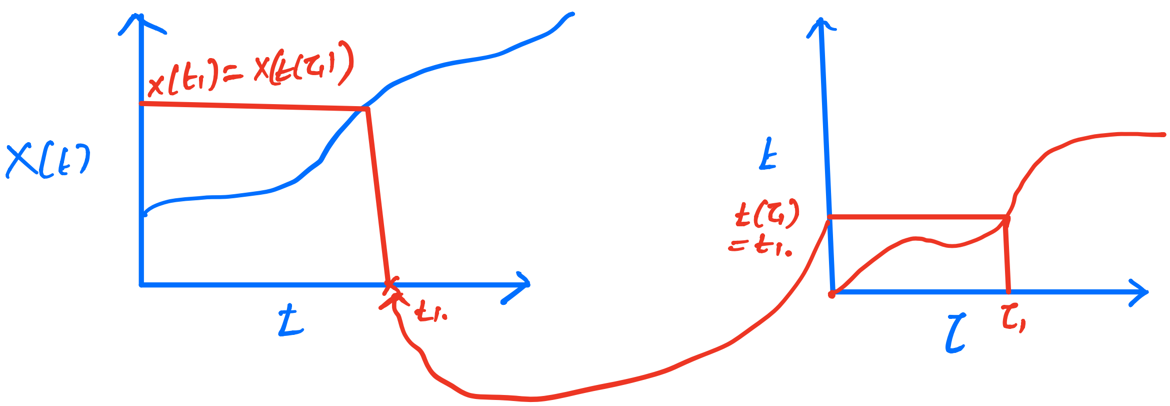
Time Scaling:

Consider a time parametrized function implicitly defined as $x(t)$

Now what happens if t is some function of a new variable τ. such that $t(\tau) = g$.



$$\dot{x}(t(\tau)) \leftarrow \frac{dx}{dt} \cdot \frac{dt}{d\tau} = x'(t) \frac{dt}{d\tau} = x'(t)\, s. \longrightarrow (1)$$

$s = \frac{dt}{d\tau}$ is the scaling function. $\longrightarrow (2)$.

$$x'(t) = \dot{x}(\tau)\, s' \longrightarrow (3).$$

$$\dot{x}(\tau) = \frac{dx(\tau)}{d\tau}, \quad s' = \frac{1}{s} \longrightarrow (4)$$

Write $x'(t)$ for $\frac{dx(t)}{dt}$ and $\dot{x}(\tau)$ for $\frac{dx(\tau)}{d\tau}$.

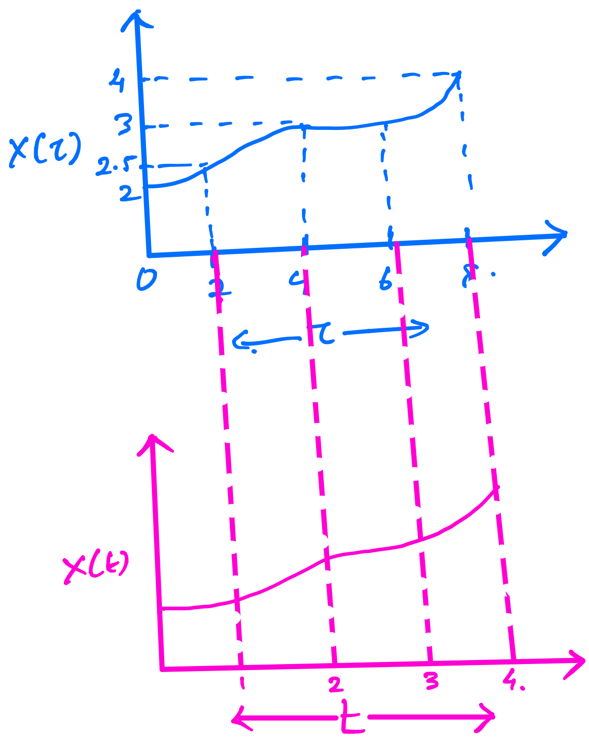
Let $t(\tau) = \frac{\tau}{2} \longrightarrow (5).$

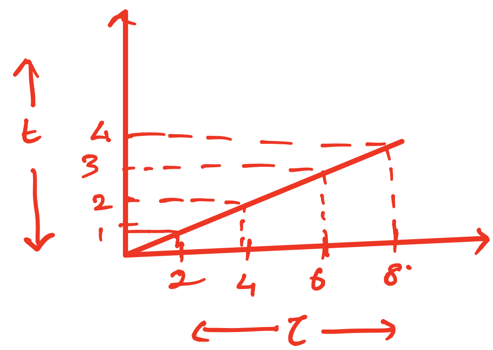
Then $\dot{x}(t(\tau)) = x'(t)(1/2) = \frac{1}{2}x'(t) \longrightarrow (6).$

$s = \frac{1}{2}$ is a constant scaling function

What does this mean?

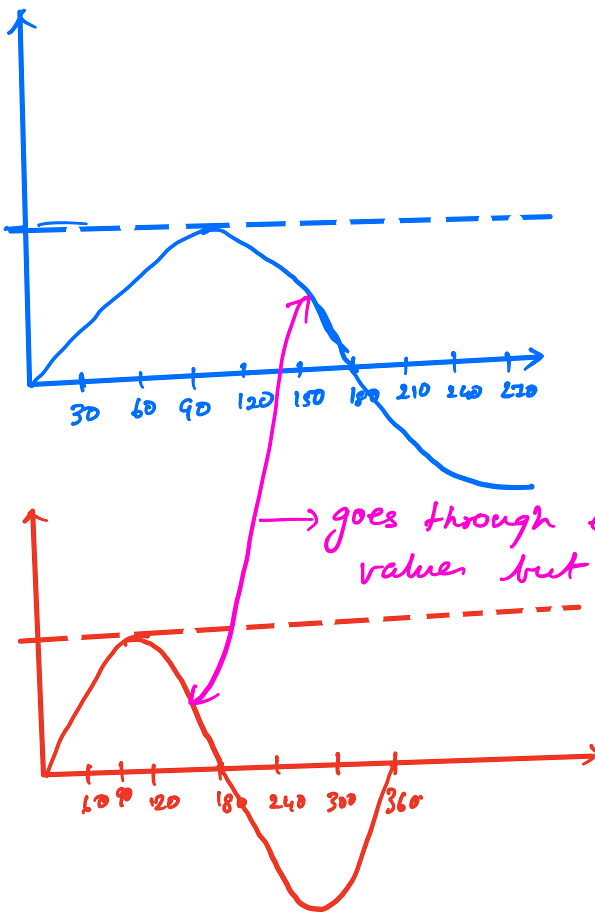
Let $x(\tau)$ be defined for time $\tau \in [0, 8]$





$x(t)$ passes precisely through the same set of points as $x(\tau)$ but at TWO times FASTER.

This has precisely the same connotation to the two functions $\sin(x)$ and $\sin(y)$, $y = 2x$



→ goes through exactly the same values but one travels faster than the other

$$\frac{d}{dx}\left(\sin(y(x))\right) = \cos y \cdot 2 = 2\cos y$$

↳ scaling fn

So how does one traverse the curve?

Let $\hat{x}(t) = 3t^2 + 4t + 5 \longrightarrow (1).$

and $t = 2\tau. \longrightarrow (2)$

Then $x(\tau) = 3(2\tau)^2 + 4(2\tau) + 5 = 12\tau^2 + 8\tau + 5 \longrightarrow (2)$

Let $\hat{x}(t)$ be defined in the interval $[2, 7]$.

Then $\tau \in [1, 7/2]$.

$\hat{x}'(t) = 6t+4 \longrightarrow (3).$

$\dot{x}(\tau) = 24\tau + 8 \longrightarrow (4).$

$dt = 2\,d\tau \longrightarrow (5).$

Let $dt = 0.1$ then $d\tau = 0.05$

$\hat{x}(t+dt) = \hat{x}(t) + \hat{x}'(t)\,dt \longrightarrow (6).$

at $t=2$, $\hat{x}'(t) = 6(2)+4 = 16. \longrightarrow (7)$

$\hat{x}(2) = 3(2^2)+4(2)+5 = 25. \longrightarrow (8).$

$\hat{x}(2.1) = \hat{x}(2) + \hat{x}'(2).(0.1)$

$= 25 + 16(0.1) = 25 + 1.6 = 26.6 \longrightarrow (9).$

$x(\tau)$ at $\tau = 1$, $= 12(1)^2 + 8(1) + 5 = 25 \longrightarrow (10)$

$x(\tau + d\tau) = x(\tau) + \dot{x}(\tau)\,d\tau.$

for $dt = 0.1$, $d\tau = 0.05.$

$\dot{x}(1) = 24(1)+8 = 32.$

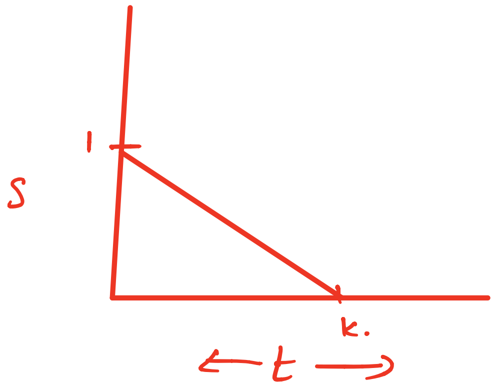
$x(1+0.05) = 25 + 32(0.05) = 25 + 1.6 = 26.6 \longrightarrow (11)$

(9) & (11) are same

Linear Time Scaling:

$s(t) = \dfrac{dt}{d\tau} = \max\left(1 - \dfrac{\tau}{k}, 0\right). \longrightarrow (12).$

Scale reduces from 1 to 0 in k time steps.
as t ↑'es from 0 to k if the step size is 1.



Let $\hat{x}(t) = 3t^2+4t+5 \longrightarrow (1)$

as before.

$$\frac{dt}{d\tau} = 1-\frac{\tau}{k} \longrightarrow (2)$$

$$dt = \left(1-\frac{\tau}{k}\right)d\tau \longrightarrow (3)$$

$$t = \tau - \frac{\tau^2}{2k} \longrightarrow (4)$$

From (1) $$x(\tau) = 3\left(\tau - \frac{\tau^2}{2k}\right)^2 + 4\left(\tau - \frac{\tau^2}{2k}\right) + 5 \longrightarrow (5)$$

Let $\tau \in [0,4]$. then $t \in \left[0, 4 - \frac{4^2}{2(4)}\right]$ $(k=4)$.

$t \in [0,2] \longrightarrow (6)$

$$\dot{x}(\tau) = 6\left(\tau - \frac{\tau^2}{2k}\right)\left(1-\frac{\tau}{k}\right) + 4\left(1-\frac{\tau}{k}\right)$$

$$= (1-\tau/k)\left[6\left(\tau - \frac{\tau^2}{2k}\right)+4\right] \longrightarrow (7)$$

Let $\tau_0 = 2$, $x(2) = 3\left(2 - \frac{4}{8}\right)^2 + 4\left(2-\frac{4}{8}\right)+5$

$x(2) = 3\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 5 = 11 + \frac{27}{4} = 17.75 \longrightarrow (8)$

When $\tau = 2$, $t = 2 - \frac{2^2}{2(4)} = 2 - \frac{1}{2} = \frac{3}{2} \longrightarrow (9)$

$\hat{x}(3/2) = 3(1.5)^2 + 4(1.5) + 5 = 6.75 + 6 + 5 = 17.75 \longrightarrow (9)$

$\dot{x}(2) = \left(1-\frac{2}{4}\right)\left(6\left(2-\frac{4}{8}\right)+4\right)$ (from (7))

$= \frac{1}{2}\left(6 \cdot \frac{3}{2} + 4\right) = \frac{1}{2}(13) = 6.5 \longrightarrow (10)$

$$\hat{x}'(1.5) = 6(1.5)+4 = 13. \longrightarrow (11).$$

Let $d\tau = 0.2$, then $dt = \left(1-\frac{\tau}{k}\right)d\tau$ $(\tau=2)$.

$$dt = \left(1-\frac{2}{4}\right)(0.2) = \frac{1}{2}(0.2) = 0.1 \longrightarrow (12).$$

Traversing the curve:

$$x(2+0.2) = x(2) + \dot{x}(2)(0.2) = (11+6.75) + 6.5(0.2)$$

$$= 17.75 + 1.3 = 19.05 \longrightarrow (13)$$

$$\begin{array}{r} 17.75 \\ +\ 1.3 \\ \hline 19.05 \end{array}$$

$$\hat{x}(1.5+0.1) = \hat{x}(1.5) + \hat{x}'(1.5)(0.1)$$

$$= 17.75 + 13(0.1) = 17.75 + 1.3$$

$$= 19.05 \longrightarrow (14)$$

$$\therefore (13) = (14)$$