Unit 15

Vector calculus

Introduction

In Unit 2 you saw how a physical quantity can be categorised as either a scalar or a vector, and we spent some time adding, scaling and multiplying vectors. In this unit we consider scalars and vectors that have different values at different points in a region of space. For instance, you know from practical experience that the temperature in a bowl of hot soup varies quite significantly from the middle to the edge of the bowl. The temperature depends on where in the soup it is measured. Temperature is a scalar quantity, and the temperature distribution is an example of a *scalar field*. Similarly, the force of the Earth's gravity on a body in space depends on where the body is in relation to the Earth. The direction of the force is always towards the Earth's centre, and the magnitude of the force depends on the distance of the body from the Earth's centre. This dependence of force on position is an example of a *vector field*. Recall that you have already met vector fields in Subsection 1.2 of Unit 12, where they were introduced as a way of describing systems of differential equations.

For bodies close to the Earth's surface, we assume that the magnitude of the acceleration due to gravity is constant.

The main focus of this unit, and the next, is the differential calculus of scalar and vector fields, that is, the study of how scalar and vector fields vary from one point to another. Many physical laws are expressed in terms of the spatial variations of fields. One important example is the flow of heat by conduction. You will see that we can describe the spatial pattern of heat flow in a conducting material as a vector field. The rate of heat flow at any point is proportional to the negative of the temperature *gradient*. You will see in Section 3 how we can express this relationship in vector form.

Section 1 is a brief introduction to the properties of orthogonal matrices that we need in Sections 3 and 4.

Section 2 introduces pictorial representations of scalar and vector fields, such as contour curves and vector field lines, and describes how we use functions of two or three spatial variables to represent, or model, scalar and vector fields mathematically. In Section 3 we extend the discussion of the *gradient function* of a scalar field from Unit 7.

While most of the calculations are carried out using Cartesian coordinates, it is often easier to use other coordinate systems that are more suited to the symmetry of the problem. Section 4 introduces *cylindrical* and *spherical coordinate systems* for specifying points in three dimensions. We find expressions for the gradient function in each of these systems.

These coordinate systems are generalisations of the plane polar coordinate system.

The *gradient function* and the associated vector operator 'del' are fundamental tools of vector calculus: they are used to solve a range of physical and engineering problems, especially problems involving heat flow, fluid flow, dynamics and electromagnetism.

You will meet the gradient function again in Unit 16, in connection with 'conservative' forces.

1 Orthogonal matrices

Later in the unit we will study some very useful coordinate systems in three dimensions. However, utility comes at a price, and some of the calculations involved in using these systems can be quite complicated. Fortunately, much of the complication can be removed by judicious use of matrix algebra, so in this short first section we study the properties of a particular type of matrix.

Two vectors are **orthogonal** if they are perpendicular to each other.

One way of defining a set of coordinates is by specifying three unit vectors to form a right-handed set. The set of Cartesian unit vectors **i**, **j** and **k** is one example, but there are other possibilities. Recall that you can think of **i**, **j** and **k** as, respectively, the unit vectors in the directions of increasing x, y and z. Similarly, for the polar coordinates, we can introduce two orthogonal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Figure 1(a) shows these vectors at a point P. In the same way that **i** and **j** are unit vectors in the directions of increasing x and increasing y, respectively, $\mathbf{e}_r$ is a unit vector in the direction of increasing r (the **radial direction**), and $\mathbf{e}_\theta$ is a unit vector in the direction of increasing θ (the **tangential direction**). Although the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ have constant magnitude (they are *unit* vectors), their directions depend on the position of P. This is illustrated in Figure 1(b), which shows two different positions of P (both positions are on a circle).

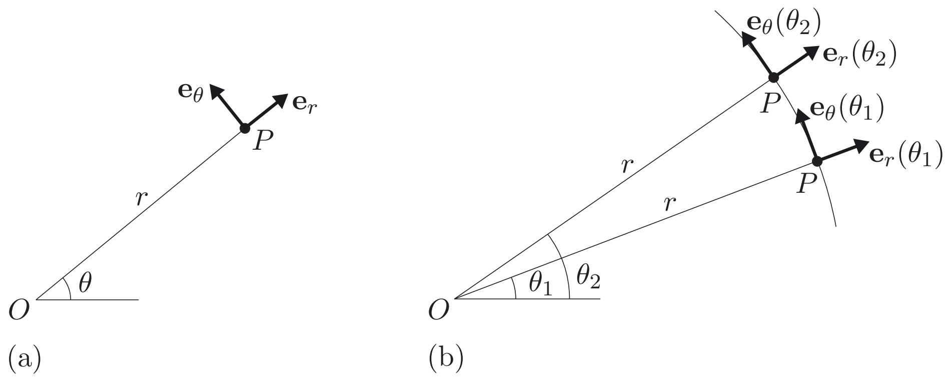

Figure 1 (a) Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ at (r, θ) in polar coordinates. (b) Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ at two points (r, θ_1) and (r, θ_2) on a circle of radius r.

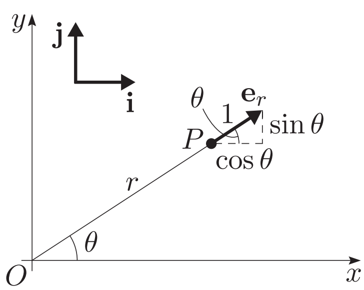

Figure 2 Unit vector $\mathbf{e}_r$ at point P with polar coordinates (r, θ)

In order to relate the coordinates of a point in one system to those in another, it will be useful to have a systematic way of transforming the corresponding sets of unit vectors. Thus we need to express each of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as a linear combination of **i** and **j**. Take the Cartesian coordinate system to have its origin at O, the centre of a circle with circumference passing through a point P, and take the positive x-axis to lie along the line from O from which the polar coordinate θ is measured, as shown in Figure 2.

First, let us consider $\mathbf{e}_r$. From Figure 2, the x-component of the *unit* vector $\mathbf{e}_r$ is $\cos\theta$, whereas its y-component is $\sin\theta$, so

$$\mathbf{e}_r = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}. \tag{1}$$

Exercise 1

Find an expression for the unit vector $\mathbf{e}_\theta$ in terms of $\mathbf{i}$, $\mathbf{j}$ and θ.

In Exercise 1 you showed that

$$\mathbf{e}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}. \tag{2}$$

Equations (1) and (2) show that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the polar coordinate angle θ.

Exercise 2

Show explicitly that the vectors

$$\mathbf{e}_r = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j} \quad \text{and} \quad \mathbf{e}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}$$

form an orthogonal pair of unit vectors, that is, show that they satisfy the conditions

$$\mathbf{e}_r \cdot \mathbf{e}_r = 1, \quad \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1, \quad \mathbf{e}_r \cdot \mathbf{e}_\theta = 0.$$

To summarise, we have found that for polar coordinates,

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}. \end{aligned}$$

These equations can be written in matrix form as

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix},$$

This is a new use of vectors, in which the components of vectors are vector quantities.

and doing so enables us to see quickly how to express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

$$\begin{aligned} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}. \end{aligned}$$

It is an interesting fact that the inverse of the matrix is the same as its transpose.

Of course, matrices can be thought of in another context, as you saw in Unit 4: any 3×3 matrix represents a linear transformation of space. However, the type of transformation with which we are concerned here is rather special. A general linear transformation will alter lengths and angles, so the images of a right-handed set of unit vectors will usually be neither of unit length nor mutually perpendicular. In changing coordinates we wish to ensure that a right-handed set of unit vectors remains as such (which, incidentally, rules out reflections).

That is, a linear transformation fixes the origin and transforms lines into lines.

This means that the distance between any two points is preserved.

It may help to think of the two sets of unit vectors as having arrowheads that lie on the surface of a unit sphere (of radius 1) – see Figure 3. It is not hard to see (though we do not prove it) that any transformation taking one set to the other is simply a rotation of three-dimensional space about the origin.

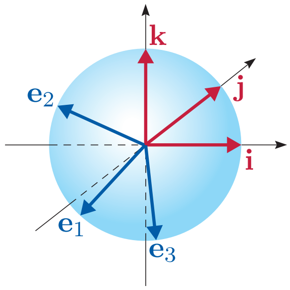

Figure 3 Two sets of orthogonal unit vectors: here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ form one right-handed set of unit vectors, and $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ form another

So we concentrate on rotations. Consider a pair of points P_1 and P_2, and their rotated images P_1' and P_2'. A rotation does not change the distance between two points, so the distance between P_1 and P_2 is the same as the distance between P_1' and P_2'. A similar remark applies to the angle between two lines – think of the angle between two straight lines l_1 and l_2. After a rotation, the images l_1' and l_2' will be inclined at the same angle to one another. In summary, a rotation preserves distance and angle. One consequence of this observation is that since

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and since a rotation preserves each term on the right, a rotation must also preserve the dot product.

Suppose that the rotation is represented by the matrix $\mathbf{A}$. Now take any two vectors $\mathbf{v}$ and $\mathbf{w}$, and form their dot product $\mathbf{v} \cdot \mathbf{w}$. Since this is preserved by the rotation, we have $\mathbf{v} \cdot \mathbf{w} = (\mathbf{A}\mathbf{v}) \cdot (\mathbf{A}\mathbf{w})$. Therefore

Recall from Unit 4 that for any two matrices, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= \mathbf{A}\mathbf{v} \cdot \mathbf{A}\mathbf{w} \\ &= (\mathbf{A}\mathbf{v})^T(\mathbf{A}\mathbf{w}) \\ &= \mathbf{v}^T(\mathbf{A}^T\mathbf{A})\mathbf{w} \\ &= \mathbf{v} \cdot (\mathbf{A}^T\mathbf{A}\mathbf{w}).\end{aligned}$$

Because this is true for any vector $\mathbf{v}$, it must hold when $\mathbf{v}$ is, in turn, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Thus the components of $\mathbf{A}^T\mathbf{A}\mathbf{w}$ are the same as those of $\mathbf{w}$, so $\mathbf{A}^T\mathbf{A}\mathbf{w} = \mathbf{w}$. This is true for any vector $\mathbf{w}$, so we are forced to conclude that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, the identity matrix. This confirms what we saw above in one (two-dimensional) case, namely that the inverse of the transformation matrix is its transpose.

A matrix $\mathbf{A}$ that satisfies the property $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ is called **orthogonal**. Any orthogonal matrix must preserve lengths and angles, since (according to the above argument in reverse) it preserves the dot product.

Exercise 3

What is the determinant of an orthogonal matrix $\mathbf{A}$?

Exercise 4

Which of the following matrices are orthogonal?

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$

In Exercise 3 you saw that the determinant of an orthogonal matrix is ± 1. However, the converse is not true: a matrix with determinant 1 may not be orthogonal, as you saw in Exercise 4(a).

If the determinant of an orthogonal matrix is -1, then the matrix represents a rotated reflection, though we do not prove this.

Exercise 5

Construct a 3×3 matrix $\mathbf{A}$ that is not orthogonal but is such that $\det \mathbf{A} = 1$.

2 Scalar and vector fields

In this section we introduce scalar and vector fields, and use ideas from Unit 7 to show how they can be described mathematically.

2.1 Scalar fields

Figure 4 shows the steady-state temperature, in degrees Celsius, at different points on a square plate. Temperature is a scalar quantity, and its value depends on where on the plate it is measured. The distribution of temperatures on the plate is an example of a *scalar field*. If we define Cartesian coordinate x- and y-axes as shown, then the variable Θ, representing the temperature, will be a function of x and y, that is, $\Theta = \Theta(x, y)$. (This is an example of a function of more than one variable, as introduced in Unit 7.)

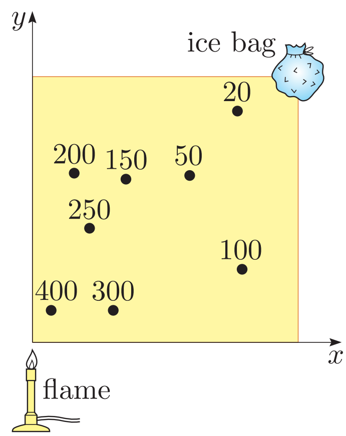

Figure 4 Steady-state temperature marked at various points on an unevenly heated plate

Another example of a scalar field is the atmospheric pressure distribution, in millibars, at ground level over the British Isles at midday on a certain day, as shown in Figure 5. Atmospheric pressure is a scalar quantity, and its value depends on where in the British Isles it is measured. The curves in Figure 5 are *isobars* and join up places where the atmospheric pressure at ground level is the same. Each isobar is labelled with the value of the atmospheric pressure in millibars. The isobars in Figure 5 give an overall picture of the atmospheric pressure across the country.

On weather maps, the pressure is usually given in millibars. The SI unit of pressure is the pascal (Pa). (1 bar $= 10^3$ millibars $= 10^5\,\text{Pa} = 10^5\,\text{N m}^{-2}$.)

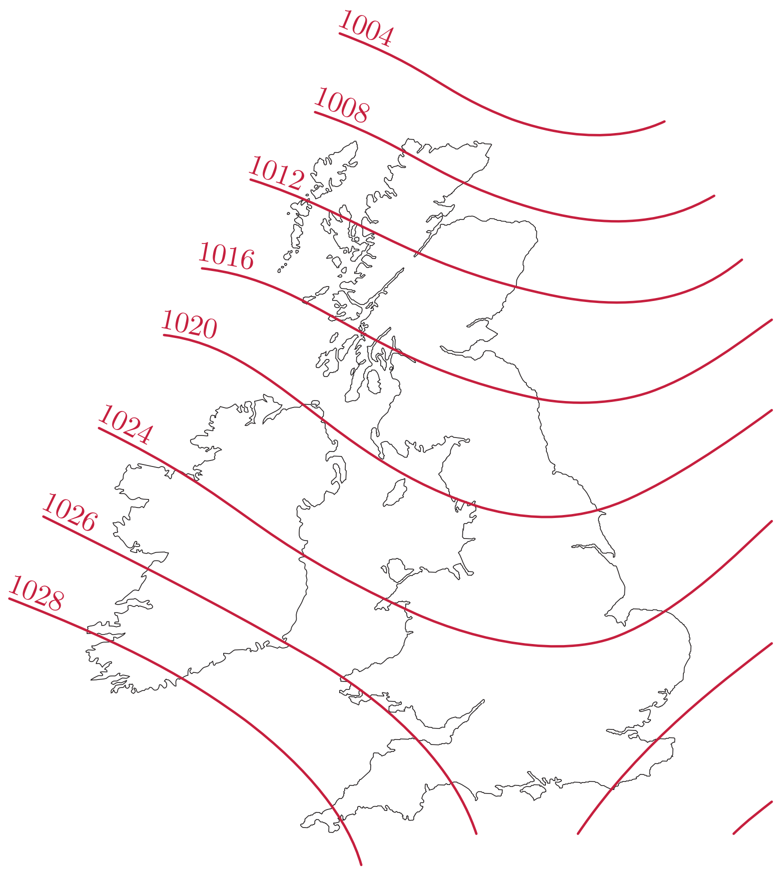

Figure 5 Isobars at ground level over the British Isles at midday

Note that the curvature of the Earth's surface is being ignored here.

If we choose London as an origin and set up a Cartesian coordinate system with the x-axis pointing East and the y-axis pointing North, then each place in the British Isles can be represented by its coordinates (x, y). The atmospheric pressure distribution can then be represented by a function of two variables $P = P(x, y)$. Of course, atmospheric pressure varies over time. But in this unit we consider only the spatial variations of scalar fields, not the time variations.

In each of the above cases, the variable representing the scalar quantity Θ or P is a function of the two coordinates x and y. These scalar distributions are examples of *two-dimensional scalar fields*, that is, they are scalar fields on a two-dimensional domain.

The steady-state temperature distribution in a room is an example of a *three-dimensional scalar field*. (The atmospheric pressure at a particular instant in time is a three-dimensional scalar field, since it varies with altitude as well as with position. In Figure 5 only the ground-level pressures at midday are shown, so this is effectively a two-dimensional scalar field.)

Each point in the room, which can be located by three coordinates, has a temperature.

In any scalar field we have:

- some scalar quantity (e.g. temperature)
- a region of a plane or a region of space, i.e. a domain, over which the scalar is defined (e.g. a flat plate or the interior of a room)
- the values of the scalar quantity at all points of the region.

The mathematical model of a two-dimensional scalar field is a function $f(x, y)$ of two spatial variables. For a three-dimensional field, a function $f(x, y, z)$ of three spatial variables is required.

Functions of two and three variables were introduced, and discussed extensively, in Unit 7.

Scalar field

A **scalar field** is a distribution of scalar values in a two- or three-dimensional region and is represented mathematically by a function of two or three spatial variables, respectively.

The domain of the field is the given two-dimensional or three-dimensional region in which the field exists. We will use the terms *scalar field* and *function of two (or three) variables* interchangeably, according to whether we want to emphasise the physical or the mathematical properties.

For example, the domain of the atmospheric pressure field is the map of the British Isles.

Example 1

The light intensity I in the region outside a spherical lamp of radius a, such as a light bulb, is inversely proportional to the square of the distance r from the centre of the lamp. The intensity 1 m away from the centre is I_0.

Specify a scalar field, as a function of three Cartesian variables, that models the light intensity in the space outside the lamp, and state the domain of the function.

Solution

The light intensity at a point a distance $r > a$ from the centre of the lamp is $I \propto 1/r^2$, with $I = I_0$ when $r = 1$. Thus $I = I_0/r^2$. Using a Cartesian coordinate system with origin at the centre of the lamp, we have $r = (x^2 + y^2 + z^2)^{1/2}$, so the scalar field function is

$$I(x, y, z) = \frac{I_0}{x^2 + y^2 + z^2} \quad ((x^2 + y^2 + z^2)^{1/2} > a).$$

The domain of the function is the statement in parentheses, expressing the fact that the function describes light intensity in the region outside the lamp.

The intensity *inside* the lamp would be represented by a different function.

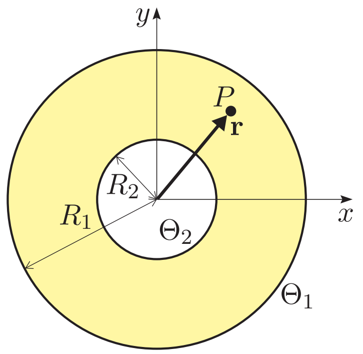

Figure 6 Annular sheet of metal with outer and inner edges kept at constant temperatures Θ_1 and Θ_2, respectively

Exercise 6

A thin circular sheet of metal of radius R_1 (in metres) has a concentric hole of radius R_2 (in metres) cut out of it (see Figure 6). The outer perimeter is maintained at a constant temperature Θ_1, and the inner perimeter is maintained at a constant temperature Θ_2. At any point P on the sheet, specified by a position vector $\mathbf{r}$ measured from the centre, the temperature (in kelvins) is given by

$$\Theta = \Theta_1 + \frac{\Theta_2 - \Theta_1}{\ln(R_2/R_1)} \ln\left(\frac{|\mathbf{r}|}{R_1}\right).$$

(a) Choose a Cartesian coordinate system with origin at the centre of the circular metal sheet, and hence specify a function of x and y that models the temperature. Describe the region of the plane over which the function is defined, and state the domain.

(b) Express the function found in part (a) in polar coordinates (r, θ).

2.2 Contour curves and contour surfaces

The scalar fields in the previous subsection were presented in three different ways. For the first scalar field, of temperature on a plate as given in Figure 4, we represented the field by showing the values of the temperature at a few points on the plate. For the second scalar field, the atmospheric pressure in the British Isles as shown in Figure 5, we represented the field by isobars (lines joining points at which the atmospheric pressure is the same) on a map of the British Isles. In both cases, the representation of the scalar field is not complete because, for instance, there is a temperature associated with *every* point of the flat plate, and not just the points shown in Figure 4. Third, in Exercise 6 and Example 1, the scalar fields were represented by functions of two or three Cartesian variables.

Of these modes of representation, the isobar example gives a good way of visualising the field pictorially. Figure 7 shows the temperature distribution (in degrees Celsius) over an unevenly heated flat plate similar to that shown in Figure 4. However, instead of giving the values of the temperature at different points as we did before, we have drawn curves through those points on the plate where the temperature is the same. Of course we cannot show all possible curves because that would require a curve through every point on the plate, so we choose certain values of the temperature. Then the curves give us a pictorial idea of the scalar field. So, for example, for each point on the curve AB in Figure 7, the temperature is 300°C.

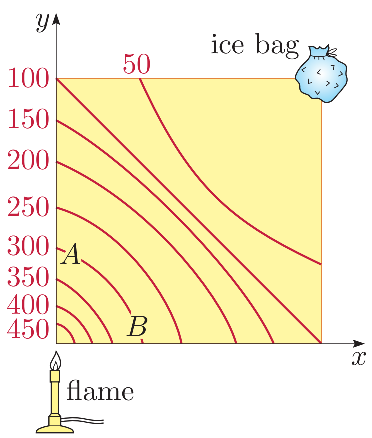

Figure 7 Isotherms on an unevenly heated flat plate

For a temperature field, the curves are called *isotherms*, and in this example we have drawn the isotherms for multiples of 50°C. In one part of the plate the isotherms are much closer together than those on the rest of the plate. The temperature is changing more rapidly in the region where the isotherms are close together.

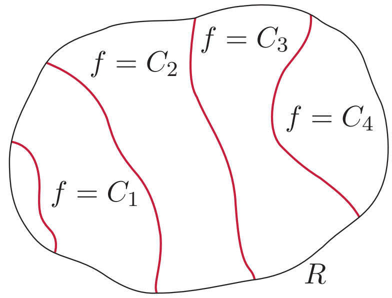

Figure 8 Contours of a function f over a region R

As was shown in Unit 7, this pictorial representation for pressure and temperature can be extended to more general two-dimensional scalar fields. Suppose that we have a two-dimensional scalar field $f(x, y)$, defined over a region R of the (x, y)-plane. We can represent this scalar field pictorially by drawing curves in the (x, y)-plane through those points that have equal values of f. Figure 8, for example, shows four such curves, where C_1, C_2, C_3 and C_4 are constants. The curves that we have drawn are called *contour curves*. (Contour curves were introduced in Unit 7.) Each contour curve is labelled by a scalar value.

The isobars in a pressure field are contour curves of equal pressure, and isotherms in a temperature field are contour curves of equal temperature.

For example, consider the scalar field

$$f(x, y) = x^2 + y^2$$

defined over the rectangular domain $-1 \leq x \leq 1$, $-2 \leq y \leq 2$ (see Figure 9). The contour curves are defined by

$$f(x, y) = \text{constant},$$

or, in this case,

$$x^2 + y^2 = \text{constant}.$$

These curves are circles centred on the origin and are drawn as parts of circles within the rectangular region for which f is defined. Figure 9 shows the contour curves

$$f(x, y) = 1, \quad f(x, y) = 2, \quad f(x, y) = 3, \quad f(x, y) = 4.$$

Again we cannot draw *all* possible curves, only some of them.

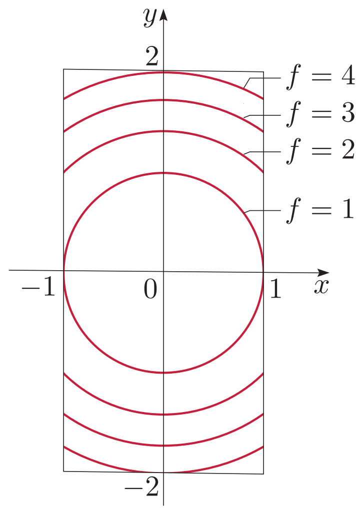

Figure 9 Contours of the function $f(x, y) = x^2 + y^2$

Contour curves

The family of curves given by $f(x, y) = C$, for different values of the constant C, are the **contour curves** of the two-dimensional scalar field f.

Exercise 7

Sketch the contour curves $f(x, y) = \frac{1}{2}$, $f(x, y) = -\frac{1}{4}$ and $f(x, y) = 1$ of the scalar field $f(x, y) = xy$, defined over the domain $x^2 + y^2 \leq 4$.

You are probably familiar with the word 'contour' from Ordnance Survey or other topographical maps, where the contour curves join points that are at the same height above sea level. The contours can give us information about the shape of the land. Figure 10 provides an example where the contour curves join points at the specified number of metres above sea level.

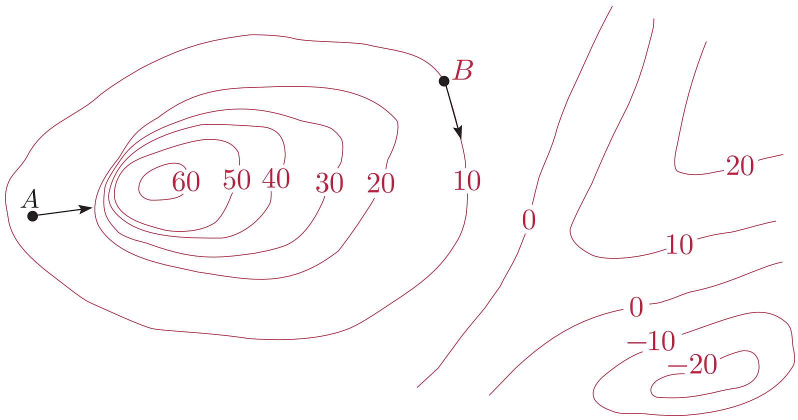

Figure 10 Topographical map showing contours indicating height above sea level

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Exercise 8

Consider Figure 10.

(a) Where would you expect to find (i) a hill and (ii) a lake?

(b) Where is the land fairly level?

(c) Imagine that you set off from A and walk in the direction of the arrow. Would you expect an easy walk or a hard climb?

Exercise 8 illustrates an important feature of a scalar field that can be deduced from its contour curves. For any scalar field, the change in value of the field depends on the direction in which you move. For example, if you set off from B in Figure 10 and walk in the direction of the arrow, then initially the field values will not change (you are contour walking). However, for movement as in Exercise 8(c), which is approximately perpendicular to the contours, the scalar field changes (increases or decreases) with changing position.

So far in this subsection we have considered two-dimensional fields. Now we turn our attention to three-dimensional scalar fields.

We can represent a three-dimensional scalar field $f(x, y, z)$ in an analogous way. If we join up points for which the three-dimensional scalar field $f(x, y, z)$ has the same constant value C, then we obtain the *surface* with equation $f(x, y, z) = C$.

Contour surfaces

The family of surfaces given by $f(x, y, z) = C$, for different values of the constant C, are the **contour surfaces** of the scalar field f.

Contour surfaces give a pictorial representation of a three-dimensional scalar field. For example, the contour surfaces of the scalar field $I = I_0/(x^2 + y^2 + z^2)$, given in Example 1, are concentric spheres with centres at the origin. Two of the contour surfaces, for $I = I_0/4$ and $I = I_0/9$, are shown in Figure 11.

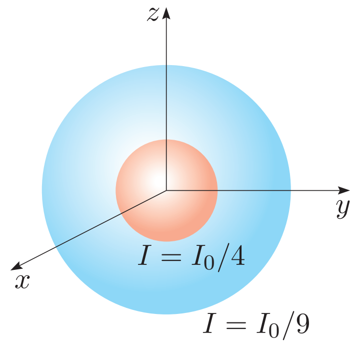

Figure 11 Contour surfaces of the function $I = I_0/(x^2 + y^2 + z^2)$

Exercise 9

The density σ of air is $1.205\,\mathrm{kg\,m^{-3}}$ at sea level. Assume that σ decreases exponentially with altitude and is reduced to $(1.205/e)\,\mathrm{kg\,m^{-3}}$ at an altitude of $9.5 \times 10^3\,\mathrm{m}$.

Assuming that the Earth's surface is flat, choose a Cartesian coordinate system and hence express the density σ as a scalar field $\sigma(x, y, z)$. What shape are the contour surfaces?

Applied mathematicians and physicists often need to know how a quantity 'e-folds', that is, they need to know the distance (or time) in which the quantity reduces to $1/e$ of its original value.

2.3 Vector fields

Figure 12(a) shows the pattern produced when iron filings are placed near a bar magnet. The iron filings align themselves in a symmetric pattern. The same pattern is revealed if we place small compasses at various points near the magnet. We find that the arrow on each compass aligns itself in the same direction as the iron filings. The arrow shows the direction of the magnetic field at each point. The magnitude of the field can also be measured, and is found to decrease as we move away from the magnet.

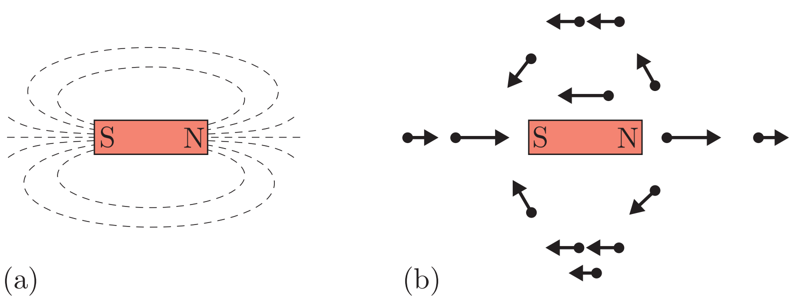

Figure 12 (a) Iron filing pattern near a bar magnet. (b) Magnetic field near a bar magnet.

The lengths and directions of the arrows in Figure 12(b) represent the magnitudes and directions of the magnetic field due to the bar magnet at various points on a plane through the magnet. There is a unique vector called the *magnetic field vector* at each point in the region of space around the magnet, and these vectors form a three-dimensional *vector field*.

Figure 13 shows an example of a two-dimensional vector field. The arrows represent the surface velocity at various points on a river.

We assume that the flow is steady. It is not changing with time.

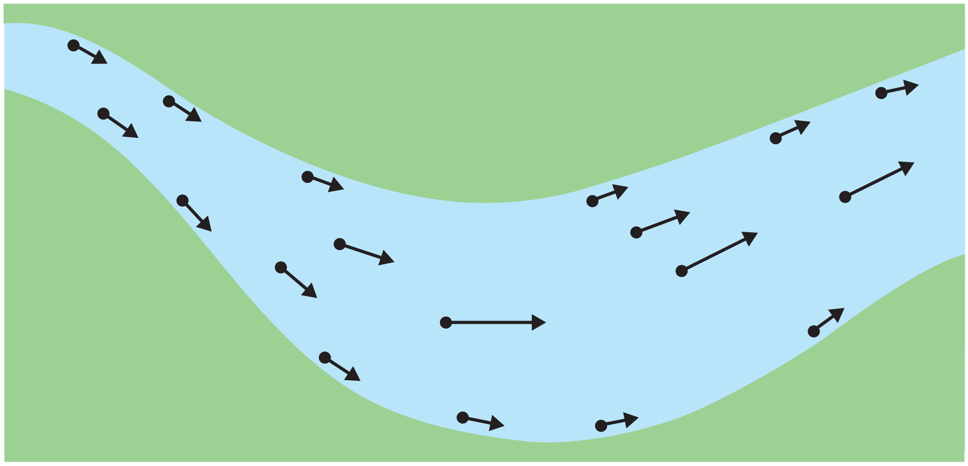

Figure 13 Surface velocity at various points along a river

The surface velocity at a particular point is the velocity that a small floating object, such as a leaf, would have at that point. Near the river banks, the water is almost at rest, so the arrows are shorter in length (indicating a smaller speed near the edges of the river). At each point on the river surface there is a unique velocity, thus defining a two-dimensional vector field of velocity vectors on the surface. If we were interested in the velocity below the surface as well, then we would need to consider the velocity vectors existing at points in the three-dimensional region forming the river itself, and this would be a three-dimensional vector field.

In any vector field we have:

- some vector quantity (e.g. a surface velocity)
- a region of a plane or a region of space, i.e. a domain, over which the vector is defined (e.g. the surface of a river)
- the magnitudes and directions of the vectors at all points in the region.

We can model a two-dimensional vector field mathematically by introducing a vector function $\mathbf{F}$ such that a vector $\mathbf{F}(x, y)$ is defined at each point (x, y) of a two-dimensional domain. In three dimensions, $\mathbf{F}(x, y, z)$ is a vector at each point (x, y, z) in a three-dimensional domain.

You met the idea of a vector field in Unit 12.

Vector field

A **vector field** is a distribution of vectors in a two- or three-dimensional region and is represented mathematically by a vector function of two or three spatial variables.

The domain of the vector function is the given region of a plane or of space in which the field exists. We use the terms *vector field* and *vector function* interchangeably, depending on whether we want to emphasise the physical or mathematical properties. Often we use the name of the physical quantity itself, such as *force field*, *velocity field* or *magnetic field*.

Example 2

The Earth's gravitational field is an example of a vector field in space. The gravitational force on a body (i.e. the weight of the body) on or above the Earth's surface at a distance r from the centre of the Earth has magnitude inversely proportional to r^2 and is directed towards the Earth's centre.

Specify the vector field as a function of three Cartesian variables, with the domain over which it is defined.

Solution

The vector quantity is the gravitational force $\mathbf{F}$ acting on the body due to the Earth's gravity. We introduce a Cartesian coordinate system with origin O at the centre of the Earth. The magnitude $F = |\mathbf{F}|$ is proportional to $1/r^2$, so

$$F(x, y, z) = \frac{C}{r^2} = \frac{C}{x^2 + y^2 + z^2},$$

for some constant C. The gravitational force on a body at P (see Figure 14) is $-F\,\widehat{\mathbf{r}}$, where $\widehat{\mathbf{r}}$ is a unit vector in the direction from O to P. Now $\widehat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ is the position vector of P, so

$$\widehat{\mathbf{r}}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}.$$

Thus the vector function that we need is

$$\mathbf{F}(x, y, z) = -F(x, y, z)\,\widehat{\mathbf{r}}(x, y, z) = \frac{-C(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}.$$

The domain is the region over which the vector field $\mathbf{F}$ is defined, that is, the region of space on or outside the Earth's surface, specified by

$$(x^2 + y^2 + z^2)^{1/2} \geq R,$$

where R is the radius of the (spherical) Earth.

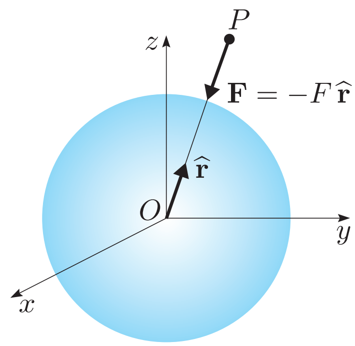

Figure 14 Gravitational force $\mathbf{F}$ acting at point P above the surface of the Earth, which is centred at the origin

At the surface of the Earth, where the magnitude of the acceleration due to gravity is measured as g, this field can be approximated by the constant-magnitude vector field $\mathbf{F} = -mg\,\widehat{\mathbf{r}}$, where m is the mass of the body, so $C = mgR^2$.

The components of a vector field $\mathbf{F}$, relative to a given Cartesian coordinate system, are themselves *scalar* fields defined over the same region as $\mathbf{F}$. Thus if $\mathbf{F}$ is a three-dimensional vector field defined on a domain D and

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k},$$

then, separately, the three components $F_1(x, y, z) = \mathbf{F} \cdot \mathbf{i}$, $F_2(x, y, z) = \mathbf{F} \cdot \mathbf{j}$ and $F_3(x, y, z) = \mathbf{F} \cdot \mathbf{k}$ are scalar fields defined on D. For example, the x-component of the vector field $\mathbf{F}$ in Example 2 is the scalar field

$$F_1(x, y, z) = \mathbf{F} \cdot \mathbf{i} = \frac{-Cx}{(x^2 + y^2 + z^2)^{3/2}} \quad ((x^2 + y^2 + z^2)^{1/2} \geq R).$$

Alternatively, we could use the column vector notation of Unit 4 and write a vector field in three dimensions as

$$\mathbf{F}(x, y, z) = \begin{pmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{pmatrix}.$$

Exercise 10

The weight $\mathbf{F}$ of a body of mass m inside the Earth (e.g. in a mine shaft) has a magnitude given by mgr/R, where r is the distance from the Earth's centre and R is the Earth's radius. The direction of $\mathbf{F}$ is towards the Earth's centre.

Express $\mathbf{F}$ as a vector function $\mathbf{F}(x, y, z)$ using the same Cartesian coordinate system as in Example 2. What is the weight of the body at the Earth's centre?

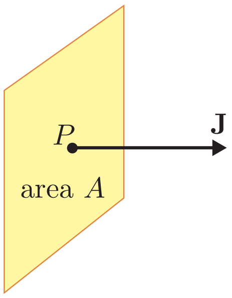

Figure 15 Heat flow $\mathbf{J}$ at point P in a conductor

The pattern of heat flow in a heat-conducting material is another example of a vector field. Heat is a form of energy (and is therefore measured in joules) that flows through a conducting body due to the motion and interactions of the constituent particles (e.g. molecules) of the body. When the magnitude and direction of the heat flow vary from place to place in a heat conductor, the heat flow rate per unit area at a point P in the conductor is a vector $\mathbf{J}$ whose direction is the direction of the heat flow at P. The magnitude of $\mathbf{J}$ is the rate at which heat flows across a very small plane surface containing P and oriented at right angles to the direction of flow, divided by the area A of the surface (see Figure 15). So $|\mathbf{J}|$ is the heat flow rate per unit area at P and has units of joules per second per square metre (i.e. $\mathrm{J\,s^{-1}\,m^{-2}}$, or $\mathrm{W\,m^{-2}}$, since 1 watt (W) equals $1\,\mathrm{J\,s^{-1}}$). (The vector $\mathbf{J}$ is called the *heat flow vector* or the *heat conduction vector*.) To define $|\mathbf{J}|$ at P formally, we take the limit as the area A of the plane surface approaches zero, that is,

$$|\mathbf{J}| = \lim_{A\to 0} \frac{\text{heat flow rate across surface area } A}{A}.$$

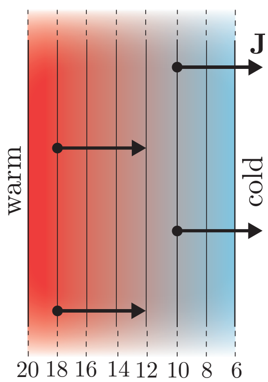

Figure 16 Cross-section of a wall in a house showing heat flow (arrows) and temperature contours

Figure 16 shows a selection of arrows representing a heat flow vector field $\mathbf{J}$ in a cross-section of a wall of a heated house. Shown also are vertical lines representing the scalar temperature contours. These lines are the intersections of the temperature contour surfaces with the plane of the figure. The contours are labelled by the temperature, in degrees Celsius, for the case where the inside surface is at 20°C and the outside surface is at 6°C. Hence Figure 16 depicts two fields: a scalar (temperature) field and a vector (heat flow) field. It is assumed that the temperature has a different constant value on each face of the wall and that the wall is homogeneous. Then the vector $\mathbf{J}$ has the same magnitude and direction everywhere inside the wall.

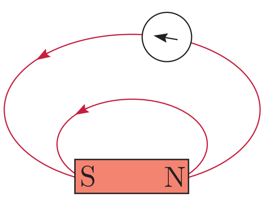

Figure 17 Magnetic field lines of a bar magnet

If we draw continuous curves in the domain such that at any point, the tangent to the curve is parallel to the direction of the vector field at that point, then the curves are called the **vector field lines** of the vector field. For instance, the orientation of iron filings near a bar magnet suggests a family of continuous curves. Two such curves are shown in Figure 17. The arrow on a vector field line specifies the sense, that is, the direction along the line, of the vector field. If we draw a curve showing the path of a leaf floating on the surface of a river, then at each point on the path, the tangent to the path is parallel to the direction of the velocity vector of the leaf. This line is a vector field line for the velocity vector field. If we were to show the vector field lines of the heat flow vector $\mathbf{J}$ in Figure 16, they would be at right angles to the temperature contour surfaces shown as lines in that figure.

Exercise 11

Figure 18 shows the cross-section of a long cylindrical central heating pipe carrying hot water. The heat flow vector $\mathbf{J}$ in the metal of the pipe is given (in the steady state) by

$$\mathbf{J}(x, y, z) = \frac{C(x\mathbf{i} + y\mathbf{j})}{x^2 + y^2} \quad (a \leq \sqrt{x^2 + y^2} \leq b),$$

where a, b and C are positive constants, and the z-axis is the central axis of the pipe.

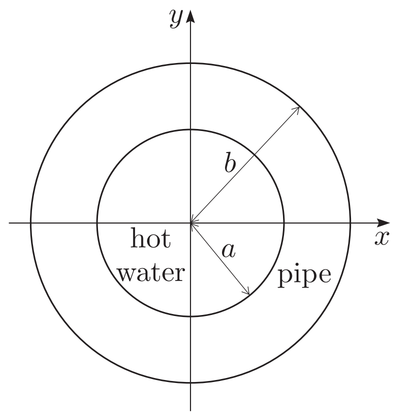

Figure 18 Cross-section of a central heating pipe

(a) Specify the x-component of $\mathbf{J}$.

(b) Describe the vector field lines of $\mathbf{J}$.

(c) How does the magnitude $|\mathbf{J}|$ vary with distance $\rho = \sqrt{x^2 + y^2}$ from the z-axis?

(d) Consider an imaginary cylindrical surface of length h with its axis on the z-axis and radius ρ such that $a \leq \rho \leq b$. By making use of your answers to parts (b) and (c), show that the rate at which heat flows outwards across the whole curved surface of this section of the cylinder is independent of the radius ρ. What does this result mean in terms of conservation of heat energy?

Note that although the vector field $\mathbf{J}$ of Exercise 11 exists in three dimensions, its z-component is zero and there are no variations in the z-direction. The field could therefore be represented by a two-dimensional vector function

$$\mathbf{J}(x, y) = \frac{C(x\mathbf{i} + y\mathbf{i})}{x^2 + y^2} \quad (a \leq \sqrt{x^2 + y^2} \leq b).$$

Exercise 12

When a long thin straight wire carries a uniform surface distribution of static positive electric charge, the electric field vector $\mathbf{E}$ at any point outside the wire has magnitude $|\mathbf{E}|$ that is inversely proportional to the perpendicular distance from the centre of the wire, and direction pointing directly away from the wire.

Describing the wire as 'thin' implies that it has negligible radius. It is just a straight line.

Introduce a Cartesian coordinate system and hence specify the vector field $\mathbf{E}(x, y, z)$. Take $|\mathbf{E}|$ to be the constant E_0 at unit distance from the wire. Describe the vector field lines.

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3 Gradient of a scalar field

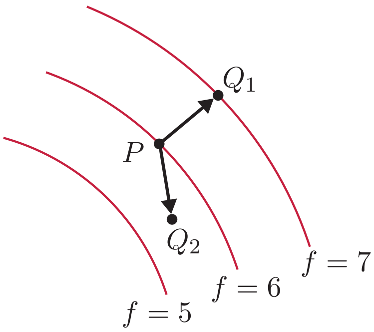


Figure 19 Contours of a function f

An important property of any field is the way in which the field value changes from one point to another. These spatial variations can be quite complicated even for a scalar field, since the change of field value in going from a point P to a nearby point Q may depend on the direction as well as the magnitude of the displacement $\overrightarrow{PQ}$. Consider Figure 19, which shows three contour curves of a scalar field f. The change in f is $7 - 6 = 1$ when the displacement $\overrightarrow{PQ_1}$ is made, and about $5.5 - 6 = -0.5$ for a displacement $\overrightarrow{PQ_2}$ of the same magnitude as $\overrightarrow{PQ_1}$ but in a different direction. We will show in this section that the way in which a scalar field varies in *any* direction can be found from the *gradient function*, which was introduced in Unit 7. There is a gradient function at each point in a scalar field, so the gradient function is a vector field.

In Unit 7 you saw how to use the gradient function to find the magnitude of the steepest upward slope at a point on a surface (and the direction in which that steepest slope occurs). The gradient function for a general two-dimensional scalar field $f(x, y)$ was expressed in terms of partial derivatives of f. You also saw in Unit 7 how the gradient function can be used to calculate slopes in arbitrary directions. In Subsections 3.1 and 3.2 we review those ideas, and in Subsection 3.3 we consider how the expression for the gradient function can be generalised to three dimensions.

You met del in Unit 7, in conjunction with the gradient function.

Subsection 3.4 discusses the *vector differential operator* called 'del', which plays a unifying role in vector calculus. Finally, in Subsection 3.5, we see how to define del in polar coordinates.

3.1 Gradient function

A two-dimensional scalar field $f(x, y)$ is defined only in the (x, y)-plane. This leaves the z-axis of the coordinate system free for showing the scalar values f, that is, we can put $z = f(x, y)$. Thus the set of points $(x, y, f(x, y))$ is a surface above (or below) the (x, y)-plane, giving a graphical picture of the scalar field. In general, the surface $(x, y, f(x, y))$ is a graphical representation of the scalar field f and shows how f varies with position in the (x, y)-plane.

You saw examples of such surfaces in Unit 7.

Consider a general two-dimensional scalar field f and a path in the (x, y)-plane parametrised by the arc length s – that is, as s varies, the point $(x(s), y(s))$ moves along the path in the (x, y)-plane. The height z of the surface will also vary as s varies, that is, $z = z(s)$. The rate of change of z with s at a point P in the (x, y)-plane is the slope or steepness of the surface for movement in the direction of the path, that is, it is the vertical rise of z over a small horizontal change in arc length δs in the direction of the path at P, and is given by the chain rule, which we can write as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}, \tag{3}$$

where all the derivatives and partial derivatives are to be evaluated at P.

Let us first consider a small change in length along the path at P in the (x, y)-plane. Since it is small, we can take it to be in the direction of the tangent to the path at P, a direction given by a unit vector $\widehat{\mathbf{d}}$, say, as shown in Figure 20. If α is the angle at P between $\widehat{\mathbf{d}}$ and the positive direction of the x-axis, then $\cos\alpha = dx/ds$ and $\sin\alpha = dy/ds$, and equation (3) may be rewritten as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}(a, b)\cos\alpha + \frac{\partial f}{\partial y}(a, b)\sin\alpha, \tag{4}$$

where we have shown explicitly that the partial derivatives are to be evaluated at the point P whose coordinates are (a, b). While the angle α depends on the direction of the path at P, the partial derivatives in equation (4) are independent of any particular path through P, and depend only on the scalar field f and the coordinates (a, b) of P. We can take advantage of this by using the **gradient function** in the (x, y)-plane at the point (a, b):

$$\mathbf{grad}\, f(a, b) = \frac{\partial f}{\partial x}(a, b)\,\mathbf{i} + \frac{\partial f}{\partial y}(a, b)\,\mathbf{j}.$$

The unit vector $\widehat{\mathbf{d}}$ may be written in terms of α as

$$\widehat{\mathbf{d}} = (\cos\alpha)\,\mathbf{i} + (\sin\alpha)\,\mathbf{j}.$$

So we can now write equation (4) as the dot product of $\mathbf{grad}\, f(a, b)$ and $\widehat{\mathbf{d}}$:

$$\frac{dz}{ds} = \mathbf{grad}\, f(a, b) \cdot \widehat{\mathbf{d}} = |\mathbf{grad}\, f(a, b)| \cos\theta, \tag{5}$$

where θ is the angle between $\mathbf{grad}\, f$ and $\widehat{\mathbf{d}}$, as shown in Figure 21.

You can see from equation (5) that dz/ds, the slope or steepness of the surface (the graph of f), has its maximum value (when $\cos\theta = 1$) for movement parallel to the (x, y)-plane in a direction $\widehat{\mathbf{d}}$ that coincides with the direction of the gradient function. Then we have

$$\left(\frac{dz}{ds}\right)_{\max} = |\mathbf{grad}\, f(a, b)|.$$

Hence the direction of the gradient function at (a, b) shows the direction of maximum slope of the surface, and the magnitude of the gradient function is equal to the magnitude of the maximum slope.

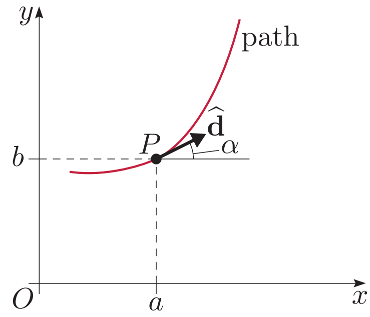


Figure 20 Unit vector at the point P

In Unit 7 you met the alternative notation $\boldsymbol{\nabla} f$ for $\mathbf{grad}\, f$, but we will postpone using that notation until we have defined $\boldsymbol{\nabla}$ in Subsection 3.4.

As with vectors, in handwritten work, **grad** should be denoted with a straight or wavy underline.

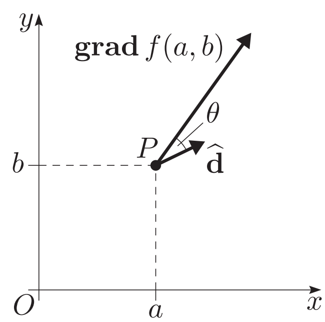


Figure 21 Finding the dot product of $\mathbf{grad}\, f(a, b)$ and $\widehat{\mathbf{d}}$

Maximum derivative of a scalar field

The value of the **maximum derivative** of the scalar field $f(x, y)$ at the point (a, b) is given by $|\mathbf{grad}\, f(a, b)|$, and the **direction** of this maximum derivative is given by the unit vector

$$\frac{\mathbf{grad}\, f(a, b)}{|\mathbf{grad}\, f(a, b)|}.$$

There is such a vector at each point in the (x, y)-plane, so $\mathbf{grad}\, f(x, y)$ is a vector field, often written simply as $\mathbf{grad}\, f$. In terms of the Cartesian unit vectors, we define it as follows.

This definition is the same as that given in Unit 7, but expressed in terms of fields. As the gradient is a vector field, i.e. a vector function, it is sometimes referred to as the *gradient function*.

Gradient in two dimensions

The **gradient in Cartesian coordinates** (x, y) **of a two-dimensional scalar field** f is the vector field

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}. \qquad (6)$$

Thus to find the gradient function at a particular point, it is necessary to find the two partial derivatives at that point and substitute them into equation (6).

For a temperature field measured in kelvins, the slope has units of kelvins per metre, i.e. $\mathrm{K\,m^{-1}}$.

The function f in the definition can represent any two-dimensional scalar field. For example, suppose that $\Theta(x, y)$ is the temperature field of a flat plate in the (x, y)-plane. Then $\mathbf{grad}\,\Theta$ is a vector parallel to the plane of the plate pointing in the direction in which the temperature increases most rapidly with distance, and $|\mathbf{grad}\,\Theta|$ is the value of the maximum rate of temperature change with distance.

The gradient function of a two-dimensional scalar field $f(x, y)$ is always parallel to the (x, y)-plane. It is a vector having only $\mathbf{i}$ and $\mathbf{j}$ components. You can think of it as lying in the (x, y)-plane, or you may prefer to imagine it positioned at the point $(x, y, f(x, y))$ and lying in a plane parallel to the (x, y)-plane. Equation (5) shows that we can use the gradient function to work out the slope, or steepness, dz/ds of a scalar field for movement in any direction parallel to the (x, y)-plane. This slope is the *derivative* of f in the *specified direction*.

This derivative is often referred to as a *directional derivative* or *slope*, the latter being particularly apt for a two-dimensional scalar field.

Derivative of a scalar field

The **derivative of a scalar field** f in a direction specified by a unit vector $\widehat{\mathbf{d}}$ is given by

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}}.$$

3.2 Calculating gradients

We can calculate gradient vectors of two-dimensional scalar fields $f(x, y)$, and derivatives in specified directions, by using the two definitions above.

Throughout this unit, when calculating $\mathbf{grad}\, f$ we assume that f is differentiable on its domain (in two and three dimensions).

Example 3

Evaluate $\mathbf{grad}\, f$ at the point $(0,1)$ when $f(x,y) = \ln(x+2y)$. Determine the magnitude of $\mathbf{grad}\, f$ and the unit vector in the direction of $\mathbf{grad}\, f$ at the point $(0,1)$. What is the derivative of f in the direction of $\mathbf{i}+\mathbf{j}$ at $(0,1)$?

Solution

The vector field $\mathbf{grad}\, f$ is found from the first partial derivatives of f:

$$\frac{\partial f}{\partial x} = \frac{1}{x+2y}, \quad \frac{\partial f}{\partial y} = \frac{2}{x+2y}.$$

So from equation (6),

$$\mathbf{grad}\, f = \left(\frac{1}{x+2y}\right)\mathbf{i} + \left(\frac{2}{x+2y}\right)\mathbf{j}.$$

This is the gradient vector field.

Putting $x=0$ and $y=1$, we have

$$\mathbf{grad}\, f(0,1) = \tfrac{1}{2}\mathbf{i} + \mathbf{j}.$$

So the magnitude of $\mathbf{grad}\, f$ at $(0,1)$ is

$$|\mathbf{grad}\, f(0,1)| = \left(\tfrac{1}{4}+1\right)^{1/2} = \sqrt{5}/2.$$

The unit vector in the direction of $\mathbf{grad}\, f$ at $(0,1)$ is

$$\frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{\mathbf{i}+2\mathbf{j}}{\sqrt{5}}.$$

The unit vector $\widehat{\mathbf{d}}$ in the direction of $\mathbf{i}+\mathbf{j}$ is given by

$$\widehat{\mathbf{d}} = \frac{\mathbf{i}+\mathbf{j}}{|\mathbf{i}+\mathbf{j}|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}.$$

Hence the derivative of f in the direction of $\mathbf{i}+\mathbf{j}$ is given at $(0,1)$ by the dot product

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}} = \frac{3}{2\sqrt{2}}.$$

Exercise 13

Evaluate $\mathbf{grad}\, f$ at the point $(-1,2)$ when $f(x,y) = x^2y$. Find the derivative of f in the x-direction at $(-1,2)$.

Exercise 14

A two-dimensional scalar field has the form $f(x,y) = (x^2 + y^2)^{1/2}$.

(a) Find the vector field $\mathbf{grad}\, f$, and evaluate $\mathbf{grad}\, f$ at the point $(1,1)$.

(b) Specify the unit vector $\widehat{\mathbf{d}}$ at $\frac{\pi}{6}$ to the positive x-direction and at $\frac{\pi}{3}$ to the positive y-direction. What is the derivative of f in the direction of $\widehat{\mathbf{d}}$ at $(1,1)$?

Exercise 15

Determine the vector field $\mathbf{grad}\, g$, where $g(x,y) = x^2y - y^2x$. Find the magnitude and direction of the steepest slope on the surface $z = g(x,y)$ at the point $(1,1)$.

Example 4

Evaluate $\mathbf{grad}\, f$ at the point $(1,2)$ when $f(x,y) = x^2 + y^2$. Show that the direction of $\mathbf{grad}\, f$ at $(1,2)$ is normal to the tangent line to the contour curve of f at the point $(1,2)$.

Solution

The first partial derivatives of f and their values at the point $(1,2)$ are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial x}(1,2) = 2,$$
$$\frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial y}(1,2) = 4.$$

Thus from equation (6),

$$\mathbf{grad}\, f(1,2) = \frac{\partial f}{\partial x}(1,2)\,\mathbf{i} + \frac{\partial f}{\partial y}(1,2)\,\mathbf{j} = 2\mathbf{i} + 4\mathbf{j}.$$

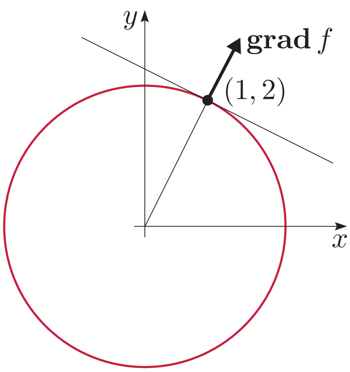


Figure 22 The contour of the function $f(x,y) = x^2 + y^2$ that passes through the point $(1,2)$

The contour curves are obtained by setting f equal to a constant and are circles centred on the origin. The contour curve through the point $(1,2)$ is shown in Figure 22, together with the tangent line and the normal to the tangent line through $(1,2)$. The position vector of $(1,2)$ is $\mathbf{i} + 2\mathbf{j}$, and this is normal to the tangent line. But $\mathbf{grad}\, f(1,2) = 2\mathbf{i} + 4\mathbf{j} = 2(\mathbf{i} + 2\mathbf{j})$. Hence $\mathbf{grad}\, f(1,2)$ is normal to the tangent to the contour curve at $(1,2)$.

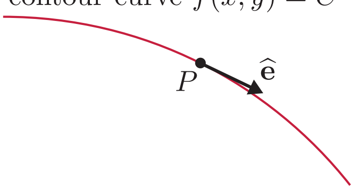


Figure 23 A contour curve

As was shown in Unit 7, it is true generally that the gradient function at a point is normal to the tangent line to the contour curve at that point. This is clear from Figure 23, which shows a contour curve $f(x,y) = C$ passing through a point P, and a unit vector $\widehat{\mathbf{e}}$ on the tangent line at P. The scalar field f does not change along the contour curve, so the derivative of f in the direction tangential to the contour curve is zero, and we must have $\mathbf{grad}\, f \cdot \widehat{\mathbf{e}} = 0$ at P. Hence $\mathbf{grad}\, f$ must be normal to $\widehat{\mathbf{e}}$.

Exercise 16

Find a vector whose direction is normal to the curve $x^2 - 2xy + y^2 = 9$ at the point $(0, 3)$.

(*Hint*: The curve is a contour of the scalar field $f(x, y) = x^2 - 2xy + y^2$.)

3.3 Gradient function in three dimensions

You have seen that for a two-dimensional scalar field $f(x, y)$, we can use the z-axis for showing the scalar values $z = f(x, y)$ and so construct the surface $(x, y, f(x, y))$. When we have a three-dimensional scalar field $f(x, y, z)$, such as the temperature distribution in a room, all three spatial coordinates (x, y, z) are needed to specify the domain of the function, and there is no 'fourth spatial dimension' for showing the function values. However, we can still consider a parametrised path in space and use the chain rule in three dimensions to give the rate of change of f along the path. Proceeding as before, we find that the derivative of $f(x, y, z)$ in the direction of $\widehat{\mathbf{d}}$, a unit vector tangential to the path, parametrised by the arc length s at a point (a, b, c) is

$$\frac{df}{ds} = \mathbf{grad}\, f(a, b, c) \cdot \widehat{\mathbf{d}},$$

where the three-dimensional vector

$$\mathbf{grad}\, f(a, b, c) = \frac{\partial f}{\partial x}(a, b, c)\,\mathbf{i} + \frac{\partial f}{\partial y}(a, b, c)\,\mathbf{j} + \frac{\partial f}{\partial z}(a, b, c)\,\mathbf{k}$$

is the gradient vector of f at the point (a, b, c).

There is a gradient vector at each point in the domain of f, so $\mathbf{grad}\, f$ is a vector field in three dimensions.

Gradient and derivative in three dimensions

The **gradient in Cartesian coordinates** (x, y, z) **of a three-dimensional scalar field** f is the vector field

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \tag{7}$$

The **derivative** of f in a direction specified by a unit vector $\widehat{\mathbf{d}}$ is given by

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}}.$$

This derivative may also be referred to as the *directional derivative*.

In three dimensions, the gradient vector at a point is always normal to the tangent plane to the contour surface passing through that point.

Example 5

Consider the scalar field

$$f(x,y,z) = -C/(x^2+y^2+z^2)^{1/2},$$

where C is a positive constant.

The vector field $\mathbf{F}$ was considered in Example 2. In that case, the complete specification of the scalar field f would include a statement of the domain $(x^2+y^2+z^2)^{1/2} \geq R$, where R is the radius of the Earth. We often omit the domain statement for convenience.

Show that the gravitational vector field $\mathbf{F} = -C\,\widehat{\mathbf{r}}/r^2$, where $\widehat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, can be expressed in terms of f by the relationship $\mathbf{F} = -\mathbf{grad}\, f$.

Solution

We have

$$f(x,y,z) = \frac{-C}{(x^2+y^2+z^2)^{1/2}},$$

so

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{Cx}{(x^2+y^2+z^2)^{3/2}},\\ \frac{\partial f}{\partial y} &= \frac{Cy}{(x^2+y^2+z^2)^{3/2}},\\ \frac{\partial f}{\partial z} &= \frac{Cz}{(x^2+y^2+z^2)^{3/2}}.\end{aligned}$$

Thus from equation (7),

$$\begin{aligned}\mathbf{grad}\, f &= \frac{C(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}\\ &= \frac{C\,\widehat{\mathbf{r}}}{x^2+y^2+z^2}.\end{aligned}$$

Hence $-\mathbf{grad}\, f$ is the vector function describing the gravitational vector field $\mathbf{F} = -C\,\widehat{\mathbf{r}}/r^2$.

In Example 5, a vector field $\mathbf{F}$ is a scalar multiple of the gradient of a scalar field f. Scalar and vector fields that are used to model physical quantities in the real world are often related in this way. A similar relationship holds between the heat flow vector field $\mathbf{J}$ and the gradient function of a temperature field Θ in a heat-conducting material. This relationship, known as **Fourier's law**, is given by

$$\mathbf{J} = -\kappa\, \mathbf{grad}\, \Theta, \tag{8}$$

where κ is the thermal conductivity of the material. Equation (8) has a natural physical interpretation. We are used to the notion that in a conducting material, heat flows from points of higher temperature to points of lower temperature. Thus we would expect the heat flow vector field $\mathbf{J}$ to point in the direction where the temperature field Θ decreases most rapidly. This is given by Fourier's law as expressed in equation (8).

As with two-dimensional scalar fields, $\mathbf{grad}\,\Theta$ points in the direction where Θ increases most rapidly, therefore $-\mathbf{grad}\,\Theta$ points in the direction where Θ *decreases* most rapidly.

Exercise 17

Evaluate $\mathbf{grad}\,\Theta$ at the point $(-1,1,0)$ for the temperature field

$$\Theta(x,y,z) = A - B\ln\left(\frac{\sqrt{x^2+y^2}}{a}\right),$$

where A, B and a are positive constants, $\sqrt{x^2+y^2} \geq a$ and $0 < a < \sqrt{2}$.

A scalar field of this form can be used to model the temperature distribution in the wall of a conducting cylindrical pipe with its axis along the z-axis and inner radius a (as in Exercise 11). The pipe carries hot water at temperature A.

You have seen examples in which a vector field is equal to a constant multiple of the gradient of a scalar field. However, note that not all vector fields can be obtained as gradients of scalar fields.

We will return to this topic in Unit 16.

Exercise 18

Consider $f(x,y,z) = x^2y^2z^2$.

(a) Find the maximum value of the derivative of f at the point $(-1,1,1)$, and the unit vector that specifies the direction in which this maximum occurs.

(b) Find the derivative of f at the point $(2,1,-1)$ in the direction of $3\mathbf{i}+4\mathbf{k}$.

3.4 Gradient as a vector operator

We can write the partial derivative $\partial f/\partial x$ of a scalar field f as $(\partial/\partial x)f$, where the symbol $\partial/\partial x$ is a **differential operator**. The operator acquires meaning only when given an **operand** on its right-hand side, that is, a function f on which to operate. Thus the differential operator $\partial/\partial x$ acts on a scalar function f to give another scalar function $\partial f/\partial x$.

Consider the quantity $\mathbf{i}\,\partial/\partial x$. This, too, is a differential operator. It acts on a scalar function f to give the vector function $(\mathbf{i}\,\partial/\partial x)f = \mathbf{i}\,\partial f/\partial x$. For example, when $f(x,y,z) = x^3 - 2xy^2 + z$, we have the vector function

$$(\mathbf{i}\,\partial/\partial x)f = \mathbf{i}\,\partial f/\partial x = \mathbf{i}(3x^2 - 2y^2).$$

The operator $\mathbf{i}\,\partial/\partial x$ is an example of a **vector differential operator**.

Consider the expression for $\mathbf{grad}\,f$ in equation (7). We can write it as

$$\mathbf{grad}\,f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)f, \qquad (9)$$

where the quantity in brackets is a vector differential operator commonly known as **del** (or **nabla** in some texts) and denoted by the bold symbol $\boldsymbol{\nabla}$. (The symbol $\boldsymbol{\nabla}$ was introduced in Unit 7. Recall that in handwritten work it should be denoted with a straight or wavy underline.)

The del operator

The vector differential operator **del** is denoted by $\boldsymbol{\nabla}$ and in Cartesian coordinates is given by

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{10}$$

We used the alternative notation for the gradient function in two dimensions in Unit 7. In two dimensions the third component of $\boldsymbol{\nabla}$ is redundant, and we have

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

We can now write the gradient function of f as $\mathbf{grad}\, f$ or as $\boldsymbol{\nabla} f$, that is,

$$\mathbf{grad}\, f = \boldsymbol{\nabla} f.$$

This use of del may appear to be nothing more than a notational device. However, as you will see later, this vector differential operator takes on the significance of a powerful unifying concept in vector calculus.

Example 6

Find the vector field $\boldsymbol{\nabla} f$, where f is the scalar field

$$f(x, y, z) = x^2 - 2xz.$$

Solution

$\boldsymbol{\nabla} f$ is the same as $\mathbf{grad}\, f$. Hence, using equation (9),

$$\begin{aligned}\boldsymbol{\nabla} f &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2 - 2xz)\\ &= \mathbf{i}(2x - 2z) + \mathbf{j}(0) + \mathbf{k}(-2x)\\ &= 2(x - z)\mathbf{i} - 2x\mathbf{k}.\end{aligned}$$

Exercise 19

Consider the scalar field $f(x, y, z) = e^{-a(x^2+y^2)-bz}$, where a and b are positive constants. Show that

$$\boldsymbol{\nabla} f = -(2a(x\mathbf{i} + y\mathbf{j}) + b\mathbf{k})f,$$

and evaluate $\boldsymbol{\nabla} f$ at the origin. Find the x-component of $\boldsymbol{\nabla} f(1, 2, 3)$.

Notice that we can use matrix notation to write the operator del as

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}. \tag{11}$$

This idea is useful when we need a formula for del in other coordinate systems, as you will see in the next subsection.

3.5 Gradient function in polar coordinates

It is sometimes more convenient to express a scalar field in terms of polar coordinates (r, θ). (One example is when the field is symmetric under rotations about the origin, so it has the form $f(r)$.) In order to calculate the gradient of a field in these coordinates, we need an expression for $\mathbf{grad}\, f$ in polar coordinates. The first step is to recall from Section 1 how the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are related to the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ (see Figure 24):

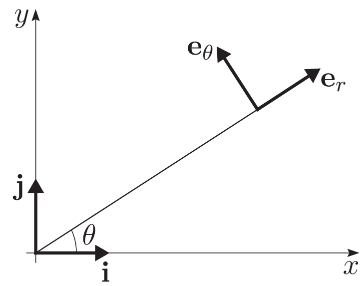


Figure 24 Illustrating the geometrical relationship between the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ and the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j},$$

or, in matrix form,

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix}.$$

We can express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by inverting this orthogonal matrix, obtaining

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}.$$

Now we use the chain rule from Unit 7 to calculate $\partial f/\partial r$ in terms of $\partial f/\partial x$ and $\partial f/\partial y$. The form of the chain rule that we need is

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}.$$

But since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{\partial x}{\partial r} = \cos\theta \quad \text{and} \quad \frac{\partial y}{\partial r} = \sin\theta.$$

Hence

$$\frac{\partial f}{\partial r} = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y},$$

and since this is true for any function f, we have

$$\frac{\partial}{\partial r} = \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}.$$

Exercise 20

Derive a similar expression for $\partial/\partial\theta$ in terms of r, $\cos\theta$, $\sin\theta$, $\partial/\partial x$ and $\partial/\partial y$.

These two pieces of information may be summarised in matrix form as

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

Although the matrix is not orthogonal (because of the presence of r), it can be inverted to give

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}.$$

Now we can find the formula for del in polar coordinates:

$$\begin{aligned}
\boldsymbol{\nabla} &= \begin{pmatrix} \mathbf{i} & \mathbf{j} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \\
&= \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} \right)^T \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \mathbf{e}_\theta \frac{\partial}{\partial \theta}.
\end{aligned}$$

Gradient function in polar coordinates

The **gradient function in polar coordinates** of a scalar field f is

$$\mathbf{grad}\, f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}, \tag{12}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors in the r and θ directions.

Exercise 21

Find the gradient vector at a point (r, θ) for each of the following scalar functions.

(a) $f(r, \theta) = r^2 - 2r\cos\theta$ (b) $f(r, \theta) = \dfrac{e^{-r\sin\theta}}{r^2}, \quad r > 0$

4 Three-dimensional polar coordinate systems

Often the first step in solving a problem is to choose an appropriate coordinate system, one that will express the problem in the simplest possible form. This may involve choosing the origin and the orientation of a Cartesian coordinate system. On the other hand, there are many problems for which other types of coordinate system are more appropriate. For example, you will see in Unit 20 how useful a polar coordinate system can be for problems involving circular motion.

This section introduces two non-Cartesian coordinate systems, which are especially useful for representing three-dimensional fields that have cylindrical or spherical symmetry. Such symmetries are usually easy to recognise. A three-dimensional scalar field has **cylindrical symmetry** when the field values depend only on the distance from a fixed straight line. If we call this line the z-axis, then the field depends only on the single variable $\rho = \sqrt{x^2 + y^2}$. The scalar field of Exercise 17 is an example of a cylindrically symmetric field. A three-dimensional scalar field has **spherical symmetry** when the field values depend only on the distance from a fixed point, which may be taken as the origin. The light intensity field I of Example 1, which varies only with $r = \sqrt{x^2 + y^2 + z^2}$, is an example of a spherically symmetric field.

4.1 Cylindrical coordinates

A cylindrical coordinate system extends the familiar polar coordinates (r, θ) to three dimensions. The coordinate r is now labelled ρ, since it represents the distance from the z-axis (rather than from the origin), and the coordinate θ is now labelled ϕ for consistency with spherical coordinates, as will become clear in Subsection 4.2. The z-coordinate of the Cartesian coordinate system provides the third variable, to give the following system.

Some texts continue to use r in place of ρ and/or θ in place of ϕ.

Cylindrical coordinates

Any point P can be represented by the triple (ρ, ϕ, z), where z is the distance from P to the (x, y)-plane, and (ρ, ϕ) are the polar coordinates of the projection N of P onto the (x, y)-plane (see Figure 25). Cylindrical coordinates are related to the Cartesian coordinates (x, y, z) by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z, \tag{13}$$

$$\rho = (x^2 + y^2)^{1/2}, \quad \cos\phi = x/\rho, \quad \sin\phi = y/\rho. \tag{14}$$

We require that

$$\rho \geq 0, \quad -\pi < \phi \leq \pi, \quad z \in \mathbb{R}.$$

The value of ϕ for points on the z-axis ($\rho = 0$) is undefined. By convention, we put $\phi = 0$ for such points.

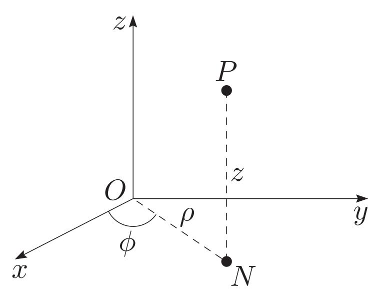


Figure 25 Cylindrical coordinates

The convention in this module is that the angle ϕ is measured anticlockwise from the x-axis and is in the range $-\pi < \phi \leq \pi$. In some texts the range of ϕ is taken as $0 \leq \phi < 2\pi$.

In three dimensions we use the symbol ρ for the distance $\sqrt{x^2 + y^2}$ from the z-axis. The symbol r is reserved for the distance $\sqrt{x^2 + y^2 + z^2}$ from the origin.

In Cartesian coordinates, the surfaces $x = \text{constant}$, $y = \text{constant}$ and $z = \text{constant}$ are planes. In cylindrical coordinates, the surfaces $\rho = \text{constant}$ (> 0) are circular cylinders with axis the z-axis and radius ρ (see Figure 26(a)). The surfaces $\phi = \text{constant}$ ($\neq 0$) are half-planes that do not contain the z-axis (see Figure 26(b)). By convention, only the half-plane $\phi = 0$ contains the z-axis on its boundary. The surfaces $z = \text{constant}$ are planes perpendicular to the z-axis (see Figure 26(c)).

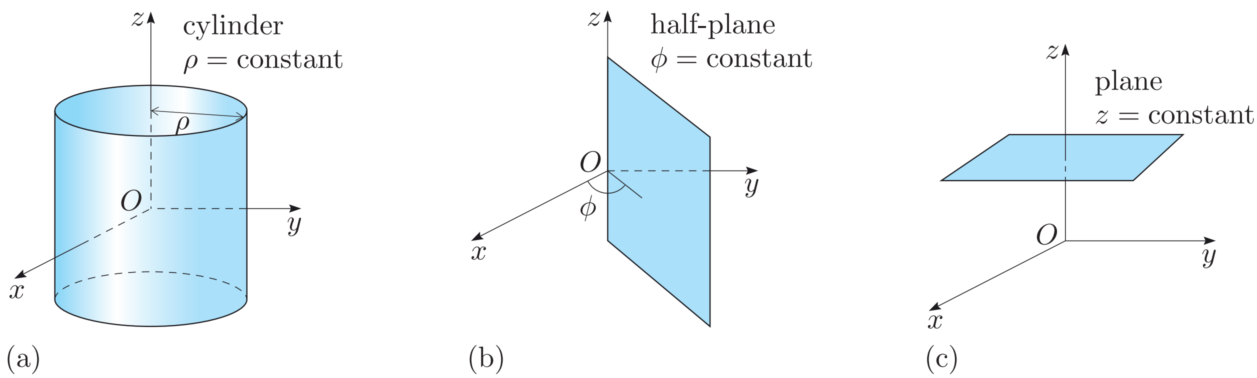


Figure 26 (a) Surface with cylindrical coordinate $\rho = \text{constant}$. (b) Surface with cylindrical coordinate $\phi = \text{constant}$. (c) Surface with cylindrical coordinate $z = \text{constant}$.

Example 7

(a) Determine the cylindrical coordinates (ρ, ϕ, z) of the following points given in Cartesian coordinates (x, y, z):

$$(5, 0, 0), \quad (0, 5, 0), \quad (-3, 3, 2), \quad (1, \sqrt{3}, 3)$$

(b) Express the scalar field

$$f(x, y, z) = \frac{x\left(x^2 + y^2 + 3\right)}{x^2 + y^2} \quad (x^2 + y^2 > 0,\ z > 1)$$

in cylindrical form, and determine, if possible, the field value at the point where $\rho = 4$, $\phi = \frac{\pi}{3}$ and $z = 2$.

Solution

(a) All the z-values remain unchanged. The first two points are both at a distance 5 from the z-axis and lie in the (x, y)-plane, so $\rho = 5$ and $z = 0$ for both of them. The angle ϕ is measured in the (x, y)-plane anticlockwise from the x-axis (see Figure 25), so $\phi = 0$ for the first point and $\phi = \frac{\pi}{2}$ for the second point.

The third point is at a distance $((-3)^2 + 3^2)^{1/2} = 3\sqrt{2}$ from the z-axis, so $\rho = 3\sqrt{2}$. The angle ϕ is in the second quadrant of the (x, y)-plane and has the value $\frac{3\pi}{4}$.

More formally, we could use equations (14) to obtain

$$\rho = ((-3)^2 + 3^2)^{1/2} = 3\sqrt{2}, \quad \cos\phi = -\tfrac{3}{3\sqrt{2}}, \quad \sin\phi = \tfrac{3}{3\sqrt{2}},$$

from which $\phi = \frac{3\pi}{4}$.

Using equations (14) for the fourth point,

$$\rho = (1^2 + (\sqrt{3})^2)^{1/2} = 2, \quad \cos\phi = \tfrac{1}{2}, \quad \sin\phi = \tfrac{\sqrt{3}}{2},$$

from which $\phi = \frac{\pi}{3}$.

So the four points in cylindrical coordinates are, respectively,

$$(5, 0, 0), \quad (5, \tfrac{\pi}{2}, 0), \quad (3\sqrt{2}, \tfrac{3\pi}{4}, 2), \quad (2, \tfrac{\pi}{3}, 3).$$

(b) Using equations (13), we see that

$$x^2 + y^2 + 3 = \rho^2(\cos^2\phi + \sin^2\phi) + 3\rho^2 + 3$$

and

$$\frac{x}{x^2 + y^2} = \frac{\rho\cos\phi}{\rho^2} = \frac{\cos\phi}{\rho}.$$

Hence

$$f(\rho, \phi, z) = \frac{(\rho^2 + 3)\cos\phi}{\rho} \quad (\rho > 0,\ z > 1).$$

First we confirm that the point with cylindrical coordinates $(4, \frac{\pi}{3}, 2)$ is within the domain of the function. It clearly satisfies $\rho > 0$ and $z > 1$. No condition is explicitly placed on ϕ, so we assume that ϕ can have any value within its range. Thus the point is within the domain of f, and

$$f(4, \tfrac{\pi}{3}, 2) = \frac{(4^2 + 3)\cos\frac{\pi}{3}}{4} = \tfrac{19}{8}.$$

Exercise 22

Consider the scalar field

$$\lambda(\rho, \phi, z) = \frac{\cos(\frac{\pi}{2}z)}{\rho} \quad (\rho > 0,\ -1 \le z \le 1).$$

(a) Describe, in geometric terms, the domain of the function.

(b) Determine, where possible, the values of λ at points P, Q and R on the positive x-, y- and z-axes, respectively, each at unit distance from the origin.

(c) Determine the value of λ at all points on a circle of radius 2, with its centre on the z-axis, in the plane $z = \frac{1}{4}$.

The cylindrical coordinate system can also be used to represent vector fields. However, before we can do this, we need to specify a set of three unit vectors in cylindrical coordinates that play the same role as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in Cartesian systems. Figure 27(a) shows three mutually perpendicular unit vectors at a point P with cylindrical coordinates (ρ, ϕ, z), as follows.

In the polar coordinate notation (r, θ) of Section 1 and Subsection 3.5, the unit vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are $\mathbf{e}_r$ and $\mathbf{e}_\theta$, respectively.

- The unit vector $\mathbf{e}_\rho$ at P points in the direction of increasing ρ, i.e. perpendicularly away from the z-axis at P.
- The unit vector $\mathbf{e}_\phi$ at P points in the direction of increasing ϕ, i.e. tangential to the circle through P and centred at $(0, 0, z)$.
- The unit vector $\mathbf{e}_z$ is the same as the Cartesian unit vector $\mathbf{k}$, i.e. it is in the positive z-direction.

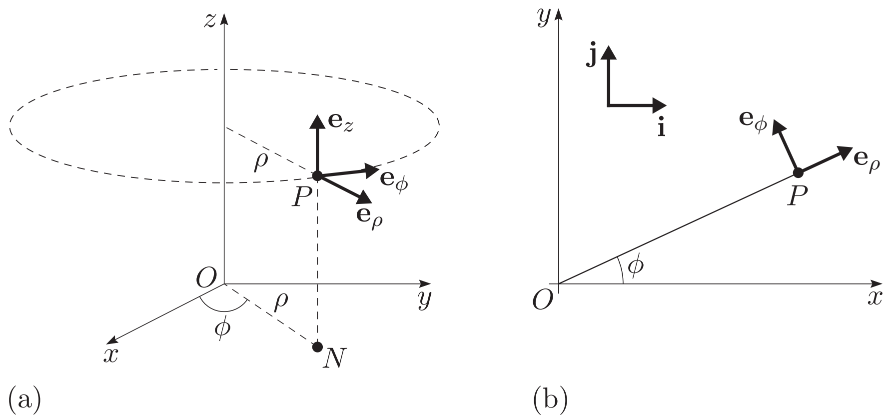


Figure 27 (a) Unit vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ lie in a plane parallel to the (x, y)-plane; $\mathbf{e}_\rho$ points away from the z-axis, $\mathbf{e}_\phi$ is perpendicular to $\mathbf{e}_\rho$ in the direction of increasing ϕ, and $\mathbf{e}_z$ is the same as the Cartesian unit vector $\mathbf{k}$. (b) Projection of $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ onto the (x, y)-plane.

You can see from Figure 27(b) that these unit vectors are related to the Cartesian unit vectors in the following way.

Cylindrical unit vectors and Cartesian unit vectors

The cylindrical unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ can be expressed in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as

$$\mathbf{e}_\rho = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}, \tag{15}$$

$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}, \tag{16}$$

$$\mathbf{e}_z = \mathbf{k}. \tag{17}$$

We can write this information more economically in matrix form as

$$\begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}. \tag{18}$$

The matrix is orthogonal, so its inverse is its transpose:

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix}. \tag{19}$$

Unlike the Cartesian unit vectors, the cylindrical unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ are not all constant vectors, since the directions of $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ depend on the position of P. In other words, $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are non-constant vector fields.

Since the cylindrical unit vectors are mutually perpendicular at each point, we have the following definition.

Cylindrical components

A vector $\mathbf{F}$ can be expressed in terms of the cylindrical unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ as

$$\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z, \tag{20}$$

where $F_\rho = \mathbf{F} \cdot \mathbf{e}_\rho$, $F_\phi = \mathbf{F} \cdot \mathbf{e}_\phi$ and $F_z = \mathbf{F} \cdot \mathbf{e}_z$. The scalar quantities F_ρ, F_ϕ and F_z are the **cylindrical components** of $\mathbf{F}$.

Example 8

Consider the vector field

$$\mathbf{F}(x, y, z) = \frac{C(x\mathbf{i} + y\mathbf{j})}{x^2 + y^2} \quad (a \le \sqrt{x^2 + y^2} \le b),$$

where a, b and C are positive constants.

(a) Express this vector field in cylindrical coordinates.

(b) Assuming that $C = 1$, $a = 2$ and $b = 4$, determine the magnitude and direction of the vector field at the point with cylindrical coordinates $(3, \pi, 7)$. What are the cylindrical components of $\mathbf{F}$ at this point?

Solution

(a) We can write

$$\mathbf{F}(x, y, z) = \frac{C}{\sqrt{x^2 + y^2}} \, \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} \quad (a \le \sqrt{x^2 + y^2} \le b).$$

Thus using equations (14) and (15),

$$\mathbf{F}(\rho, \phi, z) = \frac{C}{\rho}\, \mathbf{e}_\rho \quad (a \le \rho \le b). \tag{21}$$

(b) From equation (21), the magnitude of the vector field at the point with cylindrical coordinates $(3, \pi, 7)$ is $|\mathbf{F}(3, \pi, 7)| = |C/\rho| = \frac{1}{3}$, and the direction of the vector field is given by $\mathbf{e}_\rho$, that is, the direction perpendicularly away from the z-axis through the point. Again from equation (21), the cylindrical components of $\mathbf{F}$ at $(3, \pi, 7)$ are $F_\rho = \frac{1}{3}$ and $F_\phi = F_z = 0$.

Exercise 23

Consider the vector field

$$\mathbf{u}(x, y, z) = \frac{(xe^{-z^2} - y)\mathbf{i} + (ye^{-z^2} + x)\mathbf{j}}{x^2 + y^2}.$$

Express $\mathbf{u}$ in cylindrical coordinates, and hence specify its cylindrical components.

Grad in cylindrical coordinates

If we are given a scalar field f in cylindrical coordinates, then we can work out the vector field $\mathbf{grad}\, f$ without first having to convert the cylindrical coordinates into Cartesian coordinates. In order to do this, we need an expression for $\mathbf{grad}\, f$ in terms of partial derivatives with respect to the cylindrical variables. We start by finding relationships between $\partial/\partial\rho$, $\partial/\partial\phi$, $\partial/\partial z$ and $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$. This is done easily using the chain rule.

First we recall the relationship between Cartesian coordinates and cylindrical coordinates from equations (13):

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

Now applying the chain rule gives, for any function f,

$$\begin{aligned}\frac{\partial f}{\partial \rho} &= \frac{\partial x}{\partial \rho}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial f}{\partial z}\\ &= \cos\phi\,\frac{\partial f}{\partial x} + \sin\phi\,\frac{\partial f}{\partial y}.\end{aligned}$$

We deduce that

$$\frac{\partial}{\partial \rho} = \cos\phi\,\frac{\partial}{\partial x} + \sin\phi\,\frac{\partial}{\partial y}.$$

Similarly, we can find an expression for $\partial/\partial\phi$:

$$\begin{aligned}\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z}\\ &= -\rho\sin\phi\,\frac{\partial}{\partial x} + \rho\cos\phi\,\frac{\partial}{\partial y}.\end{aligned}$$

Fortunately, there is no additional work for $\partial/\partial z$ because

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$

We can write these three operator equations in matrix form as

$$\begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

This is very similar to the operator equations derived for polar coordinates in Subsection 3.5.

This matrix is not orthogonal for $\rho \neq 1$, but we can deduce that its inverse is

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi & 0 \\ \sin\phi & \frac{1}{\rho}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To see this, note that the 3×3 matrix that we require to invert consists of the 2×2 matrix

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{pmatrix}$$

coinciding with the first two elements of the diagonal, with 1 being the remaining element of the diagonal, and all the remaining off-diagonal elements being zero. We can then use the method described in Unit 4 for finding the inverse of a 2×2 matrix, to obtain

This is an example of a matrix said to be in *block-diagonal* form.

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{pmatrix}^{-1} = \begin{pmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi \\ \sin\phi & \frac{1}{\rho}\cos\phi \end{pmatrix},$$

from which the inverse of the 3×3 matrix now follows. Hence we can now express $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ in terms of $\partial/\partial\rho$, $\partial/\partial\theta$ and $\partial/\partial z$:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi & 0 \\ \sin\phi & \frac{1}{\rho}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix}. \tag{22}$$

Recall from equation (11) that

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

Now from equation (19) we have

$$\begin{aligned} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} = \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}^T &= \left(\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} \right)^T \\ &= \begin{pmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}^T. \end{aligned}$$

Substituting for $(\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z)^T$ from equation (22), we see that

$$\begin{aligned}
\boldsymbol{\nabla} &= \begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z\end{pmatrix} \begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}^T \begin{pmatrix}\cos\phi & -\frac{1}{\rho}\sin\phi & 0\\ \sin\phi & \frac{1}{\rho}\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}\frac{\partial}{\partial\rho}\\ \frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z}\end{pmatrix}\\
&= \begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z\end{pmatrix} \begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}\cos\phi & -\frac{1}{\rho}\sin\phi & 0\\ \sin\phi & \frac{1}{\rho}\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}\frac{\partial}{\partial\rho}\\ \frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z}\end{pmatrix}\\
&= \begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z\end{pmatrix} \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{\rho} & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}\frac{\partial}{\partial\rho}\\ \frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z}\end{pmatrix}.
\end{aligned}$$

Thus

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z \frac{\partial}{\partial z}. \tag{23}$$

Gradient function in cylindrical coordinates

The **gradient function in cylindrical coordinates** of a scalar field f is

$$\mathbf{grad}\, f = \mathbf{e}_\rho \frac{\partial f}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial\phi} + \mathbf{e}_z \frac{\partial f}{\partial z}, \tag{24}$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ are unit vectors in the ρ-, ϕ- and z-directions, respectively.

When the scalar field has cylindrical symmetry, the scalar field does not vary with ϕ or z, so the equation simplifies to

$$\mathbf{grad}\, f = \mathbf{e}_\rho \frac{\partial f}{\partial\rho}. \tag{25}$$

Example 9

Consider the scalar field

$$f(\rho,\phi,z) = \ln\rho \quad (\rho > 0).$$

Determine the vector field $\mathbf{grad}\, f$. Specify the magnitude and direction of the gradient vector at a point that is 5 units perpendicularly away from the z-axis. Check your answer for $\mathbf{grad}\, f$ by expressing the scalar function f in Cartesian coordinates and using the Cartesian expression for the gradient function.

Solution

We can use the form for $\mathbf{grad}\, f$ in equation (25), since the function varies with ρ only. Hence the vector field is

$$\mathbf{grad}\, f = \mathbf{e}_\rho \frac{\partial}{\partial \rho}(\ln \rho) = \mathbf{e}_\rho \left(\frac{1}{\rho}\right) = \frac{1}{\rho}\,\mathbf{e}_\rho \quad (\rho > 0).$$

At a perpendicular distance of 5 units from the z-axis, we have $\rho = 5$, so $|\mathbf{grad}\, f| = \frac{1}{5}$. The direction of $\mathbf{grad}\, f$ is along the line perpendicularly out from the z-axis, passing through the point.

Since $\rho = (x^2 + y^2)^{1/2}$, in Cartesian coordinates the scalar field is

$$f(x, y, z) = \ln((x^2 + y^2)^{1/2}) = \tfrac{1}{2} \ln(x^2 + y^2).$$

Hence, from equation (7), as before,

$$\mathbf{grad}\, f = \tfrac{1}{2} \left(\frac{2x}{x^2 + y^2}\,\mathbf{i} + \frac{2y}{x^2 + y^2}\,\mathbf{j} \right) = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2} = \frac{1}{\rho}\,\mathbf{e}_\rho.$$

Exercise 24

(a) Describe the geometric shapes of the surfaces defined in a cylindrical system by the following.

(i) $\rho = 2$ (ii) $\phi = \frac{\pi}{2}$

(b) Show that the cylindrical unit vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are perpendicular.

(*Hint*: Use the fact that the Cartesian unit vectors are mutually perpendicular.)

(c) Find the gradient of the scalar field $F(\rho, \phi, z) = \rho e^{-z}$ at a point on the positive x-axis that is 5 units from the origin.

Exercise 25

Consider the scalar field

$$U(x, y, z) = 1/(x^2 + y^2), \quad \text{where } x^2 + y^2 > 0.$$

Find $\boldsymbol{\nabla} U$ using the Cartesian form of the gradient function. Express U in cylindrical coordinates, and hence confirm your answer for the gradient function. Sketch some contour surfaces of U and vector field lines for $\boldsymbol{\nabla} U$.

Exercise 26

Determine the cylindrical components of the vector field $\mathbf{grad}\, f$ for the scalar field $f(\rho, \phi, z) = (\cos z)/\rho$, where $\rho > 0$.

4.2 Spherical coordinates

In the spherical coordinate system, one of the three coordinates of any point is specified as the magnitude r of the position vector $\mathbf{r}$ of the point. Hence for fixed r, all points with coordinate r lie on the surface of a sphere of radius r centred on the origin. To specify position on the sphere, two angle coordinates θ and ϕ are used, based on the idea of locating a point on the Earth's surface by giving its latitude and longitude (see Figure 28(a)).

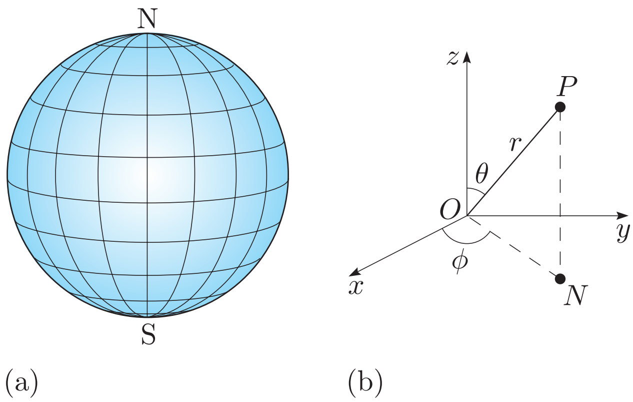


Figure 28 (a) Surface of the Earth, showing curves of constant latitude and longitude. (b) Spherical coordinates r, θ and ϕ.

The **polar angle** θ is a measure of latitude on the sphere, with θ increasing from 'north pole' to 'south pole', so $\theta = 0$ for points on the positive z-axis, $\theta = \frac{\pi}{2}$ for points in the (x, y)-plane, and $\theta = \pi$ for points on the negative z-axis. The **azimuthal angle** ϕ is a measure of longitude on the sphere and is the same as the coordinate ϕ in the cylindrical coordinate system.

It may help you to think of ϕ as increasing 'west to east'; compare Figures 27(a) and 28(b).

Spherical coordinates

Any point P (see Figure 28(b)) can be represented by the triple (r, θ, ϕ), where r is the distance from P to the origin, and θ and ϕ are the polar and azimuthal angles, respectively. In Figure 28(b), N is the projection of P onto the (x, y)-plane.

The spherical coordinates of P are related to the Cartesian coordinates (x, y, z) by

$$x = ON\cos\phi = r\sin\theta\cos\phi, \tag{26}$$

$$y = ON\sin\phi = r\sin\theta\sin\phi, \tag{27}$$

$$z = r\cos\theta, \tag{28}$$

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{29}$$

We require that

$$r \geq 0, \quad -\pi < \phi \leq \pi, \quad 0 \leq \theta \leq \pi.$$

There are explicit expressions for θ and ϕ in terms of the other variables, but we do not need them in this module.

The surface $r =$ constant is a sphere (see Figure 29(a)). The surface $\theta =$ constant is a cone (see Figure 29(b)), or a plane if $\theta = \frac{\pi}{2}$, or a half-line if $\theta = 0$ or $\theta = \pi$. The surface $\phi =$ constant ($\neq 0$) is a half-plane that does not contain the z-axis (see Figure 29(c)). By convention, only the half-plane $\phi = 0$ contains the z-axis on its boundary.

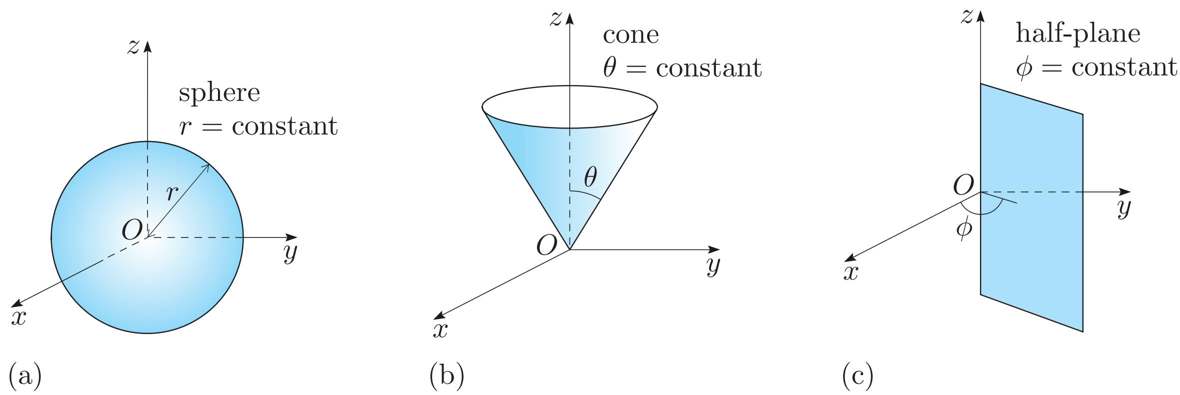


Figure 29 (a) Surface with spherical coordinate $r =$ constant. (b) Surface with spherical coordinate $\theta =$ constant. (c) Surface with spherical coordinate $\phi =$ constant.

Exercise 27

Give the spherical coordinates of points with the following Cartesian coordinates.

(a) $(5, 0, 0)$

(b) $(0, 5, 0)$

The mutually perpendicular unit vectors for the spherical coordinate system are shown in Figure 30. They are related to the Cartesian unit vectors in the following way.

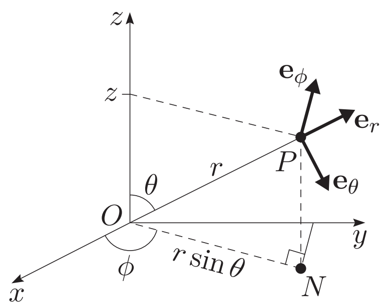


Figure 30 Orthogonal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ lie in the plane containing the triangle ONP with $\mathbf{e}_r$ and $\mathbf{e}_\theta$ pointing in the directions of increasing r and θ, respectively. Unit vector $\mathbf{e}_\phi$ is perpendicular to the plane containing the triangle ONP and points in the direction of increasing ϕ.

Spherical unit vectors and Cartesian unit vectors

The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ can be expressed in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as

$$\mathbf{e}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta, \tag{30}$$

$$\mathbf{e}_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta, \tag{31}$$

$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi. \tag{32}$$

The directions of the unit vectors for the spherical coordinate system at a point P depend on the θ- and ϕ-coordinates of P (but not on r). Once again, this information is more compactly expressed as a matrix equation

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}. \tag{33}$$

Since the matrix is orthogonal, its inverse is its transpose, so we have

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix}. \tag{34}$$

Since the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are mutually perpendicular, we have the following definition.

Spherical components

A vector $\mathbf{F}$ can be expressed in terms of the spherical unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as

$$\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi, \tag{35}$$

where $F_r = \mathbf{F} \cdot \mathbf{e}_r$, $F_\theta = \mathbf{F} \cdot \mathbf{e}_\theta$ and $F_\phi = \mathbf{F} \cdot \mathbf{e}_\phi$. The scalars F_r, F_θ and F_ϕ are the **spherical components** of $\mathbf{F}$.

Exercise 28

Express the following vector field in terms of spherical coordinates and unit vectors, and hence determine its spherical components:

$$\mathbf{F}(x, y, z) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2 + z^2} \quad (x^2 + y^2 + z^2 > 0).$$

Grad in spherical coordinates

The derivation is similar to (but more complicated than) that for cylindrical coordinates. We omit the details here.

The gradient function can be expressed in spherical coordinates as follows.

Gradient function in spherical coordinates

The **gradient function in spherical coordinates** of a scalar field f is

$$\textbf{grad}\, f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta} \frac{\partial f}{\partial \phi}, \tag{36}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the r-, θ- and ϕ-directions, respectively.

For spherically symmetric fields, in which the field values do not vary with θ or ϕ, the gradient simplifies to

$$\mathbf{grad}\, f = \mathbf{e}_r \frac{\partial f}{\partial r}. \tag{37}$$

Example 10

Consider the scalar field expressed in Cartesian coordinates as

$$V(x, y, z) = \frac{-I_0}{(x^2 + y^2 + z^2)^{1/2}} \quad ((x^2 + y^2 + z^2)^{1/2} > a > 0),$$

where I_0 is a positive constant.

Express this field in spherical coordinates, and hence show that $|\mathbf{grad}\, V| = I$, where I is the light intensity field specified in Example 1 as $I = I_0/(x^2 + y^2 + z^2)$.

Solution

We know that $x^2 + y^2 + z^2 = r^2$, so

By definition, $r \geq 0$.

$$V(r, \theta, \phi) = -I_0/r \quad (r > a).$$

The field is spherically symmetric, so we can use equation (37) to obtain

$$\mathbf{grad}\, V = \mathbf{e}_r \frac{\partial}{\partial r}\left(\frac{-I_0}{r}\right) = \frac{I_0}{r^2}\,\mathbf{e}_r.$$

Hence

$$|\mathbf{grad}\, V| = \frac{I_0}{r^2} = \frac{I_0}{x^2 + y^2 + z^2},$$

which is the light intensity field I of Example 1.

Exercise 29

Specify the spherical components of the vector field $\mathbf{grad}\, V$ in Example 10.

Exercise 30

(a) Describe the geometric shape of the surface defined in a spherical coordinate system by $\theta = \frac{\pi}{6}$.

(b) Use equation (30) to confirm that the spherical unit vector $\mathbf{e}_r$ is of unit magnitude.

(c) Find the gradient in spherical coordinates of the scalar field $U(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the spherical coordinate point $(5, \frac{\pi}{2}, \pi)$. Give your answer in terms of the Cartesian unit vectors.

Exercise 31

Consider the scalar field

$$V(r,\theta,\phi) = \frac{M(3\cos^2\theta - 1)}{r^3} \quad (r > 0),$$

On the z-axis, where ϕ is not defined, by convention we have $\phi = 0$.

where M is a positive constant. Three points A, B and C have the following positions: A is on the positive y-axis 3 units from the origin, B has Cartesian coordinates $(0,0,1)$, and C has spherical coordinates $(1,\pi,0)$.

(a) Determine, where possible, the value of the scalar field V at the three points A, B and C.

This result, relating force and potential energy, is obtained in Unit 16.

(b) If $V(r,\theta,\phi)$ represents the potential energy of a particle at a point, then the force acting on the particle at that point is given by

$$\mathbf{F} = -\mathbf{grad}\, V.$$

Determine the magnitude and direction of the force on the particle when the particle is at A.

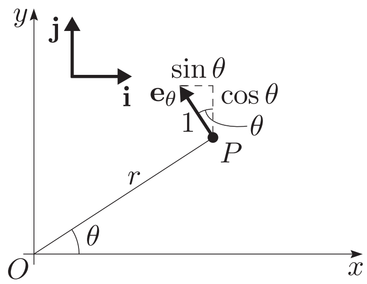
Learning outcomes

After studying this unit, you should be able to:

- appreciate the concept of a scalar field
- interpret and sketch contour curves for a given scalar field
- appreciate the concept of a vector field
- interpret and sketch vector field lines for a given vector field
- determine the gradient function of a scalar field, and calculate the derivative of a scalar field in a specified direction
- convert between the Cartesian coordinates of a point and the cylindrical or spherical coordinates of the point
- express a scalar or vector field in cylindrical or spherical coordinates, given the field in Cartesian coordinates.

Solutions to exercises

Solution to Exercise 1



From the figure above, the x-component of the unit vector $\mathbf{e}_\theta$ is $-\sin\theta$, and the y-component is $\cos\theta$. Hence

$$\mathbf{e}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}.$$

Solution to Exercise 2

Since $\mathbf{i}$ and $\mathbf{j}$ are themselves an orthogonal pair of unit vectors, we know from Unit 2 that

$$\mathbf{i}\cdot\mathbf{i} = 1, \quad \mathbf{j}\cdot\mathbf{j} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0.$$

Hence

$$\begin{aligned}
\mathbf{e}_r\cdot\mathbf{e}_r &= ((\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j})\cdot((\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j})\\
&= (\cos^2\theta)\,\mathbf{i}\cdot\,\mathbf{i} + (2\sin\theta\cos\theta)\,\mathbf{i}\cdot\mathbf{j} + (\sin^2\theta)\,\mathbf{j}\cdot\mathbf{j}\\
&= \cos^2\theta + \sin^2\theta = 1,\\
\mathbf{e}_\theta\cdot\mathbf{e}_\theta &= (-\sin\theta)^2 + (\cos\theta)^2\\
&= \sin^2\theta + \cos^2\theta = 1,\\
\mathbf{e}_r\cdot\mathbf{e}_\theta &= (\cos\theta)(-\sin\theta) + (\sin\theta)(\cos\theta)\\
&= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0.
\end{aligned}$$

Note that the result for $\mathbf{e}_\theta\cdot\mathbf{e}_r$ does not need to be explicitly stated since the dot product is commutative (see Unit 2).

Solution to Exercise 3

Since $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, we have

$$\det(\mathbf{A}^T\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{A}) = \det(\mathbf{I}) = 1.$$

However, $\det(\mathbf{A}^T) = \det(\mathbf{A})$, so this becomes

$$(\det(\mathbf{A}))^2 = 1.$$

We can therefore deduce that $\det(\mathbf{A}) = \pm 1$.

(For rotations, $\det(\mathbf{A}) = 1$; the minus sign arises only for a reflection. We do not investigate reflections since we are concentrating on right-handed systems.)

Solution to Exercise 4

(a) In each case we compute $\mathbf{A}^T\mathbf{A}$ and compare it with the identity matrix $\mathbf{I}$.

Here, $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} \neq \mathbf{I}$.

Hence this matrix is not orthogonal.

(b) $\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \mathbf{I}$.

Hence this matrix is orthogonal.

(c) $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix} \neq \mathbf{I}$.

Hence this matrix is not orthogonal.

(d) $\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix} = \mathbf{I}$.

Hence this matrix is orthogonal.

(e) $\begin{pmatrix}\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}\\ 0 & 1 & 0\\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2}\\ 0 & 1 & 0\\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\end{pmatrix} = \mathbf{I}$.

Hence this matrix is orthogonal.

Solution to Exercise 5

There are many possible choices – in fact, infinitely many. One example is

$$\begin{pmatrix}2 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

which certainly has determinant 1. However this matrix cannot be orthogonal because when it multiplies the vector $(1 \quad 0 \quad 0)^T$, the result is $(2 \quad 1 \quad 0)^T$, a vector of different length.

Solution to Exercise 6

(a) Let the Cartesian coordinate system be in the plane of the sheet with $|\mathbf{r}| = \sqrt{x^2+y^2}$. Then the scalar field function is

$$\Theta(x,y) = \Theta_1 + \frac{(\Theta_2-\Theta_1)\ln(\sqrt{x^2+y^2}/R_1)}{\ln(R_2/R_1)}.$$

The scalar field is defined over the region of the sheet. So the domain of the function is $R_2 \leq \sqrt{x^2+y^2} \leq R_1$.

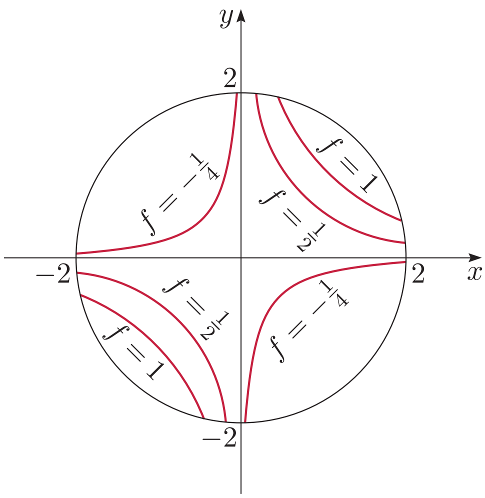
(b) In polar coordinates, $|\mathbf{r}| = r$, so the scalar field function is

$$\Theta(r,\theta) = \Theta_1 + \frac{(\Theta_2-\Theta_1)\ln(r/R_1)}{\ln(R_2/R_1)},$$

with domain $R_2 \leq r \leq R_1$. Note that in this case, the field function is independent of θ.

Solution to Exercise 7

The domain of f is the area of the plane inside and on the circle $x^2 + y^2 = 4$ with radius 2. The contour curves are hyperbolas given by $xy = C$, where C is a constant, or equivalently, by $y = C/x$. The figure below shows the contour curves for $C = \frac{1}{2}$, $C = -\frac{1}{4}$ and $C = 1$.



Solution to Exercise 8

(a) (i) There is a hill summit inside the 60 m contour.

(ii) There might be a lake in the depression in the region of the −10 m and −20 m contours.

(b) The land is fairly level in the vicinity of the 0 m and 10 m contours, since the contour curves are widely spaced in this region.

(c) Starting at A and walking in the direction of the arrow, you would soon experience a hard climb as you ascend across the closely spaced contours leading to the top of the hill. Then there is a fairly gentle descent to the relatively level area, with a low point near the 0 m contour, and finally a gentle climb.

Solution to Exercise 9

Choose a coordinate system with the Earth's (flat) surface in the (x, y)-plane and the z-axis pointing vertically upwards. Then $\sigma(x, y, z) = Ae^{-\alpha z}$, for constants $A > 0$ and $\alpha > 0$. Knowing that $\sigma = 1.205$ at $z = 0$ tells us that $A = 1.205$. We are also told that $\sigma = 1.205/e$ when $z = 9.5 \times 10^3$. This gives $1.205/e = 1.205e^{-\alpha \times 9.5 \times 10^3}$, from which $\alpha = 1/(9.5 \times 10^3)$. Hence the scalar field is

$$\sigma(x, y, z) = 1.205e^{-z/(9.5 \times 10^3)}.$$

The domain is $z \geq 0$.

Since the field depends only on z, the contour surfaces are horizontal planes, that is, $z =$ constant.

Solution to Exercise 10

We have

$$\mathbf{F} = \frac{mgr}{R}(-\widehat{\mathbf{r}}) = -\frac{mg}{R}\mathbf{r},$$

since $r\,\widehat{\mathbf{r}} = \mathbf{r}$, so

$$\mathbf{F}(x,y,z) = \frac{-mg}{R}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

At the Earth's centre, we have $\mathbf{F}(0,0,0) = \mathbf{0}$.

Solution to Exercise 11

(a) The x-component of the vector field $\mathbf{J}$ is

$$J_1(x,y,z) = \mathbf{J} \cdot \mathbf{i} = \frac{Cx}{x^2+y^2}.$$

(b) The vector $x\mathbf{i} + y\mathbf{j}$ points perpendicularly away from the z-axis. Hence the vector field lines of $\mathbf{J}$ are directed radially out from the axis of the pipe.

(c) The magnitude of $\mathbf{J}$ is given by

$$|\mathbf{J}| = \left|\frac{C(x\mathbf{i} + y\mathbf{j})}{x^2+y^2}\right| = \frac{C}{(x^2+y^2)^{1/2}} = \frac{C}{\rho} \quad (a \le \rho \le b).$$

Hence $|\mathbf{J}|$ is inversely proportional to ρ.

(d) The rate of flow outwards is the outward flow rate per unit area times the area of the surface, that is,

$$|\mathbf{J}|\,(2\pi\rho h) = (C/\rho)(2\pi\rho h) = 2\pi Ch,$$

which is independent of ρ. This means that heat energy is being conducted through the material of the pipe, from the water surface to the outside surface of the pipe, without any loss or gain of heat in the material of the pipe.

Solution to Exercise 12

Introduce a Cartesian coordinate system with the z-axis along the axis of the wire. Then the magnitude $|\mathbf{E}|$ is inversely proportional to the distance $\rho = (x^2+y^2)^{1/2}$ from the z-axis. The direction of $\mathbf{E}$ is directly away from the wire, in the direction of the vector $x\mathbf{i} + y\mathbf{j}$. The unit vector in this direction is $\mathbf{e}_\rho = (x\mathbf{i} + y\mathbf{j})/(x^2+y^2)^{1/2}$. Hence the electric field vector at distance ρ is $\mathbf{E} = E_0\mathbf{e}_\rho/\rho$, which has magnitude E_0 when $\rho = 1$, as required. Thus in Cartesian coordinates we have the vector field

$$\mathbf{E}(x,y,z) = E_0(x\mathbf{i} + y\mathbf{j})/(x^2+y^2) \quad (x^2+y^2>0).$$

The domain statement excludes the z-axis, where the function is not defined.

The vector field lines are straight lines perpendicular to the z-axis and directed away from it.

Solution to Exercise 13

The scalar field is $f(x, y) = x^2 y$, so we have

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial x}(-1, 2) = -4,$$
$$\frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial y}(-1, 2) = 1.$$

Thus from equation (6),

$$\mathbf{grad}\, f(-1, 2) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = -4\mathbf{i} + \mathbf{j}.$$

The derivative in the x-direction is $\mathbf{grad}\, f \cdot \mathbf{i} = \partial f/\partial x$, which has the value -4 at $(-1, 2)$. Note that here we have $\widehat{\mathbf{d}} = \mathbf{i}$.

Solution to Exercise 14

(a) We have

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}},$$
$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

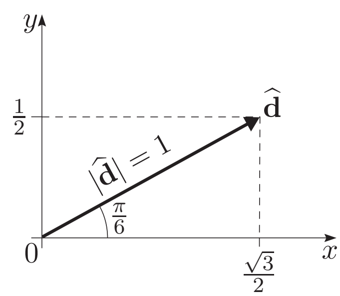
Thus from equation (6),

$$\mathbf{grad}\, f = \frac{1}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j})$$

and

$$\mathbf{grad}\, f(1, 1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

(b) The unit vector $\widehat{\mathbf{d}}$ is shown in the figure in the margin.



We have

$$\widehat{\mathbf{d}} = (\cos \tfrac{\pi}{6})\mathbf{i} + (\sin \tfrac{\pi}{6})\mathbf{j} = \tfrac{\sqrt{3}}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}.$$

The required derivative is

$$\begin{aligned}\mathbf{grad}\, f(1, 1) \cdot \widehat{\mathbf{d}} &= \left(\tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})\right) \cdot \left(\tfrac{\sqrt{3}}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}\right) \\ &= \tfrac{\sqrt{3}}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}} \\ &= \tfrac{1+\sqrt{3}}{2\sqrt{2}} \\ &\simeq 0.9659.\end{aligned}$$

Solution to Exercise 15

We have

$$\frac{\partial g}{\partial x} = 2xy - y^2, \quad \frac{\partial g}{\partial y} = x^2 - 2xy.$$

Thus from equation (6),

$$\mathbf{grad}\, g = (2xy - y^2)\mathbf{i} + (x^2 - 2xy)\mathbf{j}$$

and

$$\mathbf{grad}\, g(1,1) = \mathbf{i} - \mathbf{j}.$$

This gradient vector specifies the magnitude and direction of the steepest slope of the surface $z = g(x,y)$ at the point $(1,1)$. Hence the magnitude of the required steepest slope is

$$|\mathbf{grad}\, g(1,1)| = |\mathbf{i} - \mathbf{j}| = \sqrt{2},$$

and the direction of steepest slope is specified by the unit vector $(\mathbf{i} - \mathbf{j})/\sqrt{2}$.

Solution to Exercise 16

The curve $x^2 - 2xy + y^2 = 9$ is a contour of $f(x,y) = x^2 - 2xy + y^2$ passing through $(0,3)$, and $\mathbf{grad}\, f$ evaluated at $(0,3)$ is normal to this contour. We have

$$\mathbf{grad}\, f = (2x - 2y)\mathbf{i} + (-2x + 2y)\mathbf{j}$$

and

$$\mathbf{grad}\, f(0,3) = -6\mathbf{i} + 6\mathbf{j}.$$

This, or any non-zero scalar multiple of it, is the required vector.

Solution to Exercise 17

The partial derivatives are

$$\frac{\partial \Theta}{\partial x} = \frac{-Ba}{\sqrt{x^2+y^2}} \times \frac{\frac{1}{2}(x^2+y^2)^{-1/2}}{a} \times 2x = -\frac{Bx}{x^2+y^2},$$

$$\frac{\partial \Theta}{\partial y} = -\frac{By}{x^2+y^2},$$

$$\frac{\partial \Theta}{\partial z} = 0.$$

Thus using equation (7),

$$\mathbf{grad}\, \Theta = \left(-\frac{Bx}{x^2+y^2}\right)\mathbf{i} + \left(-\frac{By}{x^2+y^2}\right)\mathbf{j}$$

and

$$\mathbf{grad}\, \Theta(-1,1,0) = \tfrac{1}{2}B\mathbf{i} - \tfrac{1}{2}B\mathbf{j} = \tfrac{1}{2}B(\mathbf{i} - \mathbf{j}).$$

Solution to Exercise 18

(a) We need the magnitude and direction of $\mathbf{grad}\, f$ at $(-1,1,1)$. From equation (7),

$$\mathbf{grad}\, f = 2xy^2z^2\mathbf{i} + 2x^2yz^2\mathbf{j} + 2x^2y^2z\mathbf{k}.$$

The maximum value of the derivative is $|\mathbf{grad}\, f|$, and it occurs in the direction of the unit vector $\mathbf{grad}\, f/|\mathbf{grad}\, f|$. At the point $(-1,1,1)$, we have

$$|\mathbf{grad}\, f| = |-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}| = 2\sqrt{3},$$
$$\frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = -\tfrac{1}{\sqrt{3}}\mathbf{i} + \tfrac{1}{\sqrt{3}}\mathbf{j} + \tfrac{1}{\sqrt{3}}\mathbf{k}.$$

(b) The unit vector in the direction of $3\mathbf{i} + 4\mathbf{k}$ is given by

$$\widehat{\mathbf{d}} = (3\mathbf{i} + 4\mathbf{k})/|3\mathbf{i} + 4\mathbf{k}| = (3\mathbf{i} + 4\mathbf{k})/5 = \tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{k}.$$

Hence the derivative of f in the direction of $3\mathbf{i} + 4\mathbf{k}$ is

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}} = \tfrac{6}{5}xy^2z^2 + \tfrac{8}{5}x^2y^2z.$$

At the point $(2,1,-1)$, this gives

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}} = -4.$$

Solution to Exercise 19

We have

$$\frac{\partial f}{\partial x} = e^{-a(x^2+y^2)-bz} \times (-2ax) = (-2ax)\, f(x,y,z),$$
$$\frac{\partial f}{\partial y} = e^{-a(x^2+y^2)-bz} \times (-2ay) = (-2ay)\, f(x,y,z),$$
$$\frac{\partial f}{\partial z} = e^{-a(x^2+y^2)-bz} \times (-b) = (-b)\, f(x,y,z).$$

Thus from equation (7),

$$\begin{aligned}\boldsymbol{\nabla} f &= (-2ax)\, f(x,y,z)\,\mathbf{i} + (-2ay)\, f(x,y,z)\,\mathbf{j} + (-b)\, f(x,y,z)\,\mathbf{k} \\ &= -(2a(x\mathbf{i} + y\mathbf{j}) + b\mathbf{k})\, f(x,y,z),\end{aligned}$$

as required. At the origin,

$$\boldsymbol{\nabla} f(0,0,0) = -be^0\mathbf{k} = -b\mathbf{k}.$$

The x-component of $\boldsymbol{\nabla} f(1,2,3)$ is

$$\begin{aligned}\boldsymbol{\nabla} f(1,2,3) \cdot \mathbf{i} &= (-2a)\, f(1,2,3) \\ &= -2ae^{-a(1^2+2^2)-3b} \\ &= -2ae^{-5a-3b}.\end{aligned}$$

Solution to Exercise 20

We have

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}.$$

Hence

$$\frac{\partial}{\partial \theta} = -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}.$$

Solution to Exercise 21

(a) We have $f(r,\,\theta) = r^2 - 2r\cos\theta$. Thus

$$\begin{aligned}\mathbf{grad}\, f &= \left(\mathbf{e}_r\,\frac{\partial}{\partial r} + \frac{1}{r}\,\mathbf{e}_\theta\,\frac{\partial}{\partial \theta}\right) f \\ &= \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \\ &= (2r - 2\cos\theta)\mathbf{e}_r + 2\sin\theta\,\mathbf{e}_\theta.\end{aligned}$$

(b) We have $f(r,\theta) = e^{-r\sin\theta}/r^2$. Thus

$$\frac{\partial f}{\partial r} = -\frac{e^{-r\sin\theta}\sin\theta}{r^2} - \frac{2e^{-r\sin\theta}}{r^3} = -\frac{e^{-r\sin\theta}}{r^2}\left(\sin\theta + \frac{2}{r}\right)$$

and

$$\frac{\partial f}{\partial \theta} = -\frac{\cos\theta}{r}\,e^{-r\sin\theta}.$$

Hence

$$\begin{aligned}\mathbf{grad}\, f &= \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \\ &= -\frac{e^{-r\sin\theta}}{r^2}\left(\left(\sin\theta + \frac{2}{r}\right)\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta\right).\end{aligned}$$

Solution to Exercise 22

(a) The domain is the slab of space between (and including) the planes $z = 1$ and $z = -1$, but excluding the z-axis.

(b) At P we have $\lambda(1,0,0) = (\cos 0)/1 = 1$.

At Q we have $\lambda(1,\frac{\pi}{2},0) = (\cos 0)/1 = 1$.

At R we have $\rho = 0$, so R is not in the domain of λ.

(c) The points on the circle all have $\rho = 2$, so the required field value is $\lambda(2,\phi,\frac{1}{4}) = (\cos\frac{\pi}{8})/2 \simeq 0.4619$.

Solution to Exercise 23

We can replace $x = \rho\cos\phi$ and $y = \rho\sin\phi$ in the equation for $\mathbf{u}$, or we can write $\mathbf{u}$ as

$$\mathbf{u}(x, y, z) = \frac{e^{-z^2}(x\mathbf{i} + y\mathbf{j})}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2}} + \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2}}.$$

Then equations (14), (15) and (16) give

$$\mathbf{u}(\rho, \phi, z) = \frac{e^{-z^2}}{\rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\,\mathbf{e}_\phi.$$

Hence the cylindrical components are

$$u_\rho = \frac{e^{-z^2}}{\rho}, \quad u_\phi = \frac{1}{\rho}, \quad u_z = 0.$$

Solution to Exercise 24

(a) (i) $\rho = 2$ defines a cylindrical surface of radius 2 with its axis along the z-axis.

(ii) $\phi = \frac{\pi}{2}$ defines a half-plane perpendicular to the x-axis, containing the positive y-axis but not the z-axis.

(b) We have

$$\mathbf{e}_\rho \cdot \mathbf{e}_\phi = \left(\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}\right) \cdot \left(\frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}\right) = \frac{-xy + xy}{x^2 + y^2} = 0,$$

so $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are perpendicular.

(c) The partial derivatives of F are

$$\frac{\partial F}{\partial \rho} = e^{-z}, \quad \frac{\partial F}{\partial \phi} = 0, \quad \frac{\partial F}{\partial z} = -\rho e^{-z}.$$

Hence, from equation (24), the gradient of F is

$$\boldsymbol{\nabla} F = \mathbf{grad}\, F = e^{-z}\mathbf{e}_\rho - \rho e^{-z}\mathbf{e}_z = e^{-z}(\mathbf{e}_\rho - \rho\mathbf{e}_z).$$

At the point $\rho = 5$, $\phi = 0$, $z = 0$, we have

$$\boldsymbol{\nabla} F(5, 0, 0) = \mathbf{e}_\rho - 5\mathbf{e}_z.$$

Solution to Exercise 25

The partial derivatives of U are

$$\frac{\partial U}{\partial x} = -\frac{2x}{(x^2 + y^2)^2}, \quad \frac{\partial U}{\partial y} = -\frac{2y}{(x^2 + y^2)^2}, \quad \frac{\partial U}{\partial z} = 0.$$

Hence, since $\boldsymbol{\nabla} U = \mathbf{grad}\, U$, from equation (7),

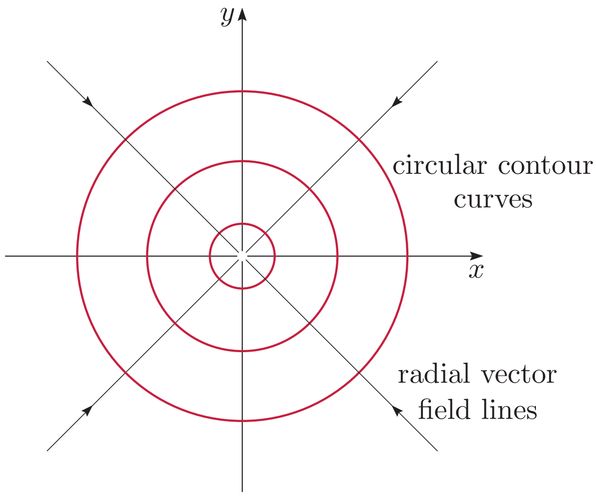
$$\boldsymbol{\nabla} U = -\frac{2}{(x^2 + y^2)^2}(x\mathbf{i} + y\mathbf{j}).$$

Converting to cylindrical coordinates using equations (14), $U(\rho,\phi,z)=1/\rho^2$. This is a cylindrically symmetric scalar field, since it depends only on ρ. So we can use equation (25) to obtain

$$\nabla U=\mathbf{e}_\rho\,\frac{\partial U}{\partial\rho}=\mathbf{e}_\rho(-2)\frac{1}{\rho^3}=-\frac{2}{\rho^3}\,\mathbf{e}_\rho.$$

But from equation (15) $\mathbf{e}_\rho=(x\mathbf{i}+y\mathbf{j})/\sqrt{x^2+y^2}$, and from equations (14) $1/\rho^3=(x^2+y^2)^{-3/2}$, hence the two expressions for ∇U are identical.

The contour surfaces of U are found by putting $U=1/\rho^2$ equal to a constant. Hence the contours are the surfaces where ρ is constant. Each such surface is cylindrical with its axis along the z-axis. We can sketch the intersections (circles) of the contours with a plane $z=$ constant (see the figure below). The vector field lines of ∇U are directed perpendicularly inwards towards the z-axis. (The vector field lines are not continuous through the z-axis, which is not in the domain of U.)



Solution to Exercise 26

The partial derivatives are

$$\frac{\partial f}{\partial\rho}=-\frac{\cos z}{\rho^2},\quad \frac{\partial f}{\partial\phi}=0,\quad \frac{\partial f}{\partial z}=-\frac{\sin z}{\rho}.$$

Hence, using equation (24),

$$\mathbf{grad}\,f=\mathbf{e}_\rho\left(-\frac{\cos z}{\rho^2}\right)+\mathbf{e}_z\left(-\frac{\sin z}{\rho}\right).$$

So the components $(\mathbf{grad}\,f)_\rho$, $(\mathbf{grad}\,f)_\phi$ and $(\mathbf{grad}\,f)_z$ are $(-\cos z)/\rho^2$, 0 and $(-\sin z)/\rho$, respectively.

Solution to Exercise 27

(a) Here $r = \sqrt{5^2 + 0^2 + 0^2} = 5$. The point is in the (x, y)-plane, so $\theta = \frac{\pi}{2}$. The point is on the x-axis, so $\phi = 0$. Hence the spherical coordinates are $(5, \frac{\pi}{2}, 0)$.

(b) The point is on the positive y-axis 5 units from the origin, so the spherical coordinates are $(5, \frac{\pi}{2}, \frac{\pi}{2})$.

Solution to Exercise 28

From equations (26), (27), (29) and (32), we have

$$\begin{aligned}
\mathbf{F}(r, \theta, \phi) &= \frac{-r\sin\theta\sin\phi\,\mathbf{i} + r\sin\theta\cos\phi\,\mathbf{j}}{r^2} \\
&= \frac{\sin\theta}{r}\,(-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) \\
&= \frac{\sin\theta}{r}\,\mathbf{e}_\phi,
\end{aligned}$$

so using equation (35), the spherical components of $\mathbf{F}$ are

$$F_r = F_\theta = 0, \quad F_\phi = \frac{\sin\theta}{r}.$$

Solution to Exercise 29

Using equation (35), the spherical components are

$$(\mathbf{grad}\,V)_r = I_0/r^2, \quad (\mathbf{grad}\,V)_\theta = (\mathbf{grad}\,V)_\phi = 0.$$

Solution to Exercise 30

(a) $\theta = \frac{\pi}{6}$ defines a cone with its axis along the positive z-axis, its apex at the origin, and half-angle $\frac{\pi}{6}$.

(b) We have

$$\begin{aligned}
|\mathbf{e}_r|^2 &= \mathbf{e}_r \cdot \mathbf{e}_r \\
&= (\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta) \\
&\quad \cdot (\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta) \\
&= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta \\
&= \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta = 1.
\end{aligned}$$

(c) In spherical coordinates, $U = r$ and the only non-zero partial derivative is $\partial U/\partial r = 1$. Hence $\mathbf{grad}\,U = (1)\mathbf{e}_r = \mathbf{e}_r$. The point $(5, \frac{\pi}{2}, \pi)$ is on the negative x-axis, so $\mathbf{e}_r = -\mathbf{i}$ at this point.

(Alternatively, using equation (30),

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta \\
&= \mathbf{i}(-1) + \mathbf{0} + \mathbf{0} = -\mathbf{i}.)
\end{aligned}$$

Solution to Exercise 31

(a) At A, $r = 3$, $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$ and $V(3, \frac{\pi}{2}, \frac{\pi}{2}) = -M/3^3 = -M/27$.

At B, $r = 1$, $\theta = 0$, $\phi = 0$ and $V(1, 0, 0) = 2M$. (B is on the z-axis, where $\phi = 0$ by convention.)

At C, $r = 1$, $\theta = \pi$, $\phi = 0$ and $V(1, \pi, 0) = 2M$.

(b) To find $\mathbf{grad}\, V$, we first determine the partial derivatives

$$\frac{\partial V}{\partial r} = -\frac{3M(3\cos^2\theta - 1)}{r^4}, \quad \frac{\partial V}{\partial \theta} = \frac{M(6\cos\theta(-\sin\theta))}{r^3}, \quad \frac{\partial V}{\partial \phi} = 0.$$

Evaluating these at A yields

$$\frac{\partial V}{\partial r}(3, \tfrac{\pi}{2}, \tfrac{\pi}{2}) = M/27, \quad \frac{\partial V}{\partial \theta}(3, \tfrac{\pi}{2}, \tfrac{\pi}{2}) = 0.$$

Hence, at A, $\mathbf{grad}\, V = \mathbf{e}_r\, M/27$. The magnitude of the force $\mathbf{F}$ is $|-\mathbf{grad}\, V| = M/27$, and the direction is given by the direction of $-\mathbf{e}_r$ at A, that is, the negative y-direction.