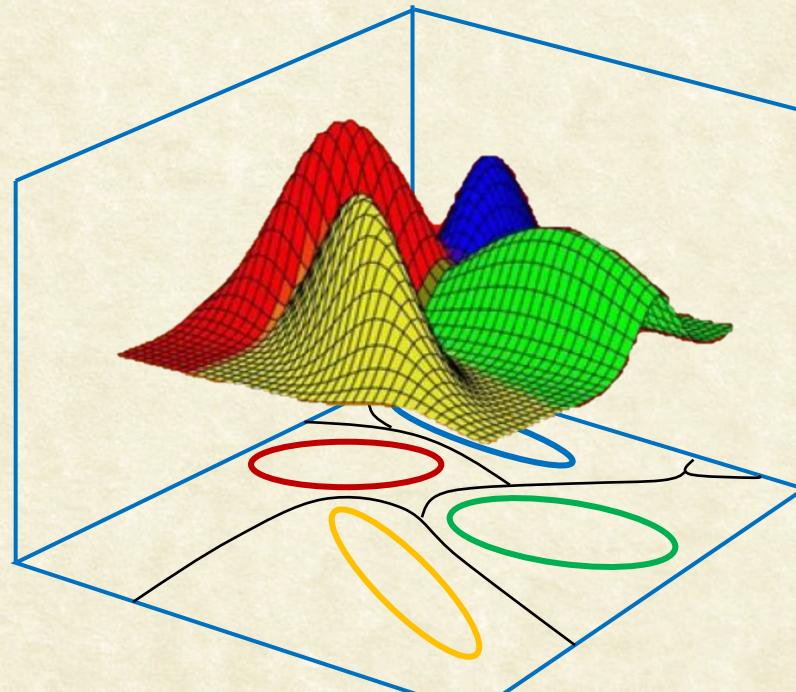




CS7.404: Digital Image Processing

Monsoon 2023: DFT and FFT



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DFT as Matrix Multiplication

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \quad m = 0, 1, 2, \dots, M-1$$

Let $\omega = e^{-j2\pi/M}$

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{M-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(M-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \omega^{3(M-1)} & & \omega^{(M-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{M-1} \end{bmatrix}$$



The FFT Algorithm

- Assume the data vector, f , is of length 2^{k+1} , and let \mathcal{F}_d represent the complete DFT matrix of dimensions: $d \times d$.
- Reorder the terms of the data vector f into even and odd groups
- The DFT matrix can be split into two matrices:

$$\begin{bmatrix} F_1 \\ \vdots \\ F_{2^{k+1}} \end{bmatrix} = [\mathcal{F}_{2^{k+1}}] \begin{bmatrix} f_1 \\ \vdots \\ f_{2^{k+1}} \end{bmatrix} = [\mathbf{I} \quad \mathbf{D}_\omega] \begin{bmatrix} \mathcal{F}_{2^k} & \mathbf{0} \\ \mathbf{0} & \mathcal{F}_{2^k} \end{bmatrix} \begin{bmatrix} f_{2^k}^{even} \\ f_{2^k}^{odd} \end{bmatrix},$$

where \mathbf{D}_ω is a diagonal matrix of powers of ω , and \mathbf{I} is the identity matrix.

- Apply the split Recursively to get the FFT algorithm.
- DFT vs FFT: $4N^2$ vs. $2 N \log_2 N$.



(Some) Properties of FT

	Name:	Condition:	Property:
1	Amplitude scaling	$f(t) \leftrightarrow F(\omega)$, constant K	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, ...	$f(t) + g(t) + \dots \leftrightarrow F(\omega) + G(\omega) + \dots$
3	Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	Even	Real and even $f(t)$	Real and even $F(\omega)$
5	Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$
6	Symmetry	$f(t) \leftrightarrow F(\omega)$	$F(t) \leftrightarrow 2\pi f(-\omega)$
7	Time scaling	$f(t) \leftrightarrow F(\omega)$, real s	$f(st) \leftrightarrow \frac{1}{ s } F(\frac{\omega}{s})$
8	Time shift	$f(t) \leftrightarrow F(\omega)$	$f(t - t_o) \leftrightarrow F(\omega)e^{-j\omega t_o}$
9	Frequency shift	$f(t) \leftrightarrow F(\omega)$	$f(t)e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$
10	Modulation	$f(t) \leftrightarrow F(\omega)$	$f(t) \cos(\omega_o t) \leftrightarrow \frac{1}{2}F(\omega - \omega_o) + \frac{1}{2}F(\omega + \omega_o)$
11	Time derivative	Differentiable $f(t) \leftrightarrow F(\omega)$	$\frac{df}{dt} \leftrightarrow j\omega F(\omega)$
12	Freq derivative	$f(t) \leftrightarrow F(\omega)$	$-jtf(t) \leftrightarrow \frac{d}{d\omega} F(\omega)$
13	Time convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$
14	Freq convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t)g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$
15	Compact form	Real $f(t)$	$f(t) = \frac{1}{2\pi} \int_0^\infty 2 F(\omega) \cos(\omega t + \angle F(\omega)) d\omega$
16	Parseval, Energy W	$f(t) \leftrightarrow F(\omega)$	$W \equiv \int_{-\infty}^{\infty} f(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) ^2 d\omega$



DFT and IDFT

$$F_{[m]} = \sum_{n=0}^{(M-1)} f_n e^{\frac{-j2\pi nm}{M}}, \quad m = 0, 1, \dots, (M-1)$$

$$f_n = \frac{1}{M} \sum_{m=0}^{(M-1)} F_{[m]} e^{\frac{j2\pi nm}{M}}, \quad n = 0, 1, \dots, (M-1)$$

$F_{[m]}$

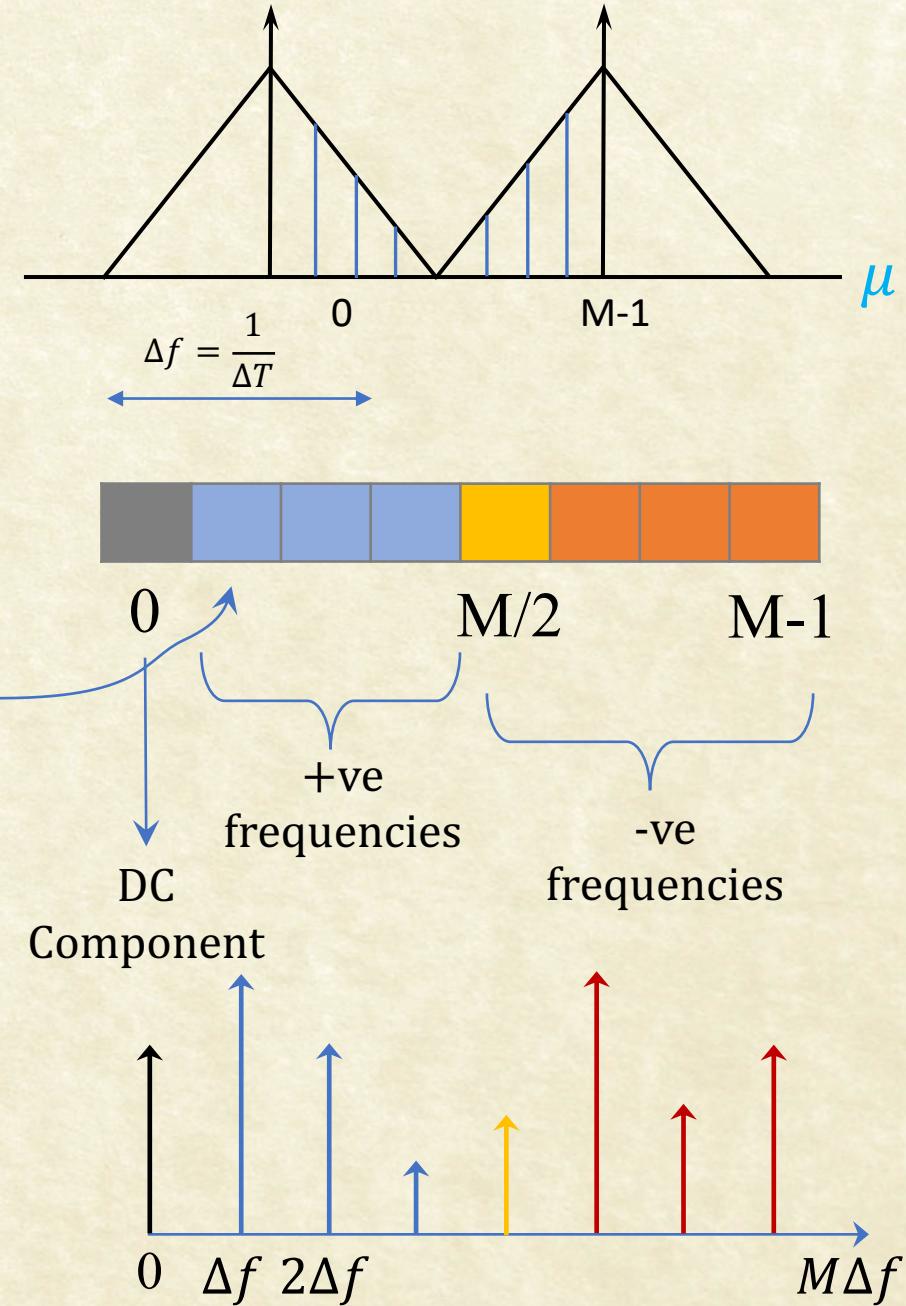
- A complex value
- Represents amplitude, phase of function f_n 's content at angular frequency $2\pi m/M$

DFT: A Record of 'energy' portion at various frequency bands present in input function: f_n



DFT (in practice)

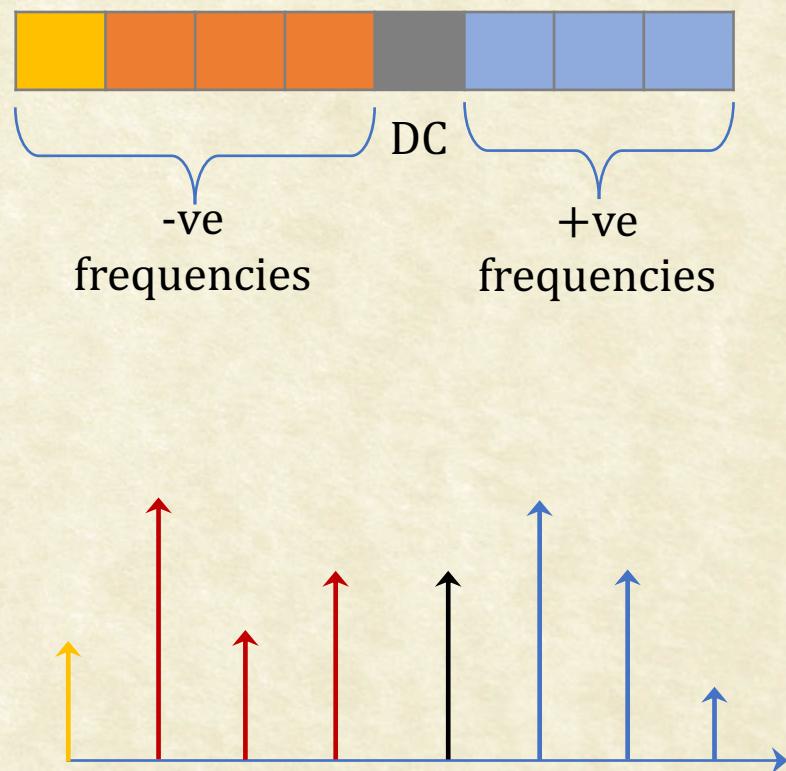
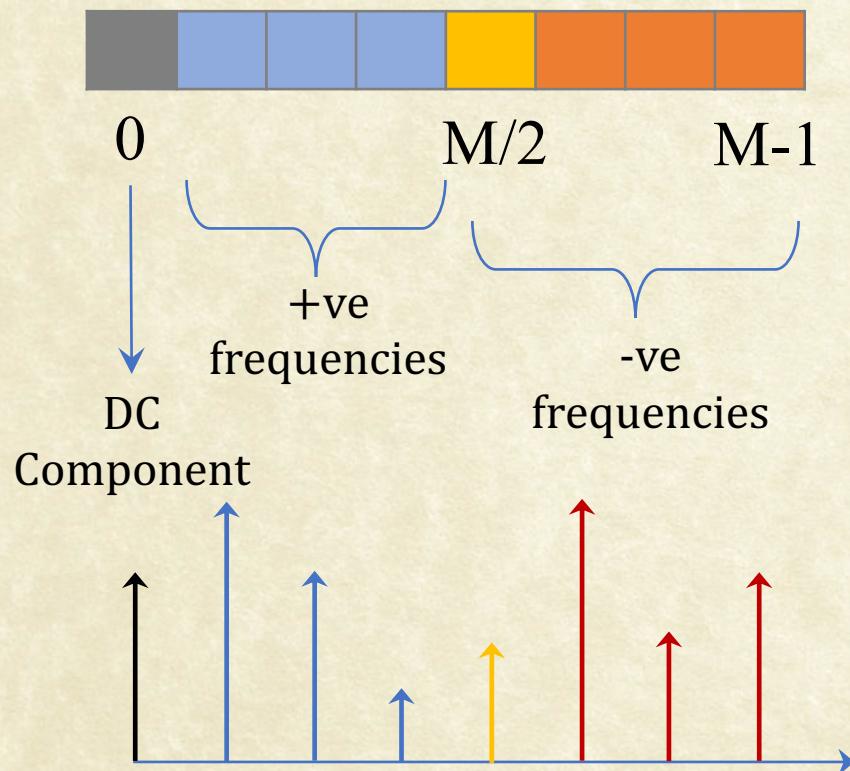
$$F[m] = \sum_{n=0}^{(M-1)} f_n e^{\frac{-j2\pi nm}{M}}, \quad m = 0, 1, \dots, (M-1)$$





DFT – Center Shifted (for plotting)

$$f[x]e^{\frac{j2\pi u_o x}{M}} \leftrightarrow F(u - u_o)$$



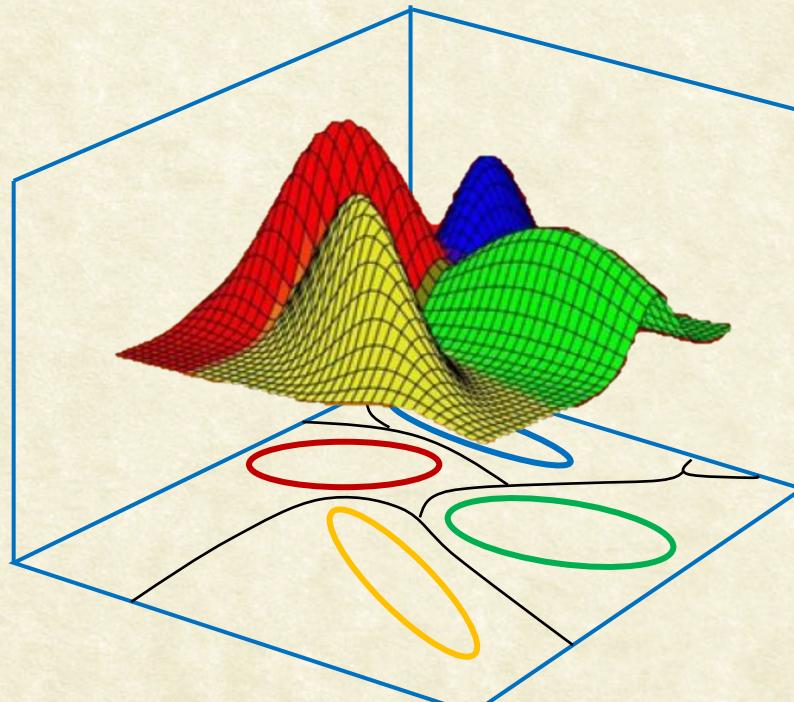


Questions?



CS7.404: Digital Image Processing

Monsoon 2023: Fourier Transform for 2D Data



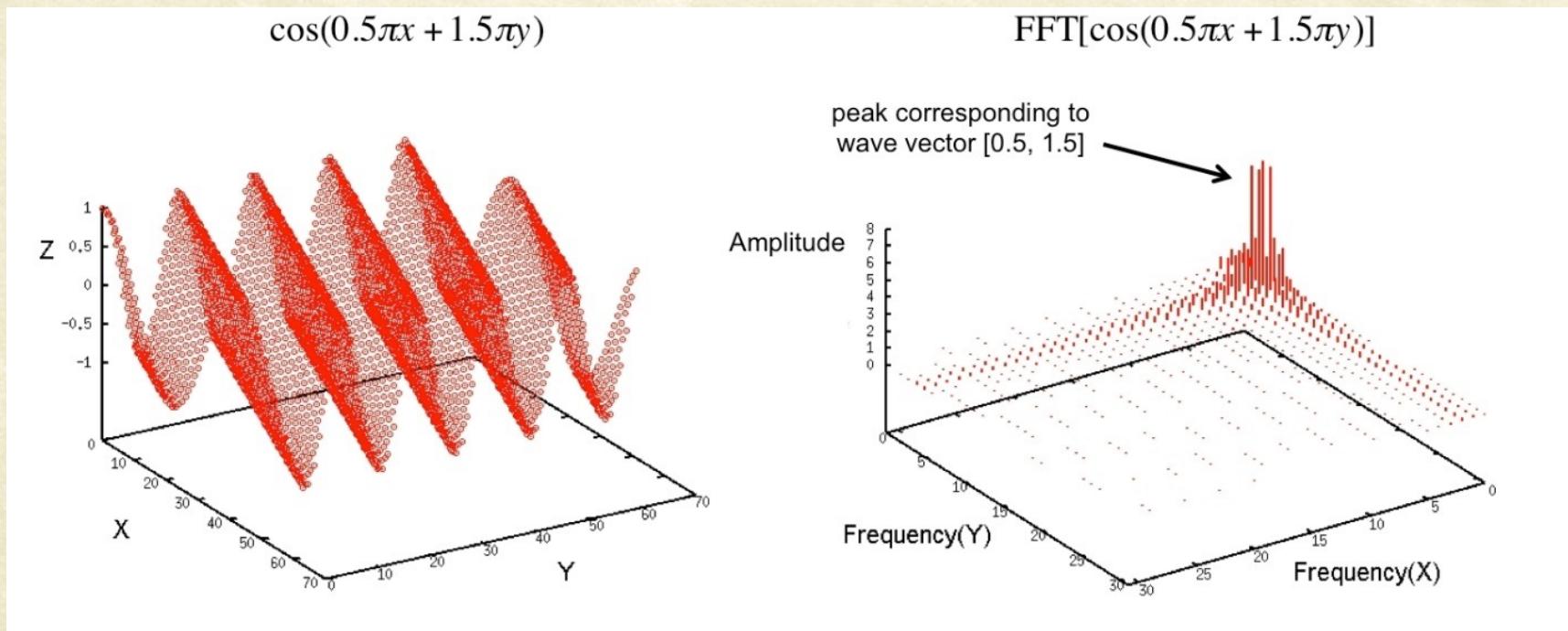
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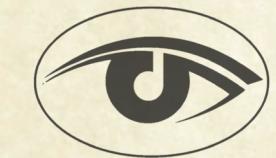


2D DFT and IDFT

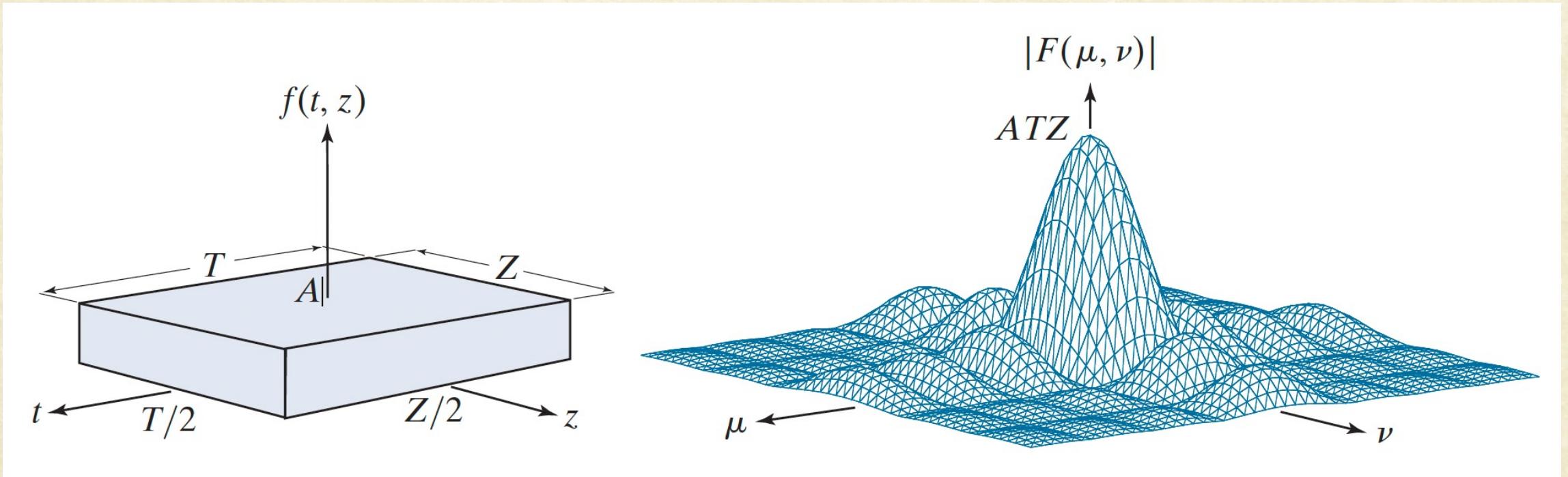
$$F_{[m,n]} = \sum_{y=0}^{(N-1)} \sum_{x=0}^{(M-1)} f_{[x,y]} e^{-2\pi j \left(\frac{mx}{M} + \frac{ny}{N} \right)}$$



$$f_{[x,y]} = \frac{1}{MN} \sum_{n=0}^{(N-1)} \sum_{m=0}^{(M-1)} F_{[m,n]} e^{2\pi j \left(\frac{mx}{M} + \frac{ny}{N} \right)}$$



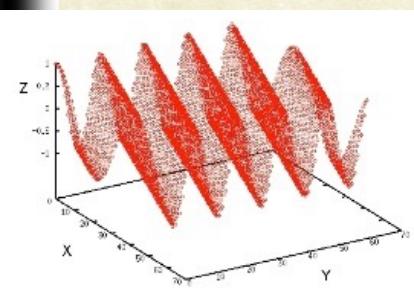
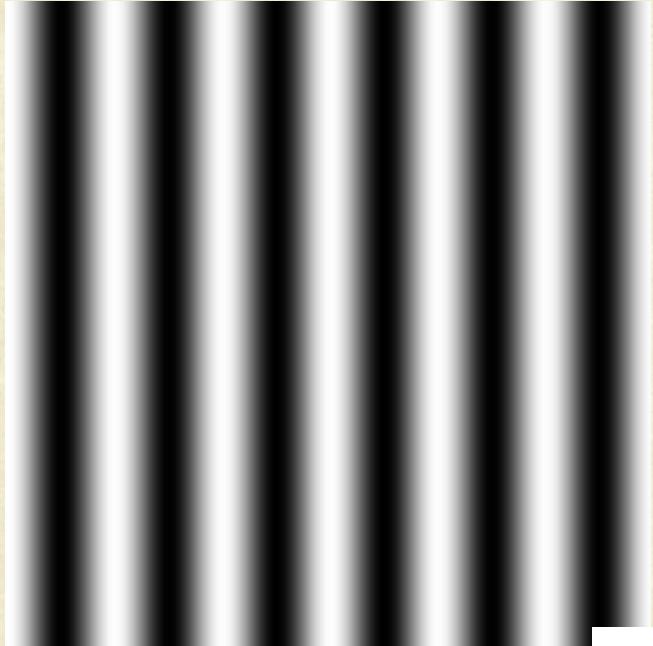
2D DFT of Box Filter



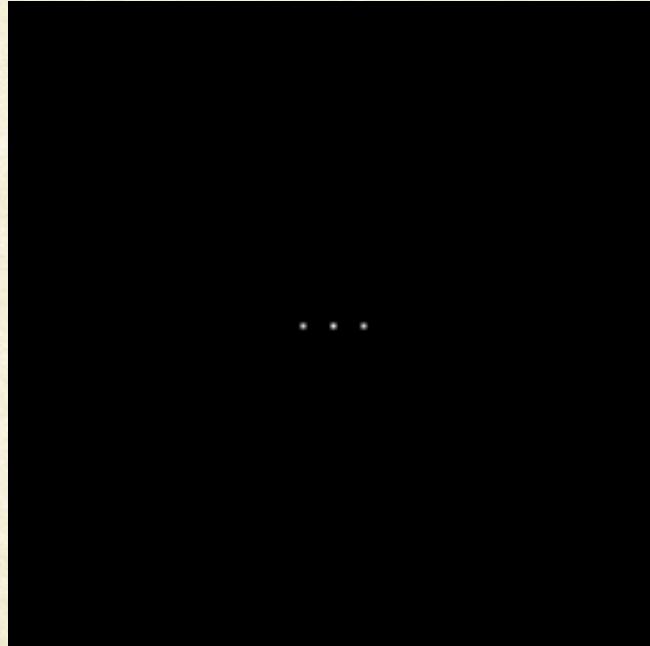


DFT for simple spatial patterns

Brightness Image

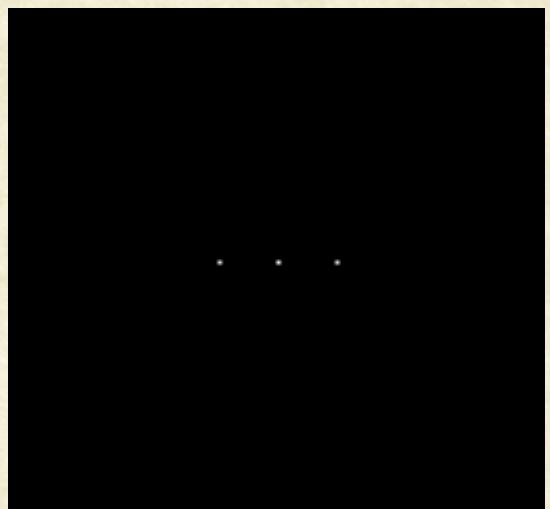
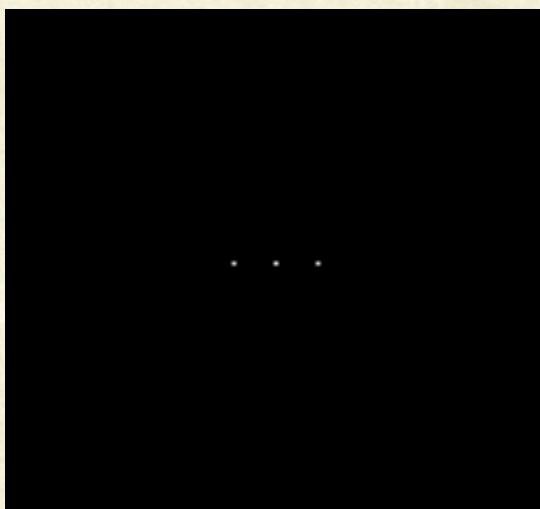
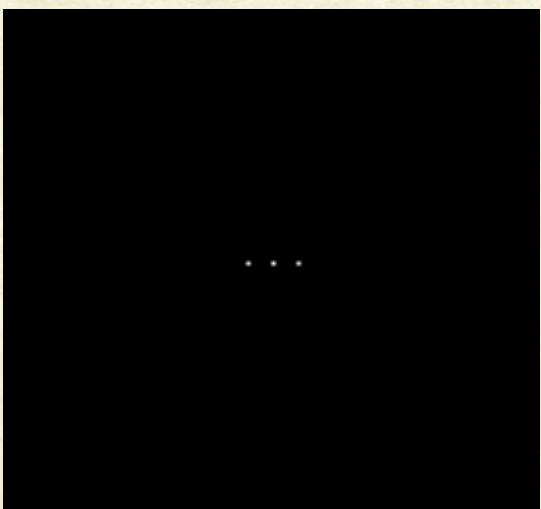
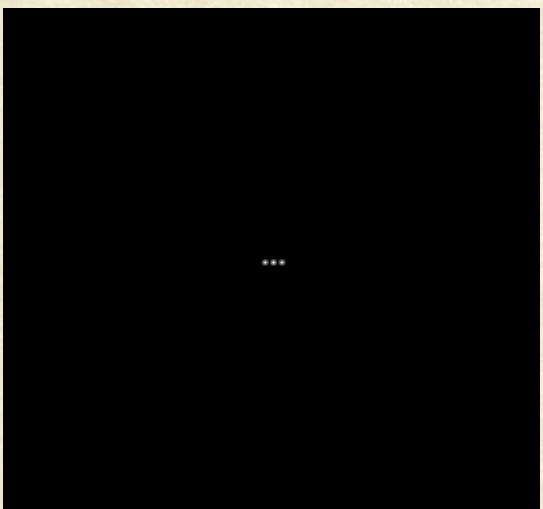
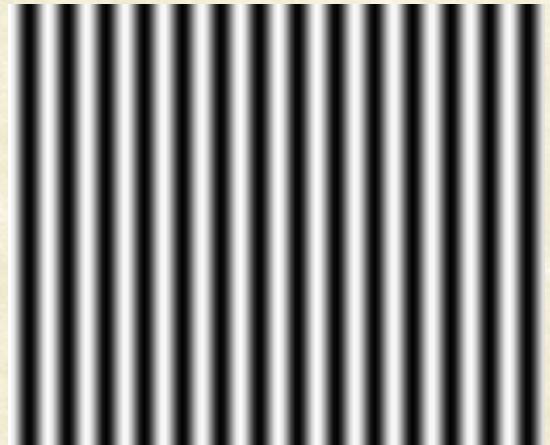
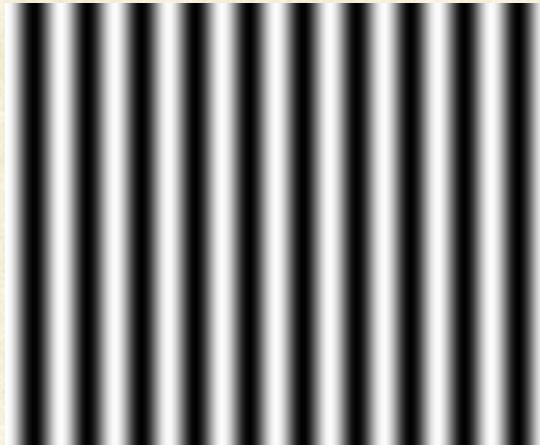
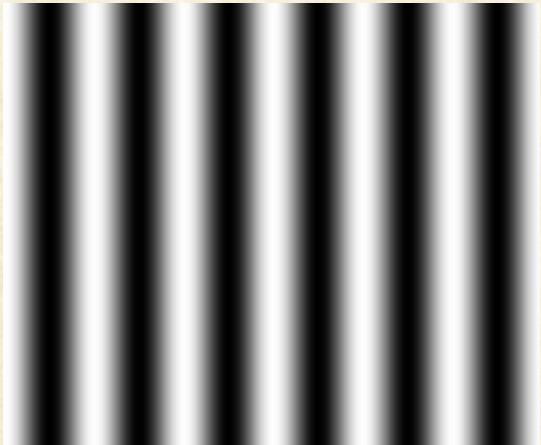
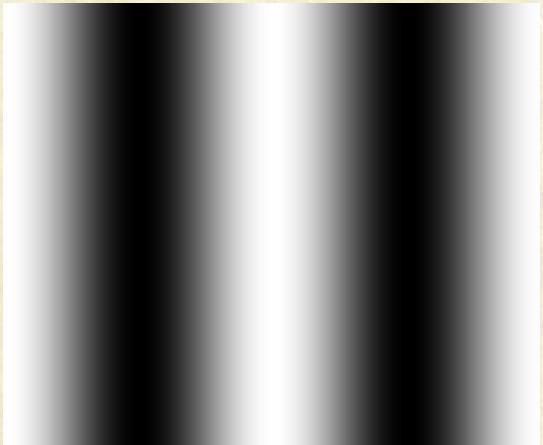


Fourier transform spectrum



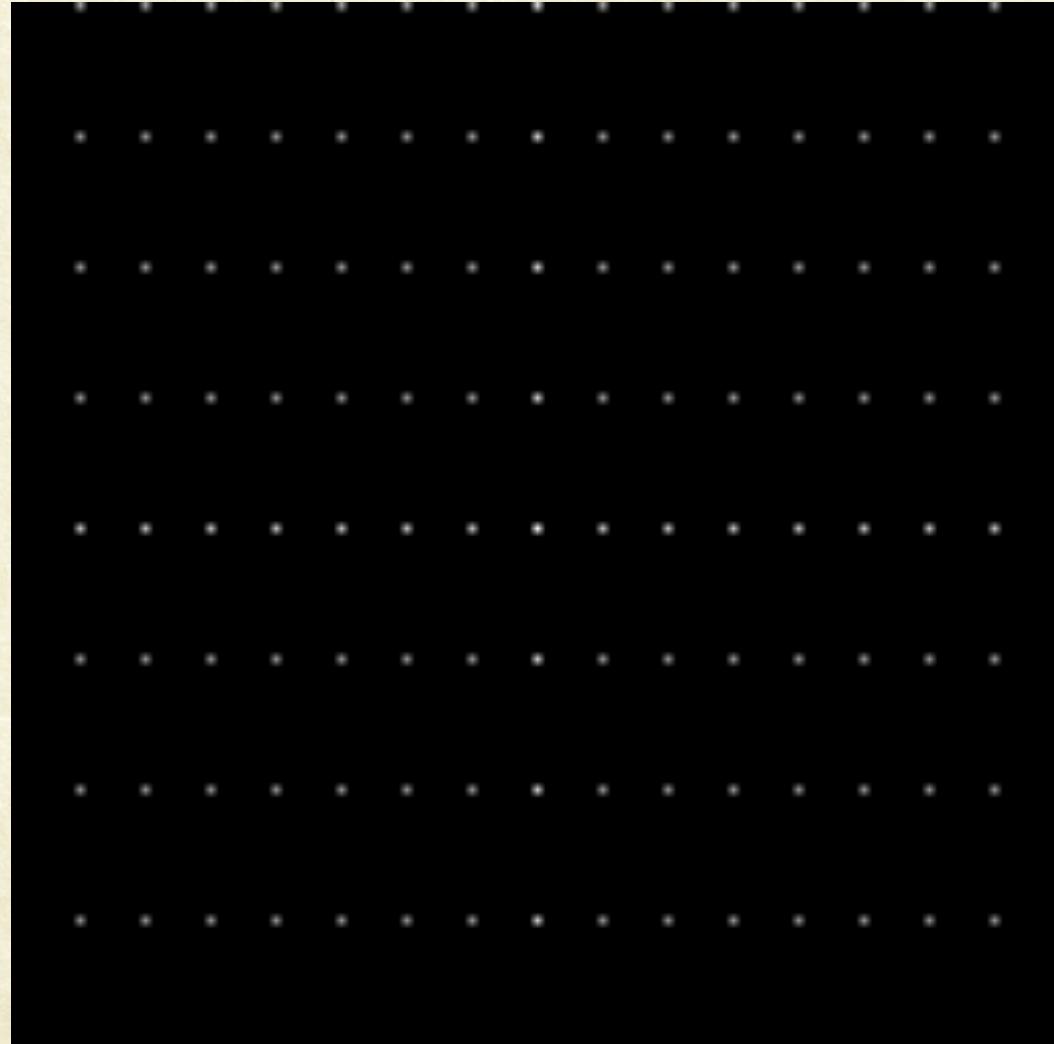
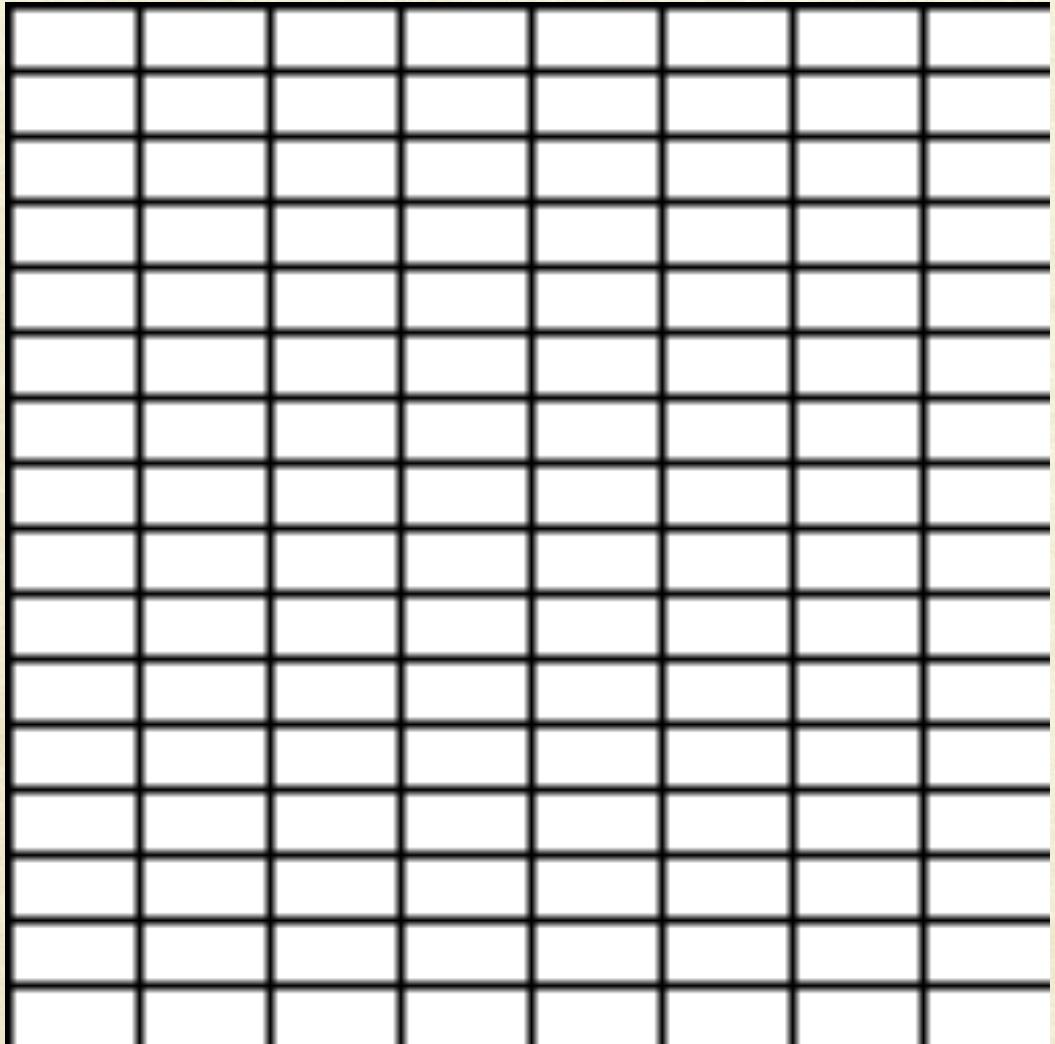


DFT Example



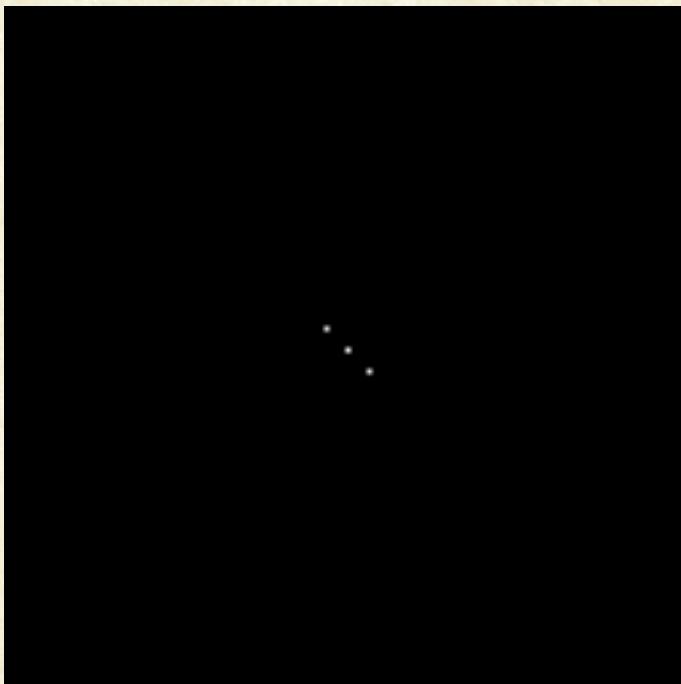
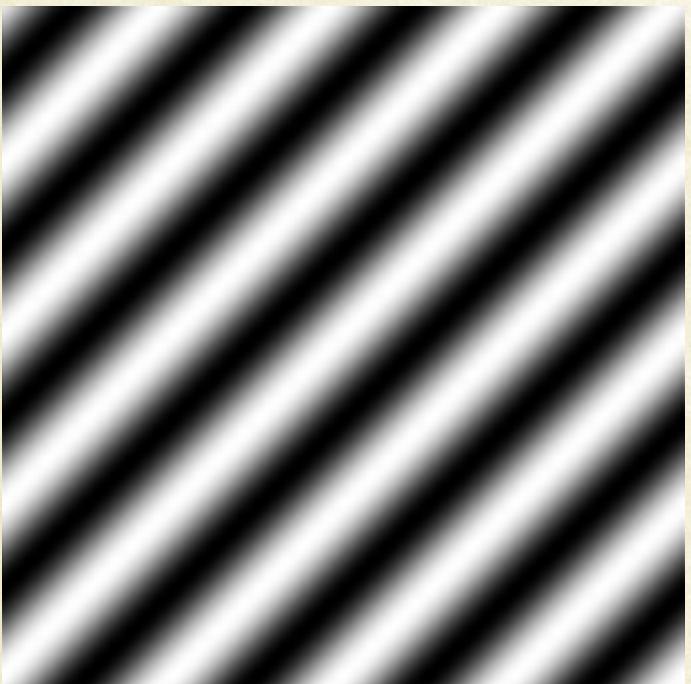


Example



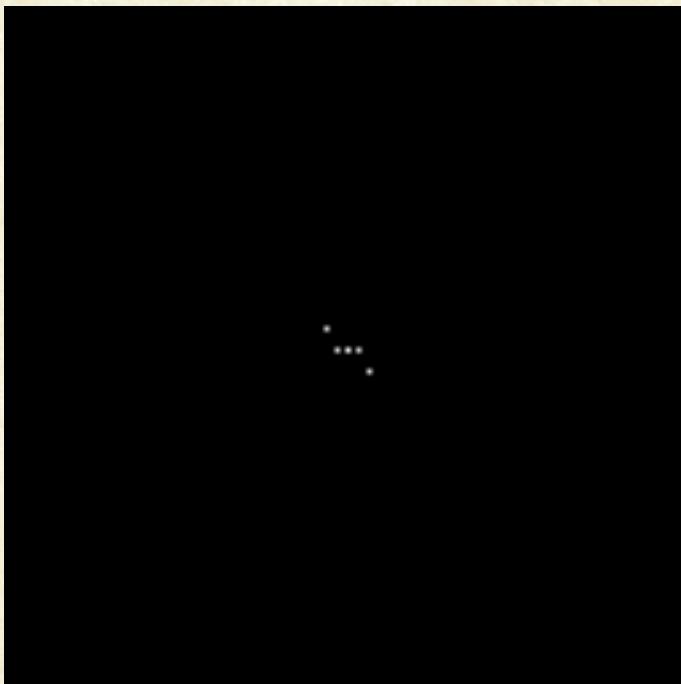
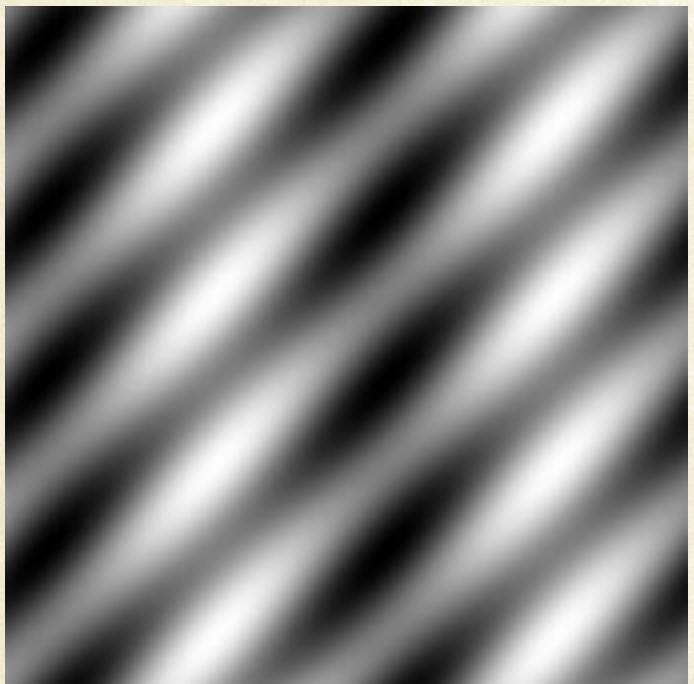


DFT Example (Rotation)



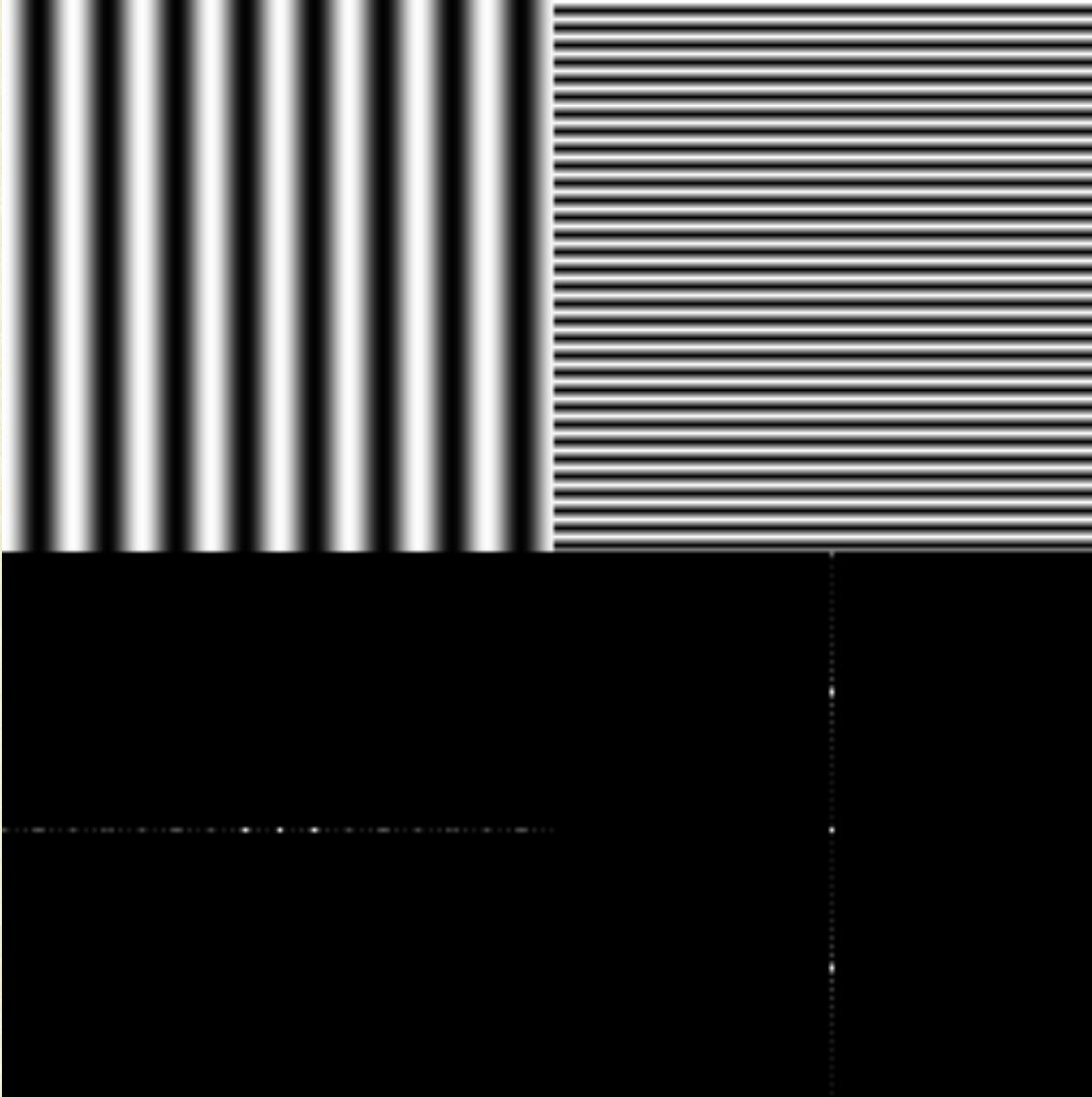


DFT Example (Sum of Signals)



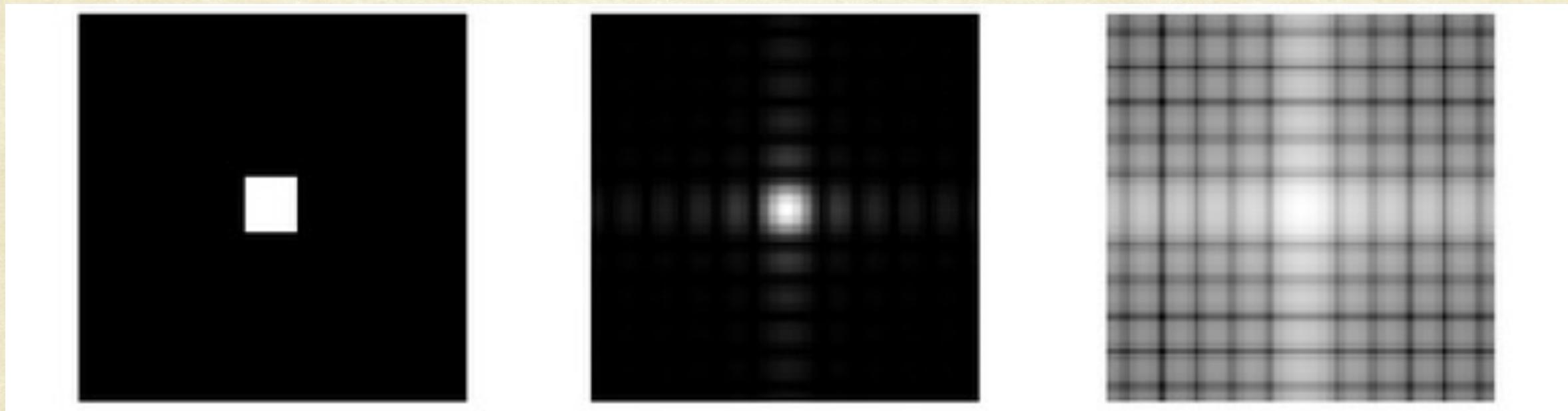
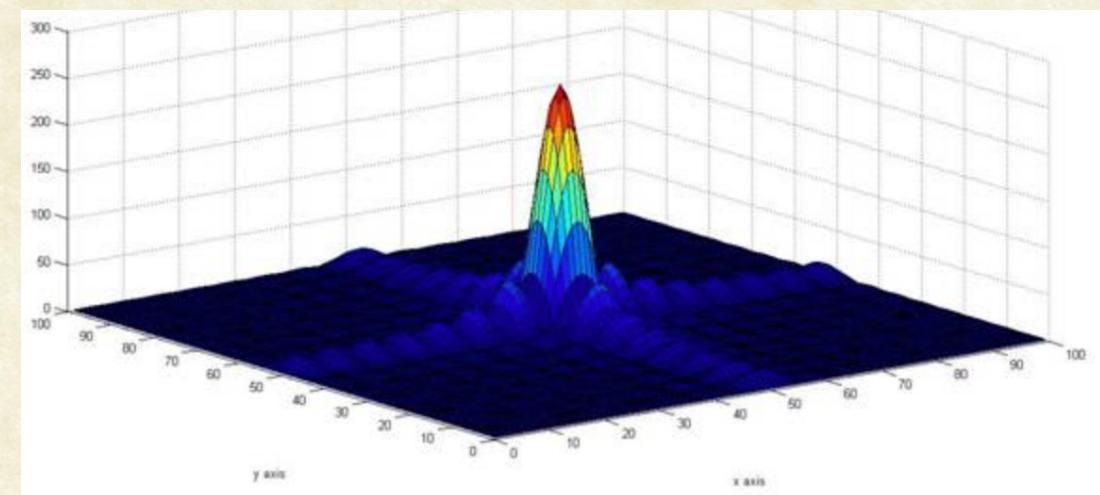
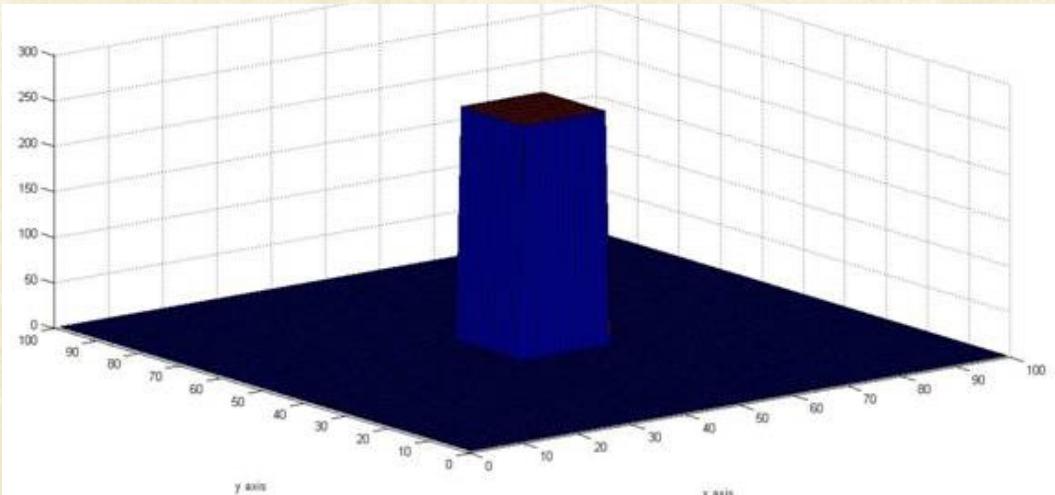


DFT for simple ‘spatial’ patterns



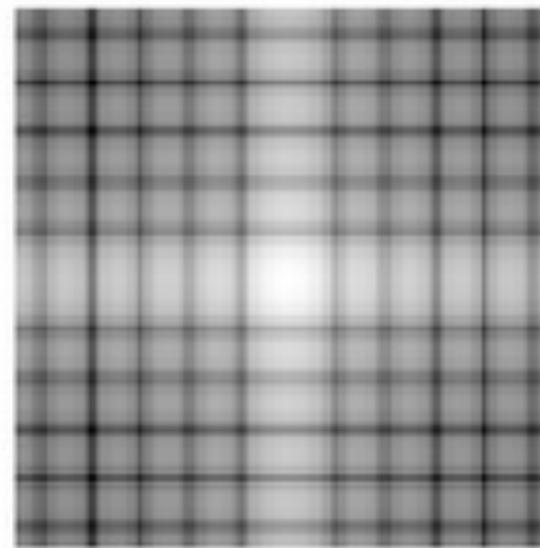
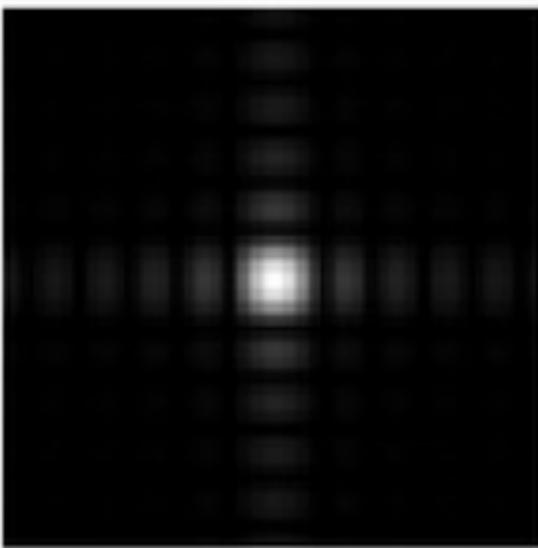
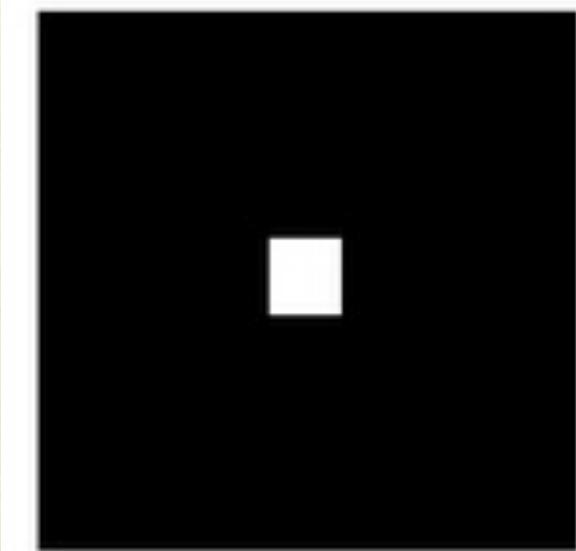


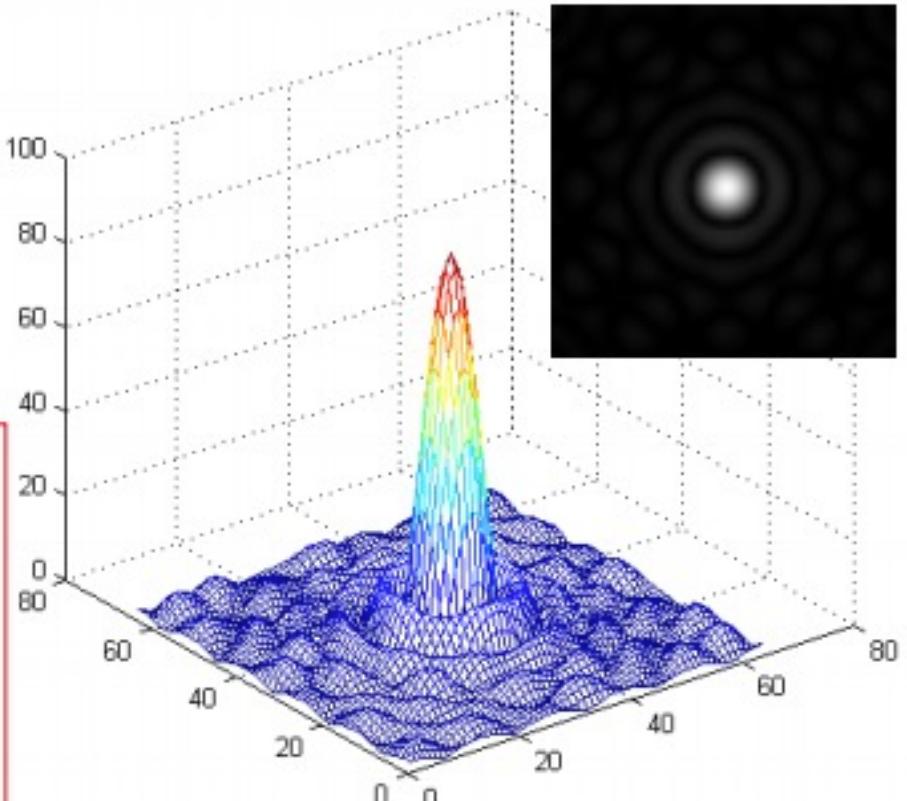
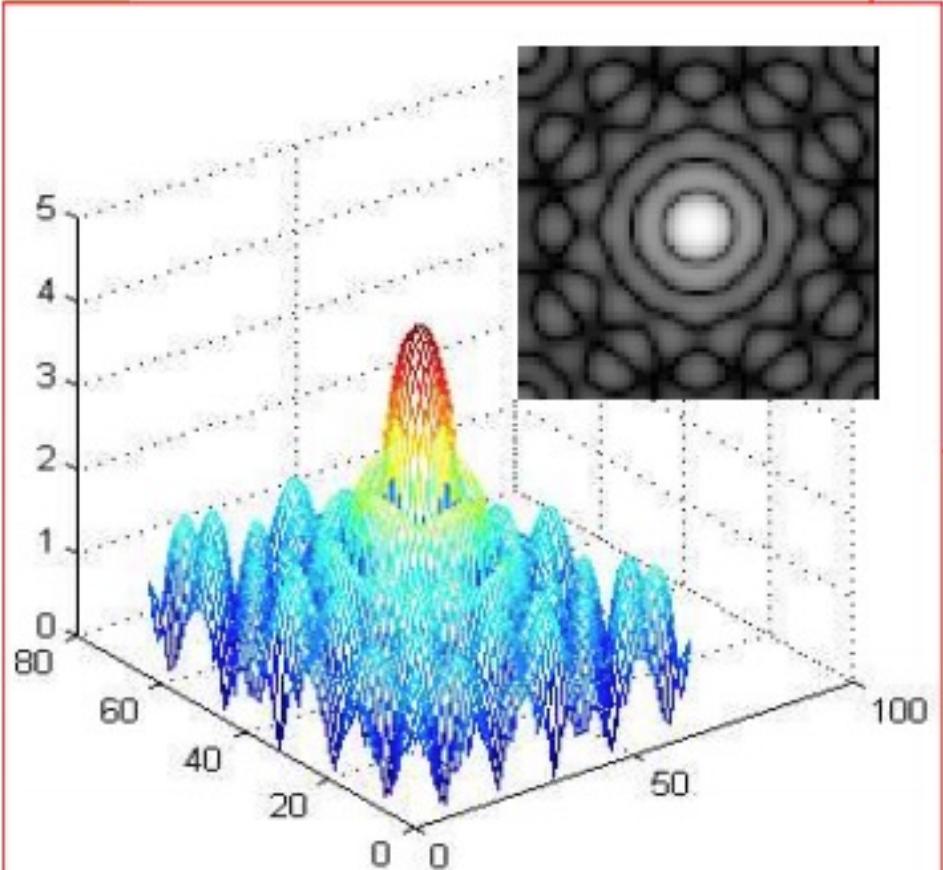
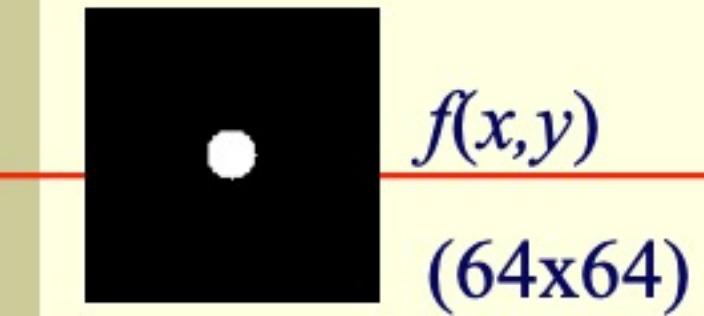
DFT Example (Rect, Log Transformation)





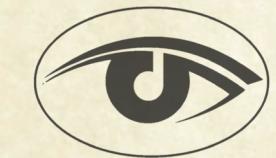
DFT Example (Rect, Log Transformation)



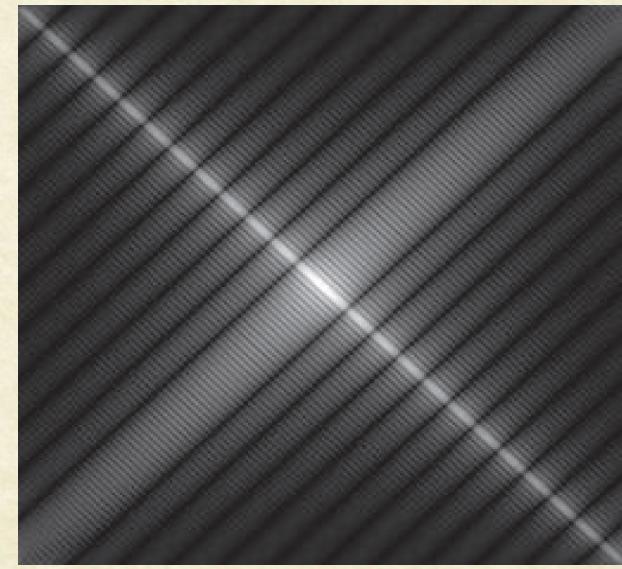
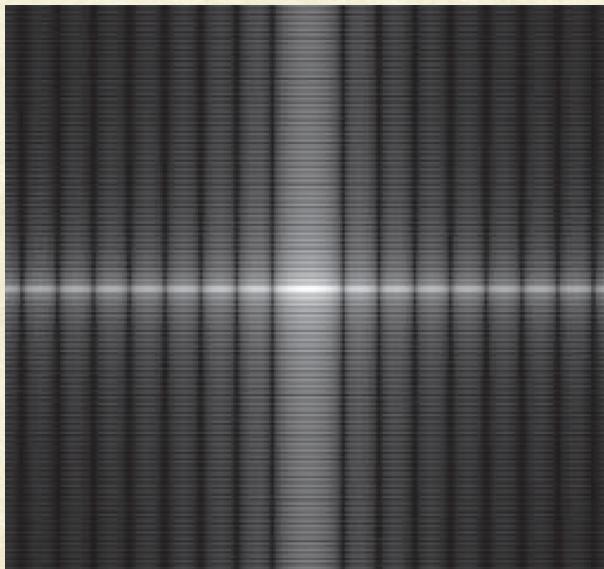
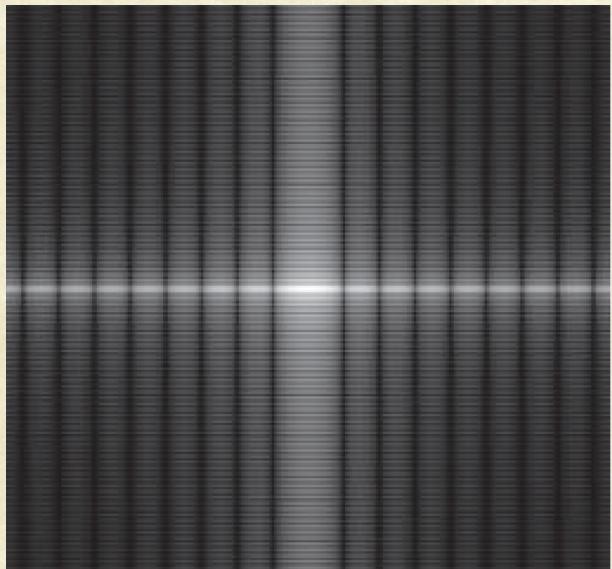
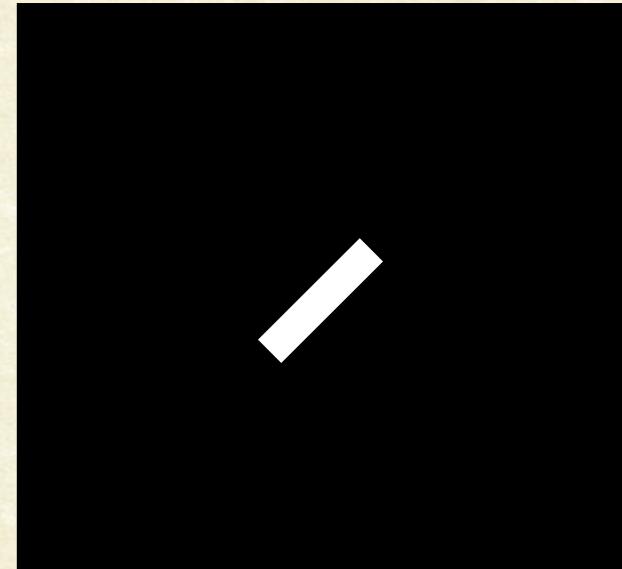
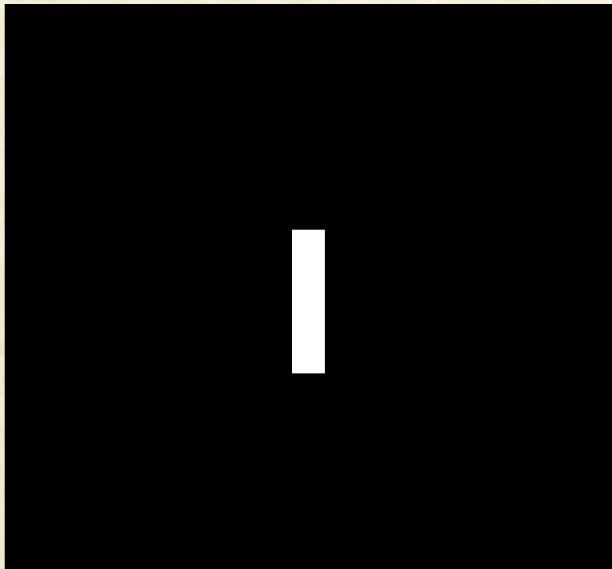


$$|F(u,v)|$$

$$\log(1+|F(u,v)|)$$

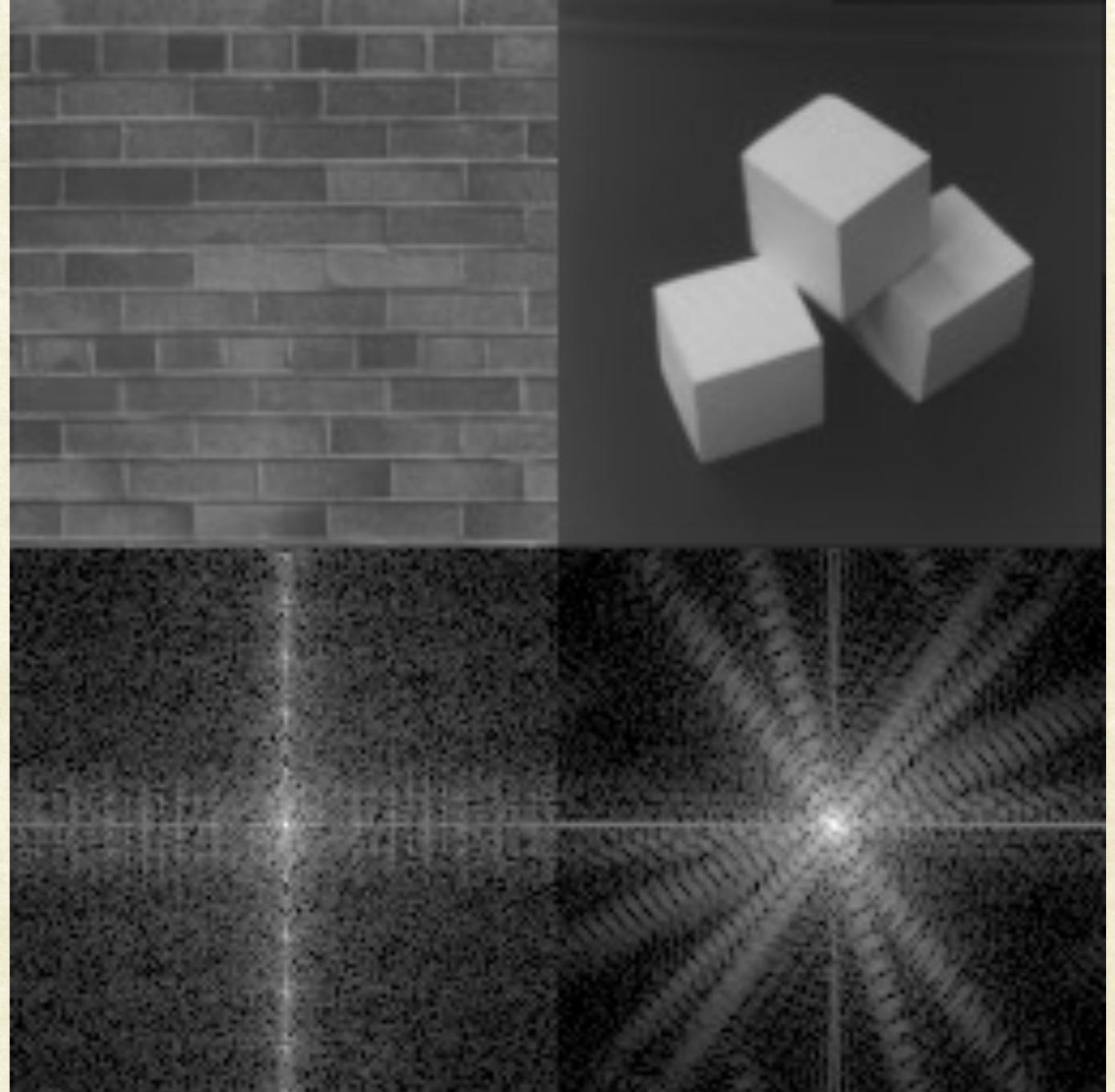


DFT Example (Translation, Rotation)



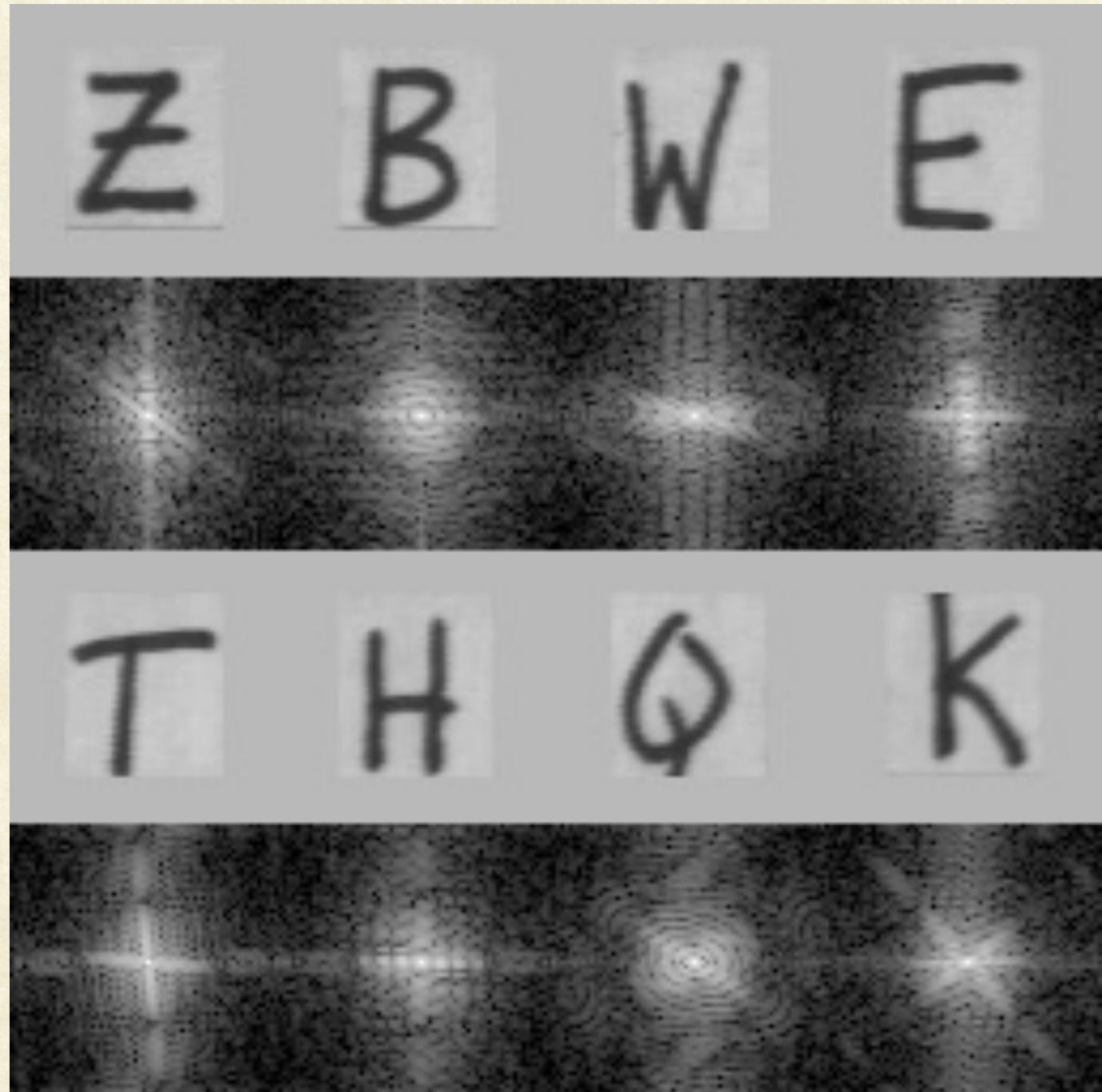


Images and their Spectra



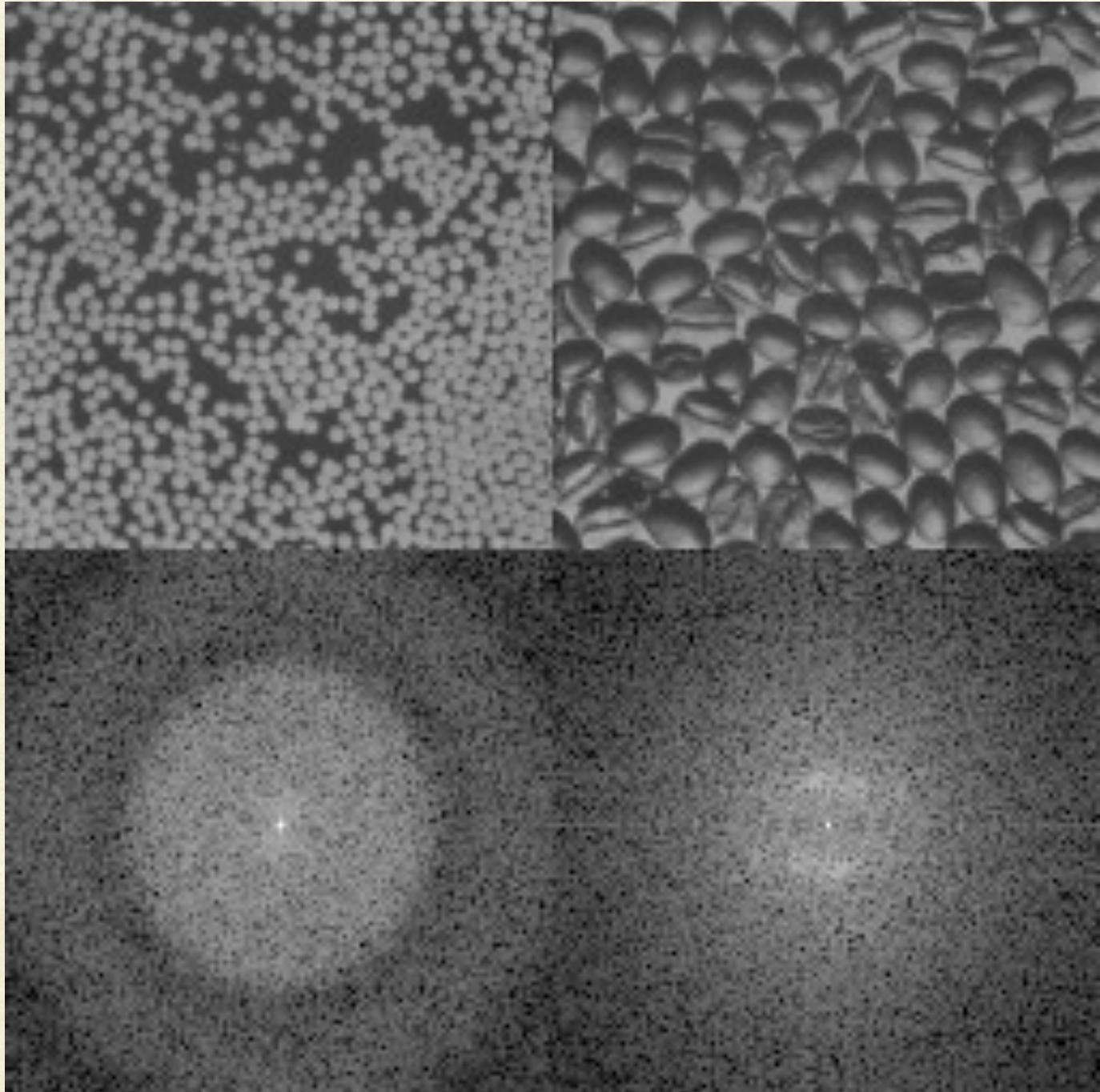


Images and their Spectra



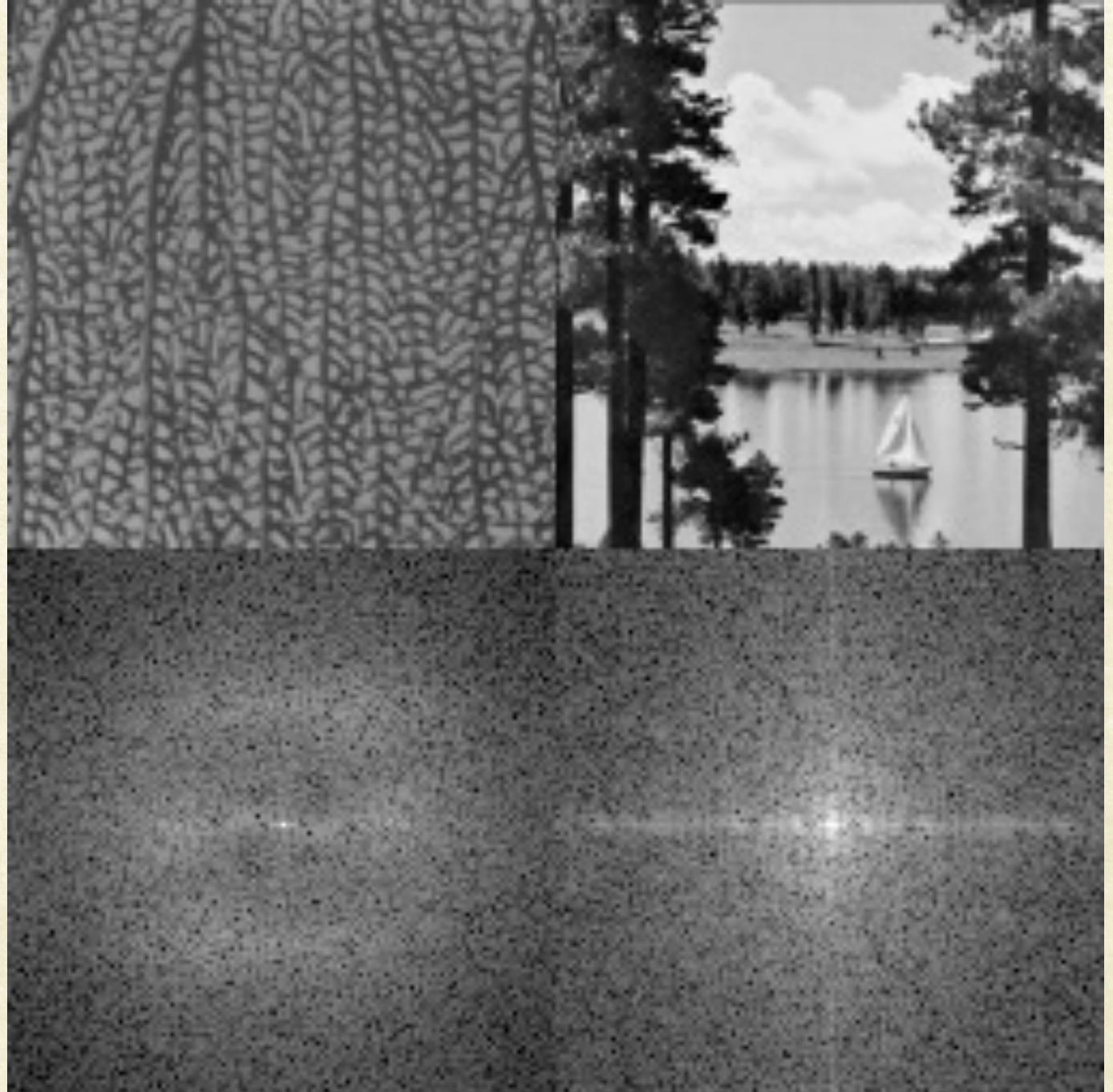


Images and their Spectra





Images and their Spectra



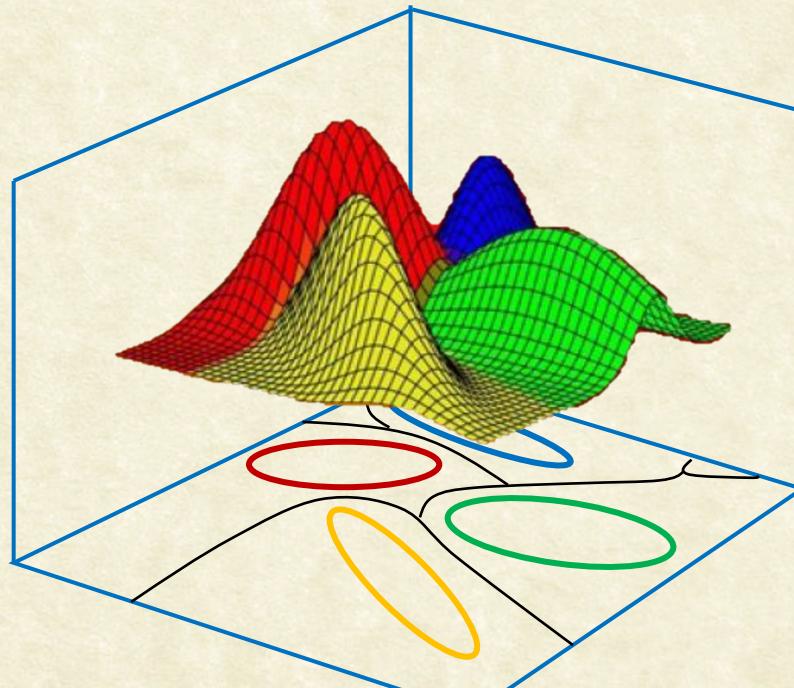


Questions?



CS7.404: Digital Image Processing

Monsoon 2023: Sampling and Aliasing

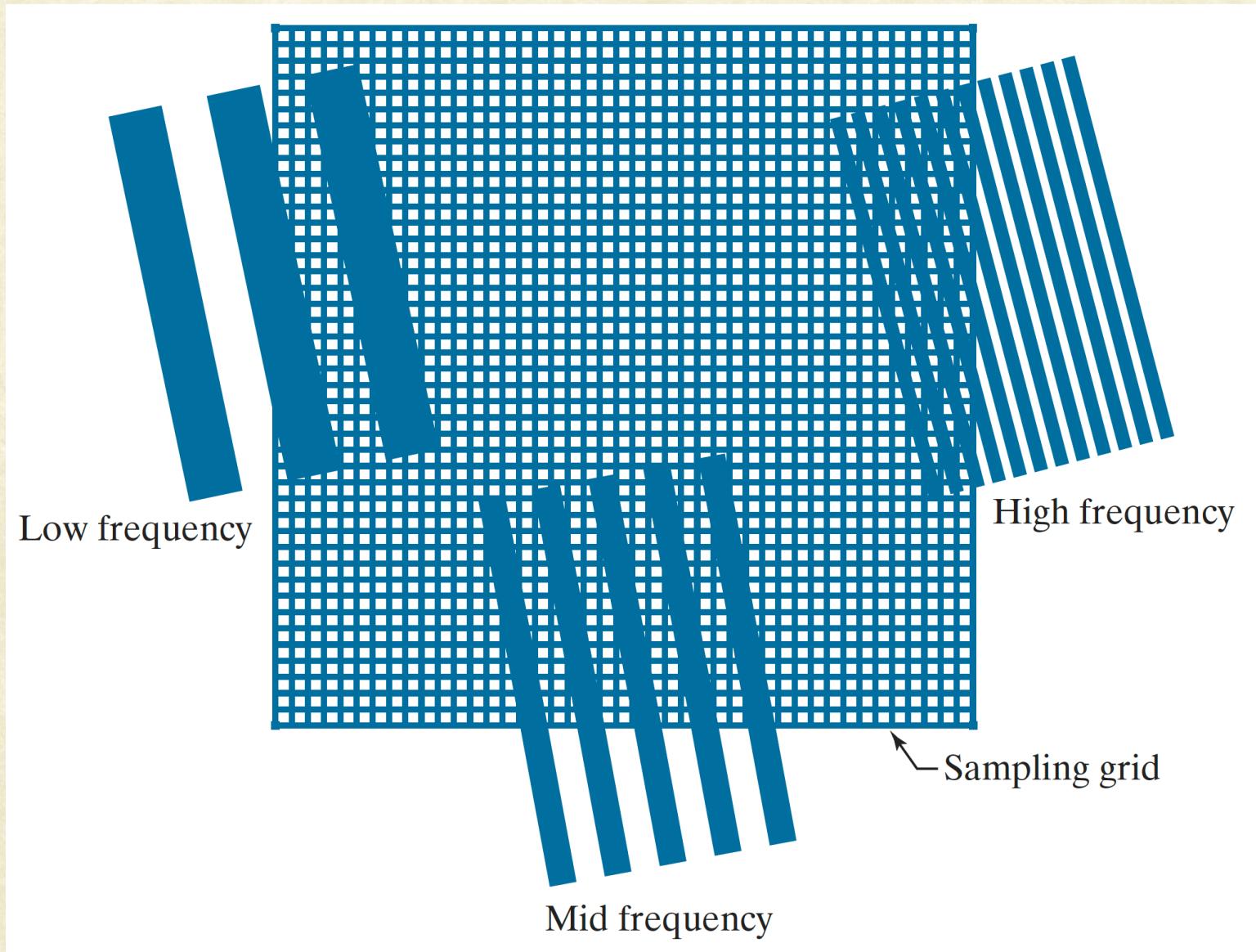


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Sampling Frequency and Aliasing





Sampling Frequency and Aliasing



Original (Minimal Aliasing)



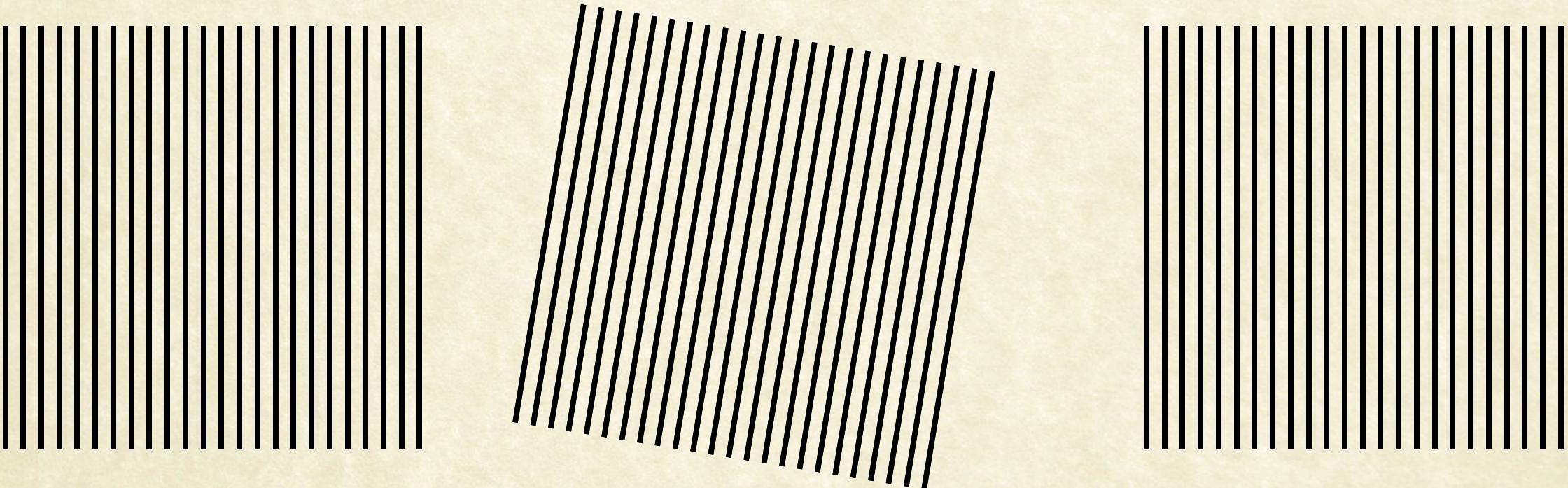
Down sampled (Strong Aliasing)



Lowpass filtered & Down sampled

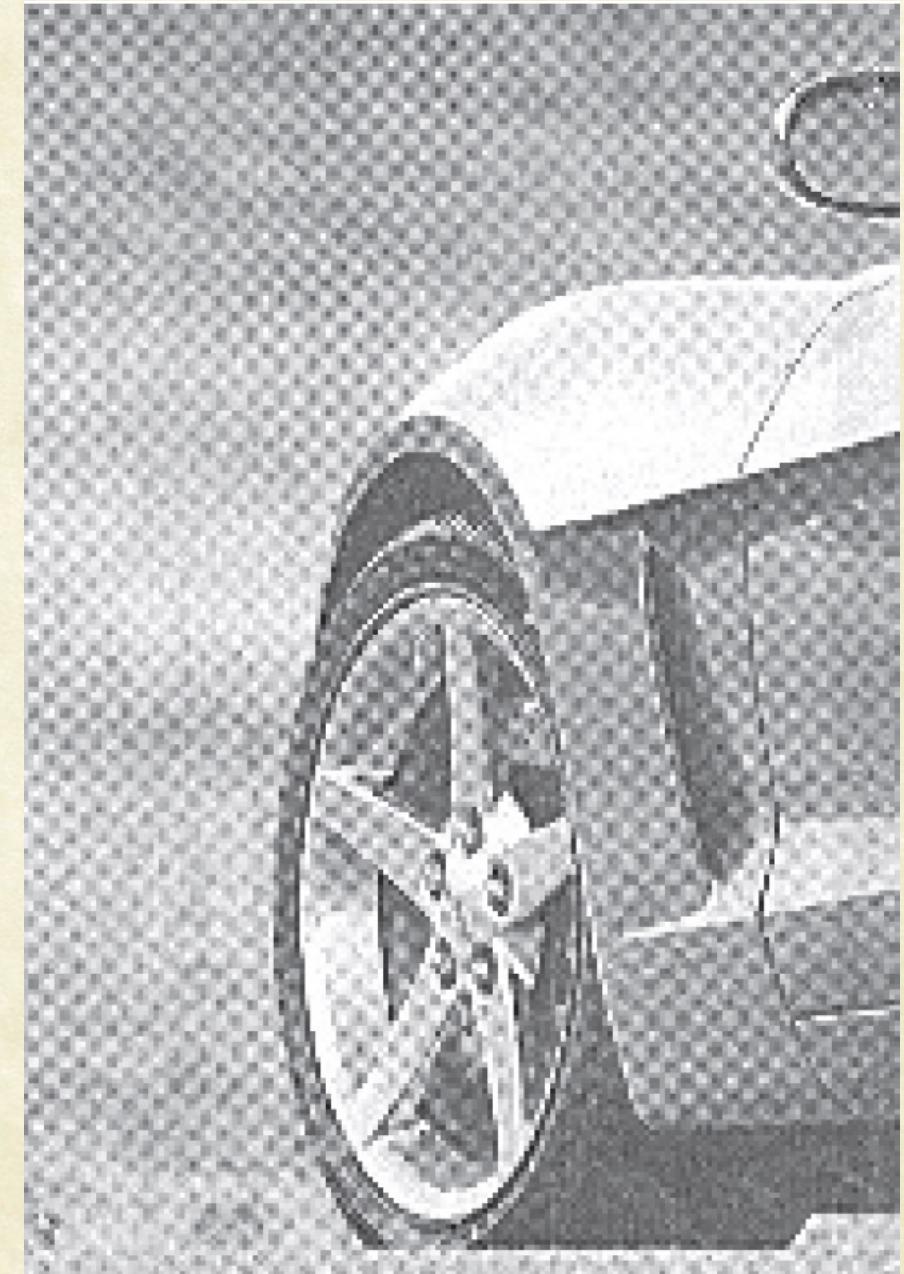
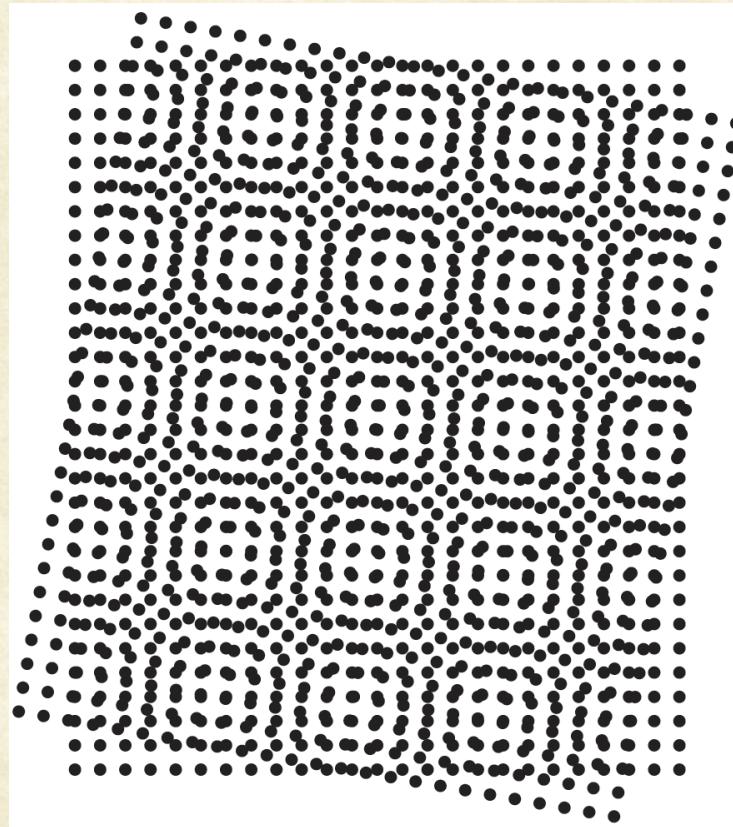
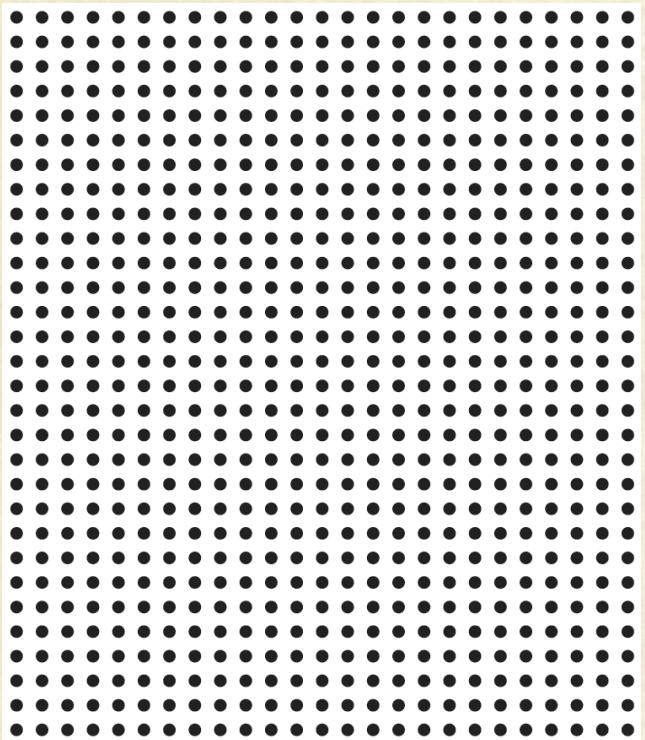


Moiré effect





Moiré effect in Print



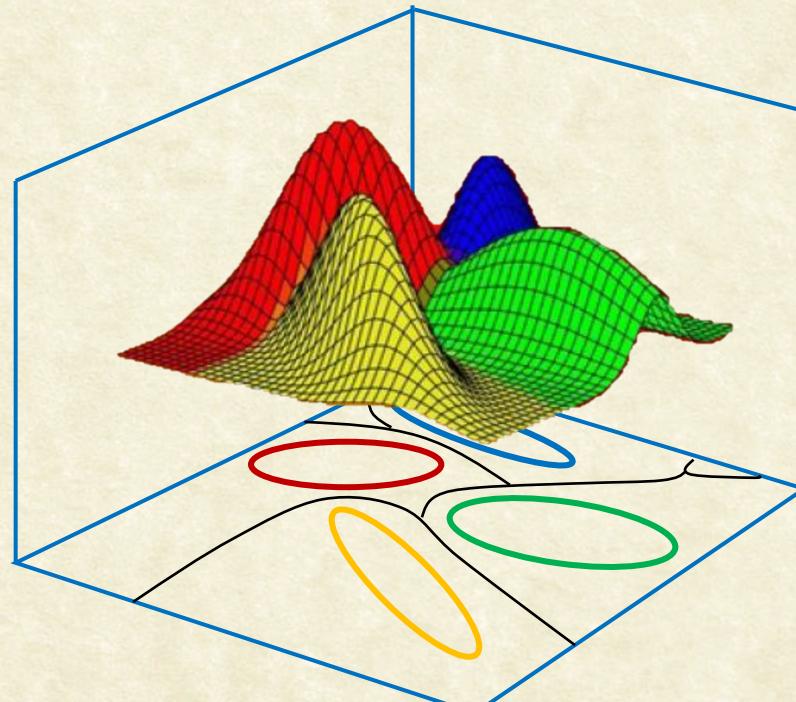


Questions?



CS7.404: Digital Image Processing

Monsoon 2023: Phase and Spectrum



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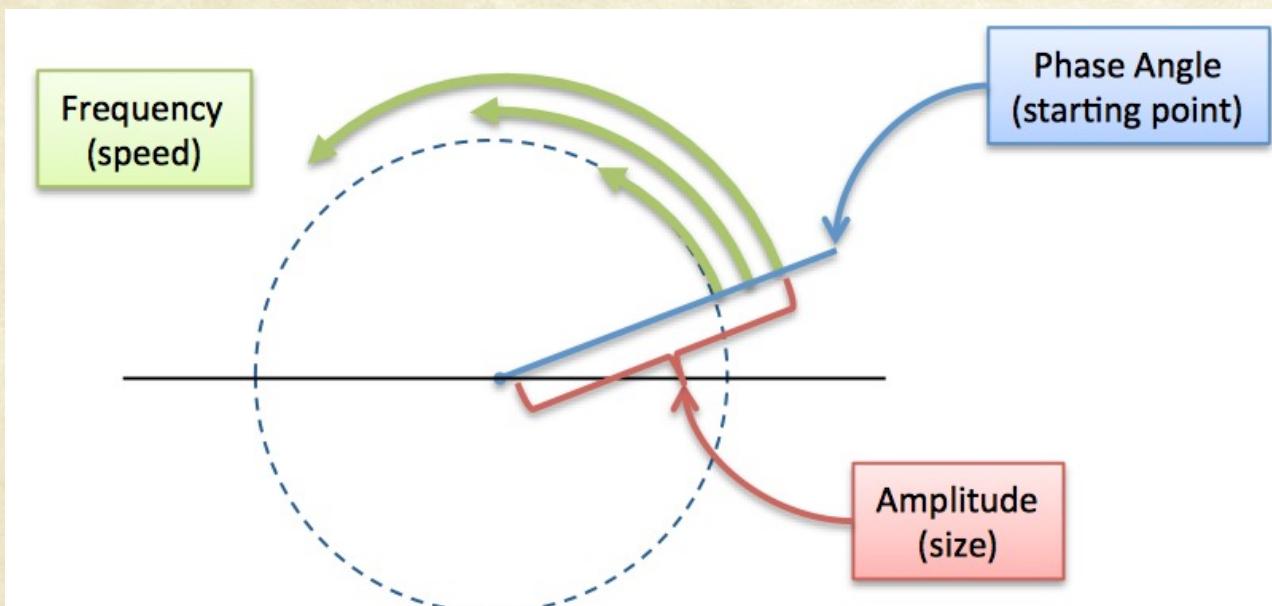
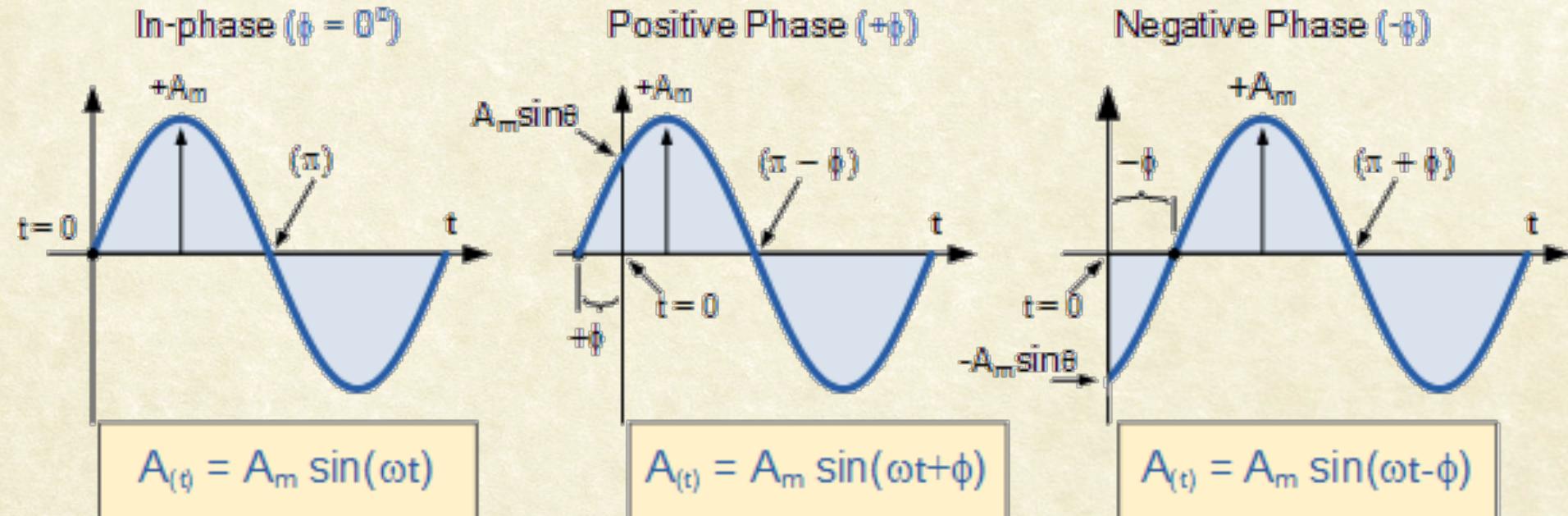


Important Terms

- Magnitude Spectrum: $|F(u)| = [R^2(u) + I^2(u)]^{1/2}$
- Phase Spectrum: $\Phi(u) = \tan^{-1} \left(\frac{I(u)}{R(u)} \right)$
- Power Spectrum: $P(u) = |F(u)|^2$



Phase





Magnitude and Phase Spectra



Figure 4a

Original

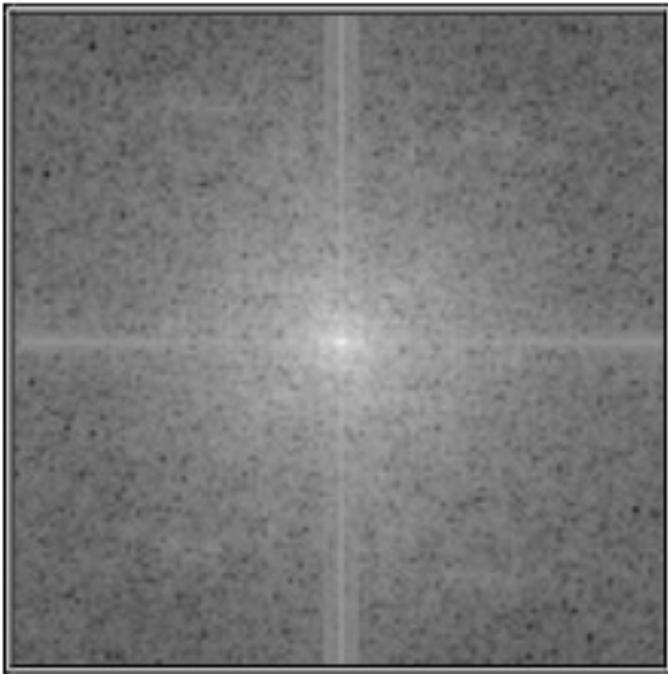


Figure 4b

$\log(|A(\Omega, \Psi)|)$

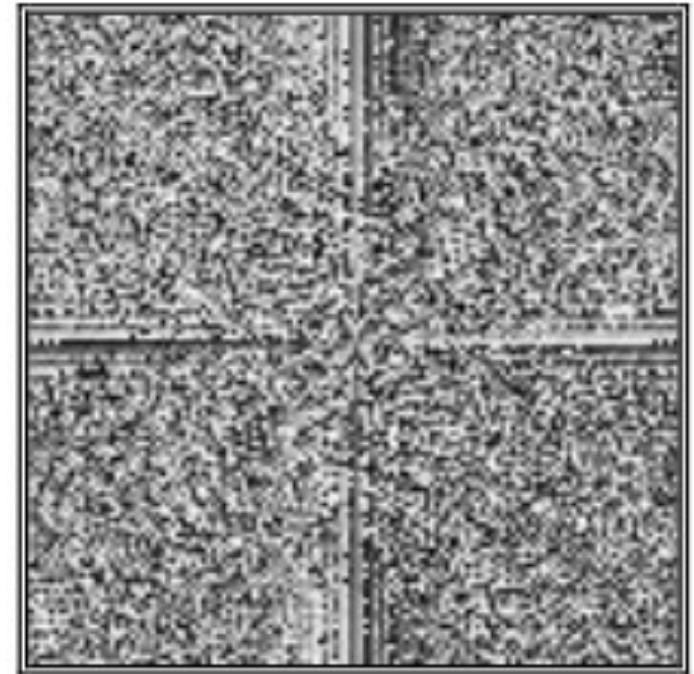


Figure 4c

$\phi(\Omega, \Psi)$

$$\tan^{-1} \left(\frac{Im}{Re} \right)$$

We generally do not display PHASE images because most people who see them shortly thereafter succumb to hallucinogenics or end up in a Tibetan monastery – John Brayer



Magnitude and Phase Spectra for reconstruction



Figure 4a

Original

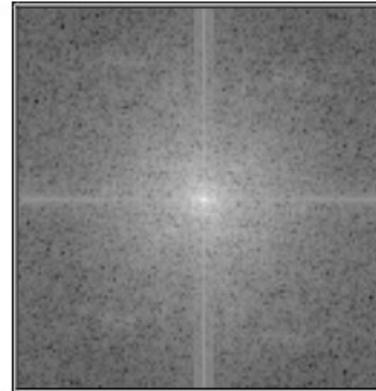


Figure 4b

$\log(|A(\Omega, \Psi)|)$

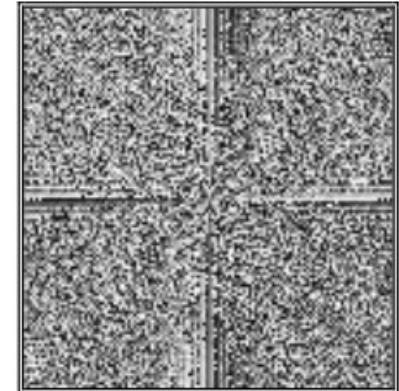


Figure 4c

$\varphi(\Omega, \Psi)$

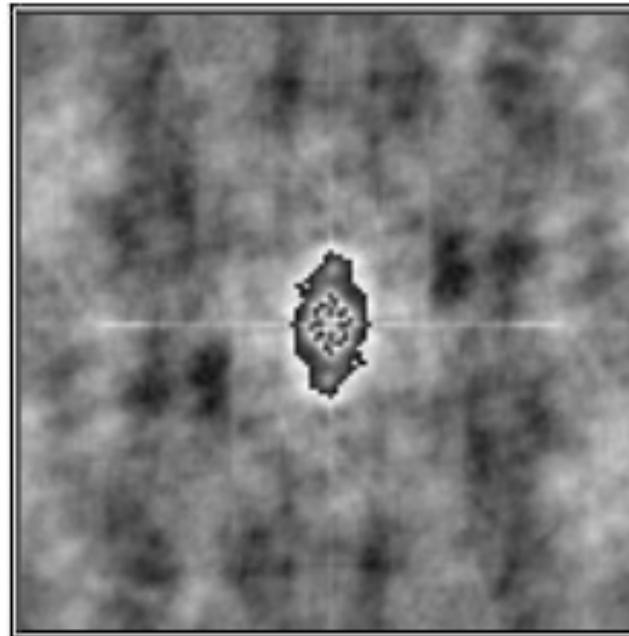


Figure 5a

$\varphi(\Omega, \Psi) = 0$



Figure 5b

$|A(\Omega, \Psi)| = \text{constant}$



Magnitude vs. Phase

Boy

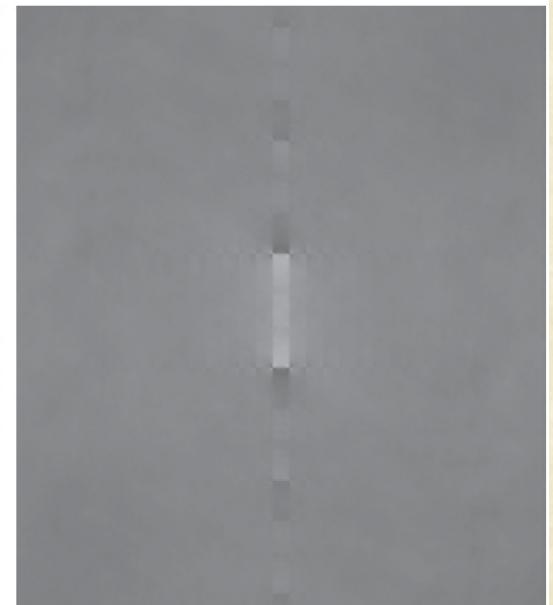
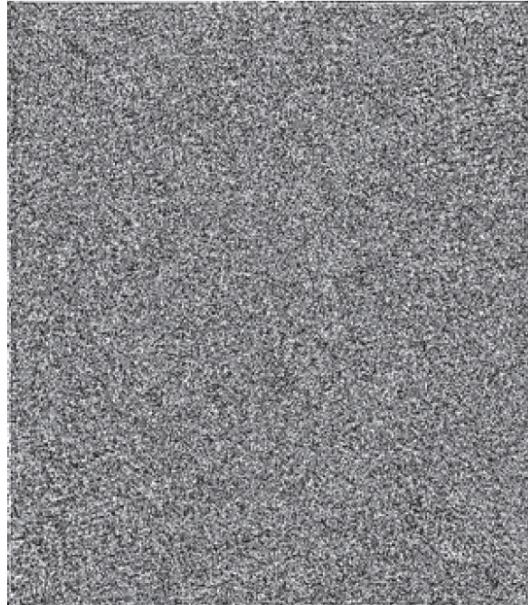
Φ_B

IDFT($0 + \Phi_B$)

IDFT($M_B + 0$)

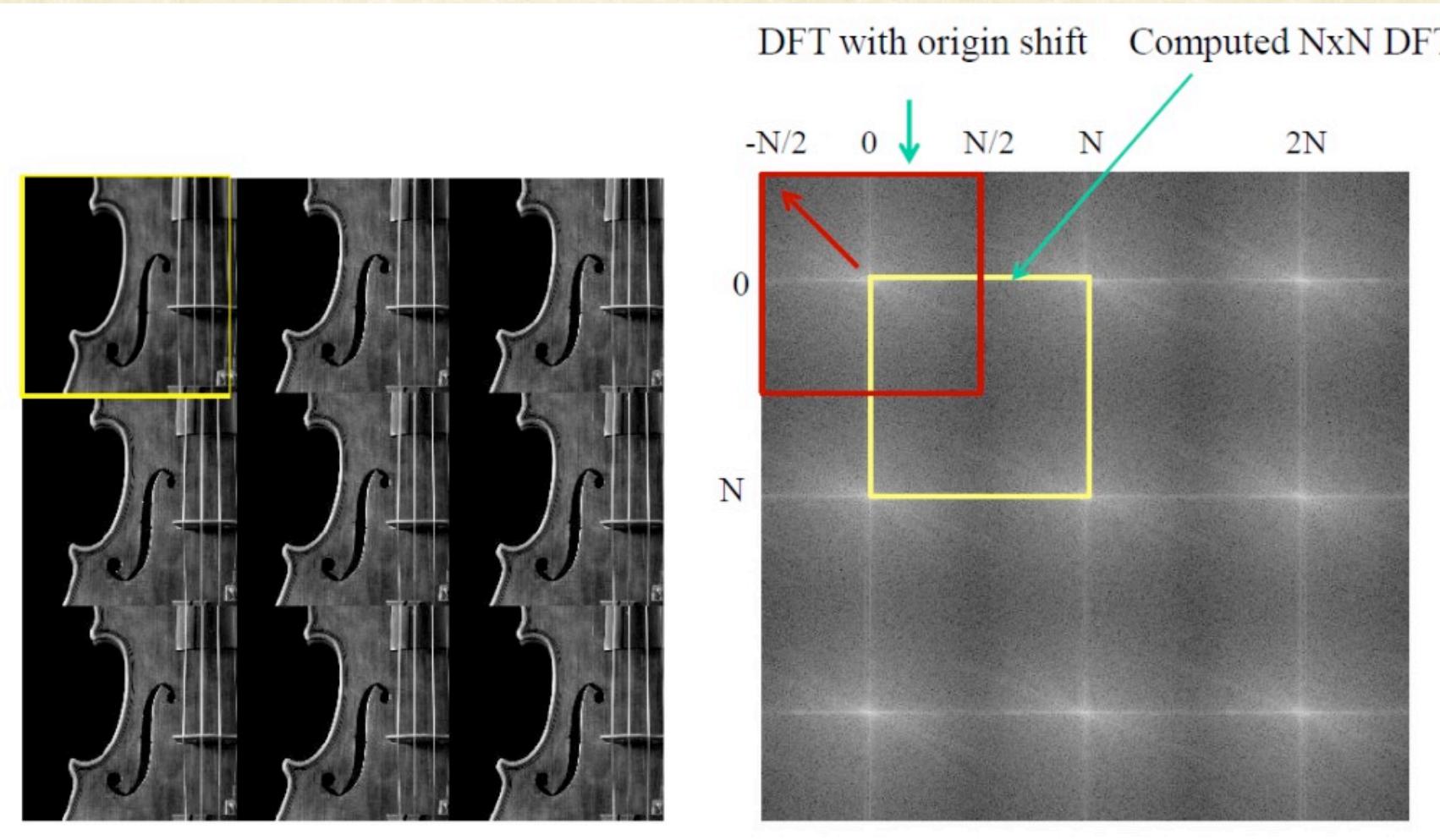
IDFT($M_R + \Phi_B$)

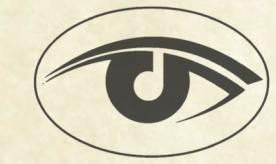
IDFT($M_B + \Phi_R$)

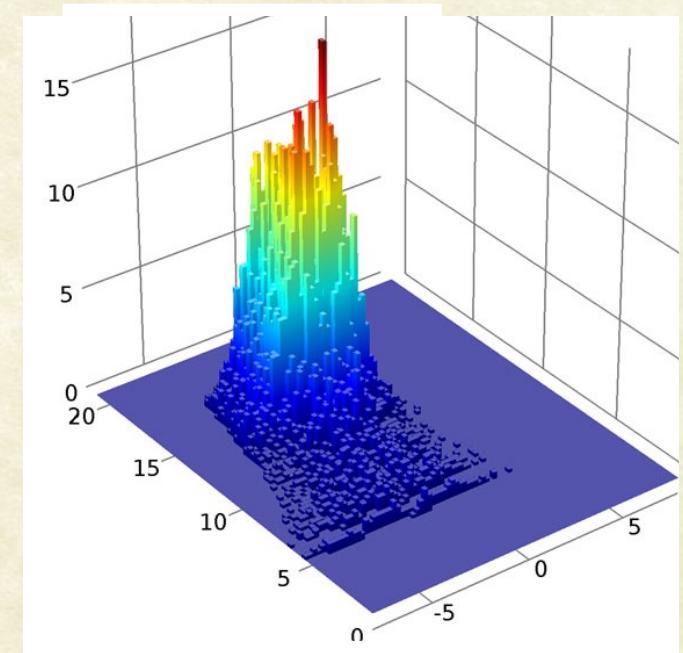




Shifting Origin







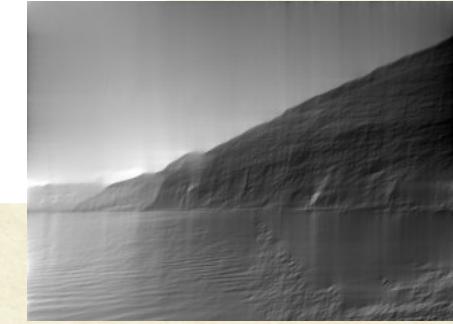
$$F[m, n] = \sum_{y=0}^{y=(N-1)} \sum_{x=0}^{x=(M-1)} f[x, y] e^{-2\pi j \left(\frac{mx}{M} + \frac{ny}{N} \right)}$$

$$f[x, y] = \frac{1}{MN} \sum_{n=0}^{n=(N-1)} \sum_{m=0}^{m=(M-1)} F[m, n] e^{2\pi j \left(\frac{mx}{M} + \frac{ny}{N} \right)}$$

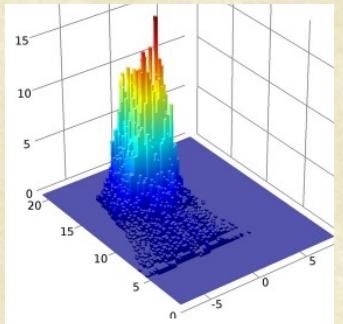


$$f[x, y] = \frac{1}{MN} \sum_{n=0}^{n=(N-1)} \sum_{m=0}^{m=(M-1)} F[m, n] e^{2\pi j \left(\frac{mx}{M} + \frac{ny}{N} \right)}$$


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From 50% of the lowest frequencies

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 $+$
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Adding up to 50% lowest frequencies





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Questions?