Q 1.

(a) Can you deduce that the following statement is true?

Charlie is not a student.

Let P be the event that Charlie is a student, and Q be the event that Charlie reads a lot. Assuming the implication $P \implies Q$ is true, then if Charlie does not read a lot the statement "Charlie is not a student" must also be true.

(b) Show, by giving a counter-example, that the following statement is false:

For all positive integers n, we have $4^n \geq n^4$.

When $n = 3$, $4^3 = 64 < 81 = 3^4$, therefore 3 is a counter-example and the statement "for all positive integers n, $4^n \geq n^4$" is false.

Q 2.

(a) Prove that the following statement is true by using a sequence of equivalences:

$x^2 \leq (x-2)(2x-3)$, for all $x \geq 6$.

Expanding the brackets on the right-hand side gives

$$\begin{aligned} & x^2 \leq (x-2)(2x-3) \\ \iff & x^2 \leq 2x^2 - 7x + 6 \\ \iff & 0 \ \leq x^2 - 7x + 6 \\ \iff & 0 \ \leq (x-1)(x-6). \end{aligned}$$

As this final inequality is true when $x \geq 6$, the statement $x^2 \leq (x-2)(2x-3)$, for all $x \geq 6$ is true.

(b) Prove that the following statement is true:

f is a polynomial function if and only if f' is a polynomial function.

Let P represent the statement "f is a polynomial function" and Q represent the statement "f' is a polynomial function".

If P is true, then f is a polynomial function of the form

$$a_0 + a_1 x + \cdots + a_n x^n.$$

The derivative f' has the form

$$a_1 + 2a_2 x + \cdots + n \cdot a_n x^{n-1}.$$

As $a_{0,1,\cdots,n}$ are constants, this derivative can be simplified to

$$b_0 + b_1 x + \cdots + b_n x^n$$

where $b_{0,1,\cdots,n}$ are constants. As this is also a polynomial function, the implication $P \implies Q$ is true.

If Q is true, then f' is a polynomial function of the form

$$b_0 + b_1x + \cdots + b_nx^n.$$

The antiderivatives of f' are of the form

$$b_0x + \frac{1}{2}b_1x^2 + \cdots + \frac{1}{n+1}b_nx^{n+1} + c.$$

As $b_{0,1,\cdots,n}$ and c are constants, this can be simplified to

$$d_0 + d_1x + \cdots + \frac{1}{n+1}d_nx^{n+1}.$$

As $d_{0,1,\cdots,n}$ are constants, these antiderivatives are also polynomial functions and the implication $Q \implies P$ is true.

As the statements $P \implies Q$ and $Q \implies P$ are both true, it follows that $P \iff Q$ is true, and so f is a polynomial function if and only if f' is a polynomial function.

Q 3.

Use mathematical induction to prove that the following statement is true:

$$2 \times 3 + 5 \times 6 + \cdots + (3n-1)(3n) = 3n^2(n+1), \text{ for all } n \geq 1.$$

Let $P(n)$ be the statement $2 \times 3 + 5 \times 6 + \cdots + (3n-1)(3n) = 3n^2(n+1)$, then $P(1)$ is true because $\big(3(1)-1\big)\big(3(1)\big) = 6$ and $3(1)^2(1+1) = 6$.

Now let $k \geq 1$, and assume that $P(k)$ is true, then we aim to show the following statement is true:

$$2 \times 3 + 5 \times 6 + \cdots + \big(3(k+1)-1\big)\big(3(k+1)\big) = 3(k+1)^2\big((k+1)+1\big)$$

We start by using the fact $P(k)$ is true:

$$\begin{aligned}
&2 \times 3 + 5 \times 6 + \cdots + \big(3(k+1)-1\big)\big(3(k+1)\big) \\
&= 3k^2(k+1) + \big(3(k+1)-1\big)\big(3(k+1)\big) \\
&= 3k^3 + 3k^2 + 9k^2 + 15k + 6 \\
&= 3(k^3 + 4k^2 + 5k + 2) \\
&= 3(k^2 + 2k + 1)(k+2) \\
&= 3(k+1)^2\big((k+1)+1\big)
\end{aligned}$$

Therefore, $P(k) \implies P(k+1)$ for all $n \geq 1$ and so the statement

$$2 \times 3 + 5 \times 6 + \cdots + (3n-1)(3n) = 3n^2(n+1) \text{ is true for all } n \geq 1.$$

Q 4.

(a) Use proof by contradiction to show that there is no positive real number x such that

$$x + \frac{4}{x} < 4.$$

Suppose that the statement is not true and that there exists a positive real number x such that $x + \frac{4}{x} < 4$. Then rearranging the inequality gives

$$\begin{aligned} x^2 + 4 &< 4x \\ x^2 - 4x + 4 &< 0 \end{aligned}$$

The quadratic equation $x^2 - 4x + 4 = 0$ has a discriminant of 0, and so has only a single root, and has a positive coefficient on the x^2 term. These two facts demonstrate that there are no real numbers for which the LHS of this equation is negative, and so the original statement that there exists a positive real number x such that $x + \frac{4}{x} < 4$ must be false.

(b) Use proof by contraposition to prove that the following statement is true for all positive integers n: If n^3 is even, then n is even.

The contrapositive implication for this statement is "if n is odd then n^3 is odd". Let n be odd, then $n = 2k + 1$ for some integer k and so

$$\begin{aligned} n^3 &= (2k + 1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

As k is an integer, this is equivalent to saying $n^3 = 2c + 1$ for some other integer c. Thus, the contrapositive statement "if n is odd, then n^3 is odd" is true and so is the original statement that "if n^3 is even, then n is even".

Q 5.

(a) Calculate the acceleration of the train as it slows down, and the distance that it travels before reaching the station, giving your answers to two significant figures.

Let x and v be the position and velocity of the train, respectively at time t, with $x = 0$, and $v = v_0$ at $t = 0$. Then the following equation of motion relates these quantities for an object moving along a straight line at constant acceleration:

$$x = \frac{1}{2}(v_0 + v)t.$$

Substituting $v_0 = 30$, $v = 0$ (as the train comes to rest at two minutes) and $t = 120$ gives

$$\begin{aligned} x &= \frac{1}{2}(30 + 0) \times 120 \\ &= 1800 \end{aligned}$$

Therefore, the position of the train after two minutes is 1800m.

The equation of motion

$$x = v_0 t + \frac{1}{2}at^2$$

relates the starting velocity, acceleration, and position at time t of an object constantly accelerating along a straight line. Substituting $v_0 = 30$, $t = 120$, and $x = 1800$ gives

$$\begin{aligned} 1800 &= 30 \times 120 + \frac{1}{2}120^2 a \\ &= 3600 + 7200a \\ 7200a &= -3600 \\ a &= -\frac{1}{4} \end{aligned}$$

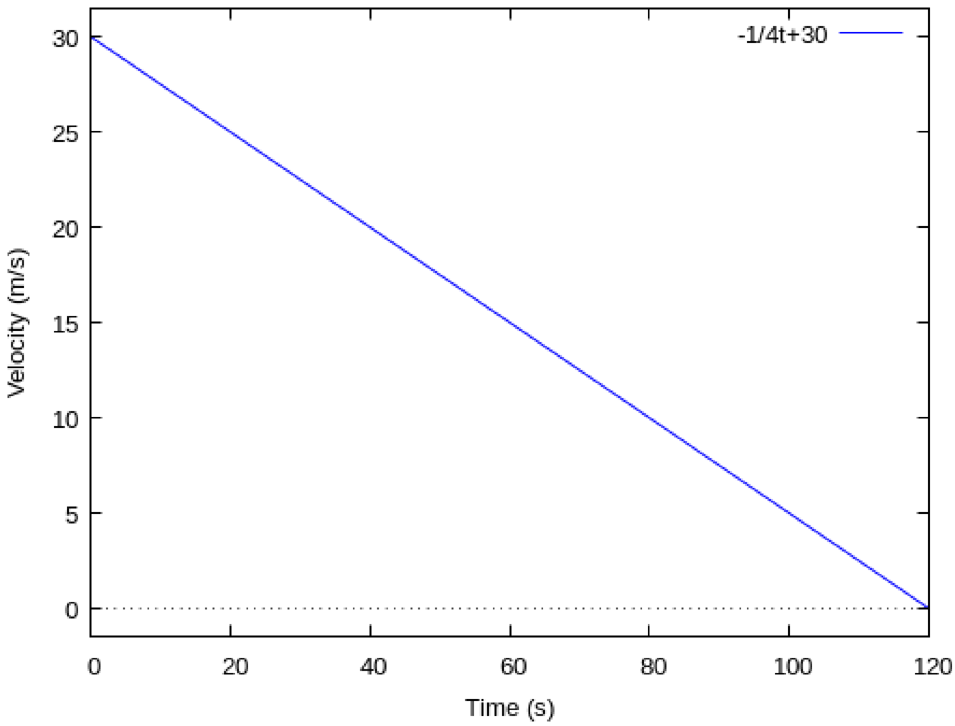
Therefore, the train is accelerating at -0.25ms^{-2} and travels 1800m before reaching the station.

(b) Draw a graph of the velocity v (in metres per second) of the train against the time t (in seconds), from the time at which the train first brakes until it arrives at the station.

Integrating the train's acceleration a with respect to time t gives an expression for its velocity v in terms of time and an arbitrary constant c:

$$v = -\frac{1}{4}t + c.$$

As the velocity of the train at $t = 0$ is 30ms^{-1}, $c = 30$ and the velocity of the train at time t is $v = -\frac{1}{4}t + 30$. The figure below shows the graph of velocity (in metres per second) of the train against the time(in seconds), from the time at which the train first brakes until it arrives at the station.



Q 6.

(a) Show that the acceleration of the coin is 0.27ms^{-2} to two significant figures.

Let v_0 be the starting velocity of the coin, and let v be the velocity of the coin at position x. As the coin is moving at constant acceleration along a straight line, the equation that relates acceleration to these quantities is

$$v^2 = v_0^2 + 2ax.$$

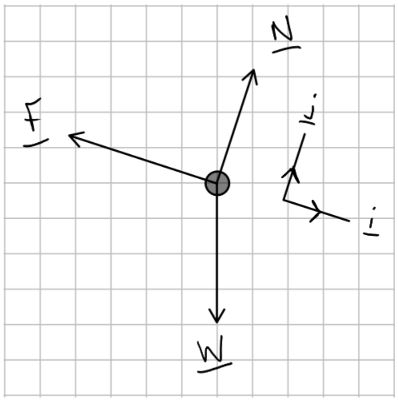
Substituting $v_0 = 0$, $v = 0.8$, and $x = 1.2$ gives

$$\begin{aligned} 0.8^2 &= 0^2 + 2 \times 1.2a \\ 0.64 &= 2.4a \\ a &= 0.2666... \end{aligned}$$

Thus, the acceleration of the coin is 0.27ms^{-2} to two significant figures.

(b) Draw a force diagram showing the forces acting on the coin when it is sliding down the board, including the directions of these forces. Define the symbols you use to denote the forces. Choose appropriate directions for the Cartesian unit vectors **i** and **j**, and mark these vectors on your diagram.

Let **W**, **N**, and **F** be the vectors representing the weight of the coin, the normal reaction of the slope, and the friction force, respectively. In the diagram below the coin is modelled as a particle with arrows indicating the directions of **W**, **N**, and **F**. The directions of the Cartesian unit vectors i and j are shown and are chosen to point in the direction the coin is moving and in the direction of the normal reaction, respectively. Note that all vectors are underlined in this hand-drawn diagram.



(c) Find μ, giving your answer to two significant figures.

Let $N = |\mathbf{N}|$, $F = |\mathbf{F}|$, and $W = |\mathbf{W}|$, then each force is component form is

$$\begin{aligned}\mathbf{F} &= -\mu N\mathbf{i}\\ \mathbf{N} &= N\mathbf{j}\\ \mathbf{W} &= mg\cos 298^\circ\mathbf{i} + mg\sin 298^\circ\mathbf{j}\end{aligned}$$

where g is the acceleration due to gravity. Let **R** be the resultant force acting on the coin, then:

$$\begin{aligned}\mathbf{R} &= \mathbf{F} + \mathbf{N} + \mathbf{W}\\ &= (mg\cos 298^\circ - \mu N)\mathbf{i} + (N + mg\cos 298^\circ)\mathbf{j}\end{aligned}$$

As the coin moves only in the **i** direction, then by Newton's second law

$$m\mathbf{ai} = (mg\cos 298^\circ - \mu N)\mathbf{i} + (N + mg\cos 298^\circ)\mathbf{j}$$

Resolving this equation in the **i** and **j** directions gives

$$ma = mg\cos 298^\circ - \mu N \tag{6.1}$$
$$0 = N + mg\sin 298^\circ \tag{6.2}$$

Rearranging equation 6.2 gives $N = -mg\sin 298°$. Substituting $a = 0.27$ and $N = -mg\sin 298°$ into equation 6.1 gives

$$0.27m = mg\cos 298° + \mu mg\sin 298°.$$

Dividing both sides of the equation by m and solving for μ gives

$$\begin{aligned} 0.27 &= g\cos 298° + \mu g\sin 298° \\ \mu g\sin 298° &= 0.27 - g\cos 298° \\ \mu &= \frac{0.27 - g\cos 298°}{g\sin 298°} \\ &= 0.5005... \end{aligned}$$

Therefore, μ is 0.50 to two significant figures.

Q 7.

(a) Show that the position r (in metres) of the water t seconds after leaving the hosepipe is given by

$$\mathbf{r} = v_0 t\cos 35°\mathbf{i} + (v_0 t\sin 35° - \tfrac{1}{2}gt^2)\mathbf{j}.$$

Assuming the only force acting on the water is gravity, the acceleration $\mathbf{a}$ of the water is given by $\mathbf{a} = -g\mathbf{j}$, where g is the acceleration due to gravity. The velocity $\mathbf{v}$ of the water is an antiderivative of $\mathbf{a}$ with respect to time:

$$\begin{aligned} \mathbf{v} &= \int -g\mathbf{j}\ \mathrm{d}t \\ &= -gt\mathbf{j} + \mathbf{c} \end{aligned}$$

where $\mathbf{c}$ is a vector quantity. Let $|\mathbf{v_0}| = v_0$, then the initial velocity of the water v_0 at $t = 0$ is

$$\mathbf{v_0} = v_0\cos 35°\mathbf{i} + v_0\sin 35°\mathbf{j},$$

and so $\mathbf{c} = v_0\cos 35°\mathbf{i} + v_0\sin 35°\mathbf{j}$. The velocity of a particle of water is therefore given by the equation

$$\mathbf{v} = v_0\cos 35°\mathbf{i} + (v_0\sin 35° - gt)\mathbf{j}.$$

The position of a particle of water at time t is an antiderivative of its velocity with respect to time, and so we have

$$\begin{aligned} \mathbf{r} &= \int v_0\cos 35°\mathbf{i} + (v_0\sin 35° - gt)\mathbf{j}\ \mathrm{d}t \\ &= v_0 t\cos 35°\mathbf{i} + (v_0 t\sin 35° - \tfrac{1}{2}gt^2)\mathbf{j} + \mathbf{d} \end{aligned}$$

Taking the position of the water at $t = 0$ to be $\mathbf{r} = \mathbf{0}$, then $\mathbf{d} = \mathbf{0}$ and the position of the water after leaving the hosepipe is given by

$$\mathbf{r} = v_0 t \cos 35^\circ \mathbf{i} + (v_0 t \sin 35^\circ - \tfrac{1}{2} g t^2)\mathbf{j}.$$

(b) Let t_1 be the time at which the water reaches the ground. Show that the numerical value of the product $v_0 t_1$ is 6.1 to two significant figures.

At t_1 the position of a particle of water is $5\mathbf{i} - \mathbf{j}$. Substituting these values into the vector equation for $\mathbf{r}$ gives

$$5\mathbf{i} - \mathbf{j} = v_0 t_1 \cos 35^\circ \mathbf{i} + (v_0 t_1 \sin 35^\circ - \tfrac{1}{2} g t_1^2)\mathbf{j}.$$

Resolving just the $\mathbf{i}$ component gives

$$\begin{aligned} 5 &= v_0 t_1 \cos 35^\circ \\ v_0 t_1 &= \frac{5}{\cos 35^\circ} \\ &= 6.1038... \end{aligned}$$

Therefore, the numerical value of the product $v_0 t_1$ is 6.1 to two significant figures.

(c) Determine t_1 and v_0 , giving your answers to two significant figures.

Substituting the position of a particle of water at t_1 into the equation for $\mathbf{r}$ gives

$$5\mathbf{i} - \mathbf{j} = v_0 t_1 \cos 35^\circ \mathbf{i} + (v_0 t_1 \sin 35^\circ - \tfrac{1}{2} g t_1^2)\mathbf{j}.$$

Resolving this equation in the $\mathbf{i}$ and $\mathbf{j}$ directions gives

$$5 = v_0 t_1 \cos 35^\circ \tag{7.1}$$
$$-1 = v_0 t_1 \sin 35^\circ - \tfrac{1}{2} g t_1^2 \tag{7.2}$$

Multiplying equation 7.1 by $\sin 35^\circ$ and equation 7.2 by $\cos 35^\circ$ gives

$$5 \sin 35^\circ = v_0 t_1 \cos 35^\circ \cdot \sin 35^\circ \tag{7.3}$$
$$-\cos 35^\circ = v_0 t_1 \sin 35^\circ \cdot \cos 35^\circ - \tfrac{1}{2} g t_1^2 \cdot \cos 35^\circ \tag{7.4}$$

Subtracting equation 7.4 from equation 7.3 to eliminate v_0 and solving for t_1 gives

$$\begin{aligned} 5 \sin 35^\circ + \cos 35^\circ &= \tfrac{1}{2} g t_1^2 \cdot \cos 35^\circ \\ 5 \sin 35^\circ + 1 &= \tfrac{1}{2} g t_1^2 \\ 10 \sin 35^\circ + 2 &= g t_1^2 \\ t_1^2 &= \frac{10 \sin 35^\circ + 2}{g} \\ t_1 &= 0.8884... \end{aligned}$$

Substituting this value of t_1 into equation 7.1 and solving for v_0 gives

$$\begin{aligned} 5 &= v_0 \times 0.8884... \times \cos 35° \\ v_0 &= \frac{5}{0.8884... \times \cos 35°} \\ &= 6.8701... \end{aligned}$$

Therefore, $t_1 = 0.89$ seconds and $v_0 = 6.9\text{ms}^{-1}$, both to two significant figures.

Q 8.

(a) Find the eigenvalues of $\mathbf{A}$ and, for each eigenvalue, find a corresponding eigenvector of the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where a, b are integers and $b > 0$.

The characteristic equation for a 2x2 matrix $\mathbf{A}$ is $\lambda^2 - (\text{tr}\mathbf{A})\lambda + \det \mathbf{A} = 0$, where $\text{tr}\mathbf{A}$ and $\det \mathbf{A}$ are the trace and determinant of $\mathbf{A}$, respectively. Substituting $\text{tr}\mathbf{A} = 18$ and $\det \mathbf{A} = -40$ and factorising gives

$$\begin{aligned} \lambda^2 - 18\lambda + 40 &= 0 \\ (\lambda + 2)(\lambda - 20) &= 0 \end{aligned}$$

Thus, $\mathbf{A}$ has eigenvalues of -2 and 20. Eigenvectors can be found for each eigenvalue using the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{0}$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Substituting $\lambda = -2$ gives

$$\begin{pmatrix} 10 - (-2) & 40 \\ 3 & 8 - (-2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

which represents the pair of simultaneous equations:

$$\begin{aligned} 12x + 40y &= 0 \\ 3x + 10y &= 0 \end{aligned}$$

As $x = -10$ and $y = 3$ satisfy both equations, the eigenvector of $\mathbf{A}$ corresponding to eigenvalue -2 is $\begin{pmatrix} -10 \\ 3 \end{pmatrix}$.

Substituting $\lambda = 20$ gives

$$\begin{pmatrix} 10 - 20 & 40 \\ 3 & 8 - 20 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

which represents the pair of simultaneous equations:

$$\begin{aligned} -10x + 40y &= 0 \\ 3x - 12y &= 0 \end{aligned}$$

As $x = 4$ and $y = 1$ satisfy both equations, the eigenvector of $\mathbf{A}$ corresponding to eigenvalue 20 is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

(b) Hence express $\mathbf{A}$ in the form $\mathbf{PDP^{-1}}$, where $\mathbf{P}$ is an invertible matrix and $\mathbf{D}$ is a diagonal matrix, stating the matrices $\mathbf{P}$, $\mathbf{P^{-1}}$ and $\mathbf{D}$.

Let $\mathbf{P}$ be the matrix $\begin{pmatrix} -10 & 4 \\ 3 & 1 \end{pmatrix}$, whose columns are the eigenvectors of $\mathbf{A}$. Let $\mathbf{D}$ be the matrix $\begin{pmatrix} -2 & 0 \\ 0 & 20 \end{pmatrix}$, whose diagonal elements are the eigenvalues of $\mathbf{A}$. The inverse of $\mathbf{P}$ is given by

$$\begin{aligned} \mathbf{P}^{-1} &= \frac{1}{\det \mathbf{P}} \begin{pmatrix} 1 & -4 \\ -3 & -10 \end{pmatrix} \\ &= -\frac{1}{22} \begin{pmatrix} 1 & -4 \\ -3 & -10 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{22} & \frac{2}{11} \\ \frac{3}{22} & \frac{5}{11} \end{pmatrix} \end{aligned}$$

Then, $\mathbf{A}$ can be expressed as

$$\begin{aligned} \mathbf{A} &= \mathbf{PDP}^{-1} \\ &= \begin{pmatrix} -10 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} -\frac{1}{22} & \frac{2}{11} \\ \frac{3}{22} & \frac{5}{11} \end{pmatrix} \end{aligned}$$

(c) Use your answer to part (b) to calculate $\mathbf{A}^4$.

A diagonalisable matrix $\mathbf{A}$ raised to the nth power is given by $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$. Using the matrices from part (b) we have

$$\mathbf{D}^4 = \begin{pmatrix} (-2)^4 & 0 \\ 0 & 20^4 \end{pmatrix}$$

and thus

$$\begin{aligned} \mathbf{A}^4 &= \begin{pmatrix} -10 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} (-2)^4 & 0 \\ 0 & 20^4 \end{pmatrix} \begin{pmatrix} -\frac{1}{22} & \frac{2}{11} \\ \frac{3}{22} & \frac{5}{11} \end{pmatrix} \\ &= \begin{pmatrix} 87,280 & 290,880 \\ 21,816 & 72,736 \end{pmatrix} \end{aligned}$$

(d) Use Maxima to check your answers to parts (a) and (c). Include a printout or screenshot of the Maxima output in your answer, and explain carefully how to interpret it.

Define A

(%i1) A: matrix(
 [10,40],
 [3,8]
);

(%o1) $\begin{pmatrix} 10 & 40 \\ 3 & 8 \end{pmatrix}$

Find eigenvalues and eigenvectors of AFind eigenvalues and eigenvectors of A

(%i2) eigenvectors(A);

(%o2) $\left[[[-2,20],[1,1]],\left[\left[\left[1,-\frac{3}{10}\right]\right],\left[\left[1,\frac{1}{4}\right]\right]\right]\right]$

Find the 4th power of A

(%i3) A^^4;

(%o3) $\begin{pmatrix} 87280 & 290880 \\ 21816 & 72736 \end{pmatrix}$

The matrix $\mathbf{A}$ is defined in cell %in1 and cell %in2 shows how the command `eigenvectors` is used to calculate eigenvalues and their eigenvectors. The output of cell %o2 is a list of lists where the first element is a list whose first element gives the distinct eigenvalues of $\mathbf{A}$ and whose second element gives their respective multiplicities. In other words, $\mathbf{A}$ has eigenvalues -2 and 20, and each of these is repeated only once. The second element of the main list gives eigenvectors corresponding to the list of eigenvalues. The eigenvectors given in part (a) are scalar multiples of those returned by Maxima. Cell %i3 shows the command for raising $\mathbf{A}$ to the 4th power, and cell %o3 shows the resultant matrix.

(e) Use your answer to part (a) to find the general solution of the following system of linear differential equations, expressing both x and y as functions of t:

$$\dot{x} = 10x + 40y,$$
$$\dot{y} = 3x + 8y.$$

The solution of a system of differential equations given by a matrix with two distinct real eigenvalues has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ce^{\lambda_1 t}\mathbf{X}_1 + De^{\lambda_2 t}\mathbf{X}_2,$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2, respectively, and C and D are arbitrary constants. Substituting the eigenvalues and corresponding eigenvectors calculated in part (a) gives the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ce^{-2t}\begin{pmatrix} -10 \\ 3 \end{pmatrix} + De^{20t}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$