Unit 3

Functions

Introduction

In mathematics you often work with situations in which one quantity depends on another. For example:

- The distance walked by a woman at a particular speed depends on the time that she's been walking.
- The height of a gondola on a Ferris wheel depends on the angle through which the wheel has rotated since the gondola was in its lowest position.
- The number of 5-litre tins of a particular type of paint needed by a decorator depends on the area that he intends to paint.

A Ferris wheel

Whenever one quantity depends on another, we say that the first quantity is a **function** of the second quantity. The idea of a function is fundamental in mathematics, and in particular it forms the foundation for *calculus*, which you'll begin to study in Unit 6.

In this unit you'll be introduced to the terminology and notation that are used for functions. You'll learn about some standard, frequently-arising types of functions, and how to use graphs to visualise properties of functions. You'll also learn how you can use your knowledge about a few standard functions to help you understand and work with a wide range of related functions. Later in the unit you'll revise *exponential functions* and *logarithms*, and practise working with them. In the final section you'll revise *inequalities*, and see how working with functions and their graphs can help you understand and solve some quite complicated inequalities.

This is a long unit. The study calendar allows extra time for you to study it.

1 Functions and their graphs

This section introduces you to the idea of a function and its graph, and shows you some standard functions. You'll start by learning about *sets*, which are needed when you work with functions and also in many other areas of mathematics.

1.1 Sets of real numbers

In mathematics a **set** is a collection of objects. The objects could be anything at all: they could be numbers, points in the plane, equations or anything else. For example, each of the following collections of objects forms a set:

- all the prime numbers less than 100
- all the points on any particular line in the plane
- all the equations that represent vertical lines
- the solutions of any particular quadratic equation.

A set can contain any number of objects. It could contain one object, two objects, twenty objects, infinitely many objects, or even no objects at all.

Each object in a set is called an **element** or **member** of the set, and we say that the elements of the set **belong to** or are **in** the set.

There are many ways to specify a set. If there are just a few elements, then you can list them, enclosing them in curly brackets. For example, you can specify a set S as follows:

$$S = \{3, 7, 9, 42\}.$$

Another simple way to specify a set is to describe it. For example, you can say 'let T be the set of all even integers' or 'let U be the set of all real numbers greater than 5'. We usually denote sets by capital letters.

The set that contains no elements at all is called the **empty set**, and is denoted by the symbol $\varnothing$.

It's often useful to state that a particular object is or is not a member of a particular set. You can do this concisely using the symbols $\in$ and $\notin$, which mean 'is in' and 'is not in', respectively. For example, if S is the set specified above, then the following statements are true:

$$7 \in S \quad \text{and} \quad 10 \notin S.$$

Activity 1 *Understanding set notation*

Let $X = \{1, 2, 3, 4\}$ and let Y be the set of all odd integers. Which of the following statements are true?

(a) $1 \in X$ (b) $1 \in Y$ (c) $2 \notin X$ (d) $2 \notin Y$

It's often useful to construct 'new sets out of old sets'. For example, if A and B are any two sets, then you can form a new set whose members are all the objects that belong to *both* A and B. This set is called the **intersection** of A and B, and is denoted by $A \cap B$. For instance, if

$$A = \{1, 2, 3, 4\} \quad \text{and} \quad B = \{3, 4, 5\},$$

then

$$A \cap B = \{3, 4\}.$$

Similarly, if A and B are any two sets, then you can form a new set whose members are all the objects that belong to *either* A or B (or both). This set is called the **union** of A and B, and is denoted by $A \cup B$. For example, if A and B are as specified above, then

$$A \cup B = \{1, 2, 3, 4, 5\}.$$

You might find it helpful to visualise intersections and unions of sets by using diagrams like those in Figure 1, which are known as **Venn diagrams**. The Venn diagrams in the figure show the intersection and union of the particular sets A and B above.

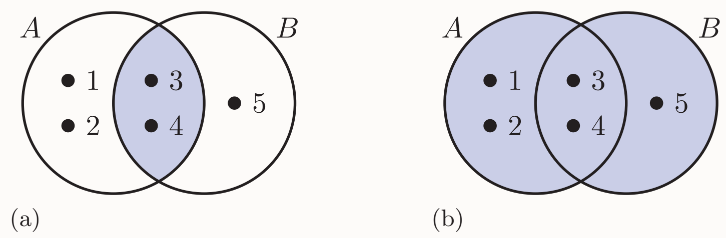
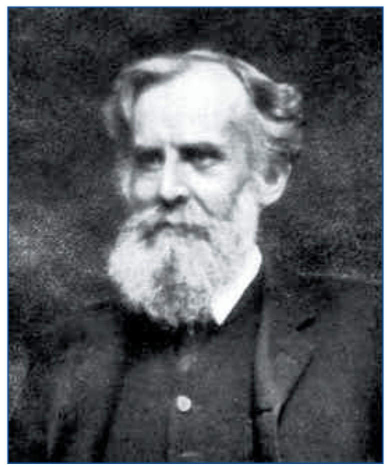


Figure 1 (a) The intersection (shaded) and (b) the union (shaded) of two sets

Venn diagrams are named after the logician John Venn, who used them in publications starting in 1880. However, the idea of using diagrams in this way did not originate with Venn. The prolific Swiss mathematician Leonhard Euler (pronounced 'oiler') used them in his *Letters to a German Princess* (1760–62). Venn acknowledged Euler's influence by calling his own diagrams 'Eulerian circles'. He extended Euler's idea, using the diagrams to analyse more complex logical problems. As well as working on logic at Cambridge University, Venn was for some time a priest and later a historian. There is more about Euler on page 214.



John Venn (1834–1923)

You can form intersections and unions of more than two sets in a similar way. In general, the **intersection** of two or more sets is the set of all objects that belong to *all* of the original sets, and the **union** of two or more sets is the set of all objects that belong to *any* of the original sets. For example, if A and B are as specified above and

$$C = \{20, 21\},$$

then

$$A \cap B \cap C = \varnothing \quad \text{and} \quad A \cup B \cup C = \{1, 2, 3, 4, 5, 20, 21\}.$$

Activity 2 *Understanding unions and intersections of sets*

Let $P = \{1, 2, 3, 4, 5, 6\}$, let $Q = \{2, 4, 6, 8, 10, 12\}$ and let R be the set of all integers divisible by 3. Specify each of the following sets.

(a) $P \cap Q$ (b) $Q \cap R$ (c) $P \cap Q \cap R$ (d) $P \cup Q$

The set membership symbol $\in$ is a stylised version of the Greek letter ε (epsilon). The Italian mathematician Giuseppe Peano (1858–1932), the founder of symbolic logic, used ε to indicate set membership in a text published in 1889. He stated that it was an abbreviation for the Latin word 'est', which means 'is'. The symbol was then adopted by the logician Bertrand Russell (1872–1970) in a text published in 1903, but it was typeset in a form that looks like the modern symbol $\in$, and this form has remained in use to the present day. Peano also introduced the symbols $\cap$ and $\cup$ for intersection and union.

The empty set symbol $\varnothing$ was introduced in 1939 by the influential French mathematician André Weil (1906–1998). It was inspired by the letter ø in the Norwegian alphabet.

Sometimes every element of a set A is also an element of a set B. In this case we say that A is a **subset** of B, and we write $A \subseteq B$. For example:

- $\{1, 3\}$ is a subset of $\{1, 2, 3\}$ (as shown in Figure 2)
- the set of integers is a subset of the set of real numbers.

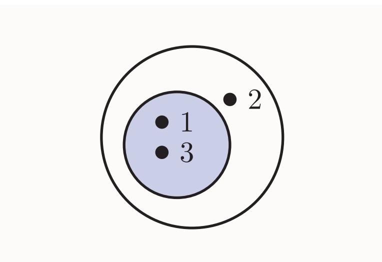


Figure 2 A subset of a set (shaded)

Every set is a subset of itself, and the empty set is a subset of every set.

In this module, and particularly in this unit, you'll mostly be working with sets whose elements are real numbers. In the rest of this subsection, you'll meet some useful ways to visualise and represent sets of this type.

The set of *all* real numbers is denoted by $\mathbb{R}$. You can handwrite this as:

$\mathbb{R}$.

You saw in Unit 1 that you can visualise the real numbers as points on an infinitely long straight line, called the **number line** or the **real line**. Part of the number line is shown in Figure 3. Although only the integers are marked in the diagram, every point on the line represents a real number.

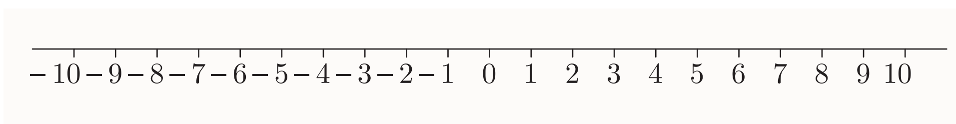


Figure 3 The number line

You can use the number line to visualise sets of real numbers. For example:

- Figure 4(a) shows the set $\{-1, 0, 1\}$.
- Figure 4(b) shows the set of real numbers that are greater than or equal to 2 and also less than or equal to 6.
- Figure 4(c) shows the set of real numbers that are greater than -5.
- Figure 4(d) shows the set of real numbers that are less than $\frac{1}{2}$ or greater than or equal to 3.

In these kinds of diagrams, a solid dot indicates a number that's included in the set, and a hollow dot indicates a number that isn't included. A heavy line that continues to the left or right end of the diagram indicates that the set extends indefinitely in that direction.

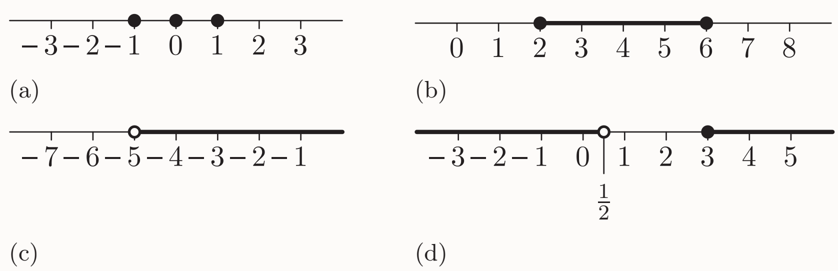


Figure 4 Sets of real numbers

The sets in Figure 4(b) and (c) are examples of a special type of set of real numbers, called an *interval*. An **interval** is a set of real numbers that corresponds to a part of the number line that you can draw 'without lifting your pen from the paper'. The sets in Figure 4(a) and (d) aren't intervals, as they have 'gaps' in them. In fact, the set in Figure 4(a) is the union of three intervals (each containing a single number), and the set in Figure 4(d) is the union of two intervals.

A number that lies at an end of an interval is called an **endpoint** of the interval. For example, the interval in Figure 4(b) has two endpoints, namely 2 and 6, and the interval in Figure 4(c) has one endpoint, namely -5. The whole set of real numbers, $\mathbb{R}$, is an interval with no endpoints.

An interval that includes all of its endpoints is said to be **closed**, and one that doesn't include any of its endpoints is said to be **open**. For example, the interval in Figure 4(b) is closed (since it includes both its endpoints), and the one in Figure 4(c) is open (since it excludes its single endpoint). An interval that includes one endpoint and excludes another, such as the interval in Figure 5, is said to be **half-open** (or **half-closed**). Since the interval $\mathbb{R}$ has no endpoints, it's both open and closed! This fact may seem strange at the moment, but it will make more sense if you go on to study pure mathematics at higher levels.

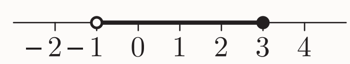
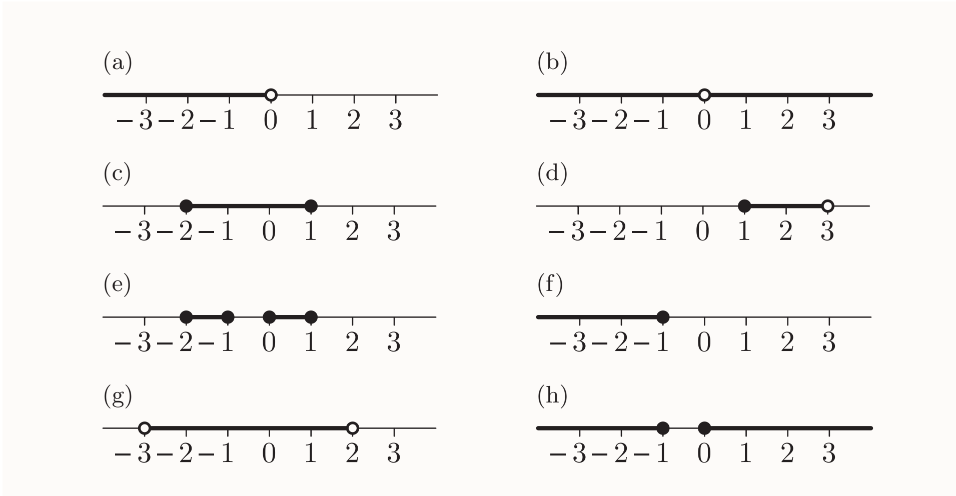


Figure 5 A half-open (or half-closed) interval

Activity 3 *Recognising intervals*

State whether each of the sets below is an interval. For each set that is an interval, state whether it's open, closed or half-open.



A convenient way to describe most intervals is to use **inequality signs**. These are listed below, with their meanings. (Note that some texts use slightly different inequality signs: $\leqslant$ and $\geqslant$ instead of $\leq$ and $\geq$.)

Inequality signs

$<$ is less than

$\leq$ is less than or equal to

$>$ is greater than

$\geq$ is greater than or equal to

For example, the interval in Figure 6(a) is the set of real numbers x such that $x > 2$ (that is, such that x is greater than 2).

Similarly, the interval in Figure 6(b) is the set of real numbers x such that $x > 1$ and $x \leq 4$ (that is, such that x is greater than 1 and x is less than or equal to 4). We usually write this description slightly more concisely, as follows: the interval is the set of real numbers x such that $1 < x \leq 4$ (that is, such that 1 is less than x, which is less than or equal to 4).

Figure 6 Intervals

It might help you to remember the meanings of the inequality signs if you notice that when you use either of the signs $<$ or $>$, the lesser quantity is on the smaller, pointed side of the sign. The same is true for the signs $\leq$ and $\geq$, except that one quantity is less than or equal to the other, rather than definitely less than it.

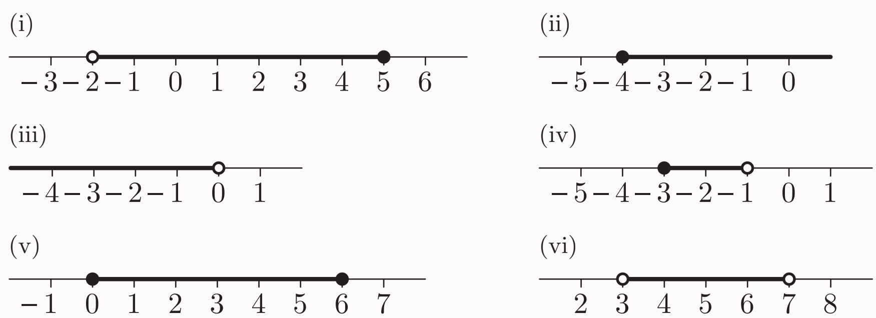
The statement '$x > 2$' is called an *inequality*. In general, an **inequality** is a mathematical statement that consists of two expressions with an inequality sign between them. A statement such as '$1 < x \leq 4$' is called a **double inequality**. The two inequality signs in a double inequality always point in the same direction as each other.

Activity 4 *Using inequality signs to describe intervals*

(a) Draw diagrams similar to those in Figure 6 to illustrate the intervals described by the following inequalities and double inequalities.

(i) $0 < x < 1$ (ii) $-3 \leq x < 2$ (iii) $x \leq 5$ (iv) $x > 4$

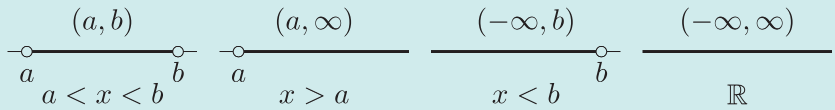
(b) For each of the following diagrams, write down an inequality or double inequality that describes the interval illustrated.



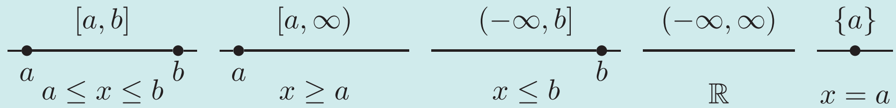
Another useful way to describe intervals is to use **interval notation**. For example, the interval described by the double inequality $4 \leq x < 7$ is denoted in interval notation by $[4, 7)$. The square bracket indicates an included endpoint, and the round bracket indicates an excluded one. An interval that extends indefinitely is denoted by using the symbol ∞ (which is read as 'infinity'), or its 'negative', $-\infty$ (which is read as 'minus infinity'), in place of an endpoint. For example, the interval described by the inequality $x \geq 5$ is denoted by $[5, \infty)$, and the interval described by the inequality $x < 6$ is denoted by $(-\infty, 6)$. We always use a round bracket next to ∞ or $-\infty$ in interval notation. Here's a summary of the notation.

Interval notation

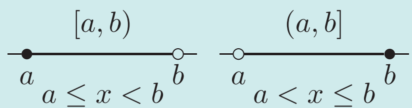
Open intervals



Closed intervals



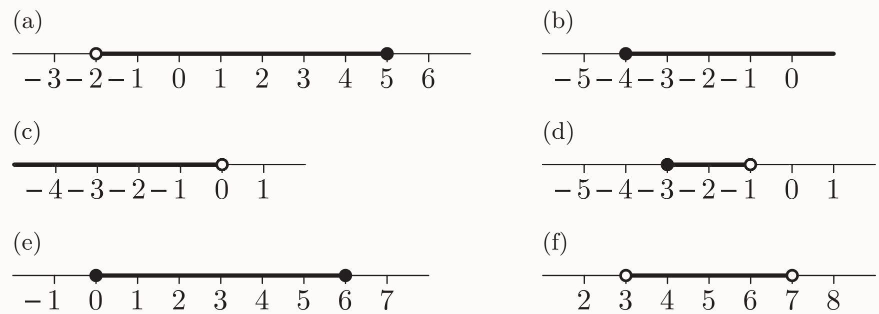
Half-open (or half-closed) intervals



Notice that you've now seen two different meanings for the notation (a, b), where a and b are real numbers. It can mean either an open interval, or a point in the coordinate plane. The meaning is usually clear from the context.

Activity 5 *Using interval notation*

Write each of the intervals below in interval notation.



Sometimes you need to work with sets of real numbers that are *unions* of intervals, like those in Figure 7.

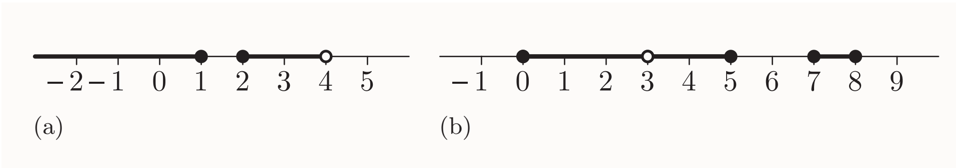


Figure 7 Two unions of intervals

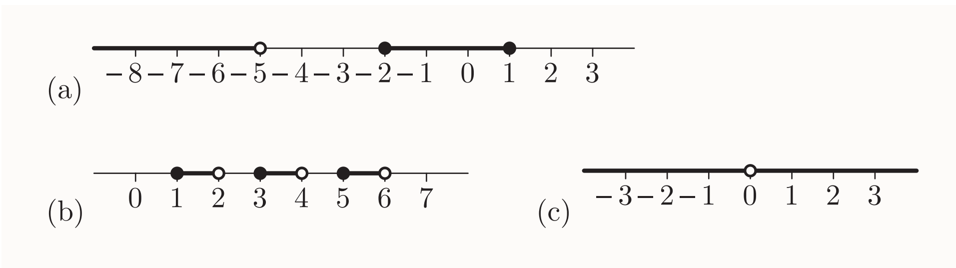
You can denote a union of intervals in interval notation by using the usual notation for intervals together with the union symbol $\cup$. For example, the sets in Figure 7 can be written as

$$(-\infty, 1] \cup [2, 4) \quad \text{and} \quad [0, 3) \cup (3, 5] \cup [7, 8],$$

respectively.

Activity 6 *Denoting unions of intervals*

For each of the following diagrams, write the set illustrated in interval notation.



It's often useful to state that a particular number lies in, or doesn't lie in, a particular interval or union of intervals. You can do this concisely using the symbols $\in$ and $\notin$ in the usual way. For example, as illustrated in Figure 8,

$$1 \in [0, 4] \quad \text{and} \quad -1 \notin [0, 4].$$

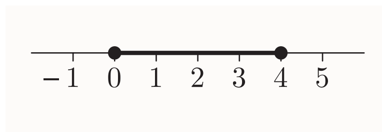


Figure 8 The interval $[0, 4]$

In the next subsection you'll begin your study of *functions*.

1.2 What is a function?

As mentioned in the introduction to this unit, whenever one quantity depends on another, we say that the first quantity is a **function** of the second quantity. Here are some more examples.

- If a car is driving along a straight road, then its displacement s (in km) from some reference point depends on the time t (in hours) that has elapsed since the start of its journey. So s is a function of t.

- The formula

  $$C = 2\pi r$$

  expresses the circumference C of a circle in terms of its radius r (with both C and r measured in the same units). So the value of C depends on the value of r, and hence C is a function of r.

- The electrical voltage between two points on a person's skin either side of his or her heart (which can be measured using electrodes) changes rhythmically with every heartbeat. So the voltage V (in volts, say) depends on the time t (in seconds, say) that has elapsed since some point in time, and hence V is a function of t. There's no simple formula for the relationship between t and V, but it's often displayed as an *electrocardiogram* (ECG), like the one in Figure 9.

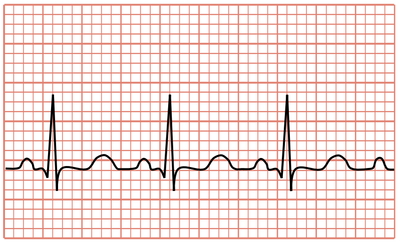


Figure 9 An electrocardiogram (each high peak in voltage corresponds to a heartbeat)

In each of these examples, there's a rule that converts each value of one variable (such as t, in the car example) to a value of the other variable (such as s, in the car example). You can think of the rule as a kind of processor that takes input values and produces output values, as illustrated in Figure 10.

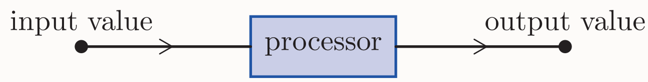


Figure 10 A processor that takes input values and produces output values

For instance, in the car example, an input value of 1.2 (a time, in hours) might be converted by the processor to an output value of 60 (a displacement, in kilometres). Similarly, in the circle example, an input value of 3 (a radius, in centimetres) would be converted by the processor to an output value of $2\pi \times 3 = 6\pi$ (a circumference, in centimetres). Sometimes the rule associated with a function can be expressed using a formula, and sometimes it can't.

In each of the three examples in the list above, there's also a set of allowed input values, and a set of values within which every output value lies. For instance, with the car example, if the journey lasts three hours, then the allowed input values are the real numbers between 0 and 3 inclusive (the possible elapsed times, in hours), and the output values lie in the set $\mathbb{R}$ of real numbers (they are displacements of the car from the reference point, in kilometres).

A *function* is a mathematical object that describes a situation like those listed above. It's defined as follows.

A **function** consists of:

- a set of allowed input values, called the **domain** of the function
- a set of values in which every output value lies, called the **codomain** of the function
- a process, called the **rule** of the function, for converting each input value into *exactly one* output value.

It's often useful to denote a function by a letter. If a function is denoted by f, say, then for any input value x, the corresponding output value is denoted by $f(x)$, which is read as 'f of x'.

For example, suppose that we denote the function associated with the car example by f. If the rule of this function converts the input value 1.2 to the output value 60, then we write

$$f(1.2) = 60.$$

Similarly, suppose that we denote the function associated with the circle example by g. The rule of this function converts the input value 3 to the output value 6π, so we write

$$g(3) = 6\pi.$$

This type of notation is known as **function notation**.

One use of function notation is for specifying the rule of a function, when this can be done using a formula. For example, suppose that h is the function whose domain and codomain each consist of all the real numbers, and whose rule is 'square the input number'. Then, for example,

$$h(2) = 4, \quad h(5) = 25 \quad \text{and} \quad h(-1) = 1,$$

and the rule of h can be written as

$$h(x) = x^2.$$

Similarly, the rule of the function associated with the circle example can be written as $g(r) = 2\pi r$.

When you write down the rule of a function, it doesn't matter what letter you use to represent the input value. So the rule of the function h above could also be written as, for example,

$$h(t) = t^2 \quad \text{or} \quad h(u) = u^2.$$

The variable used to denote the input value of a function is sometimes called the **input variable**.

It's traditional to use the letters f, g and h for functions, and the letters x, t and u for input variables. Although you can use any letters, these ones are often used in general discussions about functions. The most standard letters are f for a function and x for an input variable.

Activity 7 *Understanding function notation*

(a) Suppose that f is the function whose domain and codomain each consist of all the real numbers, and whose rule is $f(t) = 4t$. Write down the values of $f(5)$ and $f(-3)$.

(b) Suppose that g is the function whose domain and codomain each consist of all the real numbers, and whose rule can be written in words as 'multiply the input number by 2 and then subtract 1'. Write down the rule of g using the notation $g(x)$.

It's important to appreciate that *every* value in the domain of a function must have a corresponding output value, given by the rule of the function. So, for example, a function f that has the rule

$$f(x) = \sqrt{x}$$

can't have any negative numbers in its domain, because if x is negative, then $\sqrt{x}$ isn't defined.

As another example, if a function f describes how the displacement in kilometres of a car from a particular point depends on the time in hours since it started a 3-hour journey, then we'd normally take the domain of f to be the interval $[0, 3]$.

In contrast, not every value in the codomain of a function actually has to occur as an output value. For instance, with the car example, we'd normally take the codomain to be the whole set of real numbers, $\mathbb{R}$. It's good enough that this set contains every possible output value: it doesn't matter that it also contains many values that couldn't be output values.

The set of values in the codomain of a function that *do* occur as output values is called the **image set** of the function. For example, if f is the function whose domain and codomain are each the whole set of real numbers, $\mathbb{R}$, and whose rule is $f(x) = x^2$, then the image set of f is the interval $[0, \infty)$.

Here's another fact about functions that it's important to appreciate. Not only must every value in the domain of a function have a corresponding output value, given by the rule of the function, but it must have *exactly one* output value. For example, a function f can't have the rule $f(x) = \pm\sqrt{x}$, because this rule assigns *two* output values to every input value (except zero).

You can visualise the facts about functions described above by using a type of diagram known as a **mapping diagram**, which is based on Venn diagrams. (The word **mapping** is another name for *function*.) For example, the mapping diagram in Figure 11 illustrates the function f that has domain $\{1, 2, 3\}$, codomain $\{2, 4, 6, 8, 10\}$ and rule $f(x) = 2x$. The arrows indicate which input value goes to which output value. Notice that *exactly one* arrow comes out of each input value. This corresponds to the fact that each input value has exactly one output value. Notice also that the image set consists of all the values that have arrows going in to them, and that (for this particular function f) the codomain contains other values too.

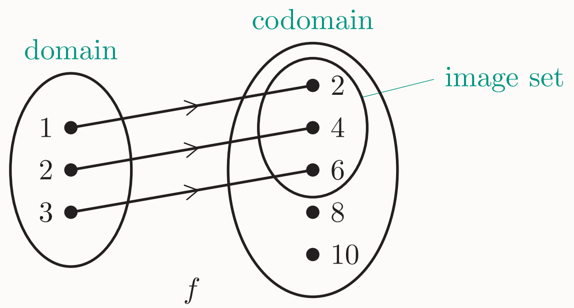


Figure 11 A function f illustrated by a mapping diagram

Here's some more terminology associated with functions. If f is a function, and x is any value in its domain, then the value $f(x)$ is called the **image of x under f**, or the **value of f at x**. This is illustrated in Figure 12. We also say that f **maps** x to $f(x)$.

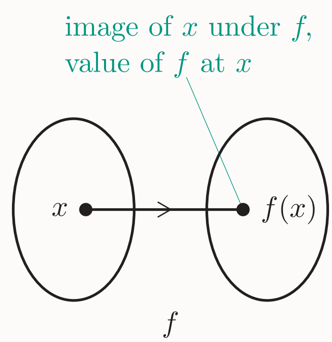


Figure 12 The image of a value x under a function f

For example, $f(2) = 4$ for the function f in Figure 11 above, so we can say that the image of 2 under f is 4, or f takes the value 4 at 2, or f maps 2 to 4.

Activity 8 *Understanding function terminology*

Suppose that f is the function whose domain and codomain each consist of all the real numbers, and whose rule is $f(t) = 4t$. Write down the following numbers.

(a) The image of 2 under f

(b) The image of -1 under f

(c) The value of f at 0.5

(d) The value of f at -0.2

(e) The number whose image under f is 44

(f) The number whose image under f is 1

(g) The number to which f maps 4

(h) The number that is mapped by f to -8

In this module you'll be working only with functions whose domains and codomains are sets of real numbers. Such functions are called **real functions**. You can also have other types of functions, such as a function whose domain and/or codomain is a set of another type of numbers (complex numbers, for instance), or a set of points in the plane. You'll meet many more types of functions if you go on to study mathematics beyond this module.

Since we'll be working only with real functions in this module, we'll make some simplifying assumptions.

In this module:

- we use the word 'function' to mean 'real function'
- we take the codomain of every function to be the whole set of real numbers, since this set contains every possible output value.

These assumptions allow you to specify any function by stating its domain and its rule. It's important to remember that to specify a function, *a domain must be stated*, as well as a rule. Two functions with the same rule but different domains are different functions.

Leonhard Euler (1707–83)

The concept of a function was first formally defined by the Swiss mathematician Johann Bernoulli (1667–1748) in 1718. But the mathematician who gave prominence to the concept, and who was responsible for the notation $f(x)$, was Bernoulli's compatriot Leonhard Euler. Euler was one of the most talented and productive mathematicians of all time. He became blind in the early 1770s but his output, rather than stopping, actually increased. His work covers almost every area of mathematics, and his collected works run to over 70 volumes, with further volumes still to appear.

1.3 Specifying functions

You've seen that to specify a function you have to state its domain and its rule. There are various ways to state the domain and rule of a function. Here's the format that we'll usually use in this module. For example, to specify the function f whose rule is $f(x) = x^2 + 1$ and whose domain is the interval consisting of the real numbers between 0 and 6, inclusive, we'll write either

$$f(x) = x^2 + 1 \quad (0 \leq x \leq 6)$$

or

$$f(x) = x^2 + 1 \quad (x \in [0, 6]).$$

We'll be even more concise when we want to specify a function whose domain is the largest possible set of real numbers for which its rule is applicable. For example, the function

$$g(x) = \sqrt{x} \quad (x \in [0, \infty))$$

is such a function: its domain is as large as it can be, because $\sqrt{x}$ is defined only for non-negative values of x. We'll usually specify a function like this by stating just its rule. This is because of the following convention, which is widely used in mathematics.

Domain convention

When a function is specified by *just a rule*, it's understood that the domain of the function is the largest possible set of real numbers for which the rule is applicable.

For example, if you read 'the function $h(x) = 1/x$', and no domain is stated, then you can assume that the domain of h is the set of all real numbers except 0.

Notice that we say, for example, 'the function $h(x) = 1/x$', when we really mean 'the function h with rule $h(x) = 1/x$'. This is another convenient convention, which is used throughout this module and throughout mathematics in general.

Activity 9 *Using the domain convention*

Describe the domain of each of the following functions, both in words and using interval notation.

(a) $f(x) = \dfrac{1}{x-4}$ (b) $g(x) = \dfrac{1}{(x-2)(x+3)}$ (c) $h(x) = \sqrt{x-1}$

Another situation where we sometimes specify a function by giving just a rule, rather than a rule and a domain, is where the domain is clear from the context. For example, if the function f is such that $f(t)$ is the displacement in kilometres of a car at time t (in hours) after it began a 3-hour journey, then we assume that the domain of f is the interval $[0, 3]$, since in this context these are the values that t can take.

Functions specified by equations for one variable in terms of another

Functions don't have to be specified using function notation. Sometimes it's convenient to express a function using an equation that expresses one variable in terms of another variable. For example, as mentioned earlier, the circumference C of a circle is given in terms of its radius r by the formula

$$C = 2\pi r. \tag{1}$$

Here C is a function of r, and, as you've seen, we can write this function as

$$g(r) = 2\pi r \quad (r > 0).$$

But there's no need to use function notation: equation (1) is a perfectly good specification of the rule of the function.

In general, any equation that expresses one variable in terms of another variable specifies the rule of a function. If we wish to specify a domain that's not the largest possible set of real numbers for which the equation is applicable, then we can do so in the usual way. For example, we can write

$$C = 2\pi r \quad (r > 0).$$

When a function is specified using an equation that expresses one variable in terms of another variable, the output variable is called the **dependent variable**, because its value depends on the value of the input variable. The input variable is called the **independent variable**. For example, for the function discussed above whose rule is $C = 2\pi r$, the dependent variable is C and the independent variable is r.

We often refer to an equation that specifies a function *as* a function. For example, we might say 'the function $y = x^2 + 1$', when we really mean 'the function specified by the rule $y = x^2 + 1$'. This is another convenient convention.

Both types of notation for functions – function notation and equations relating input and output variables – are used throughout this module.

Piecewise-defined functions

Sometimes it's useful to specify the rule of a function by using different formulas for different parts of its domain. For example, you can specify a function f as follows:

$$f(x) = \begin{cases} x^2 & (x \geq 0) \\ x + 5 & (x < 0). \end{cases}$$

To find the image of a number x under this function f, you use the rule $f(x) = x^2$ if x is greater than or equal to zero, and the rule $f(x) = x + 5$ if x is less than zero. For example,

$$f(2) = 2^2 = 4 \quad \text{and} \quad f(-2) = -2 + 5 = 3.$$

A function defined in this way is called a **piecewise-defined function**.

Such piecewise-defined functions can be used to construct curves with a great variety of shapes, so they are used extensively in the design of objects such as car bodies and roads.

1.4 Graphs of functions

A convenient way to visualise many of the properties of a function is to draw or plot its *graph*. The **graph** of a function f is the graph of the equation $y = f(x)$, for all the values of x that are in the domain of f. In other words, it's the set of points (x, y) in the coordinate plane such that x is in the domain of f and $y = f(x)$.

For example, the graph of the function $f(x) = x^2 + 1$ is the graph of the equation $y = x^2 + 1$, which is shown in Figure 13(a).

Similarly, the graph of the function $g(x) = x^2 + 1$ $(0 < x \le 2)$ is the graph of the equation $y = x^2 + 1$ for values of x in the interval $(0, 2]$, which is shown in Figure 13(b).

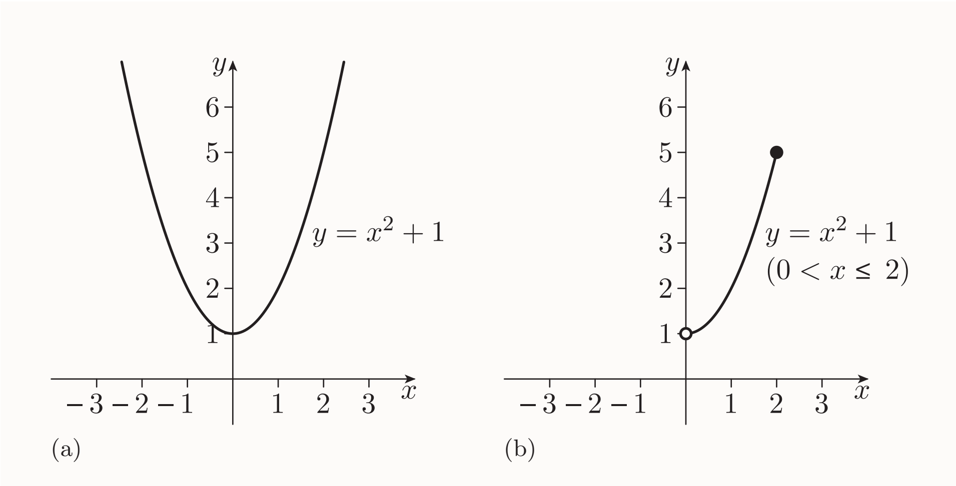


Figure 13 The graphs of (a) $f(x) = x^2 + 1$ (b) $f(x) = x^2 + 1$ $(x \in (0, 2])$

Notice that when we draw graphs we use similar conventions to those that we use for illustrations of sets on the number line. For example, we use solid and hollow dots to indicate whether points at the 'ends' of a graph do or don't lie on the graph. In Figure 13(b) the graph is labelled with its rule and also with its domain, but we often omit the latter.

A function whose rule you can't express using a formula still has a graph. For example, Figure 14(a) shows the graph of a function f that describes the displacement of a car along a road from its starting point during a 3-hour journey. Similarly, the electrocardiogram that you saw earlier, which is repeated in Figure 14(b), is the graph of a function (with the axes omitted). This graph represents the changing voltage between two points on a person's skin over a time period of about two seconds.

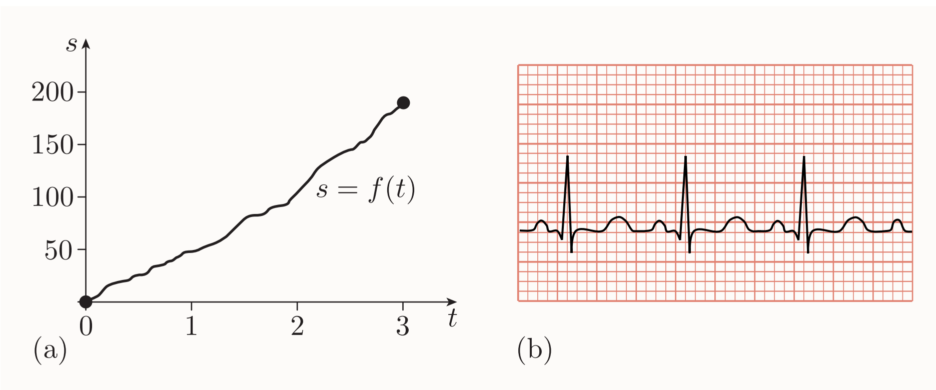


Figure 14 (a) The graph of $s = f(t)$, where $f(t)$ is the displacement of a car in kilometres at time t (in hours) (b) an electrocardiogram

The graph of a function is normally drawn with the input numbers on the horizontal axis and the output numbers on the vertical axis. (So, if the axes are labelled with variables, then the variable on the horizontal axis is the independent variable, and the variable on the vertical axis is the dependent variable.) In this module we'll assume that graphs of functions are always drawn like this.

You can 'read off' the output number corresponding to any particular input number by drawing a vertical line from the input number on the horizontal axis to the graph and then a horizontal line across to the vertical axis. For example, for the function f whose graph is shown in Figure 15, the value of $f(3)$ is about 5.

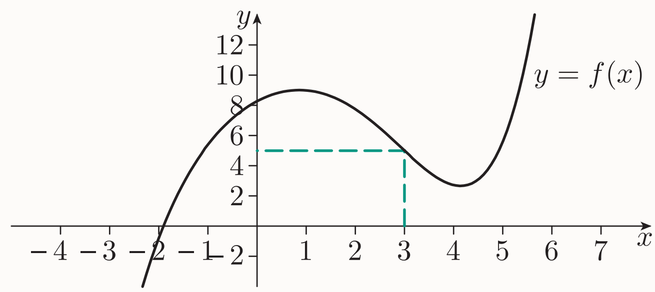


Figure 15 The graph of a function f

One way to produce a graph of a function is to use a table of values, in the way that you saw in Unit 2. You're asked to do this in the next activity.

Activity 10 *Plotting the graph of a function using a table of values*

Consider the function $f(x) = x^3$.

(a) Use your calculator to complete the following table of values for this function.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^3									

(b) Plot the points given by your completed table on a pair of axes.

(c) Draw a smooth curve through the points.

If you use a table of values to plot the graph of a function whose domain isn't the whole set of real numbers, remember to choose input values that lie in the domain, and to make sure that you don't extend the graph beyond the endpoints of the domain. Where appropriate, you should mark the ends of the graph with solid or hollow dots.

A quicker way to obtain a graph of a function is to use a computer. You can learn how to do that in the next activity.

Activity 11 *Plotting graphs of functions on the computer*

Work through Section 4 of the *Computer algebra guide*.

As mentioned in Unit 2, one disadvantage of using a table of values to plot a graph is that you can't be entirely sure about the shape of the graph between the values in the table, or to the right or left of them. A graph produced by a computer has similar disadvantages, as in essence it's plotted using a large table of values. In general, it's useful to become familiar with the shapes of the graphs of a variety of functions and function types, and to learn how to sketch such graphs. You'll have opportunities to do that throughout this module.

You saw in Unit 2 how to sketch the graphs of equations of the form $y = ax + b$ and $y = ax^2 + bx + c$, where a, b and c are constants. So you already know how to sketch the graphs of functions of the form $f(x) = ax + b$ and $f(x) = ax^2 + bx + c$. The next example illustrates how to adapt the methods in Unit 2 in order to sketch the graph of a function whose rule has one of these forms, but whose domain isn't the largest possible set of real numbers for which the rule is applicable.

Example 1 *Sketching the graph of a function whose domain is not the largest set of numbers for which its rule is applicable*

Sketch the graph of the function

$$f(x) = \tfrac{1}{4}x^2 - 2x + 6 \quad (5 \le x < 7).$$

Solution

First sketch the graph of $y = \frac{1}{4}x^2 - 2x + 6$, by using any of the methods from Unit 2. Also include on the sketch the points corresponding to the endpoints of the domain of f, plotted as solid or hollow dots as appropriate, and labelled with their coordinates.

The required graph is part of the graph of $y = \frac{1}{4}x^2 - 2x + 6$, which is a u-shaped parabola. Completing the square gives

$$\begin{aligned} f(x) &= \tfrac{1}{4}x^2 - 2x + 6 \\ &= \tfrac{1}{4}(x^2 - 8x) + 6 \\ &= \tfrac{1}{4}((x-4)^2 - 16) + 6 \\ &= \tfrac{1}{4}(x-4)^2 - 4 + 6 \\ &= \tfrac{1}{4}(x-4)^2 + 2. \end{aligned}$$

The least value taken by $(x-4)^2$ is 0, so the least value taken by $(x-4)^2 + 2$ is 2. This occurs when $x - 4 = 0$, that is, when $x = 4$.

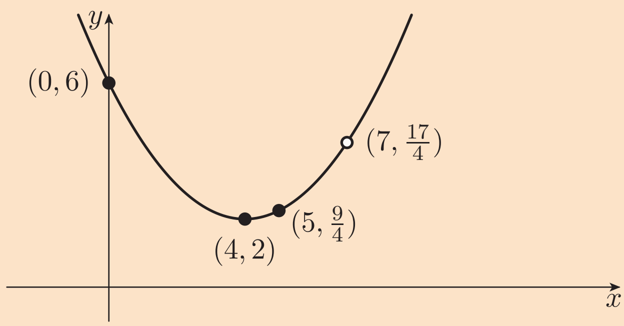
So the parabola has vertex $(4, 2)$. Also, since the expression $\frac{1}{4}(x-4)^2 + 2$ is always positive, the parabola has no x-intercepts. Its y-intercept is 6.

The endpoints of the domain of f are 5 and 7. We have

$$f(5) = \tfrac{1}{4} \times 5^2 - 2 \times 5 + 6 = \tfrac{9}{4} \quad \text{and}$$
$$f(7) = \tfrac{1}{4} \times 7^2 - 2 \times 7 + 6 = \tfrac{17}{4}.$$

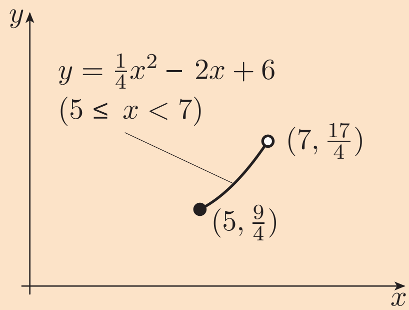
So the points $(5, \frac{9}{4})$ and $(7, \frac{17}{4})$ lie on the graph of $y = \frac{1}{4}x^2 - 2x + 6$.

These features give the following graph.



Erase the parts of the graph that don't lie between the points $(5, \frac{9}{4})$ and $(7, \frac{17}{4})$ (or draw a new graph).

So the graph of f is as follows.



With a little practice, you should be able to sketch the graph of a function like the one in Example 1 without having to sketch a larger graph first. It's straightforward to do this for a simple graph, such as a straight line.

Activity 12 *Sketching the graphs of functions whose domains are not the largest sets of numbers for which their rules are applicable*

Sketch the graphs of the following functions.

(a) $f(x) = 3 - 2x \quad (-1 < x < 4)$

(b) $f(x) = -\frac{1}{2}x^2 - 2x - 5 \quad (x \geq -5)$

You can use the graph of a function to visualise its domain on the horizontal axis. The domain consists of all the possible input numbers of the function, that is, all points on the horizontal axis that lie directly below or above a point on the graph, as illustrated in Figure 16.

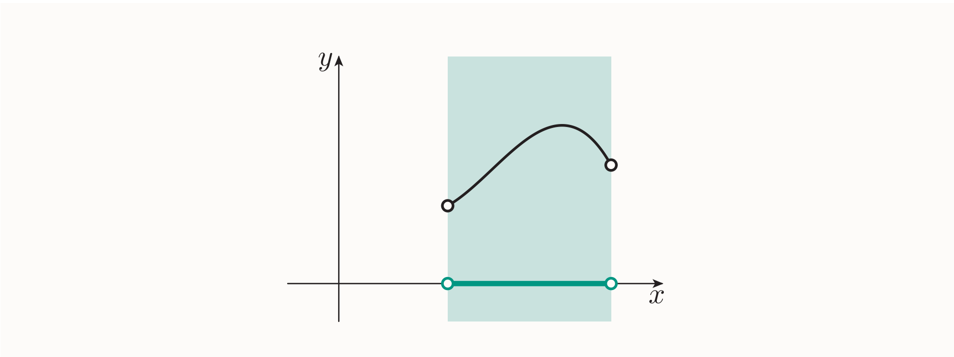
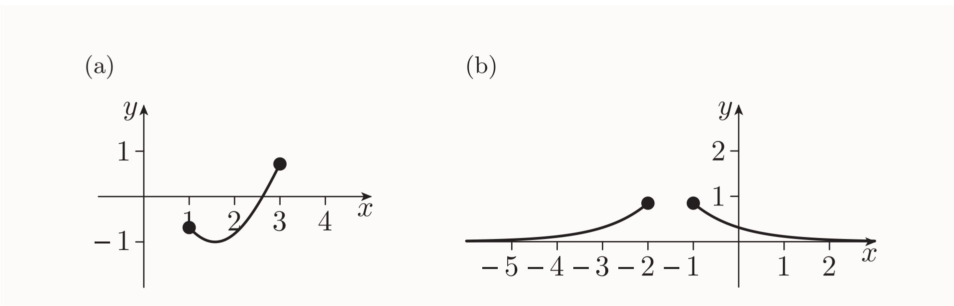


Figure 16 The domain of a function marked on the horizontal axis

Activity 13 *Identifying the domains of functions*

Write down the domains of the functions whose graphs are shown below, using interval notation. All the endpoints of the intervals involved are integers, and in part (b) the graph continues indefinitely to the left and right.



As you've seen, a function has exactly one output number for every input number. So if you draw the vertical line through any number in the domain of a function on the horizontal axis, then it will cross the graph of the function *exactly once*, as illustrated in Figure 17(a). If you can draw a vertical line that crosses a curve more than once, then the curve isn't the graph of a function. For example, the curve in Figure 17(b) isn't the graph of a function.

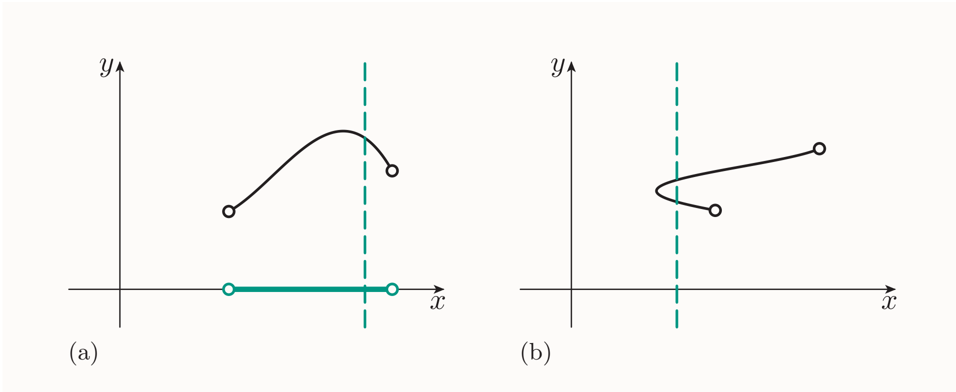
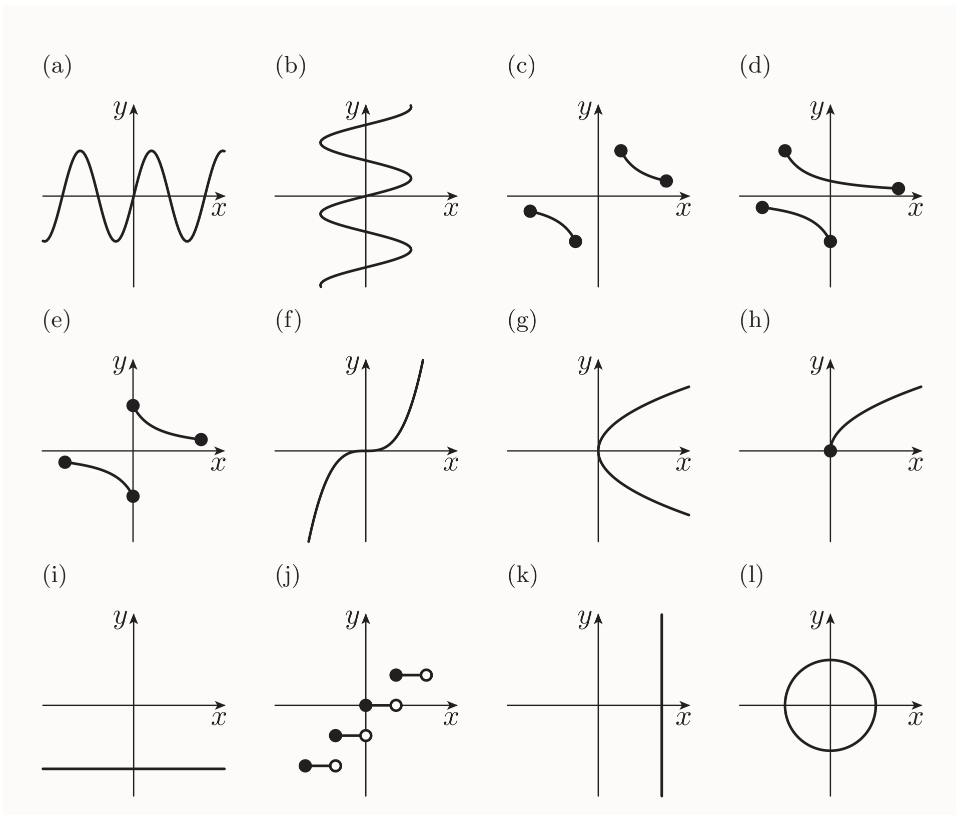


Figure 17 (a) The graph of a function (b) a curve that isn't the graph of a function

Activity 14 *Identifying graphs of functions*

Which of the following diagrams are the graphs of functions?



Increasing and decreasing functions

Figure 18 shows the graph of a function with domain $[-1, 9]$. As x increases, the graph first slopes up, then slopes down, then slopes up again. It changes from sloping up to sloping down when $x = 2$, and it changes from sloping down to sloping up again when $x = 6$.

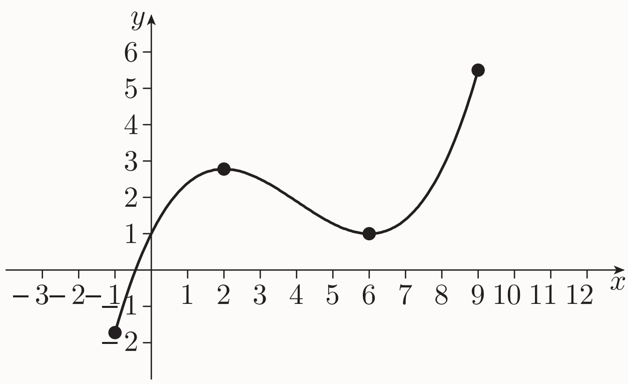


Figure 18 The graph of a function

To express these facts about the function f, we say that f is *increasing on the interval* $[-1, 2]$, *decreasing on the interval* $[2, 6]$, and increasing again on the interval $[6, 9]$. Here are the formal definitions of these terms. The definitions are illustrated in Figure 19.

Functions increasing or decreasing on an interval

A function f is **increasing on the interval** I if for all values x_1 and x_2 in I such that $x_1 < x_2$,

$$f(x_1) < f(x_2).$$

A function f is **decreasing on the interval** I if for all values x_1 and x_2 in I such that $x_1 < x_2$,

$$f(x_1) > f(x_2).$$

(The interval I must be part of the domain of f.)

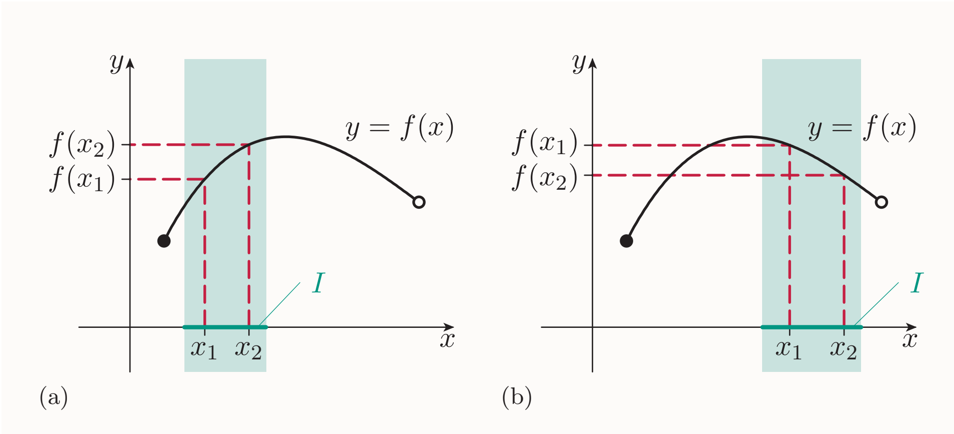


Figure 19 (a) A function increasing on an interval I (b) a function decreasing on an interval I

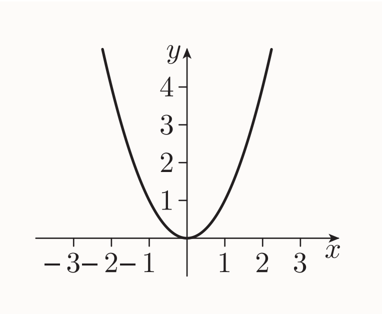
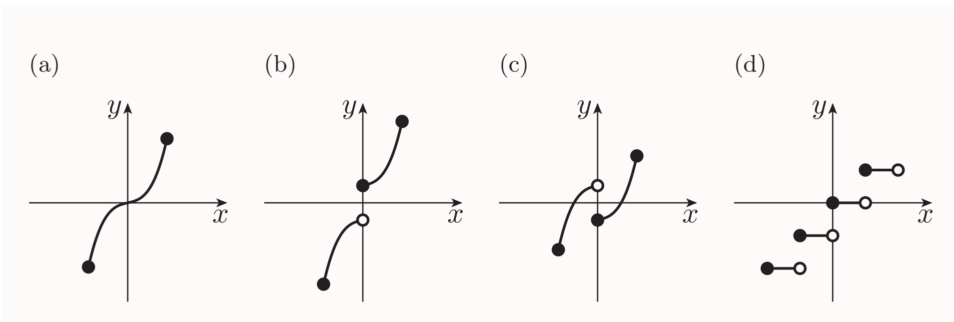


Figure 20 The graph of the function $f(x) = x^2$

For example, the function $f(x) = x^2$, whose graph is shown in Figure 20, is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

Activity 15 *Identifying increasing functions*

Which of the following graphs show functions that are increasing on their whole domains?



1.5 Image sets of functions

Remember that the *image set* of a function is the set consisting of all the values in its codomain that occur as output numbers. For example, the image set of the function $f(x) = x^2$ is the interval $[0, \infty)$, because all non-negative numbers occur as output numbers of this function, but no negative numbers do. You saw in the last subsection that if you have a graph of a function, then you can visualise the *domain* of the function on the horizontal axis, as illustrated in Figure 21(a). In the same way, you can visualise the *image set* of the function on the vertical axis.

The image set consists of all the possible output numbers, that is, all the points on the vertical axis that lie directly to the right or left of a point on the graph, as illustrated in Figure 21(b).

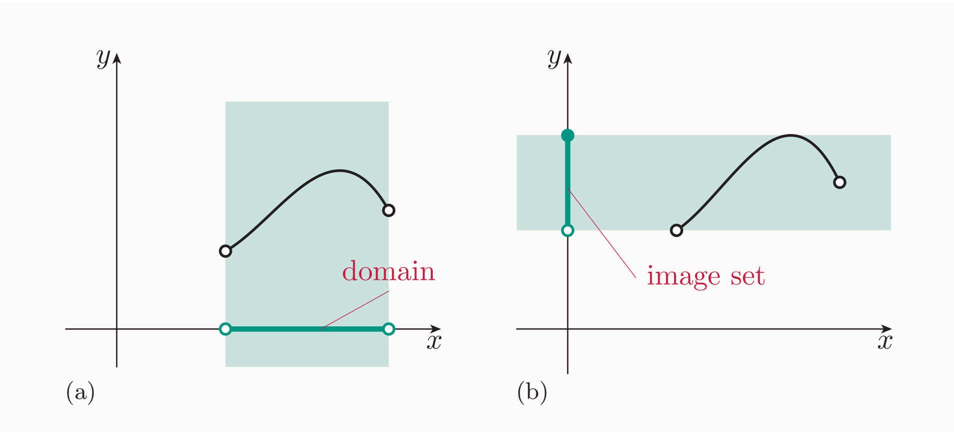


Figure 21 (a) The domain of a function marked on the horizontal axis (b) the image set marked on the vertical axis

So you can use the graph of a function to help you find its image set, as demonstrated in the next example. Remember that a 'play button' icon in the margin next to a worked example indicates that a tutorial clip is available for the example.

Example 2 *Finding the image set of a function*

Find the image set of the function

$$f(x) = x^2 + 6x + 14 \quad (-6 < x < 2).$$

Solution

Obtain a sketch, plot or computer plot of the graph of the function. Remember to 'stop' the graph at the endpoints of its domain, and to mark the resulting ends of the graph with solid or hollow dots, as appropriate. There's no need to find the intercept(s).

The parabola is u-shaped. Completing the square gives

$$\begin{aligned} f(x) &= x^2 + 6x + 14 \\ &= (x+3)^2 - 9 + 14 \\ &= (x+3)^2 + 5. \end{aligned}$$

The least value taken by $(x+3)^2$ is 0, so the least value taken by $(x+3)^2 + 5$ is 5. This occurs when $x + 3 = 0$, that is, when $x = -3$.

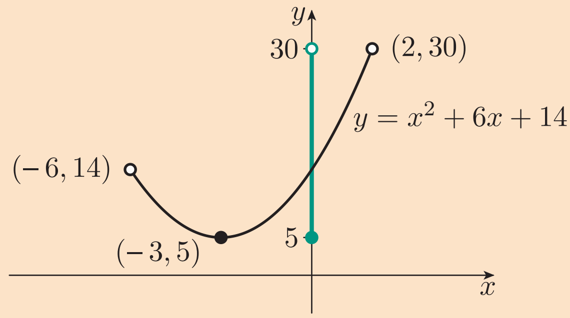
So the vertex is $(-3, 5)$.

Also

$$f(-6) = (-6)^2 + 6 \times (-6) + 14 = 14 \quad \text{and}$$
$$f(2) = 2^2 + 6 \times 2 + 14 = 30.$$

So the graph stops at the points $(-6, 14)$ and $(2, 30)$, both of which are excluded.

These features give the following graph.



The graph shows that the smallest value in the image set is the y-coordinate of the vertex, and that the image set contains all the values larger than this number, up to but not including $f(2)$.

The graph shows that the image set of f is $[5, 30)$.

You might have expected that if the domain of a function f is the interval $(-6, 2)$, then its image set is the interval $(f(-6), f(2))$. Example 2 shows that this isn't necessarily true.

Activity 16 *Finding image sets of functions*

Find the image sets of the following functions.

(a) $f(x) = -x^2 + 10x - 24 \quad (3 \leq x < 6)$

(b) $f(x) = 2 - 2x \quad (-2 < x < 0)$

(c) $f(x) = x^2 - 1$

(d) $f(x) = 1/x^2$

Hint for part (d): try to work out the answer by thinking about what output numbers are possible for this function. If you're still not sure, try plotting the graph of the function on a computer.

1.6 Some standard types of functions

As mentioned earlier, it's useful to become familiar with some standard types of functions and their graphs. You'll meet some types of functions in this subsection, and further types later in the unit and in Unit 4. If you study mathematics beyond this module, then you'll meet many more types of functions.

Linear functions

First consider any function whose rule is of the form

$$f(x) = mx + c,$$

where m and c are constants. Its graph is the graph of the equation $y = mx + c$, which, as you saw in Unit 2, is the straight line with gradient m and y-intercept c. For this reason, any function of the form above is called a **linear function**.

Figure 22 shows the graphs of some linear functions.

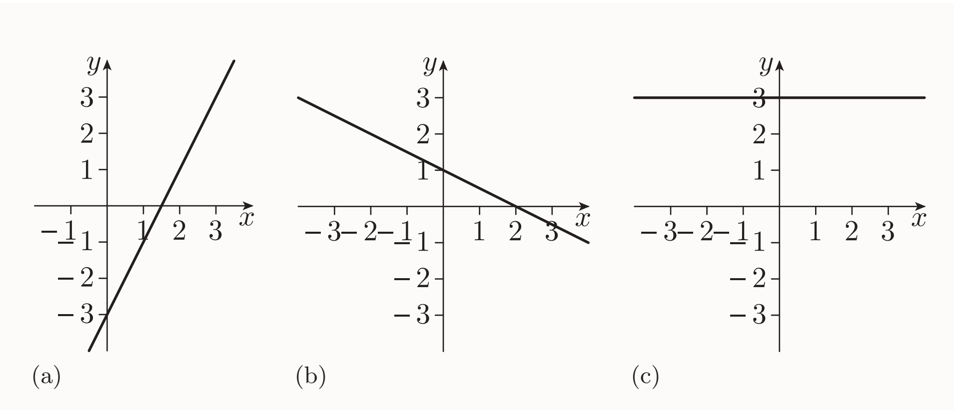


Figure 22 The graphs of the linear functions (a) $f(x) = 2x - 3$ (b) $f(x) = -\frac{1}{2}x + 1$ (c) $f(x) = 3$

A linear function whose rule is of the form

$$f(x) = c,$$

where c is a constant, is called a **constant function**. Its graph is a horizontal line. For example, the function $f(x) = 3$, whose graph is shown in Figure 22(c), is a constant function.

Quadratic functions

From what you saw in Unit 2, you also know that the graph of any function of the form

$$f(x) = ax^2 + bx + c, \qquad (2)$$

where a, b and c are constants with $a \neq 0$, is a parabola. You saw how to find various features of the parabola, such as its vertex and intercepts, from the values of a, b and c. Any function whose rule is of form (2) is called a **quadratic function**. The graphs of some quadratic functions are shown in Figure 23.

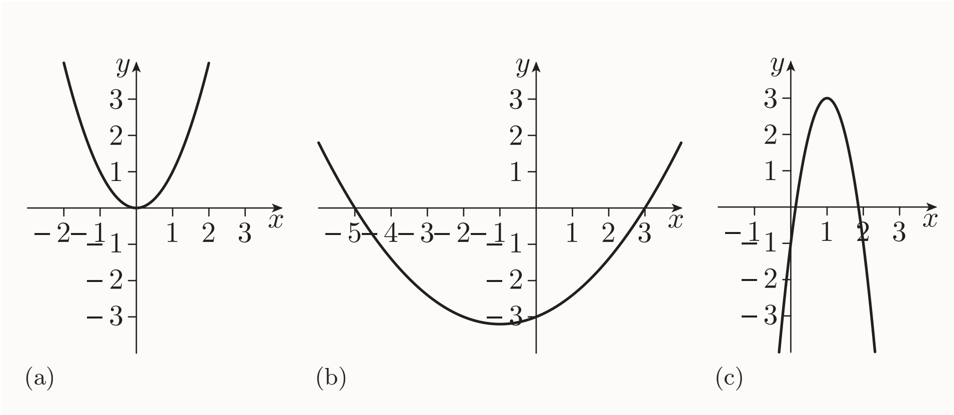


Figure 23 The graphs of the functions (a) $f(x) = x^2$ (b) $f(x) = \frac{1}{5}x^2 + \frac{2}{5}x - 3$ (c) $f(x) = -4x^2 + 8x - 1$

Polynomial functions

Linear functions and quadratic functions are particular types of *polynomial functions*. Here are some more polynomial functions:

$$f(x) = 2x^4 - 5x^3 + x^2 + 2x - 2$$
$$g(x) = x^3$$
$$h(x) = -\tfrac{1}{7}x^7 + \tfrac{1}{3}x^6 + x^5 - \tfrac{5}{2}x^4 - \tfrac{4}{3}x^3 + 4x^2 + 1.$$

Polly, no meal

In general, if an expression is a sum of finitely many terms, each of which is of the form ax^n where a is a number and n is a non-negative integer, then the expression is called a **polynomial expression in** x. If the right-hand side of the rule of a function is a polynomial expression in x, then the function is called a **polynomial function**.

> The word 'polynomial' appears to be a hybrid word meaning 'many names' that is a mixture of Greek and Latin.

Note that the terms of a polynomial expression must all have powers that are non-negative integers; for example, $\sqrt{x}$ (which is the same as $x^{1/2}$) is *not* a polynomial expression.

The highest power of the variable x in a polynomial expression or function is called the **degree** of the polynomial expression or function. For example, the highest power of x in the rule of the polynomial function f above is x^4, so the degree of this polynomial function is 4. Similarly, the polynomial functions g and h above have degrees 3 and 7, respectively.

Quadratic functions are polynomial functions of degree 2, since the highest power of x in the rule of a quadratic function is x^2.

Linear functions are polynomial functions of degree 1, 0 or no degree at all. If a linear function is of the form $f(x) = ax + b$ where $a \neq 0$, then the highest power of x is x^1, so the degree is 1. If it is of the form $f(x) = c$ where $c \neq 0$, then the highest power of x is x^0 (since the function can be expressed as $f(x) = cx^0$), so the degree is 0. The particular linear function $f(x) = 0$ is usually regarded as not having a degree at all (or sometimes as having degree $-\infty$), for technical reasons.

Polynomial functions of degrees 3, 4 and 5 are called **cubic**, **quartic** and **quintic functions**, respectively.

Figure 24 shows the graphs of the three polynomial functions above.

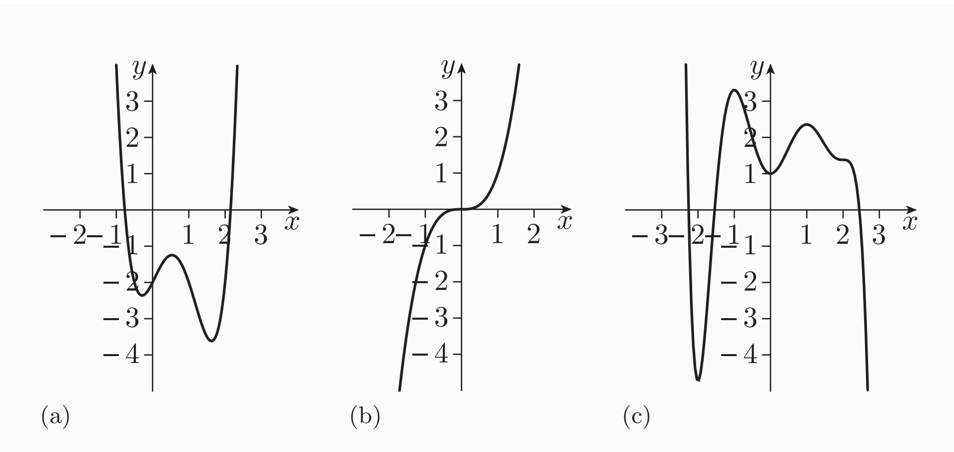
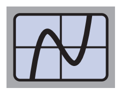


Figure 24 The graphs of the functions
(a) $f(x) = 2x^4 - 5x^3 + x^2 + 2x - 2$ (b) $g(x) = x^3$
(c) $h(x) = -\frac{1}{7}x^7 + \frac{1}{3}x^6 + x^5 - \frac{5}{2}x^4 - \frac{4}{3}x^3 + 4x^2 + 1$

Activity 17 *Investigating graphs of polynomial functions*



Use the module computer algebra system (CAS) to experiment with plotting the graphs of some polynomial functions, to obtain a general idea of the sorts of shapes that they have.

For example, you might like to try plotting the following polynomial functions: $f(x) = -x^3$, $f(x) = x^4$, $f(x) = x^4 - 4x^3$, $f(x) = x^5 - 4x^3$.

In Activity 17 you should have seen evidence of the following. Every polynomial function has a graph that's a smooth, unbroken curve (or a straight line; we normally consider a straight line to be a particular type of curve). The graph often has a 'wiggly' section, but (unless the function is a constant function) if you trace your pen tip along the graph towards the right, then eventually the wiggles stop and your pen either keeps moving 'uphill' or keeps moving 'downhill'. The same happens if you trace your pen tip towards the left.

In fact, the graph of every polynomial function (with domain $\mathbb{R}$) that isn't a constant function **tends to infinity** or **tends to minus infinity** at the left and right. In other words, no matter how large a positive number you choose, as you trace your pen tip along the graph, the y-values of the graph either eventually exceed your chosen number (if the graph tends to infinity) or are eventually less than the negative of your chosen number (if the graph tends to minus infinity).

You can tell whether the graph of a polynomial function tends to infinity or tends to minus infinity at each end by looking at the term in its rule that has the highest power of x. This term is called the **dominant term**, because for large values of x, the value taken by the dominant term 'outweighs' (*dominates*) the sum of the values taken by all the other terms. For example, the dominant term in the rule

$$f(x) = 2x^4 - 5x^3 + x^2 + 2x - 2$$

is $2x^4$. If the dominant term has a plus sign and

- contains an even power of x, then the graph tends to infinity at both ends
- contains an odd power of x, then the graph tends to minus infinity at the left and to infinity at the right.

If the dominant term has a minus sign, then similar facts hold, but with infinity replaced by minus infinity and vice versa.

To see examples of these facts, look at the graphs in Figure 24 above, and at the graphs that you plotted in Activity 17.

The modulus function

All the functions that you've met so far have graphs that are smooth curves. The *modulus function* has a graph that's smooth except at one point, where it turns a corner!

As you saw in Unit 2, the **modulus** of a real number (also known as its **magnitude** or **absolute value**) is its 'distance from zero', or its 'value without its sign'. For example, the modulus of 3 is 3, and the modulus of -3 is also 3. The modulus of a real number x is denoted by $|x|$. So, for example, $|-3| = 3$.

The **modulus function** is

$$f(x) = |x|.$$

It follows from the definition of modulus that

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

So the graph of the modulus function is the same as the graph of $y = x$ when $x \geq 0$, and the same as the graph of $y = -x$ when $x < 0$. It's shown in Figure 25. It has a corner at the origin, and the image set is $[0, \infty)$.

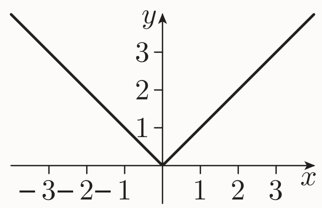


Figure 25 The graph of the modulus function, $f(x) = |x|$

The reciprocal function

Remember that if x is any non-zero number, then the **reciprocal** of x is $1/x$. The **reciprocal function** is the function

$$f(x) = \frac{1}{x}.$$

Its domain consists of all real numbers except 0. That is, its domain is the set $(-\infty, 0) \cup (0, \infty)$. The graph of the reciprocal function is shown in Figure 26.

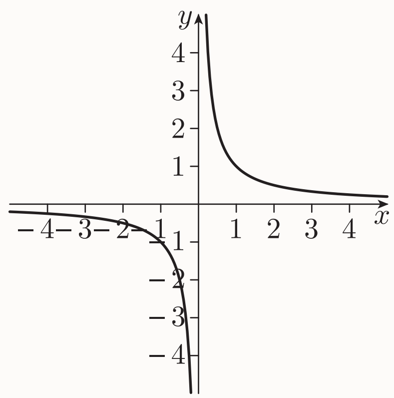


Figure 26 The graph of the reciprocal function, $f(x) = 1/x$

The graph consists of two separate pieces, each of which gets closer and closer to one of the coordinate axes at each end. To see why this is, first think about the piece of the graph to the right of the y-axis, that is, the piece that shows the values of $1/x$ for positive values of x. Its shape can be explained as follows. As x gets larger and larger, the value of $1/x$ gets closer and closer to zero. Similarly, as x gets closer and closer to zero, the value of $1/x$ gets larger and larger. The shape of the piece of the graph to the left of the y-axis, for negative values of x, has a similar explanation: you might like to think it through.

If a curve has the property that, as you trace your pen tip along it further and further from the origin, it gets *arbitrarily close* to a straight line, then that line is called an **asymptote** of the curve. The phrase 'arbitrarily close' here has the following meaning: no matter how small a distance you choose, if you trace your pen tip along the curve far enough, then eventually the curve lies within that distance of the line, and stays within that distance of the line.

So the coordinate axes are asymptotes of the graph of the reciprocal function. Asymptotes are often drawn as dashed lines on graphs, when they don't coincide with the coordinate axes.

The word 'asymptote' comes from a Greek word meaning 'not coinciding', used to describe a straight line that a curve approaches arbitrarily closely but doesn't meet.

Rational functions

The reciprocal function and all polynomial functions are particular examples of *rational functions*. In general, a **rational function** is a function whose rule is of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions. If q is a constant function, then f is a polynomial function, and if $p(x) = 1$ and $q(x) = x$, then f is the reciprocal function. Here are some more examples of rational functions:

$$f(x) = \frac{x^2+1}{2x+4}, \quad f(x) = \frac{2x^2-6x-8}{x^2-x-6}, \quad f(x) = \frac{7x+5}{x^2+1}.$$

The graphs of these rational functions are shown in Figure 27. The dashed lines are asymptotes. The third graph has the x-axis as an asymptote.

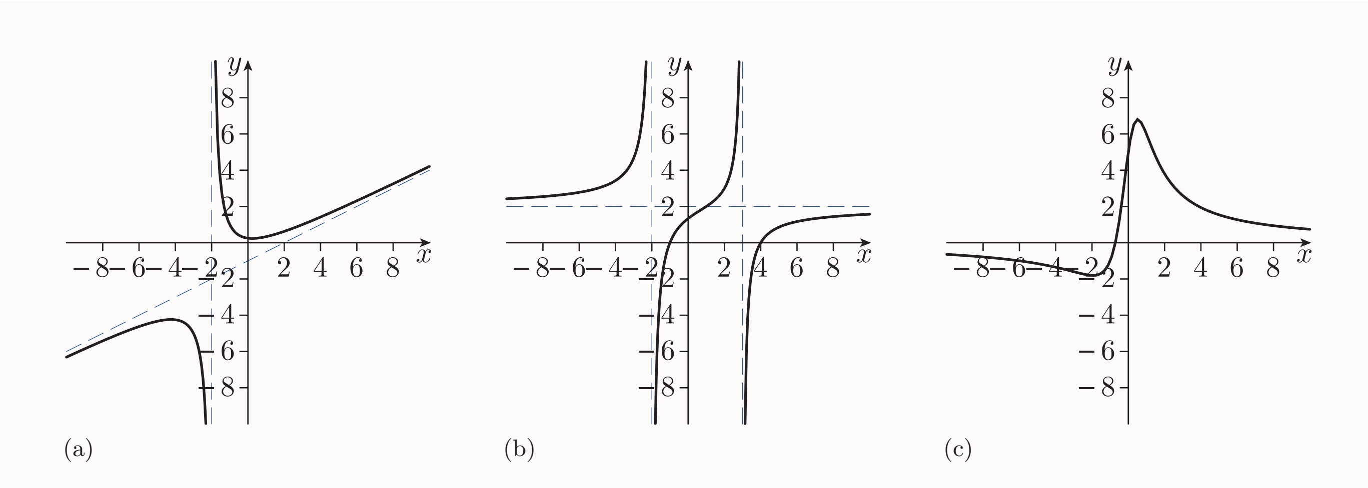


Figure 27 The graphs of (a) $f(x) = \dfrac{x^2+1}{2x+4}$ (b) $f(x) = \dfrac{2x^2-6x-8}{x^2-x-6}$ (c) $f(x) = \dfrac{7x+5}{x^2+1}$

Every rational function has a graph that consists of one or more pieces, each of which is a smooth curve. The graphs of many rational functions have asymptotes, which can be horizontal, vertical or slant. For example, the graph in Figure 27(a) has one vertical asymptote and one slant asymptote.

A detailed study of the graphs of rational functions is beyond the scope of this module, but you can learn more about them in the follow-on module to this one, *Essential mathematics 2* (MST125).

2 New functions from old functions

In this section, you'll learn how to use your knowledge about the graphs of a few functions to deduce facts about the graphs of many more functions. For example, you can use the graph of the function $f(x) = x^2$, which is shown in Figure 28, to deduce the appearance of the graphs of the functions $g(x) = x^2 + 1$ and $h(x) = 3x^2$. The rules of these functions are obtained from the rule for f simply by adding 1 and by multiplying by 3, respectively.

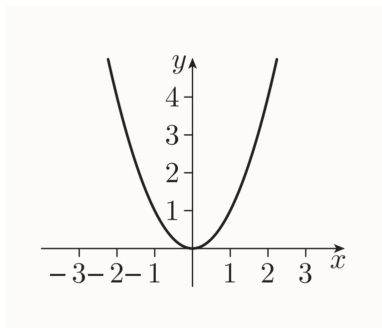


Figure 28 The graph of $f(x) = x^2$

2.1 Translating the graphs of functions

Let's start by considering what happens to the graph of a function when you add a constant to the right-hand side of its rule. You can see some instances of this in the next activity.

Activity 18 *Investigating graphs of equations of the form $y = f(x) + c$*

Open the *Translating and scaling graphs* applet. Make sure that the $y = f(x) + c$ option is selected, and that the original function is $f(x) = x^2$.

Change the value of c to display the graph of $y = x^2 + c$ for various values of c, and observe how the new graphs are related to the original graph. In particular, notice the effect of positive values of c, and the effect of negative values of c.

Now change the original function to a different function of your choice, and repeat the process above.

The effects that you saw in Activity 18 are examples of *translations* of graphs. **Translating** a shape means sliding it to a different position, without rotating or reflecting it, or distorting it in any way.

You saw that if you add any constant c to the right-hand side of the rule of a function, then its graph is translated *vertically*. Specifically, it's translated up by c units (the translation is down if c is negative). This is because when you add the constant c, the y-values all increase by c.

There's another fairly simple change that you can make to the rule of a function, which causes its graph to be translated *horizontally*. To do this, you replace each occurrence of the input variable x in the right-hand side of the rule of the function by an expression of the form $x - c$, where c is a constant. For example, if you start with the function $f(x) = x^2$, then you can replace x by $x - 3$, say, to obtain the new function $g(x) = (x - 3)^2$. You're asked to investigate changes of this sort in the next activity.

Activity 19 *Investigating graphs of equations of the form $y = f(x - c)$*

In the *Translating and scaling graphs* applet, select the $y = f(x - c)$ option, and make sure that the original function is $f(x) = x^2$.

Change the value of c to display the graph of $y = (x - c)^2$ for various values of c, and observe how the new graphs are related to the original graph. In particular, notice the effect of positive values of c, and the effect of negative values of c.

Now change the original function to a different function of your choice, and repeat the process above.

In Activity 19 you saw that if you replace every occurrence of the input variable x in the right-hand side of the rule of a function by the expression $x - c$, where c is a constant, then the graph of the function is translated horizontally. Specifically it's translated to the right by c units (the translation is to the left if c is negative). For example, if you replace x by $x - 3$, then the graph is translated to the right by 3 units (here $c = 3$). Similarly, if you replace x by $x + 3$, then the graph is translated to the left by 3 units (here $c = -3$).

To see why this happens, let's think about what happens when you translate the graph of a particular function to the right by c units (where c might be positive, negative or zero).

For example, Figure 29 shows the graph of the equation $y = x^2$ (in black), and the graph that's obtained by translating it to the right by 3 units (in green). Let's try to work out the equation of this second graph. To do this, we have to find a relationship between x and y that holds for every point (x, y) on the second graph. Now, whenever the point (x, y) lies on the second graph, the point $(x - 3, y)$ lies on the original graph, so the second coordinate, y, is the square of the first coordinate, $x - 3$. Hence the following equation holds:

$$y = (x - 3)^2.$$

This equation expresses a relationship between x and y, so it is the equation of the second graph.

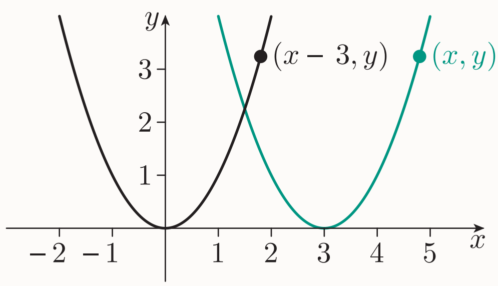


Figure 29 The graph of $y = x^2$, and the graph obtained by translating it by 3 units to the right

More generally, consider any function f, and suppose that you translate its graph to the right by c units, as illustrated in Figure 30 in a case where $c > 0$. Then whenever the point (x, y) lies on the second graph, the point $(x - c, y)$ lies on the original graph, so the following equation holds:

$$y = f(x - c).$$

So this equation is the equation of the second graph.

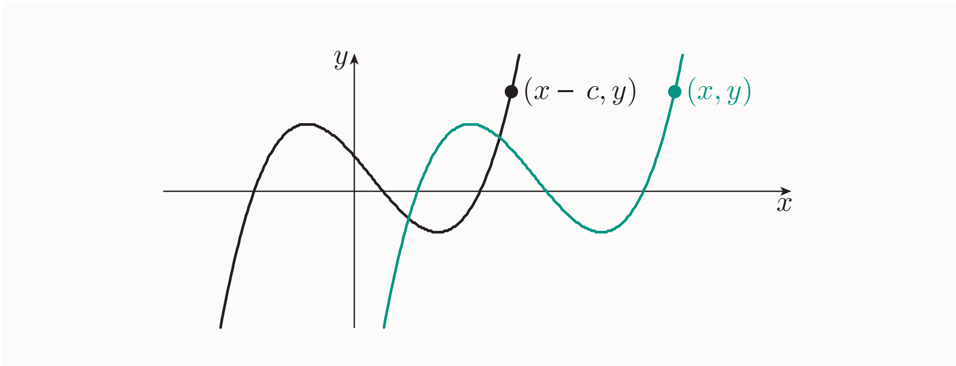


Figure 30 The graph of an equation of the form $y = f(x)$, and the graph obtained by translating it by c units to the right, where $c > 0$

This reasoning explains the effects that you saw in Activity 19.

Here's a summary of what you've seen so far in this subsection.

Translations of graphs

Suppose that f is a function and c is a constant. To obtain the graph of:

- $y = f(x) + c$, translate the graph of $y = f(x)$ up by c units (the translation is down if c is negative)
- $y = f(x - c)$, translate the graph of $y = f(x)$ to the right by c units (the translation is to the left if c is negative).

These effects are illustrated in Figure 31.

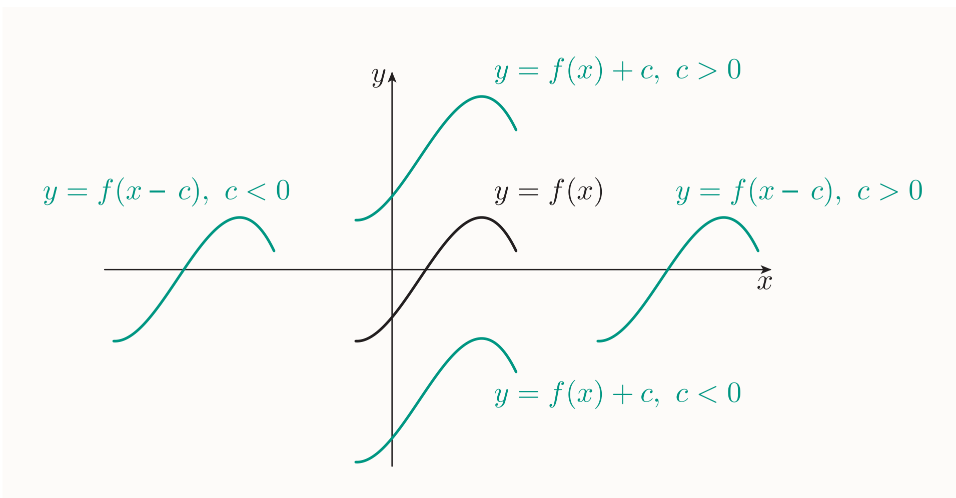


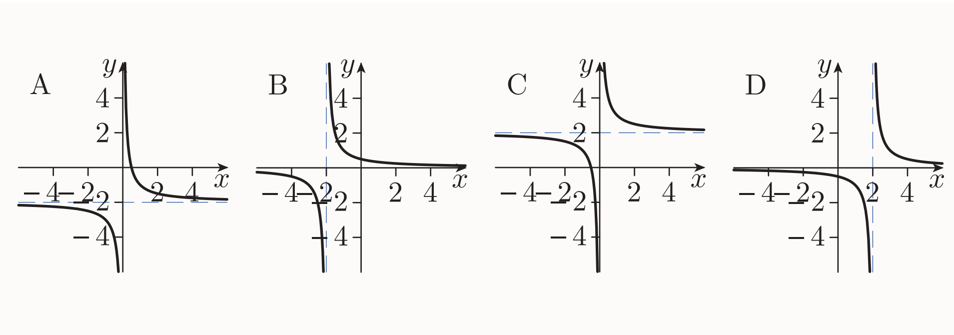
Figure 31 Pieces of graphs of equations of the form $y = f(x) + c$ and $y = f(x - c)$

Activity 20 *Understanding translations of graphs*

You saw the graph of $y = 1/x$ in the previous section, and it's repeated in Figure 32. Using this graph, and without using a computer, match up the equations below with their graphs.

(a) $y = \dfrac{1}{x-2}$ (b) $y = \dfrac{1}{x} - 2$ (c) $y = \dfrac{1}{x} + 2$ (d) $y = \dfrac{1}{x+2}$

Graphs:



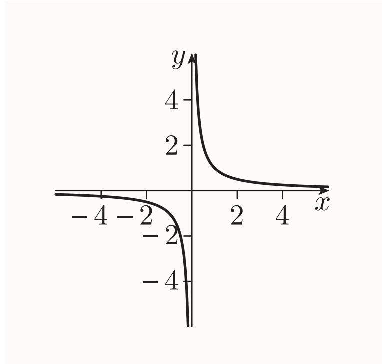


Figure 32 The graph of $y = \dfrac{1}{x}$

Now suppose that you change the rule of a function in such a way that its graph is translated horizontally, and then you change the rule of the *new* function in such a way that *its* graph is translated vertically. The final result is that the graph of the original function is translated both horizontally and vertically. For example, consider the equation $y = x^2$, whose graph is shown in Figure 33(a). If you replace x by $x - 4$, then you obtain the equation

$$y = (x-4)^2,$$

and the graph is translated to the right by 4 units, as shown in Figure 33(b). If you now add the constant 2, then you obtain the equation

$$y = (x-4)^2 + 2,$$

and the original graph is now translated to the right by 4 units and up by 2 units, as shown in Figure 33(c).

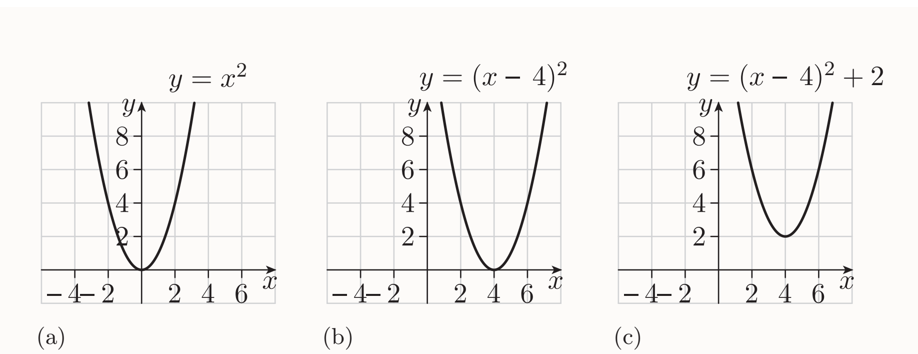


Figure 33 The graphs of (a) $y = x^2$ (b) $y = (x-4)^2$ (c) $y = (x-4)^2 + 2$

In general, suppose that you start with an equation $y = f(x)$. If you first replace x by $x - c$, where c is a constant, then you obtain the equation $y = f(x - c)$, and the graph is translated to the right by c units. If you then add the constant d to the right-hand side, then you obtain the equation

$$y = f(x - c) + d,$$

and the original graph is translated to the right by c units and up by d units.

In fact, the order in which you make the two changes doesn't matter. One way to see this is to think about the situation geometrically. If you translate a graph to the right by c units and then up by d units, then the overall effect will be the same as if you had translated it up by d units and then to the right by c units. You can also confirm it algebraically, as follows. Suppose that you carry out the two changes to the equation $y = f(x)$ in the opposite order to the order used above. Adding d to the right-hand side of the equation $y = f(x)$ gives the equation $y = f(x) + d$, and then replacing x in this equation by $x - c$ gives the final equation $y = f(x - c) + d$, which is the same as the final equation obtained above.

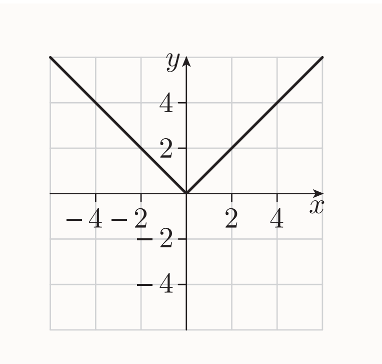


Figure 34 The graph of $y = |x|$

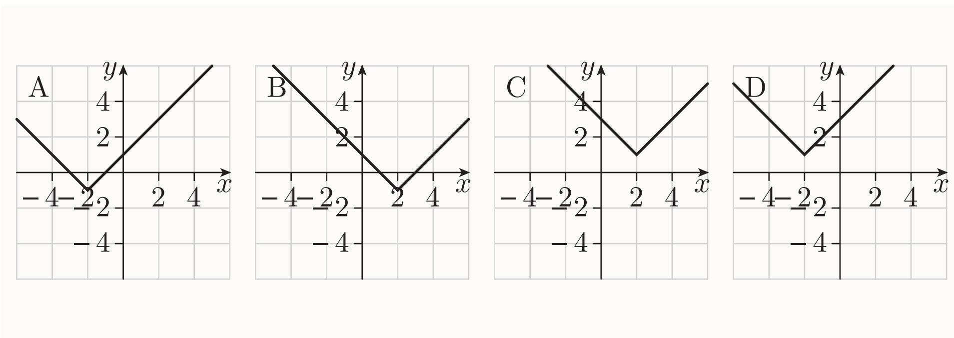
Activity 21 *Understanding successive horizontal and vertical translations of graphs*

You saw the graph of $y = |x|$ in the previous section, and it's repeated in Figure 34. Using this graph, and without using a computer, match up the equations below with their graphs.

(a) $y = |x - 2| + 1$ (b) $y = |x + 2| + 1$ (c) $y = |x - 2| - 1$

(d) $y = |x + 2| - 1$

Graphs:



2.2 Scaling the graphs of functions vertically

In this subsection you'll see what happens to the graph of a function when you multiply the right-hand side of its rule by a constant. The new function that you obtain is called a **constant multiple** of the original function. For example, the function $g(x) = 3x^2$ is a constant multiple of the function $f(x) = x^2$.

Activity 22 *Investigating graphs of equations of the form $y = cf(x)$*

In the *Translating and scaling graphs* applet, select the $y = cf(x)$ option, and make sure that the original function is $f(x) = x^2 - 1$.

Change the value of c to display the graph of $y = c(x^2 - 1)$ for various values of c, and observe how the new graphs are related to the original graph. In particular, notice the effect of positive values of c, and the effect of negative values of c. Also notice the effect of values of c such that $|c| < 1$, and the effect of values of c such that $|c| > 1$.

Now change the original function to $y = x^3$, and repeat the process above. If you wish, also try another function of your choice as the original function.

The effects that you saw in Activity 22 are called **vertical scalings**. **Scaling** a graph **vertically** by a **factor** of c means the following.

- If c is positive, then move each point on the graph vertically, in the direction away from the x-axis, until it's c times as far from the x-axis as it was before.
- If c is negative, then move each point on the graph vertically, in the direction away from the x-axis, until it's $|c|$ times as far from the x-axis as it was before, and then reflect it in the x-axis.
- If c is zero, then move each point on the graph vertically until it lies on the x-axis.

(In each of the first two cases, if $|c|$ is less than 1, then each point is actually moved *closer* to the x-axis than it was before.)

Informally, when you scale a graph vertically by a factor of c, you stretch or squash it parallel to the y-axis (depending on whether $|c|$ is greater than or less than 1), and if c is negative, you also reflect it in the x-axis.

In Activity 22 you should have seen evidence of the following.

Vertical scalings of graphs

Suppose that c is a constant. To obtain the graph of $y = cf(x)$,

scale the graph of $y = f(x)$ vertically by a factor of c.

These effects are illustrated in Figure 35. They occur because when you multiply the right-hand side of the rule of a function by the constant c, the y-value corresponding to each x-value is multiplied by c.

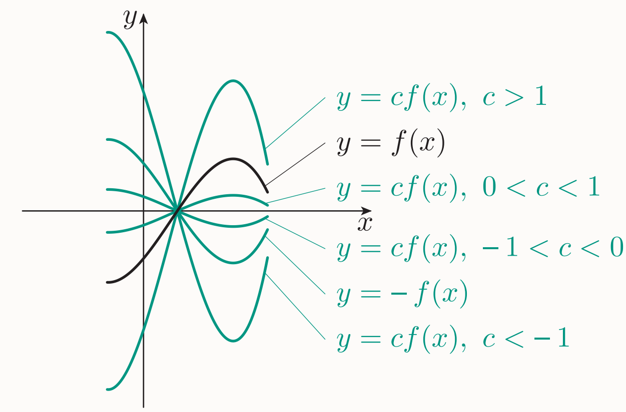


Figure 35 Pieces of graphs of equations of the form $y = cf(x)$

Notice in particular what happens when $c = -1$. For any function f, the graph of $y = -f(x)$ is the same shape as the graph of $y = f(x)$, but reflected in the x-axis. The function that results from multiplying the right-hand side of the rule of a function f by -1 is called the **negative** of the function f.

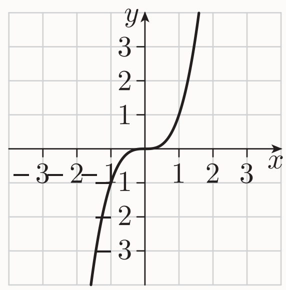


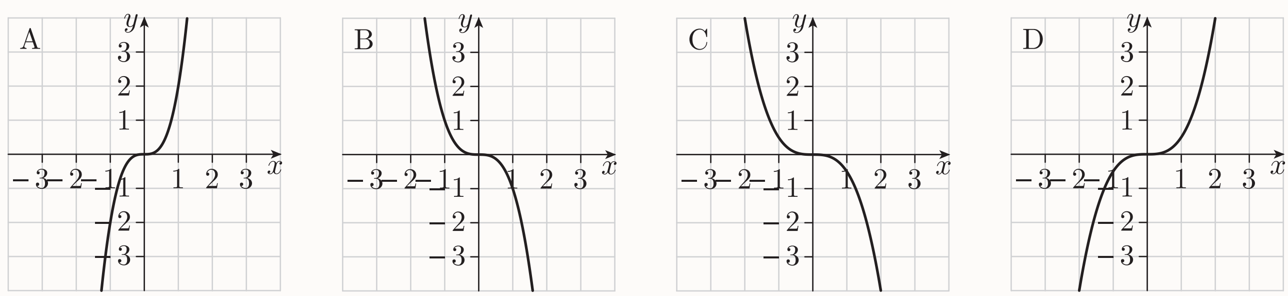
Figure 36 The graph of $y = x^3$

Activity 23 *Understanding vertical scalings of graphs*

You saw the graph of $y = x^3$ in Subsection 1.6, and it's repeated in Figure 36. Using this graph, and without using a computer, match up the equations below with their graphs.

(a) $y = 2x^3$ (b) $y = \frac{1}{2}x^3$ (c) $y = -x^3$ (d) $y = -\frac{1}{2}x^3$

Graphs:



You can combine vertical scalings of graphs with vertical and/or horizontal translations of graphs, in the same way that horizontal and vertical translations of graphs were combined in the previous subsection. For example, suppose that you start with the equation $y = x^3$, whose graph is shown in Figure 37(a). If you multiply the right-hand side of this equation by 4, then you obtain the equation $y = 4x^3$, and the graph is scaled vertically by a factor of 4, as illustrated in Figure 37(b). If you then add the constant 1 to the right-hand side of this new equation, then you obtain the final equation $y = 4x^3 + 1$, and the original graph is first scaled vertically by a factor of 4, then translated up by 1 unit, as illustrated in Figure 37(c).

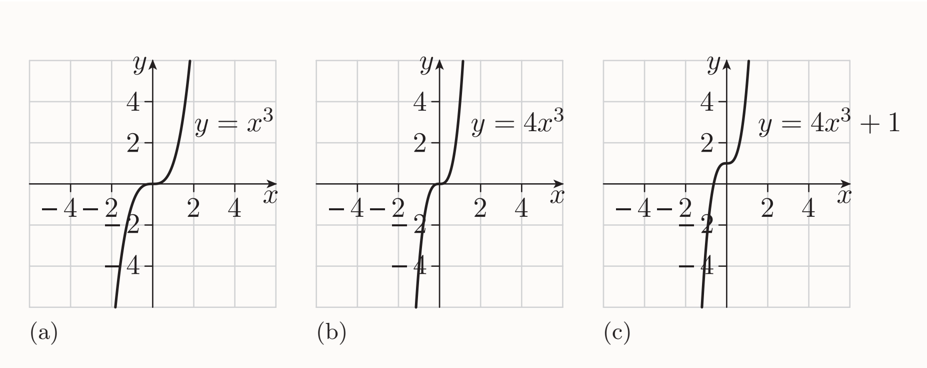


Figure 37 The graphs of (a) $y = x^3$ (b) $y = 4x^3$ (c) $y = 4x^3 + 1$

When you combine vertical scalings and translations in this way, the order in which you carry out the changes to the rule of the function *does* often matter. Different orders can give different results. For example, suppose that you start with the equation $y = x^3$, as above, and you make the same two changes as above, but in the opposite order. Adding the constant 1 to the right-hand side of the equation $y = x^3$ gives the intermediate equation $y = x^3 + 1$, and the graph is translated up by 1 unit, as illustrated in Figure 38(b). Then multiplying the right-hand side by 4 gives the final equation $y = 4(x^3 + 1)$, and the original graph is first translated up by 1 unit and then scaled vertically by a factor of 4, as illustrated in Figure 38(c). You can see that the final equation and graph are different from those obtained above.

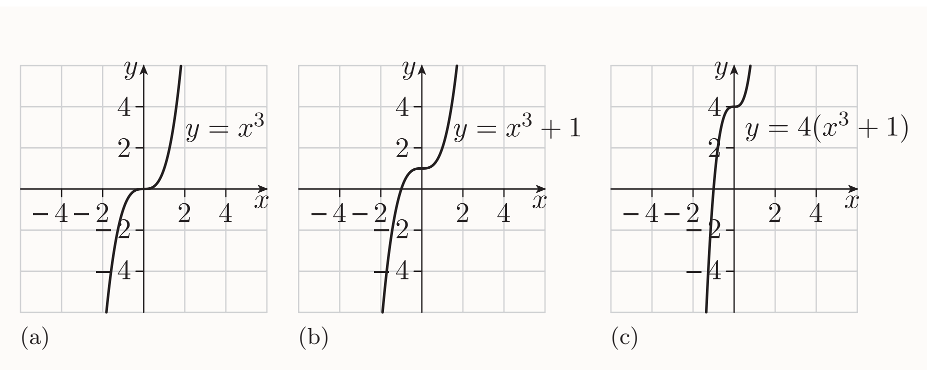


Figure 38 The graphs of (a) $y = x^3$ (b) $y = x^3 + 1$ (c) $y = 4(x^3 + 1)$

In general, you can make any number of successive changes to the rule of a function to scale and translate its graph in various ways, but you have to be careful about the order in which you carry out the changes. Sometimes the order matters, and sometimes it doesn't.

If you change the rule of a function to carry out a horizontal translation, a vertical translation and a vertical scaling, then you can make the changes in any order, except that the changes for the vertical translation and vertical scaling must be made in the correct order relative to each other.

Example 3 *Understanding successive scalings and translations of graphs*

For each of the following functions, describe how you could obtain its graph by applying scalings and translations to the graph of the function $f(x) = x^3$.

(a) $g(x) = \frac{1}{2}(x+3)^3$ (b) $h(x) = \frac{1}{2}(x+3)^3 - 2$

Solution

(a) Try to work out how the equation $y = \frac{1}{2}(x+3)^3$ is obtained from the equation $y = x^3$ by making two or more changes of the types that you've seen, one after another.

Consider the equation $y = x^3$. If you multiply the right-hand side by $\frac{1}{2}$, you obtain the equation $y = \frac{1}{2}x^3$. If you then replace x by $x+3$, you obtain the equation $y = \frac{1}{2}(x+3)^3$.

So the graph of the equation $y = \frac{1}{2}(x+3)^3$ is obtained by starting with the graph of $y = x^3$, scaling it vertically by the factor $\frac{1}{2}$, and then translating it to the left by 3 units.

(b) Use the same method as in part (a). Here you can recognise that the given equation is obtained from the equation in part (a) by making a simple change.

The equation $y = \frac{1}{2}(x+3)^3 - 2$ is obtained from the final equation in part (a) by adding -2 to the right-hand side.

So the graph of the equation $y = \frac{1}{2}(x+3)^3 - 2$ is obtained by starting with the graph of $y = x^3$, scaling it vertically by the factor $\frac{1}{2}$, then translating it to the left by 3 units, and finally translating it down by 2 units.

Figure 39 shows the graph of the function $f(x) = x^3$ and the results of applying the scalings and translations in Example 3 to this graph.

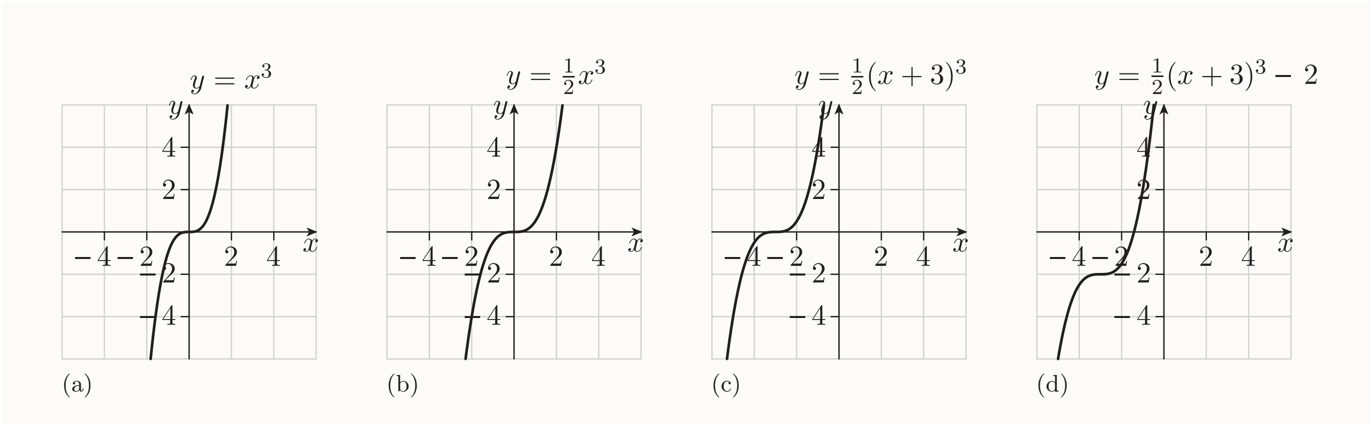


Figure 39 Three graphs obtained by scaling and/or translating the graph of $y = x^3$

Activity 24 *Understanding successive scalings and translations of graphs*

For each of the following functions, describe how you could obtain its graph by applying scalings and translations to the graph of the function $f(x) = |x|$ (which is shown in Figure 40).

(a) $g(x) = 2|x| + 3$ (b) $h(x) = 2|x+2| + 3$ (c) $j(x) = \frac{1}{2}|x-3| - 4$

(d) $k(x) = -|x-1| + 1$

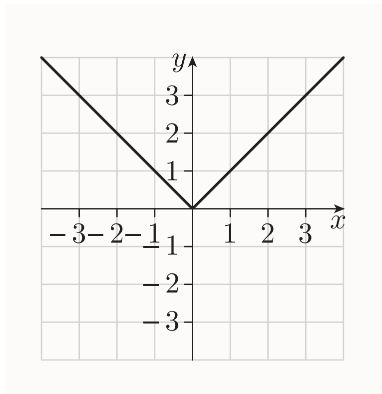


Figure 40 The graph of $y = |x|$

In the final activity of this subsection, you'll see how the new ideas that you've met can give you a deeper understanding of the shapes of the graphs of quadratic functions.

Activity 25 *Understanding the graph of a quadratic function*

Consider the quadratic function $f(x) = 2x^2 + 12x + 19$.

(a) Complete the square in the quadratic expression on the right-hand side.

(b) Hence describe how you could obtain the graph of this function by applying scalings and translations to the graph of the function $f(x) = x^2$ (which is shown in Figure 41).

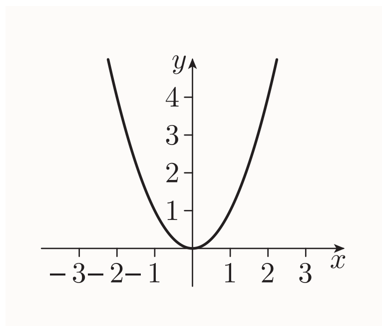


Figure 41 The graph of $y = x^2$

You can see that the method that you were asked to use in Activity 25 can be applied to *any* quadratic function. You just need to start by completing the square in the quadratic expression that forms the right-hand side of its rule.

This tells you the following enlightening fact: the graph of any quadratic function is the same basic shape as the graph of $y = x^2$, but scaled vertically, and then translated horizontally and/or vertically.

2.3 Scaling the graphs of functions horizontally

In the first activity of this subsection, you're asked to investigate a change to the rule of a function that results in its graph being scaled *horizontally*. The change is that you replace each occurrence of the input variable x in the right-hand side of the rule of the function by an expression of the form x/c, where c is a constant.

Activity 26 *Investigating graphs of equations of the form* $y = f\left(\dfrac{x}{c}\right)$

In the *Translating and scaling graphs* applet, select the $y = f(x/c)$ option, and make sure that the original function is $y = x^3$.

Change the value of c to display the graph of $y = (x/c)^3$ for various non-zero values of c, and observe how the new graphs are related to the original graph. In particular, notice the effect of positive values of c, and the effect of negative values of c. Also notice the effect of values of c such that $|c| < 1$, and the effect of values of c such that $|c| > 1$.

Now change the original function to $y = x^2$, and repeat the process above. If you wish, also try another function of your choice as the original function.

The effects that you saw in Activity 26 are called **horizontal scalings**. **Scaling** a graph **horizontally** by a **factor** of c means the following.

- If c is positive, then move each point on the graph horizontally, in the direction away from the y-axis, until it's c times as far from the y-axis as it was before.
- If c is negative, then move each point on the graph horizontally, in the direction away from the y-axis, until it's $|c|$ times as far from the y-axis as it was before, and then reflect it in the y-axis.
- If c is zero, then move each point on the graph horizontally until it lies on the y-axis.

(In each of the first two cases, if $|c|$ is less than 1, then each point is actually moved *closer* to the y-axis than it was before.)

Informally, when you scale a graph horizontally by a factor of c, you stretch or squash it parallel to the x-axis (depending on whether $|c|$ is greater than or less than 1), and if c is negative, you also reflect it in the y-axis.

In Activity 26 you should have seen evidence of the following.

Horizontal scalings of graphs

Suppose that c is a non-zero constant. To obtain the graph of

$$y = f\left(\frac{x}{c}\right),$$

scale the graph of $y = f(x)$ horizontally by a factor of c.

These effects are illustrated in Figure 42.

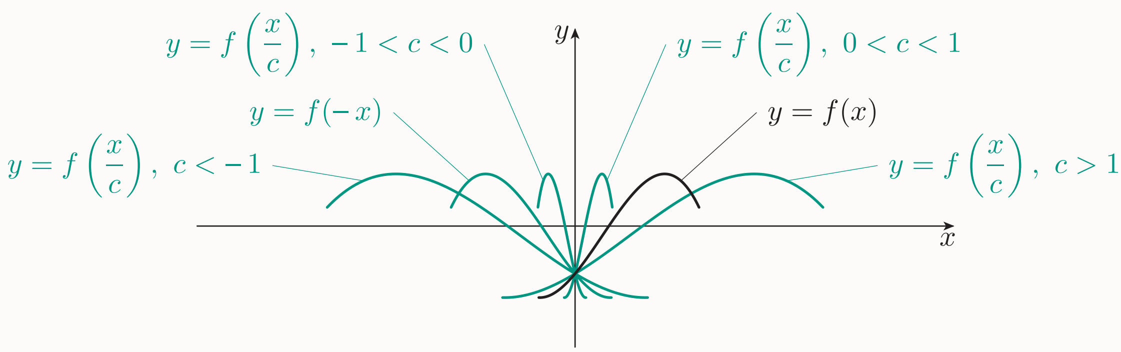


Figure 42 Pieces of graphs of equations of the form $y = f(x/c)$

To see why these effects occur, let's suppose that you scale the graph of a particular function horizontally by a factor of c (where c might be positive or negative), and let's try to work out how this affects the rule of the function.

For example, Figure 43 shows the graph of the equation $y = x^2$ (in black), and the graph that's obtained by scaling it horizontally by a factor of 3 (in green). Let's try to work out the equation of this second graph.

To do this, we have to find a relationship between x and y that holds for every point (x, y) on the second graph. Now whenever the point (x, y) lies on the second graph, the point $(x/3, y)$ lies on the original graph, so the following equation holds:

$$y = \left(\frac{x}{3}\right)^2.$$

This equation expresses a relationship between x and y, so it is the equation of the second graph.

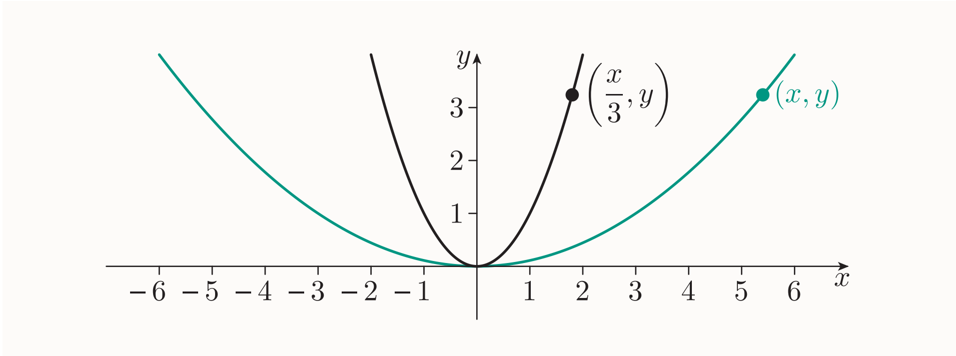


Figure 43 The graph of $y = x^2$, and the graph obtained by scaling it horizontally by a factor of 3

More generally, consider any function f, and suppose that you scale its graph horizontally by a factor of c, as illustrated in Figure 44 in a case where $c > 1$. Then whenever the point (x, y) lies on the second graph, the point $(x/c, y)$ lies on the original graph, so the following equation holds:

$$y = f\left(\frac{x}{c}\right).$$

So this equation is the equation of the second graph.

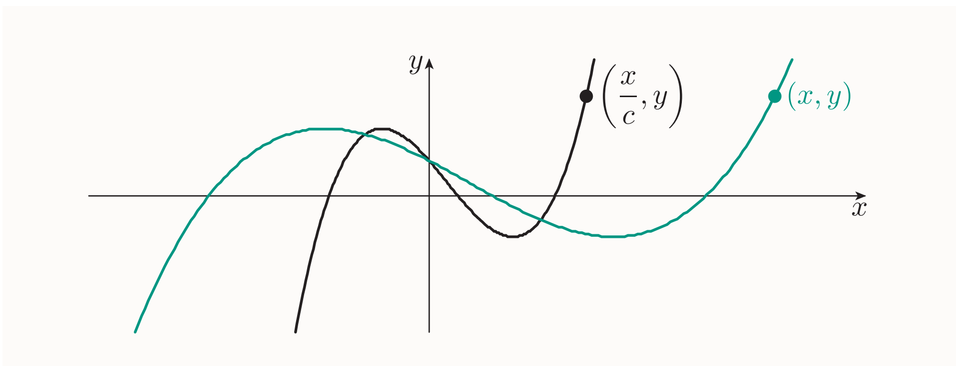


Figure 44 The graph of an equation $y = f(x)$, and the graph obtained by scaling it horizontally by a factor of c, where $c > 1$

Notice in particular what happens when $c = -1$. For any function f, the graph of $y = f(-x)$ is the same shape as the graph of $y = f(x)$, but reflected in the y-axis.

For convenience, the two facts that you've seen (in this subsection and the previous one) about reflections of graphs in the coordinate axes are summarised below, and illustrated in Figure 45.

Reflections of graphs in the coordinate axes

To obtain the graph of $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

To obtain the graph of $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.

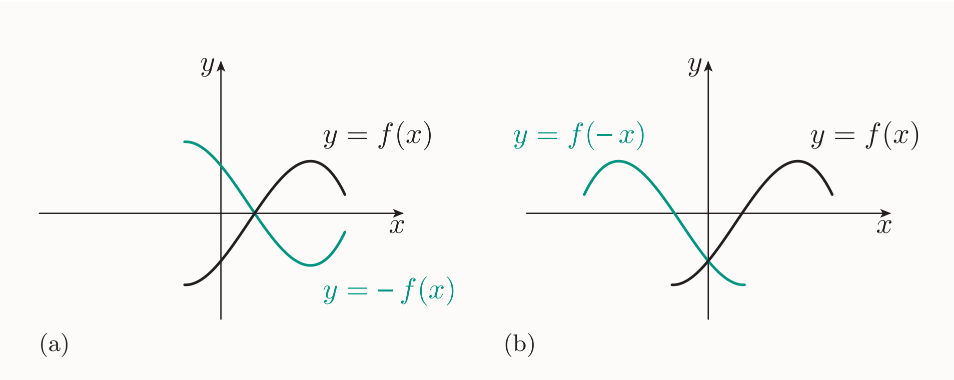


Figure 45 Pieces of graphs of equations of the form (a) $y = -f(x)$ and (b) $y = f(-x)$

In the final two activities of this section, you'll need to put together all the facts and skills about the effects of changing the equations of graphs that you've learned in this section.

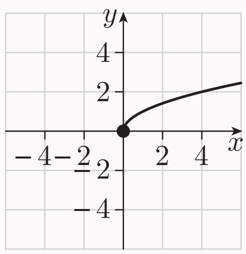


Figure 46 The graph of $y = \sqrt{x}$

Activity 27 *Understanding horizontal and vertical translations and scalings of graphs*

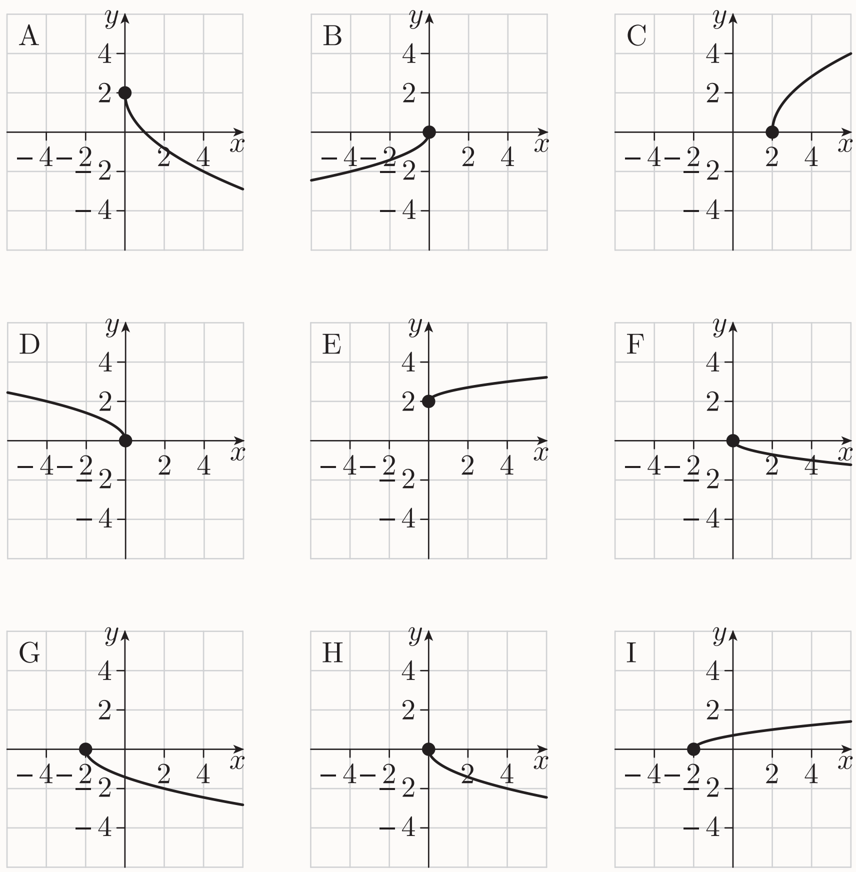
The graph of the function $f(x) = \sqrt{x}$ is shown in Figure 46. Using this graph, and without using a computer, match up the equations below with their graphs.

(a) $y = \sqrt{-x}$ (b) $y = -\sqrt{x}$ (c) $y = 2\sqrt{x-2}$

(d) $y = \frac{1}{2}\sqrt{x} + 2$ (e) $y = -\frac{1}{2}\sqrt{x}$ (f) $y = -\sqrt{x+2}$

(g) $y = -\sqrt{-x}$ (h) $y = \frac{1}{2}\sqrt{x+2}$ (i) $y = -2\sqrt{x} + 2$

Graphs:



Activity 28 *Understanding horizontal and vertical translations and scalings of graphs, again*

By considering graphs of functions that you met in Subsection 1.6, and without using a computer, draw sketch graphs of the following functions. In particular, mark the values of the intercepts, and in part (c) draw the asymptotes, and label the vertical one with its equation. (You met the idea of an *asymptote* on page 232.)

(a) $f(x) = 2|x| + 1$ (b) $h(x) = (x-1)^3$ (c) $g(x) = \dfrac{1}{x+3}$

Hint: in part (a), think about the graph of $y = |x|$; in part (b), think about the graph of $y = x^3$; in part (c), think about the graph of $y = 1/x$.

3 More new functions from old functions

In this section you'll meet some ways in which you can combine the rules of two or more functions to obtain the rule of a new function. You'll also see that for some functions there's a related function, called the *inverse function* of the original function, which 'reverses' the effect of the original function.

3.1 Sums, differences, products and quotients of functions

In this short subsection you'll see how to combine functions by forming sums, differences, products and quotients of them. These combinations mean just what the names suggest.

Suppose that f and g are functions. The **sum** of f and g has the rule

$$h(x) = f(x) + g(x).$$

There are two **differences** of f and g, with rules

$$h(x) = f(x) - g(x) \quad \text{and} \quad h(x) = g(x) - f(x).$$

The **product** of f and g has the rule

$$h(x) = f(x)g(x).$$

There are two **quotients** of f and g, with rules

$$h(x) = \frac{f(x)}{g(x)} \quad \text{and} \quad h(x) = \frac{g(x)}{f(x)}.$$

For example, if $f(x) = x^2$ and $g(x) = x$, then the sum of f and g has the rule $h(x) = x^2 + x$. The domain of each of the combined functions above is the intersection of the domain of f and the domain of g, with the additional requirement for the first quotient of f and g that the numbers x such that $g(x) = 0$ are removed, since it's not possible to divide by zero, and a similar additional requirement for the second quotient.

Activity 29 *Understanding sums, differences, products and quotients of functions*

Consider the functions $f(x) = 2x - 1$ and $g(x) = x + 3$. Find the rules of the sum, the two differences, the product and the two quotients of f and g, and state the domain of each of these functions.

You can form sums and products of three or more functions. For example, the sum of the functions $f(x) = x^2$, $g(x) = x$ and $h(x) = 1$ is the function $s(x) = x^2 + x + 1$.

There isn't much more to be said about sums, products, differences and quotients of functions, at this stage. It's usually not easy to deduce the shape of the graph of any one of these functions from the shapes of the graphs of the original functions. These types of combinations of functions will be important later in the module.

3.2 Composite functions

There's another useful way to combine two functions to obtain a new function. This is to apply one function after the other.

For example, suppose that f and g are functions. Consider any value that lies in the domain of f. If you input this value to the function f, then you obtain an output value. If this output value is in the domain of the function g, then you can, in turn, input it to the function g, to obtain a final output value. This process is illustrated in Figure 47.

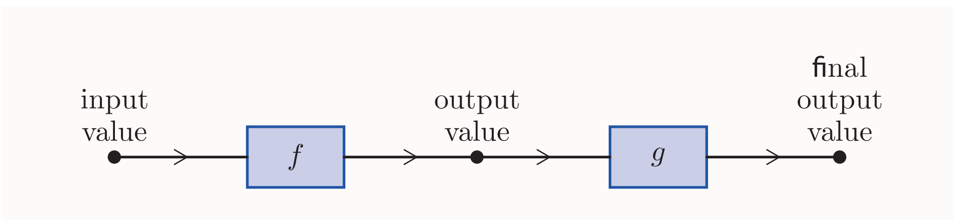


Figure 47 An input value processed by two functions f and g one after the other

The function whose rule is given by this process, and whose domain is the largest set of real numbers to which you can apply the process, is called a **composite function**, or just **composite**, of f and g. It's denoted by $g \circ f$ (the symbol $\circ$ is read as 'circle'). Note that the function that's applied *first* is written *second* in this notation – you'll see why shortly.

The largest set of real numbers for which you can apply the process is the set of all numbers in the domain of f such that $f(x)$ lies in the domain of g. For example, Figure 48 illustrates the process of finding the image of the number 3 under the composite function $g \circ f$, where $f(x) = x^2$ and $g(x) = x + 1$. It shows that $(g \circ f)(3) = 10$.

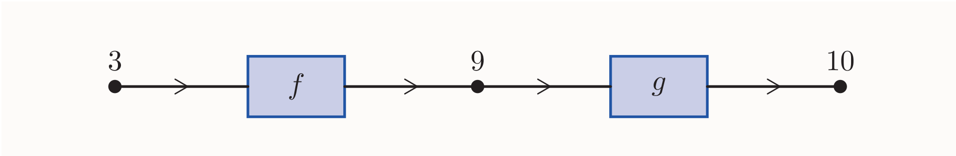


Figure 48 The image of 3 under $g \circ f$, where $f(x) = x^2$ and $g(x) = x + 1$

Activity 30 *Understanding composite functions*

Suppose that $f(x) = x^2$ and $g(x) = x + 1$, as in the paragraph above. Find the image of 5 under the composite function $g \circ f$.

In general, for any two functions f and g, the process of finding the image of an input value x under the composite function $g \circ f$ is as shown in Figure 49.

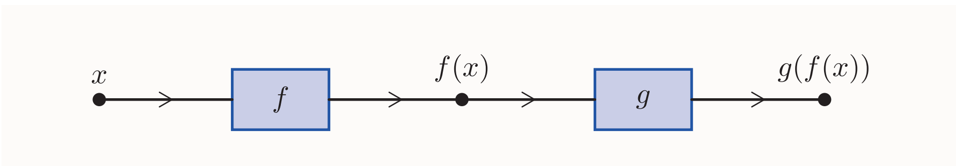


Figure 49 The image of x under $g \circ f$, for any functions f and g

So a composite function can be defined concisely as follows.

Composite functions

Suppose that f and g are functions. The **composite function** $g \circ f$ is the function whose rule is

$$(g \circ f)(x) = g(f(x)),$$

and whose domain consists of all the values x in the domain of f such that $f(x)$ is in the domain of g.

It's important to remember that $g \circ f$ means f followed by g, not the other way round, as you might at first expect. To understand why the notation is this way round, consider the equation in the box above:

$$(g \circ f)(x) = g(f(x)).$$

It would be confusing if f and g were in different orders on the two sides of the equation.

The process of forming a composite function from two functions is called **composing** the functions. You can compose two functions f and g in either order, so they have *two* composite functions: $g \circ f$, which means first f and then g, and $f \circ g$, which means first g and then f. These two composite functions are illustrated in Figure 50.

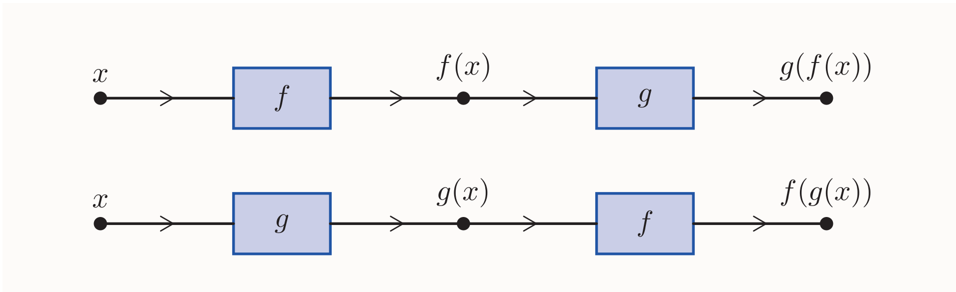


Figure 50 The image of x under $g \circ f$ and $f \circ g$, respectively

The next example shows you how to work out the rules of composite functions.

Example 4 *Composing functions*

Suppose that $f(x) = x^2$ and $g(x) = x + 1$. Find the rules of the composite functions $g \circ f$ and $f \circ g$.

Solution

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$$

and

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2.$$

Figure 51 illustrates the composite functions in Example 4.

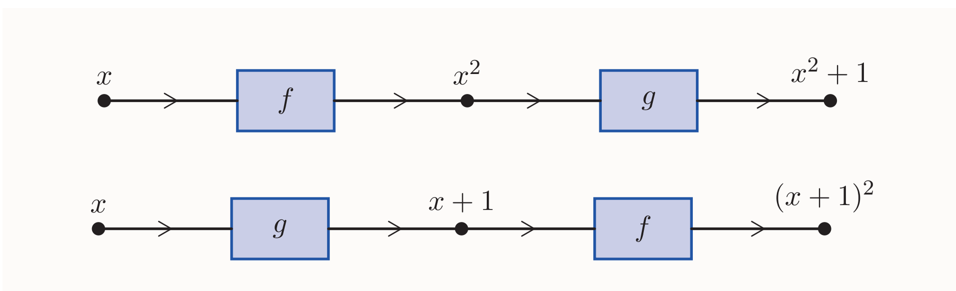


Figure 51 The image of x under particular composite functions $g \circ f$ and $f \circ g$

Notice that the composite functions $g \circ f$ and $f \circ g$ in Example 4 are *different functions*. If f and g are two different functions, then the composite functions $g \circ f$ and $f \circ g$ are usually different from each other.

Activity 31 *Composing functions*

(a) Suppose that $f(x) = x - 3$ and $g(x) = \sqrt{x}$. Find the rules of the following composite functions.

(i) $g \circ f$ (ii) $f \circ g$ (iii) $f \circ f$ (iv) $g \circ g$

(b) Determine the domain of the function $g \circ f$.

You can compose more than two functions. For example, if f, g and h are functions, then you can form a composite function whose rule is given by first applying f, then g, then h. This composite function is denoted by $h \circ g \circ f$, and its rule can be stated as

$$(h \circ g \circ f)(x) = h(g(f(x))).$$

Example 5 *Composing three functions*

Suppose that $f(x) = x + 2$, $g(x) = 1/x$ and $h(x) = \sqrt{x}$. Find the rule of the composite function $h \circ g \circ f$.

Solution

$$\begin{aligned}(h \circ g \circ f)(x) &= h(g(f(x))) \\ &= h(g(x+2)) \\ &= h\left(\frac{1}{x+2}\right) \\ &= \sqrt{\frac{1}{x+2}} \\ &= \frac{1}{\sqrt{x+2}}.\end{aligned}$$

Activity 32 *Composing three functions*

Suppose that $f(x) = x + 2$, $g(x) = 1/x$ and $h(x) = \sqrt{x}$, as in Example 5. Find the rules of the following composite functions.

(a) $f \circ g \circ h$ (b) $g \circ h \circ f$ (c) $f \circ h \circ g$ (d) $f \circ g \circ f$

3.3 Inverse functions

In this subsection you'll learn what's meant by the *inverse* of a function. To illustrate the idea, let's consider the function $f(x) = 2x$. The mapping diagram in Figure 52 shows *some* of the inputs and corresponding outputs of this function, linked by arrows.

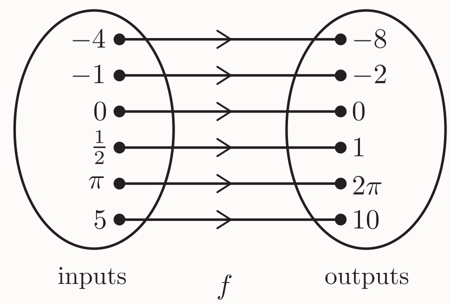


Figure 52 Some inputs and outputs of the function $f(x) = 2x$

Imagine a full version of the mapping diagram in Figure 52, which shows *all* the inputs and outputs of the function f. You can't actually draw such a diagram, of course, because f has infinitely many inputs and outputs.

The *inverse function*, or simply *inverse*, of the function f, which is denoted by f^{-1}, is the function whose mapping diagram is obtained by reversing the directions of all the arrows in this full version of the diagram. For example, Figure 53 shows some of the inputs and outputs of f^{-1}.

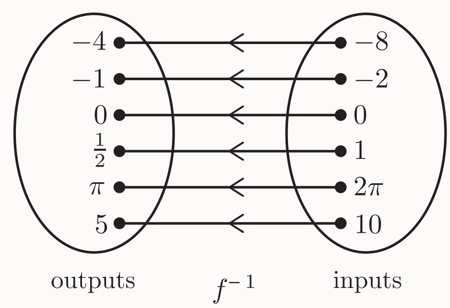


Figure 53 Some inputs and outputs of the function f^{-1}, where $f(x) = 2x$

This figure shows that, for example,

$$f^{-1}(-8) = -4, \quad f^{-1}(1) = \tfrac{1}{2} \quad \text{and} \quad f^{-1}(10) = 5.$$

You can imagine a full version of the mapping diagram in Figure 53, which shows *all* the inputs and outputs of f^{-1}. You can think of the inverse functions of other functions in the same way. Essentially, the inverse function f^{-1} of a function f is the function that 'has the reverse effect' of f. That is, if inputting a number x to f gives the number y, then inputting the number y to f^{-1} gives the original number x, as illustrated in Figure 54. For example, if f is the function $f(x) = 2x$, as above, then inputting 5 to f gives 10, and inputting 10 to f^{-1} gives 5.

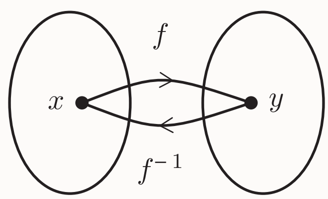


Figure 54 A mapping diagram illustrating a function f and its inverse function f^{-1}

Another way to think of the inverse function f^{-1} of a function f is that f^{-1} 'undoes' the effect of f. For example, if f is the function $f(x) = 2x$, as above, and you input the number 5 to f, then you get 10; and if you then take this output 10 and input it to the inverse function f^{-1}, then you get 5 back again.

You can sometimes write down the rule of an inverse function by thinking about it in this way. For example, consider once more the function $f(x) = 2x$. This function *doubles* numbers, so the function that undoes its effect *halves* numbers. So the rule of the inverse function of this function f is

$$f^{-1}(x) = \tfrac{1}{2}x.$$

In the next activity you're asked to write down the rules of the inverse functions of some other simple functions.

Activity 33 *Finding the rules of inverse functions of simple functions*

(a) Write down the rules of the inverse functions of the following functions.

(i) $f(x) = x + 1$ (ii) $f(x) = x - 3$ (iii) $f(x) = \tfrac{1}{3}x$

(b) Can you think of a function with an inverse function that has the same rule as the original function? Can you think of another such function?

Some functions don't have inverse functions. For example, consider the function $f(x) = x^2$. Some of the inputs and outputs of this function are shown in Figure 55.

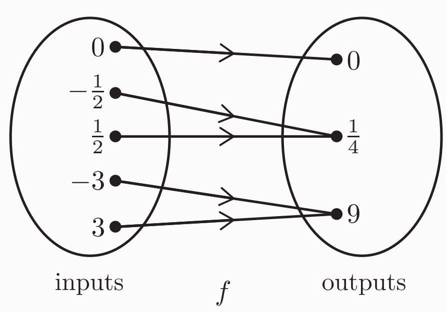


Figure 55 Some inputs and outputs of the function $f(x) = x^2$

If you reverse the directions of all the arrows in the full version of this mapping diagram, then the new diagram that you get *isn't the mapping diagram of a function*. That's because, in the new diagram, some of the input numbers have more than one output number, as illustrated in Figure 56. Remember that, for a function, every input number must have *exactly one* output number.

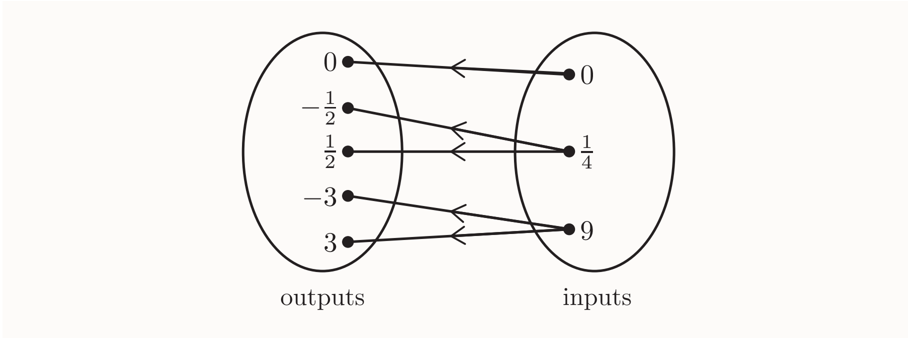


Figure 56 After reversing the directions of the arrows in Figure 55

You can see that this problem will arise whenever, for the original function f, there are two or more different input values that have the same output value. A function for which this *doesn't* happen, and which therefore *does* have an inverse function, is said to be **one-to-one**. In other words, we have the following definition.

One-to-one functions

A function f is **one-to-one** if for all values x_1 and x_2 in its domain such that $x_1 \neq x_2$,

$$f(x_1) \neq f(x_2).$$

For example, the function $f(x) = x^3$ is one-to-one, because no two different numbers have the same cube. On the other hand, the function $f(x) = x^2$ isn't one-to-one, because, for instance,

$$f(3) = 3^2 = 9 \quad \text{and} \quad f(-3) = (-3)^2 = 9,$$

so $f(3) = f(-3)$.

Activity 34 *Recognising whether functions are one-to-one*

Which of the following functions are one-to-one? For each function that isn't one-to-one, state two input numbers that have the same output number.

(a) $f(x) = |x|$ (b) $f(x) = x + 1$ (c) $f(x) = x^4$ (d) $f(x) = x^5$

(e) $f(x) = -x$ (f) $f(x) = 1$

A useful way to recognise whether a function is one-to-one is to look at its graph. You've seen that, for a one-to-one function, every output number is obtained from exactly one input number. So if you draw any horizontal line that crosses the graph of the function, then it crosses it *exactly once*, as illustrated in Figure 57(a). If you can draw a horizontal line that crosses a graph more than once, then the graph isn't the graph of a one-to-one function. For example, the function whose graph is shown in Figure 57(b) isn't one-to-one, since the dashed horizontal line shows that the two input numbers marked as x_1 and x_2 have the same output number.

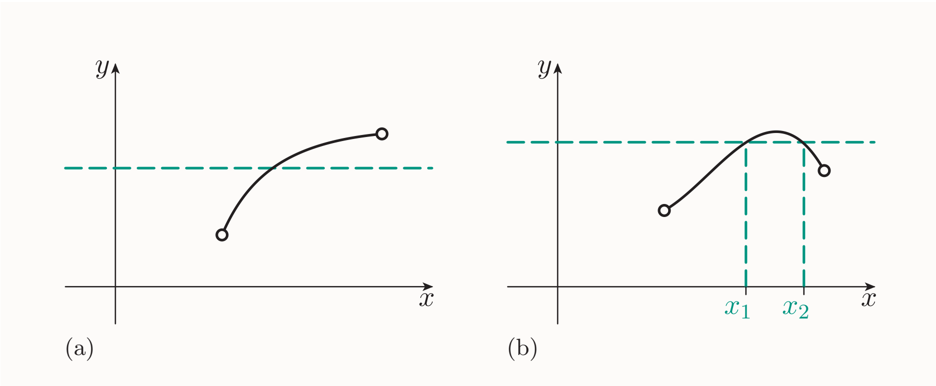
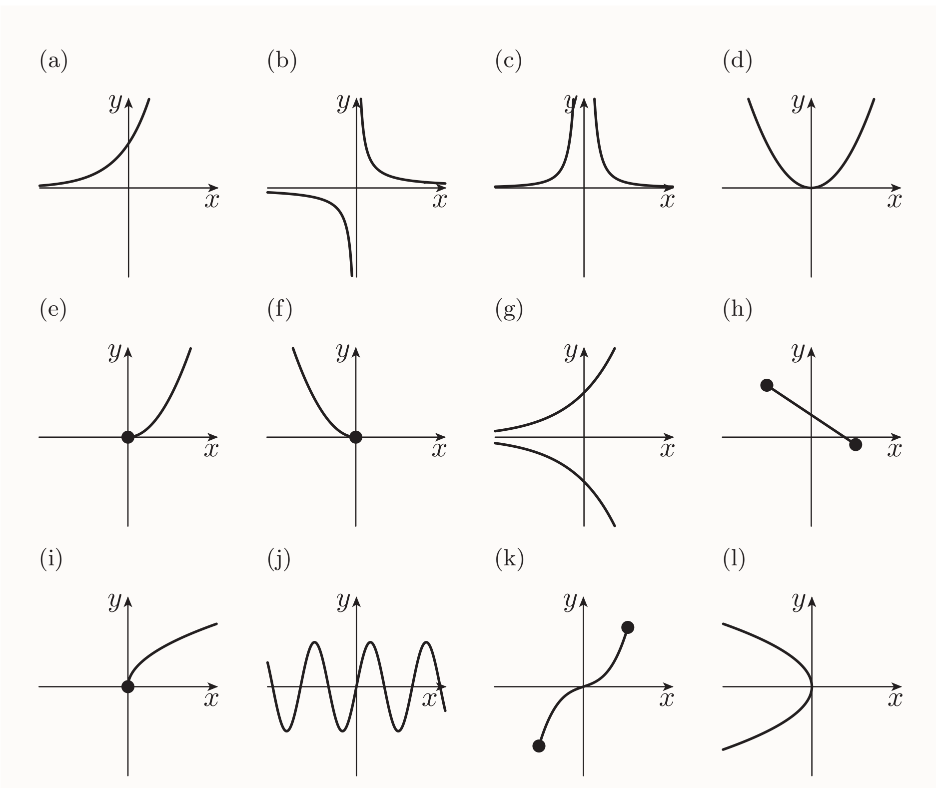


Figure 57 The graphs of (a) a one-to-one function and (b) a function that isn't one-to-one

Activity 35 *Recognising the graphs of one-to-one functions*

For each of the following diagrams, state whether it's the graph of a one-to-one function, the graph of a function that isn't one-to-one, or not even the graph of a function.



The following important fact summarises the ideas that you've just met.

Only one-to-one functions have inverse functions.

A function that has an inverse function is said to be **invertible**. So 'invertible function' means the same as 'one-to-one function'.

Now let's consider the domains of inverse functions. If f is any one-to-one function, then the domain of f^{-1} is the image set of the original function f. To see this, think of the mapping diagram for the original function f. Now imagine reversing the directions of all the arrows, to obtain the mapping diagram for the inverse function f^{-1}. The numbers that have arrows starting from them are the numbers in the image set of the original function f. So the domain of f^{-1} is the image set of f.

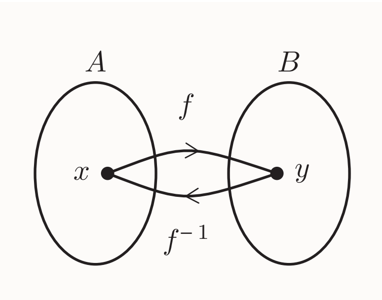


Figure 58 A mapping diagram illustrating a function f and its inverse function f^{-1}. The set B is the *image set* of f, not necessarily its whole codomain.

You can also see, by thinking about these diagrams, that the image set of an inverse function f^{-1} is the domain of the original function f. Here's a concise definition of an inverse function, which summarises what you've seen so far. It's illustrated in Figure 58.

Inverse functions

Suppose that f is a one-to-one function, with domain A and image set B. Then the **inverse function**, or simply **inverse**, of f, denoted by f^{-1}, is the function with domain B whose rule is given by

$$f^{-1}(y) = x, \quad \text{where} \quad f(x) = y.$$

The image set of f^{-1} is A.

The next example illustrates how you can use this definition to find an inverse function, even when the rule of the original function is more complicated than those that you've seen so far.

Example 6 *Finding an inverse function*

Find the inverse function of the function

$$f(x) = 2x + 1.$$

Solution

To find the rule of f^{-1}, rearrange the equation $f(x) = y$ to express x in terms of y.

The equation $f(x) = y$ gives

$$2x + 1 = y$$
$$2x = y - 1$$
$$x = \tfrac{1}{2}(y - 1).$$

Hence the rule of f^{-1} is

$$f^{-1}(y) = \tfrac{1}{2}(y - 1).$$

Usually, change the input variable from y to x, as this is the letter normally used for the input variable of a function.

That is, it is

$$f^{-1}(x) = \tfrac{1}{2}(x - 1).$$

The domain of f^{-1} is the image set of f, which is $\mathbb{R}$. By the domain convention, you don't need to write down this domain.

The domain of f^{-1} is the whole of $\mathbb{R}$, so the rule above completely specifies f^{-1}.

The method used to find the rule of the inverse function in Example 6 is summarised below.

Strategy:
To find the rule of the inverse function of a one-to-one function f

- Write $y = f(x)$ and rearrange this equation to express x in terms of y.
- Use the resulting equation $x = f^{-1}(y)$ to write down the rule of f^{-1}. (Usually, change the input variable from y to x.)

If it's *possible* to rearrange an equation $y = f(x)$ to express x in terms of y (so that each value of y gives exactly one value of x), then this shows that the function f *has* an inverse function, whose rule is given by the rearranged equation. Remember that to fully specify the function f^{-1}, you also have to indicate its domain, which, as you've seen, is the image set of f. (As always, if the domain of f^{-1} is the largest set of real numbers for which its rule is applicable, then there's no need to state its domain explicitly.)

Activity 36 *Finding inverse functions*

Find the inverse functions of the following functions. (Remember to find the domain of the inverse function in each case.)

(a) $f(x) = 3x - 4$ (b) $f(x) = 2 - \frac{1}{2}x$ (c) $f(x) = 5 + \dfrac{1}{x}$

In cases that are trickier than those in Activity 36, it often helps to obtain a graph of the original function f. For example, this can be useful if the domain of the function f isn't the largest set of real numbers for which its rule is applicable, or if you're not sure whether f is one-to-one. Here's an example.

Example 7 *Finding another inverse function*

Does the function

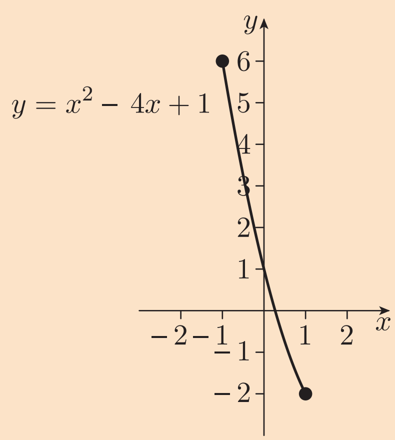
$$f(x) = x^2 - 4x + 1 \quad (x \in [-1, 1])$$

have an inverse function? If so, find it.

Solution

Obtain a sketch or computer plot of the graph of f.

The graph of f is shown below.



Think about whether every horizontal line that crosses the graph of f does so exactly once.

The graph shows that f is one-to-one and therefore has an inverse function.

Try to find the rule of f^{-1} in the usual way, by rearranging the equation $f(x) = y$. For a quadratic function like f, it helps to begin by completing the square.

The equation $f(x) = y$ gives

$$x^2 - 4x + 1 = y$$
$$(x-2)^2 - 4 + 1 = y$$
$$(x-2)^2 - 3 = y$$
$$(x-2)^2 = y + 3$$
$$x - 2 = \pm\sqrt{y+3}$$
$$x = 2 \pm \sqrt{y+3}$$

Decide whether the + or the − applies. Remember that the final equation above is a rearrangement of the equation $f(x) = y$, so x is an element of the domain of f, which is $[-1, 1]$. Now 2 *plus* the positive square root of something can't be equal to a number in this interval, but 2 *minus* the positive square root of something can, so the correct sign is −.

Since the domain of f is $[-1, 1]$, each input value x of f is less than 2. So

$$x = 2 - \sqrt{y+3}.$$

Hence the rule of f^{-1} is

$$f^{-1}(y) = 2 - \sqrt{y+3};$$

that is,

$$f^{-1}(x) = 2 - \sqrt{x+3}.$$

To find the domain of f^{-1}, find the image set of f, using the graph to help you.

The domain of f^{-1} is the image set of f. The graph shows that this is

$$[f(1), f(-1)] = [-2, 6].$$

Finally, specify f^{-1} by stating its domain and rule.

So the inverse function of f is the function

$$f^{-1}(x) = 2 - \sqrt{x+3} \quad (x \in [-2, 6]).$$

Activity 37 *Finding more inverse functions*

In each of parts (a)–(c), determine whether the function has an inverse function. If it does, then find the inverse function.

(a) $f(x) = x^2 + 2x + 2 \quad (x \in (-2, 2))$

(b) $f(x) = x^2 + 2x + 2 \quad (x \in (0, 2))$

(c) $f(x) = 1 - x \quad (x \in [-3, 1])$

Here's a useful fact that sometimes gives you a quick way of confirming that a function has an inverse function.

> If a function is either increasing on its whole domain, or decreasing on its whole domain, then it is one-to-one and so has an inverse function.

This fact holds because if a function is either increasing on its whole domain or decreasing on its whole domain, then any horizontal line drawn on its graph will cross the graph at most once, as illustrated in Figure 59.

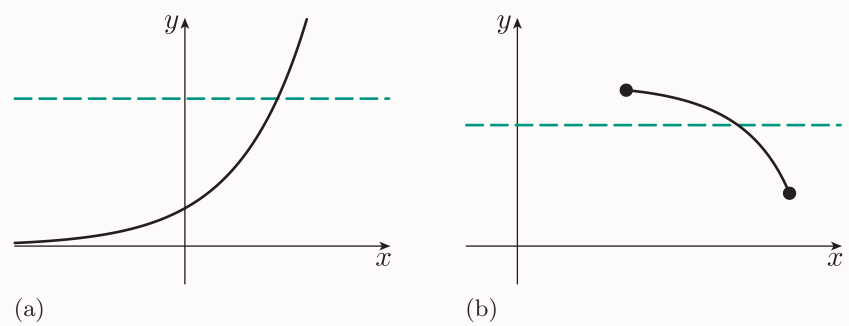


Figure 59 (a) A function that's increasing on its whole domain (b) a function that's decreasing on its whole domain

Here's another useful property of inverse functions, which you can understand by thinking about mapping diagrams. If a function f has an inverse function f^{-1}, then f^{-1} also has an inverse function, namely f. In other words, f and f^{-1} are inverses of each other. This is because the mapping diagram for each of these functions is obtained by reversing the directions of the arrows in the mapping diagram for the other function. In particular, each of the functions f and f^{-1} 'undoes' the effect of the other. So if you take any value x in the domain of f, input it to f, and then input the resulting output value to f^{-1}, then you get the value x back again; and similarly if you take any value x in the domain of f^{-1}, input it to f^{-1}, and then input the resulting output value to f, then you get the value x back again. These two facts can be stated concisely as follows, using the notation for composite functions.

For any pair of inverse functions f and f^{-1},

$$(f^{-1} \circ f)(x) = x, \quad \text{for every value } x \text{ in the domain of } f \text{, and}$$
$$(f \circ f^{-1})(x) = x, \quad \text{for every value } x \text{ in the domain of } f^{-1}.$$

A warning

When you're working with the notation f^{-1}, where f is a function, it's important to appreciate that it *doesn't* mean the function g with rule

$$g(x) = (f(x))^{-1}; \quad \text{that is,} \quad g(x) = \frac{1}{f(x)}.$$

This function g is called the **reciprocal** of the function f, and it's never denoted by f^{-1}. For example, consider the function $f(x) = x + 5$. Its inverse function is

$$f^{-1}(x) = x - 5,$$

whereas its reciprocal is

$$g(x) = \frac{1}{x+5}.$$

Graphs of inverse functions

There's a useful geometric connection between the graph of a function and the graph of its inverse function. Figure 60(a) shows the graphs of the function $f(x) = x^2 - 1$ $(x \geq 0)$ and its inverse function $f^{-1}(x) = \sqrt{x+1}$ $(x \geq -1)$, drawn on axes with equal scales. Similarly, Figure 60(b) shows the graphs of the function $f(x) = 2x + 1$ $(x \in [-2, 1])$ and its inverse function $f^{-1}(x) = \frac{1}{2}(x - 1)$ $(x \in [-3, 3])$, again drawn on axes with equal scales.

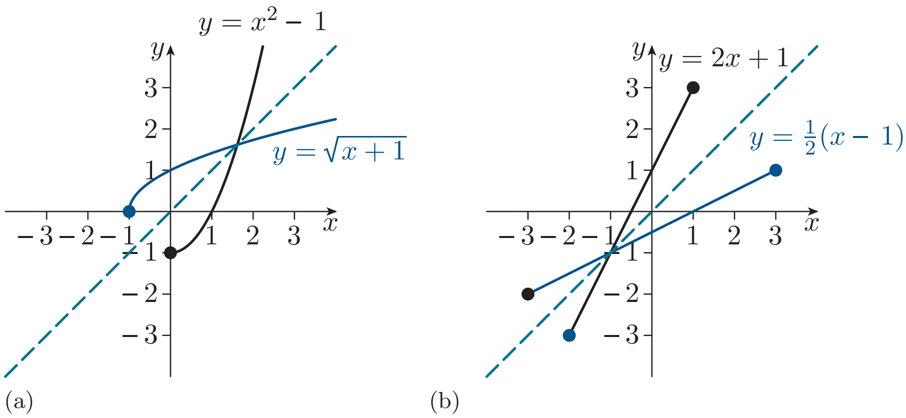


Figure 60 Graphs of pairs of inverse functions

In each case the graphs of f and f^{-1} are the reflections of each other in the line $y = x$, which is shown as a green dashed line.

This happens for every pair of inverse functions, when their graphs are drawn on axes *with equal scales*. To see why, let's start by considering any point on the graph of the function $f(x) = x^2 - 1$ $(x \geq 0)$. For example, the point $(2, 3)$ lies on this graph, because inputting 2 to this function f gives the output 3. It follows that inputting 3 to the inverse function f^{-1} gives the output 2, and so the point $(3, 2)$ lies on the graph of f^{-1}. You can see that, for any pair of inverse functions f and f^{-1}, if you swap the coordinates of any point on the graph of f, then you'll get the coordinates of a point on the graph of f^{-1}, and vice versa.

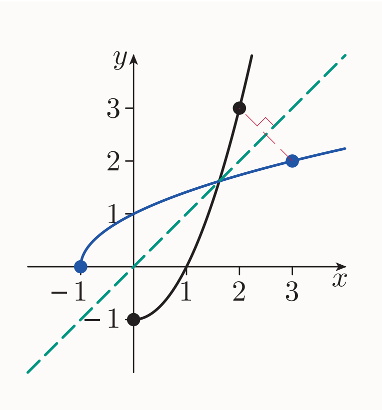


Figure 61 The points $(2, 3)$ and $(3, 2)$ are reflections of each other in the line $y = x$

Now when you swap the coordinates of a point, the resulting point is the reflection of the original point in the line $y = x$ (provided the axes have equal scales). This is illustrated in Figure 61, for the example discussed above. This reasoning explains the connection between the graphs of a function and its inverse function, which is summarised below.

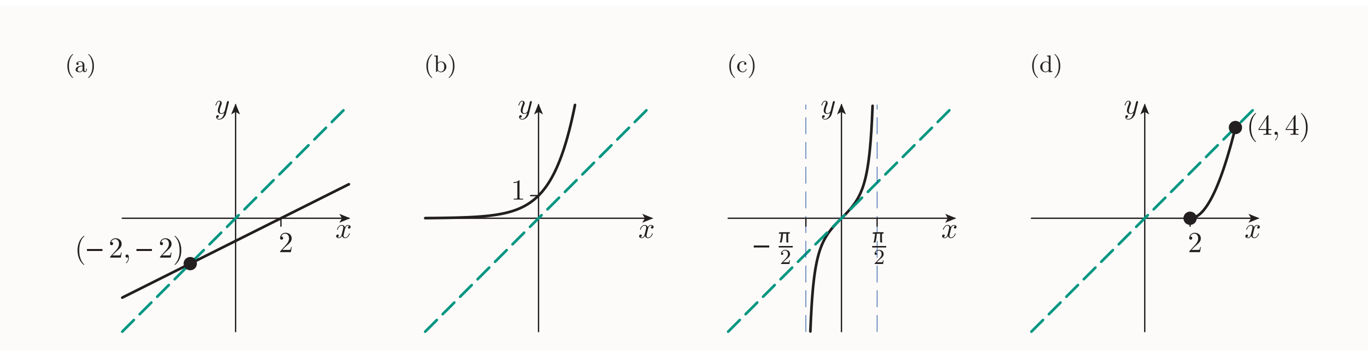
Graphs of inverse functions

The graphs of a pair of inverse functions are the reflections of each other in the line $y = x$ (when the coordinate axes have equal scales).

Activity 38 *Sketching graphs of inverse functions*

Each of the following diagrams shows the graph of a function, drawn on axes with equal scales. The line $y = x$ is shown as a green dashed line. The vertical dashed lines in graph (c) are asymptotes.

For each graph, sketch the graph of the inverse function.



Functions that aren't one-to-one

Finally in this subsection, let's consider functions that don't have inverses, because they're not one-to-one. An example of such a function is $f(x) = x^2$, whose graph is shown in Figure 62.

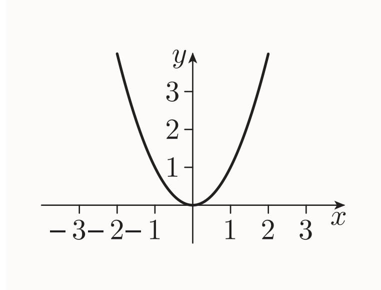


Figure 62 The graph of the function $f(x) = x^2$

When you have a function like this, it's sometimes useful to consider 'some sort of inverse' of the function. For example, you'd probably consider the function $h(x) = \sqrt{x}$ to be 'some sort of inverse' of the function $f(x) = x^2$.

Here's the approach that we take in situations like this. Starting with the function f that's not one-to-one, we specify a new function that has the same rule as f, but a smaller domain. We choose the new domain to make sure that the following two conditions are satisfied:

- the new function is one-to-one and therefore has an inverse
- the image set of the new function is the same as the image set of the original function.

For example, for the function $f(x) = x^2$, we could take the new function to be the function

$$g(x) = x^2 \quad (x \in [0, \infty)),$$

whose graph is shown in Figure 63(a). This function g is one-to-one, and therefore has an inverse function, namely

$$g^{-1}(x) = \sqrt{x},$$

whose graph is shown in Figure 63(b).

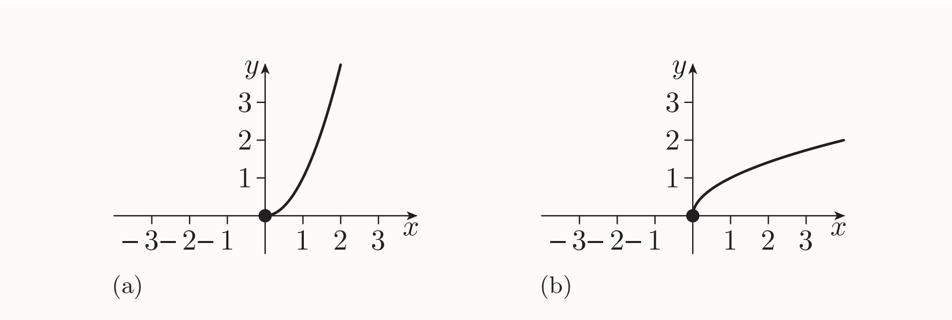


Figure 63 The graphs of (a) the function $g(x) = x^2$ $(x \in [0, \infty))$ and (b) its inverse function $g^{-1}(x) = \sqrt{x}$

A function that's obtained from another function f by keeping the rule the same but removing some numbers from the domain is called a **restriction** of the original function f. The process of obtaining such a function is called **restricting** the domain of f, or **restricting** f. So the function g above is a restriction of the function $f(x) = x^2$. The graph of a restriction of a function is obtained by erasing part of the graph of the original function.

When you want to restrict the domain of a function that isn't one-to-one to enable you to find an inverse function, there's always more than one possibility for the new domain. For example, for the function $f(x) = x^2$, you could have chosen the new domain $(-\infty, 0]$ instead of $[0, \infty)$. This would have given you the new function g whose graph is shown in

Figure 64(a). The inverse function of this function is $g^{-1}(x) = -\sqrt{x}$, whose graph is shown in Figure 64(b).

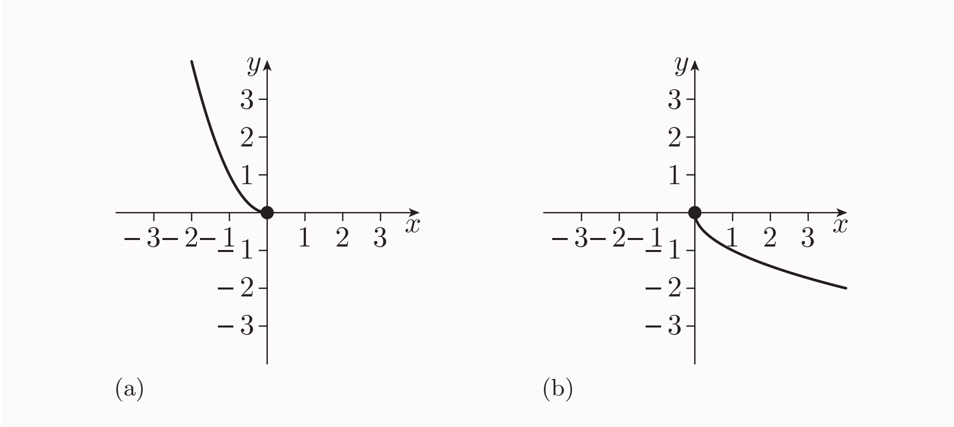


Figure 64 The graphs of (a) the function $g(x) = x^2$ $(x \in (-\infty, 0])$ and (b) its inverse function $g^{-1}(x) = -\sqrt{x}$

Usually when we restrict the domain of a function to enable us to find an inverse function, we choose the new domain that seems to be the most convenient.

Activity 39 *Restricting a function to find an inverse function*

Consider the function $f(x) = (x-1)^2$, whose graph is shown in Figure 65. Specify a one-to-one function g that is a restriction of f and has the same image set as f. Find the inverse function g^{-1} of g, and sketch its graph.

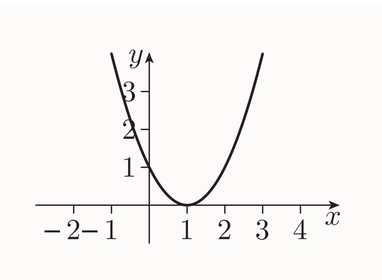


Figure 65 The graph of $y = (x-1)^2$

In Unit 4 you'll see some more examples of this process of restricting the domain of a function to enable you to find an inverse function. It's useful in particular for *trigonometric functions*, which you'll meet in that unit.

4 Exponential functions and logarithms

In this section you'll revise exponential functions and logarithms. In particular, you'll have the opportunity to practise working with logarithms. It's important that you can do this fluently and correctly.

4.1 Exponential functions

An **exponential function** is a function whose rule is of the form

$$f(x) = b^x,$$

where b is a positive constant, not equal to 1. (Note that in some other texts the case $b = 1$ is not excluded.) The number b is called the **base number**, or just **base**, of the exponential function. For example, $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are exponential functions, with base numbers 2 and $\frac{1}{2}$, respectively. The graphs of these two functions are shown in Figure 66. The y-intercept is 1 in each case.

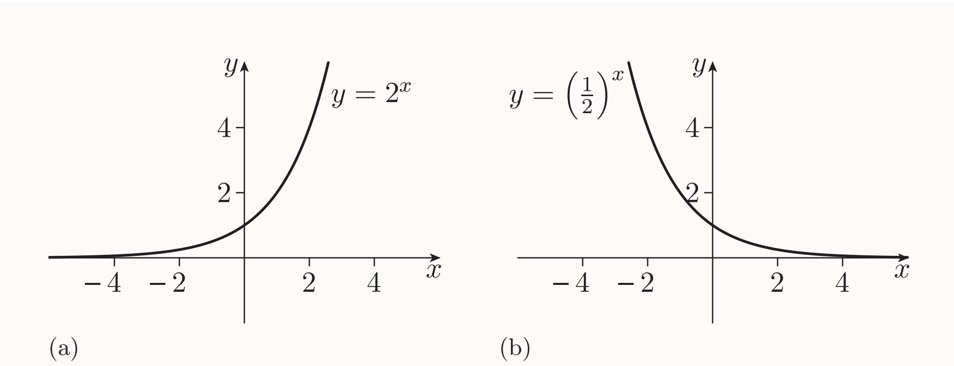


Figure 66 The graphs of (a) $y = 2^x$ (b) $y = \left(\frac{1}{2}\right)^x$

To see why these graphs have the shapes that they do, first consider the function $f(x) = 2^x$. Some values of this function are given in Table 1. The corresponding points are shown in Figure 67 (except that the final two points are off the scale).

Table 1 Values of 2^x

x	-4	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

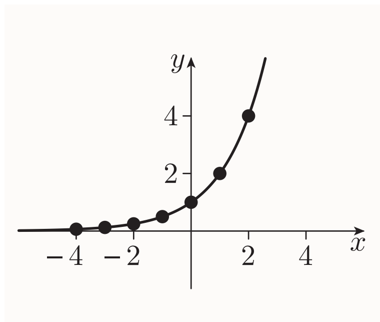


Figure 67 Points on the graph of $y = 2^x$ with integer values of x

Notice that $f(0) = 2^0 = 1$, which explains why the y-intercept is 1.

Notice also that each time the value of x increases by 1 unit to the next integer up, the value of $f(x)$ *doubles*. So as x takes values that are further and further along the number line to the right, the value of $f(x) = 2^x$ increases, and increases more and more rapidly. This explains the shape of the graph as x increases.

Similarly, each time the value of x decreases by 1 unit to the next integer down, the value of $f(x)$ *halves*. So, as x takes values that are further and further along the number line to the left, the value of $f(x) = 2^x$ gets closer and closer to zero, but never reaches zero. This gives the shape of the graph as x decreases.

The shape of the graph of the function $g(x) = \left(\frac{1}{2}\right)^x$ can be explained in a similar way, and you might like to think it through for yourself.

Alternatively, you can deduce it from the shape of the graph of the function $f(x) = 2^x$. Notice that, for any number x,

$$\left(\tfrac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}.$$

So the rule of the function $g(x) = \left(\frac{1}{2}\right)^x$ can be written as $g(x) = 2^{-x}$, and hence, by what you saw in Subsection 2.3, its graph is the same shape as the graph of $f(x) = 2^x$, but reflected in the y-axis.

In the next activity you're asked to investigate the shapes of the graphs of some more exponential functions.

Activity 40 *Investigating the graphs of exponential functions*

Open the *Exponential functions* applet. Display the graph of $y = b^x$ for various positive values of b, and observe the shapes of the graphs.

In particular, notice how the graphs obtained when $0 < b < 1$ differ from those obtained when $b > 1$.

In Activity 40 you should have observed the following facts, which are illustrated in Figure 68.

Graphs of exponential functions

The graph of the function $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has the following features.

- The graph lies entirely above the x-axis.
- If $b > 1$, then the graph is increasing, and it gets steeper as x increases.
- If $0 < b < 1$, then the graph is decreasing, and it gets less steep as x increases.
- The x-axis is an asymptote.
- The y-intercept is 1.
- The closer the value of b is to 1, the flatter is the graph.

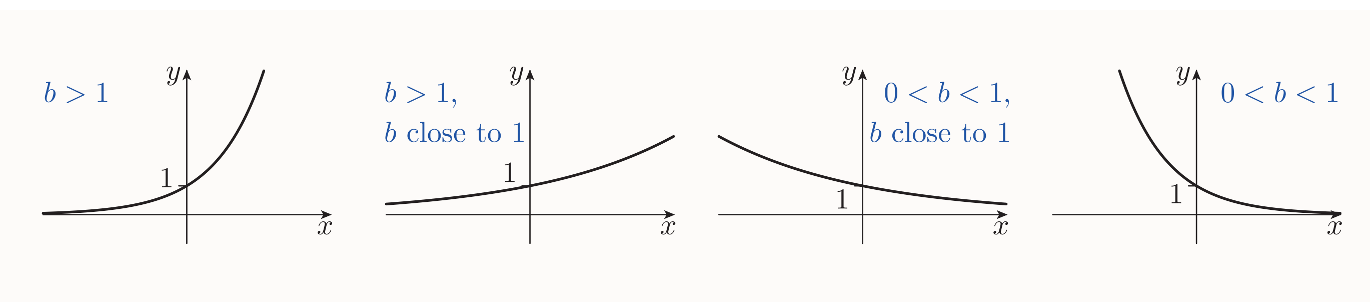


Figure 68 Graphs of equations of the form $y = b^x$

A helpful way to remember the final feature listed in the box above is to notice that when the value of b is *exactly* 1, the function $f(x) = b^x$ is the function $f(x) = 1^x$, that is, $f(x) = 1$, and hence its graph is the horizontal line with y-intercept 1, as shown in Figure 69. Remember, though, that this function f isn't an exponential function – it's a constant function.

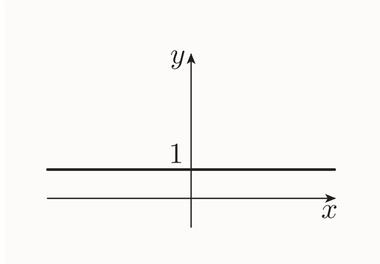


Figure 69 The graph of $y = 1$

The exponential function

There's a particular exponential function that's crucially important in mathematics. You've seen that the graph of every exponential function $f(x) = b^x$ passes through the point $(0, 1)$. The steepness of the graph at this point depends on the value of b, as you can see from the examples in Figure 70.

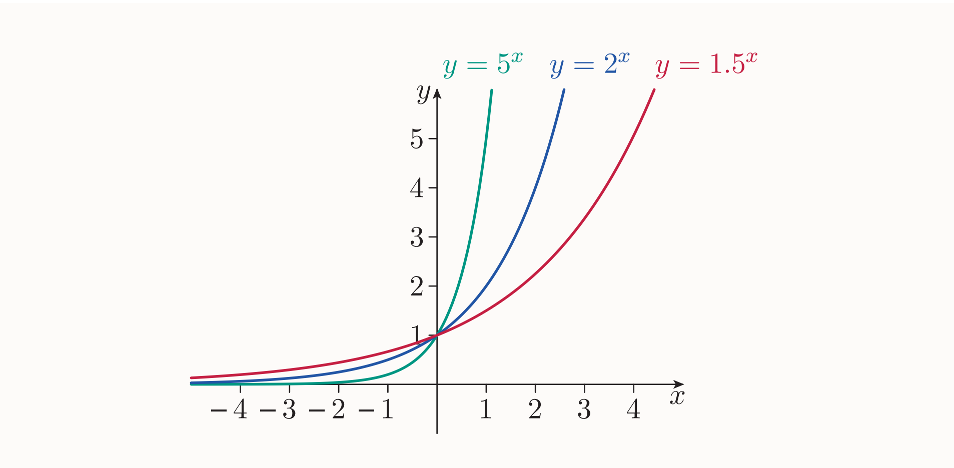


Figure 70 Graphs of exponential functions

In the next activity you're asked to investigate the steepness of the graphs of exponential functions at the point $(0, 1)$.

Activity 41 *Investigating the gradient of the graph of $y = b^x$ at $(0, 1)$*

Use the *Exponential functions* applet to do the following.

(a) Reset the applet and choose the option to show a grid. Zoom in on the point $(0, 1)$ until the graph looks like a straight line. Keep the scales on the axes the same as each other.

(b) Change the value of b until the gradient of the graph at $(0, 1)$ appears to be exactly 1; that is, until the graph goes up by the same distance vertically as it goes along horizontally to the right. What value of b seems to achieve this?

In Activity 41 you should have found that the value of b that gives a gradient of 1 at $(0, 1)$ seems to be about 2.7. In fact, the precise value is a special number, usually denoted by the letter e, whose first few digits are $2.718\,28\ldots$. The number e is irrational, like π, so its digits have no repeating pattern, and it can't be written down exactly as a fraction or a terminating decimal. It occurs frequently in mathematics, and you'll learn more about it, and why it's so important, later in the module.

So the exponential function with the rule $f(x) = e^x$ has the special property that its gradient is exactly 1 at the point $(0, 1)$. Its graph is shown in Figure 71. This function is important both in applications of mathematics and in pure mathematics, and because of its importance it's sometimes referred to as ***the* exponential function**. The expression e^x is sometimes written as $\exp x$, or $\exp(x)$. An approximate value for e is available from your calculator keypad, just as for π, and you can also work out values of e^x by using a function button on your calculator.

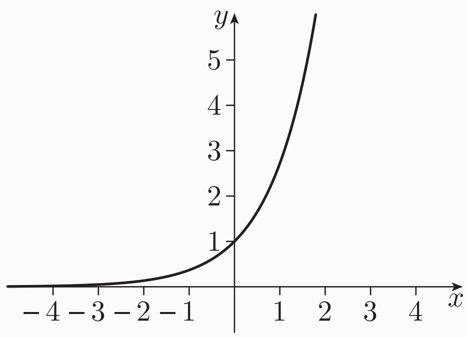


Figure 71 The graph of $y = e^x$

The use of the letter e for the base of *the* exponential function was introduced by Leonhard Euler (see page 214).

4.2 What is a logarithm?

In this subsection, you'll revise *logarithms*, which are closely related to exponential functions. Logarithms are sometimes called *logs*, for short.

The first thing to remember about logarithms is that whenever you're working with them, you're always using logarithms to a particular *base* (also called *base number*). Let's start by considering *logarithms to base* 10, which are known as **common logarithms**. These are defined as follows. The **logarithm to base** 10 of a number x is the power to which 10 must be raised to give the number x.

For example,

the logarithm to base 10 of 100 is 2, because $100 = 10^2$.

Similarly,

the logarithm to base 10 of 1000 is 3, because $1000 = 10^3$, and
the logarithm to base 10 of $\frac{1}{10}$ is -1, because $\frac{1}{10} = 10^{-1}$.

The logarithm to base 10 of a number x is denoted by $\log_{10} x$, so the three logarithms found above can be written as follows:

$$\log_{10} 100 = 2$$
$$\log_{10} 1000 = 3$$
$$\log_{10}\left(\tfrac{1}{10}\right) = -1.$$

You can see that if you can easily write a number as a power of 10, then it's straightforward to find its logarithm to base 10. For other numbers, you can use your calculator to find an approximate value. For example, a calculator gives

$$\log_{10} 42 = 1.623\,249\,2\ldots,$$

which is the same as saying that

$$10^{1.623\,249\,2\ldots} = 42.$$

The button on a calculator for finding common logarithms is usually labelled 'log'.

Activity 42 *Understanding logarithms to base 10*

(a) Find the following numbers, without using your calculator.

(i) $\log_{10} 10\,000$ (ii) $\log_{10} \frac{1}{100}$ (iii) $\log_{10} 10$ (iv) $\log_{10} 1$

(b) If the number x is such that $\log_{10} x = \frac{1}{2}$, what is x?

(c) Use your calculator to find the following numbers, to three decimal places.

(i) $\log_{10} 3700$ (ii) $\log_{10} 370$ (iii) $\log_{10} 37$

(iv) $\log_{10} 3.7$ (v) $\log_{10} 0.37$ (vi) $\log_{10} 0.037$

Notice that only positive numbers have logarithms to base 10. For example, the negative number -2 has no logarithm to base 10, because there's no power to which 10 can be raised to give -2. Similarly, 0 has no logarithm to base 10, because there's no power to which 10 can be raised to give 0.

However, logarithms themselves can be positive, negative or zero. For example, you've seen that

$$\log_{10} 100 = 2, \quad \log_{10}\left(\tfrac{1}{10}\right) = -1 \quad \text{and} \quad \log_{10} 1 = 0.$$

Now let's consider logarithms to other bases. Like the base of an exponential function, the base of a logarithm can be any positive number except 1. Logarithms to other bases work in the same way as logarithms to base 10. Here's a general definition of logarithms, to any base.

Logarithms

The **logarithm to base** b of a number x, denoted by $\log_b x$, is the power to which the base b must be raised to give the number x. So the two equations

$$y = \log_b x \quad \text{and} \quad x = b^y$$

are equivalent.

Remember that:

- the base b must be positive and not equal to 1
- only positive numbers have logarithms, but logarithms themselves can be any number.

For example, $\log_6 36 = 2$ because $36 = 6^2$.

Activity 43 *Understanding logarithms to any base*

(a) Find the following numbers without using your calculator.

(i) $\log_3 9$ (ii) $\log_2 8$ (iii) $\log_4 64$ (iv) $\log_5 25$ (v) $\log_4 2$

(vi) $\log_8 2$ (vii) $\log_2 \frac{1}{2}$ (viii) $\log_2 \frac{1}{8}$ (ix) $\log_3 \frac{1}{27}$

(x) $\log_8 \frac{1}{8}$ (xi) $\log_3 3$ (xii) $\log_4 \frac{1}{4}$ (xiii) $\log_6 6$

(xiv) $\log_5 \sqrt{5}$ (xv) $\log_7 \sqrt[3]{7}$ (xvi) $\log_2 1$ (xvii) $\log_{15} 1$

(b) Find the solution of each of the following equations in x.

(i) $\log_2 x = 5$ (ii) $\log_8 x = \frac{1}{3}$ (iii) $\log_7 x = 1$

You've seen that it's straightforward to write down the logarithm to base b of a number if you can express the number as a power of b. In particular, for any base b, it's straightforward to write down the logarithm to base b of 1, and the logarithm to base b of b itself, because

$$1 = b^0 \quad \text{and} \quad b = b^1.$$

This gives the following useful facts.

Logarithm of the number 1 and logarithm of the base

For any base b,

$$\log_b 1 = 0 \quad \text{and} \quad \log_b b = 1.$$

John Napier (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier for the purpose of easing the labour involved in astronomical and navigational calculations. Napier's rather awkward initial formulation, published in 1614, was greeted enthusiastically by the English mathematician Henry Briggs (1561–1630), who immediately set about trying to improve it. The following year Briggs visited Napier, and together they invented logarithms to base 10, the first publication of which appeared in 1617 shortly after Napier's death.

The most common choices for the base of logarithms are 10, 2 and e. (Remember that e is the important constant whose value is approximately 2.718.) Usually, once you've chosen a base, you use the same base for all the logarithms in your calculations (otherwise, your calculations may be wrong!).

In university-level mathematics, the number most commonly used as the base for logarithms is e. As you'll see later in the module, logarithms to base e turn out to be easier to work with in many ways than logarithms to any other base.

Logarithms to base e are called **natural logarithms**. The notation 'ln' is often used in place of '$\log_e$', and this is the notation that will be used in this module. For example, the natural logarithm of 5 is written as

$$\ln 5 \quad \text{rather than} \quad \log_e 5.$$

(The first symbol in 'ln' is the letter l, not the digit 1.)

There's no consensus about how the notation 'ln' should be pronounced, but some common pronunciations are 'log', 'ell enn', 'linn' and 'lawn'. The box below summarises the definition of a natural logarithm, using this notation.

Natural logarithms

The **natural logarithm** of a number x, denoted by $\ln x$, is the power to which the base e must be raised to give the number x. So the two equations

$$y = \ln x \quad \text{and} \quad x = e^y$$

are equivalent.

Scientific calculators have a button for finding natural logarithms, usually labelled ‘ln’.

The properties that $\log_b 1 = 0$ and $\log_b b = 1$ for any base b give the following two useful properties of natural logarithms.

$$\ln 1 = 0 \quad \text{and} \quad \ln e = 1.$$

Here are some calculations involving natural logarithms for you to try.

Activity 44 *Understanding natural logarithms*

(a) Find the following numbers without using your calculator.

(i) $\ln e^4$ (ii) $\ln e^2$ (iii) $\ln e^{3/5}$ (iv) $\ln \sqrt{e}$

(v) $\ln\left(\dfrac{1}{e}\right)$ (vi) $\ln\left(\dfrac{1}{e^3}\right)$

(b) If the number x is such that $\ln x = -\frac{1}{2}$, what is x?

(c) Use your calculator to find the following numbers to three decimal places.

(i) $\ln 5100$ (ii) $\ln 510$ (iii) $\ln 51$

(iv) $\ln(51e)$ (v) $\ln(51e^2)$

The natural logarithm of a number x is sometimes denoted by $\log x$, rather than by $\ln x$ or $\log_e x$. For example, this notation is used by some computer algebra systems. Confusingly, the same notation, $\log x$, is also sometimes used to denote $\log_{10} x$, the common logarithm of x. For example, as mentioned earlier, the button on a calculator for finding common logarithms is usually labelled ‘log’. Even more confusingly, the notation $\log x$ is also sometimes used to denote the logarithm of x with no specific base (but the same base for each use of the notation). So, wherever you see the notation $\log x$ used, it’s important to check its meaning. The notation $\log x$ isn’t used at all in the main books of this module, but you may have to use it with your calculator or computer.

Logarithms to base e were first described as 'natural' logarithms by the Danish mathematician Nicolaus Mercator (1620–87), in his treatise *Logarithmotechnica* published in 1668. Mercator's treatise was difficult to follow, so later writers tended to refer to the exposition of it given by the English mathematician John Wallis (1616–1703), published in the same year. The first published use of the notation 'ln' for a natural logarithm was in a book written by an American mathematician, Irving Stringham, which was published in 1893. He explained his choice, in a textbook published a little later, as follows: 'In place of ${}^{e}\log$ we shall henceforth use the shorter symbol ln, made up of the initial letters of *logarithm* and of *natural* or *Napierian*.'

4.3 Logarithmic functions

A **logarithmic function** is a function whose rule is of the form

$$f(x) = \log_b x,$$

where b is a positive constant, not equal to 1. The number b is called the **base** or **base number** of the logarithmic function. For example, $f(x) = \log_2 x$ and $g(x) = \ln x$ are logarithmic functions, with bases 2 and e, respectively.

For any positive constant b not equal to 1, the logarithmic function $f(x) = \log_b x$ is the inverse function of the exponential function $g(x) = b^x$. This is because, as you've seen, the two equations

$$y = \log_b x \quad \text{and} \quad x = b^y$$

are rearrangements of each other.

In particular, the function $f(x) = \ln x$ is the inverse function of the function $g(x) = e^x$, which explains why these two functions usually share the same button on a calculator.

So the graphs of $f(x) = \log_b x$ and $g(x) = b^x$ are reflections of each other in the line $y = x$ (provided that the coordinate axes have equal scales).

For example, Figure 72 shows the graphs of $y = \ln x$ and $y = e^x$.

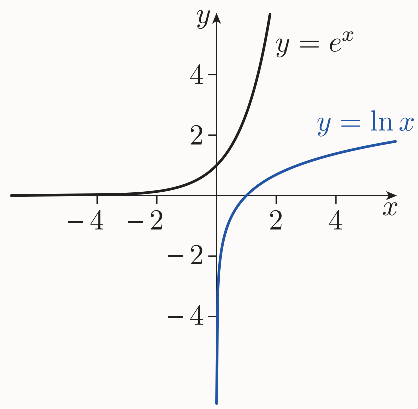


Figure 72 The graphs of $y = \ln x$ and $y = e^x$

You can deduce the following general properties of the graphs of logarithmic functions from the properties of the graphs of exponential functions, using the fact that they're reflections of each other in the line $y = x$. These properties are illustrated in Figure 73.

Graphs of logarithmic functions

The graph of the function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$, has the following features.

- The graph lies entirely to the right of the y-axis.
- If $b > 1$, then the graph is increasing, and it gets less steep as x increases.
- If $0 < b < 1$, then the graph is decreasing, and it gets less steep as x increases.
- The y-axis is an asymptote.
- The x-intercept is 1.
- The closer the value of b is to 1, the steeper is the graph.

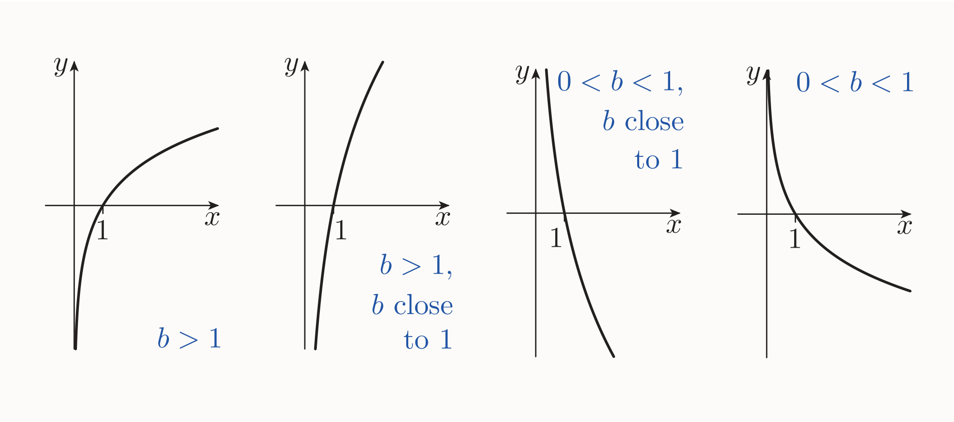


Figure 73 Graphs of equations of the form $y = \log_b x$

The properties in the box below are often useful when you're working with logarithms. They're just the two facts about composing a pair of inverse functions that are stated in the box on page 263, in the particular case when the functions are $f(x) = b^x$ and $f^{-1}(x) = \log_b x$.

For any base b,

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b x} = x.$$

In particular,

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x.$$

These properties hold for all appropriate values of x.

Activity 45 *Simplifying expressions involving e and $\ln$*

Simplify the following expressions.

(a) $e^{\ln(7x)}$ (b) $\ln(e^{8x})$ (c) $\ln(e^{2x}) + \ln(e^{3x})$ (d) $\ln(e^2) - \ln e$

(e) $\ln(e^{x/2}) + 3\ln 1$ (f) $e^{2\ln c}$ (g) $e^{\ln(3a)} + 4e^0$

(h) $\ln(e^{y+2}) + 2\ln(e^{y-1})$ (i) $e^{3\ln B}$ (j) $e^{2+\ln x}$

Hint: in some parts you may need to use the index laws from Unit 1.

4.4 Logarithm laws

In this subsection you'll revise three laws for logarithms, which are often useful when you're working with logarithms. These laws are really just the same as the following three index laws that were given in Unit 1, rewritten using logarithm notation.

Three index laws from Unit 1

$$b^m b^n = b^{m+n} \qquad \frac{b^m}{b^n} = b^{m-n} \qquad (b^m)^n = b^{mn}$$

Here are the three logarithm laws.

Three logarithm laws

$$\log_b x + \log_b y = \log_b (xy)$$

$$\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$$

$$r \log_b x = \log_b (x^r)$$

As with the index laws, these logarithm laws apply to all appropriate numbers. So the base b of the logarithms can be any positive number except 1, the numbers x and y must be positive (since only positive numbers have logarithms), and r can be any number (in particular, it can be fractional and/or negative).

To see how these three logarithm laws are deduced from the three index laws above, let's write $m = \log_b x$ and $n = \log_b y$. This is the same as saying that $x = b^m$ and $y = b^n$.

So,

$$xy = b^m b^n = b^{m+n},$$

from which it follows that

$$\log_b(xy) = m + n = \log_b x + \log_b y.$$

This is the first logarithm law.

Also,

$$\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n},$$

from which it follows that

$$\log_b \left(\frac{x}{y}\right) = m - n = \log_b x - \log_b y.$$

This is the second logarithm law.

Finally,

$$x^r = (b^m)^r = b^{mr},$$

from which it follows that

$$\log_b(x^r) = mr = r\log_b x.$$

This is the third logarithm law.

Example 8 *Using the logarithm laws*

Write the expression $3\ln 6 - 2\ln 2$ as the logarithm of a single number.

Solution

$$3\ln 6 - 2\ln 2 = \ln 6^3 - \ln 2^2 = \ln\left(\frac{6^3}{2^2}\right) = \ln 54$$

Here are some examples for you to try.

Activity 46 *Using the logarithm laws*

(a) Write each of the following expressions as the logarithm of a single number or expression.

(i) $\ln 5 + \ln 3$ (ii) $\ln 2 - \ln 7$ (iii) $3\ln 2$

(iv) $\ln 3 + \ln 4 - \ln 6$ (v) $\ln 24 - 2\ln 3$ (vi) $\frac{1}{3}\log_{10} 27$

(vii) $3\log_2 5 - \log_2 3 + \log_2 6$ (viii) $\frac{1}{2}\ln(9x) - \ln(x+1)$

(b) Simplify the following expressions.

(i) $\ln c^3 - \ln c$ (ii) $3\ln(p^2)$ (iii) $\ln(y^2) + 2\ln y - \frac{1}{2}\ln(y^3)$

(iv) $\ln(3u) - \ln(2u)$ (v) $\ln(4x) + 3\ln x - \ln(e^6)$ (vi) $\frac{1}{2}\ln(u^8)$

(c) Can you explain the pattern in the answers to Activity 42(c)? (This activity is on page 271.)

Hint: notice that each number in Activity 42(c) is of the form $\log_{10}(37 \times 10^n)$, for some integer n.

(d) Suppose that the multiplication button on your scientific calculator doesn't work. Can you use the remaining buttons to find the value of 1567×2786, at least approximately?

Hint: start by writing $1567 \times 2786 = e^{\ln(1567\times 2786)}$.

Solving exponential equations

An **exponential equation** is one in which the unknown is in the exponent, such as

$$5^x = 130.$$

Equations like this often arise when you use exponential functions to model real-life situations. You'll see some examples in the next subsection.

You can solve exponential equations by using the third of the three logarithm laws given earlier in this subsection. For this purpose it's best to think of the law with its left and right sides swapped:

$$\log_b(x^r) = r\log_b x. \quad \text{In particular,} \quad \ln(x^r) = r\ln x.$$

Example 9 *Solving an exponential equation*

Solve the equation $2 \times 1.5^{3x} = 45$, giving the solution to three significant figures.

Solution

The equation can be solved as follows.

$$2 \times 1.5^{3x} = 45$$

Rearrange it into the form $(\text{number})^{(\text{expression in } x)} = \text{number}$.

$$1.5^{3x} = 22.5$$

Take the natural logarithm of both sides.

$$\ln(1.5^{3x}) = \ln 22.5$$

Use the fact that $\ln(x^r) = r\ln x$.

$$3x\ln 1.5 = \ln 22.5$$

Divide both sides by the coefficient of the unknown.

$$x = \frac{\ln 22.5}{3\ln 1.5}$$

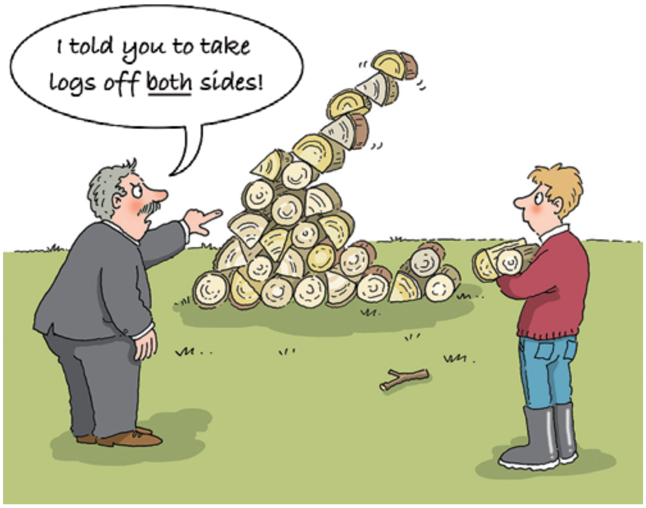
Use your calculator to evaluate the answer.

$$x = 2.56 \text{ (to 3 s.f.)}.$$

The solution is approximately $x = 2.56$.

(Check: when $x = 2.56$,
LHS $= 2 \times 1.5^{3\times 2.56} = 45.020\,557\ldots \approx 45 =$ RHS.)

The crucial step in the method demonstrated in Example 9 is to 'take logs of both sides' of the exponential equation. Then you can use the property in the box above to turn the exponent into a factor, which makes it straightforward to solve the equation. You don't have to use natural logarithms in this method – you can use logarithms to any base, as long as you're consistent.



In fact, if your calculator has a button for finding logarithms to any base, then you can make the working in Example 9 slightly shorter by proceeding as follows, starting from the second equation in the solution:

$$1.5^{3x} = 22.5$$
$$3x = \log_{1.5} 22.5$$
$$x = \tfrac{1}{3} \log_{1.5} 22.5$$
$$x = 2.56 \text{ (to 3 s.f.).}$$

Activity 47 *Solving exponential equations*

Solve the following exponential equations, giving your answers to three significant figures.

(a) $5^x = 0.5$ (b) $4e^{7t} = 64$ (c) $5 \times 2^{u/2} + 30 = 600$

(d) $2^{3x-5} = 100$

Remember that you can always check a solution that you've found for an equation by substituting it into the equation.

In the final activity of this subsection you can learn how to use the computer to work with expressions involving exponentials and logarithms.

Activity 48 *Working with exponentials and logarithms on the computer*

Work through Section 5 of the *Computer algebra guide*.

4.5 Alternative form for exponential functions

You've seen that an *exponential function* is one whose rule is of the form

$$f(x) = b^x,$$

where b is a positive constant, not equal to 1.

There's a useful alternative way to express a rule of this form. If you let k be the number such that $e^k = b$ (in other words, if you take $k = \ln b$), then

$$f(x) = b^x = (e^k)^x = e^{kx}.$$

For example, another way to express the rule

$$f(x) = 5^x$$

is, approximately,

$$f(x) = e^{1.609\,438x},$$

because $\ln 5 = 1.609\,437\,912\ldots$.

In general, we have the following fact.

Any exponential function $f(x) = b^x$, where b is a positive constant not equal to 1, can be written in the alternative form

$$f(x) = e^{kx},$$

where k is a non-zero constant. The constant k is given by $k = \ln b$.

Activity 49 *Understanding the alternative form of an exponential function*

Consider the exponential function $f(x) = 3^x$. Write its rule in the form $f(x) = e^{kx}$, giving the constant k to seven significant figures. Use each form of the rule in turn to work out $f(1.5)$ to three significant figures, and check that you get the same answer.

The fact in the box above gives us the following alternative definition of an exponential function.

An **exponential function** is a function whose rule is of the form

$$f(x) = e^{kx},$$

where k is a non-zero constant. This alternative form is the one that's usually used in university-level mathematics, as it turns out to be easier to work with, for reasons that you'll see later in this module.

The properties of the graphs of exponential functions that you saw in Subsection 4.1 can be stated in terms of this alternative form as in the box below. These properties are illustrated in Figure 74.

Graphs of exponential functions

The graph of the function $f(x) = e^{kx}$, where $k \neq 0$, has the following features.

- The graph lies entirely above the x-axis.
- If $k > 0$, then the graph is increasing, and it gets steeper as x increases.
- If $k < 0$, then the graph is decreasing, and it gets less steep as x increases.
- The x-axis is an asymptote.
- The y-intercept is 1.
- The closer the value of k is to 0, the flatter is the graph.

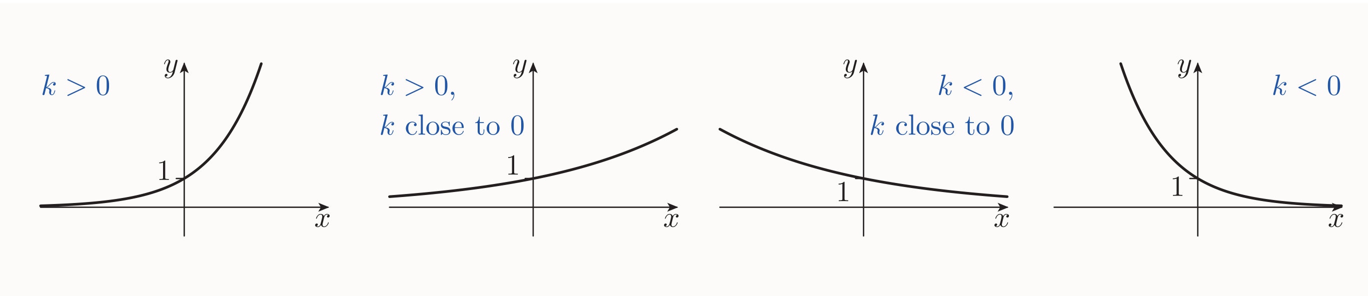


Figure 74 Graphs of equations of the form $y = e^{kx}$

The fact that every exponential function can be written in the form $f(x) = e^{kx}$ also tells you the following enlightening fact.

The graph of every exponential function is a horizontal scaling of the graph of the exponential function $f(x) = e^x$.

That's because, from what you saw in Subsection 2.3, if f is any function, then the graph of $y = f(kx)$ is a horizontal scaling of the graph of $y = f(x)$ by the factor $c = 1/k$. So the graphs of all exponential functions have the same basic shape, just stretched or squashed horizontally by different amounts, and possibly reflected in the y-axis.

It follows that the graphs of all logarithmic functions are *vertical* scalings of each other, since the graphs of logarithmic functions are reflections of the graphs of exponential functions in the line $y = x$.

4.6 Exponential models

Functions with rules of the form $f(x) = ab^x$, where a is a non-zero number and b is a positive number not equal to 1, are useful for modelling some types of real-life situations. Models of this type are called **exponential models**.

From what you saw in the previous subsection, rules of this form are the same as rules of the form $f(x) = ae^{kx}$, where a and k are non-zero numbers. We'll use this alternative form in this subsection.

From your work in Subsection 2.2, you know that the graph of the function $f(x) = ae^{kx}$ is obtained by vertically scaling the graph of the function $g(x) = e^{kx}$ by a factor of a. Also, as you saw in Subsection 4.5, the graph of the function $g(x) = e^{kx}$ is itself obtained by horizontally scaling the graph of the function $h(x) = e^x$ by a factor of $1/k$. So the graph of any function of the form $f(x) = ae^{kx}$ is obtained by scaling the graph of the function $h(x) = e^x$ both horizontally and vertically. In particular, since the graph of $h(x) = e^x$ has y-intercept 1, the graph of $f(x) = ae^{kx}$ has y-intercept a. Figure 75 shows the graphs of some functions of this form.

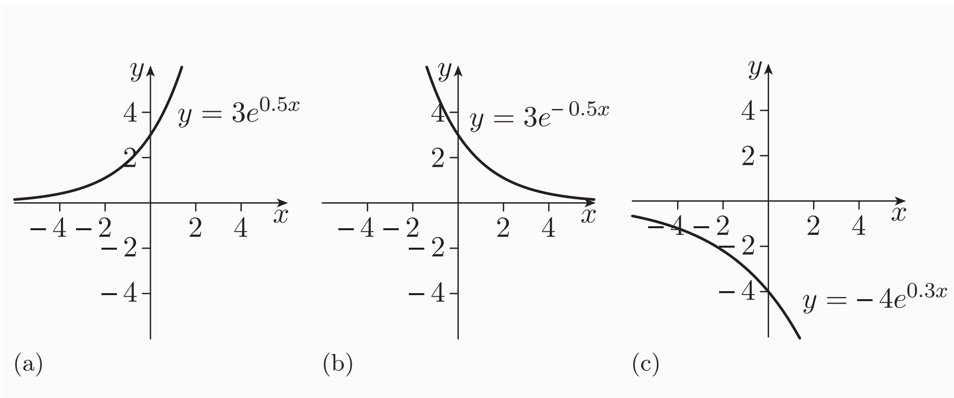


Figure 75 The graphs of three functions of the form $f(x) = ae^{kx}$

A quantity that changes in a way that can be modelled by a function whose rule is of the form $f(x) = ae^{kx}$, where a and k are non-zero constants, is said to **change exponentially**. If a and k are both positive, then the graph of f looks like the graph in Figure 76(a), or a part of it. In this case the quantity is said to **grow exponentially**, the function is called an **exponential growth function**, and the graph is called an **exponential growth curve**.

Similarly, if a is positive as before but k is negative, then the graph of f looks like the graph in Figure 76(b), or a part of it. In this case the quantity is said to **decay exponentially**, the function is called an **exponential decay function**, and the graph is called an **exponential decay curve**.

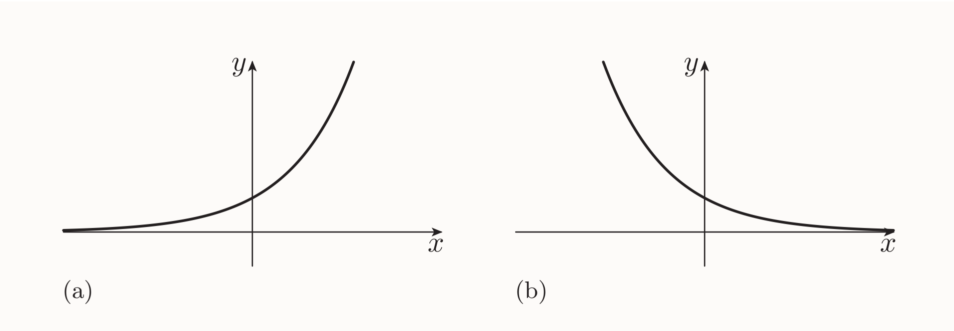


Figure 76 (a) An exponential growth curve (b) an exponential decay curve

An example of a real-life situation that can often be modelled by an exponential decay function is the concentration of a prescription drug in a patient's bloodstream. The concentration always peaks shortly after the drug is administered, and then falls, quickly at first but more slowly later, as the drug is metabolised or eliminated from the body. For example, Figure 77 shows such a model. The model covers the period of time after the concentration of the drug peaks. The unit 'μg/ml' is micrograms per millilitre.

Prescription drugs

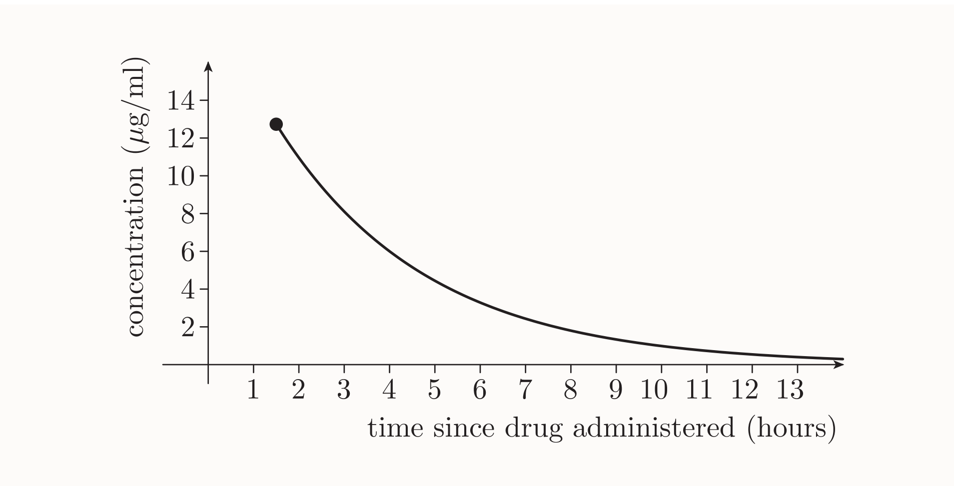


Figure 77 An exponential decay curve modelling the concentration of a particular prescription drug in a patient's bloodstream

The next example is about a model of this type. The solution to part (a) of the example involves solving a pair of *simultaneous exponential equations*.

Example 10 *Using an exponential model*

A drug is administered to a patient, and blood tests show that after one and a quarter hours the concentration of the drug in the patient's bloodstream is 105 ng/ml (nanograms per millilitre), and after two and three quarter hours it is 86.0 ng/ml. Assume that the concentration of the drug in the patient's bloodstream can be modelled by an exponential decay function f, where $f(t)$ is the concentration (in ng/ml) at time t (in hours) after the drug was administered, for $t \geq 1$.

(a) Find the function f, giving each of the two constants in it to three significant figures.

(b) What is the predicted concentration of the drug after 10 hours?

Solution

(a) Let $f(t) = ae^{kt}$, where a and k are constants.

Use the information that you know about f to find the values of a and k.

From the information given in the question, $f(1.25) = 105$ and $f(2.75) = 86.0$, so

$$ae^{1.25k} = 105 \quad \text{and} \quad ae^{2.75k} = 86.0. \tag{3}$$

These are simultaneous exponential equations in a and k. To solve them, first eliminate a, by dividing one equation by the other.

Hence

$$\frac{ae^{2.75k}}{ae^{1.25k}} = \frac{86}{105},$$

which gives

$$\begin{aligned} e^{2.75k-1.25k} &= \frac{86}{105} \\ e^{1.5k} &= \frac{86}{105} \\ 1.5k &= \ln\left(\frac{86}{105}\right) \\ k &= \frac{\ln(86/105)}{1.5} \\ k &= -0.133\,075\ldots. \end{aligned}$$

Now find a, by substituting into one of equations (3).

Substituting $k = -0.133\,075\ldots$ into the first of equations (3) gives

$$ae^{1.25\times(-0.133\,075\ldots)} = 105$$

$$a = \frac{105}{e^{1.25\times(-0.133\,075\ldots)}}$$

$$a = 124.002\ldots.$$

So $a = 124$ and $k = -0.133$, both to three significant figures.

Hence the required function f is given, approximately, by

$$f(t) = 124e^{-0.133t} \quad (t \geq 1).$$

(b) The predicted concentration after 10 hours, in ng/ml, is

$$f(10) = (124.002\ldots)e^{(-0.133\,075\ldots)\times 10} = 32.7712\ldots.$$

That is, the concentration is predicted to be about 33 ng/ml.

There are many other types of real-life situations that can be modelled by exponential growth and decay functions. These include the level of radioactivity in a sample of radioactive material, which decreases over time, and, sometimes, the size of a population of organisms, such as bacteria, plants, animals or even human beings, which often increases over a period of time. The next activity is about an exponential model for the growth of a population of bacteria.

Radioactive waste

Activity 50 *Using an exponential model*

The fluid in a test tube was inoculated with a sample of bacteria, which began to divide after 8 hours. Tests after 9 hours and 12 hours showed that the test tube contained about 300 and 4200 bacteria per millilitre, respectively. Assume that the number of bacteria per millilitre can be modelled by an exponential growth function f, where $f(t)$ is the number of bacteria per millilitre at time t (in hours), for $8 \leq t \leq 24$.

(a) Find the function f, giving each of the two constants in it to three significant figures.

(b) What is the predicted number of bacteria per millilitre after 24 hours? Give your answer in scientific notation, to two significant figures.

Bacteria dividing

Exponential growth and decay functions have an interesting characteristic property. If f is such a function, and you start with any value of x and *add* a number to it, then the value of $f(x)$ is *multiplied* by a factor. This factor doesn't depend on the value of x that you started with, but only on the number that you added.

For example, if f is such a function, and $f(5)$ happens to be 3 times larger than $f(1)$, then also $f(20)$ will be 3 times larger than $f(16)$ (because you need to add the same number of units, namely 4, to get from 16 to 20 as from 1 to 5). Similarly, $f(154)$ will be 3 times larger than $f(150)$, and $f(2)$ will be 3 times larger than $f(-2)$, and so on.

To see why this happens, consider any exponential growth or decay function $f(x) = ae^{kx}$. When you start with a particular value of x, and add p units, say, the value of $f(x)$ changes from

$$f(x) = ae^{kx} \quad \text{to} \quad f(x+p) = ae^{k(x+p)}.$$

Now,

$$ae^{k(x+p)} = ae^{kx+kp} = ae^{kx}e^{kp} = f(x) \times e^{kp}.$$

So the value of $f(x)$ is multiplied by the factor e^{kp}.

Here's a concise statement of the fact discussed above.

A characteristic property of exponential growth and decay functions

If $f(x) = ae^{kx}$, then whenever p units are added to the value of x, the value of $f(x)$ is multiplied by e^{kp}.

For any exponential function f, the factor by which $f(x)$ is multiplied when a particular amount is added to the value of x is called a **growth factor** or a **decay factor**, according to whether f is an exponential growth or decay function. Growth factors are greater than 1, and decay factors are between 0 and 1, exclusive. The next example illustrates how to find growth or decay factors.

Example 11 *Finding growth factors for exponential growth*

Suppose that the number $f(t)$ of bacteria per millilitre of fluid in a test tube at time t (in hours) after it was inoculated is modelled by the exponential growth function

$$f(t) = 3e^{0.4t} \quad (9 \leq t \leq 30).$$

By what factor is the number of bacteria predicted to multiply

(a) every hour? (b) every two and a half hours?

Give your answers to three significant figures.

Solution

Use the property in the box above.

(a) In every hour, the number of bacteria is predicted to multiply by the factor

$$e^{0.4\times 1} = e^{0.4} = 1.49 \text{ (to 3 s.f.)}.$$

(b) In every period of 2.5 hours, the number of bacteria is predicted to multiply by the factor

$$e^{0.4\times 2.5} = e^{1} = 2.72 \text{ (to 3 s.f.)}.$$

Here are some examples of exponential growth and decay for you to analyse.

Activity 51 *Finding growth factors for exponential growth*

Suppose that the number of trees of a particular variety in a region, at time t (in decades) after the variety was introduced, is modelled by the exponential growth function

$$f(t) = 700e^{0.06t} \quad (10 \le t \le 50).$$

By what factor is the size of the tree population predicted to multiply

(a) every decade? (b) every century? (c) every five years?

Give your answers to three significant figures.

The phrase 'grown exponentially' is used frequently in the media. However the intended meaning is nearly always that a quantity has grown a lot, or has grown quickly, rather than the true meaning. It's possible for a quantity to grow slowly but exponentially. For example, this is true of the size of the population of trees in Activity 51.

Modelling tree populations is a long-term project

Activity 52 *Finding decay factors for exponential decay*

Suppose that the radioactivity level $r(t)$ (in becquerels) of a sample of radioactive material at time t (in years) after the level was first measured is modelled by the function

$$r(t) = 2800e^{-0.035t} \quad (t \ge 0).$$

By what factor is the radioactivity level predicted to multiply

(a) every year? (b) every 25 years? (c) every century?

Give your answers to two significant figures.

The characteristic property of exponential growth and decay functions in the box above gives us a useful way to describe how quickly a particular instance of exponential growth or decay is taking place. For example, consider again the situation in Example 11, where the number $f(t)$ of bacteria per millilitre of fluid in a test tube at time t (in hours) after it was inoculated is modelled by the exponential growth function

$$f(t) = 3e^{0.4t} \quad (9 \leq t \leq 30).$$

You can see that, by the property in the box, if p is the number such that

$$e^{0.4p} = 2, \tag{4}$$

then whenever you add p to t, the value of $f(t)$ is multiplied by *exactly* 2. This value of p is called the *doubling period* for the function f. Solving equation (4) gives

$$0.4p = \ln 2; \quad \text{that is,} \quad p = (\ln 2)/0.4 = 1.732\,867\ldots.$$

So the number of bacteria per millilitre doubles every 1.7 hours, approximately. In general we make the following definitions.

Doubling and halving periods

Suppose that f is an exponential growth function. Then p is the **doubling period** of f if whenever you add p to x, the value of $f(x)$ doubles.

Similarly, suppose that f is an exponential decay function. Then p is the **halving period** of f if whenever you add p to x, the value of $f(x)$ halves.

Here's a summary of how to find doubling or halving periods.

Strategy:
To find a doubling or halving period

If $f(x) = ae^{kx}$ is an exponential growth function (so $k > 0$), then the doubling period of f is the solution p of the equation $e^{kp} = 2$; that is, $p = (\ln 2)/k$.

Similarly, if $f(x) = ae^{kx}$ is an exponential decay function (so $k < 0$), then the halving period of f is the solution p of the equation $e^{kp} = \frac{1}{2}$; that is, $p = (\ln \frac{1}{2})/k = -(\ln 2)/k$.

In many exponential models, such as the ones in Example 11 and Activities 51 and 52, the input variable represents time. When this is the case, the doubling or halving period is more usually called the **doubling** or **halving time**. In the particular case of radioactive decay, the halving time is usually called the **half-life**.

Activity 53 *Finding doubling and halving periods*

Find the doubling time of the exponential growth in Activity 51, and the half-life of the exponential decay in Activity 52. Give your answers to three significant figures.

Radiocarbon dating is a method of estimating the age of material that originates from a living organism, such as an animal or a plant. A living organism absorbs the radioactive isotope carbon-14 from the atmosphere. When it dies, the amount of carbon-14 in its remains decays exponentially, with a half-life of 5730 years. A measurement of the amount of carbon-14 that's left can be used to estimate when the organism was alive, up to about 60 000 years ago. For example, radiocarbon dating of organic material at Stonehenge was used in 2008 to determine that the monument was built in about 2300 BC.

Radiocarbon dating was developed by Willard Libby at the University of Chicago in 1950. He was awarded the Nobel Prize for Chemistry in 1960.

Willard Libby (1908–80)

Stonehenge, Wiltshire

5 Inequalities

In the module so far you've worked with equations of various types. However, sometimes you need to work not with equations, but with *inequalities*. Whereas an equation expresses the fact that two quantities are equal, an inequality expresses the fact that one quantity is greater than, less than, greater than or equal to, or less than or equal to, another quantity.

5.1 Terminology for inequalities

You saw some examples of inequalities in Section 1 of this unit. Here are a few more:

$$x \geq 5, \quad 3a - 2 > b + 1, \quad p^2 - 5p + 6 \leq 0, \quad 2^t > 10.$$

In general, an **inequality** is the same as an equation, except that instead of an equals sign it contains one of the four inequality signs, $<$, $\leq$, $>$ and $\geq$. In other words, an inequality is made up of two expressions, with one of the four inequality signs between them.

Inequality signs

$<$ is less than

$\leq$ is less than or equal to

$>$ is greater than

$\geq$ is greater than or equal to

Much of the terminology that applies to equations also applies to inequalities. For example:

- an inequality **in** x is one that contains the variable x and no other variables (the first inequality above is an example)
- the **solutions** of an inequality are the values of its variables that **satisfy** it – in other words, they are the real numbers for which it is true (for instance, the values $x = 6$ and $t = 4$ are solutions of the first and fourth inequalities above, respectively, and there are many other solutions)
- **solving** an inequality means finding all its solutions
- two inequalities are **equivalent** if they contain the same variables and are satisfied by the same values of those variables
- **rearranging** an inequality means transforming it into an equivalent inequality.

Most inequalities have either infinitely many solutions or no solutions. A useful way to specify all the solutions of an inequality is to state the set that they form. This set is called the **solution set** of the inequality. For example, the solution set of the simple inequality $x \geq 5$ is the interval $[5, \infty)$, and the solution set of the inequality $x^2 < 0$ is the empty set $\varnothing$.

In this section you'll learn how to rearrange inequalities, and you'll see how you can use this technique to help you solve some types of inequality in one variable. You'll also see how the graphs of functions can help you visualise the solution sets of inequalities, and you'll learn some further techniques that you can use to extend the range of inequalities in one variable that you can solve.

5.2 Rearranging inequalities

You can rearrange inequalities using methods similar to those that you use for rearranging equations. However there are some important differences. Here are the three main ways to rearrange an inequality.

Rearranging inequalities

Carrying out any of the following operations on an inequality gives an equivalent inequality.

- Rearrange the expressions on one or both sides.
- Swap the sides, *provided you reverse the inequality sign.*
- Do any of the following things to both sides:
  - add or subtract something
  - multiply or divide by something that's positive
  - multiply or divide by something that's negative, *provided you reverse the inequality sign.*

To understand why these rules make sense, consider, for example, the simple, true inequality $1 < 2$.

- You can swap the sides of this inequality to obtain another true inequality, *provided you reverse the inequality sign.* This gives $2 > 1$.
- You can multiply both sides of the original inequality $1 < 2$ by the positive number 3, say, to obtain another true inequality. This gives $3 < 6$.
- You can multiply both sides of the original inequality $1 < 2$ by the negative number -3, say, to obtain another true inequality, *provided you reverse the inequality sign.* This gives $-3 > -6$.

When you're rearranging an inequality, you should not multiply or divide both sides by a variable, or by an expression containing a variable, unless you know that the variable or expression takes only positive values or takes only negative values. That's because in other cases you can't follow the rule about when to reverse the inequality sign, so usually the inequality that you obtain won't be equivalent to the original one.

5.3 Linear inequalities

As you'd expect, a **linear inequality in one unknown** is the same as a linear equation in one unknown, but with one of the four inequality signs in place of the equals sign. You can solve such an inequality by using the same methods that you use to solve a linear equation in one unknown (see Subsection 5.3 of Unit 1), except that you need to use the rules for rearranging inequalities instead of the rules for rearranging equations. Here's an example.

Example 12 *Solving a linear inequality*

Solve the inequality

$$\frac{5x}{2} - 1 > 4x + \frac{7}{2}.$$

Give your answer as a solution set in interval notation.

Solution

The inequality

$$\frac{5x}{2} - 1 > 4x + \frac{7}{2}$$

can be rearranged as follows.

Clear the fractions, by multiplying through by 2. This is a positive number, so leave the direction of the inequality sign unchanged.

$$5x - 2 > 8x + 7$$

Get all the terms in the unknown on one side, and all the other terms on the other side. Collect like terms.

$$5x - 8x > 7 + 2$$
$$-3x > 9$$

Obtain x by itself on one side, by dividing through by -3. This is a negative number, so reverse the inequality sign.

$$x < -3$$

The solution set is the interval $(-\infty, -3)$.

Activity 54 *Solving linear inequalities*

(a) Solve the following linear inequalities. Give your answers as solution sets in interval notation.

(i) $5x + 2 < 3x - 1$ (ii) $6 - 3x \geq \dfrac{x}{2} - 1$

(b) An employee has achieved 54%, 69% and 72% in the first three of her four assignments in a workplace training course. She has to achieve an average of at least 60% over all four assignments (which are equally-weighted) to pass the course. Let x% be the score that she will achieve for her final assignment. Write down an inequality that x must satisfy if the employee is to pass the course, and solve it to find the acceptable values of x.

5.4 Quadratic inequalities

Sometimes you have to solve a **quadratic inequality**. As you'd expect, this is an inequality that's the same as a quadratic equation, but with one of the inequality signs in place of the equals sign.

The first step in solving a quadratic inequality is to simplify it, if possible, in the same ways that you'd simplify a quadratic equation (this was covered in Subsection 4.3 of Unit 2). In particular, you should get all the terms on one side of the inequality sign, leaving just the number zero on the other side. Of course, you have to simplify the inequality using the rules for rearranging inequalities.

Once you've simplified the inequality, you can solve it by considering the graph of the function whose rule is given by the quadratic expression on one side of the inequality, as illustrated in the next example.

Example 13 *Solving a quadratic inequality*

Solve the inequality

$$x^2 + 3 \geq 4x.$$

Solution

Get all the terms on one side. Simplify the inequality in other ways if possible – in this case there's no further simplification to be done.

Rearranging the inequality $x^2 + 3 \geq 4x$ gives

$$x^2 - 4x + 3 \geq 0.$$

Roughly sketch the graph of $y = x^2 - 4x + 3$. The only features that you need to show are the x-intercepts and whether the parabola is u-shaped or n-shaped. In particular, there's no need to find the vertex.

The x-intercepts of the graph of $f(x) = x^2 - 4x + 3$ are given by

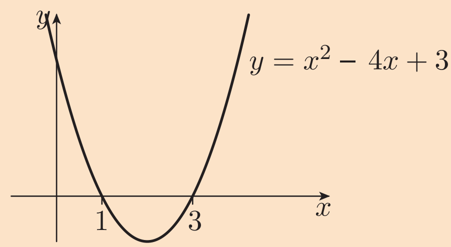
$$x^2 - 4x + 3 = 0;$$

that is,

$$(x - 1)(x - 3) = 0,$$

so they are 1 and 3.

Also, the graph is u-shaped. So the graph is as follows.



From the sketch you can see that the value of the expression $x^2 - 4x - 3$ is greater than or equal to zero precisely when $x \leq 1$ or $x \geq 3$ (since the parabola lies above or on the x-axis for these values of x).

The solution set is $(-\infty, 1] \cup [3, \infty)$.

As you become more familiar with the method in Example 13, you might find that you don't need to sketch the graph – instead you can just work out the intercepts, note whether the graph is u-shaped or n-shaped, and picture the sketch in your head. You might like to try this in the later parts of the next activity.

If you prefer, you can always make sure that the coefficient of x^2 is *positive* by, if necessary, multiplying the inequality through by -1 and reversing the inequality sign. Then you don't need to think about whether the parabola is u-shaped or n-shaped, as it will always be u-shaped.

Activity 55 *Solving quadratic inequalities*

Solve the following inequalities.

(a) $x^2 + x < 2$ (b) $-x^2 + 7x < 10$ (c) $-x^2 \geq 2x$

The next example illustrates an alternative method for solving a quadratic inequality. This method starts in the same way as the method that you've just seen – you rearrange the inequality to obtain a quadratic expression on one side. Then, instead of using a graph to determine the values of x that make the value of the quadratic expression greater than, less than or equal to zero, you obtain the same information by constructing a type of table known as a *table of signs*. You can use this alternative method whenever you can factorise the quadratic expression. It might seem a little more complicated than the method that you've just practised, but it's worth learning, as you can use it to solve more complicated inequalities. You'll see this in the next subsection.

Example 14 *Solving a quadratic inequality using a table of signs*

Solve the inequality

$$2x^2 + x - 6 \geq 0.$$

Solution

Make sure that all the terms are on one side, and simplify the inequality in other ways if possible – here there's no simplification to be done. Next, factorise the quadratic expression on the left-hand side, and find the values of x for which the resulting factors are equal to zero.

Factorising gives

$$(x+2)(2x-3) \geq 0.$$

The factors $x+2$ and $2x-3$ are equal to zero when $x=-2$ and when $x=\frac{3}{2}$, respectively.

Construct a table, as follows. In the top row, write, in increasing order, the values of x for which the factors are equal to zero, and also the largest open intervals to the left and right of, and between, these values. In the left-most column, write the factors $x+2$ and $2x-3$, and then their product $(x+2)(2x-3)$.

We have the following table.

x	$(-\infty,-2)$	-2	$(-2,\frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2},\infty)$
$x+2$					
$2x-3$					
$(x+2)(2x-3)$					

The factor $x+2$ is zero when $x=-2$, negative when $x<-2$ and positive when $x>-2$, so fill in its row appropriately. Use similar thinking to fill in the row for the factor $2x-3$. Finally, use the signs of $x+2$ and $2x-3$ to find the signs of $(x+2)(2x-3)$ for the various values of x, and enter these in the bottom row. For example, if $x+2$ and $2x-3$ are both negative, then their product is positive.

x	$(-\infty,-2)$	-2	$(-2,\frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2},\infty)$
$x+2$	$-$	0	$+$	$+$	$+$
$2x-3$	$-$	$-$	$-$	0	$+$
$(x+2)(2x-3)$	$+$	0	$-$	0	$+$

Use the entries in the bottom row to help you solve the inequality. Remember that you're looking for the values of x such that $(x+2)(2x-3)$ is positive or zero.

The solution set is $(-\infty,-2] \cup [\frac{3}{2},\infty)$.

Activity 56 *Solving quadratic inequalities using tables of signs*

Solve the following inequalities using tables of signs.

(a) $2x^2 - 5x - 3 < 0$ (b) $-2x^2 + 4x + 16 \leq 0$

5.5 More complicated inequalities

In this final subsection you'll see how to use tables of signs to solve more complicated inequalities. In particular, this method is useful for some inequalities that contain algebraic fractions, such as

$$\frac{3}{x-1} \leq 2x + 3.$$

When you're trying to solve an inequality that contains an algebraic fraction, remember that you're not allowed to multiply it through by a variable or expression, unless you know that the variable or expression takes only positive values or takes only negative values. That's because otherwise you can't follow the rule about when to reverse the inequality sign. So, for example, you can't simplify the inequality above by multiplying through by $x - 1$. (You could consider the two cases $x - 1 > 0$ and $x - 1 < 0$ separately, but it's more straightforward to use the method illustrated in the following example.)

Example 15 *Solving an inequality containing algebraic fractions*

Solve the inequality

$$\frac{3}{x-1} \leq 2x + 3.$$

Solution

Get all the terms on one side, leaving only 0 on the other side.

$$\frac{3}{x-1} - (2x + 3) \leq 0$$

Combine the terms into a single algebraic fraction, and simplify it.

$$\frac{3}{x-1} - \frac{(2x+3)(x-1)}{x-1} \leq 0$$
$$\frac{3-(2x+3)(x-1)}{x-1} \leq 0$$
$$\frac{3-(2x^2+x-3)}{x-1} \leq 0$$
$$\frac{-2x^2-x+6}{x-1} \leq 0$$
$$\frac{2x^2+x-6}{x-1} \geq 0$$

Factorise the numerator and denominator, where possible. Find the values of x for which the factors are equal to 0.

$$\frac{(2x-3)(x+2)}{x-1} \geq 0$$

A factor is equal to 0 when $x=-2$, $x=1$ or $x=\frac{3}{2}$.

Construct a table of signs to help you find the values of x for which the whole fraction is positive, negative or zero. You need a row for each of the three factors.

x	$(-\infty,-2)$	-2	$(-2,1)$	1	$(1,\frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2},\infty)$
$2x-3$							
$x+2$							
$x-1$							
$\dfrac{(2x-3)(x+2)}{x-1}$							

Fill in the row for each factor. Then use the signs of the factors to find the signs of the whole fraction, and enter these in the bottom row. Note that where a factor in the *denominator* takes the value 0, the fraction is undefined. Use the symbol $*$ to indicate this.

x	$(-\infty,-2)$	-2	$(-2,1)$	1	$(1,\frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2},\infty)$
$2x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$\dfrac{(2x-3)(x+2)}{x-1}$	$-$	0	$+$	$*$	$-$	0	$+$

Use the entries in the bottom row to help you solve the inequality. Remember that you're looking for the values of x such that $(2x-3)(x+2)/(x-1)$ is positive or zero.

The solution set is $[-2,1) \cup [\frac{3}{2},\infty)$.

Activity 57 *Solving inequalities containing algebraic fractions*

Solve the following inequalities by using tables of signs.

(a) $\dfrac{3x-4}{2x+1} \leq 1$ (b) $\dfrac{2x^2+5x-8}{x-3} \geq 2$

You can check the solution set that you've found for an inequality by obtaining the graph of an appropriate function. For example, consider again the inequality in Example 15. It was rearranged into the form

$$\frac{(2x-3)(x+2)}{x-1} \geq 0.$$

Figure 78 shows the graph of the equation

$$y = \frac{(2x-3)(x+2)}{x-1},$$

as a computer would plot it. The expression in x here is the left-hand side of the inequality above. The graph shows that this expression takes values greater than or equal to zero roughly when x is in the set $[-2, 1) \cup [\frac{3}{2}, \infty)$. This accords with the solution set found in Example 15.

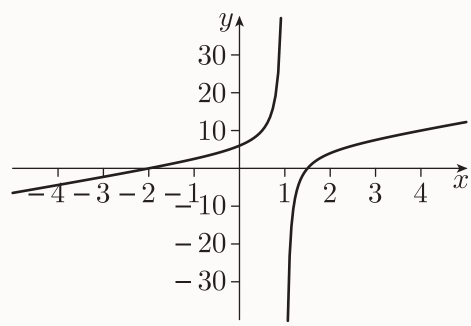


Figure 78 The graph of $y = (2x-3)(x+2)/(x-1)$

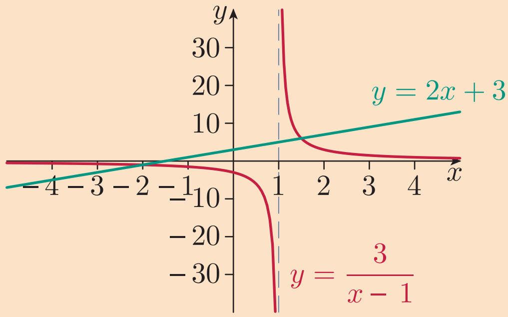
You can also use graphs to check the solution set of an inequality directly from the original, un-rearranged version of the inequality. This provides a more thorough check on your working. To do this, you usually have to obtain *two* graphs on the same axes.

This method is illustrated in the next example, in which the solution set of the original version of the inequality in Example 15 is estimated from a graph.

Example 16 *Estimating solutions from a graph*

Use the graph below to estimate the solution set of the inequality

$$\frac{3}{x-1} \le 2x + 3.$$



Solution

Estimate the values of x for which the graph of $y = 3/(x-1)$ lies below or on the graph of $y = 2x + 3$.

The graph shows that $3/(x-1)$ is less than or equal to $2x+3$ roughly when x is in the set $[-2, 1) \cup [\frac{3}{2}, \infty)$. So this set is the solution set of the inequality, at least approximately.

This agrees with the solution set found in Example 15.

Estimating solution sets from graphs in the way illustrated in Example 16 can be useful not only as a check on algebraic working, but also when you need only an approximate solution set, or when you don't know a method for solving an inequality or equation algebraically.

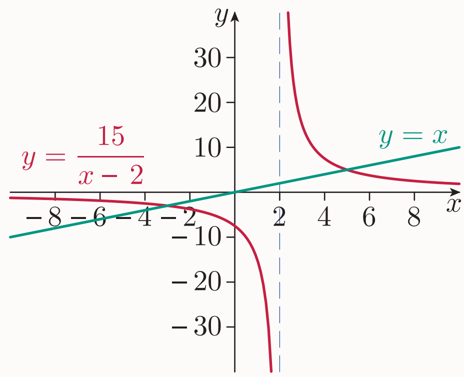
Activity 58 *Estimating solutions from a graph*

Use the graph below to estimate the following.

(a) The solutions of the equation $x = \dfrac{15}{x-2}$.

(b) The solution set of the inequality $x \leq \dfrac{15}{x-2}$.

(c) The solution set of the inequality $x > \dfrac{15}{x-2}$.



Learning outcomes

After studying this unit, you should be able to:

- understand and use the terminology and notation associated with functions
- work with graphs of functions
- work with a range of standard types of functions, and understand their properties and graphs
- understand the changes to the rules of functions that cause their graphs to be translated or scaled, horizontally or vertically
- form sums, differences, products, quotients and composites of functions
- understand what's meant by the inverse function of a one-to-one function, and find the inverse in some cases
- understand the properties and graphs of exponential and logarithmic functions
- work fluently and correctly with logarithms
- work with exponential models
- solve some types of inequalities in one variable
- construct tables of signs.

Solutions to activities

Solution to Activity 1

(a) True (b) True (c) False (d) True

Solution to Activity 2

(a) $P \cap Q = \{2, 4, 6\}$

(b) $Q \cap R = \{6, 12\}$

(c) $P \cap Q \cap R = \{6\}$

(d) $P \cup Q = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$

Solution to Activity 3

(a) This set is an open interval.

(b) This set is not an interval.

(c) This set is a closed interval.

(d) This set is a half-open interval.

(e) This set is not an interval.

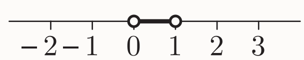
(f) This set is a closed interval. (It has only one endpoint, and it includes it.)

(g) This set is an open interval.

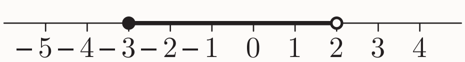
(h) This set is not an interval.

Solution to Activity 4

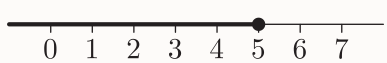
(a) (i)



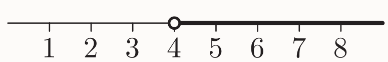
(ii)



(iii)



(iv)



(b) (i) $-2 < x \leq 5$

(ii) $x \geq -4$

(iii) $x < 0$

(iv) $-3 \leq x < -1$

(v) $0 \leq x \leq 6$

(vi) $3 < x < 7$

Solution to Activity 5

(a) $(-2, 5]$

(b) $[-4, \infty)$

(c) $(-\infty, 0)$

(d) $[-3, -1)$

(e) $[0, 6]$

(f) $(3, 7)$

Solution to Activity 6

(a) $(-\infty, -5) \cup [-2, 1]$

(b) $[1, 2) \cup [3, 4) \cup [5, 6)$

(c) $(-\infty, 0) \cup (0, \infty)$

Solution to Activity 7

(a) $f(5) = 4 \times 5 = 20$ and $f(-3) = 4 \times (-3) = -12$

(b) $g(x) = 2x - 1$

Solution to Activity 8

(a) The image of 2 is 8, because $f(2) = 8$.

(b) The image of -1 is -4, because $f(-1) = -4$.

(c) The value of f at 0.5 is 2, because $f(0.5) = 2$.

(d) The value of f at -0.2 is -0.8, because $f(-0.2) = -0.8$.

(e) The number 11 has image 44 under f, because $f(11) = 44$.

(f) The number $\frac{1}{4}$ has image 1 under f, because $f(\frac{1}{4}) = 1$.

(g) The function f maps 4 to $f(4) = 16$.

(h) The function f maps -2 to -8 because $f(-2) = -8$.

Solution to Activity 9

(a) The domain of f is the set of all real numbers except 4, that is, the set $(-\infty, 4) \cup (4, \infty)$.

(b) The domain of g is the set of all real numbers except 2 and -3, that is, the set $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

(c) The domain of h is the set of all real numbers x such that $x - 1$ is non-negative. This set can be described more concisely as the set of all real numbers greater than or equal to 1. In interval notation this set is denoted by $[1, \infty)$.

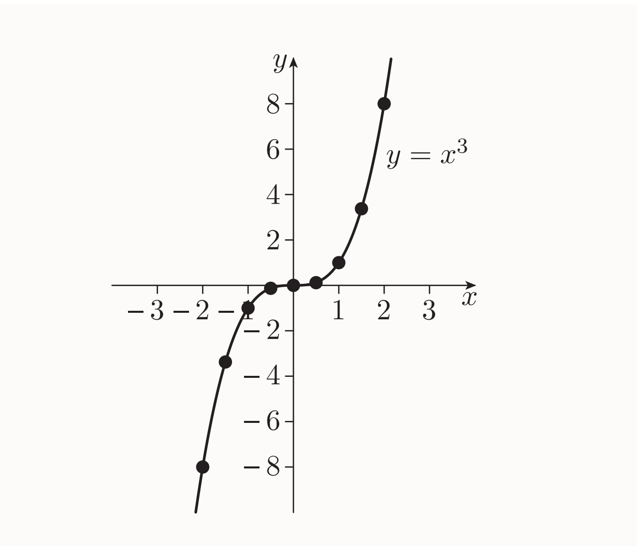
Solution to Activity 10

(a)

x	-2	-1.5	-1	-0.5
x^3	-8	-3.375	-1	-0.125

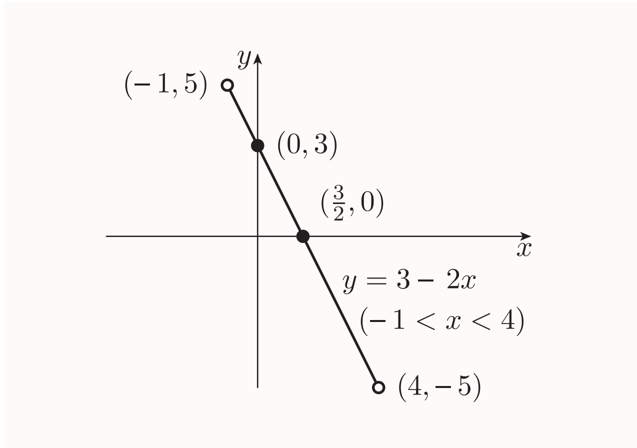
x	0	0.5	1	1.5	2
x^3	0	0.125	1	3.375	8

(b) and (c)



Solution to Activity 12

(a) The graph is part of the straight-line graph of $y = 3 - 2x$.



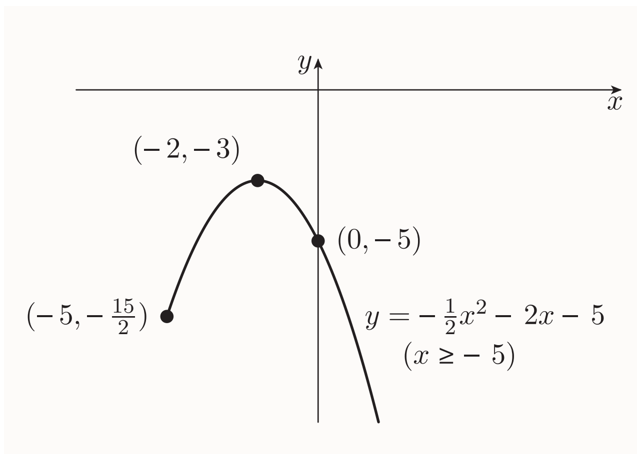
(b) The graph is part of an n-shaped parabola. Completing the square gives

$$\begin{aligned} f(x) &= -\tfrac{1}{2}x^2 - 2x - 5 \\ &= -\tfrac{1}{2}(x^2 + 4x) - 5 \\ &= -\tfrac{1}{2}((x+2)^2 - 4) - 5 \\ &= -\tfrac{1}{2}(x+2)^2 + 2 - 5 \\ &= -\tfrac{1}{2}(x+2)^2 - 3 \end{aligned}$$

So the vertex is $(-2, -3)$. Also

$$f(-5) = -\tfrac{1}{2}(-5)^2 - 2 \times (-5) - 5 = -\tfrac{15}{2}.$$

So the graph stops at the point $(-5, -\frac{15}{2})$ (which is included). These features give the following sketch.



Solution to Activity 13

(a) The domain is $[1,3]$.

(b) The domain is $(-\infty,-2]\cup[-1,\infty)$.

Solution to Activity 14

Diagrams (a), (c), (f), (h), (i) and (j) are the graphs of functions.

Solution to Activity 15

Graphs (a) and (b) show functions that are increasing on their whole domains.

(For the function in graph (c), if you take x_1 and x_2 to be values slightly less than 0 and slightly greater than 0, respectively, then the function takes a smaller value at x_2 than it does at x_1, so it is not increasing on its whole domain.

For the function in graph (d), you can find values x_1 and x_2 with $x_1 < x_2$ such that the function takes the same value at x_2 as it does at x_1, so it is not increasing on its whole domain.)

Solution to Activity 16

(a) The graph of f is part of an n-shaped parabola. Completing the square gives

$$\begin{aligned} f(x) &= -x^2+10x-24 \\ &= -(x^2-10x)-24 \\ &= -((x-5)^2-25)-24 \\ &= -(x-5)^2+25-24 \\ &= -(x-5)^2+1. \end{aligned}$$

So the vertex is $(5,1)$. Also

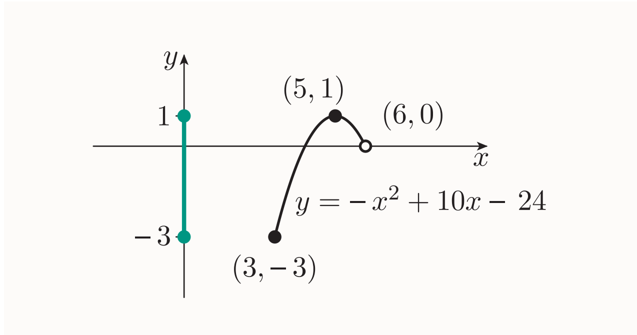
$$f(3) = -3^2+10\times 3-24=-3$$

and

$$f(6) = -6^2+10\times 6-24=0.$$

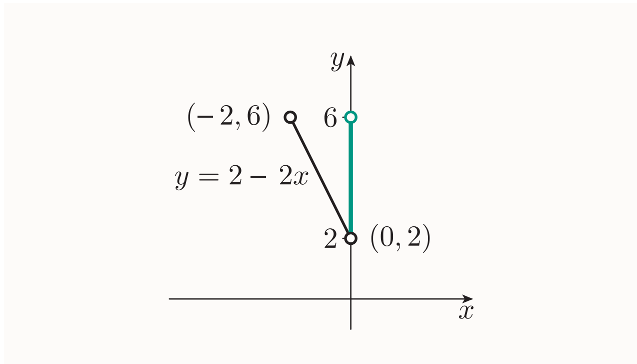
So the graph stops at the points $(3,-3)$ (which is included) and $(6,0)$ (which is excluded).

These features give the graph below. The image set is shown on the y-axis.



The graph shows that the image set of f is the interval $[-3,1]$.

(b) The graph of f is below, with the image set shown on the y-axis.

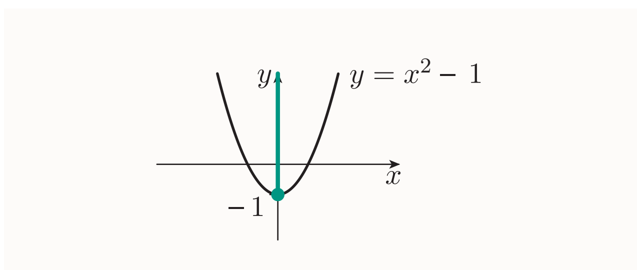


The graph shows that the image set is $(2,6)$.

(c) The image set of f is the interval $[-1,\infty)$.

This is because the image set of the function $g(x)=x^2$ is $[0,\infty)$, since every non-negative number can be expressed as the square of a number. Hence the image set of the function $f(x)=x^2-1$ is $[-1,\infty)$.

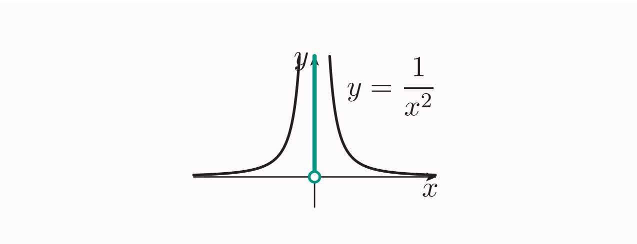
(The graph of f is shown below, with the image set shown on the y-axis.)



(d) The image set of f is the interval $(0, \infty)$.

This is because the image set of f doesn't contain any negative numbers, since the value of $1/x^2$ can't be negative. Similarly, the image set doesn't contain 0. However, the image set does contain every positive number, because every positive number can be expressed as $1/x^2$ for some number x.

(The graph of f is shown below, with the image set shown on the y-axis.)



Solution to Activity 18

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 19

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 20

(a) $y = \dfrac{1}{x-2}$ is the equation of graph D.

(b) $y = \dfrac{1}{x} - 2$ is the equation of graph A.

(c) $y = \dfrac{1}{x} + 2$ is the equation of graph C.

(d) $y = \dfrac{1}{x+2}$ is the equation of graph B.

Solution to Activity 21

(a) $y = |x-2| + 1$ is the equation of graph C.

(b) $y = |x+2| + 1$ is the equation of graph D.

(c) $y = |x-2| - 1$ is the equation of graph B.

(d) $y = |x+2| - 1$ is the equation of graph A.

Solution to Activity 22

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 23

(a) $y = 2x^3$ is the equation of graph A.

(b) $y = \frac{1}{2}x^3$ is the equation of graph D.

(c) $y = -x^3$ is the equation of graph B.

(d) $y = -\frac{1}{2}x^3$ is the equation of graph C.

Solution to Activity 24

(a) The graph of $g(x) = 2|x| + 3$ can be obtained from the graph of $f(x) = |x|$ by first scaling it vertically by the factor 2 and then translating it up by 3 units.

(b) The graph of $h(x) = 2|x+2| + 3$ can be obtained from the graph of $f(x) = |x|$ by first scaling it vertically by the factor 2, then translating it to the left by 2 units, and finally translating it up by 3 units.

(You can carry out the operations in any order, except that you have to do the vertical scaling before the vertical translation.)

(c) The graph of $j(x) = \frac{1}{2}|x-3| - 4$ can be obtained from the graph of $f(x) = |x|$ by first scaling it vertically by the factor $\frac{1}{2}$, translating it to the right by 3 units, and finally translating it down by 4 units.

(You can carry out the operations in any order, except that you have to do the vertical scaling before the vertical translation.)

(d) The graph of $k(x) = -|x-1| + 1$ can be obtained from the graph of $f(x) = |x|$ by first reflecting it in the x-axis (that is, scaling it vertically by the factor -1), then translating it to the right by 1 unit, and finally translating it up by 1 unit.

(You can carry out the operations in any order, except that you have to do the reflection before the vertical translation.)

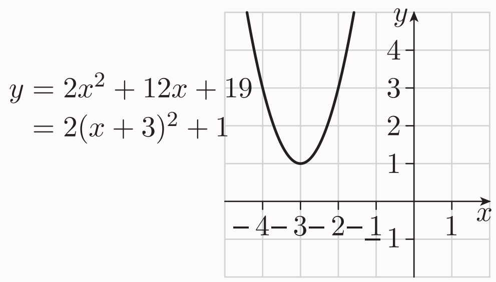
Solution to Activity 25

(a) Completing the square gives

$$\begin{aligned} f(x) &= 2x^2 + 12x + 19 \\ &= 2(x^2 + 6x) + 19 \\ &= 2((x+3)^2 - 9) + 19 \\ &= 2(x+3)^2 - 18 + 19 \\ &= 2(x+3)^2 + 1. \end{aligned}$$

(b) This equation is obtained from the equation $y = x^2$ by first multiplying the right-hand side by 2, then replacing x by $x + 3$, and finally adding 1 to the right-hand side. So its graph is obtained from the graph of $y = x^2$ by first scaling vertically by a factor of 2, then translating to the left by 3 units, and finally translating up by 1 unit.

(You can carry out the operations in any order, except that you have to do the vertical scaling before the vertical translation. The resulting graph is shown below.)



Solution to Activity 26

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 27

(a) $y = \sqrt{-x}$ is the equation of graph D.

(b) $y = -\sqrt{x}$ is the equation of graph H.

(c) $y = 2\sqrt{x-2}$ is the equation of graph C.

(d) $y = \frac{1}{2}\sqrt{x} + 2$ is the equation of graph E.

(e) $y = -\frac{1}{2}\sqrt{x}$ is the equation of graph F.

(f) $y = -\sqrt{x+2}$ is the equation of graph G.

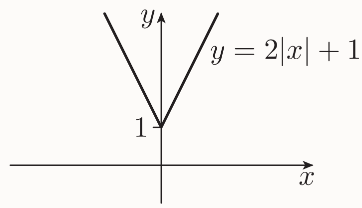
(g) $y = -\sqrt{-x}$ is the equation of graph B.

(h) $y = \frac{1}{2}\sqrt{x+2}$ is the equation of graph I.

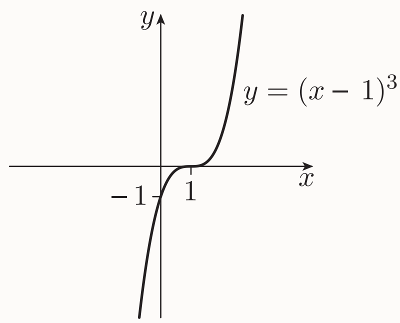
(i) $y = -2\sqrt{x} + 2$ is the equation of graph A.

Solution to Activity 28

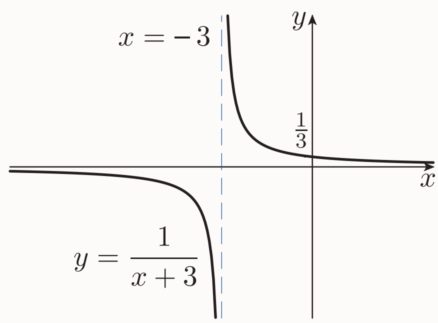
(a) The graph of the function $f(x) = 2|x| + 1$ can be obtained from the graph of $y = |x|$ by first scaling it vertically by a factor of 2 and then translating it up by 1 unit.



(b) The graph of the function $h(x) = (x-1)^3$ can be obtained from the graph of $y = x^3$ by translating it to the right by 1 unit.



(c) The graph of the function $g(x) = 1/(x+3)$ can be obtained from the graph of $y = 1/x$ by translating it to the left by 3 units.



(You can find the y-intercept of this graph simply by substituting $x = 0$ into its equation.)

Solution to Activity 29

The sum of f and g has rule

$$h(x) = 2x - 1 + x + 3,$$

which can be simplified to

$$h(x) = 3x + 2.$$

One difference of f and g has rule

$$h(x) = 2x - 1 - (x + 3),$$

which can be simplified to

$$h(x) = x - 4.$$

The other difference of f and g has rule

$$h(x) = x + 3 - (2x - 1),$$

which can be simplified to

$$h(x) = -x + 4.$$

The product of f and g has rule

$$h(x) = (2x - 1)(x + 3),$$

which can also be expressed as

$$h(x) = 2x^2 + 5x - 3.$$

The two quotients of f and g have rules

$$h(x) = \frac{2x - 1}{x + 3}$$

and

$$h(x) = \frac{x + 3}{2x - 1}.$$

All of these functions have domain $\mathbb{R}$, except the final two functions, which have domains $(-\infty, -3) \cup (-3, \infty)$ and $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$, respectively.

Solution to Activity 30

The function f maps 5 to 25, and the function g maps 25 to 26, so $(g \circ f)(5) = 26$.

Solution to Activity 31

(a) (i) $(g \circ f)(x) = g(f(x)) = g(x - 3) = \sqrt{x - 3}$

(ii) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \sqrt{x} - 3$

(iii)
$$\begin{aligned}(f \circ f)(x) &= f(f(x)) = f(x - 3)\\ &= (x - 3) - 3 = x - 6\end{aligned}$$

(iv)
$$\begin{aligned}(g \circ g)(x) &= g(g(x)) = g(\sqrt{x}) = \sqrt{\sqrt{x}}\\ &= (x^{1/2})^{1/2} = x^{1/4}\end{aligned}$$

(b) If x is in the domain of f, then $f(x) = x - 3$. For $x - 3$ to be in the domain of g, the value of $x - 3$ must be greater than or equal to zero, which means that the value of x must be greater than or equal to 3. That is, the domain of $g \circ f$ is $[3, \infty)$.

Solution to Activity 32

(a)
$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x)))\\ &= f(g(\sqrt{x}))\\ &= f\left(\frac{1}{\sqrt{x}}\right)\\ &= \frac{1}{\sqrt{x}} + 2\end{aligned}$$

(b)
$$\begin{aligned}(g \circ h \circ f)(x) &= g(h(f(x)))\\ &= g(h(x + 2))\\ &= g(\sqrt{x + 2})\\ &= \frac{1}{\sqrt{x + 2}}\end{aligned}$$

(c)
$$\begin{aligned}(f \circ h \circ g)(x) &= f(h(g(x)))\\ &= f\left(h\left(\frac{1}{x}\right)\right)\\ &= f\left(\sqrt{\frac{1}{x}}\right)\\ &= f\left(\frac{1}{\sqrt{x}}\right)\\ &= \frac{1}{\sqrt{x}} + 2\end{aligned}$$

(d)
$$\begin{aligned}(f \circ g \circ f)(x) &= f(g(f(x)))\\ &= f(g(x + 2))\\ &= f\left(\frac{1}{x + 2}\right)\\ &= \frac{1}{x + 2} + 2\\ &= \frac{1 + 2(x + 2)}{x + 2}\\ &= \frac{2x + 5}{x + 2}\end{aligned}$$

Solution to Activity 33

(a) (i) The rule of the inverse function of $f(x) = x + 1$ is $f^{-1}(x) = x - 1$.

(ii) The rule of the inverse function of $f(x) = x - 3$ is $f^{-1}(x) = x + 3$.

(iii) The rule of the inverse function of $f(x) = \frac{1}{3}x$ is $f^{-1}(x) = 3x$.

(b) Some possible answers are $f(x) = x$, $f(x) = -x$ and $f(x) = 1/x$. There are many others, such as $f(x) = 3 - x$ and $f(x) = 8/x$.

Solution to Activity 34

(a) The function $f(x) = |x|$ is not one-to-one. For example, $f(1) = f(-1) = 1$.

(b) The function $f(x) = x + 1$ is one-to-one.

(c) The function $f(x) = x^4$ is not one-to-one. For example, $f(1) = f(-1) = 1$.

(d) The function $f(x) = x^5$ is one-to-one.

(e) The function $f(x) = -x$ is one-to-one.

(f) The function $f(x) = 1$ is not one-to-one. For example, $f(0) = f(1) = 1$.

Solution to Activity 35

Diagrams (a), (b), (e), (f), (h), (i) and (k) are the graphs of one-to-one functions.

Diagrams (c), (d) and (j) are the graphs of functions that aren't one-to-one.

Diagrams (g) and (l) are not the graphs of functions.

Solution to Activity 36

(a) The equation $f(x) = y$ gives

$$3x - 4 = y$$
$$3x = y + 4$$
$$x = \tfrac{1}{3}(y + 4).$$

Since the equation $f(x) = y$ can be rearranged to express x as a function of y, the function f has an inverse function f^{-1}, with rule

$$f^{-1}(y) = \tfrac{1}{3}(y + 4);$$

that is,

$$f^{-1}(x) = \tfrac{1}{3}(x + 4).$$

The domain of f^{-1} is the image set of f, which is $\mathbb{R}$. So the inverse function f^{-1} is given by

$$f^{-1}(x) = \tfrac{1}{3}(x + 4).$$

(b) The equation $f(x) = y$ gives

$$2 - \tfrac{1}{2}x = y$$
$$2 - y = \tfrac{1}{2}x$$
$$x = 2(2 - y).$$

Since the equation $f(x) = y$ can be rearranged to express x as a function of y, the function f has an inverse function f^{-1}, with rule

$$f^{-1}(y) = 2(2 - y);$$

that is,

$$f^{-1}(x) = 2(2 - x).$$

The domain of f^{-1} is the image set of f, which is $\mathbb{R}$. So the inverse function f^{-1} is given by

$$f^{-1}(x) = 2(2 - x).$$

(c) The equation $f(x) = y$ gives

$$5 + \frac{1}{x} = y$$
$$\frac{1}{x} = y - 5$$
$$x = \frac{1}{y - 5}.$$

Hence f has an inverse function f^{-1}, with rule

$$f^{-1}(y) = \frac{1}{y - 5};$$

that is,

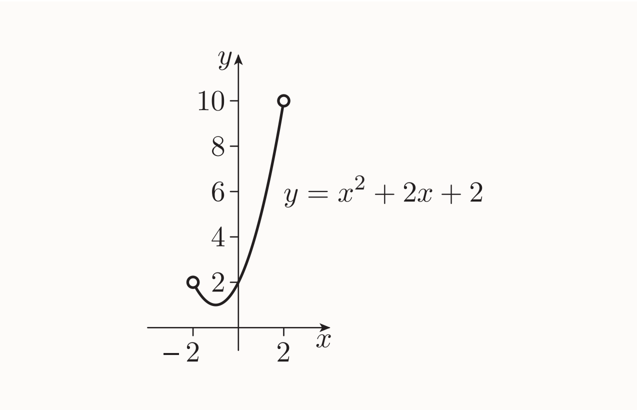
$$f^{-1}(x) = \frac{1}{x - 5}.$$

The domain of f^{-1} is the image set of f, which is $(-\infty, 5) \cup (5, \infty)$. This is the largest set of real numbers for which the rule of f^{-1} is applicable. So the inverse function f^{-1} is given by

$$f^{-1}(x) = \frac{1}{x - 5}.$$

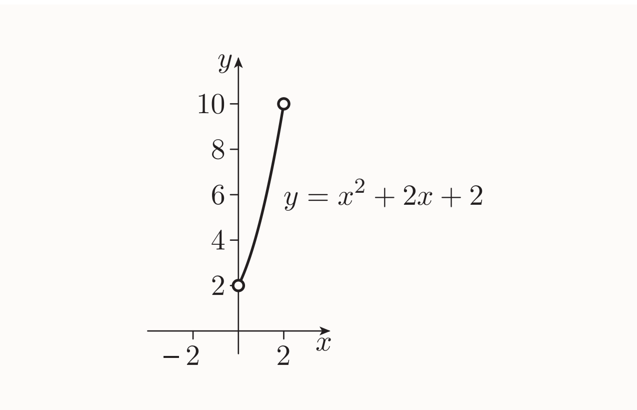
Solution to Activity 37

(a) The graph of f is shown below.



The graph shows that f isn't one-to-one. So it doesn't have an inverse function.

(b) The graph of f is shown below.



The graph shows that f is one-to-one and therefore has an inverse function. The equation $f(x) = y$ gives

$$x^2 + 2x + 2 = y$$
$$(x+1)^2 - 1 + 2 = y$$
$$(x+1)^2 + 1 = y$$
$$(x+1)^2 = y - 1$$
$$x + 1 = \pm\sqrt{y-1}$$
$$x = -1 \pm \sqrt{y-1}.$$

Since the domain of f is $(0, 2)$, each input value x of f is greater than -1. So

$$x = -1 + \sqrt{y-1}.$$

Hence the rule of f^{-1} is

$$f^{-1}(y) = -1 + \sqrt{y-1};$$

that is,

$$f^{-1}(x) = -1 + \sqrt{x-1}.$$

The domain of f^{-1} is the image set of f. The graph shows that this is

$$(f(0), f(2)) = (2, 10).$$

So the inverse function of f is the function

$$f^{-1}(x) = -1 + \sqrt{x-1} \quad (x \in (2, 10)).$$

(c) The equation $f(x) = y$ gives

$$1 - x = y$$
$$x = 1 - y.$$

Hence f has an inverse function f^{-1}, with rule

$$f^{-1}(y) = 1 - y;$$

that is,

$$f^{-1}(x) = 1 - x.$$

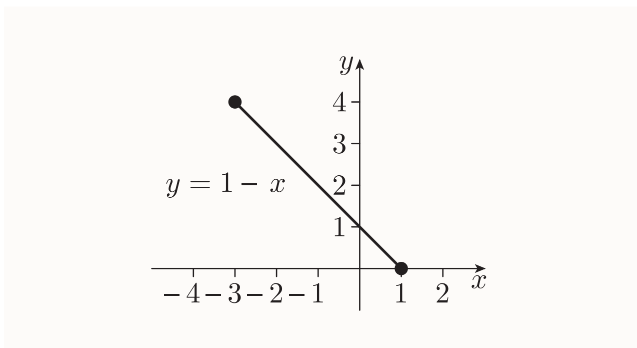
The domain of f^{-1} is the image set of f, which is

$$[f(1), f(-3)] = [0, 4].$$

Hence the inverse function of f is

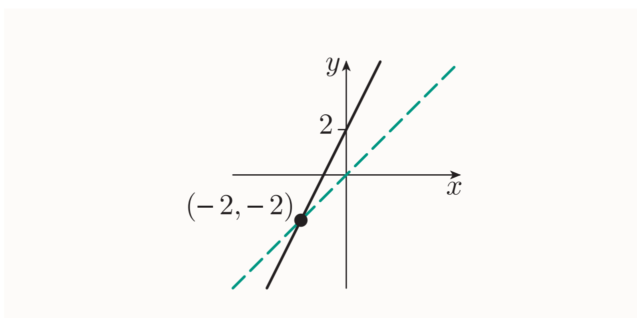
$$f^{-1}(x) = 1 - x \quad (x \in [0, 4]).$$

(The graph of f, shown below, might help you find the image set of f.)

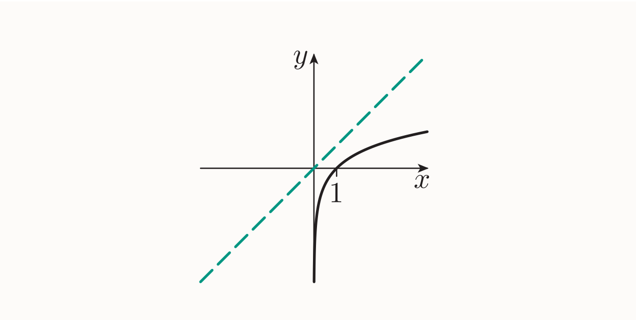


Solution to Activity 38

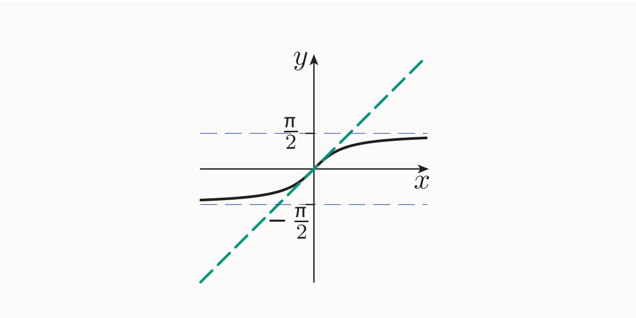
(a)



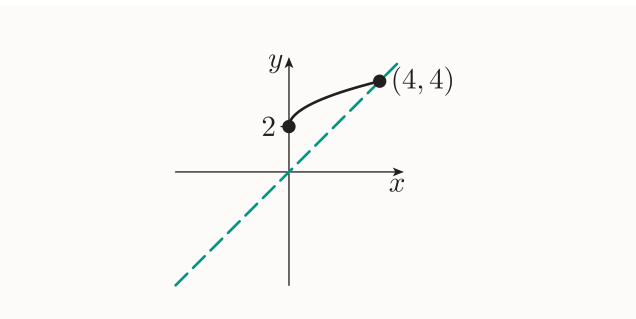
(b)



(c)



(d)



Solution to Activity 39

The image set of f is $[0, \infty)$.

A one-to-one function g that is a restriction of f and has the same image set as f is

$$g(x) = (x - 1)^2 \quad (x \in [1, \infty)).$$

The equation $g(x) = y$ gives

$$(x - 1)^2 = y$$
$$x - 1 = \pm\sqrt{y}$$
$$x = 1 \pm \sqrt{y}.$$

Since the domain of g is $[1, \infty)$, each input value x of f is greater than or equal to 1. So

$$x = 1 + \sqrt{y}.$$

Hence the rule of g^{-1} is

$$g^{-1}(y) = 1 + \sqrt{y};$$

that is,

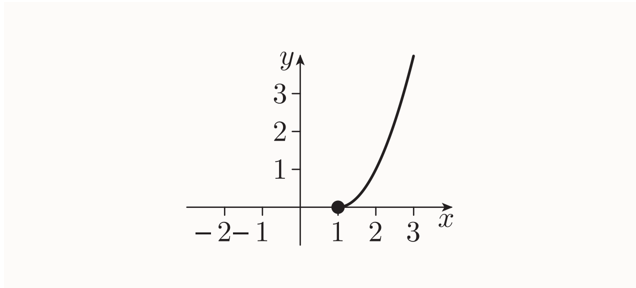
$$g^{-1}(x) = 1 + \sqrt{x}.$$

The domain of g^{-1} is the image set of g, which is $[0, \infty)$.

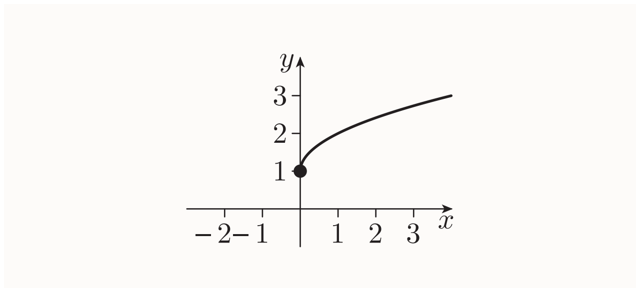
So the inverse function of g is the function

$$g^{-1}(x) = 1 + \sqrt{x} \quad (x \in [0, \infty)).$$

The graph of the function g is as shown below.



Hence the graph of g^{-1} is as shown below.



(Alternatively, you can take the domain of g to be $(-\infty, 1]$. Then g^{-1} is the function

$$g^{-1}(x) = 1 - \sqrt{x} \quad (x \in [0, \infty).)$$

Solution to Activity 40

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 41

(The answer is given in the text after the activity.)

Solution to Activity 42

(a) (i) $\log_{10} 10\,000 = 4$, since $10\,000 = 10^4$.

(ii) $\log_{10} \frac{1}{100} = -2$, since $\frac{1}{100} = 10^{-2}$.

(iii) $\log_{10} 10 = 1$, since $10 = 10^1$.

(iv) $\log_{10} 1 = 0$, since $1 = 10^0$.

(b) If the number x is such that $\log_{10} x = \frac{1}{2}$, then $10^{1/2} = x$; that is, $x = \sqrt{10}$.

(c) (i) $\log_{10} 3700 = 3.568$ (to 3 d.p.)

(ii) $\log_{10} 370 = 2.568$ (to 3 d.p.)

(iii) $\log_{10} 37 = 1.568$ (to 3 d.p.)

(iv) $\log_{10} 3.7 = 0.568$ (to 3 d.p.)

(v) $\log_{10} 0.37 = -0.432$ (to 3 d.p.)

(vi) $\log_{10} 0.037 = -1.432$ (to 3 d.p.)

(You'll be asked to explain the pattern in the answers to part (c) later in this section.)

Solution to Activity 43

(a) (i) $\log_3 9 = 2$, since $3^2 = 9$.

(ii) $\log_2 8 = 3$, since $2^3 = 8$.

(iii) $\log_4 64 = 3$, since $4^3 = 64$.

(iv) $\log_5 25 = 2$, since $5^2 = 25$.

(v) $\log_4 2 = \frac{1}{2}$, since $4^{1/2} = 2$.

(vi) $\log_8 2 = \frac{1}{3}$, since $8^{1/3} = 2$.

(vii) $\log_2 \frac{1}{2} = -1$, since $2^{-1} = \frac{1}{2}$.

(viii) $\log_2 \frac{1}{8} = -3$, since $2^{-3} = \frac{1}{8}$.

(ix) $\log_3 \frac{1}{27} = -3$, since $3^{-3} = \frac{1}{27}$.

(x) $\log_8 \frac{1}{8} = -1$, since $8^{-1} = \frac{1}{8}$.

(xi) $\log_3 3 = 1$, since $3^1 = 3$.

(xii) $\log_4 \frac{1}{4} = -1$, since $4^{-1} = \frac{1}{4}$.

(xiii) $\log_6 6 = 1$, since $6^1 = 6$.

(xiv) $\log_5 \sqrt{5} = \frac{1}{2}$, since $5^{1/2} = \sqrt{5}$.

(xv) $\log_7 \sqrt[3]{7} = \frac{1}{3}$, since $7^{1/3} = \sqrt[3]{7}$.

(xvi) $\log_2 1 = 0$, since $2^0 = 1$.

(xvii) $\log_{15} 1 = 0$, since $15^0 = 1$.

(b) (i) If $\log_2 x = 5$, then $x = 2^5 = 32$.

(ii) If $\log_8 x = \frac{1}{3}$, then $x = 8^{1/3} = 2$.

(iii) If $\log_7 x = 1$, then $x = 7^1 = 7$.

Solution to Activity 44

(a) (i) $\ln e^4 = 4$

(ii) $\ln e^2 = 2$

(iii) $\ln e^{3/5} = \frac{3}{5}$

(iv) $\ln \sqrt{e} = \ln(e^{1/2}) = \frac{1}{2}$

(v) $\ln\left(\dfrac{1}{e}\right) = \ln(e^{-1}) = -1$

(vi) $\ln\left(\dfrac{1}{e^3}\right) = \ln(e^{-3}) = -3$

(b) If the number x is such that $\ln x = -\frac{1}{2}$, then $x = e^{-1/2} = 1/\sqrt{e}$.

(c) (i) $\ln 5100 = 8.537$ (to 3 d.p.)

(ii) $\ln 510 = 6.234$ (to 3 d.p.)

(iii) $\ln 51 = 3.932$ (to 3 d.p.)

(iv) $\ln(51e) = 4.932$ (to 3 d.p.)

(v) $\ln(51e^2) = 5.932$ (to 3 d.p.)

Solution to Activity 45

(a) $e^{\ln(7x)} = 7x$

(b) $\ln(e^{8x}) = 8x$

(c) $\ln(e^{2x}) + \ln(e^{3x}) = 2x + 3x = 5x$

(d) $\ln(e^2) - \ln e = 2 - 1 = 1$

(e) $\ln(e^{x/2}) + 3\ln 1 = \dfrac{x}{2} + 3 \times 0 = \dfrac{x}{2}$

(f) $e^{2\ln c} = e^{(\ln c)\times 2} = \left(e^{\ln c}\right)^2 = c^2$

(g) $e^{\ln(3a)} + 4e^0 = 3a + 4 \times 1 = 3a + 4$

(h)
$$\begin{aligned} &\ln(e^{y+2}) + 2\ln(e^{y-1}) \\ &= y + 2 + 2(y-1) \\ &= y + 2 + 2y - 2 \\ &= 3y \end{aligned}$$

(i) $e^{3\ln B} = e^{(\ln B)\times 3} = \left(e^{\ln B}\right)^3 = B^3$

(j) $e^{2+\ln x} = e^2 e^{\ln x} = e^2 x$

Solution to Activity 46

(a) (i) $\ln 5 + \ln 3 = \ln(5\times 3) = \ln 15$

(ii) $\ln 2 - \ln 7 = \ln(\frac{2}{7})$

(iii) $3\ln 2 = \ln(2^3) = \ln 8$

(iv) $\ln 3 + \ln 4 - \ln 6 = \ln\left(\frac{3\times 4}{6}\right) = \ln 2$

(v)
$$\begin{aligned}\ln 24 - 2\ln 3 &= \ln 24 - \ln 3^2\\ &= \ln\left(\frac{24}{3^2}\right)\\ &= \ln(\tfrac{8}{3})\end{aligned}$$

(vi) $\frac{1}{3}\log_{10} 27 = \log_{10} 27^{1/3} = \log_{10} 3$

(vii)
$$\begin{aligned}&3\log_2 5 - \log_2 3 + \log_2 6\\ &= \log_2 5^3 - \log_2 3 + \log_2 6\\ &= \log_2\left(\frac{5^3\times 6}{3}\right)\\ &= \log_2 250\end{aligned}$$

(viii)
$$\begin{aligned}&\tfrac{1}{2}\ln(9x) - \ln(x+1)\\ &= \ln(9x)^{1/2} - \ln(x+1)\\ &= \ln\left(\frac{(9x)^{1/2}}{x+1}\right)\\ &= \ln\left(\frac{3\sqrt{x}}{x+1}\right)\end{aligned}$$

(b) (i) $\ln c^3 - \ln c = \ln\left(\frac{c^3}{c}\right) = \ln(c^2)$

(The final answer $2\ln c$ is just as acceptable.)

(ii) $3\ln(p^2) = \ln(p^6)$

(The answer $6\ln p$ is just as acceptable.)

(iii)
$$\begin{aligned}&\ln(y^2) + 2\ln y - \tfrac{1}{2}\ln(y^3)\\ &= \ln(y^2) + \ln(y^2) - \ln\left((y^3)^{1/2}\right)\\ &= \ln(y^2) + \ln(y^2) - \ln(y^{3/2})\\ &= \ln\left(\frac{y^2\times y^2}{y^{3/2}}\right)\\ &= \ln(y^{5/2})\end{aligned}$$

(The final answer $\frac{5}{2}\ln y$ is just as acceptable.)

(iv) $\ln(3u) - \ln(2u) = \ln\left(\frac{3u}{2u}\right) = \ln(\frac{3}{2})$

(The final answer $\ln 3 - \ln 2$ is just as acceptable.)

(v)
$$\begin{aligned}\ln(4x) + 3\ln x - \ln(e^6) &= \ln(4x) + \ln x^3 - 6\\ &= \ln(4x\times x^3) - 6\\ &= \ln(4x^4) - 6\end{aligned}$$

(Other acceptable final answers include $\ln 4 + \ln(x^4) - 6$ and $\ln 4 + 4\ln x - 6$.)

(vi) $\frac{1}{2}\ln(u^8) = \ln(u^4)$

(The answer $4\ln u$ is just as acceptable.)

(c) The pattern can be explained as follows. It follows from the logarithm laws that, for any value of n,
$$\begin{aligned}\log_{10}(37\times 10^n) &= \log_{10} 37 + \log_{10}(10^n)\\ &= \log_{10} 37 + n.\end{aligned}$$
So if you multiply 37 by 10^n, then its common logarithm is increased by n.

(d)
$$\begin{aligned}1567\times 2786 &= e^{\ln(1567\times 2786)}\\ &= e^{\ln 1567 + \ln 2786}\\ &\approx e^{7.356\,918\,242 + 7.932\,362\,154}\\ &\approx e^{15.289\,280\,4}\\ &\approx 4\,365\,662\end{aligned}$$
(This is the exact answer.)

Solution to Activity 47

(a)
$$\begin{aligned}5^x &= 0.5\\ \ln(5^x) &= \ln 0.5\\ x\ln 5 &= \ln 0.5\\ x &= \frac{\ln 0.5}{\ln 5}\\ x &= -0.430\,676\ldots\end{aligned}$$
The solution is $x = -0.431$ (to 3 s.f.).

(Alternatively, you can proceed as follows:
$$\begin{aligned}5^x &= 0.5\\ x &= \log_5(0.5)\\ x &= -0.430\,676\ldots.)\end{aligned}$$

(b)
$$\begin{aligned}4e^{7t} &= 64\\ e^{7t} &= 16\\ \ln(e^{7t}) &= \ln 16\\ 7t &= \ln 16\\ t &= \tfrac{1}{7}\ln 16\\ t &= 0.396\,084\ldots\end{aligned}$$
The solution is $t = 0.396$ (to 3 s.f.).

(c)
$$\begin{aligned}5 \times 2^{u/2} + 30 &= 600\\ 5 \times 2^{u/2} &= 570\\ 2^{u/2} &= 114\\ \ln(2^{u/2}) &= \ln 114\\ \tfrac{1}{2}u \ln 2 &= \ln 114\\ u \ln 2 &= 2 \ln 114\\ u &= \frac{2 \ln 114}{\ln 2}\\ u &= 13.665\,780\ldots\end{aligned}$$

The solution is $u = 13.7$ (to 3 s.f.).

(Alternatively, you can proceed as follows from the third equation above:

$$\begin{aligned}2^{u/2} &= 114\\ \frac{u}{2} &= \log_2 114\\ u &= 2\log_2 114\\ u &= 13.665\,780\ldots.)\end{aligned}$$

(d)
$$\begin{aligned}2^{3x-5} &= 100\\ \ln(2^{3x-5}) &= \ln 100\\ (3x-5)\ln 2 &= \ln 100\\ 3x - 5 &= \frac{\ln 100}{\ln 2}\\ 3x &= \frac{\ln 100}{\ln 2} + 5\\ x &= \tfrac{1}{3}\left(\frac{\ln 100}{\ln 2} + 5\right)\\ x &= 3.881\,285\,396\ldots\end{aligned}$$

The solution is $x = 3.88$ (to 3 s.f.).

(Alternatively, you can proceed as follows:

$$\begin{aligned}2^{3x-5} &= 100\\ 3x - 5 &= \log_2(100)\\ 3x &= \log_2(100) + 5\\ x &= \tfrac{1}{3}(\log_2(100) + 5)\\ x &= 3.881\,285\,396\ldots.)\end{aligned}$$

Solution to Activity 49

Since $\ln 3 = 1.098\,612$ (to 7 s.f.), the rule of f can be written, approximately, as

$$f(x) = e^{1.098\,612x}.$$

Using the original form of the rule gives

$$f(1.5) = 3^{1.5} = 5.20 \text{ (to 3 s.f.)}.$$

Using the alternative form gives

$$f(1.5) = e^{1.098\,612\times 1.5} = 5.20 \text{ (to 3 s.f.)}.$$

Solution to Activity 50

(a) Let $f(t) = ae^{kt}$, where a and k are constants.

Then $f(9) = 300$ and $f(12) = 4200$, so

$$ae^{9k} = 300 \quad\text{and}\quad ae^{12k} = 4200. \qquad (5)$$

Hence

$$\frac{ae^{12k}}{ae^{9k}} = \frac{4200}{300},$$

which gives

$$\begin{aligned}e^{12k-9k} &= 14\\ e^{3k} &= 14\\ 3k &= \ln 14\\ k &= \tfrac{1}{3}\ln 14\\ k &= 0.879\,685\ldots.\end{aligned}$$

The first of equations (5) can be written as $a(e^{3k})^3 = 300$ and substituting $e^{3k} = 14$ into this equation gives

$$a \times 14^3 = 300,$$

so

$$\begin{aligned}a &= \frac{300}{14^3} = \frac{75}{686}\\ &= 0.109\,329\ldots.\end{aligned}$$

So $a = 0.109$ and $k = 0.880$, both to three significant figures.

Hence the required function f is given, approximately, by

$$f(t) = 0.109e^{0.880t} \quad (8 \le t \le 24).$$

(b) The predicted number of bacteria per millilitre after 24 hours is

$$\begin{aligned}f(24) &= (0.109\,329\ldots)e^{(0.879\,685\ldots)\times 24}\\ &= 1.6 \times 10^8 \text{ (to 2 s.f.)}.\end{aligned}$$

Solution to Activity 51

(a) Every decade the size of the tree population is predicted to multiply by the factor

$$e^{0.06\times 1} = 1.06 \text{ (to 3 s.f.)}.$$

(b) Every century (10 decades) the size of the tree population is predicted to multiply by the factor

$$e^{0.06\times 10} = 1.82 \text{ (to 3 s.f.)}.$$

(c) Every five years (0.5 decades) the size of the tree population is predicted to multiply by the factor

$$e^{0.06\times 0.5} = 1.03 \text{ (to 3 s.f.)}.$$

Solution to Activity 52

(a) Every year the level of radioactivity is predicted to multiply by the factor

$$e^{-0.035\times 1} = 0.97 \text{ (to 2 s.f.)}.$$

(b) Every 25 years the level of radioactivity is predicted to multiply by the factor

$$e^{-0.035\times 25} = 0.42 \text{ (to 2 s.f.)}.$$

(c) Every century (100 years) the level of radioactivity is predicted to multiply by the factor

$$e^{-0.035\times 100} = 0.030 \text{ (to 2 s.f.)}.$$

Solution to Activity 53

(a) The exponential growth function in Activity 51 is

$$f(t) = 700e^{0.06t} \quad (10 \le t \le 50),$$

where $f(t)$ is the number of trees at time t (in decades) after the variety was introduced.

The doubling time p (in decades) for this exponential growth is given by

$$p = (\ln 2)/0.06 = 11.6 \text{ (to 3 s.f.)}.$$

So the number of trees doubles every 11.6 decades (116 years), approximately.

(b) The exponential decay function in Activity 52 is

$$r(t) = 2800e^{-0.035t} \quad (t \ge 0),$$

where $r(t)$ (in becquerels) is the level of radioactivity of the sample of radioactive material at time t (in years) after the level was first measured.

The half-life p (in years) for this exponential decay is given by

$$p = (\ln \tfrac{1}{2})/(-0.035) = 19.8 \text{ (to 3 s.f.)}.$$

So the level of radioactivity halves every 19.8 years, approximately.

Solution to Activity 54

(a) (i) Rearranging the inequality gives

$$\begin{aligned} 5x + 2 &< 3x - 1 \\ 2x &< -3 \\ x &< -\tfrac{3}{2}. \end{aligned}$$

The solution set is the interval $(-\infty, -\frac{3}{2})$.

(ii) Rearranging the inequality gives

$$\begin{aligned} 6 - 3x &\ge \frac{x}{2} - 1 \\ 12 - 6x &\ge x - 2 \\ -7x &\ge -14 \\ x &\le 2. \end{aligned}$$

The solution set is the interval $(-\infty, 2]$.

(b) Each acceptable value of x satisfies the inequality

$$\frac{54 + 69 + 72 + x}{4} \ge 60.$$

Solving this inequality gives

$$\begin{aligned} \frac{195 + x}{4} &\ge 60 \\ 195 + x &\ge 240 \\ x &\ge 45. \end{aligned}$$

So the employee must score at least 45% in her final assignment to pass the course.

Solution to Activity 55

(a) The inequality can be rearranged as follows:

$$\begin{aligned} x^2 + x &< 2 \\ x^2 + x - 2 &< 0. \end{aligned}$$

The graph of $f(x) = x^2 + x - 2$ is u-shaped. Its intercepts are given by

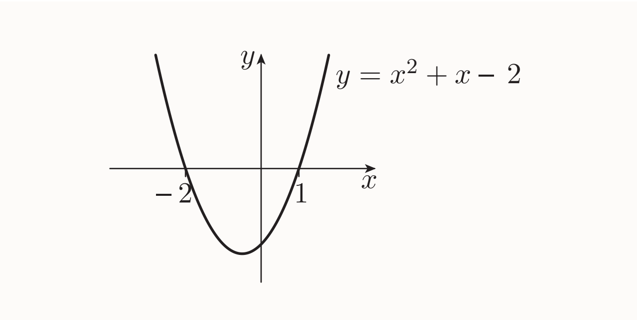
$$x^2 + x - 2 = 0;$$

that is,

$$(x + 2)(x - 1) = 0.$$

So they are $x = -2$ and $x = 1$.

Hence the graph is as shown below.



The solution set is $(-2, 1)$.

(b) The inequality can be rearranged as follows:

$$-x^2 + 7x < 10$$
$$-x^2 + 7x - 10 < 0.$$

The graph of $f(x) = -x^2 + 7x - 10$ is n-shaped. Its intercepts are given by

$$-x^2 + 7x - 10 = 0;$$

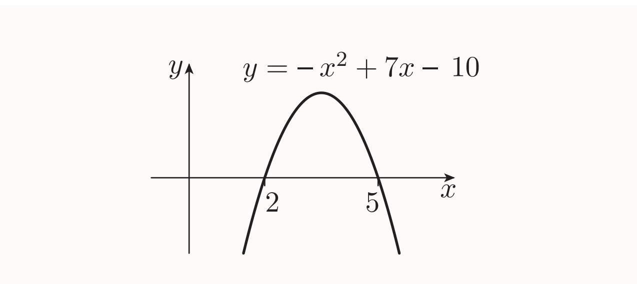
that is,

$$x^2 - 7x + 10 = 0,$$

or

$$(x - 5)(x - 2) = 0.$$

So they are $x = 5$ and $x = 2$. Hence the graph is as shown below.



The solution set is $(-\infty, 2) \cup (5, \infty)$.

(c) The inequality can be rearranged as follows:

$$-x^2 \geq 2x,$$
$$-x^2 - 2x \geq 0$$
$$x^2 + 2x \leq 0.$$

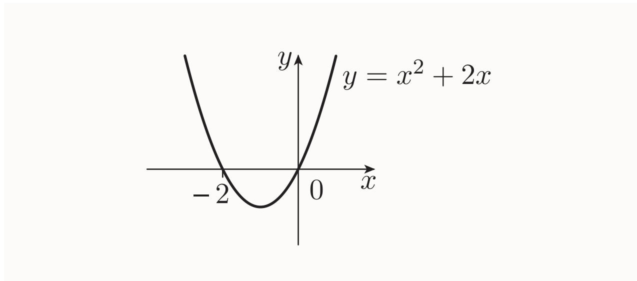
The graph of $f(x) = x^2 + 2x$ is u-shaped. Its x-intercepts are given by

$$x^2 + 2x = 0;$$

that is,

$$x(x + 2) = 0$$

So they are $x = 0$ and $x = -2$. Hence the graph is as shown below.



The solution set is $[-2, 0]$.

Solution to Activity 56

(a) The inequality is

$$2x^2 - 5x - 3 < 0,$$

which can be factorised as

$$(2x + 1)(x - 3) < 0.$$

A factor is equal to 0 when $x = -\frac{1}{2}$ or $x = 3$.

A table of signs for the expression on the left-hand side of the inequality is given below.

x	$(-\infty, -\frac{1}{2})$	$-\frac{1}{2}$	$(-\frac{1}{2}, 3)$	3	$(3, \infty)$
$2x + 1$	$-$	0	$+$	$+$	$+$
$x - 3$	$-$	$-$	$-$	0	$+$
$(2x + 1)$ $\times(x - 3)$	$+$	0	$-$	0	$+$

The solution set is $(-\frac{1}{2}, 3)$.

(b) The inequality is

$$-2x^2 + 4x + 16 \leq 0;$$

that is,

$$x^2 - 2x - 8 \geq 0.$$

Factorising gives

$$(x + 2)(x - 4) \geq 0.$$

A factor is equal to 0 when $x = -2$ or $x = 4$.

A table of signs for the expression on the left-hand side of the inequality is given below.

x	$(-\infty,-2)$	-2	$(-2,4)$	4	$(4,\infty)$
$x+2$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$
$(x+2)$ $\times(x-4)$	$+$	0	$-$	0	$+$

The solution set is $(-\infty,-2]\cup[4,\infty)$.

(Here's a slightly different way to solve the inequality in part (b).

Unlike the working above, this alternative working doesn't involve multiplying through by a negative number to simplify the inequality, so the inequality sign stays the same way round as in the original inequality. You might find that this approach helps you to avoid errors.

The inequality is

$$-2x^2+4x+16\leq 0.$$

Factorising gives

$$-2(x+2)(x-4)\leq 0.$$

A factor is equal to 0 when $x=-2$ or $x=4$.

A table of signs for the expression on the left-hand side of the inequality is given below.

x	$(-\infty,-2)$	-2	$(-2,4)$	4	$(4,\infty)$
-2	$-$	$-$	$-$	$-$	$-$
$x+2$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$
$-2(x+2)$ $\times(x-4)$	$-$	0	$+$	0	$-$

The solution set is

$(-\infty,-2]\cup[4,\infty)$.)

Solution to Activity 57

(a) The inequality can be rearranged as follows:

$$\frac{3x-4}{2x+1}\leq 1$$

$$\frac{3x-4}{2x+1}-1\leq 0$$

$$\frac{3x-4}{2x+1}-\frac{2x+1}{2x+1}\leq 0$$

$$\frac{3x-4-2x-1}{2x+1}\leq 0$$

$$\frac{x-5}{2x+1}\leq 0.$$

A factor of the numerator or denominator of the expression on the left-hand side is equal to 0 when $x=5$ or $x=-\frac{1}{2}$.

A table of signs for the expression follows.

x	$(-\infty,-\frac{1}{2})$	$-\frac{1}{2}$	$(-\frac{1}{2},5)$	5	$(5,\infty)$
$x-5$	$-$	$-$	$-$	0	$+$
$2x+1$	$-$	0	$+$	$+$	$+$
$\frac{x-5}{2x+1}$	$+$	$*$	$-$	0	$+$

The solution set is $(-\frac{1}{2},5]$.

(b) The inequality can be rearranged as follows:

$$\frac{2x^2+5x-8}{x-3}\geq 2$$

$$\frac{2x^2+5x-8}{x-3}-2\geq 0$$

$$\frac{2x^2+5x-8}{x-3}-\frac{2(x-3)}{x-3}\geq 0$$

$$\frac{2x^2+5x-8-2x+6}{x-3}\geq 0$$

$$\frac{2x^2+3x-2}{x-3}\geq 0$$

$$\frac{(2x-1)(x+2)}{x-3}\geq 0.$$

A factor of the numerator or denominator of the expression on the left-hand side is equal to 0 when $x=-2$, $x=\frac{1}{2}$ or $x=3$.

A table of signs for the expression follows.

To save space, the last row uses the notation

$$f(x) = \frac{(2x-1)(x+2)}{x-3}.$$

x	$(-\infty,-2)$	-2	$(-2,\frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2},3)$	3	$(3,\infty)$
$2x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	$*$	$+$

The solution set is $[-2,\frac{1}{2}] \cup (3,\infty)$.

Solution to Activity 58

(a) The solutions of the equation $x = \dfrac{15}{x-2}$ are roughly -3 and 5.

(b) The solution set of the inequality $x \leq \dfrac{15}{x-2}$ is roughly $(-\infty,-3] \cup (2,5]$.

(c) The solution set of the inequality $x > \dfrac{15}{x-2}$ is roughly $(-3,2) \cup (5,\infty)$.

(The answers given here are in fact exact.)

Acknowledgements

Grateful acknowledgement is made to the following sources:

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