Complex numbers

Introduction

As you know, every positive real number has two square roots, one positive and one negative. For example, the positive real number 9 has square roots 3 and -3. Another way to express this fact is to say that the equation

$$x^2 = 9$$

has solutions 3 and -3. Suppose now that you want to find a square root of a *negative* number, such as -1. That is, you want to solve the equation

$$x^2 = -1.$$

You may believe that this equation has no solutions; after all, the square of any real number is positive or zero. In this unit you'll learn about a system of numbers, known as the *complex numbers*, in which -1 and all other negative numbers have square roots.

The complex numbers are created by first introducing a new number, written as i, with the property that $i^2 = -1$ (so i is a square root of -1). It may seem like cheating to simply define i in this way, but you'll see that the resulting new system of numbers is incredibly powerful and useful. The other complex numbers are created by multiplying i by any real number, and then adding any real number. For example, the following are complex numbers:

$$3 + 4i, \quad -\sqrt{2} + 99i \quad \text{and} \quad 0.7 - \pi i.$$

Each real number can also be considered to be a complex number (3 is $3 + 0i$, for example). The number i is a solution of the equation $x^2 = -1$, and there's a second solution of this equation, namely $-i$.

The first publication to include a reference to complex numbers was the book *Ars Magna* (1545) by Gerolamo Cardano (1501–1576). The possibility of using complex numbers first emerged when Italian mathematicians were developing methods for solving cubic equations, such as $x^3 + x^2 + 6x + 3 = 0$. Both Scipione del Ferro (1465–1526) and Niccolò Fontana Tartaglia (1499/1500–1557) independently discovered how to solve any cubic equation, using methods that sometimes involve complex numbers. Tartaglia revealed his method in secret to Cardano, who later published it in his *Ars Magna*. This angered Tartaglia, who insulted Cardano for revealing the method. Cardano, in his defence, claimed to have also seen del Ferro's method, which was unpublished, and so he no longer felt obliged to keep the method of solving cubic equations secret.

Niccolò Fontana Tartaglia

At first, complex numbers may seem abstract, because they don't obviously represent physical quantities in the way that real numbers do. However, they're of fundamental importance in mathematics – as you'll begin to see in this unit – and they're an essential tool in many scientific disciplines, such as electromagnetism, fluid dynamics and quantum mechanics.

Figure 1   A transistor

Quantum mechanics, for instance, is about the motion of very small objects, such as atoms. The foundational equations of the subject involve complex numbers.

Discoveries in quantum mechanics led to the development of the modern transistor, midway through the last century. Transistors (one is shown in Figure 1) are devices used to control current in circuits, and are an essential part of electronic systems, such as those found in cars, computers and portable media players.

It's instructive to think of complex numbers geometrically, using the *complex plane*, which is a plane such as that shown in Figure 2. Each complex number is represented by a point on the plane. For instance, the complex number $2 + 3i$ is represented by the point with coordinates $(2, 3)$. You'll learn more about the complex plane in Section 2.
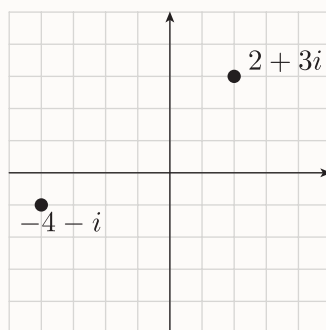

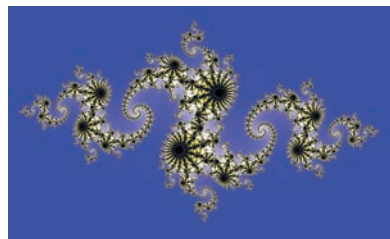**Figure 2**   The complex plane

Figure 3   A fractal in the complex plane

In higher-level modules involving complex numbers you can find out how some simple formulas involving complex numbers give rise to *fractals* in the complex plane, which are intricate shapes with repetitive structures, such as that shown in Figure 3.

Not only do complex numbers have fascinating geometric properties that give rise to beautiful fractals, but the system of complex numbers also has some useful algebraic properties that the system of real numbers lacks. For instance, you saw in Unit 2 that, if you're working only with the real numbers, then some quadratic equations have no solutions. An example is the equation $x^2 - 4x + 5 = 0$: if you write this equation as $(x - 2)^2 + 1 = 0$, then you can see that for every real number x the left-hand side is greater than or equal to 1, and hence the equation has no solutions that are real numbers. You'll see in Section 3 that if you're allowed to use complex numbers, then every quadratic equation has at least one solution. In fact, you'll meet an even stronger result: if you're allowed to use complex numbers, then every *polynomial equation* has at least one solution. A polynomial equation is an equation of the form 'polynomial expression $= 0$' (where the polynomial expression has degree at least 1), such as

$$5x^6 + 8 = 0 \quad \text{or} \quad x^7 - 13x^5 + \tfrac{5}{3}x^2 - 2 = 0.$$

In Section 4 you'll be shown *Euler's formula*,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is a hugely important equation that relates the exponential function to the trigonometric functions. Euler's formula allows you to write complex numbers in a helpful manner, called *exponential form*, which is widely used in mathematics, engineering and other scientific subjects.

Some of the blue boxes in this unit give you a taste of some of the ways in which complex numbers are used in higher-level mathematics. Remember that these boxes are only for your interest; you won't be assessed on their contents.

1 Arithmetic with complex numbers

In this section you'll learn the details of what complex numbers are, and how you can add, subtract, multiply and divide them. You'll also learn about another arithmetic operation, called *complex conjugation*, which is particular to the complex numbers.

1.1   What are complex numbers?

To define the complex numbers we start by considering the equation

$$x^2 = -1.$$

This equation has no solutions that are real numbers, because the square of any real number is non-negative. To overcome this problem, we introduce a new number, which we call i, and declare that i is a solution of the equation.

> The number i is defined to have the property $i^2 = -1$.

So i is a square root of -1. You learned in Section 4 of Unit 1 that the symbol $\sqrt{\ }$ is used to denote the non-negative square root of a non-negative real number. In other texts you may see i written as $\sqrt{-1}$, even though neither -1 nor i is a non-negative real number. This notation isn't used in this module, as it can be misleading. To see why, remember the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$, also from Unit 1, which is true when a and b are non-negative real numbers. If you try to apply this rule with $a = b = -1$, then you obtain the incorrect statement

$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)\times(-1)} = \sqrt{1} = 1.$$

This statement is wrong because $\sqrt{-1}\sqrt{-1}$ should equal -1, not 1.

To avoid this kind of pitfall, it's best not to use the notation $\sqrt{-1}$, except in certain particular circumstances, such as those described in Subsection 3.1, where you'll meet expressions involving $\pm\sqrt{\ }$.

$\sqrt{-1}$ ♥ maths
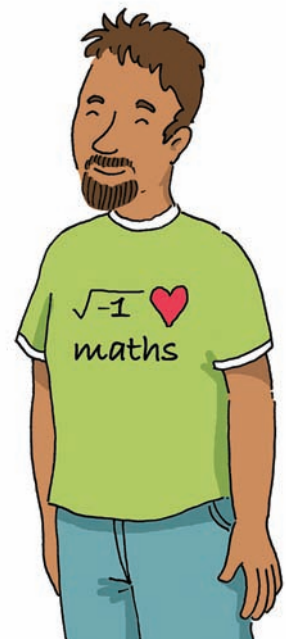
Let's now look at how the complex numbers are obtained by combining this new number i with the real numbers. For instance, you can multiply i by a real number such as 5 to give another number, $5i$. You can then add $5i$ to a real number such as 3 to give another number, $3 + 5i$. The complex numbers are all the numbers that you can obtain in this way.

Complex numbers

A **complex number** is a number of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$.

For example, $-7 + 3i$ and $\frac{3}{5} + 19i$ are complex numbers. Using the usual conventions of algebra, we can write some complex numbers in a form simpler than $a + bi$. For example, we write

$$
\begin{array}{rcl}
45 + (-11)i & \text{as} & 45 - 11i \\
3 + 1i & \text{as} & 3 + i \\
6 + 0i & \text{as} & 6 \\
0 + (-1)i & \text{as} & -i \\
0 + 0i & \text{as} & 0.
\end{array}
$$

Note that since i is a square root of -1, the number $-i$ is also a square root of -1, as you'd expect. You'll see this confirmed later.

In the rest of this section, you'll meet some of the basic properties of complex numbers, and practise manipulating them. To start with, here are two important definitions.

> ### Real and imaginary parts
>
> For any complex number $z = a + bi$, the real number a is called the **real part** of z and the real number b is called the **imaginary part** of z. We write
>
> $$\mathrm{Re}(z) = a \quad \text{and} \quad \mathrm{Im}(z) = b.$$

For example,

$$\mathrm{Re}(-7 + 3i) = -7 \quad \text{and} \quad \mathrm{Im}(-7 + 3i) = 3.$$

Also,

$$\mathrm{Re}(5i) = 0 \quad \text{and} \quad \mathrm{Im}(5i) = 5.$$

Notice that it is b, and not bi, that is the imaginary part of $a + bi$. For example, the imaginary part of $-7 + 3i$ is 3, not $3i$.

If the real part of a complex number is 0, then the complex number is sometimes called an **imaginary number** or a **purely imaginary number**. For instance, the complex numbers $5i$ and $-i$ are imaginary numbers.

If the imaginary part of a complex number is 0, then that complex number is in fact a real number. For instance, the complex number 5 (which you could write as $5 + 0i$) is also a real number. In the same way, *every* real number is a complex number. This implies that the set of real numbers is a subset of the set of complex numbers; that is, $\mathbb{R} \subseteq \mathbb{C}$.

Activity 1 *Identifying real and imaginary parts*

Write down the real and imaginary part of each of the following complex numbers.

(a) $2 + 9i$ (b) 4 (c) $-7i$ (d) 0 (e) i (f) $1 - i$

You can also write complex numbers in the form $a + ib$, which is equivalent to $a + bi$, and both forms are used. Your choice may depend on the nature of the real number b. For instance, you may prefer to write $1 + i\sqrt{2}$ rather than $1 + \sqrt{2}i$, to avoid possible confusion with $1 + \sqrt{2i}$.
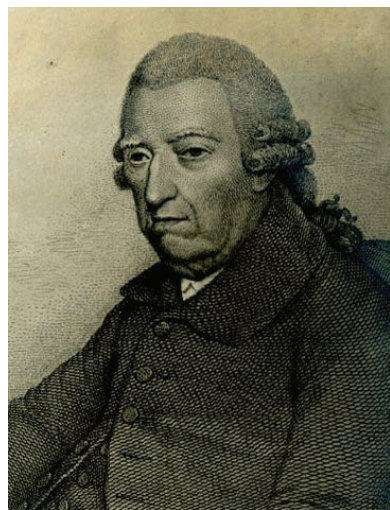
Francis Maseres (1731–1824)

Over the centuries, mathematicians have struggled with definitions of numbers. Few people have trouble understanding the natural numbers $1, 2, 3, \ldots$, but the concept of zero was more troublesome. One of the first people to bring the idea to Europe was Fibonacci (Leonardo of Pisa) in his *Liber Abaci* (1202), the text that spread the Hindu–Arabic numeral system through Europe. Even so, the symbol 0 was not widely used in Europe until the seventeenth century, and in 1759 the English mathematician Francis Maseres wrote of the negative numbers that they

> darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple.

Likewise, many mathematicians at first doubted the validity of complex numbers, which Cardano had described in *Ars Magna* as 'some recondite third kind of thing'. In fact, the renowned French philosopher and mathematician René Descartes (1596–1650) coined the term *imaginary numbers* for real number multiples of i because he considered them to be illusory.

Today we're aware of the many uses of different types of numbers, and the foundations of mathematics are well established, so there's no longer any doubt about the validity of zero, negative numbers or complex numbers.

1.2 Adding and subtracting complex numbers

You can add or subtract complex numbers by adding or subtracting their real and imaginary parts separately. For example,

$$(5 + 6i) + (3 + 2i) = \underbrace{(5 + 3)}_{\substack{\text{add real} \\ \text{parts}}} + \underbrace{(6 + 2)}_{\substack{\text{add imaginary} \\ \text{parts}}} i \quad = 8 + 8i,$$

and

$$(5 + 6i) - (3 + 2i) = \underbrace{(5 - 3)}_{\substack{\text{subtract} \\ \text{real parts}}} + \underbrace{(6 - 2)}_{\substack{\text{subtract} \\ \text{imaginary parts}}} i \quad = 2 + 4i.$$

In essence, to add or subtract complex numbers you treat i as a variable and add or subtract in the normal way.

Example 1   *Adding and subtracting complex numbers*

Let $z = 7 + 19i$ and $w = 13 - 10i$. Work out $z + w$ and $z - w$.

Solution

$$(7 + 19i) + (13 - 10i) = (7 + 13) + (19 + (-10))i = 20 + 9i$$
$$(7 + 19i) - (13 - 10i) = (7 - 13) + (19 - (-10))i = -6 + 29i$$

Activity 2 *Adding and subtracting complex numbers*

Work out $z + w$ and $z - w$ for each case below.

(a) $z = 2 + 5i$, $w = -7 + 13i$ (b) $z = -4i$, $w = -9i$

(c) $z = 3 - 7i$, $w = 3 - 7i$ (d) $z = 3 + 7i$, $w = 3 - 7i$

(e) $z = \frac{1}{6} - \frac{1}{3}i$, $w = -\frac{1}{3} + \frac{1}{6}i$ (f) $z = 1.2$, $w = 3.4i$

Many familiar rules for adding and subtracting real numbers also apply to complex numbers. For instance, the order in which you add two complex numbers z and w doesn't matter:

$$z + w = w + z.$$

Also, any three complex numbers u, v and w satisfy

$$(u + v) + w = u + (v + w).$$

You've met similar rules before, when you learned about addition of vectors, in Unit 5, and addition of matrices, in Unit 9. Recall that an operation (in this case, addition of complex numbers) that obeys the first rule is said to be *commutative*, and an operation that obeys the second rule is said to be *associative*. Together the two rules tell you that you can add several complex numbers in any order that you choose.

The number 0 has the same role in the arithmetic of complex numbers as it does in the arithmetic of real numbers, in that adding 0 to any number leaves that number unchanged.

You should approach the next activity using the usual rules of algebra, remembering to treat i like a variable.

Activity 3 *Adding and subtracting several complex numbers*

Let $u = 4 + 6i$, $v = -3 + 5i$ and $w = 2 - i$. Work out the following.

(a) $u + v + w$ (b) $w + v + u$ (c) $u - (v + w)$ (d) $u - (v - w)$

1.3 Multiplying complex numbers

You can multiply two complex numbers using the usual rules of algebra and the fact that $i^2 = -1$, as shown in the next example.

Example 2 *Multiplying complex numbers*

Find the product of $3 + 2i$ and $5 + i$.

Solution

Multiply out the brackets.

$$(3 + 2i)(5 + i) = 15 + 3i + 10i + 2i^2$$
$$= 15 + 13i + 2i^2$$

Simplify using $i^2 = -1$.

$$= 15 + 13i + 2(-1)$$
$$= 15 + 13i - 2$$
$$= 13 + 13i$$

As usual, you can also write products using the $\times$ symbol. For instance, the product $(3 + 2i)(5 + i)$ can also be written as

$$(3 + 2i) \times (5 + i).$$

Activity 4 *Multiplying complex numbers*

Find the following products of complex numbers.

(a) $(1 + 3i)(2 + 4i)$ (b) $(-2 + 3i)(4 - 7i)$ (c) $3i(4 - 5i)$

(d) $7(-2 + 5i)$ (e) $(2 - 3i)(2 + 3i)$ (f) $\left(\frac{1}{2} + i\right)\left(1 + \frac{1}{2}i\right)$

Multiplication of complex numbers, like addition of complex numbers, is both *commutative* and *associative*. This means that, for any complex numbers z and w,

$$z \times w = w \times z,$$

and, for any complex numbers u, v and w,

$$u \times (v \times w) = (u \times v) \times w.$$

Multiplication of complex numbers is also *distributive* over addition of complex numbers; that is, for any complex numbers u, v and w,

$$u \times (v + w) = u \times v + u \times w.$$

When you add and multiply real numbers, you probably apply the commutative, associative and distributive laws without thinking about them. You should be comfortable doing the same for complex numbers.

The numbers 0 and 1 play the same roles in the multiplication of complex numbers as they do in the multiplication of real numbers. That is, multiplying a number by 0 gives the answer 0, and multiplying a number by 1 leaves that number unchanged.

Activity 5 *Adding and multiplying several complex numbers*

Let $u = 1 + 2i$, $v = 4 - 3i$ and $w = -i$. Work out the following.

(a) $u(v + w)$ (b) $uv + uw$ (c) uvw

When multiplying a complex number z by itself, you should write z^2, rather than zz or $z \times z$, just as for real numbers. Other positive integer powers of a complex number z, such as z^3 or z^{100}, are defined in the usual way. When z is not zero, the zeroth power z^0 of z is defined to be 1, just as for real numbers. You'll learn about negative integer powers later on.

The index laws from Section 4 of Unit 1 continue to hold when the base numbers are complex numbers and the powers are positive integers. For example, one of these laws for complex numbers states that, if z and w are complex numbers and n is a positive integer, then

$$(zw)^n = z^n w^n.$$

In fact, once you learn about negative powers you'll see that these laws hold when the powers are any integers (not necessarily positive integers).

Activity 6 *Working out powers of i*

Work out i^0, i^1, i^2, i^3, i^4, i^5 and i^6. Predict the pattern that would emerge if you continued to work out higher powers of i.

As you learn more about complex numbers, you'll come to see that the system of complex numbers has richer properties than the system of real numbers. One example of this is in taking square roots of numbers.

You know that each positive real number has two square roots. For example, the number 3 has the two square roots $\pm\sqrt{3}$. If you're working with the complex numbers, then it's also true that each *negative* real number has two square roots. For example, the number -3 has the two square roots $\pm i\sqrt{3}$, because

$$\left(i\sqrt{3}\right)^2 = i^2\left(\sqrt{3}\right)^2 = (-1)\times 3 = -3$$

and

$$\left(-i\sqrt{3}\right)^2 = \left(i\times\left(-\sqrt{3}\right)\right)^2 = i^2\left(-\sqrt{3}\right)^2 = (-1)\times 3 = -3.$$

As you'll see later, these are the only two square roots of -3. In general, we have the following useful fact.

Square roots of a negative real number

If d is a positive real number, then the square roots of $-d$ are $\pm i\sqrt{d}$.

For example, the square roots of -4 are $\pm i\sqrt{4}$, that is, $\pm 2i$.

Activity 7   *Checking the square roots of a negative number*

Show that $3i$ and $-3i$ are both square roots of -9.

Gauss in his prime

Gaussian integers

As you know, an integer p greater than 1 whose only integer factors are ± 1 and $\pm p$ is called a *prime number*, or just a *prime*. The first few primes are

2, 3, 5, 7, 11, ...

Notice that even though 2 is a prime,

$$(1+i)(1-i) = 1 - i + i - i^2 = 2.$$

This shows that $1+i$ and $1-i$ are both factors of 2, so if we allow not only integer factors but also factors that are complex numbers whose real and imaginary parts are integers, then 2 is no longer a prime! A complex number $a + bi$ for which a and b are integers is known as a *Gaussian integer*, after the German mathematician Carl Friedrich Gauss (1777–1855), who first developed them.

Just as any ordinary integer x can be factorised in two 'trivial' ways as $x = 1\times x$ and $x = (-1)\times(-x)$, so any Gaussian integer z can be factorised in four trivial ways as

$$z = 1\times z = (-1)\times(-z) = i\times(-iz) = (-i)\times(iz).$$

If these are the only ways in which a Gaussian integer z can be factorised as a product of two Gaussian integers, then z is called a *Gaussian prime*. You've seen that 2 is not a Gaussian prime, and neither is 5, because

$$(2+i)(2-i) = 5.$$

However, it can be shown that 3 is a Gaussian prime, as is $1+i$, and in fact there are infinitely many Gaussian primes.

You can learn about the Gaussian integers in more advanced modules on number theory.

1.4 Complex conjugation

Before you find out how to divide complex numbers, it's useful for you to learn about another operation that you can perform on complex numbers, called *complex conjugation*.

Complex conjugation

The **complex conjugate** of $a + bi$ is $a - bi$.

The complex conjugate of z is denoted by $\overline{z}$.

The operation of transforming z to $\overline{z}$ is called **complex conjugation**.

For example,

$$\text{if} \quad z = -3 + 2i \quad \text{then} \quad \overline{z} = -3 - 2i, \quad \text{and}$$
$$\text{if} \quad z = 5 - 3i \quad \text{then} \quad \overline{z} = 5 + 3i.$$

Activity 8   *Finding complex conjugates*

Find the complex conjugate of each of the following complex numbers.

(a) $4 + 2i$ (b) $-3 - 8i$ (c) $9i$ (d) 5

Notice that if z is a real number, then $\overline{z} = z$. So complex conjugation has no effect on real numbers.

In the next activity you're asked to show that to undo complex conjugation, you just apply complex conjugation again.

Activity 9   *Applying complex conjugation twice*

Let $z = a + bi$ and $w = \overline{z}$. Show that $\overline{w} = z$.

There are some more properties of complex conjugation in the box below. The last property involves division of complex numbers, which you'll meet shortly.

These properties, and all the other complex number properties stated in this unit, hold for all complex numbers for which the expressions in them are defined. For example, the fourth property below holds for all complex numbers z and w with $w \neq 0$.

> ### Some properties of complex conjugation
>
> $$\overline{z + w} = \overline{z} + \overline{w}$$
> $$\overline{z - w} = \overline{z} - \overline{w}$$
> $$\overline{zw} = \overline{z}\,\overline{w}$$
> $$\overline{z/w} = \overline{z}/\overline{w}$$

You can prove these properties by writing $z = a + bi$ and $w = c + di$. This gives

$$
\begin{aligned}
\overline{z + w} &= \overline{(a + bi) + (c + di)} \\
&= \overline{(a + c) + (b + d)i} \\
&= (a + c) - (b + d)i \\
&= (a - bi) + (c - di) \\
&= \overline{z} + \overline{w}.
\end{aligned}
$$

In the same way you can show that $\overline{z - w} = \overline{z} - \overline{w}$.

To prove the third property, let's work out the sides of the equation separately. We have

$$
\begin{aligned}
\overline{z}\,\overline{w} &= (a - bi)(c - di) \\
&= ac - adi - bci + bdi^2 \\
&= (ac - bd) - (ad + bc)i.
\end{aligned}
$$

Also,

$$
\begin{aligned}
zw &= (a + bi)(c + di) \\
&= ac + adi + bci + bdi^2 \\
&= (ac - bd) + (ad + bc)i,
\end{aligned}
$$

which gives

$$\overline{zw} = (ac - bd) - (ad + bc)i.$$

So both $\overline{z}\,\overline{w}$ and $\overline{zw}$ are equal to $(ac - bd) - (ad + bc)i$, which implies that $\overline{zw} = \overline{z}\,\overline{w}$.

The fourth property will be proved in the next subsection, once you've seen how to divide complex numbers.

In the next activity you're asked to prove two further properties of complex conjugation.

Activity 10   *Proving two identities involving complex numbers*

By writing $z = a + bi$, prove the following identities.

(a) $z + \overline{z} = 2\operatorname{Re}(z)$ (b) $z - \overline{z} = 2i\operatorname{Im}(z)$

Here's the property of complex conjugation that's useful when you want to divide complex numbers: whenever you multiply a complex number by its complex conjugate, you obtain a real number. To see why this is, recall the formula for the difference of two squares from Unit 1:

$$(A + B)(A - B) = A^2 - B^2.$$

Given any complex number $z = a + bi$, we can apply the difference of two squares formula with $A = a$ and $B = bi$ to give

$$\underbrace{(a + bi)}_{z}\,\underbrace{(a - bi)}_{\overline{z}} = a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 + b^2.$$

So we have the useful fact below.

> For any complex number $z = a + bi$,
> $$z\overline{z} = a^2 + b^2.$$

For example,

$$(1 + 2i)(1 - 2i) = 1^2 + 2^2 = 1 + 4 = 5.$$

Activity 11   *Multiplying complex numbers by their complex conjugates*

Multiply the following complex numbers by their complex conjugates.

(a) $2 + 3i$ (b) $-1 - 2i$ (c) $5i$ (d) -2

1.5 Dividing complex numbers

In this subsection you'll learn how to divide complex numbers, making use of the complex conjugation operation that you met in the previous subsection.

Suppose, for example, that you want to divide $5 + 3i$ by $1 + 2i$. You can write the result as

$$\frac{5 + 3i}{1 + 2i}.$$

To see that this really is a complex number, you can simplify it by multiplying the top and bottom of the fraction by the complex conjugate of the denominator. For our fraction, the denominator is $1 + 2i$, so its complex conjugate is $1 - 2i$. Multiplying the top and bottom of the fraction by $1 - 2i$ gives

$$\frac{5 + 3i}{1 + 2i} = \frac{(5 + 3i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 10i + 3i - 6i^2}{1^2 + 2^2} = \frac{11 - 7i}{5}.$$

You can write this number as

$$\tfrac{1}{5}(11 - 7i) \quad \text{or} \quad \tfrac{11}{5} - \tfrac{7}{5}i.$$

The second alternative has the form $a + bi$, which shows that it is indeed a complex number. However, it's fine to leave the answer in any of the final three forms above.

Simplifying fractions in this way may remind you of simplifying fractions involving surds, such as

$$\frac{1 + 5\sqrt{3}}{\sqrt{2} + \sqrt{3}},$$

which you met in Section 4 of Unit 1. To do this, you used a conjugate of an expression involving surds, which performs a similar role to the complex conjugate.

Example 3   *Dividing complex numbers*

Write the fraction $\dfrac{1 + 3i}{5 - i}$ as a single complex number.

Solution

💭 Multiply the top and bottom of the fraction by the complex conjugate of the denominator, which is $5 + i$, and simplify the resulting expression. 💬

$$\frac{1 + 3i}{5 - i} = \frac{(1 + 3i)(5 + i)}{(5 - i)(5 + i)}$$

$$= \frac{5 + i + 15i + 3i^2}{5^2 + (-1)^2}$$

$$= \frac{2 + 16i}{26}$$

$$= \frac{1 + 8i}{13}$$

You can check that the answer that you obtain from a division of complex numbers is correct by using multiplication. In the example that you've just seen, for instance, you find that

$$\tfrac{1}{13}(1 + 8i) \times (5 - i) = \tfrac{1}{13}(5 + 39i - 8i^2) = \tfrac{1}{13}(13 + 39i) = 1 + 3i,$$

as expected.

Activity 12   *Dividing complex numbers*

Write each of the following fractions as a single complex number.

(a) $\dfrac{1}{-2 + 3i}$ (b) $\dfrac{2i}{1 + i}$ (c) $\dfrac{11 - 8i}{2i}$ (d) $\dfrac{1}{i}$ (e) $\dfrac{1}{-i}$

(f) $\dfrac{4 + 7i}{-1 + 2i}$ (g) $\dfrac{8 + 3i}{1 + 3i}$ (h) $\dfrac{-2 + 5i}{-4 - i}$

It isn't possible to divide a complex number by 0, just as it isn't possible to divide a real number by 0.

As for real numbers, if z is a non-zero complex number, then the number $1/z$ is called the **reciprocal** of z. Also, as for real numbers,

$$\frac{1}{z} \text{ is denoted by } z^{-1}, \quad \frac{1}{z^2} \text{ is denoted by } z^{-2}, \quad \frac{1}{z^3} \text{ is denoted by } z^{-3},$$

and so on.

As mentioned earlier, the index laws from Unit 1 hold when the base numbers are complex numbers and the powers are any integers, positive, negative or zero. For example, the following index laws hold for all complex numbers z and w and all integers m and n:

$$z^m z^n = z^{m+n}, \quad \frac{z^m}{z^n} = z^{m-n} \quad \text{and} \quad (zw)^n = z^n w^n.$$

When you're using complex numbers as base numbers, you should be careful to apply index laws only in cases in which the indices (such as m and n above) are integers. This is because the definitions of expressions such as $z^{1/2}$, where z is a complex number, are beyond the scope of this module. As you'll see in Section 3, all complex numbers other than 0 have two square roots, so an expression such as $z^{1/2}$ is potentially ambiguous since it could refer to either of the square roots.

Finally, as promised, here's the proof of the fourth property of complex conjugates stated in the box on page 186. The property is

$$\overline{z/w} = \overline{z}/\overline{w}.$$

If $w \neq 0$, then

$$(z/w) \times w = z, \quad \text{so} \quad \overline{(z/w) \times w} = \overline{z}.$$

Hence, by the third property in the box on page 186,

$$\overline{z/w} \times \overline{w} = \overline{z}.$$

Dividing both sides by $\overline{w}$ gives the property stated above.

2 Geometry with complex numbers

In this section you'll learn how complex numbers can be represented by points in a plane, called the *complex plane*, which gives a rich geometric way of interpreting the arithmetic operations described in the previous section. You'll see that you can add and subtract complex numbers represented by points in the complex plane in a similar way to how you add and subtract vectors.

2.1 The complex plane

You saw in Unit 1 that the real numbers can be represented as points on a line, called the **real line** or the **number line**, as shown in Figure 4.

$$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$
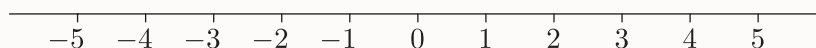
Figure 4   Part of the real line

Since a complex number is made up of two real numbers, its real part and its imaginary part, it can be represented by a point in a plane.

> **The complex plane**
>
> The **complex plane** is a plane in which the complex number $a + bi$ is represented by the point (a, b).
>
> The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**.

For example, the complex number $3 + 4i$ is represented in the complex plane by the point $(3, 4)$. This complex number is shown, with others, in Figure 5. Notice that the origin represents the complex number 0.
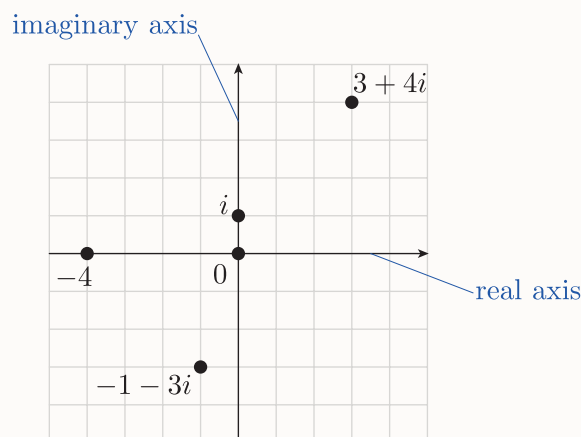
Figure 5   Complex numbers in the complex plane

The grid in Figure 5 is a **unit grid**, which means that adjacent horizontal lines in the grid are separated by one unit, and adjacent vertical lines are also separated by one unit. All the grids that you'll see in this unit are unit grids.

Note that for simplicity we usually don't distinguish between complex numbers and the points that represent them in the complex plane. For example, we say that the complex number i lies on the imaginary axis, rather than that the point representing the complex number i lies on the imaginary axis.

Activity 13   *Marking complex numbers in the complex plane*

Mark the following complex numbers on a diagram of the complex plane:

$$2 - 4i, \quad -3 + 2i, \quad 3, \quad -i \quad \text{and} \quad 4i.$$

The complex plane is also known as the **Argand diagram**, after a French mathematician with the surname Argand, who in 1806 wrote an essay on representing complex numbers geometrically in a plane.

Recent research has shown that reliable biographical information about Argand is extremely limited; not even his first name is known! There is no evidence that he was a certain Swiss-born man called Jean-Robert Argand, as was previously believed. Information in one of Argand's publications suggests that he was a scientifically oriented technician, based in the Parisian clock-making industry.

The idea of introducing a complex plane had been proposed before, by the English mathematician John Wallis (1616–1703), and separately by the Norwegian–Danish mathematician Caspar Wessel (1745–1818), but these earlier proposals failed to gain popular acceptance.

Let's consider what happens in the complex plane when you add two complex numbers. By the usual rule for adding complex numbers,

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

So if you add the complex numbers represented by the points (a, b) and (c, d), then you get the complex number represented by the point $(a + c, b + d)$. That is, you just add the coordinates separately, in the same way that you add the components separately when you add vectors.

Because of this, you can add two complex numbers in the complex plane by drawing (or imagining) a parallelogram, as illustrated in Figure 6. This is the **parallelogram law** for the addition of complex numbers, which is similar to the parallelogram law for vector addition.
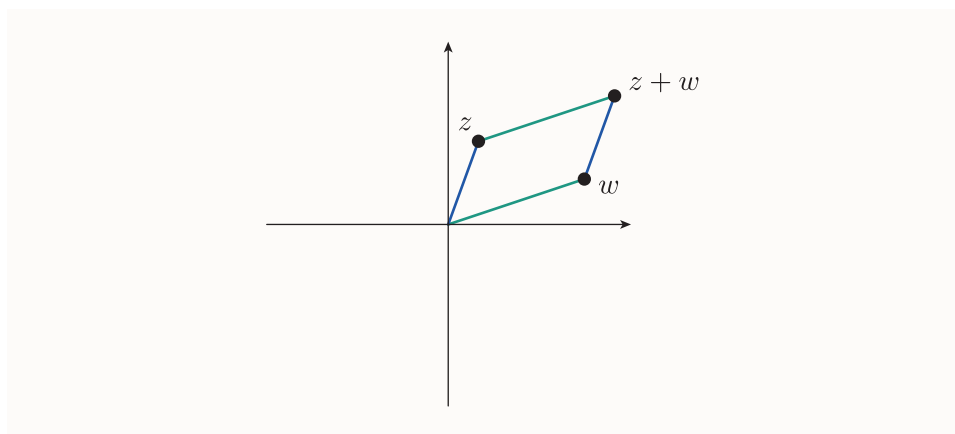
Figure 6 The parallelogram law for the addition of complex numbers

Next let's consider what happens in the complex plane when you multiply a complex number by a real number. We have

$$m(a + bi) = ma + mbi.$$

So if you multiply the complex number represented by the point (a, b) by the real number m, then you get the complex number represented by the point (ma, mb). That is, you just multiply the coordinates separately by the real number, in the same way that you multiply the components separately when you multiply a vector by a scalar.

The effect of this is that when you multiply a complex number z by a real number m, the resulting complex number mz remains on the line that passes through the origin and z, but is $|m|$ times as far from the origin as z is. If m is positive, then mz lies on the same side of the origin as z, and if m is negative, then mz lies on the opposite side of the origin from z. These effects are illustrated in Figure 7. They're similar to the effects of multiplying a vector by a scalar.
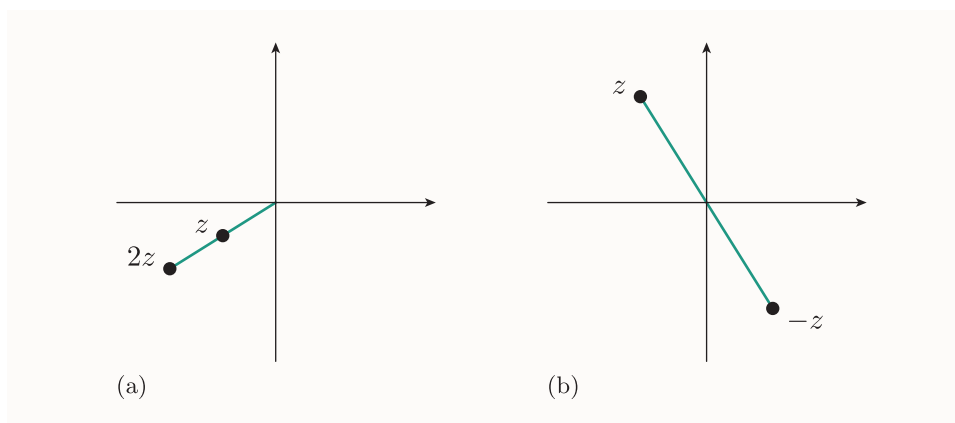
Figure 7 (a) Multiplication by 2 (b) Multiplication by -1

Complex conjugation also has a simple geometric interpretation in the complex plane: the number $\overline{z} = a - bi$ is the image of $z = a + bi$ under reflection in the real axis, as shown in Figure 8.
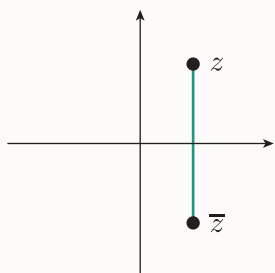
Figure 8   Complex conjugation in the complex plane

Activity 14   *Adding complex numbers, multiplying complex numbers by real numbers, and complex conjugation in the complex plane*

On a copy (or several copies) of the diagram below, mark the complex numbers in parts (a)–(f). Do this by thinking geometrically, in the ways described above, rather than by first writing z and w in the form $a + bi$ and using the algebraic methods from Section 1.

(a) $z + w$ (b) $2z$ (c) $\overline{z}$ (d) $z + \overline{z}$ (e) $-w$ (f) $z - w$

(Remember that $z - w = z + (-w)$.)
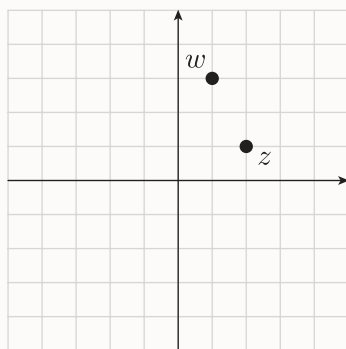
You can check your answers to Activity 14 by writing z and w in the form $a + bi$ and using the methods from Section 1. You can see from the diagram that $z = 2 + i$ and $w = 1 + 3i$. Therefore, for example,

$$z + w = (2 + i) + (1 + 3i) = (2 + 1) + (1 + 3)i = 3 + 4i.$$

If you plot this answer $3 + 4i$ in the complex plane, as shown in Figure 9, then you obtain the same point that you found in Activity 14(a) by using the parallelogram law.
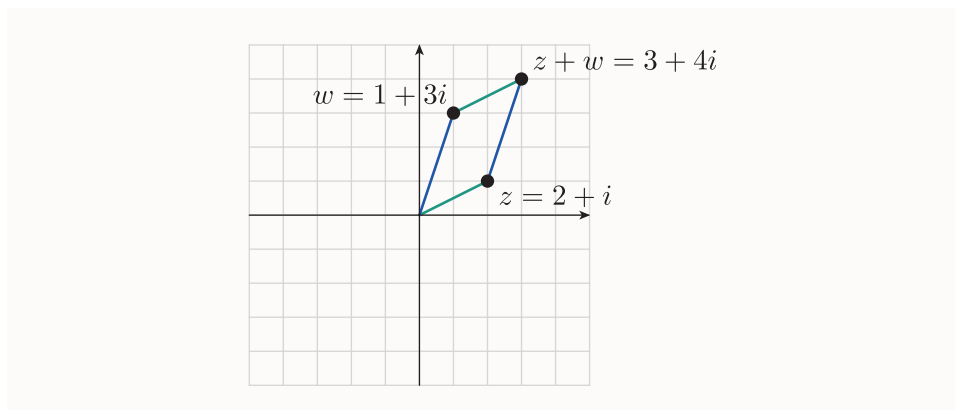
Figure 9   Adding $2 + i$ and $1 + 3i$ in the complex plane

2.2 Modulus of a complex number

You saw in Section 2 of Unit 2 that the *modulus* of a real number x, which is denoted by $|x|$ and is also called the *magnitude* or *absolute value* of x, is the distance of x from zero. For example, $|3| = 3$ and $|-3| = 3$. Essentially, the modulus of a real number x is the 'size' of x.

In the same way, the **modulus** (or **absolute value**) of a complex number z, which is denoted by $|z|$, is its distance from the origin, and is a measure of its size. (We don't use the word 'magnitude' for complex numbers.) Pythagoras' theorem gives the formula below for the modulus of a complex number, which is illustrated in Figure 10.

> **Modulus of a complex number**
>
> Let $z = a + bi$. The **modulus** $|z|$ of z is its distance from the origin, given by
>
> $$|z| = \sqrt{a^2 + b^2}.$$
>
> The plural of 'modulus' is *moduli*.
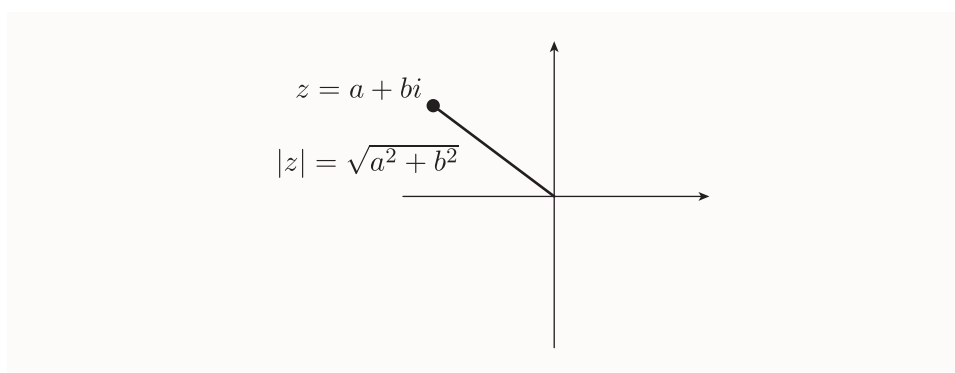
Figure 10   The distance between the origin and $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Let's calculate the moduli of some particular complex numbers. For example,

$$|3 - 4i| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

and

$$|-7| = \sqrt{(-7)^2} = \sqrt{49} = 7.$$

As illustrated by the second example, if z is a real number, then $|z|$ is just the usual modulus of this real number.

Activity 15   *Finding the moduli of complex numbers*

Find the moduli of the following complex numbers.

(a) $3 + 4i$ (b) $-4 + 3i$ (c) 0 (d) -31 (e) $17i$

(f) $7 - i\sqrt{15}$ (g) $1 + i$ (h) $19 + 19i$ (i) $-i$ (j) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

In Subsection 1.4 you saw that, for any complex number $z = a + bi$,

$$z\overline{z} = a^2 + b^2.$$

The expression $a^2 + b^2$ is equal to $|z|^2$, so we have the useful identity

$$z\overline{z} = |z|^2.$$

We can obtain another useful identity by replacing z by zw in the identity $|z|^2 = z\overline{z}$, to give

$$|zw|^2 = zw\overline{zw}.$$

You learned earlier that $\overline{zw} = \overline{z}\,\overline{w}$, so

$$|zw|^2 = zw\overline{zw} = zw\overline{z}\,\overline{w} = z\overline{z}w\overline{w} = |z|^2|w|^2.$$

Hence $|zw|^2 = |z|^2|w|^2$, and taking square roots gives the useful identity

$$|zw| = |z||w|.$$

From this identity you can deduce a third useful identity, as follows. If $w \neq 0$, then

$$|z| = \left|\frac{z}{w} \times w\right| = \left|\frac{z}{w}\right||w|, \quad \text{so} \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

Properties of modulus

$$z\overline{z} = |z|^2$$

$$|zw| = |z||w|$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

Activity 16 *Proving identities involving modulus*

By writing $z = a + bi$, show that $|-z| = |z|$ and $|\overline{z}| = |z|$.

If you divide both sides of the identity $z\overline{z} = |z|^2$ by $\overline{z}$, then you obtain

$$z = \frac{|z|^2}{\overline{z}}.$$

Taking the reciprocal of both sides of this equation gives you a neat formula for finding $1/z$, as follows.

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

For example, the reciprocal of $1 + 3i$ is

$$\frac{1}{1+3i} = \frac{\overline{1+3i}}{|1+3i|^2} = \frac{1-3i}{1^2+3^2} = \frac{1-3i}{10}.$$

Activity 17 *Working out the reciprocals of complex numbers*

Find the reciprocals of the following complex numbers.

(a) $2 + i$ (b) $-1 - 3i$ (c) $2i$

2.3 Argument of a complex number

You can completely specify a particular complex number by stating its modulus, that is, its distance from the origin in the complex plane, together with the direction in which it lies from the origin. This direction is usually specified by stating a particular angle associated with the complex number, which we call an *argument* of the complex number. You'll learn about the arguments of a complex number in this subsection. Together, the modulus and an argument of a complex number can be used to express the complex number in a helpful form, called *polar form*, which you'll meet in the next subsection.

Arguments of a complex number

An **argument** of a non-zero complex number z is an angle in radians measured anticlockwise from the positive real axis to the line between the origin and z.

The number 0 doesn't have an argument.

For example, one argument of the complex number $-4 + 4i$ is $3\pi/4$, as shown in Figure 11(a). You can see that this angle is $3\pi/4$ because it's three-quarters of a half-turn, and a half-turn is π radians. Other arguments of the same complex number are $-5\pi/4$ and $11\pi/4$, as shown in Figures 11(b) and (c). The angle $-5\pi/4$ is negative because it's measured clockwise, rather than anticlockwise.
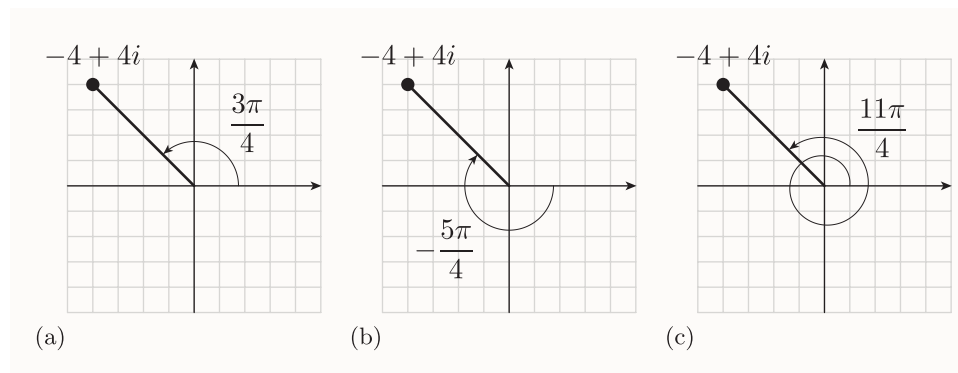
Figure 11   Three different arguments of $-4 + 4i$

You've encountered this sort of situation before, in which the same direction is specified by different angles. For example, it occurred when you worked with the directions of vectors in Section 6 of Unit 5. The angles $-5\pi/4$ and $11\pi/4$ each have the same rotational effect as $3\pi/4$, because they each differ from $3\pi/4$ by an integer multiple of 2π:

$$-\frac{5\pi}{4} = \frac{3\pi}{4} - 2\pi \quad \text{and} \quad \frac{11\pi}{4} = \frac{3\pi}{4} + 2\pi.$$

In fact, you can see that each non-zero complex number has infinitely many arguments, because you can add any integer multiple of 2π to an argument to obtain another argument. Exactly one of these arguments lies in the interval $(-\pi, \pi]$, and it's often the simplest one to use.

Principal argument of a complex number

The **principal argument** of a non-zero complex number z is the argument of z that lies in the interval $(-\pi, \pi]$. This angle is denoted by $\text{Arg}(z)$.

For example, the principal argument of $z = -4 + 4i$ is $3\pi/4$. That is,

$$\text{Arg}(-4 + 4i) = \frac{3\pi}{4}.$$

In some other texts, the principal argument is called the **principal value of the argument**, and sometimes the interval $[0, 2\pi)$ is used rather than $(-\pi, \pi]$.

Let's now consider how to find the principal argument of a complex number. If the complex number lies on one of the axes, then you can find the principal argument just from a sketch, as the next example demonstrates.

Example 4 *Finding the principal argument of a complex number that lies on one of the axes*

Find the principal argument of the complex number $3i$.

Solution

🔍 Sketch $3i$ in the complex plane. 💭
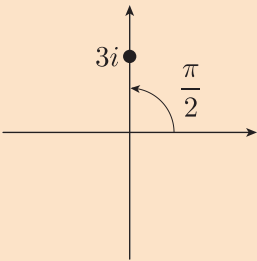
From the diagram, $\mathrm{Arg}(3i) = \pi/2$.

Activity 18 *Finding the principal argument of a complex number that lies on one of the axes*

Find the principal argument of each of the following complex numbers.

(a) $\frac{7}{2}i$ (b) $-4i$ (c) -3 (d) 2

The method for finding the principal argument of a complex number that doesn't lie on one of the axes is similar to the method for finding the direction of a two-dimensional vector from its components, which you met in Unit 5. It involves finding values of inverse tangent, so you may find it helpful to refer to Table 1, which contains the tangents of special angles that you met in Unit 4.

Table 1 Tangents of special angles

θ in radians	$\tan\theta$
$\dfrac{\pi}{6}$	$\dfrac{1}{\sqrt{3}}$
$\dfrac{\pi}{4}$	1
$\dfrac{\pi}{3}$	$\sqrt{3}$

Example 5   *Finding the principal argument of a complex number that doesn't lie on one of the axes*

Find the principal argument of the complex number $-1 + i\sqrt{3}$.

Solution

🔍 Sketch $-1 + i\sqrt{3}$ in the complex plane. The important thing is to get it in the correct quadrant. Label the principal argument θ, and label by ϕ the acute angle between the real axis and the line from the origin to z. 💬

🔍 Draw a line from z to the real axis that is perpendicular to the real axis to form a right-angled triangle. Mark the lengths of the horizontal and vertical sides of the triangle. 💬

🔍 Use the triangle to work out the acute angle ϕ, and hence work out the principal argument θ. 💬

From the diagram,

$$\tan \phi = \frac{\sqrt{3}}{1} = \sqrt{3}.$$
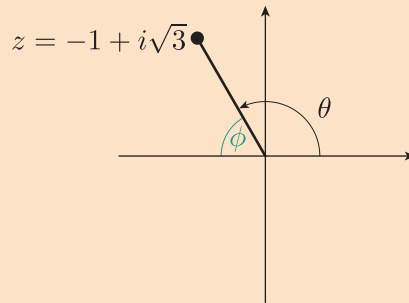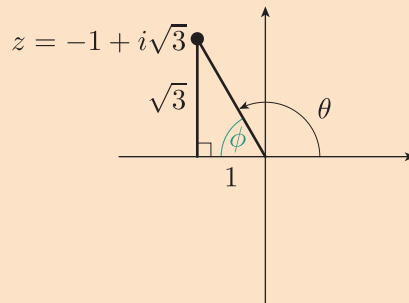
Therefore $\phi = \pi/3$. So the principal argument is

$$\theta = \pi - \phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

> **Activity 19**   *Finding the principal arguments of complex numbers that don't lie on one of the axes*
>
> Find the principal argument of each of the following complex numbers.
>
> (a) $1 + i$ (b) $1 - i\sqrt{3}$ (c) $-\sqrt{3} - i$ (d) $2\sqrt{3} - 2i$
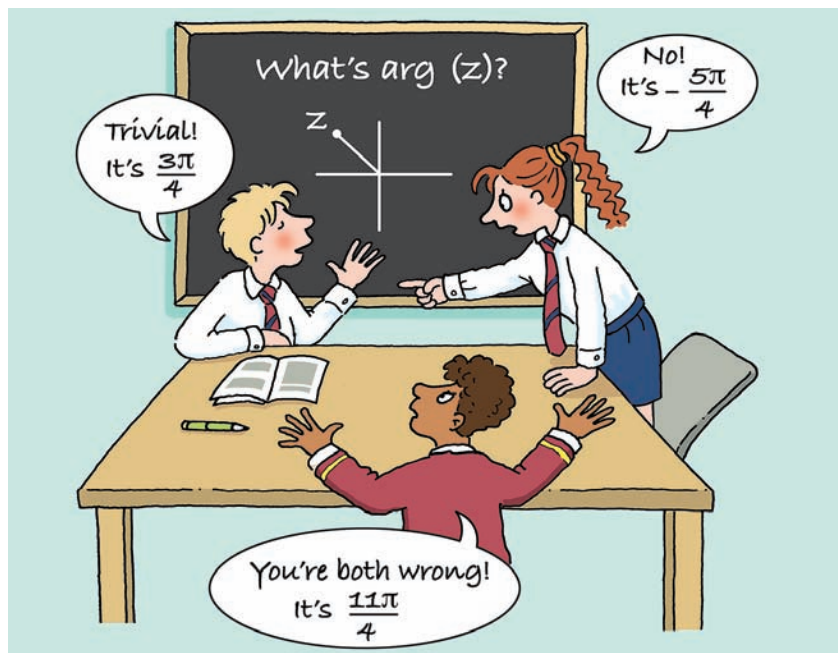
Possible origin of the term 'argument'

2.4 Polar form

All the complex numbers that you've met so far have been written in the form $a + bi$, for real numbers a and b. This form is known as the **Cartesian form** of a complex number. In this subsection you'll learn about an alternative way to write complex numbers, using the modulus and argument, known as *polar form*. In fact, the idea is not entirely new to you, as you met a similar procedure in Unit 5 when you saw how to find the components of a vector from its magnitude and direction.

To find the polar form of a non-zero complex number z, we begin with the Cartesian form $z = a + bi$, and express a and b in terms of the modulus and one of the arguments of z. Let's write r for the modulus $|z|$, and θ for one of the arguments, as shown in Figure 12. In this case θ happens to be the principal argument of z, but the discussion that follows is valid no matter what the choice of argument.
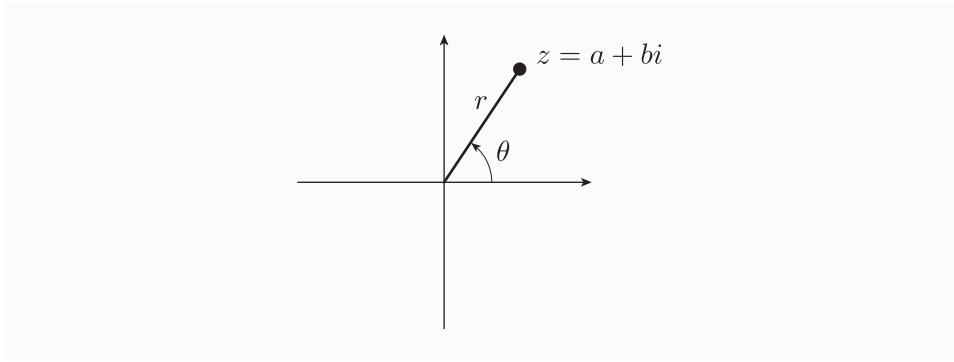
Figure 12   A complex number $z = a + bi$ with modulus r and argument θ

To see how to write a and b in terms of r and θ, first consider the complex number w that has the same argument as z, namely θ, but whose modulus is 1, as shown in Figure 13(a). This complex number w lies on the unit circle, and hence it follows from what you saw in Unit 4, Subsection 2.2, that its coordinates are $(\cos\theta, \sin\theta)$. This is true whatever the size of the argument θ.
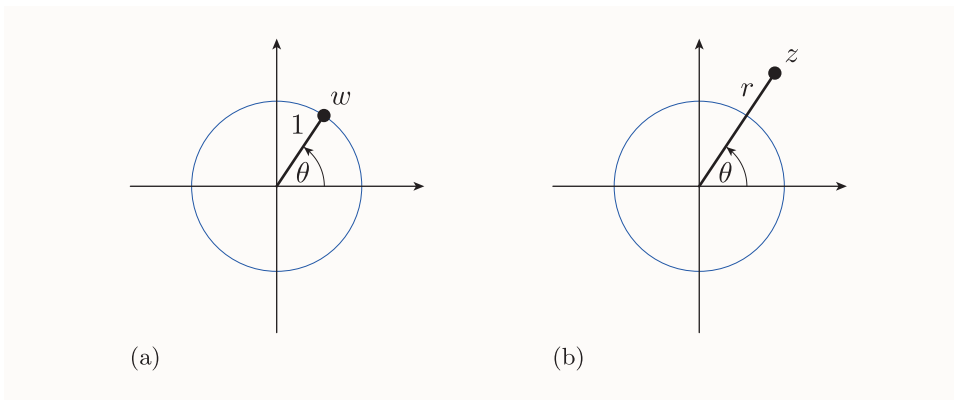
Figure 13   The complex numbers (a) w, with modulus 1 and argument θ (b) z, with modulus r and argument θ

In other words,

$$w = \cos\theta + i\sin\theta.$$

Since $z = rw$, as illustrated in Figure 13(b), it follows that

$$z = r(\cos\theta + i\sin\theta).$$

This is the *polar form* of z, in which z is written in terms of its modulus and one of its arguments.

Polar form of a complex number

A non-zero complex number z is in **polar form** if it is expressed as

$$z = r(\cos\theta + i\sin\theta).$$

Here r is the modulus of z, and θ is an argument of z.

For example, the complex number z with modulus 2 and argument $3\pi/4$, which is shown in Figure 14, can be written in polar form as

$$z = 2 \left(\cos \left(\frac{3\pi}{4} \right) + i \sin \left(\frac{3\pi}{4} \right) \right).$$
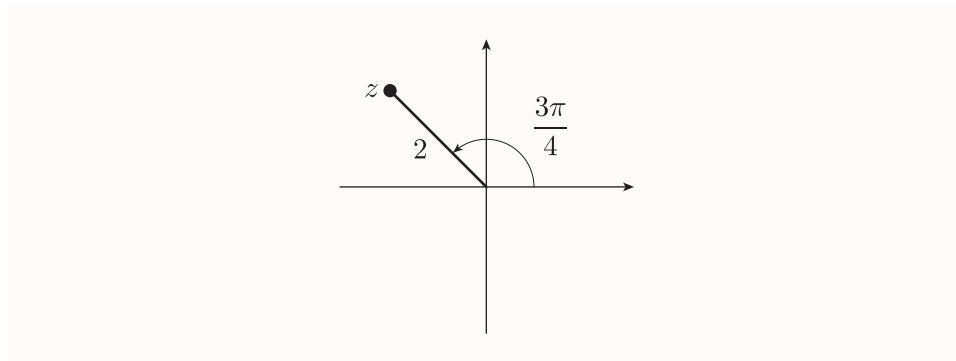
Figure 14   The complex number z with modulus 2 and argument $3\pi/4$

As you saw in the previous subsection, any angle that differs from $3\pi/4$ by an integer multiple of 2π is also an argument of the complex number z in Figure 14. For instance, $-5\pi/4$ and $11\pi/4$ are also arguments of z, so you can alternatively write z in polar form as

$$z = 2 \left(\cos \left(-\frac{5\pi}{4} \right) + i \sin \left(-\frac{5\pi}{4} \right) \right)$$

or

$$z = 2 \left(\cos \left(\frac{11\pi}{4} \right) + i \sin \left(\frac{11\pi}{4} \right) \right).$$

However, unless there's a good reason not to do so, you should use the principal argument when writing a complex number in polar form, which in this case is our original argument $3\pi/4$.

The number 0 doesn't have an argument, so it doesn't have a polar form either. If you want to write the number zero, then you should just use the symbol 0, whether you're working with Cartesian or polar forms.

Let's now consider how to convert between the Cartesian and polar forms of a complex number. To convert a complex number from polar form to Cartesian form you need to work out values of sine and cosine. You can use your calculator to do this, but you'll develop a better understanding of the procedure if instead you use techniques from Unit 4 for working out sines and cosines of simple fractions of π, and refer to Table 2, which contains the sines and cosines of special angles from Section 1 of Unit 4.

Table 2   Sines and cosines of special angles

θ in radians	$\sin\theta$	$\cos\theta$
0	0	1
$\dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$
$\dfrac{\pi}{4}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{\sqrt{2}}$
$\dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$
$\dfrac{\pi}{2}$	1	0

Example 6   *Converting from polar form to Cartesian form*

Write the complex number

$$2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

in Cartesian form.

Solution

Evaluate the cosine and sine, then expand the brackets.

$$2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -\sqrt{2} + i\sqrt{2}$$

Activity 20   *Converting from polar form to Cartesian form*

Write the following complex numbers in Cartesian form.

(a) $3(\cos 0 + i\sin 0)$ (b) $7\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

(c) $6(\cos\pi + i\sin\pi)$ (d) $\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

(e) $5(\cos(-\pi) + i\sin(-\pi))$ (f) $4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

(g) $2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ (h) $\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$

To convert a complex number from Cartesian form to polar form, you need to calculate its modulus and principal argument.

Example 7   *Converting from Cartesian form to polar form*

Write the complex number $-2 - 2\sqrt{3}\,i$ in polar form.

Solution

🔍 First find the modulus. 💭

The modulus is

$$r = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{4 + 12} = \sqrt{16} = 4.$$

🔍 To find the principal argument, sketch $z = -2 - 2\sqrt{3}\,i$ in the complex plane. The important thing is to get it in the correct quadrant. Label the principal argument θ, and label by ϕ the acute angle between the real axis and the line from the origin to z. 💭
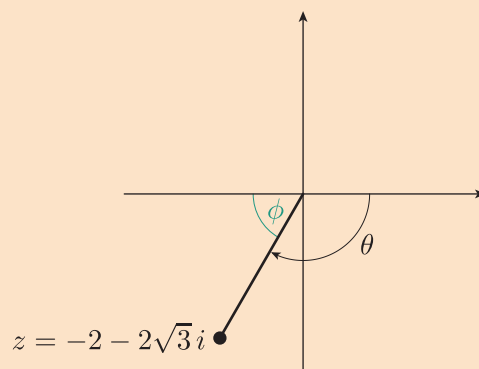
🔍 Draw a line from z to the real axis that is perpendicular to the real axis, to form a right-angled triangle. Mark the lengths of the horizontal and vertical sides of the triangle. 💭
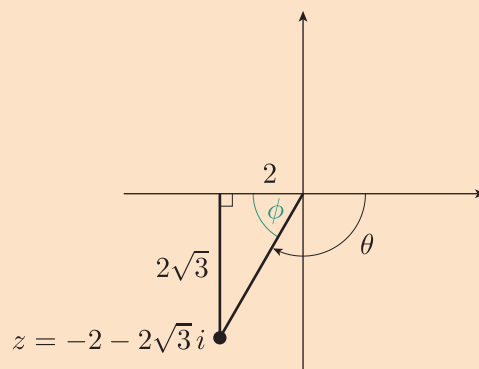
  Use the triangle to work out the acute angle ϕ, and hence work out the principal argument θ.

From the diagram,

$$\tan \phi = \frac{2\sqrt{3}}{2} = \sqrt{3}.$$

Therefore $\phi = \pi/3$. So

$$\theta = -(\pi - \phi) = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}.$$

  Write z in polar form.

Hence

$$z = 4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right).$$

When you convert a complex number from Cartesian form to polar form, you can readily check your answer by converting back. For example, for the complex number in Example 7,

$$z = 4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right) = 4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -2 - 2\sqrt{3}\,i.$$

You can use your solutions to Activities 18 and 19 to help you with the next activity.

Activity 21   *Converting from Cartesian form to polar form*

Write the following complex numbers in polar form.

(a) $\frac{7}{2}i$ (b) $-4i$ (c) -3 (d) 2 (e) $1 + i$ (f) $1 - i\sqrt{3}$

(g) $-\sqrt{3} - i$ (h) $2\sqrt{3} - 2i$

2.5 Multiplication and division in polar form

The Cartesian form of a complex number is convenient for adding and subtracting complex numbers, because to add or subtract complex numbers you just add or subtract their real and imaginary parts separately. In this subsection you'll see that polar form is more suited for multiplication and division.

Consider two complex numbers in polar form, $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. Then

$$zw = r(\cos\theta + i\sin\theta) \times s(\cos\phi + i\sin\phi)$$
$$= rs(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi)$$
$$= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\big).$$

Recall the angle sum identities for sine and cosine, from Section 4 of Unit 4:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Using these identities we obtain

$$zw = rs(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

This shows that to multiply two complex numbers in polar form, you *multiply their moduli* and *add their arguments.*

Product of complex numbers in polar form

Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. Then

$$zw = rs\big(\cos(\theta + \phi) + i\sin(\theta + \phi)\big).$$

Example 8   *Finding the product of complex numbers in polar form*

Let

$$z = 10\left(\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right)$$

and

$$w = 5\left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right).$$

Find zw in polar form.

Solution

Multiply the moduli and add the arguments.

$$zw = 50\left(\cos\left(\frac{3\pi}{5} + \frac{4\pi}{5}\right) + i\sin\left(\frac{3\pi}{5} + \frac{4\pi}{5}\right)\right)$$
$$= 50\left(\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right)$$

The argument $7\pi/5$ doesn't lie in the interval $(-\pi, \pi]$, so it isn't the principal argument of zw. To find the principal argument, subtract an integer multiple of 2π from $7\pi/5$ to obtain an angle that lies in the interval $(-\pi, \pi]$.

Since

$$\frac{7\pi}{5} - 2\pi = \frac{7\pi}{5} - \frac{10\pi}{5} = -\frac{3\pi}{5},$$

it follows that

$$zw = 50\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right).$$

In the next activity, remember to use the *principal* argument in the polar form in your answers.

Activity 22   *Finding the product of complex numbers in polar form*

Find zw in polar form in each of the following cases.

(a) $z = 12 \left(\cos \left(\dfrac{\pi}{5} \right) + i \sin \left(\dfrac{\pi}{5} \right) \right),\quad w = 4 \left(\cos \left(\dfrac{\pi}{10} \right) + i \sin \left(\dfrac{\pi}{10} \right) \right)$

(b) $z = 8 \left(\cos \left(\dfrac{7\pi}{8} \right) + i \sin \left(\dfrac{7\pi}{8} \right) \right),\quad w = 5 \left(\cos \left(\dfrac{3\pi}{4} \right) + i \sin \left(\dfrac{3\pi}{4} \right) \right)$

(c) $z = \cos \left(\dfrac{8\pi}{9} \right) + i \sin \left(\dfrac{8\pi}{9} \right),\quad w = \cos \left(\dfrac{8\pi}{9} \right) + i \sin \left(\dfrac{8\pi}{9} \right)$

You've seen that to find the product of two complex numbers in polar form you multiply their moduli and add their arguments. You do the same to find the product of three or more complex numbers in polar form. For example, the product of

$$u = 3 \left(\cos \left(\frac{4\pi}{5} \right) + i \sin \left(\frac{4\pi}{5} \right) \right),$$

$$v = 2 \left(\cos \left(-\frac{2\pi}{5} \right) + i \sin \left(-\frac{2\pi}{5} \right) \right) \text{ and}$$

$$w = 7 \left(\cos \left(\frac{\pi}{5} \right) + i \sin \left(\frac{\pi}{5} \right) \right)$$

is

$$uvw = (3 \times 2 \times 7) \left(\cos \left(\frac{4\pi}{5} - \frac{2\pi}{5} + \frac{\pi}{5} \right) + i \sin \left(\frac{4\pi}{5} - \frac{2\pi}{5} + \frac{\pi}{5} \right) \right)$$

$$= 42 \left(\cos \left(\frac{3\pi}{5} \right) + i \sin \left(\frac{3\pi}{5} \right) \right).$$

Activity 23   *Finding the product of several complex numbers in polar form*

Find the product of the following three complex numbers in polar form:

$$u = 4 \left(\cos \left(\frac{3\pi}{7} \right) + i \sin \left(\frac{3\pi}{7} \right) \right)$$

$$v = \cos \left(\frac{\pi}{7} \right) + i \sin \left(\frac{\pi}{7} \right)$$

$$w = 2 \left(\cos \left(\frac{5\pi}{7} \right) + i \sin \left(\frac{5\pi}{7} \right) \right).$$

Let's now look at how to divide two complex numbers in polar form. As before, let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. At the end of Subsection 2.2, you saw a formula for the reciprocal of a non-zero complex number, which gives

$$\frac{1}{w} = \frac{\overline{w}}{|w|^2}.$$

Remember that $|w| = s$. Multiplying both sides of the equation above by z gives

$$\begin{aligned}
\frac{z}{w} &= \frac{z\overline{w}}{|w|^2}\\
&= \frac{r(\cos\theta + i\sin\theta) \times s(\cos\phi - i\sin\phi)}{s^2}\\
&= \frac{r}{s}\left(\cos\theta\cos\phi - i\cos\theta\sin\phi + i\sin\theta\cos\phi - i^2\sin\theta\sin\phi\right)\\
&= \frac{r}{s}\left((\cos\theta\cos\phi + \sin\theta\sin\phi) + i(\sin\theta\cos\phi - \cos\theta\sin\phi)\right).
\end{aligned}$$

Recall the angle difference identities for sine and cosine, from Section 4 of Unit 4:

$$\begin{aligned}
\sin(\theta - \phi) &= \sin\theta\cos\phi - \cos\theta\sin\phi\\
\cos(\theta - \phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi.
\end{aligned}$$

Using these identities we obtain

$$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi)).$$

This shows that to divide two complex numbers in polar form, you *divide their moduli* and *subtract their arguments*.

Quotient of complex numbers in polar form

Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. Then

$$\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi)).$$

Example 9   *Finding the quotient of complex numbers in polar form*

Let

$$z = 10\left(\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right)$$

and

$$w = 5\left(\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right).$$

Find z/w in polar form.

Solution

🗨 Divide the moduli and subtract the arguments. 💭

$$\frac{z}{w} = 2\left(\cos\left(\frac{\pi}{5} - \frac{7\pi}{5}\right) + i\sin\left(\frac{\pi}{5} - \frac{7\pi}{5}\right)\right)$$

$$= 2\left(\cos\left(-\frac{6\pi}{5}\right) + i\sin\left(-\frac{6\pi}{5}\right)\right).$$

🗨 The argument $-6\pi/5$ doesn't lie in the interval $(-\pi, \pi]$, so it isn't the principal argument of z/w. To find the principal argument, add an integer multiple of 2π to $-6\pi/5$ to obtain an angle that lies in the interval $(-\pi, \pi]$. 💭

Since

$$-\frac{6\pi}{5} + 2\pi = -\frac{6\pi}{5} + \frac{10\pi}{5} = \frac{4\pi}{5},$$

it follows that

$$\frac{z}{w} = 2\left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right).$$

Activity 24   *Finding the quotient of complex numbers in polar form*

Find z/w in polar form in each of the following cases.

(a) $z = 12\left(\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right), \quad w = 4\left(\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right)$

(b) $z = 8\left(\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right), \quad w = 4\left(\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right)$

(c) $z = \cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right), \quad w = \cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)$

Using the polar form of complex numbers you can now visualise in the complex plane how to multiply one complex number by another, as the following example demonstrates.

Example 10   *Visualising multiplication by $2i$ geometrically*

Describe the geometric effect of multiplying a complex number by $2i$.

Solution

🗨 Start by finding the polar form of $2i$. 💭
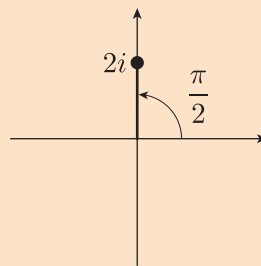
From the diagram,

$$2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).$$

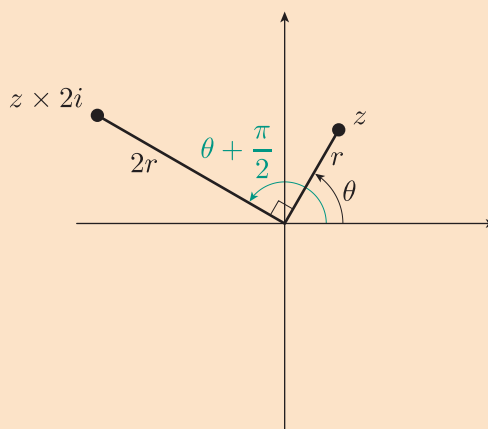Consider a general complex number z in polar form, to be multiplied by $2i$.

Now let $z = r(\cos\theta + i\sin\theta)$.

Find the polar form of $z \times 2i$.

Then

$$z \times 2i = r(\cos\theta + i\sin\theta) \times 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$
$$= 2r\left(\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right).$$

If you find it helpful, sketch z and $z \times 2i$ in the complex plane to help you understand how $z \times 2i$ is obtained from z geometrically.

Therefore, multiplying z by $2i$ corresponds to an anticlockwise rotation of z through a quarter turn ($\pi/2$ radians) and a scaling of z by the factor 2.

Activity 25   *Multiplying a complex number by $-i$*

Describe the geometric effect of multiplying a complex number by $-i$.

2.6 De Moivre's formula

You've seen how to multiply two or more complex numbers in polar form, so it's now possible to consider powers of complex numbers in polar form. There's a helpful formula for working out powers, known as *de Moivre's formula*, named after the French mathematician Abraham de Moivre. ('De Moivre' is pronounced as 'de mwa-vr', where the 'e' is spoken like the 'u' in 'number'.) To obtain this formula, we apply the formulas for products and quotients of complex numbers found in the previous subsection, as follows.

You saw that to find the product of two or more complex numbers in polar form, you multiply their moduli (to give the modulus of the product) and add their arguments (to give an argument of the product). Applying this procedure to n copies of the complex number $z = r(\cos\theta + i\sin\theta)$ gives

$$z^n = \underbrace{r \times r \times \cdots \times r}_{\text{multiply moduli}} \left(\cos(\underbrace{\theta + \theta + \cdots + \theta}_{\text{add arguments}}) + i\sin(\underbrace{\theta + \theta + \cdots + \theta}_{\text{add arguments}}) \right).$$

That is,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

This is de Moivre's formula, for positive integers. In fact the formula is valid for *all* integers n, as stated below.

De Moivre's formula

Let $z = r(\cos\theta + i\sin\theta)$. Then, for any integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

So far you've only seen why de Moivre's formula is true for positive integers. When $n = 0$ it's certainly true, because then both sides of the formula are equal to 1. To see why the formula is true for negative integers, begin with any positive integer n, and write the numbers 1 and z^n in polar form:

$$1 = \cos 0 + i\sin 0 \quad \text{and} \quad z^n = r^n(\cos n\theta + i\sin n\theta).$$
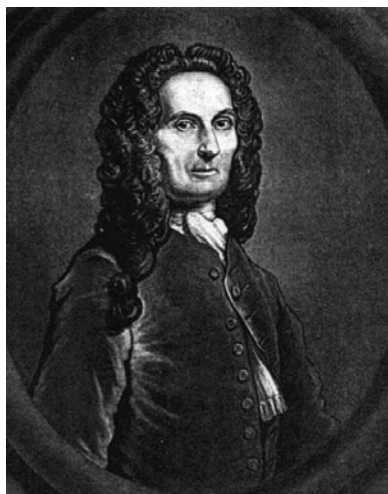
Now apply the formula for quotients of complex numbers in polar form to the complex numbers 1 and z^n. Dividing the moduli and subtracting the arguments gives

$$\frac{1}{z^n} = \frac{1}{r^n}(\cos(0 - n\theta) + i\sin(0 - n\theta)) = r^{-n}(\cos(-n\theta) + i\sin(-n\theta)).$$

Since z^{-n} is the same as $1/z^n$, we obtain

$$z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta)).$$

Because $-n$ can represent any negative integer, this confirms that de Moivre's formula is also true for negative integers.

Arguably Abraham de Moivre's greatest contributions to mathematics were in the theory of probability. In 1733, in a paper published in Latin, he was the first person to discuss the normal distribution, an important mathematical tool for modelling probabilities. In 1738 he included an English translation of the paper in the second edition of his book *The Doctrine of Chances: A method of calculating the probabilities of events in play.* It is said that de Moivre correctly predicted the day of his own death after observing that he was sleeping 15 minutes longer each night. Assuming that this would continue, he predicted that he would die on the day that he slept for 24 hours, and did indeed die on that day, 27 November 1754.

Abraham de Moivre
(1667–1754)

De Moivre's formula allows you to find powers of complex numbers in polar form quickly. To find powers of complex numbers in Cartesian form, you can convert to polar form, find the power, then convert back to Cartesian form, as the following example demonstrates.

Example 11   *Working out powers of complex numbers*

Find $\left(-\sqrt{3} + i\right)^5$ in Cartesian form.

Solution

Write $-\sqrt{3} + i$ in polar form. To do this, first find the modulus.

The modulus of $-\sqrt{3} + i$ is

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2.$$

Then find the principal argument, by first sketching $-\sqrt{3} + i$ in the complex plane.
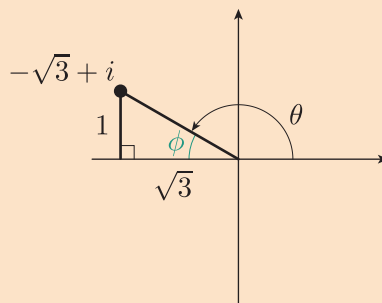
From the diagram,

$$\tan \phi = \frac{1}{\sqrt{3}}.$$

Therefore $\phi = \pi/6$. So
$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Hence
$$-\sqrt{3} + i = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right).$$

🔍 Apply de Moivre's formula. 💬

Therefore
$$\left(-\sqrt{3} + i\right)^5 = 2^5\left(\cos\left(5 \times \frac{5\pi}{6}\right) + i\sin\left(5 \times \frac{5\pi}{6}\right)\right)$$
$$= 32\left(\cos\left(\frac{25\pi}{6}\right) + i\sin\left(\frac{25\pi}{6}\right)\right).$$

🔍 Convert to Cartesian form. To help do this by hand, first rewrite the polar form using the principal argument. 💬

Since
$$\frac{25\pi}{6} = 4\pi + \frac{\pi}{6},$$

we obtain
$$\left(-\sqrt{3} + i\right)^5 = 32\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$
$$= 32\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$
$$= 16\sqrt{3} + 16i.$$

Activity 26 *Working out powers of complex numbers*

Work out the Cartesian forms of the following complex numbers.

(a) $(1+i)^3$ (b) $\left(2^{1/3}\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right)\right)^{12}$ (c) $\left(-1 + i\sqrt{3}\right)^7$

(d) $\left(\sqrt{3} + i\right)^{-6}$ (e) $(2 + 2i)^{-5}$

In the final activity of this section you can learn how to work with complex numbers on the computer.

Activity 27 *Manipulating complex numbers with the CAS*

Work through Subsection 13.1 of the *Computer algebra guide*.

3 Polynomial equations

A **polynomial equation** is an equation of the form
'polynomial expression $= 0$', where the polynomial expression has degree
at least 1. For example,

$$5x^6 + 8 = 0 \quad \text{and} \quad x^7 - 13x^5 + \tfrac{5}{3}x^2 - 2 = 0$$

are polynomial equations, and every linear or quadratic equation is a
polynomial equation.

At the start of this unit you were told the remarkable fact that by
introducing a solution i of the quadratic equation

$$x^2 + 1 = 0$$

to form the complex numbers, you make it possible to find at least one
solution of *any* quadratic equation. In this subsection, you'll learn how to
find all the solutions of any quadratic equation, including any solutions
that are not real numbers. You'll also learn about *the fundamental theorem
of algebra*, which shows that in fact every *polynomial* equation has at least
one solution in the set of complex numbers.

3.1   Quadratic equations

As you know, a *quadratic equation* is an equation of the
form $az^2 + bz + c = 0$, where $a \neq 0$. For example,

$$2z^2 - 5z - 1 = 0 \quad \text{and} \quad z^2 - 6z + 25 = 0$$

are quadratic equations. We use the variable z here (rather than the more
familiar x, say) because we want to work with complex numbers, and it's
traditional to use the letter z for a variable that represents a complex
number. For the moment, let's restrict our attention to quadratic
equations in which the coefficients a, b and c are real numbers, and leave
the possibility that a, b and c might be complex numbers that aren't
necessarily real to the end of this subsection.

You saw in Section 4 of Unit 2 that the number of **real solutions**
(solutions that are real numbers) of the quadratic equation
$az^2 + bz + c = 0$ depends on the value of the *discriminant* $b^2 - 4ac$. The
equation has

- two real solutions if $b^2 - 4ac > 0$
- one real solution if $b^2 - 4ac = 0$
- no real solutions if $b^2 - 4ac < 0$.

For example, the quadratic equation $2z^2 - 5z - 1 = 0$ has discriminant

$$(-5)^2 - 4 \times 2 \times (-1) = 25 + 8 = 33,$$

so it has two real solutions. In contrast, the quadratic
equation $z^2 - 6z + 25 = 0$ has discriminant

$$(-6)^2 - 4 \times 1 \times 25 = 36 - 100 = -64,$$

so it has no real solutions.

Although some quadratic equations have no real solutions, every quadratic equation has either one or two solutions that are complex numbers, which we call **complex solutions**. Complex solutions can be real numbers (because a real number is a special type of complex number), but they can also be complex numbers that are not real, like $1 + i$ or $-7i$.

One way to find the complex solutions of a quadratic equation is to use the method of completing the square, in much the same way as in Unit 2. The only difference is that now you have to allow square roots of negative numbers.

Example 12   *Solving a quadratic equation that has no real solutions by completing the square*

Solve the quadratic equation $z^2 - 6z + 25 = 0$ by completing the square.

Solution

Completing the square gives

$$(z - 3)^2 + 16 = 0;$$

that is,

$$(z - 3)^2 = -16.$$

🔍 Take the square root of both sides. Remember from Subsection 1.3 that if d is a positive real number, then the square roots of $-d$ are $\pm i\sqrt{d}$. 💬

Therefore

$$z - 3 = \pm i\sqrt{16};$$

that is,

$$z - 3 = \pm 4i,$$

so

$$z = 3 \pm 4i.$$

Remember that you can check the solutions of a quadratic equation by substituting them back into the equation. For the quadratic equation in Example 12, we have

$$(3 + 4i)^2 - 6(3 + 4i) + 25 = (3 + 4i)(3 + 4i) - 6(3 + 4i) + 25$$
$$= 9 + 12i + 12i + 16i^2 - 18 - 24i + 25$$
$$= (9 - 16 - 18 + 25) + i(12 + 12 - 24)$$
$$= 0.$$

You can check the other solution, $3 - 4i$, in the same way.

Activity 28 *Solving quadratic equations that have no real solutions by completing the square*

Solve the following quadratic equations by completing the square.

(a) $z^2 - 2z + 2 = 0$ (b) $z^2 + 4z + 13 = 0$ (c) $z^2 + 25 = 0$

Another way to find the complex solutions of a quadratic equation is to use the usual quadratic formula.

The quadratic formula

The solutions of the quadratic equation $az^2 + bz + c = 0$, where a, b and c are real numbers, are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

As you saw earlier, the quadratic formula gives two real solutions of the quadratic equation if $b^2 - 4ac > 0$ and one real solution if $b^2 - 4ac = 0$.

When $b^2 - 4ac < 0$, there's a problem with the quadratic formula, because the square root sign $\sqrt{\ }$ cannot be used with a negative number inside it. For convenience, we make an exception here, and when d is a positive real number we allow the notation $\pm\sqrt{-d}$ to mean the two square roots of the negative number $-d$, which you learned in Subsection 1.3 are $\pm i\sqrt{d}$. This practice doesn't cause errors, because the $\pm$ symbol means that you'll always consider both square roots of a negative number at once.

For example, if $b^2 - 4ac = -16$ then the quadratic formula involves the term

$$\pm\sqrt{-16},$$

which means the two square roots of -16, namely $\pm 4i$.

Example 13 *Solving a quadratic equation that has no real solutions using the quadratic formula*

Solve the quadratic equation $z^2 + 10z + 34 = 0$ using the quadratic formula.

Solution

🗨 Use the quadratic formula. 💭

$$z = \frac{-10 \pm \sqrt{10^2 - 4 \times 1 \times 34}}{2 \times 1}$$

$$= \frac{-10 \pm \sqrt{100 - 136}}{2}$$

$$= \frac{-10 \pm \sqrt{-36}}{2}$$

🗨 The term $\pm\sqrt{-36}$ means the two square roots of -36, namely $\pm 6i$. 💭

$$= \frac{-10 \pm 6i}{2}$$

$$= -5 \pm 3i$$

Activity 29 *Solving quadratic equations using the quadratic formula*

Solve the following quadratic equations by using the quadratic formula, or a simpler method if possible.

(a) $z^2 + 2z + 2 = 0$ (b) $z^2 + 6z + 9 = 0$ (c) $3z^2 + 5 = 0$

(d) $z^2 - 4z + 8 = 0$ (e) $z^2 + 3z = 0$ (f) $2z^2 - 3z + 5 = 0$

You can see from the quadratic formula that when a, b and c are real numbers such that $b^2 - 4ac < 0$, the two complex solutions of the quadratic equation $az^2 + bz + c = 0$ are complex conjugates of each other. For example, the two solutions $-5 + 3i$ and $-5 - 3i$ of the quadratic equation $z^2 + 10z + 34 = 0$ in Example 13 are complex conjugates of each other. Two complex numbers that are complex conjugates of each other are together called a **complex conjugate pair**.

The next example shows how you can choose any complex conjugate pair and then find a quadratic equation whose solutions are that complex conjugate pair.

Example 14 *Finding a quadratic equation with a given pair of solutions*

Find, in its simplest form, a quadratic equation that has solutions $2 \pm i$.

Solution

💭 For any numbers u and v, a quadratic equation with solutions u and v is $(z - u)(z - v) = 0$. 💭

A quadratic equation with solutions $2 \pm i$ is

$$(z - (2 + i))(z - (2 - i)) = 0.$$

💭 To simplify this equation, expand the brackets. Do this directly, or, to do it slightly more efficiently, first write the expression $(z - (2 + i))(z - (2 - i))$ as a difference of two squares, by rewriting $z - (2 + i)$ as $(z - 2) - i$ and $z - (2 - i)$ as $(z - 2) + i$. 💭

Simplifying gives

$$((z - 2) - i)((z - 2) + i) = 0$$

💭 Apply the difference of two squares formula $(A - B)(A + B) = A^2 - B^2$ with $A = z - 2$ and $B = i$. 💭

$$(z - 2)^2 - i^2 = 0$$
$$z^2 - 4z + 4 + 1 = 0$$
$$z^2 - 4z + 5 = 0.$$

So a quadratic equation with solutions $2 \pm i$ is $z^2 - 4z + 5 = 0$.

The procedure in Example 14 always gives a quadratic equation in which the coefficient of z^2 is 1. However, you can multiply the equation through by any non-zero number to give a quadratic equation that has the same solutions but in which the coefficient of z^2 is not 1.

Activity 30   *Finding quadratic equations with given pairs of solutions*

Find, in their simplest forms, quadratic equations with the following pairs of solutions.

(a) $1 \pm 2i$ (b) $-3 \pm 4i$ (c) $\pm 7i$ (d) $1 \pm \frac{1}{2}i$

So far all the quadratic equations $az^2 + bz + c = 0$ that you've met have had real coefficients a, b and c, and this is the only type of quadratic equation that you'll solve in this module. However, the quadratic formula also gives the solutions of quadratic equations in which a, b and c are complex numbers that aren't necessarily real. For such an equation, the expression $\pm\sqrt{b^2 - 4ac}$ represents the two square roots of a complex number. You'll learn how to find square roots, and other roots, of complex numbers in the next subsection.

3.2   Roots of complex numbers

As you know, if a is any number, then any solution x of the equation

$$x^2 = a$$

is called a *square root* of a. For example, the equation

$$x^2 = 4$$

has two solutions; in other words, the number 4 has two square roots, namely 2 and -2. Similarly, you've seen that the equation

$$x^2 = -9$$

has two solutions; in other words, the number -9 has two square roots, namely $3i$ and $-3i$.

Notice that in both the cases $a = 4$ and $a = -9$ the pairs of square roots lie symmetrically on opposite sides of the origin, as shown in Figure 15.
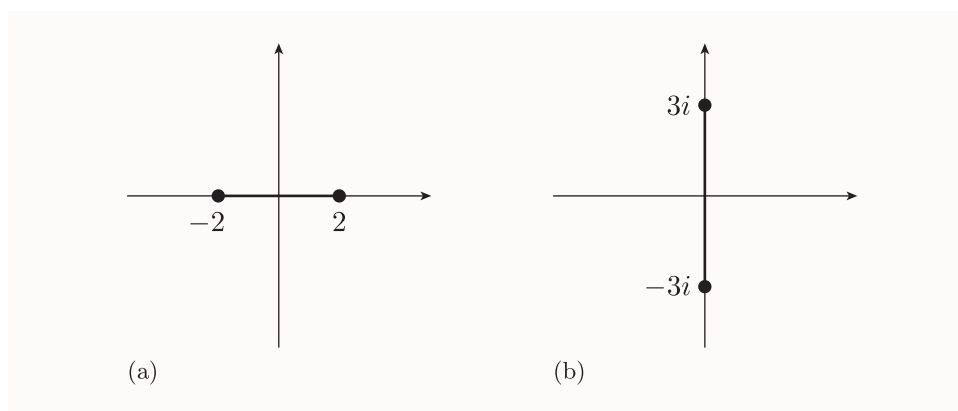
Figure 15   The square roots of (a) 4 (b) -9

You can investigate this symmetry property of square roots further in the next activity.

Activity 31   *Investigating the symmetry of square roots*

Use the *Square roots of real numbers* applet to investigate the solutions of the equation $x^2 - a = 0$, that is, the square roots of a, as a varies.

In general, any solution z of the equation

$$z^n = a,$$

where a is a complex number and n is a positive integer, is called an **nth root** of a. (We say 'square root' rather than '2nd root', and 'cube root' rather than '3rd root', as we do for real numbers.) In this subsection you'll learn how to find all the solutions of equations of this form. You'll see that, if a is non-zero, then there are exactly n solutions, positioned symmetrically around the origin. If a is 0, then there's only one solution, namely 0.

Let's start by looking at equations of the form $z^n = 1$, such as $z^3 = 1$ and $z^4 = 1$. Solutions of equations of this form are called **roots of unity**; the word 'unity' refers to the number 1. More specifically,

- the solutions of the equation $z^2 = 1$ are the **square roots of unity**,

- the solutions of the equation $z^3 = 1$ are the **cube roots of unity**,

- the solutions of the equation $z^4 = 1$ are the **fourth roots of unity**,

and so on.

Let's consider the cube roots of unity. You know that 1 is a cube root of unity, because $1^3 = 1$. There are two other cube roots though, namely

$$\tfrac{1}{2}\left(-1 + i\sqrt{3}\right) \quad \text{and} \quad \tfrac{1}{2}\left(-1 - i\sqrt{3}\right).$$

You'll see how to find such roots shortly, but for now let's check that these numbers really are cube roots of unity. To check the first one, let's first find the square and then the cube of the expression inside the brackets:

$$\left(-1 + i\sqrt{3}\right)^2 = 1 - 2i\sqrt{3} + i^2\left(\sqrt{3}\right)^2$$
$$= 1 - 2i\sqrt{3} - 3$$
$$= -2 - 2i\sqrt{3},$$

so

$$\left(-1 + i\sqrt{3}\right)^3 = \left(-1 + i\sqrt{3}\right)^2\left(-1 + i\sqrt{3}\right)$$
$$= \left(-2 - 2i\sqrt{3}\right)\left(-1 + i\sqrt{3}\right)$$
$$= 2 - 2i\sqrt{3} + 2i\sqrt{3} - 2i^2\left(\sqrt{3}\right)^2$$
$$= 2 + 6$$
$$= 8.$$

Therefore

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^3 = \frac{\left(-1 + i\sqrt{3}\right)^3}{2^3} = \frac{8}{8} = 1.$$

So $\tfrac{1}{2}\left(-1 + i\sqrt{3}\right)$ is indeed a cube root of unity.

Activity 32   *Checking a cube root of unity*

Check that $\tfrac{1}{2}\left(-1 - i\sqrt{3}\right)$ is a cube root of unity.

The three cube roots of unity all have modulus 1, as you can check directly if you wish. To see why, suppose that z is a cube root of unity, so $z^3 = 1$.

Taking the modulus of both sides of this equation gives

$$|z^3| = 1.$$

Now

$$|z^3| = |z \times z \times z| = |z| \times |z| \times |z| = |z|^3,$$

so it follows that

$$|z|^3 = 1.$$

Therefore $|z| = 1$; that is, z has modulus 1.

You can see that a similar argument will hold for any nth root of unity, for any positive integer n. So, for any positive integer n, the nth roots of unity all have modulus 1.

The three cube roots of unity are shown in the complex plane in Figure 16. Because they all have modulus 1, they all lie on the **unit circle**, which is the circle of radius 1 that is centred on the origin.
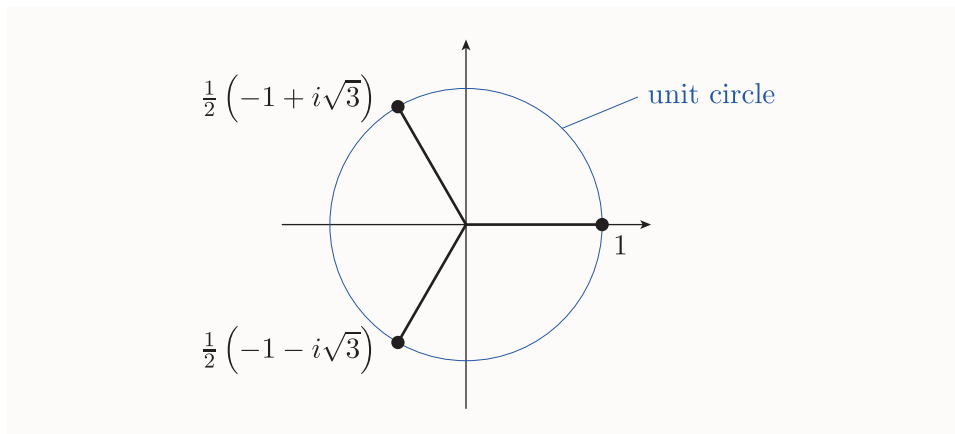
Figure 16   The cube roots of unity

It appears from Figure 16 that the cube roots of unity are equally spaced around the unit circle, and you'll see shortly that this is indeed so. More generally, you'll see that for any positive integer n, there are n solutions of the equation $z^n = 1$ and they're equally spaced around the unit circle. One of these solutions is the number 1, of course, which lies on the positive part of the real axis.

The next example illustrates a method for finding the nth roots of unity, for any positive integer n, using the case $n = 5$ as an example. You'll see later in the subsection that you can use essentially the same method to find the nth roots of any complex number.

You might find it particularly helpful to watch the tutorial clip for the following example.

Example 15   *Finding the fifth roots of unity*

Solve the equation $z^5 = 1$. Sketch the solutions in the complex plane.

Solution

🔍 Write the unknown z in polar form, in terms of an unknown modulus r and an unknown argument θ. Also write the number on the right-hand side of the equation in polar form. 💭

Let $z = r(\cos\theta + i\sin\theta)$. Also, $1 = \cos 0 + i\sin 0$. So the equation is

$$(r(\cos\theta + i\sin\theta))^5 = \cos 0 + i\sin 0.$$

🔍 Use de Moivre's formula to find the polar form of the left-hand side. 💭

De Moivre's formula gives

$$r^5(\cos 5\theta + i\sin 5\theta) = \cos 0 + i\sin 0.$$

🔍 Find r by comparing the moduli of the two sides of this equation. The left-hand side has modulus r^5, and the right-hand side has modulus 1. 💭

Comparing moduli gives

$$r^5 = 1, \quad \text{so} \quad r = 1.$$

🔍 Now find θ by comparing the arguments of the two sides of the equation. The left-hand side has argument 5θ, and the right-hand side has argument 0. However, it *doesn't* follow that $5\theta = 0$. Instead it follows that $5\theta = 0 + 2m\pi$, for some integer m. So there are infinitely many possible values of 5θ, and hence infinitely many possible values of θ. 💭

Comparing arguments gives

$$5\theta = 0 + 2m\pi = 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{2m\pi}{5}, \quad \text{where } m \text{ is an integer.}$$

🔍 Find the five values of θ given by $m = 0, 1, 2, 3, 4$. (You'll see after the example that other integers m don't give further solutions. In general, if the exponent in the original equation is n, then you should find the values of θ given by n consecutive integer values of m, starting with $m = 0$.) 💭

Taking $m = 0, 1, 2, 3, 4$ gives the following values of θ:

$$\theta = 0, \ \frac{2\pi}{5}, \ \frac{4\pi}{5}, \ \frac{6\pi}{5}, \ \frac{8\pi}{5}.$$

🔍 Write out the solutions. It's convenient to label them as z_0, z_1, z_2, z_3, z_4. 💬

The solutions are

$$z_0 = \cos 0 + i \sin 0 = 1$$

$$z_1 = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$$

$$z_2 = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)$$

$$z_3 = \cos\left(\frac{6\pi}{5}\right) + i \sin\left(\frac{6\pi}{5}\right)$$

$$z_4 = \cos\left(\frac{8\pi}{5}\right) + i \sin\left(\frac{8\pi}{5}\right).$$

🔍 Sketch the solutions in the complex plane. Each solution has modulus 1, so they all lie on the unit circle. Plot the solution $z_0 = 1$, and then sketch z_1, z_2, z_3 and z_4, in that order, by marking regularly spaced points separated by the angle $2\pi/5$ anticlockwise around the unit circle. Just estimate each angle $2\pi/5$; there's no need for a precise drawing. 💬
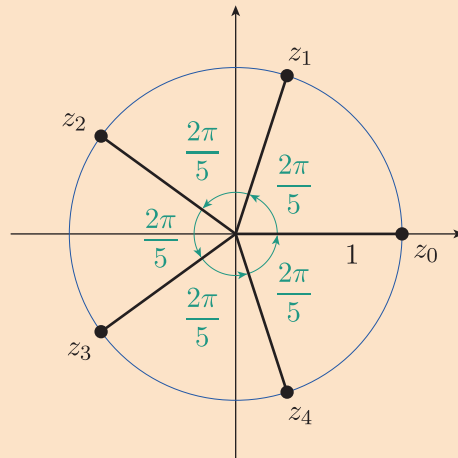
In the solution to Example 15 the fifth roots of unity were given in polar form, with arguments in the interval $[0, 2\pi)$ rather than principal arguments (remember that principal arguments lie in the interval $(-\pi, \pi]$). In general when you're finding roots of complex numbers, it's usually more convenient to use arguments in the interval $[0, 2\pi)$ rather than principal arguments.

To help you see why values of m other than 0, 1, 2, 3 and 4 don't give further solutions of the equation $z^5 = 1$ in Example 15, let's try some of these other values of m, to see what happens.

For example, if you take $m = 5$, then you obtain the solution

$$z = \cos\left(\frac{10\pi}{5}\right) + i\sin\left(\frac{10\pi}{5}\right).$$

Since $10\pi/5 = 2\pi = 2\pi + 0$, this solution is the same as the solution z_0 found in the example. Similarly, if you take $m = 6$, then you obtain the solution

$$z = \cos\left(\frac{12\pi}{5}\right) + i\sin\left(\frac{12\pi}{5}\right).$$

Since $12\pi/5 = 2\pi + 2\pi/5$, this solution is the same as the solution z_1 found in the example. Once you've taken m to be 0, 1, 2, 3 and 4, the solutions that you obtain just start to repeat. The same thing happens if you take negative values of m.

Here's an algebraic explanation of this repeating behaviour. When you use the method in Example 15 to find the nth roots of unity for some value of n, each possible value of θ is of the form

$$\theta = \frac{2m\pi}{n}, \quad \text{where } m \text{ is an integer.}$$

Consider any integer m. It can be written in the form

$$m = qn + r,$$

where q is an integer and r is the remainder after m is divided by n. (For example, if $n = 5$, then the integer $m = 14$, for instance, can be written as $14 = 2 \times 5 + 4$.) So the value of θ corresponding to m can be written as

$$\theta = \frac{2(qn + r)\pi}{n} = q \times 2\pi + \frac{2r\pi}{n},$$

where q is an integer, and r is one of the integers $0, 1, \ldots, n - 1$. That is, the value of θ corresponding to m differs from the value of θ corresponding to one of the integers $0, 1, \ldots, n - 1$ by an integer multiple of 2π. Hence the solution z arising from the integer m is the same as the solution arising from one of the integers $0, 1, \ldots, n - 1$.

The following box summarises the main steps of the method used in Example 15. The method is stated for a general equation $z^n = a$, where a is any non-zero complex number, rather than just for equations of the form $z^n = 1$, because it applies in this more general situation, as you'll see later in this subsection.

Strategy:
To find the complex solutions of the equation $z^n = a$, where $a \neq 0$

1.   Write the unknown z in polar form, in terms of an unknown modulus r and an unknown argument θ, and write the number a in polar form.

2.   Substitute the polar forms of z and a into the equation, and apply de Moivre's formula to find the polar form of the left-hand side.

3.   Compare moduli to find the value of r.

4.   Compare arguments to find n successive possible values of θ.

5.   Hence write down the n possible values of z.

It is usually convenient to use arguments in the interval $[0, 2\pi)$.

The solutions found in Example 15 were left in polar form because in that form they're exact, and it's difficult to find the exact Cartesian form of all of them. Also, the polar form helps you to see that the solutions give five equally spaced points around the unit circle. The solution z_0 was given in both polar form and Cartesian form. This is because its Cartesian form is so simple ($z_0 = 1$) that it would be strange not to mention this.

In the next activity you're asked to find roots of unity in both polar form *and* Cartesian form, because it's fairly straightforward to give exact solutions in both forms. You saw the solutions of the equation in part (a) of this activity earlier in the subsection, but you should obtain them again, using the strategy.

Activity 33   *Finding roots of unity*

Solve the following equations. Give your answers in both polar form and Cartesian form. Sketch the solutions in the complex plane.

(a) $z^3 = 1$ (b) $z^4 = 1$ (c) $z^6 = 1$

Now let's look at equations of the form

$$z^n = a,$$

where a is any non-zero complex number. Before we use the strategy above to solve an equation like this, let's look at a particular example of such an equation.

Consider the equation

$$z^3 = -8.$$

It has one real solution, namely -2, because $(-2)^3 = -8$. It also has two other complex solutions, namely $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$, which you can check yourself. All three solutions are shown in the complex plane in Figure 17.
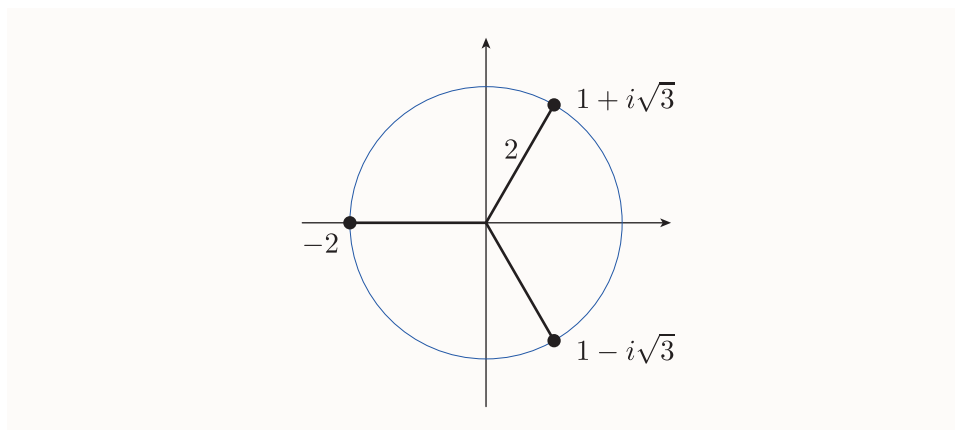
Figure 17   The cube roots of -8

The three solutions don't lie on the unit circle; instead they lie on the circle of radius 2 centred at the origin. To see why, suppose that z is any solution of the equation, so $z^3 = -8$. Taking the modulus of both sides of this equation gives

$$|z^3| = 8, \quad \text{so} \quad |z|^3 = 8.$$

Therefore

$$|z| = 8^{1/3} = 2.$$

That is, z has modulus 2, and therefore lies on the circle of radius 2 centred on the origin.

It appears from Figure 17 that the three solutions are equally spaced around this circle, and indeed they are. In general, the following fact holds.

> For each non-zero number a, the equation $z^n = a$ has n complex solutions, and these are equally spaced around a circle centred on the origin.

The next example illustrates how to apply the strategy that you've seen in this subsection to find all the complex solutions of an equation of the form $z^n = a$, where a is non-zero. Again, you might find it helpful to watch the tutorial clip.

Example 16   *Finding the roots of a complex number*

Solve the equation $z^4 = 4i$. Sketch the solutions in the complex plane.

Solution

🔍 Write the unknown z in polar form, in terms of an unknown modulus r and an unknown argument θ. Also write the number on the right-hand side of the equation in polar form. 💬

Let $z = r(\cos\theta + i\sin\theta)$. The complex number $4i$ has modulus 4 and principal argument $\pi/2$, so its polar form is

$$4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

So the equation is

$$(r(\cos\theta + i\sin\theta))^4 = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

💭 Use de Moivre's formula to find the polar form of the left-hand side. 💭

De Moivre's formula gives

$$r^4(\cos 4\theta + i\sin 4\theta) = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

💭 Find r by comparing the moduli of each side of the equation. 💭

Comparing moduli gives

$$r^4 = 4, \quad\text{so}\quad r = 4^{1/4} = \sqrt{2}.$$

💭 Find θ by comparing the arguments of the two sides of the equation. 💭

Comparing arguments gives

$$4\theta = \frac{\pi}{2} + 2m\pi, \quad\text{where } m \text{ is an integer.}$$

Therefore

$$\theta = \frac{\pi}{8} + \frac{m\pi}{2}, \quad\text{where } m \text{ is an integer.}$$

💭 Find the values of θ given by $m = 0, 1, 2, 3$. 💭

The values of θ for $m = 0, 1, 2, 3$ are

$$\frac{\pi}{8}, \quad \frac{5\pi}{8}, \quad \frac{9\pi}{8}, \quad \frac{13\pi}{8}.$$

💭 Write out the solutions. It's convenient to label them as z_0, z_1, z_2 and z_3. 💭

The solutions are

$$z_0 = \sqrt{2}\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)$$

$$z_1 = \sqrt{2}\left(\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right)$$

$$z_3 = \sqrt{2}\left(\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)\right).$$

Sketch the solutions in the complex plane. Each solution has modulus $\sqrt{2}$, so they all lie on the circle of radius $\sqrt{2}$ centred at the origin. Sketch the solution z_0, and then sketch z_1, z_2 and z_3, in that order, by marking regularly spaced points separated by angles $\pi/2$ anticlockwise around the circle.
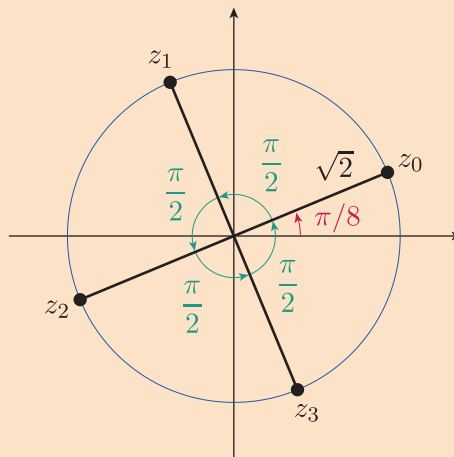
As expected, the solutions of the equation $z^4 = 4i$ in Example 16 are equally spaced round a circle centred on the origin. Note that the infinitely many possible values of the argument, $\theta = \pi/8 + m\pi/2$, where m is an integer, give repeating solutions of the equation $z^4 = 4i$ in a similar way to the solutions of $z^n = 1$ described earlier.

Activity 34   *Finding roots of complex numbers*

Solve the following equations. Give the solutions to parts (a) and (b) in both polar form and Cartesian form, and give the solutions to part (c) in polar form only. Sketch the solutions in the complex plane.

(a) $z^6 = 64$ (b) $z^3 = -8$ (c) $z^5 = -1 - i$

3.3 The fundamental theorem of algebra

As you learned earlier, a *polynomial equation* is an equation of the form 'polynomial expression = 0', where the polynomial expression has degree at least 1. The coefficients in the polynomial expression can be complex numbers. For example, the following equations are all polynomial equations:

$$z^5 + 1 + i = 0, \quad 3z^2 + \tfrac{11}{4}z - 18 = 0 \quad \text{and} \quad z^3 - (17 + 3i)z + 2 = 0.$$

The first of these equations can be rearranged as $z^5 = -1 - i$, which is an equation of the type that you learned how to solve in the previous subsection. The second equation is a quadratic equation. **Cubic equations** (such as the third equation above) and **quartic equations** are polynomial equations in which the polynomial expressions have degrees 3 and 4, respectively. This subsection is about *the fundamental theorem of algebra*, which shows that every polynomial equation has at least one solution.

Before you meet the fundamental theorem of algebra, let's first consider an example that will help you to understand it. In Example 12 on page 215 you saw that the solutions of the quadratic equation $z^2 - 6z + 25 = 0$ are $3 + 4i$ and $3 - 4i$. It follows that you can factorise the quadratic $z^2 - 6z + 25$ as

$$z^2 - 6z + 25 = \big(z - (3 + 4i)\big)\big(z - (3 - 4i)\big).$$

More generally, if z_1 and z_2 are the solutions of the quadratic equation $az^2 + bz + c = 0$ (there may be only one solution, in which case $z_1 = z_2$), then you can factorise the quadratic $az^2 + bz + c$ as

$$az^2 + bz + c = a(z - z_1)(z - z_2).$$

The fundamental theorem of algebra says that it's not only quadratics that can be factorised like this, but in fact any polynomial expression can be factorised in a similar way.

The fundamental theorem of algebra

Every polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

of degree $n \geq 1$ has a factorisation

$$a_n(z - z_1)(z - z_2) \cdots (z - z_n),$$

where $z_1, z_2, \ldots, z_n$ are complex numbers, some of which may be equal to others.

The fundamental theorem of algebra was first proved by Carl Friedrich Gauss, and a proof was also given by Argand, who was mentioned in Subsection 2.1. The proof requires techniques more advanced than those in this module and is given in higher-level modules.

The fundamental theorem of algebra tells you that any polynomial equation, say

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

can be written in the form

$$a_n(z - z_1)(z - z_2) \cdots (z - z_n) = 0.$$

It follows that the polynomial equation has solutions $z_1, z_2, \ldots, z_n$, some of which may be equal to others. In particular, *every polynomial equation has at least one complex solution.*

Although the fundamental theorem of algebra tells you that every polynomial equation has at least one complex solution, it doesn't tell you how to *find* any solutions. Often finding solutions is difficult, and you have to use a computer. You can learn how to do that in the next activity.
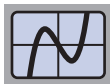
Activity 35   *Solving polynomial equations with the CAS*

Work through Subsection 13.2 of the *Computer algebra guide.*

Insolvability of the quintic

Not only is there a formula for the solutions of a quadratic equation in terms of its coefficients, but there are also formulas for the solutions of cubic equations and quartic equations in terms of their coefficients. These formulas are long, and difficult to evaluate without a computer. For polynomial equations of higher degree, however, it's impossible to find formulas involving the usual arithmetic operations that give all the solutions of all the equations. For example, the quintic equation

$$z^5 + 3z + 6 = 0$$

is said to be *insolvable* because, although it has five solutions, it's impossible to obtain these solutions from the coefficients 1, 3 and 6 by using only the operations $+$, $-$, $\times$, $\div$ and $\sqrt[n]{\ }$.

One of the first mathematicians to develop the theory of insolvable polynomial equations was the French Republican Évariste Galois ('Galois' is pronounced 'Gal-wah'). Galois' discoveries led to the development of the subject now known as *Galois theory*, in which the solutions of polynomial equations are studied systematically. Sadly, Galois didn't receive full recognition for his work in his lifetime, because he died aged only twenty, as a result of a duel.

Évariste Galois (1811–1832)

4   Exponential form

In this section you'll learn about a more concise version of polar form, called *exponential form.* You'll see that when you write complex numbers in exponential form, some of their properties, such as the formula for multiplication and de Moivre's formula, become more intuitive.

4.1 Euler's formula

To find the exponential form of a complex number from its polar form, you use a formula called *Euler's formula*. This formula gives a meaning for the expression $e^{i\theta}$, where θ is a real number. So far you've learned what the expression e^x means only when x is a real number, but you can get some idea of what the natural meaning of $e^{i\theta}$ might be by considering Taylor series. Recall from Unit 11 that the Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots .$$

Let's try substituting $i\theta$ for x in this series. Since you've learned about Taylor series only for *real* numbers, you haven't been shown that you're allowed to do this (in fact you are: it's justified in more advanced modules). Nonetheless, you'll see that the substitution can help you understand why Euler's formula makes sense. We obtain

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \cdots .$$

To simplify this expression, notice the following pattern:

$$\underbrace{i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i,}_{\text{This pattern } 1, i, -1, -i \text{ repeats.}} \quad i^4 = 1, \quad i^5 = i, \ldots$$

(You may remember this pattern from Activity 6.) It follows that

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right).$$

You also saw in Unit 11 that the Taylor series about 0 for $\cos\theta$ and $\sin\theta$ are

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \quad \text{and} \quad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots .$$

Therefore

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This equation is **Euler's formula**. It's named after its discoverer, the Swiss mathematician Leonard Euler (1707–1783), who was mentioned in Unit 3. Euler obtained the formula in much the way that we have.

The manipulation above shows that if the Taylor series about 0 for e^x is valid with $i\theta$ instead of x, then Euler's formula must hold. So Euler's formula seems to be the natural definition of $e^{i\theta}$, and hence we use it to define $e^{i\theta}$.

Euler's formula

Let θ be a real number. Then $e^{i\theta}$ is defined by **Euler's formula**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

When $\theta = \pi$, Euler's formula says that $e^{i\pi} = \cos\pi + i\sin\pi$; that is, $e^{i\pi} = -1$. This equation can be rearranged to give the equation below, which is known as **Euler's equation**.

Euler's equation

$$e^{i\pi} + 1 = 0$$

This equation, which is also sometimes called *Euler's identity*, is one of the most famous equations in mathematics, because it relates the five fundamental numbers 0, 1, i, e and π.

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ tells us that a complex number z in polar form

$$z = r(\cos\theta + i\sin\theta)$$

can be written in a more concise way as

$$z = re^{i\theta}.$$

This is the *exponential form* of a complex number, promised at the start of this subsection.

Exponential form of a complex number

A non-zero complex number z is in **exponential form** if it is expressed as

$$z = re^{i\theta},$$

where r is the modulus of z and θ is an argument of z.

For example, because the complex number $3i$ has modulus 3 and principal argument $\pi/2$, it has exponential form $3e^{i\pi/2}$. It also has other exponential forms, corresponding to different choices of argument of $3i$, such as $3e^{5\pi i/2}$ and $3e^{-3\pi i/2}$. However, you should use the principal argument when writing a complex number in exponential form, unless there's a good reason not to do so.

The number 0 doesn't have a polar form, as you've seen, and it doesn't have an exponential form either. When working with complex numbers in exponential form, you should write the number zero using the usual symbol 0.

Because the exponential form of a complex number is just a concise way to write its polar form, the process of converting between the Cartesian and exponential forms of a complex number is much the same as the process of converting between its Cartesian and polar forms, which you learned in Section 2.

Example 17   *Converting from exponential form to Cartesian form*

Write the complex number $4e^{-5\pi i/6}$ in Cartesian form.

Solution

🔍 Write the number in polar form, evaluate the cosine and sine, and multiply out the brackets. 💬

$$4e^{-5\pi i/6} = 4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$
$$= 4\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$
$$= -2\sqrt{3} - 2i$$

You can use your solutions to Activity 20 on page 203 to help you with the next activity.

Activity 36   *Converting from exponential form to Cartesian form*

Write the following complex numbers in Cartesian form.

(a) $3e^{i0}$ (b) $7e^{i\pi/2}$ (c) $6e^{i\pi}$ (d) $e^{-i\pi/2}$ (e) $5e^{-i\pi}$

(f) $4e^{i\pi/3}$ (g) $2e^{-i\pi/4}$ (h) $\sqrt{3}e^{5\pi i/6}$

Example 18   *Converting from Cartesian form to exponential form*

Write the complex number $1 - i$ in exponential form.

Solution

🔍 First find the modulus. 💬

The modulus is

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}.$$

To find the principal argument, sketch $1 - i$ in the complex plane.
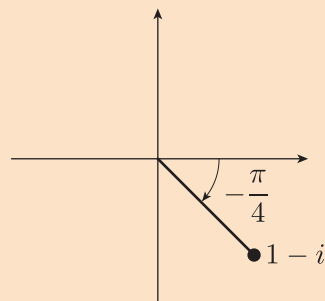
From the diagram, the principal argument is $-\pi/4$. Therefore

$$1 - i = \sqrt{2}e^{-i\pi/4}.$$

You can use your solutions to Activity 21 on page 205 to help you with the next activity.

Activity 37   *Converting from Cartesian form to exponential form*

Write the following complex numbers in exponential form.

(a) $7i/2$ (b) $-4i$ (c) -3 (d) 2 (e) $1 + i$ (f) $1 - i\sqrt{3}$

(g) $-\sqrt{3} - i$ (h) $2\sqrt{3} - 2i$

It's often better to use the exponential form $re^{i\theta}$ of a complex number, rather than its polar form $r(\cos\theta + i\sin\theta)$, because it's shorter. Another reason for preferring the exponential form is that some of the formulas involving polar form that you met earlier become more intuitive when complex numbers are expressed in exponential form.

Consider, for example, the formula for the product of two complex numbers in polar form:

$$\underbrace{r(\cos\theta + i\sin\theta)}_{re^{i\theta}} \times \underbrace{s(\cos\phi + i\sin\phi)}_{se^{i\phi}} = \underbrace{rs(\cos(\theta + \phi) + i\sin(\theta + \phi))}_{rse^{i(\theta+\phi)}}.$$

If you write the complex numbers in exponential form, then this formula becomes

$$re^{i\theta} \times se^{i\phi} = rse^{i(\theta+\phi)}.$$

This version of the formula is more intuitive, and hence easier to remember and use, because it agrees with the usual index laws for multiplying powers of a real number.

In particular, when $r = s = 1$ you obtain the following equation, which looks like it follows from a familiar index law, but involves complex numbers.

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

Similarly, de Moivre's formula,

$$\underbrace{\left(r(\cos\theta + i\sin\theta)\right)^n}_{\left(re^{i\theta}\right)^n} = \underbrace{r^n(\cos n\theta + i\sin n\theta)}_{r^n e^{in\theta}},$$

is simpler and more intuitive when written using exponential form:

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}.$$

Choosing $r = 1$ gives the rule below, which again looks like it follows from a familiar index law, but involves complex numbers.

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

Activity 38   *Writing a formula for the quotient of two complex numbers in exponential form*

The formula for the quotient of two complex numbers in polar form is

$$\frac{r(\cos\theta + i\sin\theta)}{s(\cos\phi + i\sin\phi)} = \frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi)).$$

Write this formula using exponential form.

Activity 39   *Working with complex numbers in exponential form*

Express each of the following products, quotients and powers of complex numbers as a single complex number in exponential form.

(a) $e^{i\pi/8} \times e^{3\pi i/8}$ (b) $\left(e^{i\pi/8}\right)^4$ (c) $\dfrac{e^{3\pi i/8}}{e^{i\pi/8}}$ (d) $\dfrac{e^{i\pi/8}}{e^{3\pi i/8}}$

You can use Euler's formula to obtain further useful formulas, by combining it with properties of sine and cosine. For example, if you replace θ by $-\theta$ in Euler's formula then you obtain

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta).$$

You saw in Unit 4 that

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta,$$

which give the following formula for $e^{-i\theta}$.

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Since $\cos\theta - i\sin\theta$ is the complex conjugate of $\cos\theta + i\sin\theta$, we can make the following useful observation.

The complex conjugate of $e^{i\theta}$ is $e^{-i\theta}$.

The equation $e^{-i\theta} = \cos\theta - i\sin\theta$ is used in the next subsection to obtain formulas for $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

Activity 40 *Proving an identity by using Euler's formula and properties of sine and cosine*

Using Euler's formula, prove the identity

$$e^{i(\theta + 2\pi)} = e^{i\theta}.$$

Complex impedance

In an electrical circuit powered by a direct current (from a battery, for example), the current J, measured in amperes, and the voltage V, measured in volts, are related by *Ohm's law*, which says that

$$V = JR,$$

where R is the resistance of the circuit, measured in ohms.

The electric power in our homes is not supplied by a direct current. Instead it's supplied by an alternating current, generated by an alternating voltage. The intensities of the alternating current J and voltage V oscillate with time, and are typically given by equations such as

$$J = J_0 \sin\omega t \quad \text{and} \quad V = V_0 \sin(\omega t + \phi),$$

where J_0 and V_0 are the maximum values of the current and voltage, respectively, ω determines the rate of oscillation of both the current and the voltage, and ϕ measures how far the current and voltage are from being synchronised.

Most large wind turbines generate an alternating current

The graphs of these equations are illustrated in Figure 18.
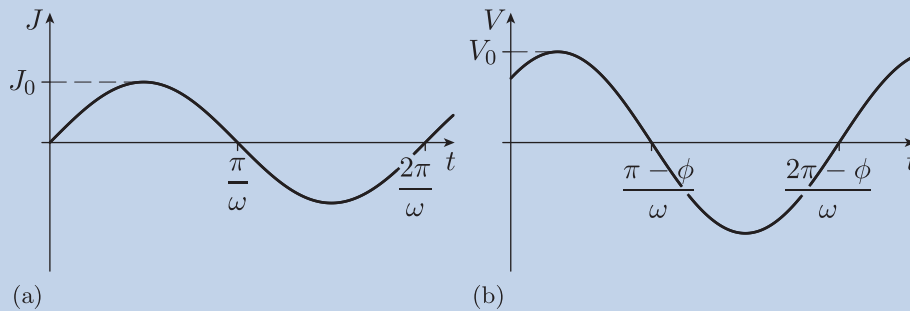
Figure 18   The graphs of (a) $J = J_0 \sin \omega t$ (b) $V = V_0 \sin(\omega t + \phi)$

Electrical engineers use complex numbers in exponential form to manipulate equations arising from alternating currents. They define the *complex current* **J** to equal $J_0 e^{i\omega t}$ and the *complex voltage* **V** to equal $V_0 e^{i(\omega t + \phi)}$. It follows that

$$\mathrm{Im}(\mathbf{J}) = \mathrm{Im}(J_0 e^{i\omega t}) = J_0 \sin \omega t$$

and

$$\mathrm{Im}(\mathbf{V}) = \mathrm{Im}(V_0 e^{i(\omega t + \phi)}) = V_0 \sin(\omega t + \phi).$$

So the current J is the imaginary part of the complex current **J** and the voltage V is the imaginary part of the complex voltage **V**. The *complex impedance* **Z** is defined to be the ratio **V**/**J**. It can be shown that **Z** measures the various ways in which an electrical circuit resists the flow of an alternating current. By rearranging the equation **Z** = **V**/**J** you obtain a version of Ohm's law for alternating currents:

$$\mathbf{V} = \mathbf{ZJ}.$$

The actual alternating voltage can be found by comparing the imaginary parts of each side of this equation.

4.2   Trigonometric identities from de Moivre's formula and Euler's formula

In Unit 4 you saw some trigonometric identities, such as

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad \text{and} \quad \cos 2\theta = 2 \cos^2 \theta - 1.$$

In this final subsection you'll learn how you can use de Moivre's formula and Euler's formula, together with the binomial theorem from Unit 10, to obtain new trigonometric identities. This illustrates the fact that techniques involving complex numbers can often be used to deduce results about real functions.

One way to find new trigonometric identities is to use the special case of de Moivre's formula given in the box below. It's obtained by taking the modulus r to be 1 in the general formula on page 211 and swapping the sides.

> **Special case of de Moivre's formula**
>
> $$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

Example 19   *Using de Moivre's formula to find trigonometric identities*

Use de Moivre's formula to obtain the following trigonometric identities:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Solution

Use the special case of de Moivre's formula. Because the required identities involve 3θ, take $n = 3$.

By de Moivre's formula,

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

Apply the binomial theorem.

$$
\begin{aligned}
&= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 \\
&\quad + (i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).
\end{aligned}
$$

To obtain the first identity, use the fact that the real part of the left-hand side is equal to the real part of the right-hand side. To express $\sin^2\theta$ in terms of $\cos\theta$, use the identity $\sin^2\theta + \cos^2\theta = 1$.

Therefore

$$
\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

To obtain the second identity, use the fact that the imaginary part of the left-hand side is equal to the imaginary part of the right-hand side.

and
$$\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta$$
$$= 3(1 - \sin^2 \theta)\sin \theta - \sin^3 \theta$$
$$= 3\sin \theta - 4\sin^3 \theta.$$

When you've obtained a new trigonometric identity, it's worth trying it out with some specific values of the variable θ, to check that you haven't made an error. Consider, for example, the identity obtained in Example 19:
$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta.$$

When $\theta = 0$,
$$\text{LHS} = 1 \quad \text{and} \quad \text{RHS} = 4 - 3 = 1.$$

When $\theta = \pi/3$,
$$\text{LHS} = -1 \quad \text{and} \quad \text{RHS} = 4 \times \left(\tfrac{1}{2}\right)^3 - 3 \times \tfrac{1}{2} = -1.$$

When $\theta = \pi/2$, both LHS and RHS are 0.

So the identity certainly holds for these particular values of θ, as expected.

Activity 41   *Using de Moivre's formula to find trigonometric identities*

Use de Moivre's formula to obtain the trigonometric identities
$$\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$$
$$\sin 4\theta = 4\sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta).$$

To keep your working short, you may find it helpful to write $c = \cos \theta$ and $s = \sin \theta$.

Another way to obtain new trigonometric identities is to start by finding formulas for $\sin \theta$ and $\cos \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, where θ is a real number. To do this, recall Euler's formula and the complex conjugate form of Euler's formula, obtained in the previous subsection:
$$e^{i\theta} = \cos \theta + i\sin \theta$$
$$e^{-i\theta} = \cos \theta - i\sin \theta.$$

Adding the two equations gives
$$e^{i\theta} + e^{-i\theta} = 2\cos \theta, \quad \text{so} \quad \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos \theta.$$

Subtracting the bottom equation from the top equation gives
$$e^{i\theta} - e^{-i\theta} = 2i\sin \theta, \quad \text{so} \quad \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin \theta.$$

These are the formulas that we need.

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The next example shows you how you can use the first of these formulas to obtain a trigonometric identity. (In fact this trigonometric identity is one of the identities from Example 19, in a rearranged form.) You also have to use the rules below, which you met in the previous subsection.

$$e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$$

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

Example 20   *Using the formula* $\cos \theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ *to obtain a trigonometric identity*

Use the formula $\cos \theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ to obtain the identity

$$\cos^3 \theta = \tfrac{1}{4}\left(\cos 3\theta + 3\cos \theta\right).$$

Solution

We have $\cos \theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. Therefore

$$\cos^3 \theta = \frac{1}{2^3}\left(e^{i\theta} + e^{-i\theta}\right)^3$$

🔍 Use the binomial theorem. 💬

$$= \frac{1}{8}\left((e^{i\theta})^3 + 3(e^{i\theta})^2 e^{-i\theta} + 3e^{i\theta}(e^{-i\theta})^2 + (e^{-i\theta})^3\right)$$

🔍 Simplify by using the rule $(e^{i\theta})^n = e^{in\theta}$. 💬

$$= \frac{1}{8}\left(e^{3i\theta} + 3e^{2i\theta} e^{-i\theta} + 3e^{i\theta} e^{-2i\theta} + e^{-3i\theta}\right)$$

🔍 Simplify by using the rule $e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$. 💬

$$= \frac{1}{8}\left(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}\right)$$

🔍 Rearrange to obtain expressions of the form $\frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$, and then use the formula given in the question again. 💬

$$= \frac{1}{4} \left(\left(\frac{e^{3i\theta} + e^{-3i\theta}}{2} \right) + 3 \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right) \right)$$

$$= \frac{1}{4} \left(\cos 3\theta + 3 \cos \theta \right).$$

Activity 42   *Using the formulas* $\cos \theta = \frac{1}{2} \left(e^{i\theta} + e^{-i\theta} \right)$
and $\sin \theta = \frac{1}{2i} \left(e^{i\theta} - e^{-i\theta} \right)$ *to obtain trigonometric identities*

Use the formulas

$$\cos \theta = \frac{1}{2} \left(e^{i\theta} + e^{-i\theta} \right) \quad \text{and} \quad \sin \theta = \frac{1}{2i} \left(e^{i\theta} - e^{-i\theta} \right)$$

to obtain the following identities.

(a) $\sin^3 \theta = \frac{1}{4} \left(3 \sin \theta - \sin 3\theta \right)$ (b) $\cos^4 \theta = \frac{1}{8} \left(\cos 4\theta + 4 \cos 2\theta + 3 \right)$

The examples and activities in this section have shown you how to use complex numbers to deduce new trigonometric identities that express $\sin n\theta$ and $\cos n\theta$ in terms of powers of $\sin \theta$ and $\cos \theta$, and vice versa. This should have given you some idea of the power of complex numbers, and an appreciation of the fact that their uses are far from 'imaginary'.

Quaternions

You've seen that when you work with complex numbers, you're performing arithmetic with pairs of real numbers. The Irish mathematician William Rowan Hamilton was the first to realise how to perform arithmetic with *quadruples* of real numbers. He did this, in a flash of inspiration, while walking beside the Royal Irish Canal near Dublin on 16 October 1843. His idea was to use numbers of the form $a + bi + cj + dk$, where a, b, c and d are real numbers, and

$$i^2 = j^2 = k^2 = ijk = -1.$$

Hamilton scratched this foundational equation on to Broom Bridge as he passed by. His scratches are no longer visible and instead a plaque records the event. Hamilton called his new system of numbers the **quaternions**. You can perform all the usual arithmetic operations with quaternions, with one crucial difference: multiplication is not commutative. For example, you can show that

$$ij = k, \quad \text{whereas} \quad ji = -k.$$

Soon after the discovery of the quaternions, a system of arithmetic with *octuples* of real numbers was developed, called the **octonions** (or **Cayley numbers**). Multiplication of octonions is neither commutative nor associative. This makes them tricky to manipulate!
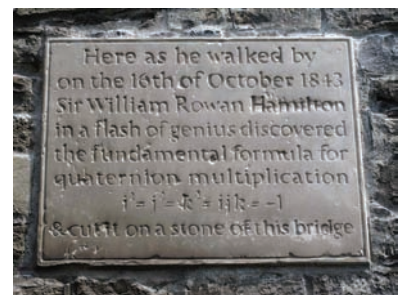
The plaque on Broom Bridge, Dublin, that marks Hamilton's discovery

Learning outcomes

After studying this unit, you should be able to:

- understand what complex numbers are, and carry out arithmetical operations on them
- work with the complex plane
- understand modulus and argument
- understand the polar form of a complex number, and convert between Cartesian form and polar form
- multiply and divide complex numbers in polar form
- understand and use de Moivre's formula
- find all solutions of quadratic equations, including complex solutions
- find roots of complex numbers
- understand the fundamental theorem of algebra
- state Euler's formula
- understand the exponential form of a complex number, and convert between this form and other forms
- use de Moivre's formula and Euler's formula to obtain trigonometric identities.
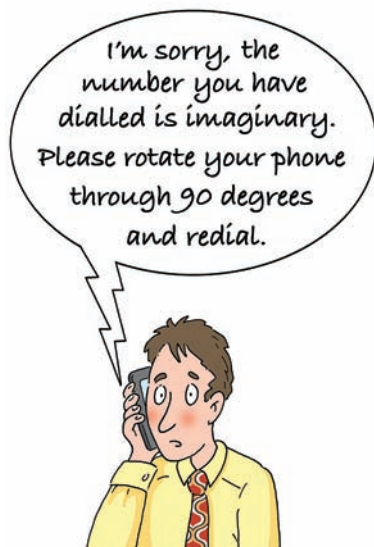
Closing remarks

Well done for completing MST124 (or skipping to the last page)! You've covered a broad range of topics, which will provide you with the essential mathematical skills that you need to develop as an engineer, scientist, economist or mathematician.

Let's review some of the key topics that you met, and see how they fit together. You learned a good deal about functions, which are among the most fundamental objects in mathematics. The exponential function is of particular significance, not least because of its use in exponential models of real-life situations. The trigonometric functions are also of great importance, and their many practical applications include measuring distances and modelling waves. In this final unit you met Euler's remarkable formula $e^{i\theta} = \cos\theta + i\sin\theta$, which brings together the exponential and trigonometric functions by using complex numbers.

With compound interest, savings increase exponentially

Another class of functions that you studied are the polynomial functions. You saw that you can approximate functions by Taylor polynomials, which is often extremely useful for understanding properties of functions and performing calculations with them.

You saw that functions and sequences can be represented geometrically by their graphs. Graphs help to quickly communicate the key properties of functions and sequences, and they also provide a simple way to represent physical quantities such as velocity and acceleration.

Through studying gradients of graphs you came to learn about differentiation, and then about integration, which is the reverse of differentiation. The theory surrounding these two concepts, which is known as calculus, is an essential part of almost every discipline that involves mathematics, from finance to medicine. You'll make good use of calculus in higher-level modules involving mathematics.

In Unit 5 you learned about another way of modelling real-life phenomena, namely by using vectors. Trigonometry is crucial when you're working with vectors, because you can use it to calculate the directions of vectors. You also learned some geometry in Unit 5, and then, in this final unit, you saw how by using the complex plane you can associate the rich structure of the complex numbers with two-dimensional geometry.

The acceleration of a diving peregrine falcon can be represented by a graph and analysed using calculus

Yet another central topic that you encountered is matrices, which have a huge range of applications, in modelling networks, solving linear equations, higher-dimensional calculus, and many other topics.

The MST124 authors hope that you feel inspired to continue to study mathematics, whether it's through engineering, science, statistics or any other subject with a mathematical component.

Solutions to activities

Solution to Activity 1

(a) $\mathrm{Re}(2+9i) = 2, \quad \mathrm{Im}(2+9i) = 9$

(b) $\mathrm{Re}(4) = 4, \quad \mathrm{Im}(4) = 0$

(c) $\mathrm{Re}(-7i) = 0, \quad \mathrm{Im}(-7i) = -7$

(d) $\mathrm{Re}(0) = 0, \quad \mathrm{Im}(0) = 0$

(e) $\mathrm{Re}(i) = 0, \quad \mathrm{Im}(i) = 1$

(f) $\mathrm{Re}(1-i) = 1, \quad \mathrm{Im}(1-i) = -1$

Solution to Activity 2

(a) $(2+5i) + (-7+13i) = (2+(-7)) + (5+13)i$
$$= -5 + 18i$$

$(2+5i) - (-7+13i) = (2-(-7)) + (5-13)i$
$$= 9 - 8i$$

(b) $(-4i) + (-9i) = ((-4)+(-9))i$
$$= -13i$$

$(-4i) - (-9i) = ((-4)-(-9))i$
$$= 5i$$

(c) $(3-7i) + (3-7i) = (3+3) + ((-7)+(-7))i$
$$= 6 - 14i$$

$(3-7i) - (3-7i) = (3-3) + ((-7)-(-7))i$
$$= 0$$

(d) $(3+7i) + (3-7i) = (3+3) + (7+(-7))i$
$$= 6$$

$(3+7i) - (3-7i) = (3-3) + (7-(-7))i$
$$= 14i$$

(e) $\left(\frac{1}{6} - \frac{1}{3}i\right) + \left(-\frac{1}{3} + \frac{1}{6}i\right)$
$$= \left(\frac{1}{6} + \left(-\frac{1}{3}\right)\right) + \left(\left(-\frac{1}{3}\right) + \frac{1}{6}\right)i$$
$$= -\frac{1}{6} - \frac{1}{6}i$$

$\left(\frac{1}{6} - \frac{1}{3}i\right) - \left(-\frac{1}{3} + \frac{1}{6}i\right)$
$$= \left(\frac{1}{6} - \left(-\frac{1}{3}\right)\right) + \left(\left(-\frac{1}{3}\right) - \frac{1}{6}\right)i$$
$$= \frac{1}{2} - \frac{1}{2}i$$

(f) $z + w = 1.2 + 3.4i$

$z - w = 1.2 - 3.4i$

Solution to Activity 3

(a) $u + v + w = (4+6i) + (-3+5i) + (2-i)$
$$= (4+(-3)+2) + (6+5+(-1))i$$
$$= 3 + 10i$$

(b) By part (a),
$$w + v + u = u + v + w = 3 + 10i.$$

(c) $u - (v+w) = u - v - w$
$$= (4+6i) - (-3+5i) - (2-i)$$
$$= (4-(-3)-2) + (6-5-(-1))i$$
$$= 5 + 2i$$

(d) $u - (v-w) = u - v + w$
$$= (4+6i) - (-3+5i) + (2-i)$$
$$= (4-(-3)+2) + (6-5+(-1))i$$
$$= 9$$

Solution to Activity 4

(a) $(1+3i)(2+4i) = 2 + 4i + 6i + 12i^2$
$$= 2 + 4i + 6i - 12$$
$$= -10 + 10i$$

(b) $(-2+3i)(4-7i) = -8 + 14i + 12i - 21i^2$
$$= -8 + 14i + 12i + 21$$
$$= 13 + 26i$$

(c) $3i(4-5i) = 12i - 15i^2$
$$= 12i + 15$$
$$= 15 + 12i$$

(d) $7(-2+5i) = -14 + 35i$

(e) $(2-3i)(2+3i) = 4 + 6i - 6i - 9i^2$
$$= 4 + 6i - 6i + 9$$
$$= 13$$

(f) $\left(\frac{1}{2}+i\right)\left(1+\frac{1}{2}i\right) = \frac{1}{2} + \frac{1}{4}i + i + \frac{1}{2}i^2$
$$= \frac{1}{2} + \frac{1}{4}i + i - \frac{1}{2}$$
$$= \frac{5}{4}i$$

Solution to Activity 5

(a) $u(v + w) = (1 + 2i)((4 - 3i) + (-i))$
$= (1 + 2i)(4 - 4i)$
$= 4 - 4i + 8i - 8i^2$
$= 4 - 4i + 8i + 8$
$= 12 + 4i$

(b) By part (a),
$uv + uw = u(v + w) = 12 + 4i.$

(c) $uvw = (1 + 2i)((4 - 3i)(-i))$
$= (1 + 2i)(-4i + 3i^2)$
$= (1 + 2i)(-3 - 4i)$
$= -3 - 4i - 6i - 8i^2$
$= -3 - 4i - 6i + 8$
$= 5 - 10i$

Solution to Activity 6

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = (i^2)i = (-1)i = -i$
$i^4 = (i^3)i = (-i)i = -i^2 = 1$
$i^5 = (i^4)i = (1)i = i$
$i^6 = (i^5)i = (i)i = -1$

The powers of i are, in order,

$$\underbrace{1, \, i, \, -1, \, -i,}_{} \underbrace{1, \, i, \, -1, \, -i,}_{} \underbrace{1, \, i, \, -1, \, -i,}_{} \ldots .$$

This pattern repeats

Solution to Activity 7

Since
$$(3i)^2 = 3^2 i^2 = 9 \times (-1) = -9,$$
it follows that $3i$ is a square root of -9. Since
$$(-3i)^2 = (-3)^2 i^2 = 9 \times (-1) = -9,$$
it follows that $-3i$ is also a square root of -9.

Solution to Activity 8

(a) $4 - 2i$

(b) $-3 + 8i$

(c) $-9i$

(d) 5

Solution to Activity 9

As $z = a + bi$, it follows that $w = \overline{z} = a - bi$.
Therefore
$$\overline{w} = \overline{a - bi} = a + bi = z.$$

Solution to Activity 10

Let $z = a + bi$.

(a) $z + \overline{z} = (a + bi) + (a - bi) = 2a = 2\operatorname{Re}(z)$

(b) $z - \overline{z} = (a + bi) - (a - bi) = 2bi = 2i\operatorname{Im}(z)$

Solution to Activity 11

(a) $(2 + 3i)(2 - 3i) = 2^2 + 3^2 = 4 + 9 = 13$

(b) $(-1 - 2i)(-1 + 2i) = (-1)^2 + (-2)^2 = 1 + 4 = 5$

(c) $(5i) \times (-5i) = 5^2 = 25$

(d) $(-2) \times (-2) = 4$

Solution to Activity 12

(a) $\dfrac{1}{-2 + 3i} = \dfrac{-2 - 3i}{(-2 + 3i)(-2 - 3i)}$
$= \dfrac{-2 - 3i}{(-2)^2 + 3^2}$
$= \dfrac{-2 - 3i}{13}$

(b) $\dfrac{2i}{1 + i} = \dfrac{2i(1 - i)}{(1 + i)(1 - i)}$
$= \dfrac{2i - 2i^2}{1^2 + 1^2}$
$= \dfrac{2 + 2i}{2}$
$= 1 + i$

(c) To simplify this fraction, you could multiply the top and bottom of the fraction by the complex conjugate of the denominator, namely $-2i$. However, it's simpler to multiply the top and bottom of the fraction by $-i$, as follows.
$$\frac{11 - 8i}{2i} = \frac{(11 - 8i)(-i)}{2i(-i)}$$
$$= \frac{-11i + 8i^2}{-2i^2}$$
$$= \frac{-11i - 8}{2}$$
$$= \frac{-8 - 11i}{2}$$

(d) $\dfrac{1}{i} = \dfrac{-i}{i \times (-i)} = \dfrac{-i}{1} = -i$

(e) $\dfrac{1}{-i} = \dfrac{i}{-i \times i} = \dfrac{i}{1} = i$

(f) $\dfrac{4 + 7i}{-1 + 2i} = \dfrac{(4 + 7i)(-1 - 2i)}{(-1 + 2i)(-1 - 2i)}$

$= \dfrac{-4 - 8i - 7i - 14i^2}{(-1)^2 + 2^2}$

$= \dfrac{10 - 15i}{5}$

$= 2 - 3i$

(g) $\dfrac{8 + 3i}{1 + 3i} = \dfrac{(8 + 3i)(1 - 3i)}{(1 + 3i)(1 - 3i)}$

$= \dfrac{8 - 24i + 3i - 9i^2}{1^2 + 3^2}$

$= \dfrac{17 - 21i}{10}$

(h) $\dfrac{-2 + 5i}{-4 - i} = \dfrac{(-2 + 5i)(-4 + i)}{(-4 - i)(-4 + i)}$

$= \dfrac{8 - 2i - 20i + 5i^2}{(-4)^2 + (-1)^2}$

$= \dfrac{3 - 22i}{17}$

Solution to Activity 13
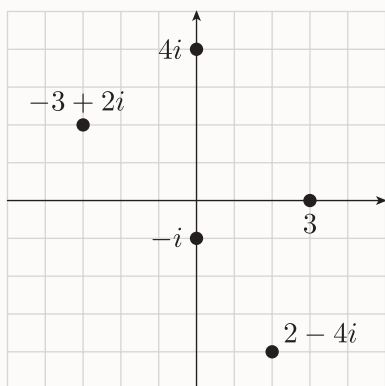
Solution to Activity 14

(The lines in the diagrams below are included to help you see how the answers are obtained. For example, in part (a) the parallelogram law is used to find $z + w$. Your solutions needn't include such lines.)

(a)
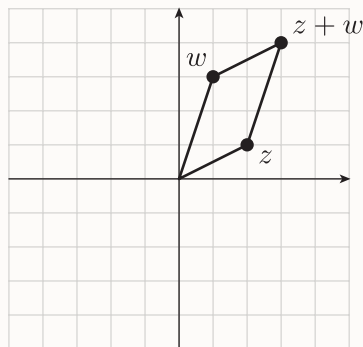
(b)
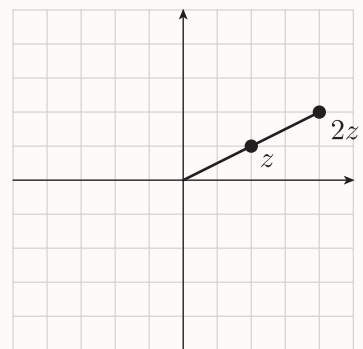
(c)
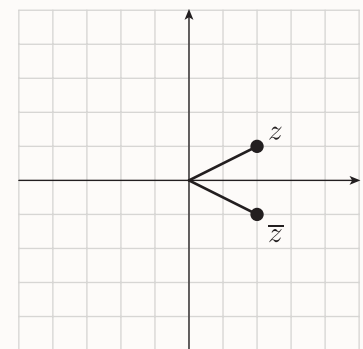
(d)
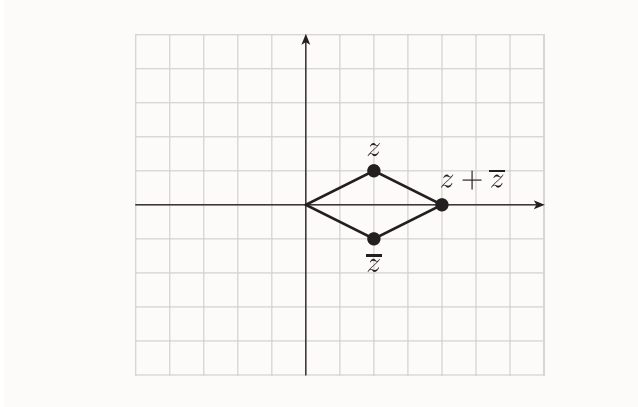
(e)
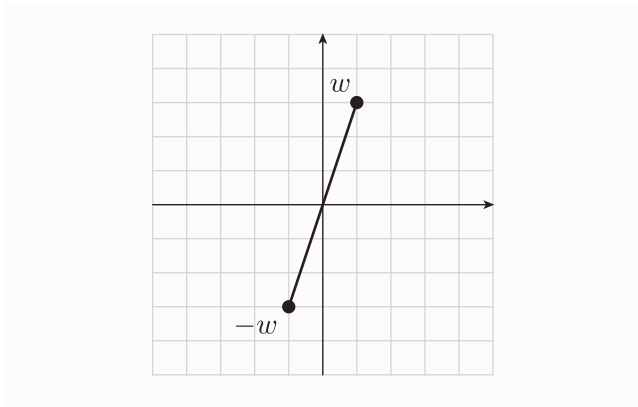
(f)
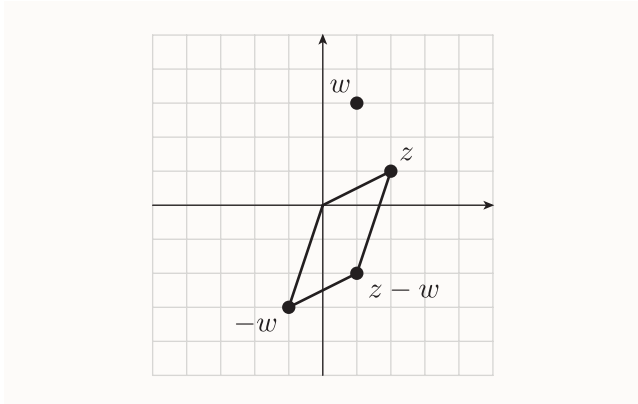
Solution to Activity 15

(a) $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

(b) $|-4 + 3i| = \sqrt{(-4)^2 + 3^2}$
$= \sqrt{16 + 9}$
$= \sqrt{25}$
$= 5$

(c) $|0| = 0$

(d) $|-31| = 31$

(e) $|17i| = \sqrt{17^2} = 17$

(f) $\left|7 - i\sqrt{15}\right| = \sqrt{7^2 + \left(-\sqrt{15}\right)^2}$
$= \sqrt{49 + 15}$
$= \sqrt{64}$
$= 8$

(g) $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

(h) $|19 + 19i| = \sqrt{19^2 + 19^2} = \sqrt{19^2 \times 2} = 19\sqrt{2}$

(i) $|-i| = \sqrt{(-1)^2} = 1$

(j) $\left|-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}$
$= \sqrt{\tfrac{1}{4} + \tfrac{3}{4}}$
$= \sqrt{1}$
$= 1$

Solution to Activity 16

Since $z = a + bi$, it follows that
$$-z = -a - bi$$
and
$$\overline{z} = a - bi.$$
Therefore $|z|$, $|-z|$ and $|\overline{z}|$ are all equal to $\sqrt{a^2 + b^2}$, so they are all equal.

Solution to Activity 17

(a) $\dfrac{1}{2 + i} = \dfrac{\overline{2 + i}}{|2 + i|^2} = \dfrac{2 - i}{2^2 + 1^2} = \dfrac{2 - i}{5}$

(b) $\dfrac{1}{-1 - 3i} = \dfrac{\overline{-1 - 3i}}{|-1 - 3i|^2}$
$= \dfrac{-1 + 3i}{(-1)^2 + (-3)^2}$
$= \dfrac{-1 + 3i}{10}$

(c) You could use the formula
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$
to find $1/2i$, but it's easier to multiply the top and bottom of the fraction $1/2i$ by i to give
$$\frac{1}{2i} = \frac{1}{2i} \times \frac{i}{i} = \frac{i}{2i^2} = -\frac{i}{2}.$$

Solution to Activity 18

(a)
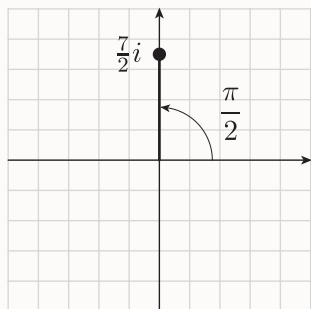
From the diagram, $\mathrm{Arg}(7i/2) = \pi/2$.

(b)
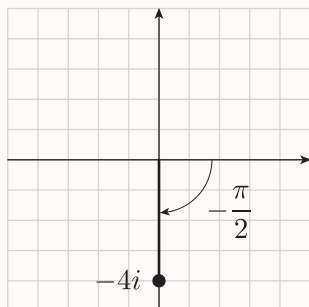
From the diagram, $\mathrm{Arg}(-4i) = -\pi/2$.

(c)
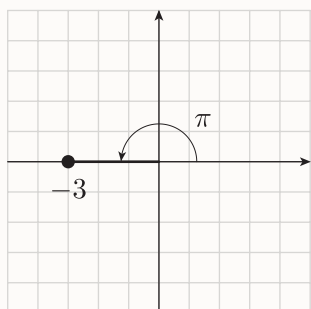
From the diagram, $\mathrm{Arg}(-3) = \pi$.

(d)
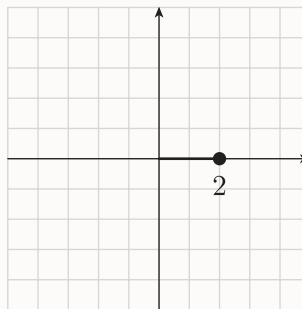
From the diagram, $\mathrm{Arg}(2) = 0$.

Solution to Activity 19

(a)
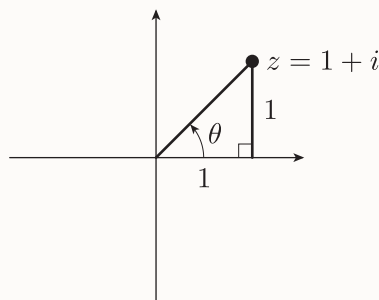
From the diagram,
$$\tan \theta = \frac{1}{1} = 1.$$
So the principal argument is $\theta = \pi/4$.

Alternatively, you may see immediately from the diagram that the principal argument is $\pi/4$, because it's half a right angle ($\pi/2$ radians).

(b)
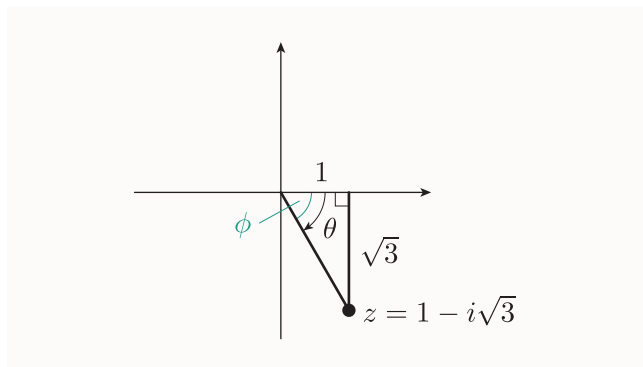
From the diagram,
$$\tan \phi = \frac{\sqrt{3}}{1} = \sqrt{3}.$$
Therefore $\phi = \pi/3$. So the principal argument is
$$\theta = -\phi = -\frac{\pi}{3}.$$

(c)
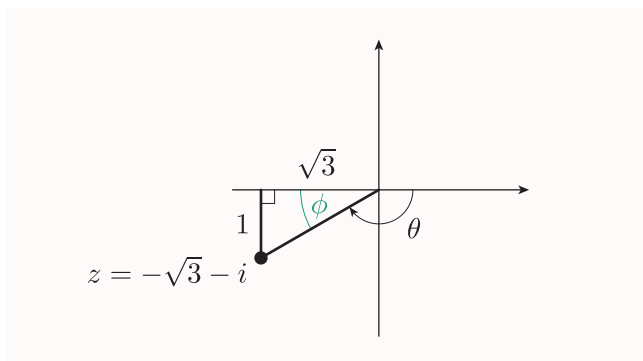
From the diagram,
$$\tan \phi = \frac{1}{\sqrt{3}}.$$
Therefore $\phi = \pi/6$. So the principal argument is
$$\theta = -(\pi - \phi) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}.$$

(d)
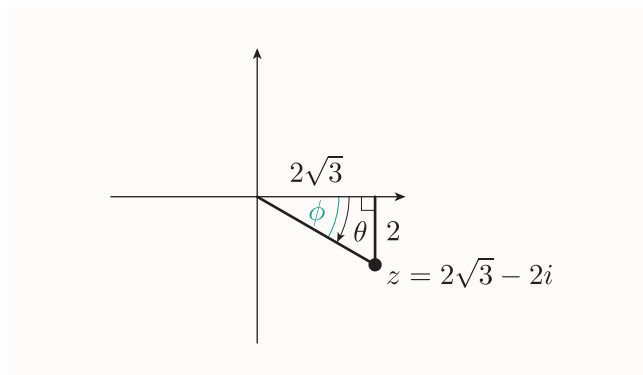
From the diagram,
$$\tan \phi = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}.$$
Therefore $\phi = \pi/6$. So the principal argument is
$$\theta = -\phi = -\frac{\pi}{6}.$$

Solution to Activity 20

(a) $3(\cos 0 + i \sin 0) = 3(1 + 0) = 3$

(b) $7\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 7(0 + i) = 7i$

(c) $6(\cos \pi + i \sin \pi) = 6(-1 + 0) = -6$

(d) $\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 - i = -i$

(e) $5(\cos(-\pi) + i\sin(-\pi)) = 5(-1 + 0) = -5$

(f) $4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$
$$= 2 + 2\sqrt{3}\, i$$

(g) $2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$
$$= \sqrt{2} - i\sqrt{2}$$

(h) $\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$
$$= \sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$
$$= -\frac{3}{2} + i\frac{\sqrt{3}}{2}$$

Solution to Activity 21

(a) The modulus is $7/2$. From Activity 18(a), the principal argument is $\pi/2$. Therefore
$$\frac{7}{2}i = \frac{7}{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).$$

(b) The modulus is 4. From Activity 18(b), the principal argument is $-\pi/2$. Therefore
$$-4i = 4\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right).$$

(c) The modulus is 3. From Activity 18(c), the principal argument is π. Therefore
$$-3 = 3(\cos \pi + i \sin \pi).$$

(d) The modulus is 2. From Activity 18(d), the principal argument is 0. Therefore
$$2 = 2(\cos 0 + i \sin 0).$$

(e) The modulus is
$$r = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}.$$
From Activity 19(a), the principal argument is $\pi/4$. Therefore
$$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

(f) The modulus is
$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$$
From Activity 19(b), the principal argument is $-\pi/3$. Therefore
$$1 - i\sqrt{3} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right).$$

(g) The modulus is
$$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2.$$
From Activity 19(c), the principal argument is $-5\pi/6$. Therefore
$$-\sqrt{3} - i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right).$$

(h) The modulus is
$$r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = \sqrt{16} = 4.$$
From Activity 19(d), the principal argument is $-\pi/6$. Therefore
$$2\sqrt{3} - 2i = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$$

Solution to Activity 22

(a) $zw = 48\left(\cos\left(\frac{\pi}{5} + \frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{5} + \frac{\pi}{10}\right)\right)$
$$= 48\left(\cos\left(\frac{3\pi}{10}\right) + i\sin\left(\frac{3\pi}{10}\right)\right)$$

(b) $zw = 40\left(\cos\left(\frac{7\pi}{8} + \frac{3\pi}{4}\right) + i\sin\left(\frac{7\pi}{8} + \frac{3\pi}{4}\right)\right)$
$$= 40\left(\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)\right)$$
The angle $13\pi/8$ lies outside the interval $(-\pi, \pi]$, so it isn't the principal argument of zw. The principal argument is given by
$$\frac{13\pi}{8} - 2\pi = \frac{13\pi}{8} - \frac{16\pi}{8} = -\frac{3\pi}{8}.$$
Therefore
$$zw = 40\left(\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)\right).$$

(c) $zw = \cos\left(\frac{8\pi}{9} + \frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9} + \frac{8\pi}{9}\right)$
$$= \cos\left(\frac{16\pi}{9}\right) + i\sin\left(\frac{16\pi}{9}\right)$$
The angle $16\pi/9$ lies outside the interval $(-\pi, \pi]$, so it isn't the principal argument of zw. The principal argument is given by
$$\frac{16\pi}{9} - 2\pi = \frac{16\pi}{9} - \frac{18\pi}{9} = -\frac{2\pi}{9}.$$
Therefore
$$zw = \cos\left(-\frac{2\pi}{9}\right) + i\sin\left(-\frac{2\pi}{9}\right).$$

Solution to Activity 23

The product uvw equals
$$8\left(\cos\left(\frac{3\pi}{7} + \frac{\pi}{7} + \frac{5\pi}{7}\right) + i\sin\left(\frac{3\pi}{7} + \frac{\pi}{7} + \frac{5\pi}{7}\right)\right)$$
$$= 8\left(\cos\left(\frac{9\pi}{7}\right) + i\sin\left(\frac{9\pi}{7}\right)\right)$$
$$= 8\left(\cos\left(-\frac{5\pi}{7}\right) + i\sin\left(-\frac{5\pi}{7}\right)\right).$$

Solution to Activity 24

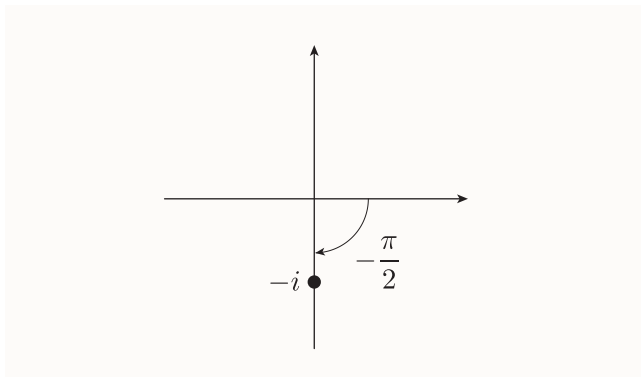(a) $\dfrac{z}{w} = 3\left(\cos\left(\frac{\pi}{5} - \frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{5} - \frac{\pi}{10}\right)\right)$
$$= 3\left(\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right)$$

(b) $\dfrac{z}{w} = 2\left(\cos\left(\frac{3\pi}{8} - \frac{5\pi}{8}\right) + i\sin\left(\frac{3\pi}{8} - \frac{5\pi}{8}\right)\right)$
$$= 2\left(\cos\left(-\frac{2\pi}{8}\right) + i\sin\left(-\frac{2\pi}{8}\right)\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

(c) $\dfrac{z}{w} = \cos\left(\frac{8\pi}{9} - \frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9} - \frac{8\pi}{9}\right)$
$$= \cos 0 + i\sin 0$$
In this case, the Cartesian form of z/w is 1 (as you would expect since $z = w$), which is simpler than the polar form.
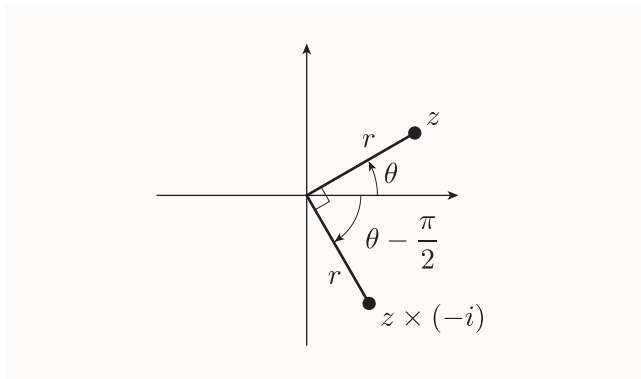
Solution to Activity 25

From the diagram,
$$-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right).$$
Let $z = r(\cos\theta + i\sin\theta)$. Then
$$z \times (-i)$$
$$= r(\cos\theta + i\sin\theta) \times \left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$
$$= r\left(\cos\left(\theta - \frac{\pi}{2}\right) + i\sin\left(\theta - \frac{\pi}{2}\right)\right).$$

Therefore multiplying z by $-i$ corresponds to a clockwise rotation through a quarter turn $(-\pi/2$ radians$)$.

Solution to Activity 26

(a) The modulus of $1 + i$ is
$$\sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}.$$
The principal argument of $1 + i$ was found to be $\pi/4$ in Activity 19(a). Therefore
$$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

Hence
$$(1 + i)^3 = (\sqrt{2})^3\left(\cos\left(3 \times \frac{\pi}{4}\right) + i\sin\left(3 \times \frac{\pi}{4}\right)\right)$$
$$= 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$
$$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$
$$= -2 + 2i.$$

(b) $\left(2^{1/3}\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right)\right)^{12}$
$$= 2^{12/3}\left(\cos\left(\frac{12\pi}{9}\right) + i\sin\left(\frac{12\pi}{9}\right)\right)$$
$$= 2^4\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$
$$= 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= -8 - 8\sqrt{3}\,i.$$

(c) The modulus of $-1 + i\sqrt{3}$ is
$$\sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2.$$
The principal argument of $-1 + i\sqrt{3}$ was found to be $2\pi/3$ in Example 5 . Therefore
$$-1 + i\sqrt{3} = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right).$$
Hence
$$(-1 + i\sqrt{3})^7$$
$$= 2^7\left(\cos\left(7 \times \frac{2\pi}{3}\right) + i\sin\left(7 \times \frac{2\pi}{3}\right)\right)$$
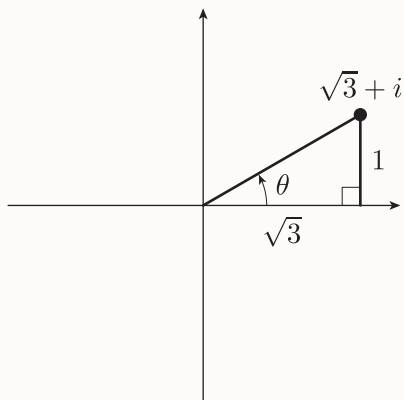$$= 128\left(\cos\left(\frac{14\pi}{3}\right) + i\sin\left(\frac{14\pi}{3}\right)\right).$$
Since
$$\frac{14\pi}{3} = 4\pi + \frac{2\pi}{3},$$
we obtain
$$(-1 + i\sqrt{3})^7 = 128\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$
$$= 128\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= 64(-1 + i\sqrt{3}).$$

(d) The modulus of $\sqrt{3} + i$ is

$$\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2.$$

From the diagram,

$$\tan\theta = \frac{1}{\sqrt{3}}.$$
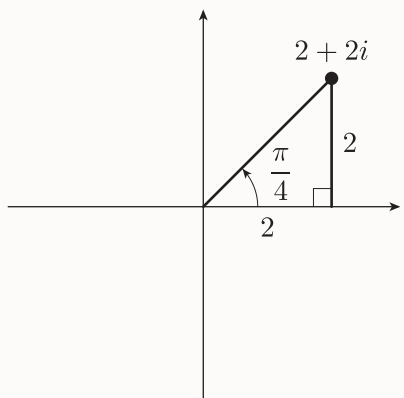
Therefore the principal argument is $\theta = \pi/6$. So

$$\sqrt{3} + i = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

Therefore

$$(\sqrt{3}+i)^{-6}$$
$$= 2^{-6}\left(\cos\left(-6\times\frac{\pi}{6}\right) + i\sin\left(-6\times\frac{\pi}{6}\right)\right)$$
$$= \frac{1}{64}(\cos(-\pi) + i\sin(-\pi))$$
$$= \frac{1}{64}(-1 + 0)$$
$$= -\frac{1}{64}.$$

(e) The modulus of $2 + 2i$ is

$$\sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}.$$

From the diagram, the principal argument is $\pi/4$. Therefore

$$2 + 2i = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

Hence

$$(2+2i)^{-5}$$
$$= (2\sqrt{2})^{-5}\left(\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right).$$

Since

$$-\frac{5\pi}{4} + 2\pi = \frac{3\pi}{4},$$

we obtain

$$(2+2i)^{-5}$$
$$= (2\sqrt{2})^{-5}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$
$$= \frac{1}{(2\sqrt{2})^5}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$
$$= \frac{1}{256}(-1 + i).$$

Solution to Activity 28

(a) Completing the square on the left-hand side gives

$$(z-1)^2 + 1 = 0;$$

that is,

$$(z-1)^2 = -1.$$

Taking square roots of both sides gives

$$z - 1 = \pm i, \quad \text{so} \quad z = 1 \pm i.$$

(b) Completing the square on the left-hand side gives

$$(z+2)^2 + 9 = 0;$$

that is,

$$(z+2)^2 = -9.$$

Taking square roots of both sides gives

$$z + 2 = \pm 3i, \quad \text{so} \quad z = -2 \pm 3i.$$

(c) Subtracting 25 from both sides gives

$$z^2 = -25.$$

Taking square roots of both sides gives

$$z = \pm 5i.$$

Solution to Activity 29

(a)
$$z = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 2}}{2 \times 1}$$
$$= \frac{-2 \pm \sqrt{-4}}{2}$$
$$= \frac{-2 \pm 2i}{2}$$
$$= -1 \pm i$$

(b) You could solve this quadratic equation by using the quadratic formula, but it's easier to solve it by factorising, as follows:
$$z^2 + 6z + 9 = 0$$
$$(z + 3)^2 = 0$$
$$z + 3 = 0$$
$$z = -3.$$

(c) You could solve this quadratic equation by using the quadratic formula, but it's easier to solve it by using a simple rearrangement.

Subtract 5 from both sides of the equation to give
$$3z^2 = -5; \quad \text{that is} \quad z^2 = -\frac{5}{3}.$$
Therefore
$$z = \pm i\sqrt{\frac{5}{3}} = \pm i\frac{\sqrt{5}}{\sqrt{3}}.$$
Rationalising the denominator gives
$$z = \pm i\frac{\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \pm i\frac{\sqrt{15}}{3}.$$

(d)
$$z = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 8}}{2 \times 1}$$
$$= \frac{4 \pm \sqrt{-16}}{2}$$
$$= \frac{4 \pm 4i}{2}$$
$$= 2 \pm 2i$$

(e) You could solve this quadratic equation by using the quadratic formula, but it's easier to solve it by factorising, as follows:
$$z^2 + 3z = 0$$
$$z(z + 3) = 0$$
$$z = 0 \quad \text{or} \quad z + 3 = 0$$
$$z = 0 \quad \text{or} \quad z = -3.$$

(f)
$$z = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 2 \times 5}}{2 \times 2}$$
$$= \frac{3 \pm \sqrt{-31}}{4}$$
$$= \frac{3 \pm i\sqrt{31}}{4}$$

Solution to Activity 30

(a) A suitable quadratic equation is
$$(z - (1 + 2i))(z - (1 - 2i)) = 0.$$
Simplifying gives
$$((z - 1) - 2i)((z - 1) + 2i) = 0$$
$$(z - 1)^2 - (2i)^2 = 0$$
$$z^2 - 2z + 1 + 4 = 0$$
$$z^2 - 2z + 5 = 0.$$

(b) A suitable quadratic equation is
$$(z - (-3 + 4i))(z - (-3 - 4i)) = 0.$$
Simplifying gives
$$((z + 3) - 4i)((z + 3) + 4i) = 0$$
$$(z + 3)^2 - (4i)^2 = 0$$
$$z^2 + 6z + 9 + 16 = 0$$
$$z^2 + 6z + 25 = 0.$$

(c) A suitable quadratic equation is
$$(z - 7i)(z - (-7i)) = 0.$$
Simplifying gives
$$z^2 - (7i)^2 = 0$$
$$z^2 + 49 = 0.$$

(d) A suitable quadratic equation is
$$(z - (1 + \tfrac{1}{2}i))(z - (1 - \tfrac{1}{2}i)) = 0.$$
Simplifying gives
$$((z - 1) - \tfrac{1}{2}i)((z - 1) + \tfrac{1}{2}i) = 0$$
$$(z - 1)^2 - (\tfrac{1}{2}i)^2 = 0$$
$$z^2 - 2z + 1 + \tfrac{1}{4} = 0$$
$$z^2 - 2z + \tfrac{5}{4} = 0$$
$$4z^2 - 8z + 5 = 0.$$

Solution to Activity 32

$$\left(-1 - i\sqrt{3}\right)^2 = 1 + 2i\sqrt{3} + i^2\left(\sqrt{3}\right)^2$$
$$= 1 + 2i\sqrt{3} - 3$$
$$= -2 + 2i\sqrt{3}.$$

Therefore

$$\left(-1 - i\sqrt{3}\right)^3 = \left(-1 - i\sqrt{3}\right)^2\left(-1 - i\sqrt{3}\right)$$
$$= \left(-2 + 2i\sqrt{3}\right)\left(-1 - i\sqrt{3}\right)$$
$$= 2 + 2i\sqrt{3} - 2i\sqrt{3} - 2i^2\left(\sqrt{3}\right)^2$$
$$= 2 + 6$$
$$= 8.$$

Hence

$$\left(\frac{-1 - i\sqrt{3}}{2}\right)^3 = \frac{\left(-1 - i\sqrt{3}\right)^3}{2^3} = \frac{8}{8} = 1.$$

(There's another, quicker solution to this activity. Let $w = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$, so that $\overline{w} = \frac{1}{2}\left(-1 - i\sqrt{3}\right)$. You've already seen that w is a cube root of unity, and now you're asked to show that $\overline{w}$ is also a cube root of unity. To do this, take the complex conjugate of each side of the equation

$$w^3 = 1$$

to give

$$\overline{w^3} = 1.$$

Since

$$\overline{w^3} = \overline{w \times w \times w} = \overline{w} \times \overline{w} \times \overline{w} = \overline{w}^3,$$

it follows that

$$\overline{w}^3 = 1.)$$

Solution to Activity 33

(a) Let $z = r(\cos\theta + i\sin\theta)$. Then
$$r^3(\cos 3\theta + i\sin 3\theta) = \cos 0 + i\sin 0.$$
Comparing moduli gives $r^3 = 1$, so $r = 1$.

Comparing arguments gives

$$3\theta = 0 + 2m\pi = 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{2m\pi}{3}, \quad \text{where } m \text{ is an integer.}$$
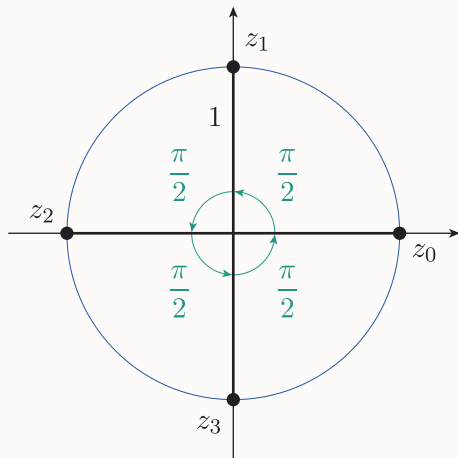
Taking $m = 0, 1, 2$ gives the solutions

$$z_0 = \cos 0 + i\sin 0 = 1$$

$$z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

All other values of m give repetitions of these three solutions.
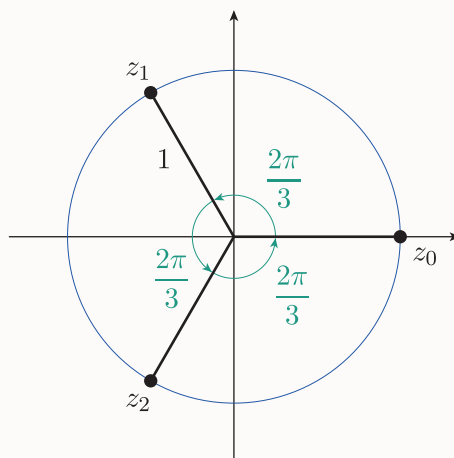
(b) Let $z = r(\cos\theta + i\sin\theta)$. Then

$$r^4(\cos 4\theta + i\sin 4\theta) = \cos 0 + i\sin 0.$$

Comparing moduli gives $r^4 = 1$, so $r = 1$.

Comparing arguments gives

$$4\theta = 0 + 2m\pi = 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{m\pi}{2}, \quad \text{where } m \text{ is an integer.}$$

Taking $m = 0, 1, 2, 3$ gives the solutions

$$z_0 = \cos 0 + i\sin 0 = 1$$
$$z_1 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$
$$z_2 = \cos\pi + i\sin\pi = -1$$
$$z_3 = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.$$

All other values of m give repetitions of these four solutions.

(c) Let $z = r(\cos\theta + i\sin\theta)$. Then

$$r^6(\cos 6\theta + i\sin 6\theta) = \cos 0 + i\sin 0.$$

Comparing moduli gives $r^6 = 1$, so $r = 1$.

Comparing arguments gives

$$6\theta = 0 + 2m\pi = 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{m\pi}{3}, \quad \text{where } m \text{ is an integer.}$$

Taking $m = 0, 1, 2, 3, 4, 5$ gives the solutions

$$z_0 = \cos 0 + i\sin 0 = 1$$
$$z_1 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$z_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$z_3 = \cos\pi + i\sin\pi = -1$$
$$z_4 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$
$$z_5 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

All other values of m give repetitions of these six solutions.
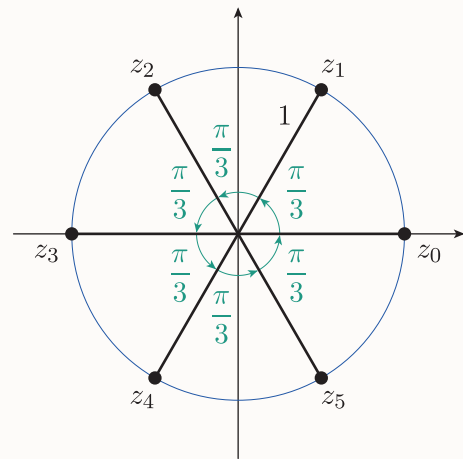
Solution to Activity 34

(a) A polar form of the complex number 64
is $64(\cos 0 + i \sin 0)$. Let $z = r(\cos\theta + i\sin\theta)$.
Then

$$r^6(\cos 6\theta + i\sin 6\theta) = 64(\cos 0 + i\sin 0).$$

Comparing moduli gives $r^6 = 64$, so $r = 2$.

Comparing arguments gives

$$6\theta = 0 + 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{m\pi}{3}, \quad \text{where } m \text{ is an integer.}$$

Taking $m = 0, 1, 2, 3, 4, 5$ gives the solutions

$$z_0 = 2(\cos 0 + i\sin 0) = 2$$

$$z_1 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 1 + i\sqrt{3}$$

$$z_2 = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -1 + i\sqrt{3}$$

$$z_3 = 2\left(\cos\pi + i\sin\pi\right) = -2$$

$$z_4 = 2\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -1 - i\sqrt{3}$$

$$z_5 = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = 1 - i\sqrt{3}.$$

All other values of m give repetitions of these
six solutions.
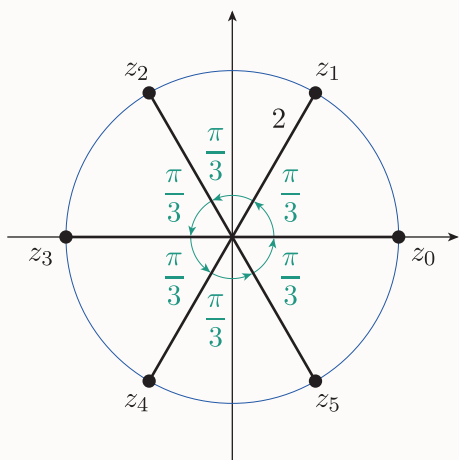
(b) A polar form of the complex number -8
is $8(\cos\pi + i\sin\pi)$. Let $z = r(\cos\theta + i\sin\theta)$.
Then

$$r^3(\cos 3\theta + i\sin 3\theta) = 8(\cos\pi + i\sin\pi).$$

Comparing moduli gives $r^3 = 8$, so $r = 2$.

Comparing arguments gives

$$3\theta = \pi + 2m\pi = (2m+1)\pi,$$

where m is an integer.

Hence

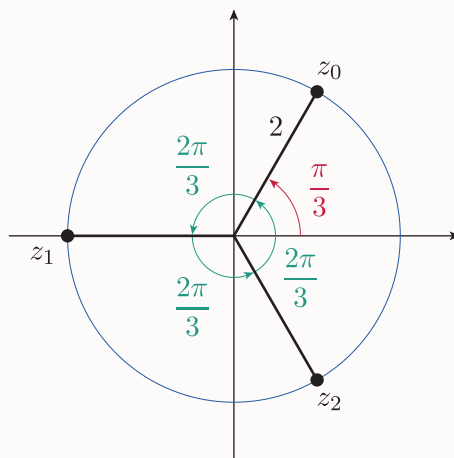$$\theta = \frac{(2m+1)\pi}{3}, \quad \text{where } m \text{ is an integer.}$$

Taking $m = 0, 1, 2$ gives the solutions

$$z_0 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 1 + i\sqrt{3}$$
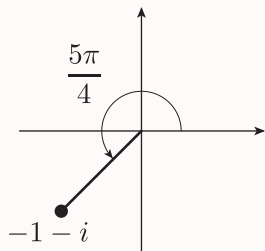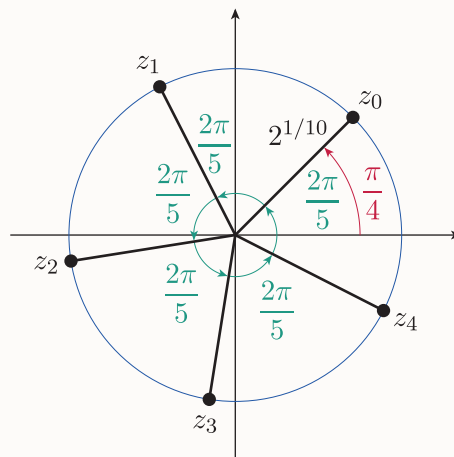
$$z_1 = 2(\cos\pi + i\sin\pi) = -2$$

$$z_2 = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) = 1 - i\sqrt{3}.$$

All other values of m give repetitions of these
three solutions.

(c) The modulus of $-1-i$ is
$$\sqrt{(-1)^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}.$$

From the diagram,
$$\text{Arg}(-1-i)=\frac{5\pi}{4}.$$
Therefore
$$-1-i=\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right).$$
Let $z=r(\cos\theta+i\sin\theta)$. Then
$$r^5(\cos 5\theta+i\sin 5\theta)$$
$$=\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right).$$
Comparing moduli gives $r^5=\sqrt{2}$, so $r=2^{1/10}$.

Comparing arguments gives
$$5\theta=5\pi/4+2m\pi,\quad\text{where }m\text{ is an integer.}$$
Hence
$$\theta=\frac{\pi}{4}+\frac{2m\pi}{5},\quad\text{where }m\text{ is an integer.}$$
Taking $m=0,1,2,3,4$ gives the solutions
$$z_0=2^{1/10}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$$
$$z_1=2^{1/10}\left(\cos\left(\frac{13\pi}{20}\right)+i\sin\left(\frac{13\pi}{20}\right)\right)$$
$$z_2=2^{1/10}\left(\cos\left(\frac{21\pi}{20}\right)+i\sin\left(\frac{21\pi}{20}\right)\right)$$
$$z_3=2^{1/10}\left(\cos\left(\frac{29\pi}{20}\right)+i\sin\left(\frac{29\pi}{20}\right)\right)$$
$$z_4=2^{1/10}\left(\cos\left(\frac{37\pi}{20}\right)+i\sin\left(\frac{37\pi}{20}\right)\right).$$
All other values of m give repetitions of these five solutions.

Solution to Activity 36

(a) $3e^{i0}=3(\cos 0+i\sin 0)$
$$=3(1+0)$$
$$=3$$

(b) $7e^{i\pi/2}=7\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$
$$=7(0+i)$$
$$=7i$$

(c) $6e^{i\pi}=6(\cos\pi+i\sin\pi)$
$$=6(-1+0)$$
$$=-6$$

(d) $e^{-i\pi/2}=\cos(-\pi/2)+i\sin(-\pi/2)$
$$=0-i$$
$$=-i$$

(e) $5e^{-i\pi}=5(\cos(-\pi)+i\sin(-\pi))$
$$=5(-1+0)$$
$$=-5$$

(f) $4e^{i\pi/3}=4\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$
$$=4\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$=2+2\sqrt{3}\,i$$

(g) $2e^{-i\pi/4}=2\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$
$$=2\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$
$$=\sqrt{2}-i\sqrt{2}$$

(h) $\sqrt{3}e^{5\pi i/6} = \sqrt{3}\left(\cos\left(\dfrac{5\pi}{6}\right) + i\sin\left(\dfrac{5\pi}{6}\right)\right)$

$= \sqrt{3}\left(-\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)$

$= -\dfrac{3}{2} + i\dfrac{\sqrt{3}}{2}$

Solution to Activity 37

(a) From Activity 21(a), the modulus is $7/2$ and the principal argument is $\pi/2$. Therefore
$$\frac{7}{2}i = \frac{7}{2}e^{i\pi/2}.$$

(b) From Activity 21(b), the modulus is 4 and the principal argument is $-\pi/2$. Therefore
$$-4i = 4e^{-i\pi/2}.$$

(c) From Activity 21(c), the modulus is 3 and the principal argument is π. Therefore
$$-3 = 3e^{i\pi}.$$

(d) From Activity 21(d), the modulus is 2 and the principal argument is 0. Therefore
$$2 = 2e^{i0}.$$

(e) From Activity 21(e), the modulus is $\sqrt{2}$ and the principal argument is $\pi/4$. Therefore
$$1 + i = \sqrt{2}e^{i\pi/4}.$$

(f) From Activity 21(f), the modulus is 2 and the principal argument is $-\pi/3$. Therefore
$$1 - i\sqrt{3} = 2e^{-i\pi/3}.$$

(g) From Activity 21(g), the modulus is 2 and the principal argument is $-5\pi/6$. Therefore
$$-\sqrt{3} - i = 2e^{-5\pi i/6}.$$

(h) From Activity 21(h), the modulus is 4 and the principal argument is $-\pi/6$. Therefore
$$2\sqrt{3} - 2i = 4e^{-i\pi/6}.$$

Solution to Activity 38
Because
$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$s(\cos\phi + i\sin\phi) = se^{i\phi}$$
$$\frac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi)) = \frac{r}{s}e^{i(\theta-\phi)},$$
the formula for the quotient of two complex numbers in polar form can be written as
$$\frac{re^{i\theta}}{se^{i\phi}} = \frac{r}{s}e^{i(\theta-\phi)}.$$

Solution to Activity 39
We use the formulas in the boxes above the activity, and the formula from Activity 38. That is, we use the usual index laws.

(a) $e^{i\pi/8} \times e^{3\pi i/8} = e^{i(\pi/8 + 3\pi/8)}$
$$= e^{i(4\pi/8)}$$
$$= e^{i\pi/2}$$

(b) $\left(e^{i\pi/8}\right)^4 = e^{i(4\times\pi/8)} = e^{i\pi/2}$

(c) $\dfrac{e^{3\pi i/8}}{e^{\pi i/8}} = e^{i(3\pi/8 - \pi/8)}$
$$= e^{i(2\pi/8)}$$
$$= e^{i\pi/4}$$

(d) $\dfrac{e^{i\pi/8}}{e^{3\pi i/8}} = e^{i(\pi/8 - 3\pi/8)}$
$$= e^{i(-2\pi/8)}$$
$$= e^{-i\pi/4}$$

Solution to Activity 40
Euler's formula gives
$$e^{i(\theta+2\pi)} = \cos(\theta + 2\pi) + i\sin(\theta + 2\pi).$$
Since
$$\cos(\theta + 2\pi) = \cos\theta \quad\text{and}\quad \sin(\theta + 2\pi) = \sin\theta,$$
it follows that
$$e^{i(\theta+2\pi)} = \cos\theta + i\sin\theta = e^{i\theta}.$$

Solution to Activity 41

For brevity, write $c = \cos\theta$ and $s = \sin\theta$. Then, by de Moivre's formula and the binomial theorem,

$$
\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\
&= (c + is)^4 \\
&= c^4 + 4c^3(is) + 6c^2(is)^2 \\
&\quad + 4c(is)^3 + (is)^4 \\
&= c^4 + 4ic^3 s - 6c^2 s^2 - 4ics^3 + s^4 \\
&= (c^4 - 6c^2 s^2 + s^4) + 4ics(c^2 - s^2).
\end{aligned}
$$

Comparing real and imaginary parts, and using the identity $c^2 + s^2 = 1$, gives

$$
\begin{aligned}
\cos 4\theta &= c^4 - 6c^2 s^2 + s^4 \\
&= c^4 - 6c^2(1 - c^2) + (1 - c^2)(1 - c^2) \\
&= c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4 \\
&= 8c^4 - 8c^2 + 1 \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}
$$

and

$$
\begin{aligned}
\sin 4\theta &= 4sc(c^2 - s^2) \\
&= 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta),
\end{aligned}
$$

which are the required identities.

(Another way to obtain the identities in this activity is to apply the double-angle identities

$$
\sin 2\theta = 2\sin\theta\cos\theta \quad\text{and}\quad \cos 2\theta = \cos^2\theta - \sin^2\theta
$$

repeatedly. For example,

$$
\begin{aligned}
\sin 4\theta &= \sin(2(2\theta)) \\
&= 2\sin 2\theta\cos 2\theta \\
&= 2(2\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta) \\
&= 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta).)
\end{aligned}
$$

Solution to Activity 42

(a) By the given formula for $\sin\theta$ and the binomial theorem,

$$
\begin{aligned}
\sin^3\theta &= \frac{1}{(2i)^3}\left(e^{i\theta} - e^{-i\theta}\right)^3 \\
&= \frac{1}{-8i}\left((e^{i\theta})^3 + 3(e^{i\theta})^2(-e^{-i\theta})\right. \\
&\qquad\left. + 3e^{i\theta}(-e^{-i\theta})^2 + (-e^{-i\theta})^3\right) \\
&= \frac{1}{-8i}\left(e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} - e^{-3i\theta}\right) \\
&= -\frac{1}{8i}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right) \\
&= \frac{1}{8i}\left(3(e^{i\theta} - e^{-i\theta}) - (e^{3i\theta} - e^{-3i\theta})\right) \\
&= \frac{1}{4}\left(3\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) - \frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right) \\
&= \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right).
\end{aligned}
$$

(b) By the given formula for $\cos\theta$ and the binomial theorem,

$$
\begin{aligned}
\cos^4\theta &= \frac{1}{2^4}\left(e^{i\theta} + e^{-i\theta}\right)^4 \\
&= \frac{1}{16}\left((e^{i\theta})^4 + 4(e^{i\theta})^3 e^{-i\theta} + 6(e^{i\theta})^2(e^{-i\theta})^2\right. \\
&\qquad\left. + 4e^{i\theta}(e^{-i\theta})^3 + (e^{-i\theta})^4\right) \\
&= \frac{1}{16}\left(e^{4i\theta} + 4e^{3i\theta}e^{-i\theta} + 6e^{2i\theta}e^{-2i\theta}\right. \\
&\qquad\left. + 4e^{i\theta}e^{-3i\theta} + e^{-4i\theta}\right) \\
&= \frac{1}{16}\left(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right) \\
&= \frac{1}{16}\left((e^{4i\theta} + e^{-4i\theta}) + 4(e^{2i\theta} + e^{-2i\theta}) + 6\right) \\
&= \frac{1}{8}\left(\frac{e^{4i\theta} + e^{-4i\theta}}{2} + 4\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + 3\right) \\
&= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).
\end{aligned}
$$

Acknowledgements