Digital Image Processing: Image Reconstruction from Projections

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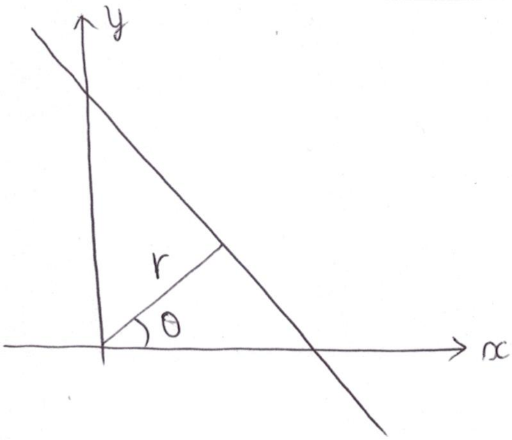
Parameterization of a line



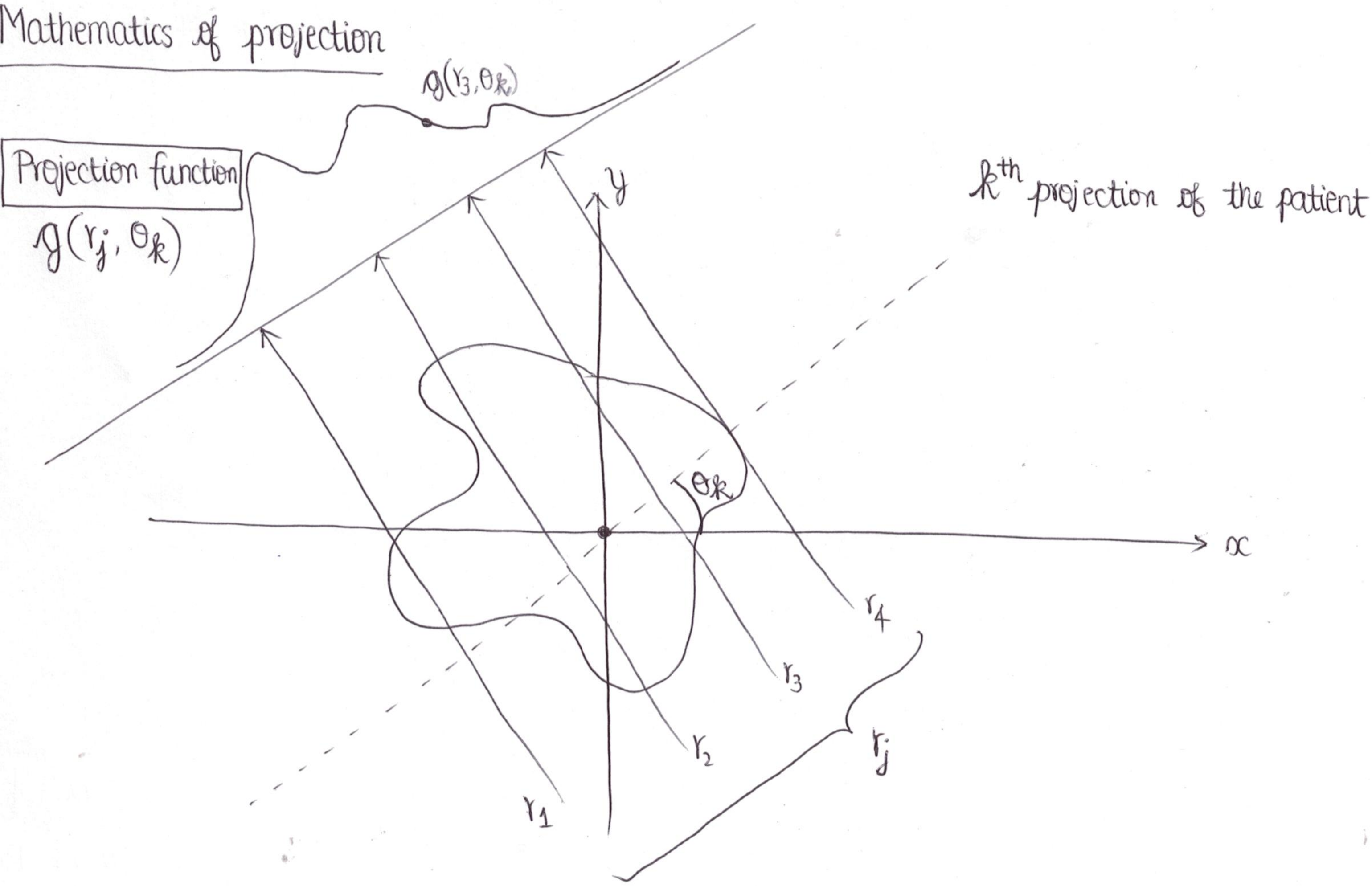
* Slope-intercept form

$$y = ax + b$$

* Normal representation

$$x\cos\theta + y\sin\theta = r$$

Mathematics of projection



How do we get the projection?

The Radon Transform

It's an integral along a line going through the patient $\longrightarrow$ "Line integral".

$$g(r,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{f(x,y)}_{\text{image fn.}} \ \underbrace{\delta(x\cos\theta + y\sin\theta - r)}_{\text{delta fn.}} \ \overbrace{dx\,dy}^{\text{integral across } x\text{-}y \text{ plane}}$$

image fn.

delta fn. (If inner condition true, it fires up, otherwise not)

taking an image and only integrating it along the line.

Sinogram $\longrightarrow$ visualize $g(r,\theta)$ as an image.

METHOD #01 : BACK PROJECTION

What would happen if we take these projections from diff. angles, smear them (backproject) and add them up?

We have $g(r, \theta_k)$ for a set of angles $\{\theta_k\}$ between $\overbrace{0 \text{ and } \pi}^{\text{not } 2\pi \text{ (same projection } 180^\circ \text{ opp.)}}$.

(Ex.)

For a fixed value $g(r_j, \theta_k)$, just copy this value into the image along the line $x\cos\theta_k + y\sin\theta_k = r_j$

Backprojected image at θ

$$b_\theta(x,y) = g(x\cos\theta + y\sin\theta, \theta) = g(r,\theta)$$

Sum of backprojections $\longrightarrow \displaystyle\int_0^{\pi} b_\theta(x,y)\, d\theta$ Laminogram

Problem: blur

$\boxed{\text{METHOD \#02: DIRECT FOURIER RECONSTRUCTION}}$

Fourier-Slice Theorem (aka. Projection-Slice Theorem)

$\left.\begin{array}{l}\text{Image has a 2D FT.} \\ \text{Each projection has a 1D FT.}\end{array}\right\}$ how are they related?

Consider a fixed angle θ's projection and take its Fourier transform w.r.t. r.

$$g(r,\theta) \longrightarrow G(k,\theta)$$

"time domain" "frequency domain".

Fourier transform

$$G(k,\theta) = \int_{-\infty}^{+\infty} g(r,\theta)\, e^{-2\pi j k r}\, dr$$

From Radon transform,

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\, \delta(x\cos\theta + y\sin\theta - r)\, dx\, dy \right] e^{-2\pi j k r}\, dr$$

Taking x and y out,

$$G(k,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y) \left[\int_{-\infty}^{+\infty} \delta(x\cos\theta + y\sin\theta - r)\, e^{-2\pi j k r}\, dr \right] dx\, dy$$

Delta fn. fires up only when inner condition is true. $(r = x\cos\theta + y\sin\theta)$

$$G(k,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi j k (x\cos\theta + y\sin\theta)}\, dx\, dy$$

Consider $k_x = k\cos\theta$ and $k_y = k\sin\theta$.

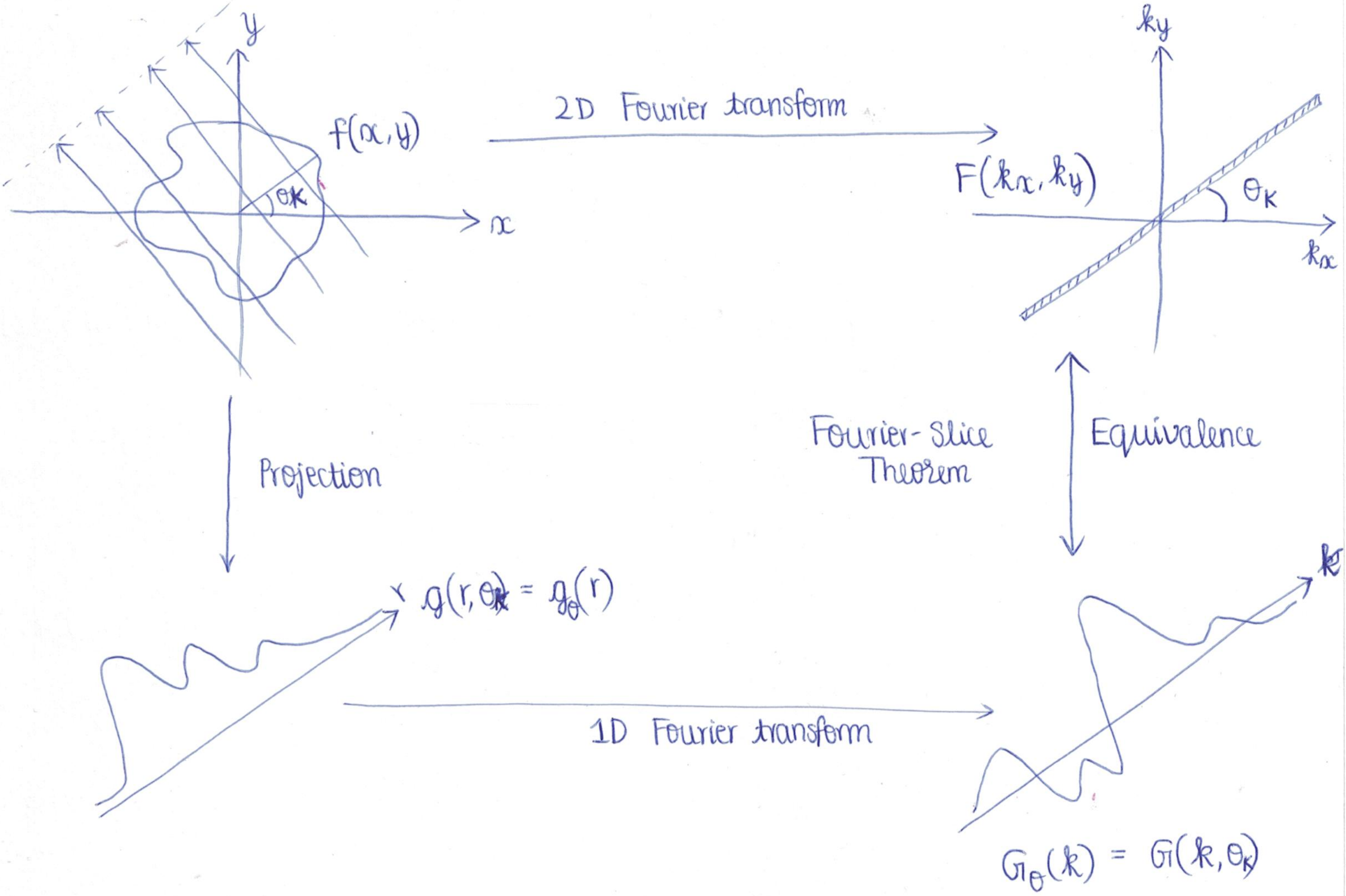
$$G(k,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi j (k_x x + k_y y)}\, dx\, dy$$

This looks like a 2D Fourier transform! FT of original image along the line $k_x = k\cos\theta$ and $k_y = k\sin\theta$.

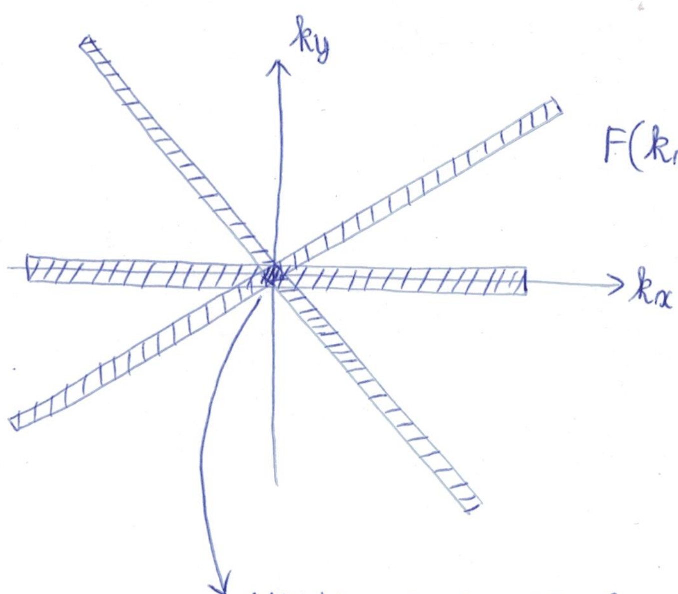
1D FT of projection	=	slice through the 2D FT of image at angle θ

Every time we are getting a projection, we are getting another slice through the image's Fourier transform.

→ gives intuition why we should be able to reconstruct from projections and why we need a lot of θ's to make it work.



Why do we have the bluriness problem?



Adding low frequency stuff to the image that shouldn't be there causes blurring.

Middle of 2D FT is oversampled.

Summing up backprojections directly is inaccurate.

→ downweight the middle s.t. when we add them up, we get exactly the contribution we need in the middle.

Algorithm - Direct Fourier Reconstruction

1. Calculate 1D FT of all projections $g_\theta(r) \equiv G_\theta(r)$.

   put all values of 1D fn. $G_\theta(r)$ on a polar grid to obtain 2D fn. $G(r,\theta)$.

2. Interpolate polar grid data samples to obtain $F(k_x, k_y)$ data on Cartesian grid.

3. Calculate 2D IFT of $F(k_x, k_y) \equiv f(x,y)$.

<u>Problem:</u> Interpolation.

METHOD #03: FILTERED BACK PROJECTION

Is it possible to avoid interpolation in reconstructing $f(x,y)$? Yes.

Fourier transform - recap

Let k and r be conjugate variables in the Fourier domain and the original domain, respectively. 'r' represents spatial position, 'k' represents spectral frequency.

<u>Forward Fourier transform</u>

$$F(\vec{k}) = \int_{-\infty}^{+\infty} f(\vec{r})\, e^{-2\pi j \vec{k}\cdot\vec{r}}\, d\vec{r}$$

<u>Inverse Fourier transform</u>

$$f(\vec{r}) = \int_{-\infty}^{+\infty} F(\vec{k})\, e^{+2\pi j \vec{k}\cdot\vec{r}}\, d\vec{k}$$

Polar form of Fourier transform - recap

Given a fn. in polar form, obtain its FT in Cartesian form.

Using polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$.

$$F(k_x, k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi j (k_x x + k_y y)}\, dx\, dy$$

$$= \int_{0}^{2\pi}\int_{0}^{+\infty} f(r,\theta)\, e^{-2\pi j (k_x r\cos\theta + k_y r\sin\theta)}\, r\, dr\, d\theta$$

Since $f(k, \theta+\pi) = f(-k,\theta)$,

$$= \int_{0}^{\pi}\int_{-\infty}^{+\infty} f(r,\theta)\, e^{-2\pi j (k_x r\cos\theta + k_y r\sin\theta)}\, |r|\, dr\, d\theta$$

Analogously, polar form of inverse Fourier transform,

Let $k_x = k\cos\theta$ and $k_y = k\sin\theta$.

$$f(x,y) = \int_{0}^{\pi}\int_{-\infty}^{+\infty} F(k,\theta)\, e^{+2\pi j (xk\cos\theta + yk\sin\theta)}\, |k|\, dk\, d\theta.$$

from Fourier-Slice theorem.

We want original image $f(x,y)$

$$f(x,y) = \int_{0}^{\pi} \underbrace{\left[\left. \int_{-\infty}^{+\infty} |k|\, G(k,\theta)\, e^{+2\pi j k r}\, dk \right|_{r = x\cos\theta + y\sin\theta} \right]} d\theta$$

Looks like inverse FT of each of the projections - but with $|k|$.

It tells us that it <u>modifies</u> each of the projections before adding them up.

$$\int_{-\infty}^{+\infty} |k| \, G(k,\theta) \, e^{+2\pi j k r} \, dk$$ is the IFT of $G(k,\theta)$, but...

multiplied by a <u>filter function</u> $|k|$.

$\hookrightarrow$ These are, thus, <u>filtered back projections</u>.

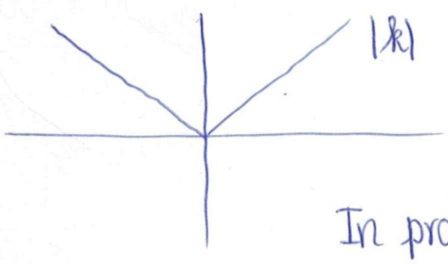
$\boxed{\text{Algorithm - Filtered Back Projection}}$

1. Compute 1D FT of each projection $g_\theta(r) \equiv G_\theta(r)$.
2. Multiply each FT $G(k, \theta_k)$ by $|k|$.
3. Take 1D IFT.
4. Integrate over all angles to get the original image $f(x,y)$.

Assumes we have infinite number of projections.

In practice, we need to have <u>enough</u> projections to get close to ideal result.

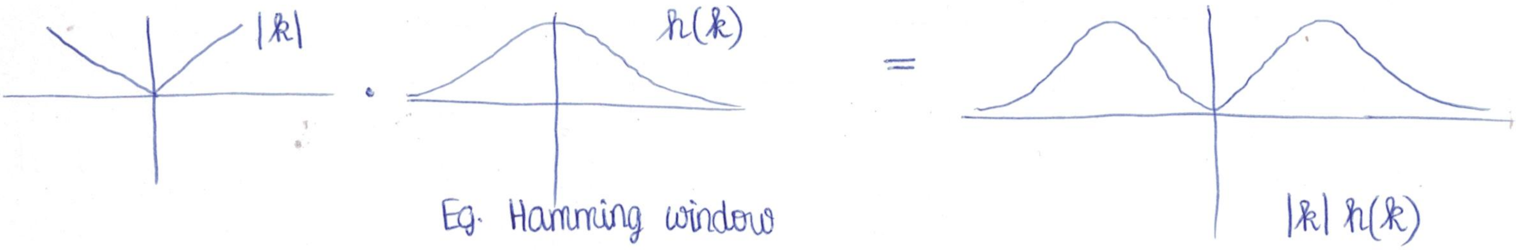
What does the fn. $|k|$ look like?



Can't take FT of a fn. like this

$\hookrightarrow$ it integrates to ∞.

In practice, we multiply it w/ some sort of fn. like window.



Eg. Hamming window

– to prevent possible FT numerical issues.

MATLAB does all this computation with FFTs but most CT systems do the computation in <u>spatial domain with convolutions</u>

$$f(x,y) = \int_0^{\pi} \left[\left[\int_{-\infty}^{+\infty} |k|\, h(k)\; G(k, \theta_k)\; e^{+2\pi j k r}\, dk \right] \right]_{r = x\cos\theta + y\sin\theta} d\theta$$

$$= \int_0^{\pi} \left[\left[s(r) * g(r,\theta) \right] \right]_{r = x\cos\theta + y\sin\theta} d\theta$$

where $s(r) \equiv$ Inverse Fourier transform of $|k|\, h(k)$.

$$\boxed{f(x,y) = \int_0^{\pi} \int_{-\infty}^{+\infty} g(r,\theta) \cdot s(x\cos\theta + y\sin\theta - r)\; dr\, d\theta}$$

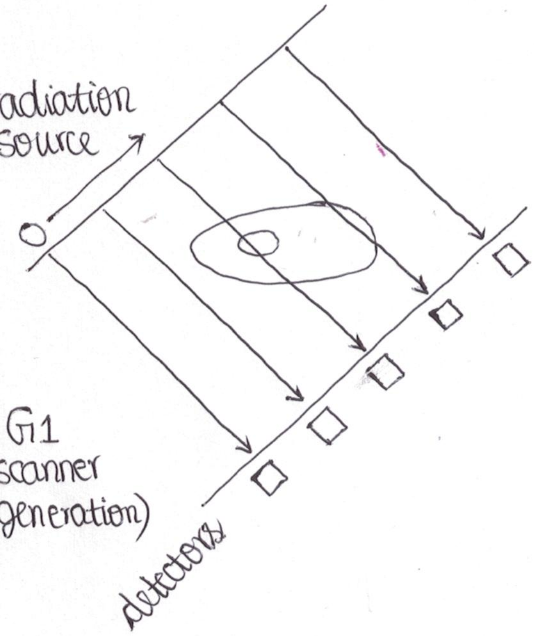
1D spatial convolution

Practical tips

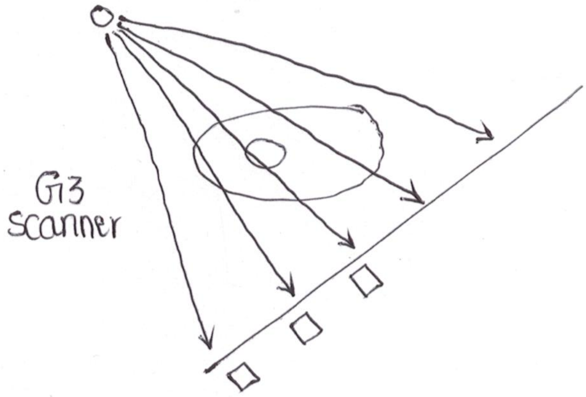
(i) No need to store all backprojections
- Just add them up as they come in.

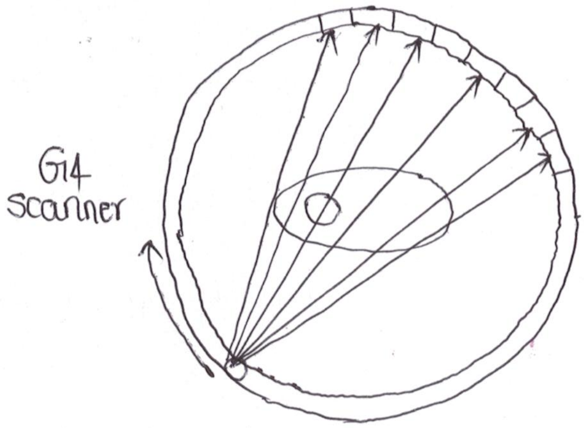
(ii) Since $|k| = 0$ at $k = 0$, we lose the DC value in the FBP algorithm.
- Scale or guess the value.

Parallel beam reconstruction



Fan-beam reconstruction

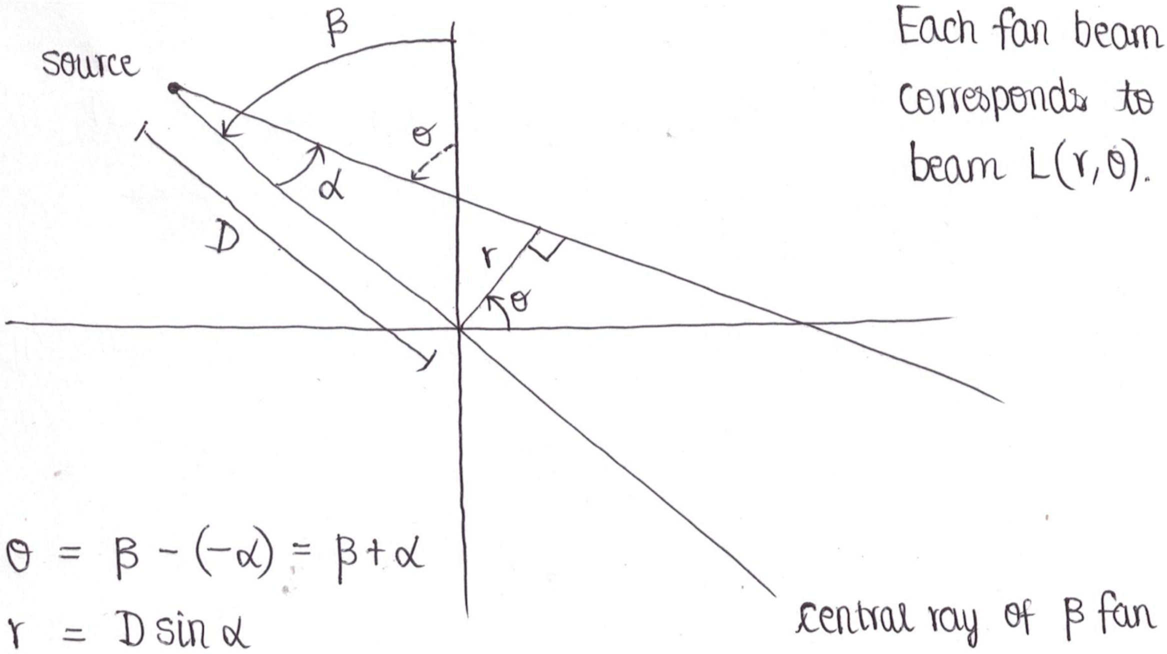




Fan-beam Reconstruction

We need different reconstruction equations since the geometry is different.

Each fan beam $p(\alpha, \beta)$ corresponds to some parallel beam $L(r, \theta)$.



$\theta = \beta - (-\alpha) = \beta + \alpha$

$r = D \sin \alpha$

From filtered back projection of parallel rays,

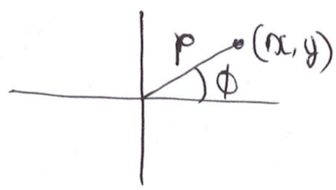
$$f(x,y) = \int_0^{\pi} \underbrace{\int_{-\infty}^{+\infty} \overbrace{g(r,\theta)}^{\text{Radon transform}} \cdot \underbrace{\delta(x\cos\theta + y\sin\theta - r)}_{\text{window function}}\, dr}_{\text{1D spatial convolution w/ window function.}}\, d\theta$$

Since r is bounded (patient has a finite width), $\longrightarrow$ ($-T$ to $+T$)

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-T}^{+T} g(r,\theta)\cdot \delta(x\cos\theta + y\sin\theta - r)\, dr\, d\theta$$

Now, convert (x,y) in Cartesian to (p,ϕ) in polar coordinates.

$$x = p\cos\phi$$
$$y = p\sin\phi$$



$$x\cos\theta + y\sin\theta = p\cos\phi\cos\theta + p\sin\phi\sin\theta$$
$$= p\cos(\theta-\phi)$$

Trigonometry - recap

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$f(x,y) \longrightarrow f(p,\phi)$$

$$f(p,\phi) = \frac{1}{2}\int_0^{2\pi}\int_{-T}^{+T} g(r,\theta)\cdot \delta\left(p\cos(\theta-\phi) - r\right)\, dr\, d\theta$$

Convert every (r, θ) ray into (α, β) ray.

$$dr\, d\theta = \left| \det\left(\begin{bmatrix} 1 & 1 \\ D\cos\alpha & 0 \end{bmatrix} \right) \right| d\alpha\, d\beta = D\cos\alpha\; d\alpha\, d\beta.$$

$$\alpha = \sin^{-1}\left(\frac{P}{D}\right)$$

Maximum value $\quad +\alpha_m = \sin^{-1}\left(\frac{T}{D}\right)$

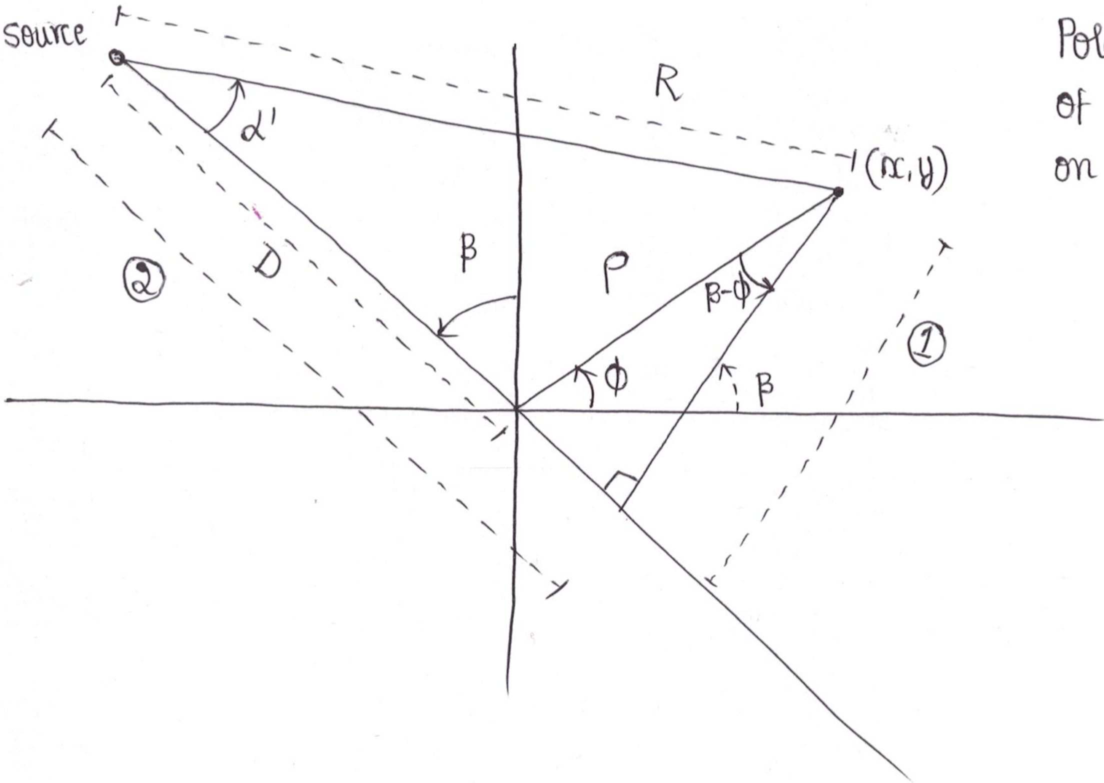
Minimum value $\quad -\alpha_m = -\sin^{-1}\left(\frac{T}{D}\right)$

Change (r, θ) integral to (α, β) integral.

$$f(P, \phi) = \frac{1}{2} \int_{-\alpha_m}^{+\alpha_m} \int_{0}^{2\pi} \underbrace{P(\alpha, \beta)}_{\text{integral of the fan beam}} \cdot \delta\left(P\cos(\alpha+\beta-\phi) - D\sin\alpha\right) D\cos\alpha\; d\beta\, d\alpha$$

Because,

$$f(P, \phi) = \frac{1}{2} \int_{-\sin^{-1}(T/D)}^{+\sin^{-1}(T/D)} \int_{-\alpha}^{2\pi-\alpha} g\left(D\sin\alpha, \alpha+\beta\right) \cdot \delta\left(P\cos(\alpha+\beta-\phi) - D\sin\alpha\right) \cdot D\cos\alpha\; d\beta\, d\alpha$$



Polar representation of an arbitrary point on a fan-beam ray.

① $\rho \cos(\beta - \phi) = R \sin \alpha'$

② $\rho \sin(\beta - \phi) + D = R \cos \alpha'$

$$\rho \cos(\alpha + \beta - \phi) - D \sin \alpha$$

$$= \rho \cos(\beta - \phi) \cos \alpha - \rho \sin(\beta - \phi) \sin \alpha - D \sin \alpha$$

$$= \rho \cos(\beta - \phi) \cos \alpha - \left[\rho \sin(\beta - \phi) + D\right] \sin \alpha$$

$$= R \sin \alpha' \cos \alpha - R \cos \alpha' \sin \alpha$$

$$= R \sin(\alpha' - \alpha)$$

R and α' are both functions of $\underbrace{\rho, \phi}_{\text{point in the image we are trying to reconstruct}}, \overbrace{\beta}^{\text{rotation around the patient}}$

$$f(\rho, \phi) = \frac{1}{2} \int_{-\alpha_m}^{+\alpha_m} \int_{0}^{2\pi} P(\alpha, \beta) \cdot g\left(\rho \cos(\alpha + \beta - \phi) - D \sin \alpha\right) D \cos \alpha \, d\beta \, d\alpha$$

$$= \frac{1}{2} \int_{-\alpha_m}^{+\alpha_m} \int_{0}^{2\pi} P(\alpha, \beta) \cdot \underbrace{g\left(R \sin(\alpha' - \alpha)\right)} D \cos \alpha \, d\beta \, d\alpha$$

recall, g was defined as the inverse Fourier transform of the window fn.

Putting it all together,

$$f(\rho, \phi) = \int_{0}^{2\pi} \underbrace{\frac{1}{R^2} \left[\int_{-\alpha_m}^{+\alpha_m} q(\alpha, \beta) \cdot h(\alpha' - \alpha) \, d\alpha \right]} d\beta$$

convolution of a projection for a fixed β and a window function h.
(fan-beam filtered backprojection)

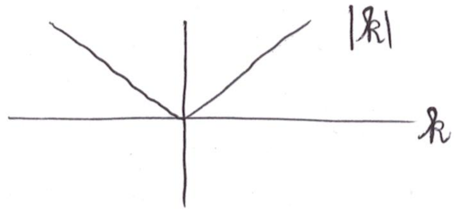
$$h(\alpha) = \frac{1}{2} \left(\frac{\alpha}{\sin \alpha} \right)^2 g(\alpha)$$

$$q(\alpha, \beta) = P(\alpha, \beta) \, D \cos \alpha$$

More detail:

s was defined as the IFT of the window function.

$$s(t) = \int_{-\infty}^{+\infty} |k| \, e^{+2\pi j k t} \, dk$$



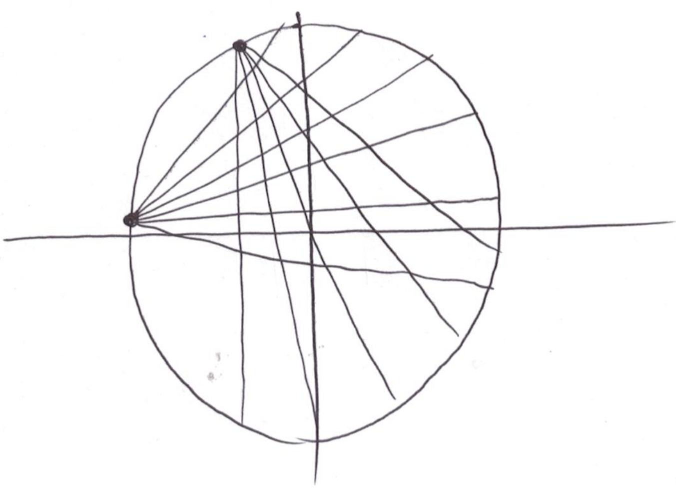
$$s(R \sin \gamma) = \int_{-\infty}^{+\infty} |k| \, e^{+2\pi j k (R \sin \gamma)} \, dk$$

Let $k' = \dfrac{k R \sin \gamma}{\gamma}$, then $dk' = \dfrac{R \sin \gamma}{\gamma} \, dk$

$$s(R \sin \gamma) = \int_{-\infty}^{+\infty} |k'| \cdot \left(\frac{\gamma}{R \sin \gamma}\right)^2 e^{+2\pi j k' \gamma} \, dk'$$

$$= \left(\frac{\gamma}{R \sin \gamma}\right)^2 s(\gamma)$$

Next: Less accurate, much easier way of dealing w/ fan-beam projections



Idea: Re-sort fan beams from different β into collections of parallel beam.

A fan-beam corresponds to some parallel beam.

$$p(\alpha, \beta) = g(r, \theta)$$
$$= g(D \sin\alpha, \alpha + \beta)$$

If fan beams are equally spaced,

$$\Delta\alpha = \Delta\beta = \gamma$$

$$p(n\gamma, m\gamma) = g(D \sin n\gamma, (n+m)\gamma)$$

since α and β corresponds to some multiple of γ.

$$\boxed{\begin{aligned} &n^{th} \text{ ray in the } m^{th} \text{ radial projection} \\ &= n^{th} \text{ ray in the } (n+m)^{th} \text{ parallel projection} \end{aligned}}$$

Problem: r is not uniformly spaced in γ,

so we need to do some interpolation to get all the values of r we need.

$\rightarrow$ not an issue if γ is small enough.

References

01. Digital Image Processing (Gonzalez & Woods)
02. Fundamentals of Medical Imaging (Paul Suetens)
03. EC5.405 Medical Image Analysis (Jayanthi Sivaswamy)

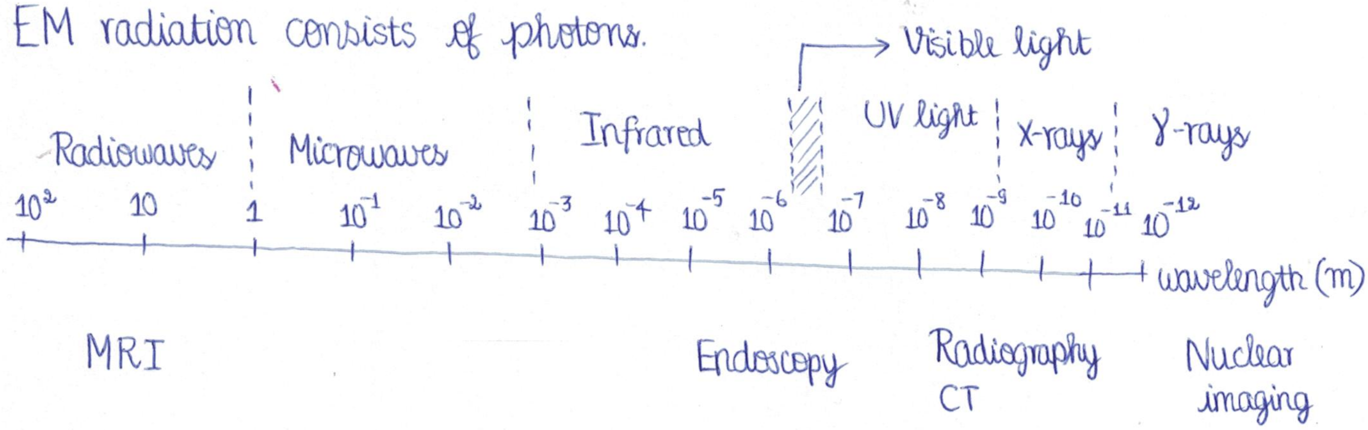
Acknowledgement

Professor Rich Radke

ECSE-4540 Intro to DIP

Electomagnetic spectrum and X-rays

EM radiation consists of photons.



X-rays → wavelength (in order of Angstroms)
X-rays → photon energy (in order of keV)

- accidental discovery in 1895 by Wilhelm Roentgen
- first Nobel Prize in physics (1901)

Imaging Physics

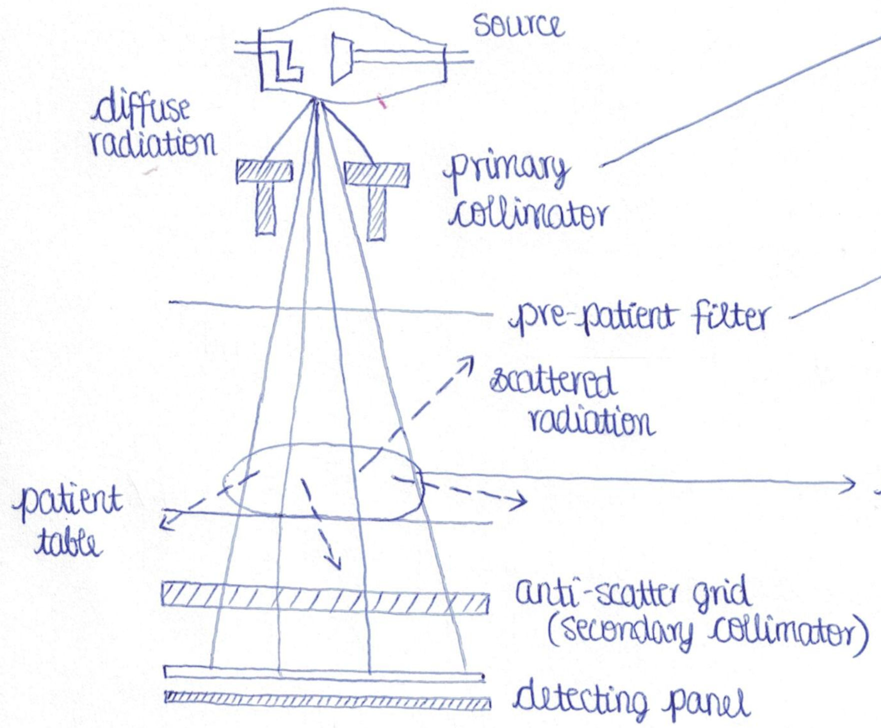
Ionizing photons can interact with matter in different ways.

1. Scattering
   - Rayleigh : coherent and does not lead to ionization.
   - Compton : incoherent and leads to ionization.

2. Photoelectric absorption
   - Ionization of the atom

3. Electron-positron pair production
   - Applications in nuclear medicine

Radiographic Imaging Chain



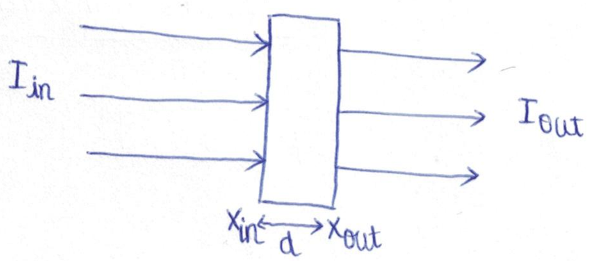
primary collimator → to limit the patient area to be irradiated

pre-patient filter → removes low-energy photons, thus, increasing mean energy of photon beam. (beam hardening)

patient attenuates X-ray beam and produces scatter.

anti-scatter grid (secondary collimator) → absorbs scatter photons

X-ray Imaging



I : intensity of X-ray beam

d : thickness of medium

μ : linear attenuation coefficient

σ : Compton coefficient

	Homogenous medium	Non-homogenous medium
Single energy (ideal case)	$I_{out} = I_{in}\, e^{-\mu d}$	$I_{out} = I_{in}\, e^{-\int_{x_{in}}^{x_{out}} \mu(x)\,dx}$
Multiple energy (real case)	—	$I_{in} = \int_0^\infty \sigma(E)\, dE$ $I_{out} = \int_0^\infty \sigma(E)\, e^{-\int_{x_{in}}^{x_{out}} \mu(E,x)\,dx}\, dE$

Tid-bits

Radon transform (1917)
↳ Johann Radon (pure mathematics)

Computer-assisted tomography
↳ Nobel Prize in Physiology/Medicine (1979)
↳ A.M. Cormack and G.N. Hounsfield