Unit 10

Forcing, damping and resonance

Introduction

In Unit 9 we studied models of vibrating systems based on the concept of a model spring. These models predicted simple harmonic motion. In particular, we looked at a simple experiment with a home-made oscillating system, using steel nuts hanging from a metal ruler (Figure 1). In Unit 9 certain qualitative features of the motion of the ruler were modelled well by assuming that the ruler could be modelled as a model spring. There was one feature of the motion that was not satisfactorily modelled in Unit 9 – the fact that the oscillations die away until the ruler is at its equilibrium position. This feature is known as *damping* and is the first topic studied in this unit.

Figure 1 The vibrating ruler experiment

Rather than continuing to model this experiment, you will meet some new oscillating system experiments in Section 1, which we will attempt to model in this unit. Section 1 introduces a linear model for damping, then Section 2 introduces the *model damper*, which provides a convenient diagrammatic and algebraic representation of linear damping, just as the model spring provides a convenient representation of a linear restoring force. After application to some everyday examples, the various types of motion brought about by the damping of an oscillatory system are summarised mathematically.

Section 3 looks at models of systems that undergo sinusoidal *forcing*, that is, an additional sinusoidal force applied to the system. Forcing is needed to keep a system vibrating in spite of any damping. For example, a pendulum clock may have a wound spring to force its oscillations, and a wave machine in a swimming pool is designed to keep the water vibrating. A vehicle driving over a bumpy surface, formed perhaps by cobbles or a cattle grid, is forced to vibrate by the bumps that its wheels encounter.

Resonance refers to the fact that the amplitude of the vibrations with which a system responds to periodic forcing may be much greater at some frequencies of the forcing than at others. This effect may be desired, as in the tuning of a radio. However, resonant oscillations can be unwanted and even destructive in mechanical systems, bridges and tall buildings. For this reason, platoons of soldiers are often ordered not to march in step over bridges. Designers try to choose spring stiffnesses and damping devices within system components so as to eliminate unwanted resonant behaviour over the range of frequencies that are likely to be encountered. The amplitude of the forced oscillations of a vehicle going over a set of bumps depends very much on the speed or frequency with which it encounters the bumps. Rumble strips on roads are therefore designed to take this into account, so that resonant vibrations are likely to occur at higher speeds and drivers are encouraged to slow down. Section 4 shows that resonance is predicted under certain circumstances by the model developed earlier for forced oscillations, and that the experimental particle–spring system also exhibits resonance.

1 Damping and vibrations

The model of a vibrating system that was developed in Unit 9 led to the prediction of *simple harmonic motion*. For a particle with position $x\mathbf{i}$ at time t, the corresponding graph of $x(t)$ is shown in Figure 2(a). The motion, about an equilibrium position $x = x_{\text{eq}}$, has amplitude A and period τ. However, in the real world no system when left to its own devices maintains oscillations of constant amplitude as predicted by this model. In reality the amplitude *decreases* with time, as illustrated in Figure 2(b), and the motion eventually dies away. This section therefore introduces a new ingredient whose inclusion in a revised model leads to predictions that match better the observed behaviour of *damped vibration*. This ingredient is a resistive force that depends on the velocity of the particle.

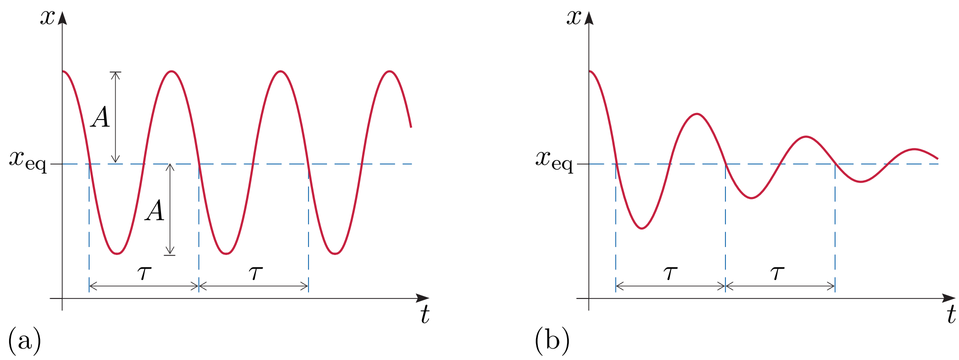


Figure 2 Graphs of (a) an undamped vibration, (b) a damped vibration

Subsection 1.1 introduces a linear model for damping, which is applied in Subsection 1.2 to an experiment that involves a magnet falling down tubes with differing resistance properties. Subsection 1.3 then models a damped particle–spring system. The model is tested experimentally in Subsection 1.4.

1.1 A linear damping model

The reason for the decreasing amplitude of vibrations shown in Figure 2(b) is the effect of some form of *resistance* due either to the air in which the system is vibrating or to some other form of friction. In the context of vibrations, such frictional effects are commonly referred to as **damping** and the corresponding motion is called a **damped vibration**.

Damping can be due to a number of physical causes and can be incorporated in a variety of mathematical models. As you saw in Unit 3, air resistance is commonly modelled as a force opposed to motion, of magnitude proportional to some power of the speed of the body – for example, on an object falling from the Clifton Suspension Bridge.

The emphasis here is not to model one particular physical cause of resistance, but to set up a general mathematical model that could apply in many situations.

In Unit 3 linear and quadratic models for air resistance were investigated: the magnitude of the resistance force was assumed to be proportional either to the magnitude of the velocity or to its square, where in each case the resistance force opposes the motion. In mathematical modelling it is a good principle to start with the *simplest* available mathematical model that appears reasonable, and not to complicate matters unless this initial model turns out to be inadequate. Consequently, we will begin here by assuming that a linear model applies, so we model the resistance to motion as being proportional to the velocity, but in the opposite direction.

If the velocity of the particle is $\dot{x}\mathbf{i}$, then the resistance force is assumed to be $\mathbf{R} = r(-\dot{x}\mathbf{i}) = -r\dot{x}\mathbf{i}$, where r is a positive constant, and the vector $-\dot{x}\mathbf{i}$ shows that the direction of $\mathbf{R}$ opposes the motion. This model is called *linear damping*, and the constant r is called the *damping constant*. The model has the virtue of simplicity in that it is mathematically easy to handle, which is such an advantage that the model is sometimes used even in situations where it may not be strictly appropriate. In such circumstances you would not expect the model to give results that are accurate in detail, but it could still show the effects of resistance and friction in general terms.

In Unit 3 when air resistance was considered, r was taken to be Dc_1, where D is the effective diameter of the object and c_1 is a constant.

The SI units of the damping constant r are $\mathrm{N\,s\,m^{-1}}$ (force divided by speed), or $\mathrm{kg\,s^{-1}}$.

The **linear damping** model assumes that the resistance force acting on a particle is proportional to its velocity, but in the opposite direction. If the velocity is $\dot{x}\mathbf{i}$, then the resistance force is modelled by

$$\mathbf{R} = -r\dot{x}\mathbf{i}, \qquad (1)$$

where r is a positive constant, called the **damping constant**.

The following example should help you to recall from Unit 3 how the linear damping model applies to motion with air resistance. It also provides some revision of the methods for solving second-order differential equations from Unit 1. The result obtained will be of use in Subsection 1.2.

Example 1

An object of mass m falls vertically under gravity. The object is to be modelled as a particle that experiences linear damping, with damping constant r, due to air resistance. Suppose that the particle has fallen a distance x after time t.

(a) Derive the equation of motion of the particle.

(b) Find the particular solution of the equation of motion for which the particle is dropped from the origin at time $t = 0$.

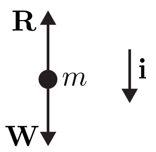


Figure 3 Force diagram

Solution

(a) The choice of direction for the x-axis (downwards) is determined by the fact that x is the distance fallen. The force diagram is shown in Figure 3. The weight of the particle is $\mathbf{W} = mg\mathbf{i}$ (acting vertically downwards). According to the linear damping model, the air resistance force is given by $\mathbf{R} = r(-\dot{x}\mathbf{i}) = -r\dot{x}\mathbf{i}$, which opposes the motion as required.

By Newton's second law,

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{R} = mg\mathbf{i} - r\dot{x}\mathbf{i}.$$

Resolving in the $\mathbf{i}$-direction gives

$$m\ddot{x} + r\dot{x} = mg.$$

(b) This differential equation may be solved in several different ways. Since it is linear, of second order and with constant coefficients, the methods of Unit 1 may be applied.

The differential equation is inhomogeneous, so the solution will be the sum of the complementary function (the general solution of the associated homogeneous differential equation) and a particular integral.

The associated homogeneous differential equation is

$$m\ddot{x} + r\dot{x} = 0,$$

whose auxiliary equation is

$$m\lambda^2 + r\lambda = 0.$$

This has solutions $\lambda = -r/m$ and $\lambda = 0$. The complementary function is therefore

$$x_{\mathrm{c}} = Ae^{-(r/m)t} + Be^{0t} = Ae^{-rt/m} + B,$$

where A and B are arbitrary constants.

Exceptional cases are discussed in Unit 1, Subsection 3.3.

The right-hand side of the original differential equation is mg. Since this is a constant, we might think of trying a constant for the particular integral. However, there is already a constant term, B, in the complementary function, so we need here to use a trial solution of the form $x_{\mathrm{p}} = pt$, where p is a constant to be determined. Substituting this into the differential equation gives $0 + rp = mg$, so $p = mg/r$. Thus $x_{\mathrm{p}} = mgt/r$ is a particular integral. On adding the complementary function to this, we obtain the general solution of the equation of motion as

$$x = x_{\mathrm{c}} + x_{\mathrm{p}} = Ae^{-rt/m} + B + \frac{mgt}{r}.$$

'Dropped' here means that the particle is initially at rest.

In order to find the appropriate particular solution, we need to refer to the initial conditions. Since the object is dropped from the origin at time $t = 0$, these conditions are $x(0) = 0$ and $\dot{x}(0) = 0$. Substituting the first of these into the general solution gives $0 = A + B + 0$, so $B = -A$.

The derivative of the general solution is

$$\dot{x} = -\frac{r}{m}\,Ae^{-rt/m} + \frac{mg}{r},$$

so the second initial condition gives $0 = -(r/m)A + mg/r$. Hence we obtain $A = m^2g/r^2$ and $B = -m^2g/r^2$. We conclude that

$$x = \frac{mgt}{r} - \frac{m^2g}{r^2}\left(1 - e^{-rt/m}\right)$$

is the required particular solution.

Another approach was adopted in Example 7(a) of Unit 3, giving a solution that agrees with the one obtained here.

Look at the form of the particular solution obtained in this example, and think about its interpretation. What does the model predict will happen?

The second term in the solution contains a negative exponential. As t increases, this exponential becomes negligibly small, so the second term in the solution tends to a constant. The first term is linear in t. In the absence of any other significant term involving t, this represents motion at the constant speed mg/r. So the model predicts that for large values of t, the motion will be at the constant velocity $(mg/r)\mathbf{i}$, where mg/r is the terminal speed of the object.

Terminal speed was defined in Unit 3.

1.2 An experiment with damping

When a magnet moves inside a copper tube, an electromagnetic effect causes its motion to be damped. The level of damping depends on the thickness and diameter of the tube. This subsection examines the results of a number of experiments carried out by the module team, starting with the motion of a magnet falling down three different copper tubes under gravity, and down a glass tube where only air resistance counteracts the descent (Figure 4). In each case, the resistive force may be modelled by linear damping as in Subsection 1.1.

Phenomena like electromagnetic damping are beyond the scope of this module. For current purposes, you do not need to understand how this effect comes about.

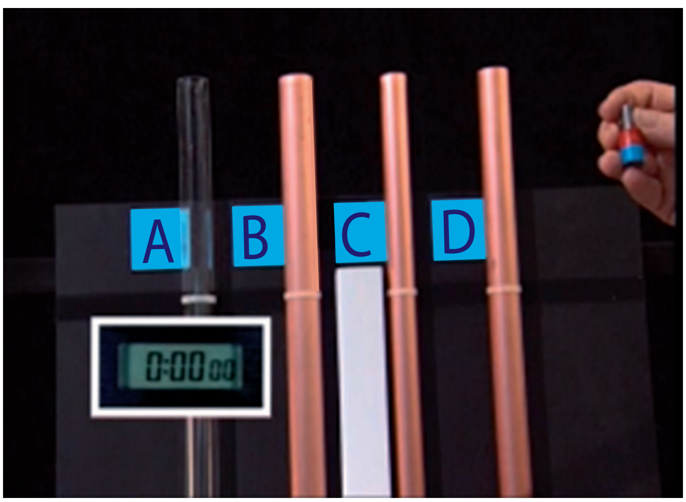


Figure 4 A magnet is dropped down each of four tubes

The equation of motion that arises is that of Example 1(a), and the appropriate particular solution is given by the equation found in Example 1(b). Using this equation and other measurements made, it is possible to determine an estimate for the damping constant r for each tube.

The magnet is then attached to a particle–spring system, like those that you met in Unit 9, and the system is set in motion. The model of motion of an undamped system is modified in order to take the damping into account. A more detailed account of this modelling is provided in Subsection 1.3.

Let $\mathbf{x}$ denote the position vector for the magnet, and let $\mathbf{i}$ be a unit vector pointing down the tube. Since the motion is one-dimensional, we have $\mathbf{x} = x\mathbf{i}$, $\dot{\mathbf{x}} = \dot{x}\mathbf{i}$ and $\ddot{\mathbf{x}} = \ddot{x}\mathbf{i}$. Thus the model for linear damping is

$$\mathbf{R} = -r\dot{x}\mathbf{i}.$$

The motion of the magnet down a tube is modelled with the origin at the top of the tube and the x-direction down the tube. This leads to the equation of motion

$$m\ddot{x} + r\dot{x} = mg,$$

which you met in Example 1(a). Since the magnet is dropped from the origin, the initial conditions are the same as in Example 1. Hence the particular solution is, once more,

$$x = \frac{mgt}{r} - \frac{m^2 g}{r^2}\left(1 - e^{-rt/m}\right). \tag{2}$$

The time t for the magnet to fall the length of each tube was measured. The length of each tube was 1 m. The magnet's mass was found to be 0.038 kg, and we take the magnitude of the acceleration due to gravity to be $g = 9.81\,\mathrm{m\,s^{-2}}$. Using these values, equation (2) can be solved numerically, to find the value of r for each tube. The results are in Table 1.

Table 1

Tube	Material	Time of descent (s)	Damping constant r ($\mathrm{N\,s\,m^{-1}}$)
A	Glass	0.64	0.15
B	Copper	2.52	0.92
C	Copper	3.60	1.33
D	Copper	22.60	8.42

These values for the damping constant r will be used to predict the motion of the magnet in each tube when it is attached to a particle–spring system.

1.3 Damping the spring motion

The next set of experiments with the magnet and tubes involves the magnet being attached to a particle–spring system (Figure 5).

Figure 5 A magnet attached to a spring, and the four tubes

The model of Unit 9 is modified by the inclusion of a linear damping force $\mathbf{R}$ as well as the model spring force $\mathbf{H}$ and the weight $\mathbf{W}$ of the particle, to give the equation

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{H} + \mathbf{R}. \tag{3}$$

In this subsection, we will show that with an appropriate choice of origin for x, the equation of motion can be written in the homogeneous form

$$m\ddot{x} + r\dot{x} + kx = 0,$$

where m is the mass of the particle, r is the damping constant, and k is the spring stiffness. To do this, we look again at the modelling involved.

The experimental apparatus and the corresponding force diagram are shown in Figure 6. This force diagram corresponds to the situation in which the spring is extended and the particle is descending, at which time both the spring force $\mathbf{H}$ and the damping force $\mathbf{R}$ will be directed upwards.

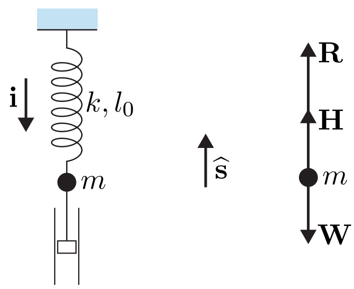


Figure 6 The magnet in the tube and the force diagram

As before, we take the x-axis to be directed downwards, but for the moment we leave open the choice of origin. Regardless of this choice, the acceleration of the particle is $\ddot{x}\mathbf{i}$ and its velocity is $\dot{x}\mathbf{i}$, so the linear damping force is $\mathbf{R} = -r\dot{x}\mathbf{i}$. The weight of the particle, $\mathbf{W} = mg\mathbf{i}$, is independent of where the origin is chosen. In fact, only the expression for the spring force $\mathbf{H}$ in terms of x will depend on the choice of origin.

Hooke's law states that the force exerted by a model spring is given by

Hooke's law was introduced in Unit 9.

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}},$$

where k, l and l_0 are respectively the stiffness, length and natural length of the spring, and $\widehat{\mathbf{s}}$ is a unit vector in the direction from the end where the particle is attached towards the centre of the spring. Here we have $\widehat{\mathbf{s}} = -\mathbf{i}$, so $\mathbf{H} = k(l - l_0)(-\mathbf{i})$.

Newton's second law (equation (3)) therefore gives

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{H} + \mathbf{R} = mg\mathbf{i} - k(l - l_0)\mathbf{i} - r\dot{x}\mathbf{i},$$

which after resolution in the **i**-direction leads to

$$m\ddot{x} + r\dot{x} + kl = mg + kl_0. \tag{4}$$

When the system is in equilibrium, we have $\dot{x} = 0$ and $\ddot{x} = 0$, so the equilibrium length l_{eq} of the spring is given by

$$0 + 0 + kl_{\text{eq}} = mg + kl_0,$$

that is,

$$l_{\text{eq}} = l_0 + \frac{mg}{k}.$$

This is the expression for the equilibrium spring length that was derived in Unit 9, Example 3. It is the sum of the natural length of the spring, l_0, and its equilibrium extension mg/k.

Hence equation (4) can be written as

$$m\ddot{x} + r\dot{x} + kl = kl_{\text{eq}}.$$

It follows that if we take $x = l - l_{\text{eq}}$, then the equation of motion takes the homogeneous form

$$m\ddot{x} + r\dot{x} + kx = 0.$$

This choice for x corresponds to taking $x = 0$ at the point where $l = l_{\text{eq}}$, that is, choosing the origin to be *at the equilibrium position* of the particle.

Exercise 1

In the next set of experiments, the total mass suspended from a spring of stiffness $23\,\text{N}\,\text{m}^{-1}$ is $0.711\,\text{kg}$. Find the equilibrium extension of the spring, taking $g = 9.81\,\text{m}\,\text{s}^{-2}$.

Exercise 2

Use equation (4) to write down the equation of motion for the case in which the origin for x is chosen to be at the fixed upper end of the spring.

Exercise 2 shows that the equation of motion takes the form

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}}, \tag{5}$$

where x_{eq} is the equilibrium displacement of the particle relative to the chosen origin. We will normally use the simplest form,

$$m\ddot{x} + r\dot{x} + kx = 0, \tag{6}$$

This equation (using different notation) was investigated in Unit 1, Subsection 2.2.

for which $x_{\text{eq}} = 0$, where the origin is chosen to be the equilibrium point of the system.

Notice that all the terms on the left-hand side of equation (6) (or equation (5)) have the same sign, and that if damping were to be removed (by putting r equal to zero), then the resulting equation would revert to that of simple harmonic motion, as derived in Unit 9.

In order to use our model to predict the motion of the damped particle–spring system, we need first to solve the equation of motion (6). This is a linear constant-coefficient second-order differential equation, so once again the methods of Unit 1 may be applied to find the form of its solution. The auxiliary equation is

The process of solving equation (6) will be revisited in Section 2, so do not dwell on the mathematical details here. The main point is that both graphical (Figure 7 below) and numerical (Table 2 below) predictions of the behaviour of the damped particle–spring system can be obtained from the model.

$$m\lambda^2 + r\lambda + k = 0,$$

whose roots are

$$\lambda_1 = \frac{-r + \sqrt{r^2 - 4mk}}{2m} \quad \text{and} \quad \lambda_2 = \frac{-r - \sqrt{r^2 - 4mk}}{2m}.$$

Provided that $\lambda_1 \neq \lambda_2$, the solution is

$$x(t) = Be^{\lambda_1 t} + Ce^{\lambda_2 t},$$

where B and C are arbitrary constants that depend on the initial conditions. If λ_1 and λ_2 are real and distinct, which they will be if $r^2 - 4mk$ is positive, then the solution is the sum of two real exponential terms. As you may recall from Unit 1, if the roots of the auxiliary equation are complex (in this case, when $r^2 - 4mk$ is negative), then the solution can be written in real form as an exponential times a sinusoid, that is, as

$$x(t) = e^{-\rho t}(B\cos\nu t + C\sin\nu t),$$

where $\rho = r/(2m)$, $\nu = \sqrt{4mk - r^2}/(2m)$, and B and C are arbitrary constants that depend on the initial conditions. As in Unit 9, the sine and cosine terms can be combined to write this solution in the form

$$x(t) = Ae^{-\rho t}\cos(\nu t + \phi), \tag{7}$$

Algebraic manipulation reveals that $B = A\cos\phi$ and $C = -A\sin\phi$ (see Unit 9).

where A and ϕ are arbitrary constants.

It can be seen that under certain circumstances, the model predicts oscillations (due to the sinusoidal factor) of decreasing amplitude (since the exponential term has a negative exponent).

The measured values $m = 0.711$ and $k = 23$ are the same in each case, and the damping constant r varies with each tube. Where $r^2 < 4mk = 65.41$, so $r < 8.09$, the model suggests that the magnet will oscillate with decreasing amplitude. The calculated values for r for tubes A to D are 0.15, 0.92, 1.33 and 8.42, so the model predicts that the magnet will oscillate for tubes A to C but not for tube D.

See Table 1.

In the cases of tubes A–C, the model can be used to predict the period of each oscillation and the rate at which the oscillations decay. For these three tubes the period of oscillations (in seconds) is

$$\tau = \frac{2\pi}{\nu} = \frac{4\pi m}{\sqrt{4mk - r^2}},$$

and that over each complete oscillation, the amplitude decays by a factor $e^{-\rho\tau}$, where $\rho = r/(2m)$. The predicted values are given in the last two columns of Table 2.

Table 2

Tube	Damping constant r (N s m^{-1})	Period τ (s)	Amplitude decay factor per cycle $e^{-\rho\tau}$
A	0.15	1.10	0.89
B	0.92	1.11	0.49
C	1.33	1.12	0.35

With these values we can sketch the graphs of the behaviour of the magnet in each of the four tubes (Figure 7).

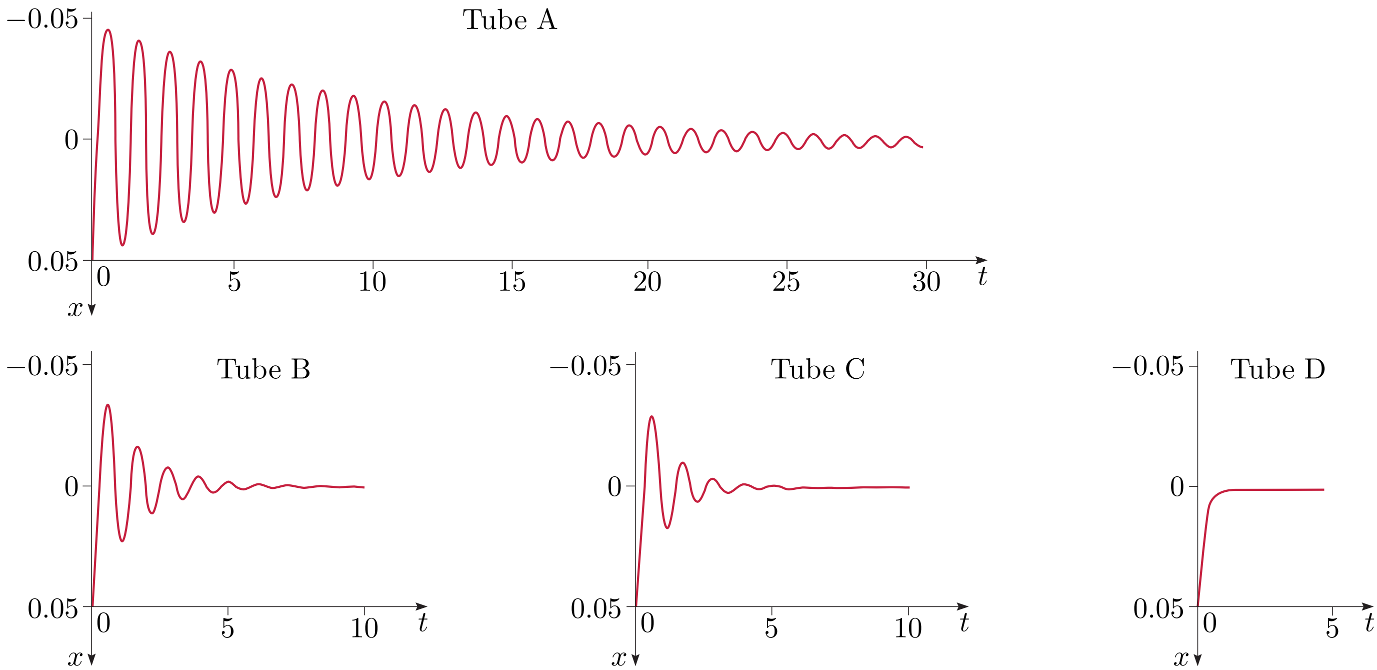


Figure 7 Predictions from the model for each tube

What are the most significant features in each case?

Tube A: The graph of the solution clearly shows the amplitude of the oscillations decreasing exponentially. The amplitude corresponds to the exponential factor $Ae^{-\rho t}$ identified earlier, while the sinusoidal factor gives oscillations of constant frequency. The model predicts that the oscillations die away relatively slowly.

See equation (7).

Tube B: The model again predicts decaying oscillations, though these die down more quickly than for tube A.

Tube C: The model predicts motion very similar to that for tube B.

Tube D: For this tube, with $r = 8.42$, the expression $4mk - r^2$ is negative, that is, $r^2 - 4mk$ is positive. In this case the auxiliary equation has two negative real roots and the solution is purely exponential. Hence there are no oscillations in this case, as confirmed by the corresponding graph in Figure 7. The model predicts no oscillations at all and a rapid return directly to the equilibrium position.

1.4 Comparing predictions with reality

Figure 7 provides predictions for the behaviour of the damped particle–spring system for each of the tubes. Do these predictions bear any relationship to reality?

The match between the model's graphical predictions in Figure 7 and the traces from the experiment in Figure 8 is striking.

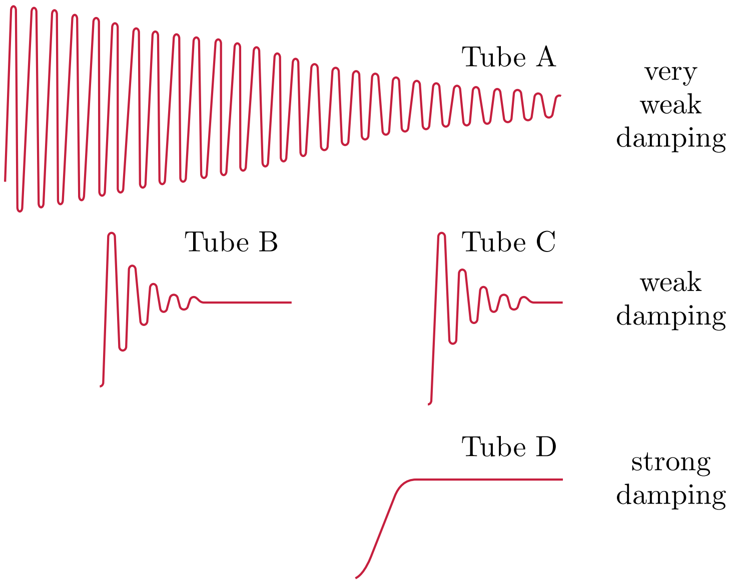


Figure 8 Traces of experimental results for each tube

These traces do not give an indication of the timescale involved. Timings provide an estimate of 1.1 seconds for the period of oscillations for tube A. The corresponding periods for tubes B and C are very similar, but harder to measure due to the few oscillations that occur. This period estimate is in agreement with those given in Table 2.

The tabulated predictions for the amplitude decay factor also appear to be borne out by the experiments, to the accuracy with which measurements can be made from the traces. For example, the prediction for tube A is that the amplitude should halve over about six cycles (since $0.89^6 \simeq 0.5$), whereas for tube B the amplitude is predicted to halve approximately every cycle.

We conclude that both qualitatively and quantitatively, the model is validated quite well by the experimental results. (There are nevertheless some discrepancies. For example, the actual decay of the amplitude in tube A does not look exponential throughout the range shown.)

When the magnet is in tube A, the damping is very light. With the magnet in either tube B or tube C, the damping level is higher but there are still some oscillations. The even higher level of damping in tube D prevents oscillations altogether. Damping like this, for which there are no oscillations, is called **strong damping**. Damping where there are decaying oscillations, as for the other three tubes, is called **weak damping**.

For a system in which the level of damping can be varied continuously, there will be a crossover point between weak and strong damping, and the corresponding level of damping is called **critical damping**. In the case of the experiments, critical damping lies between the damping levels of tubes C and D. Critical damping gives the most rapid return to the equilibrium position, given an initial displacement, whereas very strong damping can cause a significant delay in the return to equilibrium.

In terms of the mass m, spring stiffness k and damping constant r, we have weak damping if $r^2 - 4mk < 0$, strong damping if $r^2 - 4mk > 0$, and critical damping if $r^2 - 4mk = 0$. More is said about the mathematical aspects of damping in Section 2. To conclude this section, we ask you to consider briefly what type of damping might be required in various real-world systems.

For the particle–spring system discussed in Subsection 1.3, critical damping occurs when $r = 8.09$.

Different levels of damping are appropriate in different situations. A baby bouncer (which we will consider in Example 3) is more fun for the child the longer its oscillations continue, following an initial displacement, so here very weak damping is desirable.

For kitchen scales to be useful, any oscillations should die down quickly, so that readings may be taken. In this case the damping should be close to critical damping. For given kitchen scales, the values of r and k are fixed. There is then only one value m of the mass placed on the scales for which critical damping can be achieved exactly, since in this case $r^2 = 4mk$. Scales are usually designed to give a speedy return to the equilibrium position for a specified range of values of the mass, and since a few small oscillations initially are acceptable, damping that is weak but close to critical is preferred. Strong damping would result in the scales taking a longer time to return to the equilibrium position, and hence hold up the taking of a reading.

Vehicle suspension systems are designed to smooth out the ride, so here strong damping is better. However, the damping should not be *too* strong. The spring needs to be returned close to its equilibrium position in order to be able to absorb the next jolt from the road. The time between jolts will obviously depend on the road surface and the speed at which the vehicle is travelling. When the vehicle is carrying a heavy load, m will be larger. Designers must aim to include a level of damping that is appropriate for both the heaviest and the lightest loads envisaged, as well as for the different terrains and speeds that are likely to be encountered.

Exercise 3

What level of damping would be appropriate for each of the following mechanisms?

(a) Buffers at the end of a railway line, which are intended to halt a train that comes into the station too fast

(b) A device that prevents a door from slamming shut

(c) A mechanism linking the fuel gauge of a vehicle and a float in the fuel tank, which is designed to damp fluctuations in the gauge reading caused by travel over an uneven surface

(d) A tow-bar mechanism, which is designed to minimise the transfer of jolts from the towing vehicle to the towed vehicle, or vice versa

(e) Bathroom scales

2 Spring–damper models of motion

This section is designed to give you practice in modelling systems where both model spring forces and linear damping forces act. As in earlier units in this module, this involves drawing appropriate diagrams and deriving the equation of motion. The emphasis is on setting up the model, including appropriate initial conditions, and interpreting solutions in terms of the physical system concerned.

In order to represent linear damping diagrammatically, we introduce in Subsection 2.1 the concept of a *model damper*. This is applied to various modelling examples described in Subsection 2.2. A mathematical summary of the various types of motion that can be caused by model springs and dampers is given in Subsection 2.3.

2.1 The model damper

In Unit 9 we introduced the concept of a *model spring*. This is a convenient means of indicating diagrammatically and describing algebraically the presence of a force that depends linearly on the length l of the spring. The force exerted on a particle connected to an end of the model spring is given by Hooke's law,

See Unit 9, Subsection 1.1.

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}, \tag{8}$$

where k is the stiffness and l_0 is the natural length of the spring. The vector $\widehat{\mathbf{s}}$ is a unit vector in the direction from the particle towards the centre of the model spring (see Figure 9).

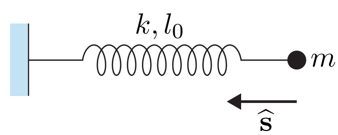


Figure 9 A model spring

In Unit 9 you saw this force specification applied to model springs that had one end attached to a particle and the other kept fixed. However, the same expression for the force on the particle applies even when the other end of the model spring is in motion. You have seen applications of this type in Unit 9, and will see further applications in Section 3 of this unit and in Unit 18.

We now define a *model damper*. Like the model spring, this is a hypothetical one-dimensional system component that can be included in diagrams and with which a certain vector expression for force is associated.

When one end of the damper is fixed, it embodies the linear damping model defined in Subsection 1.1.

A **model damper**, with a particle attached to one of its ends, provides a force on the particle that opposes its motion relative to the other end. The magnitude of this resistance force is proportional to the rate of change of length of the model damper. The force provided by the model damper (when compressing or extending) is therefore

$$\mathbf{R} = r\dot{l}\,\widehat{\mathbf{s}}, \tag{9}$$

where $\dot{l}$ is the rate of change of length of the model damper, r is a positive constant (called the **damping constant**), and $\widehat{\mathbf{s}}$ is a unit vector in the direction from the particle towards the centre of the model damper.

A model damper has zero mass. It is represented diagrammatically as shown in Figure 10.

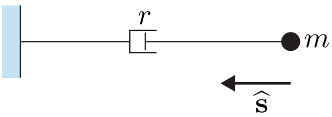


Figure 10 A model damper

The diagrammatic representation of a model damper, as shown in Figure 10, is a cross-sectional picture of an actual physical device known as a **dashpot**, which involves the motion of a piston within a circular cylinder. However, you could also think of it as a picture of the magnet within a copper tube from Section 1, where the resistance was electromagnetic.

One form of dashpot is used as the shock-absorber on a car, where the cylinder is full of oil and the relative motion of the piston causes oil to flow through the small annular gap between piston and cylinder. This provides resistance to the relative motion.

When used in a diagram, the model damper simply indicates the presence of a force of the type described by equation (9), just as a model spring in a diagram stands for a force of the type given by equation (8). Thus we could represent the modelling of the damped particle–spring system as shown in Figure 11. This is a more abstract form of the system in Figure 6, but it indicates clearly the assumed presence of linear damping, which the earlier diagram does not do.

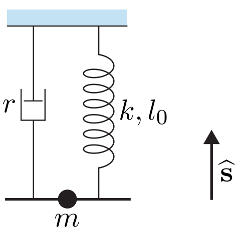


Figure 11 Modelling a damped particle–spring system

Although we represent a model spring and a model damper as independent elements, they may correspond to a single physical entity (a real spring, say) that exhibits to some extent both types of force behaviour. For example, the metal ruler from which nuts were hung in Unit 9 shows spring-like behaviour but also seems to exhibit considerable ‘internal friction’, which may be a more significant factor than air resistance in reducing the amplitude of oscillations. If we were to model this internal friction as being linear, then Figure 11 as it stands would suffice to represent our model for the nuts hanging from the ruler.

Exercise 4

(a) Suppose that a model damper is attached to a particle at one end and to a fixed point at the other, where the damper lies along the direction of motion of the particle (described by an x-axis). Show that if $\mathbf{i}$ is a unit vector in the positive x-direction, then equation (9) leads in this case to the expression

$$\mathbf{R} = -r\dot{x}\mathbf{i}$$

(as in the definition of linear damping in Subsection 1.1) for either of the possible choices of x-direction.

(b) Suppose now that the end of the model damper not attached to the particle is made to move in such a way that its position on the x-axis is given at time t by $y(t)\,\mathbf{i}$. Show that equation (9) now leads to

$$\mathbf{R} = -r(\dot{x} - \dot{y})\mathbf{i}.$$

Note that the model damper is very much a first model of damping effects, and may not describe accurately what occurs in real systems except over small ranges of the relative velocity. However, its simplicity makes it convenient to use, and it is capable (as you saw in Section 1) of providing reasonable representations of certain damped systems.

2.2 Applying model dampers

The examples and exercises in this subsection present a number of situations where there is some damping. In each case you are invited to consider carefully the setting up of the model, by drawing diagrams and then using Newton's second law to obtain the equation of motion. Do not focus too much here on how to solve the differential equations that arise. Concentrate rather on the modelling, including the specification of initial conditions and the way in which they enable values for the arbitrary constants to be found. Think also about the interpretation of the solutions.

The methods required to solve these differential equations are from Unit 1.

Example 2

A toy train of mass 2 kg, travelling on a straight horizontal track, freewheels into buffers at a speed of $0.25\,\mathrm{m\,s^{-1}}$.

(a) Model the buffers as a model spring with stiffness $25\,\mathrm{N\,m^{-1}}$ together with a model damper with damping constant $15\,\mathrm{N\,s\,m^{-1}}$. Derive the equation of motion, and determine its general solution.

(b) Use this model to predict the subsequent motion. In particular, by how much will the buffers be compressed, and what happens to the train thereafter?

Solution

(a) The model of the buffers is shown in Figure 12. The train is represented by a particle of mass m, and the model spring of stiffness k and natural length l_0, and model damper of damping constant r, are assumed to be joined at their non-fixed ends. The origin is taken to be where the free end of the spring is situated when the length of the spring is equal to its natural length. This is the point at which the front of the train will first come into contact with the buffers. The train's initial direction of travel is taken to be the positive x-direction.

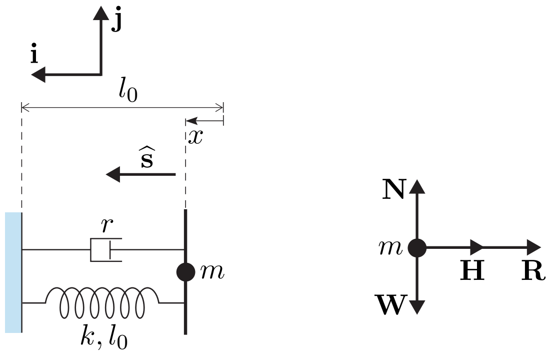


Figure 12 The buffers and the force diagram

The corresponding force diagram is also shown in Figure 12. Since the weight $\mathbf{W}$ of the train and normal reaction $\mathbf{N}$ on it from the track are vertical forces, they balance each other, as there is no motion in this direction. As with model springs, there is in general no single choice of direction for the model damper force that is correct at all times, but for the purposes of the force diagram, *either of the possible choices will do*. The directions for $\mathbf{H}$ and $\mathbf{R}$ in Figure 12 correspond to the model spring being compressed and the model damper shortening. The same unit vector $\widehat{\mathbf{s}} = \mathbf{i}$ can be used for the spring and damper, as the directions from the particle to the centre of the spring and centre of the damper are the same. The length of the spring is $l = l_0 - x$, so $\dot{l} = -\dot{x}$.

From Hooke's law, the model spring force is

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}} = k(l_0 - x - l_0)\mathbf{i} = -kx\mathbf{i},$$

and from equation (9) and Exercise 4(a), the resistance force is

$$\mathbf{R} = r\dot{l}\,\widehat{\mathbf{s}} = r(-\dot{x})\mathbf{i} = -r\dot{x}\mathbf{i}.$$

The equation of motion is therefore

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{H} + \mathbf{R} + \mathbf{W} + \mathbf{N} \\ &= -kx\mathbf{i} - r\dot{x}\mathbf{i} - mg\mathbf{j} + |\mathbf{N}|\,\mathbf{j}, \end{aligned}$$

which after resolution in the $\mathbf{i}$-direction gives

This is equation (6) once more.

$$m\ddot{x} + r\dot{x} + kx = 0.$$

Substituting in the given values $m = 2$, $k = 25$ and $r = 15$, we have

$$2\ddot{x} + 15\dot{x} + 25x = 0.$$

Note that in this case $r^2 - 4mk = 25$, which is positive. According to the criteria obtained near the end of Section 1, this confirms that the buffers will provide strong damping, with real exponential terms in the solution, rather than decaying oscillations.

The corresponding auxiliary equation is $2\lambda^2 + 15\lambda + 25 = 0$, which has roots $\lambda = -5$ and $\lambda = -2.5$. Hence the general solution of the differential equation is

$$x = Ae^{-5t} + Be^{-2.5t},$$

where A and B are arbitrary constants that depend on the initial conditions. In order to find the appropriate particular solution, we need to formulate these initial conditions.

(b) Choose the origin of time as the instant at which the particle representing the train makes contact with the buffers. The initial conditions are then $x(0) = 0$ and $\dot{x}(0) = 0.25$. Substituting the first of these into the general solution gives $0 = A + B$, so we have $B = -A$. To apply the second initial condition, we first need to differentiate the general solution, obtaining

$$\dot{x} = -5Ae^{-5t} - 2.5Be^{-2.5t}.$$

Substituting $\dot{x}(0) = 0.25$ and $B = -A$ here gives $0.25 = -5A + 2.5A$, so $A = -0.1$ and $B = 0.1$. Putting these values for A and B into the general solution gives the required particular solution as

$$x = 0.1(e^{-2.5t} - e^{-5t}).$$

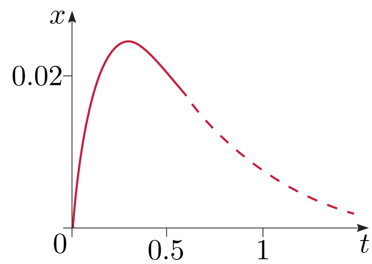


Figure 13 Graph of the position function x

The graph of this position function is shown in Figure 13.

We now need to interpret this solution in order to predict the motion of the train after it meets the buffers. Clearly, the buffers will be compressed. The maximum compression is achieved when x is a maximum, for which $\dot{x} = 0$ (the train's velocity is zero). This occurs when

$$\dot{x} = -0.25e^{-2.5t} + 0.5e^{-5t} = 0, \quad \text{that is,} \quad 1 = 2e^{-2.5t}.$$

Here we have divided through by $e^{-2.5t}$ and noted that $e^{-5t} = (e^{-2.5t})^2$.

The corresponding time is $t = 0.4 \ln 2 \simeq 0.28$, and (putting this time into the particular solution) the maximum compression is $x = 0.1\left(\frac{1}{2} - \frac{1}{4}\right) = 0.025$. So the model predicts that the buffers will be compressed by 2.5 cm.

For this to be reasonable, the spring must have a natural length longer than this! Otherwise, the model predicts that the train will compress the spring to zero length without being brought to a halt, which might result in the train breaking through the buffers or becoming derailed, or being brought to a standstill very quickly.

It remains to consider what happens to the train after it has been instantaneously brought to rest at the point of maximum compression for the model spring. As expected, and as the graph in Figure 13 indicates, it then starts to move back in the direction from which it arrived. However, the spring–damper is not attached to the train thus can only 'push' it, not 'pull' it. If at some time the model predicts a total force on the particle that is in the positive x-direction, then the model has become invalid. This will correspond to a moment at which the train has lost contact with the buffers.

According to Newton's second law, the total force on the particle is equal to $m\ddot{x}\mathbf{i}$, so the train leaves the buffers where $\ddot{x}$ becomes positive. This is where the graph in Figure 13 changes to a broken line. Differentiation of the expression above for $\dot{x}$ gives

Notice that the curvature of the graph reverses where the graph changes to a broken line; there is a point of inflection, where $\ddot{x}$ changes sign.

$$\ddot{x} = 0.625e^{-2.5t} - 2.5e^{-5t} = 0.625e^{-2.5t}(1 - 4e^{-2.5t}).$$

This expression becomes positive when $4e^{-2.5t}$ is less than 1, which first happens when $t = 0.4\ln 4 \simeq 0.55$. Therefore the corresponding position of the particle is $x = 0.1\left(\frac{1}{4} - \frac{1}{16}\right) \simeq 0.019$, which is 0.6 cm back from the point of maximum compression. The velocity at this time is $\dot{x}\mathbf{i}$, where $\dot{x} = -0.25 \times \frac{1}{4} + 0.5 \times \frac{1}{16} \simeq -0.031$.

We conclude that provided that the buffers are long enough to sustain the maximum compression, they are predicted to turn an incoming speed of $0.25\,\mathrm{m\,s^{-1}}$ into an outgoing speed of $0.031\,\mathrm{m\,s^{-1}}$. This is the sort of effect that buffers are intended to have!

The speed decrease is also evidence of a very significant decrease in the train's kinetic energy. There is no space in this unit to focus further on the topic of energy, beyond pointing out that the presence of damping in a system will always entail loss of energy, and that heavier damping means a greater rate of loss of energy.

In the example above, we chose the x-axis to be directed from the track towards the buffers, with origin at the point where the train first comes into contact with the buffers. If the opposite direction were to be chosen for the x-axis, with the same origin, then the equation of motion would not be altered. The initial condition $x(0) = 0$ is unchanged, while that for $\dot{x}$ has the opposite sign, that is, $\dot{x}(0) = -0.25$. The solution for x, correspondingly, has its sign reversed.

If, on the other hand, we selected a different point for the origin (with either direction for the x-axis), the equation of motion would become

This is equation (5) once more.

$$m\ddot{x} + r\dot{x} + kx = kx_{\mathrm{eq}},$$

where x_{eq} is the value of x where the train meets the buffers. The initial conditions are now $x(0) = x_{\mathrm{eq}}$, $\dot{x}(0) = \pm 0.25$, where the sign for $\dot{x}(0)$ depends on the choice of x-direction, as discussed above. The solution for x is altered by the addition of x_{eq} to the expression obtained previously (when the origin is at the point where the train meets the buffers).

As you would expect, the eventual answers to the problem in Example 2 do not depend on the choices of origin or direction for the x-axis. The interpretation is the same in each case.

Exercise 5

A miniature train of mass 40 kg, travelling on a straight horizontal track, freewheels into buffers at a speed of $1\,\mathrm{m\,s^{-1}}$. The buffers are to be modelled by a model spring with stiffness $140\,\mathrm{N\,m^{-1}}$, together with a model damper with damping constant $180\,\mathrm{N\,s\,m^{-1}}$. The x-axis is chosen directed away from the buffers down the track (in the direction opposite to the incoming train), with origin at the fixed end of the model spring. The natural length of the model spring is 0.5 m.

(a) Show that the equation of motion can be written as

$$4\ddot{x} + 18\dot{x} + 14x = 7.$$

(b) Write down a pair of initial conditions for the motion of the train while it is in contact with the buffers.

(c) The solution of the equation of motion that satisfies the initial conditions of part (b) is

$$x = 0.5 - 0.4e^{-t} + 0.4e^{-3.5t}.$$

What is the maximum compression of the buffers? At what point, and with what speed, does the train leave the buffers?

Example 3

A baby bouncer's suspension (see Figure 14) is modelled by a model spring with stiffness $200\,\mathrm{N\,m^{-1}}$. (This is equivalent to having two equal springs of stiffness $100\,\mathrm{N\,m^{-1}}$; see Unit 9, Exercise 4.) There is some internal damping in the suspension and there is some air resistance, which together can be modelled by a model damper with damping constant $0.2\,\mathrm{N\,s\,m^{-1}}$. The bouncer is designed for a baby whose mass is about $10\,\mathrm{kg}$.

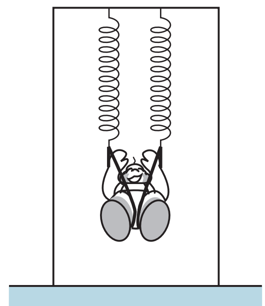


Figure 14 A baby bouncer

Set up the equation of motion for the model, and find the particular solution for the case in which the baby is released from rest when $0.3\,\mathrm{m}$ above its equilibrium position. Use the model to predict how long it takes before the amplitude of the oscillations drops below $0.05\,\mathrm{m}$, if the baby is not pushed in any way.

Solution

The system and force diagram are shown in Figure 15. We model the baby as a particle of mass m. Take the origin at the top of the spring, with the x-axis pointing downwards and the unit vector $\mathbf{i}$ pointing in the positive x-direction, so $l = x$ and $\dot{l} = \dot{x}$.

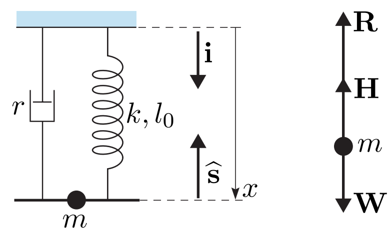


Figure 15 Spring–damper system and force diagram

The forces acting on the baby are the weight $\mathbf{W} = mg\mathbf{i}$, the model spring force $\mathbf{H} = k(x - l_0)(-\mathbf{i})$, and the resistance force $\mathbf{R} = r\dot{x}(-\mathbf{i})$. Newton's second law gives

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{H} + \mathbf{R} = mg\mathbf{i} - k(x - l_0)\mathbf{i} - r\dot{x}\mathbf{i},$$

which leads, after resolution in the $\mathbf{i}$-direction, to the equation of motion

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0.$$

The right-hand side can be written as kx_{eq}, where $x_{\mathrm{eq}} = mg/k + l_0$ is the equilibrium position of the baby. Hence $x_{\mathrm{p}} = x_{\mathrm{eq}}$ is a particular integral of the differential equation.

After substituting the given values for m, k and r, the associated homogeneous differential equation has auxiliary equation

$$10\lambda^2 + 0.2\lambda + 200 = 0,$$

which has solutions $\lambda = -0.01 \pm 4.5i$ (to two significant figures). This leads to the complementary function

$$x_c = Ae^{-0.01t}\cos(4.5t + \phi),$$

where A and ϕ are arbitrary constants, so the general solution is

$$x = x_c + x_p = Ae^{-0.01t}\cos(4.5t + \phi) + x_{eq}.$$

This represents decaying oscillations about the equilibrium position.

In order to find the particular solution, we need the initial conditions. Since the baby is released from rest at 0.3 m above the equilibrium position, the initial conditions are $x(0) = x_{eq} - 0.3$ and $\dot{x}(0) = 0$. These give

$$-0.3 = A\cos\phi \quad \text{and} \quad 0 = A(-0.01\cos\phi - 4.5\sin\phi).$$

Solving these equations leads to $\phi \simeq \pi - 0.0022$ and $A \simeq 0.30$, so the amplitude of the decaying oscillations is about $0.30e^{-0.01t}$ m. This reduces to 0.05 m when $0.30e^{-0.01t} = 0.05$, or $t = 100\ln 6 \simeq 180$ (about 3 minutes).

Exercise 6

A sit-ski is a seat attached above a ski, with a spring–damper suspension (see Figure 16). The spring is chosen according to the weight of the skier.

Figure 16 A sit-ski on a ski slope

The damper can be adjusted according to the skier's weight and the terrain. Suppose that a particular sit-ski suspension is modelled as a model spring–damper system, with a spring of natural length 0.2 m and stiffness $30\,000\,\mathrm{N\,m^{-1}}$, and damping constant $6300\,\mathrm{N\,s\,m^{-1}}$. The skier (plus seat) is modelled as a single particle of mass 60 kg. Take the magnitude of the acceleration due to gravity as $g = 9.81\,\mathrm{m\,s^{-2}}$. Figure 17 represents the sit-ski with skier.

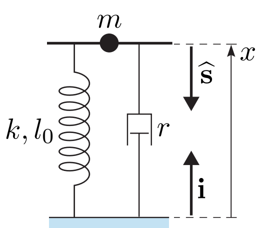


Figure 17 Spring–damper system for the sit-ski and skier

(a) Draw the diagram for the forces acting on the skier. Take the x-axis to be directed upwards, with origin at ground level, and obtain the corresponding equation of motion.

(b) Suppose that the skier is lowered onto the seat and then released from rest when the spring has its natural length. Write down the corresponding initial conditions for the skier's subsequent motion.

(c) Given that the particular solution of the equation of motion is $x = x_{\text{eq}} + 0.021e^{-5t} - 0.001e^{-100t}$ (in metres), find approximately how long the model predicts that it will take before the skier's displacement is within $0.001\,\text{m}$ ($1\,\text{mm}$) of the equilibrium position.

We look next at a different type of situation, although it leads to an equation of motion that is very similar to those seen already.

The fuel gauge for a vehicle is connected to a mechanism that monitors the level of fuel in the tank. This mechanism is designed to damp oscillations in the gauge reading after disturbances such as those caused by travel over bumps on the road. The mechanism includes a float on the surface of the liquid in the fuel tank, and the buoyancy force of the liquid on this float acts in an analogous way to a spring.

You first met buoyancy in Unit 9, Exercise 19 and the text above it, in the context of simple harmonic motion.

A buoyancy force, sometimes called an *upthrust*, is experienced by any object that is wholly or partly immersed in a liquid. If the object floats in equilibrium on the surface of the liquid, then the upthrust from the liquid balances the weight of the object (see Figure 18). If you push the object down further into the liquid, then there is a greater upthrust pushing it back up. When you lift the object up a little from its equilibrium position, the upthrust is less and the weight of the object pulls it down again.

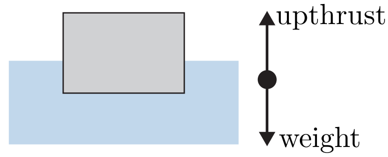


Figure 18 A float on the surface of a liquid

According to Archimedes' principle (see Unit 9), the upthrust is directed vertically upwards and is equal in magnitude to the weight of liquid displaced by the object. It follows that if the object has a constant horizontal cross-section, then the magnitude of the upthrust on it is proportional to the depth of its base below the surface of the liquid. In fact, if the displacement of the base (measured downwards from the surface of the liquid) is $x\mathbf{i}$, then the upthrust is $-kx\mathbf{i}$, where k is a constant that depends on the density of the liquid and the cross-sectional area of the floating object. (For this model to be valid, the object must be at least partly immersed in the liquid, but not wholly submerged.)

In fact, $k = \rho A g$, where ρ is the liquid density, A is the cross-sectional area, and g is the magnitude of the acceleration due to gravity.

Hence the float system in a vehicle's fuel tank can be modelled by a model spring–damper system, provided that the length of the spring is regarded as being equal to its natural length when its non-fixed end is at the level specified by the liquid surface. However, it may be more straightforward to write down the upthrust force directly, as in Example 4 below. The return to equilibrium should be rapid, for ease of reading the fuel gauge, hence the system requires near-critical damping.

Example 4

The mechanism in a particular fuel tank is to be modelled by a model spring–damper system. The upthrust from the liquid fuel on the float is equal in magnitude to twice the weight of the float times the proportion of the float below the surface. The mass of the float is $m = 0.1$, its vertical height is $d = 0.01$, and the damping constant is $r = 21$. Take $g = 9.81\,\mathrm{m\,s^{-2}}$.

This formula for the upthrust is specific to this example. It corresponds to the equilibrium position of the float being half in and half out of the fuel.

Find the equation of motion, and solve it for the case in which the motion begins with the base of the float 0.01 m below the surface, with zero velocity. Hence predict when the float will be less than 0.001 m (1 mm) from its equilibrium position, assuming that there is no further disturbance.

Solution

Take x as the downward displacement of the bottom of the float from the surface of the liquid (see Figure 19). Then the unit vector $\mathbf{i}$ points downwards, and the unit vector $\widehat{\mathbf{s}}$, for both the upthrust (spring) and the damper, points upwards.

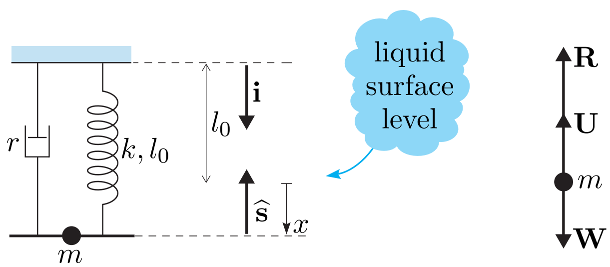


Figure 19 Spring–damper system and force diagram

The upthrust from the liquid is $\mathbf{U} = 2mg(x/d)\widehat{\mathbf{s}} = 2mg(x/d)(-\mathbf{i})$, the weight is $\mathbf{W} = mg\mathbf{i}$, and the damping resistance force is $\mathbf{R} = r\dot{x}\widehat{\mathbf{s}} = r\dot{x}(-\mathbf{i})$. Hence, using Newton's second law, we obtain

From the form of $\mathbf{U}$, the stiffness of the model spring equivalent to the upthrust is $k = 2mg/d$.

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{U} + \mathbf{R} = mg\mathbf{i} - (2mg/d)x\mathbf{i} - r\dot{x}\mathbf{i},$$

which leads to the equation of motion

$$m\ddot{x} + r\dot{x} + (2mg/d)x = mg.$$

Once the parameter values have been substituted, the general solution is found to be

$$x = Ae^{-200t} + Be^{-9.80t} + 0.005,$$

where A and B are constants.

The initial conditions are $x(0) = 0.01$ and $\dot{x}(0) = 0$, which give the particular solution

$$x = 0.005\,26\,e^{-9.80t} - 0.000\,257\,e^{-200t} + 0.005.$$

There are three terms in the solution. The last represents the equilibrium position of 0.005 m. The second term involves e^{-200t}, which dies away very quickly, and the other term is the dominant term in the variable part of the solution, namely $0.005\,26\,e^{-9.80t}$. This will reduce to 0.001 m when $e^{-9.80t} = 0.190$, that is, when $t = -(\ln 0.190)/9.80 \simeq 0.17$.

Hence the model predicts that the displacement of the float will be within 1 mm of its equilibrium position in less than a fifth of a second.

Exercise 7

Modify the model from Example 4 for a mechanism where the upthrust from the liquid fuel is again equal in magnitude to twice the weight of the float times the proportion of the float below the surface level, and the mass of the float is again $m = 0.1$, but now the vertical height of the float is $d = 0.02$ and the damping constant is $r = 11$. Take $g = 9.81\,\mathrm{m\,s^{-2}}$.

(a) Derive the equation of motion, and write down the initial conditions for the case in which the motion begins with the base of the float at the liquid surface with zero velocity.

(b) Given that the particular solution for the initial conditions described in part (a) is $x = 0.001\,08\,e^{-100.2t} - 0.0111\,e^{-9.79t} + 0.01$, predict when the float will be less than 0.001 m (1 mm) from its equilibrium position, assuming that there is no further disturbance.

Exercise 8

Bathroom scales can be modelled by a model spring–damper system, as shown in Figure 20. If the stiffness of the spring is $k = 50\,000$ and the damping constant is $r = 5000$, what mass of person standing on the scales will give critical damping? What would you expect to happen if a slightly heavier or lighter person stood on the scales?

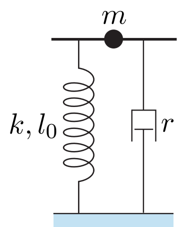


Figure 20 Spring–damper system for bathroom scales

All the models encountered in this subsection have led to equations of motion of the form

$$m\ddot{x} + r\dot{x} + kx = kx_{\mathrm{eq}}, \tag{10}$$

where x_{eq} is the equilibrium position of the particle. The following points should now be apparent.

- In equation (10), only the expression for x_{eq} depends on the choices of origin and direction for the x-axis.
- If the origin is chosen to be the equilibrium position of the particle, then equation (10) reduces to its homogeneous form

$$m\ddot{x} + r\dot{x} + kx = 0. \tag{11}$$

- Given any solution of equation (11), there is a corresponding solution of equation (10) obtainable by adding the constant x_{eq}.

In fact, $x_{\mathrm{p}} = x_{\mathrm{eq}}$ is a particular integral for equation (10).

Hence equation (11) always describes the motion of the system *relative to the equilibrium position*. In the next subsection, we summarise the mathematical possibilities that arise when solving this differential equation.

We have concentrated on systems with a single model spring and a single model damper, but it is easy to extend the results to situations where more than one model spring or damper is present, acting as before along an x-axis. All that needs to be done is to add an appropriate force term to the right-hand side of Newton's second law for each component present. For example, if there are two model springs, with stiffnesses k_1 and k_2, then for motion relative to the equilibrium position we again obtain equation (11), with $k = k_1 + k_2$. In other words, the combined effect of the two model springs is equivalent to that of a single model spring with stiffness $k_1 + k_2$. A similar result holds for model dampers: the combined effect of two model dampers, with damping constants r_1 and r_2, is equivalent to that of a single model damper with damping constant $r_1 + r_2$.

2.3 Weak, critical and strong damping

A mechanical system whose equation of motion is of the form (10) or (11) is called a **damped linear harmonic oscillator**, or **damped harmonic oscillator** for short. In the absence of damping, the equation reduces to that of a simple harmonic oscillator, whose motion you studied in Unit 9.

During this subsection, you may like to refer to Procedure 5 in Unit 1.

In this subsection we return to the types of behaviour that can occur for a damped harmonic oscillator that we saw in Section 1. We concentrate on the equation of motion (11), for which the motion of the particle is described relative to its equilibrium position. The auxiliary equation for equation (11) is

$$m\lambda^2 + r\lambda + k = 0,$$

whose roots are

$$\lambda_1 = \frac{-r + \sqrt{r^2 - 4mk}}{2m} \quad \text{and} \quad \lambda_2 = \frac{-r - \sqrt{r^2 - 4mk}}{2m}. \tag{12}$$

The form of solution falls into one of three types, depending on whether the expression $r^2 - 4mk$ is positive, negative or zero. These three cases correspond respectively to strong, weak and critical damping.

Before examining each case in turn, we introduce the **damping ratio**

Note that by definition, $\alpha > 0$.

$$\alpha = \sqrt{\frac{r^2}{4mk}} = \frac{r}{2\sqrt{mk}}, \tag{13}$$

which makes some of the mathematical descriptions more transparent. For example, since $r^2 - 4mk = 4mk(\alpha^2 - 1)$, the conditions for strong, weak and critical damping can be expressed in terms of the damping ratio as $\alpha > 1$, $\alpha < 1$ and $\alpha = 1$, respectively. Whereas the damping *constant* r provides an absolute value for the damping force per unit speed exerted on the particle, the damping *ratio* α gives a measure of how important damping is *relative* to the mass m and spring stiffness k of the system.

Note that α is a dimensionless quantity, since the dimensions of r are the same as those of $\sqrt{mk}$.

Exercise 9

Increasing the damping constant r (while keeping the mass m and spring stiffness k fixed) will increase the damping ratio α. What other changes in parameters will increase α?

Exercise 10

For the spring–damper system in Exercise 1, the mass was $m = 0.711$ and the spring stiffness was $k = 23$. The damping constants r for tubes A–D were, respectively, 0.15, 0.92, 1.33, 8.42. Find the corresponding damping ratios, and verify that tubes A–C provide weak damping while tube D provides strong damping.

We will now look in turn at each of the three cases of damping identified above, but first, here is a reminder of the situation with no damping that you saw in Unit 9.

No damping

If there is no damping, then we have $r = 0$ and $\alpha = 0$. From equations (12), the roots of the auxiliary equation are

$$\lambda_1 = \frac{\sqrt{-4mk}}{2m} = i\sqrt{\frac{k}{m}} \quad \text{and} \quad \lambda_2 = -\frac{\sqrt{-4mk}}{2m} = -i\sqrt{\frac{k}{m}}.$$

The motion is simple harmonic, as described by the solution

$$x(t) = A\cos(\omega t + \phi),$$

where A and ϕ are arbitrary constants, and $\omega = \sqrt{k/m}$ is the *natural* (undamped) *angular frequency.* In practice, we restrict A to be positive, and call it the *amplitude* of the motion. The *period* of the motion is $\tau = 2\pi/\omega$, and ϕ (restricted to the range $-\pi < \phi \le \pi$) is called the *phase angle.* A graph of $x(t)$, with $\phi = 0$, is shown in Figure 21.

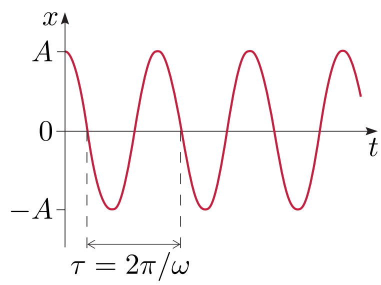


Figure 21 No damping

Weak damping

For weak damping, $r^2 - 4mk < 0$ and $\alpha < 1$. The solution of equation (11) is the product of a decaying exponential and a sinusoidal function, namely

$$x(t) = Ae^{-\rho t}\cos(\nu t + \phi), \tag{14}$$

where $\rho = r/(2m)$, $\nu = \sqrt{4mk - r^2}/(2m)$, and A and ϕ are arbitrary constants. (As before, we restrict A to be positive, and the phase angle ϕ to be within the range $-\pi < \phi \le \pi$.) A graph of this motion is shown in Figure 22.

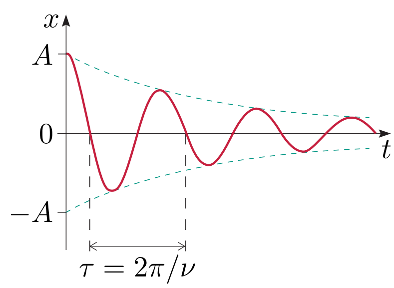


Figure 22 Weak damping

For simple harmonic motion, the period is the time to complete one cycle. Here the motion does not repeat itself, so we generalise the definition. The **period** τ of the motion is the time between successive zeros of x where the particle is moving in the same direction. In equation (14), the angular frequency is ν and the period is $\tau = 2\pi/\nu$.

To investigate equation (14) further, we write the parameters ρ and ν in terms of the damping ratio $\alpha = r/(2\sqrt{mk})$ and the natural angular frequency $\omega = \sqrt{k/m}$. To do this, we first note that

$$\omega\alpha = \sqrt{\frac{k}{m}} \times \frac{r}{2\sqrt{mk}} = \frac{r}{2m} = \rho.$$

The angular frequency ν can also be written in terms of ω and α:

$$\nu = \frac{\sqrt{4mk - r^2}}{2m} = \frac{2\sqrt{mk}\sqrt{1 - r^2/(4mk)}}{2m} = \sqrt{\frac{k}{m}}\sqrt{1-\alpha^2}$$
$$= \omega\sqrt{1-\alpha^2}.$$

The broken curves in Figure 22, corresponding to the two graphs $x = \pm Ae^{-\omega\alpha t}$, indicate how quickly the oscillations decay (larger α gives more rapid decay). The angular frequency $\nu = \omega\sqrt{1-\alpha^2}$ is less than the natural angular frequency ω, so the period $\tau = 2\pi/\nu$ is greater than the period of undamped oscillations (becoming larger as α increases). If α is close to zero, then the period τ is very close to its undamped value $2\pi/\omega$ (because τ depends on the square of α, namely $\tau = 2\pi/(\omega\sqrt{1-\alpha^2})$).

The negative exponent $-\omega\alpha t$ ensures that $Ae^{-\omega\alpha t}$ is a decreasing function, since $A > 0$.

This effect was apparent in Table 2, where the periods for tubes A–C were very close to the undamped value of 1.10 seconds.

For simple harmonic motion, the amplitude is the constant maximum displacement from the mean position. In this motion, the maximum displacement from the mean position is not constant, so we define the **amplitude** to be the positive and continually changing quantity $Ae^{-\omega\alpha t}$. Over one cycle, of period τ, the amplitude of the motion decreases from $Ae^{-\omega\alpha t}$ to $Ae^{-\omega\alpha(t+\tau)}$, which is equivalent to $Ae^{-\omega\alpha t}e^{-\omega\alpha\tau}$, so $Ae^{-\omega\alpha t}$ is multiplied by the factor

$$e^{-\omega\alpha\tau} = \exp\left(-\frac{2\pi\alpha}{\sqrt{1-\alpha^2}}\right).$$

This is the amplitude decay factor per cycle, which was used to make predictions for the magnet motion within tubes A–C in Table 2.

Exercise 11

Suppose that the period of a weakly damped harmonic oscillator is greater by 10% than the corresponding undamped period $2\pi/\omega$. Show that the amplitude of the motion decays by a factor of about 0.056 per cycle.

A conclusion from the result of Exercise 11 is that if the period is very different from the undamped value, then few oscillations will be visible before the motion dies away.

Strong damping

Now consider the solution of equation (11) when $r^2 - 4mk > 0$, that is, when $\alpha > 1$. Here the auxiliary equation has real roots given by equations (12), namely

$$\lambda_1 = \frac{-r + \sqrt{r^2 - 4mk}}{2m} \quad \text{and} \quad \lambda_2 = \frac{-r - \sqrt{r^2 - 4mk}}{2m},$$

and the corresponding solution is

$$x(t) = Be^{\lambda_1 t} + Ce^{\lambda_2 t}, \tag{15}$$

where B and C are arbitrary constants. In terms of the natural angular frequency ω and the damping ratio α, the roots can be written as

$$\lambda_1 = \omega(-\alpha + \sqrt{\alpha^2 - 1}\,) \quad \text{and} \quad \lambda_2 = \omega(-\alpha - \sqrt{\alpha^2 - 1}\,).$$

Both λ_1 and λ_2 are negative since $\alpha > \sqrt{\alpha^2 - 1}$. Hence solution (15) is a sum of two *decaying* exponentials. Solutions of this type predict a return to the equilibrium position without oscillation (see Figure 23), although the graph of x against t may cross the t-axis once.

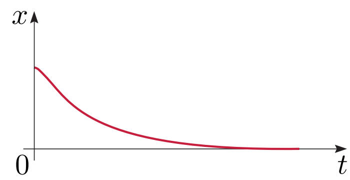


Figure 23 Strong damping

Exercise 12

As α increases from 1, what happens to the values of λ_1 and λ_2? Which of the exponential terms in equation (15) will be the dominant term for very strongly damped systems?

Critical damping

Finally, we consider the solution of equation (11) when $r^2 - 4mk = 0$, that is, when $\alpha = 1$. Here we have equal roots of the auxiliary equation, namely

$$\lambda_1 = \lambda_2 = -\frac{r}{2m} = -\omega\alpha = -\omega.$$

From Procedure 5 of Unit 1, the corresponding solution is

$$x(t) = (Bt + C)e^{-\omega t},$$

where B and C are arbitrary constants. Solutions of this form do not represent oscillations, although the graph of x against t may cross the t-axis once if the initial conditions are such that $t = -C/B > 0$. The graph (see Figure 24) resembles that for strong damping, but the system returns more quickly to close to the equilibrium position.

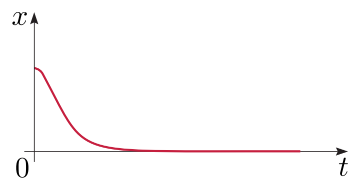


Figure 24 Critical damping

In conclusion, note that there is a continuum of behaviour from weak damping, with marked decaying oscillations, through near-critical damping, with a rapid return towards the equilibrium position, to strong damping, with a slower return towards the equilibrium position. Figures 21 to 24 show snapshots of the possible behaviour along this continuum.

3 Forcing the oscillations

All the damped solutions obtained in Section 2 have the property that they tend over time towards the equilibrium position of the particle, with the motion dying away. Sometimes, in addition to the effect of the model spring and damper, we need to model a further force that acts to keep the system moving. For example, the sit-ski may go over bumpy terrain or an adult may push the child in the baby bouncer. The motion of a model spring–damper system that is subjected to an additional time-dependent force is said to be **forced**.

There are several distinct ways of providing the forcing to a damped particle–spring system. In Subsection 3.1 we look at the case in which a periodic force is applied to the particle itself. This could, for example, model the regular forcing of the baby bouncer motion by pushing the child. In Subsection 3.2 we turn to alternative possibilities in which the force arises due to the prescribed displacement of some point of the system other than the particle itself. The end of the model spring or damper not attached to the particle would be such a point. For example, the baby bouncer motion could be forced by the action of a motor that moves the top end of the spring up and down. For the sit-ski suspension, it is the base of the model spring–damper that is displaced, due to contact with an uneven surface beneath. In Subsection 3.3 we summarise mathematically how the amended equation of motion may be solved.

3.1 Direct forcing

As a first model, we assume that the forcing is not just periodic, but sinusoidal. In Unit 13 you will see that any periodic force can be expressed as a sum of sinusoidal terms of different frequencies. It follows from the principle of superposition that if we can find a particular integral of the equation of motion for a 'typical' sinusoidal input, then by taking an appropriate sum of such solutions, we obtain the particular integral for any periodic input. The assumption of sinusoidal forcing is not therefore as restrictive as it might seem initially.

See Unit 1, Theorem 2.

Example 5

Consider the baby bouncer described in Example 3, with spring stiffness $k = 200$ and damping constant $r = 0.2$. The baby plus parts of the apparatus suspended from the spring have mass 10 kg. Suppose that by alternately pushing downwards and pulling upwards on the baby, an adult exerts a direct sinusoidal force of amplitude 10 newtons and frequency 1 hertz (1 cycle per second).

(a) Modelling the baby as a particle, formulate the equation of motion for the baby.

(b) Find the general solution of the equation of motion, and interpret this to predict what motion the baby will undergo in the long term. Is this affected by the initial conditions?

Solution

(a) The model assumes that the baby does not touch the ground and that the motion is completely vertical. The model spring–damper diagram for the system is shown in Figure 25, along with the corresponding force diagram. The origin is taken, as in Example 3, to be at the top of the spring, with the x-axis and unit vector $\mathbf{i}$ pointing downwards. The unit vector $\widehat{\mathbf{s}}$ for both the spring and the damper points upwards.

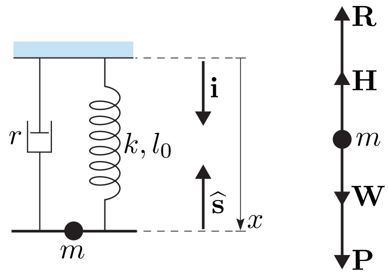


Figure 25 Spring–damper system and force diagram for the baby bouncer

There are four forces to consider. Three of these are the same as in Example 3, namely the weight $\mathbf{W} = mg\mathbf{i}$, the model spring force $\mathbf{H} = k(x - l_0)\,\widehat{\mathbf{s}} = k(x - l_0)(-\mathbf{i})$, and the resistance force $\mathbf{R} = r\dot{x}\,\widehat{\mathbf{s}} = r\dot{x}(-\mathbf{i})$. In addition, we have the sinusoidal force provided by the adult. With a suitable choice of time origin, this can be represented as $\mathbf{P} = P\cos(\Omega t)\,\mathbf{i}$, where the amplitude is $P = 10$ and the angular frequency is $\Omega = 2\pi$ (corresponding to 1 hertz). Newton's second law gives

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{H} + \mathbf{R} + \mathbf{P} = mg\mathbf{i} - k(x - l_0)\mathbf{i} - r\dot{x}\mathbf{i} + P\cos(\Omega t)\,\mathbf{i},$$

which leads, after resolution in the $\mathbf{i}$-direction, to the equation of motion

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0 + P\cos(\Omega t).$$

The first two terms on the right-hand side can again be written as kx_{eq}, where $x_{\text{eq}} = mg/k + l_0$ is the equilibrium position of the baby in the absence of forcing. Substituting the given values for m, k, r, P and Ω in the equation of motion, we obtain

$$10\ddot{x} + 0.2\dot{x} + 200x = 200x_{\text{eq}} + 10\cos(2\pi t). \qquad (16)$$

If the equilibrium position were chosen as the origin, then there would be no constant term on the right-hand side of the equation of motion.

(b) The complementary function is the same as that found in Example 3, which was

$$x_{\text{c}} = Ae^{-0.01t}\cos(4.5t + \phi),$$

where A and ϕ are arbitrary constants that are determined by the initial conditions. The particular integral will be the sum of two terms, the first of which (again as in Example 3) is the constant x_{eq}. According to the principle of superposition, we need to add to this a particular integral corresponding to the sinusoidal term on the right-hand side of equation (16). With a trial function of the form

See Unit 1, Subsection 3.2.

$$x_{\text{p}} = B\cos(2\pi t) + C\sin(2\pi t),$$

where B and C are constants, we find $B = -5.1337 \times 10^{-2}$ and $C = 3.3120 \times 10^{-4}$, so

$$x_{\text{p}} = -5.1337 \times 10^{-2}\cos(2\pi t) + 3.3120 \times 10^{-4}\sin(2\pi t).$$

This can also be written in the alternative sinusoidal form as

$$x_{\text{p}} = 5.1338 \times 10^{-2}\cos(2\pi t - 3.1351).$$

When the numerical values are rounded to two significant figures, the general solution of the equation of motion becomes

$$x = Ae^{-0.01t}\cos(4.5t + \phi) + x_{\text{eq}} + 5.1 \times 10^{-2}\cos(2\pi t - 3.1).$$

Whatever the initial conditions are, the magnitude of the complementary function will decay gradually towards zero, as observed in Section 2. The initial conditions therefore have no influence on the long-term behaviour. After a long time, the motion will be given simply by the remainder of the general solution,

$$x = x_{\text{eq}} + 5.1 \times 10^{-2} \cos(2\pi t - 3.1).$$

The predicted motion settles down to steady oscillations about the mean position x_{eq}, with amplitude 5.1×10^{-2} m (about 5 cm). These oscillations have the same angular frequency 2π as the input sinusoidal force $10\cos(2\pi t)\,\mathbf{i}$, but the output is out of phase with the input by almost π. This means that when the displacement is at its maximum (at the lowest point for the baby), the **i**-component of the force exerted by the adult is at its minimum, and vice versa. The baby reaches the highest point as the adult pushes down hardest, reaches the lowest point as the adult pulls up hardest, and passes through the equilibrium position as the adult momentarily exerts no force.

This example shows several features that are typical of such forcing problems. The complementary function dies away with time, regardless of the initial conditions, since in all cases the complementary function corresponds to one of the damped but unforced systems seen in Section 2. For this reason, the complementary function is referred to in this context as the **transient** part of the solution, while the remainder (corresponding to the particular integral, which does not die away) is called the **steady-state** solution of the equation of motion. The particular integral does not depend on the initial conditions, so neither does the steady-state behaviour. Another common feature is that the frequency of the steady-state solution is the same as that of the sinusoidal input force, but the output is out of phase with the input.

Exercise 13

(a) Without performing any detailed calculations or algebra, say how the solution to Example 5 would alter if the amplitude of the forcing oscillations were reduced to 2 N, say, by the baby pushing with its feet on the floor rather than being pushed by an adult.

(b) Suppose that a heavier or lighter baby is placed in the baby bouncer. Without going into details, say what aspects of the long-term motion predicted in Example 5 will alter, and what will remain the same.

Exercise 14

Bathroom scales are modelled by a model spring–damper system. The spring stiffness is k, and the damping constant is r. A girl of mass m is on the scales. By alternately pushing down against and pulling up on an adjacent towel rail, she manages to alter her effective weight on the scales by an amount modelled as an input force $\mathbf{P} = mg\cos(\Omega t)\,\mathbf{i}$, where the **i**-direction is upwards.

Draw the force diagram for this situation. Taking the origin at the base of the model spring, obtain the equation of motion. What long-term behaviour is predicted by this model?

You have seen that the effect of a directly applied sinusoidal force on a model spring–damper system is modelled by an equation of motion of the form

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + P\cos(\Omega t), \tag{17}$$

where P and Ω are the amplitude and angular frequency of the applied force, and x_{eq} is the equilibrium position of the particle in the absence of forcing.

Initially, the behaviour of the system depends on both the particular integral and the complementary function of equation (17), with the arbitrary constants in the complementary function being determined by the initial conditions of the situation. However, the complementary function dies away with time (is transient), and the particular solution then takes the form (equal to the particular integral)

$$x = x_{\text{eq}} + B\cos(\Omega t) + C\sin(\Omega t) = x_{\text{eq}} + A\cos(\Omega t + \phi).$$

This represents sinusoidal oscillations about the equilibrium position, which are independent of the initial conditions.

The model predicts that the steady-state output oscillations have the same frequency as the input forcing, but a different phase. The values of the output amplitude A and phase angle ϕ depend on the configuration of the system, and on the values of the parameters m, k, r, P and Ω.

3.2 Forcing by displacement

In this subsection we look at examples of spring–damper systems in which the forcing is not applied directly to the particle where the mass of the system is concentrated. Suppose, for instance, that the baby bouncer from Example 3 is adapted by the addition of a motor at the top, which moves the top of the spring up and down sinusoidally with time. This situation is represented in Figure 26. The force is not applied directly to the baby, but is applied to the top end of the spring in such a way that the top end of the spring is forced to oscillate.

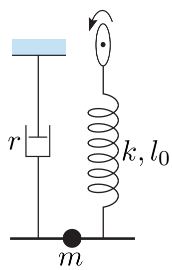


Figure 26 The top of the spring is forced to vibrate

The precise form of the model spring force exerted on the particle that represents the baby and bouncer must differ from that seen before, because now both ends of the spring are in motion. However, Hooke's law continues to apply, so the force on the particle due to the model spring is

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}},$$

where the spring has stiffness k and natural length l_0, and $\widehat{\mathbf{s}}$ is a unit vector in the direction from the particle towards the centre of the spring. The novel aspect introduced by this new situation is how the spring length l is changing. The following example illustrates how this is used.

Example 6

Consider the baby bouncer described in Examples 3 and 5, with spring stiffness $k = 200$ and damping constant $r = 0.2$. As before, the baby plus parts of the apparatus suspended from the spring has mass 10 kg. Suppose that a small motor causes the top of the spring to oscillate sinusoidally, with amplitude 0.04 m and period 1 s. Suppose also that the damping is regarded as due to air resistance alone, so that the top of the model damper remains fixed.

(a) Derive the equation of motion for the particle representing the baby and bouncer.

(b) Find the general solution of the equation of motion, and interpret this to predict what motion the baby will undergo in the long term.

(c) Say in general terms what would happen if the period of the oscillations at the top of the spring were to be changed to 2 s.

Solution

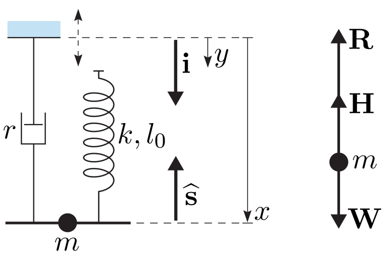


Figure 27 Forcing the top of the baby bouncer to vibrate

(a) As before, the model assumes that the baby does not touch the ground. The configuration of the apparatus is shown in Figure 27, along with the force diagram. The origin is chosen to be at the mean position of the top of the spring, and the directions of the x-axis and unit vector $\mathbf{i}$ are downwards. The position of the top of the spring is then given by $y\mathbf{i}$, where we take $y = a\cos(\Omega t)$. The particular given oscillations have amplitude $a = 0.04$ and angular frequency $\Omega = 2\pi$. (Note that we are using a to denote amplitude here, not acceleration.)

There are three forces to consider. Two of these are as in the previous analyses of the baby bouncer, namely, the weight $\mathbf{W} = mg\mathbf{i}$ and the resistance force $\mathbf{R} = r\dot{x}(-\mathbf{i})$. Since the model spring has length $l = x - y$, the force that it exerts on the particle is

$$\mathbf{H} = k(l - l_0)\widehat{\mathbf{s}} = k(x - y - l_0)(-\mathbf{i}).$$

Newton's second law gives

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{H} + \mathbf{R} = mg\mathbf{i} - k(x - y - l_0)\mathbf{i} - r\dot{x}\mathbf{i},$$

which leads to the equation of motion

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0 + ky.$$

On putting $x_{\text{eq}} = mg/k + l_0$ and $y = a\cos(\Omega t)$, this becomes

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + ak\cos(\Omega t). \tag{18}$$

On substituting in the given values for m, k, r, a and Ω, we obtain

$$10\ddot{x} + 0.2\dot{x} + 200x = 200x_{\text{eq}} + 8\cos(2\pi t).$$

(b) The equation of motion is almost identical to that obtained in Example 5 (see equation (16)). The only difference is that the amplitude of the sinusoidal term on the right-hand side is 8, rather than 10. Thus the solution here is obtained by taking that in Example 5(b) but scaling the particular integral x_{p} for the sinusoidal term by $\frac{8}{10}$. This gives the steady-state solution

A similar approach was adopted in Exercise 13(a).

$$x = x_{\text{eq}} + 4.1 \times 10^{-2} \cos(2\pi t - 3.1).$$

The baby undergoes oscillations as before, but now of amplitude about 4 cm.

(c) If the period changes to 2 s, then the forcing angular frequency becomes $\Omega = \pi$. The corresponding steady-state solution will also have this angular frequency, but its amplitude and phase angle depend on Ω as well as on m, k, r and a, so it is not possible to deduce what these are from the previous solution. We would need to use a fresh trial function, of the form $B\cos(\pi t) + C\sin(\pi t)$.

Note that in comparison with Example 5, the equation of motion in Example 6 does not involve an additional force. The effect of the motor is modelled entirely by the inclusion of the term y in the expression for the spring force, and the fact that y is a function of time. Although the mechanism is different, Example 6(b) demonstrates that the effect of the motor displacing the top of the model spring is the same as that obtainable by direct sinusoidal forcing with a suitable amplitude and the same angular frequency. Leaving aside the particular values of the parameters, the equation of motion for direct forcing with amplitude P is equation (17), while the equation of motion for prescribed displacement oscillations of amplitude a is equation (18). The two match, provided that we put $P = ak$, and the mathematical solution and interpretation are then essentially the same in either case.

The model of the baby bouncer in Example 6 assumed that the top of the model spring moved, but that the top of the model damper was fixed. Suppose instead that the chief cause of damping is not air resistance but the internal damping in the spring. Then it is appropriate to model the situation by assuming that the top end of the model damper performs the same motion as the top end of the model spring (see Figure 28).

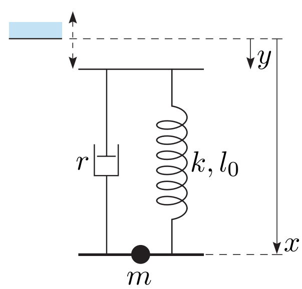


Figure 28 Forcing the spring–damper system to vibrate

This leads to an amended model in which a new expression is required for the damping resistance force. You showed in Exercise 4(b) that if the model damper extends from $x\mathbf{i}$ (where the particle is) to $y\mathbf{i}$, so that the length is $l = x - y$, then the corresponding resistance force on the particle is given by

$$\mathbf{R} = r\dot{l}\,\widehat{\mathbf{s}} = r(\dot{x} - \dot{y})(-\mathbf{i}).$$

Exercise 15

(a) Modify the model of the baby bouncer with motor, from Example 6, to represent the top of the model damper being attached to the top of the model spring and hence experiencing the same forced sinusoidal displacement. How does the equation of motion compare with that for direct forcing?

(b) Describe how you would find the steady-state solution, and hence the type of motion predicted by the model (but do not do the detailed calculations).

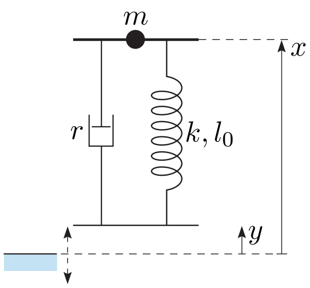


Figure 29 The sit-ski on a rough surface

Exercise 16

The model sit-ski that you considered in Exercise 6 has a spring of natural length 0.2 m and stiffness 30 000 N m^{-1}, and damping constant 6300 N s m^{-1}. During testing, the sit-ski carries a skier plus seat of mass 60 kg. To simulate the effect of travelling over uneven ground, the base of the sit-ski is subjected to regular sinusoidal oscillations, with amplitude 0.1 m and angular frequency π rad s^{-1}, as shown in Figure 29.

Derive the equation of motion, taking the origin to be at the mean level of the sit-ski base. Hence describe in general terms the long-term motion predicted by the model.

See Exercise 19.

You have now seen a case where the model spring alone was subjected to a forcing displacement at the end not attached to the particle, and other cases where the forcing point was located on both the model spring and damper together. A third possibility is where the model damper alone is displaced by a forcing displacement (and you are asked to consider this case at the end of the section). Here again, a very similar equation of motion results, with a sinusoid on the right-hand side that is similar to the corresponding term for direct forcing.

Under certain circumstances, the position x of the particle relative to some fixed point may be of less interest than its position z relative to the forcing point. Since $z = x - y$, it is a simple matter to obtain the solution for z from that for x, since the forcing input term y is a known function. Alternatively, we can set up a differential equation directly for z rather than for x.

For example, the equation

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + r\dot{y} + ky$$

arose in Exercises 15 and 16. On replacing x by $y + z$, this becomes

$$m\ddot{z} + r\dot{z} + kz = kx_{\text{eq}} - m\ddot{y},$$

which with $y = a\cos(\Omega t)$ gives

$$m\ddot{z} + r\dot{z} + kz = kx_{\text{eq}} + ma\Omega^2\cos(\Omega t).$$

This is yet another occurrence of a differential equation of the form (17), so similar comments apply as to the method of solution and output behaviour.

Whether the forcing is applied via a direct sinusoidal force or by sinusoidal displacement to one end of the spring and/or damper, and whether the output is measured relative to a fixed point (x) or to the forcing point (z), the solution of the equation of motion is the sum of two parts: the complementary function (transient), which dies away and becomes negligible, and the particular integral, representing a steady-state sinusoidal oscillation about the equilibrium position, with the same frequency as the input forcing.

The process of finding a particular integral that corresponds to a sinusoidal forcing function is time-consuming, as described so far, and we have gone over this step in full for only one example. A computer can be used to undertake this task, but in the next subsection we indicate how it is also possible to calculate the output amplitude and phase angle more rapidly from suitable formulas.

Exercise 17

Consider the baby bouncer problem of Example 6. The equation of motion is

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + ky,$$

where x is measured from the fixed point (see Figure 27), and the baby bouncer is forced to vibrate using a motor attached to the top of the spring that causes the top of the spring to vibrate sinusoidally with $y = a\cos(\Omega t)$.

Let $z = x - y$ denote the length of the spring. Determine the equation of motion for the baby in terms of z, and find the form of its general solution.

3.3 The steady-state solution

It is tedious to look from scratch for sinusoidal particular integrals for inhomogeneous differential equations. In this subsection we will solve this problem once for a large number of possible cases. While this requires a fair amount of algebraic manipulation, the result obtained saves much further work. It is also a starting point for the discussion in Section 4.

We start from the differential equation

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + P\cos(\Omega t), \tag{19}$$

where m, r, k, x_{eq}, P and Ω are constants. This equation can be regarded as representing all the equations of motion that arose earlier in this section, provided that all of the constants that appear in it (except possibly x_{eq}) are positive. Where there is forcing applied to a model damper, the sinusoidal term on the right-hand side includes a phase shift and hence appears initially as $P\cos(\Omega t + \psi)$. However, such a phase shift can readily be 'transformed away' by moving the time origin from $t = 0$ to $t = -\psi/\Omega$. In a similar way, the kx_{eq} term on the right-hand side of equation (19) can be 'transformed away' by choosing a new origin for x at the equilibrium position, $x = x_{\text{eq}}$.

See Exercises 15, 16 and 19.

This permits us to concentrate on the slightly simpler equation

$$m\ddot{x} + r\dot{x} + kx = P\cos(\Omega t). \qquad (20)$$

Once a solution of this equation has been obtained, we may, if it is desired, express it in terms of the original x- and t-coordinates, by adding x_{eq} to it and by reversing the phase shift (adding ψ to the phase angle obtained).

The general solution of equation (20) will be the sum of a complementary function and a particular integral. The complementary function was found in Subsection 2.3 and, as shown there, it dies away in all cases (is transient). We concentrate here on finding general expressions that determine the particular integral of equation (20). These provide a complete description of the steady-state behaviour of any system represented by equation (20).

The right-hand side of this differential equation is a sinusoid with angular frequency Ω. This means that the particular integral will also be sinusoidal, with the same frequency, so we start with a trial solution

$$x_{\text{p}} = B\cos(\Omega t) + C\sin(\Omega t),$$

where B and C are constants to be found in terms of m, r, k, P and Ω. The first two derivatives of the trial function are

$$\dot{x}_{\text{p}} = -\Omega B\sin(\Omega t) + \Omega C\cos(\Omega t),$$
$$\ddot{x}_{\text{p}} = -\Omega^2 B\cos(\Omega t) - \Omega^2 C\sin(\Omega t).$$

Substituting into equation (20) gives

$$m\big(-\Omega^2 B\cos(\Omega t) - \Omega^2 C\sin(\Omega t)\big) + r\big(-\Omega B\sin(\Omega t) + \Omega C\cos(\Omega t)\big) + k\big(B\cos(\Omega t) + C\sin(\Omega t)\big) = P\cos(\Omega t).$$

We can compare coefficients here because the cosine and sine functions are *linearly independent*: if $c_1\sin c_3x + c_2\cos c_3x = 0$ for all x, then $c_1 = c_2 = 0$.

On equating the coefficients of $\cos(\Omega t)$ and of $\sin(\Omega t)$, we have

$$-m\Omega^2 B + r\Omega C + kB = P, \quad -m\Omega^2 C - r\Omega B + kC = 0.$$

The second of these equations gives $B = (k - m\Omega^2)C/(r\Omega)$, and by substituting this expression for B into the first equation, we obtain

$$-\frac{m\Omega^2(k - m\Omega^2)C}{r\Omega} + r\Omega C + \frac{k(k - m\Omega^2)C}{r\Omega} = P,$$

that is,

$$\frac{\left((k - m\Omega^2)^2 + r^2\Omega^2\right)C}{r\Omega} = P.$$

The required expressions for B and C are therefore

$$C = \frac{Pr\Omega}{(k - m\Omega^2)^2 + r^2\Omega^2} \quad \text{and} \quad B = \frac{P(k - m\Omega^2)}{(k - m\Omega^2)^2 + r^2\Omega^2}. \qquad (21)$$

The alternative formulation for the sinusoidal particular integral of equation (20) is $x_{\text{p}} = A\cos(\Omega t + \phi)$, where the amplitude A of the motion is given by $A = \sqrt{B^2 + C^2}$. Noting that

$$(Pr\Omega)^2 + (P(k - m\Omega^2)^2 = P^2((k - m\Omega^2)^2 + r^2\Omega^2),$$

we have

$$A = \frac{P}{\sqrt{(k - m\Omega^2)^2 + r^2\Omega^2}}. \tag{22}$$

The phase angle ϕ satisfies the pair of equations

$$A\cos\phi = B, \quad A\sin\phi = -C.$$

Now the expression for C is always positive, so $\sin\phi < 0$ and ϕ lies in the third or fourth quadrant. Since arccos is defined to give values in the first or second quadrant, it follows that the formula $\phi = -\arccos(B/A)$ will apply in all cases, that is,

The fact that ϕ is negative shows that the output lags behind the input.

$$\phi = -\arccos\left(\frac{k - m\Omega^2}{\sqrt{(k - m\Omega^2)^2 + r^2\Omega^2}}\right). \tag{23}$$

For any values of the constants m, r, k, P and Ω, the steady-state solution

$$x = B\cos(\Omega t) + C\sin(\Omega t) = A\cos(\Omega t + \phi)$$

is completely determined by either equations (21) (first form) or equations (22) and (23) (second form).

Exercise 18

(a) In Example 5 we studied the baby bouncer with direct forcing applied, for which $m = 10$, $k = 200$, $r = 0.2$, $P = 10$ and $\Omega = 2\pi$. Use equations (22) and (23) to check the values quoted in Example 5 for the amplitude and phase angle of the steady-state solution.

(b) Example 6 concerned the baby bouncer with forcing at the top of the model spring alone. The values of m, k and r were as in part (a). Use equation (22), with $P = ak$ and $a = 0.04$, to find whether the forcing period of 2 s referred to in Example 6(c) would give a greater or lesser steady-state amplitude than the 4 cm that was found for a forcing period of 1 s.

(c) Use equation (22) to estimate the steady-state output amplitude for the sit-ski testing scenario described in Exercise 16. Here $m = 60$, $k = 30\,000$, $r = 6300$, $a = 0.1$ and $\Omega = \pi$. Take $P = a\sqrt{k^2 + r^2\Omega^2}$ (as explained in the solution to Exercise 15(a)).

Suppose that we are looking at a forced displacement of the model spring alone, as in Example 6. Then we have $P = ak$ in equation (22), which gives

In other cases P will also depend on the amplitude a of the input, but in a different manner.

$$\frac{A}{a} = \frac{k}{\sqrt{(k - m\Omega^2)^2 + r^2\Omega^2}}. \tag{24}$$

Now A/a is the ratio of the amplitude A of the steady-state output oscillations to the amplitude a of the input forcing displacement. In other words, $M = A/a$ is the amplitude **magnification factor** caused by the forcing process.

Note that M itself, as the ratio of two lengths, is also dimensionless.

It is possible to write the right-hand side of equation (24) in terms of two dimensionless constants:

- the damping ratio $\alpha = r/(2\sqrt{mk})$, which was introduced in Subsection 2.3
- the ratio $\beta = \Omega/\omega$ of the forcing angular frequency Ω and the natural angular frequency of the system $\omega = \sqrt{k/m}$.

After some algebraic manipulation, we find that the magnification factor $M = A/a$ is given by

$$M = \left((1-\beta^2)^2 + 4\alpha^2\beta^2\right)^{-1/2}. \tag{25}$$

From this formula, we can predict the extent to which input oscillations are magnified in amplitude for any values of α and β. Thus we can also examine how M varies with changes to α and β, which corresponds to investigating how the system response depends on the input forcing frequency and on other parameters of the system. This is done in the next section.

In this section you have seen how to model the behaviour of systems such as damped harmonic oscillators maintained in a state of vibration by some external sinusoidal force or displacement. There are many examples of forced spring–damper systems, in addition to those that have been modelled so far, for instance: a vehicle suspension going over rumble strips, cobbles or a cattle grid; an idling engine causing the body of a stationary bus to vibrate; people walking in step over a bridge. The model predicts that a sinusoidal forcing input produces a sinusoidal output vibration of the same frequency.

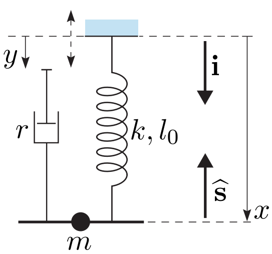


Figure 30 The damper forced to vibrate

Exercise 19

Modify the model of the baby bouncer with motor, from Example 6, to represent the top of the model damper alone being made to undergo a sinusoidal motion, with the top of the model spring held fixed (see Figure 30). Find the form of the equation of motion, without substituting in numerical values. How does this equation of motion compare with the direct forcing equation (19)?

4 Forced vibrations and resonance

At the end of Section 3 we derived alternative formulas (equations (24) and (25)) for the magnification factor by which input oscillations are enlarged in amplitude by a particular type of spring–damper system. As well as providing amplitude output values in specific cases, this enables us to study how the magnification factor depends on the input parameters, and especially on the forcing angular frequency.

It turns out that for some systems, there is a marked peak in amplitude magnification close to a certain forcing angular frequency. This phenomenon is known as *resonance*, and we take a look at this in Subsection 4.1. In Subsection 4.2, the particle–spring system with magnetic damping from Section 1 is modified by a forcing motor at the top of the spring. The model constructed in Section 3 can be applied to predict what magnification factor would occur for certain input frequencies, and the estimates obtained may be compared with experimental outcomes.

4.1 Resonance

Consider once more a spring–damper system in which the forcing point is attached to the model spring only, while the model damper has one end fixed. As pointed out at the end of Section 3, the magnification factor $M = A/a$, from the input forced displacement amplitude a to the steady-state output amplitude A, is given for such a system by

The baby bouncer of Example 6 is such a system.

$$M = \frac{k}{\sqrt{(k - m\Omega^2)^2 + r^2\Omega^2}} = \left((1-\beta^2)^2 + 4\alpha^2\beta^2\right)^{-1/2}, \qquad (26)$$

See equations (24) and (25).

where $\alpha = r/(2\sqrt{mk})$ and $\beta = \Omega/\omega$. Here the system has mass m, spring stiffness k, damping constant r and natural angular frequency $\omega = \sqrt{k/m}$. The input forcing angular frequency is Ω.

For a given damping ratio α (and hence for given values of the system parameters m, k and r), the magnification factor M is a function of β. From the graph of this function, you can read off the rough magnification factor for the system for any input forcing angular frequency. Several graphs of M against β, for different fixed values of α, are shown in Figure 31.

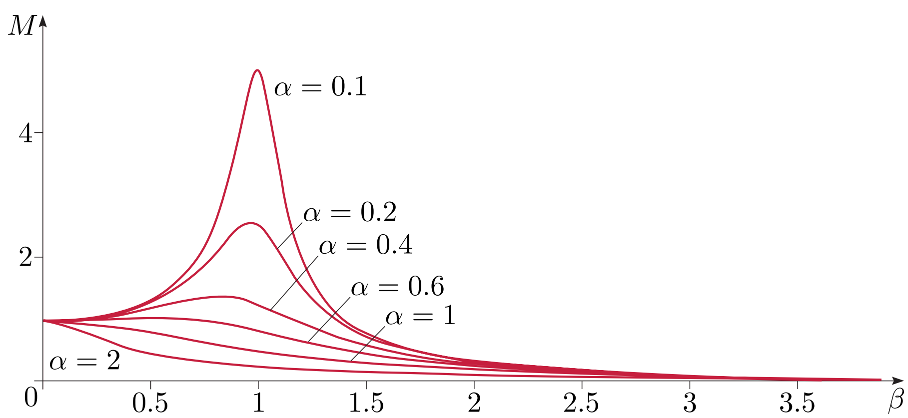


Figure 31 Graphs of the magnification factor M against the ratio $\beta = \Omega/\omega$ for different value of the damping ratio α

The graphs predict that for some (but not all) damping ratios, there is a maximum magnification factor at a certain positive forcing frequency, and this phenomenon is called **resonance**.

If the damping is strong ($\alpha > 1$) or critical ($\alpha = 1$), then no resonance occurs, and the magnification factor decreases throughout as β increases. Nor does resonance occur with weak damping, unless the value of α is beneath a particular threshold level. For smaller values of α, however, there is resonance, and its effect becomes more and more significant as α decreases towards zero. Note that for small values of α, resonance occurs in the vicinity of $\beta = 1$, that is, when the forcing angular frequency Ω is close to the natural angular frequency ω of the system.

You will see below that this threshold value is
$$\alpha = 1/\sqrt{2} \simeq 0.7.$$

Now that we have seen from Figure 31 that the model predicts the phenomenon of resonance, let us see if we can derive this analytically directly from equation (26). We consider $M = M(\beta)$ to be a function of β, and we wish to find the maximum magnification as β varies. The easiest way to do this is to recognise $M(\beta)$ as a composite function, that is, $M(\beta) = f(g(\beta^2))$, where

Another approach is to differentiate $M(\beta)$ directly.

$$f(u) = u^{-1/2} \quad \text{and} \quad g(x) = (1-x)^2 + 4\alpha^2 x.$$

Now $f(u)$ is a strictly decreasing function for positive u, with no local maxima or minima (as shown in Figure 32). Also, $g(\beta^2)$ is always positive, since $\alpha > 0$ and $\beta^2 > 0$. So a *minimum* of $g(x)$ corresponds to a *maximum* of $f(g(x))$. Thus we need to find the minima of $g(x)$.

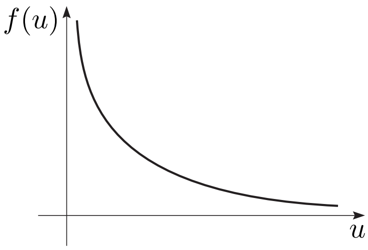


Figure 32 Graph of $f(u) = u^{-1/2}$

Now consider $g(x)$, which is a quadratic with a positive coefficient of x^2 and so has a single local minimum (which is also the global minimum). To find the location of the minimum, we differentiate $g(x)$:

$$\begin{aligned} g'(x) &= 2(1-x) \times (-1) + 4\alpha^2 \\ &= 2x + 4\alpha^2 - 2. \end{aligned}$$

So the derivative is zero when $x = 1 - 2\alpha^2$. (Alternatively, the minimum can be found without calculus, by completing the square.)

Putting the above results together gives that $f(g(x))$ has a unique global maximum at $x = 1 - 2\alpha^2$. So $M(\beta) = f(g(\beta^2))$ has a single maximum when $\beta^2 = 1 - 2\alpha^2$. Note that if $1 - 2\alpha^2 < 0$, then there are no real values of β satisfying this equation. These results are worth remembering.

Resonance frequency

The frequency at which resonance occurs is given by

$$\beta = \sqrt{1 - 2\alpha^2}. \tag{27}$$

Resonance can occur when (and only when) $1 - 2\alpha^2 > 0$.

By substituting in this value, we can show that the maximum magnification factor is given by
$$M = \frac{1}{2\alpha\sqrt{1-\alpha^2}}.$$

Let us now return to the baby bouncer example.

Exercise 20

The baby bouncer from Example 6, with a baby plus seat of mass 10 kg, has damping ratio

$$\alpha = \frac{r}{2\sqrt{mk}} = \frac{0.2}{2\sqrt{10 \times 200}} \simeq 0.002,$$

and natural angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{10}} \simeq 4.47.$$

What amplitude of oscillations for the baby is predicted for a forcing input (at the top of the spring) with amplitude $a = 0.04$ and angular frequency ω? Comment on your result.

Resonance is found in many physical systems, and in some is desirable. Thus a radio receiver may be tuned to an input signal of a particular carrier frequency, and inputs of other carrier frequencies have much lower magnification factor at that frequency. However, there are other situations where resonance is most undesirable. In the case of a vehicle suspension, a large magnification factor when the vehicle goes over regular bumps would certainly be uncomfortable, and might also be dangerous and destructive.

4.2 Forcing in practice

Consider again the experiments discussed in Section 1. Here we consider the addition of a motor to the particle–spring system with magnetic damping.

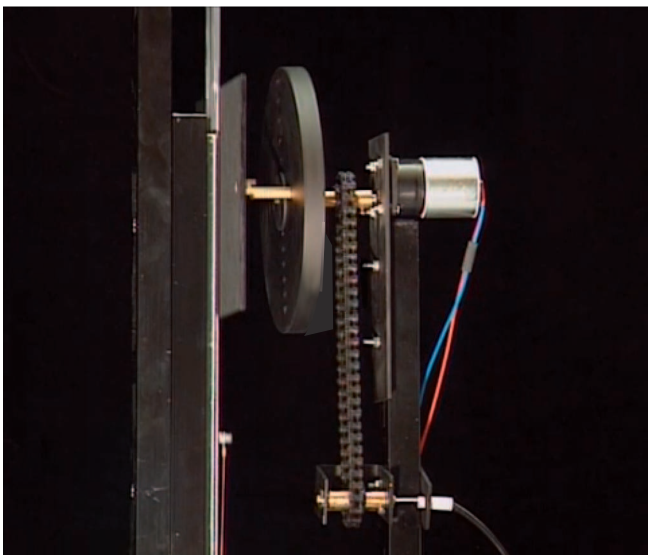


Figure 33 The addition of a motor to force the spring–mass system to oscillate

This motor has the effect of forcing the top of the spring to undergo a sinusoidal displacement, while the tubes that cause the damping remain fixed. An appropriate model for this situation is the same as that developed for the baby bouncer in Example 6 (see Figure 27). The corresponding equation of motion is as before,

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + ak\cos(\Omega t),$$

if the origin is taken at the mean position of the top of the spring. However, with the alternative choice of origin at the equilibrium position of the particle, this becomes

$$m\ddot{x} + r\dot{x} + kx = ak\cos(\Omega t),$$

which is of the form of equation (20) with $P = ak$. Equation (26) gives the magnification factor, which is the ratio of the amplitude of the sinusoidal particular integral to the amplitude of the input forcing displacement.

The experiments indicated that the results for tubes B and C were similar, so we look at only tube C here.

Examine the forced motion for each of tubes A, C and D, for different sets of initial conditions. The model predicts that the transient part of the solution dies away with time, leaving a sinusoidal steady-state solution. This is illustrated in Figure 34 for each of the tubes, where the input forcing has angular frequency $\Omega = \frac{4}{3}\pi$ (equivalent to $\frac{2}{3}$ hertz or 40 cycles per minute).

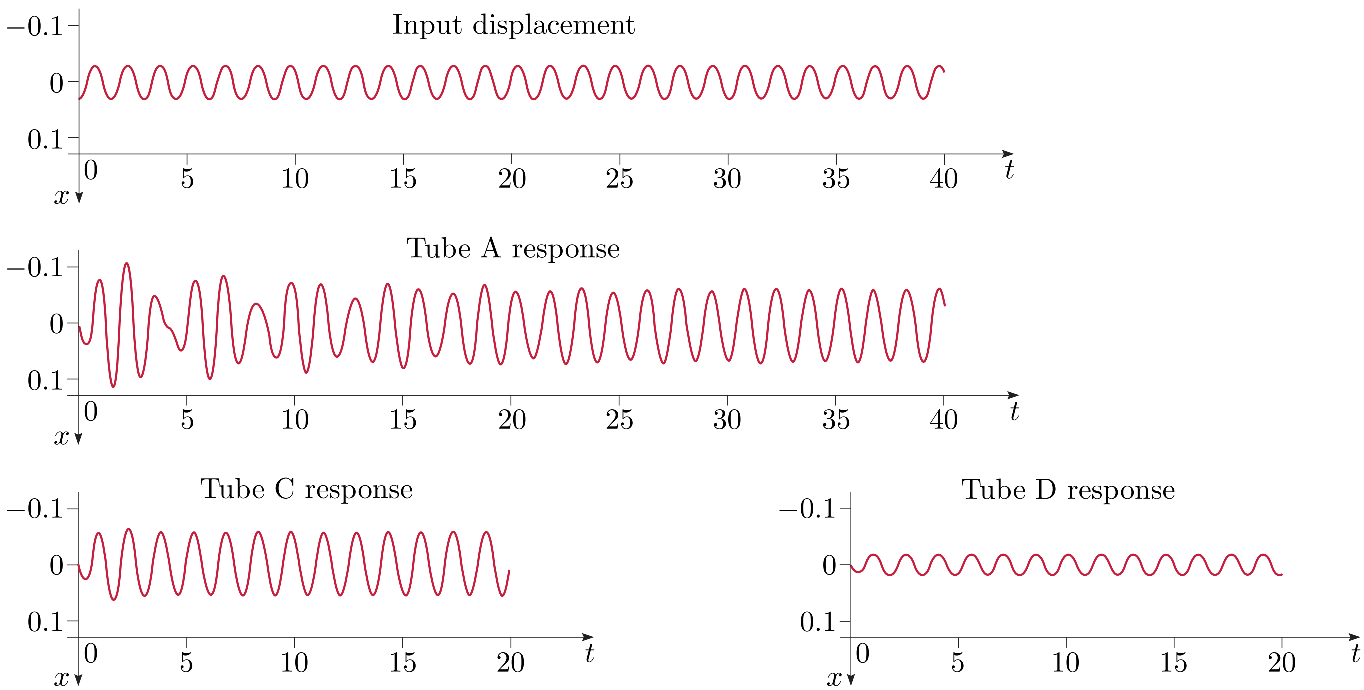


Figure 34 Predictions from the model for tubes A, C and D, with $\Omega = \frac{4}{3}\pi$, $x(0) = 0$ and $\dot{x}(0) = 0$

As before, the experimental apparatus has particle mass $m = 0.711$ and model spring stiffness $k = 23$. The amplitude of the input forcing is $a = 0.03$, hence the value of $P = ak$ is 0.69 N. The damping constants (as found for Table 1) are

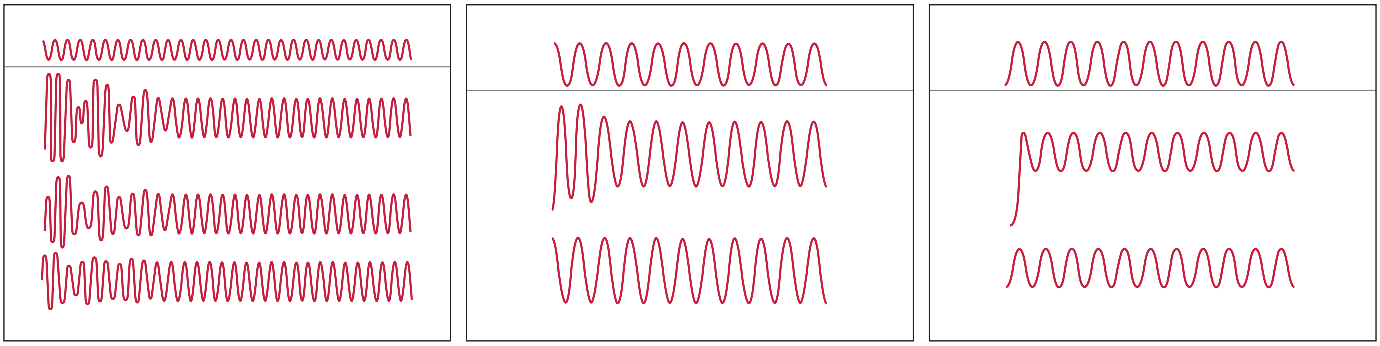
$$r = 0.15 \text{ (tube A)}, \quad r = 1.33 \text{ (tube C)}, \quad r = 8.42 \text{ (tube D)}.$$

The corresponding damping ratios were found in Exercise 10. According to equation (26), the magnification factors for the steady-state output amplitude as compared with the input amplitude are

$$M = 2.2 \text{ (tube A)}, \quad M = 1.9 \text{ (tube C)}, \quad M = 0.6 \text{ (tube D)}. \tag{28}$$

These magnifications are visible on the graphs in Figure 34.

The graphs and the values (28) for M are predictions of the model, which may be compared with the outcomes of actual experiments, shown in Figure 35.



(a) (b) (c)

Figure 35 Input (at the top) and output traces with $\Omega = \frac{4}{3}\pi$ for (a) tube A, (b) tube C, (c) tube D

These experiments confirm the predictions of the model in the following respects.

- For each tube, there is an apparently sinusoidal steady-state solution, independent of the specific initial conditions (see Figure 35), together with a transient part that dies away.
- The transient part of the motion dies away more rapidly when the damping is stronger.
- The angular frequency of the steady-state solution is identical to that of the input forcing.
- The amplitude magnification factors for this forcing frequency (which can be measured from Figure 35) decline as the amount of damping increases, from tube A through tube C to tube D.
- For tubes A and C, the output has larger amplitude than the input, while for tube D there is attenuation (lower output than input amplitude).

It is possible to perform these measurements, and to compare the values obtained with the values (29) given below.

The measured values of the magnification factor are approximately given by

$$M = 2 \text{ (tube A)}, \quad M = 1.8 \text{ (tube C)}, \quad M = 0.8 \text{ (tube D)}, \tag{29}$$

which may be compared with the predicted values (28) of the model.

For each tube, the steady-state oscillations of the suspended mass alter with changes to the input forcing frequency. The experiment is run with the forcing oscillations at 40, 60 and 80 cycles per minute, for which the angular frequency Ω (in $\text{rad}\,\text{s}^{-1}$) has the respective values $\frac{4}{3}\pi$, 2π and $\frac{8}{3}\pi$. The corresponding predictions of the model for the magnification factors are given in Table 3. These values may be obtained from equation (26).

Table 3 Magnification factors predicted by the model

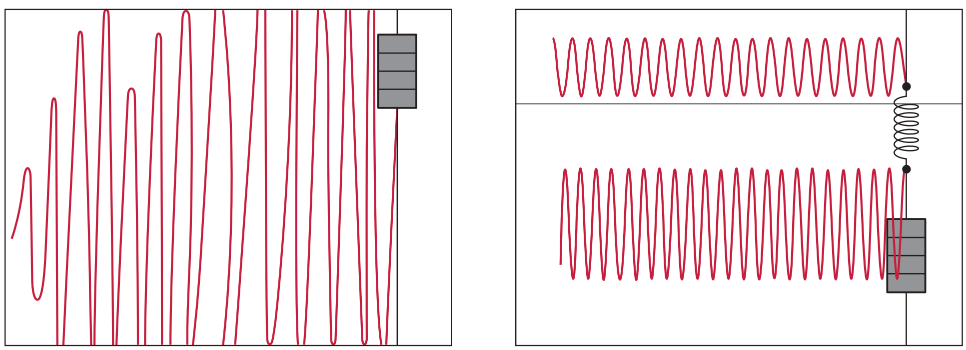
Tube	Damping constant r ($\text{N}\,\text{s}\,\text{m}^{-1}$)	Damping ratio α	Magnification factor M $\Omega = \frac{4}{3}\pi$	$\Omega = 2\pi$	$\Omega = \frac{8}{3}\pi$
A	0.15	0.02	2.2	4.5	0.9
C	1.33	0.16	1.9	2.4	0.8
D	8.42	1.04	0.6	0.4	0.3

Note that the values predicted at the intermediate angular frequency, $\Omega = 2\pi$, for tubes A and C are higher than those at the higher or lower frequency. This amounts to a prediction that resonance will occur. Indeed, the model predicts resonance in these cases close to the natural angular frequency of the system, which is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{23}{0.711}} \simeq 5.7.$$

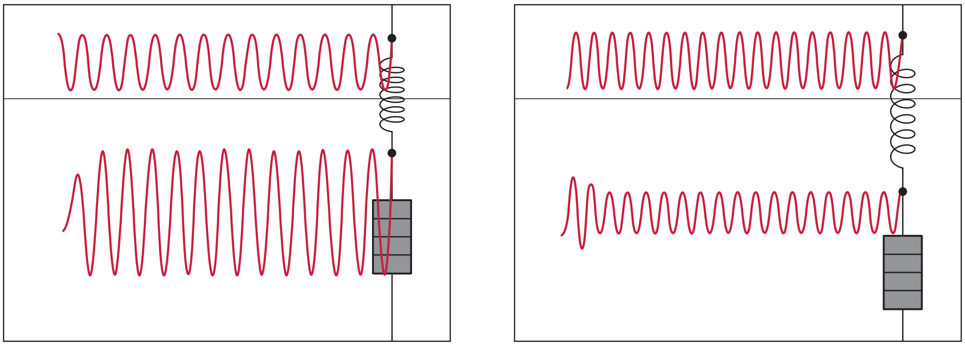
This corresponds to forcing at a frequency of about 54 cycles per minute.

The results for the experiments with forcing at angular frequency $\Omega = \frac{4}{3}\pi$ are as shown earlier, in Figure 35, with magnification factors as given in (29). The traces for the other experiments are shown in Figures 36–38.



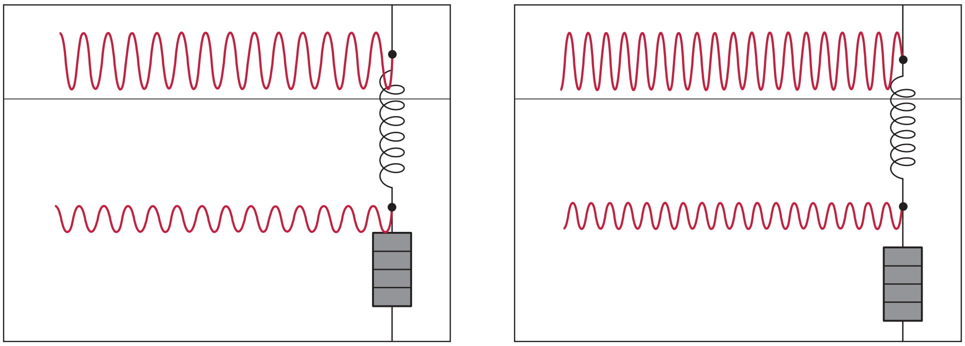
(a) (b)

Figure 36 Output traces for tube A with (a) $\Omega = 2\pi$, (b) $\Omega = \frac{8}{3}\pi$



(a) (b)

Figure 37 Output traces for tube C with (a) $\Omega = 2\pi$, (b) $\Omega = \frac{8}{3}\pi$



(a) (b)

Figure 38 Output traces for tube D with (a) $\Omega = 2\pi$, (b) $\Omega = \frac{8}{3}\pi$

From these traces, the approximate magnification factors in Table 4 may be derived. (The trace for tube A with $\Omega = 2\pi$ in Figure 36(a) shows that the motion went off the scale, indicating an amplitude magnification greater than 4.)

It is possible to estimate these experimental magnification factors by taking measurements directly from the traces.

Table 4 Magnification factors obtained from experiment

Tube	Magnification factor M		
	$\Omega = \frac{4}{3}\pi$	$\Omega = 2\pi$	$\Omega = \frac{8}{3}\pi$
A	2	>4	2
C	1.8	2.3	0.7
D	0.8	0.5	0.5

Comparing these values with those predicted by the model in Table 3, the qualitative agreement is quite good. The model predicts correctly that the magnification factors for the weakly damped systems of tubes A and C will be greater than 1 at certain frequencies, and that those for the strongly damped motion of tube D will be less than 1 in each case. The predictions of resonance for tubes A and C, and more markedly for A, are borne out by the actual experiments.

The experiments were carried out once more for tubes A and C, at a forcing frequency of 50 cycles per minute (for which $\Omega = \frac{5}{3}\pi$). The observed magnification factor for tube C was about 2.7, while the behaviour for tube A was again beyond the limits of the apparatus. The corresponding predictions from the model are, respectively, $M = 2.95$ and $M = 6.4$.

The model makes a number of simplifying assumptions: the model spring behaviour is one assumption, linear damping is another, and a pure sinusoidal input forcing displacement is a third. Hence it is not surprising that the numerical predictions are somewhat at odds with the experimental values obtained.

Despite this, there is a significant degree of qualitative agreement between the behaviour of the model and that of the real system. In particular, the phenomenon of resonance was observed as predicted.

Understanding resonant behaviour in oscillating systems and how it can affect performance, and how it can be avoided, is very important in many practical situations. The experiments carried out for this unit have clearly demonstrated that resonance can and does occur at particular forcing frequencies, when large vibrations can be observed.

Learning outcomes

After studying this unit, you should be able to:

- understand the meanings of damping, forcing and resonance
- explain and apply the linear damping model
- distinguish between weak, critical and strong damping, and be aware of the distinctive features of each case
- appreciate the role of the damping ratio
- apply the model spring and model damper force specifications to situations in which either spring or damper, or both, may undergo a forced displacement at the end away from the particle
- model direct forcing to the particle where appropriate
- derive an equation of motion, based on Newton's second law, for models that feature model springs and model dampers, with or without forcing, and formulate the initial conditions for a particular motion
- formulate and solve the equation of motion with the origin either at the equilibrium position of the particle or at some other fixed point
- interpret the solutions of an equation of motion in terms of the situation from which the model arose
- understand the terms transient and steady-state, as applied to the solutions for forced and damped harmonic oscillators, and explain the essential features of each of these
- find via formulas the amplitude and phase angle of a steady-state solution in terms of the amplitude and angular frequency of the input forcing and other parameters of the system
- find the magnification factor for the steady-state output amplitude as compared with an input forced displacement amplitude for the model spring
- identify whether resonance may occur or will not occur in a system, and where it may occur, say what approximate input angular frequency will cause it for small damping ratios.

Solutions to exercises

Solution to Exercise 1

We have

$$\frac{mg}{k} = \frac{0.711 \times 9.81}{23} \simeq 0.30,$$

so the equilibrium extension is about 0.3 m.

Solution to Exercise 2

Equation (4) is

$$m\ddot{x} + r\dot{x} + kl = mg + kl_0.$$

If the origin for x is chosen at the fixed top end of the spring, then the displacement of the particle from that origin is $x = l$, and the equation of motion takes the inhomogeneous form

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0.$$

This can also be written as

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}},$$

where $x_{\text{eq}} = l_0 + mg/k$ is the equilibrium displacement of the particle. (With this choice of origin, we have $x_{\text{eq}} = l_{\text{eq}}$.)

Solution to Exercise 3

(a) Strong damping is needed, as the train should bounce back as little as possible, with the buffers absorbing most of the energy.

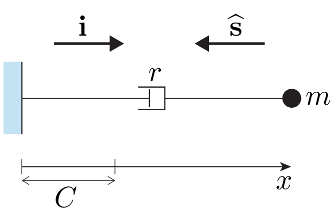
(b) Strong damping is called for, but not so strong as to make the door close too slowly.

(c) Near-critical damping is appropriate, as the fuel gauge should revert quickly to a true reading and not oscillate much.

(d) Strong damping is needed, as oscillations between the vehicles could be dangerous and should be avoided.

(e) Near-critical damping is required, perhaps slightly on the weak side of critical to allow for the envisaged range of weights. This situation is similar to that of kitchen scales, discussed earlier.

Solution to Exercise 4



(a) Suppose that the x-axis is in the horizontal direction from left to right, as shown in the figure in the margin. The direction of $\widehat{\mathbf{s}}$ is from the particle towards the centre of the damper, so $\widehat{\mathbf{s}} = -\mathbf{i}$. We have $l = C + x$, where C is a constant that depends on the position of the origin for x, so $\dot{l} = \dot{x}$ regardless of the choice of origin. Hence the resistance force is

$$\mathbf{R} = r\dot{l}\,\widehat{\mathbf{s}} = r\dot{x}(-\mathbf{i}) = -r\dot{x}\mathbf{i}.$$

If, on the other hand, the x-axis is in the opposite direction, from right to left, with the unit vector $\mathbf{i}$ in the positive x-direction, then $\widehat{\mathbf{s}} = \mathbf{i}$. Choosing the origin to be a distance L from the fixed point, we have $x = L - l$, where l is the length of the damper. Thus $\dot{x} = -\dot{l}$, so the same expression for $\mathbf{R}$ results.

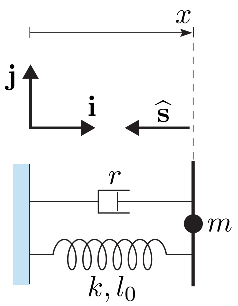
(b) The damper has length $l = x - y$, and $\widehat{\mathbf{s}}$ is in the opposite direction to $\mathbf{i}$. Hence we have

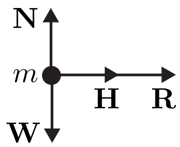
$$\mathbf{R} = r\dot{l}\,\widehat{\mathbf{s}} = r(\dot{x} - \dot{y})(-\mathbf{i}) = -r(\dot{x} - \dot{y})\mathbf{i}.$$

(This approach also provides an alternative way of tackling part (a), where y is constant thus $\dot{y} = 0$.)

Solution to Exercise 5

(a) A diagram of the buffer system and a force diagram are given below.





The train of mass m hits the buffers of stiffness k, damping constant r and natural length l_0, at time $t = 0$. The length of the buffer is $l = x$, and the unit vector $\mathbf{i}$ points in the direction of the x-axis.

From Hooke's law, with $\widehat{\mathbf{s}}$ pointing to the left, the model spring force is

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}} = k(x - l_0)(-\mathbf{i}).$$

From Exercise 4(a), the resistance force is

$$\mathbf{R} = r\dot{l}\,\widehat{\mathbf{s}} = r\dot{x}(-\mathbf{i}) = -r\dot{x}\mathbf{i}.$$

The equation of motion is therefore

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{H} + \mathbf{R} + \mathbf{W} + \mathbf{N} \\ &= -k(x - l_0)\mathbf{i} - r\dot{x}\mathbf{i} - mg\mathbf{j} + |\mathbf{N}|\,\mathbf{j}, \end{aligned}$$

which after resolution in the $\mathbf{i}$-direction gives

$$m\ddot{x} + r\dot{x} + kx = kl_0.$$

Substituting in $k = 140$, $r = 180$, $m = 40$ and $l_0 = 0.5$, we have

$$40\ddot{x} + 180\dot{x} + 140x = 70,$$

that is,

$$4\ddot{x} + 18\dot{x} + 14x = 7.$$

(b) The train meets the buffers first at $x = l_0 = 0.5$, at time $t = 0$. It is then moving in the negative x-direction, with speed $1\,\mathrm{m\,s^{-1}}$. Hence the appropriate initial conditions are

$$x(0) = 0.5, \quad \dot{x}(0) = -1.$$

(c) The solution is given as

$$x = 0.5 - 0.4e^{-t} + 0.4e^{-3.5t}.$$

Maximum compression of the buffers occurs when

$$\dot{x} = 0.4e^{-t} - 1.4e^{-3.5t} = 0,$$

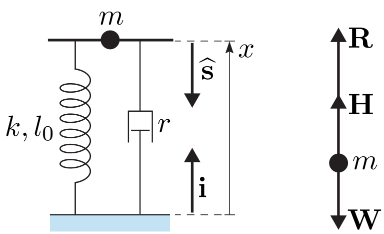
that is, when $e^{-2.5t} = \frac{2}{7}$ or $t = 0.4\ln 3.5 \simeq 0.50$. The corresponding value of x is $0.33\,\mathrm{m}$, so the maximum compression is $0.50 - 0.33 = 0.17\,\mathrm{m}$.

As explained in Example 2, the train leaves the buffers when

$$\ddot{x} = -0.4e^{-t} + 4.9e^{-3.5t} = 0,$$

that is, when $e^{-2.5t} = \frac{4}{49}$ or $t = 0.8\ln 3.5 \simeq 1.00$. The corresponding value of x is $0.37\,\mathrm{m}$, and the corresponding velocity (with which the train leaves the buffers) is $\dot{x}\mathbf{i}$, where $\dot{x} = 0.10$, that is, velocity $0.10\,\mathrm{m\,s^{-1}}$.

Solution to Exercise 6



(a) The figure in the margin shows the set-up and the force diagram.

With $\mathbf{i}$ pointing upwards, $\widehat{\mathbf{s}}$ pointing downwards, $l = x$ and $\dot{l} = \dot{x}$, the model spring force is $\mathbf{H} = k(x - l_0)\widehat{\mathbf{s}} = k(x - l_0)(-\mathbf{i})$, the weight is $\mathbf{W} = -mg\mathbf{i}$, and the resistance force is $\mathbf{R} = r\dot{x}\,\widehat{\mathbf{s}} = r\dot{x}(-\mathbf{i}) = -r\dot{x}\mathbf{i}$. Newton's second law gives

$$m\ddot{x}\mathbf{i} = \mathbf{H} + \mathbf{W} + \mathbf{R} = -k(x - l_0)\mathbf{i} - mg\mathbf{i} - r\dot{x}\mathbf{i},$$

which leads to the equation of motion

$$m\ddot{x} + r\dot{x} + kx = kx_{\mathrm{eq}},$$

where $x_{\mathrm{eq}} = l_0 - mg/k$. On substituting the values given for the parameters, we have

$$x_{\mathrm{eq}} = 0.2 - \frac{60 \times 9.81}{30\,000} \simeq 0.180$$

and

$$60\ddot{x} + 6300\dot{x} + 30\,000x = 0.2 \times 30\,000 - 60 \times 9.81 = 5411.4,$$

that is,

$$\ddot{x} + 105\dot{x} + 500x = 90.19.$$

(b) The required initial conditions are

$$x(0) = 0.2, \quad \dot{x}(0) = 0.$$

(c) As given, but with $x_{\mathrm{eq}} = 0.180$, the particular solution is

$$x = 0.180 + 0.021e^{-5t} - 0.001e^{-100t}.$$

The third term decays very quickly, so we are most interested in the second term. This reduces to 0.001 when

$$e^{-5t} = 0.001/0.021 \simeq 0.048,$$

that is, when

$$t = -0.2 \ln 0.048 \simeq 0.61.$$

The model predicts that the sit-ski will take just over half a second for the amplitude to reduce to 0.001 m and hence for the displacement of the skier to be that close to the equilibrium position.

Solution to Exercise 7

(a) Take x as the downward displacement of the bottom of the float from the surface of the liquid, as shown in Figure 19, so that $l = x$ and $\dot{l} = \dot{x}$. The upthrust from the liquid is $\mathbf{U} = -2mg(x/d)\mathbf{i}$, the weight is $\mathbf{W} = mg\mathbf{i}$, and the damping resistance force is $\mathbf{R} = -r\dot{x}\mathbf{i}$.

Hence, using Newton's second law, we obtain

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{W} + \mathbf{U} + \mathbf{R} \\ &= mg\mathbf{i} - (2mg/d)x\mathbf{i} - r\dot{x}\mathbf{i}, \end{aligned}$$

which leads to the equation of motion

$$m\ddot{x} + r\dot{x} + (2mg/d)x = mg.$$

Once the parameter values have been substituted, this becomes

$$0.1\ddot{x} + 11\dot{x} + 98.1x = 0.981,$$

that is,

$$\ddot{x} + 110\dot{x} + 981x = 9.81.$$

The initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.

(b) The particular solution is given as

$$x = 0.001\,08\, e^{-100.2t} - 0.0111\, e^{-9.79t} + 0.01.$$

The last term represents the equilibrium position. The first term involves $e^{-100.2t}$, which dies away very quickly, and the other term is the dominant term in the variable part of the solution, namely $-0.0111\, e^{-9.79t}$. The magnitude of this will reduce to 0.001 m when $e^{-9.79t} = 0.090$, that is, when $t = -(\ln 0.090)/9.79 \simeq 0.25$.

Hence the model predicts that the displacement of the float will be within 1 mm of its equilibrium position in about a quarter of a second.

Solution to Exercise 8

Critical damping will occur when $r^2 - 4mk$ is zero. The corresponding mass is therefore

$$m = \frac{r^2}{4k} = 125.$$

So a person whose mass is exactly 125 kg will produce critical damping when he or she stands on the scales. However, if a person with a slightly larger mass stands on the scales, then $r^2 - 4mk$ will be negative and there will be weak damping (decaying oscillations). If a person with a slightly smaller mass stands on the scales, then $r^2 - 4mk$ will be positive and there will be strong damping.

(This is a correct answer, but in reality you would not be able to detect any noticeable difference in the way the scales behaved for slight variations around critical damping. In the case that is technically weak damping, any oscillations would die down so quickly that they would be imperceptible. In the case that is technically strong damping, the return towards the equilibrium position would be slightly slower than with critical damping, but imperceptibly so.)

Solution to Exercise 9

The damping ratio $\alpha = r/(2\sqrt{mk})$ will be increased if either m is decreased (with r and k fixed) or k is decreased (with r and m fixed).

Solution to Exercise 10

For tube A, we have the damping ratio

$$\alpha = \frac{r}{2\sqrt{mk}} = \frac{0.15}{2\sqrt{0.711 \times 23}} \simeq 0.02.$$

Similarly, we obtain $\alpha \simeq 0.11$ for tube B, $\alpha \simeq 0.16$ for tube C, and $\alpha \simeq 1.04$ for tube D. The first three values satisfy $\alpha < 1$, so the motion in tubes A–C is weakly damped. For tube D, we see that $\alpha > 1$, confirming strong damping. (However, this is not far from critical damping, for which $\alpha = 1$.)

Solution to Exercise 11

If $\tau = 2\pi/\nu = 1.1 \times 2\pi/\omega$, where $\nu = \omega\sqrt{1-\alpha^2}$, then we have $1.1\sqrt{1-\alpha^2} = 1$, with solution $\alpha \simeq 0.417$. The amplitude decay factor per cycle is therefore

$$\exp\left(-\frac{2\pi\alpha}{\sqrt{1-\alpha^2}}\right) \simeq 0.056.$$

Solution to Exercise 12

As α increases from 1, the value of $\sqrt{\alpha^2 - 1}$ becomes increasingly close to α, so the magnitude of λ_1 decreases towards zero, while λ_2 tends towards $-2\omega\alpha$, which increases in magnitude with α.

Since both λ_1 and λ_2 are negative, it is the exponential with the exponent of smaller magnitude that dominates for large α, that is, the $e^{\lambda_1 t}$ term.

(In fact, for large α, we have $\lambda_1 \simeq -\omega/(2\alpha)$, since the product of the two roots of the auxiliary equation is $\lambda_1\lambda_2 = k/m = \omega^2$.)

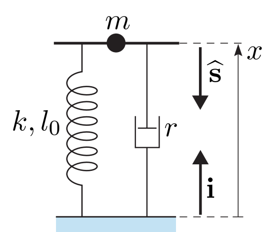
Solution to Exercise 13

(a) The solution would proceed in the same way as in Example 5, only this time P would be $2\,\mathrm{N}$ instead of $10\,\mathrm{N}$. Since the differential equation is linear, this change has the effect of scaling the constants B and C by the factor 0.2. Hence the predicted motion about the equilibrium position in the long term would consist of oscillations with the same angular frequency and phase angle as before, but with one-fifth of the amplitude. The baby would then bounce with an amplitude of about $10^{-2}\,\mathrm{m}$, or $1\,\mathrm{cm}$.

(b) If the mass m of the baby is changed, then this will change the complementary function, but the latter still dies away with time. The value of x_{eq} will be changed (becoming smaller if the baby plus seat is lighter than $10\,\mathrm{kg}$, or larger if the baby is heavier). The steady-state behaviour will still consist of sinusoidal oscillations about the (new) equilibrium position, with the same angular frequency, but the values of the constants B and C will be different. The amplitude and phase angle of the output oscillations may both be different.

Solution to Exercise 14

The set-up and force diagram are shown in the figure in the margin.



The forces acting are the weight $\mathbf{W} = -mg\mathbf{i}$, the spring force $\mathbf{H} = k(x - l_0)(-\mathbf{i})$, the damping resistance $\mathbf{R} = r\dot{x}(-\mathbf{i})$, and the girl's input force $\mathbf{P} = mg\cos(\Omega t)\,\mathbf{i}$, so Newton's second law gives

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{W} + \mathbf{H} + \mathbf{R} + \mathbf{P} \\ &= -mg\mathbf{i} - k(x - l_0)\mathbf{i} - r\dot{x}\mathbf{i} + mg\cos(\Omega t)\,\mathbf{i}. \end{aligned}$$

This leads to the equation of motion

$$\begin{aligned} m\ddot{x} + r\dot{x} + kx &= kl_0 - mg + mg\cos(\Omega t) \\ &= kx_{\mathrm{eq}} + mg\cos(\Omega t), \end{aligned}$$

where $x_{\mathrm{eq}} = l_0 - mg/k$. This differential equation is of the same form as that derived for the baby bouncer in Example 5. The values of the parameters will differ, but the model predicts the same overall long-term behaviour, namely, oscillations of angular frequency Ω about the equilibrium position, with a steady-state displacement function of the form

$$x = x_{\mathrm{eq}} + A\cos(\Omega t + \phi).$$

The values of A and ϕ here may be calculated (using a sinusoidal trial function in the manner of Example 5) from the values for m, k, r and Ω.

Solution to Exercise 15

(a) The weight $\mathbf{W} = mg\mathbf{i}$ and model spring force $\mathbf{H} = k(x - y - l_0)(-\mathbf{i})$ are as in Example 6. The damping resistance force is now $\mathbf{R} = r(\dot{x} - \dot{y})(-\mathbf{i})$. This change leads to the amended equation of motion

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0 + r\dot{y} + ky.$$

On putting $x_{\text{eq}} = mg/k + l_0$ and $y = a\cos(\Omega t)$ (so that also $\dot{y} = -a\Omega\sin(\Omega t)$), this becomes

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + ka\cos(\Omega t) - ra\Omega\sin(\Omega t).$$

On substituting in the values for m, k, r, a and Ω given in Example 6, we obtain

$$10\ddot{x} + 0.2\dot{x} + 200x = 200x_{\text{eq}} + 8\cos(2\pi t) - 0.016\pi\sin(2\pi t).$$

Comparing the equation with parameters with the direct forcing equation (17), the form of the sinusoid on the right-hand side is different, in that a sine appears as well as the cosine term. This is to some extent a superficial difference, since we can re-express the sinusoid on the right-hand side using its alternative form, that is,

$$\begin{aligned} ka\cos(\Omega t) - ra\Omega\sin(\Omega t) &= P\cos(\Omega t + \psi) \\ &= P\cos\Omega t\cos\psi - P\sin\Omega t\sin\psi. \end{aligned}$$

Hence the connection between the two forms is given by

$$P\cos\psi = ka, \quad P\sin\psi = ra\Omega,$$

so that

$$P = a\sqrt{k^2 + r^2\Omega^2}, \quad \psi = \arctan(r\Omega/k).$$

The equation of motion then becomes

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} + P\cos(\Omega t + \psi),$$

and this now differs from the form of the direct forcing equation (17) only by a shift of phase.

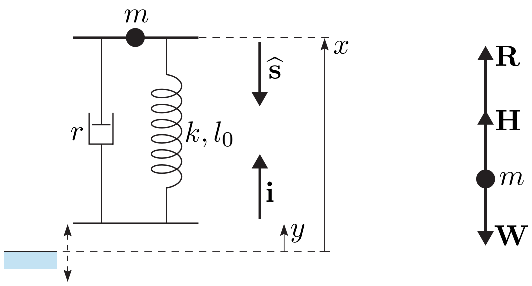
(b) The steady-state solution will be the constant x_{eq} plus a sinusoid of the form $B\cos(\Omega t) + C\sin(\Omega t)$, which can also be written in the form $A\cos(\Omega t + \phi)$. In order to find the values for B and C (and hence subsequently A and ϕ), substitute the trial particular integral $B\cos(\Omega t) + C\sin(\Omega t)$ into the differential equation and equate coefficients of $\cos(\Omega t)$ and $\sin(\Omega t)$, to obtain simultaneous equations in B and C.

Then $A = \sqrt{B^2 + C^2}$ gives the amplitude of the vibration (with $A > 0$), and the phase angle ϕ is the solution of the pair of equations $\cos\phi = B/A$, $\sin\phi = -C/A$. Hence the model again predicts steady-state oscillations, of the same frequency as the input forcing displacement.

(In fact, because the given damping constant r is so small compared with the mass m and stiffness k, the solution for the case here is almost indistinguishable from that obtained in Example 6.)

Solution to Exercise 16

The set-up and force diagram are shown in the figure below.



The forces acting are the weight $\mathbf{W} = -mg\mathbf{i}$, the model spring force $\mathbf{H} = k(l - l_0)\widehat{\mathbf{s}} = k(x - y - l_0)(-\mathbf{i})$, and the damping resistance $\mathbf{R} = r(\dot{x} - \dot{y})\widehat{\mathbf{s}} = r(\dot{x} - \dot{y})(-\mathbf{i})$, where $y = a\cos(\Omega t)$. This leads to the equation of motion

$$\begin{aligned} m\ddot{x} + r\dot{x} + kx &= kl_0 - mg + r\dot{y} + ky \\ &= kx_{\text{eq}} + ka\cos(\Omega t) - ra\Omega\sin(\Omega t), \end{aligned}$$

where $x_{\text{eq}} = l_0 - mg/k$. With the numerical values inserted (and taking $g = 9.81\,\mathrm{m\,s^{-2}}$), this becomes

$$60\ddot{x} + 6300\dot{x} + 30\,000x = 30\,000x_{\text{eq}} + 3000\cos(\pi t) - 630\pi\sin(\pi t),$$

or

$$\ddot{x} + 105\dot{x} + 500x = 90.2 + 50\cos(\pi t) - 10.5\pi\sin(\pi t).$$

The form of this equation is the same as that considered in Exercise 15, hence the same approach applies to finding a particular integral and general conclusions. The model predicts steady-state oscillations of the same angular frequency $\pi\,\mathrm{rad\,s^{-1}}$ as the input forcing displacement.

Solution to Exercise 17

We have $\dot{z} = \dot{x} - \dot{y}$ and $\ddot{z} = \ddot{x} - \ddot{y}$. Replacing x with $z + y$ gives

$$m(\ddot{z} + \ddot{y}) + r(\dot{z} + \dot{y}) + k(z + y) = kx_{\text{eq}} + ky,$$

so

$$m\ddot{z} + r\dot{z} + kz = kx_{\text{eq}} - m\ddot{y} - r\dot{y}.$$

The left-hand side of the differential equation for z is the same as that for x, so the complementary function will be the same. The particular integral will be of the form $z_{\text{p}} = x_{\text{eq}} + C\cos(\Omega t + \phi)$, which is the same form as x_{p} with different amplitude and phase angle, but with the same frequency Ω.

Of course, we could just have said that $z = x - y = x_{\text{c}} + x_{\text{p}} - a\cos(\Omega t)$, so $z_{\text{p}} = x_{\text{p}} - a\cos(\Omega t)$.

Solution to Exercise 18

(a) Equation (22) gives $A \simeq 0.051\,338$, and equation (23) gives $\phi \simeq -3.1351$, in agreement with the result quoted in the solution to Example 5.

(b) Here we have $P = ak = 8$. For a period of 2 s, the angular frequency is $\Omega = \pi$. According to equation (22), the corresponding output amplitude in the steady state is $A \simeq 0.079$. This is about 8 cm, which is twice the amplitude that was found for a forcing period of 1 s.

(c) Here we have

$$P = 0.1\sqrt{30\,000^2 + 6300^2\pi^2} \simeq 3594,$$

which leads to $A \simeq 0.101$. This output amplitude is almost the same as that of the input forcing.

Solution to Exercise 19

The solution is very similar to that for Exercise 15. As there, the weight is $\mathbf{W} = mg\mathbf{i}$ and the damping resistance is $\mathbf{R} = r(\dot{x} - \dot{y})(-\mathbf{i})$, but now the model spring force is $\mathbf{H} = k(x - l_0)(-\mathbf{i})$. The resulting equation of motion is

$$m\ddot{x} + r\dot{x} + kx = mg + kl_0 + r\dot{y}.$$

On putting $x_{\text{eq}} = mg/k + l_0$ and $y = a\cos(\Omega t)$ (so that $\dot{y} = -a\Omega\sin(\Omega t)$), this becomes

$$m\ddot{x} + r\dot{x} + kx = kx_{\text{eq}} - ra\Omega\sin(\Omega t).$$

This is of the same form as equation (19) except for a phase shift, since

$$-ra\Omega\sin(\Omega t) = P\cos(\Omega t + \psi)$$

provided that $P = ra\Omega$ and $\psi = \frac{\pi}{2}$.

Solution to Exercise 20

We have $\alpha \simeq 0.002$ and $\beta = \Omega/\omega = 1$. According to equation (26), the corresponding magnification factor is

$$M = \left((1-\beta^2)^2 + 4\alpha^2\beta^2\right)^{-1/2} = (2\alpha)^{-1} \simeq 250.$$

The steady-state output oscillations are therefore predicted to have amplitude

$$A = Ma \simeq 250 \times 0.04 = 10.$$

The system will not in fact be able to sustain an oscillation of this amplitude! For one thing, the natural length of the model spring used to represent a baby bouncer will be nowhere near 10 m. However, the model may be useful to the extent of predicting a potential catastrophe that needs to be avoided.

(Note that the forcing angular frequency of $4.47\,\text{rad}\,\text{s}^{-1}$, which is predicted to cause such a breakdown, lies between $\pi\,\text{rad}\,\text{s}^{-1}$ and $2\pi\,\text{rad}\,\text{s}^{-1}$, for which we found earlier that the predicted output amplitudes were only 8 cm and 4 cm, respectively. The large magnification factors occur for a fairly narrow range of values of the input forcing frequency.)