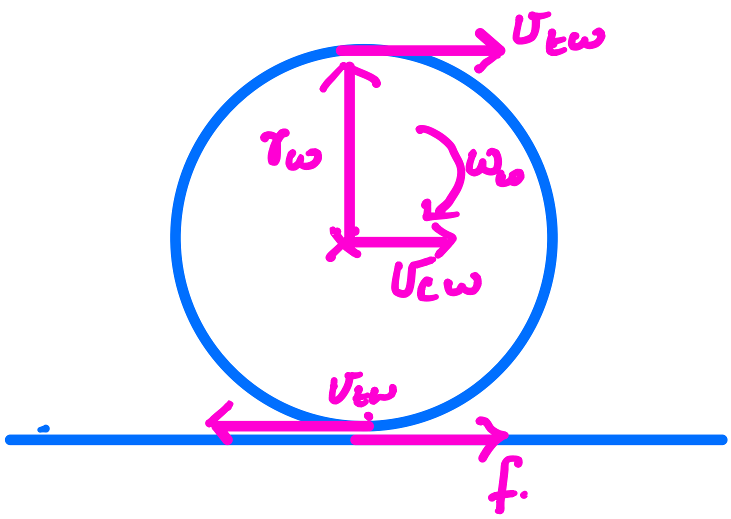
Forward Kinematics:

If the motors of a robot rotate with commanded angular velocities where would the robot reach in a given time interval.

Differential Drive:

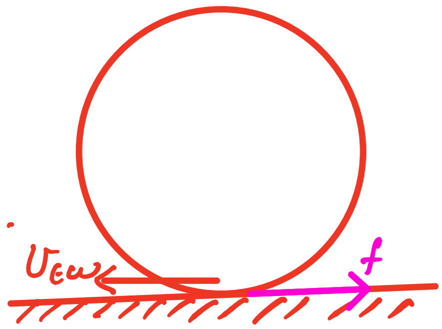
→ Rolling W/O Slipping



Consider a wheel that rotates freely about the center in air. When this wheel is placed on the ground, aided by the force of friction, f, begins to move forward.

In other words in the absence of such a force the wheel would rotate, but its velocities at the top and bottom are in opposite direction and of equal magnitude that it would not move. $\vec{v}_{tw}$ direction at the top of the wheel is $\rightarrow$ and at the bottom (point of contact with the ground) is $\leftarrow$, cancelling each other and preventing any translation

However at the point of contact with the ground the tendency of the wheel is to move in $\leftarrow$ direction.



f opposes this tendency by acting in $\rightarrow$ direction

This opposing nature of f causes the wheel to move in $\rightarrow$ direction.

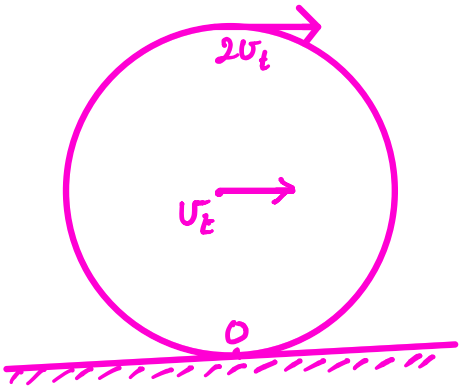
Let $\vec{v}_{cw}$ be the translational velocity of the wheel centre. Then velocity at the top of the wheel is $\vec{v}_{cw} + \vec{v}_{tw}$. Note $\vec{v}_{cw}$ is the effect of f.

The velocity at the point of contact with the ground is $\vec{v}_{cw} - \vec{v}_{tw}$

Rolling without slipping or pure rolling entails: $v_{cw} - v_{tw} = 0$

Or $v_{c\omega} = v_{t\omega} = r_\omega \omega_\omega$. or $r\omega$ dropping the suffix ω denoting wheel.

Hence

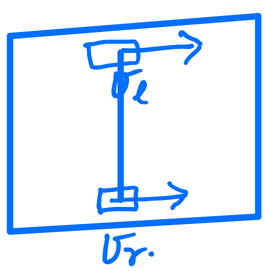


at the point of contact with the ground the wheel is instantaneously at rest.

NOTE: The translational velocity of the wheel is $v_{c\omega}$ is the same at all points on the wheel. However the velocity due to the rotation of the wheel center v_t is different at different location of the wheel.

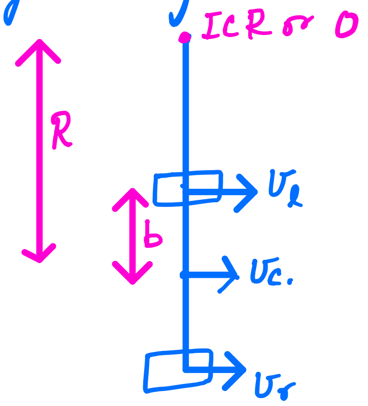
DIFFERENTIAL DRIVE ROBOT:

Differential drive kinematics: left and right wheel independently controlled.



Let the linear velocity of the left wheel center (translational velocity) be v_l and the right wheel center be v_r.

The entire robot rotates about an instantaneous center of rotation ICR that lies on the line joining the left and right wheel (the wheel base)



Let ω be the angular velocity with which it rotates about O. Then

$$\vec{v_c} = \vec{\omega} \times \vec{r} \quad \text{or} \quad |v_c| = R\omega. \longrightarrow (1).$$

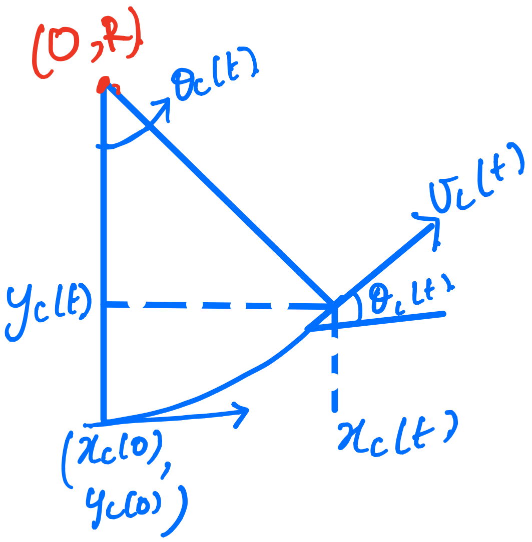
$$v_l = (R-b)\omega \longrightarrow (2)$$

$$v_r = (R+b)\omega \longrightarrow (3).$$

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = f(\underbrace{v_l, v_r}).$$

$$v_p = f(v_w)$$

$$v_c = \frac{v_l + v_r}{2} \longrightarrow (4). \qquad \omega = \frac{v_r - v_l}{2b}. \longrightarrow (5)$$



$$x_c(t) = \int_0^t v_c(t)\cos(\theta_c(t)).dt.$$

$$= \int_0^t v_c \cos(\theta_c)\,dt \longrightarrow (6).$$

$$y_c(t) = \int_0^t v_c \sin(\theta_c)\,dt \longrightarrow (7).$$

$$\theta_c(t) = \int_0^t \omega\,dt \longrightarrow (8).$$

$$\theta_c(t) = \omega t \longrightarrow (9).$$

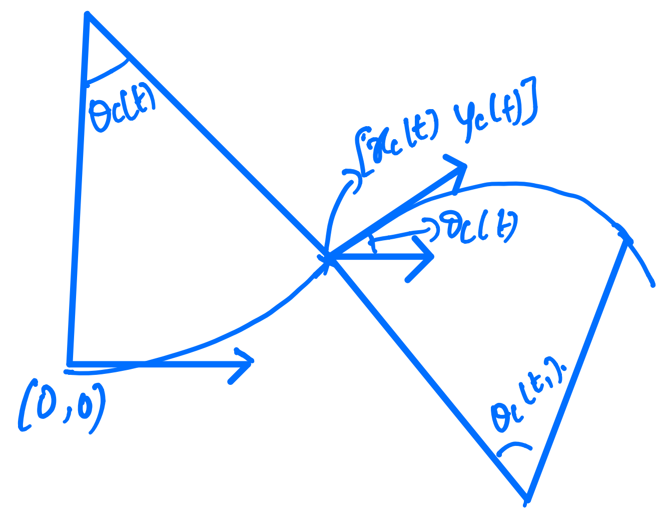
Then $x_c(t) = \int_0^t v_c \cos(\omega t)\,dt = \left.\frac{v_c \sin(\omega t)}{\omega}\right|_0^t = \frac{v_c \sin(\omega t)}{\omega}$ $\longrightarrow (9)$.

$$y_c(t) = \int_0^t v_c \sin(\omega t)\,dt = -\left.\frac{v_c}{\omega}\cos(\omega t)\right|_0^t = -\frac{v_c}{\omega}\left[\cos(\omega t) - 1\right]$$
$\longrightarrow (10)$

From (9) & (10) we get

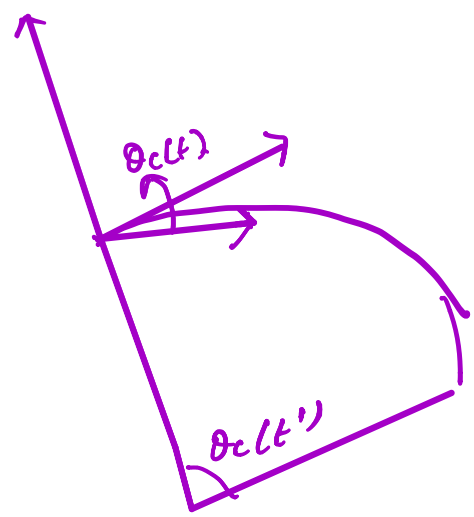
$$x_c^2 + \left(y_c(t) - \frac{v_c}{\omega}\right)^2 = \frac{v_c^2}{\omega^2}\left(\cos^2\omega t + \sin^2\omega t\right) = R^2$$

or $\underline{x_c^2(t) + (y_c(t) - R)^2 = R^2}$



How do you aggregate multiple such segment over time?

Use coordinate transform.



Let $t=0$ translate the spatial and temporal origin to $(x_c(t), y_c(t), t)$.

$t' = t_1 - t$.

Then $x_c(t') = \frac{v_c'}{\omega'}\sin(\omega t') \rightarrow (11)$.

$y_c(t') = -\frac{v_c'}{\omega'}\left[1 - \cos(\omega t')\right] \rightarrow (12)$.

where (v_c', ω') is the linear and angular velocity from $t \rightarrow t'$.

Where is the robot now with respect to the original origin?

$$\begin{bmatrix} x_c(t_1) \\ y_c(t_1) \end{bmatrix} = \begin{bmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} x_c(t') \\ y_c(t') \end{bmatrix} + \begin{bmatrix} x_c(t) \\ y_c(t) \end{bmatrix} \rightarrow (13).$$

$$\theta_c(t_1) = \theta_c(t) + \theta_c(t') \rightarrow (14).$$

Kinematics in the presence of accelerations:

$$x_c(t) = x_c(0) + \int_0^t (v_c + at)\cos\left(\theta_c(0) + \omega t + \frac{\alpha t^2}{2}\right) dt. \rightarrow (15).$$

$$y_c(t) = y_c(0) + \int_0^t (v_c + at)\sin\left(\theta_c(0) + \omega t + \frac{\alpha t^2}{2}\right) dt \rightarrow (16).$$

The above integrals need to be numerically computed by the method of Fresnal Integrals.

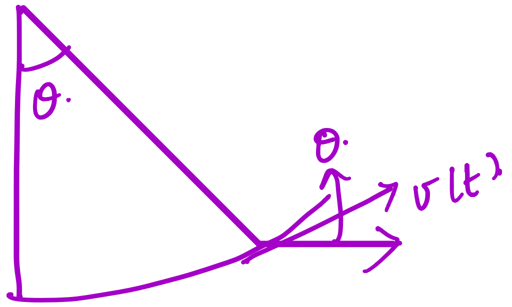
The resulting curve obtained from numerical integration is called a CLOTHOID.

Non Holonomic Robot:

The Differential Drive robot is Non Holonomic.

$$\left.\begin{aligned} \dot{x}_c(t) &= v\cos(\omega t). \\ \dot{y}_c(t) &= v\sin(\omega t). \end{aligned}\right\} \quad \boxed{\dot{y}_c(t) = \dot{x}_c(t)\tan\theta.}$$

v_y and v_x are not Decoupled. They are coupled through the robots instantaneous direction / heading θ.

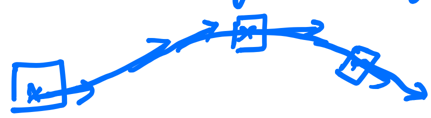


The robot's tangential velocity $v(t)$ is always along the heading direction $\theta(t)$

No INDEPENDENT CONTROL of V_x and V_y.

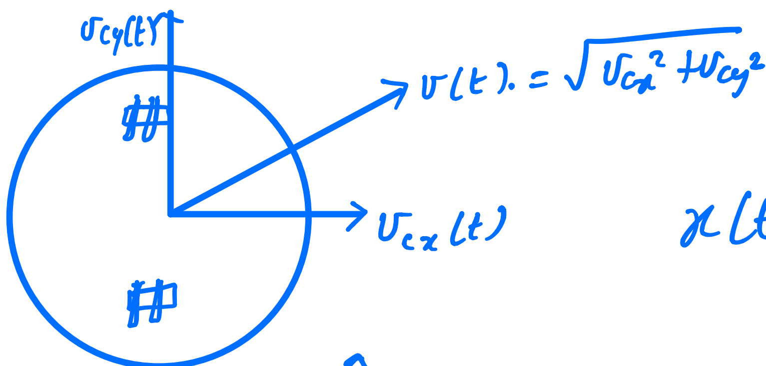
Control degrees of freedom (v, ω).

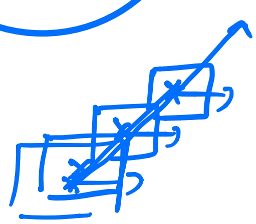
Configuration / cartesian degrees of freedom (x, y, θ).



The degrees of control is less than the degrees of configuration accesible by the robot.

HOLONOMIC KINEMATICS:



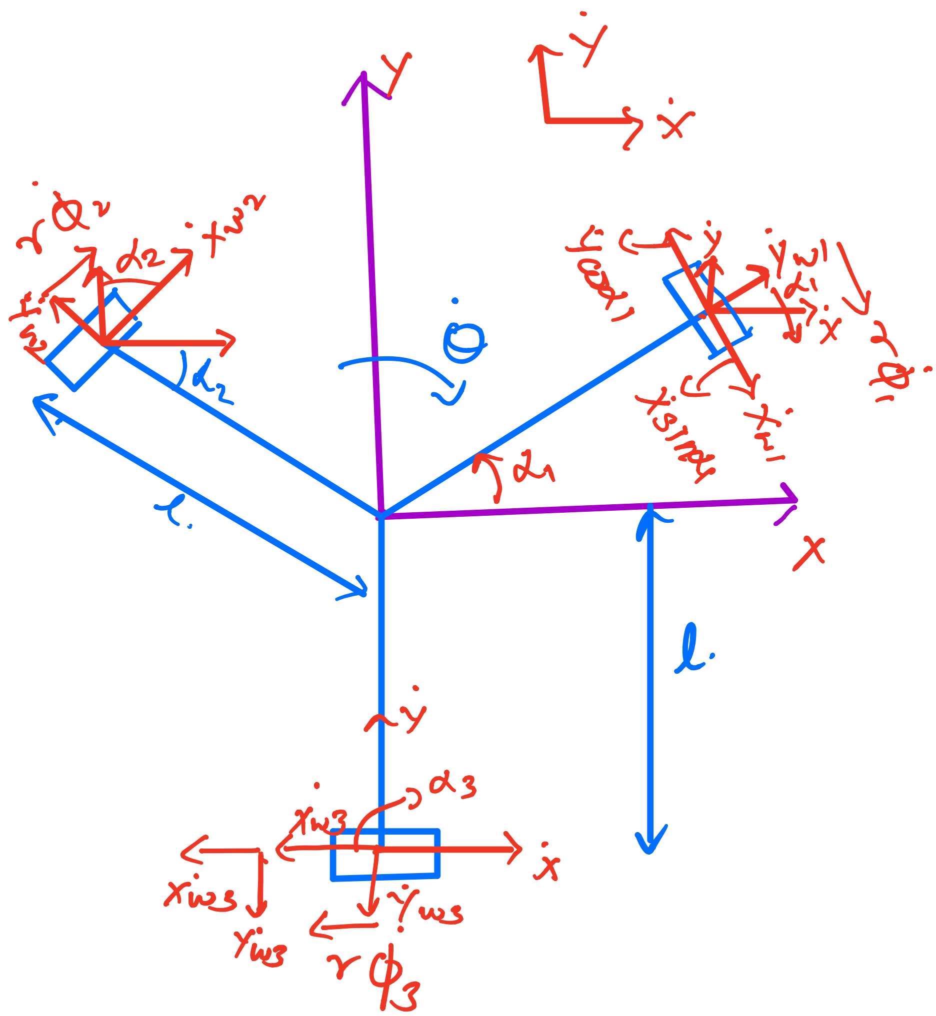


Independent control of v_x, v_y.

$$x(t) = x(0) + \int_0^t v_x \, dt \longrightarrow (17)$$

$$y(t) = y(0) + \int_0^t v_y \, dt \longrightarrow (18)$$

Omnidirectional Kinematics:



$\dot{X} = [\dot{x}\ \dot{y}\ \dot{\theta}]$ is the body center velocities.

$\dot{x}_{wi}, \dot{y}_{wi}$ is the wheel velocity of the ith wheel.

$\dot{\phi}_i$ is the angular velocity of the ith wheel.

Wheel 1: $\dot{x}\sin\alpha_1 - \dot{y}\cos\alpha_1 + l\dot{\theta} = r\dot{\phi}_1 \longrightarrow (1).$

$\dot{x}\cos\alpha_1 + \dot{y}\sin\alpha_1 = 0 \longrightarrow (2).$

(No lateral sliding constraint).

Wheel 2: $\dot{x}\sin\alpha_2 + \dot{y}\cos\alpha_2 + l\dot{\theta} = r\dot{\phi}_2 \longrightarrow (3).$

$\dot{x}\cos\alpha_2 - \dot{y}\sin\alpha_2 = 0 \longrightarrow (4).$

(No lateral sliding constraint)

Wheel 3:

$$\dot{x}\cos(90+\alpha_3) + \dot{y}\cos\alpha_3 + l\dot{\theta} = r\dot{\phi}_3 \rightarrow (5)$$

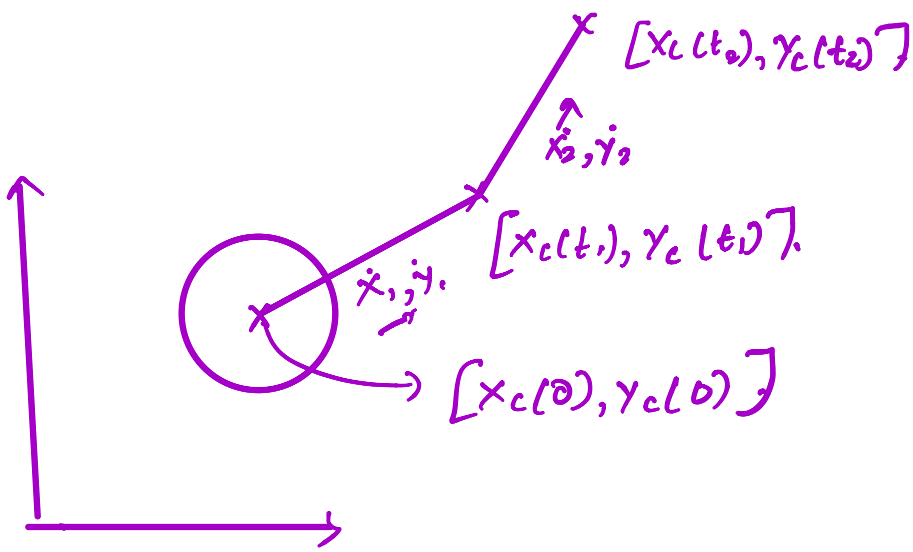
$$\dot{x}\sin(90+\alpha_3) + \dot{y}\sin\alpha_3 = 0 \rightarrow (6)$$

Taking together 1, 3 & 5 we get

$$\begin{bmatrix} \sin\alpha_1 & -\cos\alpha_1 & l \\ \sin\alpha_2 & \cos\alpha_2 & l \\ \cos(90+\alpha_3) & \cos\alpha_3 & l \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} r\dot{\phi}_1 \\ r\dot{\phi}_2 \\ r\dot{\phi}_3 \end{bmatrix} \rightarrow (7)$$

or $J\dot{X}_p = \dot{X}_w \rightarrow (8)$

$$\dot{X}_p = [\dot{x} \ \ \dot{y} \ \ \dot{\theta}]^T \qquad \dot{X}_w = [r\dot{\phi}_1 \ \ r\dot{\phi}_2 \ \ r\dot{\phi}_3]^T$$



$$\dot{x}_1, \dot{y}_1 \xrightarrow{a_x, a_y} \dot{x}_2, \dot{y}_2$$

$\dot{x}_1(t) = \dot{x}_1(t_0) + a_x(t - t_0)$ until you reach $\dot{x}_2$

$\dot{y}_1(t) = \dot{y}_1(t_0) + a_y(t - t_0)$ until you reach $\dot{y}_2$

$$x_1(t) = \int_{t_0}^{t} \dot{x}_1(t)\,dt = x_1(t_0) + \dot{x}_1(t-t_0) + \frac{a_x}{2}(t-t_0)^2$$

$$y_1(t) = \int_{t_0}^{t} \dot{y}_1(t)\,dt = y_1(t_0) + \dot{y}_1(t-t_0) + \frac{a_y}{2}(t-t_0)^2$$