MST2102206F1PV2

MST210

Module Examination 2022

Mathematical methods, models and modelling

Friday 10 June 2022

There are **three sections** in this examination.

In **Section 1** you should **submit answers to all 19 questions**. Each question is worth 2% of the total mark. A wholly incorrectly answered question will get zero marks. Answers to Section 1 should be submitted using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 6 questions**. Each question is worth 5% of the total mark.

In **Section 3** you should **attempt both questions**. Each question is worth 16% of the total mark.

For **Sections 2 and 3**:

Include all your working, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST210.

PLAGIARISM WARNING – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

You should not:

- provide any assessment question to a website, online service, social media platform or any individual or organisation, as this is an infringement of copyright.
- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

The University actively monitors websites, online services and social media platforms for answers and solutions to assessment questions, and for assessment questions posted by students. Work submitted by students for assessment is also monitored for plagiarism.

A student who is found to have posted a question or answer to a website, online service or social media platform and/or to have used any resulting, or otherwise obtained, output as if it is their own work has committed a disciplinary offence under **Section SD 1.2** of our Code of Practice for Student Discipline. **This means the academic reputation and integrity of the University has been undermined.**

The Open University's Plagiarism policy defines plagiarism in part as:

- using text obtained from assignment writing sites, organisations or private individuals.
- obtaining work from other sources and submitting it as your own.

If it is found that you have used the services of a website, online service or social media platform, or that you have otherwise obtained the work you submit from another person, this is considered serious academic misconduct and you will be referred to the Central Disciplinary Committee for investigation.

Section 1

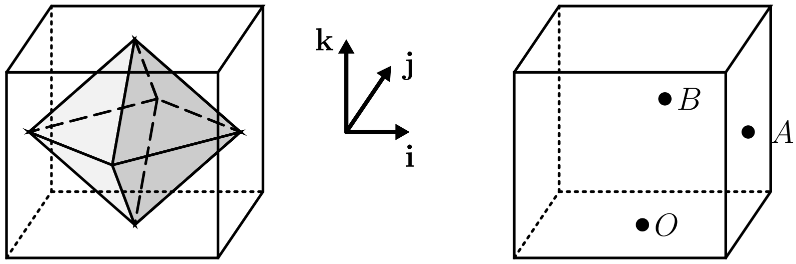
The questions for this section are contained within the iCME on the website.

Section 2

You should ***attempt all questions****.* ***Each question is worth 5%.***

Question 20

For every polyhedron there is a construction for a so called dual polyhedron by connecting together the centres of the faces. The diagram on the left below shows this construction for a cube that constructs an octahedron inside it.



Choose an origin O to be the centre of the base of the cube and choose the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ directions to be parallel to the faces of the cube. Let A be the centre of the face with positive $\mathbf{i}$-coordinate and B be the centre of the face with positive $\mathbf{j}$- coordinate, as shown in the right-hand diagram.

Let the cube have side length 2.

(a) Write down $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. *[1]*

(b) Calculate $\overrightarrow{OA} \cdot \overrightarrow{OB}$ and hence calculate the angle $\widehat{AOB}$. *[2]*

(c) Calculate $\overrightarrow{OA} \times \overrightarrow{OB}$ and hence calculate the ratio of the surface area of the octahedron to the surface area of the cube. *[2]*

Question 21

Consider the following simultaneous linear differential equations.

$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = -3y.$$

(a) Express the system of equations in matrix form. *[1]*

(b) Find the eigenvalues and eigenvectors of the matrix of coefficients. *[3]*

(c) Hence write down the general solution to this system. *[1]*

Question 22

A function f has partial derivatives

$$f_x = 4x + y^2 - 6y + 5, \quad f_y = 2xy - 6x.$$

(a) Find the stationary points of f. *[3]*

(b) Classify the stationary point in the region $x > 0$, $y > 0$. *[2]*

Question 23

Consider the function $f(t) = 0.01e^{2t}$ for $0 \leq t \leq \pi$).

(a) Sketch the even extension, f_{even}, and the odd extension f_{odd} of the function $f(t)$ over the range $-3\pi \leq t \leq 3\pi$, clearly indicating each case. *[3]*

(b) Now consider the function $g(t) = f_{\text{even}}(t) + f_{\text{odd}}(t)$, which has Fourier series $G(t)$ given by

$$G(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi t) + \sum_{n=1}^{\infty} B_n \sin(n\pi t).$$

Calculate the constant term A_0. *[2]*

Question 24

A vector field $\mathbf{F}$ is given by

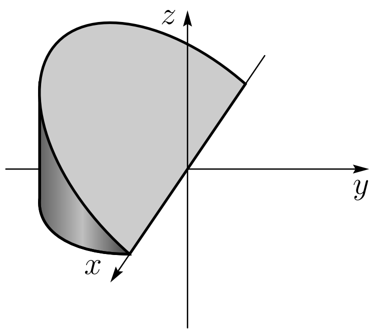
$$\mathbf{F} = (-6x - y)\,\mathbf{i} + (-x - 2y)\,\mathbf{j}.$$

(a) Show that $\mathbf{F}$ is conservative. *[2]*

(b) Find a potential function for $\mathbf{F}$. *[3]*

Question 25

Find the volume of the semicircular wedge shown in the following diagram, which is the region where the following three inequalities are satisfied: $x^2 + y^2 \leq 9$, $z \geq 0$ and $z \leq -y$.



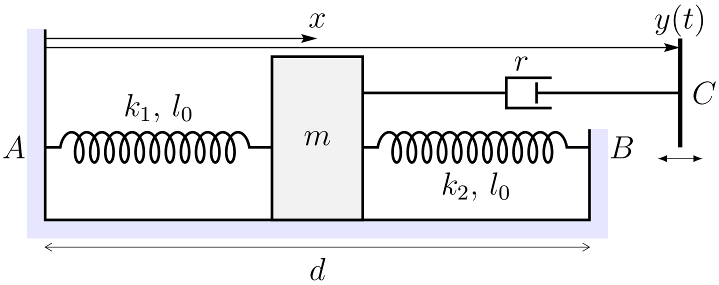
[5]

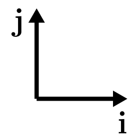
Section 3

You should ***attempt all questions****.* ***Each question is worth 16%.***

Question 26

A particle of mass m slides on a smooth horizontal surface and is attached to two model springs and a model damper. The two springs connect the particle to two fixed points A and B that are a fixed distance d apart. The spring connected to point A has stiffness k_1 and natural length l_0 and the spring connecting to point B has stiffness k_2 and natural length l_0. The model damper has damping constant r and is connected to a point C that is a distance $y(t)$ from point A. Choose the origin at the fixed point A and let x be the position of the particle. Choose unit vectors $\mathbf{i}$ horizontal in the direction of increasing x and $\mathbf{j}$ vertical, as shown below.





(a) Draw a force diagram showing all forces acting on the particle. Define each force you introduce. *[3]*

(b) Express each force in terms of the given unit vectors and the variables and parameters defined above. *[4]*

Now suppose $m = 1\,\mathrm{kg}$, $k_1 = 2\,\mathrm{N\,m^{-1}}$, $k_2 = 3\,\mathrm{N\,m^{-1}}$, $l_0 = 1\,\mathrm{m}$, $r = 2\,\mathrm{N\,m^{-1}\,s}$, $d = 3\,\mathrm{m}$ and $y(t) = 5 + \sin(2t)\,\mathrm{m}$.

(c) Show that the motion of the particle is described by the differential equation

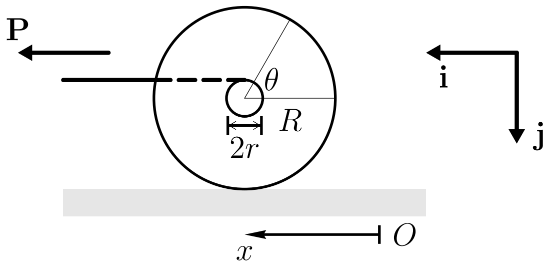
$$\ddot{x} + 2\dot{x} + 5x = 4\cos(2t) + 8.$$

[3]

(d) The complementary function for this mechanical system is transient and so the steady state amplitude is determined by the particular integral. Find the particular integral by using the trial function $x = p\cos(2t) + q\sin(2t) + m_0$, where p, q and m_0 are constants. Hence determine the steady state amplitude of the oscillations. *[6]*

Question 27

A yo-yo rests on a rough horizontal surface and rolls without slipping. The yo-yo is a cylindrical shape with a groove cut in it so that a string can be wound around a circle of radius r. Model the yo-yo as a solid cylinder of radius R and mass M and the string as a model string. Assume that the string unwinds without slipping from the top of the groove. After the yo-yo has moved a distance x from a fixed point O the axle has turned through an angle θ, as shown in the following diagram. Choose a unit vector **i** in the direction of motion and a unit vector **j** vertically downwards, as shown in the diagram.



The free end of the string is subjected to a constant horizontal force of magnitude P.

(a) Draw a force diagram showing all of the forces acting on the yo-yo and mark the point of action of each force. *[4]*

(b) Apply Newton's second law to obtain an equation for the magnitude of the friction force at the point of contact F in terms of the acceleration $\ddot{x}$ and other parameters. *[3]*

(c) Write down the rolling condition for this motion and use it to obtain a relationship between $\ddot{x}$ and $\ddot{\theta}$. *[2]*

(d) Calculate the torques of each force about the axis of symmetry of the yo-yo. *[3]*

(e) Show that the acceleration of the yo-yo is given by

$$\ddot{x} = \frac{2P(R+r)}{3MR}. \qquad [4]$$

[END OF QUESTION PAPER]