MST125
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Section 2

Question 19

a) A hyperbola in standard position has foci at $(\pm ae, 0)$, so $ae = 2\sqrt{3}$ and $a = \frac{2\sqrt{3}}{e}$. Its directrices are at $x = \pm \frac{a}{e}$ so

$$\sqrt{3} = \frac{2\sqrt{3}/e}{e}$$

$$\sqrt{3}e^2 = 2\sqrt{3}$$

$$e = \sqrt{2}$$

And

$$a = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$$

So the hyperbola has eccentricity $\sqrt{2}$ and positive x intercept $a = \sqrt{6}$

b) The eccentricity is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

So

$$\sqrt{2} = \sqrt{1 + \frac{b^2}{6}}$$

$$2 = 1 + \frac{b^2}{6}$$

$$b^2 = 6$$

By the equation for a hyperbola in standard position

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{6} - \frac{y^2}{6} = 1$$

So the equation for this hyperbola is

$$x^2 - y^2 = 6$$

c) The asymptotes are given by

$$y = \pm \frac{b}{a} x$$
$$= \pm \frac{\sqrt{6}}{\sqrt{6}} x$$
$$= \pm x$$

d) The hyperbola is rectangular as its asymptotes are perpendicular.

Question 20

a) If $f = g \circ h$ then

$$h(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \underline{x}$$

and

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \underline{x}$$

where $\underline{x}$ is the vector $\binom{x}{y}$.
Then the matrix A is

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$$

b) $\det A = 6$. As $\det A \neq 0$, A is invertible and therefore f^{-1} exists.
The matrix of f^{-1} is given by

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

c) The image of $(-1,4)$ under f is given by

$$f(-1,4) = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

So the image of $(-1,4)$ under f is $(-2,8)$

The point with image $(-2,2)$ under f is given by

$$f^{-1}(-2,2) = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

So $(-1,2)$ has the image $(-2,2)$ under f.

d) The area of the image of a circle with area 12π under f^{-1} is scaled by the determinant of A^{-1}.
As $\det A^{-1} = \frac{1}{6}$, the area of $f^{-1}(C) = 2\pi$

Question 21

d) $\int \sinh^2(5t)\cosh^2(5t)\,dt$

$= \int \sinh(5t)\cosh(5t) \times \sinh(5t)\cosh(5t)\,dt$

$= \int \frac{1}{2}\sinh(10t) \times \frac{1}{2}\sinh(10t)\,dt$

$= \frac{1}{4}\int \sinh^2(10t)\,dt$

$= \frac{1}{4}\int \frac{1}{2}\left(\cosh(20t) - 1\right)\,dt$

$= \frac{1}{8}\int \cosh(20t) - 1 \quad dt$

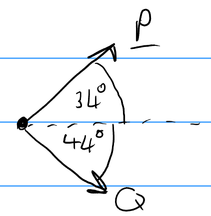
$= \frac{1}{8}\left(\frac{1}{20}\sinh(20t) - t + c\right)$

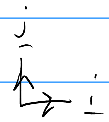
Where c is a constant of integration.

Question 22

a) $\underline{Q} = 21\cos(-44)\,\underline{i} + 21\sin(-44)\,\underline{j}$

$\underline{P} = P\cos(34)\,\underline{i} + P\sin(34)\,\underline{j}$





b) Let $\underline{F}$ be the resultant force acting on the particle, then applying Newton's 2^{nd} law gives

$$\underline{F} = \underline{Q} + \underline{P}$$

$$m\underline{a} = \underline{Q} + \underline{P}$$

where m is the mass of the particle in kg.

Resolving in the $\underline{i}$ and $\underline{j}$ directions gives

$$ma = 21\cos(-44) + P\cos(34) \qquad (1)$$

$$0 = 21\sin(-44) + P\sin(34) \qquad (2)$$

Rearranging (2) gives

$$P = \frac{-21\sin(-44)}{\sin(34)}$$

$$= 26.087\ldots$$

Substituting $P = 26.087\ldots$ into (1) gives

$$ma = 21\cos(-44) + 26.087\ldots \times \cos(34)$$

$$= 36.733\ldots$$

$$a = 2.448\ldots$$

So the particle is accelerating at $2.4\,ms^{-2}$ (2 s.f.).

c) As the particle has constant acceleration in a straight line, we can use the equation

$$v^2 = v_0^2 + 2ax$$

$$v^2 = 2 \times 2.448\ldots \times 2$$

$$v = 3.129\ldots$$

After 2m, the velocity is $3.1\,ms^{-1}$ (2 s.f.)

Question 23

a) i) An eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ and corresponding eigenvalue λ of a matrix A satisfy

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

In this case

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A and its corresponding eigenvalue is 3.

ii) Factorising the characteristic equation gives

$$\lambda^2 - 6\lambda + 9 = 0$$
$$(\lambda - 3)^2 = 0$$

As there is a single solution to this equation, $\lambda = 3$ is a repeated eigenvalue

b) i) Let $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$, $P^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

Then $B = PDP^{-1}$

ii) If $D^4 = \begin{pmatrix} 256 & 0 \\ 0 & 1 \end{pmatrix}$, then $B^4 = PD^4P^{-1}$

$$B^4 = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 256 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} \frac{768}{5} & -\frac{256}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 154 & -51 \\ -306 & 103 \end{pmatrix}$$

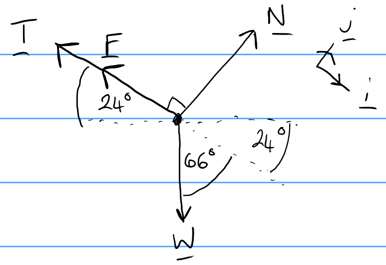
Section 3

Question 24

a) The forces acting on the crate are:

$\underline{W}$, the weight of the crate under gravity
$\underline{T}$, the tension in the rope
$\underline{N}$, the normal reaction of the plane
$\underline{F}$, the friction force

Here is a force diagram showing the forces



b) $\underline{T} = -52\,\underline{i}$

$\underline{F} = -0.27\,N\,\underline{i}$

$\underline{N} = N\,\underline{j}$

$\underline{W} = mg\cos(-66)\,\underline{i} + mg\sin(-66)\,\underline{j}$

c) As the crate is at rest, the vector equation for the forces acting on it is

$$\underline{0} = \underline{T} + \underline{F} + \underline{N} + \underline{W}$$

Resolving in $\underline{i}$ and $\underline{j}$ directions gives

$$0 = mg\cos(-66) - 52 - 0.27N \quad (1)$$

$$0 = mg\sin(-66) + N \quad (2)$$

Therefore $N = -mg\sin(-66)$. Substituting into (1) gives

$$0 = mg\cos(-66) - 52 + 0.27\,mg\sin(-66)$$

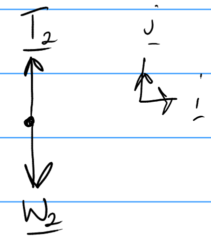
$$52 = m\left(g\cos(-66) + 0.27\,g\sin(-66)\right)$$

$$m = \frac{52}{1.568\ldots}$$

$$= 33.146\ldots$$

Therefore the mass of the crate is 33 kg (2 s.f.).

d) Let $\underline{T_2}$ be the tension of the rope attached to the weight, and let $\underline{W_2}$ be its weight under gravity, then



as $|\underline{T}| = |\underline{T_2}|$,

$$\underline{T_2} = T\,\underline{j}$$

$$\underline{W_2} = -m_2 g\,\underline{j}$$

Then

$$0 = 52 - m_2 g$$

$$m_2 = \frac{52}{g}$$

$$= 5.306\ldots$$

Therefore the mass of the weight is 5.3 kg (2 s.f.)

Question 25

a) Taking the derivative of both sides gives

$$\frac{d}{dx}\frac{1}{y^2} + e^y = \frac{d}{dx}\cos(x^2+x)$$

By the chain rule

$$-2y^{-3} + e^y \frac{dy}{dx} = \frac{d}{dx}\cos(x^2+x)$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{y^3(2x+1)\sin(x^2+x)}{2-y^3e^y}$$

b) i) Differentiating both sides gives

$$\frac{dy}{dx} = -3(x+1)^{-4} - 3\sin(3x) - e^{-x}$$

ii) The equation above is directly integrable, so integrating both sides gives

$$y = (x+1)^{-3} + \cos(3x) + e^{-x} + C$$

If the initial condition is that $y=10$ when $x=0$ this would give

$$10 = 1+1+1+C$$
$$C = 10-3$$
$$= 7$$

So an initial condition that satisfies the particular solution is $y=10$ and $x=0$