Unit 8

Integration methods

Introduction

In this unit you'll continue your study of integration. In Unit 7 you saw that integration is the reverse of differentiation. You saw that you can use it to solve problems in which you know the values taken by the rate of change of a continuously changing quantity, and you want to work out the values taken by the quantity, or how much the quantity changes over some period. For example, you saw that if you have a formula for the velocity of an object in terms of time, then you can use integration to find a formula for the displacement of the object in terms of time, or work out the change in the displacement of the object over some period of time.

In Sections 1 and 2 of this unit, you'll meet a different way to think about integration. This way of thinking about it involves areas of regions of the plane, rather than rates of change. You'll see that the link between the two ways of thinking about integration helps you to solve problems both about areas and about rates of change.

In Sections 3 to 5 of the unit, you'll learn some more methods for integrating functions, which you can use in addition to the methods that you learned in Unit 7. These further methods will allow you to integrate a much wider variety of functions than you've learned to integrate so far.

Many students find some of the material in this unit quite challenging when they first meet it, so don't worry if that's the case for you too. As with most mathematics, it will seem more straightforward once you're more familiar with it. If you find a particular technique difficult, then make sure that you watch the associated tutorial clips, and use the practice quiz and exercise booklet for this unit for more practice.

This unit is likely to take you more time to study than most of the other units, so the study planner allows extra time for it.

1 Areas, signed areas and definite integrals

In this section you'll look at a type of mathematical problem that you might think has little connection with the mathematics of rates of change that you studied in Units 6 and 7. In Section 2 you'll see that in fact the two areas of mathematics are closely related.

Throughout Sections 1 and 2, we'll work with *continuous* functions. Remember that a **continuous** function is a function whose graph has no breaks in it, over its whole domain. Informally, it's a function whose graph you can draw without taking your pen off the paper (though you might need an infinitely large piece of paper, as the graph might be infinitely long!).

1.1 Areas and signed areas

Sometimes it's useful to calculate the area of a flat shape with a curved boundary. For example, suppose that an architect is designing a large building shaped as shown in Figure 1(a). The curve of the roof is given by the graph of the function

$$f(x) = 9 - \tfrac{1}{36}x^2 \quad (-18 \le x \le 18),$$

where x and $f(x)$ are measured in metres. This graph is shown in Figure 1(b).

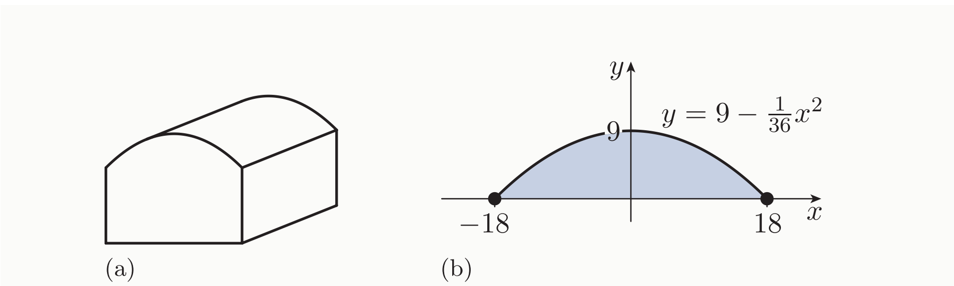


Figure 1 (a) The shape of a building (b) the graph that gives the curve of the roof

Suppose that the architect wants to calculate the area shaded in Figure 1(b), so that she can determine the amounts of materials needed to build the end wall of the building.

Here's a way of calculating an approximate value for this area, which you can make as accurate as you wish. You start by dividing the interval $[-18, 18]$, which is the interval of x-values that corresponds to the curve, into a number of subintervals of equal width.

For example, in Figure 2 the interval has been divided into twelve subintervals. The right endpoint of the first subinterval is equal to the left endpoint of the next subinterval, and so on.

For each subinterval, you approximate the shape of the curve on that subinterval by a horizontal line segment whose height is the value of the function at the left endpoint of the subinterval, as shown in Figure 2. You calculate the areas of the rectangles between these line segments and the x-axis (simply by multiplying their heights by their widths in the usual way), and add up all these areas to give you an approximate value for the required area. The more subintervals you use, the better will be the approximation.

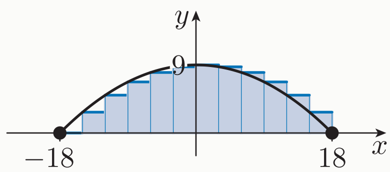
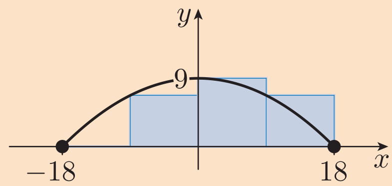


Figure 2 A collection of rectangles whose total area is approximately the area in Figure 1(b)

In the next example this method is used, with four subintervals, to find an approximate value for the area discussed above. Notice that, in both the example and in Figure 2 above, the rectangle corresponding to the first subinterval has zero height and therefore zero area – this happens because the value of the function at the left endpoint of this subinterval is zero.

Example 1 *Calculating an approximate value for an area*

Use the method described above, with four subintervals as shown below, to find an approximate value for the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the x-axis.



Solution

We divide the interval $[-18, 18]$ into four subintervals of equal width. The whole interval has width 36, so each subinterval has width $36/4 = 9$.

The left endpoints of the four subintervals are

$$-18, \quad -18+9, \quad -18+2\times 9, \quad -18+3\times 9,$$

that is

$$-18, \quad -9, \quad 0, \quad 9.$$

So the heights of the rectangles are

$$f(-18), \quad f(-9), \quad f(0), \quad f(9),$$

that is,

$$9 - \tfrac{1}{36}\times(-18)^2, \quad 9 - \tfrac{1}{36}\times(-9)^2, \quad 9 - \tfrac{1}{36}\times 0^2, \quad 9 - \tfrac{1}{36}\times 9^2,$$

which evaluate to

$$0, \quad \tfrac{27}{4}, \quad 9, \quad \tfrac{27}{4}.$$

So we obtain the following approximate value for the area:

$$(0\times 9) + (\tfrac{27}{4}\times 9) + (9\times 9) + (\tfrac{27}{4}\times 9) = \frac{405}{2} = 202.5.$$

The calculation in Example 1 shows that a (rather crude) approximate value for the area of the cross-section of the roof discussed at the beginning of this subsection is $202.5\,\mathrm{m}^2$.

Note that the units for the area of a region on a graph are, as you'd expect, the units on the vertical axis times the units on the horizontal axis. For example, if the units on both axes are metres, then the units for area are square metres (m^2). If a graph has no specific units on the axes, then the units for area are simply 'square units', and we usually omit them when we state an area.

In the next activity you're asked to find another approximation for the area of the cross-section of the roof discussed earlier, by using the subinterval method with six subintervals instead of four.

Activity 1 *Calculating an approximate value for an area*

Use the method described above, with six subintervals as shown below, to find an approximate value for the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the x-axis.

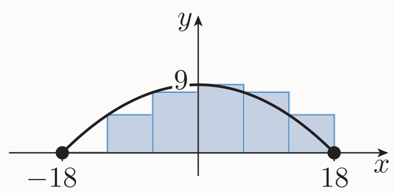


Table 1 shows the approximate values for the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the x-axis that were found in Example 1 and in the solution to Activity 1. It also shows, to three decimal places, some further approximate values that were found in the same way, but using larger numbers of subintervals. The calculations were carried out using a computer.

Table 1 Approximate values for the area in Figure 1(b)

Number of subintervals	4	6	12	50	100	500
Approximation obtained	202.5	210	214.5	215.914	215.978	215.999

You can see that as the number of subintervals gets larger and larger, the approximation obtained seems to be getting closer and closer to a particular number. We say that this number is the **limit** of the approximations as the number of subintervals **tends to infinity**. It's the *exact value* of the area between the graph of the function $f(x) = 9 - \frac{1}{36}x^2$ and the x-axis. From Table 1, it looks as if the exact value of this area is 216, or perhaps a number very close to 216.

In general, if you have a continuous function f whose graph lies on or above the x-axis throughout an interval $[a, b]$, then you can use the method demonstrated above to find, as accurately as you want, the area between the graph of f and the x-axis, from $x = a$ to $x = b$. Figure 3(a) illustrates an area of this type, and Figure 3(b) illustrates the approximation to the area that's obtained by using 8 subintervals.

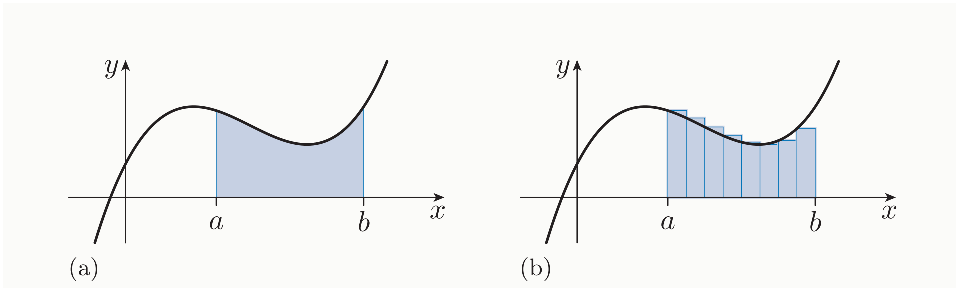


Figure 3 (a) The area between a graph and the x-axis, from $x = a$ to $x = b$ (b) A collection of rectangles whose total area is approximately this area

Here's a summary of the method that you've met for finding an approximate value for an area of this type. You start by dividing the interval $[a, b]$ into a number, say n, of subintervals of equal width. The width of each subinterval is then $(b - a)/n$. For each subinterval you calculate the product

$$f\begin{pmatrix}\text{left endpoint}\\ \text{of subinterval}\end{pmatrix} \times \frac{b-a}{n}, \tag{1}$$

and you add up all these products. In general, the larger the number n of subintervals, the closer your answer will be to the area between the graph of f and the x-axis, from $x = a$ to $x = b$.

Now suppose that you have a continuous function f whose graph lies on or *below* the x-axis throughout an interval $[a, b]$, and you want to calculate the area between the graph of f and the x-axis, from $x = a$ to $x = b$, as illustrated in Figure 4(a).

You can use the method above, with a small adjustment at the end, to calculate an approximate value for an area like this. Consider what happens when you apply the method to a graph like the one in Figure 4(a). You start by dividing the interval $[a, b]$ into n subintervals of equal width, then you calculate all the products of form (1) and add them all up. For a graph like this, because the value of f at the left endpoint of each subinterval is *negative* (or possibly zero), each product of form (1) will also be negative (or zero). In fact, each product of form (1) will be the *negative* of the area between the line segment that approximates the curve and the x-axis, as illustrated in Figure 4(b). So when you add up all the products of form (1), you'll obtain an approximate value for the *negative* of the area between the curve and the x-axis, from $x = a$ to $x = b$.

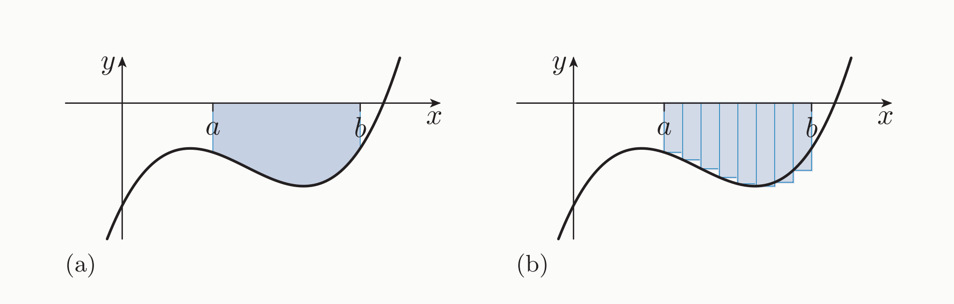


Figure 4 (a) The area between another graph and the x-axis, from $x = a$ to $x = b$ (b) A collection of rectangles whose total area is approximately this area

That's not a problem, because you can simply remove the minus sign to obtain the approximate value for the area that you want. However, to help us deal with situations like this, it's useful to make the following definitions. These definitions will be important throughout the unit.

Consider any region on a graph that lies either entirely above or entirely below the x-axis. The **signed area** of the region is its area with a plus or minus sign according to whether it lies above or below the x-axis, respectively. For example, in Figure 5 the two shaded regions above the x-axis have signed areas $+4$ and $+6$, respectively, which you can write simply as 4 and 6, and the shaded region below the x-axis has signed area -3.

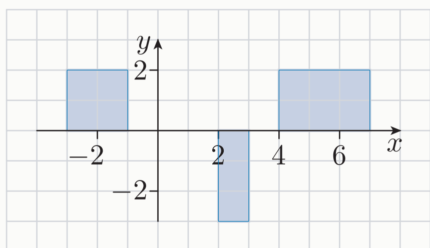


Figure 5 Regions on a graph

If you have a collection of regions on a graph, where each region in the collection lies either entirely above or entirely below the x-axis, then the total signed area of the collection is the sum of the signed areas of the individual regions. For example, the total signed area of the collection of three regions in Figure 5 is $4 + (-3) + 6 = 7$.

The units for signed area on a graph are, as you'd expect, the same as the units for area on the graph. That is, they're the units on the vertical axis times the units on the horizontal axis. If there are no specific units on the axes, then we don't use any specific units for signed area.

Positive signed area

Negative signed area

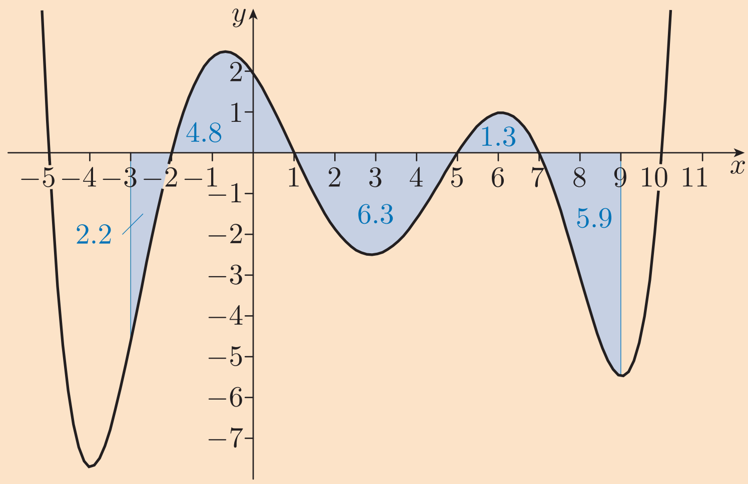
Example 2 *Understanding signed areas*

The areas of some regions on a graph are marked below.

In each of parts (a)–(c), use these areas to find the signed area between the graph and the x-axis, from the first value of x to the second value of x.

(a) From $x = -3$ to $x = -2$. (b) From $x = -2$ to $x = 1$.

(c) From $x = -3$ to $x = 1$.



Solution

(a) The signed area from $x = -3$ to $x = -2$ is -2.2.

(b) The signed area from $x = -2$ to $x = 1$ is $+4.8 = 4.8$.

(c) The signed area from $x = -3$ to $x = 1$ is $-2.2 + 4.8 = 2.6$.

Of course, the signed area value found in Example 2(c) doesn't correspond to any actual area on the graph, as it's the sum of a positive signed area and a negative signed area.

Activity 2 *Understanding signed areas*

Consider again the graph in Example 2. In each of parts (a)–(d) below, use the given areas to find the signed area between the graph and the x-axis, from the first value of x to the second value of x.

(a) From $x = 1$ to $x = 5$. (b) From $x = 5$ to $x = 7$.

(c) From $x = 1$ to $x = 7$. (d) From $x = 1$ to $x = 9$.

With the definition of signed area that you've now seen, the method that you've met in this subsection can be described concisely as follows.

Strategy:
To find an approximate value for the signed area between the graph of a continuous function f and the x-axis, from $x = a$ to $x = b$

Divide the interval between a and b into n subintervals, each of width $(b - a)/n$. For each subinterval, calculate the product

$$f\left(\begin{matrix}\text{endpoint of subinterval}\\ \text{nearest } a\end{matrix}\right) \times \frac{b-a}{n},$$

and add up all these products.

In general, the larger the number n of subintervals, the closer your answer will be to the required signed area.

Notice that the box above uses the phrase 'endpoint of subinterval nearest a', rather than 'left endpoint of subinterval', which means the same thing. This will be convenient later in this subsection. (Note that, in this phrase, it's the endpoint of the subinterval, not the subinterval itself, that's nearest a!)

Figures 6 and 7 illustrate the strategy above. If you add up the signed areas of all the rectangles in Figure 6(a), then you'll obtain an approximate value for the signed area in Figure 6(b).

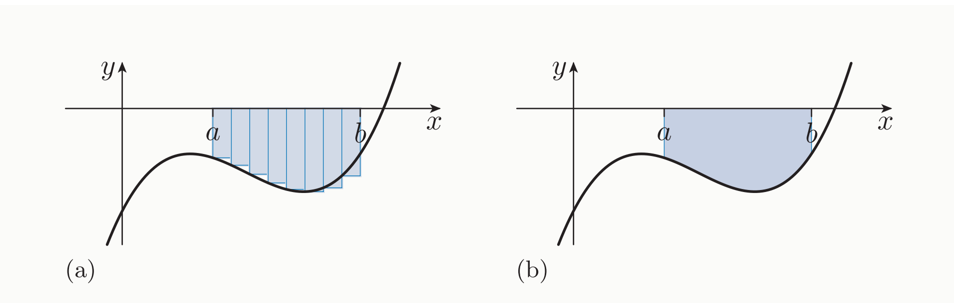


Figure 6 (a) A collection of rectangles whose total signed area is approximately the signed area shown in (b)

Similarly, if you add up the signed areas of all the rectangles in Figure 7(a), then you'll obtain an approximate value for the signed area in Figure 7(b).

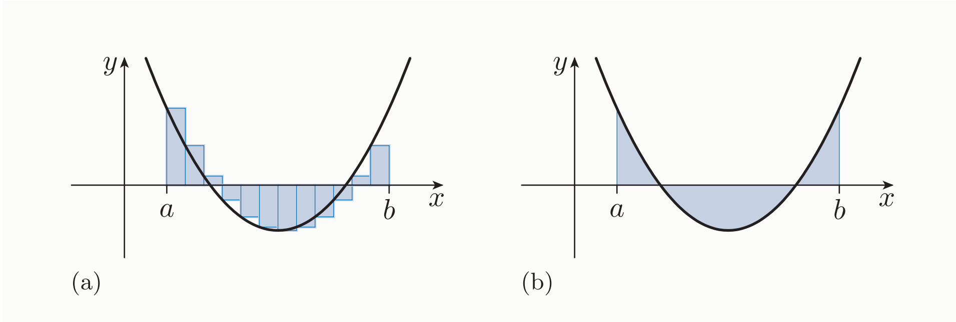
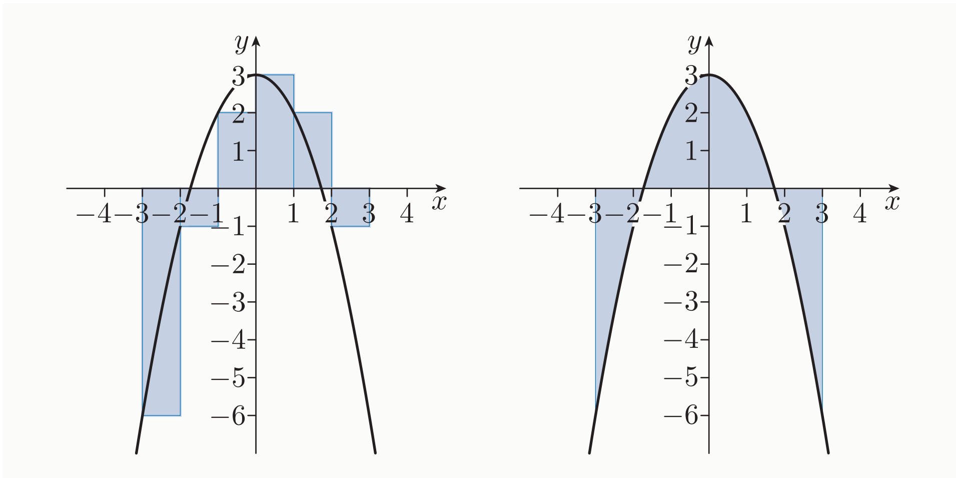


Figure 7 (a) Another instance of a collection of rectangles whose total signed area is approximately the signed area shown in (b)

Activity 3 *Calculating an approximate value for a signed area*

Use the method described in the box above, with six subintervals as shown on the left below, to find an approximate value for the signed area between the graph of the function $f(x) = 3 - x^2$ and the x-axis from $x = -3$ to $x = 3$, as shown on the right below.



In the next activity you can use a computer to explore approximations to signed areas found by using subintervals in the way that you've seen.

Activity 4 *Calculating more approximate values for signed areas*

Open the applet *Approximations for signed areas*. Check that the function is set to $f(x) = 3 - x^2$, the values of a and b are set to -3 and 3 respectively, and the number of subintervals is set to 6. The resulting approximation for the signed area between the graph of f and the x-axis, from $x = a$ to $x = b$, should then be the value found in the solution to Activity 3.

Now increase the number of subintervals, and observe the effect on the approximation. What do you think is the exact value of the signed area?

Experiment by changing the function, the values of a and b, and the number of subintervals, to find approximate values for some other signed areas.

So far in this subsection, whenever a signed area from $x = a$ to $x = b$ has been discussed, the value of b has been greater than the value of a.

The value of b can also be *equal* to the value of a, and this gives a signed area of zero, as illustrated in Figure 8. In cases like this, the method that you've seen for using subintervals to find approximate values for signed areas gives the answer zero, since the width $(b - a)/n$ of each subinterval is zero.

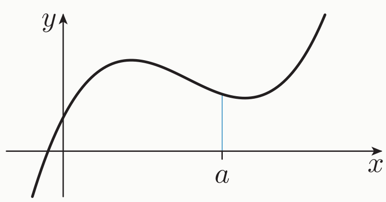


Figure 8 A signed area equal to zero

In fact, for reasons that you'll see later in the unit, it's useful to extend the definition of signed area a little, to give a meaning to the phrase 'the signed area between the graph of f and the x-axis from $x = a$ to $x = b$' when b is *less* than a. It's not immediately obvious what the phrase means in this case, but it turns out that the natural meaning is as follows. This definition will be important throughout the unit.

Suppose that f is a continuous function whose domain includes the interval $[b, a]$, as illustrated in Figure 9. Then the **signed area** between the graph of f and the x-axis from $x = a$ to $x = b$ is defined to be the *negative* of the signed area between the graph of f and the x-axis from $x = b$ to $x = a$.

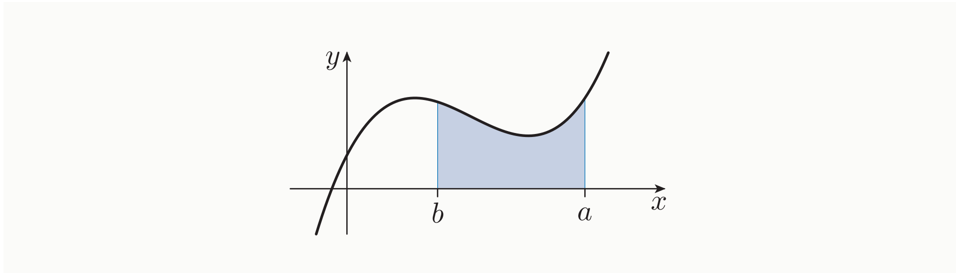


Figure 9 The graph of a continuous function f whose domain includes the interval $[b, a]$

For example, consider the graph in Figure 10. The signed area from $x = 1$ to $x = 3$ is 4, so the signed area from $x = 3$ to $x = 1$ is -4. Similarly, the signed area from $x = 6$ to $x = 8$ is -5, so the signed area from $x = 8$ to $x = 6$ is $-(-5) = 5$.

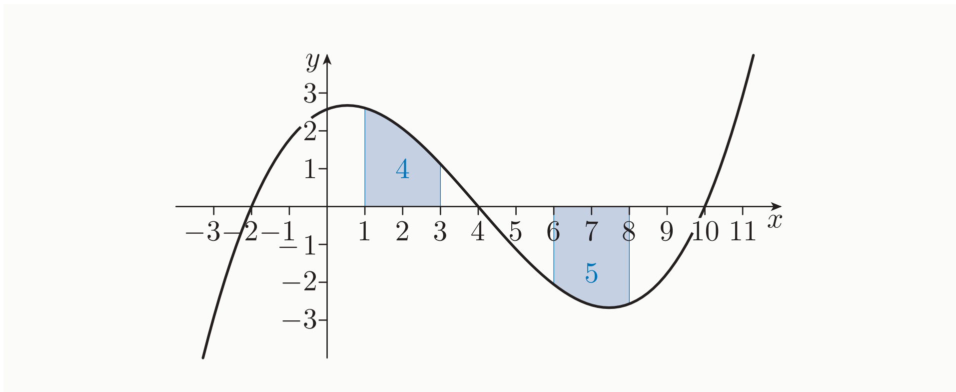


Figure 10 Areas on a graph

Because of this extended definition of signed area, you now have to think a little more carefully when you read or use the phrase 'signed area' (and the signed area isn't zero). If there's no mention of 'from $x = a$ to $x = b$', then to decide whether the signed area is positive or negative you just need to consider whether it lies above or below the x-axis. However, if the text is of the form 'the signed area from $x = a$ to $x = b$', then to decide whether the signed area is positive or negative you also have to consider whether b is greater than or less than a.

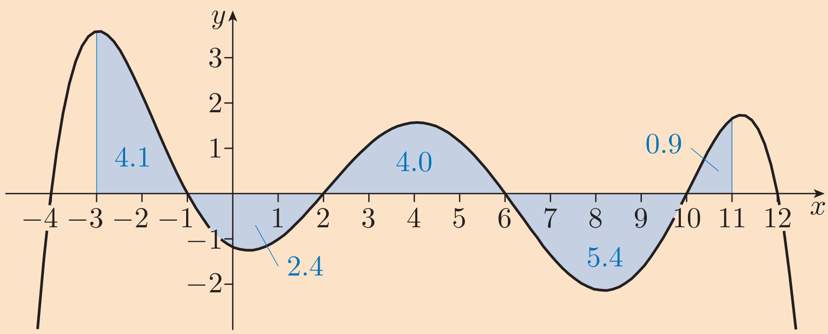
This is illustrated in the next example. A helpful way to think about whether b is greater than or less than a is to think about whether x moves forward or backward as it changes from $x = a$ to $x = b$.

Example 3 *Understanding the extended definition of signed area*

The areas of some regions on a graph are marked below. In each of parts (a)–(d), use the given areas to find the signed area between the graph and the x-axis, from the first value of x to the second value of x.

(a) From $x = 3$ to $x = 3$. (b) From $x = 6$ to $x = 10$.

(c) From $x = 10$ to $x = 6$. (d) From $x = 6$ to $x = -1$.



Solution

(a) From 3 to 3 is from a number to the same number.

The signed area from $x = 3$ to $x = 3$ is 0.

(b) From 6 to 10 is forward.

The signed area from $x = 6$ to $x = 10$ is -5.4.

(c) From 10 to 6 is backward.

The signed area from $x = 6$ to $x = 10$ is -5.4,
so the signed area from $x = 10$ to $x = 6$ is 5.4.

(d) From 6 to -1 is backward.

The signed area from $x = -1$ to $x = 6$ is $-2.4 + 4.0 = 1.6$,
so the signed area from $x = 6$ to $x = -1$ is -1.6.

In each part of the next activity, start by thinking about whether x moves forward or backward (or stays fixed) as it moves from the first value of x to the second value of x.

Activity 5 *Understanding the extended definition of signed area*

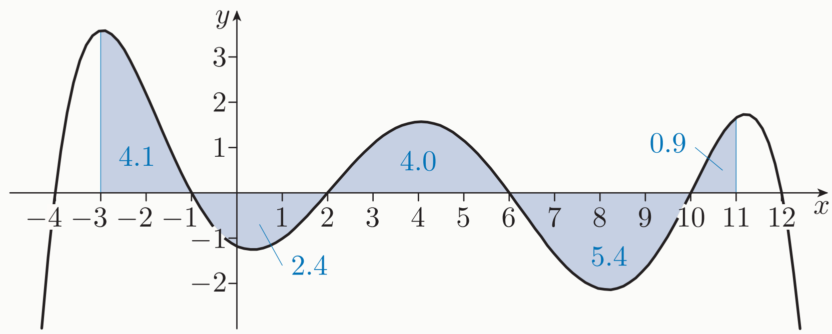
The graph in Example 3 is repeated below. In each of parts (a)–(g), use the given areas to find the signed area between the graph and the x-axis, from the first value of x to the second value of x.

(a) From $x = 2$ to $x = 6$. (b) From $x = 6$ to $x = 2$.

(c) From $x = 2$ to $x = 10$. (d) From $x = 10$ to $x = 2$.

(e) From $x = 8$ to $x = 8$. (f) From $x = 2$ to $x = -3$.

(g) From $x = 10$ to $x = -1$.



The subinterval method for finding approximate values for signed areas, which is summarised in the box on page 113, applies no matter whether the value of b is greater than, less than, or equal to the value of a. The reason why it works in cases where b is less than a is that in these cases the 'width' $(b - a)/n$ of each subinterval is *negative*.

You might like to use the applet *Approximations for signed areas* to see examples of the method giving approximate values for signed areas in cases where b is less than a.

Of course, if you want to work out an area between a curve and the x-axis over an interval $[a, b]$ by using the method that you've seen in this subsection, then normally you would work out the signed area from a to b, rather than the signed area from b to a. However, it's important to understand what's meant by the signed area from b to a, for reasons that you'll see later in the unit.

1.2 Definite integrals

Before you learn more about why signed areas are important, it's useful for you to learn some terminology and notation that are used when working with them.

If f is a continuous function and a and b are numbers in its domain, then the signed area between the graph of f and the x-axis from $x = a$ to $x = b$ is called the **definite integral** of f from a to b, and is denoted by

$$\int_a^b f(x)\,\mathrm{d}x.$$

This notation is read as 'the integral from a to b of f of x, d x'. The numbers a and b are called the **lower** and **upper limits of integration**, respectively.

For example, for the function f in Figure 11,

$$\int_3^7 f(x)\,\mathrm{d}x = -9, \quad \int_7^9 f(x)\,\mathrm{d}x = 2,$$

$$\text{and} \quad \int_3^9 f(x)\,\mathrm{d}x = -9 + 2 = -7.$$

Similarly, for the same function,

$$\int_4^4 f(x)\,\mathrm{d}x = 0, \quad \text{and} \quad \int_7^3 f(x)\,\mathrm{d}x = -(-9) = 9.$$

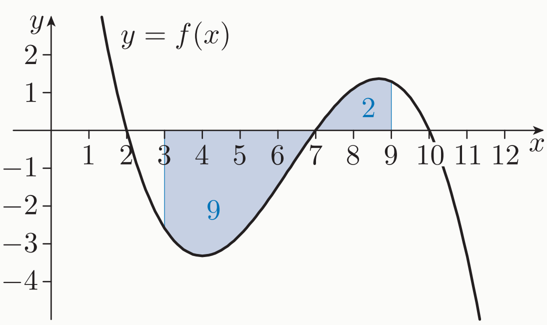


Figure 11 The graph of a function, and some areas

You'll see a little later in this subsection where this notation for a definite integral comes from.

The symbol $\displaystyle\int$ is called the **integral sign**. It sometimes appears in a smaller form, $\int$, when it's in a line of typed text.

As with Leibniz notation for derivatives, the 'd' in the notation for definite integrals has no independent meaning. In many texts, including this one, it's printed in upright type, rather than italic type, to emphasise that it's not a variable. You don't need to do anything special when you handwrite it.

When you use the notation $\int_a^b f(x)\,\mathrm{d}x$, remember that you must include not only the $\int_a^b$ at the beginning, but also the $\mathrm{d}x$ at the end. Try not to forget the $\mathrm{d}x$!

You've now seen that a *definite integral* is a type of signed area, and you saw in Unit 7 that an *indefinite integral* is a type of general antiderivative. These two seemingly unrelated concepts are, as you probably suspect, closely related. You'll see how in Section 2.

The box below lists some standard properties of definite integrals, which come from their definition as signed areas. These properties hold for all values of a, b and c in the domain of the continuous function f. The second property comes from the extended definition of signed area, to cases where b is less than a. The third property expresses the fact that the signed area from a to c is equal to the signed area from a to b plus the signed area from b to c. Figure 12 illustrates this property in a case where $a < b < c$, but the extended definition of signed area ensures that the property holds even when a, b and c aren't in this order. This is one reason why the definition of signed area is extended in the way that you've seen.

Standard properties of definite integrals

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

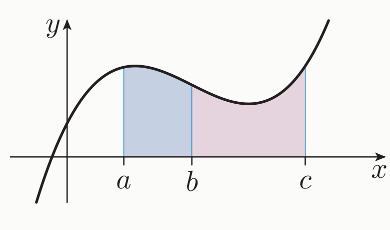


Figure 12 Two adjacent signed areas

Activity 6 *Understanding definite integrals*

Consider the function f whose graph is shown below. The areas of some regions are marked. By using these areas, write down the values of the following definite integrals.

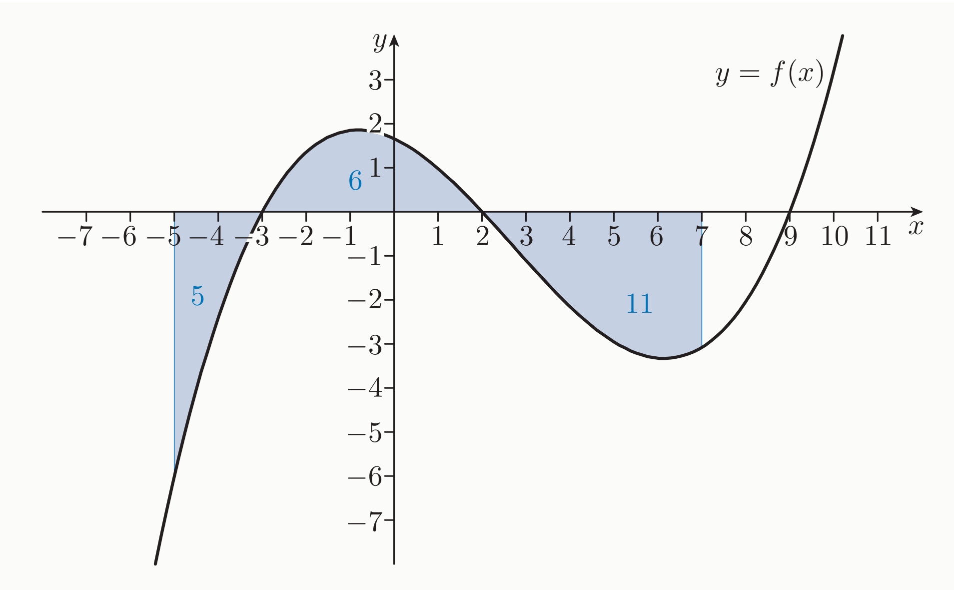
Hint: notice that in some of these definite integrals the upper limit of integration is less than the lower limit of integration.

(a) $\displaystyle\int_{-5}^{-3} f(x)\,\mathrm{d}x$ (b) $\displaystyle\int_{-3}^{2} f(x)\,\mathrm{d}x$ (c) $\displaystyle\int_{2}^{7} f(x)\,\mathrm{d}x$

(d) $\displaystyle\int_{-5}^{2} f(x)\,\mathrm{d}x$ (e) $\displaystyle\int_{-3}^{7} f(x)\,\mathrm{d}x$ (f) $\displaystyle\int_{-5}^{7} f(x)\,\mathrm{d}x$

(g) $\displaystyle\int_{5}^{5} f(x)\,\mathrm{d}x$ (h) $\displaystyle\int_{2}^{-3} f(x)\,\mathrm{d}x$ (i) $\displaystyle\int_{7}^{2} f(x)\,\mathrm{d}x$

(j) $\displaystyle\int_{-3}^{-5} f(x)\,\mathrm{d}x$ (k) $\displaystyle\int_{7}^{-3} f(x)\,\mathrm{d}x$ (l) $\displaystyle\int_{7}^{-5} f(x)\,\mathrm{d}x$



Activity 7 *Using a standard property of definite integrals*

Consider again the graph in Activity 6.

Given that $\displaystyle\int_{2}^{9} f(x)\,\mathrm{d}x = -15$, find $\displaystyle\int_{7}^{9} f(x)\,\mathrm{d}x$.

As you'd expect, you can replace the expression $f(x)$ in the notation for a definite integral by the formula for a particular function. For example, the signed areas in Figure 13 are denoted by

$$\int_{-1}^{1} x^2\,\mathrm{d}x \quad \text{and} \quad \int_{0}^{1} (x^3-1)\,\mathrm{d}x,$$

respectively.

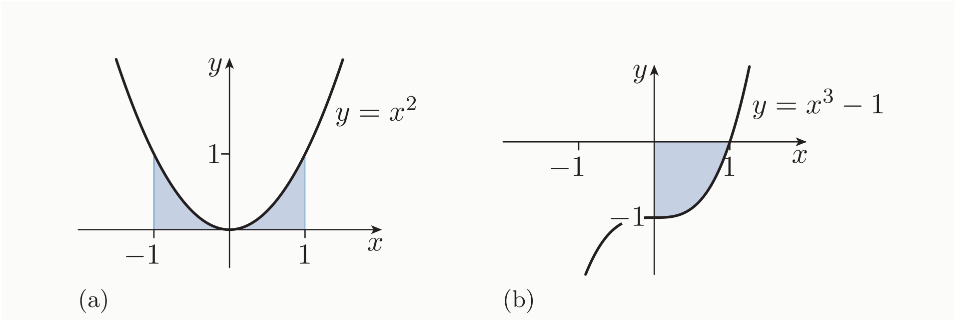


Figure 13 Signed areas between the graphs of particular functions and the x-axis

As always with algebraic notation, the notation for a definite integral can be used with letters other than the standard ones. For example, if g is a continuous function whose domain contains the numbers p and q, and you use t to denote the input variable of g, then the definite integral of g from $t=p$ to $t=q$ is denoted by

$$\int_{p}^{q} g(t)\,\mathrm{d}t.$$

In fact, the input variable of the function in a definite integral is what's known as a **dummy variable** – you can change its name to any other variable name that you like, without affecting the value of the definite integral. For example, if f is a continuous function whose domain contains the numbers a and b, then

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \int_{a}^{b} f(t)\,\mathrm{d}t = \int_{a}^{b} f(u)\,\mathrm{d}u,$$

and so on. As a particular example,

$$\int_{-1}^{1} x^2\,\mathrm{d}x = \int_{-1}^{1} t^2\,\mathrm{d}t = \int_{-1}^{1} u^2\,\mathrm{d}u,$$

since all these definite integrals denote the signed area shown in Figure 13(a).

If f is any continuous function, and a and b are numbers in its domain, then you can find an approximate value for the definite integral $\int_a^b f(x)\,\mathrm{d}x$, as accurately as you like, by using the method that you met in the last subsection. Here's the method expressed algebraically.

Suppose that you want to find $\int_a^b f(x)\,\mathrm{d}x$, as illustrated in Figure 14(a). You divide the interval between a and b into n subintervals, each of width $(b-a)/n$, as illustrated in Figure 14(b). We'll denote $(b-a)/n$ by w here, for conciseness.

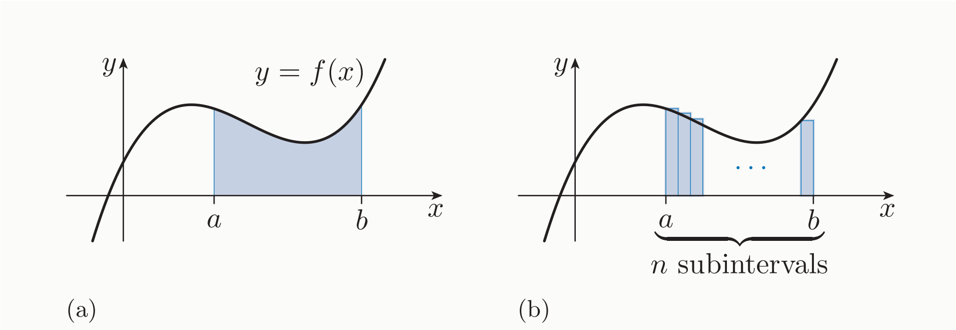


Figure 14 (a) A definite integral $\int_a^b f(x)\,\mathrm{d}x$ (b) A collection of n rectangles whose total signed area is approximately this definite integral

The endpoints nearest a of the subintervals are

$$a+0w, \quad a+1w, \quad a+2w, \quad \ldots, \quad a+(n-1)w.$$

(Remember that the strategy on page 113 specifies 'endpoint of subinterval nearest a', rather than 'left endpoint of subinterval'. The reason for this is that it ensures that the algebraic expressions for the endpoints stated above are correct when b is less than a, as well as when b is greater than a or equal to a.)

So an approximate value for the definite integral of f from $x=a$ to $x=b$ is given by

$$f(a+0w)\times w + f(a+1w)\times w + f(a+2w)\times w + \cdots$$
$$\cdots + f(a+(n-1)w)\times w.$$

This expression can be simplified slightly by taking out the common factor w. Doing this (and writing the common factor at the end) gives

$$\Big(f(a+0w)+f(a+1w)+f(a+2w)+\cdots+f(a+(n-1)w)\Big)w.$$

As the value of n gets larger and larger, the value of this expression gets closer and closer to the value of $\int_a^b f(x)\,\mathrm{d}x$.

The *limit* of the value of the expression, as the value of n tends to infinity, is the *exact* value of $\int_a^b f(x)\,\mathrm{d}x$. This idea is summarised below. The notation '$\lim\limits_{n\to\infty}$' means 'the limit, as n tends to infinity, of'.

Algebraic definition of a definite integral

Suppose that f is a continuous function and a and b are numbers in its domain. Then the **definite integral** of f from $x = a$ to $x = b$ is given by the equation

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\Big(f(a+0w) + f(a+1w) + f(a+2w) + \cdots \\ \cdots + f(a+(n-1)w)\Big)w$$

where $w = (b-a)/n$.

If f is a particular continuous function, and a and b are numbers in its domain, then you can sometimes calculate an exact value for $\int_a^b f(x)\,\mathrm{d}x$ by using an algebraic method to find the limit in the box above. In some cases you might even be able to obtain a general formula for $\int_a^b f(x)\,\mathrm{d}x$, in terms of a and b. However, this is usually a complicated and difficult process, so normally we don't do it. Instead, if you want to evaluate a definite integral $\int_a^b f(x)\,\mathrm{d}x$, then you have two main options.

The first option is to obtain an approximate value by using subintervals in the way that you've seen. We usually use a computer to carry out the calculations. The computer algebra system that you're using in this module includes a command that instructs the computer to find an approximate value for a definite integral, to a particular level of precision, by using a version of the subinterval method that you've seen. You'll learn about this command in the final subsection of this unit.

The other option for evaluating a definite integral $\int_a^b f(x)\,\mathrm{d}x$ is to use an algebraic method that arises from the *fundamental theorem of calculus*, which you'll meet in Section 2. This algebraic method involves much less calculation than using subintervals, and it has the advantage of giving an exact answer. However, sometimes it's not possible or not practicable to use this method, as you'll see.

Now here's an explanation of where the notation and terminology for definite integrals come from. They arise directly from the method that you've seen for calculating approximate values for definite integrals. To see how, consider applying the method to a function f, as illustrated in Figure 15(a).

Consider any one of the subintervals. You could denote its endpoint nearest a by x, and its width by δx, as shown in Figure 15(b). As you saw in Unit 6, the notation δx represents a small change in x.

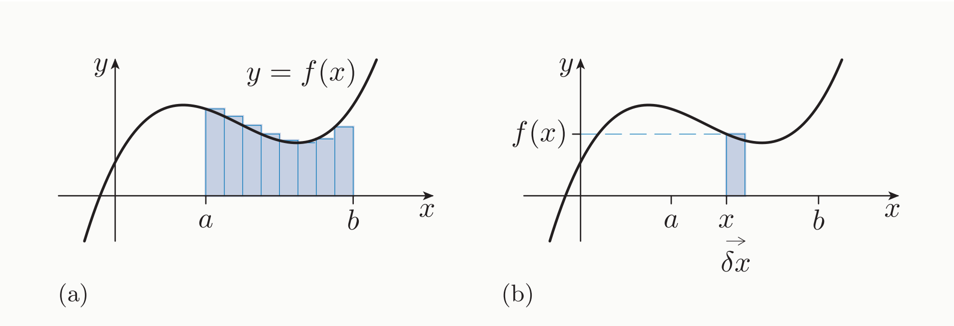


Figure 15 (a) The subinterval method applied to a function f from $x = a$ to $x = b$ (b) One of the subintervals and its corresponding rectangle

With this notation, the signed area of the rectangle corresponding to the subinterval, from x to $x + \delta x$, is

$$f(x)\,\delta x. \tag{2}$$

So the total signed area of all the rectangles is the sum of a number of terms, each of form (2). As the number of subintervals gets larger and larger, the size of δx gets smaller and smaller, and the total signed area of the rectangles gets closer and closer to the definite integral of f from a to b. So, loosely, you can think of this definite integral as the sum of infinitely many terms of form (2), where the quantity δx is infinitely small. Historically, as you saw in Unit 6, the quantity δx was denoted by $\mathrm{d}x$ when it becomes infinitely small, so the sum described above was denoted by

$$\int_a^b f(x)\,\mathrm{d}x,$$

where the symbol $\int$ is an elongated 'S', which stands for 'sum'.

Johann Bernoulli (1667–1748)

To see where the term 'integral' comes from, remember that the verb 'to integrate' means 'to join together'. Loosely, you can think of a definite integral as being obtained by joining together infinitely many signed areas, each infinitely narrow, into a single signed area.

The name and principal symbol for integral calculus were discussed by Gottfried Wilhelm Leibniz and the Swiss mathematician Johann Bernoulli. Leibniz preferred 'calculus summatorius' as the name, and $\int$, the elongated S, as the symbol. Bernoulli preferred 'calculus integralis' as the name, and the capital letter I as the symbol. Eventually they agreed to compromise, adopting Bernoulli's name 'integral calculus' and Leibniz's symbol $\int$. The first appearance in print of the word 'integral' was in a work by Johann's brother, Jacob Bernoulli, in 1690, though Johann insisted that he had been the one to introduce the term.

Jacob Bernoulli (1654–1705)

Joseph Fourier (1768–1830)

The modern notation for a definite integral, with limits at the bottom and top of the integral sign, is due to the French mathematician Joseph Fourier, who introduced it in his pioneering book on heat diffusion *Théorie analytique de la chaleur* (The analytic theory of heat) of 1822. He first published the notation four years previously in an announcement for the book. Fourier accompanied Napoleon Bonaparte on his Egyptian expedition of 1798, and later supported Jean-François Champollion, the decipherer of the Rosetta Stone. He is remembered today for initiating the investigation of *Fourier series.* These are infinite trigonometric series, which are now named after him, and which arose in the context of his work on heat diffusion. You'll meet the idea of a *series* in Unit 10.

To finish this section, here's a summary of the main idea that you've met.

Definite integrals

Suppose that f is a continuous function, and a and b are numbers in its domain. The signed area between the graph of f and the x-axis from $x = a$ to $x = b$ is called the **definite integral** of f from a to b, and is denoted by

$$\int_a^b f(x)\,\mathrm{d}x.$$

2 The fundamental theorem of calculus

In this section you'll discover how the ideas about signed areas that you met in Section 1 are connected with the ideas about rates of change that you met in Units 6 and 7. The two areas of mathematics are linked by an important fact known as *the fundamental theorem of calculus.*

This theorem is stated in the first subsection of this section, preceded by an explanation of why it's true. It's a good idea to read the explanation, to increase your understanding, particularly if you intend to study more mathematics after this module. However, if you find the explanation hard to understand, then you can skip it, and go straight to the box headed 'Fundamental theorem of calculus', near the end of Subsection 2.1. Read the contents of this box, and continue reading to the end of the subsection. Make sure that you understand the fundamental theorem of calculus, and the other important facts stated there, even if you don't know why they're true. The two subsections that follow, Subsections 2.2 and 2.3, show you how to use the fundamental theorem of calculus. You might like to try reading Subsection 2.1 again later, as some of the ideas that you found difficult the first time might seem easier once you're more familiar with the ways in which you can use the fundamental theorem of calculus.

2.1 What is the fundamental theorem of calculus?

The first step in obtaining a link between signed areas and rates of change is to define, for any continuous function f, a new function whose values are signed areas between the graph of f and the x-axis. Here's how we do that.

Consider any continuous function f, as illustrated in Figure 16.

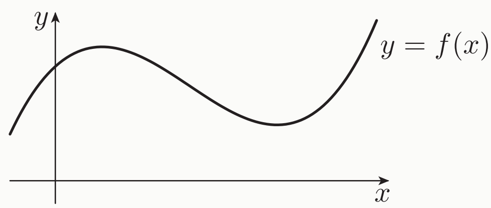


Figure 16 The graph of a continuous function f

Let's choose any number, say s, in the domain of f, and define a new function, A, to have the same domain as f and the following rule:

$A(x) =$ signed area between the graph of f and the x-axis, from s to x.

This definition is illustrated in Figure 17.

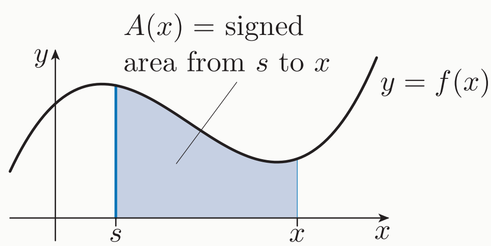


Figure 17 A continuous function f and a signed area $A(x)$

We call the function A the **signed-area-so-far function** for the function f, with starting point s. Its rule can be expressed concisely as follows:

$$A(x) = \int_s^x f(t)\,\mathrm{d}t.$$

(Here the variable t has been used in the definite integral instead of the usual variable x, to avoid confusion with the input variable x of the function A.)

Note that if a and b are any two numbers in the domain of f, then the signed area between the graph of f and the x-axis, from $x = a$ to $x = b$, is given by

$$A(b) - A(a).$$

In other words,

$$\int_a^b f(x)\,\mathrm{d}x = A(b) - A(a).$$

This fact is illustrated in Figure 18, in a case where $s < a < b$. The third property of definite integrals in the box on page 120 tells you that the fact holds even when s, a and b aren't in this order.

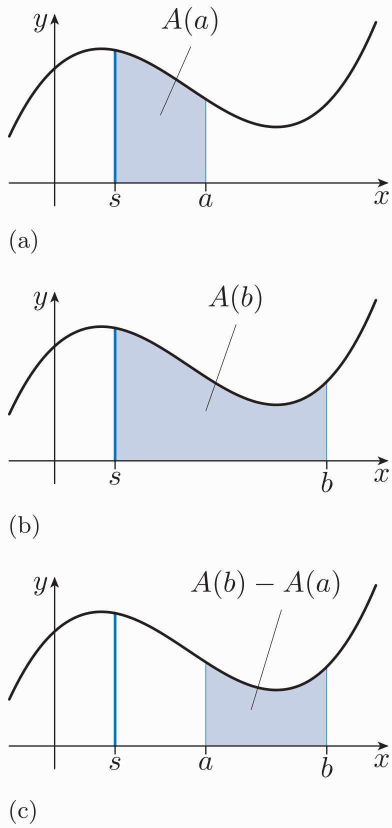


Figure 18 An illustration of the fact that $\int_a^b f(x)\,\mathrm{d}x = A(b) - A(a)$

Now let's consider how the value of a signed-area-so-far function changes as the value of the input variable x changes. Let's start by looking at what happens when the function f is a *constant* function.

For example, consider the constant function $f(x) = 3$, whose graph is shown in Figure 19. Let's take the starting point s for the signed-area-so-far function A to be 1, as indicated in Figure 19.

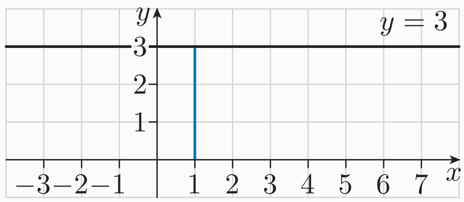


Figure 19 The graph of the constant function $f(x) = 3$

Figure 20 shows some values of $A(x)$.

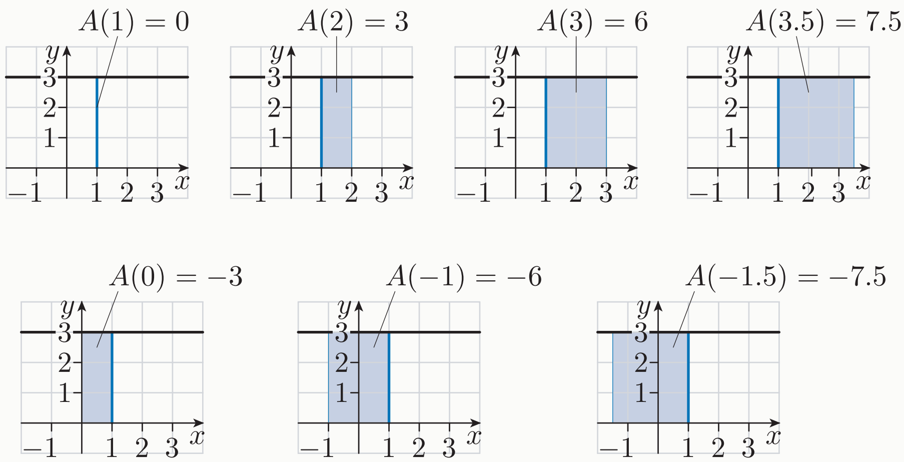


Figure 20 Some values of $A(x)$ when f is the constant function $f(x) = 3$ and the starting point s is 1

You can see that the signed area $A(x)$ changes at the rate of 3 square units for every unit by which x changes. For example,

if x increases by 1, then $A(x)$ increases by 3
if x increases by 0.5, then $A(x)$ increases by $0.5 \times 3 = 1.5$
if x increases by -1, then $A(x)$ increases by $-1 \times 3 = -3$.

(Remember that a negative increase is a decrease.)

In other words, the *rate of change* of $A(x)$ with respect to x is 3.

More generally, you can see that for any constant function $f(x) = k$, and any signed-area-so-far function A for f, the value of $A(x)$ changes at the rate of k square units for every unit by which x changes. In other words, the rate of change of $A(x)$ is k.

There are two more examples of this in Figure 21. In Figure 21(a), the function is $f(x) = 1.5$, so the signed area $A(x)$ changes at the rate of 1.5 square units for each unit by which x changes. In Figure 21(b), the function is $f(x) = -2$, so the signed area $A(x)$ changes at the rate of -2 square units for each unit by which x changes.

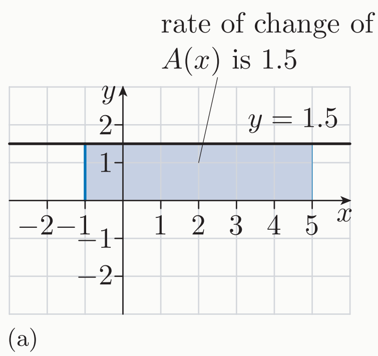


Figure 21 Signed-area-so-far functions for constant functions, with starting points $s = -1$ in (a) and $s = 3$ in (b)

Now let's look at what happens for a function f that isn't a constant function. Consider, for example, the function f whose graph is shown in Figure 22. The starting point s for the signed-area-so-far function A has been chosen to be 2.

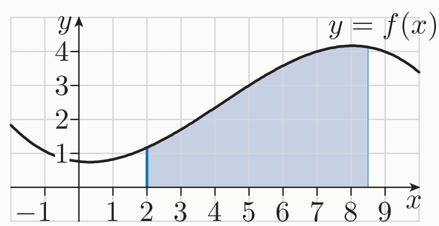


Figure 22 A signed-area-so-far function for a function f that isn't a constant function

For this function, as the value of x changes, the value of $A(x)$ doesn't change at a constant rate. For example, as illustrated in Figure 23(a), when x is close to 5 the value of $A(x)$ changes at the rate of about 3 square units for every unit by which x increases. Similarly, as illustrated in

Figure 23(b), when x is close to 7 the value of $A(x)$ changes at the rate of about 4 square units for every unit by which x increases.

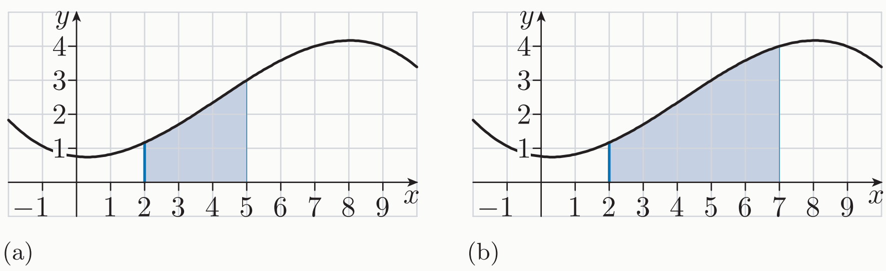


Figure 23 (a) When x is close to 5, the rate of change of $A(x)$ is about 3 (b) When x is close to 7, the rate of change of $A(x)$ is about 4

More precisely, when $x = 5$ the *instantaneous* rate of change of $A(x)$ is 3, and when $x = 7$ the *instantaneous* rate of change of $A(x)$ is 4. In fact – and this is the crucial thing to realise – at any value of x, the instantaneous rate of change of $A(x)$ is $f(x)$, as illustrated in Figure 24.

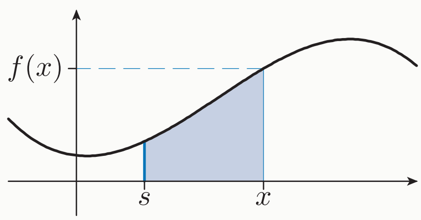


Figure 24 The instantaneous rate of change of $A(x)$ at x is $f(x)$

In other words, the derivative of the function A is the function f, and so the function A is an antiderivative of the function f.

You can see that this will be true for any continuous function f and any signed-area-so-far function A for f. That is, we have the important fact below. Because of its importance, it deserves to be called a *theorem*.

Theorem

Suppose that f is a continuous function, and s is any number in its domain. Let A be the signed-area-so-far function for f with starting point s. Then A is an antiderivative of f.

It's this fact that leads to the fundamental theorem of calculus. Consider any continuous function f. Let s be any number in its domain, and let A be the signed-area-so-far function for f with starting point s.

Suppose that a and b are any numbers in the domain of f. Then, as you saw earlier in this subsection, the value of $\int_a^b f(x)\,\mathrm{d}x$, the signed area between the graph of f and the x-axis from $x = a$ to $x = b$, is given by

$$\int_a^b f(x)\,\mathrm{d}x = A(b) - A(a). \tag{3}$$

Now let F be *any* antiderivative of f. Then, since A and F are both antiderivatives of f, the value of $F(b) - F(a)$ is the same as the value of $A(b) - A(a)$. That's because, as you saw in Unit 7, the value of $F(b) - F(a)$ is the same no matter which antiderivative of f you take F to be.

So you can replace $A(b) - A(a)$ by $F(b) - F(a)$ in equation (3) above. This gives the crucial fact that's known as the **fundamental theorem of calculus**. It's stated concisely below.

Fundamental theorem of calculus

Suppose that f is a continuous function whose domain contains the numbers a and b, and that F is an antiderivative of f. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

This theorem can be stated in words as follows.

Fundamental theorem of calculus (in words)

Suppose that f is a continuous function whose domain contains the numbers a and b, and that F is an antiderivative of f. Then the signed area between the graph of f and the x-axis from $x = a$ to $x = b$ is equal to the change in the value of F as x changes from $x = a$ to $x = b$.

The equation in the fundamental theorem of calculus is true no matter whether the value a is less than, equal to or greater than the value b. This is the main reason why the definition of signed area is extended in the way that you saw earlier in this unit.

Here's another important fact that's come out of the discussion in this subsection. The theorem in the box on page 131 tells you immediately that the following fact is true.

Every continuous function has an antiderivative.

In fact there's some disagreement among mathematicians about exactly which theorem should be called 'the fundamental theorem of calculus'. Everyone agrees that this name should be given to a theorem that links signed areas with rates of change. However, while some mathematicians use the name for the theorem that's given this name in this module, other mathematicians use it for the theorem in the box on page 131. Still other mathematicians take the view that these two theorems are *two* fundamental theorems of calculus, or two parts of a single fundamental theorem of calculus. So if you search for 'fundamental theorem of calculus' on the internet, or look it up in books, then you're likely to find a variety of different results.

2.2 Using the fundamental theorem to find signed areas

The fundamental theorem of calculus gives you a quick way to find the signed area between the graph of a continuous function f and the x-axis, from $x = a$ to $x = b$, in cases *where you know a formula for an antiderivative F of f*. The theorem tells you that this signed area, which is the definite integral

$$\int_a^b f(x)\,\mathrm{d}x,$$

is given simply by $F(b) - F(a)$.

For example, suppose that you want to find the area between the graph of the function $f(x) = \cos x$ and the x-axis, from $x = 0$ to $x = \pi/2$, as illustrated in Figure 25. This area is the definite integral

$$\int_0^{\pi/2} \cos x\,\mathrm{d}x.$$

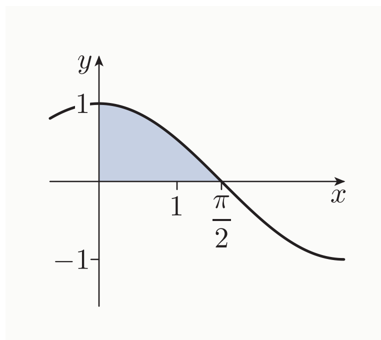


Figure 25 The area between the graph of $f(x) = \cos x$ and the x-axis, from $x = 0$ to $x = \pi/2$

Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$ (as you saw in Unit 7), the fundamental theorem of calculus tells you that

$$\int_0^{\pi/2} \cos x\,\mathrm{d}x = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1 - 0 = 1.$$

So the area in Figure 25 is 1 square unit.

You may find it magical that a complicated area like the one in Figure 25 can be found from such a simple calculation. In particular, it may seem amazing that although the area in Figure 25 depends on infinitely many values of $\cos x$ (namely all the values of $\cos x$ for $x = 0$ to $x = \pi/2$), it can be calculated from the values of an antiderivative of $\cos x$ at just *two* values of x, namely $x = 0$ and $x = \pi/2$.

Notice that in the calculation above it doesn't matter which of the infinitely many antiderivatives of $f(x) = \cos x$ you use, because they all differ by a constant and hence they all lead to the same final answer. For example, using the antiderivative $F(x) = \sin x + 7$ gives

$$\int_0^{\pi/2} \cos x \, dx = \left(\sin\left(\frac{\pi}{2}\right) + 7\right) - (\sin 0 + 7) = 1 + 7 - 0 - 7 = 1.$$

It's usually best to use the simplest antiderivative in calculations of this kind.

You'll see another example of the fundamental theorem of calculus used to find a signed area shortly, and you'll then have the opportunity to find some signed areas in this way yourself. First, however, it's helpful for you to learn a type of notation that's convenient in calculations of this kind.

For any function F, the expression

$$F(b) - F(a)$$

can be denoted by

$$\big[F(x)\big]_a^b.$$

For example,

$$\big[\sin x\big]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1 - 0 = 1.$$

You can practise evaluating expressions of this type in the next activity.

Activity 8 *Evaluating expressions of the form* $[F(x)]_a^b$

Evaluate the following expressions. In part (c), give your answer to three significant figures.

(a) $\left[\frac{1}{2}x^2\right]_3^5$ (b) $\big[\cos x\big]_0^{2\pi}$ (c) $\big[e^x\big]_{-1}^1$

With the square bracket notation introduced above, the fundamental theorem of calculus can be restated as follows.

Fundamental theorem of calculus (square bracket notation)

Suppose that f is a continuous function whose domain contains the numbers a and b, and F is an antiderivative of f. Then

$$\int_a^b f(x) \, dx = \big[F(x)\big]_a^b.$$

In the next example, the fundamental theorem of calculus is used, together with the square bracket notation, to find a signed area – that is, to evaluate a definite integral.

Example 4 *Evaluating a definite integral*

Evaluate the definite integral

$$\int_1^2 x^3 \, dx.$$

(This definite integral is shown in Figure 26.)

Solution

Use the fundamental theorem of calculus. By the general formula for the indefinite integral of a power function, an antiderivative of x^3 is $\frac{1}{4}x^4$.

$$\int_1^2 x^3 \, dx = \left[\tfrac{1}{4}x^4\right]_1^2$$

Evaluate this expression.

$$\begin{aligned} &= \left(\tfrac{1}{4} \times 2^4\right) - \left(\tfrac{1}{4} \times 1^4\right) \\ &= 4 - \tfrac{1}{4} = \tfrac{15}{4}. \end{aligned}$$

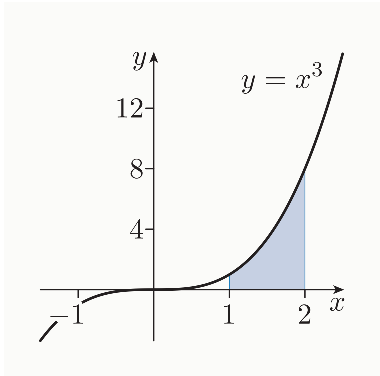


Figure 26 The definite integral in Example 4

You've now seen that, when you want to find a signed area, it's much quicker and simpler to use the fundamental theorem of calculus than to use the method involving subintervals that you saw in Subsection 1.1. Another advantage of using the fundamental theorem of calculus is that it gives you an *exact* answer, rather than an approximate one. However, remember that you can use the fundamental theorem of calculus only when you can find a formula for the antiderivative that you need. Sometimes it's difficult or impossible to find such a formula.

In the next activity, you can try evaluating some definite integrals by using the fundamental theorem of calculus. Use the square bracket notation, as in Example 4, since you need to get used to this notation. You can find the antiderivatives that you need by using the table of standard indefinite integrals given near the end of Unit 7. This table is also included in the *Handbook*.

Activity 9 *Evaluating definite integrals*

Evaluate the following definite integrals (which are shown in Figure 27). Give exact answers, and (except in parts (a) and (d), where the answer is a simple fraction or an integer) also give answers to three significant figures.

(a) $\displaystyle\int_{-1}^{1} x^2\,dx$ (b) $\displaystyle\int_{0}^{2} e^t\,dt$ (c) $\displaystyle\int_{1}^{4} \frac{1}{u}\,du$

(d) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2\theta\,d\theta$ (e) $\displaystyle\int_{-1}^{1} \frac{1}{1+u^2}\,du$

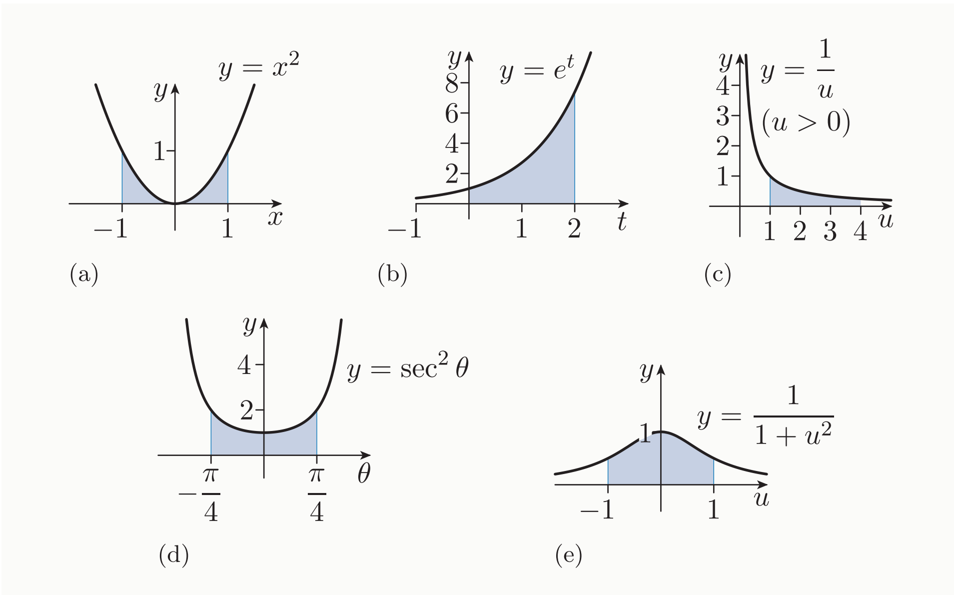


Figure 27 The definite integrals in Activity 9

The next example demonstrates how to evaluate another definite integral by using the fundamental theorem of calculus. The only difference from the examples that you've seen so far is that the antiderivative that's needed for the method can't be found directly from the table of standard indefinite integrals. Instead, we have to start by manipulating the given function to express it as a sum of constant multiples of functions that we can integrate, then use the constant multiple rule and sum rule for antiderivatives, together with the table of standard indefinite integrals. You practised finding antiderivatives in this way in Unit 7.

As shown in the example, a convenient way to carry out these sorts of manipulations is to manipulate the expression inside the notation $\int_a^b \dots\,dx$. This expression is known as the **integrand**. For example, the definite integral $\int_0^1 x^2\,dx$ has integrand x^2.

Example 5 *Evaluating a more complicated definite integral*

Evaluate the definite integral

$$\int_2^4 \frac{(2x+1)(x-4)}{x}\,\mathrm{d}x.$$

(It is shown in Figure 28.)

Solution

Manipulate the integrand to get it into a form that you can integrate.

$$\begin{aligned}
&\int_2^4 \frac{(2x+1)(x-4)}{x}\,\mathrm{d}x \\
&= \int_2^4 \frac{2x^2-7x-4}{x}\,\mathrm{d}x \\
&= \int_2^4 \left(2x-7-\frac{4}{x}\right)\mathrm{d}x \\
&= \int_2^4 \left(2x-7-4\times\frac{1}{x}\right)\mathrm{d}x
\end{aligned}$$

Apply the fundamental theorem of calculus, and evaluate the result.

$$\begin{aligned}
&= \left[2\times\tfrac{1}{2}x^2-7x-4\ln x\right]_2^4 \\
&= \left[x^2-7x-4\ln x\right]_2^4 \\
&= (4^2-7\times 4-4\ln 4)-(2^2-7\times 2-4\ln 2) \\
&= 16-28-4\ln 4-4+14+4\ln 2 \\
&= -2-4(\ln 4-\ln 2) \\
&= -2-4\ln(4/2) \\
&= -2-4\ln 2
\end{aligned}$$

There's no need to find a decimal approximation, as the question didn't ask for one.

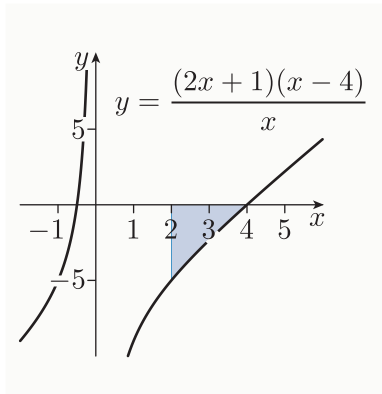


Figure 28 The definite integral in Example 5

There are other acceptable ways to leave the final answer in Example 5. For example, you could write it as $-2(1+2\ln 2)$.

Activity 10 *Evaluating more complicated definite integrals*

Evaluate the following definite integrals (which are shown in Figure 29). Give exact answers.

(a) $\displaystyle\int_1^4 3\sqrt{x}\,\mathrm{d}x$ (b) $\displaystyle\int_{-1}^0 x(1+x)\,\mathrm{d}x$ (c) $\displaystyle\int_2^4 \frac{r+5}{r}\,\mathrm{d}r$

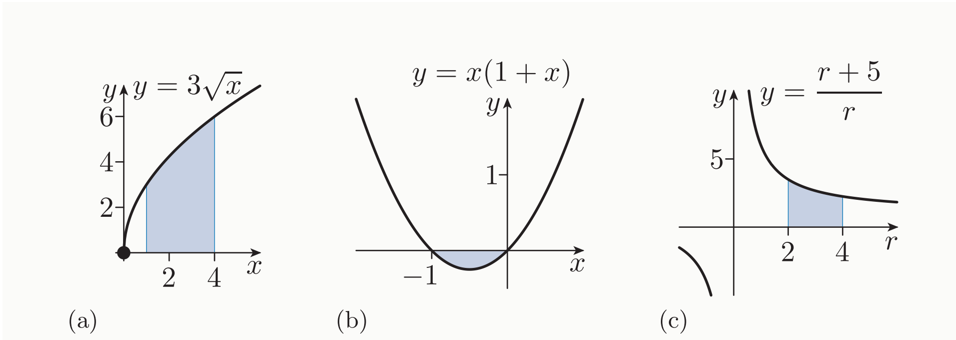


Figure 29 The definite integrals in Activity 10

When you use the square bracket notation, you can sometimes simplify your working by using the following rules.

Constant multiple rule and sum rule for the square bracket notation

$$\big[kF(x)\big]_a^b = k\big[F(x)\big]_a^b, \quad \text{where } k \text{ is a constant}$$

$$\big[F(x)+G(x)\big]_a^b = \big[F(x)\big]_a^b + \big[G(x)\big]_a^b$$

For example, the constant multiple rule for the square bracket notation tells you that

$$\big[5\sin x\big]_1^2 = 5\big[\sin x\big]_1^2$$

and the sum rule for the square bracket notation tells you that

$$\big[\sin x + \cos x\big]_1^2 = \big[\sin x\big]_1^2 + \big[\cos x\big]_1^2.$$

To see why these rules hold, you just need to use the definition of the square bracket notation. For example,

$$\begin{aligned}\big[5\sin x\big]_1^2 &= 5\sin 2 - 5\sin 1\\ &= 5(\sin 2 - \sin 1)\\ &= 5\big[\sin x\big]_1^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\left[\sin x + \cos x\right]_1^2 &= (\sin 2 + \cos 2) - (\sin 1 + \cos 1)\\ &= (\sin 2 - \sin 1) + (\cos 2 - \cos 1)\\ &= \left[\sin x\right]_1^2 + \left[\cos x\right]_1^2.\end{aligned}$$

As always when a constant multiple rule and a sum rule hold, the sum rule for the square bracket notation also holds if you replace the plus sign on each side with a minus sign. To prove this, you combine the sum rule with the case $k = -1$ of the constant multiple rule. You saw this done for the constant multiple and sum rules for derivatives in Unit 6, Subsection 2.3.

The next example illustrates how you can use the constant multiple rule and the sum rule for the square bracket notation when you're evaluating a definite integral.

Example 6 *Using the constant multiple rule and sum rule for the square bracket notation*

Evaluate the definite integral

$$\int_1^2 (1 + x^3)\,\mathrm{d}x.$$

(It is shown in Figure 30.)

Solution

Use the fundamental theorem of calculus.

$$\int_1^2 (1 + x^3)\,\mathrm{d}x = \left[x + \tfrac{1}{4}x^4\right]_1^2$$

Use the sum rule.

$$= \left[x\right]_1^2 + \left[\tfrac{1}{4}x^4\right]_1^2$$

Use the constant multiple rule, and simplify the result.

$$\begin{aligned}&= \left[x\right]_1^2 + \tfrac{1}{4}\left[x^4\right]_1^2\\ &= (2 - 1) + \tfrac{1}{4}(2^4 - 1^4)\\ &= 1 + \tfrac{1}{4} \times 15\\ &= \tfrac{19}{4}.\end{aligned}$$

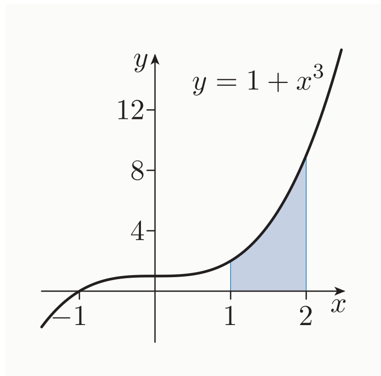


Figure 30 The definite integral in Example 6

You can practise using the constant multiple rule and sum rule for the square bracket notation in the next activity.

Activity 11 *Using the constant multiple rule and sum rule for the square bracket notation*

Evaluate the following definite integrals. (They're shown in Figure 31.) Give your answer to part (a) to three significant figures.

(a) $\displaystyle\int_3^5 x^{3/2}\,\mathrm{d}x$ (b) $\displaystyle\int_{-1/2}^{1} (u - 2u^2)\,\mathrm{d}u$

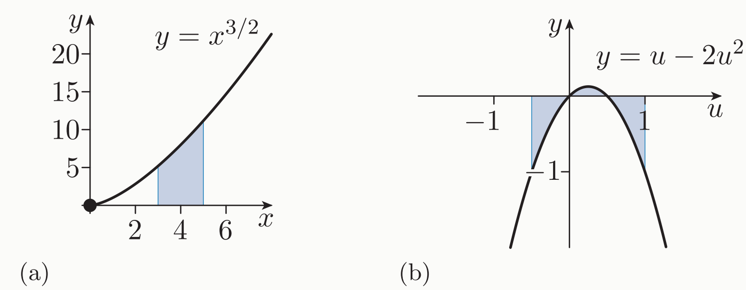


Figure 31 The definite integrals in Activity 11

Note that when you're evaluating a definite integral it's sometimes more convenient *not* to use the constant multiple rule and/or the sum rule for the square bracket notation, even if there's an opportunity to use them. You can choose whether or not to use them, as seems convenient.

A constant multiple rule and a sum rule also hold for definite integrals, as follows.

Constant multiple rule and sum rule for definite integrals

$$\int_a^b k\,f(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x, \quad \text{where } k \text{ is a constant}$$

$$\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

These rules follow from the fundamental theorem of calculus and the constant multiple rule and sum rule for antiderivatives, which you met in Unit 7. To see this for the constant multiple rule, suppose that f is a continuous function whose domain contains the numbers a and b, and that k is a constant. Let F be an antiderivative of f. Then, by the constant multiple rule for antiderivatives, $kF(x)$ is an antiderivative of $kf(x)$.

Hence, by the fundamental theorem of calculus and the constant multiple rule for the square bracket notation,

$$\begin{aligned}\int_a^b k\,f(x)\,\mathrm{d}x &= \big[k\,F(x)\big]_a^b \\ &= k\big[F(x)\big]_a^b \\ &= k\int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$

You can prove the sum rule for definite integrals in a similar way.

Remember that the sum rule for definite integrals also holds if you replace the plus sign on each side with a minus sign, because, as mentioned earlier, this is always the case when a constant multiple rule and a sum rule hold.

The constant multiple rule for definite integrals tells you that, for example, the area in Figure 32(b) is twice the area in Figure 32(a).

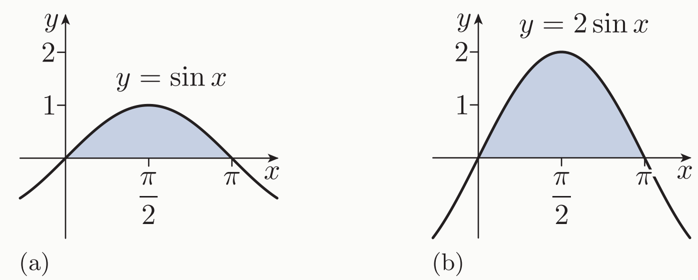


Figure 32 The area in (b) is twice the area in (a)

Similarly, the sum rule for definite integrals tells you that, for example, the area in Figure 33(c) is the sum of the areas in Figure 33(a) and (b).

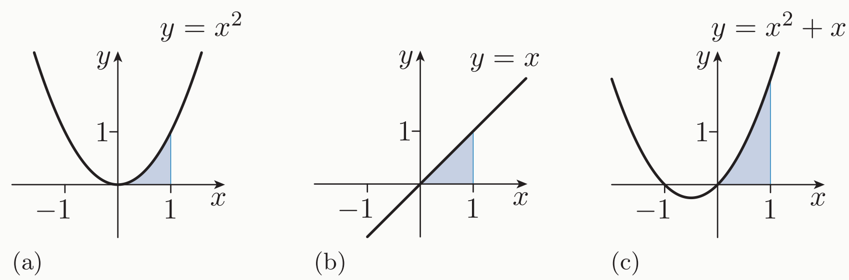


Figure 33 The area in (c) is the sum of the areas in (a) and (b)

The next example illustrates how you can use the constant multiple rule and the sum rule for definite integrals when you're evaluating a definite integral.

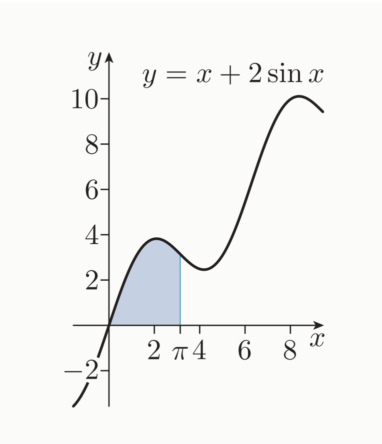


Figure 34 The definite integral in Example 7

Example 7 *Using the constant multiple rule and sum rule for definite integrals*

Evaluate the definite integral

$$\int_0^{\pi} (x + 2\sin x)\,\mathrm{d}x.$$

(It is shown in Figure 34.)

Solution

Use the fundamental theorem of calculus. Use the constant multiple rule and sum rule, for definite integrals and for the square bracket notation, as convenient.

$$\begin{aligned}\int_0^{\pi} (x + 2\sin x)\,\mathrm{d}x &= \int_0^{\pi} x\,\mathrm{d}x + \int_0^{\pi} 2\sin x\,\mathrm{d}x \\ &= \int_0^{\pi} x\,\mathrm{d}x + 2\int_0^{\pi} \sin x\,\mathrm{d}x \\ &= \left[\tfrac{1}{2}x^2\right]_0^{\pi} + 2\left[-\cos x\right]_0^{\pi} \\ &= \tfrac{1}{2}\left[x^2\right]_0^{\pi} - 2\left[\cos x\right]_0^{\pi} \\ &= \tfrac{1}{2}(\pi^2 - 0^2) - 2(\cos\pi - \cos 0) \\ &= \tfrac{1}{2}\pi^2 - 2(-1-1) \\ &= \tfrac{1}{2}\pi^2 + 4.\end{aligned}$$

You can practise using the constant multiple rule and sum rule for definite integrals in the next activity.

Activity 12 *Using the constant multiple rule and sum rule for definite integrals*

Evaluate the following definite integrals. (They are shown in Figure 35.) Give exact answers.

(a) $\displaystyle\int_{\sqrt{3}}^{\sqrt{7}} \tfrac{1}{2}x^3\,\mathrm{d}x$ (b) $\displaystyle\int_{-\pi/4}^{\pi/4} (\sin x + \cos x)\,\mathrm{d}x$

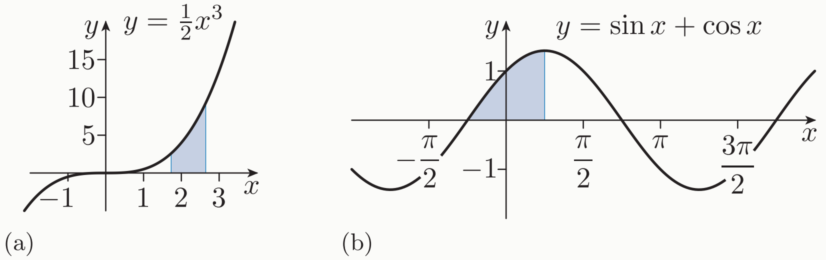


Figure 35 The definite integrals in Activity 12

As with the sum rule and constant multiple rule for the square bracket notation, you can choose whether or not to use the sum rule and the constant multiple rule for definite integrals in your working, as seems convenient. Often it's more convenient not to use them, even if there's an opportunity to do so.

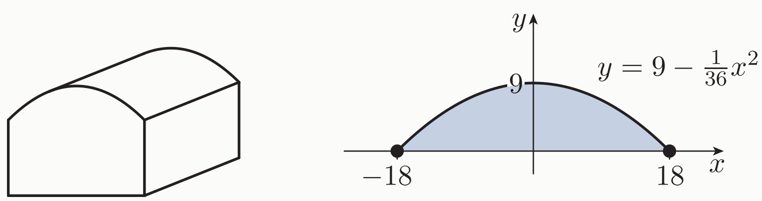
In the next activity, you're asked to use the fundamental theorem of calculus to find the exact area of the cross-section of a roof that was discussed at the beginning of Subsection 1.1.

Activity 13 *Using the fundamental theorem for a practical problem*

The curve of the roof of a building is given by the graph of the function

$$f(x) = 9 - \tfrac{1}{36}x^2,$$

where x and $f(x)$ are measured in metres. Use the ideas that you've met in this section to find the area between this graph and the x-axis from $x = -18$ to $x = 18$, as shown below. Hence state the area of the cross-section of the roof.



You may remember that in Subsection 1.1 we used subintervals to work out that the area in Activity 13 is approximately $216\,\mathrm{m}^2$. The solution to Activity 13 confirms that the area is *exactly* $216\,\mathrm{m}^2$.

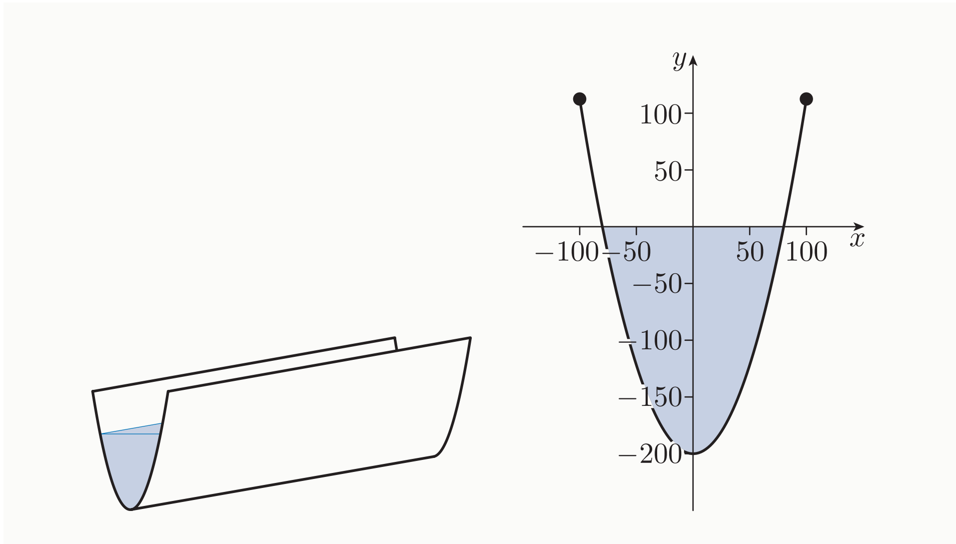
Here's another practical example for you to try. In this example the graph lies below the x-axis, so the signed area that you'll obtain by using the fundamental theorem of calculus will be negative. You have to remove the minus sign to obtain the required area.

Activity 14 *Solving another practical problem about area*

An engineer is planning the construction of an open channel with a parabolic cross-section, as illustrated below. The shape of the cross-section is given by the equation

$$y = \tfrac{1}{32}x^2 - 200 \quad (-100 \le x \le 100),$$

where x and y are measured in centimetres, as shown in the graph below.



The maximum height of liquid permissible in the channel is the height of the x-axis on the graph; that is, 200 cm above the lowest point of the channel. To ensure that the channel is adequate for the expected flow of liquid, the engineer needs to know the area shaded on the graph.

(a) Find the x-intercepts of the graph.

(b) Hence calculate the shaded area, to the nearest square centimetre.

The history of the fundamental theorem of calculus

You've seen that the fundamental theorem of calculus relates rates of change – that is, gradients of tangents – to signed areas.

One of the earliest mathematicians to relate the problem of calculating tangents to the problem of calculating areas was Isaac Barrow, the first Lucasian professor at the University of Cambridge. However, his method was presented in the geometrical style of the day and was not computationally useful for solving problems. Isaac Newton was one of Barrow's students, and in 1669 Barrow resigned his professorship so that Newton could take it up.

Newton based his definition of quadrature (determining area) on the idea that it is the converse of finding tangents. So what for some people was, and for modern-day mathematicians is, a *theorem* that finding areas and tangents are inverse processes – the fundamental theorem of calculus – was taken by Newton as the very *definition* of quadrature. Newton's way of regarding area questions as the opposite of tangency questions is set down clearly in his *De analysi* of 1669.

Gottfried Wilhelm Leibniz published his first accounts of differential and integral calculus in the newly founded journal *Acta Eruditorum Lipsiensium* (Acts of the scholars of Leipzig) in 1684 and 1686 respectively, and his proof of the fundamental theorem of calculus followed in 1693.

The first rigorous proof of the fundamental theorem of calculus was provided in the early decades of the 19th century by the French mathematician Augustin Louis Cauchy, who included it in his course at the École Polytechnique in Paris. Cauchy was one of the most productive and influential mathematicians of his generation. His reformulation of the foundations of calculus, while very thorough, was also very hard to understand – so much so that not only his students but also his fellow professors protested!

Isaac Barrow (1630–1677)

Augustin Louis Cauchy (1789–1857)

2.3 Using the fundamental theorem to find changes in the values of antiderivatives

The fundamental theorem of calculus isn't useful just for solving problems related to finding areas. It's also useful for solving problems involving rates of change.

As you saw in Subsection 5.3 of Unit 7, in integral calculus you often want to solve problems of the following type. You have a continuous function f, and two numbers a and b in its domain, and you want to work out the amount by which an antiderivative of f changes from $x = a$ to $x = b$. (Remember that all the antiderivatives of f change by the same amount.)

If you can find a formula for an antiderivative F of f, then it's straightforward to do this: you just calculate $F(b) - F(a)$. You saw examples like this in Unit 7.

If you can't find a formula for an antiderivative, then you can use the fundamental theorem of calculus to help you find an approximate answer. The theorem tells you that, for any antiderivative F of f,

$$F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x.$$

So the answer that you want is equal to the signed area between the graph of f and the x-axis, from $x = a$ to $x = b$. You can use the method that you met in Subsection 1.1 to find an approximate value for this signed area. You would normally do that by using a computer, as mentioned earlier.

For example, suppose that you have a velocity-time graph for an object moving along a straight line, such as the graph in Example 8 below. Since displacement is an antiderivative of velocity, the fundamental theorem of calculus tells you that the change in the displacement of the object, from any point in time to any other point in time, is equal to the signed area between the graph and the time axis, from the first point in time to the second point in time. This fact is used in Example 8.

As with any graph, the units for signed area on a velocity-time graph are the units on the vertical axis times the units on the horizontal axis. So, for example, the units for signed area on the graph in Example 8 are

$$\mathrm{m\,s^{-1}} \times \mathrm{s} = \mathrm{m};$$

that is, metres. These are units for displacement, as you'd expect.

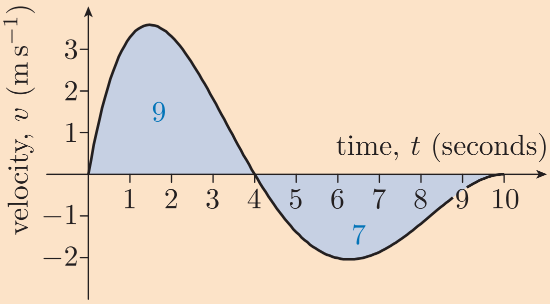
Example 8 *Using the fundamental theorem of calculus for a velocity-time graph*

The graph below is the velocity–time graph of an object moving along a straight line. The areas (in m) of some regions are marked on the graph.

In each of parts (a) to (c), use the given areas to find the amount by which the displacement of the object changes from the first time to the second time.

(a) From $t = 0$ to $t = 4$. (b) From $t = 4$ to $t = 10$.

(c) From $t = 0$ to $t = 10$.



Solution

(a) The change in displacement from $t = 0$ to $t = 4$ is $9\,\text{m}$.

(b) The change in displacement from $t = 4$ to $t = 10$ is $-7\,\text{m}$.

(c) The change in displacement from $t = 0$ to $t = 10$ is $9\,\text{m} + (-7)\,\text{m} = 2\,\text{m}$.

Notice that the velocity of the object in Example 8 changes from positive to negative at time $t = 4$. That is, at this time the object changes from moving in the positive direction to moving in the negative direction, as illustrated in Figure 36. The negative displacement that occurs after this time cancels out most of the positive displacement that occurred initially. Since the object was displaced by $9\,\text{m}$ in the positive direction, and then by $7\,\text{m}$ in the negative direction, its final displacement from its starting position is $2\,\text{m}$, as found in Example 8(c). The object travels a total distance of $9\,\text{m} + 7\,\text{m} = 16\,\text{m}$ during the 10 seconds of its motion.

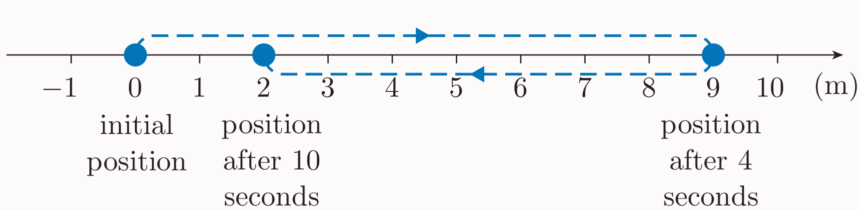
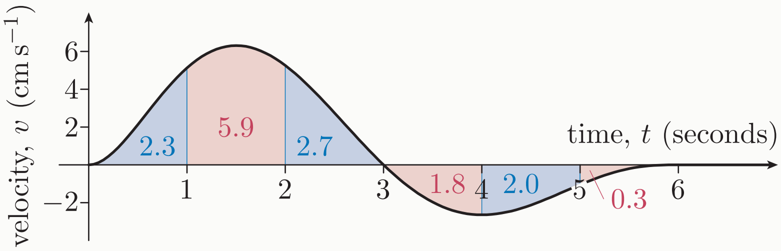


Figure 36 The object in Example 8 starts moving in the positive direction, then turns and moves in the negative direction

Activity 15 *Using the fundamental theorem of calculus for a velocity–time graph*

The velocity–time graph of an object moving along a straight line is shown below. The areas of some regions are marked on the graph. Use these areas to find the answers in parts (a) to (c) below.



(a) In each of parts (i) to (iv), find the amount by which the displacement of the object changes from the first time to the second time.

(i) From $t = 0$ to $t = 1$. (ii) From $t = 3$ to $t = 4$.

(iii) From $t = 0$ to $t = 3$. (iv) From $t = 3$ to $t = 6$.

(b) What is the displacement of the object from its starting position at each of the following times?

(i) $t = 2$ (ii) $t = 6$

(c) What is the total distance travelled by the object during the 6 seconds of its motion?

If you want to solve a problem that involves finding a change in the value of an antiderivative, and you *can* find a formula for an antiderivative, then it's usually convenient to use the notation for a definite integral, and the square bracket notation, in your working. Here's Example 22 from Subsection 5.3 of Unit 7 again, but with the working carried out using this notation.

Example 9 *Finding a change in the value of an antiderivative*

Suppose that the rate of change of a quantity is given by the function $f(x) = \frac{1}{5}x^2$ $(x \geq 0)$, as shown in Figure 37. By what amount does the quantity change from $x = 3$ to $x = 6$?

Solution

The change in the quantity from $x = 3$ to $x = 6$ is

$$\begin{aligned}\int_3^6 \tfrac{1}{5}x^2\,\mathrm{d}x &= \left[\tfrac{1}{5} \times \tfrac{1}{3}x^3\right]_3^6\\ &= \left[\tfrac{1}{15}x^3\right]_3^6\\ &= \tfrac{1}{15}\left[x^3\right]_3^6\\ &= \tfrac{1}{15}(6^3 - 3^3)\\ &= \tfrac{1}{15} \times 189\\ &= \tfrac{63}{5}\\ &= 12.6.\end{aligned}$$

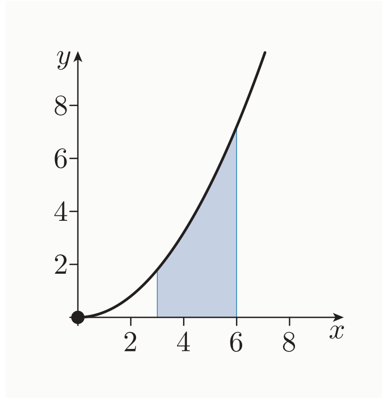


Figure 37 The graph of $y = \frac{1}{5}x^2$ $(x \geq 0)$

In the next activity, you're asked to repeat Activity 36 from Subsection 5.3 of Unit 7, but using the notation for a definite integral.

Activity 16 *Finding a change in displacement*

Suppose that an object moves along a straight line, and its velocity v (in $\mathrm{m\,s^{-1}}$) at time t (in seconds) is given by $v = 3 - \frac{3}{2}t$ $(0 \leq t \leq 6)$ (as shown in Figure 38).

(a) Write down a definite integral that gives the change in the displacement of the object from time $t = 0$ to time $t = 1$. Evaluate it, and hence state how far the object travels in that time.

(b) Repeat part (a), but from time $t = 4$ to time $t = 5$.

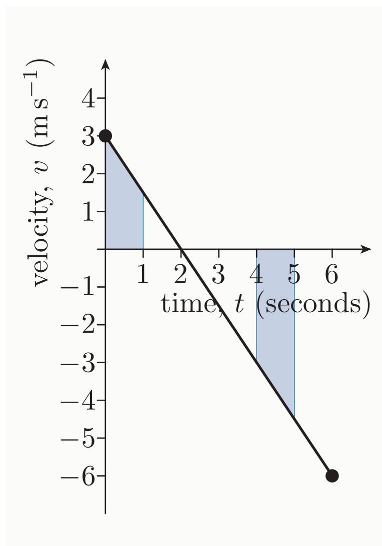


Figure 38 The graph of $v = 3 - \frac{3}{2}t$ $(0 \leq t \leq 6)$

Try using the notation for a definite integral to solve the problem in the next activity.

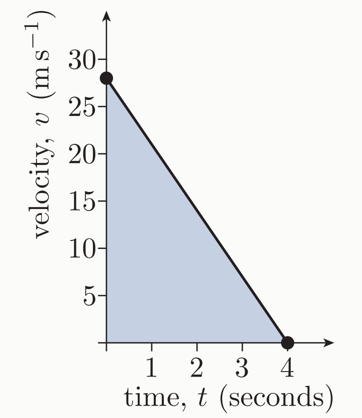


Figure 39 The graph of $v = 28 - 7t$ $(0 \le t \le 4)$

Activity 17 *Finding another change in displacement*

The driver of a car travelling at $28\,\mathrm{m\,s^{-1}}$ along a straight road applies the brakes. It takes four seconds for the car to stop, and its decreasing velocity v (in $\mathrm{m\,s^{-1}}$) during that time is given by the equation

$$v = 28 - 7t,$$

where t is the time in seconds since the brakes were applied (as shown in Figure 39). How far does the car travel during those four seconds?

In this subsection you've seen examples of the following two useful facts.

For the velocity-time graph of any object, and for any two points in time $t = a$ and $t = b$ in the time period covered by the graph, the following hold.

- The change in the displacement of the object from $t = a$ to $t = b$ is the total *signed area* between the graph and the time axis from $t = a$ to $t = b$.
- The total distance travelled by the object from $t = a$ to $t = b$ is the total *area* between the graph and the time axis from $t = a$ to $t = b$ (this applies when $a \le b$).

Note that if a graph is a straight line, or consists of several line segments, then integration may not be the easiest way to work out an area or a signed area between the graph and the horizontal axis. For example, you can work out the area shaded in Figure 39 by using the formula for the area of a triangle, as shown at the end of the solution to Activity 17. Try using a similar geometric method in the next activity, which is a repeat of Example 23 in Unit 7.

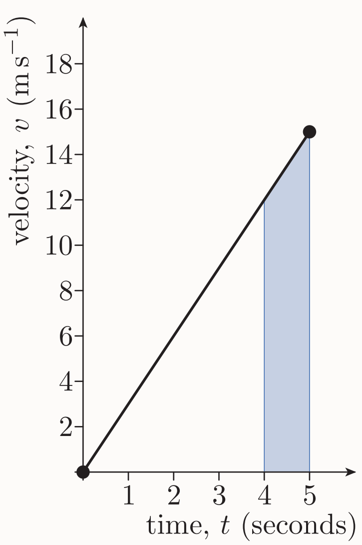


Figure 40 The graph of $v = 3t$ $(0 \le t \le 5)$

Activity 18 *Finding yet another change in displacement*

Suppose that a car begins to move along a straight road, and its velocity v (in $\mathrm{m\,s^{-1}}$) during the first five seconds of its journey is given by the equation

$$v = 3t,$$

where t is the time in seconds since it began moving, as shown in Figure 40. How far does the car travel from the end of the fourth second to the end of the fifth second?

2.4 Notation for indefinite integrals

Until now, when we've been working with antiderivatives and indefinite integrals, we've been using a lower-case letter, such as f, to denote the function that we start with, and the corresponding upper-case letter, such as F, to denote any of its antiderivatives, or its indefinite integral.

For example, some antiderivatives of the function $f(x) = x^2$ are

$$F(x) = \tfrac{1}{3}x^3, \quad F(x) = \tfrac{1}{3}x^3 + 7, \quad \text{and} \quad F(x) = \tfrac{1}{3}x^3 - \tfrac{2}{9},$$

and the indefinite integral of this function f is

$$F(x) = \tfrac{1}{3}x^3 + c.$$

However, there's a standard notation for indefinite integrals, which arises from the link between antiderivatives and definite integrals. For any function f that has an antiderivative, the indefinite integral of $f(x)$ is denoted by

$$\int f(x)\,\mathrm{d}x.$$

(This notation is read as 'the integral of f of x, d x'.)

For example,

$$\int x^2\,\mathrm{d}x = \tfrac{1}{3}x^3 + c,$$

and similarly

$$\int \sin x\,\mathrm{d}x = -\cos x + c.$$

(You saw in Unit 7 that the indefinite integral of $\sin x$ is $-\cos x + c$.)

Note that this notation is used only for indefinite integrals – *it's never used for antiderivatives.* So it's important to remember the following fact.

> If an equation has the notation $\int \ldots \mathrm{d}x$ on one side only, then there must be an arbitrary constant on the other side.

For example, the equation

$$\int x^2\,\mathrm{d}x = \tfrac{1}{3}x^3$$

is *incorrect.* The correct equation is

$$\int x^2\,\mathrm{d}x = \tfrac{1}{3}x^3 + c,$$

as you saw above.

We'll continue to use the capital letter notation for antiderivatives, when it's convenient to do so.

The notation for indefinite integrals introduced above, and the similar notation for definite integrals, are together known as **integral notation**. We refer to either an indefinite integral or a definite integral as an **integral**. In the indefinite integral

$$\int f(x)\,\mathrm{d}x,$$

the expression $f(x)$ is called the **integrand**, just as in a definite integral.

The next example illustrates how integral notation is used for indefinite integrals in practice. It's similar to the examples and activities on finding indefinite integrals in the second half of Unit 7, except that it uses integral notation.

Example 10 *Finding indefinite integrals, using integral notation*

Find the following indefinite integrals.

(a) $\displaystyle\int \sec^2 x\,\mathrm{d}x$ (b) $\displaystyle\int \frac{x-5}{x}\,\mathrm{d}x$

Solution

(a) The indefinite integral of $\sec^2 x$ is given in the table of standard indefinite integrals, so you can simply write it down.

$$\int \sec^2 x\,\mathrm{d}x = \tan x + c$$

(b) Manipulate the integrand to get it into a form that allows you to use the constant multiple rule and sum rule for antiderivatives, together with standard indefinite integrals.

$$\begin{aligned}\int \frac{x-5}{x}\,\mathrm{d}x &= \int \left(1-\frac{5}{x}\right)\mathrm{d}x\\ &= \int \left(1-5\times\frac{1}{x}\right)\mathrm{d}x\\ &= x-5\ln|x|+c\end{aligned}$$

You can practise using integral notation for indefinite integrals in the following activity. Don't forget to add the '$+c$' in each part.

Activity 19 *Finding indefinite integrals, using integral notation*

Find the following indefinite integrals.

(a) $\displaystyle\int e^x\,\mathrm{d}x$ (b) $\displaystyle\int (x+3)(x-2)\,\mathrm{d}x$ (c) $\displaystyle\int \frac{1}{x^3}\,\mathrm{d}x$

(d) $\displaystyle\int (\sin\theta - \cos\theta)\,\mathrm{d}\theta$ (e) $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x$ (f) $\displaystyle\int \frac{-3}{1+u^2}\,\mathrm{d}u$

(g) $\displaystyle\int \frac{1}{2(1+r^2)}\,\mathrm{d}r$ (h) $\displaystyle\int \frac{(1+x^2)}{2}\,\mathrm{d}x$

Since an indefinite integral is just an antiderivative with an arbitrary constant added on, the constant multiple rule and sum rule for antiderivatives, from Unit 7, can be written in terms of indefinite integrals. These forms of the rules are stated in the following box, in integral notation.

Constant multiple rule and sum rule for indefinite integrals

$$\int k\,f(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x, \quad \text{where } k \text{ is a constant}$$

$$\int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

As with the other constant multiple rules and sum rules, you can choose whether or not to use these rules in your working, as seems convenient. The next example repeats Example 10(b), but uses the constant multiple rule and sum rule for indefinite integrals in the forms stated above.

Example 11 *Using the constant multiple rule and sum rule for indefinite integrals*

Find the indefinite integral

$$\int \frac{x-5}{x}\,\mathrm{d}x.$$

Solution

Manipulate the integrand to get it into a form that allows you to use the constant multiple rule and sum rule for indefinite integrals, together with standard indefinite integrals.

$$\begin{aligned}\int \frac{x-5}{x}\,\mathrm{d}x &= \int \left(1-\frac{5}{x}\right)\mathrm{d}x\\ &= \int 1\,\mathrm{d}x - 5\int \frac{1}{x}\,\mathrm{d}x\\ &= x - 5\ln|x| + c\end{aligned}$$

Here's something to notice about Example 11. You might have thought that each of the two indefinite integrals in the second line of the working should have been replaced by an expression containing an arbitrary constant. That would lead to a solution something like this:

$$\begin{aligned}\int \frac{x-5}{x}\,\mathrm{d}x &= \int \left(1-\frac{5}{x}\right)\mathrm{d}x\\ &= \int 1\,\mathrm{d}x - 5\int \frac{1}{x}\,\mathrm{d}x\\ &= (x+c) - 5(\ln|x| + d)\\ &= x - 5\ln|x| + c - 5d,\end{aligned}$$

where c and d are arbitrary constants.

However, since the arbitrary constants c and d can take any value, the expression $c-5d$ can also take any value, so we may as well replace it by a single letter that can take any value. It's convenient to use the usual letter, c. In general, if an expression consists of a sum of constant multiples of indefinite integrals, then it needs only a single arbitrary constant.

You can practise using the constant multiple rule and sum rule for indefinite integrals in the next activity.

Activity 20 *Using the constant multiple rule and sum rule for indefinite integrals*

Find the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{3}{x} - \frac{1}{1+x^2}\right) \mathrm{d}x$ (b) $\displaystyle\int (\operatorname{cosec} x)(\operatorname{cosec} x + \cot x)\, \mathrm{d}x$

You've now reached the end of the material in this module that tells you what integration is all about. In the rest of this unit you'll concentrate on actually finding indefinite and definite integrals.

Before you go on to that, you might like to watch the 25-minute video *The birth of calculus*, which is available on the module website. It tells you more about the history of the development of calculus.

3 Integration by substitution

To allow you to fully exploit all the ideas about integration that you've met, it's important that you're able to find formulas for the indefinite integrals of as wide a range of functions as possible. In the rest of this unit, you'll learn to integrate a much wider variety of functions than you learned to integrate in Unit 7. In the final subsection of the unit, you'll learn how to use the module computer algebra system to find indefinite and definite integrals.

Unfortunately, as mentioned in Unit 7, it's generally more tricky to integrate functions than it is to differentiate them. That's because there are no rules for antiderivatives that are similar to the product, quotient and chain rules for derivatives, so you can't usually integrate functions in the systematic way that you can differentiate them.

However, there are still many useful techniques that you can use to integrate functions. Each of these techniques applies to functions with particular characteristics. You'll meet some of these techniques in the rest of this unit. You'll learn how to recognise functions to which each technique can be applied, and how to use the techniques.

The technique that you'll learn in this section is *integration by substitution*. This technique might seem quite complicated when you first meet it, but you should find it more straightforward once you've had some practice with it. If you find it difficult, then make sure that you watch the tutorial clips, and try some extra practice. Remember that there are more practice exercises in the practice quiz and exercise booklet for this unit.

3.1 Basic integration by substitution

Integration by substitution is based on reversing the chain rule for differentiation.

Consider what happens when you differentiate an expression by using the chain rule. Remember that you start by recognising that the expression is a function of 'something'. You differentiate the expression with respect to the 'something', then you multiply the result by the derivative of the 'something' with respect to the input variable (usually x). For example, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x^2)\right) = \left(\cos(x^2)\right)(2x).$$

Since differentiation is the reverse of integration, this equation tells you that

$$\int \left(\cos(x^2)\right)(2x)\,\mathrm{d}x = \sin(x^2) + c.$$

Whenever you differentiate an expression by using the chain rule, you always obtain an expression of the form

$f(\text{something}) \times$ the derivative of the something,

where f is a function that you can integrate. The essential idea of integration by substitution is to recognise an expression that you want to integrate as having this form, and then perform the integration by reversing the chain rule.

To help you reverse the chain rule, it's helpful to denote the 'something' by an extra variable, in the way that you did when you first learned to use the chain rule in Unit 7. The usual choice of letter for the extra variable is u.

Here's how you can use this method to find the integral

$$\int \left(\cos(x^2)\right)(2x)\,\mathrm{d}x,$$

if you hadn't first seen the integrand obtained as an 'output' of the chain rule.

The first step is to recognise that the integrand has the form

$f(\text{something}) \times$ the derivative of the something,

where f is a function that you can integrate. You can see that it does, since it is

$\cos(\text{something}) \times$ the derivative of the something,

where the 'something' is x^2.

Then, since the 'something' is x^2, you put $u = x^2$. This gives $\mathrm{d}u/\mathrm{d}x = 2x$. Hence, in the integral you can replace $\cos(x^2)$ by $\cos u$, and $2x$ by $\mathrm{d}u/\mathrm{d}x$, to give

$$\int \cos(u)\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

Now here's the crucial step that you need to apply at this stage. Any integral of the form

$$\int f(u)\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

is equal to the simpler integral

$$\int f(u)\,\mathrm{d}u.$$

This is the step that uses the chain rule, and you'll see an explanation of why it's correct later in this subsection. Notice that it's easy to remember, because it looks like we've simply cancelled a '$\mathrm{d}x$' in a denominator with a '$\mathrm{d}x$' in a numerator. (Of course that's *not* what we've done, since $\mathrm{d}u/\mathrm{d}x$ isn't a fraction.)

So the integral that we're trying to find here can be expressed in terms of u as the simpler integral

$$\int \cos u\,\mathrm{d}u,$$

where $u = x^2$. You can now do the integration, which gives

$$\sin u + c.$$

The final step is to express this answer in terms of x, using the fact that $u = x^2$. This gives the final answer

$$\sin(x^2) + c.$$

So we've now worked out that

$$\int \left(\cos(x^2)\right)(2x)\,\mathrm{d}x = \sin(x^2) + c,$$

as expected.

Here's a summary of the method used above, which is the method known as **integration by substitution**.

Integration by substitution

1. Recognise that the integrand is of the form

   $$f(\text{something}) \times \text{the derivative of the something},$$

   where f is a function that you can integrate.
2. Set the something equal to u, and find $\mathrm{d}u/\mathrm{d}x$.
3. Hence write the integral in the form

   $$\int f(u)\,\mathrm{d}u,$$

   by using the fact that $\displaystyle\int f(u)\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$.
4. Do the integration.
5. Substitute back for u in terms of x.

As illustrated by the example above, the effect of substituting the variable u in place of the 'something' in the integral is to reduce the integral to a simpler integral in terms of u, which you may be able to find more easily.

You'll see another example of integration by substitution shortly, and you'll then have a chance to try it for yourself, but first it's useful for you to learn about a convenient way to carry out step 3 of the method.

Consider once more the integral above:

$$\int \left(\cos(x^2)\right)(2x)\,\mathrm{d}x.$$

Remember that we put $u = x^2$, which gave $\mathrm{d}u/\mathrm{d}x = 2x$, and this allowed us to write the integral in terms of u as

$$\int \cos u\,\mathrm{d}u.$$

It's convenient to think of this new form of the integral as being obtained from the original form by making two replacements, as shown below:

$$\int \underbrace{\left(\cos(x^2)\right)}_{\cos u}\ \underbrace{(2x)\,\mathrm{d}x}_{\mathrm{d}u}.$$

It's straightforward to work out that you can replace $\cos(x^2)$ by $\cos u$, as that just comes from the equation $u = x^2$.

A helpful way to work out that you can replace $(2x)\,\mathrm{d}x$ by $\mathrm{d}u$ is to imagine 'cross-multiplying' in the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

to obtain

$$\mathrm{d}u = (2x)\,\mathrm{d}x.$$

This isn't a real equation, of course, since $\mathrm{d}u$ and $\mathrm{d}x$ don't have independent meanings outside the notation for a derivative or integral. But it tells you immediately that you can replace $(2x)\,\mathrm{d}x$ by $\mathrm{d}u$. (In fact, it's possible to attach a meaning to $\mathrm{d}u$ and $\mathrm{d}x$, and hence to the equation above, but this is outside the scope of this module.)

Here's another example of integration by substitution, which illustrates all the ideas above.

Example 12 *Integrating by substitution*

Find the integral

$$\int e^{\tan x}\sec^2 x\,\mathrm{d}x.$$

Solution

Since the derivative of $\tan x$ is $\sec^2 x$, the integrand is of the form

$$e^{\text{something}} \times \text{the derivative of the something.}$$

So set the something equal to u.

Let $u = \tan x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$.

Imagine 'cross-multiplying' in the second equation to obtain $\mathrm{d}u = (\sec^2 x)\,\mathrm{d}x$.

So

$$\int e^{\tan x} \sec^2 x \,\mathrm{d}x = \int e^u \,\mathrm{d}u$$

Do the integration.

$$= e^u + c$$

Substitute back for u in terms of x.

$$= e^{\tan x} + c.$$

Here are some examples for you to try. Make sure that you have your *Handbook* to hand, so you can consult the tables of standard derivatives and standard integrals.

Activity 21 *Integrating by substitution*

Find the following indefinite integrals.

(a) $\displaystyle\int e^{\cos x}(-\sin x)\,\mathrm{d}x$ (b) $\displaystyle\int \left(\sin(x^3)\right)(3x^2)\,\mathrm{d}x$

(c) $\displaystyle\int \left(\frac{1}{\sin x}\right)\cos x\,\mathrm{d}x$ (d) $\displaystyle\int \left(\frac{1}{\cos^2 x}\right)(-\sin x)\,\mathrm{d}x$

(e) $\displaystyle\int \sin^4 x \cos x\,\mathrm{d}x$ (f) $\displaystyle\int \left(\frac{1}{1+\sin^2 x}\right)\cos x\,\mathrm{d}x$

In the examples that you've seen so far, it's been fairly straightforward to decide what the 'something' should be, so that the integrand is of the form

$$f(\text{something}) \times \text{the derivative of the something.}$$

However, sometimes you need to think about this more carefully, as illustrated in the next example.

Example 13 *Choosing the 'something' when you integrate by substitution*

Find the integral

$$\int \left(\frac{1}{2+\sin x}\right) \cos x \, dx.$$

Solution

The integrand is of the form

$$\frac{1}{2+\text{something}} \times \text{the derivative of the something},$$

where the something is $\sin x$.

It's also of the form

$$\frac{1}{\text{something}} \times \text{the derivative of the something},$$

where the something is $2+\sin x$.

Choosing the first option would lead to the first part of the integrand being replaced by $1/(2+u)$, which isn't straightforward to integrate. Choosing the second option would lead to the first part of the integrand being replaced by $1/u$, which you can integrate using a result from the table of standard integrals.

So choose the second option: take the something to be $2+\sin x$, and set it equal to u.

Let $u = 2+\sin x$; then $\dfrac{du}{dx} = \cos x$.

Imagine 'cross-multiplying' to obtain $du = \cos x \, dx$.

So

$$\begin{aligned}\int \left(\frac{1}{2+\sin x}\right) \cos x \, dx &= \int \frac{1}{u} \, du \\ &= \ln |u| + c \\ &= \ln |2+\sin x| + c\end{aligned}$$

In the particular case here you can remove the modulus signs, since $2+\sin x$ is always positive.

$$= \ln(2+\sin x) + c.$$

In each part of the next activity it's useful to take the 'something' to be an expression that includes a constant term, in a similar way to Example 13.

Activity 22 *Choosing the 'something' when you integrate by substitution*

Find the following indefinite integrals.

(a) $\displaystyle\int (4+\cos x)^7(-\sin x)\,\mathrm{d}x$ (b) $\displaystyle\int \sqrt{1+x^2}\,(2x)\,\mathrm{d}x$

(c) $\displaystyle\int (x^5-8)^{10}(5x^4)\,\mathrm{d}x$ (d) $\displaystyle\int \left(\frac{1}{e^x+5}\right) e^x\,\mathrm{d}x$

(e) $\displaystyle\int \big(\sin(5+2x^3)\big)\,(6x^2)\,\mathrm{d}x$

In the next activity, some of the integrals are similar to those in Activity 22, whereas others are similar to those in Activity 21. You have to decide in each case what the 'something' should be. Remember that the idea is that when you replace the something by u in the first part of the integrand, you obtain an expression in u that you can integrate.

Activity 23 *Choosing the 'something' when you integrate by substitution*

Find the following indefinite integrals.

(a) $\displaystyle\int (\cos(e^x))\,e^x\,\mathrm{d}x$ (b) $\displaystyle\int e^{1+\sin x}(\cos x)\,\mathrm{d}x$

(c) $\displaystyle\int \left(\frac{1}{x^{10}+6}\right)(10x^9)\,\mathrm{d}x$ (d) $\displaystyle\int (\cos^3 x)(-\sin x)\,\mathrm{d}x$

Finally in this subsection, as promised near the start, here's an explanation of why

$$\int f(u)\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u,$$

where the variable u is a function of x, and f is a function that you can integrate. To justify this, we let F be an antiderivative of f, and apply the chain rule to $F(u)$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(F(u)\big) = f(u)\,\frac{\mathrm{d}u}{\mathrm{d}x}.$$

It follows from this equation that

$$\int f(u)\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = F(u)+c.$$

However, because F is an antiderivative of f, we also have

$$\int f(u)\,\mathrm{d}u = F(u)+c.$$

Comparing the final two equations above proves the result.

3.2 Integration by substitution in practice

Usually when you have an integrand that you can integrate by substitution, it won't initially be exactly in the form

$f(\text{something}) \times$ the derivative of the something.

You have to start by recognising that it can be rearranged into this form, then rearrange it accordingly. Here are two examples that illustrate some things that you can do.

Example 14 *Rearranging an integral before using substitution*

Find the integral

$$\int e^x \cos(e^x)\,\mathrm{d}x.$$

Solution

The derivative of e^x is e^x, so the integrand is in the required form, except that the two factors $\cos(e^x)$ and e^x are in the 'wrong' order. If you swap them, then the integrand will be exactly in the required form.

Let $u = e^x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = e^x$.

Imagine 'cross-multiplying' to obtain $\mathrm{d}u = e^x\,\mathrm{d}x$. Then start by swapping the two factors in the integrand.

So

$$\begin{aligned}\int e^x \cos(e^x)\,\mathrm{d}x &= \int (\cos(e^x))\ e^x\,\mathrm{d}x\\ &= \int \cos u\,\mathrm{d}u\\ &= \sin(u) + c\\ &= \sin(e^x) + c.\end{aligned}$$

Example 15 *Rearranging another integral before using substitution*

Find the integral

$$\int x^2 \left(1+x^3\right)^8 \, dx.$$

Solution

The derivative of $1+x^3$ is $3x^2$, which is 'nearly' x^2 (it just has the wrong coefficient). So if you swap the order of the two factors x^2 and $(1+x^3)^8$, then the integrand will be nearly in the required form.

Let $u = 1 + x^3$; then $\dfrac{du}{dx} = 3x^2$.

Imagine 'cross-multiplying' to obtain $du = 3x^2 \, dx$. Then start by swapping the two factors in the integrand.

So

$$\int x^2 \left(1+x^3\right)^8 \, dx = \int \left(1+x^3\right)^8 x^2 \, dx$$

The only problem is that in place of x^2 you need $3x^2$. To achieve this, multiply by 3 inside the integral, and divide by 3 outside, to compensate. (The constant multiple rule for indefinite integrals tells you that it's okay to do this.)

$$= \tfrac{1}{3} \int \left(1+x^3\right)^8 \left(3x^2\right) dx$$

Now do the substitution.

$$\begin{aligned} &= \tfrac{1}{3} \int u^8 \, du \\ &= \tfrac{1}{3} \times \tfrac{1}{9} u^9 + c \\ &= \tfrac{1}{27} u^9 + c \\ &= \tfrac{1}{27} (1+x^3)^9 + c. \end{aligned}$$

Here's something to notice about Example 15. You might have thought that when the integral $\int u^8 \, du$ in the fourth-last line of the solution was replaced by $\frac{1}{9}u^9 + c$, the solution should have proceeded like this:

$$\begin{aligned} &= \tfrac{1}{3} \int u^8 \, du \\ &= \tfrac{1}{3} \left(\tfrac{1}{9} u^9 + c\right) \\ &= \tfrac{1}{27} u^9 + \tfrac{1}{3} c \\ &= \tfrac{1}{27} (1+x^3)^9 + \tfrac{1}{3} c. \end{aligned}$$

However, since the arbitrary constant c can take any value, the expression $\frac{1}{3}c$ can also take any value, so we may as well replace it by a single letter that can take any value. It's convenient to use the usual letter, c.

This sort of thing occurs frequently when you integrate. You never need to multiply or divide a term that is an *arbitrary* constant by a number – you just replace it by another arbitrary constant, and you usually use the standard letter, c.

Another point to remember from the working in Example 15 is the strategy of multiplying by a constant inside an integral, and dividing by the same constant outside to compensate. It's often helpful to do this when you use integration by substitution. It's okay to do it because the constant multiple rule for indefinite integrals tells you that for any function f that has an antiderivative, and any non-zero constant k,

$$\int f(x)\,\mathrm{d}x = \int \frac{1}{k}kf(x)\,\mathrm{d}x = \frac{1}{k}\int kf(x)\,\mathrm{d}x.$$

Remember that the constant k can be either positive or negative. In particular, it can be -1.

Remember too that k must be a *constant*. You can't do the same thing with an expression in x.

Here are some examples for you to try.

Activity 24 *Rearranging integrals before using substitution*

Find the following integrals.

(a) $\int x^2\cos(x^3)\,\mathrm{d}x$ (b) $\int x\sqrt{1+x^2}\,\mathrm{d}x$ (c) $\int \cos^4 x\sin x\,\mathrm{d}x$

The next example illustrates another way in which you can sometimes rearrange an integrand to get it into the right form for integration by substitution.

Example 16 *Rearranging another integral before using substitution*

Find the integral

$$\int \frac{x}{1-x^2}\,\mathrm{d}x.$$

Solution

The derivative of the denominator $1 - x^2$ is $-2x$, which is 'nearly' the numerator x (it just has the wrong coefficient). So if you split the numerator from the rest of the fraction, then the integrand will be nearly in the required form.

Let $u = 1 - x^2$; then $\dfrac{du}{dx} = -2x$.

Imagine 'cross-multiplying' to obtain $du = -2x\,dx$. Then start by splitting the numerator from the rest of the fraction in the integrand.

So

$$\int \frac{x}{1-x^2}\,dx = \int \left(\frac{1}{1-x^2}\right) x\,dx$$

The only problem is that in place of x you need $-2x$. To achieve this, multiply by -2 inside the integral, and divide by -2 outside to compensate.

$$= -\tfrac{1}{2} \int \left(\frac{1}{1-x^2}\right) (-2x)\,dx$$

Now do the substitution.

$$\begin{aligned} &= -\tfrac{1}{2} \int \frac{1}{u}\,du \\ &= -\tfrac{1}{2} \ln|u| + c \\ &= -\tfrac{1}{2} \ln|1 - x^2| + c. \end{aligned}$$

The type of integral in Example 16 occurs quite frequently, so it's worth learning to recognise it. You can use integration by substitution to find any integral of the form

$$\int \frac{\text{the derivative of the something}}{\text{something}}\,dx,$$

or

$$\int \frac{\text{the derivative of the something}}{(\text{something})^n}\,dx, \quad \text{where } n \text{ is a constant.}$$

The constant n can be any nonzero real number. Here are some integrals of this type for you to find.

Activity 25 *Rearranging more integrals before using substitution*

Find the following integrals.

(a) $\displaystyle\int \frac{x}{1+2x^2}\,dx$ (b) $\displaystyle\int \frac{e^x}{(2+e^x)^2}\,dx$ (c) $\displaystyle\int \frac{\cos x}{3-\sin x}\,dx$

Here are three more integrals for you to find by using substitution. They're a mixture of the types in Activities 24 and 25.

Activity 26 *Rearranging even more integrals before using substitution*

Find the following integrals.

(a) $\displaystyle\int e^{\cos x}\sin x\,dx$ (b) $\displaystyle\int \frac{e^x}{3-e^x}\,dx$ (c) $\displaystyle\int (1+\tfrac{1}{2}\sin x)^2\cos x\,dx$

You can use integration by substitution to find the indefinite integral of the tangent function. You're asked to do that in the next activity.

Activity 27 *Finding the indefinite integral of tan*

By writing

$$\tan x = \frac{\sin x}{\cos x},$$

and using integration by substitution, find

$$\int \tan x\,dx.$$

Remember that when you've found an indefinite integral, you can always check your answer by differentiating it. This can be particularly helpful when you found the integral by using integration by substitution.

3.3 Integrating functions of linear expressions

This subsection is about a particular type of integral that occurs frequently, and which you can find by using integration by substitution.

The integrals

$$\int \sin(5x-2)\,\mathrm{d}x, \quad \int \frac{1}{4-x}\,\mathrm{d}x, \quad \text{and} \quad \int e^{-9x}\,\mathrm{d}x$$

all have integrands of the form $f(ax+b)$, where f is a function, and a and b are constants with $a \neq 0$. In other words, each of the integrands is of the form $f(\text{a linear expression in } x)$.

You can always find an integral of this form by using integration by substitution, provided that you can integrate the function f. Here's an example.

Example 17 *Integrating a function of a linear expression*

Find the integral

$$\int \sin(5x-2)\,\mathrm{d}x.$$

Solution

The derivative of $5x-2$ is 5, so the integral will be in the required form if you rearrange it to obtain 5 inside.

Let $u = 5x-2$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 5$. So

$$\begin{aligned}
\int \sin(5x-2)\,\mathrm{d}x &= \tfrac{1}{5}\int \sin(5x-2)\times 5\,\mathrm{d}x \\
&= \tfrac{1}{5}\int \sin(5x-2)\times 5\,\mathrm{d}x \\
&= \tfrac{1}{5}\int \sin u\,\mathrm{d}u \\
&= -\tfrac{1}{5}\cos u + c \\
&= -\tfrac{1}{5}\cos(5x-2) + c.
\end{aligned}$$

Here are some similar integrals for you to find.

Activity 28 *Integrating functions of linear expressions*

Find the following integrals.

(a) $\displaystyle\int e^{6x-1}\,dx$ (b) $\displaystyle\int \sin(4x)\,dx$ (c) $\displaystyle\int e^{-9x}\,dx$

(d) $\displaystyle\int e^{-x/3}\,dx$ (e) $\displaystyle\int \cos(3-7x)\,dx$ (f) $\displaystyle\int \frac{1}{4-x}\,dx$

(g) $\displaystyle\int \frac{1}{(2x+1)^2}\,dx$ (h) $\displaystyle\int \frac{1}{(1-x)^3}\,dx$ (i) $\displaystyle\int \sec^2(6x)\,dx$

The next example uses exactly the same ideas as Example 17 and Activity 28, but is a little more complicated.

Example 18 *Integrating another function of a linear expression*

Find the integral

$$\int \frac{1}{1+9x^2}\,dx.$$

Solution

You can write the integrand as $\dfrac{1}{1+(3x)^2}$, which is of the form $f(3x)$ where $f(x)=\dfrac{1}{1+x^2}$. The integral of this function f is a standard integral. Since the derivative of $3x$ is 3, you need to rearrange the given integral to obtain 3 inside.

Let $u=3x$; then $\dfrac{du}{dx}=3$. So

$$\begin{aligned}\int \frac{1}{1+9x^2}\,dx &= \int \frac{1}{1+(3x)^2}\,dx\\ &= \tfrac{1}{3}\int \frac{1}{1+(3x)^2}\times 3\,dx\\ &= \tfrac{1}{3}\int \frac{1}{1+u^2}\,du\\ &= \tfrac{1}{3}\tan^{-1}u + c\\ &= \tfrac{1}{3}\tan^{-1}(3x) + c.\end{aligned}$$

Here are some similar integrals for you to find.

Activity 29 *Integrating more functions of linear expressions*

Find the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{1}{1+2x^2}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{1}{4+9x^2}\,\mathrm{d}x$

Hint for part (c): start by writing

$$\frac{1}{4+9x^2} \quad \text{as} \quad \frac{1}{4} \times \frac{1}{1+\frac{9}{4}x^2}.$$

The answers to Activities 28 and 29 all follow the same general pattern, as follows. Suppose that you have an integral of the form

$$\int f(ax+b)\,\mathrm{d}x,$$

where a and b are constants with $a \neq 0$, and you know an antiderivative F of the function f. To find the integral by substitution, you put $u = ax + b$, which gives $\dfrac{\mathrm{d}u}{\mathrm{d}x} = a$. Then

$$\begin{aligned}\int f(ax+b)\,\mathrm{d}x &= \frac{1}{a}\int f(ax+b)\,a\,\mathrm{d}x\\ &= \frac{1}{a}\int f(u)\,\mathrm{d}u\\ &= \frac{1}{a}F(u) + c\\ &= \frac{1}{a}F(ax+b) + c.\end{aligned}$$

So we have the following general fact.

Indefinite integral of a function of a linear expression

If f is a function with antiderivative F, and a and b are constants with $a \neq 0$, then

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + c.$$

It's worth remembering this fact, and using it to deal with integrals where the integrand is a simple function of a linear expression, rather than using integration by substitution every time. For example, the fact above tells you immediately that

$$\int \cos(3x-2)\,\mathrm{d}x = \tfrac{1}{3}\sin(3x-2) + c.$$

The following special case of the fact in the box above is useful especially often.

Indefinite integral of a function of a multiple of the variable

If f is a function with antiderivative F, and a is a non-zero constant, then

$$\int f(ax)\,\mathrm{d}x = \frac{1}{a}F(ax) + c.$$

For example,

$$\int e^{-2x}\,\mathrm{d}x = -\tfrac{1}{2}e^{-2x} + c.$$

Try using the facts in the boxes above in the following activity.

Activity 30 *Integrating functions of linear expressions efficiently*

Find the following integrals.

(a) $\displaystyle\int \cos(10x)\,\mathrm{d}x$ (b) $\displaystyle\int \sin(2-5x)\,\mathrm{d}x$ (c) $\displaystyle\int e^{3t+4}\,\mathrm{d}t$

(d) $\displaystyle\int \cos(\tfrac{1}{2}\theta)\,\mathrm{d}\theta$ (e) $\displaystyle\int e^{-4p}\,\mathrm{d}p$ (f) $\displaystyle\int \frac{1}{7x-4}\,\mathrm{d}x$

(g) $\displaystyle\int \frac{1}{1-r}\,\mathrm{d}r$ (h) $\displaystyle\int \sec^2(5x-1)\,\mathrm{d}x$ (i) $\displaystyle\int \sin(\tfrac{1}{5}x)\,\mathrm{d}x$

(j) $\displaystyle\int e^{-x}\,\mathrm{d}x$ (k) $\displaystyle\int e^{x/2}\,\mathrm{d}x$ (l) $\displaystyle\int \sec^2(2-3p)\,\mathrm{d}p$

(m) $\displaystyle\int \sin\left(\frac{2-x}{3}\right)\,\mathrm{d}x$

The next example is of a type that you met in Unit 7, but it involves a function that you've learned how to integrate in this section.

Example 19 *Integrating in a practical situation*

The velocity of an object moving along a straight line is given by the equation

$$v = \tfrac{1}{2}\sin\left(\tfrac{3}{2}t - 2\right),$$

where t is the time in seconds since the object began moving, and v is the velocity of the object, in $\mathrm{m\,s^{-1}}$. The displacement of the object at time $t = 0$ is $3\,\mathrm{m}$.

Find an equation for the displacement s (in metres) of the object, in terms of t. Give the constant term to two decimal places.

Solution

Use the fact that displacement is an antiderivative of velocity.

The velocity v (in $\mathrm{m\,s^{-1}}$) of the object at time t (in seconds) is given by

$$v = \tfrac{1}{2}\sin\left(\tfrac{3}{2}t - 2\right).$$

Hence the displacement s (in m) of the object at time t (in seconds) is given by

$$\begin{aligned} s &= \int \tfrac{1}{2}\sin\left(\tfrac{3}{2}t - 2\right)\,\mathrm{d}t \\ &= \frac{1}{2} \times \frac{1}{3/2}\left(-\cos\left(\tfrac{3}{2}t - 2\right)\right) + c \\ &= -\tfrac{1}{3}\cos\left(\tfrac{3}{2}t - 2\right) + c, \end{aligned}$$

where c is an arbitrary constant.

Use information given in the question to find the value of c.

When $t = 0$, $s = 3$. Hence

$$3 = -\tfrac{1}{3}\cos\left(\tfrac{3}{2} \times 0 - 2\right) + c$$
$$3 = -\tfrac{1}{3}\cos(-2) + c$$
$$c = 3 + \tfrac{1}{3}\cos(-2)$$
$$c = 3 + \tfrac{1}{3}\cos 2$$
$$c = 2.86 \text{ (to 2 d.p.)}.$$

So the equation for the displacement of the object in terms of time is

$$s = -\tfrac{1}{3}\cos\left(\tfrac{3}{2}t - 2\right) + 2.86.$$

Activity 31 *Integrating in a practical example*

The velocity of an object moving along a straight line is given by the equation

$$v = 2\cos\left(\tfrac{1}{2}t + 5\right),$$

where t is the time in seconds since the object began moving, and v is the velocity of the object, in $\mathrm{m\,s^{-1}}$. The displacement of the object at time $t = 0$ is $4\,\mathrm{m}$.

(a) Find an equation for the displacement s (in metres) of the object, in terms of t. Give the constant term to two decimal places.

(b) Find the displacement of the object at time $t = 10$. Give your answer in metres to two significant figures.

3.4 Integration by substitution for definite integrals

Once you've used integration by substitution to find an indefinite integral, you can find any *definite* integral with the same integrand by using the fundamental theorem of calculus in the usual way. Here's an example.

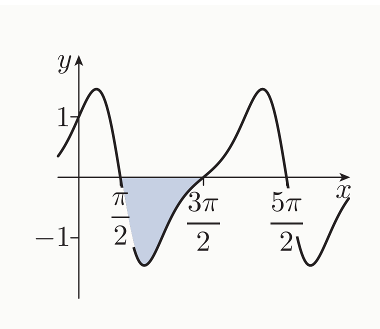


Figure 41 The definite integral in Example 20

Example 20 *Evaluating a definite integral by substitution (first method)*

Evaluate the definite integral

$$\int_{\pi/2}^{3\pi/2} e^{\sin x} \cos x \, \mathrm{d}x.$$

(It's shown in Figure 41.) Give your answer to three significant figures.

Solution

Use integration by substitution to obtain the indefinite integral of the integrand.

Let $u = \sin x$, then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$. So

$$\begin{aligned}\int e^{\sin x} \cos x \, \mathrm{d}x &= \int e^u \, \mathrm{d}u \\ &= e^u + c \\ &= e^{\sin x} + c.\end{aligned}$$

Apply the fundamental theorem of calculus.

Hence

$$\begin{aligned}\int_{\pi/2}^{3\pi/2} e^{\sin x}\cos x\,\mathrm{d}x &= \left[e^{\sin x}\right]_{\pi/2}^{3\pi/2}\\ &= e^{\sin(3\pi/2)} - e^{\sin(\pi/2)}\\ &= e^{-1} - e^{1}\\ &= \frac{1}{e} - e\\ &= -2.35 \text{ (to 3 s.f.).}\end{aligned}$$

An alternative way to use integration by substitution to evaluate a definite integral is to work with the definite integral from the start, rather than first finding an indefinite integral. If you do this, then because the limits of integration are values of x, when you change the variable from x to u you must also change the limits of integration to the corresponding values of u.

This method is demonstrated in the next example. The definite integral in this example is the same as the one in Example 20.

Example 21 *Evaluating a definite integral by substitution (second method)*

Evaluate the definite integral

$$\int_{\pi/2}^{3\pi/2} e^{\sin x}\cos x\,\mathrm{d}x.$$

Give your answer to three significant figures.

Solution

Use integration by substitution.

Let $u = \sin x$, then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$.

Find the values of u that correspond to the values of x that are the limits of integration.

Putting $x = \pi/2$ into $u = \sin x$ gives $u = \sin\left(\dfrac{\pi}{2}\right) = 1$.

Putting $x = 3\pi/2$ into $u = \sin x$ gives $u = \sin\left(\dfrac{3\pi}{2}\right) = -1$.

Change the variable from x to u, remembering that you also have to change the limits of integration.

So

$$\begin{aligned}\int_{\pi/2}^{3\pi/2} e^{\sin x}\cos x\,\mathrm{d}x &= \int_{1}^{-1} e^{u}\,\mathrm{d}u \\ &= \left[e^{u}\right]_{1}^{-1} \\ &= e^{-1} - e^{1} \\ &= -2.35 \text{ (to 3 s.f.).}\end{aligned}$$

You can try using this method in the next activity.

Activity 32 *Evaluating definite integrals by substitution*

Evaluate the following definite integrals. Give your answers to three significant figures.

(a) $\displaystyle\int_0^1 \frac{x}{1+2x^2}\,\mathrm{d}x$ (b) $\displaystyle\int_{\pi/2}^{\pi} \cos^3 x \sin x\,\mathrm{d}x$ (c) $\displaystyle\int_{-1}^{0} xe^{-3x^2}\,\mathrm{d}x$

3.5 Finding more complicated integrals by substitution

You can sometimes use integration by substitution to find an integral even if the integrand isn't obviously of the form

f(something) $\times$ the derivative of the something.

In this subsection, you'll see some examples where the integrand is nearly of this form, except that it also contains a further factor, which is an expression in x. If you can express this further factor in terms of the variable u that you use for the substitution, then you may be able to simplify the integrand into a form that you can integrate. Here's an example.

Example 22 *Expressing a further factor in terms of u when integrating by substitution*

Find the integral

$$\int (x+1)(3x+1)^9 \, dx.$$

Solution

If the integral were just

$$\int (3x+1)^9 \, dx,$$

then we could use the substitution $u = 3x+1$, and find the integral by first writing it as

$$\tfrac{1}{3}\int (3x+1)^9 \times 3 \, dx.$$

However, the integrand contains a further factor, $x+1$. Try using the substitution, expressing the further factor $x+1$ in terms of u.

Let $u = 3x+1$. Then $\dfrac{du}{dx} = 3$.

Also, rearranging the equation $u = 3x+1$ gives

$$\begin{aligned} u &= 3x+1 \\ 3x &= u-1 \\ x &= \tfrac{1}{3}(u-1). \end{aligned}$$

Hence

$$\begin{aligned} x+1 &= \tfrac{1}{3}(u-1)+1 \\ &= \tfrac{1}{3}(u+2). \end{aligned}$$

So

$$\begin{aligned} \int (x+1)(3x+1)^9 \, dx &= \tfrac{1}{3}\int (x+1)(3x+1)^9 \times 3 \, dx \\ &= \tfrac{1}{3}\int \tfrac{1}{3}(u+2)u^9 \, du \\ &= \tfrac{1}{9}\int (u+2)u^9 \, du \\ &= \tfrac{1}{9}\int (u^{10}+2u^9) \, du \\ &= \tfrac{1}{9}\left(\tfrac{1}{11}u^{11} + 2 \times \tfrac{1}{10}u^{10}\right) + c \\ &= \tfrac{1}{99}u^{11} + \tfrac{1}{45}u^{10} + c \\ &= \tfrac{1}{99}(3x+1)^{11} + \tfrac{1}{45}(3x+1)^{10} + c. \end{aligned}$$

Here are two similar integrals for you to try.

Activity 33 *Expressing further factors in terms of u*

Use integration by substitution to find the following indefinite integrals.

(a) $\displaystyle\int x\sqrt{1+x}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{x-2}{(x+3)^3}\,\mathrm{d}x$

You'll learn more about integration by substitution in the module MST125 *Essential mathematics 2*, if it's in your study programme.

4 Integration by parts

In this section you'll meet another integration technique, *integration by parts*. Like integration by substitution, this technique is useful for integrating functions that have certain characteristics. It's based on the product rule for differentiation.

4.1 Basic integration by parts

Suppose that f and g are functions, and that the function G is an antiderivative of the function g. Consider the equation that you obtain when you use the product rule to differentiate the product $f(x)G(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)G(x)) = f(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}G(x)\right) + G(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right),$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)G(x)) = f(x)g(x) + G(x)f'(x).$$

If you swap the order of $G(x)$ and $f'(x)$ in the final term, and subtract this term from both sides of the equation, then you obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)G(x)) - f'(x)G(x) = f(x)g(x).$$

Swapping the sides of this equation gives

$$f(x)g(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)G(x)) - f'(x)G(x).$$

If you now integrate both sides of this equation with respect to x, then you obtain

$$\int f(x)g(x)\,\mathrm{d}x = f(x)G(x) - \int f'(x)G(x)\,\mathrm{d}x.$$

There's no need to add an explicit arbitrary constant here, because it's included in the integral $\int f(x)g(x)\,\mathrm{d}x$ on the left-hand side, and in the integral $\int f'(x)G(x)\,\mathrm{d}x$ on the right-hand side.

The formula above is known as the **integration by parts formula**. It's stated again in the box below, for easy reference. If you've met it before, then you may have seen it stated in a different form. This is explained at the end of this section.

Integration by parts formula (Lagrange notation)

$$\int f(x)g(x)\,\mathrm{d}x = f(x)G(x) - \int f'(x)G(x)\,\mathrm{d}x.$$

Here G is an antiderivative of g.

The integration by parts formula is useful for integrating some products of the form $f(x)g(x)$. You can see that it changes the problem of integrating $f(x)g(x)$ into the problem of integrating $f'(x)G(x)$, where G is an antiderivative of g.

So it's useful for integrating products of the form $f(x)g(x)$ that have the following characteristics.

- You can find an antiderivative G of g; in other words, you can integrate the expression $g(x)$.
- The product $f'(x)G(x)$ is easier to integrate than the product $f(x)g(x)$. For example, this may be the case if $f'(x)$ is simpler than $f(x)$.

When we integrate by using the integration by parts formula, we say that we're **integrating by parts**. Here's an example.

Example 23 *Integrating by parts*

Find the integral $\int x \sin x \, dx$.

Solution

The integrand is a product of two expressions, x and $\sin x$. You can integrate the second expression, $\sin x$, and differentiating the first expression, x, makes it simpler. So try integration by parts.

Let $f(x) = x$ and $g(x) = \sin x$.

Then $f'(x) = 1$, and an antiderivative of $g(x)$ is $G(x) = -\cos x$.

So the integration by parts formula gives

$$\begin{aligned}\int x \sin x \, dx &= \int f(x)g(x) \, dx \\ &= f(x)G(x) - \int f'(x)G(x) \, dx \\ &= x \times (-\cos x) - \int 1 \times (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx\end{aligned}$$

Find the integral in this expression.

$$= -x \cos x + \sin x + c.$$

You can see that in Example 23 the use of the integration by parts formula changed the problem of integrating $x \sin x$ into the problem of integrating $\cos x$, which is easier. Here are two similar examples for you to try.

Activity 34 *Integrating by parts*

Find the following integrals.

(a) $\int x \cos x \, dx$ (b) $\int xe^x \, dx$

As you become more familiar with integration by parts, you'll probably find that you can carry it out more quickly and conveniently if you remember the informal version of the formula stated below. You can recite this version in your head as you apply integration by parts, in a similar way to the informal versions of the product and quotient rules for differentiation that you met in Unit 7. It's useful for applying integration by parts in fairly simple cases, like the ones that you've seen so far, which is usually all that you need to do.

Integration by parts formula (informal)

$$\begin{aligned}\text{integral of product} &= (\text{first}) \times \begin{pmatrix}\text{antiderivative}\\ \text{of second}\end{pmatrix}\\ &\qquad - \text{integral of}\left(\begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix} \times \begin{pmatrix}\text{antiderivative}\\ \text{of second}\end{pmatrix}\right).\end{aligned}$$

In the next example, the integration in Example 23 is repeated, using the informal version of the integration by parts formula.

Example 24 *Integrating by parts, using the informal formula*

Find the integral $\displaystyle\int x \sin x \, \mathrm{d}x$.

Solution

The integrand is a product of two expressions, x and $\sin x$. You can integrate the second expression, $\sin x$, and differentiating the first expression, x, makes it simpler. So try integration by parts.

Recite the informal version of the formula in your head. As you think 'first', write down the first expression, then as you think 'antiderivative of second', write down an antiderivative of the second expression, and so on.

Integrating by parts gives

$$\begin{aligned}\int x \sin x \, \mathrm{d}x &= x \times (-\cos x) - \int 1 \times (-\cos x) \, \mathrm{d}x\\ &= -x \cos x + \int \cos x \, \mathrm{d}x\\ &= -x \cos x + \sin x + c.\end{aligned}$$

You can practise using the informal version of the integration by parts formula in the next activity. Alternatively, you might prefer to continue using the formal version for a little longer, and move to the informal version when you feel more confident with it. You can use the exercises on integration by parts in the practice quiz and exercise booklet for this unit for further practice.

Activity 35 *Integrating by parts*

Find the following integrals.

(a) $\displaystyle\int x\sin(5x)\,dx$ (b) $\displaystyle\int xe^{2x}\,dx$ (c) $\displaystyle\int xe^{-x}\,dx$

When you use integration by parts, remember that it matters which expression in the integrand you take to be the 'first expression', and which you take to be the 'second expression'. If you're trying to use integration by parts, but you find that it leads to an integral that's more complicated than the original integral, then try swapping the two expressions in the original integrand.

4.2 Integration by parts in practice

There are several frequently occurring types of integrals that often have the right characteristics for integration by parts. In this subsection you'll meet three of these types. It's helpful for you to learn to recognise them.

Integrands of the form $x^n\,g(x)$

You can sometimes find an integral of the form

$$\int x^n\,g(x)\,dx,$$

where n is a positive integer and $g(x)$ is an expression that you can integrate, by using integration by parts n times.

All the integrals in the previous subsection were of this form, with $n = 1$, so you could find them by using integration by parts once. Here's an example where you have to use integration by parts twice.

Example 25 *Integrating by parts twice*

Find the integral $\displaystyle\int x^2e^{4x}\,dx$.

Solution

The integral is of the form described above, so try integration by parts.

Integrating by parts gives

$$\begin{aligned}\int x^2 e^{4x}\,\mathrm{d}x &= x^2 \times \tfrac{1}{4}e^{4x} - \int 2x \times \tfrac{1}{4}e^{4x}\,\mathrm{d}x \\ &= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{2}\int xe^{4x}\,\mathrm{d}x\end{aligned}$$

The integral in this expression, $\int xe^{4x}\,\mathrm{d}x$, looks easier to find than the original integral, but it's still not straightforward to find. However, it is itself of the form described above, so try integration by parts again.

$$\begin{aligned}&= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{2}\left(x \times \tfrac{1}{4}e^{4x} - \int 1 \times \tfrac{1}{4}e^{4x}\,\mathrm{d}x\right) \\ &= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{2}\left(\tfrac{1}{4}xe^{4x} - \tfrac{1}{4}\int e^{4x}\,\mathrm{d}x\right)\end{aligned}$$

Multiply out the brackets. Take care with the minus sign in front of the brackets – remember that it turns the minus sign inside the brackets into a plus sign.

$$= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{8}xe^{4x} + \tfrac{1}{8}\int e^{4x}\,\mathrm{d}x$$

Now find the integral in this expression, and simplify the resulting expression.

$$\begin{aligned}&= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{8}xe^{4x} + \tfrac{1}{8} \times \tfrac{1}{4}e^{4x} + c \\ &= \tfrac{1}{4}x^2e^{4x} - \tfrac{1}{8}xe^{4x} + \tfrac{1}{32}e^{4x} + c \\ &= \tfrac{1}{32}e^{4x}(8x^2 - 4x + 1) + c.\end{aligned}$$

Activity 36 *Integrating by parts twice*

Find the following integrals.

(a) $\int x^2 e^x\,\mathrm{d}x$ (b) $\int x^2 \cos(2x)\,\mathrm{d}x$

The method introduced above for dealing with integrands of the form $x^n g(x)$ works just as well if the expression x^n is replaced by any polynomial expression in x of degree n. For example, you can find the integral $\int (3x^2 - 1)e^{4x}\,\mathrm{d}x$ by integrating by parts twice, and you can find the integral in the next activity by integrating by parts once.

Activity 37 *Integrating an expression of the form* $(ax + b)g(x)$

Use integration by parts to find the integral $\int (5x + 2)e^{-5x}\,dx$.

Integrands of the form $x^r \ln x$

Suppose that you want to integrate an expression of the form

$$x^r \ln x,$$

where r is a constant (any real number). Your first thought, particularly if r is a positive integer, might be that this is similar to the expressions that you saw how to integrate earlier in this subsection. However, you can't apply the integration by parts formula with $\ln x$ as the second expression, because $\ln x$ isn't an expression that you've seen how to integrate.

In fact, you can find an integral of the form above by applying the integration by parts formula with $\ln x$ as the *first* expression. Here's an example.

Example 26 *Integrating an expression of the form* $x^n \ln x$

Find the integral

$$\int x^2 \ln x\,dx.$$

Solution

The integral is of the form discussed above, so swap the expressions x^2 and $\ln x$, then integrate by parts.

$$\begin{aligned}
\int x^2 \ln x\,dx &= \int (\ln x)\,x^2\,dx \\
&= (\ln x) \times \tfrac{1}{3}x^3 - \int \frac{1}{x} \times \tfrac{1}{3}x^3\,dx \\
&= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2\,dx \\
&= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\left(\tfrac{1}{3}x^3\right) + c \\
&= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + c \\
&= \tfrac{1}{9}x^3(3\ln x - 1) + c
\end{aligned}$$

Activity 38 *Integrating expressions of the form $x^r \ln x$*

Find the following integrals.

(a) $\displaystyle\int x^3 \ln x \, dx$ (b) $\displaystyle\int \ln x \, dx$

Hint for part (b): write $\ln x$ as $(\ln x) \times 1$.

You can use the ideas that you used in Activity 38 to deal with other, similar integrals that involve the natural logarithm function. Try the following activity.

Activity 39 *Integrating another expression containing the function ln*

Use integration by parts to find the integral $\displaystyle\int (x-2)\ln(3x)\, dx$.

Integrands of the form $e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$

You can integrate an expression of the form

$$e^{ax}\sin(bx) \quad \text{or} \quad e^{ax}\cos(bx),$$

where a and b are nonzero numbers, by integrating by parts twice. When you do this, you obtain an equation that expresses the original integral in terms of itself, and you can rearrange this equation to find the original integral. The method is demonstrated in the example below.

When you use this method, you can arrange the exponential expression and the trigonometric expression in the integrand in either order before you integrate by parts for the first time. For example, you can start with either $\int e^{ax}\sin(bx)\, dx$ or $\int \sin(bx)e^{ax}\, dx$. However, you must choose the *same order* (exponential before trigonometric, or trigonometric before exponential) when you integrate by parts for the second time. If you don't do that, then the equation that you'll obtain will reduce to $0 = 0$, which isn't helpful!

Example 27 *Finding an integral by expressing it in terms of itself*

Find the indefinite integral $\int e^x \cos(2x)\,dx$.

Solution

The integral is of the form discussed above, so use integration by parts twice. Start by swapping the expressions e^x and $\cos(2x)$, as this makes the working slightly easier (it avoids introducing fractions in the early stages).

We use integration by parts twice.

$$\int e^x \cos(2x)\,dx = \int \cos(2x)\,e^x\,dx.$$

Integrate by parts.

$$\begin{aligned}\int \cos(2x)\,e^x\,dx &= \cos(2x)e^x - \int (-2\sin(2x))e^x\,dx \\ &= \cos(2x)e^x + 2\int \sin(2x)e^x\,dx\end{aligned}$$

Integrate by parts again.

$$\begin{aligned}&= \cos(2x)e^x + 2\left(\sin(2x)e^x - \int (2\cos(2x))e^x\,dx\right) \\ &= \cos(2x)e^x + 2\sin(2x)e^x - 4\int \cos(2x)e^x\,dx\end{aligned}$$

The final term here contains the original integral. Move this term to the other side of the equation. When you do this, the right-hand side no longer contains an indefinite integral, so you have to add an arbitrary constant to that side.

So

$$\int \cos(2x)e^x\,dx + 4\int \cos(2x)e^x\,dx = \cos(2x)e^x + 2\sin(2x)e^x + c;$$

that is,

$$5\int \cos(2x)e^x\,dx = \cos(2x)e^x + 2\sin(2x)e^x + c.$$

To obtain the final answer, divide both sides by 5. (Remember that you leave the arbitrary constant as c rather than $c/5$.) Also, it's slightly tidier to write the exponential expression before the trigonometric expression in each term, and to take out a common factor.

This gives

$$\int e^x \cos(2x)\,dx = \tfrac{1}{5}e^x\cos(2x) + \tfrac{2}{5}e^x\sin(2x) + c$$
$$= \tfrac{1}{5}e^x(\cos(2x) + 2\sin(2x)) + c.$$

Activity 40 *Finding integrals by expressing them in terms of themselves*

Find the following integrals.

(a) $\int e^x \sin(3x)\,dx$ (b) $\int e^{2x}\cos x\,dx$

4.3 Integration by parts for definite integrals

Once you've used integration by parts to find an indefinite integral, you can find any definite integral with the same integrand by using the fundamental theorem of calculus in the usual way. Here's an example.

Example 28 *Evaluating a definite integral by parts (first method)*

Evaluate the definite integral

$$\int_0^1 xe^{7x}\,dx.$$

Solution

Use integration by parts to obtain the indefinite integral of the integrand.

Integration by parts gives

$$\int xe^{7x}\,dx = x \times \tfrac{1}{7}e^{7x} - \int 1 \times \tfrac{1}{7}e^{7x}\,dx$$
$$= \tfrac{1}{7}xe^{7x} - \tfrac{1}{7}\int e^{7x}\,dx$$
$$= \tfrac{1}{7}xe^{7x} - \tfrac{1}{7} \times \tfrac{1}{7}e^{7x} + c$$
$$= \tfrac{1}{49}e^{7x}(7x-1) + c.$$

Apply the fundamental theorem of calculus.

It follows that

$$\begin{aligned}\int_0^1 xe^{7x}\,\mathrm{d}x &= \left[\tfrac{1}{49}e^{7x}(7x-1)\right]_0^1 \\ &= \tfrac{1}{49}\left[e^{7x}(7x-1)\right]_0^1 \\ &= \tfrac{1}{49}\left(e^{7\times 1}(7\times 1-1)-e^{7\times 0}(7\times 0-1)\right) \\ &= \tfrac{1}{49}(6e^7-(-1)) \\ &= \tfrac{1}{49}(6e^7+1).\end{aligned}$$

An alternative way to use integration by parts to evaluate a definite integral is to apply the limits of integration 'as you go along'. To do this, you use the following formula, which is just the usual integration by parts formula with the limits of integration included.

Integration by parts formula for definite integrals

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \left[f(x)G(x)\right]_a^b - \int_a^b f'(x)G(x)\,\mathrm{d}x.$$

Here G is an antiderivative of g.

You can remember this formula as an informal version, by applying the limits of integration a and b in the appropriate way to the informal version of the usual integration by parts formula.

The next example demonstrates how to apply the limits of integration as you go along when you integrate by parts. The definite integral in this example is the same as the one in Example 28.

Example 29 *Evaluating a definite integral by parts (second method)*

Evaluate the definite integral

$$\int_0^1 xe^{7x}\,\mathrm{d}x.$$

Solution

Integrate by parts, applying the limits of integration.

$$\int_0^1 xe^{7x}\,\mathrm{d}x = \left[x \times \tfrac{1}{7}e^{7x}\right]_0^1 - \int_0^1 1 \times \tfrac{1}{7}e^{7x}\,\mathrm{d}x$$
$$= \tfrac{1}{7}\left[xe^{7x}\right]_0^1 - \tfrac{1}{7}\int_0^1 e^{7x}\,\mathrm{d}x$$

Evaluate the first term, and do the integration in the second term.

$$= \tfrac{1}{7}(1 \times e^{7\times 1} - 0 \times e^{7\times 0}) - \tfrac{1}{7}\left[\tfrac{1}{7}e^{7x}\right]_0^1$$
$$= \tfrac{1}{7}e^7 - \tfrac{1}{49}\left[e^{7x}\right]_0^1$$
$$= \tfrac{1}{7}e^7 - \tfrac{1}{49}(e^{7\times 1} - e^{7\times 0})$$
$$= \tfrac{1}{7}e^7 - \tfrac{1}{49}(e^7 - 1)$$
$$= \tfrac{1}{49}(7e^7 - (e^7 - 1))$$
$$= \tfrac{1}{49}(6e^7 + 1)$$

Activity 41 *Evaluating a definite integral by parts*

Use the integration by parts formula for definite integrals (the method of Example 29) to evaluate the definite integral

$$\int_0^1 xe^{3x}\,\mathrm{d}x.$$

Give your answer to four decimal places.

In the next two activities you're asked to use integration by parts to find areas.

Activity 42 *Using integration by parts to find an area*

This question is about the function $f(x) = x\sin(3x)$.

(a) Explain why the graph of f lies above the x-axis for values of x in the interval $[\pi/12, \pi/6]$.

(b) Write down a definite integral that gives the area between the graph of f and the x-axis from $x = \pi/12$ to $x = \pi/6$.

(c) Use the integration by parts formula for definite integrals to find the area described in part (b). Give your answer to four decimal places.

Activity 43 *Using integration by parts to find another area*

This question is about the function $f(x) = (x-2)(e^{-x}-1)$.

(a) Explain why the graph of f lies above the x-axis for $0 < x < 2$ and below the x-axis for $x > 2$. You might find it convenient to use a table of signs as part of your explanation.

(b) Find the total area (not the total *signed area*) between the graph of f and the x-axis, from $x = 0$ to $x = 4$. Give your answer to four significant figures.

As mentioned earlier, in some texts the integration by parts formula is expressed in a different form.

Here's the formula that you've seen in this section:

$$\int f(x)g(x)\,\mathrm{d}x = f(x)G(x) - \int f'(x)G(x)\,\mathrm{d}x.$$

It involves a function G and its derivative g.

If you denote this pair of functions by g and g' instead, then you obtain the alternative version below.

Integration by parts formula (Lagrange notation, alternative version)

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

As with all calculus formulas, the integration by parts formula can be stated in Leibniz notation.

If you put $u = f(x)$ and $v = g(x)$ in the formula in the box above, then you obtain the following form of the formula.

Integration by parts formula (Leibniz notation)

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

5 More integration

You've now met several integration techniques, and practised using them on a variety of integrals.

In this section, you'll start by meeting one further approach, which can be helpful for integrals that contain trigonometric expressions. Then you'll have an opportunity to practise identifying which of the integration techniques that you've met are appropriate for which integrals, for a variety of different types of integrals.

Finally, you'll learn how to use a computer for integration.

5.1 Trigonometric integrals

In Section 3 you saw that you can integrate some expressions that contain trigonometric functions by using integration by substitution. Another approach that's often helpful when you want to integrate an expression that contains trigonometric functions is to start by using one of the trigonometric identities that you met in Unit 4 to write the expression in a different form. This can give you an expression that's easier to integrate.

For example, you can use the identities below to help you integrate the expressions $\sin^2 x$, $\cos^2 x$ and $\sin x \cos x$, as illustrated in the next example. The first two identities are the half-angle identities, and the third identity is one of the double-angle identities.

Three trigonometric identities

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos(2\theta))$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta))$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Example 30 *Integrating by using a trigonometric identity*

Find the integral

$$\int \sin^2 x \, \mathrm{d}x.$$

Solution

By the half-angle identity for sine,

$$\begin{aligned}\int \sin^2 x \, \mathrm{d}x &= \int \tfrac{1}{2}(1 - \cos(2x)) \, \mathrm{d}x \\ &= \tfrac{1}{2} \int (1 - \cos(2x)) \, \mathrm{d}x \\ &= \tfrac{1}{2}(x - \tfrac{1}{2}\sin(2x)) + c \\ &= \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + c \\ &= \tfrac{1}{4}(2x - \sin(2x)) + c.\end{aligned}$$

In the next activity you're asked to use two of the trigonometric identities stated near the beginning of this subsection to integrate another two expressions that contain trigonometric functions.

Activity 44 *Integrating by using trigonometric identities*

(a) Use a half-angle identity to find the integral $\displaystyle\int \cos^2 \theta \, \mathrm{d}\theta$.

(b) Use the double-angle identity in the box above to find the integral $\displaystyle\int \sin x \, \cos x \, \mathrm{d}x$.

The trigonometric integral in Example 30 and the one in Activity 44(a) can't easily be found without first using a trigonometric identity to write them in a different form.

However, there's an alternative way to find the integral in Activity 44(b). Since the derivative of $\sin x$ is $\cos x$, you can use integration by substitution, taking $u = \sin x$. You're asked to do that in the next activity.

Activity 45 *Integrating a trigonometric expression by using substitution*

Use integration by substitution to find the integral $\int \sin x \, \cos x \, dx$.

In fact there are even two different ways to find the integral in Activity 44(b), namely

$$\int \sin x \cos x \, dx,$$

by using integration by substitution. One way is as suggested above: you use the fact that the derivative of $\sin x$ is $\cos x$ and hence make the substitution $u = \sin x$. Alternatively, you can use the fact that the derivative of $\cos x$ is $-\sin x$, write the integral as

$$-\int (\cos x)(-\sin x) \, dx,$$

and make the substitution $u = \cos x$. The working for each of these methods is given in the solution to Activity 45.

So you've now seen three different ways to integrate the expression $\sin x \, \cos x$: one way by using a trigonometric identity, and two ways by using substitution. These three different ways lead to three answers that look different, as you can see in the solutions to Activities 44(b) and 45. This illustrates that there can be more than one way to integrate an expression, and different ways of integrating it can lead to answers that look different, though of course these answers will be equivalent.

If an answer that you obtain for an integral is different from the answer given in a book or by a computer, but you think that you haven't made a mistake, then you may be able to check that the two answers are equivalent by using algebraic manipulation. You may need to use trigonometric identities – there are examples of this in the solution to Activity 45.

A rougher, but sometimes quicker, check is to plot the graphs of the two different answers on a computer (taking the arbitrary constants to be zero, for example), and check that they seem to be vertical translations of each other. This shows that the two different answers seem to be equivalent once the arbitrary constants are taken into account. Again, there's an example of this in the solution to Activity 45.

In this subsection you've seen just a few examples of how you can use trigonometric identities and integration by substitution to help you integrate expressions that contain trigonometric functions. There are many more possibilities, some of which are covered in the module *Essential mathematics 2* (MST125).

5.2 Choosing a method of integration

When you have an integral that you want to find, it may not be obvious which method to try. Also, sometimes you might need to combine two or more techniques – for example you might be able to use algebraic manipulation, the constant multiple rule and the sum rule to express an integral in terms of simpler integrals, then use integration by substitution or integration by parts to find the simpler integrals.

Here are some things to think about when you have an integral that you want to find.

Choosing a method for finding an integral

- Is it a standard integral? Consult the table in the *Handbook*.
- Can you rearrange the integral to express it as a sum of constant multiples of simpler integrals?
- Is the integrand of the form $f(ax)$ or $f(ax+b)$, where a and b are constants and f is a function that you can integrate? If so, use the rule for integrating a function of a linear expression, or use integration by substitution, with $u = ax$ or $u = ax + b$ as appropriate.
- Can the integrand be written in the form

  $$f(\text{something}) \times \text{the derivative of the something},$$

  where f is a function that you can integrate? If so, use integration by substitution. Start by setting the 'something' equal to u.
- Is the integrand of the form

  $$f(x)g(x),$$

  where f is a function that becomes simpler when differentiated, and g is a function that you can integrate? (For example, $f(x)$ might be x or x^2 or $\ln x$.) If so, try integration by parts.
- If the integrand contains trigonometric functions, can you use trigonometric identities to rewrite it in a form that's easier to integrate?

When you're finding an indefinite integral, try to always remember to add the arbitrary constant '$+c$'. It's easy to forget it!

In the next activity you can practise identifying which method of integration to use.

Activity 46 *Choosing a method of integration*

For each integral below, suggest a method of integration to use. Where you suggest substitution, also suggest a suitable equation for u in terms of x.

You are not asked to actually find the integrals, but solutions are included so you can do so if you wish.

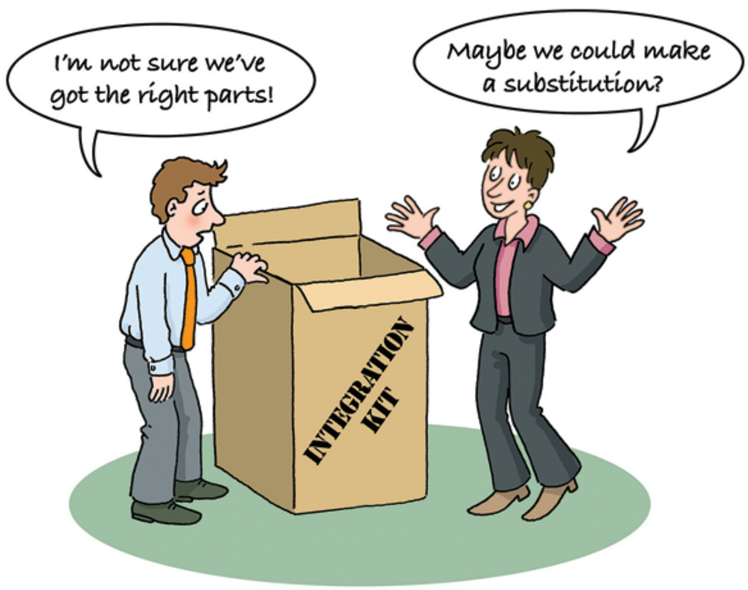
(a) $\int x^3\cos(9x^4)\,dx$ (b) $\int x\cos(5x)\,dx$ (c) $\int (x+\cos(5x))\,dx$

(d) $\int x(x^2+3)\,dx$ (e) $\int \frac{1}{1+3x^2}\,dx$ (f) $\int \frac{x^2}{(7-x^3)^7}\,dx$

(g) $\int xe^{-x/3}\,dx$ (h) $\int \sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}x\right)\,dx$ (i) $\int (x-1)^4\,dx$

(j) $\int e^x(1-2e^x)\,dx$ (k) $\int x(1+\sin(x^2))\,dx$ (l) $\int x(3x^2-2)^8\,dx$

You'll learn further useful strategies for integration in the module *Essential mathematics 2* (MST125), if it's in your study programme. In particular, you'll learn more about integration by substitution and integration by using trigonometric identities, and you'll study *partial fractions*, which are useful when you want to integrate rational functions.



5.3 Integration using a computer

There are many situations in which it's helpful to use a computer for integration, such as when you want to integrate an expression that's difficult to integrate by hand. In the first activity of this subsection you can learn how to use the module computer algebra system to find indefinite integrals and evaluate definite integrals.

Activity 47 *Using a computer for integration*

Work through Section 9 of the *Computer algebra guide*.

You can use the skills that you learned in Activity 47 in the next activity.

Activity 48 *Using a computer to work with a complicated function*

This activity is about the function

$$f(x) = \frac{x^2 - 10x + 15}{1 + \sqrt{x+1}}.$$

Do parts (b) to (d) below by using the module computer algebra system. Give your answers to parts (c) and (d) to three significant figures.

(a) What is the domain of f?

(b) Plot the graph of f.

(c) Find the two x-intercepts of f.

(d) Find the area between the graph of f and the x-axis, between the two x-intercepts.

The next two activities are the final ones of the unit. In them, you're asked to use your knowledge of integration to solve practical problems. The definite integrals involved in the problems can't easily be evaluated by using the fundamental theorem of calculus, as it's difficult to find antiderivatives of $\sqrt{x^3+1}$ and $\sin(t^2/150)$. However, you can use a computer to find approximate values.

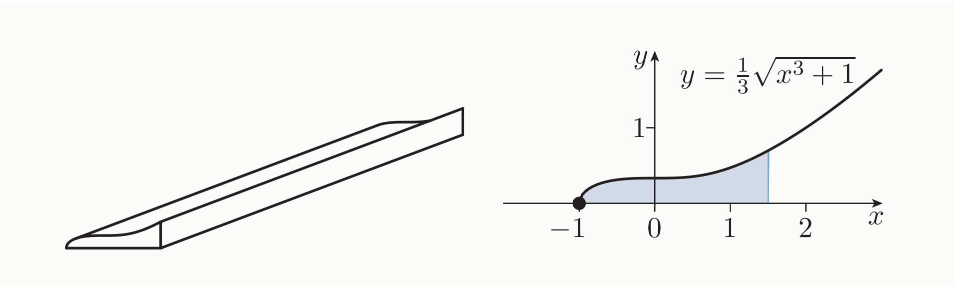
Activity 49 *Using integration to solve a practical problem*

Consider the plastic edging shown below. The shape of its cross-section is given by the equation

$$y = \tfrac{1}{3}\sqrt{x^3+1} \quad (-1 \le x \le 1.5),$$

where x and y are measured in centimetres.

Calculate the volume of plastic that is required to make one metre of the edging. Give your answer in cubic centimetres to three significant figures.



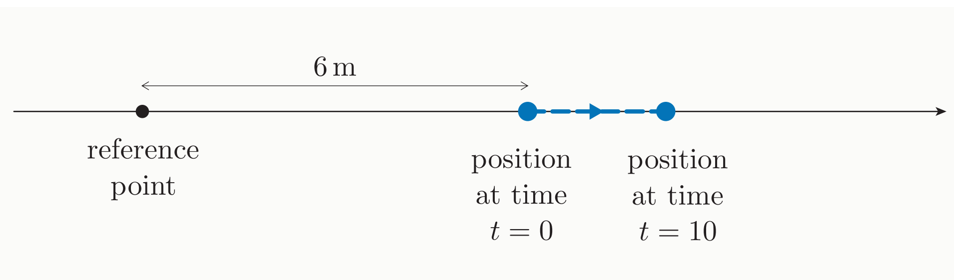
Activity 50 *Using integration to solve another practical problem*

Suppose that an object moves in a straight line, and its velocity v (in $\mathrm{m\,s^{-1}}$) at time t (in seconds) is given by the equation

$$v = \sin\left(\frac{t^2}{150}\right) \quad (0 \le t \le 20),$$

as shown in Figure 42.

If the displacement of the object from a particular reference point on the line at time $t = 0$ is 6 m, as shown below, what is the displacement of the object from this reference point at time $t = 10$? Give your answer to the nearest centimetre.



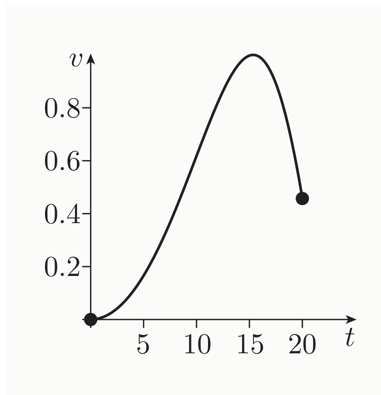


Figure 42 The graph of $v = \sin(x^2/150)$ $(0 \le t \le 20)$

You've now come to the end of the introduction to basic calculus in this module. You can use the skills that you've acquired to solve many different types of problems in mathematics, engineering and science, for example. These skills also equip you to study many further topics in these areas. You'll make a start on that in Unit 11, in which you'll use calculus in your study of *Taylor polynomials*. If your study programme includes further mathematics modules, then in some of these you may study more advanced calculus, with even more powerful applications.

Learning outcomes

After studying this unit, you should be able to:

- understand what is meant by a definite integral
- understand how to use subintervals to find an approximate value for a definite integral
- use integral notation
- understand the fundamental theorem of calculus
- evaluate some definite integrals by using the fundamental theorem of calculus
- integrate by substitution
- integrate by parts
- appreciate how to use trigonometric identities and integration by substitution to help you integrate some trigonometric expressions
- identify a suitable method of integration for some types of integrals
- use a computer for integration.

Solutions to activities

Solution to Activity 1

If we divide the interval $[-18, 18]$ into six subintervals of equal width, then each subinterval has width $36/6 = 6$.

The left endpoints of the six subintervals are

$$-18, \quad -18+6, \quad -18+2\times 6,$$
$$-18+3\times 6, \quad -18+4\times 6, \quad -18+5\times 6$$

that is

$$-18, \quad -12, \quad -6, \quad 0, \quad 6, \quad 12.$$

So the heights of the rectangles are:

$$f(-18) = 9 - \tfrac{1}{36}\times(-18)^2 = 0$$
$$f(-12) = 9 - \tfrac{1}{36}\times(-12)^2 = 5$$
$$f(-6) = 9 - \tfrac{1}{36}\times(-6)^2 = 8$$
$$f(0) = 9 - \tfrac{1}{36}\times 0^2 = 9$$
$$f(6) = 9 - \tfrac{1}{36}\times 6^2 = 8$$
$$f(12) = 9 - \tfrac{1}{36}\times 12^2 = 5.$$

So we obtain the following approximate value for the area:

$$(0\times 6)+(5\times 6)+(8\times 6)$$
$$+(9\times 6)+(8\times 6)+(5\times 6) = 210.$$

Solution to Activity 2

(a) The signed area from $x=1$ to $x=5$ is -6.3.

(b) The signed area from $x=5$ to $x=7$ is 1.3.

(c) The signed area from $x=1$ to $x=7$ is $-6.3+1.3=-5$.

(d) The signed area from $x=1$ to $x=9$ is $-6.3+1.3-5.9=-10.9$.

Solution to Activity 3

If we divide the interval $[-3, 3]$ into six subintervals of equal width, then each subinterval has width $6/6 = 1$.

The left endpoints of the six subintervals are

$$-3, -2, -1, 0, 1, 2.$$

So the heights of the rectangles are:

$$f(-3) = 3-(-3)^2 = -6$$
$$f(-2) = 3-(-2)^2 = -1$$
$$f(-1) = 3-(-1)^2 = 2$$
$$f(0) = 3-0^2 = 3$$
$$f(1) = 3-1^2 = 2$$
$$f(2) = 3-2^2 = -1.$$

So we obtain the following approximate value for the signed area:

$$(-6\times 1)+(-1\times 1)+(2\times 1)$$
$$+(3\times 1)+(2\times 1)+(-1\times 1) = -1.$$

Solution to Activity 4

The exact value of the signed area between the graph of the function $f(x) = 3-x^2$ and the x-axis from $x=-3$ to $x=3$ is 0.

Solution to Activity 5

(a) The signed area from $x=2$ to $x=6$ is 4.0.

(b) The signed area from $x=2$ to $x=6$ is 4.0, so the signed area from $x=6$ to $x=2$ is -4.0.

(c) The signed area from $x=2$ to $x=10$ is $4.0-5.4=-1.4$.

(d) From part (c), the signed area from $x=2$ to $x=10$ is -1.4, so the signed area from $x=10$ to $x=2$ is 1.4.

(e) The signed area from $x=8$ to $x=8$ is 0.

(f) The signed area from $x=-3$ to $x=2$ is $4.1-2.4=1.7$, so the signed area from $x=2$ to $x=-3$ is -1.7.

(g) The signed area from $x=-1$ to $x=10$ is $-2.4+4.0-5.4=-3.8$, so the signed area from $x=10$ to $x=-1$ is 3.8.

Solution to Activity 6

(a) $\displaystyle\int_{-5}^{-3} f(x)\,\mathrm{d}x = -5.$

(b) $\displaystyle\int_{-3}^{2} f(x)\,\mathrm{d}x = 6.$

(c) $\displaystyle\int_{2}^{7} f(x)\,\mathrm{d}x = -11.$

(d) $\displaystyle\int_{-5}^{2} f(x)\,\mathrm{d}x = -5+6 = 1.$

(e) $\displaystyle\int_{-3}^{7} f(x)\,\mathrm{d}x = 6-11 = -5.$

(f) $\displaystyle\int_{-5}^{7} f(x)\,\mathrm{d}x = -5+6-11 = -10.$

(g) $\displaystyle\int_{5}^{5} f(x)\,\mathrm{d}x = 0.$

(h) $\displaystyle\int_{2}^{-3} f(x)\,\mathrm{d}x = -\int_{-3}^{2} f(x)\,\mathrm{d}x = -6.$

(i) $\displaystyle\int_{7}^{2} f(x)\,\mathrm{d}x = -\int_{2}^{7} f(x)\,\mathrm{d}x = -(-11) = 11.$

(j) $\displaystyle\int_{-3}^{-5} f(x)\,\mathrm{d}x = -\int_{-5}^{-3} f(x)\,\mathrm{d}x = -(-5) = 5.$

(k) $$\begin{aligned}\int_{7}^{-3} f(x)\,\mathrm{d}x &= -\int_{-3}^{7} f(x)\,\mathrm{d}x\\ &= -(-5) = 5,\end{aligned}$$
by part (e).

(l) $$\begin{aligned}\int_{7}^{-5} f(x)\,\mathrm{d}x &= -\int_{-5}^{7} f(x)\,\mathrm{d}x\\ &= -(-10) = 10,\end{aligned}$$
by part (f).

Solution to Activity 7

$$\int_2^9 f(x)\,\mathrm{d}x = \int_2^7 f(x)\,\mathrm{d}x + \int_7^9 f(x)\,\mathrm{d}x$$
so
$$-15 = -11 + \int_7^9 f(x)\,\mathrm{d}x$$
and hence
$$\int_7^9 f(x)\,\mathrm{d}x = -15-(-11) = -4.$$

Solution to Activity 8

(a) $\left[\tfrac{1}{2}x^2\right]_3^5 = \tfrac{1}{2}\times 5^2 - \tfrac{1}{2}\times 3^2 = \tfrac{1}{2}(25-9) = 8.$

(b) $\left[\cos x\right]_0^{2\pi} = \cos(2\pi) - \cos 0 = 1-1 = 0.$

(c) $\left[e^x\right]_{-1}^{1} = e^1 - e^{-1} = e - (1/e) = 2.35$ (to 3 s.f.).

Solution to Activity 9

(a) An antiderivative of x^2 is $\tfrac{1}{3}x^3$, so
$$\begin{aligned}\int_{-1}^{1} x^2\,\mathrm{d}x &= \left[\tfrac{1}{3}x^3\right]_{-1}^{1}\\ &= \tfrac{1}{3}\times 1^3 - \tfrac{1}{3}\times(-1)^3\\ &= \tfrac{2}{3}.\end{aligned}$$

(b) An antiderivative of e^t is e^t, so
$$\begin{aligned}\int_0^2 e^t\,\mathrm{d}t &= \left[e^t\right]_0^2\\ &= e^2 - e^0\\ &= e^2 - 1 = 6.39 \text{ (to 3 s.f.)}.\end{aligned}$$

(c) An antiderivative of $1/u$ (for $u>0$) is $\ln u$, so
$$\begin{aligned}\int_1^4 \frac{1}{u}\,\mathrm{d}u &= \left[\ln u\right]_1^4\\ &= \ln 4 - \ln 1\\ &= \ln 4 = 1.39 \text{ (to 3 s.f.)}.\end{aligned}$$

(d) An antiderivative of $\sec^2\theta$ is $\tan\theta$, so
$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \sec^2\theta\,\mathrm{d}\theta &= \left[\tan\theta\right]_{-\pi/4}^{\pi/4}\\ &= \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\\ &= 1-(-1)\\ &= 2.\end{aligned}$$

(e) An antiderivative of $1/(1+u^2)$ is $\tan^{-1}u$, so
$$\begin{aligned}\int_{-1}^{1} \frac{1}{1+u^2}\,\mathrm{d}u &= \left[\tan^{-1}u\right]_{-1}^{1}\\ &= \tan^{-1}(1) - \tan^{-1}(-1)\\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\\ &= \pi/2\\ &= 1.57 \text{ (to 3 s.f.)}.\end{aligned}$$

Solution to Activity 10

(a)
$$\begin{aligned}\int_1^4 3\sqrt{x}\,dx &= \int_1^4 3x^{1/2}\,dx\\ &= \left[3\times\frac{1}{3/2}x^{3/2}\right]_1^4\\ &= \left[2x^{3/2}\right]_1^4\\ &= 2\times 4^{3/2} - 2\times 1^{3/2}\\ &= 2\times 8 - 2\\ &= 14.\end{aligned}$$

(b)
$$\begin{aligned}&\int_{-1}^0 x(1+x)\,dx\\ &= \int_{-1}^0 (x+x^2)\,dx\\ &= \left[\tfrac{1}{2}x^2+\tfrac{1}{3}x^3\right]_{-1}^0\\ &= \left(\tfrac{1}{2}\times 0^2+\tfrac{1}{3}\times 0^3\right) - \left(\tfrac{1}{2}(-1)^2+\tfrac{1}{3}(-1)^3\right)\\ &= -\left(\tfrac{1}{2}-\tfrac{1}{3}\right)\\ &= -\tfrac{1}{6}.\end{aligned}$$

(c)
$$\begin{aligned}\int_2^4 \frac{r+5}{r}\,dr &= \int_2^4 \left(1+\frac{5}{r}\right)\,dr\\ &= \left[r+5\ln r\right]_2^4\\ &= (4+5\ln 4)-(2+5\ln 2)\\ &= 2+5(\ln 4-\ln 2)\\ &= 2+5\ln(4/2)\\ &= 2+5\ln 2.\end{aligned}$$

Solution to Activity 11

(a)
$$\begin{aligned}\int_3^5 x^{3/2}\,dx &= \left[\frac{1}{5/2}x^{5/2}\right]_3^5\\ &= \left[\tfrac{2}{5}x^{5/2}\right]_3^5\\ &= \tfrac{2}{5}\left[x^{5/2}\right]_3^5\\ &= \tfrac{2}{5}(5^{5/2}-3^{5/2})\\ &= 16.1 \text{ (to 3 s.f.)}.\end{aligned}$$

(b)
$$\begin{aligned}&\int_{-1/2}^1 (u-2u^2)\,du\\ &= \left[\tfrac{1}{2}u^2 - 2\times\tfrac{1}{3}u^3\right]_{-1/2}^1\\ &= \left[\tfrac{1}{2}u^2 - \tfrac{2}{3}u^3\right]_{-1/2}^1\\ &= \left[\tfrac{1}{2}u^2\right]_{-1/2}^1 - \left[\tfrac{2}{3}u^3\right]_{-1/2}^1\\ &= \tfrac{1}{2}\left[u^2\right]_{-1/2}^1 - \tfrac{2}{3}\left[u^3\right]_{-1/2}^1\\ &= \tfrac{1}{2}\left(1^2-(-\tfrac{1}{2})^2\right) - \tfrac{2}{3}\left(1^3-(-\tfrac{1}{2})^3\right)\\ &= \tfrac{1}{2}(1-\tfrac{1}{4}) - \tfrac{2}{3}(1+\tfrac{1}{8})\\ &= \tfrac{1}{2}\times\tfrac{3}{4} - \tfrac{2}{3}\times\tfrac{9}{8}\\ &= \tfrac{3}{8}-\tfrac{3}{4}\\ &= -\tfrac{3}{8}.\end{aligned}$$

Solution to Activity 12

(a)
$$\begin{aligned}\int_{\sqrt{3}}^{\sqrt{7}} \tfrac{1}{2}x^3\,dx &= \tfrac{1}{2}\int_{\sqrt{3}}^{\sqrt{7}} x^3\,dx\\ &= \tfrac{1}{2}\left[\tfrac{1}{4}x^4\right]_{\sqrt{3}}^{\sqrt{7}}\\ &= \tfrac{1}{8}\left[x^4\right]_{\sqrt{3}}^{\sqrt{7}}\\ &= \tfrac{1}{8}\left((\sqrt{7})^4-(\sqrt{3})^4\right)\\ &= \tfrac{1}{8}(7^2-3^2)\\ &= \tfrac{1}{8}(49-9)\\ &= 5.\end{aligned}$$

(b)
$$\begin{aligned}&\int_{-\pi/4}^{\pi/4} (\sin x+\cos x)\,dx\\ &= \int_{-\pi/4}^{\pi/4} \sin x\,dx + \int_{-\pi/4}^{\pi/4} \cos x\,dx\\ &= \left[-\cos x\right]_{-\pi/4}^{\pi/4} + \left[\sin x\right]_{-\pi/4}^{\pi/4}\\ &= -\left[\cos x\right]_{-\pi/4}^{\pi/4} + \left[\sin x\right]_{-\pi/4}^{\pi/4}\\ &= -\left(\cos\left(\frac{\pi}{4}\right)-\cos\left(-\frac{\pi}{4}\right)\right)\\ &\quad + \sin\left(\frac{\pi}{4}\right)-\sin\left(-\frac{\pi}{4}\right)\\ &= -\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)\\ &= \frac{2}{\sqrt{2}}\\ &= \sqrt{2}.\end{aligned}$$

Solution to Activity 13

The required area is given by

$$\begin{aligned}
&\int_{-18}^{18}\left(9-\tfrac{1}{36}x^2\right)\mathrm{d}x\\
&=\left[9x-\tfrac{1}{36}\times\tfrac{1}{3}x^3\right]_{-18}^{18}\\
&=\left[9x-\tfrac{1}{108}x^3\right]_{-18}^{18}\\
&=\left[9x\right]_{-18}^{18}-\left[\tfrac{1}{108}x^3\right]_{-18}^{18}\\
&=9\left[x\right]_{-18}^{18}-\tfrac{1}{108}\left[x^3\right]_{-18}^{18}\\
&=9(18-(-18))-\tfrac{1}{108}(18^3-(-18)^3)\\
&=9\times 36-\tfrac{1}{108}(18^3-(-18^3))\\
&=9\times 36-\tfrac{1}{108}(2\times 18^3)\\
&=324-108\\
&=216.
\end{aligned}$$

So the area of the cross-section of the roof is $216\,\mathrm{m}^2$.

Solution to Activity 14

(a) The x-intercepts are given by

$$\begin{aligned}
\tfrac{1}{32}x^2-200&=0\\
\tfrac{1}{32}x^2&=200\\
x^2&=6400\\
x&=\pm\sqrt{6400}\\
x&=\pm 80.
\end{aligned}$$

So the x-intercepts are -80 and 80.

(b) The shaded area is

$$\begin{aligned}
&\int_{-80}^{80}\left(\tfrac{1}{32}x^2-200\right)\mathrm{d}x\\
&=\left[\tfrac{1}{32}\times\tfrac{1}{3}x^3-200x\right]_{-80}^{80}\\
&=\left[\tfrac{1}{96}x^3-200x\right]_{-80}^{80}\\
&=\tfrac{1}{96}\left[x^3\right]_{-80}^{80}-200\left[x\right]_{-80}^{80}\\
&=\tfrac{1}{96}(80^3-(-80)^3)-200(80-(-80))\\
&=\tfrac{1}{96}(2\times 80^3)-200\times 160\\
&=\frac{80^3}{48}-32\,000\\
&=-\frac{64\,000}{3}\\
&=-21\,333\tfrac{1}{3}.
\end{aligned}$$

Hence the shaded area is $21\,333\,\mathrm{cm}^2$, to the nearest cm^2.

Solution to Activity 15

(a) (i) From $t=0$ to $t=1$ the displacement of the object changes by $2.3\,\mathrm{cm}$.

(ii) From $t=3$ to $t=4$ the displacement of the object changes by $-1.8\,\mathrm{cm}$. That is, it changes by $1.8\,\mathrm{cm}$ in the negative direction.

(iii) From $t=0$ to $t=3$ the displacement of the object changes by

$$(2.3+5.9+2.7)\,\mathrm{cm}=10.9\,\mathrm{cm}.$$

(iv) From $t=3$ to $t=6$ the displacement of the object changes by

$$(-1.8-2.0-0.3)\,\mathrm{cm}=-4.1\,\mathrm{cm}.$$

That is, it changes by $4.1\,\mathrm{cm}$ in the negative direction.

(b) (i) The displacement of the object from its starting position at time $t=2$ is

$$(2.3+5.9)\,\mathrm{cm}=8.2\,\mathrm{cm}.$$

(ii) By (a) parts (iii) and (iv), the displacement of the object from its starting position at time $t=6$ is

$$(10.9-4.1)\,\mathrm{cm}=6.8\,\mathrm{cm}.$$

(c) By parts (a)(iii) and (a)(iv), the object travels $10.9\,\mathrm{cm}$ in the positive direction, then $4.1\,\mathrm{cm}$ in the negative direction, so the total distance that it travels is

$$(10.9+4.1)\,\mathrm{cm}=15\,\mathrm{cm}.$$

Solution to Activity 16

(a) The change in the displacement of the object from time $t=0$ to time $t=1$ is given by

$$\begin{aligned}
\int_0^1\left(3-\tfrac{3}{2}t\right)\mathrm{d}t&=\left[3t-\tfrac{3}{2}\times\tfrac{1}{2}t^2\right]_0^1\\
&=\left[3t-\tfrac{3}{4}t^2\right]_0^1\\
&=3\left[t\right]_0^1-\tfrac{3}{4}\left[t^2\right]_0^1\\
&=3(1-0)-\tfrac{3}{4}(1^2-0^2)\\
&=3-\tfrac{3}{4}\\
&=\tfrac{9}{4}=2.25.
\end{aligned}$$

That is, the object travels $2.25\,\mathrm{m}$ in that time (in the positive direction).

(b) The change in the displacement of the object from time $t = 4$ to time $t = 5$ is given by

$$\begin{aligned}\int_4^5 (3 - \tfrac{3}{2}t)\mathrm{d}t &= \Big[3t - \tfrac{3}{2} \times \tfrac{1}{2}t^2\Big]_4^5 \\ &= \Big[3t - \tfrac{3}{4}t^2\Big]_4^5 \\ &= 3\big[t\big]_4^5 - \tfrac{3}{4}\big[t^2\big]_4^5 \\ &= 3(5-4) - \tfrac{3}{4}(5^2 - 4^2) \\ &= 3 - \tfrac{3}{4} \times 9 \\ &= -\tfrac{15}{4} \\ &= -3.75.\end{aligned}$$

That is, the object travels 3.75 m in that time (in the negative direction).

Solution to Activity 17

The change in the displacement of the car during those four seconds is given by

$$\begin{aligned}\int_0^4 (28 - 7t)\,\mathrm{d}t &= \Big[28t - \tfrac{7}{2}t^2\Big]_0^4 \\ &= 28\big[t\big]_0^4 - \tfrac{7}{2}\big[t^2\big]_0^4 \\ &= 28(4-0) - \tfrac{7}{2}(4^2 - 0) \\ &= 112 - 56 \\ &= 56.\end{aligned}$$

That is, the car travels 56 m during those four seconds.

(In fact, since the required answer is the area of the shaded triangle in Figure 39, you can calculate it in the following more straightforward way.

The change in the displacement of the car during the four seconds is

$(\tfrac{1}{2} \times 4 \times 28)\,\mathrm{m} = 56\,\mathrm{m}$.)

Solution to Activity 18

The change in the displacement of the car is the area shaded in Figure 40, which is given by

$$\begin{aligned}\tfrac{1}{2} \times 5 \times 15 - \tfrac{1}{2} \times 4 \times 12 &= \tfrac{1}{2}(75 - 48) \\ &= \tfrac{1}{2} \times 27 = 13.5.\end{aligned}$$

So the car travels 13.5 m during the fifth second.

(There are various ways to work out the area shaded in Figure 40. It is worked out above by subtracting the area of one triangle from the area of another, but alternatively you could add the areas of a rectangle and a triangle, or use the formula for the area of a trapezium.)

Solution to Activity 19

(a) $\displaystyle\int e^x\,\mathrm{d}x = e^x + c.$

(b) $$\begin{aligned}\int (x+3)(x-2)\,\mathrm{d}x &= \int (x^2 + x - 6)\,\mathrm{d}x \\ &= \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 6x + c.\end{aligned}$$

(c) $$\begin{aligned}\int \frac{1}{x^3}\,\mathrm{d}x &= \int x^{-3}\,\mathrm{d}x \\ &= \frac{1}{-2}x^{-2} + c \\ &= -\frac{1}{2x^2} + c.\end{aligned}$$

(d) $\displaystyle\int (\sin\theta - \cos\theta)\,\mathrm{d}\theta = -\cos\theta - \sin\theta + c.$

(e) $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + c.$

(f) $\displaystyle\int \frac{-3}{1+u^2}\,\mathrm{d}u = -3\tan^{-1} u + c.$

(g) $\displaystyle\int \frac{1}{2(1+r^2)}\,\mathrm{d}r = \tfrac{1}{2}\tan^{-1} r + c.$

(h) $$\begin{aligned}\int \frac{(1+x^2)}{2}\,\mathrm{d}x &= \tfrac{1}{2}(x + \tfrac{1}{3}x^3) + c \\ &= \tfrac{1}{6}(3x + x^3) + c.\end{aligned}$$

Solution to Activity 20

(a) $$\begin{aligned}\int \left(\frac{3}{x} - \frac{1}{1+x^2}\right)\mathrm{d}x &= \int \frac{3}{x}\,\mathrm{d}x - \int \frac{1}{1+x^2}\,\mathrm{d}x \\ &= 3\int \frac{1}{x}\,\mathrm{d}x - \int \frac{1}{1+x^2}\,\mathrm{d}x \\ &= 3\ln|x| - \tan^{-1} x + c.\end{aligned}$$

(b) $$\begin{aligned}&\int (\operatorname{cosec} x)(\operatorname{cosec} x + \cot x)\,\mathrm{d}x \\ &= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)\,\mathrm{d}x \\ &= \int \operatorname{cosec}^2 x\,\mathrm{d}x + \int \operatorname{cosec} x \cot x\,\mathrm{d}x \\ &= -\cot x - \operatorname{cosec} x + c.\end{aligned}$$

Solution to Activity 21

(a) Let $u = \cos x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$. So

$$\begin{aligned}\int e^{\cos x}(-\sin x)\,\mathrm{d}x &= \int e^u\,\mathrm{d}u\\ &= e^u + c\\ &= e^{\cos x} + c.\end{aligned}$$

(b) Let $u = x^3$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$. So

$$\begin{aligned}\int \left(\sin(x^3)\right)(3x^2)\,\mathrm{d}x &= \int \sin(u)\,\mathrm{d}u\\ &= -\cos u + c\\ &= -\cos(x^3) + c.\end{aligned}$$

(c) Let $u = \sin x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$. So

$$\begin{aligned}\int \left(\frac{1}{\sin x}\right)\cos x\,\mathrm{d}x &= \int \frac{1}{u}\,\mathrm{d}u\\ &= \ln|u| + c\\ &= \ln|\sin x| + c.\end{aligned}$$

(d) Let $u = \cos x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$. So

$$\begin{aligned}\int \left(\frac{1}{\cos^2 x}\right)(-\sin x)\,\mathrm{d}x &= \int \frac{1}{u^2}\,\mathrm{d}u\\ &= \int u^{-2}\,\mathrm{d}u\\ &= \frac{1}{-1}u^{-1} + c\\ &= -\frac{1}{u} + c\\ &= -\frac{1}{\cos x} + c\\ &= -\sec x + c.\end{aligned}$$

(A quicker way to find this indefinite integral, without using substitution, is to write the integrand as $-\sec x \tan x$ and use the table of standard indefinite integrals.)

(e) Let $u = \sin x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$. So

$$\begin{aligned}\int \sin^4 x\cos x\,\mathrm{d}x &= \int u^4\,\mathrm{d}u\\ &= \tfrac{1}{5}u^5 + c\\ &= \tfrac{1}{5}\sin^5 x + c.\end{aligned}$$

(f) Let $u = \sin x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$. So

$$\begin{aligned}\int \left(\frac{1}{1+\sin^2 x}\right)\cos x\,\mathrm{d}x &= \int \left(\frac{1}{1+u^2}\right)\mathrm{d}u\\ &= \tan^{-1} u + c\\ &= \tan^{-1}(\sin x) + c.\end{aligned}$$

Solution to Activity 22

(a) Let $u = 4 + \cos x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$. So

$$\begin{aligned}\int (4+\cos x)^7(-\sin x)\,\mathrm{d}x &= \int u^7\,\mathrm{d}u\\ &= \tfrac{1}{8}u^8 + c\\ &= \tfrac{1}{8}(4+\cos x)^8 + c.\end{aligned}$$

(b) Let $u = 1 + x^2$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$. So

$$\begin{aligned}\int \sqrt{1+x^2}\,(2x)\,\mathrm{d}x &= \int \sqrt{u}\,\mathrm{d}u\\ &= \int u^{1/2}\,\mathrm{d}u\\ &= \frac{1}{3/2}u^{3/2} + c\\ &= \tfrac{2}{3}(1+x^2)^{3/2} + c.\end{aligned}$$

(c) Let $u = x^5 - 8$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 5x^4$. So

$$\begin{aligned}\int (x^5-8)^{10}(5x^4)\,\mathrm{d}x &= \int u^{10}\,\mathrm{d}u\\ &= \tfrac{1}{11}u^{11} + c\\ &= \tfrac{1}{11}(x^5-8)^{11} + c.\end{aligned}$$

(d) Let $u = e^x + 5$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = e^x$. So

$$\begin{aligned}\int \left(\frac{1}{e^x+5}\right)e^x\,\mathrm{d}x &= \int \frac{1}{u}\,\mathrm{d}u\\ &= \ln|u| + c\\ &= \ln|e^x+5| + c\\ &= \ln(e^x+5) + c.\end{aligned}$$

(e) Let $u = 5 + 2x^3$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 6x^2$. So

$$\begin{aligned}\int \left(\sin(5+2x^3)\right)(6x^2)\,\mathrm{d}x &= \int \sin u\,\mathrm{d}u\\ &= -\cos u + c\\ &= -\cos(5+2x^3) + c.\end{aligned}$$

Solution to Activity 23

(a) Let $u = e^x$; then $\dfrac{du}{dx} = e^x$. So

$$\begin{aligned}\int (\cos(e^x))\, e^x \, dx &= \int \cos u \, du \\ &= \sin u + c \\ &= \sin(e^x) + c.\end{aligned}$$

(b) Let $u = 1 + \sin x$; then $\dfrac{du}{dx} = \cos x$. So

$$\begin{aligned}\int e^{1+\sin x} (\cos x) \, dx &= \int e^u \, du \\ &= e^u + c \\ &= e^{1+\sin x} + c.\end{aligned}$$

(c) Let $u = x^{10} + 6$; then $\dfrac{du}{dx} = 10x^9$. So

$$\begin{aligned}\int \left(\frac{1}{x^{10}+6}\right)(10x^9) \, dx &= \int \frac{1}{u} \, du \\ &= \ln|u| + c \\ &= \ln|x^{10}+6| + c \\ &= \ln(x^{10}+6) + c.\end{aligned}$$

(d) Let $u = \cos x$; then $\dfrac{du}{dx} = -\sin x$. So

$$\begin{aligned}\int (\cos^3 x)(-\sin x) \, dx &= \int u^3 \, du \\ &= \tfrac{1}{4}u^4 + c \\ &= \tfrac{1}{4}\cos^4 x + c.\end{aligned}$$

Solution to Activity 24

(a) Let $u = x^3$; then $\dfrac{du}{dx} = 3x^2$. So

$$\begin{aligned}\int x^2 \cos(x^3) \, dx &= \int \big(\cos(x^3)\big)\, x^2 \, dx \\ &= \tfrac{1}{3}\int (\cos(x^3))(3x^2) \, dx \\ &= \tfrac{1}{3}\int \cos u \, du \\ &= \tfrac{1}{3}\sin u + c \\ &= \tfrac{1}{3}\sin(x^3) + c.\end{aligned}$$

(b) Let $u = 1 + x^2$; then $\dfrac{du}{dx} = 2x$. So

$$\begin{aligned}\int x\sqrt{1+x^2} \, dx &= \int \left(\sqrt{1+x^2}\right) x \, dx \\ &= \tfrac{1}{2}\int \left(\sqrt{1+x^2}\right) \times 2x \, dx \\ &= \tfrac{1}{2}\int \sqrt{u} \, du \\ &= \tfrac{1}{2}\int u^{1/2} \, du \\ &= \tfrac{1}{2} \times \tfrac{2}{3}u^{3/2} + c \\ &= \tfrac{1}{3}u^{3/2} + c \\ &= \tfrac{1}{3}(1+x^2)^{3/2} + c.\end{aligned}$$

(c) Let $u = \cos x$; then $\dfrac{du}{dx} = -\sin x$. So

$$\begin{aligned}\int \cos^4 x \sin x \, dx &= -\int (\cos^4 x)(-\sin x) \, dx \\ &= -\int u^4 \, du \\ &= -\tfrac{1}{5}u^5 + c \\ &= -\tfrac{1}{5}\cos^5 x + c.\end{aligned}$$

Solution to Activity 25

(a) Let $u = 1 + 2x^2$; then $\dfrac{du}{dx} = 4x$. So

$$\begin{aligned}\int \frac{x}{1+2x^2} \, dx &= \int \left(\frac{1}{1+2x^2}\right) x \, dx \\ &= \tfrac{1}{4}\int \left(\frac{1}{1+2x^2}\right)(4x) \, dx \\ &= \tfrac{1}{4}\int \frac{1}{u} \, du \\ &= \tfrac{1}{4}\ln|u| + c \\ &= \tfrac{1}{4}\ln|1+2x^2| + c \\ &= \tfrac{1}{4}\ln(1+2x^2) + c\end{aligned}$$

(since $1 + 2x^2$ is always positive).

(b) Let $u = 2 + e^x$; then $\dfrac{du}{dx} = e^x$. So

$$\begin{aligned}\int \frac{e^x}{(2+e^x)^2}\,dx &= \int \left(\frac{1}{(2+e^x)^2}\right) e^x\,dx\\ &= \int \frac{1}{u^2}\,du\\ &= \int u^{-2}\,du\\ &= -u^{-1} + c\\ &= -\frac{1}{u} + c\\ &= -\frac{1}{2+e^x} + c.\end{aligned}$$

(c) Let $u = 3 - \sin x$; then $\dfrac{du}{dx} = -\cos x$. So

$$\begin{aligned}\int \frac{\cos x}{3-\sin x}\,dx &= -\int \left(\frac{1}{3-\sin x}\right)(-\cos x)\,dx\\ &= -\int \frac{1}{u}\,du\\ &= -\ln|u| + c\\ &= -\ln|3-\sin x| + c\\ &= -\ln(3-\sin x) + c\end{aligned}$$

(since $3 - \sin x$ is always positive).

Solution to Activity 26

(a) Let $u = \cos x$; then $\dfrac{du}{dx} = -\sin x$. So

$$\begin{aligned}\int e^{\cos x}\sin x\,dx &= -\int e^{\cos x}(-\sin x)\,dx\\ &= -\int e^u\,du\\ &= -e^u + c\\ &= -e^{\cos x} + c.\end{aligned}$$

(b) Let $u = 3 - e^x$; then $\dfrac{du}{dx} = -e^x$. So

$$\begin{aligned}\int \frac{e^x}{3-e^x}\,dx &= -\int \left(\frac{1}{3-e^x}\right)(-e^x)\,dx\\ &= -\int \frac{1}{u}\,du\\ &= -\ln|u| + c\\ &= -\ln|3-e^x| + c.\end{aligned}$$

(c) Let $u = 1 + \frac{1}{2}\sin x$; then $\dfrac{du}{dx} = \frac{1}{2}\cos x$. So

$$\begin{aligned}&\int (1+\tfrac{1}{2}\sin x)^2\cos x\,dx\\ &= 2\int (1+\tfrac{1}{2}\sin x)^2(\tfrac{1}{2}\cos x)\,dx\\ &= 2\int u^2\,du\\ &= 2\times\tfrac{1}{3}u^3 + c\\ &= \tfrac{2}{3}u^3 + c\\ &= \tfrac{2}{3}(1+\tfrac{1}{2}\sin x)^3 + c.\end{aligned}$$

Solution to Activity 27

Let $u = \cos x$; then $\dfrac{du}{dx} = -\sin x$. So

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= \int \left(\frac{1}{\cos x}\right)(\sin x)\,dx\\ &= -\int \left(\frac{1}{\cos x}\right)(-\sin x)\,dx\\ &= -\int \frac{1}{u}\,du\\ &= -\ln|u| + c\\ &= -\ln|\cos x| + c.\end{aligned}$$

Solution to Activity 28

(a) Let $u = 6x - 1$; then $\dfrac{du}{dx} = 6$. So

$$\begin{aligned}\int e^{6x-1}\,dx &= \tfrac{1}{6}\int e^{6x-1}\times 6\,dx\\ &= \tfrac{1}{6}\int e^u\,du\\ &= \tfrac{1}{6}e^u + c\\ &= \tfrac{1}{6}e^{6x-1} + c.\end{aligned}$$

(b) Let $u = 4x$; then $\dfrac{du}{dx} = 4$. So

$$\begin{aligned}\int \sin(4x)\,dx &= \tfrac{1}{4}\int (\sin(4x))\times 4\,dx\\ &= \tfrac{1}{4}\int \sin u\,du\\ &= \tfrac{1}{4}(-\cos u) + c\\ &= -\tfrac{1}{4}\cos u + c\\ &= -\tfrac{1}{4}\cos(4x) + c.\end{aligned}$$

(c) Let $u = -9x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -9$. So

$$\begin{aligned}\int e^{-9x}\,\mathrm{d}x &= -\tfrac{1}{9}\int e^{-9x}\times(-9)\,\mathrm{d}x\\ &= -\tfrac{1}{9}\int e^{u}\,\mathrm{d}u\\ &= -\tfrac{1}{9}e^{u}+c\\ &= -\tfrac{1}{9}e^{-9x}+c.\end{aligned}$$

(d) Let $u = -x/3$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\tfrac{1}{3}$. So

$$\begin{aligned}\int e^{-x/3}\,\mathrm{d}x &= -3\int e^{-x/3}\times(-\tfrac{1}{3})\,\mathrm{d}x\\ &= -3\int e^{u}\,\mathrm{d}u\\ &= -3e^{u}+c\\ &= -3e^{-x/3}+c.\end{aligned}$$

(e) Let $u = 3-7x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -7$. So

$$\begin{aligned}\int \cos(3-7x)\,\mathrm{d}x &= -\tfrac{1}{7}\int (\cos(3-7x))\times(-7)\,\mathrm{d}x\\ &= -\tfrac{1}{7}\int \cos u\,\mathrm{d}u\\ &= -\tfrac{1}{7}\sin u+c\\ &= -\tfrac{1}{7}\sin(3-7x)+c.\end{aligned}$$

(f) Let $u = 4-x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -1$. So

$$\begin{aligned}\int \frac{1}{4-x}\,\mathrm{d}x &= -\int\left(\frac{1}{4-x}\right)\times(-1)\,\mathrm{d}x\\ &= -\int \frac{1}{u}\,\mathrm{d}u\\ &= -\ln|u|+c\\ &= -\ln|4-x|+c.\end{aligned}$$

(g) Let $u = 2x+1$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$. So

$$\begin{aligned}\int \frac{1}{(2x+1)^2}\,\mathrm{d}x &= \tfrac{1}{2}\int\left(\frac{1}{(2x+1)^2}\right)\times 2\,\mathrm{d}x\\ &= \tfrac{1}{2}\int \frac{1}{u^2}\,\mathrm{d}u\\ &= \tfrac{1}{2}\int u^{-2}\,\mathrm{d}u\\ &= -\tfrac{1}{2}u^{-1}+c\\ &= -\frac{1}{2}\times\frac{1}{u}+c\\ &= -\frac{1}{2(2x+1)}+c.\end{aligned}$$

(h) Let $u = 1-x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -1$. So

$$\begin{aligned}\int \frac{1}{(1-x)^3}\,\mathrm{d}x &= -\int\left(\frac{1}{(1-x)^3}\right)\times(-1)\,\mathrm{d}x\\ &= -\int \frac{1}{u^3}\,\mathrm{d}u\\ &= -\int u^{-3}\,\mathrm{d}u\\ &= -(-\tfrac{1}{2}u^{-2})+c\\ &= \frac{1}{2u^2}+c\\ &= \frac{1}{2(1-x)^2}+c.\end{aligned}$$

(i) Let $u = 6x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 6$. So

$$\begin{aligned}\int \sec^2(6x)\,\mathrm{d}x &= \tfrac{1}{6}\int (\sec^2(6x))\times 6\,\mathrm{d}x\\ &= \tfrac{1}{6}\int \sec^2 u\,\mathrm{d}u\\ &= \tfrac{1}{6}\tan u+c\\ &= \tfrac{1}{6}\tan(6x)+c.\end{aligned}$$

Solution to Activity 29

(a) Let $u = 2x$; then $\dfrac{du}{dx} = 2$. So

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \int \frac{1}{\sqrt{1-(2x)^2}}\,dx\\ &= \tfrac{1}{2}\int \frac{1}{\sqrt{1-(2x)^2}} \times 2\,dx\\ &= \tfrac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du\\ &= \tfrac{1}{2}\sin^{-1} u + c\\ &= \tfrac{1}{2}\sin^{-1}(2x) + c.\end{aligned}$$

(b) Let $u = \sqrt{2}\,x$; then $\dfrac{du}{dx} = \sqrt{2}$. So

$$\begin{aligned}\int \frac{1}{1+2x^2}\,dx &= \frac{1}{\sqrt{2}}\int \frac{1}{1+\left(\sqrt{2}\,x\right)^2} \times \sqrt{2}\,dx\\ &= \frac{1}{\sqrt{2}}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{\sqrt{2}}\tan^{-1} u + c\\ &= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\,x\right) + c.\end{aligned}$$

(c) Let $u = \frac{3}{2}x$; then $\dfrac{du}{dx} = \frac{3}{2}$. So

$$\begin{aligned}\int \frac{1}{4+9x^2}\,dx &= \tfrac{1}{4}\int \frac{1}{1+\frac{9}{4}x^2}\,dx\\ &= \tfrac{1}{4}\int \frac{1}{1+\left(\frac{3}{2}x\right)^2}\,dx\\ &= \tfrac{1}{4}\times\tfrac{2}{3}\int \frac{1}{1+\left(\frac{3}{2}x\right)^2}\times\tfrac{3}{2}\,dx\\ &= \tfrac{1}{6}\int \frac{1}{1+\left(\frac{3}{2}x\right)^2}\times\tfrac{3}{2}\,dx\\ &= \tfrac{1}{6}\int \frac{1}{1+u^2}\,du\\ &= \tfrac{1}{6}\tan^{-1} u + c\\ &= \tfrac{1}{6}\tan^{-1}\left(\tfrac{3}{2}x\right) + c.\end{aligned}$$

Solution to Activity 30

(a) $\displaystyle\int \cos(10x)\,dx = \tfrac{1}{10}\sin(10x) + c.$

(b) $$\begin{aligned}\int \sin(2-5x)\,dx &= -\tfrac{1}{5}(-\cos(2-5x)) + c\\ &= \tfrac{1}{5}\cos(2-5x) + c.\end{aligned}$$

(c) $\displaystyle\int e^{3t+4}\,dt = \tfrac{1}{3}e^{3t+4} + c.$

(d) $\displaystyle\int \cos(\tfrac{1}{2}\theta)\,d\theta = 2\sin(\tfrac{1}{2}\theta) + c.$

(e) $\displaystyle\int e^{-4p}\,dp = -\tfrac{1}{4}e^{-4p} + c.$

(f) $\displaystyle\int \frac{1}{7x-4}\,dx = \tfrac{1}{7}\ln|7x-4| + c.$

(g) $\displaystyle\int \frac{1}{1-r}\,dr = -\ln|1-r| + c.$

(h) $\displaystyle\int \sec^2(5x-1)\,dx = \tfrac{1}{5}\tan(5x-1) + c.$

(i) $\displaystyle\int \sin(\tfrac{1}{5}x)\,dx = -5\cos(\tfrac{1}{5}x) + c.$

(j) $\displaystyle\int e^{-x}\,dx = -e^{-x} + c.$

(k) $\displaystyle\int e^{x/2}\,dx = 2e^{x/2} + c.$

(l) $$\begin{aligned}\int \sec^2(2-3p)\,dp &= -\tfrac{1}{3}\tan(2-3p) + c\\ &= \tfrac{1}{3}\tan(3p-2) + c.\end{aligned}$$

(The final expression was obtained by using the trigonometric identity $\tan(-\theta) = -\tan\theta$.)

(m) $$\begin{aligned}\int \sin\left(\frac{2-x}{3}\right)dx &= \int \sin\left(\tfrac{2}{3}-\tfrac{1}{3}x\right)dx\\ &= -3\left(-\cos\left(\tfrac{2}{3}-\tfrac{1}{3}x\right)\right) + c\\ &= 3\cos\left(\frac{2-x}{3}\right) + c.\end{aligned}$$

Solution to Activity 31

(a) The velocity v (in $\mathrm{m\,s^{-1}}$) of the object at time t (in seconds) is given by

$$v = 2\cos\left(\tfrac{1}{2}t+5\right).$$

Hence the displacement s (in m) of the object at time t (in seconds) is given by

$$\begin{aligned}s &= \int 2\cos\left(\tfrac{1}{2}t+5\right)dt\\ &= 2\times 2\sin\left(\tfrac{1}{2}t+5\right) + c\\ &= 4\sin\left(\tfrac{1}{2}t+5\right) + c,\end{aligned}$$

where c is an arbitrary constant.

When $t = 0$, $s = 4$. Hence

$$4 = 4\sin\left(\tfrac{1}{2} \times 0 + 5\right) + c$$
$$4 = 4\sin(5) + c$$
$$c = 4 - 4\sin(5)$$
$$c = 7.835\ldots$$
$$c = 7.84 \text{ (to 2 d.p.)}.$$

So the equation for the displacement of the object in terms of time is

$$s = 4\sin\left(\tfrac{1}{2}t + 5\right) + 7.84.$$

(b) When $t = 10$,

$$\begin{aligned} s &= 4\sin\left(\tfrac{1}{2} \times 10 + 5\right) + 7.835\ldots \\ &= 4\sin(10) + 7.835\ldots \\ &= 5.7 \text{ (to 2 s.f.)}. \end{aligned}$$

So the displacement of the object at time 10 seconds is 5.7 m (to 2 s.f.).

Solution to Activity 32

(a) Let $u = 1 + 2x^2$; then $\dfrac{du}{dx} = 4x$.

Putting $x = 0$ gives $u = 1$ and putting $x = 1$ gives $u = 3$. So

$$\begin{aligned} \int_0^1 \frac{x}{1+2x^2}\,dx &= \tfrac{1}{4}\int_0^1 \left(\frac{1}{1+2x^2}\right)(4x)\,dx \\ &= \tfrac{1}{4}\int_1^3 \frac{1}{u}\,du \\ &= \tfrac{1}{4}\big[\ln|u|\big]_1^3 \\ &= \tfrac{1}{4}(\ln 3 - \ln 1) \\ &= \tfrac{1}{4}\ln 3 \\ &= 0.275 \text{ (to 3 s.f.)}. \end{aligned}$$

(b) Let $u = \cos x$; then $\dfrac{du}{dx} = -\sin x$. Putting $x = \pi/2$ gives $u = 0$ and putting $x = \pi$ gives $u = -1$. So

$$\begin{aligned} \int_{\pi/2}^{\pi} \cos^3 x \sin x\,dx &= -\int_{\pi/2}^{\pi} \cos^3 x(-\sin x)\,dx \\ &= -\int_0^{-1} u^3\,du \\ &= -\left[\tfrac{1}{4}u^4\right]_0^{-1} \\ &= -\tfrac{1}{4}\left[u^4\right]_0^{-1} \\ &= -\tfrac{1}{4}\left((-1)^4 - 0^4\right) \\ &= -\tfrac{1}{4} \\ &= -0.25. \end{aligned}$$

(Notice that in the second line of the manipulation above we obtain an integral, $\int_0^{-1} u^3$, in which the lower limit of integration is greater than the upper limit of integration. An alternative way to proceed from this line is to use the fact that, in general, $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. Then the manipulation continues like this:

$$\begin{aligned} \int_0^{-1} u^3\,du &= \int_{-1}^{0} u^3\,du \\ &= \left[\tfrac{1}{4}u^4\right]_{-1}^{0} \\ &= \tfrac{1}{4}\left[u^4\right]_{-1}^{0} \\ &= \tfrac{1}{4}\left(0^4 - (-1)^4\right) \\ &= -\tfrac{1}{4} \\ &= -0.25.) \end{aligned}$$

(c) Let $u = -3x^2$; then $\dfrac{du}{dx} = -6x$.

Putting $x = -1$ gives $u = -3$ and putting $x = 0$ gives $u = 0$. So

$$\begin{aligned} \int_{-1}^{0} xe^{-3x^2}\,dx &= -\tfrac{1}{6}\int_{-1}^{0}\left(e^{-3x^2}\right)(-6x)\,dx \\ &= -\tfrac{1}{6}\int_{-3}^{0} e^u\,du \\ &= -\tfrac{1}{6}\left[e^u\right]_{-3}^{0} \\ &= -\tfrac{1}{6}\left(e^0 - e^{-3}\right) \\ &= \tfrac{1}{6}\left(e^{-3} - 1\right) \\ &= -0.158 \text{ (to 3 s.f.)}. \end{aligned}$$

Solution to Activity 33

(a) Let $u = 1 + x$; then $\dfrac{du}{dx} = 1$. Also, $x = u - 1$. So

$$\begin{aligned}\int x\sqrt{1+x}\,dx &= \int (u-1)\sqrt{u}\,du\\ &= \int \left(u^{3/2} - u^{1/2}\right)\,du\\ &= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + c\\ &= \tfrac{2}{5}(1+x)^{5/2} - \tfrac{2}{3}(1+x)^{3/2} + c.\end{aligned}$$

(b) Let $u = x + 3$; then $\dfrac{du}{dx} = 1$. Also, $x = u - 3$, so $x - 2 = u - 5$. Hence

$$\begin{aligned}\int \frac{x-2}{(x+3)^3}\,dx &= \int \frac{u-5}{u^3}\,du\\ &= \int (u^{-2} - 5u^{-3})\,du\\ &= -u^{-1} + \tfrac{5}{2}u^{-2} + c\\ &= -\frac{1}{u} + \frac{5}{2u^2} + c\\ &= -\frac{1}{x+3} + \frac{5}{2(x+3)^2} + c.\end{aligned}$$

Solution to Activity 34

(a) Let $f(x) = x$ and $g(x) = \cos x$. Then $f'(x) = 1$ and an antiderivative of $g(x)$ is $G(x) = \sin x$. Hence

$$\begin{aligned}\int x\cos x\,dx &= \int f(x)g(x)\,dx\\ &= f(x)G(x) - \int f'(x)G(x)\,dx\\ &= x\sin x - \int \sin x\,dx\\ &= x\sin x - (-\cos x) + c\\ &= x\sin x + \cos x + c.\end{aligned}$$

(b) Let $f(x) = x$ and $g(x) = e^x$. Then $f'(x) = 1$ and an antiderivative of $g(x)$ is $G(x) = e^x$. Hence

$$\begin{aligned}\int xe^x\,dx &= \int f(x)g(x)\,dx\\ &= f(x)G(x) - \int f'(x)G(x)\,dx\\ &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + c\\ &= e^x(x-1) + c.\end{aligned}$$

Solution to Activity 35

(a)
$$\begin{aligned}\int x\sin(5x)\,dx &= x\left(-\tfrac{1}{5}\cos(5x)\right)\\ &\quad - \int 1 \times \left(-\tfrac{1}{5}\cos(5x)\right)\,dx\\ &= -\tfrac{1}{5}x\cos(5x) + \tfrac{1}{5}\int \cos(5x)\,dx\\ &= -\tfrac{1}{5}x\cos(5x) + \tfrac{1}{5} \times \tfrac{1}{5}\sin(5x) + c\\ &= -\tfrac{1}{5}x\cos(5x) + \tfrac{1}{25}\sin(5x) + c\\ &= \tfrac{1}{25}\sin(5x) - \tfrac{1}{5}x\cos(5x) + c.\end{aligned}$$

(b)
$$\begin{aligned}\int xe^{2x}\,dx &= x\left(\tfrac{1}{2}e^{2x}\right) - \int 1 \times \left(\tfrac{1}{2}e^{2x}\right)\,dx\\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx\\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2} \times \tfrac{1}{2}e^{2x} + c\\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + c\\ &= \tfrac{1}{4}e^{2x}(2x-1) + c.\end{aligned}$$

(c)
$$\begin{aligned}\int xe^{-x}\,dx &= x\left(-e^{-x}\right) - \int 1 \times \left(-e^{-x}\right)\,dx\\ &= -xe^{-x} + \int e^{-x}\,dx\\ &= -xe^{-x} + (-e^{-x}) + c\\ &= -xe^{-x} - e^{-x} + c\\ &= -e^{-x}(x+1) + c.\end{aligned}$$

Solution to Activity 36

(a)
$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - \int 2xe^x\,dx\\ &= x^2e^x - 2\int xe^x\,dx\\ &= x^2e^x - 2\left(xe^x - \int 1\times e^x\,dx\right)\\ &= x^2e^x - 2xe^x + 2\int e^x\,dx\\ &= x^2e^x - 2xe^x + 2e^x + c\\ &= e^x(x^2-2x+2)+c.\end{aligned}$$

(b)
$$\begin{aligned}&\int x^2\cos(2x)\,dx\\ &= x^2(\tfrac{1}{2}\sin(2x)) - \int 2x(\tfrac{1}{2}\sin(2x))\,dx\\ &= \tfrac{1}{2}x^2\sin(2x) - \int x\sin(2x)\,dx\\ &= \tfrac{1}{2}x^2\sin(2x)\\ &\quad - \left(x(-\tfrac{1}{2}\cos(2x)) - \int 1\times(-\tfrac{1}{2}\cos(2x))\,dx\right)\\ &= \tfrac{1}{2}x^2\sin(2x) + \tfrac{1}{2}x\cos(2x) - \tfrac{1}{2}\int\cos(2x)\,dx\\ &= \tfrac{1}{2}x^2\sin(2x) + \tfrac{1}{2}x\cos(2x) - \tfrac{1}{2}\times\tfrac{1}{2}\sin(2x) + c\\ &= \tfrac{1}{4}(2x^2-1)\sin(2x) + \tfrac{1}{2}x\cos(2x) + c.\end{aligned}$$

Solution to Activity 37

$$\begin{aligned}&\int (5x+2)e^{-5x}\,dx\\ &= (5x+2)\left(\frac{1}{-5}\right)e^{-5x} - \int 5\left(\frac{1}{-5}\right)e^{-5x}\,dx\\ &= -\tfrac{1}{5}(5x+2)e^{-5x} + \int e^{-5x}\,dx\\ &= -\tfrac{1}{5}(5x+2)e^{-5x} - \tfrac{1}{-5}e^{-5x}\,dx + c\\ &= -\tfrac{1}{5}e^{-5x}(5x+2+1)+c\\ &= -\tfrac{1}{5}(5x+3)e^{-5x}+c.\end{aligned}$$

(An alternative way to find this integral is to start by writing it as

$$5\int xe^{-5x}\,dx + 2\int e^{-5x}\,dx.$$

The first integral in this expression has an integrand of the form $xg(x)$, where $g(x)$ is an expression that you can integrate, so you can find this integral by using integration by parts. The second integral is straightforward to find.)

Solution to Activity 38

(a)
$$\begin{aligned}\int x^3\ln x\,dx &= \int(\ln x)x^3\,dx\\ &= (\ln x)\times\tfrac{1}{4}x^4 - \int\frac{1}{x}\times\tfrac{1}{4}x^4\,dx\\ &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}\int x^3\,dx\\ &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}\times\tfrac{1}{4}x^4 + c\\ &= \tfrac{1}{16}x^4(4\ln x - 1)+c.\end{aligned}$$

(b)
$$\begin{aligned}\int\ln x\,dx &= \int(\ln x)\times 1\,dx\\ &= (\ln x)x - \int\frac{1}{x}\times x\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + c\\ &= x(\ln x - 1)+c.\end{aligned}$$

Solution to Activity 39

$$\begin{aligned}&\int(x-2)\ln(3x)\,dx\\ &= \int(\ln(3x))(x-2)\,dx\\ &= \ln(3x)\times(\tfrac{1}{2}x^2-2x) - \int\frac{1}{x}(\tfrac{1}{2}x^2-2x)\,dx\\ &= \tfrac{1}{2}x(x-4)\ln(3x) - \int(\tfrac{1}{2}x-2)\,dx\\ &= \tfrac{1}{2}x(x-4)\ln(3x) - \tfrac{1}{2}\int(x-4)\,dx\\ &= \tfrac{1}{2}x(x-4)\ln(3x) - \tfrac{1}{2}(\tfrac{1}{2}x^2-4x)+c\\ &= \tfrac{1}{2}x(x-4)\ln(3x) - \tfrac{1}{4}x(x-8)+c.\end{aligned}$$

Solution to Activity 40

(a) We write

$$\int e^x \sin(3x)\,dx = \int \sin(3x)\,e^x\,dx.$$

Integrating by parts twice gives

$$\begin{aligned}
&\int \sin(3x)\,e^x\,dx\\
&= \sin(3x)e^x - \int (3\cos(3x))e^x\,dx\\
&= \sin(3x)e^x - 3\int \cos(3x)e^x\,dx\\
&= \sin(3x)e^x\\
&\qquad - 3\left(\cos(3x)e^x - \int(-3\sin(3x))e^x\,dx\right)\\
&= \sin(3x)e^x - 3\cos(3x)e^x - 9\int \sin(3x)e^x\,dx.
\end{aligned}$$

So

$$\begin{aligned}
&10\int \sin(3x)\,e^x\,dx\\
&= \sin(3x)e^x - 3\cos(3x)e^x + c
\end{aligned}$$

and hence

$$\begin{aligned}
&\int \sin(3x)\,e^x\,dx\\
&= \tfrac{1}{10}\sin(3x)e^x - \tfrac{3}{10}\cos(3x)e^x + c.
\end{aligned}$$

That is,

$$\begin{aligned}
&\int e^x\sin(3x)\,dx\\
&= \tfrac{1}{10}e^x\sin(3x) - \tfrac{3}{10}e^x\cos(3x) + c\\
&= \tfrac{1}{10}e^x(\sin(3x) - 3\cos(3x)) + c.
\end{aligned}$$

(b) Integrating by parts twice gives

$$\begin{aligned}
&\int e^{2x}\cos x\,dx\\
&= e^{2x}\sin x - \int (2e^{2x})\sin x\,dx\\
&= e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\\
&= e^{2x}\sin x\\
&\quad - 2\left(e^{2x}(-\cos x) - \int(2e^{2x})(-\cos x)\,dx\right)\\
&= e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\,dx.
\end{aligned}$$

So

$$5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x + c$$

and hence

$$\int e^{2x}\cos x\,dx = \tfrac{1}{5}e^{2x}\sin x + \tfrac{2}{5}e^{2x}\cos x + c.$$

That is,

$$\int e^{2x}\cos x\,dx = \tfrac{1}{5}e^{2x}(\sin x + 2\cos x) + c.$$

Solution to Activity 41

$$\begin{aligned}
\int_0^1 xe^{3x}\,dx &= \left[x\times\tfrac{1}{3}e^{3x}\right]_0^1 - \int_0^1 1\times\tfrac{1}{3}e^{3x}\,dx\\
&= \tfrac{1}{3}\left[xe^{3x}\right]_0^1 - \tfrac{1}{3}\int_0^1 e^{3x}\,dx\\
&= \tfrac{1}{3}(e^3 - 0) - \tfrac{1}{3}\left[\tfrac{1}{3}e^{3x}\right]_0^1\\
&= \tfrac{1}{3}e^3 - \tfrac{1}{9}\left[e^{3x}\right]_0^1\\
&= \tfrac{1}{3}e^3 - \tfrac{1}{9}\left(e^3 - 1\right)\\
&= \tfrac{1}{9}(3e^3 - e^3 + 1)\\
&= \tfrac{1}{9}(2e^3 + 1)\\
&= 4.5746 \text{ (to 4 d.p.)}.
\end{aligned}$$

Solution to Activity 42

(a) If $x \in [\pi/12, \pi/6]$, then $3x \in [\pi/4, \pi/2]$.

Now

$$\sin\theta > 0 \text{ when } \theta \in [\pi/4, \pi/2],$$

so

$$\sin(3x) > 0 \text{ when } x \in [\pi/12, \pi/6].$$

Also

$$x > 0 \text{ when } x \in [\pi/12, \pi/6].$$

Hence

$$x\sin(3x) > 0 \text{ when } x \in [\pi/12, \pi/6].$$

That is, the graph of $f(x) = x\sin(3x)$ lies above the x-axis for values of x in the interval $[\pi/12, \pi/6]$.

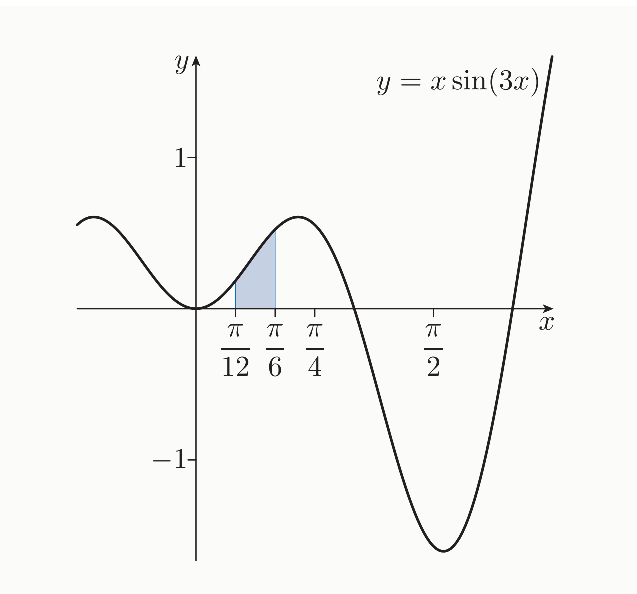
(b) It follows from part (a) that the area between this graph and the x-axis from $x = \pi/12$ to $x = \pi/6$ is

$$\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx.$$

(c) Integrating by parts gives

$$\begin{aligned}
&\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx \\
&= \Big[x(-\tfrac{1}{3}\cos(3x))\Big]_{\pi/12}^{\pi/6} - \int_{\pi/12}^{\pi/6}(-\tfrac{1}{3}\cos(3x))\,dx \\
&= -\tfrac{1}{3}\big[x\cos(3x)\big]_{\pi/12}^{\pi/6} + \tfrac{1}{3}\int_{\pi/12}^{\pi/6}\cos(3x)\,dx \\
&= -\tfrac{1}{3}\left(\frac{\pi}{6}\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{12}\cos\left(\frac{\pi}{4}\right)\right) \\
&\qquad\qquad + \tfrac{1}{3}\left[\tfrac{1}{3}\sin(3x)\right]_{\pi/12}^{\pi/6} \\
&= -\tfrac{1}{3}\left(\frac{\pi}{6}\times 0 - \frac{\pi}{12}\times\frac{1}{\sqrt{2}}\right) + \tfrac{1}{9}\big[\sin(3x)\big]_{\pi/12}^{\pi/6} \\
&= \frac{\pi}{36\sqrt{2}} + \tfrac{1}{9}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{\pi}{36\sqrt{2}} + \tfrac{1}{9}\left(1 - \frac{1}{\sqrt{2}}\right) \\
&= 0.0943 \text{ (to 4 d.p.).}
\end{aligned}$$

(The area found in this solution is shown below.)



Solution to Activity 43

(a) To help us construct a table of signs for the function $f(x) = (x-2)(e^{-x}-1)$, first we consider the two factors of the expression $(x-2)(e^{-x}-1)$ separately.

The function $y = x - 2$ is increasing on its whole domain, and the x-intercept of its graph is $x = 2$.

The function $y = e^{-x}$ is decreasing on its whole domain, so the function $y = e^{-x} - 1$ is also decreasing on its whole domain. The x-intercept of its graph is given by $e^{-x} - 1 = 0$, which gives

$$\begin{aligned}
e^{-x} &= 1 \\
-x &= \ln 1 \\
-x &= 0 \\
x &= 0.
\end{aligned}$$

So its x-intercept is $x = 0$.

So we have the following table of signs, for $x > 0$.

x	$(0,2)$	2	$(2,\infty)$
$x-2$	$-$	0	$+$
$e^{-x}-1$	$-$	$-$	$-$
$(x-2)(e^{-x}-1)$	$+$	0	$-$

The table shows that the graph of the function $f(x) = (x-2)(e^{-x}-1)$ lies above the x-axis for $0 < x < 2$ and below the x-axis for $x > 2$.

(b) Because the graph of f lies above the x-axis for $0 < x < 2$ and below the x-axis for $x > 2$, to find the total area between the graph of f and the x-axis from $x = 0$ to $x = 4$ we have to work out the signed area from $x = 0$ to $x = 2$ and the signed area from $x = 2$ to $x = 4$ separately.

The signed area between the graph of f and the x-axis from $x = 0$ to $x = 2$ is given by

$$\begin{aligned}
&\int_0^2 (x-2)(e^{-x}-1)\,dx \\
&= \Big[(x-2)(-e^{-x}-x)\Big]_0^2 - \int_0^2(-e^{-x}-x)\,dx \\
&= 0 - (-2)(-e^{-0}) + \int_0^2 (e^{-x}+x)\,dx \\
&= 2(-1) + \Big[-e^{-x} + \tfrac{1}{2}x^2\Big]_0^2 \\
&= -2 + (-e^{-2}+2) - (-e^{-0}+0) \\
&= -2 - e^{-2} + 2 + 1 \\
&= 1 - e^{-2}.
\end{aligned}$$

Since the graph of f lies above the x-axis for $0 < x < 2$, it follows that the area between the graph of f and the x-axis from $x = 0$ to $x = 2$ is $1 - e^{-2}$.

Similarly, the signed area between the graph of f and the x-axis from $x = 2$ to $x = 4$ is given by

$$\begin{aligned}
&\int_2^4 (x-2)(e^{-x}-1)\,\mathrm{d}x \\
&= \Big[(x-2)(-e^{-x}-x)\Big]_2^4 - \int_2^4 (-e^{-x}-x)\,\mathrm{d}x \\
&= 2(-e^{-4}-4) - 0 + \int_2^4 (e^{-x}+x)\,\mathrm{d}x \\
&= -2e^{-4} - 8 + \Big[-e^{-x} + \tfrac{1}{2}x^2\Big]_2^4 \\
&= -2e^{-4} - 8 + (-e^{-4}+8) - (-e^{-2}+2) \\
&= -3e^{-4} + e^{-2} - 2.
\end{aligned}$$

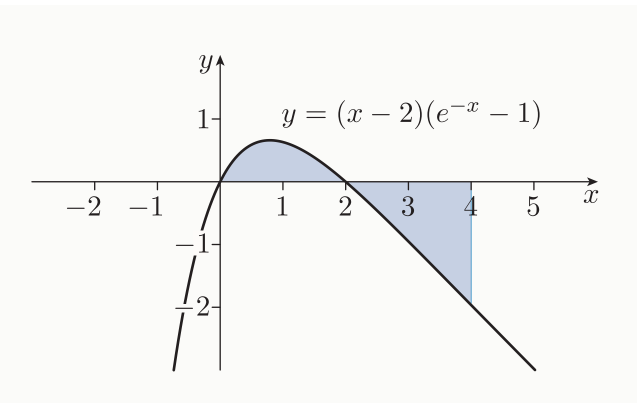
Since the graph of f lies below the x-axis for $2 < x < 4$, it follows that the area between the graph of f and the x-axis from $x = 2$ to $x = 4$ is

$$-(-3e^{-4}+e^{-2}-2) = 3e^{-4} - e^{-2} + 2.$$

So the total area between the graph of f and the x-axis from $x = 0$ to $x = 4$ is

$$\begin{aligned}
&(1-e^{-2}) + (3e^{-4} - e^{-2} + 2) \\
&= 3 - 2e^{-2} + 3e^{-4} \\
&= 2.784 \text{ (to 4 s.f.)}.
\end{aligned}$$

(The area found in this solution is shown below.)



Solution to Activity 44

(a)
$$\begin{aligned}
\int \cos^2\theta\,\mathrm{d}\theta &= \int \tfrac{1}{2}(1+\cos(2\theta))\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\int (1+\cos(2\theta))\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\big(\theta + \tfrac{1}{2}\sin(2\theta)\big) + c \\
&= \tfrac{1}{4}(2\theta + \sin(2\theta)) + c.
\end{aligned}$$

(b)
$$\begin{aligned}
\int \sin x\,\cos x\,\mathrm{d}x &= \int \tfrac{1}{2}\sin(2x)\,\mathrm{d}x \\
&= \tfrac{1}{2}\times\tfrac{1}{2}(-\cos(2x)) + c \\
&= -\tfrac{1}{4}\cos(2x) + c.
\end{aligned}$$

Solution to Activity 45

Let $u = \sin x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x$. So

$$\begin{aligned}
\int \sin x\,\cos x\,\mathrm{d}x &= \int u\,\mathrm{d}u \\
&= \tfrac{1}{2}u^2 + c \\
&= \tfrac{1}{2}\sin^2 x + c.
\end{aligned}$$

(There's an alternative approach to finding this integral, as follows. Let $u = \cos x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$. So

$$\begin{aligned}
\int \sin x\,\cos x\,\mathrm{d}x &= \int \cos x \sin x\,\mathrm{d}x \\
&= -\int (\cos x)(-\sin x)\,\mathrm{d}x \\
&= -\int u\,\mathrm{d}u \\
&= -\tfrac{1}{2}u^2 + c \\
&= -\tfrac{1}{2}\cos^2 x + c.
\end{aligned}$$

This approach gives a different answer, but of course it's equivalent to the first answer. You can confirm this by using the identity $\sin^2\theta + \cos^2\theta = 1$, as follows:

$$\begin{aligned}
-\tfrac{1}{2}\cos^2 x + c &= -\tfrac{1}{2}(1-\sin^2 x) + c \\
&= \tfrac{1}{2}\sin^2 x - \tfrac{1}{2} + c \\
&= \tfrac{1}{2}\sin^2 x + d,
\end{aligned}$$

where $d = -\frac{1}{2} + c$ is an arbitrary constant.

Notice also that the solution to Activity 44(b) gives a third different answer:

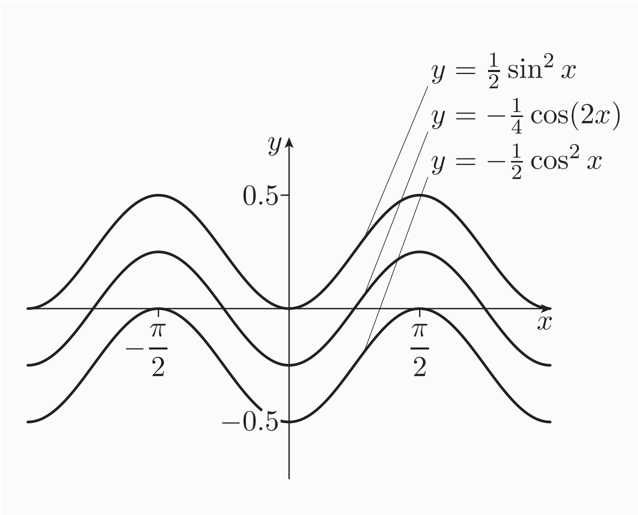
$$\int \sin x\,\cos x\,\mathrm{d}x = -\tfrac{1}{4}\cos(2x) + c.$$

You can confirm that this answer is equivalent to the first answer above by using the identity $\cos(2\theta) = 1 - 2\sin^2\theta$, as follows:

$$\begin{aligned}
-\tfrac{1}{4}\cos(2x) + c &= -\tfrac{1}{4}(1 - 2\sin^2 x) + c \\
&= \tfrac{1}{2}\sin^2 x - \tfrac{1}{4} + c \\
&= \tfrac{1}{2}\sin^2 x + d,
\end{aligned}$$

where $d = -\frac{1}{4} + c$ is an arbitrary constant.)

(The diagram below shows the graphs of the three different antiderivatives of $f(x) = \sin x \cos x$ that were found in the solutions to Activity 44(b) and Activity 45. The arbitrary constant has been taken to be zero in each case. You can see that, as you'd expect, the three graphs appear to be vertical translations of each other.)



Solution to Activity 46

(a) Since the derivative of $9x^4$ is $36x^3$, which is 'nearly' x^3, you can use the substitution $u = 9x^4$.

Let $u = 9x^4$; then $\dfrac{du}{dx} = 36x^3$. So

$$\begin{aligned}\int x^3\cos(9x^4)\,dx &= \tfrac{1}{36}\int \cos(9x^4)\times 36x^3\,dx\\ &= \tfrac{1}{36}\int \cos u\,du\\ &= \tfrac{1}{36}\sin u + c\\ &= \tfrac{1}{36}\sin(9x^4) + c.\end{aligned}$$

(b) The integrand is of the form $xg(x)$, where g is a function that you can integrate, which suggests that you should try integration by parts. To integrate the 'second' expression, you can use the fact that it is a simple function of a linear expression.

$$\begin{aligned}\int x\cos(5x)\,dx &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\int \sin(5x)\,dx\\ &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\left(-\tfrac{1}{5}\cos(5x)\right) + c\\ &= \tfrac{1}{5}x\sin(5x) + \tfrac{1}{25}\cos(5x) + c.\end{aligned}$$

(c) The integrand is the sum of two expressions that are straightforward to integrate. The second expression is a simple function of a linear expression.

$$\int (x + \cos(5x))\,dx = \tfrac{1}{2}x^2 + \tfrac{1}{5}\sin(5x) + c.$$

(d) You could use integration by substitution (taking $u = x^2 + 3$) or integration by parts, but the simplest method is to start by multiplying out the brackets.

$$\begin{aligned}\int x(x^2+3)\,dx &= \int (x^3 + 3x)\,dx\\ &= \tfrac{1}{4}x^4 + \tfrac{3}{2}x^2 + c\\ &= \tfrac{1}{4}x^2(x^2+6) + c.\end{aligned}$$

(e) The integrand can be written as $\dfrac{1}{1+\left(\sqrt{3}\,x\right)^2}$, so it's a function of the linear expression $\sqrt{3}\,x$. So you can use the substitution $u = \sqrt{3}\,x$, together with the indefinite integral of $\dfrac{1}{1+x^2}$, which is a standard integral.

Let $u = \sqrt{3}\,x$; then $du/dx = \sqrt{3}$. So

$$\begin{aligned}\int \frac{1}{1+3x^2}\,dx &= \frac{1}{\sqrt{3}}\int \frac{1}{1+\left(\sqrt{3}\,x\right)^2}\times\sqrt{3}\,dx\\ &= \frac{1}{\sqrt{3}}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{\sqrt{3}}\tan^{-1}u + c\\ &= \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\,x\right) + c.\end{aligned}$$

(f) Since the derivative of $7 - x^3$ is $-3x^2$, which is 'nearly' the numerator x^2, you can use the substitution $u = 7 - x^3$.

Let $u = 7 - x^3$; then $\dfrac{du}{dx} = -3x^2$. So

$$\begin{aligned}\int \frac{x^2}{(7-x^3)^7}\,dx &= -\tfrac{1}{3}\int \frac{1}{(7-x^3)^7}(-3x^2)\,dx\\ &= -\tfrac{1}{3}\int \frac{1}{u^7}\,du\\ &= -\tfrac{1}{3}\int u^{-7}\,du\\ &= -\frac{1}{3}\times\frac{1}{-6}u^{-6} + c\\ &= \frac{1}{18u^6} + c\\ &= \frac{1}{18(7-x^3)^6} + c.\end{aligned}$$

(g) The integrand is of the form $xg(x)$, where g is a function that you can integrate, which suggests that you should try integration by parts. To integrate the 'second' expression, you can use the fact that it's a simple function of a linear expression.

$$\begin{aligned}&\int xe^{-x/3}\,\mathrm{d}x\\&\quad= x\left(-3e^{-x/3}\right) - \int 1\times\left(-3e^{-x/3}\right)\,\mathrm{d}x\\&\quad= -3xe^{-x/3} + 3\int e^{-x/3}\,\mathrm{d}x\\&\quad= -3xe^{-x/3} - 9e^{-x/3} + c\\&\quad= -3e^{-x/3}(x+3) + c.\end{aligned}$$

(h) You can use the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$\begin{aligned}\int \sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}x\right)\,\mathrm{d}x &= \int \tfrac{1}{2}\sin x\,\mathrm{d}x\\&= \tfrac{1}{2}\int \sin x\,\mathrm{d}x\\&= -\tfrac{1}{2}\cos x + c.\end{aligned}$$

(i) You can use the fact that the integrand is a simple function of a linear expression, or you can use the substitution $u = x - 1$.

$$\int (x-1)^4\,\mathrm{d}x = \frac{(x-1)^5}{5} + c.$$

(j) You could use integration by substitution (taking $u = 1 - 2e^x$), but the simplest method is to start by multiplying out the brackets.

$$\begin{aligned}\int e^x(1-2e^x)\,\mathrm{d}x &= \int (e^x - 2(e^x)^2)\,\mathrm{d}x\\&= \int (e^x - 2e^{2x})\,\mathrm{d}x\\&= e^x - 2\times\tfrac{1}{2}e^{2x} + c\\&= e^x - e^{2x} + c\\&= e^x - (e^x)^2 + c\\&= e^x(1-e^x) + c.\end{aligned}$$

(Here's how the integration goes if you use integration by substitution.

Let $u = 1 - 2e^x$; then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -2e^x$. So

$$\begin{aligned}\int e^x(1-2e^x)\,\mathrm{d}x &= -\tfrac{1}{2}\int (1-2e^x)(-2e^x)\,\mathrm{d}x\\&= -\tfrac{1}{2}\int u\,\mathrm{d}u\\&= -\tfrac{1}{2}\times\tfrac{1}{2}u^2 + c\\&= -\tfrac{1}{4}u^2 + c\\&= -\tfrac{1}{4}(1-2e^x)^2 + c\\&= -\tfrac{1}{4}(1 - 4e^x + 4(e^x)^2) + c\\&= -\tfrac{1}{4}(1 - 4e^x + 4e^{2x}) + c\\&= e^x - e^{2x} - \tfrac{1}{4} + c\\&= e^x - (e^x)^2 - \tfrac{1}{4} + c\\&= e^x(1-e^x) - \tfrac{1}{4} + c\\&= e^x(1-e^x) + d,\end{aligned}$$

where $d = -\frac{1}{4} + c$ is an arbitrary constant.)

(k) You can start by multiplying out the brackets and applying the sum rule. Then you have two integrals to find. One of them is straightforward, and you can find the other by using integration by substitution, taking $u = x^2$.

$$\begin{aligned}&\int x(1+\sin(x^2))\,\mathrm{d}x\\&\quad= \int (x + x\sin(x^2))\,\mathrm{d}x\\&\quad= \int x\,\mathrm{d}x + \int x\sin(x^2)\,\mathrm{d}x\\&\quad= \tfrac{1}{2}x^2 + \int x\sin(x^2)\,\mathrm{d}x\end{aligned}$$

Let $u = x^2$; then $\mathrm{d}u/\mathrm{d}x = 2x$. The expression above becomes

$$\begin{aligned}&\tfrac{1}{2}x^2 + \tfrac{1}{2}\int(\sin(x^2))\times 2x\,\mathrm{d}x\\&\quad= \tfrac{1}{2}x^2 + \tfrac{1}{2}\int \sin u\,\mathrm{d}u\\&\quad= \tfrac{1}{2}x^2 + \tfrac{1}{2}(-\cos u) + c\\&\quad= \tfrac{1}{2}x^2 - \tfrac{1}{2}\cos(x^2) + c\\&\quad= \tfrac{1}{2}(x^2 - \cos(x^2)) + c.\end{aligned}$$

(l) Since the derivative of $3x^2 - 2$ is $6x$, which is 'nearly' x, you can use the substitution $u = 3x^2 - 2$.

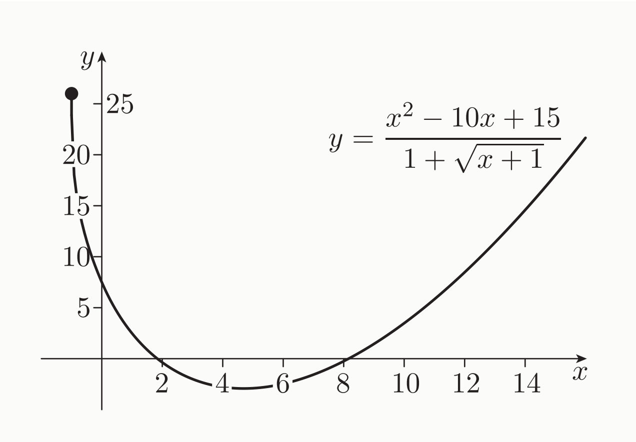
Let $u = 3x^2 - 2$; then $\frac{du}{dx} = 6x$. So

$$\begin{aligned}\int x(3x^2-2)^8\,dx &= \tfrac{1}{6}\int (3x^2-2)^8 \times 6x\,dx\\ &= \tfrac{1}{6}\int u^8\,du\\ &= \tfrac{1}{6}\times\tfrac{1}{9}u^9\\ &= \tfrac{1}{54}(3x^2-2)^9 + c.\end{aligned}$$

Solution to Activity 48

(a) The domain of f is the interval $[-1, \infty)$ (because the expression $\sqrt{x+1}$ is defined only when x is in this interval).

(b) The graph of f is shown below.



(c) The x-intercepts of f are 1.84 and 8.16 (to 3 s.f.).

(The exact values are $5 \pm \sqrt{10}$.)

(d) The area between the graph of f and the x-axis, between the two x-intercepts, is

$$-\int_{1.837\ldots}^{8.162\ldots} \frac{x^2 - 10x + 15}{1 + \sqrt{x+1}}\,dx.$$

A computer gives the value of this expression as 12.4 (to 3 s.f.).

(Details of how to use the CAS for this activity are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

Solution to Activity 49

The area of the cross-section, in square centimetres, is

$$\int_{-1}^{1.5} \tfrac{1}{3}\sqrt{x^3+1}\,dx.$$

A computer gives the value of this definite integral as 0.939 (to 3 s.f.). So the volume of plastic required to make one metre of the edging is approximately $93.9\,\text{cm}^3$.

(Details of how to use the CAS to find the indefinite integral are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

Solution to Activity 50

The change in the displacement of the object from time $t = 0$ to time $t = 10$ is

$$\int_0^{10} \sin\left(\frac{t^2}{150}\right) dt.$$

A computer gives the value of this definite integral as 2.15 (to 2 d.p.).

So the displacement of the object at time $t = 10$ is $(6 + 2.15)\,\text{m} = 8.15\,\text{m}$, to the nearest centimetre.

(Details of how to use the CAS to find the indefinite integral are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

Acknowledgements

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Page 193: Mark Hobbs, for the idea for the cartoon