Q 1. The system of equations can be represented in matrix form as

$$\begin{aligned}\mathbf{Ax} &= \mathbf{b}\\ &= \begin{pmatrix} 2 & -1 & -2 \\ -2 & 5 & 1 \\ 4 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 12 \\ 7 \end{pmatrix}\end{aligned}$$

and the coefficients and RHS can be represented together in the augmented matrix

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{rrr|r} 2 & -1 & -2 & -5 \\ -2 & 5 & 1 & 12 \\ 4 & 2 & -3 & 7 \end{array}\right) \begin{array}{l} \mathbf{R_1} \\ \mathbf{R_2} \\ \mathbf{R_3} \end{array}$$

where $\mathbf{R_1}$, $\mathbf{R_2}$, and $\mathbf{R_3}$ represent each row vector. Using the Gaussian elimination method (Subsections 1.2 and 1.3 of Unit 4, MST210), we eliminate the coefficients below the leading diagonal:

$$\begin{array}{r} \\ \mathbf{R_2} + \mathbf{R_1} \\ \mathbf{R_3} - 2\mathbf{R_1} \end{array} \left(\begin{array}{rrr|r} 2 & -1 & -2 & -5 \\ 0 & 4 & -1 & 7 \\ 0 & 4 & 1 & 17 \end{array}\right) \begin{array}{l} \mathbf{R_1} \\ \mathbf{R_{2a}} \\ \mathbf{R_{3a}} \end{array}$$

$$\begin{array}{r} \\ \\ \mathbf{R_{3a}} - \mathbf{R_{2a}} \end{array} \left(\begin{array}{rrr|r} 2 & -1 & -2 & -5 \\ 0 & 4 & -1 & 7 \\ 0 & 0 & 2 & 10 \end{array}\right) \begin{array}{l} \mathbf{R_1} \\ \mathbf{R_{2a}} \\ \mathbf{R_{3b}} \end{array}$$

Now $\mathbf{A}|\mathbf{b}$ is in upper triangular form, we express it as a system of equations

$$\begin{aligned} 2x - y - 2z &= -5 \\ 4y - z &= 7 \\ 2z &= 10 \end{aligned}$$

Therefore $z = 5$, and applying back substitution gives

$$\begin{aligned} 4y - 5 &= 7 \\ y &= 3 \end{aligned}$$

$$\begin{aligned} 2x - 3 - 10 &= -5 \\ x &= 4 \end{aligned}$$

So the system of equations has a unique solution of $\mathbf{x} = (4 \;\; 3 \;\; 5)^\top$.

Q 2. The system of equations can be represented in matrix form as

$$\begin{aligned}\mathbf{Ax} &= \mathbf{b}\\ &= \begin{pmatrix} 3 & 1 & -2\\ 6 & -1 & 2\\ -9 & 3 & -6\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} -2\\ 8\\ -18\end{pmatrix}\end{aligned}$$

and the coefficients and RHS can be represented together in the augmented matrix

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{rrr|r} 3 & 1 & -2 & -2\\ 6 & -1 & 2 & 8\\ -9 & 3 & -6 & -18\end{array}\right)\begin{array}{l}\mathbf{R_1}\\ \mathbf{R_2}\\ \mathbf{R_3}\end{array}$$

where $\mathbf{R_1}$, $\mathbf{R_2}$, and $\mathbf{R_3}$ represent each row vector. Using the Gaussian elimination method (Subsections 1.2 and 1.3 of Unit 4, MST210), we eliminate the coefficients below the leading diagonal:

$$\begin{array}{r}\\ \mathbf{R_2} + 2\mathbf{R_1}\\ \mathbf{R_3} + 3\mathbf{R_1}\end{array}\left(\begin{array}{rrr|r} 3 & 1 & -2 & -2\\ 0 & -3 & 6 & 12\\ 0 & 6 & -12 & -24\end{array}\right)\begin{array}{l}\mathbf{R_1}\\ \mathbf{R_{2a}}\\ \mathbf{R_{3a}}\end{array}$$

$$\begin{array}{r}\\ \\ \mathbf{R_{3a}} + 2\mathbf{R_{2a}}\end{array}\left(\begin{array}{rrr|r} 3 & 1 & -2 & -2\\ 0 & -3 & 6 & 12\\ 0 & 0 & 0 & 0\end{array}\right)\begin{array}{l}\mathbf{R_1}\\ \mathbf{R_{2a}}\\ \mathbf{R_{3b}}\end{array}$$

Now $\mathbf{A}|\mathbf{b}$ is in upper triangular form, we express it as a system of equations

$$\begin{aligned}3x + y - 2z &= -2\\ -3y + 6z &= 12\\ 0z &= 0\end{aligned}$$

As any value of z gives $0z = 0$, let $z = k$, where k is an arbitrary number, then applying back substitution gives

$$\begin{aligned}-3y + 6k &= 12\\ y &= 2k - 4\end{aligned}$$

$$\begin{aligned}3x + 2k - 4 - 2k &= -2\\ x &= \frac{2}{3}\end{aligned}$$

Therefore, the system of equations has an infinite number of solutions of the form $\mathbf{x} = \begin{pmatrix}\frac{2}{3} & 2k-4 & k\end{pmatrix}^\top$, where k is any arbitrary number.

Q 3.

(a) Using the method of page 32 in MST210 Book B, we expand the 3x3 determinant by considering the second row of the matrix and applying the table of signs to the determinant of each 2x2 matrix. This gives the determinant to be

$$\begin{aligned}\det \mathbf{A} &= -\begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} \\ &= -2 - (-2) \\ &= 0\end{aligned}$$

So the determinant of the 3x3 matrix is zero.

(b) $\mathbf{A}$ does not have an inverse because its determinant is zero.

Q 4.

(a) Using Procedure 3 (MST210 Book B, page 94), the characteristic equation of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ -2 & -4-\lambda & 2 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

Expanding the determinant by the first row gives

$$(4-\lambda)\begin{vmatrix} -4-\lambda & 2 \\ -3 & 3-\lambda \end{vmatrix} = 0$$
$$(4-\lambda)\big[(-4-\lambda)(3-\lambda)+6\big] = 0$$
$$(4-\lambda)(\lambda+3)(\lambda-2) = 0$$

Therefore the eigenvalues are -3, 2, and 4, as required.

We find corresponding eigenvectors by substituting the eigenvalues into the eigenvector equations $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ and solving for the vector $\mathbf{v} = (x_1 \;\; x_2 \;\; x_3)^\top$.

When $\lambda = -3$:

$$\begin{pmatrix} 4-(-3) & 0 & 0 \\ -2 & -4-(-3) & 2 \\ 1 & -3 & 3-(-3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ -2 & -1 & 2 \\ 1 & -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 = 0$$
$$-2x_1 - x_2 + 2x_3 = 0$$
$$x_1 - 3x_2 + 6x_3 = 0$$

$$x_1 = 0, \quad x_2 = 2x_3$$

Therefore, an eigenvector corresponding to the eigenvalue -3 is $(0 \;\; 2 \;\; 1)^\top$.

When $\lambda = 4$:

$$\begin{pmatrix} 4-4 & 0 & 0 \\ -2 & -4-4 & 2 \\ 1 & -3 & 3-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -8 & 2 \\ 1 & -3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 8x_2 + 2x_3 = 0$$
$$x_1 - 3x_2 - x_3 = 0$$

$$x_1 = k, \quad x_2 = 0, \quad x_3 = k$$

where k is any arbitrary number. Therefore, an eigenvector corresponding to the eigenvalue 4 is $(1 \;\; 0 \;\; 1)^\top$.

When $\lambda = 2$:

$$\begin{pmatrix} 4-2 & 0 & 0 \\ -2 & -4-2 & 2 \\ 1 & -3 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -2 & -6 & 2 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 0 \\ -2x_1 - 6x_2 + 2x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

$$x_1 = 0, \quad x_3 = 3x_2$$

Therefore, an eigenvector corresponding to the eigenvalue 2 is $(0 \;\; 1 \;\; 3)^\top$.

(b) If $\mathbf{A}$ is an arbitrary matrix with eigenvalue λ, then $p\lambda$ is an eigenvalue of $p\mathbf{A}$, for any number p. Therefore the eigenvalues of $-2\mathbf{A}$ are 6, -4, and -8.

As λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$, and λ^k is an eigenvalue of $\mathbf{A}^k$ (where k is any positive integer), the eigenvalues of $(\mathbf{A}^{-1})^2$ are $\frac{1}{9}$, $\frac{1}{4}$, and $\frac{1}{16}$.

Using the rules stated above and the fact that $\lambda - p$ is an eigenvalue of $\mathbf{A} - p\mathbf{I}$, for any number p, the eigenvalues of $2\mathbf{A}^3 + \mathbf{I}$ are -53, 17, and 129.

(c)

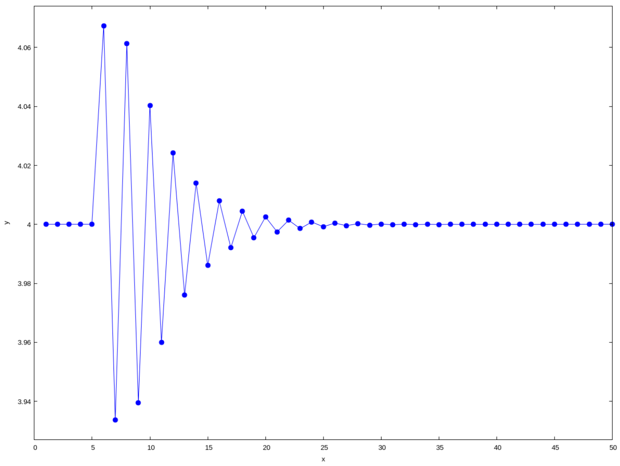
```
(%i2) /* Define function to return element vector of largest magnitude */
maxmod(v):= block([mm,i],mm: 0, for i thru length(v) do if abs(v[i][1])>abs(mm) then mm: v[i][1],mm)$
kill(z,alpha,e)$

(%i3) A:matrix(
    [ 4,  0, 0],
    [−2, −4, 2],
    [ 1, −3, 3]
)$

(%i4)  /* Define e[0], an intial guess for an eigenvector*/
e[0]:[1, 0, 0]$

(%i5) /* Define z[n] as Ae[n-1] */
z[n]:= A.e[n−1]$

(%i6) /* Define alpha[n], the component of largest magnitude of z[n] */
alpha[n]:= maxmod(z[n])$

(%i7) /* Define e[n] as the nth iteration as z[n] / alpha[n] */
e[n]:= z[n]/alpha[n]$

(%i8) /* Decimal approximation of the eigenvector corresponding to the
eigenvalue of largest magnitude, at the 50th iteration */
float(e[50]);
```

(%o8) $\begin{pmatrix} 0.9999998867356827 \\ 2.265286365578894\ 10^{-7} \\ 1.0 \end{pmatrix}$

```
(%i9) /* Decimal approximation of the eigenvalue of largest magnitude,
at the 50th iteration */
float(alpha[50]);
```

(%o9) 4.00000045305732

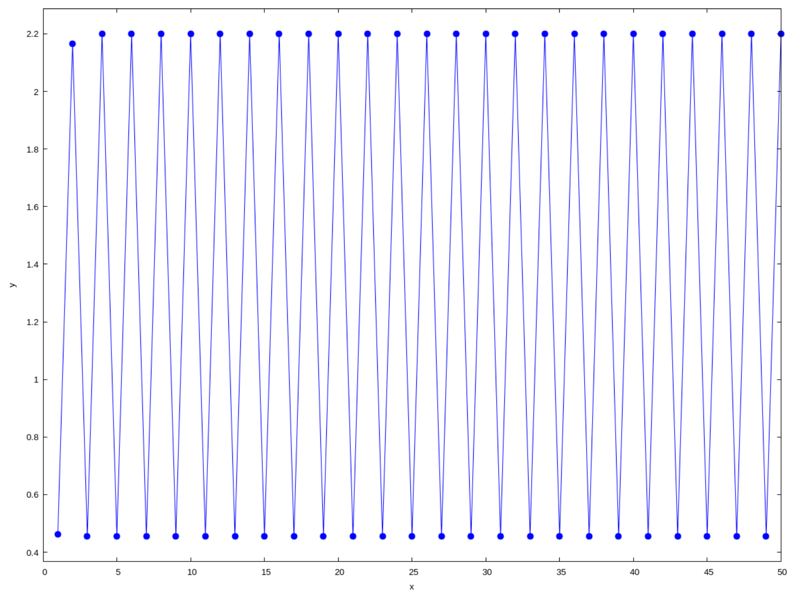
```
(%i10) wxplot2d([discrete,makelist([n, alpha[n]], n ,50)], [style, linespoints]);
```



The iteration converges on an approximation of the eigenvalue 4 and corresponding eigenvector $(1 \;\; 0 \;\; 1)^{\top}$ as found in part (a).

(d)

```
(%i2) /* Define function to return element vector of largest magnitude */
      maxmod(v):= block([mm,i],mm: 0, for i thru length(v) do if abs(v[i][1])>abs(mm) then mm: v[i][1],mm)$
      kill(z,alpha,e)$

(%i3) A:matrix(
        [ 4,  0, 0],
        [−2, −4, 2],
        [ 1, −3, 3]
      )$

(%i4) /* Define p, the value we wish to find the eigenvalue closest to */
      p:3$

(%i5) /* Define the inverse of A - 3I */
      M:invert(A − p·ident(3))$

(%i6)  /* Define e[0], an intial guess for an eigenvector */
      e[0]:[1, 0, 0]$

(%i7) /* Define z[n] as Ae[n-1] */
      z[n]:= M.e[n−1]$

(%i8) /* Define alpha[n], the component of largest magnitude of z[n] */
      alpha[n]:= maxmod(z[n])$

(%i9) /* Define e[n] as the nth iteration as z[n] / alpha[n] */
      e[n]:= z[n]/alpha[n]$

(%i12) wxplot2d([discrete,makelist([n, 1/alpha[n]], n ,50)], [style, linespoints]);
```



The iteration fails to converge because p is equidistant between two of the eigenvalues, 2 and 4. In order to converge on the eigenvalue of smallest magnitude (2), p should be set to a value that is closer to 2 than the magnitude of the other eigenvalues.

Q 5. The system of differential equations is inhomogeneous, and so we begin by finding and solving its associated homogeneous system of equations (Procedure 4, MST210 Book B page 172):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 3y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 2y$$

This homogeneous system can be represented in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and solved by applying Procedure 1 (MST210 Book B, page 156), starting with solving the characteristic equation of the coefficient matrix to find its eigenvalues:

$$\begin{vmatrix} 5-\lambda & -3 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-2-\lambda) + 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$

Therefore, the eigenvalues of the coefficient matrix are 4 and -1.

When $\lambda = 4$:

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$x - 3y = 0$$
$$2x - 6y = 0$$

$$x = 3y$$

Therefore $(3 \;\; 1)^\top$ is an eigenvector corresponding to the eigenvalue 4.

When $\lambda = -1$:

$$\begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$6x - 3y = 0$$
$$2x - y = 0$$

$$y = 2x$$

Therefore $(1 \;\; 2)^\top$ is an eigenvector corresponding to the eigenvalue -1. The solution to the associated homogeneous differential equation is therefore

$$\mathbf{x_c} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{4t} + \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

where α and β are arbitrary constants.

To find a particular solution to the system of differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, we note that $\mathbf{h} = (-17e^{-2t} \quad -4e^{-2t})^{\top}$. So we seek a particular integral of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ae^{-2t} \\ be^{-2t} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-2t}$$

where a and b are constants. We try the solution $x = ae^{-2t}, y = be^{-2t}$ and make these substitutions into the system of differential equations:

$$\begin{aligned} -2ae^{-2t} &= 5ae^{-2t} - 3be^{-2t} - 17e^{-2t} \\ -2be^{-2t} &= 2ae^{-2t} - 2be^{-2t} - 4e^{-2t} \end{aligned}$$

$$\begin{aligned} -2a &= 5a - 3b - 17 \\ -2b &= 2a - 2b - 4 \end{aligned}$$

$$\begin{aligned} a &= 2 \\ b &= -1 \end{aligned}$$

Therefore the required particular integral is

$$\mathbf{x_p} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-2t}$$

By the principle of superposition, the general solution of the system of inhomogeneous differential equations is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} \\ &= \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{4t} + \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t} \end{aligned}$$

Substituting the initial values $x(0) = 7$ and $y(0) = 4$ gives

$$\begin{aligned} 7 &= 3\alpha + \beta + 2 \\ 4 &= \alpha + 2\beta + 1 \end{aligned}$$

$$\alpha = \frac{7}{5}\ ,\ \beta = \frac{4}{5}$$

Therefore the solution to the initial value problem is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{7}{5} \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{4t} + \frac{4}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t}$$

Q 6. We begin by finding the eigenvalues corresponding to each eigenvector of the coefficient matrix:

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 10 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

So the coefficient matrix has eigenvalues 5 and 10. Applying Procedure 6 (MST210 Book B, page 178), the general solution to a system of homogeneous, second order differential equations is given by

$$\mathbf{x} = C_1\mathbf{v_1}e^{\mu_1 t} + C_2\mathbf{v_1}e^{-\mu_1 t} + \ldots\ C_{2n-1}\mathbf{v_n}e^{\mu_n t} + C_{2n}\mathbf{v_n}e^{-\mu_n t} \tag{6.1}$$

where $\mathbf{v_n}$ is an eigenvector corresponding to eigenvalue μ_n^2. Substituting the eigenvalues and corresponding eigenvectors into (6.1) gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\sqrt{5}t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-\sqrt{5}t} + C_3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{\sqrt{10}t} + C_4 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-\sqrt{10}t}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Q 7.

(a) The partial derivatives of $f(x, y)$ are

$$f_x(x, y) = y^2 + 6y + 16x - 7$$
$$f_y(x, y) = 2xy + 6x$$

A stationary point of a function $f(x, y)$ is a point $f(a, b)$ in the domain of f at which $f_x(a, b) = f_y(a, b) = 0$ (MST210 Book B page 230). So the stationary points of $f(x, y)$ can be found by solving the simultaneous equations

$$y^2 + 6y + 16x - 7 = 0 \tag{7.1}$$
$$2xy + 6x = 0 \tag{7.2}$$

As (7.2) can be rearranged as $2xy = -6x$, $x = 1$, $y = -3$ is a solution. But as $x = 0$ is also a solution to (7.2), substituting into (7.1) gives:

$$y^2 + 6y - 7 = 0$$
$$(y - 1)(y + 7) = 0$$

and thus $x = 0$, $y = 1$ is a solution, and so is $x = 0$, $y = -7$. Therefore, the stationary points of $f(x, y)$ are (1, -3), (0, 1), and (0, -7).

(b) First, we calculate the following second-order partial derivatives of $f(x, y)$:

$$\begin{aligned} A &= f_{xx}(x, y) = 16 \\ B &= f_{xy}(x, y) = 2y + 6 \\ C &= f_{yy}(x, y) = 2x \end{aligned} \tag{7.3}$$

We then apply the $AC - B^2$ test (MST210 Book B, page 237) to determine the nature of each stationary point.

For $(1, -3)$, $AC - B^2 = 32$.

As $AC - B^2 > 0$ and $A > 0$, $(1, -3)$ is a local maximum.

For $(0, 1)$ and $(0, -7)$, $AC - B^2 = -64$.

As $AC - B^2 < 0$, both $(0, 1)$ and $(0, -7)$ are saddle points.

(c) The second order Taylor polynomial for f near $(2,1)$ is given by

$$\begin{aligned} p2(x,y) &= f(2,1) + f_x(2,1)x + f_y(2,1)y \\ &\quad + \tfrac{1}{2}\big(f_{xx}(2,1)x^2 + 2f_{xy}(2,1)xy + f_{yy}(2,1)y^2\big) \\ &= 40 + 32x + 16y + \tfrac{1}{2}\big(16x^2 + 16xy + 4y^2\big) \end{aligned}$$

(d) Let $z = f(x,y)$. First, we calculate the partial derivatives of z with respect to x and y:

$$\frac{\partial z}{\partial x} = 16x + y^2 - 7$$

$$\frac{\partial z}{\partial y} = 2xy + 6x$$

We can then express small increments in z with small increments in x and y as

$$\begin{aligned} \delta z &\simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y \\ &\simeq (16x + y^2 - 7)\delta x + (2xy + 6x)\delta y \end{aligned}$$

Substituting $x = 2.98$, $y = -2.01$, $\delta x = 0.04$, and $\delta y = 0.02$ gives

$$\begin{aligned} \delta z &\simeq (44.72...)0.04 + (5.90...)0.02 \\ &\simeq 1.91... \end{aligned}$$

Therefore the estimate of 35 may compare with an actual value as low as 33.09 or as high as 36.91 (both to 2 d.p.).