Unit 11

Normal modes

Introduction

In Unit 9 you studied the simple harmonic motion of oscillating mechanical systems, in particular those involving model springs. A typical example of such a system is shown in Figure 1, where an object modelled as a particle is connected by a model spring (of natural length l_0 and stiffness k) to a fixed wall; the particle is constrained to move in a straight line on a frictionless horizontal surface. In Unit 9 the equation of motion for this system was found to be

$$m\ddot{x} + kx = kl_0.$$

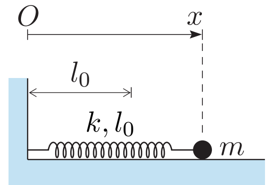


Figure 1 An oscillating system

The solution of this differential equation can be written in the form

$$x(t) = l_0 + A\cos(\omega t + \phi),$$

where x is the displacement from the wall, and ω $(= \sqrt{k/m})$, A and ϕ $(-\pi < \phi \leq \pi)$ are, respectively, the angular frequency, the amplitude and the phase angle of the oscillations executed by the particle.

Damping and forcing were incorporated in the simple harmonic motion model in Unit 10 to make it more realistic. This unit extends the basic model in a different way, to take account of another aspect of the motion of oscillating mechanical systems – the fact that usually more than one part of the system is free to move.

Damping and forcing are not considered in this unit.

A simple mechanical system that is typical of those considered in this unit, and its schematic representation, are shown in Figure 2. The diagram shows two particles connected by model springs to one another and to two fixed walls. Each particle is free to move in a straight line, but the motion of the system is not as straightforward as the simple harmonic motion considered above – unlike simple harmonic motion, it is, in general, not sinusoidal. But there are particular solutions of the equation of motion for such systems that *do* correspond to each part of the system oscillating backwards and forwards sinusoidally with the same frequency. These particular solutions are called *normal modes*. This unit is concerned with finding the normal modes of simple oscillating mechanical systems and showing how *any* motion of such a system can be built up from normal modes.

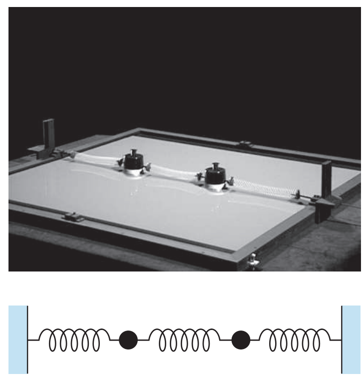


Figure 2 A two-particle system

Section 1 provides an overview of the unit and introduces many of the ideas that underpin it; notably, it contains several important concepts and definitions. Section 2 looks at systems that are confined to one dimension. Section 3 looks at a two-dimensional problem, modelling the behaviour of a guitar string.

1 Oscillations and normal modes

The important concepts of a normal mode and degrees of freedom are introduced in Subsection 1.1. Subsection 1.2 goes on to show how normal modes constitute the building blocks for modelling the motion of an oscillating mechanical system. Subsection 1.3 then explores how certain eigenvectors of a matrix can be used to determine the initial conditions for the normal mode motion of such a system.

1.1 What is a normal mode?

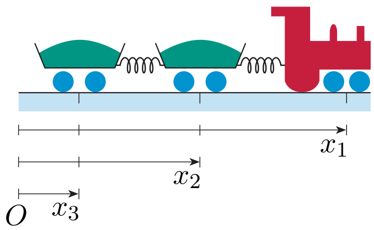


Figure 3 A railway engine and two trucks

In a real mechanical system, usually more than one part of the system is free to move. For example, in the system shown in Figure 3, a railway engine and two trucks are connected by stiff springs. As the engine starts to move along the track, so do the trucks. However, because of the springs, the motion of the trucks is different from that of the engine – the position of each truck oscillates backwards and forwards relative to the position of the engine. The system can be modelled by three particles connected by two model springs. Since the relative positions of the particles representing the trucks and engine can change, we need to specify not one but three coordinates to describe the configuration of the system – one per particle (as shown in Figure 3).

> The number of **degrees of freedom** of a system is the smallest number of coordinates needed to describe its configuration (i.e. the positions of the constituent parts of the system) at any instant in time.

The positions of the constituent parts of the system illustrated in Figure 3 – that is, the engine and the two trucks (or the particles that model them) – can be specified by giving their displacements from a fixed origin O, as shown in the figure. But there are many other ways of specifying the configuration: for example, we could use a different fixed point for the origin; or we could measure the position of each truck relative to the changing position of the engine, which, in turn, would be determined by reference to a fixed point; and so on. The important thing to note is that, in each case, three pieces of information are required to specify the configuration of the system completely – fewer would leave some element unspecified, more would be superfluous. It is for this reason that we say that this system has three degrees of freedom.

In general, any convenient fixed point can be chosen as an origin. Often, for simplicity, an equilibrium position of a system is taken to be the origin.

Exercise 1

State the number of degrees of freedom of each of the mechanical systems shown in Figure 4. Assume that the railway track is straight and flat, that the engine and trucks are modelled as particles, and that the springs are modelled as model springs.

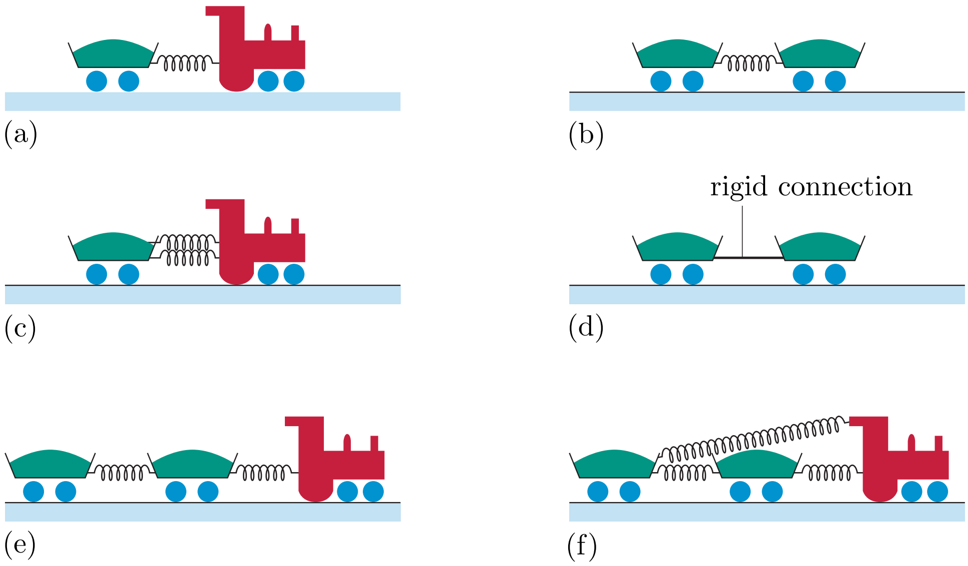


Figure 4 How many degrees of freedom?

The concept of degrees of freedom relates to other situations besides trains travelling on straight railway tracks. It applies to systems ranging from pendulums to the motion of individual molecules. However, determining the number of degrees of freedom of a system is not always as straightforward as in the case of the train considered above – more detailed analysis of the system is often necessary. For example, consider the simple pendulum shown in Figure 5. This is a mechanical system that is constrained to move in two dimensions, that is, in a vertical plane, so we could specify the position of the pendulum bob using a pair of Cartesian coordinates. You might therefore expect that the system would have two degrees of freedom. But the pendulum bob is constrained to move along a circular path in the vertical plane, so only one coordinate, the angle θ, is needed to specify its position. Hence the system has one degree of freedom. To show this, and to help you to make sense of what follows, the equation of motion for small oscillations of a simple pendulum is derived and solved.

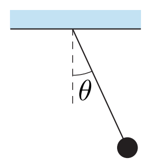


Figure 5 A simple pendulum

For the simple pendulum of length l, moving in a vertical plane, let the datum for measuring potential energy be the point O at the top of the light rod. Consider the mechanical energy of the bob of mass m. Its potential energy is $U(\theta) = -mgl\cos\theta$, where θ is measured in radians and the minus sign indicates that the bob is below the datum. The length of the pendulum is constant, and the distance s of the bob along the arc of a circle can be measured as $s = l\theta$, where $s = 0$ when the rod is vertical.

Newton's second law will be used to derive the equations of motion for a *double* pendulum in Subsection 1.2.

Differentiating this equation to get the velocity of the bob along the path gives $\dot{s} = l\dot{\theta}$. Hence the kinetic energy of the bob is given by $T(\theta) = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}ml^2\dot{\theta}^2$. By the conservation of mechanical energy, the total energy of the bob is

$$E(\theta) = T(\theta) + U(\theta) = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \text{constant}.$$

Note that using the chain rule,
$$\frac{d}{dt}(\cos\theta) = \frac{d}{d\theta}(\cos\theta)\,\frac{d\theta}{dt}.$$

Differentiating this equation with respect to t gives

$$\tfrac{1}{2}ml^2\,\frac{d}{dt}\big(\dot{\theta}^2\big) - mgl\,\frac{d}{dt}(\cos\theta) = 0,$$

so

$$ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta = 0.$$

Assuming that $\dot{\theta} \neq 0$, this equation can be divided through by $ml^2\dot{\theta}$ to obtain

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

For example, when $\theta = \pi/6 = 0.523$, $\sin\theta = 0.5$, and the approximation is within 5% of the true value. For $-\pi/6 < \theta < \pi/6$, the approximation will be within 5%, and it will be much more accurate as $\theta \to 0$.

The small-angle approximation $\sin\theta \simeq \theta$, which is a good approximation when θ is small, gives rise to a simple model for the motion of the pendulum bob:

$$\ddot{\theta} + \frac{g}{l}\theta = 0.$$

This is the equation for simple harmonic motion that you first solved in Unit 1, and studied in Unit 9. Its general solution can be written as $\theta(t) = A\cos\omega t + B\sin\omega t$, where $\omega = \sqrt{g/l}$, and A and B are arbitrary constants. This can also be written as $\theta(t) = C\cos(\omega t + \phi)$, where C and ϕ are arbitrary constants. The small-angle approximation $\sin\theta \simeq \theta$ can be used to obtain an equation for x, the horizontal displacement of the bob, as

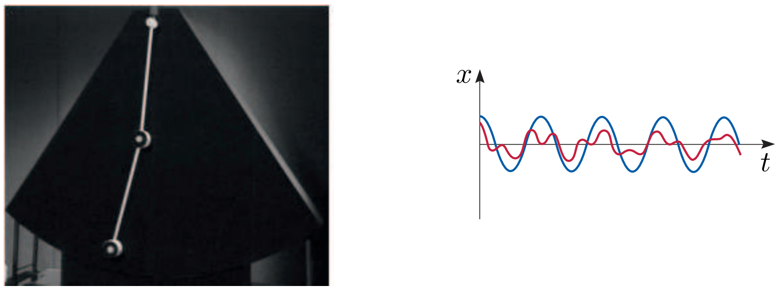
$$x = l\sin\theta \simeq l\theta = lC\cos(\omega t + \phi) = D\cos(\omega t + \phi),$$

where $D = lC$ is another arbitrary constant.

Since the motion of the pendulum can be described using one variable, it has one degree of freedom. Other systems with one degree of freedom include the spring systems that you studied in Units 9 and 10. This unit, however, concentrates on systems with more than one degree of freedom.

You could investigate the behaviour of a double pendulum at home, using two lengths of light string and two steel nuts, for example. Use one string to join the two nuts, and use the other string attached at one end to one nut and suspended from a fixed point at the other end. Hold the nuts out sideways, so that the system lies in a vertical plane, then release them simultaneously. Observe the subsequent motion of the nuts.

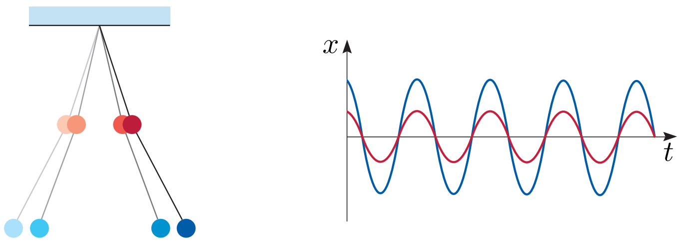
Degrees of freedom are important in the context of normal modes as they indicate the number of normal modes that a system has. Consider small oscillations for the planar motion of a double pendulum, illustrated in Figure 6(a). Two coordinates are needed to specify the configuration of this mechanical system, which therefore has two degrees of freedom. Two possible coordinates are the horizontal displacements of the two pendulum bobs: x_1 (for the upper pendulum bob) and x_2 (for the lower pendulum bob). The system usually moves in a complicated manner, as shown in Figure 6(b). This behaviour is very different from that of the mechanical systems with one degree of freedom studied in Unit 9, where the motion is always sinusoidal.



(a) (b)

Figure 6 (a) A double pendulum. (b) Motion of the upper bob (red) and lower bob (blue).

However, the motion is not always so complicated, and there are situations where both particles move sinusoidally with constant frequency. There are two possible simpler motions, the first of which is illustrated in Figure 7.



(a) (b)

The snapshots in Figure 7(a) include the most extreme points of the motion, when the pendulum bobs are both momentarily at rest.

Figure 7 (a) Snapshots, at intervals of one-sixth of a cycle, of the motion of a double pendulum system moving in a normal mode. (b) Traces of the corresponding sinusoidal motions of the pendulum bobs.

Comparing Figures 6(b) and 7(b), you can see that the motion is much more regular in the latter: the particles complete each cycle simultaneously, that is, the sinusoidal motion is of the same period (and so of the same frequency) for both. Consequently, the motion of the particles in Figure 7(b) can be modelled by a pair of equations of the form

Notice that each equation represents simple harmonic motion, as defined in Unit 9.

$$x_1(t) = A_1 \cos(\omega t + \phi), \quad x_2(t) = A_2 \cos(\omega t + \phi), \tag{1}$$

where $|A_1|$ and $|A_2|$ are the amplitudes of the motions of the particles, ω is the common angular frequency, and ϕ is the common phase angle. Such motion is known as a *normal mode* of the system.

For a normal mode, the coefficients A_1 and A_2 can be positive or negative.

A **normal mode** is a type of motion of a system of particles in which the coordinates of the particles all vary sinusoidally with the same frequency. The angular frequency of the sinusoidal motion is called the **normal mode angular frequency**.

In this context, *normal* means 'standard', not 'usual'. (In fact, normal modes are unusual in that most modes of motion are not normal modes.)

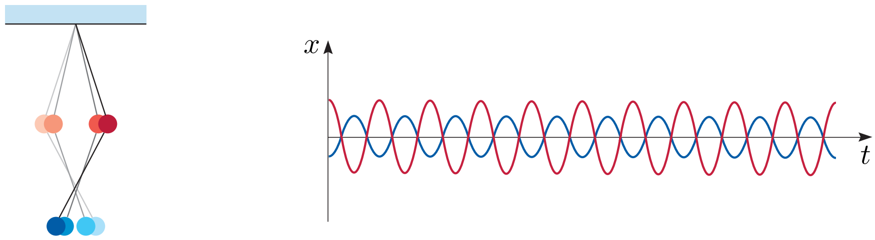
For each of the systems with *two* degrees of freedom, there are *two* normal modes. More generally, the number of distinct normal modes of a system corresponds to the number of degrees of freedom of the system (though we do not prove this here).

This correspondence occurs in *all* cases if we extend the definition of normal modes to include what is known as *rigid body motion.* Such motion is discussed in Subsection 2.2.

We turn now to another important aspect of motion, which is illustrated in Figure 7. You may have noticed that when one of the pendulum bobs moves from left to right, so does the other, and similarly when one moves from right to left, so does the other. This type of motion, in which two particles move in the same direction, is called **in-phase motion**. Not all normal mode motion is in-phase: it can also be **phase-opposed motion**, which occurs, as you might expect, when two particles move in opposite directions, as illustrated in Figure 8.

If you have set up the home experiment, you may wish to try to reproduce these two normal modes.

The snapshots in Figure 8(a) include the most extreme points of the motion, when the pendulum bobs are both momentarily at rest.



(a) (b)

Figure 8 (a) Snapshots, at intervals of one-sixth of a cycle, of the motion of a double pendulum system moving in a phase-opposed normal mode. (b) Traces of the corresponding sinusoidal motions of the pendulum bobs.

If you look at Figure 7(b) you will see that for the in-phase motion, the phase angles of the two sinusoids are the same. In contrast, you can see in Figure 8(b) that for the phase-opposed motion, the phase angles of the two sinusoids differ by π. This is true in general: the sinusoids representing normal mode motion have phase angles that either are the same (in which case the motion is in-phase), or differ by π (in which case the motion is phase-opposed). In both cases, however, equations of the form (1) can be used to represent the normal mode motion.

This is because $\cos(\omega t + \phi + \pi) = -\cos(\omega t + \phi)$, and the negative sign can be absorbed into the coefficient A_i (which can be negative or positive).

If two particles are moving in a normal mode with their motion aligned with a common axis, then there are only three situations that can occur: the particles move in-phase, or they are phase-opposed, or one or both particles are stationary. By extension, in a system with three or more particles, each pair of particles can be considered separately; thus for any particular pair, the particles can be in-phase, or phase-opposed, or stationary, relative to each other.

Exercise 2

For each of the sets of graphs in Figure 9 showing the motion of two or three particles, state whether or not the motion represented is normal mode motion.

If it is, state whether the motion of each pair of particles is in-phase or phase-opposed.

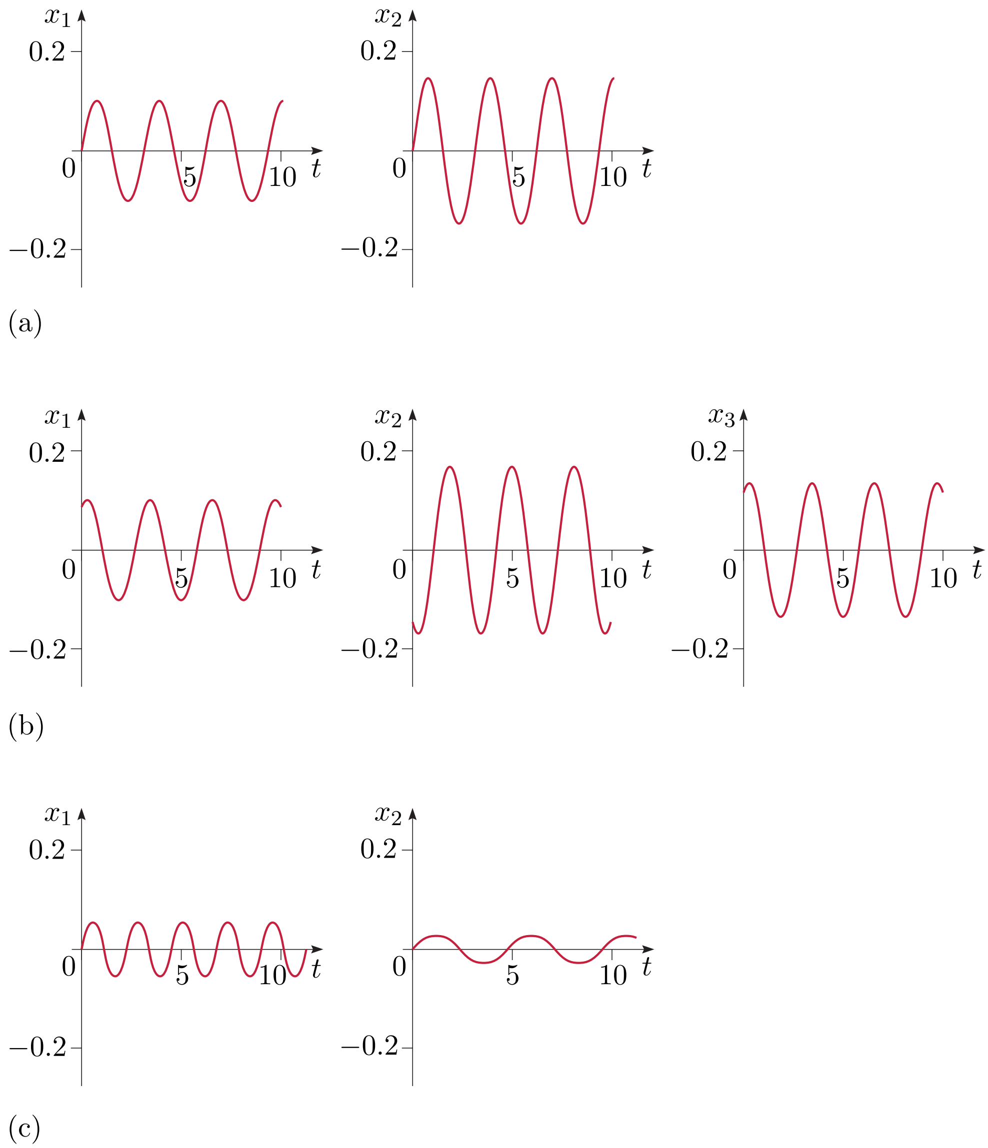


Figure 9 Motion of systems of two or three particles

1.2 Analysing oscillating mechanical systems

In this subsection you will see how to obtain the equations of normal mode motion for an oscillating mechanical system, and how these equations can be combined to model any motion of such a system. We begin by deriving the equations of motion for one of these systems, using techniques familiar from Units 2 and 9.

Example 1

Consider two particles of masses m_1 and m_2, joined to each other and to two fixed walls by three identical model springs of stiffness k and natural length l_0, as shown in Figure 10. The particles are constrained to move across a frictionless horizontal surface in the straight line joining the points of attachment (i.e. the motion is one-dimensional). To simplify the analysis, the distance between the fixed walls is assumed to be exactly three times the natural length of the springs; hence the equilibrium position of the system is that shown in Figure 10(a).

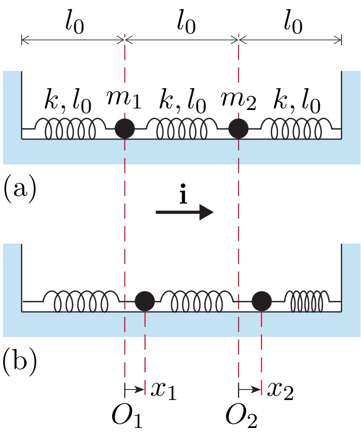


Figure 10 Two masses and three springs

Two coordinates are needed to specify the positions of the particles in the system; therefore the system has two degrees of freedom. The simplest option for the coordinates is to measure the displacement of each particle from its equilibrium position. These displacements are labelled x_1 and x_2 in Figure 10(b), and are measured from the origins O_1 and O_2, respectively.

What are the equations of motion for this oscillating system?

Solution

To obtain the equations of motion for this system, Newton's second law is applied to each of the particles separately. In order to do this, the first step is to draw two force diagrams – one for each particle (see Figure 11).

You have already met the idea (in Unit 2, in relation to particles in equilibrium) that when considering the forces acting on the particles in a system, each particle can be treated separately. This is the first time in the module that this principle has been applied to a system in motion.

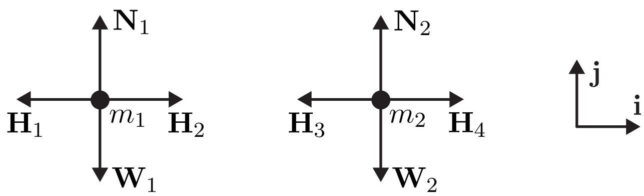


Figure 11 Force diagrams for the two particles

You have seen one-particle systems analysed in this way in Unit 9.

Here the weights of the particles are denoted by $\mathbf{W}_1$ and $\mathbf{W}_2$, the normal reactions of the surface on the particles by $\mathbf{N}_1$ and $\mathbf{N}_2$, and the forces exerted by the springs on the particles by $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$ and $\mathbf{H}_4$.

Now we apply Newton's second law to the first particle and obtain

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W}_1 + \mathbf{N}_1, \qquad (2)$$

where $\mathbf{r}_1$ is the position vector of the first particle with respect to its equilibrium position, that is, $\mathbf{r}_1 = x_1\mathbf{i}$, with $\mathbf{i}$ being a unit vector in the direction of positive x_1 and x_2. Similarly, for the second particle,

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{H}_3 + \mathbf{H}_4 + \mathbf{W}_2 + \mathbf{N}_2. \qquad (3)$$

The four forces due to the springs can be modelled using Hooke's law. First, recall from Unit 9 that according to Hooke's law, a model spring of natural length l_0 and stiffness k exerts a force on an object attached to one end of the spring, such that the force is given by $\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}$, where l is the length of the spring, and $\widehat{\mathbf{s}}$ is a unit vector directed from the point where the object is attached towards the centre of the spring.

For the force $\mathbf{H}_1$, which is the force exerted by the left-hand spring on the first particle, we have $l = l_0 + x_1$ and $\widehat{\mathbf{s}} = -\mathbf{i}$, so

$$\mathbf{H}_1 = k((l_0 + x_1) - l_0)(-\mathbf{i}) = -kx_1\mathbf{i}.$$

For the force $\mathbf{H}_2$, the distance between the two particles is given by $l = l_0 + x_2 - x_1$, and $\widehat{\mathbf{s}} = \mathbf{i}$, so

$$\mathbf{H}_2 = k((l_0 + x_2 - x_1) - l_0)\mathbf{i} = k(x_2 - x_1)\mathbf{i}.$$

For the force $\mathbf{H}_3$, we can apply Hooke's law again, or we can simply note that all the data are the same as for $\mathbf{H}_2$ (since we are considering the same spring) except that now $\widehat{\mathbf{s}} = -\mathbf{i}$, thus

Hooke's law tells us that the forces exerted at either end of a model spring are always equal in magnitude but opposite in direction. We frequently make use of this property, which is a consequence of Newton's third law. (Compare this property of model springs with that of model strings discussed in Unit 2.)

$$\mathbf{H}_3 = k(x_2 - x_1)(-\mathbf{i}).$$

For the force $\mathbf{H}_4$, we have $l = l_0 - x_2$ and $\widehat{\mathbf{s}} = \mathbf{i}$, so

$$\mathbf{H}_4 = k((l_0 - x_2) - l_0)\mathbf{i} = -kx_2\mathbf{i}.$$

The weights and normal reaction forces are vertical, so can be written as

$$\mathbf{W}_1 = -m_1 g\mathbf{j}, \quad \mathbf{W}_2 = -m_2 g\mathbf{j}, \quad \mathbf{N}_1 = |\mathbf{N}_1|\,\mathbf{j}, \quad \mathbf{N}_2 = |\mathbf{N}_2|\,\mathbf{j}.$$

Now that all the forces have been modelled, they can be substituted into equations (2) and (3), giving

$$m_1\ddot{\mathbf{r}}_1 = -kx_1\mathbf{i} + k(x_2 - x_1)\mathbf{i} - m_1 g\mathbf{j} + |\mathbf{N}_1|\,\mathbf{j}, \qquad (4)$$

$$m_2\ddot{\mathbf{r}}_2 = -k(x_2 - x_1)\mathbf{i} - kx_2\mathbf{i} - m_2 g\mathbf{j} + |\mathbf{N}_2|\,\mathbf{j}. \qquad (5)$$

But $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$, therefore $\ddot{\mathbf{r}}_1 = \ddot{x}_1\mathbf{i}$ and $\ddot{\mathbf{r}}_2 = \ddot{x}_2\mathbf{i}$. Substituting for $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$ in equations (4) and (5), resolving in the $\mathbf{i}$-direction and collecting terms gives

$$m_1\ddot{x}_1 = -2kx_1 + kx_2, \qquad (6)$$

$$m_2\ddot{x}_2 = kx_1 - 2kx_2. \qquad (7)$$

These differential equations are the equations of motion for the oscillating mechanical system depicted in Figure 10.

Equations (6) and (7) can be written in matrix form as

$$\ddot{\mathbf{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2k/m_1 & k/m_1 \\ k/m_2 & -2k/m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{A}\mathbf{x}. \qquad (8)$$

The equation $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is referred to as the (*matrix*) *equation of motion* for the mechanical system, and the matrix $\mathbf{A}$ is called the **dynamic matrix** of the system.

In Unit 6 you saw that provided that $\mathbf{A}$ has distinct real eigenvalues, the general solution of such an equation can be expressed as a linear combination of exponentials, sinusoids and linear terms. Since equation (8) models the motion of an oscillating mechanical system (without damping or forcing), we would expect only sinusoids in the general solution because sinusoids best represent this type of periodic motion.

See Procedure 6 in Unit 6.

Thus we would anticipate a general solution of the form

$$\mathbf{x}(t) = \mathbf{v}_1(D_1 \cos\omega_1 t + D_2 \sin\omega_1 t) + \mathbf{v}_2(D_3 \cos\omega_2 t + D_4 \sin\omega_2 t),$$

comprising four linearly independent terms, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to the eigenvalues λ_1 and λ_2 of $\mathbf{A}$, and $\omega_1 = \sqrt{-\lambda_1}$, $\omega_2 = \sqrt{-\lambda_2}$, while D_1, D_2, D_3 and D_4 are arbitrary constants that depend on the initial conditions. For present purposes, we rewrite this general solution in the form

You saw in Units 9 and 10 that $A\cos\omega t + B\sin\omega t$ can be rewritten in the form $C\cos(\omega t + \phi)$, where $A = C\cos\phi$ and $B = -C\sin\phi$.

$$\mathbf{x}(t) = C_1\mathbf{v}_1 \cos(\omega_1 t + \phi_1) + C_2\mathbf{v}_2 \cos(\omega_2 t + \phi_2), \tag{9}$$

where this form of the solution involves the four arbitrary constants C_1, C_2, ϕ_1 and ϕ_2.

One specific solution of equation (9), obtained by putting $C_1 = 1$ and $C_2 = 0$, is $\mathbf{x}(t) = \mathbf{v}_1 \cos(\omega_1 t + \phi_1)$, where $\mathbf{v}_1 = (v_{11} \quad v_{12})^T$. This can be written in the form

In general, any normal mode can be written in vector form as $\mathbf{x}(t) = \mathbf{v}\cos(\omega t + \phi)$, where $\mathbf{v}$ is a constant eigenvector, ω is the normal mode angular frequency, and ϕ is a constant scalar.

$$x_1(t) = v_{11} \cos(\omega_1 t + \phi_1), \quad x_2(t) = v_{12} \cos(\omega_1 t + \phi_1).$$

The interesting thing about this solution is that it represents a normal mode of the system in Example 1. To see this, look back at the general equations for normal mode motion (1), and replace A_1, A_2 and ϕ by v_{11}, v_{12} and ϕ_1, respectively.

Similarly, putting $C_1 = 0$ and $C_2 = 1$ in equation (9) gives the solution $\mathbf{x}(t) = \mathbf{v}_2 \cos(\omega_2 t + \phi_2)$, which represents a second normal mode of the system. Hence equation (9) tells us that the general solution of the equation of motion for the mechanical system in Example 1 can be expressed as a linear combination of normal modes.

This finding concerning the solution of the equation of motion is generally satisfied. This is because the motion of an oscillating mechanical system (without damping or forcing) can be modelled by a system of linear second-order differential equations, such as equation (8), whose coefficient matrix, that is, the dynamic matrix, has real non-positive eigenvalues, all of which usually are negative. If we put aside for the moment the case where some of the eigenvalues are zero, we have a dynamic matrix with negative eigenvalues. From Unit 6, this means that provided that these eigenvalues are distinct, the general solution can be written as a linear combination of sinusoids of the form $\mathbf{v}\cos(\omega t + \phi)$, that is, as a linear combination of normal modes. Furthermore, these normal modes will be linearly independent, since they correspond to linearly independent eigenvectors. Therefore we have the following important result.

We do not prove here that the eigenvalues must be non-positive.

The case where some eigenvalues are zero is considered in Subsection 2.2.

Theorem 1

The general solution of the equation of motion for an oscillating mechanical system (without damping or forcing) whose dynamic matrix has distinct non-zero eigenvalues can be written as a linear combination of linearly independent normal modes of the system.

In this case the number of linearly independent normal modes of a system is the same as the number of degrees of freedom of the system.

Theorem 1 indicates why normal modes are so important: not only are they especially simple oscillations themselves, but *every* motion can be expressed in terms of them.

Having established that the solution of the equation of motion obtained in Example 1 can be expressed as a linear combination of normal modes, we now use the techniques of Units 5 and 6 to find those normal modes, and hence the general solution of the equation of motion.

Solving equations of the form $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is dealt with in Section 4 of Unit 6, while finding eigenvalues and eigenvectors is covered in Unit 5.

Example 2

Suppose that in the mechanical system in Example 1, each particle has mass 0.1 kg, and each spring has stiffness $0.2\,\mathrm{N\,m^{-1}}$. Determine the general solution of the equation of motion.

Solution

From the given data, we have $k/m_1 = k/m_2 = 0.2/0.1 = 2$, so the dynamic matrix of the system is

$$\mathbf{A} = \begin{pmatrix} -2k/m_1 & k/m_1 \\ k/m_2 & -2k/m_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}.$$

To find each normal mode, and hence the general solution of the equation of motion for the system, the first step is to determine the eigenvalues and eigenvectors of the dynamic matrix. The eigenvalues can be found by forming the characteristic equation, and solving the resulting quadratic equation:

$$\begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} = \lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6) = 0.$$

So the eigenvalues are -2 and -6.

- For $\lambda = -2$, the eigenvector equations are

  $$\begin{pmatrix} -4-(-2) & 2 \\ 2 & -4-(-2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  or equivalently

  $$\begin{aligned} -2v_1 + 2v_2 &= 0, \\ 2v_1 - 2v_2 &= 0. \end{aligned}$$

  These equations both reduce to $v_2 = v_1$, so $(1 \quad 1)^T$ is an eigenvector.

- For $\lambda = -6$, the eigenvector equations are

  $$\begin{pmatrix} -4-(-6) & 2 \\ 2 & -4-(-6) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  or equivalently

  $$\begin{aligned} 2v_1 + 2v_2 &= 0, \\ 2v_1 + 2v_2 &= 0. \end{aligned}$$

  These equations both reduce to $v_2 = -v_1$, so $(1 \quad -1)^T$ is an eigenvector.

As you saw above, each eigenvalue/eigenvector pair gives a normal mode of the system of the form $\mathbf{v}\cos(\omega t + \phi)$, where $\mathbf{v}$ is the eigenvector and $\omega = \sqrt{-\lambda}$, with λ the eigenvalue. For $\lambda = -2$ we have the normal mode angular frequency $\omega_1 = \sqrt{2}$, and for $\lambda = -6$ we have $\omega_2 = \sqrt{6}$. Therefore two linearly independent normal modes of the system are

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{2}t + \phi_1) \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{6}t + \phi_2).$$

Hence by Theorem 1, the general solution of the equation of motion is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{2}t + \phi_1) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{6}t + \phi_2),$$

where C_1, C_2, ϕ_1 and ϕ_2 are arbitrary constants.

Exercise 3

Determine the general solution of the equation of motion for the mechanical system considered in Example 1 if the particles have masses 0.1 kg and 0.2 kg, respectively, and each spring has stiffness $0.2\,\mathrm{N\,m^{-1}}$. Give your calculations to three decimal places.

The method employed in Examples 1 and 2 can be generalised to any oscillating mechanical system (without damping or forcing), and takes the form of the following general procedure.

Procedure 1 Analysing oscillating mechanical systems

To analyse the motion of a given oscillating mechanical system (without damping or forcing) with n degrees of freedom, proceed as follows.

◀ Draw picture ▶

1. Model the system using particles in conjunction with model springs, model rods, etc. Draw a sketch of the physical situation and annotate it with any relevant information.

◀ Choose coordinates ▶

2. Choose n coordinates (denoted by, say, $x_1, \ldots, x_n$) and corresponding origins.

◀ State assumption(s) ▶

3. State any assumptions that you make about the objects and the forces acting on them.

◀ Draw force diagram(s) ▶

4. Draw a force diagram for each particle separately.

◀ Apply Newton's 2nd law ▶

5. Apply Newton's second law to each particle separately, and resolve each force in the directions of appropriate axes to obtain n linear second-order differential equations of motion.

◀ Solve equation(s) ▶

6. Write the equations in matrix form as $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the $n \times n$ dynamic matrix of the system.

7. Find the eigenvalues $\lambda_1, \ldots, \lambda_n$ and a corresponding set of linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbf{A}$. (The eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are often referred to as the **normal mode eigenvectors** of the system.)
8. Determine the normal mode angular frequencies $\omega_1, \ldots, \omega_n$ from the formula $\omega_i = \sqrt{-\lambda_i}$.
9. Write down the general solution of the equations of motion in the form

$$\mathbf{x}(t) = C_1\mathbf{v}_1 \cos(\omega_1 t + \phi_1) + \cdots + C_n\mathbf{v}_n \cos(\omega_n t + \phi_n), \qquad (10)$$

where $C_1, \ldots, C_n$ and $\phi_1, \ldots, \phi_n$ are arbitrary constants.
10. Interpret the solution in terms of the original problem.

If the eigenvalues are not distinct, then the procedure may break down. If any of them are zero, then the procedure needs to be adapted as described in Subsection 2.2. If any of them are positive or complex, then you have made a mistake!

◀ Interpret solution ▶

We make the following notes about the application of this procedure.

- As mentioned earlier, the eigenvalues of the dynamic matrix of an oscillating mechanical system (without damping and forcing) are always real and non-positive. However, if the eigenvalues are not distinct or if any of them are zero, Procedure 1 will break down as Theorem 1 does not apply in such circumstances.
- If we know $x_i(0)$ and $\dot{x}_i(0)$ for each i, then a particular solution of the equations of motion may be obtained. But obtaining such a solution can involve solving a system of $2n$ equations – even for a mechanical system with just two degrees of freedom, this could mean solving a system of four equations.
- When we do not have particular values for the C_i and ϕ_i, it is still possible to obtain some information about the behaviour of the system. For example, knowledge of the normal mode angular frequencies ω_i enables us to determine the frequencies f_i and periods τ_i of the normal modes from the formulas $f_i = \omega_i/(2\pi)$ and $\tau_i = 2\pi/\omega_i$.

As you will see later, if the mechanical system starts from rest, then each ϕ_i can be taken to be zero, and a system of only n equations then needs to be solved.

See Unit 9.

We now apply Procedure 1 to the double pendulum.

Example 3

Consider the motion of a double pendulum where the lower pendulum is attached to the bob of the upper pendulum and where both pendulums are constrained to move in the same vertical plane (see Figure 12). The stems of the pendulums have lengths l_1 and l_2, and may be modelled as light model rods. The pendulum bobs have masses m_1 and m_2, and may be modelled as particles. The angles made by the stems of the pendulums to the vertical are denoted by θ_1 and θ_2 (measured anticlockwise from the vertical). The oscillations of both pendulums are sufficiently small for the approximations $\sin\theta_i \simeq \theta_i$ and $\cos\theta_i \simeq 1$ ($i = 1, 2$) to be applicable at all times during the motion of the system.

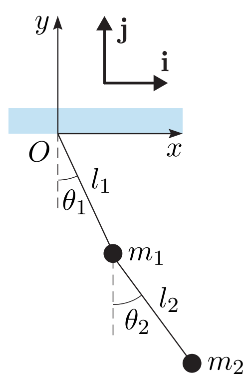


Figure 12 A double pendulum

(a) Derive linear equations of motion for the double pendulum.

(b) Suppose that $l_1 = l_2$ and $m_1 = m_2$. What is the general solution of the equations of motion?

(c) If each pendulum has length 50 cm, estimate the periods of the two normal modes.

Solution

◀ Choose coordinates ▶

(a) The system has two degrees of freedom – one associated with each pendulum. A suitable choice of coordinates would be the angles θ_1 and θ_2, which are marked on Figure 12, as these completely specify the configuration of the system at any given time. However, in order to use Newton's second law to generate equations of motion, we also need a set of Cartesian coordinate axes, which we will take to have origin O at the point of attachment of the upper pendulum, with the x-axis pointing horizontally to the right and the y-axis pointing vertically upwards, as in Figure 12.

◀ State assumption(s) ▶

We assume that the rods are model rods (with no mass) and the bobs are considered as particles. We assume that the only forces acting on the bobs are the gravitational forces and the tension forces in the rods. We further assume that the displacements are small.

◀ Draw force diagram(s) ▶

The force diagrams for the system are shown in Figure 13, with $\mathbf{W}_1 = -m_1 g\mathbf{j}$ and $\mathbf{W}_2 = -m_2 g\mathbf{j}$ denoting the weights of the particles, and

We model the forces on the particles due to the model rods as tension forces in the same way that we model forces on a particle due to model strings.

$$\mathbf{T}_1 = -|\mathbf{T}_1| \sin\theta_1\, \mathbf{i} + |\mathbf{T}_1| \cos\theta_1\, \mathbf{j},$$
$$\mathbf{T}_2 = |\mathbf{T}_2| \sin\theta_2\, \mathbf{i} - |\mathbf{T}_2| \cos\theta_2\, \mathbf{j},$$
$$\mathbf{T}_3 = -|\mathbf{T}_3| \sin\theta_3\, \mathbf{i} + |\mathbf{T}_3| \cos\theta_3\, \mathbf{j}$$

denoting the tension forces exerted on the particles by the model rods.

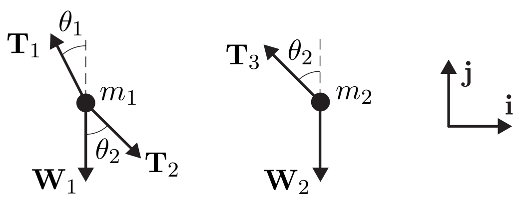


Figure 13 Force diagrams for the two particles

◀ Apply Newton's 2nd law ▶

Applying Newton's second law to the two particles separately, we obtain

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{W}_1 + \mathbf{T}_1 + \mathbf{T}_2, \quad m_2\ddot{\mathbf{r}}_2 = \mathbf{W}_2 + \mathbf{T}_3,$$

where $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$ are the position vectors of the two pendulum bobs relative to O.

Resolving in the **i**- and **j**-directions gives

$$m_1\ddot{x}_1 = -|\mathbf{T}_1|\sin\theta_1 + |\mathbf{T}_2|\sin\theta_2,$$
$$m_1\ddot{y}_1 = -m_1 g + |\mathbf{T}_1|\cos\theta_1 - |\mathbf{T}_2|\cos\theta_2,$$
$$m_2\ddot{x}_2 = -|\mathbf{T}_3|\sin\theta_2,$$
$$m_2\ddot{y}_2 = -m_2 g + |\mathbf{T}_3|\cos\theta_2.$$

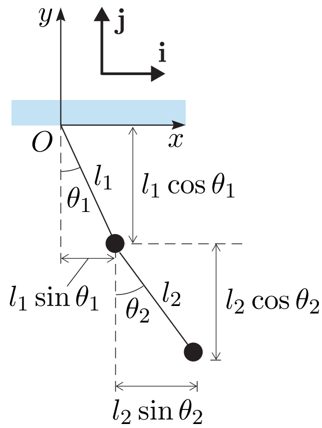


Figure 14 Relationships between linear and angular coordinates

From Figure 14 we can deduce the following relationships between the linear coordinates and the angular coordinates:

$$x_1 = l_1\sin\theta_1, \quad x_2 = l_1\sin\theta_1 + l_2\sin\theta_2,$$
$$y_1 = -l_1\cos\theta_1, \quad y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2.$$

Now since the oscillations of the pendulums are small, we can use the small-angle approximations specified in the question, that is, $\sin\theta_i \simeq \theta_i$ and $\cos\theta_i \simeq 1$ $(i = 1, 2)$, in the previous equations, to obtain

$$m_1\ddot{x}_1 \simeq -|\mathbf{T}_1|\theta_1 + |\mathbf{T}_2|\theta_2, \tag{11}$$
$$m_1\ddot{y}_1 \simeq -m_1 g + |\mathbf{T}_1| - |\mathbf{T}_2|, \tag{12}$$
$$m_2\ddot{x}_2 \simeq -|\mathbf{T}_3|\theta_2, \tag{13}$$
$$m_2\ddot{y}_2 \simeq -m_2 g + |\mathbf{T}_3|, \tag{14}$$

and

$$x_1 \simeq l_1\theta_1, \quad x_2 \simeq l_1\theta_1 + l_2\theta_2,$$
$$y_1 \simeq -l_1, \quad y_2 \simeq -l_1 - l_2.$$

(These small-angle approximations will ensure that the resulting differential equations are linear.)

From $x_1 \simeq l_1\theta_1$, an approximation for the horizontal acceleration can be derived by differentiation as $\ddot{x}_1 \simeq l_1\ddot{\theta}_1$. Similarly, differentiating $x_2 \simeq l_1\theta_1 + l_2\theta_2$ gives $\ddot{x}_2 \simeq l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2$. As the expressions above for y_1 and y_2 are constants, it follows that $\ddot{y}_1 \simeq 0$ and $\ddot{y}_2 \simeq 0$. (The physical interpretation of this is that for small oscillations the pendulum bobs do not move vertically.) These approximations for the horizontal and vertical accelerations can then be substituted into equations (11)–(14) to give

Differentiation of these approximations is valid provided that θ_1 and θ_2 not only are small, but also do not vary too rapidly.

$$m_1 l_1\ddot{\theta}_1 \simeq -|\mathbf{T}_1|\theta_1 + |\mathbf{T}_2|\theta_2, \tag{15}$$
$$0 \simeq -m_1 g + |\mathbf{T}_1| - |\mathbf{T}_2|, \tag{16}$$
$$m_2(l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2) \simeq -|\mathbf{T}_3|\theta_2, \tag{17}$$
$$0 \simeq -m_2 g + |\mathbf{T}_3|. \tag{18}$$

From equation (18) we obtain $|\mathbf{T}_3| \simeq m_2 g$. By Newton's third law we have $|\mathbf{T}_2| = |\mathbf{T}_3|$, therefore $|\mathbf{T}_2| \simeq m_2 g$. This can be substituted into equation (16) to yield $|\mathbf{T}_1| \simeq (m_1 + m_2)g$. If we then substitute for $|\mathbf{T}_1|$, $|\mathbf{T}_2|$ and $|\mathbf{T}_3|$ in equations (15) and (17), we have

Newton's third law tells us that the forces exerted at either end of a model rod must be equal in magnitude but opposite in direction. Recall the case of a model string in Unit 2.

$$m_1 l_1\ddot{\theta}_1 \simeq -(m_1 + m_2)g\theta_1 + m_2 g\theta_2,$$
$$m_2(l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2) \simeq -m_2 g\theta_2.$$

To put these equations into a standard form, that is, with only one second derivative on the left-hand side of each, first divide by the masses to get

$$l_1\ddot{\theta}_1 \simeq -\left(1+\frac{m_2}{m_1}\right)g\theta_1 + \frac{m_2}{m_1}g\theta_2, \tag{19}$$

$$l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 \simeq -g\theta_2. \tag{20}$$

Next subtract equation (19) from equation (20) to eliminate the $l_1\ddot{\theta}_1$ term from the left-hand side, giving

$$\begin{aligned} l_2\ddot{\theta}_2 &\simeq -g\theta_2 - \left(-\left(1+\frac{m_2}{m_1}\right)g\theta_1 + \frac{m_2}{m_1}g\theta_2\right) \\ &= \left(1+\frac{m_2}{m_1}\right)g\theta_1 - \left(1+\frac{m_2}{m_1}\right)g\theta_2. \end{aligned} \tag{21}$$

Now divide equation (19) by l_1, and equation (21) by l_2, to obtain equations with one second derivative on the left-hand side of each. Thus we arrive at the following linear second-order differential equations of motion for the double pendulum when it is undergoing small oscillations:

Since this is a *model* of the motion, there is no need to retain the $\simeq$ signs.

$$\ddot{\theta}_1 = -\left(1+\frac{m_2}{m_1}\right)\frac{g}{l_1}\theta_1 + \frac{m_2}{m_1}\frac{g}{l_1}\theta_2,$$

$$\ddot{\theta}_2 = \left(1+\frac{m_2}{m_1}\right)\frac{g}{l_2}\theta_1 - \left(1+\frac{m_2}{m_1}\right)\frac{g}{l_2}\theta_2.$$

◀ Solve equation(s) ▶

(b) In the case where $l_1 = l_2$ $(= l)$ and $m_1 = m_2$, the equations of motion become

$$\ddot{\theta}_1 = -\frac{2g}{l}\theta_1 + \frac{g}{l}\theta_2,$$

$$\ddot{\theta}_2 = \frac{2g}{l}\theta_1 - \frac{2g}{l}\theta_2.$$

To solve these equations, we write them in matrix form as

$$\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \frac{g}{l}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}.$$

Following Procedure 6 in Unit 6, we now need to find the eigenvalues and eigenvectors of the dynamic matrix

$$\mathbf{A} = \frac{g}{l}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}.$$

However, this is equivalent to finding the eigenvalues and eigenvectors of

Recall from Unit 5 that if λ is an eigenvalue of $\mathbf{M}$ with eigenvector $\mathbf{v}$, then $p\lambda$ is an eigenvalue of $p\mathbf{M}$ with the same eigenvector $\mathbf{v}$.

$$\mathbf{B} = \begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix},$$

provided that we remember to scale these eigenvalues by the factor g/l later. The characteristic equation of $\mathbf{B}$ is

$$\begin{vmatrix}-2-\lambda & 1\\ 2 & -2-\lambda\end{vmatrix} = \lambda^2 + 4\lambda + 2 = 0.$$

Therefore the eigenvalues of $\mathbf{B}$ are $\lambda = -2 \pm \sqrt{2}$.

- For $\lambda = -2 + \sqrt{2}$, the eigenvector equations are
$$\begin{pmatrix} -2 - (-2 + \sqrt{2}) & 1 \\ 2 & -2 - (-2 + \sqrt{2}) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
or equivalently
$$\begin{aligned} -\sqrt{2}v_1 + \quad v_2 &= 0, \\ 2v_1 - \sqrt{2}v_2 &= 0. \end{aligned}$$
Both of these equations simplify to $v_2 = \sqrt{2}v_1$, so $(1 \quad \sqrt{2})^T$ is an eigenvector.
- For $\lambda = -2 - \sqrt{2}$, the eigenvector equations are
$$\begin{pmatrix} -2 - (-2 - \sqrt{2}) & 1 \\ 2 & -2 - (-2 - \sqrt{2}) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
or equivalently
$$\begin{aligned} \sqrt{2}v_1 + \quad v_2 &= 0, \\ 2v_1 + \sqrt{2}v_2 &= 0. \end{aligned}$$
Both of these equations simplify to $v_2 = -\sqrt{2}v_1$, so $(1 \quad -\sqrt{2})^T$ is an eigenvector.

After we have scaled by g/l, we obtain the eigenvalues of the dynamic matrix $\mathbf{A}$ as $\lambda_1 = -g(2 - \sqrt{2})/l$ and $\lambda_2 = -g(2 + \sqrt{2})/l$. Since $\omega_i = \sqrt{-\lambda_i}$, the normal mode angular frequencies are

Note that $2 > \sqrt{2}$, so both eigenvalues are negative.

$$\omega_1 = \sqrt{\frac{g(2 - \sqrt{2})}{l}}, \quad \omega_2 = \sqrt{\frac{g(2 + \sqrt{2})}{l}}.$$

From equation (10), the general solution of the equations of motion can then be written as
$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_1 t + \phi_1) + C_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_2 t + \phi_2),$$
where C_1, C_2, ϕ_1 and ϕ_2 are constants that depend on the initial conditions.

(c) The information that each pendulum has length 50 cm, that is, $l_1 = l_2 = 0.5$, enables us to calculate the normal mode angular frequencies as

◀ Interpret solution ▶

$$\omega_1 = \sqrt{\frac{g(2 - \sqrt{2})}{0.5}} \simeq 3.390, \quad \omega_2 = \sqrt{\frac{g(2 + \sqrt{2})}{0.5}} \simeq 8.185.$$

The periods of oscillation of the two normal modes of the double pendulum are
$$\tau_1 = \frac{2\pi}{\omega_1} \simeq 1.853, \quad \tau_2 = \frac{2\pi}{\omega_2} \simeq 0.768.$$

So this model predicts that the periods of oscillation of the normal modes are approximately 0.8 s and 1.9 s.

Notice how, in this example, we used small-angle approximations to produce a system of *linear* differential equations; otherwise the equations would have been non-linear.

Exercise 4

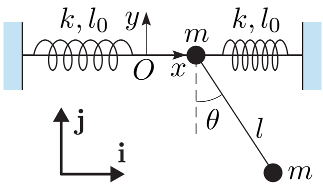


Figure 15 A spring pendulum

Figure 15 shows a mechanical system in which a particle of mass m is free to move along a smooth straight horizontal bar. The particle is subject to the forces exerted by two identical model springs of stiffness k and natural length l_0, and to the force exerted by a pendulum of length l suspended from the particle. The bar is twice the natural length of each spring, so the system is in equilibrium when the springs are at their natural lengths, that is, not extended or compressed.

The pendulum stem may be modelled as a light model rod and its bob as a particle of mass m. Assume that the oscillations of the pendulum are small, and hence that small-angle approximations (as in Example 3) are valid.

The displacement x_1 of the sliding particle is measured from its equilibrium position O, and the displacement of the pendulum is given by the angle θ (measured anticlockwise from the vertical), as shown in Figure 15.

(a) Derive linear equations of motion for the system in terms of x_1 and θ.

(b) If $k = 1$, $m = 1$ and $l = 1$, find the normal mode angular frequencies to two decimal places.

1.3 Interpretation of normal mode eigenvectors

For oscillating systems with more than one degree of freedom, how do we know the initial conditions required to make a system oscillate in a normal mode? This question is answered in this subsection, by using the normal mode eigenvectors. Before we do this, we derive a result that is useful in describing systems where all the particles start from rest (which are the easiest starting conditions in practice). We illustrate the argument in relation to a system with two degrees of freedom, though the argument can be generalised to a system with any number of degrees of freedom.

For a system with two degrees of freedom, the general solution of the equation of motion for the system (equation (10)) becomes

$$\mathbf{x}(t) = C_1\mathbf{v}_1\cos(\omega_1 t + \phi_1) + C_2\mathbf{v}_2\cos(\omega_2 t + \phi_2).$$

Differentiating this gives

$$\dot{\mathbf{x}}(t) = -\omega_1 C_1\mathbf{v}_1\sin(\omega_1 t + \phi_1) - \omega_2 C_2\mathbf{v}_2\sin(\omega_2 t + \phi_2).$$

Initially, at $t = 0$,

$$\dot{\mathbf{x}}(0) = -\omega_1 C_1\mathbf{v}_1\sin\phi_1 - \omega_2 C_2\mathbf{v}_2\sin\phi_2.$$

Applying the condition that all the particles start from rest, that is, $\dot{\mathbf{x}}(0) = \mathbf{0}$, gives

$$\mathbf{0} = -\omega_1 C_1 \mathbf{v}_1 \sin\phi_1 - \omega_2 C_2 \mathbf{v}_2 \sin\phi_2.$$

Notice that this initial condition is satisfied if $\phi_1 = \phi_2 = 0$. We will now show that we may assume that $\phi_1 = \phi_2 = 0$.

Since the two normal mode eigenvectors are linearly independent, the only way that a linear combination of them can be zero is if all the coefficients are zero, that is, if

$$\omega_1 C_1 \sin\phi_1 = 0 \quad \text{and} \quad \omega_2 C_2 \sin\phi_2 = 0.$$

Since the normal mode angular frequencies are non-zero, we have

You will see mechanical systems with zero normal mode angular frequencies later.

$$C_1 \sin\phi_1 = 0 \quad \text{and} \quad C_2 \sin\phi_2 = 0.$$

From the first equation, either $C_1 = 0$ or $\sin\phi_1 = 0$. If $C_1 = 0$, then the normal mode does not contribute to the motion $\mathbf{x}(t)$ and the value of ϕ_1 is irrelevant, so we can choose $\phi_1 = 0$. If $\sin\phi_1 = 0$, then either $\phi_1 = 0$ or $\phi_1 = \pi$. If $\phi_1 = \pi$, then the term $C_1\mathbf{v}_1\cos(\omega t + \pi)$ becomes $-C_1\mathbf{v}_1\cos(\omega t)$, and we can include the minus sign in the normal mode eigenvector $\mathbf{v}_1$ and choose $\phi_1 = 0$. So the first equation implies that either $\phi_1 = 0$ or we have a free choice and may take $\phi_1 = 0$. Similarly, we may take $\phi_2 = 0$. This is an important result that is worth remembering.

Recall that for normal modes we take $-\pi < \phi \leq \pi$.

Systems starting from rest

If one of the initial conditions of a mechanical system is that all the particles start from rest, then it follows that all the phase angles are zero.

Now we use this result to find a suitable initial condition for normal mode motion when all the particles start from rest. As before, consider a system with two degrees of freedom to illustrate the argument. If the system starts from rest, then by the above result all the phase angles are zero. So the general motion of the two-degrees-of-freedom system can be written as

$$\mathbf{x}(t) = C_1\mathbf{v}_1\cos(\omega_1 t) + C_2\mathbf{v}_2\cos(\omega_2 t). \tag{22}$$

Initially, at $t = 0$, this becomes

$$\mathbf{x}(0) = C_1\mathbf{v}_1 + C_2\mathbf{v}_2. \tag{23}$$

Suppose that the system is initially at a position corresponding to the first normal mode eigenvector, that is, at a position $\mathbf{x}(0) = k\mathbf{v}_1$ for some constant $k \neq 0$. Then substituting for $\mathbf{x}(0)$ in equation (23), we obtain

Recall, from Unit 5, that if $\mathbf{v}$ is an eigenvector, then so is $k\mathbf{v}$ for any non-zero constant k.

$$k\mathbf{v}_1 = C_1\mathbf{v}_1 + C_2\mathbf{v}_2. \tag{24}$$

Since the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we can equate their respective coefficients on either side of equation (24), thereby finding that $C_1 = k \neq 0$ and $C_2 = 0$. Substituting into equation (22) gives

$$\mathbf{x}(t) = k\mathbf{v}_1\cos(\omega_1 t). \tag{25}$$

Therefore if we start the system from rest at a position $\mathbf{x}(0) = k\mathbf{v}_1$, it will move with the normal mode motion given by equation (25). Similarly, starting the system from rest at $\mathbf{x}(0) = k\mathbf{v}_2$ results in the normal mode motion given by $\mathbf{x}(t) = k\mathbf{v}_2 \cos(\omega_2 t)$. Generalising this argument to any number of degrees of freedom gives the following result.

Normal modes of systems starting from rest

If all the particles within a mechanical system start from rest at positions given by the elements of a normal mode eigenvector, then the system will oscillate in the corresponding normal mode.

The following example illustrates how this result can be used.

Example 4

In Example 2 the following general solution was derived for the positions of two particles, each of mass 0.1 kg, joined to one another and to two fixed walls by three identical model springs of stiffness $0.2\,\mathrm{N\,m^{-1}}$ (see Figure 10):

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{2}t + \phi_1) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{6}t + \phi_2).$$

What initial conditions give rise to normal mode motion in this system when both particles start from rest?

Solution

When the two particles start from rest, so that $\dot{\mathbf{x}}(0) = \mathbf{0}$, we can deduce that $\phi_1 = \phi_2 = 0$, whatever the subsequent motion. For the normal mode with angular frequency $\sqrt{2}$, a normal mode eigenvector is $(1 \quad 1)^T$. The positions of the particles are measured from their respective equilibrium positions. So one initial condition for normal mode motion is that each particle starts from rest at a distance 1 cm, say, to the right of its equilibrium position. However, any constant multiple of $(1 \quad 1)^T$ is also an eigenvector, so we can say rather more: a normal mode results if both particles start from rest after being given the *same* displacement. Therefore suitable initial conditions include not only starting both particles from rest 1 cm to the right of their equilibrium positions, but also starting both particles 1 cm to the left, or 10 cm to the right (provided that the natural length of the spring is significantly greater than 10 cm), and so on.

For the second normal mode, a normal mode eigenvector is $(1 \quad -1)^T$. Hence another initial condition for normal mode motion is that the first particle starts from rest at a distance 1 cm to the right of its equilibrium position, and the second particle starts from rest 1 cm to the left of its equilibrium position. Again, however, we can say more: a normal mode motion results whenever the particles start from rest after being displaced from equilibrium by equal distances in opposite directions.

Exercise 5

For the double pendulum analysed in Example 3, what initial conditions give rise to normal mode motion when both bobs start from rest?

Another feature of the motion can be read from the normal mode eigenvectors. The signs of the components of the eigenvectors indicate whether the normal mode motion is in-phase or phase-opposed.

For example, consider the double pendulum and Example 3. The normal mode eigenvector corresponding to the smallest normal mode frequency is $(1 \quad \sqrt{2})^T$. The components of this normal mode eigenvector have the same sign, so the normal mode motion is in-phase.

Note that $(-1 \quad -\sqrt{2})^T$ is also a normal mode eigenvector for this motion, and its components are not positive, but they have the same sign.

The other normal mode eigenvector of the double pendulum was calculated to be $(1 \quad -\sqrt{2})^T$ in Example 3. Here the two components have opposite signs and the normal mode motion is phase-opposed. Again, this is generally true.

We summarise this as follows.

Theorem 2

For a normal mode motion, the relative motion of a pair of particles with non-zero coordinates is determined by the components of the corresponding normal mode eigenvector. The motion is:

- in-phase if the components have the same sign
- phase-opposed if the components have opposite signs.

Example 5

Consider the oscillating mechanical system with three degrees of freedom shown in Figure 16. Three particles are constrained to move in a straight line between two fixed walls. The particles are attached to one another and to the walls by four identical model springs that are aligned with the direction of movement of the particles. The positions of the particles are measured from their equilibrium positions, as shown in the figure.

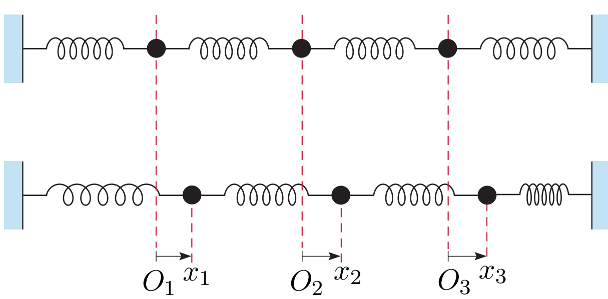


Figure 16 An oscillating system with three degrees of freedom

A normal mode eigenvector of the lowest angular frequency normal mode is given as $(1 \quad \sqrt{2} \quad 1)^T$. Draw sketches of the mechanical system at the following times while it is oscillating according to this normal mode:

(a) when all the particles are instantaneously at rest at the same time

(b) one-sixth of a cycle after the time in part (a)

(c) one-quarter of a cycle after the time in part (a)

(d) half a cycle after the time in part (a).

Is the motion in-phase or phase-opposed?

Solution

For a system starting from rest, we can take $\phi = 0$.

As a starting point for the system, consider a time instant when all three particles are instantaneously at rest at the same time. An equation of motion for the given normal mode is $\mathbf{x}(t) = k\mathbf{v}\cos\omega t$, where k is a constant and $\mathbf{v} = (1 \quad \sqrt{2} \quad 1)^T$. The required sketch is shown in Figure 17.

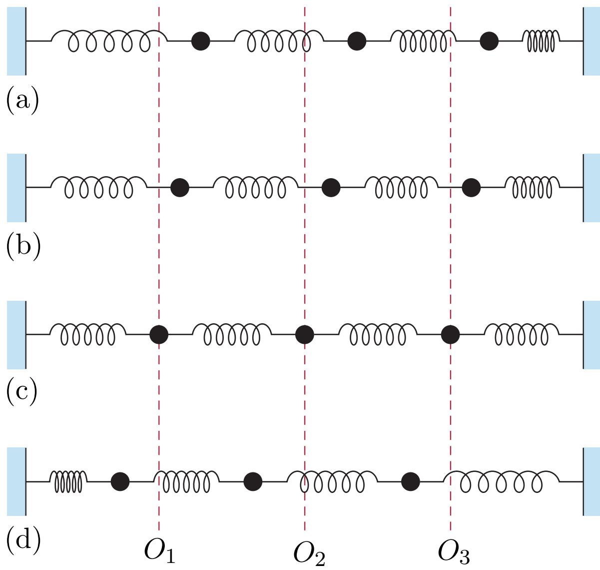


Figure 17 The oscillating system at various times during the motion

(a) In this instance the positions of the particles relative to their equilibrium positions are given by $k\mathbf{v}$, and are sketched in Figure 17(a). Thus if particle 1 is at, say, $x_1 = 2$, then particle 2 will be at $x_2 = 2\sqrt{2}$, and particle 3 will be at $x_3 = 2$, as determined by $k\mathbf{v} = 2\mathbf{v} = (2 \quad 2\sqrt{2} \quad 2)^T$.

(b) After a sixth of a cycle (i.e. when $\omega t = \frac{2\pi}{6} = \frac{\pi}{3}$), the equation of motion yields $\mathbf{x}(t) = k\mathbf{v}\cos\omega t = k\mathbf{v}\cos\frac{\pi}{3} = \frac{1}{2}k\mathbf{v}$, so each particle is half as far from its equilibrium position as it was in part (a), as sketched in Figure 17(b).

(c) After a quarter of a cycle (i.e. when $\omega t = \frac{2\pi}{4} = \frac{\pi}{2}$), $\mathbf{x}(t) = k\mathbf{v}\cos\frac{\pi}{2} = \mathbf{0}$, so each particle is at its equilibrium position, as sketched in Figure 17(c).

(d) After half a cycle (i.e. when $\omega t = \frac{2\pi}{2} = \pi$), $\mathbf{x}(t) = k\mathbf{v}\cos\pi = -k\mathbf{v}$, so each particle is the same distance from its equilibrium position as it was in part (a), but in the opposite direction, as sketched in Figure 17(d).

Note that although the particles are at their equilibrium positions, the *system* is not in equilibrium since the velocities are non-zero. (In fact, the speeds are at a maximum.)

Since all the coordinates of the given normal mode eigenvector have the same sign, all three particles move in-phase with one another, as the sketches in Figure 17 illustrate.

Exercise 6

Repeat Example 5 for the normal modes with normal mode eigenvectors $(1 \quad 0 \quad -1)^T$ and $(1 \quad -\sqrt{2} \quad 1)^T$.

This section concludes by asking you to apply the methods described in this section to another mechanical system.

Exercise 7

Consider the mechanical system with two degrees of freedom, shown in Figure 18, in which two identical particles of mass m are attached to one another and to a fixed wall by two model springs of stiffnesses $3k$ and $2k$, each with natural length l_0. The particles are constrained to move across a frictionless horizontal surface in a straight line along the line of the springs. Let x_1 and x_2 be the displacements of the two particles along this line, measured from the equilibrium positions of the particles as shown in the figure.

(a) Derive the equations of motion for this system.

(b) If $k/m = 1$, determine the normal mode eigenvectors and the angular frequencies. Hence write down the general solution of the matrix equation of motion.

(c) Write down equations from which the constants could be determined in the general solution obtained in part (b), given the initial condition that both particles start from rest, with displacements d_1 and d_2, respectively.

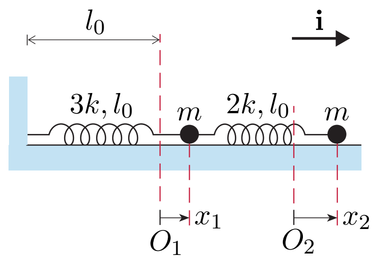


Figure 18 A system with two degrees of freedom

2 One-dimensional systems

In this section we continue to look at mechanical systems where particles are restricted to move along a line (such systems have the number of degrees of freedom equal to the number of particles).

A quicker method for deriving the equations of motion is developed in this section that has the advantage that the equations of motion can be derived without first needing to calculate the equilibrium positions of particles.

Subsection 2.1 considers systems that have one or two springs fixed at one end. The equations of motion derived are similar to those derived in Section 1 and are not investigated further. Subsection 2.2 considers free systems that have no components fixed. Here the solution of the equations of motion is different, so we investigate it.

2.1 Displacement from equilibrium

In this subsection you will see how to obtain the homogeneous equations of motion for a mechanical system (i.e. the equation obtained by taking the origin at the equilibrium position), without first having to find an equilibrium position of the system.

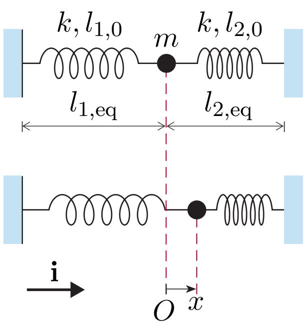


Figure 19 A particle and two springs

Consider the system made up of a particle of mass m attached to two walls by two model springs of stiffnesses k_1 and k_2, as shown in Figure 19, with the particle constrained to move in one dimension along the line of the springs (friction may be neglected). Because the springs have different stiffnesses, the equilibrium position of the system is not obvious.

We denote the forces exerted by the springs when the system is in equilibrium by $\mathbf{H}_{1,\mathrm{eq}}$ and $\mathbf{H}_{2,\mathrm{eq}}$, and the corresponding forces when the system is displaced a distance x from the equilibrium position by $\mathbf{H}_1$ and $\mathbf{H}_2$. At equilibrium,

$$\mathbf{H}_{1,\mathrm{eq}} + \mathbf{H}_{2,\mathrm{eq}} = \mathbf{0}. \tag{26}$$

Applying Hooke's law, we obtain

$$\begin{aligned} \mathbf{H}_1 &= k_1((l_{1,\mathrm{eq}} + x) - l_{1,0})(-\mathbf{i}), & \mathbf{H}_{1,\mathrm{eq}} &= k_1(l_{1,\mathrm{eq}} - l_{1,0})(-\mathbf{i}), \\ \mathbf{H}_2 &= k_2((l_{2,\mathrm{eq}} - x) - l_{2,0})\mathbf{i}, & \mathbf{H}_{2,\mathrm{eq}} &= k_2(l_{2,\mathrm{eq}} - l_{2,0})\mathbf{i}, \end{aligned}$$

where $l_{1,\mathrm{eq}}$ and $l_{2,\mathrm{eq}}$ are the equilibrium lengths of the springs, $l_{1,0}$ and $l_{2,0}$ are the natural lengths of the springs, and $\mathbf{i}$ is a unit vector in the positive x-direction marked on Figure 19. From these equations, the changes in the forces exerted by the springs when the particle is displaced a distance x from the equilibrium position are given by

The symbol Δ is an upper-case delta. It is quite commonly used to denote a change in a quantity (just as a lower-case delta, δ, denotes a small change in a quantity).

$$\Delta\mathbf{H}_1 = \mathbf{H}_1 - \mathbf{H}_{1,\mathrm{eq}} = -k_1 x\mathbf{i}, \tag{27}$$

$$\Delta\mathbf{H}_2 = \mathbf{H}_2 - \mathbf{H}_{2,\mathrm{eq}} = -k_2 x\mathbf{i}. \tag{28}$$

Then

$$\Delta\mathbf{H}_1 + \Delta\mathbf{H}_2 = (\mathbf{H}_1 + \mathbf{H}_2) - (\mathbf{H}_{1,\mathrm{eq}} + \mathbf{H}_{2,\mathrm{eq}}) = -(k_1 + k_2)x\mathbf{i},$$

and on substituting from equation (26), we obtain

$$\Delta\mathbf{H}_1 + \Delta\mathbf{H}_2 = \mathbf{H}_1 + \mathbf{H}_2 = -(k_1 + k_2)x\mathbf{i}.$$

So the total force $\mathbf{H}_1 + \mathbf{H}_2$ on the particle when it is displaced a distance x from the equilibrium position is the same as the sum of the changes in the forces, $\Delta\mathbf{H}_1 + \Delta\mathbf{H}_2$. Furthermore, if we apply Newton's second law to the particle when it is displaced a distance x from equilibrium, we find that

$$m\ddot{\mathbf{r}} = \mathbf{H}_1 + \mathbf{H}_2 = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2 = -(k_1 + k_2)x\mathbf{i},$$

where $\mathbf{r} = x\mathbf{i}$ is the position vector, and the origin is at the equilibrium position. Resolving in the $\mathbf{i}$-direction and rearranging gives the equation of motion as

$$m\ddot{x} + (k_1 + k_2)x = 0. \tag{29}$$

Thus we have found the equation of motion without knowing the equilibrium position of the particle, knowing only the changes in the forces acting as given by equations (27) and (28).

This method of obtaining an equation of motion can be generalised to any number of forces acting on any particle that has been disturbed from an equilibrium position. If forces $\mathbf{F}_i$ $(i = 1, \ldots, n)$ act on a particle disturbed from an equilibrium position, and $\mathbf{F}_{i,\text{eq}}$ $(i = 1, \ldots, n)$ denote the corresponding forces when the particle is in equilibrium, while $\Delta\mathbf{F}_i$ $(i = 1, \ldots, n)$ denote the changes in the forces from equilibrium, then

$$\sum_{i=1}^{n} \Delta\mathbf{F}_i = \sum_{i=1}^{n}(\mathbf{F}_i - \mathbf{F}_{i,\text{eq}}) = \sum_{i=1}^{n}\mathbf{F}_i - \sum_{i=1}^{n}\mathbf{F}_{i,\text{eq}} = \sum_{i=1}^{n}\mathbf{F}_i,$$

since $\sum_{i=1}^{n}\mathbf{F}_{i,\text{eq}} = \mathbf{0}$. So the total force $\sum_{i=1}^{n}\mathbf{F}_i$ acting on the particle is the same as the total change in the forces $\sum_{i=1}^{n}\Delta\mathbf{F}_i$, which leads to the following useful result based on Newton's second law.

$\sum_{i=1}^{n}\mathbf{F}_{i,\text{eq}} = \mathbf{0}$ is the equilibrium condition for particles, from Unit 2.

Particle displaced from equilibrium

If a particle of mass m, displaced from an equilibrium position, is acted on by forces $\mathbf{F}_i$ $(i = 1, \ldots, n)$, then its acceleration $\ddot{\mathbf{r}}$ is given by

$$m\ddot{\mathbf{r}} = \sum_{i=1}^{n}\Delta\mathbf{F}_i, \tag{30}$$

where $\mathbf{r}$ is the displacement of the particle from its equilibrium position, and $\Delta\mathbf{F}_i = \mathbf{F}_i - \mathbf{F}_{i,\text{eq}}$ $(i = 1, \ldots, n)$, where $\mathbf{F}_{i,\text{eq}}$ is the force corresponding to $\mathbf{F}_i$ when the particle is at its equilibrium position.

Applying equation (30) is equivalent to applying Newton's second law.

Equation (30) can provide a useful shortcut in the derivation of the equation of motion for a mechanical system involving a displacement from equilibrium, if the changes in the forces acting can be calculated easily. For a spring displaced along its length, you have seen that the changes in the forces can be calculated easily using equations (27) and (28).

The general result that applies to any coordinate system for a change of length Δl is

See Section 3 for formulas that apply in other cases.

$$\Delta \mathbf{H} = k\,\Delta l\,\widehat{\mathbf{s}}, \tag{31}$$

where k is the stiffness of the spring, $\Delta l = l - l_{\text{eq}}$ is the increase in the spring's length from its equilibrium value, and $\widehat{\mathbf{s}}$ is a constant unit vector directed from the particle to the centre of the spring.

Another force for which this shortcut is useful is the force $\mathbf{W}$ due to gravity, that is, the weight of an object, since this force is constant for a given object near the Earth's surface, so $\Delta \mathbf{W} = \mathbf{0}$. Thus the weights of the components of a mechanical system have no effect on the system's equation of motion when the components are displaced from their equilibrium positions. If a normal reaction force $\mathbf{N}$ balances the weight of a particle (such as when objects are placed on a smooth horizontal table), then $\mathbf{N}$ is constant thus $\Delta \mathbf{N} = \mathbf{0}$.

You saw in Unit 9 that the period of oscillations of a mechanical system is the same whether the system is oriented horizontally or vertically.

These results have useful consequences for one- or two-dimensional mechanical systems that can be modelled using only model springs and particles. It means that in deriving the equations of motion for such systems, provided that displacements are measured from the equilibrium positions, we can ignore all weights and normal reactions, irrespective of the orientation of the system – in effect, whatever its orientation, the system behaves as if it is in a horizontal plane. We will adopt the policy of ignoring weights and normal reactions for such systems for the remainder of this unit.

This result also applies to systems involving model rods, as you will see in Unit 19.

Let us now see how these ideas can be applied, by looking at an example similar to the one considered in Example 1.

Example 6

Consider the mechanical system shown in Figure 20. This system consists of two particles of masses m_1 and m_2, which are constrained to move in a straight line across a frictionless horizontal surface while attached by model springs of stiffnesses k_1, k_2 and k_3 to each other and to two fixed walls. Derive the equation of motion for each of the particles.

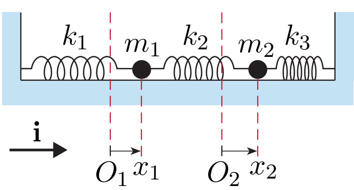


Figure 20 A spring–mass system. (Where natural lengths are not needed for calculations, we may not mark them on diagrams from here onwards.)

Solution

Unlike the situation in Example 1, the springs have different stiffnesses and we do not know the equilibrium positions of the components of the system. Hence we need to adopt a different approach from that used in Example 1.

The first step is to draw a force diagram for each particle (Figure 21). We use the usual notation for the spring forces and, as discussed above, we ignore the weights and normal reactions, which balance.

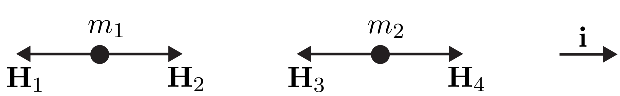


Figure 21 Force diagrams showing only the additional forces

Applying equation (30) to each particle gives

$$m_1\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2, \tag{32}$$

$$m_2\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4, \tag{33}$$

where $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$ are the position vectors of the two particles with respect to their (unknown) equilibrium positions, and $\mathbf{i}$ is a unit vector in the direction of positive x_1 and x_2.

Noting that the additional extension of the middle spring is $x_2 - x_1$, we use equation (31) to obtain

$$\Delta\mathbf{H}_1 = k_1x_1(-\mathbf{i}) = -k_1x_1\mathbf{i},$$
$$\Delta\mathbf{H}_2 = k_2(x_2 - x_1)\mathbf{i},$$
$$\Delta\mathbf{H}_3 = k_2(x_2 - x_1)(-\mathbf{i}) = -k_2(x_2 - x_1)\mathbf{i},$$
$$\Delta\mathbf{H}_4 = k_3(-x_2)\mathbf{i} = -k_3x_2\mathbf{i},$$

where the minus sign in the last expression indicates that the third spring has been compressed by the amount x_2 (or extended by an amount $-x_2$) from its equilibrium point, as in equations (27) and (28).

Substituting these expressions into equations (32) and (33) gives

$$m_1\ddot{\mathbf{r}}_1 = -k_1x_1\mathbf{i} + k_2(x_2 - x_1)\mathbf{i},$$
$$m_2\ddot{\mathbf{r}}_2 = -k_2(x_2 - x_1)\mathbf{i} - k_3x_2\mathbf{i}.$$

Resolving in the $\mathbf{i}$-direction then gives

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2, \tag{34}$$

$$m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2. \tag{35}$$

These are the equations of motion for the system illustrated in Figure 20.

Exercise 8

Use equation (30) to derive the equations of motion for the system in Exercise 7 (see Figure 18).

2.2 Free motion

The one-dimensional mechanical systems studied so far in this unit have all been constrained by being attached to fixed walls. This subsection looks at one-dimensional mechanical systems without fixed points, where free motion (in one dimension) is possible. You will see that the absence of fixed points manifests itself in the dynamic matrix having a zero eigenvalue, and you will see how this affects the normal modes of the system. We begin with an example.

Example 7

This example is concerned with the motions of a hydrogen molecule, in particular the vibrations within it. This molecule consists of two atoms of equal mass joined by a bond. We model this as two particles of equal mass m, joined by a model spring of stiffness k. The system is considered to be one-dimensional, so only those vibrations that are along the straight line joining the two atoms are considered.

(a) Derive the equation of motion for this system.

(b) Determine the general solution of this equation of motion for the system in terms of m and k.

(c) Interpret this general solution.

(d) Experimentally, the frequency of vibration for a hydrogen molecule is 1.3×10^{14} Hz. If the mass of a hydrogen atom is 1.67×10^{-27} kg, estimate the stiffness of the bond between the two hydrogen atoms in the molecule.

Solution

◀ Draw picture ▶

(a) A sketch of the model of the hydrogen molecule is shown in Figure 22.

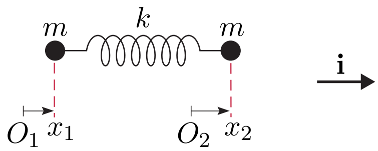


Figure 22 Model of a hydrogen molecule

◀ Choose coordinates ▶

Choose axes aligned with the spring that joins the two particles, with two separate origins at the equilibrium positions of the two particles (i.e. separated by a distance equal to the natural length of the model spring), as in Figure 22. Using the usual notation, we can draw the force diagrams (one for each particle) in Figure 23.

◀ Draw force diagram(s) ▶

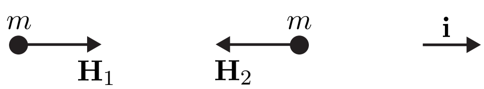


Figure 23 Force diagrams for the two atoms

◀ Apply Newton's 2nd law ▶

Applying Newton's second law (in the form of equation (30)) to each particle, we obtain

$$m\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1, \quad m\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_2, \tag{36}$$

where $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$, and $\mathbf{i}$ is a unit vector in the direction of positive x_1 and x_2.

Then from equation (31),

$$\Delta\mathbf{H}_1 = k(x_2 - x_1)\mathbf{i},$$
$$\Delta\mathbf{H}_2 = k(x_2 - x_1)(-\mathbf{i}) = -k(x_2 - x_1)\mathbf{i}.$$

Substituting these expressions into equations (36) gives

$$m\ddot{\mathbf{r}}_1 = k(x_2 - x_1)\mathbf{i},$$
$$m\ddot{\mathbf{r}}_2 = -k(x_2 - x_1)\mathbf{i}.$$

Resolving in the **i**-direction and using matrix notation, we obtain the equation of motion that represents the vibrations of a hydrogen molecule:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k/m & k/m \\ k/m & -k/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) To find the eigenvalues and eigenvectors of the dynamic matrix, it is convenient to take out the common factor k/m and work with the matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. The eigenvalues are found by solving

◄ Solve equation(s) ►

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda = 0.$$

So the eigenvalues are $\lambda = 0$ and $\lambda = -2$. We now proceed to find the eigenvectors.

- For $\lambda = 0$, the eigenvector equations are

  $$\begin{pmatrix} -1-0 & 1 \\ 1 & -1-0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  or equivalently

  $$-v_1 + v_2 = 0,$$
  $$v_1 - v_2 = 0.$$

  Both of these equations reduce to $v_2 = v_1$, so $(1 \quad 1)^T$ is an eigenvector.

- For $\lambda = -2$, the eigenvector equations are

  $$\begin{pmatrix} -1-(-2) & 1 \\ 1 & -1-(-2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  or equivalently

  $$v_1 + v_2 = 0,$$
  $$v_1 + v_2 = 0.$$

  Both of these equations reduce to $v_2 = -v_1$, so $(1 \quad -1)^T$ is an eigenvector.

To obtain the eigenvalues of the dynamic matrix, we must multiply the above eigenvalues by the factor k/m. Hence the dynamic matrix has eigenvalues 0 and $-2k/m$.

In previous examples all the eigenvalues were negative, so all the terms in the general solution of the equation of motion were sinusoidal. But this time we have one zero eigenvalue and one negative eigenvalue.

See Procedure 6 of Unit 6.

You may recall that a zero eigenvalue gives rise to a linear term in the general solution of the equation of motion, so in this case the general solution has the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(A+Bt) + C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega t+\phi),$$

where A, B, C and ϕ are constants, and $\omega = \sqrt{2k/m}$. Therefore there is just one normal mode, with normal mode angular frequency ω, which is given by the sinusoidal term, and one other 'mode', which is given by the linear term.

◀ Interpret solution ▶

(c) As just noted, the sinusoidal term in the general solution represents a normal mode. Since a corresponding normal mode eigenvector is $(1 \quad -1)^T$, the system will vibrate in this normal mode if it starts from rest with the first particle a distance d from its origin and the second particle a distance $-d$ from its origin, for any non-zero value of d. The angular frequency of the vibrations will be $\omega = \sqrt{2k/m}$.

The linear term in the general solution represents a solution of the form $\mathbf{x}(t) = \mathbf{v}(A+Bt)$, where $\mathbf{v} = (1 \quad 1)^T$; this corresponds to the linear motion of the system. Now $\dot{\mathbf{x}}(t) = \mathbf{v}B$, so if the system starts from rest, that is, $\dot{\mathbf{x}}(0) = \mathbf{v}B = 0$, it follows that $B = 0$, giving $\mathbf{x}(t) = \mathbf{v}A = (A \quad A)^T$. Thus if the particles start from rest, equidistant from their equilibrium positions and in the same direction, then they will remain static for all t. Alternatively, putting $\mathbf{x}(0) = \mathbf{0}$, we obtain $\mathbf{x}(0) = \mathbf{v}A = \mathbf{0}$, so $A = 0$, hence $\mathbf{x}(t) = \mathbf{v}Bt$ and $\dot{\mathbf{x}}(t) = \mathbf{v}B = (B \quad B)^T$. The system will move in this mode if the particles start at their equilibrium positions with the same velocity, in which case they will continue to move indefinitely at the same velocity. In both of these 'linear modes', the particles remain separated by a distance equal to the natural length of the spring (consequently there are no forces exerted by the spring, and therefore no vibrations).

More specifically in terms of the hydrogen molecule, the normal mode given by the sinusoidal term relates to the vibrations of the atoms within the molecule, while the 'mode' given by the linear term relates to the translational motion of the molecule.

(d) From the given frequency of vibration, we can calculate the normal mode angular frequency as

$$\omega = 2\pi f = 2\pi \times 1.3 \times 10^{14} \simeq 8.2 \times 10^{14}.$$

But $\omega = \sqrt{2k/m}$, so

$$k = \tfrac{1}{2}m\omega^2 = \tfrac{1}{2} \times 1.67 \times 10^{-27} \times (8.2 \times 10^{14})^2 \simeq 557.$$

Hence the stiffness of the bond in the hydrogen molecule is about $560\,\mathrm{N\,m^{-1}}$.

Hanging a 1 kg object on a spring of this stiffness would extend it by only 2 cm. This suggests that the atoms in a hydrogen molecule are very tightly bonded.

In part (c) of Example 7 you saw how a zero eigenvalue leads to a linear term in the general solution of the equation of motion, and how this term can be interpreted as representing linear motion, with the constituent parts of the system staying a fixed distance apart, and with each constituent part either moving with the same velocity or remaining at rest.

This type of motion, where all the constituent parts move as one, is given a special name.

Rigid body motion is the motion that occurs when all the particles of a system move with the distances between them remaining invariant; that is, the particles move as if part of a rigid body.

Rigid bodies were introduced in Unit 2. You may recall that the rigid body model assumes that whatever forces act on the body, it does not change shape or vibrate.

If a mechanical system is not constrained by being attached to a fixed object (e.g. the ground or a wall) but is free to move, then it is capable of executing rigid body motion. For such a system, one or more of the eigenvalues of the dynamic matrix will be zero. For one-dimensional systems, there is at most one type of rigid body motion (translation along the axis), so there is at most one zero eigenvalue for any dynamic matrix. For two-dimensional systems, there are three possible distinct (i.e. linearly independent) rigid body motions (translation along either axis, plus a rotation), so the situation is more complicated, with up to three zero eigenvalues.

The free motion of two-dimensional systems is discussed in Unit 19.

Although a zero eigenvalue does not lead to a normal mode as defined in Section 1, the resulting mode has similar properties and is often loosely referred to as a normal mode, as in the next two exercises. In this unit, if you are asked to determine or analyse normal modes, you should always include these 'linear modes' in your answer.

Sometimes a 'linear mode' is referred to as a 'trivial normal mode'.

Exercise 9

Consider a railway engine connected to two trucks, as shown in Figure 24. The trucks and the engine are modelled as particles of equal mass m, and the couplings are modelled as model springs of equal stiffness k. Assume that the train travels along a horizontal straight frictionless track and that the engine provides no motive force. The positions of the components of the system are measured from the equilibrium positions shown in the figure.

(a) Derive the equation of motion for this system.

(b) Use the following information to obtain the general solution of the equation of motion:

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}.$$

(c) For each normal mode, state the initial conditions that give rise to it.

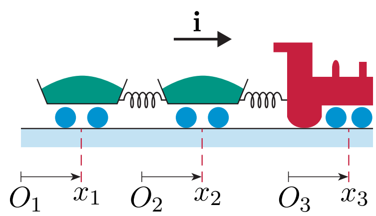


Figure 24 A railway engine and two trucks

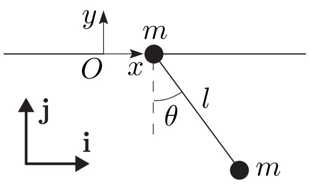


Figure 25 A slider pendulum

Exercise 10

This exercise considers a mechanical system that is not one-dimensional, but it belongs here because it is a simple system that exhibits free motion. Since the system is not one-dimensional, the method introduced in this section does not apply and you will need to use the more general approach of Section 1 (see Exercise 4).

Figure 25 shows a particle of mass m that slides freely along a smooth straight horizontal bar and has a pendulum suspended from it. Model the pendulum stem as a light model rod of length l, and the bob as a particle of mass m. Assume that the oscillations of the pendulum are small. Measure the displacement x_1 of the sliding particle from a fixed point O on the bar, and the displacement of the pendulum by the angle θ (measured anticlockwise from the vertical), as shown in the figure.

(a) Derive linear equations of motion for the system, in terms of x_1 and θ.

(b) Find the normal mode angular frequencies and eigenvectors, and hence write down the general solution of the matrix equation of motion.

(c) How does the period of this pendulum compare with the period of a simple pendulum of the same length (with equation of motion $\ddot{\theta} = -(g/l)\theta$)?

(d) For each of the normal modes, draw a sketch of the motion.

3 Modelling a guitar string

In this section a series of models that approximate the behaviour of a guitar string are analysed.

Vibrations of a stretched string, such as a guitar string, that are aligned with the string are **longitudinal**, while vibrations at right angles to the string are **transverse**. In this context, the **fundamental** is the normal mode of a system that has the lowest non-zero normal mode angular frequency.

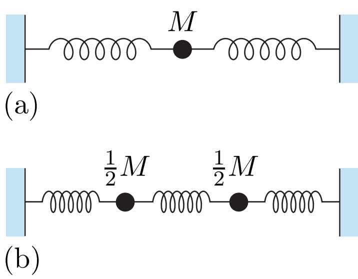


Figure 26 Modelling a guitar string

You might think that we could model a guitar string as a model spring, but this model would not be able to predict the vibrations of the string because, by definition, a model spring has no mass thus, in effect, there would be nothing to vibrate! To model the vibrations of a guitar string, the string has to be represented as having both mass and elasticity. The simplest such model is shown in Figure 26(a). In this diagram, all the mass of the string is 'lumped' together as a single particle in the middle of the string, and the elasticity is divided into two identical model springs. Figure 26(b) shows a revised model, where the string is represented as two particles of equal mass joined by three identical model springs. Since the revised model in Figure 26(b) is, in some sense, closer to the real situation where the mass is continuously distributed along the string, we would expect this to be a better model. Similarly, by increasing the number of particles and springs, we would expect to produce better and better models.

In the models of a guitar string described above, the properties of a real object are lumped together into discrete components (e.g. particles and model springs). This type of model is given a special name.

A model of a real-world system in which properties of the system are lumped together into discrete components (e.g. particles and model springs) is called a **lumped parameter model**.

To make the discussion less abstract, we will consider a particular string on the guitar, namely the E string, which has fundamental frequency 323 Hz. We will take an example of an E string that has stiffness $4000\,\mathrm{N\,m^{-1}}$, mass 0.25 g, and natural length 0.633 m. When this string is tuned to the note E, its length is 0.65 m. We will model the E string by means of lumped parameter models: the purpose of the modelling is to construct a model of the E string that successfully predicts the fundamental frequency of the string as 323 Hz.

A different sort of model of the E string, as an elastic heavy continuous string, is discussed in Unit 14.

Before we begin the modelling, we introduce a quantity that will prove useful.

The **equilibrium tension in a model spring**, T_{eq}, is defined by the formula

$$T_{\mathrm{eq}} = k(l_{\mathrm{eq}} - l_0), \tag{37}$$

where k is the stiffness of the spring, l_0 is its natural length, and l_{eq} is its equilibrium length.

The equilibrium tension in a model spring is a scalar quantity that is equal to plus or minus the magnitude of the force exerted by the spring, with the sign depending on whether the spring is extended or compressed when in equilibrium. For the E string modelled as a single model spring, we have

$$T_{\mathrm{eq}} = 4000(0.65 - 0.633) = 68,$$

so the equilibrium tension in the E string is 68 N.

It was noted above that a single-spring model is inadequate when considering the *vibrations* of a guitar string, but we can use a single-spring model for a guitar string that remains *in equilibrium*.

In modelling the E string by means of lumped parameter models, we will need to ensure that the equilibrium tension in each model spring is the same as the equilibrium tension in the E string modelled as a single spring. This means that although l_{eq}, l_0 and k will change as we introduce more springs, T_{eq} will remain the same, at 68 N. Hence as we use more springs, the stiffness of these springs increases by the same factor by which their natural length and equilibrium length decrease.

To keep the modelling simple, in any given model we will use identical model springs.

A basic model of the E string is derived in Subsection 3.1, and this is revised in Subsection 3.2. Subsection 3.3 investigates briefly how further revisions might improve the model.

3.1 A first model

We start by developing the basic model of the guitar string that lumps all the mass at the centre of the string.

Example 8

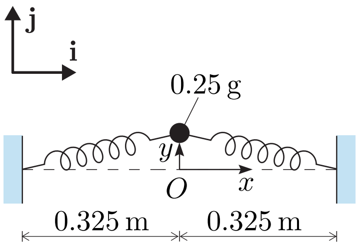


Figure 27 Guitar string modelled as a single particle and two springs

Suppose that the E string is modelled with all its mass (0.25 g) lumped together as a particle at the centre of the string, with the particle connected by model springs to endpoints 0.65 m apart, as shown in Figure 27. The equilibrium length of each of the two springs is half the equilibrium length of the E string, so $l_{\text{eq}} = 0.65/2 = 0.325$, but the equilibrium tension in each spring is the same as the equilibrium tension in the E string (68 N, as calculated above).

In order to simplify the analysis, the following assumptions are made.

- The oscillations are in a horizontal plane. This means that we have a *two-dimensional* mechanical system modelled solely by springs and particles. Consequently (by the argument in Subsection 2.1), if we measure displacements from the equilibrium position of the particle, the force of gravity can be ignored in deriving the equation of motion.
- The particle is displaced in a direction perpendicular to the equilibrium alignment of the springs (i.e. along the y-axis marked on Figure 27); therefore the vibrations are transverse. This assumption means that the system behaves as if it has one degree of freedom.
- The oscillations are small, that is, the displacement y is small compared with the length of the string.

Calculate the fundamental frequency of vibration.

Solution

◀ Choose coordinates ▶

Choose the axes that are shown in Figure 27, with the origin at the equilibrium position of the particle (i.e. the point halfway between the endpoints).

◀ State assumption(s) ▶

◀ Draw force diagram(s) ▶

The only two forces affecting the particle's motion are those due to the two springs (as gravity can be ignored), so the force diagram (using the usual notation) is as follows.

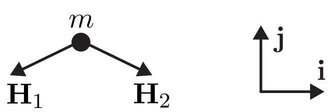


Figure 28 Force diagram for the single-particle model

◀ Apply Newton's 2nd law ▶

Newton's second law applied to the particle gives

$$m\ddot{\mathbf{r}} = \mathbf{H}_1 + \mathbf{H}_2, \tag{38}$$

where $\mathbf{r} = y\mathbf{j}$, and $\mathbf{j}$ is a unit vector in the positive y-direction.

Now the forces acting on the particle must be modelled. The displacement of the particle is not aligned with the springs, therefore the changes in the forces acting cannot be modelled using equation (31). Instead, we use the full form of Hooke's law, $\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}$.

You will see below that an alternative strategy is to make use of equation (30).

First, consider $\mathbf{H}_1$. From Figure 29 you can see that a unit vector from the particle to the centre of the spring is $\widehat{\mathbf{s}} = -(l_{\text{eq}}\mathbf{i} + y\mathbf{j})/l$, where $l = \sqrt{l_{\text{eq}}^2 + y^2}$. Thus, by Hooke's law,

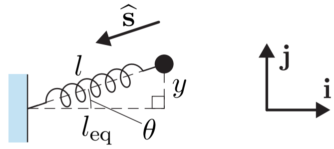


Figure 29 Defining the unit vector $\widehat{\mathbf{s}}$

$$\mathbf{H}_1 = k(l - l_0)\left(\frac{-(l_{\text{eq}}\mathbf{i} + y\mathbf{j})}{l}\right).$$

Under the assumption that the oscillations are small compared with the length of the string, we can use the approximation $l \simeq l_{\text{eq}}$ to simplify this equation to

$$\mathbf{H}_1 \simeq -k(l_{\text{eq}} - l_0)\left(\mathbf{i} + \frac{y}{l_{\text{eq}}}\mathbf{j}\right).$$

We can then simplify further by using the equation for the equilibrium tension in a model spring, $T_{\text{eq}} = k(l_{\text{eq}} - l_0)$, to obtain

$$\mathbf{H}_1 \simeq -T_{\text{eq}}\left(\mathbf{i} + \frac{y}{l_{\text{eq}}}\mathbf{j}\right). \tag{39}$$

Similarly, for $\mathbf{H}_2$ we have $\widehat{\mathbf{s}} = (l_{\text{eq}}\mathbf{i} - y\mathbf{j})/l$, hence

$$\mathbf{H}_2 \simeq T_{\text{eq}}\left(\mathbf{i} - \frac{y}{l_{\text{eq}}}\mathbf{j}\right).$$

Substituting into equation (38), we obtain the model

$$m\ddot{\mathbf{r}} = -T_{\text{eq}}\left(\mathbf{i} + \frac{y}{l_{\text{eq}}}\mathbf{j}\right) + T_{\text{eq}}\left(\mathbf{i} - \frac{y}{l_{\text{eq}}}\mathbf{j}\right) = -T_{\text{eq}}\,\frac{2y}{l_{\text{eq}}}\,\mathbf{j}.$$

As this is a *model* of the motion, we revert to = rather than $\simeq$ signs.

Resolving in the $\mathbf{j}$-direction and rearranging gives

$$\ddot{y} = -\frac{2T_{\text{eq}}}{ml_{\text{eq}}}\,y.$$

◀ Solve equation(s) ▶

Writing this in the form $\ddot{y} + \omega^2 y = 0$, where $\omega^2 = 2T_{\text{eq}}/(ml_{\text{eq}})$, we can see that this is the equation for simple harmonic motion (as in Unit 9). We can therefore write its general solution as

$$y(t) = A\cos(\omega t + \phi),$$

where the angular frequency ω is given by

$$\omega = \sqrt{\frac{2T_{\text{eq}}}{ml_{\text{eq}}}}, \tag{40}$$

and A and ϕ are constants that can be determined from the initial conditions.

◀ Interpret solution ▶

Substituting in the data in the question, we find that the angular frequency for this vibration is

$$\omega = \sqrt{\frac{2 \times 68}{(0.25 \times 10^{-3}) \times 0.325}} \simeq 1294.$$

The fundamental frequency of the guitar string is then obtained by converting the above angular frequency to a frequency:

$$f = \frac{\omega}{2\pi} = \frac{1294}{2\pi} \simeq 206.$$

So the fundamental frequency of vibration predicted by this first model of the E string is approximately 206 Hz.

The frequency predicted in Example 8 is not close to 323 Hz, the experimentally determined fundamental frequency of the E string. This suggests that the model needs modifying. Accordingly, modifications are made in the next exercise and, in a different way, in the next subsection.

Exercise 11

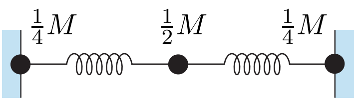


Figure 30 An alternative strategy for distributing the mass of the guitar string

The model of a guitar string used in Example 8 is not the best one-degree-of-freedom model. A better model, based on the argument that the ends of the string are not moving, involves distributing the total mass, M say, between a particle of mass $m = \frac{1}{2}M$ in the middle of the string and particles of mass $\frac{1}{4}M$ at either end, with two model springs in between, as shown in Figure 30. Only the particle of mass $\frac{1}{2}M$ in the middle vibrates.

Use equation (40) to find the fundamental frequency of vibration predicted by this model. Comment on your answer.

In Example 8 the particle was displaced by a small amount, $y\mathbf{j}$, and the force exerted by the left-hand spring on the particle was given by equation (39) as

$$\mathbf{H}_1 \simeq -T_{\text{eq}} \left(\mathbf{i} + \frac{y}{l_{\text{eq}}} \mathbf{j} \right) = -T_{\text{eq}} \mathbf{i} - \frac{T_{\text{eq}}}{l_{\text{eq}}} y \, \mathbf{j}.$$

Now $-T_{\text{eq}}\mathbf{i}$ is simply the force exerted by the spring when the system is in equilibrium, so the approximate change in the force from its equilibrium value is

$$\Delta\mathbf{H}_1 \simeq -\frac{T_{\text{eq}}}{l_{\text{eq}}} y \, \mathbf{j}.$$

We could have used this approximation and the corresponding one for $\Delta\mathbf{H}_2$, in conjunction with equation (30), to obtain the equation of motion in Example 8.

This expression for the approximate change in the force due to a spring holds for any model spring when a small displacement $y\mathbf{j}$ is made at one end and at right angles to the equilibrium alignment of the spring. Hence we have the useful approximation

$$\Delta\mathbf{H} \simeq -\frac{T_{\text{eq}}}{l_{\text{eq}}} y \, \mathbf{j}. \tag{41}$$

We will make frequent use of this approximation in the next subsection, where we try to further increase the accuracy of the fundamental frequency by modelling the guitar string as n particles and $n+1$ springs, as n increases.

3.2 Revised models

In the previous subsection we looked at two lumped parameter models of the E string of the guitar, both of which behaved as if they had one degree of freedom. In this subsection we will try to improve the accuracy of our models by considering lumped parameter models of the E string that behave as if they have two or more degrees of freedom.

Example 9

The next model of the E string has two identical particles and three identical model springs, as shown in Figure 31.

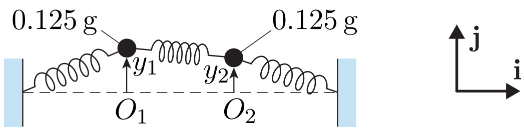


Figure 31 Two-particle model of the guitar string

The mass of the string is evenly distributed between the two particles (i.e. 0.125 g each), and the equilibrium length of each spring is $l_{\text{eq}} = 0.65/3 \simeq 0.217$. The equilibrium tension in the springs, T_{eq}, remains the same (68 N) as in previous models. The three assumptions made in Example 8 are used again here.

Calculate the fundamental frequency of vibration.

Solution

Choose the axes that are shown in Figure 31, with the origins at the equilibrium positions of the particles. ◀ Choose coordinates ▶

The only two forces affecting each particle's motion are those due to the two springs attached to it (as gravity can be ignored). ◀ State assumption(s) ▶ The force diagrams for the two particles (using the usual notation) are shown in Figure 32. ◀ Draw force diagram(s) ▶

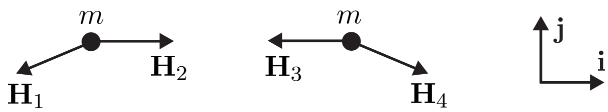


Figure 32 Force diagrams for the two particles

Applying equation (30) to the two particles in turn, we obtain ◀ Apply Newton's 2nd law ▶

$$m\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2, \tag{42}$$

$$m\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4, \tag{43}$$

where $\mathbf{r}_1 = y_1\mathbf{j}$ and $\mathbf{r}_2 = y_2\mathbf{j}$ are the displacements of the first and second particles, respectively, and $\mathbf{j}$ is a unit vector in the positive y-direction.

Now the forces must be modelled. From approximation (41) we have

Since we are modelling here, we write = rather than $\simeq$ when using (41).

$$\Delta\mathbf{H}_1 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\, y_1\,\mathbf{j}, \quad \Delta\mathbf{H}_4 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\, y_2\,\mathbf{j}.$$

If $y_1 > y_2$, the relative displacement is $(y_1 - y_2)\mathbf{j}$. If $y_2 > y_1$, the relative displacement is $(y_2 - y_1)(-\mathbf{j}) = (y_1 - y_2)\mathbf{j}$.

The computations of the other forces are complicated by the fact that both ends of the central spring are displaced. For the force $\mathbf{H}_2$, the spring is displaced at the left-hand end by $y_1 - y_2$ relative to the right-hand end, so from approximation (41) we have

$$\Delta\mathbf{H}_2 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}(y_1 - y_2)\,\mathbf{j}.$$

Similarly, we obtain

This expression for $\Delta\mathbf{H}_3$ can also be obtained by using the fact that $\mathbf{H}_3 = -\mathbf{H}_2$ (since they are forces exerted by the same model spring), hence $\Delta\mathbf{H}_3 = -\Delta\mathbf{H}_2$.

$$\Delta\mathbf{H}_3 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}(y_2 - y_1)\,\mathbf{j} = \frac{T_{\text{eq}}}{l_{\text{eq}}}(y_1 - y_2)\,\mathbf{j}.$$

Substitution into equations (42) and (43) gives

$$m\ddot{\mathbf{r}}_1 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\,y_1\,\mathbf{j} - \frac{T_{\text{eq}}}{l_{\text{eq}}}(y_1 - y_2)\,\mathbf{j},$$

$$m\ddot{\mathbf{r}}_2 = \frac{T_{\text{eq}}}{l_{\text{eq}}}(y_1 - y_2)\,\mathbf{j} - \frac{T_{\text{eq}}}{l_{\text{eq}}}\,y_2\,\mathbf{j}.$$

Resolving in the $\mathbf{j}$-direction and rearranging, we obtain

$$\ddot{y}_1 = -2\frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_1 + \frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_2,$$

$$\ddot{y}_2 = \frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_1 - 2\frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_2,$$

which can be written in matrix form as

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \frac{T_{\text{eq}}}{ml_{\text{eq}}}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

◀ Solve equation(s) ▶

To solve this, the eigenvalues of the matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ must be found by solving

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0,$$

to give $\lambda = -1$ and $\lambda = -3$. So the eigenvalues of the dynamic matrix are $-T_{\text{eq}}/(ml_{\text{eq}})$ and $-3T_{\text{eq}}/(ml_{\text{eq}})$.

Hence the normal mode angular frequencies are

$$\omega_1 = \sqrt{\frac{T_{\text{eq}}}{ml_{\text{eq}}}} \quad\text{and}\quad \omega_2 = \sqrt{\frac{3T_{\text{eq}}}{ml_{\text{eq}}}}.$$

◀ Interpret solution ▶

The objective of this analysis is to calculate the fundamental frequency of vibration. This corresponds to the smaller normal mode angular frequency, thus is given by

$$f = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{68}{(0.125 \times 10^{-3}) \times 0.217}} \simeq 252.$$

Therefore the fundamental frequency is approximately 252 Hz.

The frequency 252 Hz obtained in Example 9 is still a long way below the 323 Hz experimental value, but it is much better than the 206 Hz predicted by the model with one degree of freedom in Example 8. Further revisions of the model are considered in Exercises 12 and 13.

Exercise 12

The model in Example 9 may be improved by taking the two moving particles each to have mass $\frac{1}{3}M$, while the remaining $\frac{1}{3}M$ is modelled by particles placed at either end of the string ($\frac{1}{6}M$ at each end); see Figure 33. Only the two particles of mass $\frac{1}{3}M$ vibrate. Find the fundamental frequency of vibration predicted by this model. Comment on your answer.

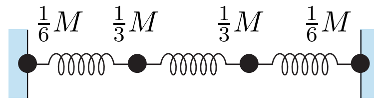


Figure 33 An alternative two-particle model of the guitar string

Exercise 13

Now consider the next refinement in developing lumped parameter models of the E string of the guitar, that is, the model shown in Figure 34, which behaves as if it has three degrees of freedom. In this model, the mass is distributed evenly between three identical particles, with four identical model springs linking the particles to one another and to the endpoints. All the particles vibrate. This model is a refinement of those in Examples 8 and 9, and uses the same simplifying assumptions and data.

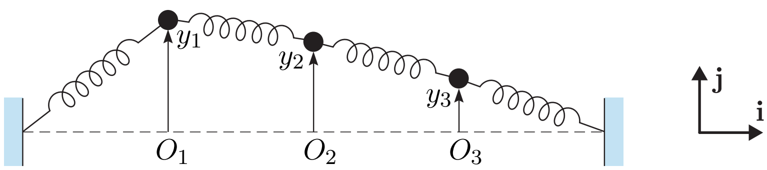


Figure 34 Three-particle model of the guitar string

(a) Use the axes shown in Figure 34 and the methods of Example 9 to derive the matrix form of the equation of motion for the system.

(b) The eigenvalues of the matrix

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

are -0.586, -2 and -3.414. Using this information, calculate the fundamental frequency of vibration predicted by this model. Comment on your answer.

3.3 Approaching the limit

So far in this section we have considered various lumped parameter models of a guitar string. This subsection links together the results of the previous two subsections and investigates the behaviour of the models as the number of degrees of freedom increases. However, we will look only at models where *all* the mass of the string is assumed to vibrate; we will not consider the models of Exercises 11 and 12 where some of the mass does not vibrate.

In Example 8, a one-particle model was studied and the fundamental frequency was calculated to be approximately 206 Hz. In Example 9, a two-particle model was studied and the fundamental frequency was calculated to be approximately 252 Hz. Exercise 13 calculated the fundamental frequency for a three-particle model as approximately 273 Hz. What happens as the number of particles increases beyond three? Does the predicted fundamental frequency become closer and closer to the experimental value of 323 Hz as the number of particles is increased?

To answer these questions, we look at the equations of motion that were derived for the various models, and see if a pattern emerges.

In Example 8, the equation of motion was

$$\ddot{y} = -\frac{2T_{\text{eq}}}{ml_{\text{eq}}}\, y.$$

In order to make the comparisons easier, we will change the notation slightly. The y here is the y-coordinate of the first particle, so we will call it y_1. The m is the total mass of the guitar string, and we will call this M. The l_{eq} is the equilibrium length of each spring, and is the distance between the endpoints, L, divided by 2. Therefore the equation of motion can be written as

$$\ddot{y}_1 = \frac{T_{\text{eq}}}{M(L/2)} \times (-2) \times y_1. \tag{44}$$

In Example 9, the equation of motion was

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \frac{T_{\text{eq}}}{ml_{\text{eq}}} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Writing this in terms of the same parameters as equation (44) gives

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \frac{T_{\text{eq}}}{(M/2)(L/3)} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{45}$$

The equation of motion from Exercise 13 can be written in terms of M and L as

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} = \frac{T_{\text{eq}}}{(M/3)(L/4)} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{46}$$

By examining the pattern emerging from equations (44), (45) and (46), you should be able to predict that the equation of motion for the corresponding four-particle model is

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{pmatrix} = \frac{T_{\text{eq}}}{(M/4)(L/5)} \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

This pattern enables a computer program to be written to compute the fundamental frequency f of the corresponding n-particle model, for any positive integer value of n, and then to plot the results on a graph. Such a graph is shown in Figure 35. It can be seen that the predicted fundamental frequency approaches the experimental value of 323 Hz. The convergence to this value is rapid at first, but slows down as n increases: a model with ten degrees of freedom predicts 307 Hz, a model with twenty degrees of freedom predicts 318 Hz, and a model with fifty degrees of freedom predicts 320 Hz.

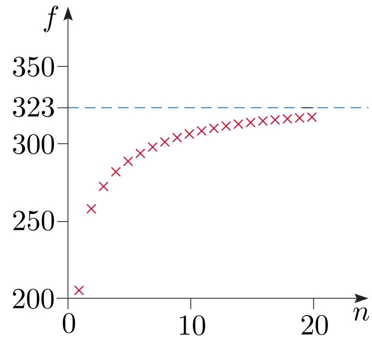


Figure 35 Predicted frequency as the number of particles increases

Up to now in this section we have looked at the vibrations of the E string of a guitar under the assumption that the string is displaced only in a direction perpendicular to its equilibrium alignment, that is, that there is only transverse vibration. But in practice, there is some *longitudinal vibration* too, that is, some vibration along the length of the string. The following exercises examine this longitudinal vibration.

Exercise 14

If you look back at the start of Subsection 2.1, you should recognise that the system shown in Figure 19 is similar to the one-particle lumped parameter model of the E string considered in Example 8. However, in the former case it was assumed that the particle was constrained to move in one dimension along the line of the springs, that is, we modelled longitudinal vibrations, whereas in Example 8 we modelled transverse vibrations. Now, by using the ideas of Subsection 2.1, we can model the longitudinal vibrations in the lumped parameter models of the E string.

Example 6 did this for what was, in effect, a two-particle lumped parameter model. In this exercise you are asked to extend the ideas of Subsection 2.1 to model the longitudinal vibrations of the three-particle lumped parameter model of the E string, as illustrated in Figure 36. (You considered the transverse vibrations in Exercise 13.)

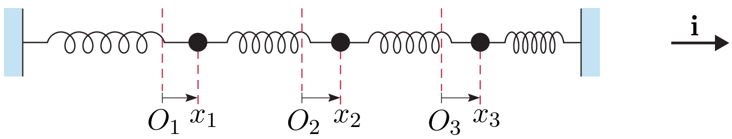


Figure 36 Three-particle system with longitudinal vibrations

(a) Derive the matrix form of the equation of motion for the longitudinal vibrations of this three-particle model.

(b) Verify that $(1 \quad \sqrt{2} \quad 1)^T$, $(1 \quad 0 \quad -1)^T$ and $(1 \quad -\sqrt{2} \quad 1)^T$ are eigenvectors of the dynamic matrix for this model, and hence deduce the eigenvalues of the dynamic matrix.

You met these eigenvectors in Example 5 and Exercise 6.

(c) Determine the normal mode angular frequencies of the longitudinal vibrations of this model, given that the mass of each particle is $\frac{1}{3}(0.25 \times 10^{-3})$ kg, and that each model spring has natural length and equilibrium length equal to a quarter of the corresponding length of the E string (so for a fixed equilibrium tension of 68 N, each model spring has stiffness $4 \times 4000 = 16\,000\,\mathrm{N\,m^{-1}}$).

From equation (37), $T_{\text{eq}} = k(l_{\text{eq}} - l_0)$, so for a *constant* T_{eq}, dividing l_{eq} and l_0 by 4 means multiplying k by 4.

Exercise 15

The aim of this question is to calculate the fundamental frequency of longitudinal vibration of the E string of a guitar by using an argument similar to the one in this subsection. Successively more sophisticated models of the longitudinal vibrations of the E string are given by equation (29) (with $k_1 = k_2 = k$ and $x_1 = x$), equations (34) and (35) (with $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$), and the equations of motion from the Solution to Exercise 14, as follows:

$$m\ddot{x}_1 = -2kx_1, \tag{47}$$

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2, \\ m\ddot{x}_2 = kx_1 - 2kx_2, \end{cases} \tag{48}$$

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2, \\ m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3, \\ m\ddot{x}_3 = kx_2 - 2kx_3. \end{cases} \tag{49}$$

Suppose that the n-degrees-of-freedom model of the longitudinal vibrations represents the guitar string (mass M, equilibrium length L, natural length L_0 and stiffness K) by $n+1$ identical model springs connecting n particles of equal mass.

(a) Write equation (47) in terms of properties of the guitar string, that is, T_{eq}, M, L and L_0.

(*Hint*: Use equation (37) to eliminate the stiffness k from the equation, bearing in mind that the equilibrium tension in each spring is the same as the equilibrium tension in the guitar string.)

(b) Repeat part (a) for the sets of equations (48) and (49), giving your answers in matrix form.

(c) Use your answers to parts (a) and (b) to deduce the equation of motion for the corresponding four-degrees-of-freedom model of the longitudinal vibrations of the guitar string.

(d) By comparing your answers with the results of this subsection, show that the ratio of the fundamental frequency for the longitudinal vibrations to the fundamental frequency for the transverse vibrations of the guitar string is $\sqrt{L/(L-L_0)}$.

If the fundamental frequency of the transverse vibrations is $323\,\text{Hz}$, calculate the fundamental frequency of the longitudinal vibrations of the guitar string.

Learning outcomes

After studying this unit, you should be able to:

- understand the terms degrees of freedom, normal mode, normal mode angular frequency, in-phase motion, phase-opposed motion and rigid body motion, all in the context of an oscillating mechanical system
- derive the equation of motion for simple one- and two-dimensional oscillating mechanical systems (without damping or forcing)
- solve the equation of motion for simple one- and two-dimensional oscillating mechanical systems by finding the eigenvalues and corresponding normal mode eigenvectors of the dynamic matrix
- interpret the normal mode eigenvectors of a simple oscillating mechanical system, in terms of the initial conditions required for the system to oscillate in a normal mode
- model a simple oscillating mechanical system by taking the equilibrium positions of the particles of the system as the origins of coordinates, and using formulas involving the changes in forces
- understand how to model a simple oscillating mechanical system as a lumped parameter model.

Solutions to exercises

Solution to Exercise 1

(a) This system has two degrees of freedom.

(b) This system has two degrees of freedom.

(c) This system has two degrees of freedom. (The additional spring does not alter the number of coordinates needed.)

(d) This system has one degree of freedom.

(e) This system has three degrees of freedom.

(f) This system has three degrees of freedom. (The additional spring does not alter the number of coordinates needed.)

Solution to Exercise 2

(a) These graphs represent normal mode motion, since both x_1 and x_2 vary sinusoidally with the same frequency (just over three cycles completed in 10 seconds) and phase angle. The two particles are in-phase.

(b) These three graphs represent normal mode motion, since the three coordinates vary sinusoidally with the same frequency (again, just over three cycles completed in 10 seconds) and phase angle. For the pairs of particles:

x_1 and x_2 are phase-opposed,
x_1 and x_3 are in-phase,
x_2 and x_3 are phase-opposed.

(c) These graphs do not represent normal mode motion. (The period of the second graph is more than twice that of the first.)

Solution to Exercise 3

From the given data, $k/m_1 = 2$ and $k/m_2 = 1$. So the dynamic matrix of the system is

$$\begin{pmatrix} -4 & 2 \\ 1 & -2 \end{pmatrix}.$$

The eigenvalues are found by solving the quadratic equation

$$\begin{vmatrix} -4-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 6 = 0,$$

which gives

$$\lambda_1 = -3 + \sqrt{3} \simeq -1.268 \quad \text{and} \quad \lambda_2 = -3 - \sqrt{3} \simeq -4.732.$$

- For $\lambda_1 = -3 + \sqrt{3}$, the eigenvector equations are

$$\begin{aligned}(-1-\sqrt{3})v_1 + \qquad 2v_2 &= 0,\\ v_1 + (1-\sqrt{3})v_2 &= 0,\end{aligned}$$

  which both reduce to the same equation. Putting $v_1 = 1$ gives $v_2 \simeq 1.366$, so $(1 \quad 1.366)^T$ is an eigenvector.

- For $\lambda_2 = -3 - \sqrt{3}$, the eigenvector equations are

$$\begin{aligned}(-1+\sqrt{3})v_1 + \qquad 2v_2 &= 0,\\ v_1 + (1+\sqrt{3})v_2 &= 0,\end{aligned}$$

  which both reduce to the same equation. Putting $v_1 = 1$ gives $v_2 \simeq -0.366$, so $(1 \quad -0.366)^T$ is an eigenvector.

The normal mode angular frequencies are

$$\omega_1 = \sqrt{-\lambda_1} \simeq 1.126 \quad \text{and} \quad \omega_2 = \sqrt{-\lambda_2} \simeq 2.175.$$

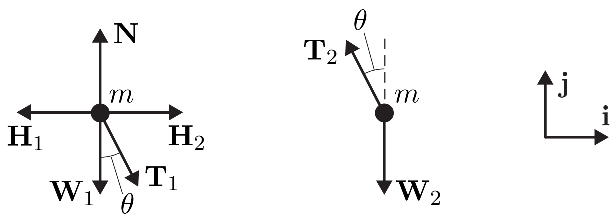
So the general solution of the equation of motion for the specified mechanical system can be written as

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 1.366 \end{pmatrix} \cos(1.126t + \phi_1) + C_2 \begin{pmatrix} 1 \\ -0.366 \end{pmatrix} \cos(2.175t + \phi_2),$$

where C_1, C_2, ϕ_1 and ϕ_2 are constants that can be determined from the initial conditions.

Solution to Exercise 4

(a) Using the usual notation, the force diagrams for the particles can be drawn as follows.



Applying Newton's second law to each particle gives

$$\begin{aligned} m\ddot{\mathbf{r}}_1 &= \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{T}_1 + \mathbf{W}_1 + \mathbf{N},\\ m\ddot{\mathbf{r}}_2 &= \mathbf{T}_2 + \mathbf{W}_2,\end{aligned}$$

where $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$ are the position vectors, relative to O, of the sliding particle and the pendulum bob, with $\mathbf{i}$ and $\mathbf{j}$ being Cartesian unit vectors in the positive x- and y-directions, respectively.

Now use Hooke's law to model the spring forces. For the force $\mathbf{H}_1$, the length of the spring is $l_0 + x_1$ (since the equilibrium position is at the natural length) and $\widehat{\mathbf{s}} = -\mathbf{i}$, so by Hooke's law,

$$\mathbf{H}_1 = k((l_0 + x_1) - l_0)(-\mathbf{i}) = -kx_1\mathbf{i}.$$

Similarly, for the force $\mathbf{H}_2$, the length of the spring is $l_0 - x_1$ and $\widehat{\mathbf{s}} = \mathbf{i}$, so by Hooke's law,

$$\mathbf{H}_2 = k((l_0 - x_1) - l_0)\mathbf{i} = -kx_1\mathbf{i}.$$

These can be substituted into the above equations, along with $\mathbf{W}_1 = \mathbf{W}_2 = -mg\mathbf{j}$, to yield

$$\begin{aligned} m\ddot{\mathbf{r}}_1 &= -kx_1\mathbf{i} - kx_1\mathbf{i} + \mathbf{T}_1 - mg\mathbf{j} + \mathbf{N}, \\ m\ddot{\mathbf{r}}_2 &= \mathbf{T}_2 - mg\mathbf{j}. \end{aligned}$$

Resolving in the **i**- and **j**-directions gives

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + |\mathbf{T}_1|\sin\theta, \\ m\ddot{y}_1 &= -|\mathbf{T}_1|\cos\theta - mg + |\mathbf{N}|, \\ m\ddot{x}_2 &= -|\mathbf{T}_2|\sin\theta, \\ m\ddot{y}_2 &= |\mathbf{T}_2|\cos\theta - mg. \end{aligned}$$

From Figure 15, we can deduce the following relationships between the linear and angular coordinates:

$$x_2 = x_1 + l\sin\theta, \quad y_1 = 0, \quad y_2 = -l\cos\theta.$$

Now we use the small-angle approximations $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$, so $x_2 \simeq x_1 + l\theta$, $y_2 \simeq -l$ (and hence $\ddot{y}_2 \simeq 0$), which on substitution in the equations of motion give

$$\begin{aligned} m\ddot{x}_1 &\simeq -2kx_1 + |\mathbf{T}_1|\theta, \\ 0 &\simeq -|\mathbf{T}_1| - mg + |\mathbf{N}|, \\ m\big(\ddot{x}_1 + l\ddot{\theta}\big) &\simeq -|\mathbf{T}_2|\theta, \\ 0 &\simeq |\mathbf{T}_2| - mg. \end{aligned}$$

From the last of these approximations we have $|\mathbf{T}_2| \simeq mg$, and since the forces exerted at either end of a model rod are equal in magnitude, $|\mathbf{T}_1| = |\mathbf{T}_2| \simeq mg$. So the first and third of the above equations become

$$\begin{aligned} m\ddot{x}_1 &\simeq -2kx_1 + mg\theta, \\ m\big(\ddot{x}_1 + l\ddot{\theta}\big) &\simeq -mg\theta. \end{aligned}$$

(As a check, if $\ddot{x}_1 = 0$ (e.g. the upper particle is fixed), then the last equation gives $\ddot{\theta} \simeq -(g/l)\,\theta$, which is the simple harmonic motion equation that is expected for a simple pendulum.)

Subtract the first displayed approximation from the second to obtain another approximation with only one second derivative on the left-hand side:

$$ml\ddot{\theta} \simeq -mg\theta - (-2kx_1 + mg\theta) = 2kx_1 - 2mg\theta.$$

Hence the linear equations of motion for the system are

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + g\theta, \quad \ddot{\theta} = \frac{2k}{lm}x_1 - \frac{2g}{l}\theta.$$

(b) Writing the equations in matrix form gives

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -2k/m & g \\ 2k/(lm) & -2g/l \end{pmatrix} \begin{pmatrix} x_1 \\ \theta \end{pmatrix}.$$

From the given data, $k/m = 1/1 = 1$ and $l = 1$, so the dynamic matrix becomes

$$\begin{pmatrix} -2 & g \\ 2 & -2g \end{pmatrix}.$$

The eigenvalues of this matrix are found by solving

$$\begin{vmatrix} -2-\lambda & g \\ 2 & -2g-\lambda \end{vmatrix} = \lambda^2 + 2(1+g)\lambda + 2g = 0,$$

to give

$$\lambda = \frac{-2(1+g) \pm \sqrt{4(1+g)^2 - 8g}}{2} = -1 - g \pm \sqrt{1+g^2}.$$

Substituting $g = 9.81$ into this gives the two eigenvalues as -0.95 and -20.67 (to two decimal places). As $\omega = \sqrt{-\lambda}$, these eigenvalues correspond to the two normal mode angular frequencies $0.97\,\mathrm{rad\,s^{-1}}$ and $4.55\,\mathrm{rad\,s^{-1}}$ (to two decimal places). The corresponding periods of normal mode oscillations, $T = 2\pi/\omega$, are $6.48\,\mathrm{s}$ and $1.38\,\mathrm{s}$, respectively.

Solution to Exercise 5

The normal mode eigenvectors derived in Example 3 were $(1 \quad \sqrt{2})^T$ and $(1 \quad -\sqrt{2})^T$. The coordinates chosen to describe the system are the angles θ_1 and θ_2 (see Figure 12), so the eigenvectors refer to these angles. The model is based on *small* angular displacements, so initial conditions where $\theta_1 = 1$ and $\theta_2 = \sqrt{2} \simeq 1.4$ (in radians) are too large. Since both of the pendulum bobs start from rest, we obtain the following possible initial conditions for normal mode motion by scaling the eigenvectors:

normal mode 1: $\theta_1 = 0.1, \quad \theta_2 = 0.14$ (to 2 d.p.);

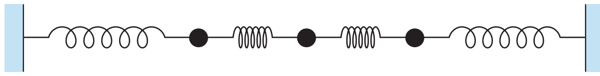
normal mode 2: $\theta_1 = 0.1, \quad \theta_2 = -0.14$ (to 2 d.p.).

Angles of 10° and 14° would also be acceptable.

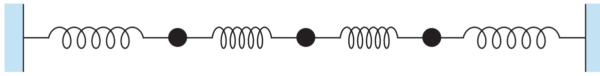
For mode 1, any initial condition in which the ratio of the initial θ_2 displacement to the initial θ_1 displacement is $\sqrt{2}$ is acceptable, provided that both θ_1 and θ_2 are small. Similarly, for mode 2, any initial condition where the ratio is $-\sqrt{2}$ is acceptable, again provided that θ_1 and θ_2 are small.

Solution to Exercise 6

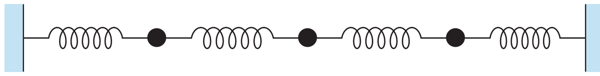
For $(1 \quad 0 \quad -1)^T$, the central particle is stationary, while the other two particles are phase-opposed.



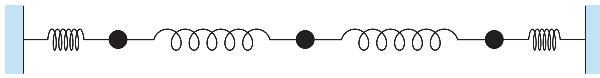
(a)



(b)

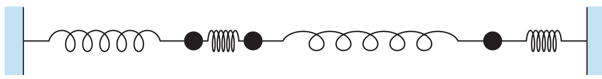


(c)

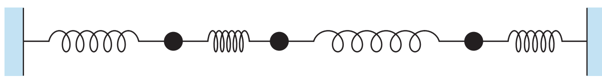


(d)

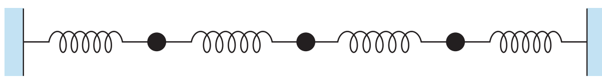
For $(1 \quad -\sqrt{2} \quad 1)^T$, the particle on the left and the central particle are phase-opposed, as are the particle on the right and the central particle. The particles on the left and right are in-phase.



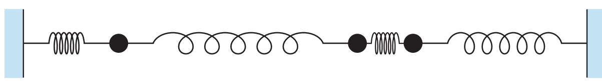
(a)



(b)



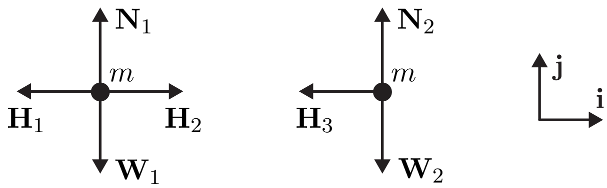
(c)



(d)

Solution to Exercise 7

(a) The force diagrams for the particles are as follows, where the symbols have their usual meanings.



Applying Newton's second law to the two particles separately gives

$$m\ddot{\mathbf{r}}_1 = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W}_1 + \mathbf{N}_1,$$
$$m\ddot{\mathbf{r}}_2 = \mathbf{H}_3 + \mathbf{W}_2 + \mathbf{N}_2,$$

where $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$, and $\mathbf{i}$ is a unit vector in the direction of positive x_1 and x_2.

Now use Hooke's law to model the spring forces. For the force $\mathbf{H}_1$, the length of the spring is $l = l_0 + x_1$ (since the equilibrium position is at the natural length), the stiffness is $3k$ and $\widehat{\mathbf{s}} = -\mathbf{i}$, so by Hooke's law we have

$$\mathbf{H}_1 = 3k((l_0 + x_1) - l_0)\,\widehat{\mathbf{s}} = -3kx_1\mathbf{i}.$$

For the force $\mathbf{H}_2$, the length of the spring is $l = l_0 + x_2 - x_1$ (since the equilibrium position is at the natural length), the stiffness is $2k$ and $\widehat{\mathbf{s}} = \mathbf{i}$, so by Hooke's law we have

$$\mathbf{H}_2 = 2k((l_0 + x_2 - x_1) - l_0)\,\widehat{\mathbf{s}} = 2k(x_2 - x_1)\mathbf{i}.$$

For the force $\mathbf{H}_3$, the length of the spring is $l = l_0 + x_2 - x_1$, the stiffness is $2k$ and $\widehat{\mathbf{s}} = -\mathbf{i}$, so by Hooke's law we have

$$\mathbf{H}_3 = 2k((l_0 + x_2 - x_1) - l_0)\,\widehat{\mathbf{s}} = -2k(x_2 - x_1)\mathbf{i}.$$

(Of course, $\mathbf{H}_3$ can most easily be obtained by using the fact that a model spring exerts forces of equal magnitude and opposite direction at either end, thus $\mathbf{H}_3 = -\mathbf{H}_2$.)

Substituting these expressions into the Newton's second law equations gives

$$m\ddot{\mathbf{r}}_1 = -3kx_1\mathbf{i} + 2k(x_2 - x_1)\mathbf{i} - mg\mathbf{j} + |\mathbf{N}_1|\,\mathbf{j},$$
$$m\ddot{\mathbf{r}}_2 = -2k(x_2 - x_1)\mathbf{i} - mg\mathbf{j} + |\mathbf{N}_2|\,\mathbf{j}.$$

Resolving in the $\mathbf{i}$-direction gives the equations of motion:

$$m\ddot{x}_1 = -5kx_1 + 2kx_2,$$
$$m\ddot{x}_2 = 2kx_1 - 2kx_2.$$

(b) If $k/m = 1$, then the equations of motion become (in matrix form)

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

To find the eigenvalues, we solve

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = 0,$$

and obtain $\lambda = -1$ or $\lambda = -6$.

- For $\lambda = -1$, the eigenvector equations are
$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
or equivalently
$$\begin{aligned} -4v_1 + 2v_2 &= 0, \\ 2v_1 - v_2 &= 0. \end{aligned}$$
These equations both reduce to $v_2 = 2v_1$, so $(1 \quad 2)^T$ is an eigenvector.
- For $\lambda = -6$, the eigenvector equations are
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
or equivalently
$$\begin{aligned} v_1 + 2v_2 &= 0, \\ 2v_1 + 4v_2 &= 0. \end{aligned}$$
These equations both reduce to $v_2 = -\frac{1}{2}v_1$, so $(2 \quad -1)^T$ is an eigenvector.

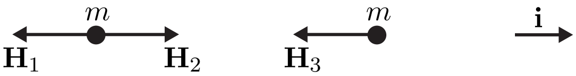
The normal mode angular frequencies are calculated from the eigenvalues by using $\omega = \sqrt{-\lambda}$, thus $\omega_1 = 1$ and $\omega_2 = \sqrt{6}$. From equation (10), the general solution of the matrix equation of motion has the form
$$\mathbf{x}(t) = C_1\mathbf{v}_1 \cos(\omega_1 t + \phi_1) + C_2\mathbf{v}_2(\cos \omega_2 t + \phi_2).$$
Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, ω_1 and ω_2 gives the general solution as
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(t + \phi_1) + C_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos(\sqrt{6}t + \phi_2).$$

(c) The given initial condition has both particles starting from rest, so we can take $\phi_1 = \phi_2 = 0$. The constants C_1 and C_2 can be determined by substituting $x_1 = d_1$ and $x_2 = d_2$ when $t = 0$ into the general solution of the equation of motion:
$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$
These can be solved to give C_1 and C_2 in terms of d_1 and d_2 as
$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = d_1 \begin{pmatrix} 0.2 \\ 0.4 \end{pmatrix} + d_2 \begin{pmatrix} 0.4 \\ -0.2 \end{pmatrix}.$$
These values of C_1 and C_2 can be substituted into the general solution to determine the particular solution satisfying the initial conditions.

Note that
$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Solution to Exercise 8

Draw a force diagram for each particle (omitting the weights and normal reactions), using the usual notation for the spring forces.



Apply equation (30) to each particle, to obtain

$$m\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2,$$
$$m\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_3,$$

where $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$, as in Exercise 7.

Using equation (31) gives

$$\Delta\mathbf{H}_1 = 3kx_1(-\mathbf{i}) = -3kx_1\mathbf{i},$$
$$\Delta\mathbf{H}_2 = 2k(x_2 - x_1)\mathbf{i},$$
$$\Delta\mathbf{H}_3 = 2k(x_2 - x_1)(-\mathbf{i}) = -2k(x_2 - x_1)\mathbf{i}.$$

Substituting into the original equations gives

$$m\ddot{\mathbf{r}}_1 = -3kx_1\mathbf{i} + 2k(x_2 - x_1)\mathbf{i},$$
$$m\ddot{\mathbf{r}}_2 = -2k(x_2 - x_1)\mathbf{i}.$$

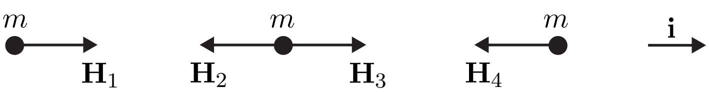
Resolving in the $\mathbf{i}$-direction gives the equations of motion:

$$m\ddot{x}_1 = -5kx_1 + 2kx_2,$$
$$m\ddot{x}_2 = 2kx_1 - 2kx_2.$$

(In this example, at equilibrium the springs have their natural lengths, so $\mathbf{H}_{1,\text{eq}} = \mathbf{H}_{2,\text{eq}} = \mathbf{H}_{3,\text{eq}} = \mathbf{0}$, hence $\Delta\mathbf{H}_1 = \mathbf{H}_1$, $\Delta\mathbf{H}_2 = \mathbf{H}_2$ and $\Delta\mathbf{H}_3 = \mathbf{H}_3$.)

Solution to Exercise 9

(a) With the usual notation, omitting the weights and normal reactions, which balance, the three force diagrams (one for each particle) are as follows.



Applying equation (30) to each particle gives

$$m\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1,$$
$$m\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_2 + \Delta\mathbf{H}_3,$$
$$m\ddot{\mathbf{r}}_3 = \Delta\mathbf{H}_4,$$

where $\mathbf{r}_1 = x_1\mathbf{i}$, $\mathbf{r}_2 = x_2\mathbf{i}$ and $\mathbf{r}_3 = x_3\mathbf{i}$, and $\mathbf{i}$ is a unit vector in the direction of positive x_1, x_2 and x_3.

Then from equation (31) we obtain

$$\begin{aligned}\Delta\mathbf{H}_1 &= k(x_2 - x_1)\mathbf{i},\\ \Delta\mathbf{H}_2 &= k(x_2 - x_1)(-\mathbf{i}) = -k(x_2 - x_1)\mathbf{i},\\ \Delta\mathbf{H}_3 &= k(x_3 - x_2)\mathbf{i},\\ \Delta\mathbf{H}_4 &= k(x_3 - x_2)(-\mathbf{i}) = -k(x_3 - x_2)\mathbf{i}.\end{aligned}$$

Substituting these expressions into the original equations gives

$$\begin{aligned}m\ddot{\mathbf{r}}_1 &= k(x_2 - x_1)\mathbf{i},\\ m\ddot{\mathbf{r}}_2 &= -k(x_2 - x_1)\mathbf{i} + k(x_3 - x_2)\mathbf{i},\\ m\ddot{\mathbf{r}}_3 &= -k(x_3 - x_2)\mathbf{i}.\end{aligned}$$

On resolving in the **i**-direction, we obtain the equations of motion

$$\begin{aligned}m\ddot{x}_1 &= -kx_1 + kx_2,\\ m\ddot{x}_2 &= kx_1 - 2kx_2 + kx_3,\\ m\ddot{x}_3 &= kx_2 - kx_3,\end{aligned}$$

or in matrix form,

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{pmatrix} = \frac{k}{m}\begin{pmatrix}-1 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}.$$

(b) The given data tell us that the matrix has eigenvalues 0, -1 and -3, with corresponding eigenvectors $(1\quad 1\quad 1)^T$, $(1\quad 0\quad -1)^T$ and $(1\quad -2\quad 1)^T$. So the normal mode angular frequencies are 0, $\sqrt{k/m}$ and $\sqrt{3k/m}$. (Do not forget the scaling factor k/m for the dynamic matrix.)

Therefore the general solution of the equation of motion is

$$\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}(A + Bt) + C_1\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) + C_2\begin{pmatrix}1\\ -2\\ 1\end{pmatrix}\cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right),$$

where A, B, C_1, C_2, ϕ_1 and ϕ_2 are constants that can be determined using the initial conditions.

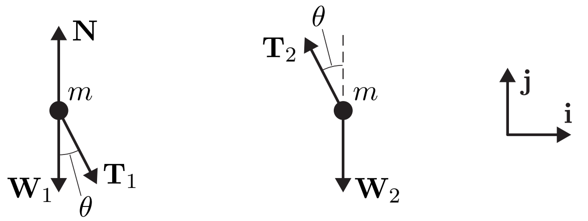
(c) For the first normal mode, that is, the linear mode with eigenvector $(1\quad 1\quad 1)^T$, a suitable initial condition is that the constituent parts all start from the given equilibrium positions with the same velocity (or alternatively that they remain at rest, each at the same displacement from its given equilibrium position).

For the second normal mode, with eigenvector $(1\quad 0\quad -1)^T$, a suitable initial condition is that the constituent parts all start from rest with the leftmost truck and the engine displaced equal distances to the right and left of their given equilibrium positions, respectively, while the central truck starts (and stays) at its given equilibrium position.

For the third normal mode, with normal mode eigenvector $(1 \quad -2 \quad 1)^T$, a suitable initial condition is that the constituent parts all start from rest with the leftmost truck and the engine both displaced the same distance to the right of their given equilibrium positions, while the central truck is displaced twice that distance to the left of its given equilibrium position.

Solution to Exercise 10

(a) With the usual notation, the force diagrams for the particles are as follows.



Applying Newton's second law to each particle gives

$$m\ddot{\mathbf{r}}_1 = \mathbf{T}_1 + \mathbf{W}_1 + \mathbf{N},$$
$$m\ddot{\mathbf{r}}_2 = \mathbf{T}_2 + \mathbf{W}_2,$$

where $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$ are the position vectors, relative to the fixed point O, of the sliding particle and the pendulum bob, respectively, and $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors in the positive x- and y-directions, respectively.

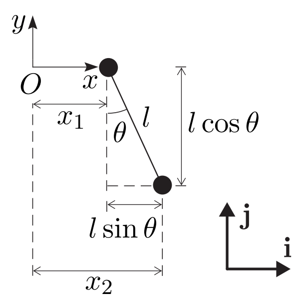
We have $\mathbf{W}_1 = \mathbf{W}_2 = -mg\mathbf{j}$, and applying Newton's second law gives

$$m\ddot{\mathbf{r}}_1 = \mathbf{T}_1 - mg\mathbf{j} + \mathbf{N},$$
$$m\ddot{\mathbf{r}}_2 = \mathbf{T}_2 - mg\mathbf{j}.$$

Resolving in the $\mathbf{i}$- and $\mathbf{j}$-directions gives

$$m\ddot{x}_1 = |\mathbf{T}_1|\sin\theta,$$
$$m\ddot{y}_1 = -|\mathbf{T}_1|\cos\theta - mg + |\mathbf{N}|,$$
$$m\ddot{x}_2 = -|\mathbf{T}_2|\sin\theta,$$
$$m\ddot{y}_2 = |\mathbf{T}_2|\cos\theta - mg.$$

Now, from the figure in the margin, the relationships between the linear and angular coordinates are



$$x_2 = x_1 + l\sin\theta, \quad y_1 = 0, \quad y_2 = -l\cos\theta.$$

Using the small-angle approximations $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$, we have $x_2 \simeq x_1 + l\theta$ and $y_2 \simeq -l$ (so $\ddot{y}_2 \simeq 0$), which on substitution in the resolved equations give

$$m\ddot{x}_1 \simeq |\mathbf{T}_1|\theta,$$
$$0 \simeq -|\mathbf{T}_1| - mg + |\mathbf{N}|,$$
$$m\big(\ddot{x}_1 + l\ddot{\theta}\big) \simeq -|\mathbf{T}_2|\theta,$$
$$0 \simeq |\mathbf{T}_2| - mg.$$

From the last of these approximations, $|\mathbf{T}_2| \simeq mg$, and since the forces exerted at either end of a model rod are equal in magnitude, $|\mathbf{T}_1| = |\mathbf{T}_2| \simeq mg$. So the first and third equations above become

$$m\ddot{x}_1 \simeq mg\theta,$$
$$m(\ddot{x}_1 + l\ddot{\theta}) \simeq -mg\theta.$$

On subtracting the first of these equations from the second, we obtain

$$ml\ddot{\theta} \simeq -mg\theta - mg\theta = -2mg\theta.$$

Hence the linear equations of motion for the system are

$$\ddot{x}_1 = g\theta, \quad \ddot{\theta} = -\frac{2g}{l}\theta.$$

(These equations could also have been obtained by putting $k = 0$ – because there is no spring here – into the result of Exercise 4(a).)

(b) Writing the equations of motion in matrix form gives

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & g \\ 0 & -2g/l \end{pmatrix} \begin{pmatrix} x_1 \\ \theta \end{pmatrix}.$$

The eigenvalues are found by solving

$$\begin{vmatrix} 0-\lambda & g \\ 0 & -2g/l-\lambda \end{vmatrix} = \lambda(\lambda + 2g/l) = 0.$$

So the eigenvalues are $\lambda = 0$ and $\lambda = -2g/l$.

- For $\lambda = 0$, the eigenvector equations are

  $$\begin{pmatrix} 0-0 & g \\ 0 & -2g/l-0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  or equivalently $gv_2 = 0$ and $2gv_2/l = 0$. Therefore $v_2 = 0$ (there is no restriction on v_1), so $(1 \quad 0)^T$ is an eigenvector.

- For $\lambda = -2g/l$, the eigenvector equations are

  $$\begin{pmatrix} 0-(-2g/l) & g \\ 0 & -2g/l-(-2g/l) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  or equivalently

  $$(2g/l)v_1 + gv_2 = 0,$$
  $$0 = 0.$$

  The first equation gives the condition $v_2 = -2v_1/l$, so $(1 \quad -2/l)^T$ is an eigenvector. The normal mode angular frequencies are calculated from the eigenvalues as $\omega_1 = 0$ and $\omega_2 = \sqrt{2g/l}$.

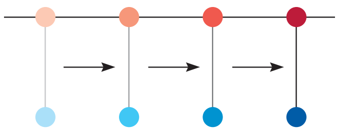
With these eigenvectors and normal mode angular frequencies, the general solution of the matrix equation of motion is

$$\begin{pmatrix} x_1(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (A + Bt) + C \begin{pmatrix} 1 \\ -2/l \end{pmatrix} \cos\left(\sqrt{\frac{2g}{l}}\, t + \phi\right),$$

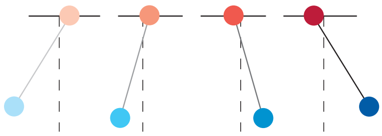
where A, B, C and ϕ are constants that can be determined from the initial conditions.

(c) The angular frequency of a simple pendulum is $\sqrt{g/l}$, so the angular frequency of this pendulum, which is $\sqrt{2g/l}$, is larger by a factor of $\sqrt{2}$. Therefore the period of this pendulum is smaller by a factor of $\sqrt{2}$ than that of the simple pendulum.

(d) For the rigid body motion of the system, the whole system is translated along the horizontal bar (with $\theta = 0$), as shown below.



For the non-zero angular frequency, the motion of both particles is oscillatory with the same frequency. The displacement ratio is negative, so the motion is phase-opposed, as shown below.



Solution to Exercise 11

The question suggests using equation (40), which is applicable because the model of the guitar string is the same as that in Example 8; only the parameter m is different. Instead of the mass of the whole spring, 0.25 g, we substitute half the mass of the spring, 0.125 g, because only the central particle (of mass $\frac{1}{2}M$) vibrates; the other parameters have the same values as before. Then

$$\omega = \sqrt{\frac{2 \times 68}{(0.125 \times 10^{-3}) \times 0.325}} \simeq 1830.$$

This gives a fundamental frequency of

$$f = \frac{\omega}{2\pi} = \frac{1830}{2\pi} \simeq 291.$$

So this model predicts a fundamental frequency of approximately 291 Hz, which is much closer to the experimental value of 323 Hz than the 206 Hz predicted in Example 8.

Solution to Exercise 12

The basic two-particle model applies here because only the two central particles vibrate. This means that the model used in Example 9 is relevant; only a parameter (the mass of the particles) has changed. Therefore the analysis is the same as in that example, leading to

$$\omega_1 = \sqrt{\frac{68}{((0.25/3) \times 10^{-3}) \times 0.217}} = 1939.$$

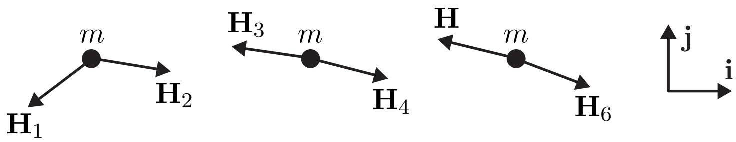
Hence the fundamental frequency is predicted to be

$$f = \frac{\omega_1}{2\pi} = \frac{1939}{2\pi} \simeq 309.$$

The predicted fundamental frequency of approximately 309 Hz is closer to the experimental value of 323 Hz than the 252 Hz predicted by Example 9, and is closer than the 291 Hz predicted in Exercise 11, so this model is an improvement on the previous ones.

Solution to Exercise 13

(a) The force diagrams for the three particles (using the usual notation) are as follows.



Applying equation (30) to each particle gives

$$m\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2,$$
$$m\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4,$$
$$m\ddot{\mathbf{r}}_3 = \Delta\mathbf{H}_5 + \Delta\mathbf{H}_6,$$

where $\mathbf{r}_1 = y_1\mathbf{j}$, $\mathbf{r}_2 = y_2\mathbf{j}$ and $\mathbf{r}_3 = y_3\mathbf{j}$ are the displacements of the particles, and $\mathbf{j}$ is a unit vector in the positive y-direction.

Using expression (41) to model the forces exerted by the outermost springs, we find

$$\Delta\mathbf{H}_1 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\, y_1\,\mathbf{j}, \quad \Delta\mathbf{H}_6 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\, y_3\,\mathbf{j}.$$

The computations of the other forces are complicated by the fact that both ends of the springs are displaced. For the force $\mathbf{H}_2$, the spring is displaced by $y_1 - y_2$ at the left-hand end relative to the right-hand end, so from equation (41),

$$\Delta\mathbf{H}_2 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_1 - y_2)\,\mathbf{j}.$$

Similarly,

$$\Delta\mathbf{H}_3 = -\Delta\mathbf{H}_2 = \frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_1 - y_2)\,\mathbf{j},$$

$$\Delta\mathbf{H}_4 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_2 - y_3)\,\mathbf{j},$$

$$\Delta\mathbf{H}_5 = -\Delta\mathbf{H}_4 = \frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_2 - y_3)\,\mathbf{j}.$$

These can be substituted into the original equations to obtain

$$m\ddot{\mathbf{r}}_1 = -\frac{T_{\text{eq}}}{l_{\text{eq}}}\,y_1\,\mathbf{j} - \frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_1 - y_2)\,\mathbf{j},$$
$$m\ddot{\mathbf{r}}_2 = \frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_1 - y_2)\,\mathbf{j} - \frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_2 - y_3)\,\mathbf{j},$$
$$m\ddot{\mathbf{r}}_3 = \frac{T_{\text{eq}}}{l_{\text{eq}}}\,(y_2 - y_3)\,\mathbf{j} - \frac{T_{\text{eq}}}{l_{\text{eq}}}\,y_3\,\mathbf{j}.$$

Resolving in the **j**-direction and rearranging gives

$$\ddot{y}_1 = -2\frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_1 + \frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_2,$$
$$\ddot{y}_2 = \frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_1 - 2\frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_2 + \frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_3,$$
$$\ddot{y}_3 = \frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_2 - 2\frac{T_{\text{eq}}}{ml_{\text{eq}}}\,y_3,$$

which can be written in matrix form as

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} = \frac{T_{\text{eq}}}{ml_{\text{eq}}} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

(b) The given eigenvalue of smallest magnitude is -0.586, and this corresponds to the fundamental frequency. Therefore the eigenvalue of the dynamic matrix that has the smallest magnitude is

$$\lambda_1 = -0.586 \times \frac{T_{\text{eq}}}{ml_{\text{eq}}}.$$

For this system, $T_{\text{eq}} = 68$, $m = (0.25 \times 10^{-3})/3$ and $l_{\text{eq}} = 0.65/4$, so the corresponding angular frequency is

$$\omega_1 = \sqrt{\frac{0.586 \times 68}{(0.25 \times 10^{-3}/3) \times (0.65/4)}} \simeq 1715,$$

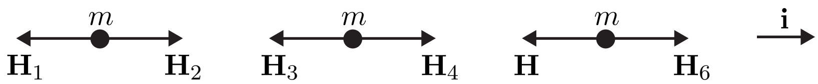
hence the fundamental frequency is

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1715}{2\pi} \simeq 273.$$

So this model has a fundamental frequency of approximately 273 Hz. This is still much smaller than the experimental value of 323 Hz, but it is better than the predictions from the models with one and two degrees of freedom in Examples 8 and 9; however, it is still not as close to the experimental value as the value found in Exercise 12, or even that in Exercise 11.

Solution to Exercise 14

(a) The first step is to draw a force diagram for each particle, using the usual notation.



Applying equation (30) to each particle gives

$$m\ddot{\mathbf{r}}_1 = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2,$$
$$m\ddot{\mathbf{r}}_2 = \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4,$$
$$m\ddot{\mathbf{r}}_3 = \Delta\mathbf{H}_5 + \Delta\mathbf{H}_6,$$

where $\mathbf{r}_1 = x_1\mathbf{i}$, $\mathbf{r}_2 = x_2\mathbf{i}$ and $\mathbf{r}_3 = x_3\mathbf{i}$ are the position vectors of the particles, and $\mathbf{i}$ is a unit vector in the positive x-direction. Modelling the forces using equation (31), we obtain

$$\Delta\mathbf{H}_1 = kx_1(-\mathbf{i}),$$
$$\Delta\mathbf{H}_2 = k(x_2 - x_1)\mathbf{i},$$
$$\Delta\mathbf{H}_3 = k(x_2 - x_1)(-\mathbf{i}),$$
$$\Delta\mathbf{H}_4 = k(x_3 - x_2)\mathbf{i},$$
$$\Delta\mathbf{H}_5 = k(x_3 - x_2)(-\mathbf{i}),$$
$$\Delta\mathbf{H}_6 = -kx_3\mathbf{i}.$$

Substituting into the original equations gives

$$m\ddot{\mathbf{r}}_1 = -kx_1\mathbf{i} + k(x_2 - x_1)\mathbf{i},$$
$$m\ddot{\mathbf{r}}_2 = -k(x_2 - x_1)\mathbf{i} + k(x_3 - x_2)\mathbf{i},$$
$$m\ddot{\mathbf{r}}_3 = -k(x_3 - x_2)\mathbf{i} - kx_3\mathbf{i}.$$

Resolving in the $\mathbf{i}$-direction gives the equations of motion for the longitudinal vibrations of the three-particle model as

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2, \\ m\ddot{x}_2 &= kx_1 - 2kx_2 + kx_3, \\ m\ddot{x}_3 &= kx_2 - 2kx_3, \end{aligned}$$

which can be written in matrix form as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(b) To verify that the given vectors are eigenvectors, evaluate the products:

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -2+\sqrt{2} \\ 2-2\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix} = (-2+\sqrt{2})\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -2-\sqrt{2} \\ 2+2\sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} = (-2-\sqrt{2}\,) \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

It follows that the eigenvalues of the matrix above are $-2+\sqrt{2}$, -2 and $-2-\sqrt{2}$, and the eigenvalues of the dynamic matrix are k/m times these.

(c) Using the given data,

$$\frac{k}{m} = \frac{16\,000}{(0.25 \times 10^{-3})/3} = 192 \times 10^6.$$

Hence, from $\omega = \sqrt{-\lambda}$, the normal mode angular frequencies are

$$\omega_1 = \sqrt{192 \times 10^6 \times (2-\sqrt{2})} \simeq 10\,605,$$
$$\omega_2 = \sqrt{192 \times 10^6 \times 2} \simeq 19\,596,$$
$$\omega_3 = \sqrt{192 \times 10^6 \times (2+\sqrt{2})} \simeq 25\,603.$$

(These angular frequencies are much larger than those for the transverse vibrations, and so the periods of vibration are much shorter.)

Solution to Exercise 15

(a) Using the hint, substitute for k from $T_{\text{eq}} = k(l_{\text{eq}} - l_0)$ in equation (47) to obtain

$$\ddot{x}_1 = \frac{-2T_{\text{eq}}}{m(l_{\text{eq}} - l_0)}\, x_1.$$

Now, in the present case, the equilibrium length of the spring is half the equilibrium length of the guitar string (i.e. $l_{\text{eq}} = L/2$), and the natural length is half the natural length of the guitar string (i.e. $l_0 = L_0/2$), while the mass of the particle is the mass of the guitar string (i.e. $m = M$). This gives the following equation of motion for the longitudinal vibration of the guitar string:

$$\ddot{x}_1 = \frac{2T_{\text{eq}}}{M(L - L_0)}\,(-2)x_1.$$

(b) As in part (a), use the formula $T_{\text{eq}} = k(l_{\text{eq}} - l_0)$ to eliminate k from the equations.

For the two-degrees-of-freedom model represented by equation (48), there are three identical springs, so their equilibrium and natural lengths are a third of the equilibrium and natural length of the guitar string, that is, $l_{\text{eq}} = L/3$ and $l_0 = L_0/3$. Therefore the stiffness of each spring is $k = 3T_{\text{eq}}/(L - L_0)$. The mass of each particle is half the mass of the guitar string, so $m = M/2$.

Thus the equations of motion become, in matrix form,

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{2 \times 3 \times T_{\text{eq}}}{M(L - L_0)} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

For the three-degrees-of-freedom model represented by equation (49), there are four springs (so $l_{\text{eq}} = L/4$ and $l_0 = L_0/4$) and three particles (so $m = M/3$). Consequently, we have

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{3 \times 4 \times T_{\text{eq}}}{M(L - L_0)} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(c) For the four-degrees-of-freedom model, there are four particles and five springs, which have the effect that the 3 in the above formula becomes a 4, and the 4 becomes a 5. The vectors have four components, and the matrix is a 4×4 matrix. So the equation of motion is

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{pmatrix} = \frac{4 \times 5 \times T_{\text{eq}}}{M(L - L_0)} \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

(d) By comparing the above equations of motion with the results derived earlier, it can be seen that the only difference between the transverse and longitudinal vibrational models lies in the constant that multiplies the matrix on the right-hand side of the equation of motion. In fact, the only difference is that the constant in the transverse case includes a factor $1/L$, whereas in the longitudinal case the corresponding factor is $1/(L - L_0)$. Hence the dynamic matrix for the longitudinal model is $(1/(L - L_0))/(1/L) = L/(L - L_0)$ times the dynamic matrix for the transverse model, and the eigenvalues are similarly related.

Writing f_{T}, ω_{T} and λ_{T} (respectively, f_{L}, ω_{L} and λ_{L}) for the fundamental frequency, the corresponding normal mode angular frequency and the corresponding eigenvalue of the dynamic matrix for the transverse model (respectively, longitudinal model), we have $\lambda_{\text{L}} = \lambda_{\text{T}} L/(L - L_0)$ from the above argument. Hence

$$\omega_{\text{L}} = \sqrt{-\lambda_{\text{L}}} = \sqrt{\frac{-\lambda_{\text{T}} L}{L - L_0}} = \sqrt{\frac{L}{L - L_0}}\,\sqrt{-\lambda_{\text{T}}} = \sqrt{\frac{L}{L - L_0}}\,\omega_{\text{T}},$$

which yields

$$f_{\text{L}} = \frac{\omega_{\text{L}}}{2\pi} = \sqrt{\frac{L}{L - L_0}}\,\frac{\omega_{\text{T}}}{2\pi} = \sqrt{\frac{L}{L - L_0}}\,f_{\text{T}},$$

that is, the ratio of f_{L} to f_{T} is $\sqrt{L/(L - L_0)}$.

In the case of the E string of the guitar, $L = 0.65$ and $L_0 = 0.633$, so the ratio is

$$f_{\text{L}}/f_{\text{T}} = \sqrt{0.65/(0.65 - 0.633)} \simeq 6.2.$$

Therefore the fundamental frequency of the longitudinal vibrations is approximately $6.2 \times 323 \simeq 2000\,\text{Hz}$.