Q 1.

(a) To find the equilibrium points of the system of differential equations, we find the point (X, Y) in the vector field at which $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$ (MST210 Handbook, page 75). For this system, the equilibrium points are the solutions of the system of equations

$$\begin{aligned} y - 3 &= 0 \\ 9x^2 - y^2 &= 0 \end{aligned} \tag{1.1}$$

So $y = 3$, and substituting this into (1.1) gives $x = \pm 1$, so the equilibrium points are at $(-1, 3)$ and $(1, 3)$.

(b) The Jacobian matrix of a system of differential equations represented by the vector field $\mathbf{u} = (u(x,y) \;\; v(x,y))^\top$ is given by

$$\mathbf{J}(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix}$$

where $u_x(x,y)$, $u_y(x,y)$, $v_x(x,y)$, and $v_y(x,y)$ are the partial derivatives of the components of $\mathbf{u}$. Calculating these partial derivatives gives

$$\mathbf{J}(x,y) = \begin{pmatrix} 0 & 1 \\ 18x & -2y \end{pmatrix}$$

At the equilibrium point $(-1,3)$ the Jacobian matrix is

$$\mathbf{J}(-1,3) = \begin{pmatrix} 0 & 1 \\ -18 & -6 \end{pmatrix}$$

with eigenvalues λ given by

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ -18 & -6-\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 6\lambda + 18 &= 0 \\ \lambda &= \frac{-6 \pm \sqrt{-36}}{2} \\ &= -3 \pm 6i \end{aligned}$$

As the eigenvalues of the Jacobian matrix are complex with a negative real part, the equilibrium point $(-1,3)$ is a spiral sink (MST210 Handbook, page 76).

At the equilibrium point $(1,3)$ the Jacobian matrix is

$$\mathbf{J}(1,3) = \begin{pmatrix} 0 & 1 \\ 18 & -6 \end{pmatrix}$$

with eigenvalues given by

$$\begin{aligned} \lambda^2 + 6\lambda - 18 &= 0 \\ \lambda &= \frac{-6 \pm \sqrt{108}}{2} \\ &= -3 \pm 6\sqrt{3} \end{aligned}$$

As the eigenvalues of the Jacobian matrix are real and distinct with one positive and one negative, the equilibrium point $(1,3)$ is a saddle.

Q 2.

(a) The equilibrium positions of the system of differential equations are the (x, y) points that satisfy the system

$$(x^2 - 1)(y + 1) = 0 \tag{2.1}$$
$$x(y^2 - 4) = 0 \tag{2.2}$$

Equation (2.1) is satisfied when $x = \pm 1$ and when $y = -1$. When $x = \pm 1$, (2.2) is satisfied only when $y = \pm 2$ so $(\pm 1, \pm 2)$ are solutions. Equation (2.2) is also satisfied when $x = 0$. Substituting $x = 0$ into (2.1) gives $(0, -1)$ as another solution to the system. Therefore there are five equilibrium positions of this system of differential equations:$(-1, -2)$, $(-1, 2)$, $(0, -1)$, $(1, -2)$, and $(1, 2)$.

(b)

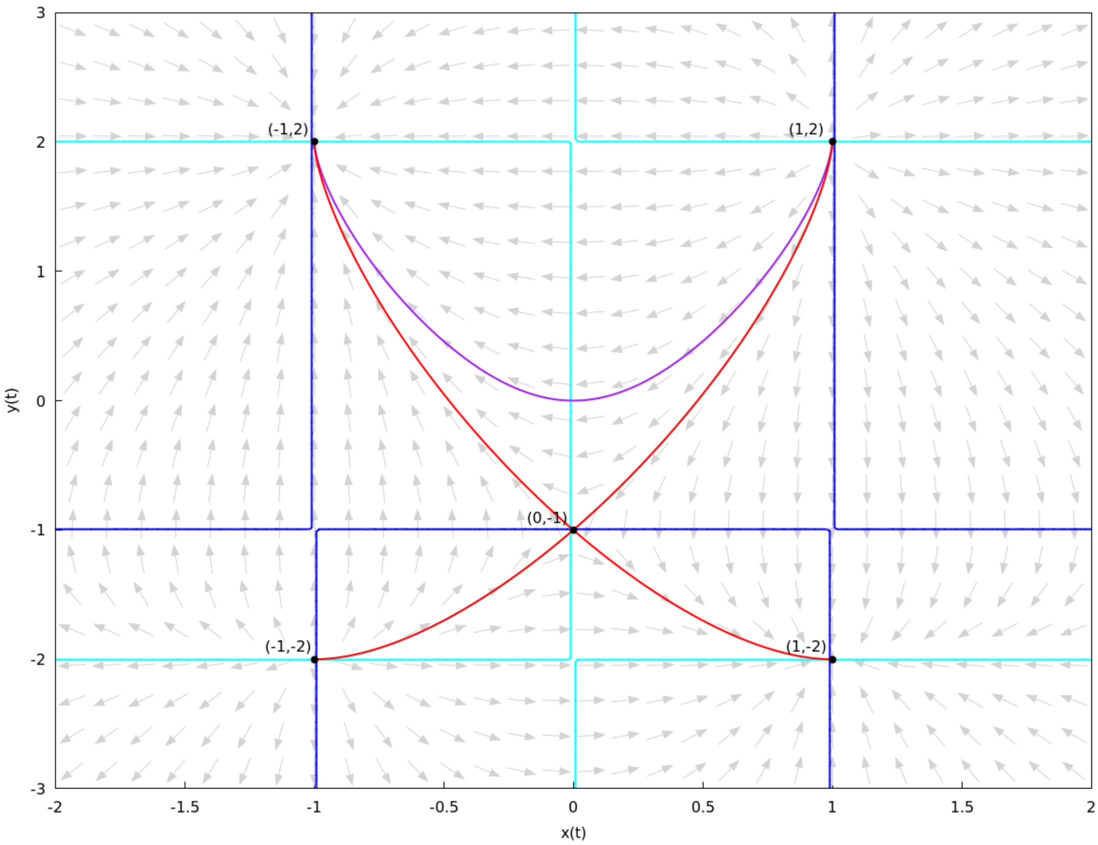

Figure 1: Phase portrait of the system of differential equations. Equilibrium points shown as black circles with coordinates. Nullclines for $\dot{x} = 0$ and $\dot{y} = 0$ shown in dark blue and cyan, respectively. Red lines indicate solutions passing through the saddle point $(0, -1)$, and the purple line indicates the solution that passes through the origin.

(c) Figure 1 shows that the phase path that passes through the origin starts at the source $(1, 2)$ and ends at the sink $(-1, 2)$.

Q 3.

(a) The equilibrium positions of the system of differential equations are the (x, y) points that satisfy the system

$$-\sin(2x) - \sin(y) = 0 \tag{3.1}$$
$$(\cos y + 2)\cos x = 0 \tag{3.2}$$

Equation (3.2) is satisfied when $x = n\pi/2$, for all non-zero integers n. Substituting $x = \pi/2$ into (3.1) gives

$$\begin{aligned} -\sin\pi - \sin y &= 0 \\ \sin y &= 0 \\ y &= k\pi \end{aligned}$$

for all integers k. Therefore $(\pi/2, \pi)$ is an equilibrium point of the system of differential equations (with $n = k = 1$).

(b) The Jacobian matrix of this system of differential equations is

$$\mathbf{J}(x,y) = \begin{pmatrix} -2\cos(2x) & -\cos y \\ -(\cos y + 2)\sin x & -\cos x \sin y \end{pmatrix}$$

Substituting $x = \pi/2$ and $y = \pi$ gives

$$\begin{aligned} \mathbf{J}\left(\frac{\pi}{2}, \pi\right) &= \begin{pmatrix} -2\cos\pi & -\cos\pi \\ -(\cos\pi + 2)\sin\left(\frac{\pi}{2}\right) & -\cos\left(\frac{\pi}{2}\right)\sin\pi \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

The eigenvalues λ of the Jacobian matrix are given by

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} &= 0 \\ -\lambda(2-\lambda) + 1 &= 0 \\ \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)^2 &= 0 \end{aligned}$$

Therefore the Jacobian matrix has a repeated eigenvalue of 1. To find the eigenvector(s) corresponding to $\lambda = 1$, we solve

$$\begin{aligned} \begin{pmatrix} 2-1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= 0 \\ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= 0 \\ x + y &= 0 \\ -x - y &= 0 \end{aligned}$$

which has a solution of $(-1 \ \ 1)^\top$. As the equilibrium point $(\pi/2, \pi)$ has a repeated, positive eigenvalue with no more than one linearly-independent eigenvector, it is an improper source.

Q 4.

(a)

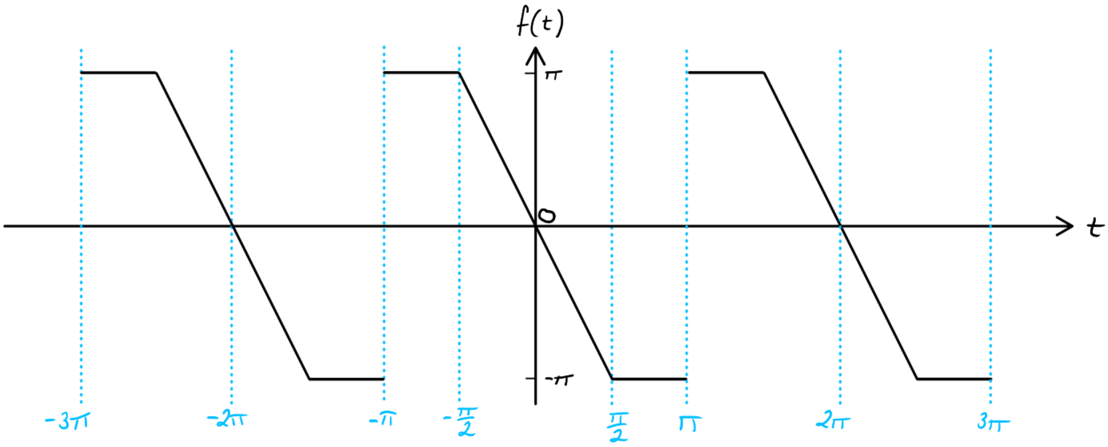

Figure 2: Sketch of the function $f(t)$ for $-3\pi \leq t \leq 3\pi$.

As figure 2 shows, f has a period τ of 2π. As $f(-t) = -f(t)$ over the interval $[-\tau/2, \tau/2]$, f is an odd function (MST210 Book D, page 71).

(b) As per MST210 Book D, page 94, a periodic function $f(t)$ with period τ and fundamental interval $[-\tau/2, \tau/2]$ has Fourier series

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right)$$

where

$$\begin{aligned} A_0 &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,\mathrm{d}t \\ A_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos\left(\frac{2n\pi t}{\tau}\right)\mathrm{d}t \qquad (n = 1, 2, ...) \\ B_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)\mathrm{d}t \qquad (n = 1, 2, ...) \end{aligned}$$

The given function is odd and piecewise so the coefficients A_0, A_n, and B_n can be determined by considering the odd periodic extension of $f(t)$ giving

$$\begin{aligned} A_0 &= \frac{1}{2\pi} int_{-\pi}^{-\pi/2} \pi\,\mathrm{d}t + \int_{-\pi/2}^{\pi/2} -2t\,\mathrm{d}t + \int_{\pi/2}^{\pi} -\pi\,\mathrm{d}t \\ &= \frac{1}{2\pi}\big([\pi t]_{-\pi}^{-\pi/2} + [-t^2]_{-\pi/2}^{\pi/2} + [-\pi t]_{\pi/2}^{\pi}\big) \\ &= \frac{1}{2\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \pi\cos nt\,\mathrm{d}t + \int_{-\pi/2}^{\pi/2} -2t\cos nt\,\mathrm{d}t + \int_{\pi/2}^{\pi} -\pi\cos nt\,\mathrm{d}t \\ &= \frac{1}{n}\big(\sin(n\pi) - \sin(n\pi/2)\big) + \frac{1}{n}\big(\sin(n\pi/2) - \sin(n\pi)\big) \\ &= 0 \end{aligned}$$

So A_0 is 0 and A_n is 0 for all n. Now we consider the B_n terms in the same way:

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \pi\sin nt\,\mathrm{d}t + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} -2t\sin nt\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{\pi} -\pi\sin nt\,\mathrm{d}t \\ &= \frac{1}{n}\big(\cos(n\pi) - \cos(n\pi/2)\big) - \frac{2}{n^2\pi}\big(2\sin(n\pi/2) - n\pi\cos(n\pi/2)\big) + \frac{1}{n}\big(\cos(n\pi) - \cos(n\pi/2)\big) \\ &= \frac{2}{n^2\pi}\Big(n\pi\cos(n\pi) - 2\sin(n\pi/2)\Big) \end{aligned}$$

As we have non-zero B_n coefficients and A_0 and A_n are o for all n, this confirms that $f(t)$ is an odd function.

(c) The first three non-zero terms in the Fourier series of $f(t)$ are

$$F(t) = -\frac{2(\pi+2)}{\pi}\sin(t) + \sin(2t) - \frac{2(3\pi-2)}{9\pi}\sin(3t)$$

Q 5.

(a) As can be seen in Figure 3, the fundamental period of $f_{\text{odd}}(t)$ is 1.

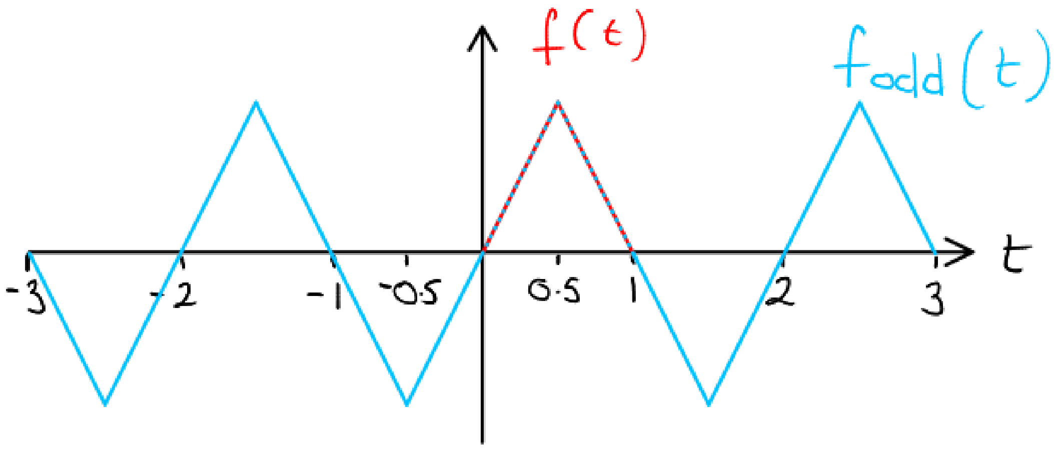

Figure 3: Sketch of the function $f(t)$ (red) and its odd periodic extension $f_{\text{odd}}(t)$ (blue).

(b) As can be seen in Figure 4, the fundamental period of $f_{\text{even}}(t)$ is 2.

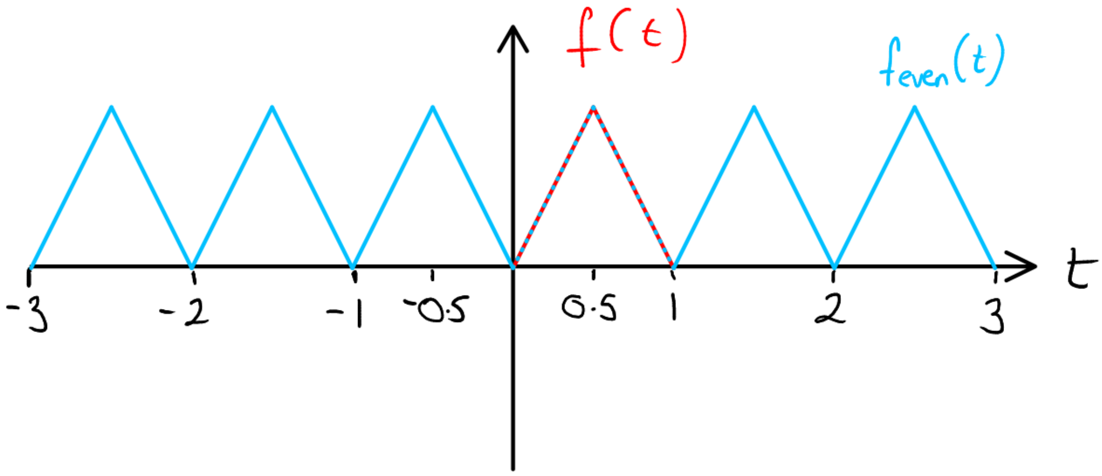

Figure 4: Sketch of the function $f(t)$ (red) and its odd periodic extension $f_{\text{even}}(t)$ (blue).

Q 6.

(a) Let $\Theta(x,t) = X(x)T(t)$ then

$$\frac{\partial \Theta}{\partial x} = X'T, \qquad \frac{\partial^2 \Theta}{\partial x^2} = X''T, \quad \text{and} \quad \frac{\partial \Theta}{\partial T} = XT'$$

Substituting these partial derivatives into the differential equation gives

$$X''T = D^{-1}T'$$

$$\frac{X''}{X} = \frac{T'}{DT}$$

Both sides must be equal to a constant, μ, so

$$\frac{X''}{X} = \mu, \quad \text{and} \quad \frac{T'}{DT} = \mu$$

and therefore $X'' - \mu X = 0$, as required, and $T' - \mu DT = 0$ is the differential equation that $T(t)$ must satisfy.

(b) As $\Theta(0,t) = 0$ and $\Theta_x(L,t) = 0$,

$$X(0)T(t) = 0, \quad \text{and} \quad X'(L)T(t) = 0$$

and so for non-trivial solutions we must have $X(0) = 0$ and $X'(L) = 0$.

If $x_n(x) = \sin(k_n x)$, with

$$k_n = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, 3, ...$$

then

$$\begin{aligned} x_n(0) &= \sin(0) \\ &= 0 \end{aligned}$$

$X_n'(L) = k_n \cos(k_n L)$, and as $k_n L$ is a multiple of $\pi/2$ for all integers n, the $\cos(k_n L)$ term is always 0 and $X_n'(L) = 0$, satisfying the boundary conditions specified.

As the RHS of $X_n(x)$ contains a sinusoid function, we try a solution of the form

$$X_n(x) = A\cos(k_n x) + B\sin(k_n x)$$

The boundary condition $X(0) = 0$ gives $A = 0$ and the boundary condition $X'(L) = 0$ gives

$$\begin{aligned} -Ak\sin(kL) + Bk\cos(kL) &= 0 \\ Bk\cos(kL) &= 0 \end{aligned}$$

As $k_n L$ is a multiple of $\pi/2$ for all n, $\cos(kL) = 0$ and $B = 1$ is a solution. Substituting $A = 0$ and $B = 1$ into the trial solution form gives

$$\begin{aligned} X_n(x) &= 0 \times \cos(k_n x) + 1 \times \sin(k_n x) \\ &= \sin(k_n x) \end{aligned}$$

as required.

Given the form of $X_n(x)$, $k_n = \sqrt{-\mu}$ and so

$$\begin{aligned} \mu &= -k_n^2 \\ &= -\left(\frac{(2n-1)\pi}{2L}\right)^2 \end{aligned}$$

(c) The differential equation that $T(t)$ must satisfy is

$$T' - \mu DT = 0$$

Substituting the separation constant μ gives

$$T' + \left(\frac{(2n-1)\pi}{2L}\right)^2 DT = 0$$

The general solution is of the form $T(t) = C\exp(-\mu Dt)$ where C is an arbitrary constant, so $T(t)$ must satisfy

$$T(t) = C\exp\left(-\frac{D(2n-1)^2\pi^2 t}{4L^2}\right)$$

(d) Multiplying the solutions for $X(x)$ and $T(t)$ gives

$$\Theta(x,t) = C\exp\left(-\frac{D(2n-1)^2\pi^2 t}{4L^2}\right)\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

There is one solution for each positive integer n, so we have a family of solutions

$$\Theta_n(x,t) = C_n\exp\left(-\frac{D(2n-1)^2\pi^2 t}{4L^2}\right)\sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n = 1, 2, 3, ...$$

As the partial differential equations and boundary conditions are homogenous and linear, any superposition of these solutions is also a solution, so

$$\Theta(x,t) = \sum_{n=1}^{\infty} C_n\exp\left(-\frac{D(2n-1)^2\pi^2 t}{4L^2}\right)\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

as required.

(e) The initial condition states that

$$\Theta(x,0) = 0.6\sin\left(\frac{3\pi x}{2L}\right)$$

Substituting $t = 0$ into the general solution given in part (d) gives

$$\begin{aligned}\Theta(x,0) &= \sum_{n=1}^{\infty} C_n e^0 \sin\left(\frac{(2n-1)\pi x}{2L}\right)\\ &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)\end{aligned}$$

Therefore

$$\Theta(x,t) = 0.6\exp\left(-\frac{9D\pi^2 t}{4L^2}\right)\sin\left(\frac{3\pi x}{2L}\right)$$

is the particular solution, with $C_n = 0.6$ and $n = 2$.

Q 7.

(a) As $\mu < 0$, the general solution takes the form

$$X(x) = A\cos(kx) + B\sin(kx)$$

where A and B are constants. Substituting the boundary conditions $X(0) = 0$ and $X'(L) = 0$ gives

$$\begin{aligned} X(0) &= A\cos(0) + B\sin(0) \\ A &= 0 \end{aligned}$$

$$\begin{aligned} X'(L) &= -Ak\sin(kL) + Bk\cos(kL) \\ &= -(0)k\sin(kL) + Bk\cos(kL) \qquad \text{(as } A = 0\text{)} \\ Bk\cos(kL) &= 0 \end{aligned}$$

As $k > 0$, for a non-trivial solution we must have

$$\begin{aligned} \cos(kL) &= 0 \\ kL &= \frac{(2n-1)\pi}{2} \\ k &= \frac{(2n-1)\pi}{2L} \end{aligned}$$

for $k > 0$. Substituting $A = 0$ and the expression for k gives the general solution as

$$X(x) = B\sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n = 1, 2, 3, ...$$

(b) Rearranging equation (4) gives

$$\ddot{T} - \mu V^2 T = 0$$

As $\mu < 0$ the solution to this differential equation takes the form

$$T(t) = C\cos(kt) + D\sin(kt)$$

where C an D are constants. Substituting the expression for k gives the general solution

$$T(t) = C\cos\left(\frac{(2n-1)\pi t}{2L}\right) + D\sin\left(\frac{(2n-1)\pi t}{2L}\right), \quad n = 1, 2, 3, ...$$

(c) Multiplying the general solutions for $X(x)$ and $T(t)$ gives

$$\begin{aligned} u(x,t) = \alpha \sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)\pi t}{2L}\right) + \\ \beta \sin\left(\frac{(2n-1)\pi x}{2L}\right)\sin\left(\frac{(2n-1)\pi t}{2L}\right) \quad n = 1, 2, 3, ... \end{aligned}$$

where $\alpha = BC$ and $\beta = BD$. There is a solution for each positive integer n so we have a family of solutions, whose superposition is also a solution:

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)\pi t}{2L}\right) + \\ \beta_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)\sin\left(\frac{(2n-1)\pi t}{2L}\right) \end{aligned}$$

(d) I'm sorry I can't get these initial conditions to make sense with my solution to (c).