TMA04

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Question 1

a) Evaluate each of the following expressions, if possible. Where evaluation is not possible, explain why not.

Solution:

i) **AB** is undefined because the number of rows in **B** does not equal the number of columns in **A**.

ii) $\mathbf{BA} = \begin{pmatrix} 3(-2)+(-4\times 6) & 3\times 3+(-4\times 2) \\ 2(-2)+5\times 6 & 2\times 3+5\times 2 \\ 3(-2)+(-2\times 6) & 3\times 3+(-2\times 2) \end{pmatrix} = \begin{pmatrix} -30 & 1 \\ 26 & 16 \\ -18 & 5 \end{pmatrix}$

iii) $2\mathbf{A} = \begin{pmatrix} 2(-2) & 2\times 3 \\ 2\times 6 & 2\times 2 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ 12 & 4 \end{pmatrix}$.

However, as $2\mathbf{A}$ and **B** have different sizes, they cannot be added together and $2\mathbf{A} + \mathbf{B}$ is undefined.

iv) $3\mathbf{B} = \begin{pmatrix} 3\times 3 & 3(-4) \\ 3\times 2 & 3\times 5 \\ 3\times 3 & 3(-2) \end{pmatrix} = \begin{pmatrix} 9 & -12 \\ 6 & 15 \\ 9 & -6 \end{pmatrix}$, and $\mathbf{BA} = \begin{pmatrix} -30 & 1 \\ 26 & 16 \\ -18 & 5 \end{pmatrix}$ (from (ii)).

Therefore 3**B** - **BA** $= \begin{pmatrix} 9-(-30) & -12-1 \\ 6-26 & 15-16 \\ 9-(-18) & -6-5 \end{pmatrix} = \begin{pmatrix} 39 & -13 \\ -20 & -1 \\ 27 & -11 \end{pmatrix}$.

v) I start by calculating $\mathbf{A}^2$ and $5\mathbf{A}$:

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix} -2 & 3 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 6 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2(-2)+3\times 6 & -2\times 3+3\times 2 \\ 6(-2)+2\times 6 & 6\times 3+2\times 2 \end{pmatrix} \\ &= \begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}\end{aligned}$$

$$5\mathbf{A} = \begin{pmatrix} 5(-2) & 5\times 3 \\ 5\times 6 & 5\times 2 \end{pmatrix} = \begin{pmatrix} -10 & 15 \\ 30 & 10 \end{pmatrix}$$

Therefore $\mathbf{A}^2 + 5\mathbf{A} = \begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}\begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix} = \begin{pmatrix} 12 & 15 \\ 30 & 32 \end{pmatrix}$

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b) Determine whether the following matrices are invertible, and in each case find the inverse if it exists.

Solution:

i) Let $\mathbf{M} = \begin{pmatrix} a=7 & b=-8 \\ c=-4 & d=5 \end{pmatrix}$. Then the determinant of $\mathbf{M}$ is given by

$$\begin{aligned}\det\mathbf{M} &= ad-bc \\ &= 7\times 5-(-8(-4)) \\ &= 3\end{aligned}$$

As $\det\mathbf{M}\neq 0$, $\mathbf{M}$ is invertible and its inverse is given by

$$\mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5 & 8 \\ 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{3} & \frac{8}{3} \\ \frac{4}{3} & \frac{7}{3} \end{pmatrix}$$

ii) Let $\mathbf{P} = \begin{pmatrix} a = -4 & b = -12 \\ c = 3 & d = 9 \end{pmatrix}$. Then the determinant of $\mathbf{P}$ is given by

$$\begin{aligned} \det \mathbf{P} &= ad - bc \\ &= -4 \times 9 - (-12 \times 3)) \\ &= 0 \end{aligned}$$

As $\det \mathbf{P} = 0$, $\mathbf{P}$ is a singular, non-invertible matrix.

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c) Use an answer from part (b) to solve the system of linear equations.

Solution:

The system of linear equations can be written in matrix form as $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{x}$ is the corresponding vector of unknowns, and $\mathbf{b}$ is the vector whose components are the corresponding right hand sides of the equations.

If $\mathbf{A}$ is invertible, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

For this system of equations, $\mathbf{A} = \begin{pmatrix} 7 & -8 \\ -4 & 5 \end{pmatrix}$ and $\mathbf{A}^{\text{-}1} = \begin{pmatrix} \frac{5}{3} & \frac{8}{3} \\ \frac{4}{3} & \frac{7}{3} \end{pmatrix}$ as determined in part (b). Therefore the solution to the system of linear equations is given by

$$\mathbf{x} = \begin{pmatrix} \frac{5}{3} & \frac{8}{3} \\ \frac{4}{3} & \frac{7}{3} \end{pmatrix} \begin{pmatrix} 22 \\ -13 \end{pmatrix}$$

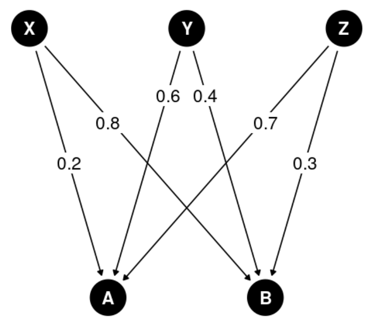
$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So $x = 2$ and $y =$ -1.

Question 2

a) i) Draw a network diagram with input nodes X, Y and Z, and output nodes A and B, that represents the proportion of their greengrocery budget spent by each shop at each wholesaler.

Solution:



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ii) Write down the matrix that represents this network.

Solution:

The matrix that represents this network is $\begin{pmatrix} 0.2 & 0.6 & 0.7 \\ 0.8 & 0.4 & 0.3 \end{pmatrix}$

whose n columns correspond to shops X, Y, and Z, from left to right, and whose m rows correspond to wholesalers A and B, from top to bottom. Each element in the matrix represents the proportion of the budget of the nth wholesaler spent at the mth wholesaler.

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iii) Use the matrix that you found in part (a)(ii) to determine how much was spent at wholesaler A in 2020.

Solution:

Multiplying the matrix of the proportions of expenditure found in (a)(ii) by a vector of the total expenditure by shops X, Y, and Z gives a vector of outputs:

$$\begin{pmatrix} 0.2 & 0.6 & 0.7 \\ 0.8 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 4500 \\ 2400 \\ 6600 \end{pmatrix} = \begin{pmatrix} 6960 \\ 6540 \end{pmatrix}$$

Therefore, the amount spent at wholesaler A in 2020 was £6960.

b) i) The matrix $\mathbf{C}$ that represents the combined matrix is the product of the matrices that represent the original networks (noting that the top network is represented second in the product):

$$\mathbf{C} = \begin{pmatrix} 0.4 & 0.3 \\ 0.05 & 0.02 \\ 0.55 & 0.68 \end{pmatrix} \begin{pmatrix} 0.2 & 0.6 & 0.7 \\ 0.8 & 0.4 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.32 & 0.36 & 0.37 \\ 0.026 & 0.038 & 0.041 \\ 0.654 & 0.602 & 0.589 \end{pmatrix}$$

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ii) Find the amounts spent on fruit, herbs and vegetables in 2020.

Solution:

The vector $\mathbf{v}$ of total expenditure on fruit, herbs, and vegetables in 2020 is given by the product of the matrix representing the combined network, and the vector of total expenditure by shops X, Y, and Z:

$$\mathbf{v} = \begin{pmatrix} 0.32 & 0.36 & 0.37 \\ 0.026 & 0.038 & 0.041 \\ 0.654 & 0.602 & 0.589 \end{pmatrix} \begin{pmatrix} 4500 \\ 2400 \\ 6600 \end{pmatrix}$$

$$= \begin{pmatrix} 4746 \\ 478.8 \\ 8275.2 \end{pmatrix}$$

Therefore, in 2020 £4746.0 was spent on fruit, £478.80 was spent on herbs, and £8275.20 was spent on vegetables.

Question 3

The problem can be written as the following system of equations:
4a + 3b + 5c = 185
7a + 4b + 3c = 236
8a + 2b + 7c = 247

where a, b, and c, represent the number of students on an arts, language, and science course, respectively.

By creating a matrix of coefficients M and a vector v whose elements are the corresponding right hand sides of the equations, the vector of class sizes is given by the product of v and the inverse of M.

(%i1)
```
M : matrix(
    [4, 3, 5],
    [7, 4, 3],
    [8, 2, 7]
);
```

(%o1) $\begin{pmatrix} 4 & 3 & 5 \\ 7 & 4 & 3 \\ 8 & 2 & 7 \end{pmatrix}$

(%i2)
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v : matrix(
    [185],
    [236],
    [274]
);
```

(%o2) $\begin{pmatrix} 185 \\ 236 \\ 274 \end{pmatrix}$

(%i3)
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invert(M).v;
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(%o3) $\begin{pmatrix} 20 \\ 15 \\ 12 \end{pmatrix}$

Therefore, there are 20 students on an arts course, 15 students on a language course, and 12 students on a science course.

Question 4

a) For the infinite geometric sequence (x_n), find the values of the first term a and the common ratio r, and write down the recurrence system for this sequence.

Solution:

The first term a is 4.0, and the common ratio r is $\frac{x_{n+1}}{x_n} = \frac{-5.2}{4.0} = -1.3$. Hence, the recurrence system for this sequence is:

$$x_1 = 4.0 \qquad x_n = -1.3x_{n-1} \qquad (n = 2, 3, 4, ...)$$

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b) Write down the closed form for this sequence.

Solution:

The closed form for this sequence is

$$x_n = 4(-1.3)^{n-1} \qquad (n = 1, 2, 3, ...)$$

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c) Calculate the 11th term of the sequence to four decimal places.

Solution:

Substituting $n = 11$ into the closed form defined in (b) gives

$$\begin{aligned} x_{11} &= 4(-1.3)^{10} \\ &= 55.1433967... \end{aligned}$$

So the 11th term of the sequence is 55.1434 (to 4 d.p.).

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d) Describe the long-term behaviour of the sequence. Justify your answer.

Solution:

The sequence alternates in sign and is unbounded as $n \to \infty$, as the common ratio r is less than -1, and the first term a is not 0.

Question 5

Work out how much money is in the account after 25 years.

Solution:

The first few terms of this sequence are £2000, £2360, £2420, £2480, £2540...

This represents an arithmetic sequence with closed form $x_n = a+(n-1)d$ where a is the starting value and d is the common difference. Taking the x_1 to be the value in the account after 1 year, the sequence can be represented as

$$x_n = 2360 + (n-1) \times 60 \qquad\qquad (n = 1, 2, 3, ...)$$

Substituting $n = 25$ gives

$$\begin{aligned} x_{25} &= 2360 + 24 \times 60 \\ &= 3800 \end{aligned}$$

Therefore there is £3800 in the account after 25 years.

Question 6

Find a fraction equivalent to the recurring decimal 0.623 762 376 237...

Solution:

Let $s = 0.623762376237...$, then

$$\begin{aligned} 10,000s &= 6237.623762376237... \\ 9,999s &= 6237 \\ s &= \frac{6237}{9999} \\ &= \frac{231}{37} \end{aligned}$$

So the fraction equivalent of 0.623 762 376 237... is $\frac{231}{37}$.

Question 7

Find the sums of the following infinite series.

Solution:

a) This is an infinite geometric series with starting value $a = 7$ and common ratio $r = -\frac{4}{17}$. The sum of an infinite geometric series is given by

$$\begin{aligned} \text{sum} &= \frac{a}{1-r} \\ &= \frac{7}{1-(-\frac{4}{17}))} \\ &= \frac{17}{3} \end{aligned}$$

Therefore, the sum of the series is $\frac{17}{3}$.

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b) This series can be represented as the difference between two infinite geometric sequences:

$$\sum_{k=1}^{\infty}\left(\left(\frac{4}{7}\right)^k - \left(\frac{5}{11}\right)^k\right) = \sum_{k=1}^{\infty}\left(\frac{4}{7}\right)^k - \sum_{k=1}^{\infty}\left(\frac{5}{11}\right)^k$$

Then using the fact that the sum of an infinite geometric sequence is given by sum $= \frac{a}{1-r}$ and substituting $a = r = \frac{4}{7}$ for the first series and $a = r = \frac{5}{11}$ for the second gives

$$\begin{aligned} \sum_{k=1}^{\infty}\left(\frac{4}{7}\right)^k - \sum_{k=1}^{\infty}\left(\frac{5}{11}\right)^k &= \frac{\frac{4}{7}}{1-\frac{4}{7}} - \frac{\frac{5}{11}}{1-\frac{5}{11}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the sum of the series is $\frac{1}{2}$.

Question 8

Find the coefficient of x^7y^4 in the expansion of $(\frac{1}{5}x - 25y)^{11}$.

Solution:

The expansion of a binomial $(a+b)^n$ is given by

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + \cdots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$$

where ${}^nC_k$ is the k^{th} binomial coefficient in the n^{th} row of Pascal's triangle (with rows numbered 0, 1, 2, ...). The binomial coefficient of for the x^7y^4 term in the expansion of $(\frac{1}{5}x - 25y)^{11}$ is ${}^{11}C_4 = 330$. Substituting $a = \frac{1}{5}x$ and $b = -25y$ gives

$$330\left(\left(\frac{1}{5}x\right)^7\left(-25y\right)^4\right) = 1650x^7y^4$$

Therefore, the coefficient of x^7y^4 in the binomial expansion of $(\frac{1}{5}x - 25y)^{11}$ is 1650.

Question 9

a) Find $\frac{z}{w}$, giving your answer in Cartesian form.

Solution:

To divide complex numbers in polar form, divide their moduli and subtract their arguments:

$$\frac{z}{w} = \frac{15}{3}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right)$$

$$= 5\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{5}{2} + i\frac{5\sqrt{3}}{2}$$

So $\frac{z}{w} = \frac{5}{2} + i\frac{5\sqrt{3}}{2}$.

b) Use de Moivre's formula to find the Cartesian form of the complex number $(-1+i)^{11}$.

Solution:

De Moivre's formula states that for a complex number in polar form $z = r(\cos\theta + i\sin\theta)$, $z^n = r^n(\cos n\theta + i\sin n\theta)$. I start by converting $(-1+i)^{11}$ into polar form:

$$\begin{aligned} r &= \sqrt{(-1)^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \theta &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4} \end{aligned}$$

$$\therefore z = \sqrt{2}\Bigg(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\Bigg)$$

Then, applying de Moivre's formula gives

$$z^n = (\sqrt{2})^{11}\Bigg(\cos\left(\frac{33\pi}{4}\right) + i\sin\left(\frac{33\pi}{4}\right)\Bigg)$$

$$= 32\sqrt{2}\Bigg(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\Bigg)$$

$$= 32\sqrt{2}\Big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\Big)$$

$$= 32 + 32i$$

So $(-1+i)^{11} = 32 + 32i$.

Question 10

a) Find the moduli and principal arguments of z*w and z/w

(%i1) z:9·(cos(5·%pi/11) + %i·sin(9·%pi/11))$

(%i2) w:4·(cos(9·%pi/11) + %i·sin(9·%pi/11))$

The modulus of z*w is:

(%i3) trigreduce(abs(z·w));

(%o3) $9\, 2^{3/2} \sqrt{-\cos\left(\frac{18\,\pi}{11}\right)+\cos\left(\frac{10\,\pi}{11}\right)+2}$

The modulus of z/w is:

(%i4) trigreduce(abs(z/w));

(%o4) $\frac{9\sqrt{-\cos\left(\frac{18\,\pi}{11}\right)+\cos\left(\frac{10\,\pi}{11}\right)+2}}{2^{5/2}}$

The principal argument of z*w is:

(%i5) trigreduce(carg(z·w));

(%o5) $\operatorname{atan}\left(\sec\left(\frac{5\,\pi}{11}\right)\sin\left(\frac{9\,\pi}{11}\right)\right)+\frac{20\,\pi}{11}$

The principal argument of z/w is:

(%i6) trigreduce(carg(z/w));

(%o6) $\operatorname{atan}\left(\sec\left(\frac{5\,\pi}{11}\right)\sin\left(\frac{9\,\pi}{11}\right)\right)+\frac{2\,\pi}{11}$

b) Solve the equation

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(%i1) s:solve(2·z^5 − 15·z^4 + 35·z^3 + 25·z^2 − 177·z + 130 = 0, z);
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(%o1) $[z=-2, z=\frac{5}{2}, z=3-2\,\%i, z=2\,\%i+3, z=1]$

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(%i2) v:makelist(rhs(s[k]), k, 1, length(s))$

(%i3) pts:makelist([realpart(v[k]), imagpart(v[k])], k, 1, length(s))$

(%i4) wxplot2d([discrete, pts], [style, points], [xlabel, "Real part"], [ylabel, "Imaginary part"]);
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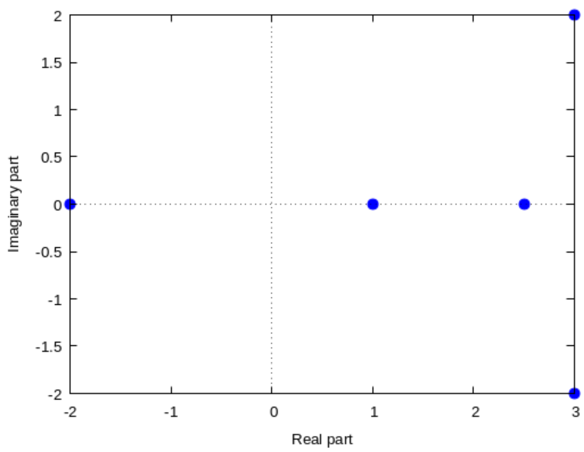
(%t4)



(%o4)

Question 11

Use $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to obtain the identity

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Solution:

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{1}$$

$$\sin^5\theta = \frac{1}{2^5 i}(e^{i\theta} - e^{-i\theta})^5 \tag{2}$$

The binomial expansion of $(e^{i\theta} - e^{-i\theta})^5$ is

$= e^{5i\theta} + 5e^{4i\theta}(-e^{-i\theta}) + 10e^{3i\theta}(-e^{-2i\theta}) + 10e^{2i\theta}(-e^{-3i\theta}) + 5e^{i\theta}(-e^{-4i\theta}) - e^{-5i\theta}$

$= e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}$

$= (2i\sin(5\theta)) - 5(2i\sin(3\theta)) + 10(2i\sin(\theta))$

Substituting $(e^{i\theta} - e^{-i\theta})^5 = (2i\sin(5\theta)) - 5(2i\sin(3\theta)) + 10(2i\sin(\theta))$ into equation (2) gives

$$\sin^5\theta = \frac{1}{32i}(2i\sin(5\theta)) - 5(2i\sin(3\theta)) + 10(2i\sin(\theta))$$

$$= \frac{1}{16}\Big(\Big(\frac{2i\sin(5\theta)}{2i}\Big) - 5\Big(\frac{2i\sin(3\theta)}{2i}\Big) + 10\Big(\frac{2i\sin\theta}{2i}\Big)\Big)$$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Therefore, $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Note: I found this question very challenging!

Question 12

a) Find a recurrence system for this sequence, give its closed form, and describe its long-term behaviour.

Solution:

The sequence shown can be represented by the recurrence system

$$x_1 = 50 \qquad x_n = -0.4x_{n-1} \qquad (n = 2, 3, 4, ...)$$

or in closed form as

$$x_n = 50(-0.4)^{n-1} \qquad (n = 1, 2, 3, ...)$$

The sequence is alternating in sign and converging to 0 as $n \to \infty$.

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b) Find the polynomial equation, giving it in its simplest form, and explain your working.

Solution:

The solutions of the polynomial equation shown are

$$z_1 = 1$$
$$z_2 = -2 + i$$
$$z_3 = -2 - i$$
$$z_4 = -3 + 2i$$
$$z_5 = -3 - 2i$$

As there are five solutions, the polynomial expression has degree five. The fundamental theorem of algebra states that any polynomial of degree $n \leq 1$ has factorisation

$$a_n(z - z_1)(z - z_2)\dots(z - z_n)$$

where $z_, z_2, \dots, z_n$ are complex solutions of the polynomial equation. Substituting the known solutions into this factorisation gives

$$\begin{aligned} a_n(z-1)(z-(-2+i))(z-(-2-i))(z-(-3+2i))(z-(-3-2i)) &= 0 \\ az^5 + 9az^4 + 32az^3 + 40az^2 - 17az - 65a &= 0 \end{aligned}$$

So when $a = 1$, the polynomial equation with the solutions shown is

$$z^5 + 9z^4 + 32z^3 + 40z^2 - 17z - 65 = 0$$

in its simplest form.