Kaustab Pal
kaustab21@gmail.com

Quadratic form

We need to write the MPC cost function in the quadratic form $\frac{1}{2}x^T P x + q^T x$ where $x = [v \quad w]^T$ and $P = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$ and $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$\therefore$ expanding we get $[v \quad w] \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} + [q_1 \quad q_2] \begin{bmatrix} v \\ w \end{bmatrix}$

$$= [v \quad w] \begin{bmatrix} Av + Bw \\ Bv + Cw \end{bmatrix} + q_1 v + q_2 w$$

$$= Av^2 + 2Bvw + Cw^2 + q_1 v + q_2 w$$

The current MPC cost function is

$$cost = (x_t - x_g)^2 + (y_t - y_g)^2 + (\theta_t - \theta_g)^2$$

x_t = x-coordinate after timestep t

y_t = y-coordinate after timestep t

θ_t = θ after timestep t

Now we know $\underbrace{x_c,\ y_c \text{ and } \theta_c}_{\text{current state}}$ and $\underbrace{x_g,\ y_g \text{ and } \theta_g}_{\text{goal state}}$.

We need to calculate x_{c+t}, y_{c+t} and θ_{c+t}

For sake of simplicity, let's consider $t = 1$ and $\underbrace{c = 0}_{\text{Initial state}}$,

$$\theta_1 = \theta_c + w_1 dt \quad \text{— Linear}$$

$$x_1 = x_c + v_1 \cos(\theta_1) dt$$

$$y_1 = y_c + v_1 \sin(\theta_1) dt$$

Linearizing x_1 we get

$$x_1 = x_c + V_1 \cos(\theta_1)\, dt$$

$$x_1 = x_c + V_{1g} \cos(\theta_0 + W_{1g} dt)\, dt + \cos(\theta_0 + W_{1g} dt)\, dt \,(V_1 - V_{1g}) - V_{1g} \sin(\theta_0 + W_{1g} dt)\, dt^2 \,(W_1 - W_{1g})$$

$$= x_c + \cancel{V_{1g} \cos(\theta_0 + W_{1g} dt)\, dt} + V_1 \cos(\theta_0 + W_{1g} dt)\, dt - \cancel{V_{1g} \cos(\theta_0 + W_{1g} dt)\, dt} - W_1 V_{1g} \sin(\theta_0 + W_{1g} dt)\, dt^2 + W_{1g} V_{1g} \sin(\theta_0 + W_{1g} dt)\, dt^2$$

$$\Rightarrow x_1 = x_c + V_1 \underbrace{\boxed{\cos(\theta_0 + W_{1g} dt)\, dt}}_{a} - W_1 \underbrace{\boxed{V_{1g} \sin(\theta_0 + W_{1g} dt)\, dt^2}}_{b} + W_{1g} \underbrace{\boxed{V_{1g} \sin(\theta_0 + W_{1g} dt)\, dt^2}}_{b}$$

$$= V_1 a - W_1 b + \underbrace{\boxed{x_c + W_{1g} b}}_{c}$$

$a = \cos(\theta_0 + W_{1g} dt)\, dt$

$b = V_{1g} \sin(\theta_0 + W_{1g} dt)\, dt^2$

$c = x_c + W_{1g} b$

$$\Rightarrow x_1 = V_1 a - W_1 b + c$$

Now $(x_1 - x_g)^2$

$$= x_1^2 + x_g^2 - 2 x_1 x_g$$

$$= (V_1 a - W_1 b + c)^2 + x_g^2 - 2(V_1 a - W_1 b + c)\, x_g$$

$$= (V_1 a - W_1 b)^2 + c^2 + 2(V_1 a - W_1 b)c + x_g^2 - 2(V_1 a - W_1 b + c)\, x_g$$

$$= \overbrace{\boxed{V_1^2 a^2 + W_1^2 b^2 - 2 V_1 W_1 ab}}^{x^T P x} + c^2 + \underline{2 V_1 a c} - \underline{2 W_1 b c} + x_g^2 - \underline{2 V_1 x_g a} + \underline{2 W_1 x_g b} - 2 c x_g$$

$$= v_1^2 a^2 + w_1^2 b^2 - 2 v_1 w_1 ab + v_1 (2ac - 2x_g a) - w_1 (2bC - 2x_g b) + \boxed{C^2 + x_g^2 - 2 C x_g}$$

K = constant (we can remove this from the cost)

$$= v_1^2 a^2 + w_1^2 b^2 - 2 v_1 w_1 ab + v_1 (2ac - 2x_g a) - w_1 (2bc - 2x_g b) \quad \text{— (1)}$$

$y_1 = y_c + v_1 \sin(\theta_1)\, dt$

Now, linearizing y_1 we get

$$y_1 = y_c + v_{1g} \sin(\theta_0 + w_{1g} dt) dt + \sin(\theta_0 + w_{1g} dt) dt (v_1 - v_{1g}) + v_{1g} \cos(\theta_0 + w_{1g} dt) dt^2 (w_1 - w_{1g})$$

$$= y_c + \cancel{v_{1g} \sin(\theta_0 + w_{1g} dt) dt} + v_1 \sin(\theta_0 + w_{1g} dt) dt - \cancel{v_{1g} \sin(\theta_0 + w_{1g} dt) dt} + w_1 v_{1g} \cos(\theta_0 + w_{1g} dt) dt^2 - w_{1g} v_{1g} \cos(\theta_0 + w_{1g} dt) dt^2$$

$$= y_c + v_1 \underbrace{\boxed{\sin(\theta_0 + w_{1g} dt) dt}}_{d} + w_1 \boxed{v_{1g} \cos(\theta_0 + w_{1g} dt) dt^2} \text{ — } e$$
$$- w_{1g} \boxed{v_{1g} \cos(\theta_0 + w_{1g} dt) dt^2} \text{ — } e$$

$$= y_c + v_1 d + w_1 e - w_{1g} e$$

$$= v_1 d + w_1 e + \underbrace{\boxed{y_c - w_{1g} e}}_{f}$$

$d = \sin(\theta_0 + w_{1g} dt) dt$

$e = v_{1g} \cos(\theta_0 + w_{1g} dt) dt^2$

$f = y_c - w_{1g} e$

$$\Rightarrow y_1 = v_1 d + w_1 e + f$$

Now $(y_1 - y_g)^2$

$= y_1^2 + y_g^2 - 2y_1 y_g$

$= (v_1 d + w_1 e + f)^2 + y_g^2 - 2(v_1 d + w_1 e + f) y_g$

$= (v_1 d + w_1 e)^2 + f^2 + 2(v_1 d + w_1 e) f + y_g^2 - 2 v_1 d y_g - 2 w_1 e y_g - 2 f y_g$

$= v_1^2 d^2 + w_1^2 e^2 + 2 v_1 w_1 de + f^2 + 2 v_1 d f + 2 w_1 e f + y_g^2 - 2 v_1 d y_g$
$\quad - 2 w_1 e y_g - 2 f y_g$

$= v_1^2 d^2 + w_1^2 e^2 + 2 v_1 w_1 de + v_1 (2df - 2dy_g) + w_1 (2ef - 2ey_g)$
$\quad + \underbrace{f^2 + y_g^2 - 2 f y_g}_{\text{Constant so we can remove this part}}$

$= v_1^2 d^2 + w_1^2 e^2 + 2 v_1 w_1 de + v_1 (2df - 2dy_g) + w_1 (2ef - 2ey_g)$ — (2)

$\theta_1 = \theta_c + w_1 dt$

Now $(\theta_1 - \theta_g)^2$

$= \theta_1^2 + \theta_g^2 - 2\theta_1 \theta_g$

$= (\theta_c + w_1 dt)^2 + \theta_g^2 - 2(\theta_c + w_1 dt)\theta_g$

$= \theta_c^2 + w_1^2 dt^2 + 2\theta_c w_1 dt + \theta_g^2 - 2\theta_c \theta_g - 2 w_1 dt \theta_g$

$= w_1^2 dt^2 + w_1 (2\theta_c dt - 2\theta_g dt) + \underbrace{\theta_c^2 + \theta_g^2 - 2\theta_c \theta_g}_{\text{constant so can be removed}}$

$= w_1^2 dt^2 + w_1 (2\theta_0 dt - 2\theta_g dt)$ — (3)

$$\text{Cost} = ① + ② + ③$$

$$= v_1^2 a^2 + w_1^2 b^2 - 2v_1 w_1 ab + v_1(2ac - 2x_g a) - w_1(2bc - 2x_g b)$$

$$+ v_1^2 d^2 + w_1^2 e^2 + 2v_1 w_1 de + v_1(2df - 2dy_g) + w_1(2ef - 2ey_g)$$

$$+ w_1^2 dt^2 + w_1(2\theta_0 dt - 2\theta_g dt)$$

$$= v_1^2 \overbrace{(a^2 + d^2)}^{A} + w_1^2 \overbrace{(b^2 + e^2 + dt^2)}^{C} + 2v_1 w_1 \overbrace{(-ab + de)}^{B}$$

$$+ v_1 \underbrace{(2ac - 2x_g a + 2df - 2dy_g)}_{q_1} + w_1 \underbrace{(-2bc + 2x_g b + 2ef - 2ey_g + 2\theta_0 dt - 2\theta_g dt)}_{q_2}$$

$$= \frac{1}{2} \begin{bmatrix} v_1 & w_1 \end{bmatrix} \underbrace{2\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{P} \begin{bmatrix} v_1 \\ w_1 \end{bmatrix} + \underbrace{\begin{bmatrix} q_1 & q_2 \end{bmatrix}}_{q^T} \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}$$

$$P = 2\begin{bmatrix} A & B \\ B & C \end{bmatrix} \qquad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Now

$$A = a^2 + d^2$$

$$B = -ab + de$$

$$C = b^2 + e^2 + d_t^2$$

$$q_1 = 2ac - 2x_g a + 2df - 2dy_g$$

$$q_2 = -2bc + 2x_g b + 2ef - 2ey_g + 2\theta_0 dt - 2\theta_g dt$$

$$a = \cos(\theta_0 + w_{1g} dt)\, df$$

$$b = v_{1g} \sin(\theta_0 + w_{1g} dt)\, df^2$$

$$c = x_0 + w_{1g} b$$

$$d = \sin(\theta_0 + W_{ig}\, dt)\, dt$$

$$e = V_{ig} \cos(\theta_0 + W_{ig}\, dt)\, dt^2$$

$$f = y_0 - W_{ig}\, e$$

Constraints

$$v_{lb} \leq v \leq v_{ub}$$

$$w_{lb} \leq w \leq w_{ub}$$

$$\left.\begin{array}{l} v_g - tb \leq v \leq v_g + tb \\ w_g - tb \leq w \leq w_g + tb \end{array}\right] \text{trust region constraints}$$

The constraints needs to be written in the form $Gx \leq h$, $Ax = b$

$$\therefore \quad G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} v_{ub} \\ v_{lb} \\ w_{ub} \\ w_{lb} \\ v_g + \text{trust\_boundary} \\ -(v_g - \text{trust\_boundary}) \\ w_g + \text{trust\_boundary} \\ -(w_g - \text{trust\_boundary}) \end{bmatrix}$$

For $t=3$, the cost becomes

$$\text{cost} = (x_3 - x_g)^2 + (y_3 - y_g)^2 + (\theta_3 - \theta_g)^2$$

Solving for x

$$x_3 = x_0 + x_1 + x_2 + v_3 \cos(\theta_3)\, dt$$

$$= x_0 + v_1 \cos(\theta_1) dt + v_2 \cos(\theta_2) dt + v_3 \cos(\theta_3) dt$$

$$\Rightarrow x_3 = x_0 + v_1 \cos(\theta_0 + w_1 dt) dt + v_2 \cos(\theta_0 + (w_1 + w_2) dt) dt$$
$$+ v_3 \cos(\theta_0 + (w_1 + w_2 + w_3) dt) dt$$

Linearizing x_3 we get

$$x_3 = \underbrace{x_0 + v_{1g} \cos(\theta_0 + w_{1g} dt) dt + v_{2g} \cos(\theta_0 + (w_{1g} + w_{2g}) dt) dt + v_{3g} \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt}_{K_1}$$
$$+ \cos(\theta_0 + w_{1g} dt) dt\, (v_1 - v_{1g})$$
$$+ \cos(\theta_0 + (w_{1g} + w_{2g}) dt) dt\, (v_2 - v_{2g})$$
$$+ \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt\, (v_3 - v_{3g})$$
$$- \Big(v_{1g} \sin(\theta_0 + w_{1g} dt) dt^2 + v_{2g} \sin(\theta_0 + (w_{1g} + w_{2g}) dt) dt^2$$
$$+ v_{3g} \sin(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt^2\Big)(w_1 - w_{1g})$$
$$- \Big(v_{2g} \sin(\theta_0 + (w_{1g} + w_{2g}) dt) dt^2$$
$$+ v_{3g} \sin(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt^2\Big)(w_2 - w_{2g})$$
$$- \Big(v_{3g} \sin(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt^2\Big)(w_3 - w_{3g})$$

$$= K_1 + v_1 \cos(\theta_0 + w_{1g} dt) dt - v_{1g} \cos(\theta_0 + w_{1g} dt) dt$$
$$+ v_2 \cos(\theta_0 + (w_{1g} + w_{2g}) dt) dt - v_{2g} \cos(\theta_0 + (w_{1g} + w_{2g}) dt) dt$$
$$+ v_3 \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt - v_{3g} \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g}) dt) dt$$

$$- v_{1g}\sin(\theta_0 + w_{1g}\,dt)\,dt^2(w_1 - w_{1g}) - v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2(w_1 - w_{1g})$$

$$- v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2(w_1 - w_{1g})$$

$$- v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2(w_2 - w_{2g})$$

$$- v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2(w_2 - w_{2g})$$

$$- v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2(w_3 - w_{3g})$$

$$\begin{aligned}
= K_1 &+ v_1 \overbrace{\boxed{\cos(\theta_0 + w_{1g}\,dt)\,dt}}^{a} - \underline{v_{1g}\cos(\theta_0 + w_{1g}\,dt)\,dt} \\
&+ v_2 \underbrace{\boxed{\cos(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt}}_{b} - \underline{v_{2g}\cos(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt} \\
&+ v_3 \underbrace{\boxed{\cos(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt}}_{c} - \underline{v_{3g}\cos(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt} \\
&- w_1 v_{1g}\sin(\theta_0 + w_{1g}\,dt)\,dt^2 + \underline{w_{1g} v_{1g}\sin(\theta_0 + w_{1g}\,dt)\,dt^2} \\
&- w_1 v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2 + \underline{w_{1g} v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2} \\
&- w_1 v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2 + \underline{w_{1g} v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2}
\end{aligned}$$

$$\begin{aligned}
&- w_2 v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2 + \underline{w_{2g} v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2} \\
&- w_2 v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2 + \underline{w_{2g} v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2} \\
&- w_3 v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2 + \underline{w_{3g} v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2}
\end{aligned}$$

$$\begin{aligned}
= &\overbrace{\boxed{\begin{aligned}
&K_1 - v_{1g}\cos(\theta_0 + w_{1g}\,dt)\,dt - v_{2g}\cos(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt - v_{3g}\cos(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt \\
&+ w_{1g} v_{1g}\sin(\theta_0 + w_{1g}\,dt)\,dt^2 + w_{1g} v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2 \\
&+ w_{1g} v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2 + w_{2g} v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2 \\
&+ w_{2g} v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2 + w_{3g} v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2
\end{aligned}}}^{K} \\
&+ v_1 a + v_2 b + v_3 c \\
&+ w_1 \Big(\overbrace{\boxed{- v_{1g}\sin(\theta_0 + w_{1g}\,dt)\,dt^2}}^{d} + \overbrace{\boxed{- v_{2g}\sin(\theta_0 + (w_{1g}+w_{2g})\,dt)\,dt^2}}^{e} \\
&\quad + \underbrace{\boxed{- v_{3g}\sin(\theta_0 + (w_{1g}+w_{2g}+w_{3g})\,dt)\,dt^2}}_{f} \Big)
\end{aligned}$$

$$+ w_2\left(\overbrace{-v_{2g}\sin\left(\theta_0+(w_{1g}+w_{2g})dt\right)dt^2}^{e}\ \underbrace{-v_{3g}\sin\left(\theta_0+(w_{1g}+w_{2g}+w_{3g})dt\right)dt^2}_{f}\right)$$

$$+ w_3\left(\underbrace{-v_{3g}\sin\left(\theta_0+(w_{1g}+w_{2g}+w_{3g})dt\right)dt^2}_{f}\right)$$

$$= K + v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f$$

Now

$$\text{cost} = (x_3 - x_g)^2$$

$$= \left(K - x_g + v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)^2$$

$$= (\overbrace{\boxed{K - x_g}}^{z})^2 + \left(v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)^2$$
$$+ 2\underbrace{\boxed{(K - x_g)}}_{z}\left(v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)$$

$$= (v_1 a + v_2 b + v_3 c)^2 + \left(w_1(d+e+f) + w_2(e+f) + w_3 f\right)^2$$
$$+ 2(v_1 a + v_2 b + v_3 c)\left(w_1(d+e+f) + w_2(e+f) + w_3 f\right)$$
$$+ 2z\left(v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)$$

$$= (v_1 a + v_2 b)^2 + v_3^2 c^2 + 2(v_1 a + v_2 b)v_3 c + \left(w_1(d+e+f) + w_2(e+f)\right)^2$$
$$+ w_3^2 f^2 + 2\left(w_1(d+e+f) + w_2(e+f)\right)w_3 f$$
$$+ 2v_1 w_1 a(d+e+f) + 2v_1 w_2 a(e+f) + 2v_1 w_3 a f$$
$$+ 2v_2 w_1 b(d+e+f) + 2v_2 w_2 b(e+f) + 2v_2 w_3 b f$$
$$+ 2v_3 w_1 c(d+e+f) + 2v_3 w_2 c(e+f) + 2v_3 w_3 c f$$

This part will be the $x^T P x$ part

$$+ 2v_1 a z + 2v_2 b z + 2v_3 c z + 2w_1(d+e+f)z + 2w_2(e+f)z + 2w_3 f z$$

This part will be the $q^T x$ part

Simplifying the $x^T P x$ part

$$v_1^2 a^2 + v_2^2 b^2 + v_3^2 c^2 + 2 v_1 v_2 ab + 2 v_1 v_3 ac + 2 v_2 v_3 bc$$
$$+ w_1^2 (d+e+f)^2 + w_2^2 (e+f)^2 + w_3^2 f^2 + 2 w_1 w_2 (d+e+f)(e+f)$$
$$+ 2 w_1 w_3 (d+e+f) f + 2 w_2 w_3 (e+f) f + 2 v_1 w_1 a (d+e+f)$$
$$+ 2 v_1 w_2 a (e+f) + 2 v_1 w_3 af + 2 v_2 w_1 b (d+e+f) + 2 v_2 w_2 b (e+f)$$
$$+ 2 v_2 w_3 bf + 2 v_3 w_1 c (d+e+f) + 2 v_3 w_3 c (e+f)$$
$$+ 2 v_3 w_3 cf.$$

$$= a^2 v_1^2 + b^2 v_2^2 + c^2 v_3^2 + (d+e+f)^2 w_1^2 + (e+f)^2 w_2^2 + f^2 w_3^2$$
$$+ 2 v_1 v_2 ab + 2 v_1 v_3 ac + 2 v_1 w_1 a(d+e+f) + 2 v_1 w_2 a(e+f)$$
$$+ 2 v_1 w_3 af + 2 v_2 v_3 bc + 2 v_2 w_1 b(d+e+f) + 2 v_2 w_2 b(e+f)$$
$$+ 2 v_2 w_3 bf + 2 v_3 w_1 c(d+e+f) + 2 v_3 w_2 c(e+f)$$
$$+ 2 v_3 w_3 cf + 2 w_1 w_2 (d+e+f)(e+f) + 2 w_1 w_3 (d+e+f) f$$
$$+ 2 w_2 w_3 (e+f) f$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} a^2 & ab & ac & a(d+e+f) & a(e+f) & af \\ ab & b^2 & bc & b(d+e+f) & b(e+f) & bf \\ ac & bc & c^2 & c(d+e+f) & c(e+f) & cf \\ a(d+e+f) & b(d+e+f) & c(d+e+f) & (d+e+f)^2 & (d+e+f)(e+f) & (d+e+f)f \\ a(e+f) & b(e+f) & c(e+f) & (d+e+f)(e+f) & (e+f)^2 & (e+f)f \\ af & bf & cf & (d+e+f)f & (e+f)f & f^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$P_x = 2*$

Simplifyin $q^T x$ part

$$2v_1 a z + 2v_2 b z + 2v_3 c z + 2w_1(d+e+f)z + 2w_2(e+f)z + 2w_3 f z$$

$$= \begin{bmatrix} 2az & 2bz & 2cz & 2(d+e+f)z & 2(e+f)z & 2fz \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$a = \cos(\theta_0 + w_{1g} dt)\, dt$$

$$b = \cos(\theta_0 + (w_{1g} + w_{2g})dt)\, dt$$

$$c = \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g})dt)\, dt$$

$$d = -v_{1g} \sin(\theta_0 + w_{1g} dt)\, dt^2$$

$$e = -v_{2g} \sin(\theta_0 + (w_{1g} + w_{2g})dt)\, dt^2$$

$$f = -v_{3g} \sin(\theta_0 + (w_{1g} + w_{2g} + w_{3g})dt)\, dt^2$$

$$z = x - x_g$$

$$x = x_1 - v_{1g} a - v_{2g} b - v_{3g} c - w_{1g}(d+e+f) - w_{2g}(e+f) - w_{3g} f$$

$$x_1 = x_0 + v_{1g} a + v_{2g} b + v_{3g} c$$

$$P_u = \begin{bmatrix} a \\ b \\ c \\ d+e+f \\ e+f \\ f \end{bmatrix} \begin{bmatrix} a & b & c & d+e+f & e+f & f \end{bmatrix}$$

Solving for y_3

$$y_3 = y_0 + y_1 + y_2 + V_3 \sin(\theta_3)\, dt$$

$$= y_0 + v_1 \sin(\theta_0 + w_1 dt)\,dt + V_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\,dt$$
$$+ V_3 \sin(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt$$

Linearizing y_3, we get

$$\boxed{\begin{aligned} & y_0 + V_{1g} \sin(\theta_0 + W_{1g}\, dt)\, dt + V_{2g} \sin(\theta_0 + (W_{1g} + W_{2g})\, dt)\, dt \\ & + V_{3g} \sin(\theta_0 + (W_{1g} + W_{2g} + W_{3g})\, dt)\, dt \end{aligned}} \quad K_1$$

$$+ \sin(\theta_0 + W_{1g}\, dt)\,dt\,(v_1 - V_{1g}) + \sin(\theta_0 + (w_{1g} + w_{2g})\, dt)\, dt\, (V_2 - V_{2g})$$
$$+ \sin(\theta_0 + (W_{1g} + W_{2g} + W_{3g})\, dt)\, dt\, (V_3 - V_{3g})$$
$$+ V_{1g} \cos(\theta_0 + W_{1g}\, dt)\, dt^2\, (w_1 - w_{1g})$$
$$+ V_{2g} \cos(\theta_0 + (w_{1g} + w_{2g})\, dt)\, dt^2\, (w_1 - w_{1g})$$
$$+ V_{3g} \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt)\, dt^2\, (w_1 - w_{1g})$$
$$+ V_{2g} \cos(\theta_0 + (w_{1g} + w_{2g})\, dt)\, dt^2\, (w_2 - w_{2g})$$
$$+ V_{3g} \cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt)\, dt^2\, (W_2 - W_{2g})$$
$$+ V_{3g} \cos(\theta_0 + (w_{1g} + W_{2g} + W_{3g})\, dt)\, dt^2\, (W_3 - W_{3g})$$

$$= K_1 + V_1 \sin(\theta_0 + W_{1g}\, dt)\, dt - V_{1g} \sin(\theta_0 + W_{1g}\, dt)\, dt$$
$$+ V_2 \sin(\theta_0 + (w_{1g} + W_{2g})\, dt)\, dt - V_{2g} \sin(\theta_0 + (w_{1g} + W_{2g})\, dt)\, dt$$
$$+ V_3 \sin(\theta_0 + (w_{1g} + W_{2g} + W_{3g})\, dt)\, dt - V_{3g} \sin(\theta_0 + (w_{1g} + W_{2g} + W_{3g})\, dt)\, dt$$
$$+ W_1 V_{1g} \cos(\theta_0 + w_{1g}\, dt)\, dt^2 - W_{1g} V_{1g} \cos(\theta_0 + W_{1g}\, dt)\, dt^2$$
$$+ W_1 V_{2g} \cos(\theta_0 + (w_{1g} + w_{2g})\, dt)\, dt^2 - W_{1g} V_{2g} \cos(\theta_0 + (w_{1g} + W_{2g})\, dt)\, dt^2$$

$$+ W_1 V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt^2 - W_{1g} V_{3g} \cos\left(\theta_0 + (W_{1g} + W_{2g} + W_{3g})\, dt\right) dt^2$$

$$+ W_2 V_{2g} \cos\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt^2 - W_{2g} V_{2g} \cos\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt^2$$

$$+ W_2 V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt^2 - W_{2g} V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + V_{3g})\, dt\right) dt^2$$

$$+ W_3 V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt^2 - W_{3g} V_{3g} \cos\left(\theta_0 + (w_{1g} + V_{2g} + w_{3g})\, dt\right) dt^2$$

$$= \overbrace{\begin{aligned} & K_1 - V_{1g} \sin(\theta_0 + w_{1g}\, dt)\, dt - V_{2g} \sin\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt \\ & - V_{3g} \sin\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt - W_{1g} V_{1g} \cos(\theta_0 + w_{1g}\, dt)\, dt^2 \\ & - w_{1g} V_{2g} \cos\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt^2 - w_{1g} V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + V_{3g})\, dt\right) dt^2 \\ & - W_{2g} V_{2g} \cos\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt^2 - W_{2g} V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + W_{3g})\, dt\right) dt^2 \\ & - W_{3g} V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt^2 \end{aligned}}^{K}$$

$$+ V_1 \underbrace{\boxed{\sin(\theta_0 + w_{1g}\, dt)\, dt}}_{a} + V_2 \underbrace{\boxed{\sin\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt}}_{b}$$

$$+ V_3 \underbrace{\boxed{\sin\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt}}_{c} + w_1 \Big(\underbrace{\boxed{V_{1g} \cos(\theta_0 + w_{1g}\, dt)\, dt^2}}_{d}$$

$$+ \underbrace{\boxed{V_{2g} \cos\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt^2}}_{e} + \underbrace{\boxed{V_{3g} \cos\left(\theta_0 + (w_{1g} + w_{2g} + w_{3g})\, dt\right) dt^2}}_{f} \Big)$$

$$+ W_2 \Big(\underbrace{\boxed{V_{2g} \cos\left(\theta_0 + (w_{1g} + w_{2g})\, dt\right) dt^2}}_{e} + \underbrace{\boxed{V_{3g} \cos\left(\theta_0 + (w_{1g} + W_{2g} + W_{3g})\, dt\right) dt^2}}_{f} \Big)$$

$$+ W_3 \underbrace{\boxed{V_{3g} \cos\left(\theta_0 + (w_{1g} + W_{2g} + W_{3g})\, dt\right) dt^2}}_{f}$$

$$= K + V_1 a + V_2 b + V_3 c + w_1(d + e + f) + w_2(e + f) + w_3 f$$

Now the cost is

$$(y_3 - y_g)^2$$

$$= \Big(\underbrace{\boxed{K - y_g}}_{Z} + V_1 a + V_2 b + V_3 c + w_1(d + e + f) + w_2(e + f) + w_3 f \Big)^2$$

$$= \overbrace{z^2}^{\text{constant}} + \left(v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)^2$$

$$+ 2z\left(v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)$$

$$= \underbrace{\boxed{(v_1 a + v_2 b + v_3 c)^2 + \left(w_1(d+e+f) + w_2(e+f) + w_3 f\right)^2 + 2(v_1 a + v_2 b + v_3 c)\left(w_1(d+e+f) + w_2(e+f) + w_3 f\right)}}_{\text{quadratic part}}$$

$$+ \underbrace{\boxed{2z\left(v_1 a + v_2 b + v_3 c + w_1(d+e+f) + w_2(e+f) + w_3 f\right)}}_{\text{linear part}}$$

Simplifying the quadratic part, we get

$$\begin{bmatrix} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{bmatrix} \underbrace{\begin{bmatrix} a^2 & ab & ac & a(d+e+f) & a(e+f) & af \\ ab & b^2 & bc & b(d+e+f) & b(e+f) & bf \\ ac & bc & c^2 & c(d+e+f) & c(e+f) & cf \\ a(d+e+f) & b(d+e+f) & c(d+e+f) & (d+e+f)^2 & (d+e+f)(e+f) & (d+e+f)f \\ a(e+f) & b(e+f) & c(e+f) & (d+e+f)(e+f) & (e+f)^2 & (e+f)f \\ af & bf & cf & (d+e+f)f & (e+f)f & f^2 \end{bmatrix}}_{P_y = 2*} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Simplifying the linear part

$$\underbrace{\begin{bmatrix} 2za & 2zb & 2zc & 2z(d+e+f) & 2z(e+f) & 2zf \end{bmatrix}}_{q_y^T} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Here

$$a = \sin(\theta_0 + w_{1g}dt)\,dt$$
$$b = \sin(\theta_0 + (w_{1g} + w_{2g})dt)\,dt$$
$$c = \sin(\theta_0 + (w_{1g} + w_{2g} + w_{3g})dt)\,dt$$
$$d = V_{1g}\cos(\theta_0 + W_{1g}dt)\,dt^2$$
$$e = V_{2g}\cos(\theta_0 + (w_{1g} + w_{2g})dt)\,dt^2$$
$$f = V_{3g}\cos(\theta_0 + (w_{1g} + w_{2g} + w_{3g})dt)\,dt^2$$
$$Z = K - y_g$$
$$K = K_1 - V_{1g}a - V_{2g}b - V_{3g}C - W_{1g}(d+e+f) - w_{2g}(e+f) - w_{3g}f$$
$$K_1 = y_0 + V_{1g}a + V_{2g}b + V_{3g}C$$

Solving for θ_3

Now $\theta_3 = \theta_0 + \theta_1 + \theta_2 + w_3\,dt$

$$\Rightarrow \theta_3 = \theta_0 + (w_1 + w_2 + w_3)\,dt$$

Cost is

$$(\theta_3 - \theta_g)^2$$

$$\therefore \left(\theta_0 + (w_1 + w_2 + w_3)dt - \theta_g\right)^2$$

$$= \underbrace{\boxed{(\theta_0 - \theta_g)}}_{z}{}^2 + (w_1 dt + w_2 dt + w_3 dt)^2 + 2z(w_1 dt + w_2 dt + w_3 dt)$$

$$= (w_1 dt + w_2 dt)^2 + w_3^2 dt^2 + 2(w_1 dt + w_2 dt)w_3 dt + 2z(w_1 dt + w_2 dt + w_3 dt)$$

$$= w_1^2 dt^2 + w_2^2 dt^2 + 2w_1 w_2 dt^2 + w_3^2 dt^2 + 2w_1 w_3 dt^2$$
$$+ 2w_2 w_3 dt^2 + 2zw_1 dt + 2zw_2 dt + 2zw_3 dt$$

$$= w_1^2 dt^2 + w_2^2 dt^2 + w_3^2 dt^2 + 2w_1 w_2 dt^2 + 2w_1 w_3 dt^2$$
$$+ 2w_2 w_3 dt^2 + 2zw_1 dt + 2zw_2 dt + 2zw_3 dt$$

$$\begin{bmatrix} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & dt^2 & dt^2 & dt^2 \\ 0 & 0 & 0 & dt^2 & dt^2 & dt^2 \\ 0 & 0 & 0 & dt^2 & dt^2 & dt^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$P_\theta = 2 *$

$$q_\theta = \begin{bmatrix} 0 & 0 & 0 & 2z\,dt & 2z\,dt & 2z\,dt \end{bmatrix}$$

$$\therefore \quad P = 2 * (P_x + P_y + P_\theta)$$

$$q = (q_x + q_y + q_\theta)$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \quad \begin{bmatrix} w_{lb} \\ 0 \quad v_{lb} \\ v_{ub} \\ w_{ub} \\ v_g + S \\ -(v_g - S) \\ w_g + 0.1 \\ -(w_g - 0.1) \end{bmatrix}$$

$w_{lb} < w$

$-w \leq -w_{lb}$

$v_{lb} \leq v$

$-v \leq -v_{lb}$

$$\begin{matrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{matrix} \quad \begin{bmatrix} v \\ w \end{bmatrix}$$

$$\begin{matrix} v_{ub} \\ w_{ub} \\ v_{lb} \\ w_{lb} \\ v_g + S \\ v_g + 0.1 \end{matrix} \qquad \begin{matrix} v_{ub} \\ v_{lb} \\ w_{ub} \\ w_{lb} \\ v_g + S \\ -(v_g - S) \\ w_g + 0.1 \\ -(w_g - 0.1) \end{matrix}$$

$$a_{min} \leq \frac{V_2 - V_1}{dt} \leq a_{max}$$

$$a_{min} \leq \frac{V_3 - V_2}{dt} \leq a_{max}$$

$$a_{min} \leq \frac{W_2 - W_1}{dt} \leq a_{max}$$

$$a_{min} \leq \frac{W_3 - W_2}{dt} \leq a_{max}$$

$a_{min} \leq V_2 - V_1$

$-(V_2 - V_1) \leq -a_{min}$

$-V_2 + V_1$

$V_1 - \cancel{V_2}$

$V_2 - V_3$

$$\begin{array}{cccccc|c}
-1 & 1 & 0 & 0 & 0 & 0 & V_1 \\
0 & -1 & 1 & 0 & 0 & 0 & V_2 \\
 & & & & & & V_3 \\
 & & & & & & W_1 \\
 & & & & & & W_2 \\
 & & & & & & W_3
\end{array}$$

$v_1 \quad v_2 \quad v_3$

$(v_2 - v_1)^2 + (v_3 - v_2)^2$

$= v_2^2 + v_1^2 - 2v_1v_2 + v_3^2 + v_2^2 - 2v_2v_3$

$= v_1^2 + v_2^2 + v_3^2 - 2v_1v_2 - 2v_2v_3$

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{array}{c c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \cancel{0} \\ 0 & \cancel{0} & 1 \end{bmatrix} & \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \end{array}$$

$v_1^2 + v_1v_2 0 v_1v_3$

$+ v_2^2$

$$v_1 \; v_2 v_3 \begin{pmatrix} v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_2 + v_3 \end{pmatrix}$$

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$v_1^2 + v_1v_2 + v_1v_2 + v_2^2 + v_2v_3 + v_2v_3 + v_3^2$

$v_1^2 + v_2^2 + v_3^2 + 2v_1v_2 + 2v_2v_3$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$v_1 + v_2$

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_3 \end{pmatrix} \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_2 + v_3 + \cancel{v}_4 \\ v_3 + v_4 \end{bmatrix}$$

Static obstacle avoidance

For $t=1$

$$d(a,o) \leq (r_a + r_o)^2$$

$$d(a,o) = (x_1 - x_0)^2 + (y_1 - y_0)^2$$

Now x_1 and y_1 are functions of v_1 and w_1

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 \leq (r_a + r_o)^2$$

$$= x_1^2 + x_0^2 - 2x_1x_0 + y_1^2 + y_0^2 - 2y_1y_0$$

$$= x_1^2 - 2x_1x_0 + y_1^2 - 2y_1y_0 \leq (r_a + r_o)^2 - x_0^2 - y_0^2$$

Now $x_1 = x_0 + v_1 \cos(\theta_0 + w_1 dt)\, dt$

$y_1 = y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt$

$$\therefore (x_0 + v_1 \cos(\theta_0 + w_1 dt)\, dt)^2 + (y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt)^2$$
$$- 2(x_0 + v_1 \cos(\theta_0 + w_1 dt)\, dt)\, x_0 - 2(y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt)\, y_0$$

$$= x_0^2 + v_1^2 \cos^2(\theta_0 + w_1 dt)\, dt^2 + \cancel{2x_0 v_1 \cos(\theta_0 + w_1 dt)\, dt}$$
$$+ y_0^2 + v_1^2 \sin^2(\theta_0 + w_1 dt)\, dt^2 + \cancel{2y_0 v_1 \sin(\theta_0 + w_1 dt)\, dt}$$
$$- 2x_0^2 - \cancel{2x_0 v_1 \cos(\theta_0 + w_1 dt)\, dt} - 2y_0^2 - \cancel{2y_0 v_1 \sin(\theta_0 + w_1 dt)\, dt}$$

$$= v_1^2 \cos^2(\theta_0 + w_1 dt)\, dt^2 + v_1^2 \sin^2(\theta_0 + w_1 dt)\, dt^2 - x_0^2 - y_0^2$$

$$= v_1^2 \cos^2(\theta_0 + w_1 dt)\, dt^2 + v_1^2 \sin^2(\theta_0 + w_1 dt)\, dt^2 \leq (r_a + r_o)^2$$

$$= v_1^2\, dt^2 \leq (r_a + r_o)^2$$

Linearizing LHS

$$v_{1g}^2 \, dt^2 + 2 v_{1g} \, dt^2 (v_1 - v_{1g})$$
$$= 2 v_1 v_{1g} \, dt^2 - v_{1g}^2 \, dt^2$$
$$\therefore \boxed{2 v_1 v_{1g} \, dt^2 \leq (r_a + r_o)^2 + v_{1g}^2 \, dt^2}$$ — for 1 tstep

For $t = 2$

$$d(a, o) = (x_2 - x_o)^2 + (y_2 - y_o)^2$$
$$x_2 = x_o + x_1 + v_2 \cos(\theta_o + (w_1 + w_2) \, dt) \, dt$$
$$y_2 = y_o + y_1 + v_2 \sin(\theta_o + (w_1 + w_2) \, dt) \, dt$$

Now

$$(x_2 - x_o)^2 + (y_2 - y_o)^2 \leq (r_a + r_o)^2$$
$$\Rightarrow x_2^2 + x_o^2 - 2 x_2 x_o + y_2^2 + y_o^2 - 2 y_2 y_o \leq (r_a + r_o)^2$$
$$\Rightarrow x_2^2 - 2 x_2 x_o + y_2^2 - 2 y_2 y_o \leq (r_a + r_o)^2 - x_o^2 - y_o^2$$

Expanding LHS

$$\left(x_0 + v_1 \cos(\theta_0 + w_1 dt)dt + v_2 \cos(\theta_0 + (w_1 + w_2)dt)\, dt\right)^2$$
$$- 2\left(x_0 + v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)dt)\, dt\right) x_0$$
$$+ \left(y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$
$$- 2\left(y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right) y_0$$

$$= x_0^2 + \left(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$
$$+ \cancel{2x_0\left(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)}$$
$$- 2x_0^2 - \cancel{2x_0\left(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)}$$
$$+ y_0^2 + \left(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$
$$+ \cancel{2y_0\left(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)}$$
$$- 2y_0^2 - \cancel{2y_0\left(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)}$$

gets cancelled out with RHS

$$= -x_0^2 + \left(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$
$$- y_0^2 + \left(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$

$$= \left(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$
$$+ \left(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt\right)^2$$

$$= v_1^2 \cos^2(\theta_0 + w_1 dt)\, dt^2 + v_2^2 \cos^2(\theta_0 + (w_1 + w_2)\, dt)\, dt^2$$
$$+ 2 v_1 v_2 \cos(\theta_0 + w_1 dt) \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt^2$$
$$+ v_1^2 \sin^2(\theta_0 + w_1 dt)\, dt^3 + v_2^2 \sin^2(\theta_0 + (w_1 + w_2)\, dt)\, dt^2$$
$$+ 2 v_1 v_2 \sin(\theta_0 + w_1 dt) \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt^2$$

$$= v_1^2 dt^2\left(\cos^2(\theta_0 + w_1 dt) + \sin^2(\theta_0 + w_1 dt)\right)$$
$$+ v_2^2 dt^2\left(\cos^2(\theta_0 + (w_1+w_2)dt) + \sin^2(\theta_0 + (w_1+w_2)dt)\right)$$
$$+ 2v_1 v_2 dt^2\Big(\cos(\theta_0 + w_1 dt)\cos(\theta_0 + (w_1+w_2)dt)$$
$$+ \sin(\theta_0 + w_1 dt)\sin(\theta_0 + (w_1+w_2)dt)\Big)$$

$$= v_1^2 dt^2 + v_2^2 dt^2 + 2v_1 v_2 dt^2\Big(\cos(\theta_0 + w_1 dt)\cos(\theta_0 + (w_1+w_2)dt)$$
$$+ \sin(\theta_0 + w_1 dt)\sin(\theta_0 + (w_1+w_2)dt)\Big)$$

$$= v_1^2 dt^2 + v_2^2 dt^2 + 2v_1 v_2 dt^2 \cos\left(\theta_0 + w_1 dt - (\theta_0 + (w_1+w_2)dt)\right)$$

$$= v_1^2 dt^2 + v_2^2 dt^2 + 2v_1 v_2 dt^2 \cos\left(\cancel{\theta_0} + \cancel{w_1 dt} - \cancel{\theta_0} - \cancel{w_1 dt} - w_2 dt\right)$$

$$= v_1^2 dt^2 + v_2^2 dt^2 + 2v_1 v_2 dt^2 \cos(w_2 dt)$$

Linearizing, we get

$$v_{1g}^2 dt^2 + v_{2g}^2 dt^2 + 2v_{1g}v_{2g}dt^2\cos(w_{2g}dt)$$
$$+ \left(2v_{1g}dt^2 + 2v_{2g}dt^2\cos(w_{2g}dt)\right)(v_1 - v_{1g})$$
$$+ \left(2v_{2g}dt^2 + 2v_{1g}dt^2\cos(w_{2g}dt)\right)(v_2 - v_{2g})$$
$$+ \left(-2v_{1g}v_{2g}dt^3\sin(w_{2g}dt)\right)(w_2 - w_{2g})$$

$$= v_{1g}^2 dt^2 + v_{2g}^2 dt^2 + \cancel{2v_{1g}v_{2g}dt^2\cos(w_{2g}dt)}$$
$$+ 2v_1 v_{1g} dt^2 + 2v_1 v_{2g} dt^2 \cos(w_{2g}dt) - 2v_{1g}^2 dt^2$$
$$- \cancel{2v_{1g}v_{2g}dt^2\cos(w_{2g}dt)}$$

$$+ 2V_2 V_{2g}\, dt^2 + 2V_2 V_{1g}\, dt^2 \cos(w_{2g}\, dt) - 2V_{2g}^2\, dt^2$$
$$- 2V_{1g} V_{2g}\, dt^2 \cos(w_{2g}\, dt)$$
$$- 2V_{1g} V_{2g} W_2\, dt^3 \sin(w_{2g}\, dt) + 2V_{1g} V_{2g} W_{2g}\, dt^3 \sin(w_{2g}\, dt)$$

$$= -V_{1g}^2\, dt^2 - V_{2g}^2\, dt^2 + 2V_1 V_{1g}\, dt^2 + 2V_1 V_{2g}\, dt^2 \cos(w_{2g}\, dt)$$
$$+ 2V_2 V_{2g}\, dt^2 + 2V_2 V_{1g}\, dt^2 \cos(w_{2g}\, dt)$$
$$- 2V_{1g} V_{2g}\, dt^2 \cos(w_{2g}\, dt)$$
$$- 2V_{1g} V_{2g} W_2\, dt^3 \sin(w_{2g}\, dt) + 2V_{1g} V_{2g} W_{2g}\, dt^3 \sin(w_{2g}\, dt)$$

constant $= K$

$$= \boxed{-V_{1g}^2\, dt^2 - V_{2g}^2\, dt^2 - 2V_{1g} V_{2g}\, dt^2 \cos(w_{2g}\, dt) + 2V_{1g} V_{2g} W_{2g}\, dt^3 \sin(w_{2g}\, dt)}$$
$$+ V_1 \left(2V_{1g}\, dt^2 + 2V_{2g}\, dt^2 \cos(w_{2g}\, dt)\right)$$
$$+ V_2 \left(2V_{2g}\, dt^2 + 2V_{1g}\, dt^2 \cos(w_{2g}\, dt)\right)$$
$$+ W_2 (-2) V_{1g} V_{2g}\, dt^3 \sin(w_{2g}\, dt)$$

For $t=3$

$$(x_3 - x_0)^2 + (y_3 - y_0)^2 \leq (r_a + r_o)^2$$

$$\Rightarrow x_3^2 + x_0^2 - 2x_3 x_0 + y_3^2 + y_0^2 - 2y_3 y_0 \leq (r_a + r_o)^2$$

$$\Rightarrow x_3^2 - 2x_3 x_0 + y_3^2 - 2y_3 y_0 \leq (r_a + r_o)^2 - x_0^2 - y_0^2$$

Now,

$$x_3 = x_0 + v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt + v_3 \cos(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt$$

$$y_3 = y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt + v_3 \sin(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt$$

Substituting x_3 and y_3, we get

$$\Big(x_0 + v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2)\, dt)\, dt + v_3 \cos(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt\Big)^2$$
$$- 2x_0^2 - 2x_0\Big(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1 + w_2) dt)\, dt + v_3 \cos(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt\Big)$$
$$+ \Big(y_0 + v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2)\, dt)\, dt + v_3 \sin(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt\Big)^2$$
$$- 2y_0^2 - 2y_0\Big(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1 + w_2) dt)\, dt + v_3 \sin(\theta_0 + (w_1 + w_2 + w_3)\, dt)\, dt\Big)$$

gets cancelled out with RHS

$$= -x_0^2 + \Big(v_1 \cos(\theta_0 + w_1 dt)\, dt + v_2 \cos(\theta_0 + (w_1+w_2)\, dt)\, dt$$
$$+ v_3 \cos(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt\Big)^2$$
$$- y_0^2 + \Big(v_1 \sin(\theta_0 + w_1 dt)\, dt + v_2 \sin(\theta_0 + (w_1+w_2)\, dt)\, dt$$
$$+ v_3 \sin(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt\Big)^2$$

$$= v_1^2 \cos^2(\theta_0 + w_1 dt)\, dt^2 + \Big(v_2 \cos(\theta_0 + (w_1+w_2)\, dt)\, dt$$
$$+ v_3 \cos(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt\Big)^2$$
$$+ 2 v_1 v_2 \cos(\theta_0 + w_1 dt) \cos(\theta_0 + (w_1+w_2)\, dt)\, dt^2$$
$$+ 2 v_1 v_3 \cos(\theta_0 + w_1 dt) \cos(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt^2$$
$$+ v_1^2 \sin^2(\theta_0 + w_1 dt)\, dt^2 + \Big(v_2 \sin(\theta_0 + (w_1+w_2)\, dt)\, dt$$
$$+ v_3 \sin(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt\Big)^2$$
$$+ 2 v_1 v_2 \sin(\theta_0 + w_1 dt) \sin(\theta_0 + (w_1+w_2)\, dt)\, dt^2$$
$$+ 2 v_1 v_3 \sin(\theta_0 + w_1 dt) \sin(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt^2$$

$$= v_1^2 dt^2 + 2 v_1 v_2 dt^2 \cos\big(\cancel{\theta_0} + \cancel{w_1 dt} - \cancel{\theta_0} - \cancel{w_1 dt} - w_2 dt\big)$$
$$+ 2 v_1 v_3 dt^2 \cos\big(\cancel{\theta_0} + \cancel{w_1 dt} - \cancel{\theta_0} - \cancel{w_1 dt} - w_2 dt - w_3 dt\big)$$
$$+ \Big(v_2 \cos(\theta_0 + (w_1+w_2)\, dt)\, dt + v_3 \cos(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt\Big)^2$$
$$+ \Big(v_2 \sin(\theta_0 + (w_1+w_2)\, dt)\, dt + v_3 \sin(\theta_0 + (w_1+w_2+w_3)\, dt)\, dt\Big)^2$$

$$= V_1^2 dt^2 + 2V_1 V_2 dt^2 \cos(w_2 dt) + 2V_1 V_3 dt^2 \cos((w_2+w_3)dt)$$
$$+ V_2^2 \cos^2(\theta_0 + (w_1+w_2)dt) dt^2 + V_3^2 dt^2 \cos^2(\theta_0 + (w_1+w_2+w_3)dt)$$
$$+ 2V_2V_3 dt^2 \cos(\theta_0 + (w_1+w_2)dt) \cos(\theta_0 + (w_1+w_2+w_3)dt)$$
$$+ V_2^2 \sin^2(\theta_0 + (w_1+w_2)dt) dt^2 + V_3^2 \sin^2(\theta_0 + (w_1+w_2+w_3)dt) dt^2$$
$$+ 2V_2V_3 dt^2 \sin(\theta_0 + (w_1+w_2)dt) \sin(\theta_0 + (w_1+w_2+w_3)dt)$$

$$= V_1^2 dt^2 + V_2^2 dt^2 + V_3^2 dt^2 + 2V_1V_2 dt^2 \cos(w_2 dt)$$
$$+ 2V_1V_3 dt^2 \cos((w_2+w_3)dt)$$
$$+ 2V_2V_3 dt^2 \cos(\cancel{\theta_0} \cancel{+w_1 dt} \cancel{+w_2 dt} - \cancel{\theta_0} \cancel{-w_1 dt} \cancel{-w_2 dt} - w_3 dt)$$

$$= V_1^2 dt^2 + V_2^2 dt^2 + V_3^2 dt^2 + 2V_1V_2 dt^2 \cos(w_2 dt)$$
$$+ 2V_1V_3 dt^2 \cos((w_2+w_3)dt) + 2V_2V_3 dt^2 \cos(w_3 dt)$$

Linearizing, we get

$$V_{1g}^2 dt^2 + V_{2g}^2 dt^2 + V_{3g}^2 dt^2 + 2V_{1g}V_{2g} dt^2 \cos(w_{2g} dt)$$
$$+ 2V_{1g}V_{3g} dt^2 \cos((w_{2g}+w_{3g})dt) + 2V_{2g}V_{3g} dt^2 \cos(w_{3g} dt)$$
$$+ \left(2V_{1g} dt^2 + 2V_{2g} dt^2 \cos(w_{2g} dt) + 2V_{3g} dt^2 \cos((w_{2g}+w_{3g})dt)\right)(V_1 - V_{1g})$$
$$+ \left(2V_{2g} dt^2 + 2V_{1g} dt^2 \cos(w_{2g} dt) + 2V_{3g} dt^2 \cos(w_{3g} dt)\right)(V_2 - V_{2g})$$
$$+ \left(2V_{3g} dt^2 + 2V_{1g} dt^2 \cos((w_{2g}+w_{3g})dt) + 2V_{2g} dt^2 \cos(w_{3g} dt)\right)(V_3 - V_{3g})$$
$$+ \left(-2V_{1g}V_{2g} dt^3 \sin(w_{2g} dt) - 2V_{1g}V_{3g} dt^3 \sin((w_{2g}+w_{3g})dt)\right)(w_2 - w_{2g})$$
$$+ \left(-2V_{1g}V_{3g} dt^3 \sin((w_{2g}+w_{3g})dt) - 2V_{2g}V_{3g} dt^3 \sin(w_{3g} dt)\right)(w_3 - w_{3g})$$

$$= -V_{1g}^2 dt^2 - V_{2g}^2 dt^2 - V_{3g}^2 dt^2 + \cancel{2V_{1g}V_{2g}\, dt^2 \cos(w_{2g}\, dt)}$$
$$+ \cancel{2V_{1g}V_{3g}\, dt^2 \cos((w_2+V_3)\, dt)} + \cancel{2V_{2g}V_{3g}\, dt^2 \cos(w_{3g}\, dt)}$$
$$+ 2V_1V_{1g}\, dt^2 + 2V_1V_{2g}\, dt^2 \cos(w_{2g}\, dt) + 2V_1V_{3g}\, dt^2 \cos((w_{2g}+w_{3g})\, dt)$$
$$- \cancel{2V_{1g}V_{2g}\, dt^2 \cos(w_{2g}\, dt)} - \cancel{2V_{1g}V_{3g}\, dt^2 \cos((w_2+V_3)\, dt)}$$
$$+ 2V_2V_{2g}\, dt^2 + 2V_2V_{1g}\, dt^2 \cos(w_{2g}\, dt) + 2V_2V_{3g}\, dt^2 \cos(w_{3g}\, dt)$$
$$- 2V_{1g}V_{2g}\, dt^2 \cos(w_{2g}\, dt) - \cancel{2V_{2g}V_{3g}\, dt^2 \cos(w_{3g}\, dt)}$$
$$+ 2V_3V_{3g}\, dt^2 + 2V_3V_{1g}\, dt^2 \cos((w_{2g}+w_{3g})\, dt) + 2V_3V_{2g}\, dt^2 \cos(w_{3g}\, dt)$$
$$- 2V_{1g}V_{3g}\, dt^2 \cos((w_{2g}+w_{2g})\, dt) - 2V_{2g}V_{3g}\, dt^2 \cos(w_{3g}\, dt)$$
$$- 2W_2V_{1g}V_{2g}\, dt^3 \sin(w_{2g}\, dt) - 2W_2V_{1g}V_{3g}\, dt^3 \sin((w_{2g}+w_{3g})\, dt)$$
$$+ 2w_{2g}V_{1g}V_{2g}\, dt^3 \sin(w_{2g}\, dt) + 2w_{2g}V_{1g}V_{3g}\, dt^3 \sin((w_{2g}+w_{3g})\, dt)$$
$$- 2W_3V_{1g}V_{3g}\, dt^3 \sin((w_{2g}+W_{3g})\, dt) - 2W_3V_{2g}V_{3g}\, dt^3 \sin(w_{3g}\, dt)$$
$$+ 2W_{3g}V_{1g}V_{3g}\, dt^3 \sin((W_{2g}+W_{3g})\, dt) + 2W_{3g}V_{2g}V_{3g}\, dt^3 \sin(w_{3g}\, dt)$$

$$= -V_{1g}^2 dt^2 - V_{2g}^2 dt^2 - V_{3g}^2 dt^2 - 2V_{1g}V_{2g}\, dt^2 \cos(w_{2g}\, dt)$$
$$- 2V_{1g}V_{3g}\, dt^2 \cos((w_{2g}+w_{3g})\, dt) - 2V_{2g}V_{3g}\, dt^2 \cos(w_{3g}\, dt)$$
$$+ 2W_{2g}V_{1g}V_{2g}\, dt^3 \sin(w_{2g}\, dt) + 2w_{2g}V_{1g}V_{3g}\, dt^3 \sin((w_{2g}+w_{3g})\, dt)$$
$$+ 2w_{3g}V_{1g}V_{3g}\, dt^3 \sin((w_{2g}+w_{3g})\, dt) + 2w_{3g}V_{2g}V_{3g}\, dt^3 \sin(w_{3g}\, dt)$$
$$+ V_1\left(2V_{1g}\, dt^2 + 2V_{2g}\, dt^2 \cos(w_{2g}\, dt) + 2V_{3g}\, dt^2 \cos((w_{2g}+w_{3g})\, dt)\right)$$
$$+ V_2\left(2V_{2g}\, dt^2 + 2V_{1g}\, dt^2 \cos(w_{2g}\, dt) + 2V_{3g}\, dt^2 \cos(w_{3g}\, dt)\right)$$

$$+ V_3\left(2V_{3g}\, dt^2 + 2V_{1g}\, dt^2 \cos((w_{2g}+w_{3g})\, dt) + 2V_{2g}\, dt^2 \cos(w_{3g}\, dt)\right)$$
$$+ W_2\left(-2V_{1g}V_{2g}\, dt^3 \sin(w_{2g}\, dt) - 2V_{1g}V_{3g}\, dt^3 \sin((w_{2g}+w_{3g})\, dt)\right)$$

$$+ w_3 \left(-2 v_{1y} v_{3y} dt^3 \sin\left((w_{2y} + w_{3y}) dt \right) - 2 v_{2y} v_{3y} dt^3 \sin\left(w_{3y} dt \right) \right)$$

$$(x_1 - x_g)^2 + (x_2 - x_g)^2 + (x_3 - x_g)^2$$

$$= x_1^2 + x_g^2 - 2x_1x_g + x_2^2 + x_g^2 - 2x_2x_g + x_3^2 + x_g^2 - 2x_3x_g$$

$$= x_1^2 + x_2^2 + x_3^2 + 3x_g^2 - 2(x_1 + x_2 + x_3)x_g$$