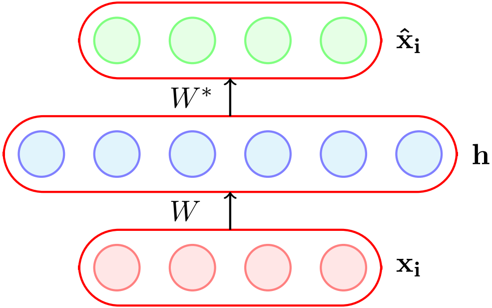
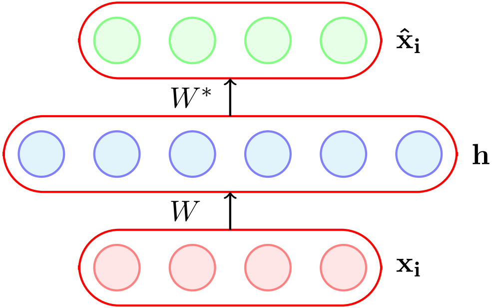
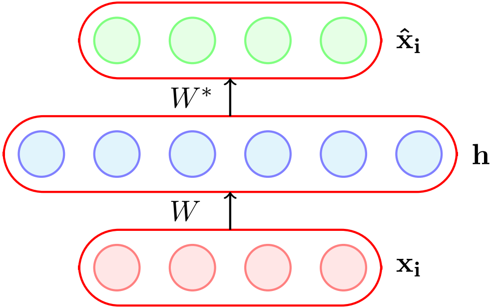


$\hat{\mathbf{x}}_{\mathbf{i}}$
W^*
$\mathbf{h}$
W
$\mathbf{x}_{\mathbf{i}}$



- While poor generalization could happen even in undercomplete autoencoders it is an even more serious problem for overcomplete auto encoders



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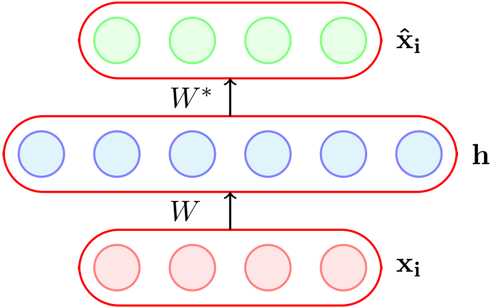

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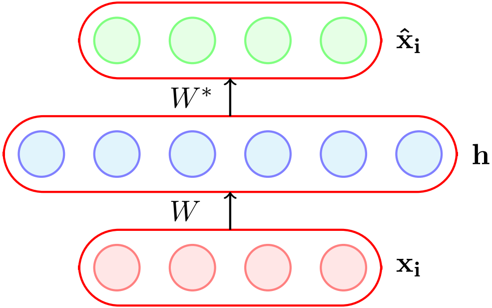
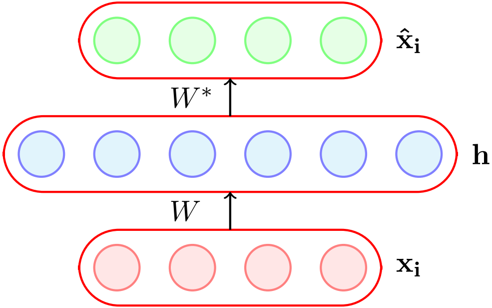
- Here, (as stated earlier) the model can simply learn to copy $\mathbf{x_i}$ to $\mathbf{h}$ and then $\mathbf{h}$ to $\hat{\mathbf{x}}_\mathbf{i}$
- To avoid poor generalization, we need to introduce regularization



- The simplest solution is to add a L_2-regularization term to the objective function

$$\min_{\theta, w, w^*, \mathbf{b}, \mathbf{c}} \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} (\hat{x}_{ij} - x_{ij})^2 + \lambda \|\theta\|^2$$



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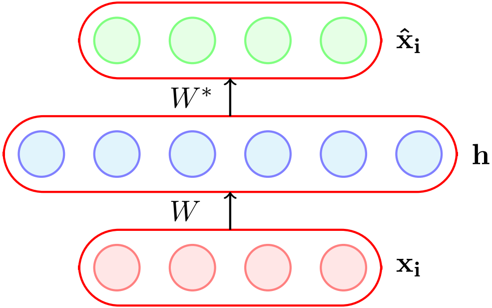
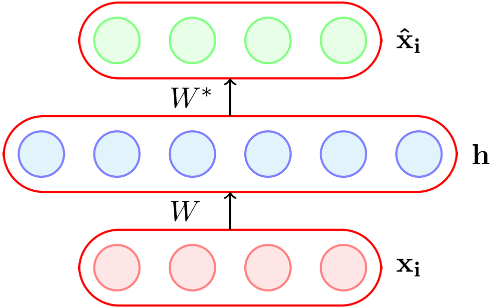
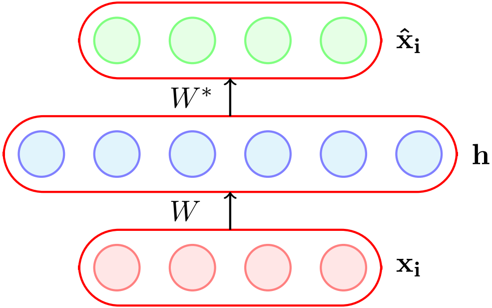
- This is very easy to implement and just adds a term λW to the gradient $\frac{\partial \mathscr{L}(\theta)}{\partial W}$ (and similarly for other parameters)



- Another trick is to tie the weights of the encoder and decoder

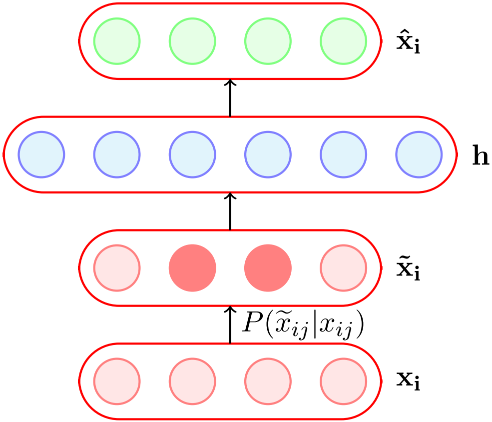


- Another trick is to tie the weights of the encoder and decoder i.e., $W^* = W^T$

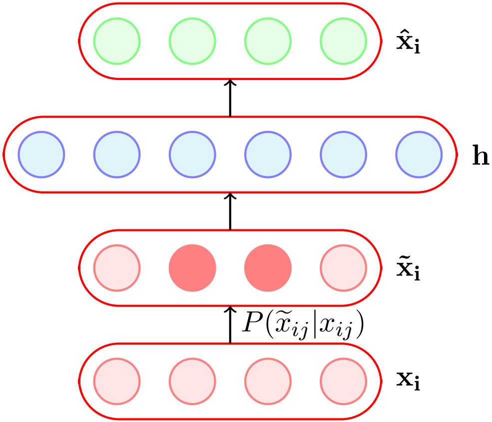


- Another trick is to tie the weights of the encoder and decoder i.e., $W^* = W^T$
- This effectively reduces the capacity of Autoencoder and acts as a regularizer

Module 7.4: Denoising Autoencoders



- A denoising encoder simply corrupts the input data using a probabilistic process ($P(\widetilde{x}_{ij}|x_{ij})$) before feeding it to the network



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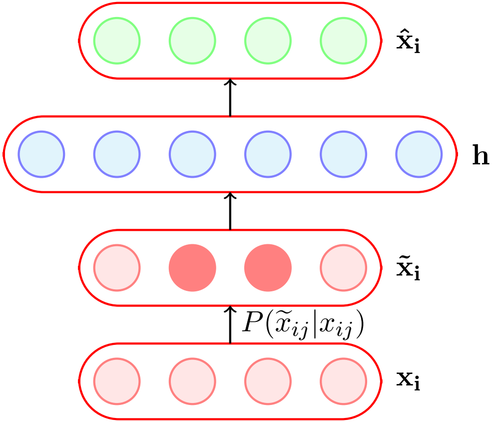

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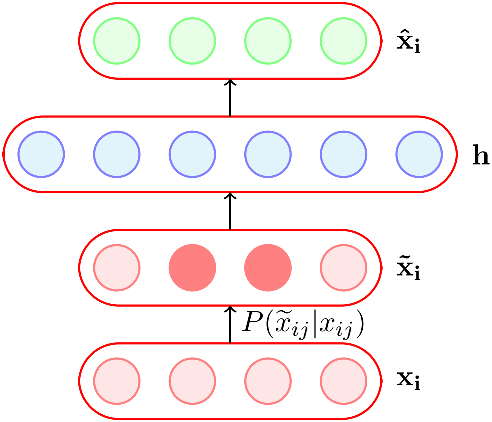

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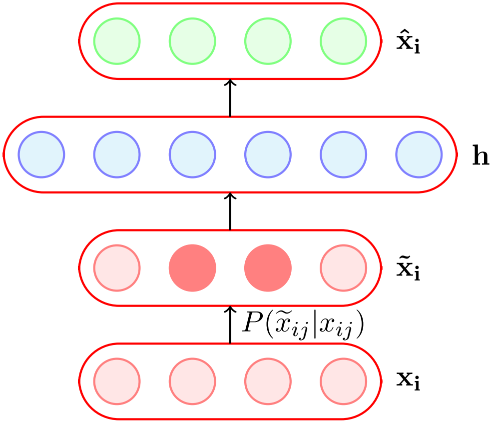

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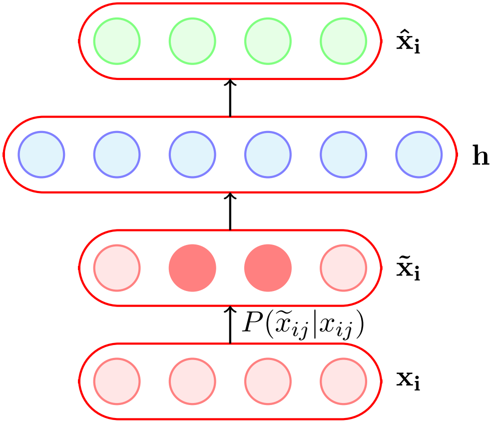
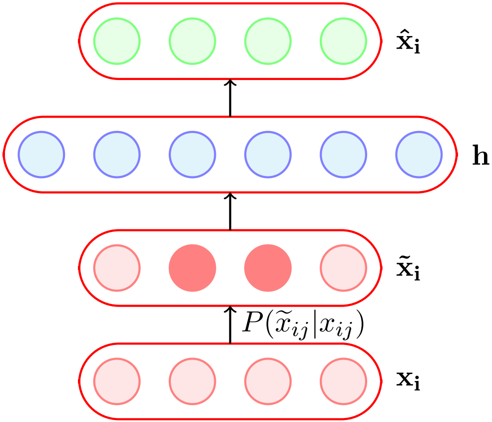
- In other words, with probability q the input is flipped to 0 and with probability $(1 - q)$ it is retained as it is

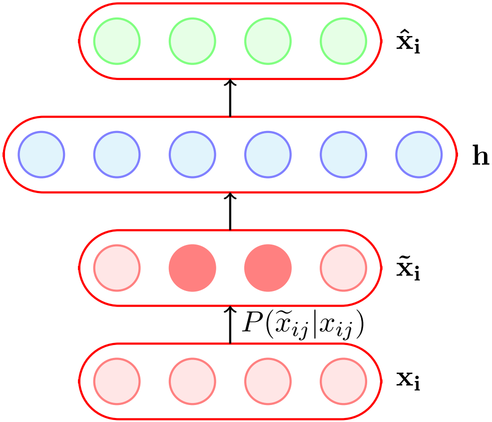


- How does this help ?



- How does this help ?
- This helps because the objective is still to reconstruct the original (uncorrupted) $\mathbf{x}_i$

$$\arg\min_{\theta} \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} (\hat{x}_{ij} - x_{ij})^2$$



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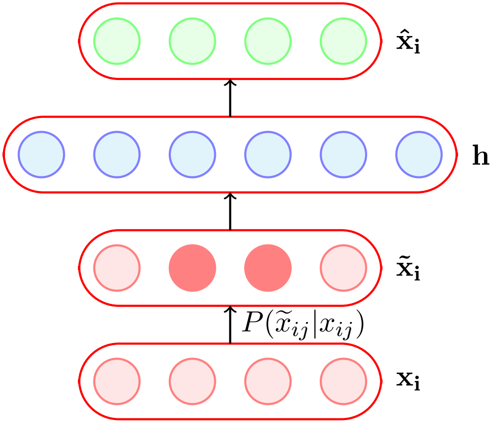

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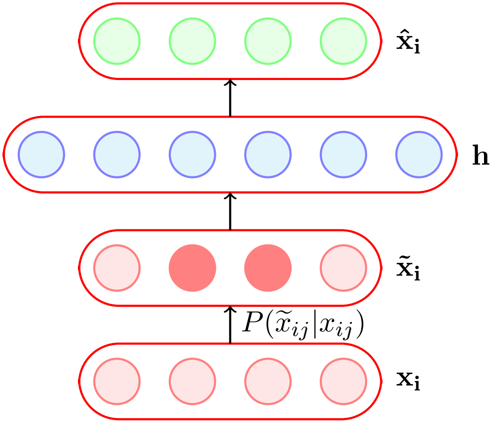

For example, it will have to learn to reconstruct a corrupted x_{ij} correctly by relying on its interactions with other elements of $\mathbf{x}_i$

- How does this help ?
- This helps because the objective is still to reconstruct the original (un-corrupted) $\mathbf{x}_i$

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We will now see a practical application in which AEs are used and then compare Denoising Autoencoders with regular autoencoders

Task: Hand-written digit recognition

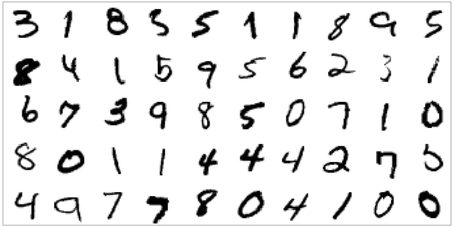


Figure: MNIST Data

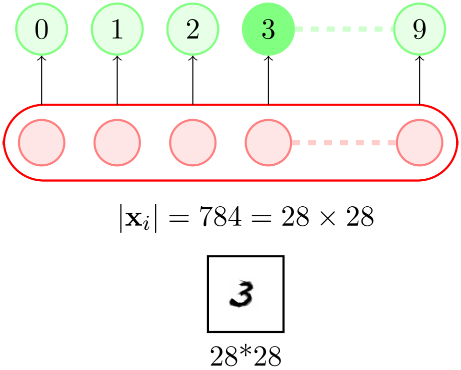
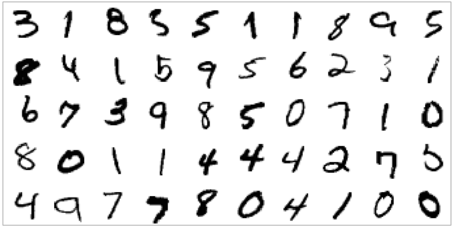


Figure: Basic approach(we use raw data as input features)

Task: Hand-written digit recognition



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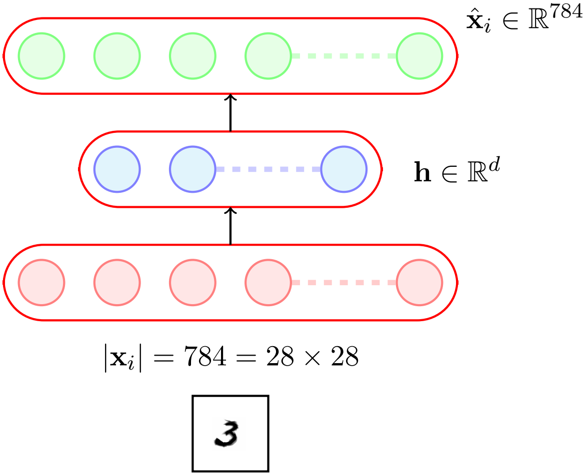
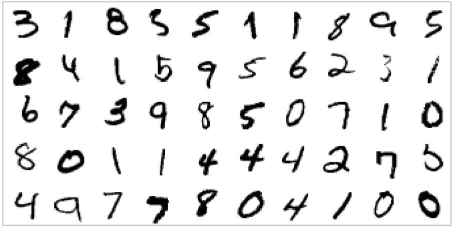


Figure: AE approach (first learn important characteristics of data)

Task: Hand-written digit recognition



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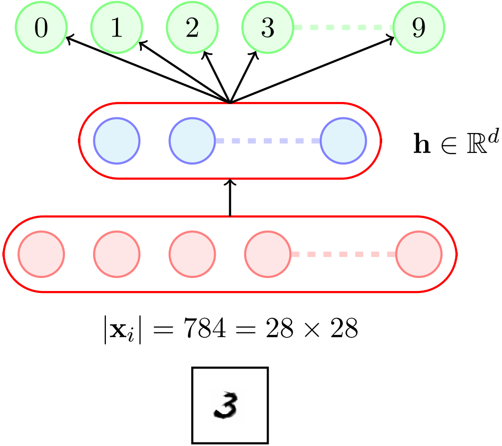
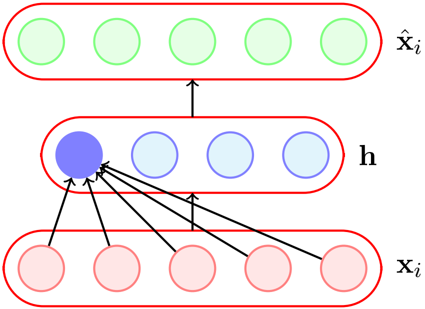
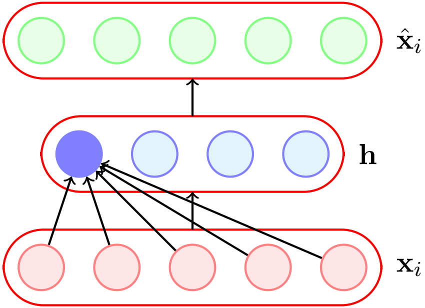


Figure: AE approach (and then train a classifier on top of this hidden representation)

We will now see a way of visualizing AEs and use this visualization to compare different AEs



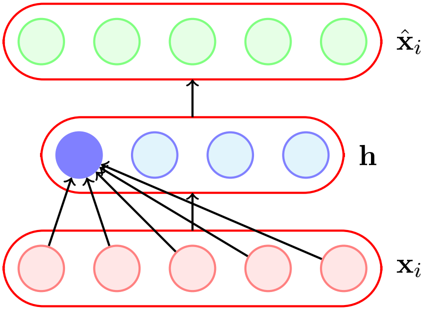
- We can think of each neuron as a filter which will fire (or get maximally) activated for a certain input configuration $\mathbf{x}_i$



- We can think of each neuron as a filter which will fire (or get maximally) activated for a certain input configuration $\mathbf{x}_i$
- For example,

$$\mathbf{h}_1 = \sigma(W_1^T \mathbf{x}_i) \ [ignoring\ bias\ b]$$

Where W_1 is the trained vector of weights connecting the input to the first hidden neuron



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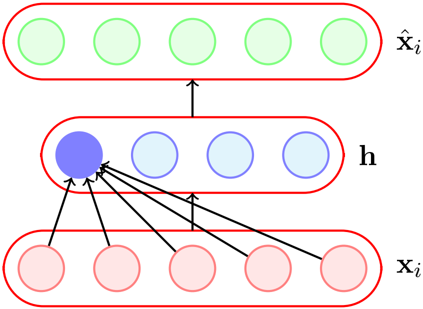
- What values of $\mathbf{x}_i$ will cause $\mathbf{h}_1$ to be maximum (or maximally activated)



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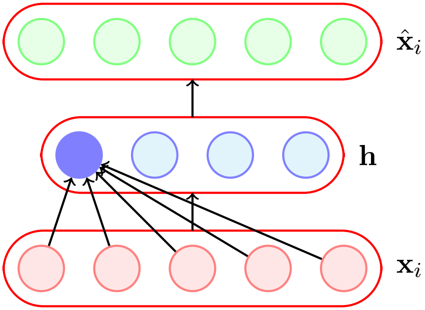
- What values of $\mathbf{x}_i$ will cause $\mathbf{h}_1$ to be maximum (or maximally activated)
- Suppose we assume that our inputs are normalized so that $\|\mathbf{x}_i\| = 1$



$$\max_{\mathbf{x}_i} \quad \{W_1^T \mathbf{x}_i\}$$
$$s.t. \quad ||\mathbf{x}_i||^2 = \mathbf{x}_i^T \mathbf{x}_i = 1$$

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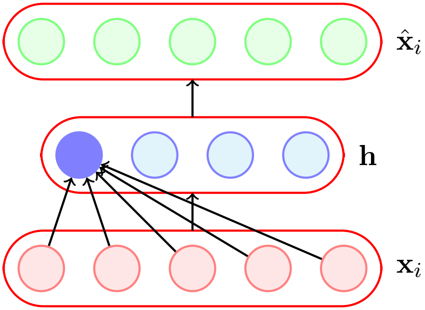

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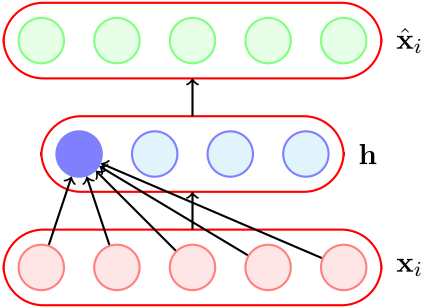

- Thus the inputs

$$\mathbf{x}_i = \frac{W_1}{\sqrt{W_1^T W_1}}, \frac{W_2}{\sqrt{W_2^T W_2}}, \ldots \frac{W_n}{\sqrt{W_n^T W_n}}$$

will respectively cause hidden neurons 1 to n to maximally fire

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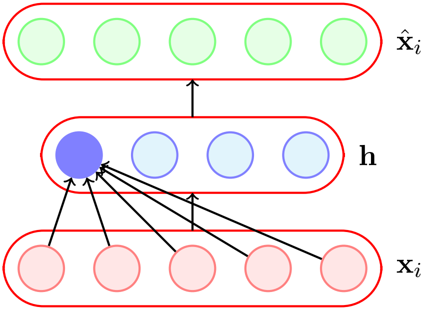

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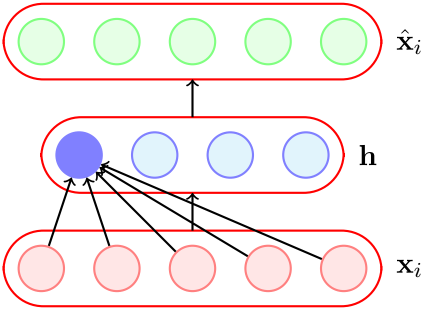
- Let us plot these images ($\mathbf{x}_i$'s) which maximally activate the first k neurons of the hidden representations learned by a vanilla autoencoder and different denoising autoencoders



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- These $\mathbf{x}_i$'s are computed by the above formula using the weights $(W_1, W_2 \ldots W_k)$ learned by the respective autoencoders

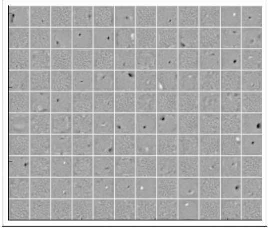


Figure: Vanilla AE (No noise)

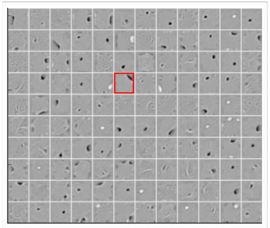


Figure: 25% Denoising AE (q=0.25)

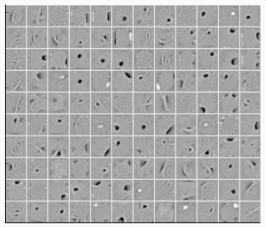
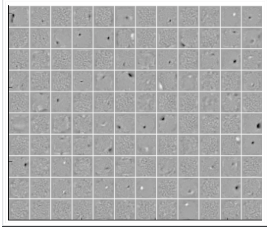


Figure: 50% Denoising AE (q=0.5)

- The vanilla AE does not learn many meaningful patterns



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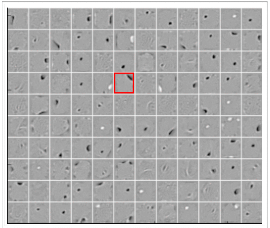

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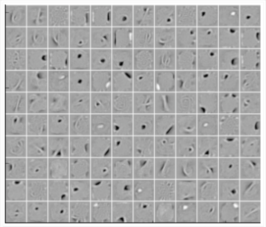

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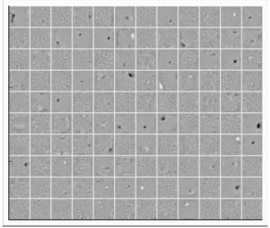
- The vanilla AE does not learn many meaningful patterns
- The hidden neurons of the denoising AEs seem to act like pen-stroke detectors (for example, in the highlighted neuron the black region is a stroke that you would expect in a '0' or a '2' or a '3' or a '8' or a '9')



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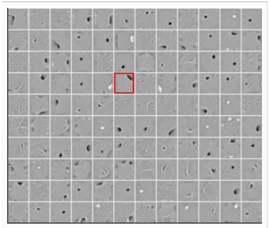

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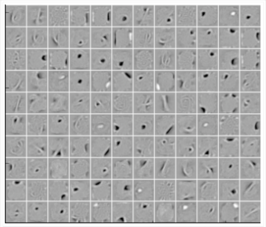

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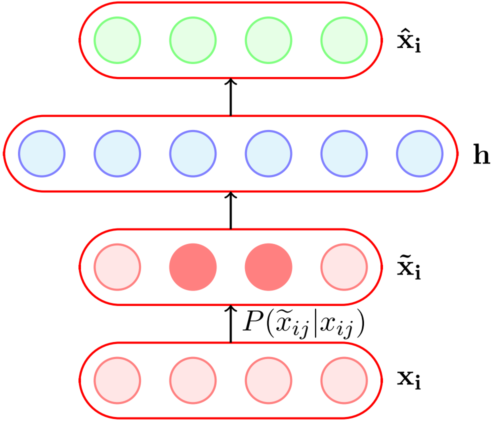
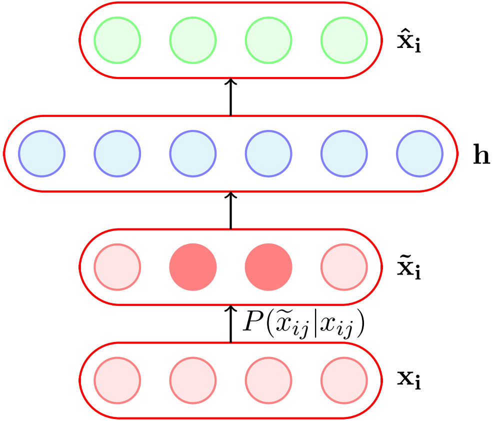
- As the noise increases the filters become more wide because the neuron has to rely on more adjacent pixels to feel confident about a stroke



- We saw one form of $P(\widetilde{x}_{ij}|x_{ij})$ which flips a fraction q of the inputs to zero



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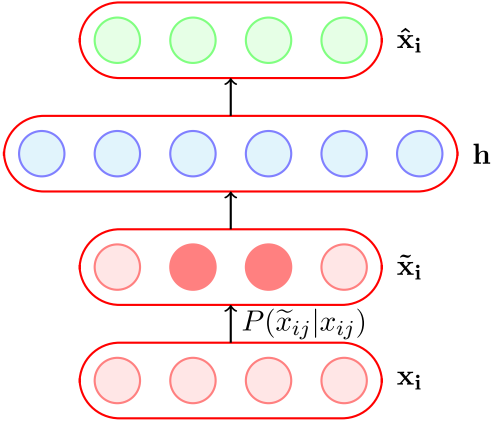
$$\widetilde{x}_{ij} = x_{ij} + \mathcal{N}(0, 1)$$



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- We will now use such a denoising AE on a different dataset and see their performance

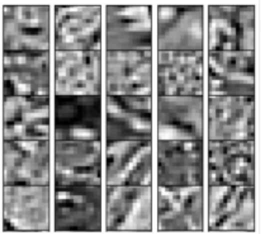


Figure: Data

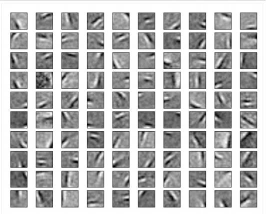


Figure: AE filters

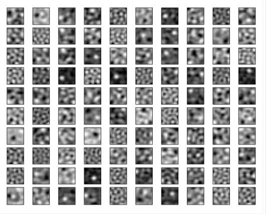


Figure: Weight decay filters

- The hidden neurons essentially behave like edge detectors

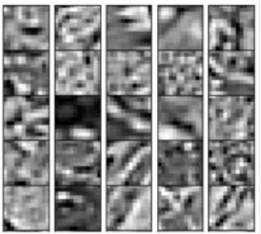


Figure: Data

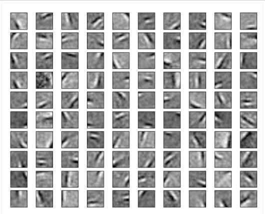


Figure: AE filters

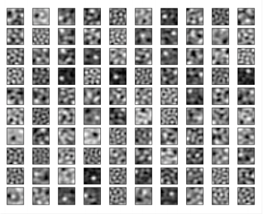
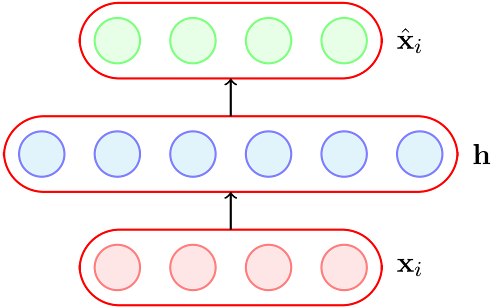
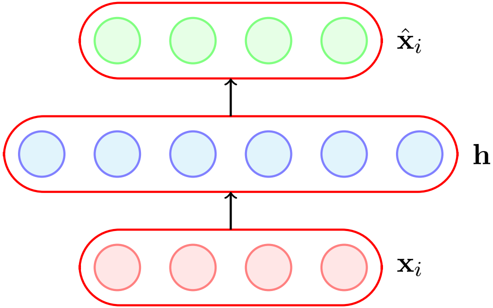


Figure: Weight decay filters

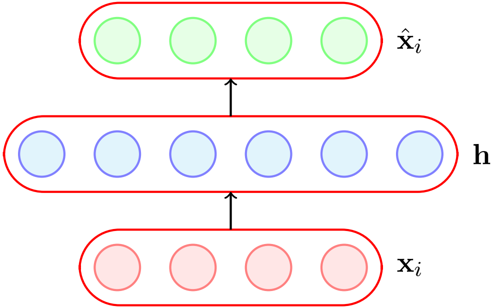
- The hidden neurons essentially behave like edge detectors
- PCA does not give such edge detectors

Module 7.5: Sparse Autoencoders

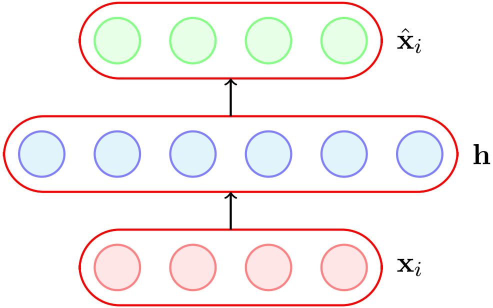

$\hat{\mathbf{x}}_i$
$\mathbf{h}$
$\mathbf{x}_i$



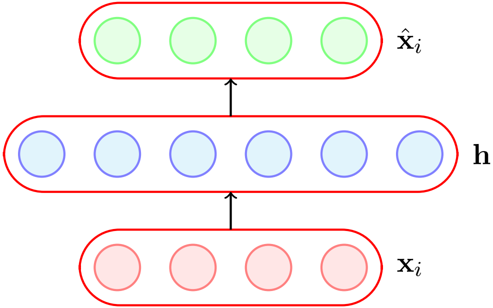
- A hidden neuron with sigmoid activation will have values between 0 and 1



- A hidden neuron with sigmoid activation will have values between 0 and 1
- We say that the neuron is activated when its output is close to 1 and not activated when its output is close to 0.



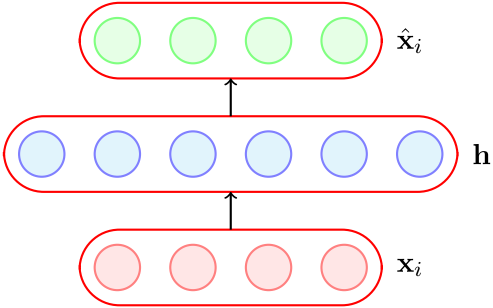
- A hidden neuron with sigmoid activation will have values between 0 and 1
- We say that the neuron is activated when its output is close to 1 and not activated when its output is close to 0.
- A sparse autoencoder tries to ensure the neuron is inactive most of the times.



The average value of the activation of a neuron l is given by

$$\hat{\rho}_l = \frac{1}{m}\sum_{i=1}^{m} h(\mathbf{x}_i)_l$$

- If the neuron l is sparse (i.e. mostly inactive) then $\hat{\rho}_l \rightarrow 0$



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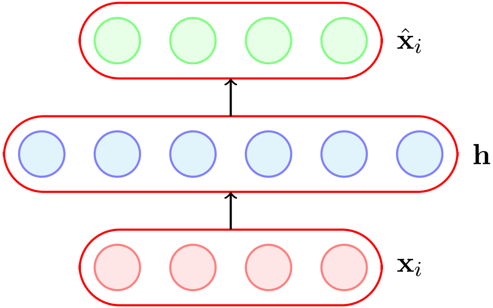
- If the neuron l is sparse (i.e. mostly inactive) then $\hat{\rho}_l \to 0$
- A sparse autoencoder uses a sparsity parameter ρ (typically very close to 0, say, 0.005) and tries to enforce the constraint $\hat{\rho}_l = \rho$



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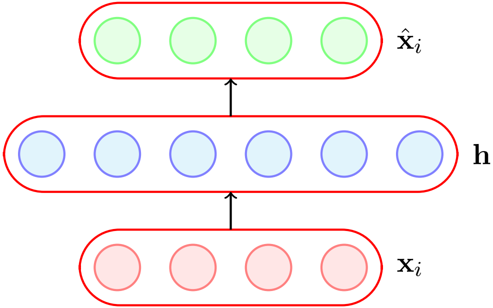
- One way of ensuring this is to add the following term to the objective function

$$\Omega(\theta) = \sum_{l=1}^{k} \rho \log \frac{\rho}{\hat{\rho}_l} + (1-\rho)\log\frac{1-\rho}{1-\hat{\rho}_l}$$



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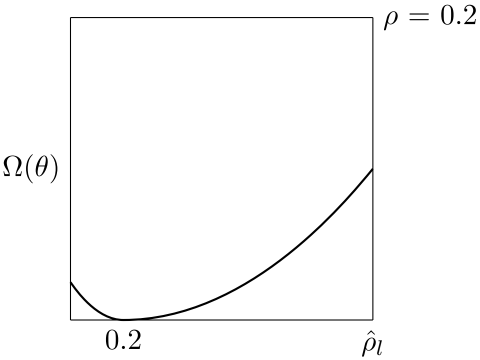
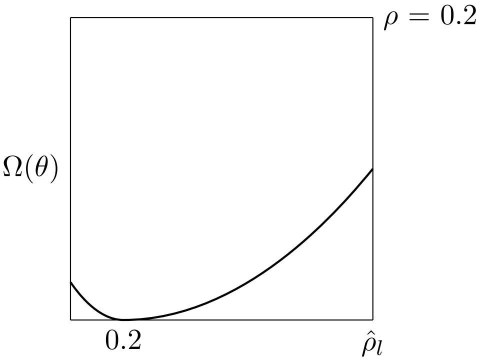
- When will this term reach its minimum value and what is the minimum value? Let us plot it and check.


$\rho = 0.2$
$\Omega(\theta)$
0.2
$\hat{\rho}_l$



- The function will reach its minimum value(s) when $\hat{\rho}_l = \rho$.

- Now,

$$\hat{\mathscr{L}}(\theta) = \mathscr{L}(\theta) + \Omega(\theta)$$

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For each neuron $l \in 1 \ldots k$ in hidden layer, we have

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and $\frac{\partial \hat{\rho}_l}{\partial W} = \mathbf{x}_i (g'(W^T \mathbf{x}_i + \mathbf{b}))^T$ (see next slide)

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- Finally,

$$\frac{\partial \hat{\mathscr{L}}(\theta)}{\partial W} = \frac{\partial \mathscr{L}(\theta)}{\partial W} + \frac{\partial \Omega(\theta)}{\partial W}$$

(and we know how to calculate both terms on R.H.S)

Derivation

$$\frac{\partial \hat{\rho}}{\partial W} = \begin{bmatrix} \frac{\partial \hat{\rho}_1}{\partial W} & \frac{\partial \hat{\rho}_2}{\partial W} \dots \frac{\partial \hat{\rho}_k}{\partial W} \end{bmatrix}$$

For each element in the above equation we can calculate $\frac{\partial \hat{\rho}_l}{\partial W}$ (which is the partial derivative of a scalar w.r.t. a matrix = matrix). For a single element of a matrix W_{jl}:-

$$\frac{\partial \hat{\rho}_l}{\partial W_{jl}} = \frac{\partial \left[\frac{1}{m} \sum_{i=1}^{m} g\left(W_{:,l}^T \mathbf{x_i} + b_l\right) \right]}{\partial W_{jl}}$$

$$= \frac{1}{m} \sum_{i=1}^{m} \frac{\partial \left[g\left(W_{:,l}^T \mathbf{x_i} + b_l\right) \right]}{\partial W_{jl}}$$

$$= \frac{1}{m} \sum_{i=1}^{m} g'\left(W_{:,l}^T \mathbf{x_i} + b_l\right) x_{ij}$$

So in matrix notation we can write it as :

$$\frac{\partial \hat{\rho}_l}{\partial W} = \mathbf{x}_i (g'(W^T \mathbf{x}_i + \mathbf{b}))^T$$