Q 1.

(a) For each of the following linear transformations, write down its matrix and describe its type.

(i) $g(x, y) = (7x, 3y)$

The matrix for the linear transformation $g(x, y)$ is $\begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$ and represents a (k, l)-scaling.

(ii) $h(x, y) = (x + 3y, y)$

The matrix for the linear transformation $h(x, y)$ is $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and represents a horizontal shear with a shear factor of 3.

(iii) $k(x, y) = (y, x)$

The matrix for the linear transformation $k(x, y)$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and represents a reflection in the line $y = x$.

(b) Use the matrices that you found in part (a) to show that the linear transformation $f = k \circ h \circ g$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 7 & 9 \end{pmatrix}.$$

Let the matrices for the transformations g, h, and k be represented by $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{K}$, respectively, then

$$\begin{aligned} \mathbf{A} &= \mathbf{KHG} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 3 \\ 7 & 9 \end{pmatrix} \end{aligned}$$

(c) Show that f is invertible, and find the matrix that represents f^{-1}.

The determinant of the matrix $\mathbf{A}$, $\det \mathbf{A}$ is $0 \times 9 - 3 \times 7 = -21$. As $\det \mathbf{A} \neq 0$, $\mathbf{A}$ has an inverse, which is given by

$$\begin{aligned} \mathbf{A}^{-1} &= -\frac{1}{21} \begin{pmatrix} 9 & -3 \\ -7 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{7} & \frac{1}{7} \\ \frac{1}{3} & 0 \end{pmatrix} \end{aligned}$$

(d) Find the equation of the image $f(C)$ of the unit circle C centred at $(0,0)$, in the form $ax2 + bxy + cy2 = d$, where a, b, c and d are integers whose values you should find.

Each point (x,y) on C is the image of the point $f^{-1}(x,y)$, which has position vector

$$\mathbf{A} = \begin{pmatrix} -\frac{3}{7} & \frac{1}{7} \\ \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{7}x & \frac{1}{7}y \\ \frac{1}{3}x & 0y \end{pmatrix}$$

Therefore, if (x,y) lies on C, then its image under $f(C)$ lies on $(-\frac{3}{7}x + \frac{1}{7}y, \frac{1}{3}x + 0y)$, and the equation of the image $f(C)$ is

$$\left(-\frac{3}{7}x + \frac{1}{7}y\right)^2 + \left(\frac{1}{3}x\right)^2 = 1$$

$$\frac{130x^2 - 54xy + 9y^2}{441} = 1$$

$$130x^2 - 54xy + 9y^2 = 441$$

(e) Calculate the area enclosed by $f(C)$.

The area enclosed by $f(C)$ is $\pi \times |\det \mathbf{A}| = 21\pi$.

Q 2.

(a) The affine transformation f maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(-5,-6)$, $(-6,-6)$ and $(-5,-7)$, respectively.

(i) Determine f in the form $f(x) = \mathbf{A}\mathbf{x} + \mathbf{a}$, where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a column vector with two components.

Let $f(0,0) = \mathbf{a}$, $f(1,0) = \mathbf{b}$, and $f(0,1) = \mathbf{c}$, then the affine transformation $f(x)$ can be written as

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{A}\mathbf{x} + \mathbf{a} \\ &= \begin{pmatrix} \mathbf{b} - \mathbf{a} & \mathbf{c} - \mathbf{a} \end{pmatrix} \mathbf{x} + \mathbf{a} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -5 \\ -6 \end{pmatrix} \end{aligned}$$

where $\mathbf{x}$ is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) Find the fixed points (if any) of f , and state whether f is a translation, rotation, reflection or glide-reflection.

We can compose the affine transformation into a set of simultaneous equations and solve to find any fixed points.

$$\begin{aligned} f\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -5 \\ -6 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} -x & -5 \\ -y & -5 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$\begin{aligned} -x - 5 &= x \\ -y - 6 &= y \end{aligned}$$

Solving this pair of simultaneous equations gives $y = -3$ and $x = -\frac{5}{2}$. As $(-\frac{5}{2}, -3)$ is the only fixed point, f represents a rotation about this point.

(b) By using the translation h that maps the point $(0, 8)$ to the origin, and its inverse h^{-1} , find the transformation k in the form $k(\mathbf{x}) = \mathbf{Bx} + \mathbf{b}$, where $\mathbf{B}$ is a 2 x 2 matrix and $\mathbf{b}$ is a column vector with two components.

Let h be the translation $h(\mathbf{x}) = \mathbf{Ix} + \begin{pmatrix} 0 \\ -8 \end{pmatrix}$ that maps the point $(0, 8)$ to the origin. As the line $y = -x$ makes an angle of $-\frac{\pi}{4}$ with the positive x axis, let g be the reflection

$$\begin{aligned} g(\mathbf{x}) &= \begin{pmatrix} \cos(-\frac{\pi}{2}) & \sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & -\cos(-\frac{\pi}{2}) \end{pmatrix}\mathbf{x} + \mathbf{0} \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \mathbf{0} \end{aligned}$$

The transformation k is then given by

$$\begin{aligned} k &= h^{-1} \circ g \circ h \\ &= h^{-1}(g(h(\mathbf{x}))) \\ &= h^{-1}\left(g\left(\mathbf{x} + \begin{pmatrix} 0 \\ -8 \end{pmatrix}\right)\right) \\ &= h^{-1}\left(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 8 \\ 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 8 \\ 8 \end{pmatrix} \end{aligned}$$

Q 3.

(a) Find the partial fraction expansion of the rational expression.

The expression

$$\frac{3x^3 - 4x^2 - 21x - 24}{x^2 - 3x - 4}$$

is an improper rational expression as the degree of the numerator is greater than the degree of the denominator. To find its partial fraction expansion, we first use polynomial long division to find the quotient and remainder.

$$
\begin{array}{rl}
 & \phantom{3x^3 - 4x^2 - 21x - {}} 3x + 5 \\
x^2 - 3x - 4 & \overline{\smash{)}\,3x^3 - 4x^2 - 21x - 24} \\
 & \underline{3x^3 - 9x^2 - 12x \phantom{{}-24}} \\
 & \phantom{3x^3 - {}} 5x^2 - 9x - 24 \\
 & \phantom{3x^3 - {}} \underline{5x^2 - 15x - 20} \\
 & \phantom{3x^3 - 4x^2 - 21x - {}} 6x - 4
\end{array}
$$

The rational expression can therefore be rearranged in terms of the product of the denominator and quotient, and the remainder:

$$3x^3 - 4x^2 - 21x - 24 = (3x + 5)(x^2 - 3x - 4) + 6x - 4$$

Dividing through by the denominator results in the sum of a polynomial expression and a proper rational expression:

$$\frac{3x^3 - 4x^2 - 21x - 24}{x^2 - 3x - 4} = 3x + 5 + \frac{6x - 4}{x^2 - 3x - 4}$$

The partial fraction expansion of the second term is given by

$$\begin{aligned} \frac{6x - 4}{x^2 - 3x - 4} &= \frac{6x - 4}{(x + 1)(x - 4)} \\ &= \frac{A}{x + 1} + \frac{B}{x - 4} \end{aligned}$$

Where A and B are constants. Using the cover-up method and substituting $x = -1$ gives

$$\begin{aligned} A &= \frac{6(-1) - 4}{-1 - 4} \\ &= 2 \end{aligned}$$

Substituting $x = 4$ gives

$$\begin{aligned} B &= \frac{6 \times 4 - 4}{4 + 1} \\ &= 4 \end{aligned}$$

Therefore the partial fraction expansion of the original, improper rational expression is given by

$$\frac{3x^3 - 4x^2 - 21x - 24}{x^2 - 3x - 4} = 3x + 5 + \frac{2}{x+1} + \frac{4}{x-4}$$

(b) Use the `partfrac` command in Maxima to verify your answer to part (a). Include a screenshot or printout of your Maxima worksheet in your answer.

(%i1) partfrac((3·x^3 − 4·x^2−21·x−24) / (x^2−3·x−4), x);

(%o1) $\frac{2}{x+1} + 3\,x + \frac{4}{x-4} + 5$

(c) Hence (without using Maxima) find the integral

$$\int \frac{3x^3 - 4x^2 - 21x - 24}{x^2 - 3x - 4} \mathrm{d}x$$

.

The expression is much simpler to integrate once rearranged into its partial fraction expansion:

$$\begin{aligned}
\int \frac{3x^3 - 4x^2 - 21x - 24}{x^2 - 3x - 4}\, \mathrm{d}x &= \int \left(3x + 5 + \frac{2}{x+1} + \frac{4}{x-4}\right) \mathrm{d}x \\
&= 3\int x\, \mathrm{d}x + \int 5\, \mathrm{d}x + 2\int \frac{1}{x+1}\, \mathrm{d}x + 4\int \frac{1}{x-4}\, \mathrm{d}x \\
&= \frac{3}{2}x^2 + 5x + 2\ln|x+1| + 4\ln|x-4| + c
\end{aligned}$$

where c is the constant of integration.

Q 4.

(a) Find the domain and intercepts of f.

As $f(x)$ is undefined when $x = 3$, the domain of f is $(-\infty, 3) \cup (3, \infty)$. The y-intercept is given by

$$\begin{aligned} f(0) &= \frac{0^2 + 12}{(-3)^2} \\ &= \frac{4}{3} \end{aligned}$$

Therefore, the y-intercept of f is $\frac{4}{3}$.

Solving for x when $f(x) = 0$ gives

$$\begin{aligned} 0 &= \frac{x^2 + 12}{(x-3)^2} \\ &= x^2 + 12 \\ x^2 &= -12 \end{aligned}$$

As x has no real solutions when $f(x) = 0$, f has no x-intercept.

(b) Find$f'(x)$.

Let $g(x) = (x-3)^2$ and $h(x) = x^2 + 12$, then by the quotient rule

$$\begin{aligned} f'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2} \\ &= \frac{(x-3)^2 \times 2x - (x^2+12)(2x-6)}{(x-3)^4} \\ &= \frac{2x^3 - 12x^2 + 18x - (2x^3 - 6x^2 + 24x - 72)}{(x-3)^4} \\ &= -\frac{6x^2 + 6x - 72}{(x-3)^4} \end{aligned}$$

(c) Find the coordinates of any stationary points of f. Construct a table of signs for $f'(x)$, determine the intervals on which f is increasing and those on which f is decreasing, and determine the nature(s) of the stationary point(s).

The x-coordinates of the stationary points of f are the solutions to the equation $f'(x) = 0$ and are found below by rearranging the numerator into completed square form, and solving for x.

$$
\begin{aligned}
0 &= f'(x)\\
&= -\frac{6x^2+6x-72}{(x-3)^4}\\
&= 6x^2+6x-72\\
&= x^2+x-12\\
&= \left(x+\frac{1}{2}\right)^2-\frac{1}{4}-12\\
\left(x+\frac{1}{2}\right)^2 &= \frac{49}{4}\\
x+\frac{1}{2} &= \pm\frac{7}{2}\\
x &= \pm\frac{7}{2}-\frac{1}{2}
\end{aligned}
$$

As $x = 3$ is outside the domain of f, the only stationary point is at $(-4, f(-4)) = \big(-4, \frac{4}{7}\big)$.

The table of signs below indicates the ranges over which $f'(x)$ is increasing, decreasing, stationary, or undefined (indicated by +, -, 0, and *, respectively).

x	$(-\infty,-4)$	-4	$(-4,3)$	3	$(3,\infty)$
$-6x^2-6x+72$	-	0	+	0	-
$(x-3)^4$	+	+	+	0	+
$f'(x)$	-	0	+	*	-

Therefore, f is decreasing on the intervals $(-\infty,-4)$ and $(3,\infty)$, increasing on the interval $\big(-4,3\big)$, and has a local minimum at $(-4,\frac{4}{7})$.

(d) Determine the equations of the asymptotes of f

As f is undefined at $x = 3$, it has a vertical asymptote with equation $x = 3$. To determine the asymptotic behaviour of f as $x \to \pm\infty$, we consider the function k that would be obtained by removing all terms from the numerator and denominator except the dominant term in each:

$$k(x) = \frac{x^2}{x^2}$$

Therefore as $x \to \pm\infty$, k and $f \to 1$. So f has a horizontal asymptote with equation $y = 1$.

(e) Determine whether f is an even or odd function, or neither.

f is neither odd nor even as

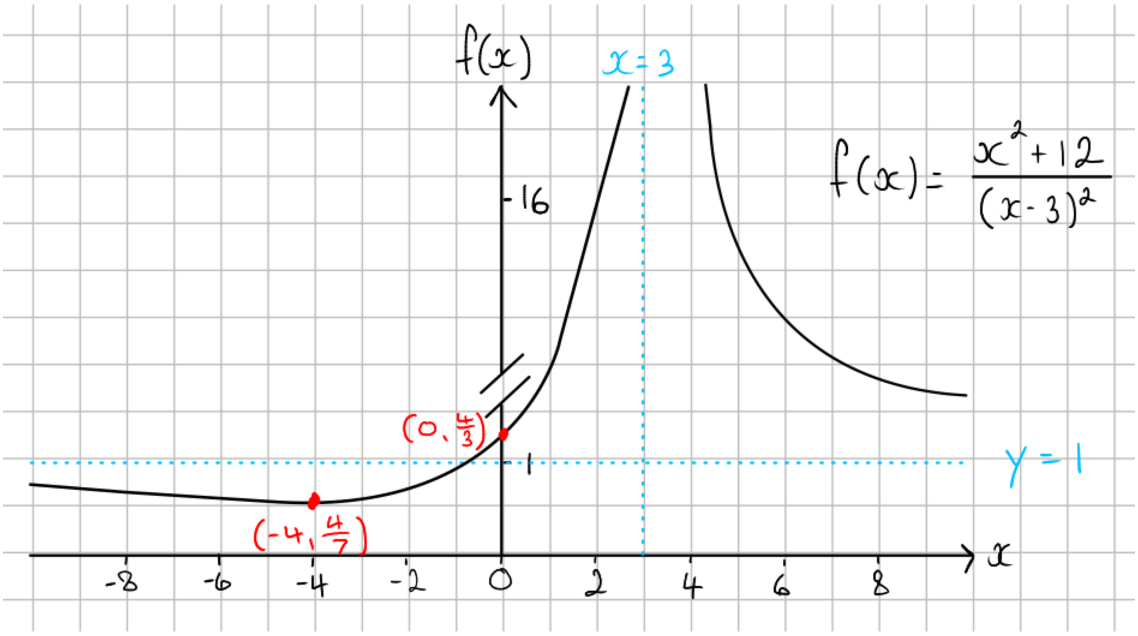
$$f(-1) \neq f(1)$$
$$\frac{13}{16} \neq \frac{13}{4}$$

and

$$f(-1) \neq -f(1)$$
$$\frac{13}{16} \neq -\frac{13}{4}$$

(f) Hence sketch the graph of f.

The stationary point $\left(-4, \frac{4}{7}\right)$, y-intercept $\left(0, \frac{4}{3}\right)$, and asymptotes $x = 3$ and $y = 1$ are shown.



Q 5.

(a)

(i) Use the chain rule to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}4^x = (\ln 4)4^x$$

Let $y = 4^x$, $u = \ln(4^x) = x\ln(4)$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = \ln(4)$, then by the chain rule

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\ \frac{\mathrm{d}}{\mathrm{d}x}e^u &= e^u \times \ln(4)\\ &= e^{\ln 4^x}(\ln(4))\\ &= (\ln 4)4^x\end{aligned}$$

(ii) Use integration by substitution to find the integral

$$\int 4^x \sinh(4^x)\ \mathrm{d}x$$

Let $u = 4^x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = (\ln 4)4^x$ and $\mathrm{d}u = (\ln 4)4^x\ \mathrm{d}x$. Substituting $u = 4^x$ into the integrand gives

$$\begin{aligned}\int 4^x \sinh(4^x)\ \mathrm{d}x &= \int u\sinh(u)\ \mathrm{d}x\\ &= \frac{1}{\ln 4}\int \sinh(u)\ \mathrm{d}u\\ &= \frac{\cosh u}{\ln 4} + c\\ &= \frac{\cosh 4^x}{\ln 4} + c\end{aligned}$$

where c is the constant of integration.

(b) Use integration by substitution and then a trigonometric identity to find the integral

$$\int 3x^2\cos^2(x^3)\ \mathrm{d}x$$

Let $u = x^3$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ and $\mathrm{d}u = 3x^2\ \mathrm{d}x$. Substituting $u = x^3$ into the integrand gives

$$\int u\cos^2(u)\ \mathrm{d}x = \int \cos^2(u)\ \mathrm{d}u$$

Using the identity $\cos^2(u) = \frac{1}{2}\big(1 + \cos(2u)\big)$ gives

$$\begin{aligned}\int \cos^2(u)\ \mathrm{d}u &= \int \frac{1}{2}\big(1 + \cos(2u)\big)\ \mathrm{d}u \\ &= \frac{1}{2}\int 1\ \mathrm{d}u + \frac{1}{2}\int \cos(2u)\ \mathrm{d}u \\ &= \frac{u}{2} + \frac{1}{4}\sin(2u) + c \\ &= \frac{2x^3 + \sin(2x^3)}{4} + c\end{aligned}$$

where c is the constant of integration.

Q 6.

(a) State the form of the differential equation and hence state which of the methods described in Unit 8 for finding solutions of differential equations you would use to solve this equation.

This is a directly integrable, first-order differential equation and can be solved by integrating both sides with respect to t.

(b) Find the general solution of the differential equation in explicit form.

Let $u = e^t + 3$, then $\frac{\mathrm{d}u}{\mathrm{d}t} = e^t$, and $\mathrm{d}u = e^t\ \mathrm{d}t$. Then, integrating by substitution gives

$$\begin{aligned}\int \frac{1}{u}\ \mathrm{d}u &= \ln|u| + c \\ &= \ln(e^t + 3) + c\end{aligned}$$

Therefore, the solution to the differential equation in explicit form is $x = \ln(e^t+3)+c$, where c is the constant of integration.

(c) Hence find the particular solution of the differential equation that satisfies the initial condition $x(0) = 17$.

Substituting $x = 17$ and $t = 0$ gives

$$\begin{aligned}17 &= \ln(e^0 + 3) + c \\ c &= 17 - \ln(4)\end{aligned}$$

Therefore, the particular solution of the differential equation that satisfies the initial condition $x(0) = 17$ is $x = \ln(e^t + 3) - \ln(4) + 17$.

Q 7.

(a) State the form of the differential equation and hence state which of the methods described in Unit 8 for finding solutions of differential equations you would use to solve this equation.

This is a separable, first-order differential equation and can be solved using the separation of variables method.

(b) Find the general solution of the differential equation in explicit form.

Applying the separation of variables method gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{y^2}{\sqrt{1-t^2}} \qquad (y > 0, -1 < t < 1) \\
&= -y^2 \times \frac{1}{\sqrt{1-t^2}} \\
-\int y^{-2}\,\mathrm{d}y &= \int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t \\
\frac{1}{y} &= \sin^{-1}(t) + b \\
y &= \frac{1}{\sin^{-1}(t)} + c
\end{aligned}$$

where b and c are arbitrary constants.

Q 8.

(a) State the form of the differential equation and hence state which of the methods described in Unit 8 for finding solutions of differential equations you would use to solve this equation.

This is a linear, first-order differential equation and can be solved using the integrating factor method.

(b) Find the general solution of the differential equation in explicit form.

First we rearrange the differential equation into the canonical form for a linear differential equation:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{9y}{x} &= x^9 \qquad (x > 0) \\
\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{9}{x} \times y &= x^9
\end{aligned}$$

Then, the integrating factor $p(x)$ is given by

$$\begin{aligned} p(x) &= \exp\left(\int -\frac{9}{x}\ \mathrm{d}x\right) \\ &= e^{-9\ln x} \\ &= x^{-9} \end{aligned}$$

Let $h(x) = x^9$, then the solution to the differential equation is given by

$$\begin{aligned} y &= \frac{1}{p(x)}\left(\int p(x)h(x)\ \mathrm{d}x\right) \\ &= x^9\left(\int x^{-9}x^9\ \mathrm{d}x\right) \\ &= x^9(x+c) \\ &= x^{10} + cx^9 \end{aligned}$$

where c is an arbitrary constant.

Q 9.

(a) Find the general solution of this differential equation in implicit form.

Let $f(x) = k$ and $g(v) = v^{-1}$, then it becomes clear this is a separable differential equation of the form

$$\frac{\mathrm{d}v}{\mathrm{d}x} = f(x)g(v)$$

Using the method of separating variables gives

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= f(x)g(v) \\ \int v\ \mathrm{d}v &= \int k\ \mathrm{d}x \\ \frac{v^2}{2} &= kx + c \end{aligned}$$

where c is the constant of integration.

(b) The speed of the object at its starting point is $8\mathrm{ms}^{-1}$. Find the particular solution that describes the speed of the object as a function of its position from its starting point.

Substituting $x = 0$ and $v = 8$ into the solution of the differential equation gives

$$\begin{aligned} \frac{8^2}{2} &= k \times 0 + c \\ c &= 32 \end{aligned}$$

Therefore, the particular solution that describes the speed of the object as a function of its position from its starting point is

$$\frac{v^2}{2} = kx + 32.$$

(c) The speed of the object is 18ms^{-1} when it is 2m from its starting point. Find the value of k.

Substituting $v = 18$ and $x = 2$ into the particular solution of the differential equation gives

$$\begin{aligned} \frac{18^2}{2} &= 2k + 32 \\ 2k &= 162 + 32 \\ k &= 65 \end{aligned}$$

Therefore, the value of k is 65.

(d) Use your particular solution to calculate the object's position when its speed is 90ms^{-1} . Give your answer to two significant figures.

Substituting $v = 90$ into the particular solution of the differential equation gives

$$\begin{aligned} \frac{90^2}{2} &= 65x + 32 \\ 65x &= 4050 - 32 \\ x &= 61.8153... \end{aligned}$$

Therefore, when the object's speed is 90ms^{-1}, its position is 62m from its starting position (to 2 s.f.).

(e) Use Maxima to find the solution of the initial value problem.

(%i1) eqn:'diff(v, x)=k/v;

(%o1) $\frac{d}{dx} v = \frac{k}{v}$

(%i2) sol:ode2(eqn, v, x);

(%o2) $\frac{v^2}{2k} = x + \%c$

(%i3) ic1(sol, v = 8, x = 0);

(%o3) $\frac{v^2}{2k} = \frac{kx+32}{k}$