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# Simple and Deterministic Matrix Sketching[1]

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[1] Authored by Edo Liberty and and won the *KDD-2013 best paper* award[3].

# Content

- Background
- Related work
- Frequent directions
- Experiments and Results
- Conclusion

# Matrices in the age of ’Big data’

# What is a sketch?

- A sketch of a matrix $A$ is another matrix $B$ which is significantly smaller than $A$, but still approximates it well.
- A good sketch matrix is one on which some computations can be performed, without *much* loss of precision.
- Formally, consider a large matrix $A \in \mathbb{R}^{n \times d}$ with $n$ rows and $d$ columns
  - a sketch matrix $B$ is one s.t. $B \in \mathbb{R}^{\ell \times d}$,
  - containing only $\ell \ll n$ rows and $A^T A \approx B^T B$.

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# How do we get a sketch?

Three existing main classes:

- Random-projection:
  - (i) use a projection matrix for dimensionality reduction (approximately preserving the lengths of, the dot products between two original vectors on average, as well as the distances in the transformed space), or
  - (ii) randomly combine matrix rows.
- Hashing: use a subspace embedding $S$ that embeds the column space of the original matrix into a lower-dimensional subspace
  - approximately preserving the norms of all vectors in that subspace
  - where $A \in \mathbb{R}^{n \times d}$, $S : \mathbb{R}^n \to \mathbb{R}^t$, for all $x \in \mathbb{R}^d$, and
  - $\|SAx\|_2 = (1 \pm \varepsilon) \|Ax\|_2$

# How do we get a sketch? (cont.)

- Sampling: Column Subset Selection Problem
  - simple solution obtained by sampling rows with probability proportional to their squared $\ell_2$ norms
  - aim is to recover a low rank matrix whose column space contains most of the space spanned by the top $k$ singular vectors of the matrix.

Proposed fourth approach, Frequent-directions

# Proposed approach: Frequent-directions

- Based on a well known existing algorithm for item frequency estimation.
- A pass-efficient algorithm, given the constraint:
  - data can be read only a constant number of times;
  - the streaming model: only one pass is permitted!

# Item frequency estimation

- Used to uncover frequent items in an item stream
- (Re-)Invented (at least!) four times [5, 1, 2, 4][2]

Goal. Use $O(\ell)$ space as opposed to $O(d)$, where $l \ll d$

- to produce estimates $g_i$, s.t.
- $|f_i - g_i| \leq n/\ell$, for all item types $i$ simultaneously.

Matrix setting. Use Frequent-directions to uncover any direction in space $x$

- for which $\|Ax\|^2 \geq \varepsilon \|A\|_2^2$,
- by taking $\ell > 2r/\varepsilon$, where $r$ is the numerical rank of $A$.

[2] [Misra and Gries, 1982; Demaine et al., 2002; Karp et al., 2003; Metwally, 2005]

# Item frequency estimation

- The algorithm:

  **Input**:

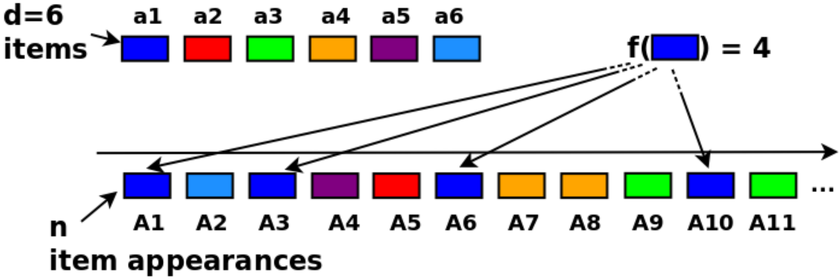
  - $d$ items $a_1, a_2, \ldots, a_d$
  - $n$ item appearances $A_1, A_2, \ldots, A_n$

  **Repeat** until there are less than $\ell$ unique items left {

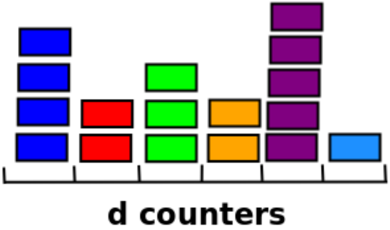
  - Get item $A_j$ from stream, for $j = 1 \ldots n$
  - *If* there are free slots among $\ell$
    - Create new bucket for item type $k$ and store the item there
  - *Else*
    - Find median count $\delta_t = f_{\ell/2}$ of items, and
    - Remove exactly $\min(\delta_t, f_i)$ appearances from each bucket $i = 1 \ldots \ell$

  }

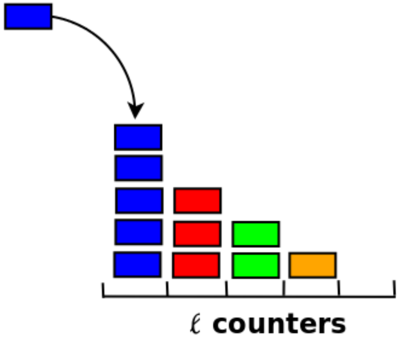
# Item frequency estimation (cont.)



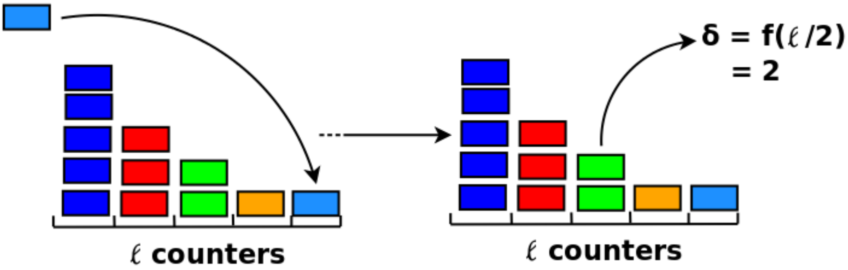
# Item frequency estimation (cont.)



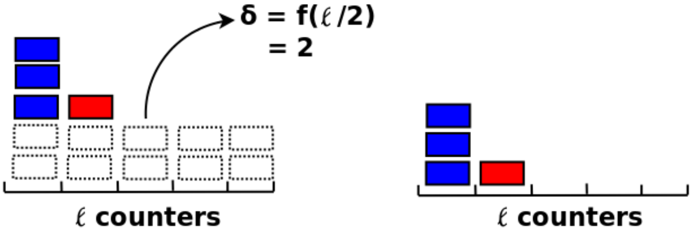
# Item frequency estimation (cont.)



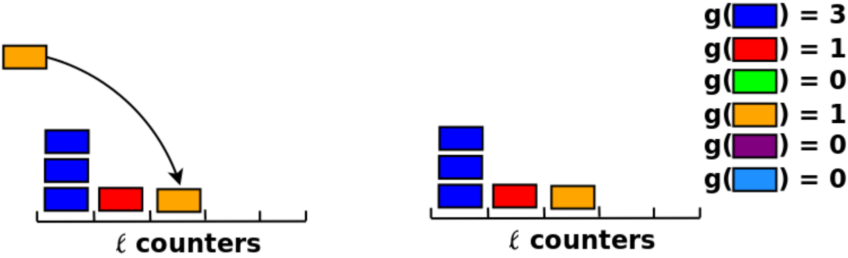
# Item frequency estimation (cont.)



# Item frequency estimation (cont.)



# Item frequency estimation (cont.)




Claim. For each item type $i$,

- $g_i$ is a good approximation for its true frequency $f_i$ (even in the case of $g_i = 0$),
- 'Good': $|f_i - g_i| \leq n/\ell$.

# Item frequency estimation (cont.)

Proof.

- Each item-type is deleted at most once per iteration:
  - $g_i \leq f_i$
- Each counter is decreased by at most $\delta_t$ at time $t$:
  - $g_i \geq f_i - \sum_t \delta_t \Leftrightarrow f_i - g_i \leq \sum_t \delta_t$
- Putting this together:
  - $0 \leq \sum_i g_i \leq \sum_t 1 - (\ell/2) \cdot \delta_t = n - (\ell/2) \cdot \sum_t \delta_t$
  - $\sum_t \delta_t \leq 2n/\ell$
- Set $\ell = 2/\varepsilon$:
  - $|f_i - g_i| \leq \varepsilon n$.

□

# Item frequency estimation (cont.)

...What is the intuition of the following? ...

- If one sets $\ell > 1/\varepsilon$,
  - Then any item that appears more than $\varepsilon n$ times in the stream must appear in the final sketch.

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# The *Frequent-directions* algorithm

Represent the frequency of a direction (unit vector):

- Assume the directions of $A$ are indicator vectors of the items:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Frequency of second item $e_2 = (0, 1, 0, 0)^T$:
  $\|Ae_2\|^2 = \left\|(0, 1, 0, 1)^T\right\|^2 = 0^2 + 1^2 + 0^2 + 1^2 = 2$.
- Generalize the directions to unit vector $\{x : \|x\| = 1\}$ and the frequency of a direction is $\|Ax\|^2$.

# The *Frequent-directions* algorithm (cont.)

Connection to SVD of $A$:

- $A = U\Sigma V^T \Leftrightarrow U^T A = \Sigma V^T \Leftrightarrow Au = \sigma v$.
- $\|Au\|^2 = \|\sigma v\|^2 = \sigma^2$.

Change $u$ to $x$:

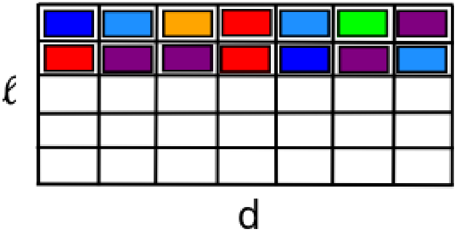
The frequency of a direction is indicated by the square of corresponding singular value $\sigma^2$.

# The *Frequent-directions* algorithm (cont.)

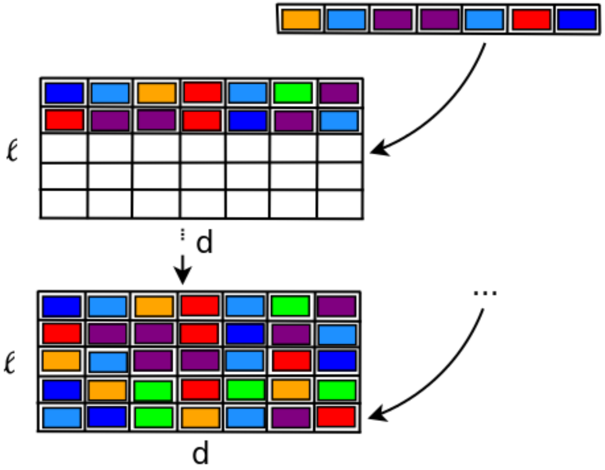
The algorithm:

**Input**: $\ell$, $A \in \mathbb{R}^{n \times d}$
$B \leftarrow$ all zeros matrix $\in \mathbb{R}^{\ell \times d}$
**for** $i = 1, \ldots, n$ **do**
  Insert $i^{th}$ row of $A$ into zero valued row of $B$
  **if** $B$ has no zero valued rows **then**
    $[U, \Sigma, V] \leftarrow SVD(B)$
    $C \leftarrow \Sigma V^T$ // for proof
    $\delta \leftarrow \sigma^2_{\ell/2}$
    $\check{\Sigma} \leftarrow \sqrt{\max(\Sigma^2 - I_\ell \delta, 0)}$
    $B \leftarrow \check{\Sigma} V^T$
  **end if**
**end for**

# The *Frequent-directions* algorithm (cont.)

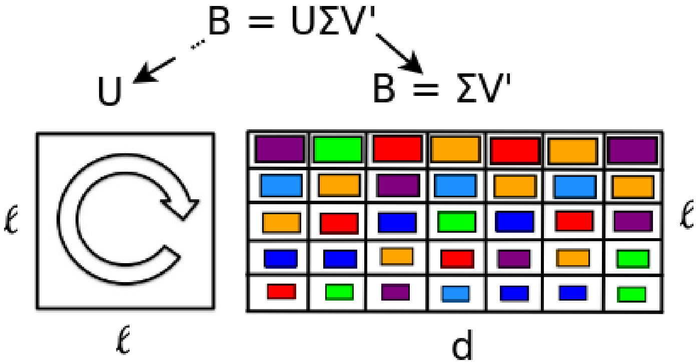


# The *Frequent-directions* algorithm (cont.)



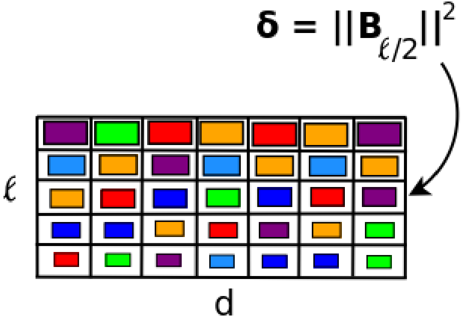
# The *Frequent-directions* algorithm (cont.)

$[U, \Sigma, V] \leftarrow SVD(B)$.



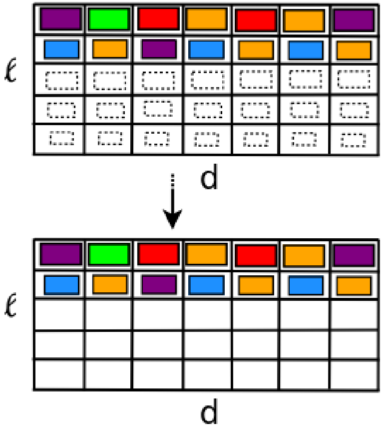
# The *Frequent-directions* algorithm (cont.)

$\delta \leftarrow \sigma^2_{\ell/2}$.

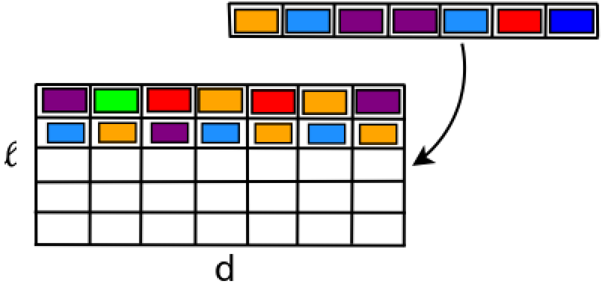


# The *Frequent-directions* algorithm (cont.)

$$\check{\Sigma} \leftarrow \sqrt{\max(\Sigma^2 - I_\ell \delta, 0)}$$
$$B \leftarrow \check{\Sigma} V^T$$



# The *Frequent-directions* algorithm (cont.)



# Properties of the sketch matrix $B$

In summary:

- $A^T A \succeq B^T B \succeq 0$.
- $||A^T A - B^T B|| \leq 2||A||_f^2/\ell$.
- Let $A = [A_1; A_2]$ and $B_1$, $B_2$ is the sketches of $A_1$ and $A_2$. A sketch $C$ of $B = [B_1; B_2]$ can be shown that:

$$||A^T A - C^T C|| \leq 2||A||_f^2/\ell.$$

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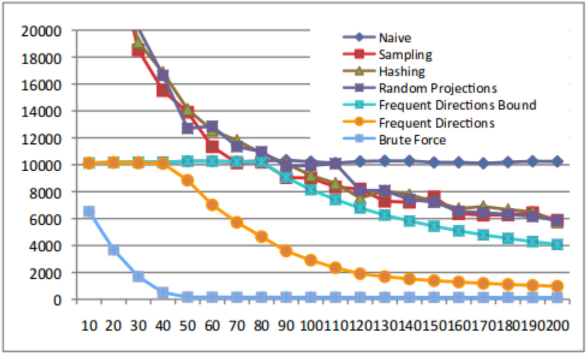
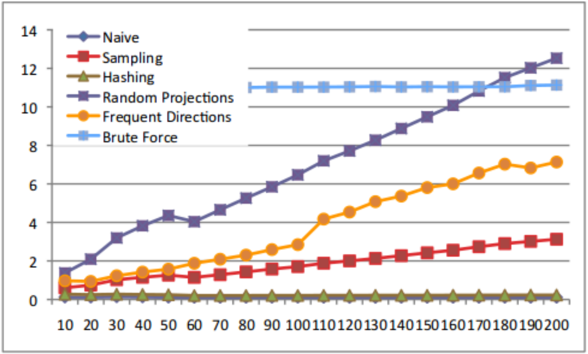
# Experiments

For a synthetic matrix $n = 10000, m = 1000$. Error $||A^T A - B^T B||$ against sketch size $\ell$ with.



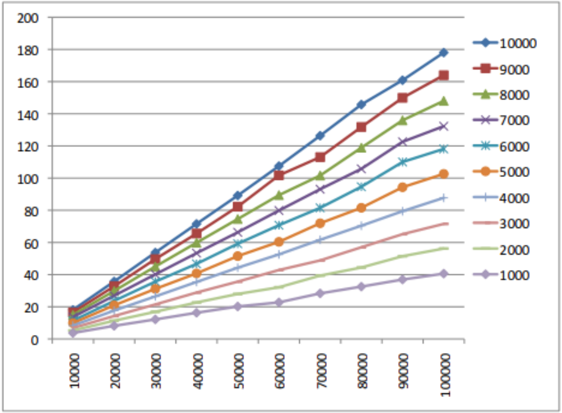
# Experiments (cont.)

Running time.

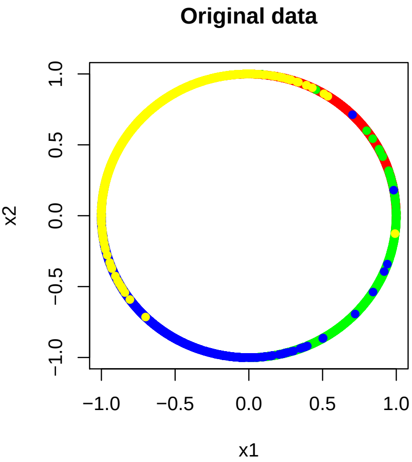


# Experiments (cont.)

Linear in $n$ and $m$ ($\ell$ fix to 100).

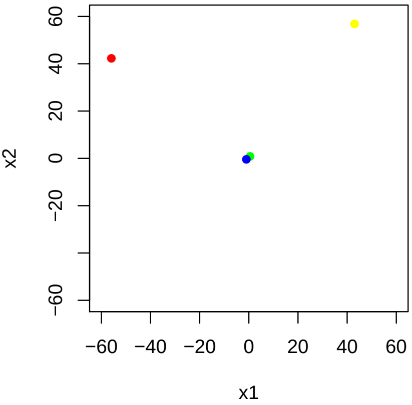


# Clustering experiment



# Clustering experiment (cont.)

K−means on Sketch



# Clustering experiment (cont.)

$$B = \begin{pmatrix} 43.0030 & 56.8110 \\ -55.9340 & 32.3390 \\ 0.5011 & 0.8654 \\ -0.9427 & -0.3336 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 4979.0 & 75.7 \\ 75.7 & 5021.0 \end{pmatrix}, B^T B = \begin{pmatrix} 4979.0 & 75.6 \\ 75.6 & 5020.9 \end{pmatrix}$$

# Conclusion

- In terms of $||A^T A - B^T B||$, the proposed sketching algorithm is more accurate than sampling, hashing and random projections.
- The proposed algorithm runs reasonably fast
  - in fact, faster than random projection, slower than sampling.
- The proposed algorithm is linear in the scale of the input size.
- Choose your sketching algorithm according to your task!

# Thank you!

- Questions?

# References

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