Q 1.

(a) I would award 4 out of 5 marks for the **Specify the purpose** section of the report. This section succinctly addresses the problem to be solved and includes the key details. I dropped one mark because, although succinct, the section is one long sentence that could be made clearer to understand.

(b) I would award 28 out of 30 marks for the **Create the model** section of the report. The outline describes qualitatively how the problem will be formulated, states assumptions clearly, states the variables and parameters (with correct SI units), and derives relationships between them, using the assumptions stated. I chose to drop two marks because the variable $umax$ could have been given a simpler name name, such as v_m, and because it isn't entirely clear that the unit of each variable/parameter is given after its description.

(c) I would award 7 out of 10 marks for the **Do the mathematics** section of the report. This section successfully derives a simple model using the variables and parameters described, however it lacks graphs and no dimensional analysis is shown.

(d) I would award 7 out of 10 marks for the **Interpret the results** section of the report. This section applies parameter values to the model to estimate a suitable value of l, and the result is clearly stated, including a table of predicted values for different maximum speeds. I dropped marks because no reference is given for the parameter values adopted for f and s, the model isn't compared to any real-world data, and the model predictions are given to an excessive number of significant figures.

Q 2.

(a) I would award 3 out of 5 marks for the **Specify the purpose** section of the report. While very succinct, the description of the purpose lacks any detail about the configuration of the scenario such as the parked cars or the position of moving cars.

(b) I would award 30 out of 30 marks for the **Create the model** section of the report. The outline describes qualitatively how the problem will be formulated, states assumptions clearly, states the variables and parameters in a clear table (with correct SI units), and derives relationships between them, using stated assumptions.

(c) I would award 10 out of 10 marks for the **Do the mathematics** section of the report. This section successfully derives a simple model using the variables and parameters described, uses graphs to perform a sense check of the relationships, and uses dimensional analysis to check for any unit errors.

(d) I would award 8 out of 10 marks for the **Interpret the results** section of the report. This section applies referenced/justified parameter values to the model to estimate a suitable value of x, and the result is clearly stated, including a table of predicted values for different maximum speeds. I dropped two marks because the model isn't compared to any real-world data, and the model predictions are not given to a high enough level of precision.

Q 3.

(a)

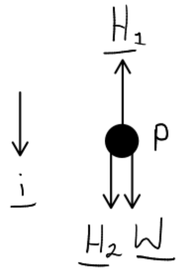

Figure 1: Force diagram of the situation described. The particle P is shown with three forces acting on it: $\mathbf{H}_1$, the force of the spring connected to the fixed point A, $\mathbf{H}_2$, the force of the spring connected to the fixed point B, and $\mathbf{W}$, the weight of the particle.

(b)

Spring	Length	Natural length	Extension	k	$\hat{\mathbf{s}}$	$\mathbf{H}$
AP	x	l_0	$x - l_0$	$2k$	$-\mathbf{i}$	$2k(x - l_0)(-\mathbf{i})$
BP	$4l_0 - x$	$2l_0$	$2l_0 - x$	k	$\mathbf{i}$	$k(2l_0 - x)\mathbf{i}$

Therefore, $\mathbf{H}_1 = 2k(x - l_0)(-\mathbf{i})$, $\mathbf{H}_2 = k(2l_0 - x)\mathbf{i}$, and $\mathbf{W} = mg\mathbf{i}$.

(c) Apply Newton's second law to the problem gives

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W} \\ &= 2k(x - l_0)(-\mathbf{i}) + k(2l_0 - x)\mathbf{i} + mg\mathbf{i} \\ &= -3kx\mathbf{i} + 4kl_0\mathbf{i} + mg\mathbf{i} \\ m\ddot{x}\mathbf{i} + 3kx\mathbf{i} &= 4kl_0\mathbf{i} + mg\mathbf{i} \end{aligned}$$

Resolving in the $\mathbf{i}$ direction gives the equation of motion as

$$m\ddot{x} + 3kx = 4kl_0 + mg$$

(d) At the equilibrium position $x = x_{eq}$ and $\ddot{x} = 0$. Substituting these values into the equation of motion gives

$$\begin{aligned} 3kx_{eq} &= 4kl_0 + mg \\ x_{eq} &= \frac{4l_0}{3} + \frac{mg}{3k} \end{aligned}$$

So the equilibrium position, in terms of the parameters given, is $\frac{4l_0}{3} + \frac{mg}{3k}$.

(e) As per the MST210 Handbook, page 66, the equation of motion

$$m\ddot{x} + 3kx = 4kl_0 + mg$$

can be written in the form

$$\ddot{x} + \omega^2 x = \omega^2 x_{eq},$$

where $\omega^2 = \frac{3k}{m}$, and $x_{eq} = \frac{4l_0}{3} + \frac{mg}{3k}$ as determined in part (d).

The general solution of this second-order, constant coefficient, differential equation can be written in the form

$$x(t) = B\cos(\omega t) + C\sin(\omega t) + x_{eq},$$

where B and C are arbitrary constants. Substituting $\omega = \sqrt{\frac{3k}{m}}$ gives the general solution as

$$x(t) = B\cos\left(\sqrt{\frac{3k}{m}}t\right) + C\sin\left(\sqrt{\frac{3k}{m}}t\right) + x_{eq}$$

(f) As shown in the MST210 Handbook, page 66, the period τ is given as

$$\tau = \frac{2\pi}{\omega},$$

where ω is the angular frequency of the oscillations. Substituting the angular frequency calculated in part (e) gives a period of

$$\tau = \frac{2\pi}{\sqrt{\frac{3k}{m}}},$$

in terms of the given parameters.

Q 4.

(a) The total energy E of a mechanical system whose force depends only on the position of the particle is constant and given by

$$E = \frac{1}{2}mv^2 + U(x),$$

where $U(x)$ is the potential energy function (MST210 Handbook, page 68). The region of motion of a particle in such a system is the range of values for x that satisfy the inequality $E - U(x) \geq 0$. Substituting $E = 3$ and $U(x) = -x^4 - x^3 + 6x^2$ gives $3 + x^4 + x^3 + 6x^2 \geq 0$. Figure 2 shows how the roots of LHS of this inequality are found using Maxima.

```
(%i1) f(x):=3 + x^4 + x^3 − 6·x^2$

(%i2) r:allroots(f(x));
(%o2) [x=0.814499876996913,x=−0.6949361557462852,x=1.809494635147004,x=−
      2.929058356397632]

(%i3) wxplot2d(
        [
          f(x),
          [discrete, [[rhs(r[1]),0]]],
          [discrete, [[rhs(r[2]),0]]],
          [discrete, [[rhs(r[3]),0]]],
          [discrete, [[rhs(r[4]),0]]]
        ],
        [x, −3.5, 2.5],
        [y, −15, 15],
        [style, lines, [points, 15, 2, 10], [points, 15, 2, 10], [points, 15, 2, 10], [points, 15, 2, 10]],
        [legend, false]
      ), wxplot_size=[500, 500];
```

(%t3)

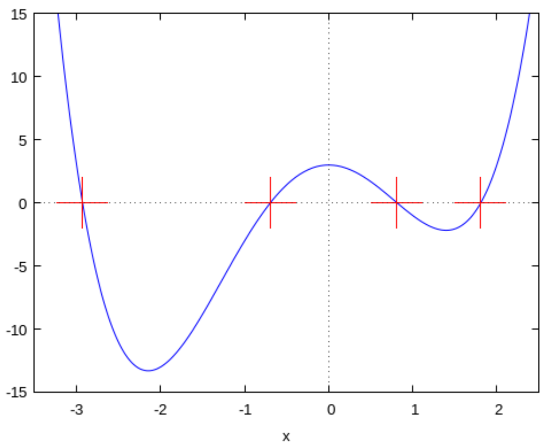

Figure 2: Maxima worksheet showing how the roots of the expression $3+x^4+x^3+6x^2$ are found. The expression is plotted in blue with its roots shown as red crosses.

Therefore, the range of x-values that could represent a motion of the system is $[-0.7, 0.8]$ (to 1 d.p.).

(b) When $E = 17$, the region of motion is the range or ranges of x-values that satisfy

$$17 + x^4 + x^3 - 6x^2 \geq 0$$

Figure 3 shows how the LHS of this inequality has no real solutions and therefore the system has no range of x-values over which it oscillates.

```
(%i1) f(x):=17 + x^4 + x^3 − 6·x^2$

(%i2) r:allroots(f(x));
(%o2) [x=0.929434402135055 %i+1.658395392608348,x=1.658395392608348−
      0.929434402135055 %i,x=0.2123620521312622 %i−2.158395392608349,x=−
      0.2123620521312622 %i−2.158395392608349]

(%i3) wxplot2d(f(x), [x, −3.5, 2.5], [legend, false]), wxplot_size=[500, 500];
```

(%t3)

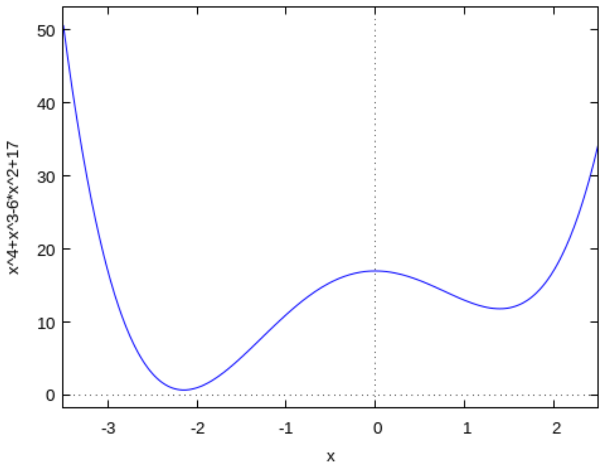

Figure 3: Maxima worksheet showing how the expression $17 + x^4 + x^3 + 6x^2$ has no real roots. The expression is plotted in blue.

Q 5.

(a)

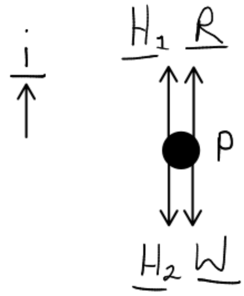

Figure 4: Force diagram of the situation described. The particle P is shown with four forces acting on it: $\mathbf{H}_1$, the force of the spring AP, $\mathbf{H}_2$, the force of the spring CP, $\mathbf{R}$ the resistance force of the model damper, and $\mathbf{W}$, the weight of the particle.

(b)

Spring	Spring length	Natural length	Extension	k	$\hat{\mathbf{s}}$	$\mathbf{H}$
AP	$h - x$	l_1	$h - x - l_1$	k_1	$\mathbf{i}$	$k_1(h - x - l_1)\mathbf{i}$
CP	x	l_2	$x - l_2$	k_2	$-\mathbf{i}$	$k_2(x - l_2)(-\mathbf{i})$

(c)

Damper length	Rate of change of length	Damping constant	$\hat{\mathbf{s}}$	$\mathbf{R}$
$h - x$	-$\dot{x}$	r	$\mathbf{i}$	$-r\dot{x}\mathbf{i}$

(d) Applying Newton's second law to the forces acting on the particle gives

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{R} + \mathbf{W} \\ &= k_1(h - x - l_1)\mathbf{i} - k_2(x - l_2)\mathbf{i} - r\dot{x}\mathbf{i} - mg\mathbf{i} \end{aligned}$$

Expanding the terms and resolving in the $\mathbf{i}$ direction gives

$$\begin{aligned} m\ddot{x}\mathbf{i} &= (k_1h - k_1x - k_1l_1 - k_2x + k_2l_2 - r\dot{x} - mg)\mathbf{i} \\ m\ddot{x} &= k_1h - k_1x - k_1l_1 - k_2x + k_2l_2 - r\dot{x} - mg \\ m\ddot{x} + r\dot{x} + (k_1 + k_2)x &= k_1(h - l_1) + k_2l_2 - mg \end{aligned}$$

as required.

(e) Substituting the provided values into the equation of motion gives

$$8\ddot{x} + 40\dot{x} + 200x = 215 - 8g$$

The natural angular frequency of a damped harmonic oscillator is $\omega = \sqrt{\frac{k}{m}}$, where k is the coefficient of the x term in the equation of motion and m is the mass of the particle (MST210 Handbook, page 70). Thus, the natural frequency of this system is

$$\begin{aligned} \omega &= \sqrt{\frac{200}{8}} \\ &= 5.00 \text{ rad}s^{-1} \end{aligned}$$

The damping ratio for a damped harmonic oscillator is $\alpha = \frac{r}{2\sqrt{mk}}$, where r is the coefficient of the $\dot{x}$ term in the equation of motion and m and k have the same definitions as above (MST210 Handbook, page 69). The damping ratio for this system is

$$\begin{aligned} \alpha &= \frac{40}{2\sqrt{8 \times 200}} \\ &= 0.50 \end{aligned}$$

As $\alpha < 1$, the system undergoes weak damping. This level of damping seems appropriate for a baby bouncer as the baby should be able to oscillate vertically for a period with a single kick, but should eventually come to rest when unpropelled.

Q 6.

(a) As shown in MST210 Handbook, page 72, the steady-state solution of a forced system with equation of motion $m\ddot{x} + r\dot{x} + kx = P\cos(\Omega t)$ has amplitude

$$A = \frac{P}{\sqrt{(k - m\Omega^2)^2 + r^2\Omega^2}}$$

Substituting $m = 4$, $r = 6$, $k = 9$, $P = 10$, and $\Omega = 3$ gives

$$\begin{aligned} A &= \frac{10}{\sqrt{(9 - 4 \times 3^2)^2 + 6^2 \times 3^2}} \\ &= \frac{10}{9\sqrt{13}} \end{aligned}$$

So the amplitude of the resulting oscillations of the particle after a long time has elapsed is 0.31 (2 d.p.).

(b) As shown in question 5(e), the damping ratio is given by

$$\begin{aligned} \alpha &= \frac{r}{2\sqrt{mk}} \\ &= \frac{6}{2\sqrt{4 \times 9}} \end{aligned}$$

So the damping ratio is 0.50. As $\alpha < \frac{1}{\sqrt{2}}$, resonance can occur in this system.

Q 7.

(a)

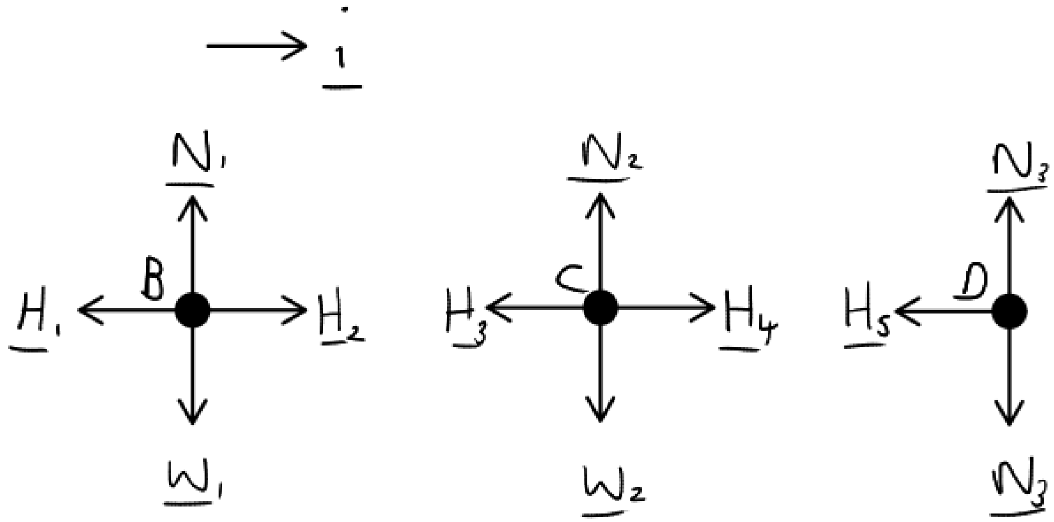

Figure 5: Force diagrams for the three particles B, C, and D. $\mathbf{H}_{1,\ldots,5}$ represent the spring forces acting on each particle, $\mathbf{W}_{1,\ldots,3}$ and $\mathbf{N}_{1,\ldots,3}$ represent the weight and normal reaction forces of each particle, respectively.

(b)

Spring	Particle	Extension	Stiffness	$\hat{\mathbf{s}}$	$\Delta\mathbf{H}$
AB	B	x_1	$2k$	$-\mathbf{i}$	$2kx_1(-\mathbf{i})$
BC	B	$x_2 - x_1$	k	$\mathbf{i}$	$k(x_2 - x_1)\mathbf{i}$
BC	C	$x_2 - x_1$	k	$-\mathbf{i}$	$k(x_2 - x_1)(-\mathbf{i})$
CD	C	$x_3 - x_2$	$3k$	$\mathbf{i}$	$3k(x_3 - x_2)\mathbf{i}$
CD	D	$x_3 - x_2$	$3k$	$-\mathbf{i}$	$3k(x_3 - x_2)(-\mathbf{i})$

(c) Applying Newton's second law to each particle separately, considering the $\mathbf{i}$ direction only gives

$$\begin{aligned} m\ddot{\boldsymbol{x}}_1 &= \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2 \\ m\ddot{\boldsymbol{x}}_2 &= \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4 \\ m\ddot{\boldsymbol{x}}_3 &= \Delta\mathbf{H}_5 \end{aligned}$$

Substituting the $\Delta\mathbf{H}$ values from the table above and resolving in the $\mathbf{i}$ direction gives

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - x_1) - 2kx_1 \\ &= k(-3x_1 + x_2) \end{aligned}$$

$$\begin{aligned} m\ddot{x}_2 &= 2\big(3k(x_3 - x_2) - k(x_2 - x_1)\big) \\ &= 2\big(k(x_1 - 4x_2 + 3x_3)\big) \end{aligned}$$

$$\begin{aligned} m\ddot{x}_3 &= 2\big(-3k(x_3 - x_2)\big) \\ &= 2\big(k(3x_2 - 3x_2)\big) \end{aligned}$$

which can be written in matrix form as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m} \begin{pmatrix} -3 & 1 & 0 \\ 2 & -8 & 6 \\ 0 & 6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Q 8.

(a) An eigenvector $\mathbf{v}$ of a matrix $\mathbf{A}$ satisfies $\mathbf{Av} = \lambda\mathbf{v}$, where λ is a scalar (MST210 Handbook, page 51). In this system of equations:

$$\begin{pmatrix} -3 & 2 & 0 \\ 4 & -6 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1.34 \\ -0.80 \\ 1.00 \end{pmatrix} = \begin{pmatrix} 2.42 \\ 1.44 \\ -1.80 \end{pmatrix} = -1.8 \begin{pmatrix} -1.34 \\ -0.80 \\ 1.00 \end{pmatrix}$$

Therefore $\lambda = -1.8$ is an eigenvalue of the eigenvector $(-1.34 \quad -0.8 \quad 1)^\top$

(b) As shown in MST210 Handbook, page 73, the normal mode angular frequency of an oscillating system is given by

$$\begin{aligned} \omega &= \sqrt{-\lambda} \\ &= \sqrt{-(-1.8)} \\ &= 1.34...\ \mathrm{rad}s^{-1} \end{aligned}$$

So the normal mode angular frequency of the system is 1.34 rads^{-1} (2 d.p.).

(c) Any eigenvector of the dynamic matrix is also an initial displacement vector that will lead to normal mode motion (MST210 Handbook, page 73). Therefore, such a displacement vector for the system in question is $(-1.34 \quad -0.80 \quad 1.00)^\top$ (2 d.p.).

(d) Particles A and B are moving in phase as their respective components of the normal mode eigenvector have the same sign.