The Open University

MST124

Essential mathematics 1

Exercise Booklet 2

1 Plotting graphs

Exercise 1

Determine whether the point $(2, -1)$ satisfies each of the following equations:

(a) $y = 3x - 7$ (b) $y = -2x - 3$

(c) $y = x^2 + 3x - 11$ (d) $y^2 = x^2 + x - 7$

Exercise 2

For each of the following equations, complete the table of values for x and y and hence plot the graph of the equation.

(a) $y = 2x + 3$

x	-3	-2	-1	0	1	2
y						

(b) $y = x^2 + x - 3$

x	-2	-1	0	1	2	3
y						

2 Straight-line graphs

Exercise 3

(a) For each of the following pairs of points, find the gradient of the line joining the two points, if the line has a gradient.

(i) $(3, 7)$ and $(5, 11)$

(ii) $(1, 7)$ and $(5, -11)$

(iii) $(3, 5)$ and $(-2, 5)$

(iv) $(2, 4)$ and $(2, 16)$

(b) For each line in part (a), say whether it slopes up from left to right, down from left to right, is parallel to the x-axis or is parallel to the y-axis.

Exercise 4

(a) Find the equation of each of the lines in Exercise 3(a).

(b) Find the x- and y-intercepts of each line.

(c) Draw each line.

Exercise 5

Find the equations of the following lines.

(a) The line with gradient 4 and y-intercept -3.

(b) The line with y-intercept 5 and x-intercept 2.

(c) The line perpendicular to the line with equation $y = 3x - 5$, passing through $(3, 1)$.

(d) The line with gradient -2, passing through $(4, 1)$.

Exercise 6

(a) Find the gradient of each of the following lines, where possible.

(i) $y = 3x + 4$ (ii) $2y = 6x - 12$

(iii) $x + 3y - 2 = 0$ (iv) $y = -3$

(v) $x - 2 = 0$

(b) Which lines in part (a) are parallel or perpendicular to other lines in part (a)?

Exercise 7

Use the equation

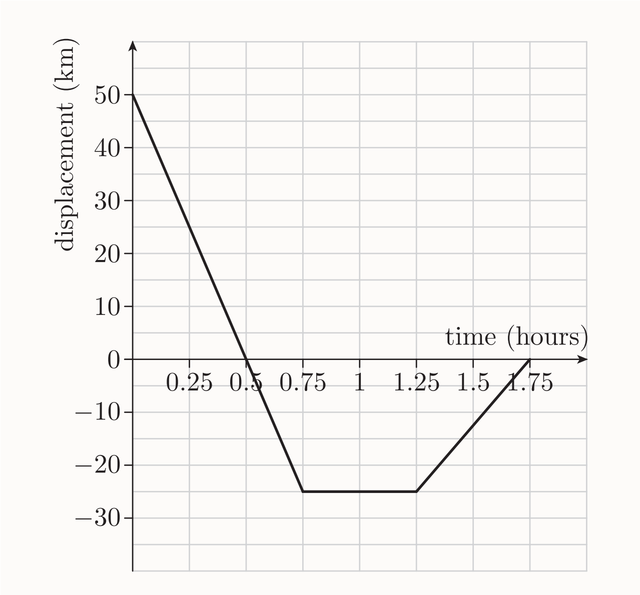
$$\text{distance travelled} = \text{constant speed} \times \text{time elapsed}$$

to fill in the gaps in the table below.

Distance	Speed	Time
	30 km/h	2 hours
	40 km/h	30 minutes
50 km		2 hours
25 km	100 km/h	

Exercise 8

The displacement-time graph below, which consists of three line segments, represents a car's journey along a straight road.



(a) Find the car's velocity and speed during each of the following time periods.

(i) The first three-quarters of an hour.

(ii) The half hour after that.

(iii) The final half hour.

(b) Describe the car's journey in words.

3 Intersections of lines

Exercise 9

For each pair of lines in Exercise 6(a), find the point of intersection if the lines intersect.

Exercise 10

Solve the following simultaneous equations by using the method of elimination.

(a) $4x - 3y = -5$
$3x - 2y = -4$

(b) $3p - 4q = 10$
$5p + 3q = 7$

(c) $4s - 3t = 1$
$6s - 2t = 1$

(d) $3x - 4y = 1$
$2x - y = \frac{2}{3}$

4 Quadratics

Exercise 11

Factorise each of the following quadratics.

(a) $x^2 + 7x + 10$ (b) $r^2 + r - 12$

(c) $p^2 - 7p + 12$ (d) $t^2 + 3t - 18$

(e) $a^2 + a - 6$ (f) $b^2 - 4b - 5$

Exercise 12

Factorise each of the following quadratics.

(a) $2x^2 + 7x + 5$ (b) $3r^2 + 5r - 12$

(c) $5p^2 - 7p - 12$ (d) $2t^2 - 5t - 18$

(e) $3a^2 + 7a - 6$ (f) $4b^2 + 8b - 5$

Exercise 13

Factorise each of the following quadratics.

(a) $a^2 + 3a$ (b) $p^2 - 6p$

(c) $4y^2 - 2y$ (d) $4t^2 - 1$

(e) $4x^2 - 9$ (f) $9r^2 - 4s^2$

Exercise 14

Use factorisation to solve the following quadratic equations. (Your solutions to Exercises 11, 12 and 13 may be helpful in some instances.)

(a) $x^2 + 7x + 10 = 0$ (b) $t^2 + 3t - 18 = 0$

(c) $5 + 4b - b^2 = 0$ (d) $2x^2 + 7x + 5 = 0$

(e) $3a^2 + 7a - 6 = 0$ (f) $2s^2 - 5s - 12 = 0$

(g) $a^2 + 3a = 0$ (h) $4y^2 - 2y = 0$

(i) $9r^2 - 4 = 0$ (j) $25p^2 - 5p - 12 = 0$

Exercise 15

Write each of the following expressions in completed-square form.

(a) $x^2 + 20x$ (b) $x^2 - 11x$ (c) $y^2 + 7y$

Exercise 16

Write each of the following expressions in completed-square form.

(a) $x^2 - 18x + 60$ (b) $3x^2 + 2x + 2$

(c) $-2x^2 + x - 5$

Exercise 17

Use the method of completing the square to solve the following quadratic equations.

(a) $x^2 + 5x + 6 = 0$ (b) $x^2 - 14x + 30 = 0$

(c) $3x^2 + 6x + 1 = 0$ (d) $3x^2 + 5x + 1 = 0$

(e) $-4x^2 + x + 2 = 0$

Exercise 18

Use the quadratic formula to confirm your answers to Exercise 17.

Exercise 19

Use the discriminant to determine whether each of the following quadratic equations has one, two or no real solutions. Find any real solutions.

(a) $x^2 + 2x + 3 = 0$ (b) $2x^2 - 4x - 5 = 0$

(c) $3x^2 + 4 = 0$ (d) $2x^2 - 5x = 0$

(e) $x^2 + 8x + 16 = 0$

Exercise 20

Solve the following equations.

(a) $\dfrac{1}{2x} = \dfrac{x}{5 - 3x}$ (b) $p^4 = 9$

(c) $u^4 - 2u^3 - 3u^2 = 0$ (d) $\dfrac{2}{1 + x} = \dfrac{3x}{1 - x}$

(e) $(b^2 - b - 2)(b^2 + 3b + 2) = 0$

Exercise 21

Sketch the graphs of the following equations.

(a) $y = x^2 + 2x + 3$

(b) $y = 2x^2 - 4x - 10$

(c) $y = 3x^2 + 4$

(d) $y = 2x^2 - 5x$

(e) $y = x^2 + 8x + 16$

Solutions to exercises

Solution to Exercise 1

(a) Setting $x = 2$ and $y = -1$ gives

$$\text{RHS} = 3x - 7 = 3 \times 2 - 7 = 6 - 7 = -1,$$
$$\text{LHS} = y = -1.$$

The LHS and RHS are equal, so the point $(2, -1)$ satisfies the equation $y = 3x - 7$.

(b) Setting $x = 2$ and $y = -1$ gives

$$\text{RHS} = -2x - 3 = -2 \times 2 - 3 = -7,$$
$$\text{LHS} = y = -1.$$

Since the LHS and RHS are not equal, the point $(2, -1)$ does not satisfy the equation $y = -2x - 3$.

(c) Setting $x = 2$ and $y = -1$ gives

$$\text{RHS} = x^2 + 3x - 11 = 2^2 + 3 \times 2 - 11 = -1,$$
$$\text{LHS} = y = -1.$$

The LHS and RHS are equal, so the point $(2, -1)$ satisfies the equation $y = x^2 + 3x - 11$.

(d) Setting $x = 2$ and $y = -1$ gives

$$\text{RHS} = 2^2 + 2 - 7 = 4 + 2 - 7 = -1,$$
$$\text{LHS} = y^2 = (-1) \times (-1) = 1.$$

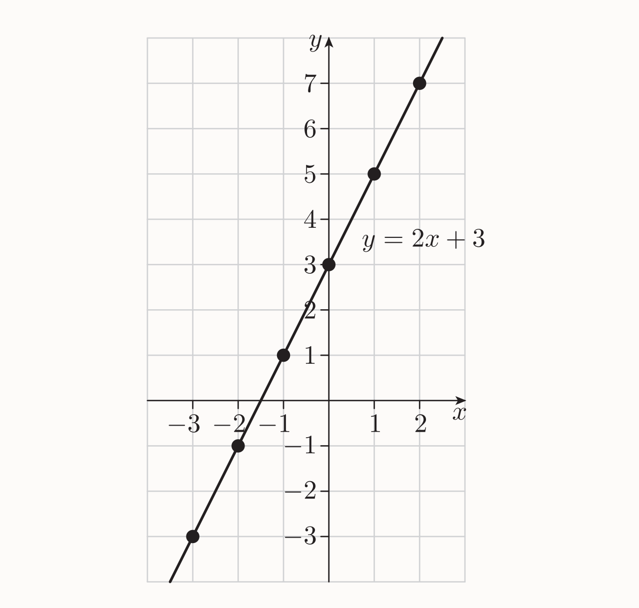
Since the LHS and RHS are not equal, the point $(2, -1)$ does not satisfy the equation $y^2 = x^2 + x - 7$.

Solution to Exercise 2

(a) Using $y = 2x + 3$ gives the following table.

x	-3	-2	-1	0	1	2
y	-3	-1	1	3	5	7

The graph is shown below.

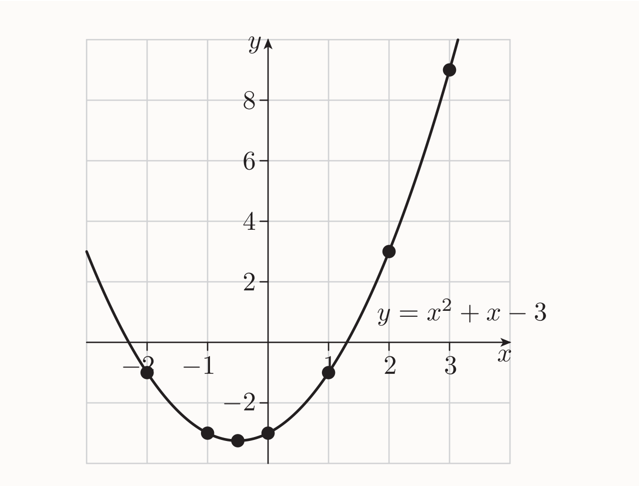


(b) Using $y = x^2 + x - 3$ gives the following table.

x	-2	-1	0	1	2	3
y	-1	-3	-3	-1	3	9

It seems from the table that the graph has a vertical line of symmetry at the x-value halfway between $x = -1$ and $x = 0$; that is, at $x = -\frac{1}{2}$. To plot the graph accurately it is helpful to calculate the value of y when $x = -\frac{1}{2}$, which is $y = -\frac{13}{4}$.

The graph is shown below.



Solution to Exercise 3

(a) (i) The gradient is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{11-7}{5-3} = \frac{4}{2} = 2.$$

(ii) The gradient is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-11-7}{5-1} = -\frac{18}{4} = -\frac{9}{2}.$$

(iii) The gradient is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-5}{-2-3} = \frac{0}{-5} = 0.$$

(iv) The two points have the same x-coordinate, so the line through them is vertical and hence has no gradient.

(b) (i) This line slopes up from left to right.

(ii) This line slopes down from left to right.

(iii) This line is parallel to the x-axis.

(iv) This line is parallel to the y-axis.

Solution to Exercise 4

(a) (i) Since the gradient of the line is 2, its equation is of the form $y = 2x + c$. It passes through the point $(3, 7)$, so $7 = 2 \times 3 + c$. Hence $c = 1$ and the equation of the line is $y = 2x + 1$.

(ii) Since the gradient of the line is $-\frac{9}{2}$, its equation is of the form $y = -\frac{9}{2}x + c$. It passes through the point $(1, 7)$, so

$$7 = -\tfrac{9}{2} \times 1 + c;$$

that is,

$$c = \tfrac{14}{2} + \tfrac{9}{2} = \tfrac{23}{2}.$$

Hence the equation of the line is $y = -\frac{9}{2}x + \frac{23}{2}$. It can be rewritten as $9x + 2y = 23$.

(iii) The y-coordinate of each point is 5, so the equation of the line is $y = 5$.

(This is the equation of a line parallel to the x-axis, as required.)

(iv) The x coordinate of each point is 2, so the equation of the line is $x = 2$.

(This is the equation of a line parallel to the y-axis, as required.)

(b) (i) Putting $x = 0$ in the equation $y = 2x + 1$ gives $y = 1$, so the y-intercept is 1.

Putting $y = 0$ in the equation gives $0 = 2x + 1$; that is, $x = -\frac{1}{2}$. So the x-intercept is $-\frac{1}{2}$.

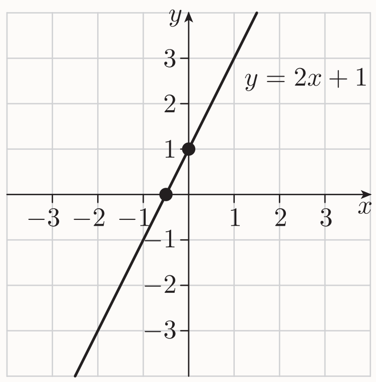
(ii) Putting $x = 0$ in the equation $9x + 2y = 23$ gives $2y = 23$; that is, $y = \frac{23}{2}$. So the y-intercept is $\frac{23}{2}$.

Putting $y = 0$ in the equation gives $9x = 23$; that is, $x = \frac{23}{9}$. So the x-intercept is $\frac{23}{9}$.

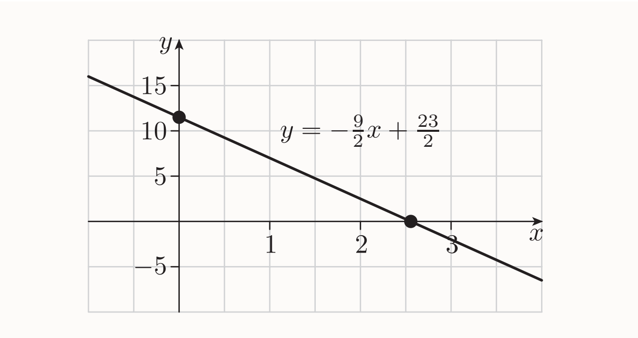
(iii) The line with equation $y = 5$ has y-intercept 5 and no x-intercept.

(iv) The line with equation $x = 2$ has x-intercept 2 and no y-intercept.

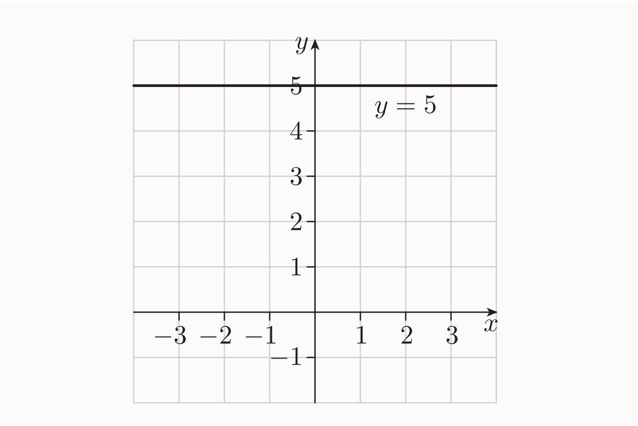
(c) (i) From part (b)(i), two points on the line are $(0, 1)$ and $(-\frac{1}{2}, 0)$. (Alternatively, you can use the two points $(3, 7)$ and $(5, 11)$ from Exercise 3(a).)



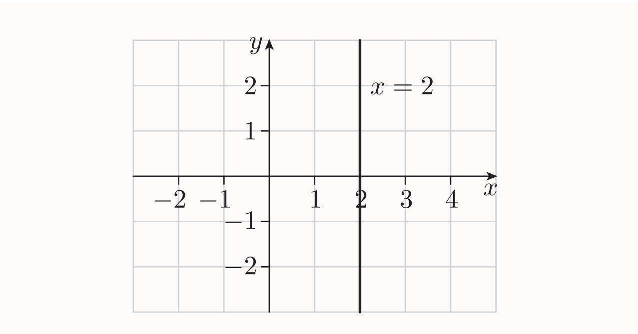
(ii) From part (b)(i), two points on the line are $(0, \frac{23}{2}) = (11.5, 0)$ and $(\frac{23}{9}, 0) \approx (2.6, 0)$. (Alternatively, you can use the two points $(1, 7)$ and $(5, -11)$ from Exercise 3(a).)



(iii) This line is the horizontal line with y-intercept 5.



(iv) This line is the vertical line with x-intercept 2.



Solution to Exercise 5

(a) The line with gradient 4 and y-intercept -3 has equation $y = 4x - 3$.

(b) The line with y-intercept 5 and x-intercept 2 passes through the points $(0, 5)$ and $(2, 0)$. So it has gradient

$$\frac{5-0}{0-2} = -\frac{5}{2},$$

and y-intercept 5. Its equation is therefore $y = -\frac{5}{2}x + 5$.

(c) The line $y = 3x - 5$ has gradient 3, so any line perpendicular to it has gradient $-\frac{1}{3}$. So the required equation is of the form $y = -\frac{1}{3}x + c$. Substituting $(x, y) = (3, 1)$ into this equation gives $1 = -\frac{1}{3} \times 3 + c$; that is, $c = 1 + 1 = 2$. So the equation of the line is $y = -\frac{1}{3}x + 2$.

(d) The line has gradient -2, so its equation is of the form $y = -2x + c$. The line passes through $(4, 1)$, so $1 = -2 \times 4 + c$; that is, $c = 9$. So the equation of the line is $y = -2x + 9$.

Solution to Exercise 6

(a) (i) The line $y = 3x + 4$ has gradient 3.

(ii) The equation $2y = 6x - 12$ can be rearranged as $y = 3x - 6$, so this line also has gradient 3.

(iii) The equation $x + 3y - 2 = 0$ can be rearranged as $y = -\frac{1}{3}x + \frac{2}{3}$, so this line has gradient $-\frac{1}{3}$.

(iv) The line $y = -3$ is parallel to the x-axis, so its gradient is 0.

(v) The line $x = 2$ is parallel to the y axis, so its gradient is undefined.

(b) The lines in parts (a)(i) and (a)(ii) are parallel to each other, since they have the same gradient.

Both these lines are perpendicular to the line in part (a)(iii), since the product of the gradients 3 and $-\frac{1}{3}$ is -1.

Also, the lines in parts (a)(iv) and (v) are perpendicular to each other.

Solution to Exercise 7

Distance	Speed	Time
60 km	30 km/h	2 hours
20 km	40 km/h	30 minutes
50 km	25 km/h	2 hours
25 km	100 km/h	15 minutes

Solution to Exercise 8

(a) (i) The car's velocity during the first three-quarters of an hour is the gradient of the first line segment, which is

$$\frac{-25-50}{0.75-0} = \frac{-75}{0.75} = -100\,\text{km/h}.$$

Its speed is 100 km/h.

(ii) The car's velocity during the next half hour is the gradient of the second line segment, which is 0. Its speed is 0.

(iii) The car's velocity during the final half hour is the gradient of the third line segment, which is

$$\frac{0-(-25)}{1.75-1.25} = \frac{25}{0.5} = 50\,\text{km/h}.$$

Its speed is 50 km/h.

(b) The car travels at a speed of 100 km/h along the road for three-quarters of an hour, then it remains stationary for half an hour, and finally it travels in the opposite direction along the road for half an hour at a speed of 50 km/h.

Solution to Exercise 9

Lines (i) and (ii):
The lines $y = 3x + 4$ and $2y = 6x - 12$ are parallel and hence do not intersect.

Lines (i) and (iii):
The equations are

$$y = 3x + 4$$
$$x + 3y - 2 = 0.$$

Using the first equation to substitute for y in the second equation gives

$$x + 3(3x + 4) - 2 = 0.$$

So $10x + 12 - 2 = 0$, giving $x = -1$.

Substituting $x = -1$ into the first equation $y = 3 \times (-1) + 4 = 1$.

Hence the point of intersection is $(-1, 1)$.

Lines (i) and (iv):
Substituting $y = -3$ into the equation $y = 3x + 4$ gives $-3 = 3x + 4$, so $3x = -7$; that is, $x = -\frac{7}{3}$. Hence the point of intersection is $(-\frac{7}{3}, -3)$.

Lines (i) and (v):
Substituting $x = 2$ into the equation $y = 3x + 4$ gives $y = 3 \times 2 + 4 = 10$, so the point of intersection is $(2, 10)$.

Lines (ii) and (iii):
The equations are

$$2y = 6x - 12$$
$$x + 3y - 2 = 0.$$

The first equation can be rearranged as $y = 3x - 6$. Using this equation to substitute for y in the second equation gives

$$x + 3(3x - 6) - 2 = 0.$$

So $10x - 18 - 2 = 0$, giving $x = 2$.

Substituting $x = 2$ into the rearranged first equation gives $y = 3 \times 2 - 6 = 0$.

Hence the point of intersection is $(2, 0)$.

Lines (ii) and (iv):
Substituting $y = -3$ into the equation $y = 3x - 6$ gives $-3 = 3x - 6$; that is, $3x = 6$, so $x = 1$. Hence the point of intersection is $(1, -3)$.

Lines (ii) and (v):
Substituting $x = 2$ into the equation $2y = 6x - 12$ gives $y = 0$. Hence the point of intersection is $(2, 0)$.

Lines (iii) and (iv):
Substituting $y = -3$ into the equation $x + 3y - 2 = 0$ gives $x + 3 \times (-3) - 2 = 0$; that is, $x - 11 = 0$, so $x = 11$. Hence the point of intersection is $(11, -3)$.

Lines (iii) and (v):
Substituting $x = 2$ into the equation $x + 3y - 2 = 0$ gives $2 + 3y - 2 = 0$; that is, $3x = 0$, so $y = 0$. Hence the point of intersection is $(2, 0)$.

Lines (iv) and (v):
These lines intersect at the point $(2, -3)$.

Solution to Exercise 10

(a) The equations are

$$4x - 3y = -5 \qquad (1)$$
$$3x - 2y = -4. \qquad (2)$$

Multiplying equation (1) by 3 and equation (2) by 4 gives

$$12x - 9y = -15 \qquad (3)$$
$$12x - 8y = -16. \qquad (4)$$

Subtracting equation (4) from equation (3) gives

$$-y = 1$$
$$y = -1.$$

Substituting $y = -1$ into equation (1) gives

$$4x - 3(-1) = -5$$
$$4x + 3 = -5$$
$$4x = -8$$
$$x = -2.$$

So the solution is $x = -2$, $y = -1$.

(b) The equations are

$$3p - 4q = 10 \qquad (5)$$
$$5p + 3q = 7. \qquad (6)$$

Multiplying equation (5) by 5 and equation (6) by 3 gives

$$15p - 20q = 50 \qquad (7)$$
$$15p + 9q = 21. \qquad (8)$$

Subtracting equation (8) from equation (7) gives

$$-29q = 29$$
$$q = -1.$$

Substituting $q = -1$ into equation (5) gives

$$3p - 4(-1) = 10$$
$$3p = 6$$
$$p = 2.$$

So the solution is $p = 2$, $q = -1$.

(c) The equations are

$$4s - 3t = 1 \qquad (9)$$
$$6s - 2t = 1. \qquad (10)$$

Multiplying equation (9) by 3 and equation (10) by 2 gives

$$12s - 9t = 3 \qquad (11)$$
$$12s - 4t = 2. \qquad (12)$$

Subtracting equation (12) from equation (11) gives

$$-5t = 1$$
$$t = -\tfrac{1}{5}.$$

Substituting $t = -\frac{1}{5}$ into equation (9) gives

$$4s - 3\left(-\tfrac{1}{5}\right) = 1$$
$$4s + \tfrac{3}{5} = 1$$
$$4s = \tfrac{2}{5}$$
$$s = \tfrac{1}{10}.$$

So the solution is $s = \frac{1}{10}$, $t = -\frac{1}{5}$.

(d) The equations are

$$3x - 4y = 1 \qquad (13)$$
$$2x - y = \tfrac{2}{3}. \qquad (14)$$

Multiplying equation (13) by 2 and equation (14) by 3 gives

$$6x - 8y = 2 \qquad (15)$$
$$6x - 3y = 2. \qquad (16)$$

Subtracting equation (16) from equation (15) gives

$$-5y = 0$$
$$y = 0.$$

Substituting $y = 0$ into equation (13) gives

$$3x = 1$$
$$x = \tfrac{1}{3}.$$

So the solution is $x = \frac{1}{3}$, $y = 0$.

Solution to Exercise 11

(a) $x^2 + 7x + 10 = (x + 2)(x + 5)$

(b) $r^2 + r - 12 = (r + 4)(r - 3)$

(c) $p^2 - 7p + 12 = (p - 3)(p - 4)$

(d) $t^2 + 3t - 18 = (t + 6)(t - 3)$

(e) $a^2 + a - 6 = (a + 3)(a - 2)$

(f) $b^2 - 4b - 5 = (b + 1)(b - 5)$

Solution to Exercise 12

(a) $2x^2 + 7x + 5 = (2x + 5)(x + 1)$

(b) $3r^2 + 5r - 12 = (r + 3)(3r - 4)$

(c) $5p^2 - 7p - 12 = (5p - 12)(p + 1)$

(d) $2t^2 - 5t - 18 = (2t - 9)(t + 2)$

(e) $3a^2 + 7a - 6 = (3a - 2)(a + 3)$

(f) $4b^2 + 8b - 5 = (2b - 1)(2b + 5)$

Solution to Exercise 13

(a) $a^2 + 3a = a(a + 3)$

(b) $p^2 - 6p = p(p - 6)$

(c) $4y^2 - 2y = 2y(2y - 1)$

(d) $4t^2 - 1 = (2t + 1)(2t - 1)$

(e) $4x^2 - 9 = (2x + 3)(2x - 3)$

(f) $9r^2 - 4s^2 = (3r + 2s)(3r - 2s)$

Solution to Exercise 14

(a) $x^2 + 7x + 10 = 0$
$(x + 2)(x + 5) = 0$
$x + 2 = 0 \quad \text{or} \quad x + 5 = 0$
$x = -2 \quad \text{or} \quad x = -5$

(b) $t^2 + 3t - 18 = 0$
$(t + 6)(t - 3) = 0$
$t + 6 = 0 \quad \text{or} \quad t - 3 = 0$
$t = -6 \quad \text{or} \quad t = 3$

(c) $5 + 4b - b^2 = 0$
$b^2 - 4b - 5 = 0$
$(b + 1)(b - 5) = 0$
$b + 1 = 0 \quad \text{or} \quad b - 5 = 0$
$b = -1 \quad \text{or} \quad b = 5$

(d) $2x^2 + 7x + 5 = 0$
$(2x + 5)(x + 1) = 0$
$2x + 5 = 0 \quad \text{or} \quad x + 1 = 0$
$x = -\frac{5}{2} \quad \text{or} \quad x = -1$

(e) $3a^2 + 7a - 6 = 0$
$(a + 3)(3a - 2) = 0$
$a + 3 = 0 \quad \text{or} \quad 3a - 2 = 0$
$a = -3 \quad \text{or} \quad a = \frac{2}{3}$

(f) $2s^2 - 5s - 12 = 0$
$(s - 4)(2s + 3) = 0$
$s - 4 = 0 \quad \text{or} \quad 2s + 3 = 0$
$s = 4 \quad \text{or} \quad s = -\dfrac{3}{2}$

(g) $a^2 + 3a = 0$
$a(a + 3) = 0$
$a = 0 \quad \text{or} \quad a + 3 = 0$
$a = 0 \quad \text{or} \quad a = -3$

(h) $4y^2 - 2y = 0$
$2y(2y - 1) = 0$
$2y = 0 \quad \text{or} \quad 2y - 1 = 0$
$y = 0 \quad \text{or} \quad y = \frac{1}{2}$

(i) $9r^2 - 4 = 0$
$(3r + 2)(3r - 2) = 0$
$3r + 2 = 0 \quad \text{or} \quad 3r - 2 = 0$
$r = -\frac{2}{3} \quad \text{or} \quad r = \frac{2}{3}$
(These solutions can be written as $r = \pm\frac{2}{3}$.)

(j) $25p^2 - 5p - 12 = 0$
$(5p - 4)(5p + 3) = 0$
$5p - 4 = 0 \quad \text{or} \quad 5p + 3 = 0$
$p = \frac{4}{5} \quad \text{or} \quad p = -\frac{3}{5}$

Solution to Exercise 15

(a) $x^2 + 20x = (x + 10)^2 - 100$

(b) $x^2 - 11x = \left(x - \dfrac{11}{2}\right)^2 - \dfrac{121}{4}$

(c) $y^2 + 7y = \left(y + \dfrac{7}{2}\right)^2 - \dfrac{49}{4}$

Solution to Exercise 16

(a)
$$\begin{aligned} x^2 - 18x + 60 &= (x - 9)^2 - 81 + 60 \\ &= (x - 9)^2 - 21 \end{aligned}$$

(b)
$$\begin{aligned} 3x^2 + 2x + 2 &= 3\left(x^2 + \frac{2}{3}x\right) + 2 \\ &= 3\left(\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right) + 2 \\ &= 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3} + 2 \\ &= 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3} + \frac{6}{3} \\ &= 3\left(x + \frac{1}{3}\right)^2 + \frac{5}{3} \end{aligned}$$

(c) $$\begin{aligned}-2x^2+x-5 &= -2\left(x^2-\frac{1}{2}x\right)-5\\ &= -2\left(\left(x-\frac{1}{4}\right)^2-\frac{1}{16}\right)-5\\ &= -2\left(x-\frac{1}{4}\right)^2+\frac{1}{8}-5\\ &= -2\left(x-\frac{1}{4}\right)^2+\frac{1}{8}-\frac{40}{8}\\ &= -2\left(x-\frac{1}{4}\right)^2-\frac{39}{8}\end{aligned}$$

Solution to Exercise 17

(a) $$\begin{aligned}&x^2+5x+6=0\\ &\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+6=0\\ &\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{24}{4}=0\\ &\left(x+\frac{5}{2}\right)^2-\frac{1}{4}=0\\ &\left(x+\frac{5}{2}\right)^2=\frac{1}{4}\\ &x+\frac{5}{2}=\frac{1}{2} \quad\text{or}\quad x+\frac{5}{2}=-\frac{1}{2}\\ &x=\frac{1}{2}-\frac{5}{2} \quad\text{or}\quad x=-\frac{1}{2}-\frac{5}{2}\\ &x=-3 \quad\text{or}\quad x=-2\end{aligned}$$

(b) $$\begin{aligned}&x^2-14x+30=0\\ &(x-7)^2-49+30=0\\ &(x-7)^2-19=0\\ &(x-7)^2=19\\ &x-7=\sqrt{19} \quad\text{or}\quad x-7=-\sqrt{19}\\ &x=7+\sqrt{19} \quad\text{or}\quad x=7-\sqrt{19}\end{aligned}$$

(c) $$\begin{aligned}&3x^2+6x+1=0\\ &3(x^2+2x)+1=0\\ &3((x+1)^2-1)+1=0\\ &3(x+1)^2-3+1=0\\ &3(x+1)^2-2=0\\ &(x+1)^2=\frac{2}{3}\\ &x+1=\sqrt{\frac{2}{3}} \quad\text{or}\quad x+1=-\sqrt{\frac{2}{3}}\\ &x=-1+\sqrt{\frac{2}{3}} \quad\text{or}\quad x=-1-\sqrt{\frac{2}{3}}\end{aligned}$$

(d) $$\begin{aligned}&3x^2+5x+1=0\\ &3\left(x^2+\frac{5}{3}x\right)+1=0\\ &3\left(\left(x+\frac{5}{6}\right)^2-\frac{25}{36}\right)+1=0\\ &3\left(x+\frac{5}{6}\right)^2-\frac{25}{12}+1=0\\ &3\left(x+\frac{5}{6}\right)^2-\frac{13}{12}=0\\ &\left(x+\frac{5}{6}\right)^2=\frac{13}{36}\\ &x+\frac{5}{6}=\frac{\sqrt{13}}{6} \quad\text{or}\quad x+\frac{5}{6}=-\frac{\sqrt{13}}{6}\\ &x=-\frac{5}{6}+\frac{\sqrt{13}}{6} \quad\text{or}\quad x=-\frac{5}{6}-\frac{\sqrt{13}}{6}\\ &x=\tfrac{1}{6}(-5+\sqrt{13}) \quad\text{or}\quad x=\tfrac{1}{6}(-5-\sqrt{13})\end{aligned}$$

(e) $$\begin{aligned}&-4x^2+x+2=0\\ &-4\left(x^2-\frac{1}{4}x\right)+2=0\\ &-4\left(\left(x-\frac{1}{8}\right)^2-\frac{1}{64}\right)+2=0\\ &-4\left(x-\frac{1}{8}\right)^2+\frac{1}{16}+2=0\\ &-4\left(x-\frac{1}{8}\right)^2+\frac{1}{16}+\frac{32}{16}=0\\ &-4\left(x-\frac{1}{8}\right)^2+\frac{33}{16}=0\\ &\left(x-\frac{1}{8}\right)^2=\frac{33}{64}\\ &x-\frac{1}{8}=\frac{\sqrt{33}}{8} \quad\text{or}\quad x-\frac{1}{8}=-\frac{\sqrt{33}}{8}\\ &x=\frac{1}{8}+\frac{\sqrt{33}}{8} \quad\text{or}\quad x=\frac{1}{8}-\frac{\sqrt{33}}{8}\\ &x=\tfrac{1}{8}\left(1+\sqrt{33}\right) \quad\text{or}\quad x=\tfrac{1}{8}\left(1-\sqrt{33}\right)\end{aligned}$$

Solution to Exercise 18

(a) The equation is $x^2 + 5x + 6 = 0$, so here $a = 1$, $b = 5$ and $c = 6$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{-5 \pm 1}{2} \\ &= \frac{-5+1}{2} \quad \text{or} \quad \frac{-5-1}{2} \\ &= -2 \quad \text{or} \quad -3. \end{aligned}$$

The solutions are $x = -2$ and $x = -3$.

(b) The equation is $x^2 - 14x + 30 = 0$, so here $a = 1$, $b = -14$ and $c = 30$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{14 \pm \sqrt{196 - 120}}{2} \\ &= \frac{14 \pm \sqrt{76}}{2} \\ &= \frac{14 \pm 2\sqrt{19}}{2} \\ &= 7 \pm \sqrt{19}. \end{aligned}$$

The solutions are $x = 7 + \sqrt{19}$ and $x = 7 - \sqrt{19}$.

(c) The equation is $3x^2 + 6x + 1 = 0$, so here $a = 3$, $b = 6$ and $c = 1$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{36 - 12}}{6} \\ &= \frac{-6 \pm \sqrt{24}}{6} \\ &= \frac{-6 \pm 2\sqrt{6}}{6} \\ &= -1 \pm \frac{\sqrt{6}}{3}. \end{aligned}$$

The solutions are $x = -1 + \frac{1}{3}\sqrt{6}$ and $x = -1 - \frac{1}{3}\sqrt{6}$. (They can also be written as $x = -1 + \sqrt{\frac{2}{3}}$ and $x = -1 - \sqrt{\frac{2}{3}}$.)

(d) The equation is $3x^2 + 5x + 1 = 0$, so here $a = 3$, $b = 5$ and $c = 1$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{25 - 12}}{6} \\ &= \frac{-5 \pm \sqrt{13}}{6}. \end{aligned}$$

The solutions are $x = \frac{1}{6}(-5 + \sqrt{13})$ and $x = \frac{1}{6}(-5 - \sqrt{13})$.

(e) The equation is $-4x^2 + x + 2 = 0$, so here $a = -4$, $b = 1$ and $c = 2$. Substituting into the quadratic formula gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1 - (-32)}}{-8} \\ &= \frac{-1 \pm \sqrt{33}}{-8} \\ &= \frac{1 \pm \sqrt{33}}{8}. \end{aligned}$$

The solutions are $\frac{1}{8}(1 + \sqrt{33})$ and $\frac{1}{8}(1 - \sqrt{33})$.

Solution to Exercise 19

(a) The equation is $x^2 + 2x + 3 = 0$, so here $a = 1$, $b = 2$ and $c = 3$. The discriminant is

$$b^2 - 4ac = 4 - 12 = -8.$$

Since the discriminant is negative, the equation has no real solutions.

(b) The equation is $2x^2 - 4x - 5 = 0$, so here $a = 2$, $b = -4$ and $c = -5$. The discriminant is

$$b^2 - 4ac = 16 - (-40) = 16 + 40 = 56.$$

Since the discriminant is positive, the equation has two real solutions.

The equation cannot be solved by factorising using integers, so we solve it by using the quadratic formula. This gives

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{56}}{4} \\ &= \frac{4 \pm 2\sqrt{14}}{4} \\ &= \frac{2 \pm \sqrt{14}}{2}. \end{aligned}$$

The solutions are $\frac{1}{2}(2 + \sqrt{14})$ and $\frac{1}{2}(2 - \sqrt{14})$.

(c) The equation is $3x^2 + 4 = 0$, so here $a = 3$, $b = 0$ and $c = 4$. The discriminant is

$$b^2 - 4ac = 0 - 48 = -48.$$

Since the discriminant is negative, the equation has no real solutions.

(d) The equation is $2x^2 - 5x = 0$, so here $a = 2$, $b = -5$ and $c = 0$. The discriminant is

$$b^2 - 4ac = 25 - 0 = 25 > 0.$$

Since the discriminant is positive, the equation has two real solutions.

The equation can be solved by factorising:

$$\begin{aligned} &2x^2 - 5x = 0 \\ &x(2x - 5) = 0 \\ &x = 0 \quad \text{or} \quad x = \frac{5}{2}. \end{aligned}$$

The solutions are $x = 0$ and $x = \frac{5}{2}$.

(e) The equation is $x^2 + 8x + 16 = 0$, so here $a = 1$, $b = 8$ and $c = 16$. The discriminant is

$$b^2 - 4ac = 64 - 64 = 0.$$

Since the discriminant is 0, there is one real solution.

The equation can be solved by factorising:

$$\begin{aligned} &x^2 + 8x + 16 = 0 \\ &(x + 4)(x + 4) = 0 \\ &x = -4. \end{aligned}$$

The solution is $x = -4$.

Solution to Exercise 20

(a) The equation is

$$\frac{1}{2x} = \frac{x}{5 - 3x}.$$

Assume that $x \neq 0$ and $x \neq \frac{5}{3}$.
Cross-multiply:

$$5 - 3x = 2x^2.$$

Solve this quadratic equation:

$$\begin{aligned} &2x^2 + 3x - 5 = 0 \\ &(2x + 5)(x - 1) = 0 \\ &x = -\frac{5}{2} \quad \text{or} \quad x = 1. \end{aligned}$$

Neither of these values is 0 or $\frac{5}{3}$, so the solutions of the original equation are $x = -\frac{5}{2}$ and $x = 1$.

(b) The equation is

$$p^4 = 9.$$

This gives

$$p^2 = 3 \quad \text{or} \quad p^2 = -3.$$

The first of these equations gives

$$p = \pm\sqrt{3},$$

and the second equation has no solutions.

So the solutions of the original equation are $p = \sqrt{3}$ and $p = -\sqrt{3}$.

(c) The equation is

$$u^4 - 2u^3 - 3u^2 = 0.$$

Factorising gives

$$\begin{aligned} &u^2(u^2 - 2u - 3) = 0 \\ &u^2(u + 1)(u - 3) = 0 \\ &u^2 = 0 \quad \text{or} \quad u = -1 \quad \text{or} \quad u = 3. \end{aligned}$$

The equation $u^2 = 0$ has the single solution $u = 0$, so the solutions of the original equation are $u = 0$, $u = -1$ and $u = 3$.

(d) The equation is

$$\frac{2}{1+x} = \frac{3x}{1-x}.$$

Assume that $x \neq 1$ and $x \neq -1$.
Cross-multiply:

$$2(1-x) = 3x(1+x).$$

Multiply out the brackets and solve the resulting quadratic equation:

$$2 - 2x = 3x + 3x^2$$
$$3x^2 + 5x - 2 = 0$$
$$(3x-1)(x+2) = 0$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2.$$

Neither of these values is 1 or -1, so the solutions of the original equation are $x = \frac{1}{3}$ and $x = -2$.

(e) The equation is:

$$(b^2 - b - 2)(b^2 + 3b + 2) = 0.$$

Factorise each quadratic:

$$(b+1)(b-2)(b+2)(b+1) = 0$$
$$(b+1)^2(b-2)(b+2) = 0$$
$$b = -1 \quad \text{or} \quad b = 2 \quad \text{or} \quad b = -2.$$

So the solutions of the original equation are $b = -2$, $b = -1$ and $b = 2$.

Solution to Exercise 21

(a) The equation is $y = x^2 + 2x + 3$.

Since the coefficient of x is positive, the graph is u-shaped.

To find the y-intercept, we put $x = 0$, which gives $y = 3$.

To find any x-intercepts, we put $y = 0$, which gives $x^2 + 2x + 3 = 0$. This quadratic expression has discriminant

$$b^2 - 4ac = 4 - 12 = -8.$$

Since the discriminant is negative, there are no solutions, and hence no x-intercepts.

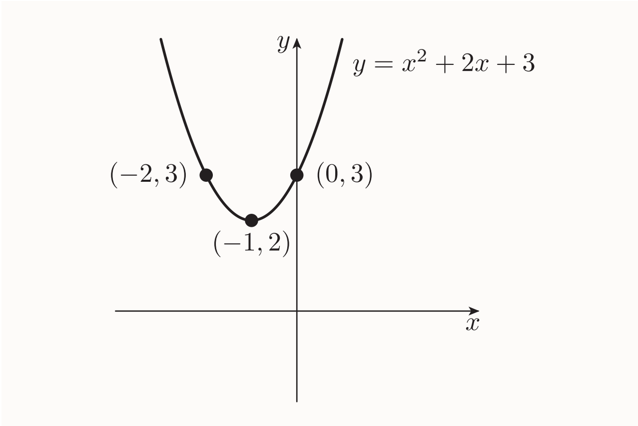
The equation $y = x^2 + 2x + 3$ can be rearranged as

$$y = x(x+2) + 3.$$

It follows that $x = 0$ and $x = -2$ give the same y-value, namely 3, so the points $(0, 3)$ and $(-2, 3)$ lie on the graph. The axis of symmetry lies halfway between these points, so its equation is $x = \frac{1}{2}(0 + (-2))$; that is, $x = -1$.

The vertex lies on the axis of symmetry, so it has x-coordinate -1 and y-coordinate $(-1)^2 + 2 \times (-1) + 3 = 2$. Hence it is $(-1, 2)$.

The graph is sketched below.



(b) The equation is $y = 2x^2 - 4x - 10$.

Since the coefficient of x is positive, the graph is u-shaped.

To find the y-intercept, we put $x = 0$, which gives $y = -10$.

To find any x-intercepts, we put $y = 0$, which gives

$$2x^2 - 4x - 10 = 0,$$

which can be simplified to

$$x^2 - 2x - 5 = 0.$$

Using the quadratic formula gives

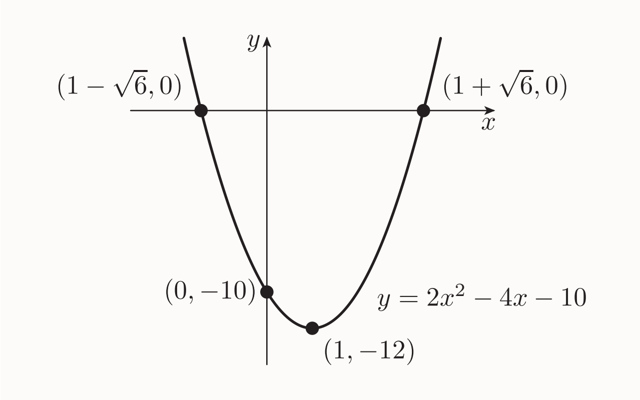
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{4 + 20}}{2} \\ &= \frac{2 \pm \sqrt{24}}{2} \\ &= \frac{2 \pm 2\sqrt{6}}{2} \\ &= 1 \pm \sqrt{6}. \end{aligned}$$

So the x-intercepts are $1 + \sqrt{6} \approx 3.5$ and $1 - \sqrt{6} \approx -1.5$.

The axis of symmetry lies halfway between the points corresponding to the x-intercepts, so its equation is $x = \frac{1}{2}((1 + \sqrt{6}) + (1 - \sqrt{6}))$; that is, $x = 1$.

The vertex lies on the axis of symmetry, so it has x-coordinate 1 and y-coordinate $2 \times 1^2 - 4 \times 1 - 10 = -12$. Hence it is $(1, -12)$.

The graph is sketched below.



(c) The equation is $y = 3x^2 + 4$.

Since the coefficient of x is positive, the graph is u-shaped.

To find the y-intercept, we put $x = 0$, which gives $y = 4$.

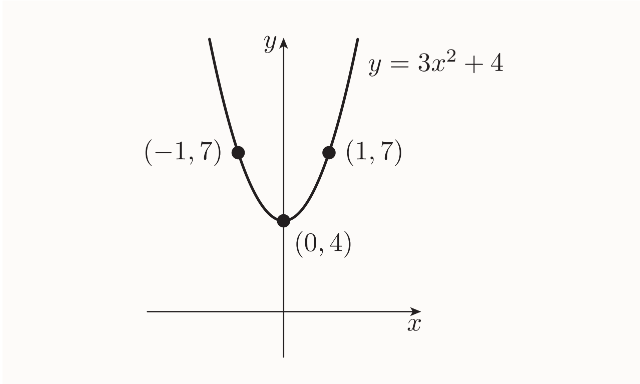
To find any x-intercepts, we put $y = 0$, which gives $3x^2 + 4 = 0$. This quadratic equation has no solutions, since $3x^2 + 4$ is positive for all values of x. So there are no x-intercepts.

It can be seen from the equation $y = 3x^2 + 4$ that any value of x and its negative give the same y-value, so the axis of symmetry is the y-axis.

The vertex lies on the axis of symmetry, so it has x-coordinate 0 and hence y-coordinate 4. So it is $(0, 4)$.

It is helpful to find another point on the graph. When $x = 1$, $y = 3 \times 1^2 + 4 = 7$, so the point $(1, 7)$ lies on the graph. Hence, since the axis of symmetry is the y-axis, the point $(-1, 7)$ also lies on the graph.

The graph is sketched below.



(d) The equation is $y = 2x^2 - 5x$.

Since the coefficient of x is positive, the graph is u-shaped.

To find the y-intercept, we put $x = 0$, which gives $y = 0$.

To find any x-intercepts, we put $y = 0$, which gives

$$2x^2 - 5x = 0$$
$$x(2x - 5) = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{5}{2}.$$

So the x-intercepts are 0 and $\frac{5}{2}$.

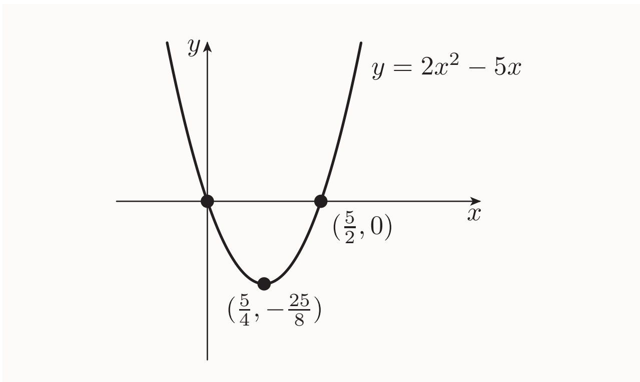
The axis of symmetry lies halfway between the points corresponding to the x-intercepts, so its equation is $x = \frac{1}{2}(0 + \frac{5}{2})$; that is, $x = \frac{5}{4}$.

The vertex lies on the axis of symmetry, so it has x-coordinate $\frac{5}{4} = 1.25$ and y-coordinate

$$2\left(\frac{5}{4}\right)^2 - 5 \times \frac{5}{4} = -\frac{25}{8} = -3.125.$$

Hence it is $(1.25, -3.125)$.

The graph is sketched below.



(e) The equation is $y = x^2 + 8x + 16$.

Since the coefficient of x is positive, the graph is u-shaped.

To find the y-intercept, we put $x = 0$, which gives $y = 16$.

To find any x-intercepts, we put $y = 0$, which gives

$$x^2 + 8x + 16 = 0$$
$$(x + 4)^2 = 0$$
$$x = -4.$$

So there is one x-intercept, namely -4.

The axis of symmetry is therefore $x = -4$.

The vertex is the point corresponding to the single x-intercept, namely $(-4, 0)$.

The graph is sketched below.

(The point 'opposite' $(0, 16)$ has been marked to make it easier to sketch the graph.)