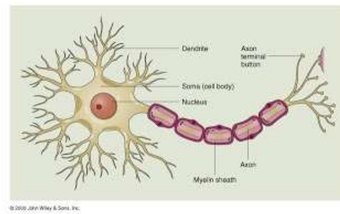
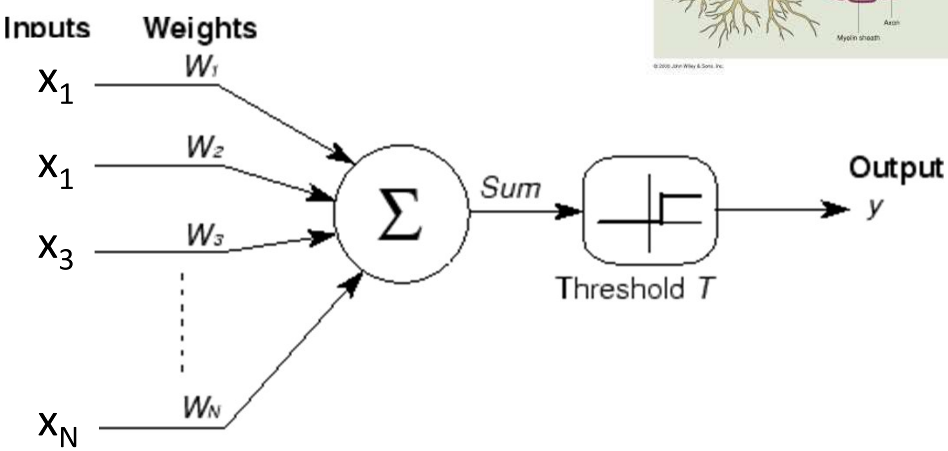
Recap: the perceptron

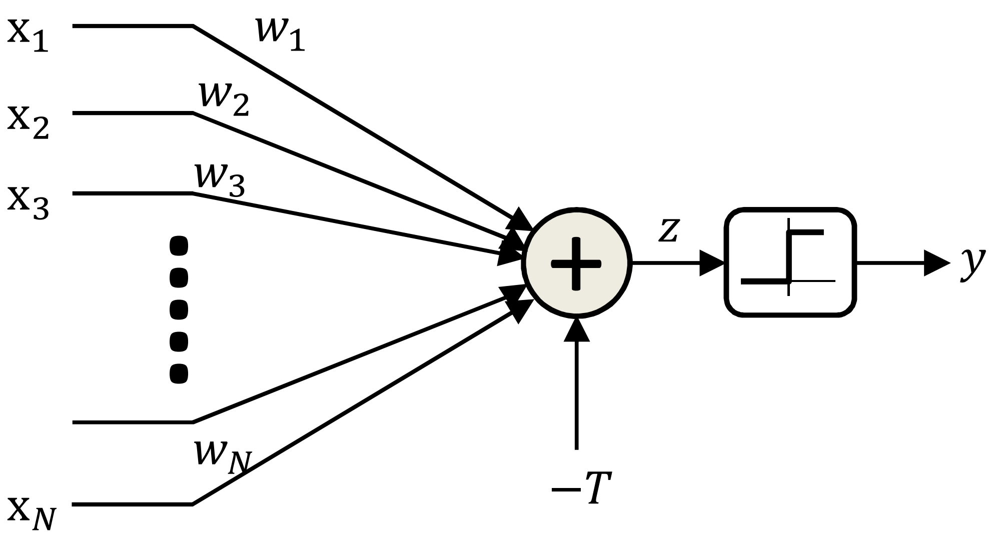




$$y = \begin{cases} 1 \; if \; \sum_i w_i x_i \geq T \\ 0 \; else \end{cases}$$

- A threshold unit
  - "Fires" if the weighted sum of inputs exceeds a threshold
  - Electrical engineers will call this a *threshold gate*
    - A basic unit of Boolean circuits

A better figure

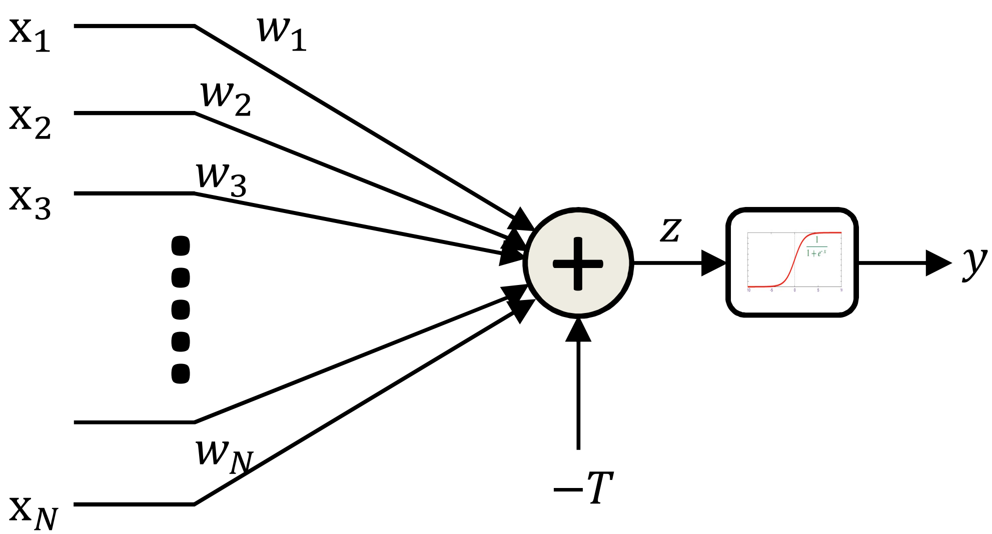


$$z = \sum_i w_i x_i - T$$

$$y = \begin{cases} 1 \; if \; z \geq 0 \\ 0 \; else \end{cases}$$

- A threshold unit
  - "Fires" if the weighted sum of inputs and the "bias" T is positive

The "soft" perceptron (logistic)

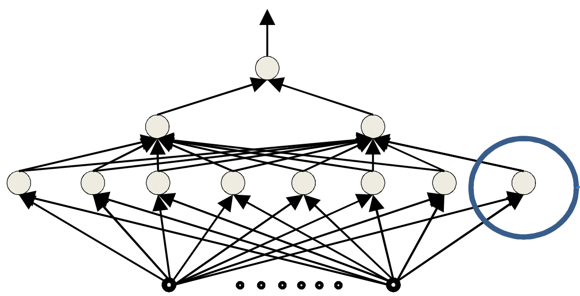


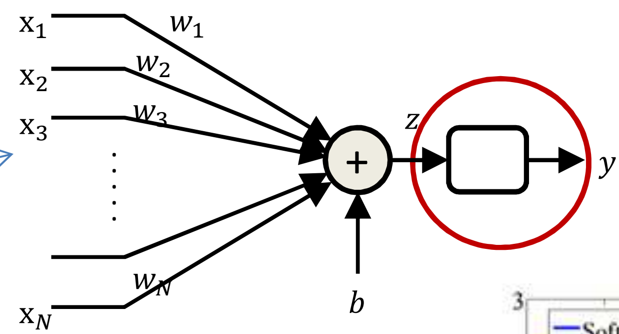
$$z = \sum_i w_i x_i - T$$

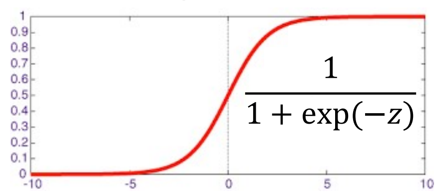
$$y = \frac{1}{1 + exp(-z)}$$

- A "squashing" function instead of a threshold at the output
  - The sigmoid "activation" replaces the threshold
    - **Activation:** The function that acts on the weighted combination of inputs (and threshold)

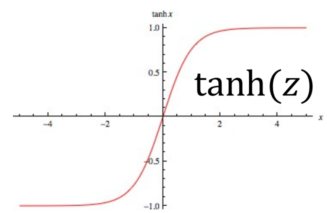
Other "activations"



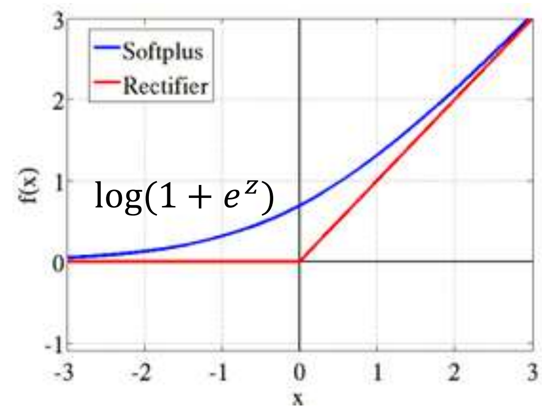




sigmoid

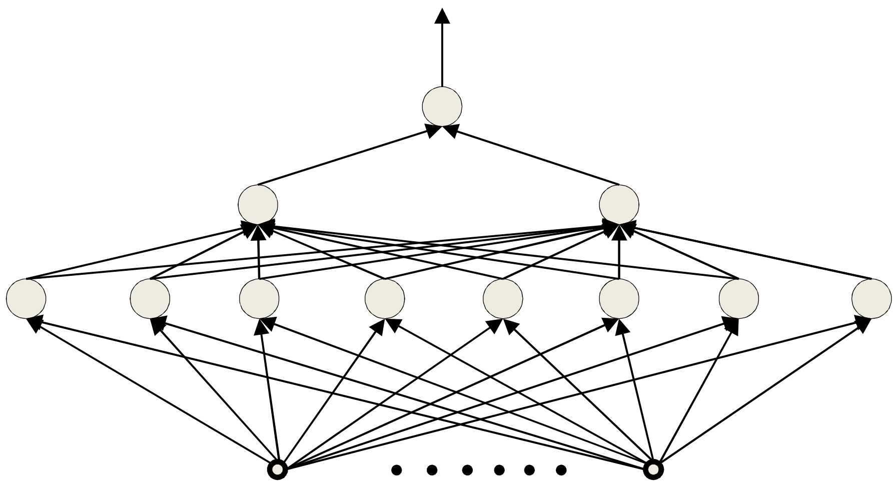


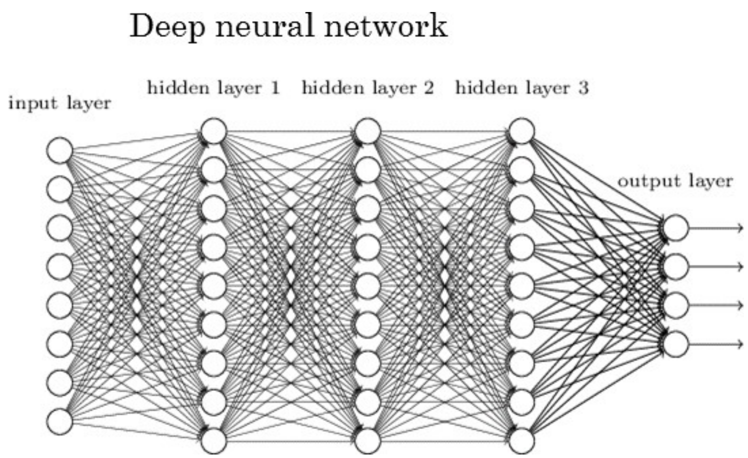
tanh

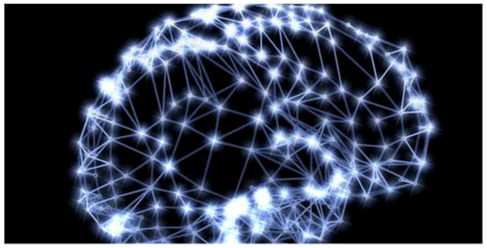


- Does not always have to be a squashing function
  - We will hear more about activations later
- We will continue to assume a "threshold" activation in this lecture

The *multi-layer* perceptron

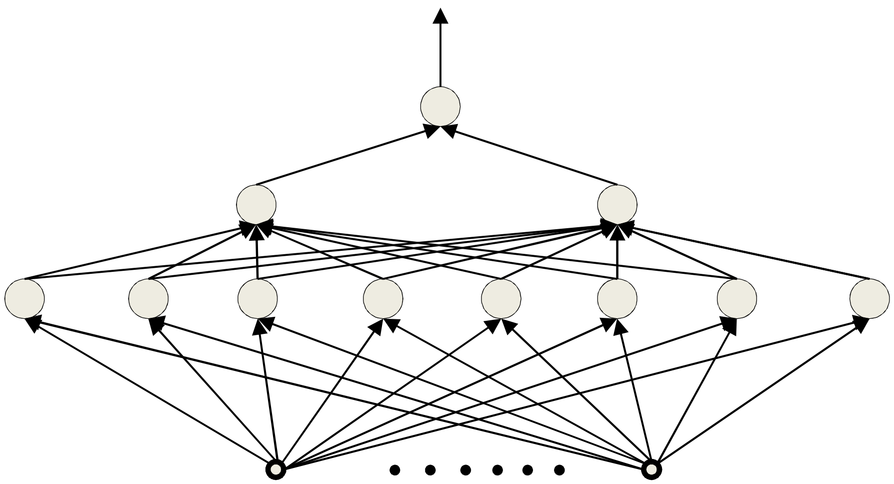


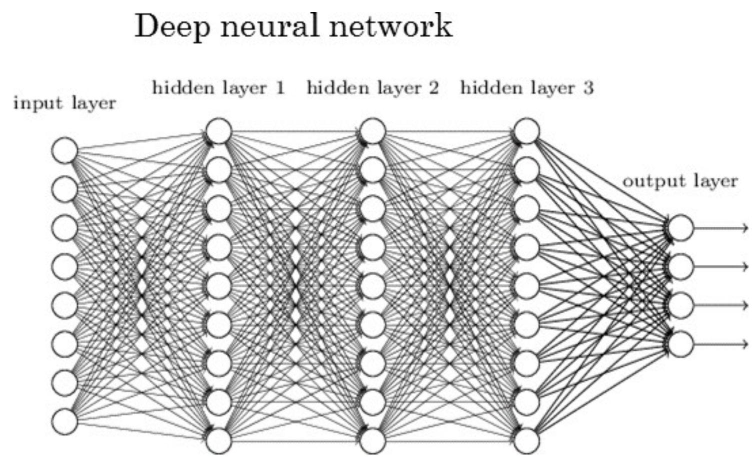




- A network of perceptrons
  - Generally "layered"

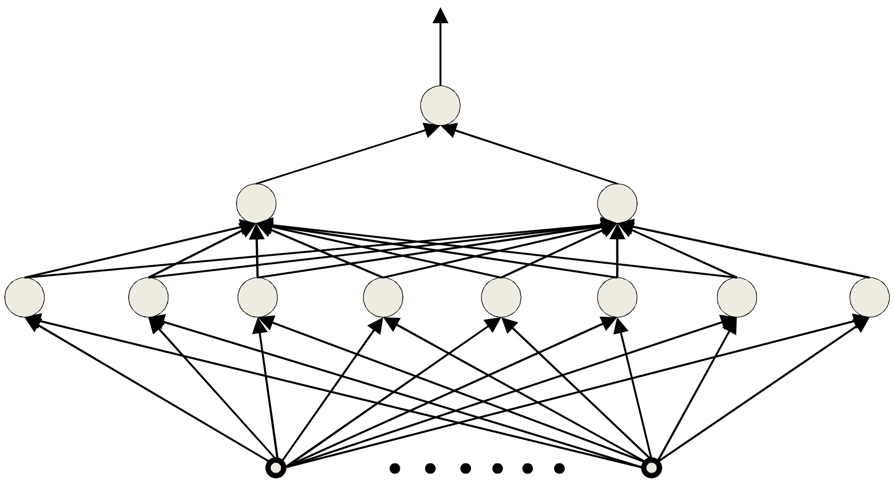
Defining "depth"

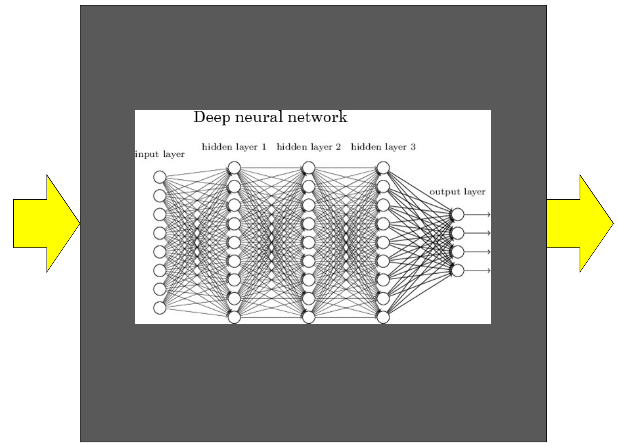




- What is a "deep" network

The multi-layer perceptron



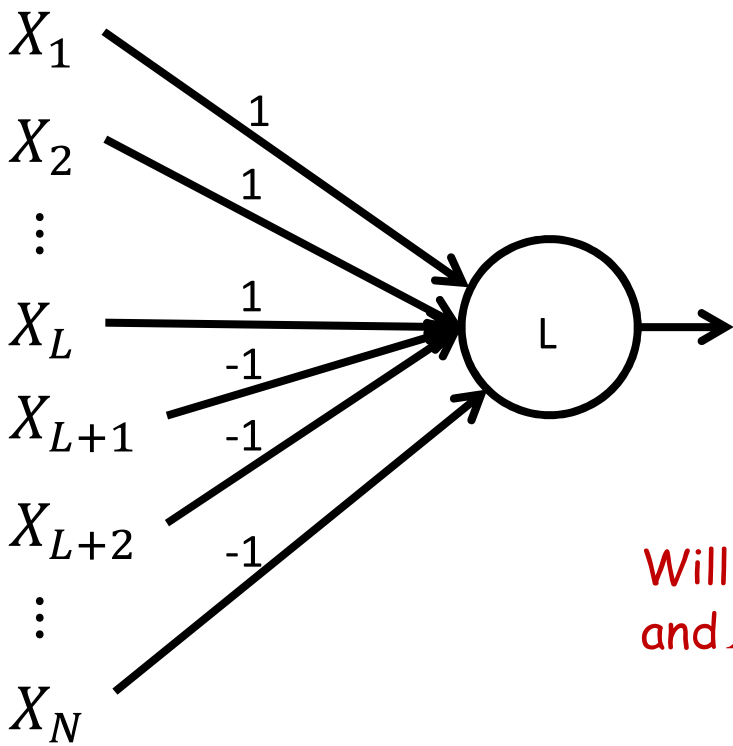


- Inputs are real or Boolean stimuli
- Outputs are real or Boolean values
  - Can have multiple outputs for a single input
- **What can this network compute?**
  - **What kinds of input/output relationships can it model?**

The MLP as a Boolean function

- How well do MLPs model Boolean functions?

Perceptron as a Boolean gate

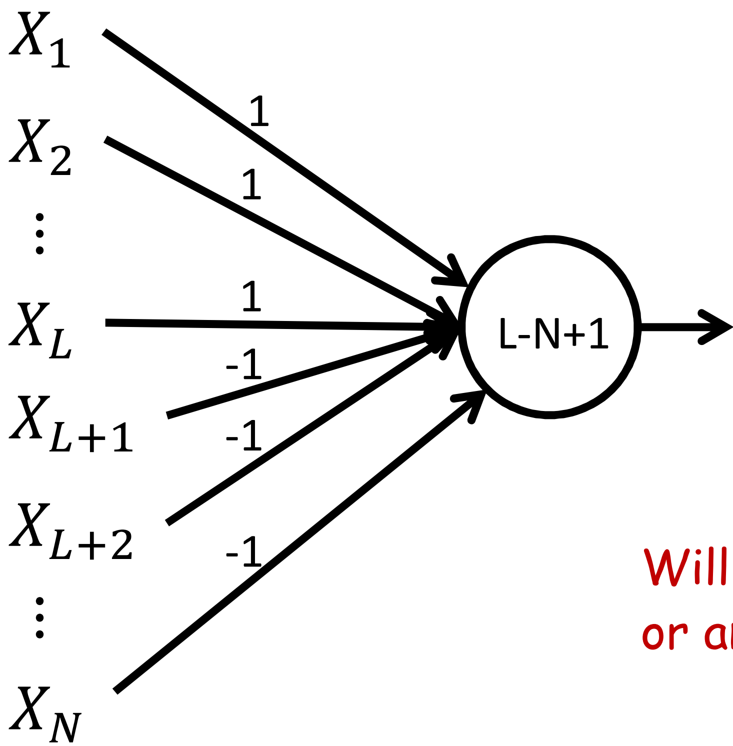


$$\left(\bigwedge_{i=1}^{L} X_i\right) \wedge \left(\bigwedge_{i=L+1}^{N} \bar{X}_i\right)$$

Will fire only if X_1 .. X_L are all 1 and X_{L+1} .. X_N are all 0

- The universal AND gate
  - AND any number of inputs
    - Any subset of who may be negated

Perceptron as a Boolean gate

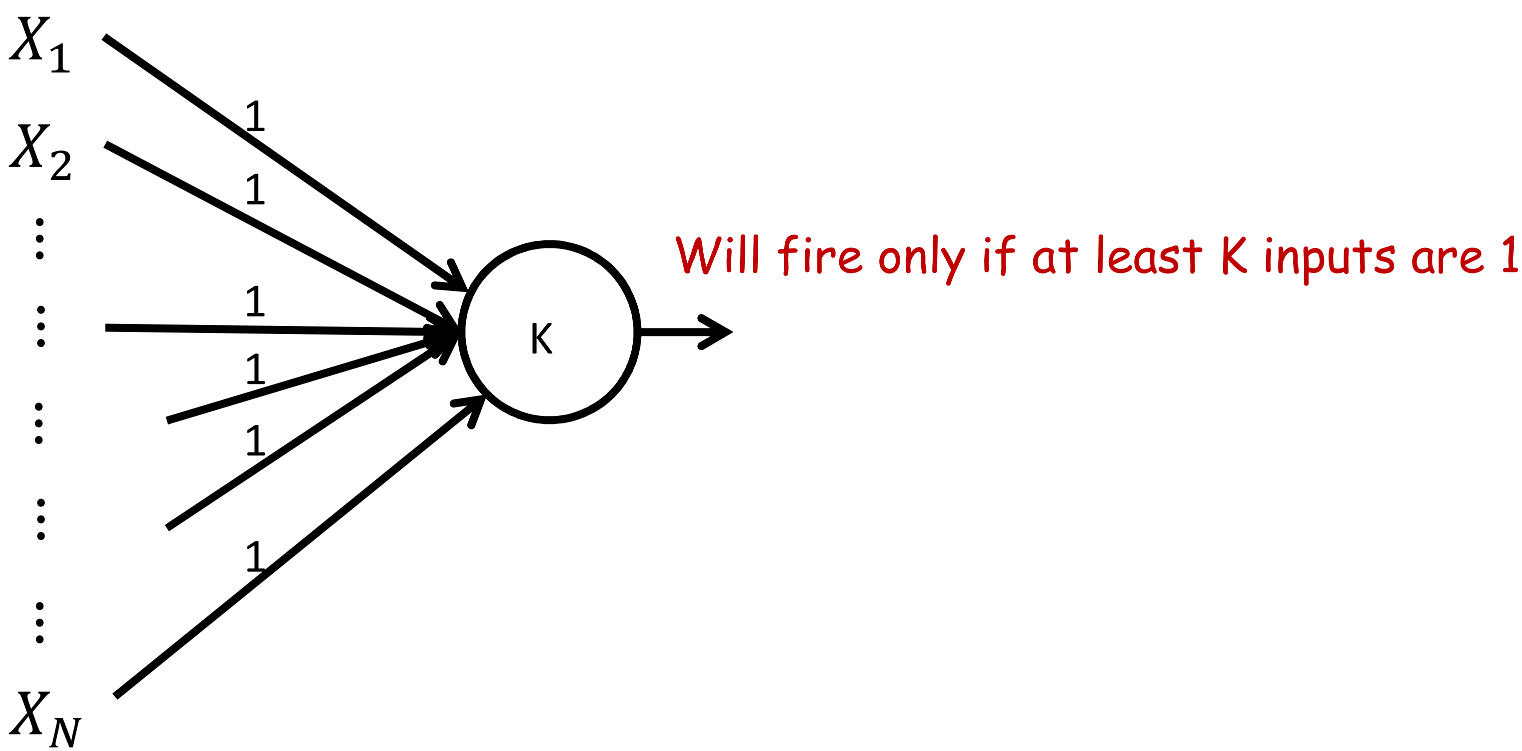


$$\left(\bigvee_{i=1}^{L} X_i\right) \vee \left(\bigvee_{i=L+1}^{N} \bar{X}_i\right)$$

Will fire only if any of X_1 .. X_L are 1 or any of X_{L+1} .. X_N are 0

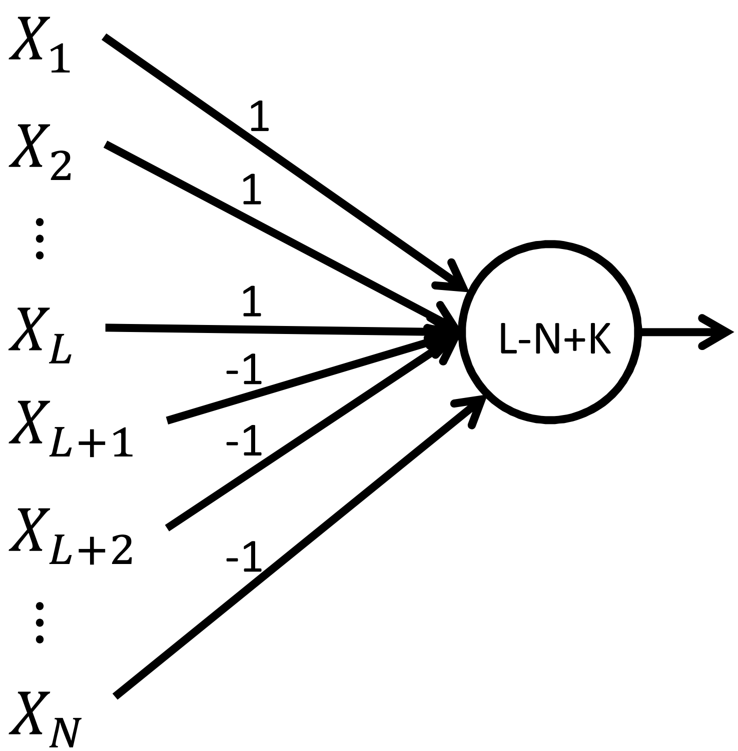
- The universal OR gate
  - OR any number of inputs
    - Any subset of who may be negated

Perceptron as a Boolean Gate



- Generalized *majority* gate
  - Fire if at least *K* inputs are of the desired polarity

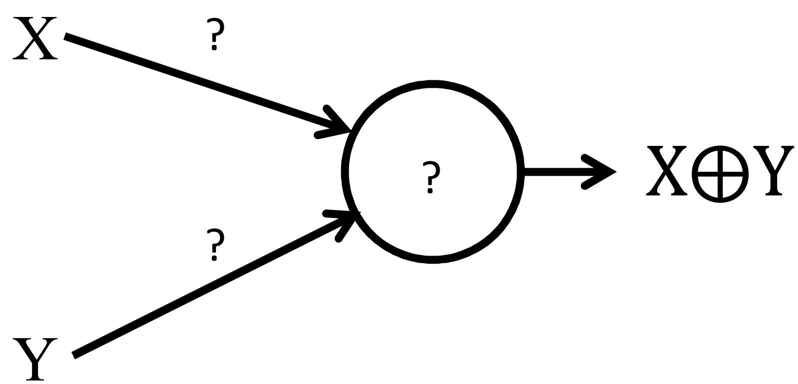
Perceptron as a Boolean Gate



Will fire only if the total number of of X_1 .. X_L that are 1 or X_{L+1} .. X_N that are 0 is at least K

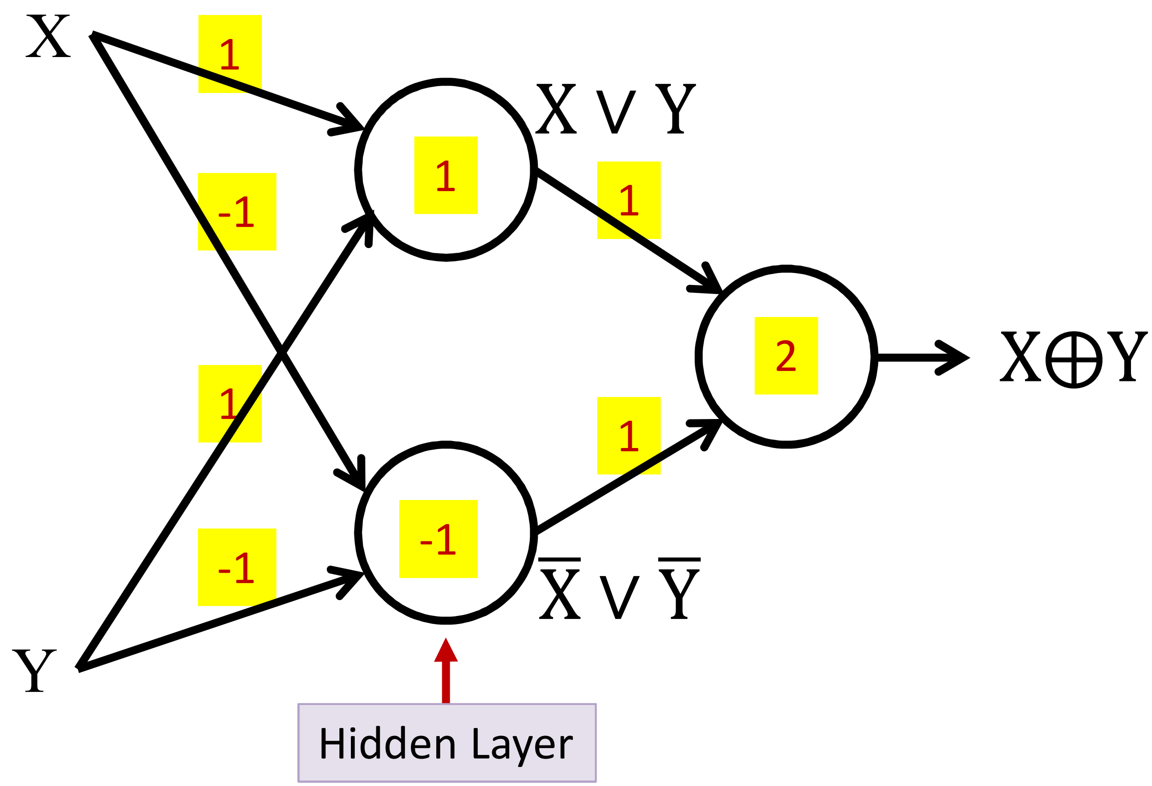
- Generalized *majority* gate
  - Fire if at least *K* inputs are of the desired polarity

The perceptron is not enough



- Cannot compute an XOR

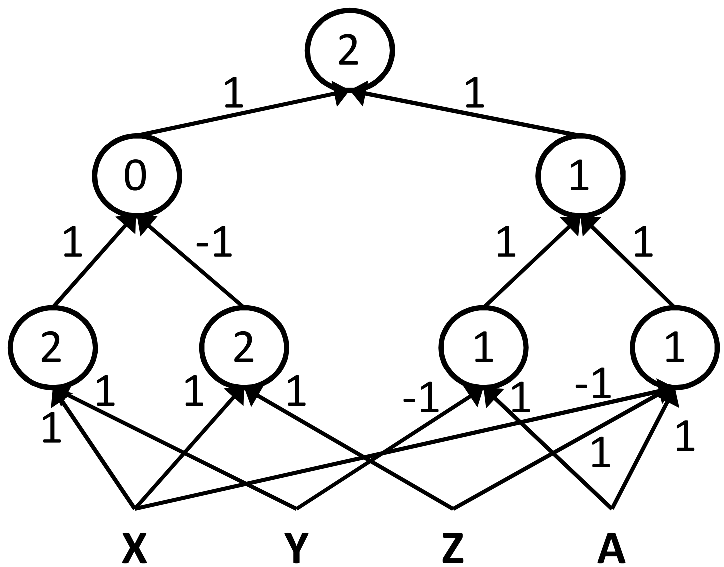
Multi-layer perceptron



- MLPs can compute the XOR

Multi-layer perceptron

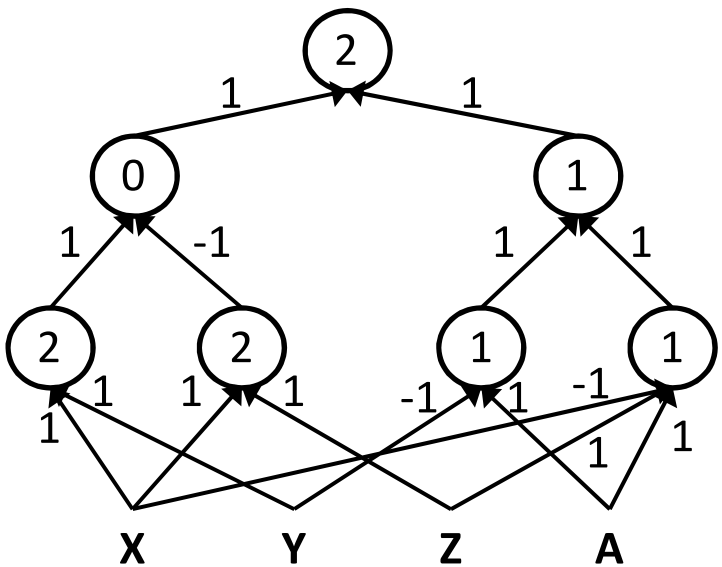
$$((A\&\bar{X}\&Z)|(A\&\bar{Y}))\&((X\ \&\ Y)|\overline{(X\&Z)})$$

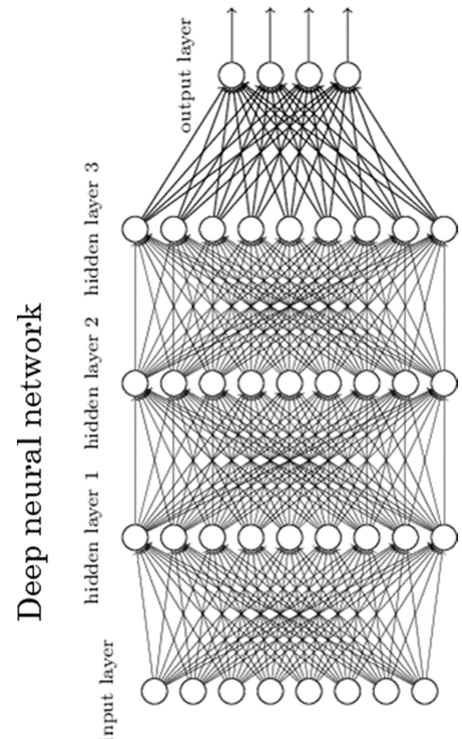


- MLPs can compute more complex Boolean functions
- MLPs can compute *any* Boolean function
  – Since they can emulate individual gates
- **MLPs are *universal Boolean functions***

MLP as Boolean Functions

$((A \& \bar{X} \& Z) | (A \& \bar{Y})) \& ((X \,\&\, Y) | \overline{(X \& Z)})$





- MLPs are universal Boolean functions
  - Any function over any number of inputs and any number of outputs
- But how many “layers” will they need?

How many layers for a Boolean MLP?

Truth table shows *all* input combinations for which output is 1

Truth Table

X_1	X_2	X_3	X_4	X_5	Y
0	0	1	1	0	1
0	1	0	1	1	1
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	1	1	1
1	1	0	0	1	1

- A Boolean function is just a truth table

How many layers for a Boolean MLP?

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$$Y = \bar{X}_1\bar{X}_2X_3X_4\bar{X}_5 + \bar{X}_1X_2\bar{X}_3X_4X_5 + \bar{X}_1X_2X_3\bar{X}_4\bar{X}_5 + X_1\bar{X}_2\bar{X}_3\bar{X}_4X_5 + X_1\bar{X}_2X_3X_4X_5 + X_1X_2\bar{X}_3\bar{X}_4X_5$$

- Expressed in disjunctive normal form

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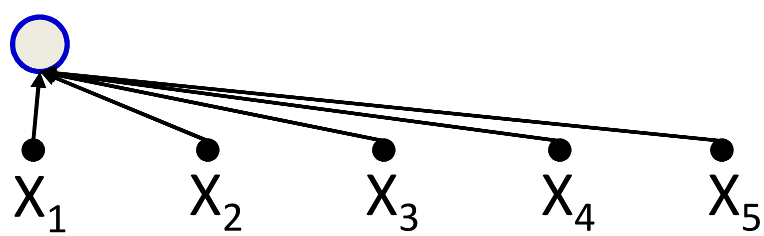

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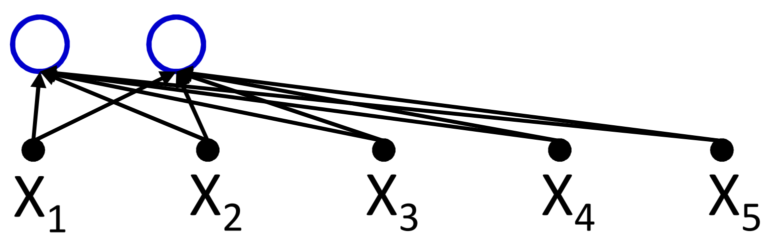

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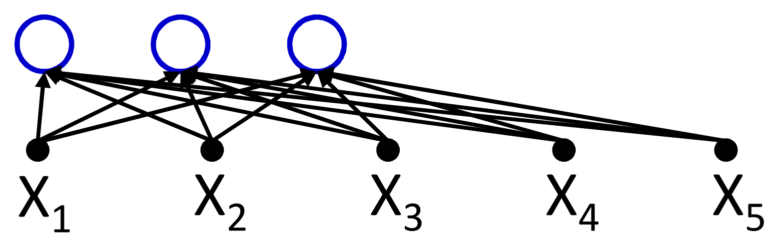

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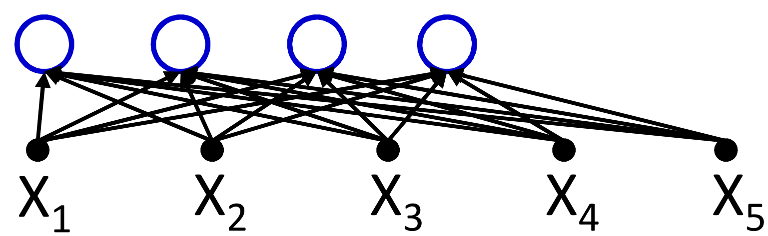

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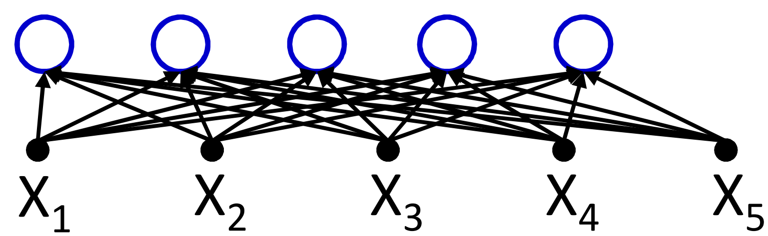

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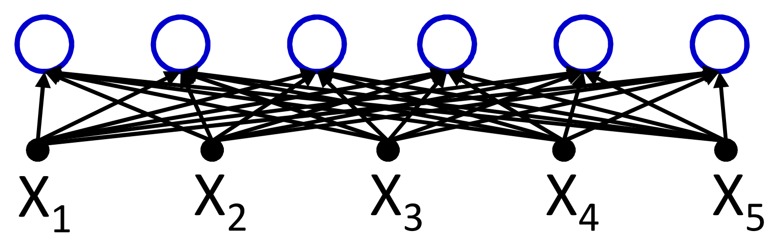

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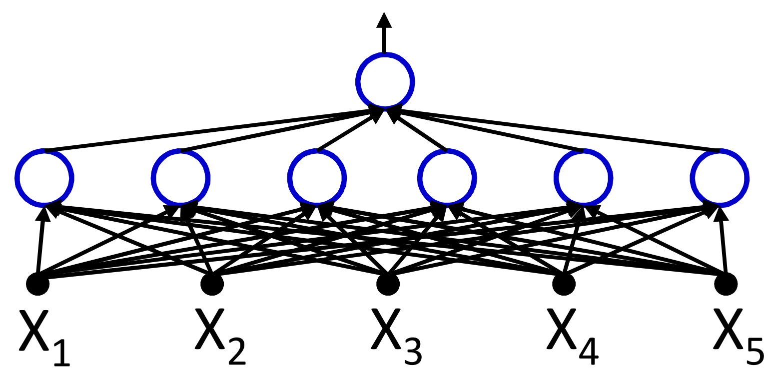

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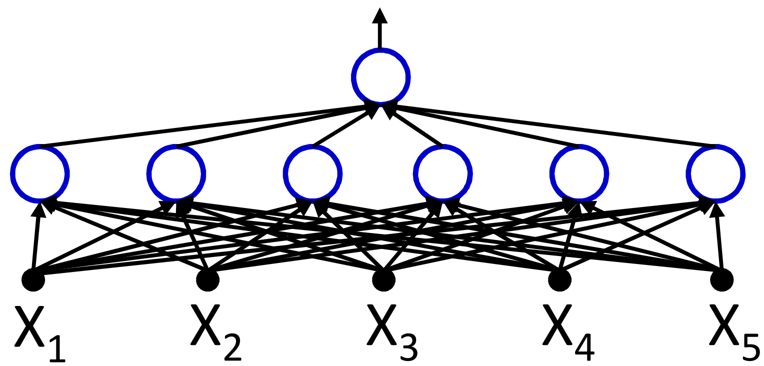

- *Any truth table can be expressed in this manner!*
- **A one-hidden-layer MLP is a Universal Boolean Function**

How many layers for a Boolean MLP?

Truth Table

X_1	X_2	X_3	X_4	X_5	Y
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0	1	0	1	1	1
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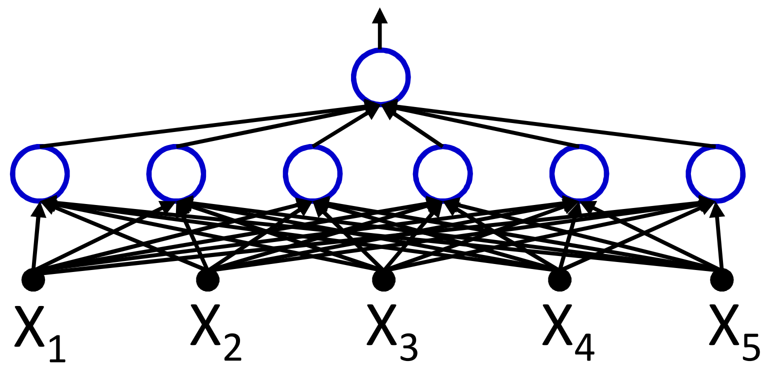
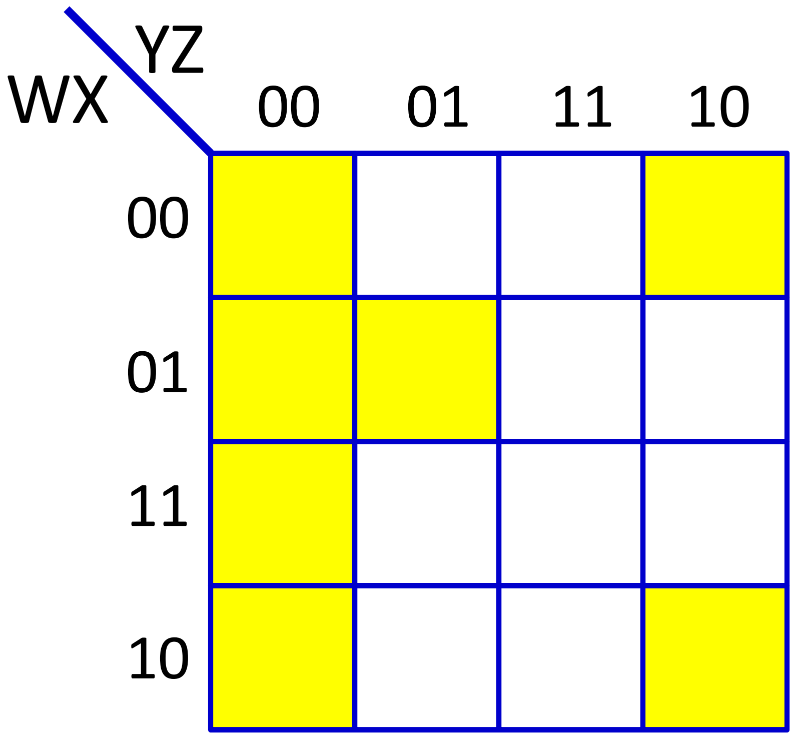


- *Any truth table can be expressed in this manner!*
- **A one-hidden-layer MLP is a Universal Boolean Function**

But what is the largest number of perceptrons required in the single hidden layer for an N-input-variable function?

Reducing a Boolean Function



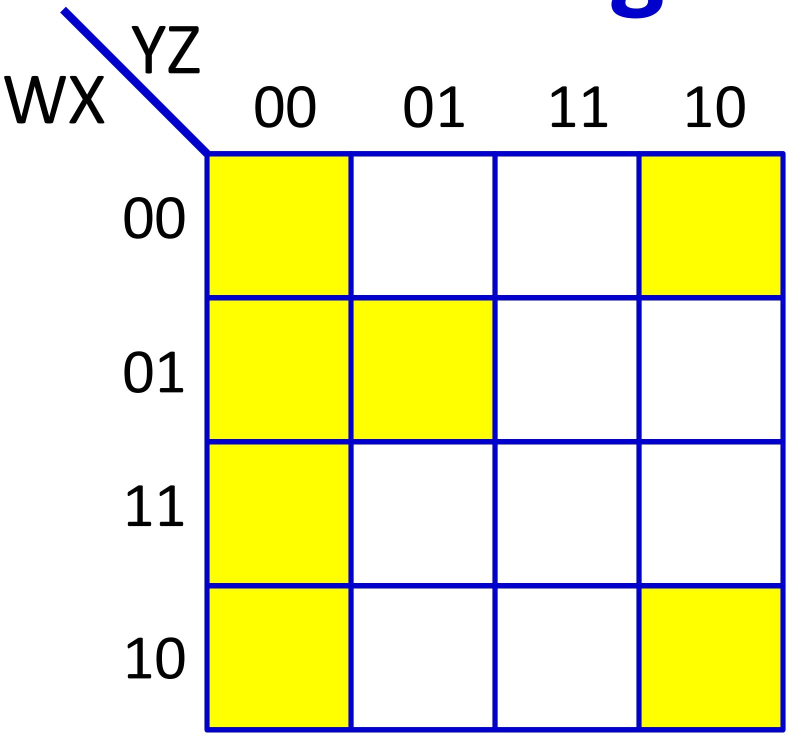
This is a "Karnaugh Map"

It represents a truth table as a grid
Filled boxes represent input combinations for which output is 1; blank boxes have output 0

Adjacent boxes can be "grouped" to reduce the complexity of the DNF formula for the table

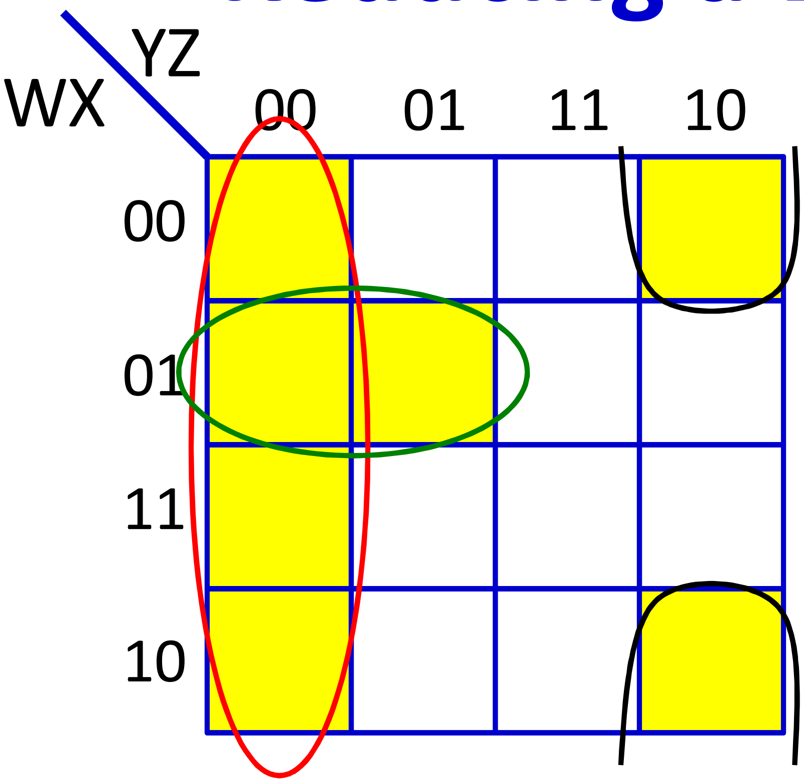
- DNF form:
  - Find groups
  - Express as reduced DNF

Reducing a Boolean Function



Basic DNF formula will require 7 terms

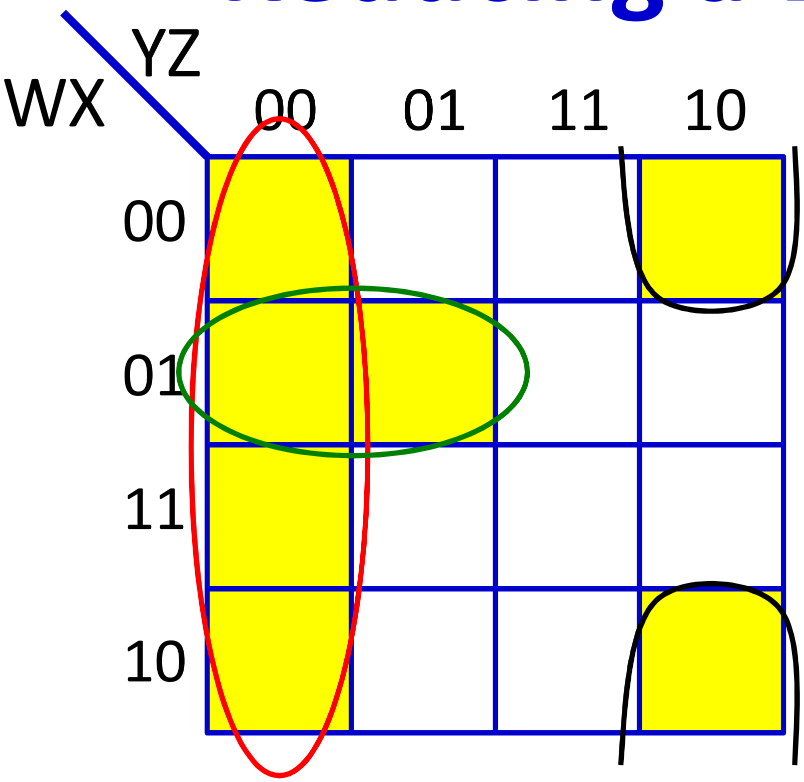
Reducing a Boolean Function



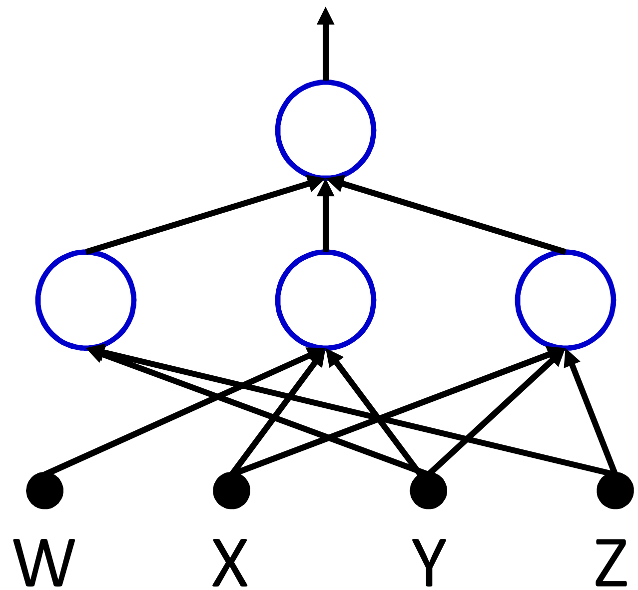
$$O = \bar{Y}\bar{Z} + \bar{W}X\bar{Y} + \bar{X}Y\bar{Z}$$

- *Reduced* DNF form:
  - Find groups
  - Express as reduced DNF

Reducing a Boolean Function



$$O = \bar{Y}\bar{Z} + \bar{W}X\bar{Y} + \bar{X}Y\bar{Z}$$



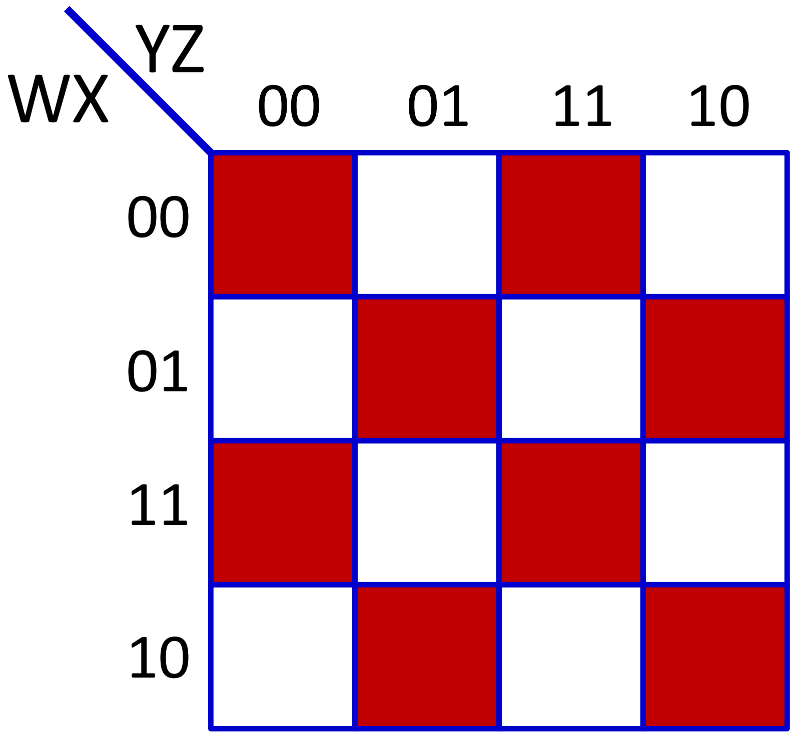
- *Reduced* DNF form:
  - Find groups
  - Express as *reduced* DNF
  - Boolean network for this function needs only 3 hidden units
    - Reduction of the DNF reduces the size of the one-hidden-layer network

Largest irreducible DNF?

WX \ YZ	00	01	11	10
00				
01				
11				
10				

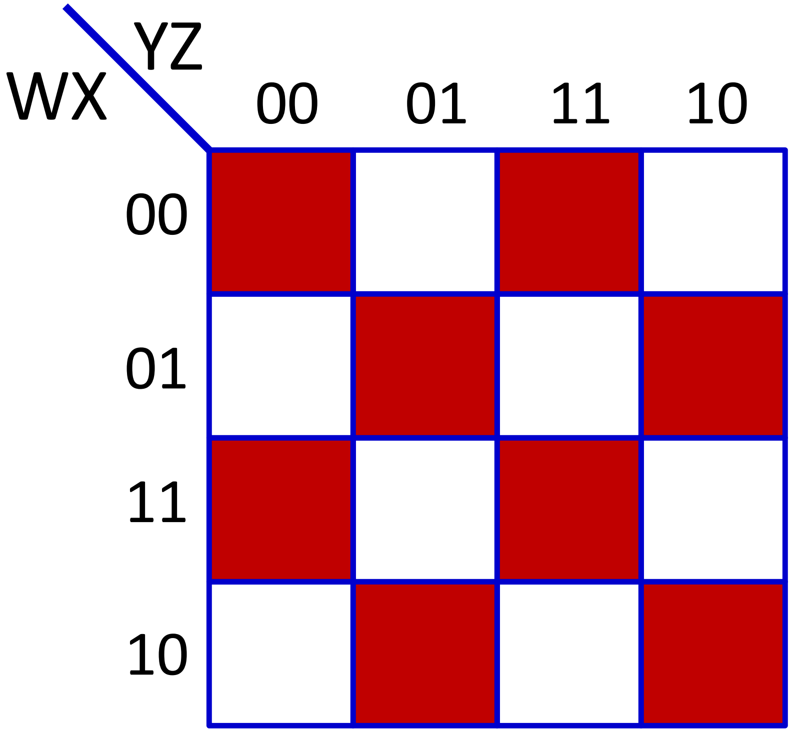
- What arrangement of ones and zeros simply cannot be reduced further?

Largest irreducible DNF?



- What arrangement of ones and zeros simply cannot be reduced further?

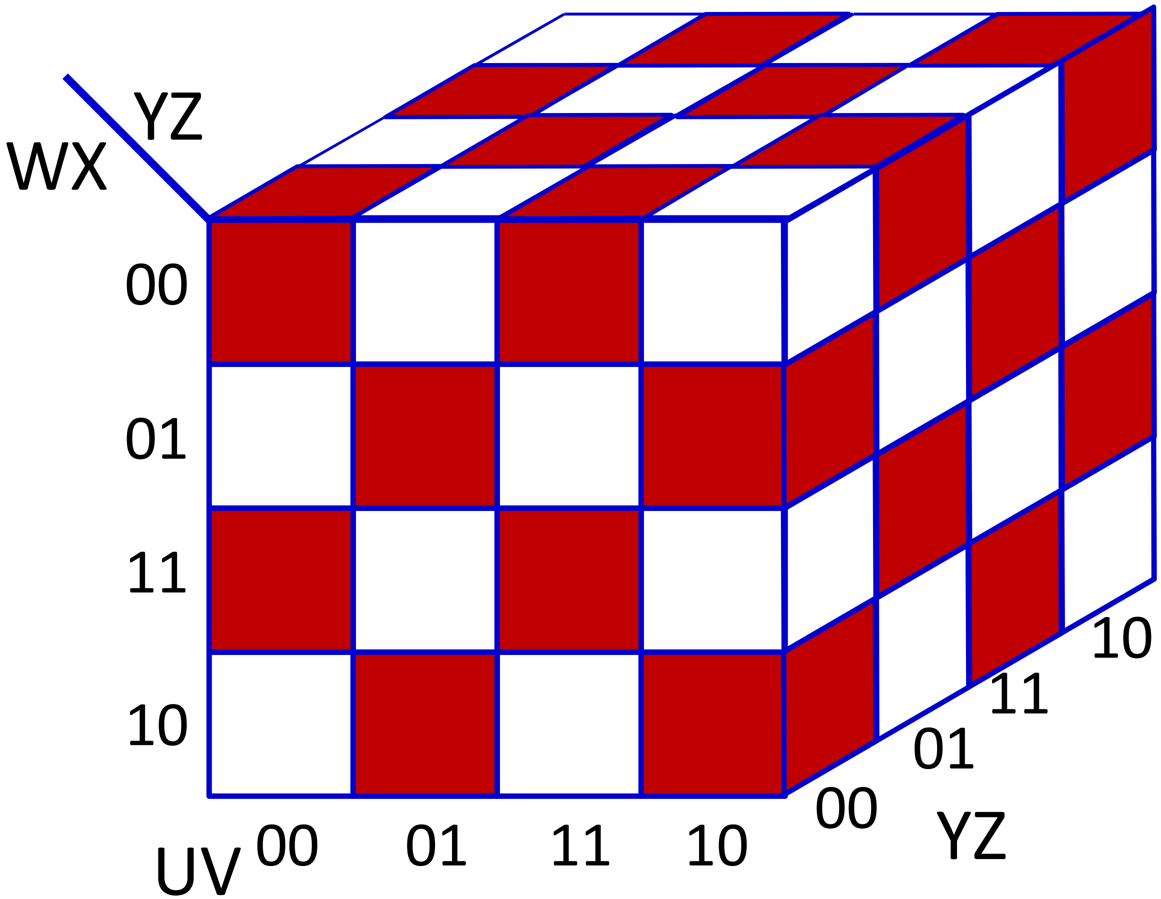
Largest irreducible DNF?



How many neurons in a DNF (one-hidden-layer) MLP for this Boolean function?

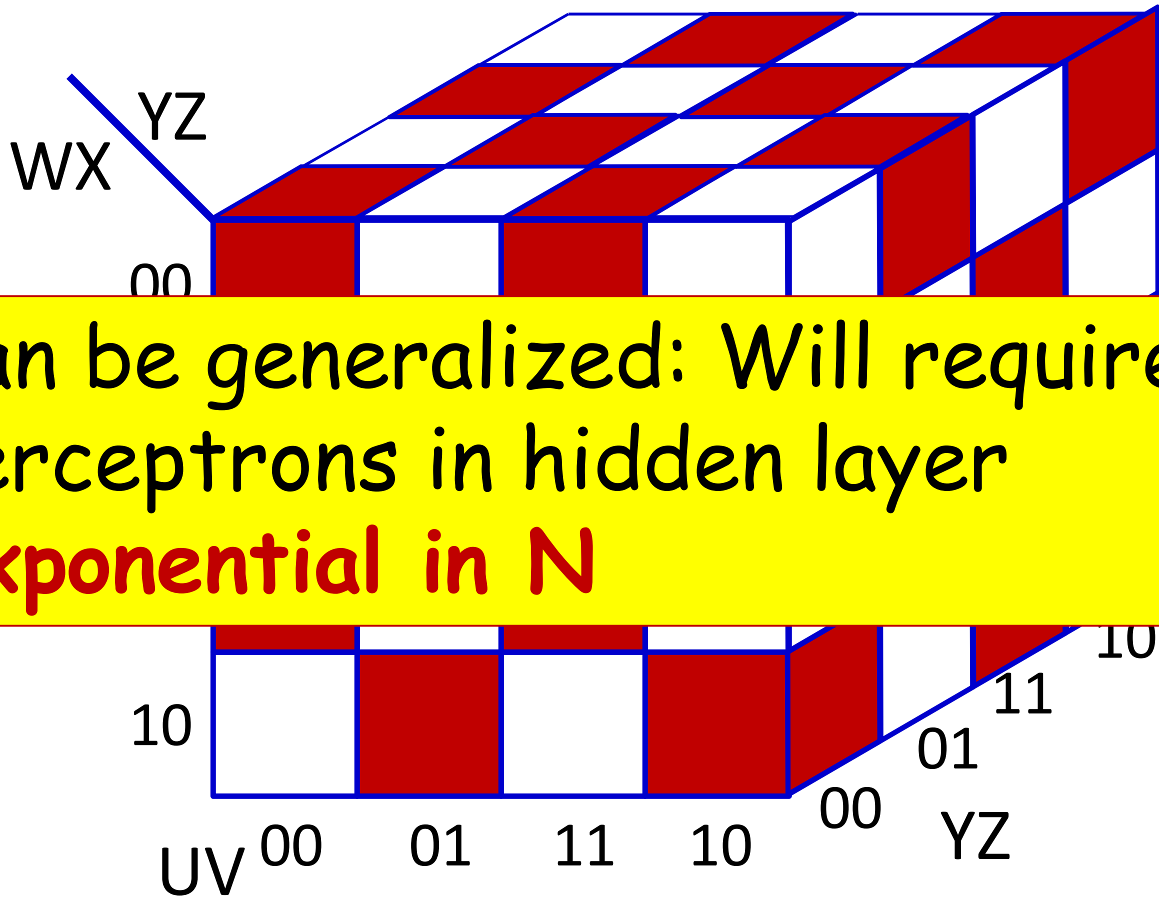
- What arrangement of ones and zeros simply cannot be reduced further?

Width of a single-layer Boolean MLP



- How many neurons in a DNF (one-hidden-layer) MLP for this Boolean function of 6 variables?

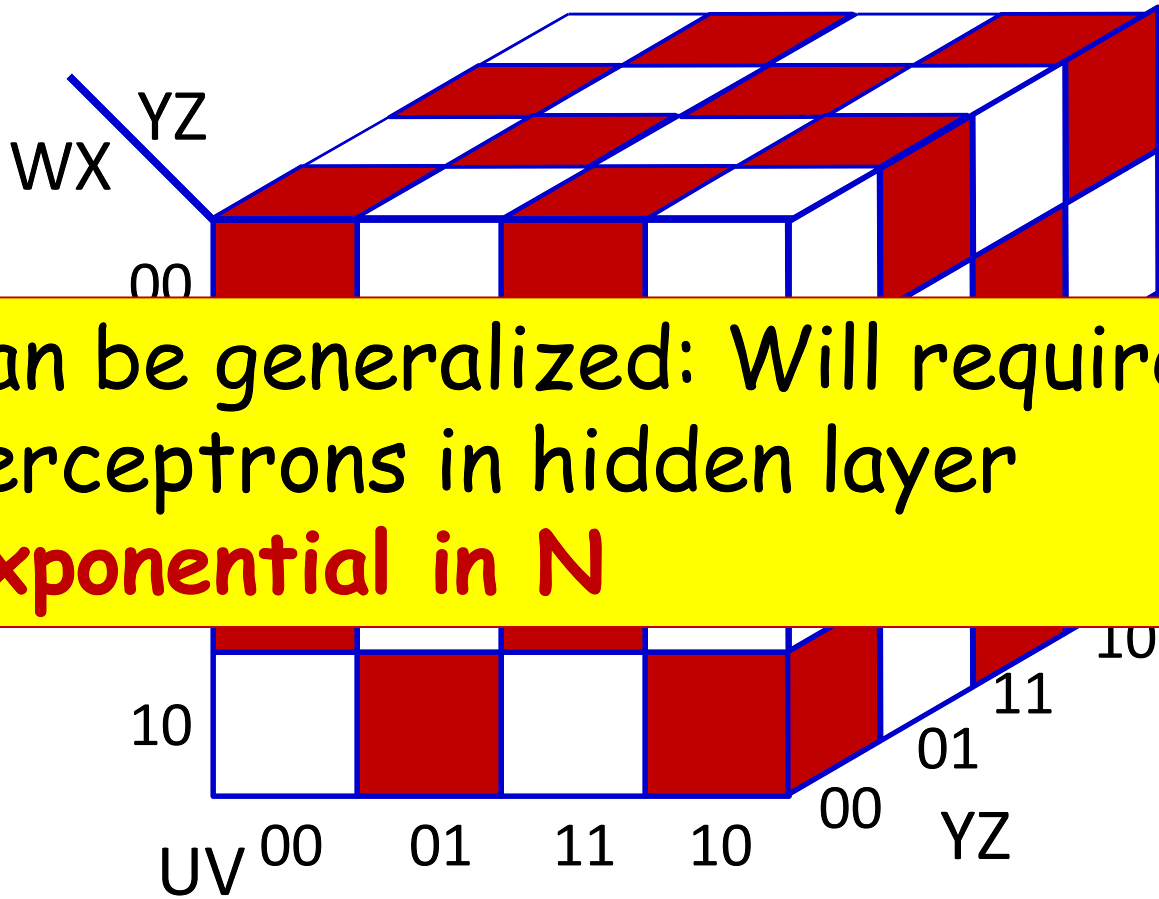
Width of a single-layer Boolean MLP



Can be generalized: Will require 2^{N-1} perceptrons in hidden layer
Exponential in N

- How many neurons in a DNF (one-hidden-layer) MLP for this Boolean function

Width of a single-layer Boolean MLP

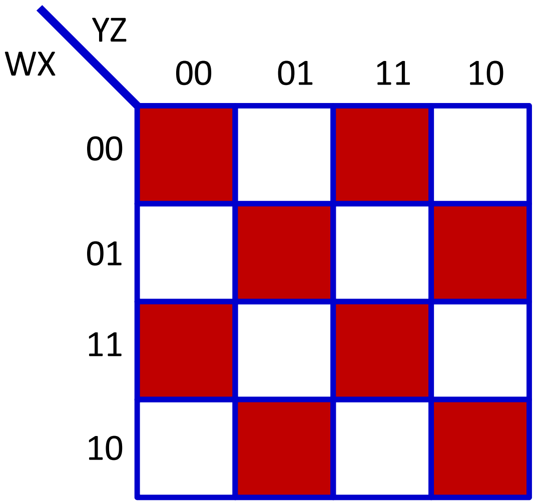


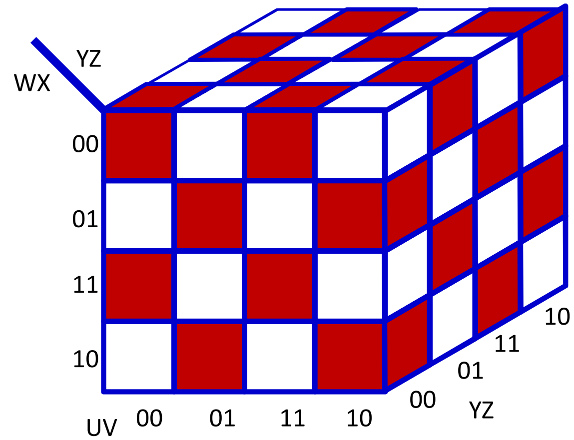
Can be generalized: Will require 2^{N-1} perceptrons in hidden layer
Exponential in N

How many units if we use multiple layers?

layer) MLP for this Boolean function

Width of a deep MLP

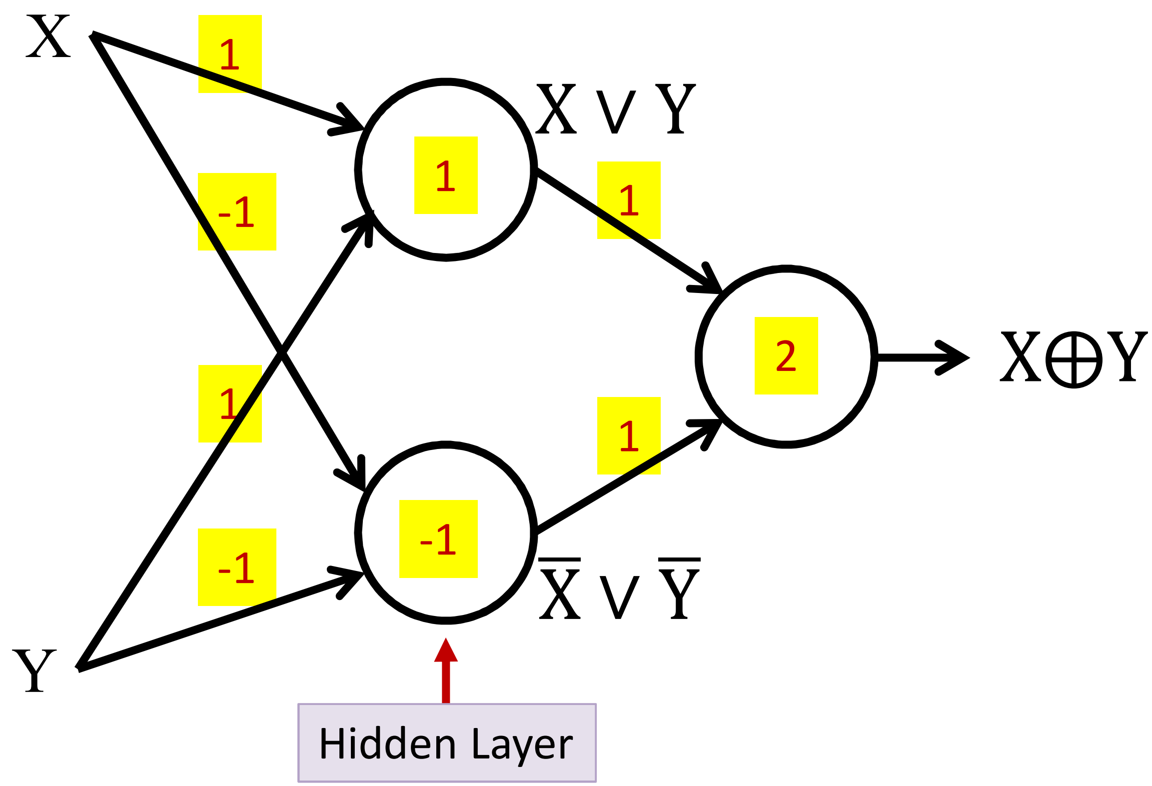




$$O = W \oplus X \oplus Y \oplus Z$$

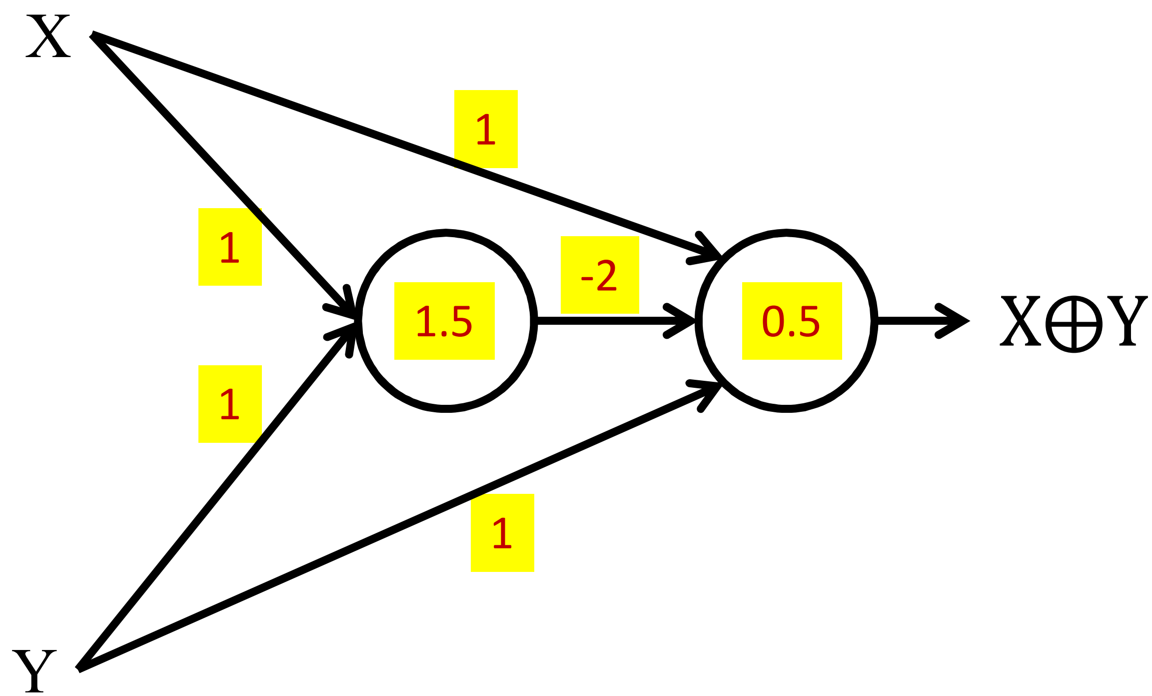
$$O = U \oplus V \oplus W \oplus X \oplus Y \oplus Z$$

Multi-layer perceptron XOR



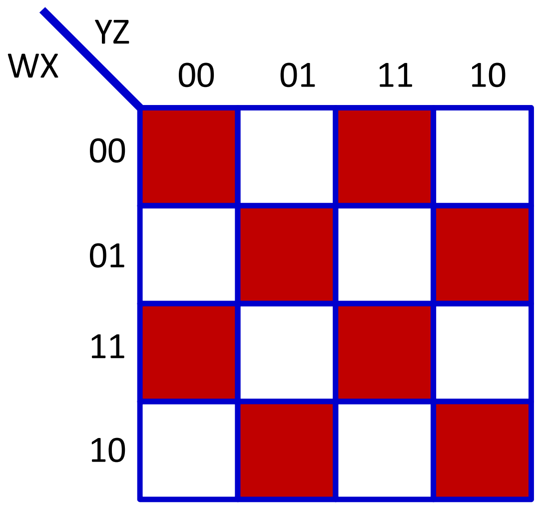
- An XOR takes three perceptrons

Multi-layer perceptron XOR

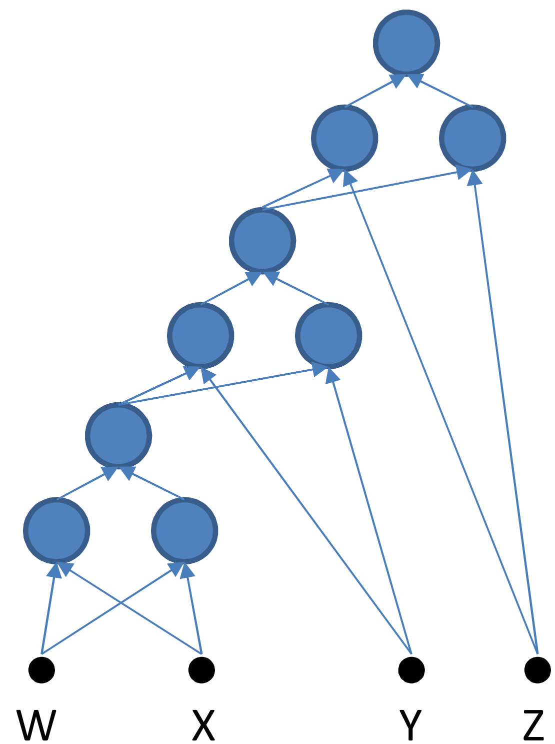


- With 2 neurons
  - 5 weights and two thresholds

Width of a deep MLP



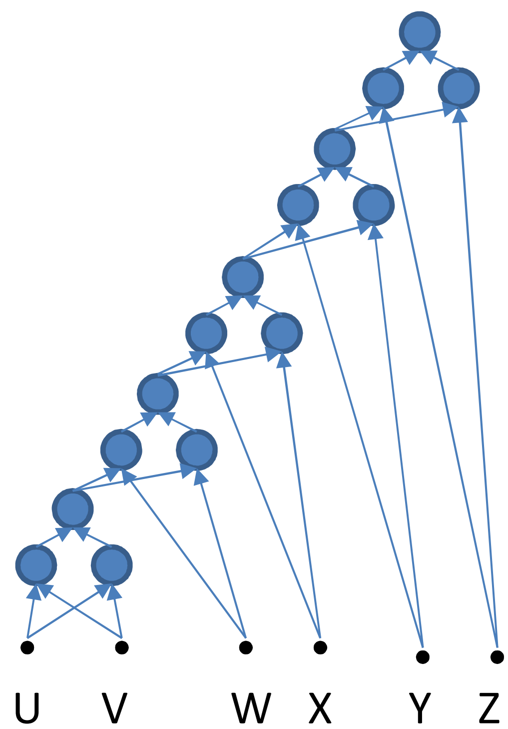
$$O = W \oplus X \oplus Y \oplus Z$$

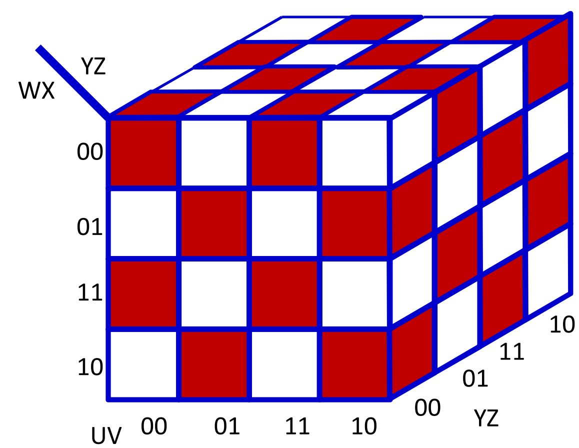


9 perceptrons

- An XOR needs 3 perceptrons
- This network will require 3x3 = 9 perceptrons

Width of a deep MLP



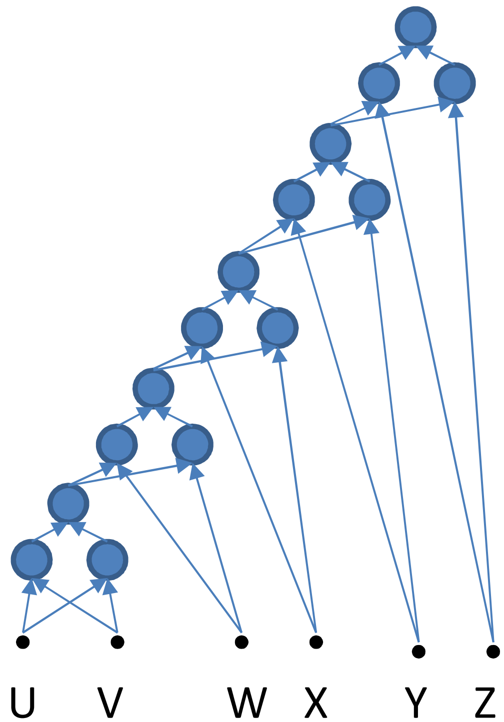


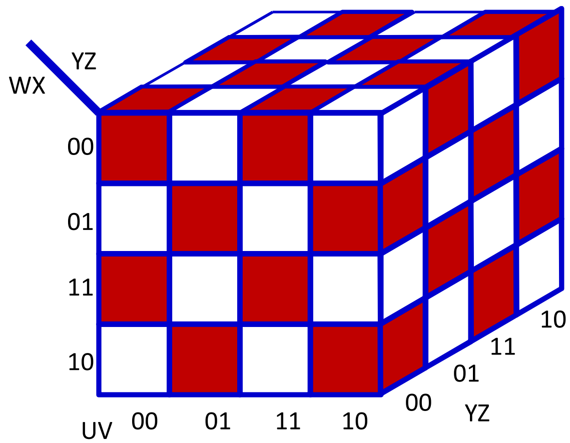
$$O = U \oplus V \oplus W \oplus X \oplus Y \oplus Z$$

15 perceptrons

- An XOR needs 3 perceptrons
- This network will require 3x5 = 15 perceptrons

Width of a deep MLP



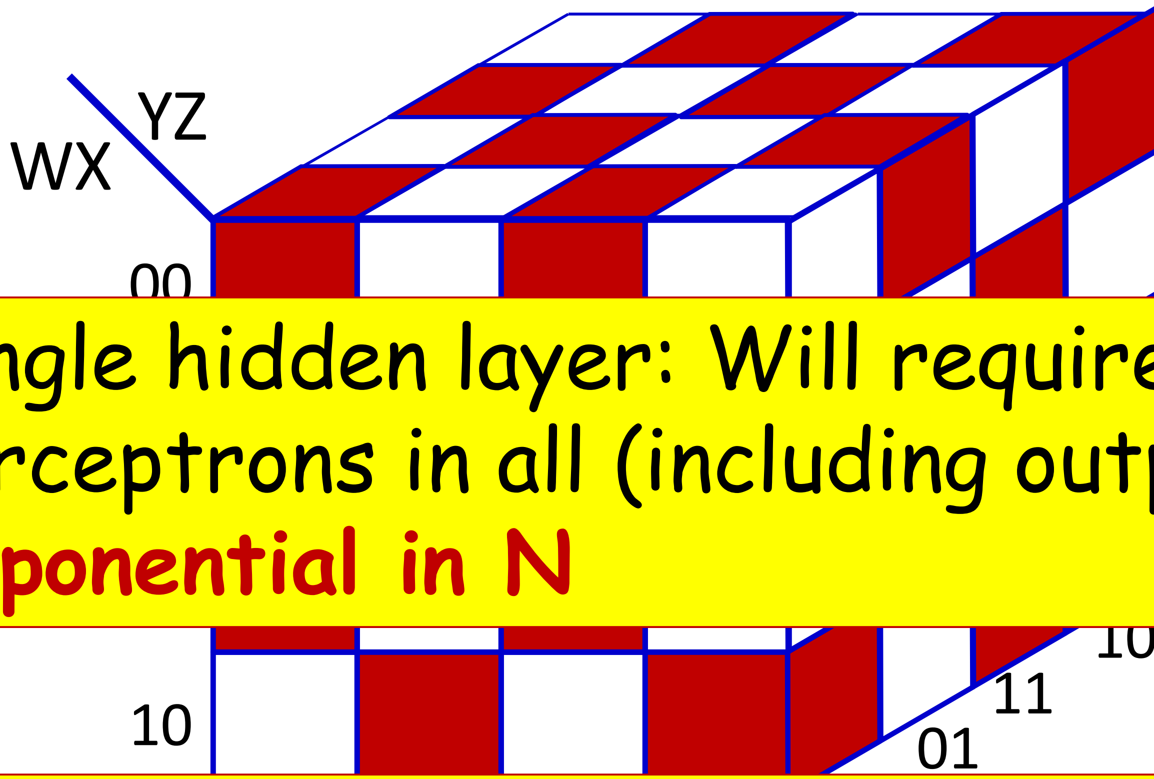


$$O = U \oplus V \oplus W \oplus X \oplus Y \oplus Z$$

More generally, the XOR of N variables will require 3(N-1) perceptrons!!

- An XOR needs 3 perceptrons
- This network will require 3x5 = 15 perceptrons

Width of a single-layer Boolean MLP



Single hidden layer: Will require $2^{N-1}+1$ perceptrons in all (including output unit)

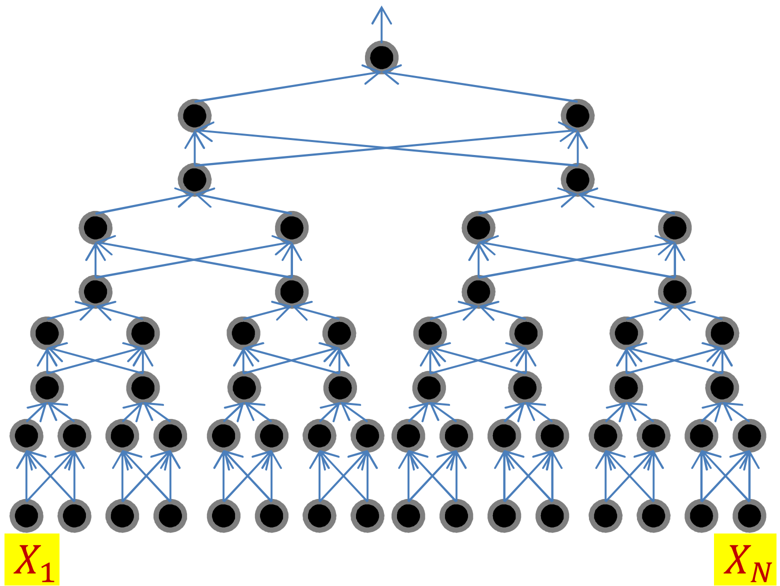
Exponential in N

Will require 3(N-1) perceptrons in a deep network

Linear in N!!!

Can be arranged in only $2\log_2(N)$ layers

A better representation



$$O = X_1 \oplus X_2 \oplus \cdots \oplus X_N$$

- Only $2\log_2 N$ layers
  - By pairing terms
  - 2 layers per XOR

$$O = (((((X_1 \oplus X_2) \oplus (X_1 \oplus X_2)) \oplus ((X_5 \oplus X_6) \oplus (X_7 \oplus X_8))) \oplus (((\ldots$$

Caveat 1: Not all Boolean functions..

- Not all Boolean circuits have such clear depth-vs-size tradeoff

- Shannon's theorem: For $n > 2$, there is Boolean function of n variables that requires at least $2^n/n$ gates
  - More correctly, for large n, almost all n-input Boolean functions need more than $2^n/n$ gates
    - Regardless of depth

- Note: If all Boolean functions over n inputs could be computed using a circuit of size that is polynomial in n, P = NP!

Caveat 2

- Used a simple "Boolean circuit" analogy for explanation
- We actually have *threshold circuit* (TC) not, just a Boolean circuit (AC)
  - Specifically composed of threshold gates
    - More versatile than Boolean gates (can compute majority function)
      - E.g. "at least K inputs are 1" is a single TC gate, but an exponential size AC
      - For fixed depth, $Boolean\ circuits \subset threshold\ circuits$ (strict subset)
  - A depth-2 TC parity circuit can be composed with $\mathcal{O}(n^2)$ weights
    - But a network of depth $\log(n)$ requires only $\mathcal{O}(n)$ weights
  - But more generally, for large n, for most Boolean functions, a threshold circuit that is polynomial in n at optimal depth d may become exponentially large at $d - 1$

- Other formal analyses typically view neural networks as *arithmetic circuits*
  - Circuits which compute polynomials over any field

- So lets consider functions over the field of reals