$\rightarrow X = \frac{-3+\sqrt{9+4y}}{2}$$

$$dX = 2 . \frac{1}{\sqrt{9+4y}} dy$$

$$f_y(y) = f_x\left(\frac{-3+\sqrt{9+4y}}{2}\right) . \frac{2}{\sqrt{9+4y}}, \quad y \in [0; +\infty)$$

за г) $X \in \chi^2(1)$ → степени на свобода $\quad Z \in N(0,1) \quad \chi^2(1) = N^2(0,1)$

$$P(X \geq c) = P(\underset{\shortparallel}{Z^2} \geq c) = 0{,}05$$

$$P(Z \geq \sqrt{c} \cup Z \leq -\sqrt{c}) = P(Z \geq \sqrt{c}) + P(Z \leq -\sqrt{c}) =$$
$$= 2 . P(Z \geq \sqrt{c})$$

→ т.е. гледаме $\sqrt{n}$ и $\alpha/2$ в χ^2 таблицата (може би не е така!!!)