MST2102106F1PV1

MST210

Module Examination 2021

Mathematical methods, models and modelling

Friday 11 June 2021

There are **three sections** in this examination.

In **Section 1** you should **attempt all 19 questions**. Each question is worth 2% of the total mark.

A wholly incorrectly answered question will get zero marks.

In **Section 2** you should **attempt all 6 questions**. Each question is worth 5% of the total mark.

In **Section 3** you should **attempt both questions**. Each question is worth 16% of the total mark.

It is important to show all working in Sections 2 and 3.

Follow the instructions in the online timed examination for how to submit your work.

For further information please refer to the Remote Exam Arrangements Handbook available from the Help Centre on StudentHome. Submit your completed exam using the iCMA system. Make sure that the name of the file you submit includes your PI and the module code e.g. X1234567X123.

Section 1

The questions for this section are contained within the iCME on the website.

Section 2

You should ***attempt all questions****.* ***Each question is worth 5%.***

Question 20

A storage tank contains a liquid chemical that slowly decomposes. Let x be the amount (in litres) of the chemical in the storage tank at time t. The input to the storage tank of the chemical is an amount α litres per second. The chemical decays at a proportional rate β per second. The output from the tank is an overflow pipe that ensures that the volume of liquid in the storage tank remains fixed at V litres.

Assume that the storage tank is well mixed, so that the proportion of the chemical in the outflow pipe is the same as the whole tank.

(a) Write down a differential equation that is satisfied by x. [*3*]

(b) What does this model predict for the amount of chemical in the storage tank after the system reaches a steady state? [*2*]

Question 21

Consider the following simultaneous linear differential equations.

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = -2y.$$

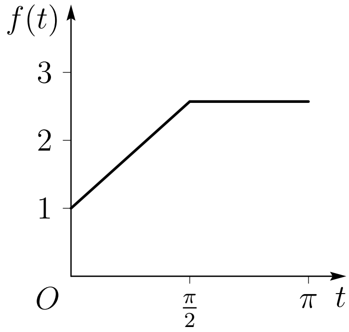
(a) Express the system of equations in matrix form. [*1*]

(b) Find the eigenvalues and eigenvectors of the matrix of coefficients. [*3*]

(c) Hence write down the general solution to this system. [*1*]

Question 22

Consider the function $f(t)$ defined on the interval $(0, \pi)$ that corresponds to the graph



The graph consists of a straight line segment from $(0, 1)$ to $(\frac{\pi}{2}, 1 + \frac{\pi}{2})$ followed by another line segment from $(\frac{\pi}{2}, 1 + \frac{\pi}{2})$ to $(\pi, 1 + \frac{\pi}{2})$.

(a) Sketch the even and odd extensions function $f(t)$ over the range $-3\pi \leq t \leq 3\pi$, clearly indicating each case. *[3]*

(b) The even extension of $f(t)$ has Fourier series $F(t)$ given by

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi t) + \sum_{n=1}^{\infty} B_n \sin(n\pi t).$$

Calculate the constant term A_0. *[2]*

Question 23

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

with the boundary conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0.$$

Applying the method of separation of variables with $u(x, t) = X(x)T(t)$ gives the differential equations

$$\frac{d^2 X}{dx^2} = kX, \quad \frac{d^2 T}{dt^2} = kT,$$

where k is the separation constant.

(a) Write down the boundary conditions that the function $X(x)$ must satisfy. *[1]*

(b) Assume that the separation constant k is negative and find non-trivial solutions to the ordinary differential equation for $X(x)$ that satisfy the boundary conditions written down in part (a). State clearly the values k can take. *[4]*

Question 24

A path C is defined by the parametric equations

$$x(t) = t^2, \quad y(t) = t^3, \quad 0 \leq t \leq 1,$$

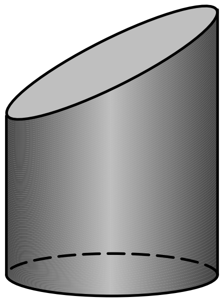
and a vector field is defined by the equation

$$\mathbf{F} = y\mathbf{i} + xy\mathbf{j}.$$

Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$. [5]

Question 25

Consider a truncated cylinder D of radius a enclosed between the planes $z = 0$ and $z = h - x$, where h is a constant with $h > a$. The midline of the cylinder has its base at the origin and extends along the positive z-axis.



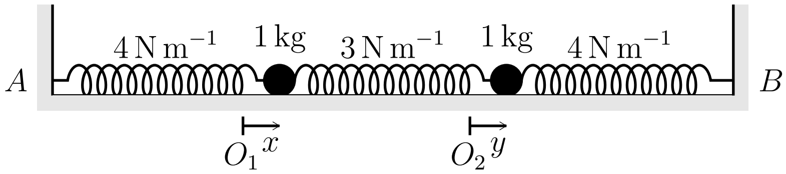
Calculate the integral $\int_D \sin\phi \, dV$, where the integrand is expressed in terms of standard cylindrical coordinates. [5]

Section 3

*You should **attempt all questions**. **Each question is worth 16%.***

Question 26

Consider the longitudinal vibrations of two particles, each of mass 1 kg. The left-hand particle is attached by a model spring of stiffness $4\,\mathrm{N\,m^{-1}}$ to a fixed point A. The two particles are connected by a model spring of stiffness $3\,\mathrm{N\,m^{-1}}$. A third spring of stiffness $4\,\mathrm{N\,m^{-1}}$ connects the second particle to a fixed point B. Choose to measure the displacements x and y of the two particles from their respective equilibrium positions O_1 and O_2, as shown in the diagram.



(a) Draw a force diagram for each particle, showing all of the forces acting on the particles. [2]

(b) Determine the changes in spring forces from their equilibrium values in terms of the above variables and parameters. [4]

(c) Hence show that the equation of motion of the system can be written as

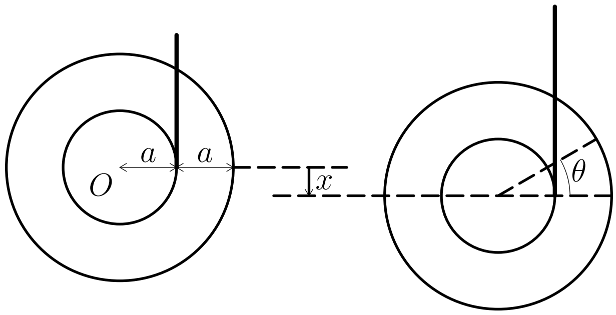
$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } \mathbf{A} = \begin{pmatrix} -7 & 3 \\ 3 & -7 \end{pmatrix}.$$ [3]

(d) Find the eigenvalues of the matrix $\mathbf{A}$ and hence write down the normal mode angular frequencies of the system. [3]

(e) Find the eigenvectors of the matrix $\mathbf{A}$ and hence write down whether each normal mode is in-phase, or phase-opposed. [4]

Question 27

Consider a model of a toy yo-yo that consists of a solid cylinder of radius $2a$ and mass m. A thin groove of depth a is cut into the cylinder and a model string is wrapped around this groove, as shown in the left-hand diagram below. Consider the downward motion of the yo-yo that occurs while the string unwraps without slipping from the groove.



The yo-yo is released from rest and falls a distance x as it rotates an angle θ, as shown in the right-hand diagram above. Assume that the top end of the string is fixed and that the effects of air resistance can be ignored.

Calculate the acceleration of the yo-yo in terms of g, the magnitude of the acceleration due to gravity. [*16*]

[END OF QUESTION PAPER]