Law of conservation of mechanical energy

If the force on a particle depends only on the particle's position, then the total mechanical energy of the particle is constant. In other words,

$$E = \tfrac{1}{2}mv^2 + U(x), \tag{28}$$

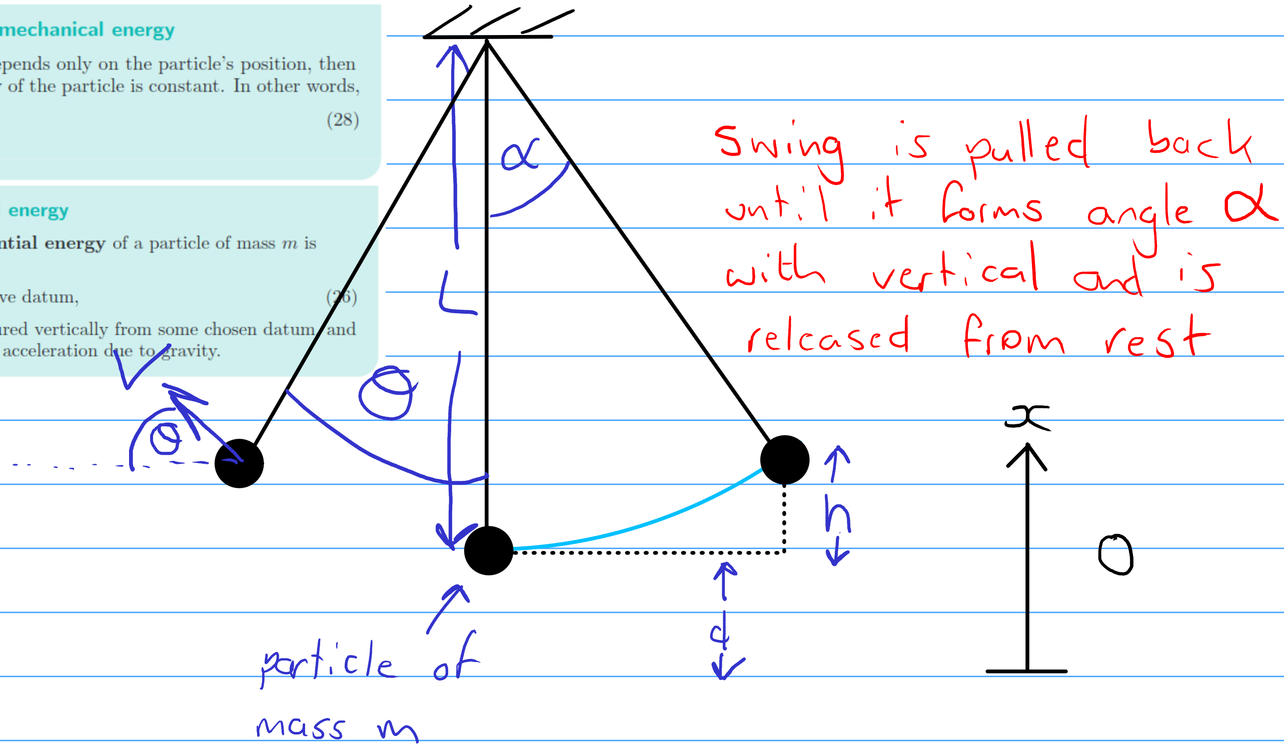
is a constant.

Gravitational potential energy

The **gravitational potential energy** of a particle of mass m is given by

$$U = mg \times \text{height above datum}, \tag{26}$$

where the height is measured vertically from some chosen datum and g is the magnitude of the acceleration due to gravity.



At $t=0$, the swing has no kinetic energy and gravitational potential energy $U = mgh$, where m is the mass, g acceleration due to gravity, and h is the vertical height above the equilibrium position, given by

$$\begin{aligned} h &= L - L\cos\alpha + d \\ &= L(1-\cos\alpha) + d \end{aligned}$$

So the total energy E at $t=0$ is

$$\begin{aligned} E_1 &= U \\ &= mgh \\ &= mg\big(L(1-\cos\alpha) + d\big) \end{aligned}$$

$$E = \frac{1}{2}mv^2 + mgh$$

$h =$

$$mg(L(1-\cos\alpha)+d) = \frac{1}{2}mv^2 + mg(L(1-\cos\theta)+d)$$

$$g(L(1-\cos\alpha)+d) = \frac{1}{2}v^2 + g(L(1-\cos\theta)+d)$$

$$\frac{1}{2}v^2 = g(L(1-\cos\alpha)+d) - g(L(1-\cos\theta)+d$$

$$= gL - gL\cos\alpha + gd - gL + gL\cos\theta - gd$$

$$= gL\cos\theta - gL\cos\alpha$$

$$v = \sqrt{2gL(\cos\theta - \cos\alpha)}$$

when in motion

$$\ddot{\underline{r}} = -g\underline{j}$$

so $\underline{v} = -gt\underline{j} + \underline{c}$

At $t=0$, $\underline{v}(0) = (v\cos\theta)\underline{i} + (v\sin\theta)\underline{j}$

So $\underline{v} = (v\cos\theta)\underline{i} + (v\sin\theta - gt)\underline{j}$

$$\underline{r} = (vt\cos\theta)\underline{i} + (-\frac{1}{2}gt^2 + vt\sin\theta)\underline{j} + \underline{D}$$

At $t=0$

$$\underline{r}(0) = (L\sin\theta)\underline{i} + (d + L - L\cos\theta)\underline{j}$$

So $\underline{r} = (L\sin\theta + vt\cos\theta)\underline{i} +$

$$(-\frac{1}{2}gt^2 + vt\sin\theta + d + L - L\cos\theta)\underline{j}$$

Projectile hits ground when $\underline{j}$ component is 0 so

$$0 = -\tfrac{1}{2}gt^2 + vt\sin\theta + d + L - L\cos\theta$$
$$\tfrac{1}{2}gt^2 = vt\sin\theta + d + L - L\cos\theta$$
$$t = \tfrac{1}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2g(H + L - L\cos\theta)}\right)$$

(using quadratic formula and considering positive solution)

So time of flight T is the above expression.
And range is

$$R = L\sin\theta + vT\cos\theta$$

Now we maximise R wrt θ

OR we go down the route of finding the maximum possible trajectory

$$R_{max} = \sqrt{L^2 + 2Lh}$$

where $L = v^2/g$ and $h = d + L - L\cos\theta$
(note the clash of notation)

$$R_{max} = \sqrt{4L^2(\cos\theta - \cos\alpha)^2 + 4L(\cos\theta - \cos\alpha)(d + L - L\cos\theta)}$$