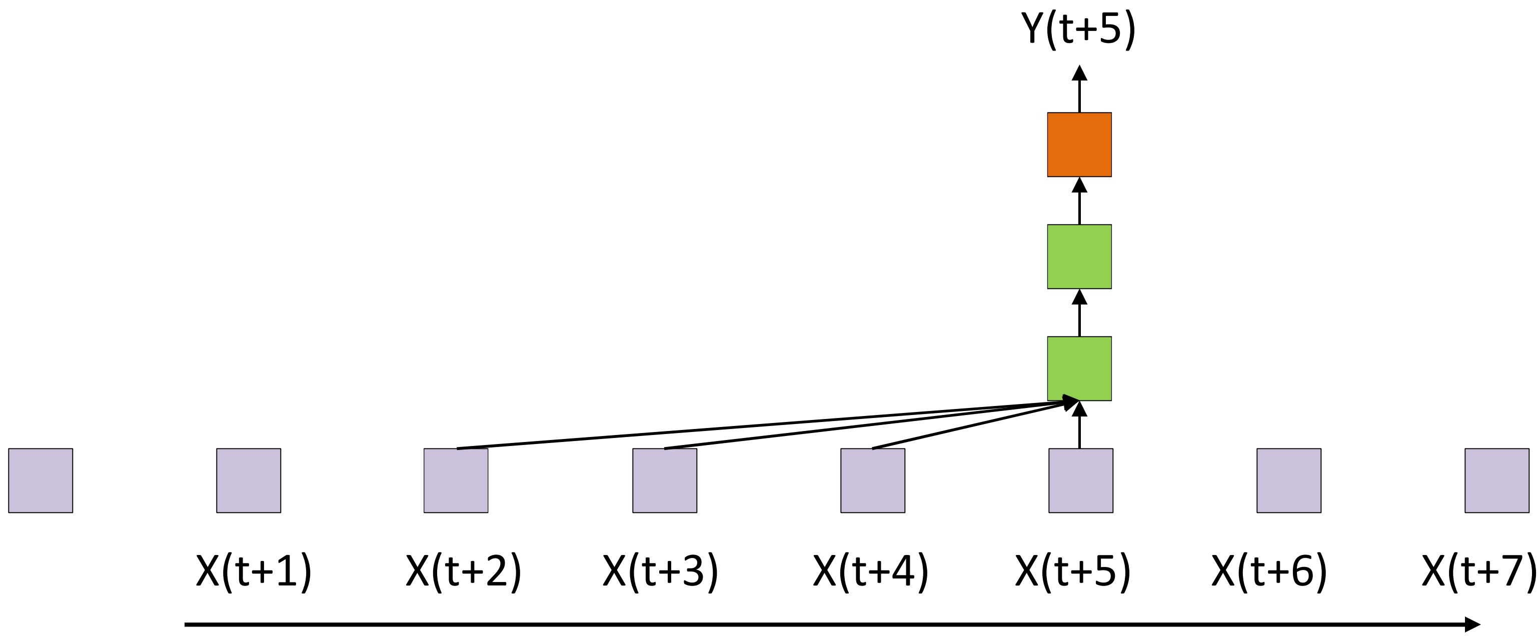
The behavior of recurrence..

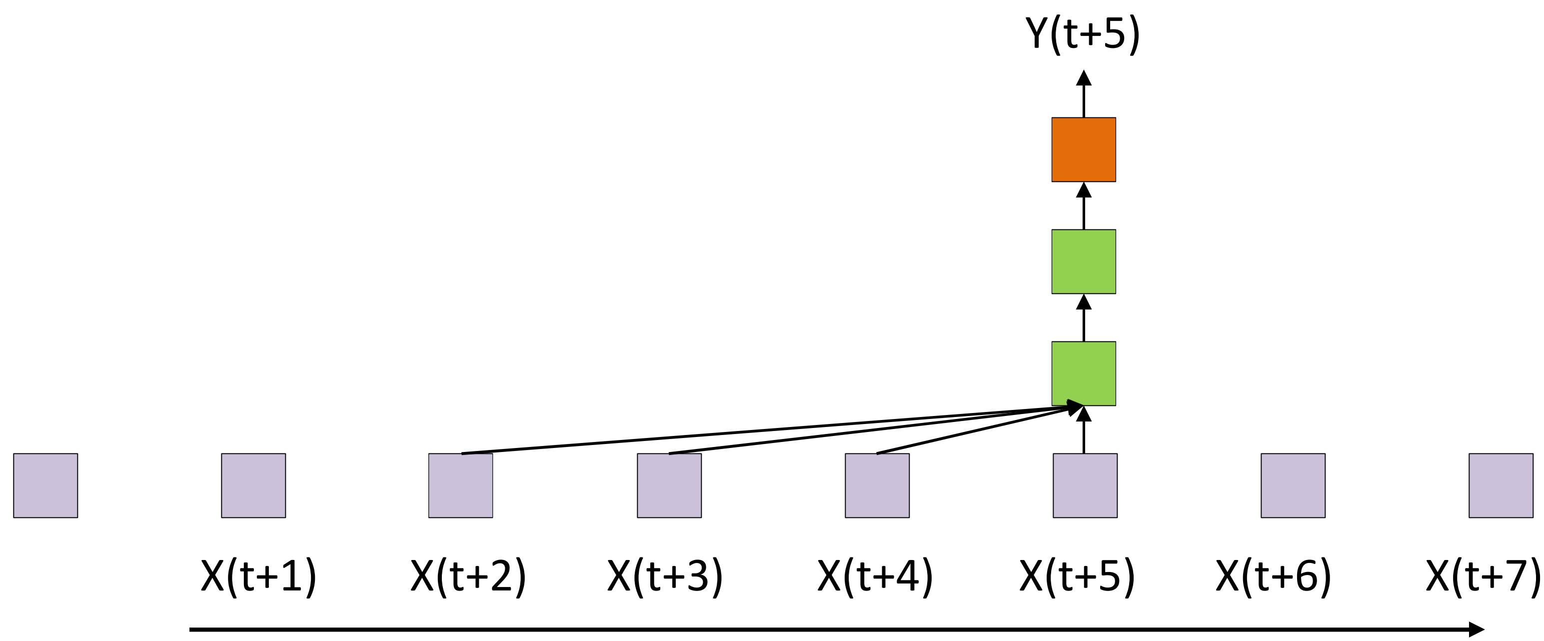


- Returning to an old model..

$$Y(t) = f(X(t-i), i = 1..K)$$

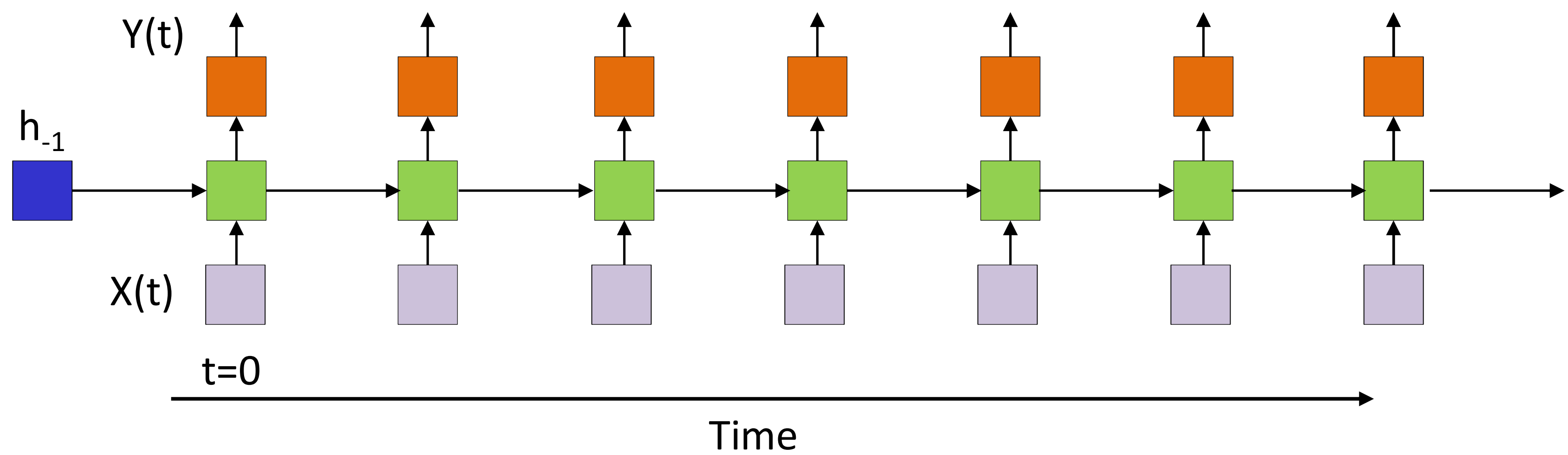
- When will the output "blow up"?

“BIBO” Stability



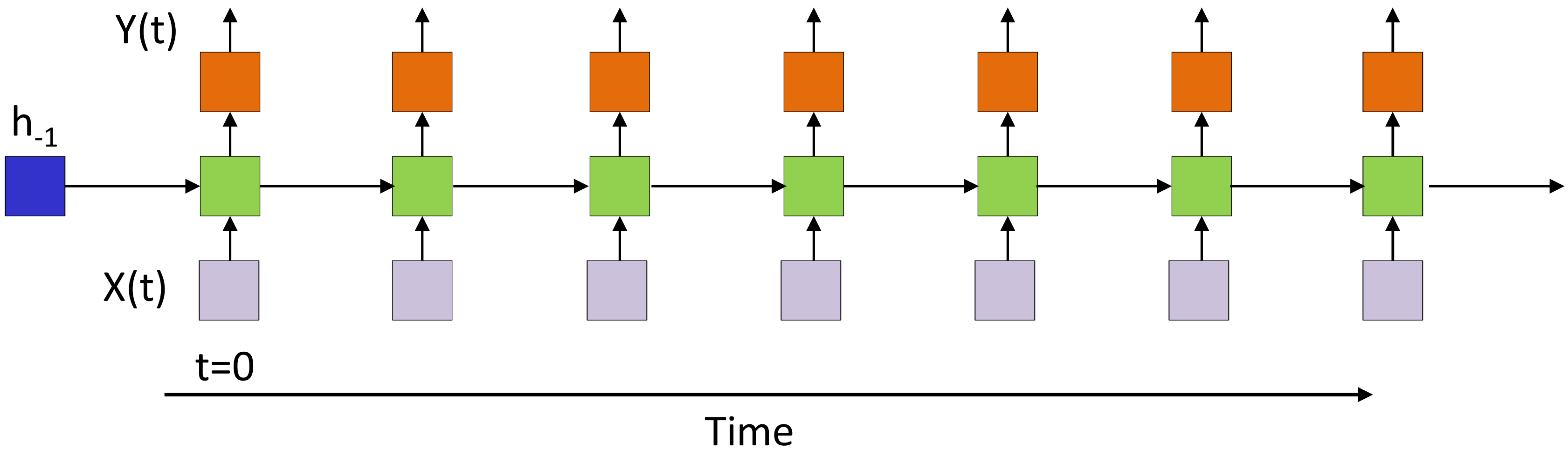
- Time-delay structures have bounded output if
  - The function $f()$ has bounded output for bounded input
    - Which is true of almost every activation function
  - $X(t)$ is bounded
- “Bounded Input Bounded Output” stability
  - This is a highly desirable characteristic

Is this BIBO?



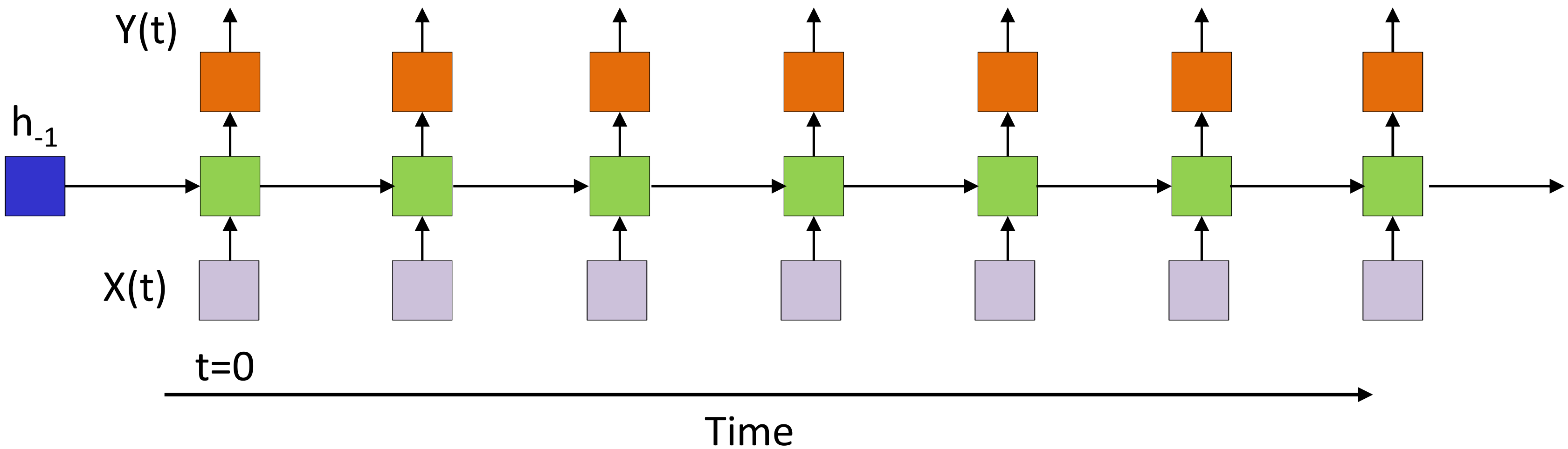
- Will this necessarily be BIBO?

Is this BIBO?



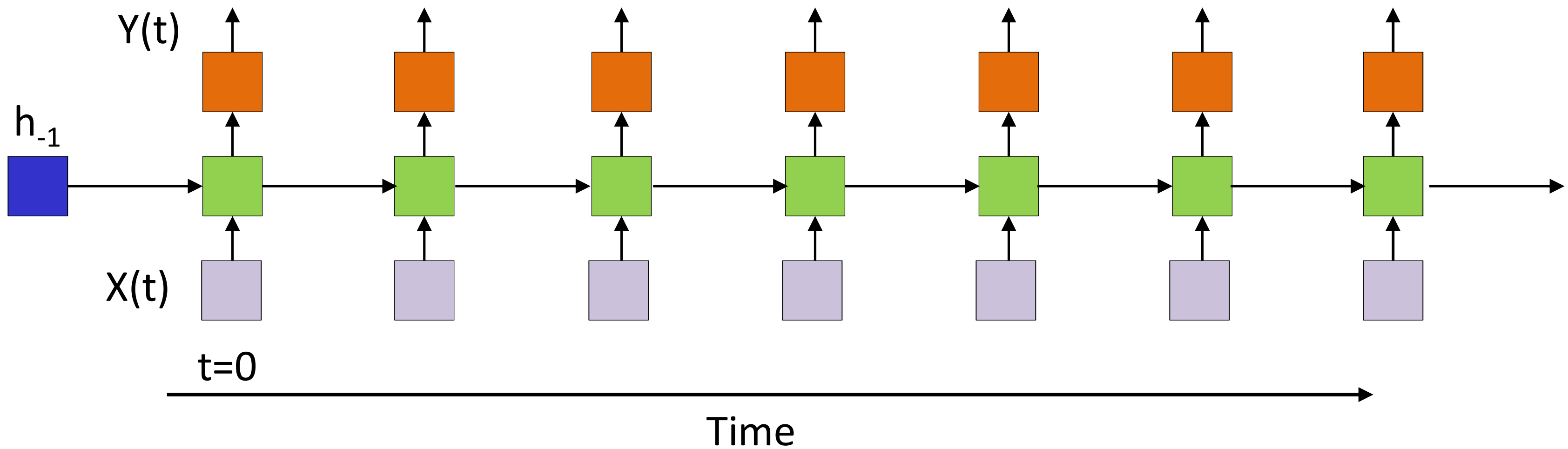
- Will this necessarily be BIBO?
  - Guaranteed if output and hidden activations are bounded
    - But will it *saturate* (and where)
  - What if the activations are linear?

Analyzing recurrence



- Sufficient to analyze the behavior of the hidden layer h_k since it carries the relevant information
  - Will assume only a single hidden layer for simplicity

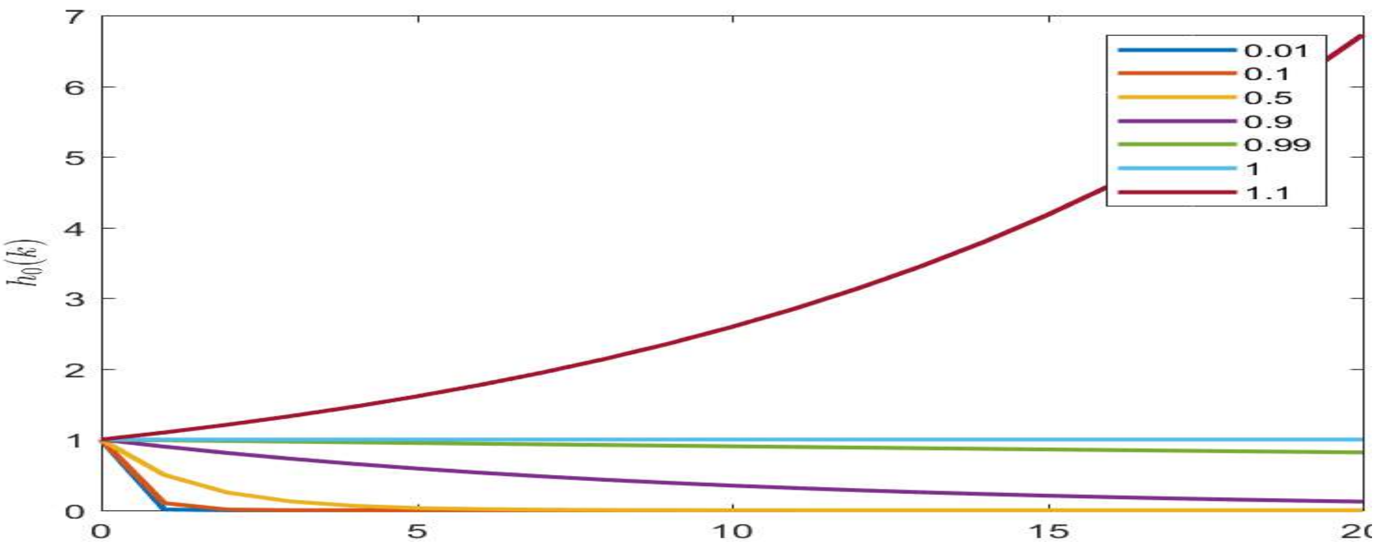
Streetlight effect



- Easier to analyze *linear* systems
  - Will attempt to extrapolate to non-linear systems subsequently
- All activations are identity functions
  - $z_k = W_h h_{k-1} + W_x x_k, \qquad h_k = z_k$

Linear recursions

- Consider simple, scalar, linear recursion (note change of notation)
  - $h(t) = wh(t-1) + cx(t)$
  - $h_0(t) = w^t cx(0)$
    - Response to a single input at 0



Linear recursions: Vector version

- Vector linear recursion (note change of notation)
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = W^t C x(0)$
    - Length of response vector to a single input at 0 is $|h_0(t)|$

- We can write $W = U \Lambda U^{-1}$
  - $W u_i = \lambda_i u_i$
  - For any vector $x' = Cx$ we can write
    - $x' = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$
    - $W x' = a_1 \lambda_1 u_1 + a_2 \lambda_2 u_2 + \cdots + a_n \lambda_n u_n$
    - $W^t x' = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$
  - $\lim_{t \to \infty} |W^t x'| = a_m \lambda_m^t u_m$ where $m = \underset{j}{\operatorname{argmax}}\, \lambda_j$

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For any input, for large t the length of the hidden vector will expand or contract according to the t −th power of the largest eigen value of the hidden-layer weight matrix

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Unless it has no component along the eigen vector corresponding to the largest eigen value. In that case it will grow according to the *second* largest Eigen value..

And so on..

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Linear recursions: Vector version

- Vector linear recursion (note change of notation)

If $|\lambda_{max}| > 1$ it will blow up, otherwise it will contract and shrink to 0 rapidly

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Linear recursions: Vector version

What about at middling values of t? It will depend on the other eigen values

If $|\lambda_{max}| > 1$ it will blow up, otherwise it will contract and shrink to 0 rapidly

- Length of response vector to a single input at 0 is $|h_0(t)|$

For any input, for large t the length of the hidden vector will expand or contract according to the t −th power of the largest eigen value of the hidden-layer weight matrix

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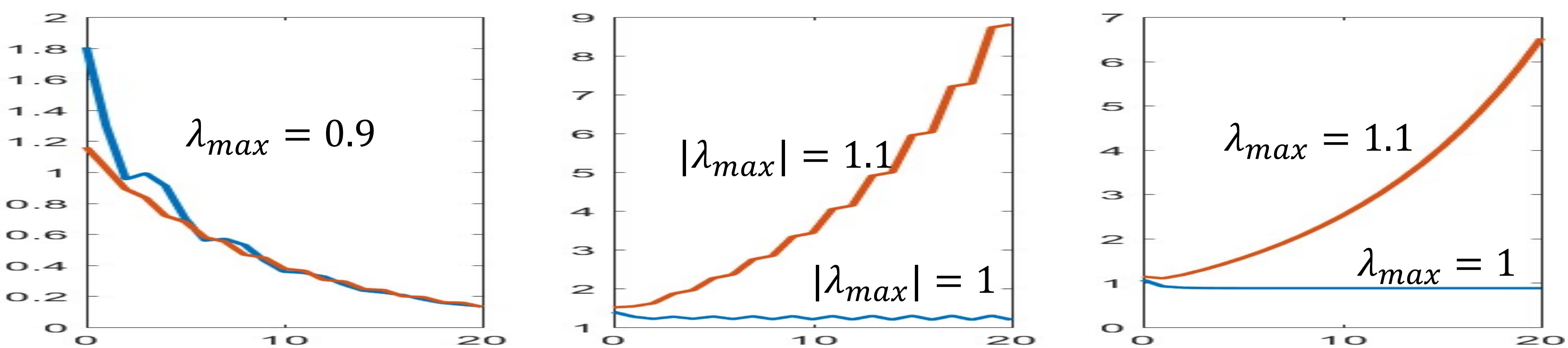
Linear recursions

- Vector linear recursion
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = W^t cx(0)$
    - Response to a single input [1 1 1 1] at 0



Linear recursions

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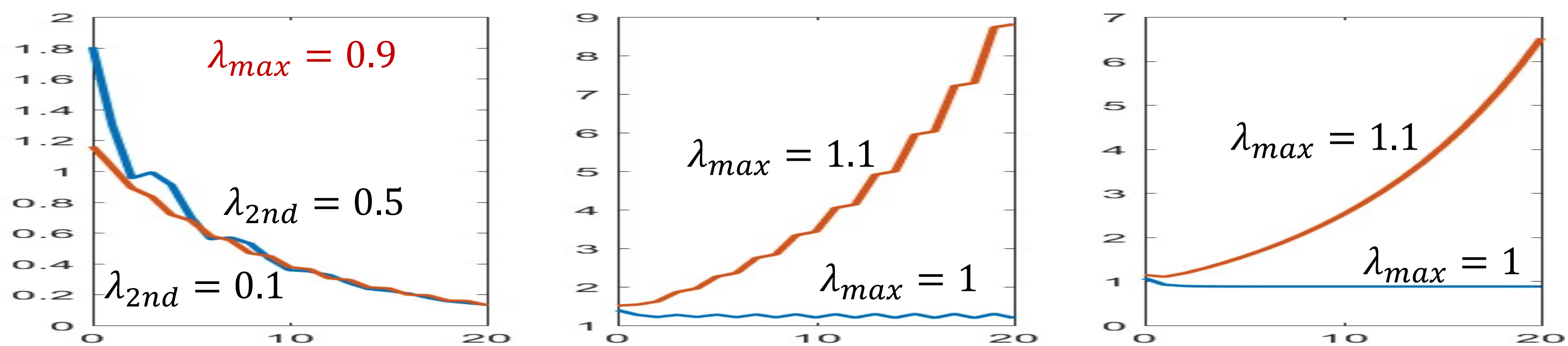

Complex Eigenvalues