Unit 21

Rotating bodies and angular momentum

Introduction

When an ice skater is performing a spin, if she brings her arms in and folds them across her chest, her rate of rotation will increase. Why is this? Take another example. Suppose that two cylindrical objects have equal size and mass, but one is hollow and the other is solid. If they are released together from the top of a slope, will they roll down at the same rate? If not, which cylinder will reach the bottom of the slope first?

To answer such questions, we need to bring together a number of ideas that have been introduced earlier in the module. In earlier mechanics units, such as Units 3 and 9, we modelled moving objects as particles. However, this approach is inadequate for dealing with the questions above, because we are now concerned with extended bodies, that is, objects that have size, and we are interested in aspects of their motion where that size is important. In Unit 19, you saw that the motion of an extended body can often be modelled by the motion of a representative particle located at the centre of mass of the body. But in the case of the spinning skater, the centre of mass may well be more or less stationary – it is the skater's rotation about the centre of mass that is of interest. For the rolling cylinders, it is perhaps less obvious that the particle model is inappropriate, but again, rotation about the centre of mass is a crucial part of the motion.

An *extended* body has one or more of length, breadth and depth.

In this unit we deal with the motion of extended bodies, and in particular with their rotational motion. In Unit 2, we considered such bodies when stationary: you learned that for a rigid body in equilibrium, the sum of all the external forces must be zero, and that the sum of the external *torques* must be zero. In Unit 20, you saw that if a non-zero torque is applied to a particle, then this changes the rotational motion of the particle. Now we combine these ideas and consider the motion of an *extended body* subject to a *non-zero torque*.

A *rigid* body is an extended body whose shape does not change.

In Section 1 we give an overview of the unit and freely apply principles that are not formally established until Section 4. Section 2 begins by reviewing concepts from earlier units, and then goes on to develop a theoretical basis for modelling the motion of extended bodies. Section 3 looks at the rotation of extended bodies about an axis that is fixed, such as the spinning ice skater. In Section 4 we explore situations where rotational motion is combined with other types of motion. For example, consider a diver in flight after leaving a high diving board (see Figure 1). From Unit 19 we would expect the diver's centre of mass to follow a parabolic trajectory (if we ignore air resistance), but the diver's rotation about her centre of mass is also a major factor in the success of the dive (as is the change of shape of the body during the dive).

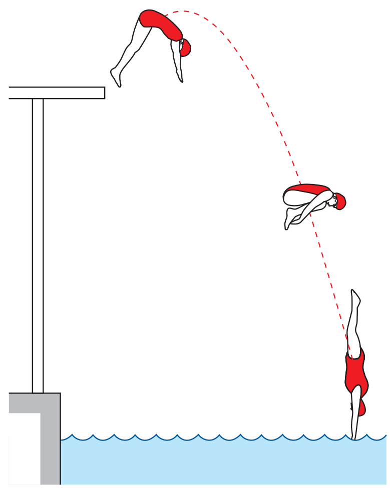


Figure 1 Diver with centre of mass following a parabolic path. Note that the diver is an extended body but *not* a rigid body.

1 Rotating bodies

Rotation is an important aspect of motion in many situations. Understanding such motion involves various mechanical concepts that you have met earlier in the module: these include torque (Unit 2), moment of inertia (Unit 17) and angular momentum (Unit 20).

To get an idea of some of the factors involved in analysing rotational motion, consider someone pushing a roundabout in a playground so as to make it move (see Figure 2). Initially the roundabout is stationary, but when it is pushed, it rotates with increasing rotational speed. Even after the person stops pushing, the roundabout will continue to rotate.

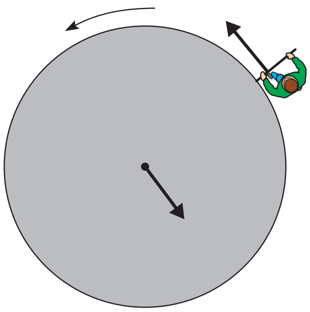


Figure 2 Playground roundabout being pushed

While the roundabout is being pushed, the total force on it is zero: the force supplied by the pusher is balanced by a force exerted by the support at the centre of the roundabout. The roundabout 'as a whole' is not going anywhere, that is, its centre of mass is not moving. However, although the two horizontal forces shown in Figure 2 are equal in magnitude and opposite in direction, they have different lines of action. As a result, there is a torque on the roundabout. This torque initiates the rotation of the roundabout and gives it angular momentum. In Unit 20, you saw that for a particle, if there is no torque being applied, then the angular momentum is constant. As you will see later, this result can be generalised to extended bodies. This means that even when the pushing stops, the roundabout will maintain its angular momentum and will continue to rotate; indeed, in the absence of resistive forces, it would go on rotating forever without the need for further pushing (but in practice resistive forces are always present).

Later in this unit we will develop quantitative models of rotational motion. But at this stage you should note the following general aspects of rotational motion.

- The motion of an extended body can be treated in two parts: the motion of an equivalent particle located at the centre of mass, and rotation about the centre of mass. This result will be established in Section 4.
- A torque applied to an extended body that is initially stationary will initiate rotation and supply angular momentum.
- Once an extended body has angular momentum, that momentum will remain constant provided that no further torque is applied to the body.
- The component L of the angular momentum along the axis of rotation of an extended body that is rotating about a fixed axis is the product of the angular speed ω and the moment of inertia I about that axis, that is,

$$L = I\omega. \tag{1}$$

This scalar equation is sufficient for the needs of this section.

You will see later, in Subsection 3.1, that this scalar equation can be obtained from a vector equation by resolving in the direction of the (fixed) axis of rotation.

- An ice skater can vary her speed of rotation during a spin by changing body shape. This is in accord with equation (1) since although the angular momentum L is constant, the moment of inertia is changing. When modelling *rigid* bodies, the moment of inertia is a constant, but the human body is flexible and its moment of inertia can change.

In the following examples and exercises we will analyse some sporting situations, thus exploring further the relationships between angular momentum, moment of inertia and rotational motion.

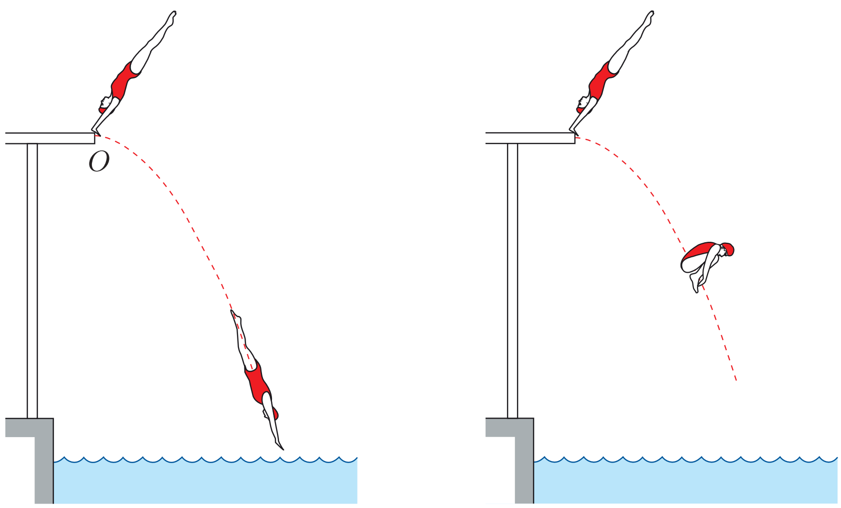
Example 1

(a) A diver is executing a simple dive in which her body shape remains constant, as illustrated in Figure 3(a). She starts in a handstand position. The subsequent motion can be divided into two phases: first, rotation about the point O while the diver remains in contact with the diving board; second, motion in flight after the diver lets go of the board, but before she enters the water.

(i) In the first phase, how is the diver's angular momentum about O changing?

(ii) In the second phase, what would you expect to happen to the angular momentum about the diver's centre of mass? Assume that resistive forces are negligible.

(b) Suppose that the diver goes into a tuck position (see Figure 3(b)) in the second phase. What aspect of the motion will be different from that in part (a)?



(a) (b)

Figure 3 Diver performing a dive: (a) with body shape constant, (b) taking a tuck position

Solution

(a) (i) In the first phase, the diver's weight provides a torque about O that will act to increase the angular momentum about O. During this phase of the motion, the angular momentum (and angular speed) about O are increasing.

(ii) In the second phase, the only force on the diver is her weight, which acts through her centre of mass. This means that there is no torque about the centre of mass, and consequently the angular momentum about the centre of mass will be constant.

(b) In the situation in part (a), the diver's body shape, and hence her moment of inertia, remain constant throughout the dive, so the angular speed of rotation about the diver's centre of mass does not change after she has let go of the board. However, if the diver adopts a tuck position after letting go of the board, this will reduce her moment of inertia, and will increase her angular speed of rotation about her centre of mass.

In Example 2 and Exercise 1, we will make use of the idea that is implicit in Example 1(a)(ii): that angular momentum about the centre of mass is constant for an (effectively rigid) extended body in flight (assuming that the body is subject only to gravity, that is, that resistive forces are negligible). This result will be established in Section 4.

Example 2

Figure 4 shows a gymnast rotating anticlockwise around a bar that is 3 m above the ground, at the point of letting go of the bar and dismounting. Assume that the gymnast does not change body shape, so he can be modelled as a rigid rod of length 2.4 m (with his arms extended). Also assume that his centre of mass is at his midpoint, that is, a distance R of 1.2 m from O, while he is contact with the bar. Just before he lets go of the bar, his centre of mass is moving in a circle at a speed of $4\,\mathrm{m\,s^{-1}}$. At the moment of release, his body makes an angle of $\frac{\pi}{3}$ with the vertical.

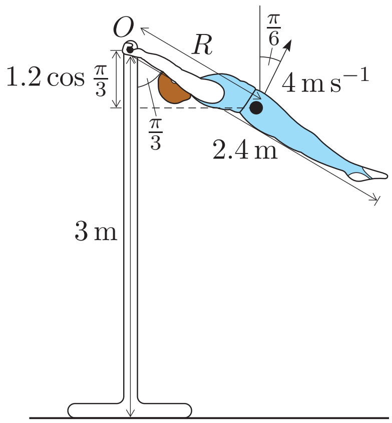


Figure 4 Gymnast rotating anticlockwise around a bar

(a) What is the gymnast's angular speed about his centre of mass just after he releases the bar?

(b) If the gymnast is to land successfully, that is, on his feet with his body in a vertical position, through what angle must he rotate while in the air? How long will it take him to rotate through this angle?

(c) Consider the vertical movement of an equivalent particle located at the centre of mass of the gymnast. What is the vertical component of the velocity of his centre of mass just after the gymnast releases the bar? Through what distance will his centre of mass fall during the time calculated in part (b)?

(d) Will the gymnast complete a dismount successfully using this approach? If not, what can he do to achieve a successful dismount?

Solution

(a) Just after releasing the bar, the gymnast's hands are stationary, and his centre of mass has the same velocity (say $\mathbf{u}$) as immediately before leaving the bar. So, relative to his centre of mass, his hands have velocity $-\mathbf{u}$. (To determine motion relative to the centre of mass, add $-\mathbf{u}$ to the velocities throughout the body, as shown in Figure 5, so that the point of view is taken in which the centre of mass is 'fixed'.) At that moment, the gymnast is rotating about his centre of mass with angular speed $|\mathbf{u}|/R$ in the direction in which he was circling the bar before letting go. (Recall from Unit 20 the relationship between speed and angular speed.) Since $|\mathbf{u}|$ is $4\,\mathrm{m\,s^{-1}}$ and R is $1.2\,\mathrm{m}$, the angular speed of rotation is $4/1.2 = \frac{10}{3}$ (in $\mathrm{rad\,s^{-1}}$).

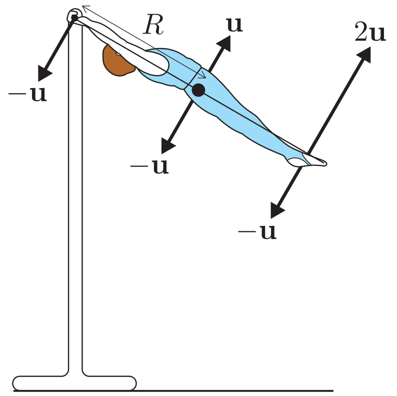


Figure 5 Velocities of parts of the gymnast

(b) At the time of release, the gymnast's body makes an angle of $\frac{\pi}{3}$ with the vertical, so the gymnast must rotate through $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ in order to achieve a vertical landing. At the angular speed calculated in part (a), this will take $\frac{5\pi}{3}/\frac{10}{3} = \frac{\pi}{2}$ (in seconds).

(c) At the moment of release, the gymnast's centre of mass has an upward vertical component of velocity of $4\cos\frac{\pi}{6}$, which is $2\sqrt{3}\,\mathrm{m\,s^{-1}}$.

Now, taking into account the constant downward acceleration of magnitude g, and substituting $x_0 = 0$, $v_0 = 2\sqrt{3}$, $a_0 = -g$ and $t = \frac{\pi}{2}$ into the constant acceleration equation $x = x_0 + v_0 t + \frac{1}{2}a_0 t^2$, we find that the vertical component of position will increase in $\frac{\pi}{2}$ seconds by

The constant acceleration equation was derived in Unit 3.

$$2\sqrt{3}\frac{\pi}{2} - \frac{1}{2}g\left(\frac{\pi}{2}\right)^2 \simeq -6.66.$$

Therefore in the time that it takes for the gymnast to attain a vertical body position for landing, his centre of mass will have fallen through a distance of about 6.66 m.

(d) The gymnast cannot complete a dismount successfully in this way. His centre of mass starts at $3 - 1.2\cos\frac{\pi}{3}$, which is 2.4 m above the ground. But as he would have to fall through 6.66 m before his body was vertical, he would hit the ground before he had completed the necessary rotation.

If the gymnast were to adjust his body position while in the air, so as to reduce his moment of inertia, he could increase his angular speed of rotation about his centre of mass. This might allow him to complete the necessary rotation before reaching the ground.

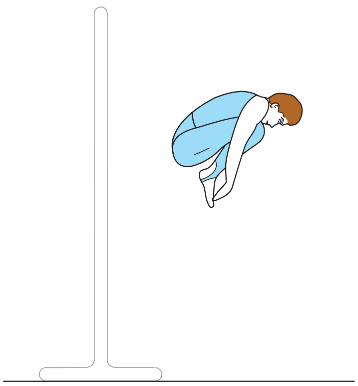


Figure 6 Gymnast in the tuck position

Exercise 1

Suppose that the gymnast in Example 2 adjusts his body position while in the air so as to achieve a successful dismount, landing on his feet in a vertical position. When he lands, his body is extended, with his centre of mass at a height of 1.2 m above the ground.

(a) How long is the gymnast in flight between letting go of the bar and landing?

(b) Assume that the gymnast is able to adjust his body position instantaneously, so that he is in a tuck position (see Figure 6) for *all* the time that he is in flight. What is the ratio of the moments of inertia of his body in the tuck and extended positions?

We end this section by discussing qualitatively, and in some detail, the Highland sport of tossing the caber. Tossing the caber can be divided into five phases (see Figure 7). Note that for the toss to be successful, the caber must land with the end originally held by the competitor pointing away from him. The five phases are as follows.

(i) The competitor runs forward with the caber held in a vertical position, and then stops.

(ii) The caber rotates forwards about the competitor's hands, which are stationary.

(iii) The competitor pushes upwards on the bottom of the caber, and releases it.

(iv) After the competitor releases the caber, it is in flight prior to hitting the ground.

(v) The end of the caber strikes the ground and remains stationary, but the motion of the rest of the caber continues until the caber falls down flat.

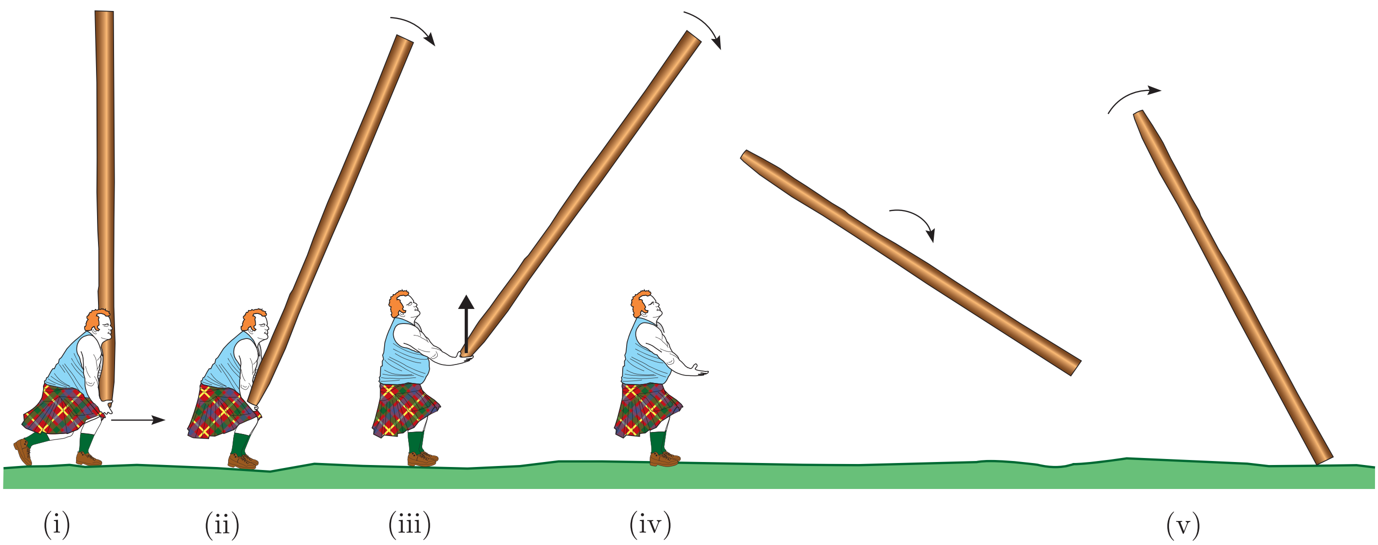


Figure 7 Tossing the caber in five phases

In the first phase, the whole caber gains a uniform horizontal speed. There is no rotation.

In the second phase, the bottom end of the caber is stationary. The upper part of the caber retains forward momentum. The result is to turn the forward motion into rotation about the bottom end of the caber. As the caber topples forwards, gravity supplies a torque about the hands, increasing the rate of rotation.

In the third phase, as the competitor pushes upwards, the resulting upward force applies a torque about the centre of mass, further increasing the rotation of the caber until the moment of release. (By waiting for the caber to rotate forward in the second phase before applying this force, the competitor increases the distance of the centre of mass from the line of action of the upward force that he applies, so increasing the torque.)

In the fourth phase, the centre of mass moves like a projectile. If resistive forces are ignored, the centre of mass follows a parabolic path and the angular speed about it is constant.

In the final phase, the end of the caber strikes the ground and becomes stationary (assuming that it does not skid or bounce). The caber will, however, retain some forward rotation about the end that hits the ground. If it lands as illustrated in Figure 7(v), with the upper end yet to reach the vertical, then the weight due to gravity will provide a torque about the end in contact with the ground that slows this rotation. Whether or not the caber will rotate past the vertical, ensuring that the toss is successful, will depend on the angle at which the caber lands, and on how much angular momentum it has from the preceding phase.

Typically the caber is tapered, with the thinner end held by the competitor. There are several good reasons for this. As the centre of mass of a tapered caber is nearer to the thicker end of the caber than to the thinner end being held by the competitor, in the third phase the centre of mass is further from the line of action of the force (applied by the competitor) than if the caber were not tapered, so increasing the applied torque about the centre of mass. Moreover, for a tapered caber, the centre of mass has further to fall during the fourth phase. This increases the time of flight, and hence increases the angle through which the caber rotates while in flight. Finally, suppose that the caber strikes the ground, thicker end first, before it has rotated past the vertical. Then, in the fifth phase, the centre of mass is closer to the ground than if the caber were symmetric, thereby reducing the torque that is slowing down the caber's rotation. Consequently the chances of a tapered caber reaching the vertical are greater.

A more quantitative mathematical treatment of various aspects of tossing the caber is given in Section 4.

2 Angular momentum

In Unit 20 we obtained the torque law for a particle,

$$\dot{\mathbf{L}} = \mathbf{\Gamma},$$

where $\mathbf{L}$ is the particle's angular momentum, and $\mathbf{\Gamma}$ is the torque applied to the particle (each taken about the same point). In this section, you will see that this result can be extended to a system of particles; in that case, $\mathbf{L}$ is the total angular momentum of all the particles in the system, and $\mathbf{\Gamma}$ is the total torque exerted by all the external forces on the particles in the system. These results can be applied to extended bodies by modelling them as systems of particles.

This section starts by reviewing a number of ideas that you have met earlier: in Subsection 2.1 we review Newton's third law of motion, and in Subsection 2.2 we review angular momentum and the torque law for a particle. Then in Subsection 2.3, we extend the torque law to a system of particles.

2.1 Newton's third law of motion revisited

In order to extend the results of Unit 20 from particles to extended bodies, we need to look again at Newton's third law, which deals with the interaction between particles. In the Introduction to Unit 2, Newton's third law was stated as follows:

> **Law III** To every action (i.e. force) by one body on another there is always opposed an equal reaction (i.e. force) – that is, the actions of two bodies on each other are always equal in magnitude and opposite in direction.

In symbols this can be expressed by considering two particles, particle 1 and particle 2. Let $\mathbf{I}_{12}$ be the force exerted on particle 1 by particle 2, and let $\mathbf{I}_{21}$ be the force exerted on particle 2 by particle 1. Then the statement of Newton's third law above can be written as $\mathbf{I}_{12} = -\mathbf{I}_{21}$. Alternatively,

$$\mathbf{I}_{12} + \mathbf{I}_{21} = \mathbf{0}. \tag{2}$$

Thus the sum of the forces is zero. Recall that these forces between the particles in a system were called *internal* forces in Unit 19.

Exercise 2 considers how pairs of inter-particle forces that satisfy equation (2) might be represented diagrammatically.

Exercise 2

Figure 8 shows examples of two forces of equal magnitude acting on particles A and B. For each pair of forces, state whether equation (2) is satisfied.

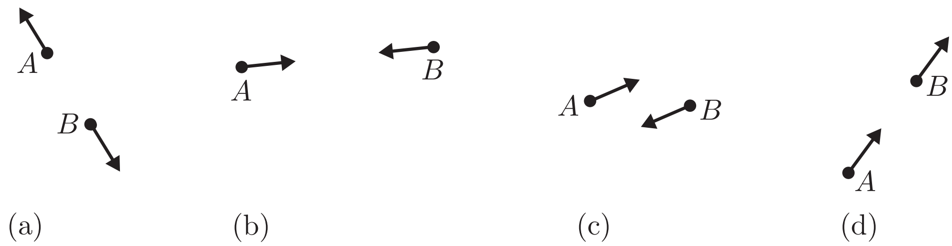


Figure 8 Pairs of forces acting on particles A and B

What we did not mention in Unit 2 was that Newton's third law also states that the pair of equal and opposite forces should have *the same line of action* (this was because the concept of line of action had yet to be introduced in Unit 2). Try the following exercise to see what this means.

Exercise 3

Look back at Figure 8 and find an example of a pair of forces that satisfy equation (2) but where the forces do *not* have the same line of action.

So there are pairs of forces that satisfy equation (2) but where the forces do *not* act in the same straight line. We therefore need a mathematical condition to test whether forces are acting in the same straight line, and hence whether Newton's third law is applicable. We will now develop such a condition.

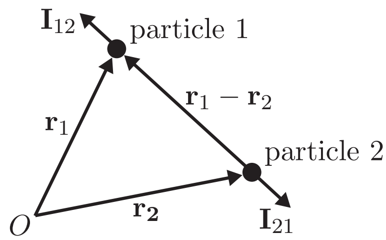


Figure 9 Application of Newton's third law

Figure 9 illustrates a situation where Newton's third law applies: the force $\mathbf{I}_{12}$ exerted on particle 1 by particle 2 is equal and opposite to the force $\mathbf{I}_{21}$ exerted on particle 2 by particle 1, with $\mathbf{I}_{12}$ and $\mathbf{I}_{21}$ having the same line of action. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of particles 1 and 2, respectively, with respect to some origin O. The vectors $\mathbf{I}_{12}$ and $\mathbf{r}_1 - \mathbf{r}_2$ have the same direction, so their cross product must be $\mathbf{0}$, that is,

Recall from Unit 2 that the cross product of parallel vectors is zero.

$$(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{I}_{12} = \mathbf{0}.$$

By equation (2) we have $\mathbf{I}_{12} = -\mathbf{I}_{21}$, so this can be rewritten as

$$\begin{aligned}(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{I}_{12} &= \mathbf{r}_1 \times \mathbf{I}_{12} - \mathbf{r}_2 \times \mathbf{I}_{12} \\ &= \mathbf{r}_1 \times \mathbf{I}_{12} + \mathbf{r}_2 \times \mathbf{I}_{21} = \mathbf{0}.\end{aligned} \tag{3}$$

Let $\mathbf{\Gamma}_{12} = \mathbf{r}_1 \times \mathbf{I}_{12}$ be the torque about O exerted on particle 1 by the force from particle 2, and similarly let $\mathbf{\Gamma}_{21} = \mathbf{r}_2 \times \mathbf{I}_{21}$ be the torque about O exerted on particle 2 by the force from particle 1. Then from equation (3) we have

Recall from Unit 2 that a force $\mathbf{F}$ whose point of action has position vector $\mathbf{r}$ exerts a torque $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$ about the origin O.

$$\mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{21} = \mathbf{0},$$

that is, the sum of the inter-particle torques is zero when equal and opposite inter-particle forces have the same line of action. In other words, when Newton's third law applies, the sum of the inter-particle torques is zero.

Conversely, if the sum of the inter-particle torques is not zero, then the line of action of the forces is not the same (by reversing the above argument). Newton's third law can therefore be expressed in terms of the properties of the forces and torques between two interacting particles, as follows.

These equations state that the sum of the inter-particle forces is zero, and the sum of the torques exerted (about the origin) by those forces is also zero.

Newton's third law re-stated

The force $\mathbf{I}_{12}$ exerted on particle 1 by particle 2 is equal in magnitude but opposite in direction to the force $\mathbf{I}_{21}$ exerted on particle 2 by particle 1, with both forces acting along the line joining the two particles. An equivalent condition in symbols is

$$\mathbf{I}_{12} + \mathbf{I}_{21} = \mathbf{0} \quad \text{and} \quad \mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{21} = \mathbf{0},$$

where $\mathbf{\Gamma}_{12} = \mathbf{r}_1 \times \mathbf{I}_{12}$ and $\mathbf{\Gamma}_{21} = \mathbf{r}_2 \times \mathbf{I}_{21}$, with $\mathbf{r}_1$ and $\mathbf{r}_2$ being the position vectors of particles 1 and 2, respectively, relative to the origin.

In electromagnetism there is a type of inter-particle force that does not obey Newton's third law, but this exception need not concern you here.

Almost all inter-particle forces conform to Newton's third law. The gravitational and electrostatic forces between particles obey this law, as do the forces in model springs or model rods joining two particles. All the systems that you will meet in this module conform to Newton's third law.

2.2 Torque law for a particle

Recall, from Unit 20, the definition of angular momentum for a particle, which is as follows: for a particle of mass m that has linear momentum $m\dot{\mathbf{r}}$ and position vector $\mathbf{r}$ relative to an origin O, its angular momentum $\mathbf{L}$ about O is

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}, \tag{4}$$

that is, the angular momentum is the cross product of the particle's position vector and its linear momentum.

Since the position vector is part of the definition, the angular momentum is dependent on the choice of origin (which is not the case for linear momentum). So the angular momentum will usually not be the same relative to different choices of origin.

Exercise 4

A particle of mass 20 has position vector $\mathbf{r} = 3\cos(2t)\mathbf{i} + 4\sin(2t)\mathbf{j} + 5\mathbf{k}$ with origin O (working in SI units), where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are Cartesian unit vectors.

What is the angular momentum of the particle about O at time $t = 0$?

Exercise 5

A particle of mass m moves anticlockwise in a circle of radius R at constant speed v. The circle has its centre at the origin O and lies in the (x, y)-plane. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be the Cartesian unit vectors.

(a) What is the angular velocity $\boldsymbol{\omega}$ of the particle?

(b) What is the angular momentum $\mathbf{L}$ of the particle about O?

(*Hint*: Work from the definition of angular momentum, and assume that the particle has position $R(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j})$, where ω is the angular speed.)

This result is also derived in Unit 20.

(c) What is the moment of inertia I of the particle about an axis through O in the $\mathbf{k}$-direction?

(d) Show that $\mathbf{L} = I\boldsymbol{\omega}$.

In Unit 20 we showed that if a particle is subject to a torque, then the rate of change of the particle's angular momentum $\mathbf{L}$ about a fixed point is equal to the applied torque $\boldsymbol{\Gamma}$ about that point, that is, $\dot{\mathbf{L}} = \boldsymbol{\Gamma}$. The derivation of this important result from Unit 20, known as the *torque law for a particle*, is repeated below.

Using equation (4), we find

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}) \\ &= (\dot{\mathbf{r}} \times m\dot{\mathbf{r}}) + (\mathbf{r} \times m\ddot{\mathbf{r}}) \\ &= \mathbf{r} \times m\ddot{\mathbf{r}}. \qquad (5)\end{aligned}$$

Since $\dot{\mathbf{r}}$ and $m\dot{\mathbf{r}}$ are parallel vectors, $\dot{\mathbf{r}} \times m\dot{\mathbf{r}} = \mathbf{0}$.

Now, by Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, where $\mathbf{F}$ is the total force on the particle. Substituting for $m\ddot{\mathbf{r}}$ in equation (5), we have

$$\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F}.$$

But the total torque on the particle (relative to the chosen origin) is $\mathbf{r} \times \mathbf{F} = \boldsymbol{\Gamma}$. So we obtain the torque law for a particle:

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}. \qquad (6)$$

An important special case of this law is when the total external torque is zero. In this case equation (6) becomes $\dot{\mathbf{L}} = \mathbf{0}$, so $\mathbf{L}$ is a constant vector. This is called the *law of conservation of angular momentum for a particle.*

As an example of a case where angular momentum is conserved, consider any particle moving at constant velocity. By Newton's first law, the total force acting on the particle must be zero, so the total torque acting on the particle will also be zero. So by the torque law, the angular momentum of the particle about *any* point will be constant. However, there is an alternative way of understanding this conservation of angular momentum that comes directly from definition (4) and proceeds as follows.

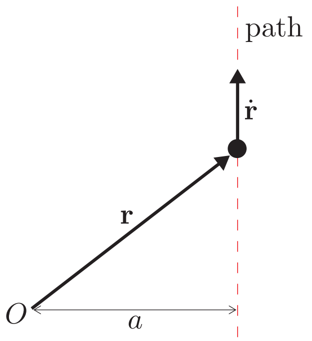


Figure 10 Particle moving with constant velocity $\dot{\mathbf{r}}$

By definition, the particle has angular momentum $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$, where $\mathbf{r}$ gives the particle's position relative to the origin. Since the velocity $\dot{\mathbf{r}}$ of the particle is constant, the particle will follow a straight-line path, as illustrated in Figure 10. The position vector $\mathbf{r}$ is varying, but wherever the particle is on the path, the vector $\mathbf{r} \times \dot{\mathbf{r}}$ will have magnitude $a|\dot{\mathbf{r}}|$, where a is the perpendicular distance from O to the particle's path, and direction normal to the plane shown in Figure 10 (and out of the page in the case illustrated).

Thus the angular momentum is constant, since both a and $|\dot{\mathbf{r}}|$ are constant and its direction is constant.

Exercise 6

(a) If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are Cartesian unit vectors and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, show that

$$\mathbf{r} \times \dot{\mathbf{r}} = (x\dot{y} - y\dot{x})\mathbf{k}.$$

(b) Suppose that a particle of mass m is moving in the (x, y)-plane. By expressing x and y in terms of polar coordinates r and θ, show that the angular momentum $\mathbf{L}$ of the particle about the origin O is

This result was also derived in Unit 20.

$$\mathbf{L} = mr^2\dot{\theta}\mathbf{k}. \qquad (7)$$

Exercise 6(b) provides an expression for the angular momentum about the origin of any particle moving in the (x, y)-plane. It generalises the result obtained in Exercise 5 for the angular momentum of a particle travelling in a circle at constant speed. In Exercise 5 the particle followed a circle whose centre was at the origin. The next example concerns motion in a circle whose centre is not the origin (indeed, the circle is not in the (x, y)-plane).

Example 3

A particle of mass m moves with a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$ in a circular path of radius R, centred on a fixed point with Cartesian coordinates $(0, 0, h)$. At $t = 0$, the particle is at the point $(R, 0, h)$. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be Cartesian unit vectors in the x-, y- and z-directions, respectively, and let O be the origin.

(a) Express the particle's position vector $\mathbf{r}$ as a function of t.

(b) (i) Calculate the particle's velocity $\dot{\mathbf{r}}$ by differentiation.

(ii) Recall from Unit 20 that a particle executing circular motion with angular velocity $\boldsymbol{\omega}$ has velocity $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. Verify that this is consistent with your result in part (b)(i).

(c) Calculate the angular momentum of the particle about O.

(d) Show that the torque about O acting to sustain the motion of the particle is $\boldsymbol{\Gamma} = -m\omega h\dot{\mathbf{r}}$.

Solution

(a) The angular velocity is in the $\mathbf{k}$-direction, so the particle must be moving parallel to the (x, y)-plane. The plane of motion is $z = h$. The particle is travelling with constant angular velocity $\omega\mathbf{k}$ in a circle of radius R, and is at $x = R$, $y = 0$ when $t = 0$. Therefore its position vector is

In Unit 20, you saw that the angular velocity is perpendicular to the plane of motion.

$$\mathbf{r} = R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j} + h\mathbf{k}. \tag{8}$$

See Unit 20.

(b) (i) Differentiating equation (8) gives the particle's velocity as

$$\dot{\mathbf{r}} = -R\omega\sin(\omega t)\mathbf{i} + R\omega\cos(\omega t)\mathbf{j}. \tag{9}$$

(ii) Using $\mathbf{r}$ from equation (8) and $\boldsymbol{\omega} = \omega\mathbf{k}$, we substitute into $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$ and obtain

$$\begin{aligned}\dot{\mathbf{r}} &= \omega\mathbf{k} \times \big(R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j} + h\mathbf{k}\big)\\ &= R\omega\cos(\omega t)\mathbf{j} - R\omega\sin(\omega t)\mathbf{i},\end{aligned}$$

which is consistent with the result in part (b)(i).

(c) By definition (equation (4)), the angular momentum of the particle about O is $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$. On substituting for $\mathbf{r}$ and $\dot{\mathbf{r}}$ from equations (8) and (9), respectively, we obtain

$$\begin{aligned}\mathbf{L} &= \big(R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j} + h\mathbf{k}\big)\\ &\qquad \times m\big(-R\omega\sin(\omega t)\mathbf{i} + R\omega\cos(\omega t)\mathbf{j}\big)\\ &= m\big(R^2\omega\cos^2(\omega t)\mathbf{k} + R^2\omega\sin^2(\omega t)\mathbf{k}\\ &\qquad - hR\omega\sin(\omega t)\mathbf{j} - hR\omega\cos(\omega t)\mathbf{i}\big)\\ &= mR^2\omega\mathbf{k} - mhR\omega\big(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\big).\end{aligned} \tag{10}$$

(d) From the torque law for a particle, we have $\boldsymbol{\Gamma} = \dot{\mathbf{L}}$. Now, by differentiating equation (10) we get

$$\dot{\mathbf{L}} = -mhR\omega^2\big(\cos(\omega t)\mathbf{j} - \sin(\omega t)\mathbf{i}\big). \tag{11}$$

Comparing this with equation (9), we see that

$$\dot{\mathbf{L}} = -mh\omega\dot{\mathbf{r}},$$

hence $\boldsymbol{\Gamma} = -mh\omega\dot{\mathbf{r}}$.

To sum up: in both Exercise 5 and Example 3 we looked at a particle moving at constant speed in a circle. In Exercise 5 you saw that, relative to the centre of the circle, angular momentum is constant and the total torque acting on the particle is zero. In Example 3, we considered angular momentum and torque relative to a point on the axis of rotation but *not* at the centre of the circle. Relative to that point, the angular momentum is not constant (although its component in the direction of the angular velocity is constant); consequently, there must be a non-zero total torque on the particle about that point if such motion is to be sustained.

Equation (10) shows that the component of $\mathbf{L}$ in the $\mathbf{k}$-direction is constant.

Exercise 7

For a particle moving as described in Example 3, calculate the acceleration $\ddot{\mathbf{r}}$. Use Newton's second law to find the total force $\mathbf{F}$ acting on the particle. Then calculate the total torque $\mathbf{\Gamma}$ on the particle by using $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$, and show that you obtain the same result as in Example 3(d).

2.3 Torque law for an n-particle system

We now move on to consider angular momentum and torque for systems involving more than one particle. First, we look at systems that have two particles, then we go on to examine a general system containing an arbitrary number of particles.

Two-particle systems

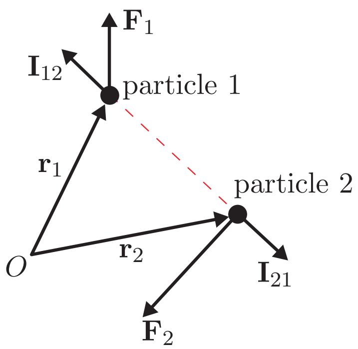


Figure 11 Two-particle system

Consider a two-particle system, as illustrated in Figure 11, to which Newton's third law applies. Particle 1, at position $\mathbf{r}_1$ relative to a fixed origin O, may be acted on by any number of external forces, but we will consider only the resultant of all these external forces, $\mathbf{F}_1$. The only other force on particle 1 is the internal force $\mathbf{I}_{12}$ exerted on it by particle 2. The total external force on particle 2, at position $\mathbf{r}_2$, is $\mathbf{F}_2$, and the only other force on particle 2 is the internal force $\mathbf{I}_{21}$ exerted on it by particle 1. Let the angular momentum of particle 1 relative to O be $\mathbf{L}_1$, and that of particle 2 relative to O be $\mathbf{L}_2$.

Applying the torque law for a particle (equation (6)) to particle 1 gives

$$\dot{\mathbf{L}}_1 = \mathbf{r}_1 \times (\mathbf{F}_1 + \mathbf{I}_{12}) = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{\Gamma}_{12},$$

where $\mathbf{\Gamma}_{12}$ is the torque about O acting on particle 1 by the internal force from particle 2. Similarly, for particle 2 we have

$$\dot{\mathbf{L}}_2 = \mathbf{r}_2 \times (\mathbf{F}_2 + \mathbf{I}_{21}) = \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{\Gamma}_{21},$$

where $\mathbf{\Gamma}_{21}$ is the torque about O acting on particle 2 by the internal force from particle 1. Adding these two equations, we obtain

$$\dot{\mathbf{L}}_1 + \dot{\mathbf{L}}_2 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{21}.$$

Newton's third law implies that $\mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{21} = \mathbf{0}$, so this reduces to

$$\dot{\mathbf{L}}_1 + \dot{\mathbf{L}}_2 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2. \tag{12}$$

In equation (12), the right-hand side represents the total torque exerted on the two-particle system by the external forces, while the left-hand side gives the rate of change with time of the total angular momentum of the system. Thus equation (12) extends the torque law to a system of two particles. The *torque law for a two-particle system* can then be stated as follows. For a two-particle system, the rate of change of the total angular momentum of the particles in the system is equal to the total torque exerted on the system by external forces (where angular momentum and torque are determined relative to the same fixed point).

Exercise 8

Figure 12 is a schematic representation of a type of children's playground roundabout. Two children sit on seats at A and B, and the roundabout is set in motion by pushing at C. It rotates horizontally about a fixed spindle at O. The distances AO, BO and CO are a, b and c, respectively. The mass of the child and seat at A is m_1, and the mass of the child and seat at B is m_2. The positions of the seats are balanced so that $m_1 a = m_2 b$. The mass of the rest of the roundabout is negligible.

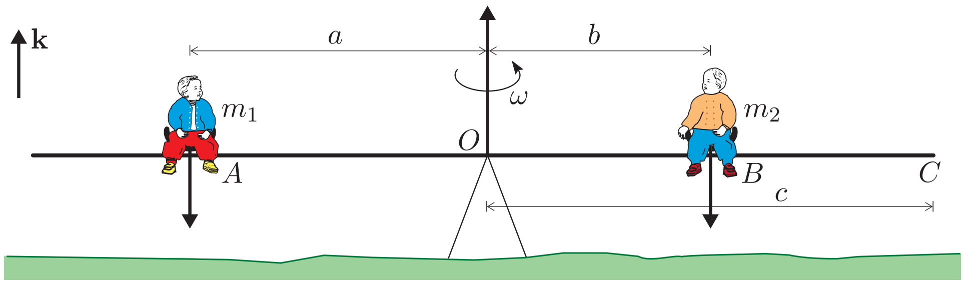


Figure 12 Two children on a playground roundabout

(a) Suppose that the roundabout is rotating anticlockwise at an angular speed ω. Model each 'child plus seat' as a particle, and show that the total angular momentum of the roundabout about O is $I\boldsymbol{\omega}$, where $I = m_1 a^2 + m_2 b^2$ and $\boldsymbol{\omega} = \omega\mathbf{k}$, with $\mathbf{k}$ being a unit vector pointing vertically upwards.

(b) Suppose that an adult pushes the roundabout so as to apply a force of constant magnitude F at C, in a horizontal direction at right angles to OC. Assuming that resistive forces are negligible, show that

$$I\dot{\omega} = cF.$$

(Note that the roundabout is rotating anticlockwise, so $\dot{\theta} \geq 0$, thus $\omega = \dot{\theta}$ and $\dot{\omega} = \ddot{\theta}$.)

(c) If the distance AO is $1\,\text{m}$, BO is $0.75\,\text{m}$ and CO is $1.5\,\text{m}$, and m_1 is $45\,\text{kg}$, m_2 is $60\,\text{kg}$ and F is $315\,\text{N}$, then for how long will the adult need to push in order to get the roundabout rotating at 0.5 revolutions per second when the roundabout has started from rest?

The next example concerns a two-particle system executing a more complicated motion.

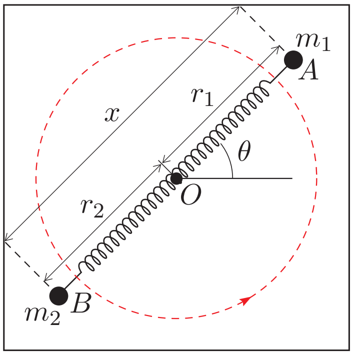


Figure 13 Two particles connected by a spring

Example 4

Consider two particles of masses m_1 and m_2, connected by a model spring and resting on a smooth horizontal table, as shown in Figure 13. The system is in motion: the two particles are rotating at the same angular velocity about the centre of mass of the system, and simultaneously they are oscillating as the spring extends and contracts. The total external force on the system is zero: the weight of each particle is balanced by the normal reaction from the table, and we assume that no other external forces are acting. The centre of mass is stationary initially. Therefore it remains stationary throughout the motion because the acceleration of the centre of mass is zero (since the total external force is zero). (This was shown in Unit 19.)

Take the origin O to be at the centre of mass, and suppose that particle A has polar coordinates (r_1, θ), and particle B has polar coordinates $(r_2, \theta - \pi)$. Let x be the distance between the two particles, and suppose that $\mathbf{k}$ is a unit vector pointing vertically upwards. Use the torque law for a two-particle system to show that the quantity $x^2\dot{\theta}$ is constant. Use this equation to describe the motion qualitatively.

Solution

The only external forces acting on the system are the weights of the particles and the normal reactions of the table balancing each weight. These normal reactions have the same points of action as the weights; consequently, not only is the total external force on the system zero, but the total external torque on the system is also zero. (The only other force acting on each particle is that from the spring, and this is an internal force.) It follows from the torque law that as the total external torque is zero, the angular momentum $\mathbf{L}$ of the system is a constant.

From equation (7), the angular momentum of particle A about O is $m_1 r_1^2 \dot{\theta}\mathbf{k}$, and that of particle B is $m_2 r_2^2 \dot{\theta}\mathbf{k}$. So the total angular momentum of the system is

$$\mathbf{L} = m_1 r_1^2 \dot{\theta}\mathbf{k} + m_2 r_2^2 \dot{\theta}\mathbf{k} = (m_1 r_1^2 + m_2 r_2^2)\dot{\theta}\mathbf{k}.$$

But $\mathbf{L}$ is a constant, as noted above, therefore

$$(m_1 r_1^2 + m_2 r_2^2)\dot{\theta} = c, \tag{13}$$

where c is a constant.

Because O is at the centre of mass of the two-particle system,

$$m_1 r_1 = m_2 r_2. \tag{14}$$

Now $x = r_1 + r_2$, so we can substitute for r_1 in equation (14) to obtain $m_1(x - r_2) = m_2 r_2$. Then solving for r_2 gives

$$r_2 = \frac{m_1 x}{m_1 + m_2}. \tag{15}$$

Substituting this into equation (14) gives r_1 in terms of x:

$$r_1 = \frac{m_2 x}{m_1 + m_2}. \tag{16}$$

Substituting from equations (15) and (16) into equation (13) yields

$$\begin{aligned}
c &= \left(m_1 \frac{m_2^2 x^2}{(m_1+m_2)^2} + m_2 \frac{m_1^2 x^2}{(m_1+m_2)^2} \right) \dot{\theta} \\
&= \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2}\, x^2 \dot{\theta} \\
&= \frac{m_1 m_2}{m_1 + m_2}\, x^2 \dot{\theta}.
\end{aligned}$$

Since the masses m_1 and m_2 are constant, we can deduce that $x^2\dot{\theta}$ must be constant.

As the system rotates, the spring extends and compresses as the particles oscillate in and out. The rotational motion of the particles is connected to these oscillations since $x^2\dot{\theta}$ is a constant. When the spring is extended (x is larger) the angular speed $|\dot{\theta}|$ decreases, and vice versa.

Exercise 9

A binary star system consists of two stars of masses m_1 and m_2. Model each star as a particle, and assume that all the forces exerted on the system, other than the gravitational attraction between the two stars, can be ignored. Assume also that the distance d between the stars is constant, and that the common centre of mass of the two stars is fixed. Let the total angular momentum of the binary system have magnitude L. Express the period of rotation of the system in terms of m_1, m_2, d and L.

---

n-particle systems

You have seen above that the torque law for a two-particle system has the same form as that for a single particle. This is because the sum of the internal torques between the two particles is zero. The situation is analogous for a system of more than two particles. Therefore we can generalise the torque law to any number of particles.

Consider a system of n particles, which we will call particles $1, 2, 3, \ldots, n$, to which Newton's third law applies. Particle 1 may be acted on by various external forces, but we will denote the resultant of all the external forces on particle 1 by $\mathbf{F}_1$. In general, for $i = 1, 2, \ldots, n$, $\mathbf{F}_i$ denotes the resultant of all the external forces on particle i. As well as being subject to forces external to the system, particle 1 may be acted on by internal forces exerted by each of the other particles in the system: denote by $\mathbf{I}_{12}$ the force exerted on particle 1 by particle 2, denote by $\mathbf{I}_{13}$ the force exerted on particle 1 by particle 3, and so on. In general, let $\mathbf{I}_{ij}$ denote the force exerted on particle i by particle j (for i and j between 1 and n, with $i \neq j$). For $i = 1, 2, \ldots, n$, let the position of particle i be $\mathbf{r}_i$ (relative to a fixed origin O), and let the angular momentum of particle i relative to O be $\mathbf{L}_i$.

The torque law for a particle when applied to particle 1 gives

$$\dot{\mathbf{L}}_1 = \mathbf{r}_1 \times (\mathbf{F}_1 + \mathbf{I}_{12} + \mathbf{I}_{13} + \mathbf{I}_{14} + \cdots + \mathbf{I}_{1n}). \tag{17}$$

If $\mathbf{\Gamma}_{ij}$ is the torque exerted on particle i by the internal force from particle j, then $\mathbf{\Gamma}_{ij} = \mathbf{r}_i \times \mathbf{I}_{ij}$, and equation (17) becomes

$$\dot{\mathbf{L}}_1 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{13} + \mathbf{\Gamma}_{14} + \cdots + \mathbf{\Gamma}_{1n}. \tag{18}$$

Similarly, the torque law for a particle when applied to particle 2 gives

$$\begin{aligned}\dot{\mathbf{L}}_2 &= \mathbf{r}_2 \times (\mathbf{F}_2 + \mathbf{I}_{21} + \mathbf{I}_{23} + \mathbf{I}_{24} + \cdots + \mathbf{I}_{2n}) \\ &= \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{\Gamma}_{21} + \mathbf{\Gamma}_{23} + \mathbf{\Gamma}_{24} + \cdots + \mathbf{\Gamma}_{2n}. \end{aligned} \tag{19}$$

For particle i, we get

$$\begin{aligned}\dot{\mathbf{L}}_i &= \mathbf{r}_i \times (\mathbf{F}_i + \mathbf{I}_{i1} + \mathbf{I}_{i2} + \mathbf{I}_{i3} + \cdots + \mathbf{I}_{in}) \\ &= \mathbf{r}_i \times \mathbf{F}_i + \mathbf{\Gamma}_{i1} + \mathbf{\Gamma}_{i2} + \mathbf{\Gamma}_{i3} + \cdots + \mathbf{\Gamma}_{in}. \end{aligned} \tag{20}$$

Note that the sum does not contain a term $\mathbf{\Gamma}_{ii}$.

Now, to obtain the total angular momentum of the system, we can sum versions of equation (20) for each of $i = 1, 2, \ldots, n$. When we do this, all the internal torques can be 'paired up': thus the term $\mathbf{\Gamma}_{12}$ from equation (18) can be paired with the term $\mathbf{\Gamma}_{21}$ from equation (19); the term $\mathbf{\Gamma}_{13}$ from equation (18) can be paired with the term $\mathbf{\Gamma}_{31}$ from the equivalent equation for particle 3; and generally, the term $\mathbf{\Gamma}_{ij}$ from the equation for particle i (where $i \neq j$) can be paired with the term $\mathbf{\Gamma}_{ji}$ from the equation for particle j. Each pair will sum to $\mathbf{0}$, since from Newton's third law, $\mathbf{\Gamma}_{ij} + \mathbf{\Gamma}_{ji} = \mathbf{0}$ (for $i \neq j$), so the total of all the internal torques for the n particles comprising the system will be $\mathbf{0}$. If $\mathbf{L}$ is the total angular momentum of the system (where $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n$), then from equation (20) (for $i = 1, 2, \ldots, n$) we have

$$\dot{\mathbf{L}} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \cdots + \mathbf{r}_n \times \mathbf{F}_n. \tag{21}$$

Here, $\mathbf{r}_1 \times \mathbf{F}_1$ is the torque exerted on particle 1 by the external forces acting on it, $\mathbf{r}_2 \times \mathbf{F}_2$ is the torque exerted on particle 2 by the external forces acting on it, and so on. Therefore the total external torque on the system is

$$\mathbf{\Gamma} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \cdots + \mathbf{r}_n \times \mathbf{F}_n.$$

So equation (21) can be written as

$$\dot{\mathbf{L}} = \mathbf{\Gamma}.$$

Thus we have shown that the rate of change of the total angular momentum of an n-particle system is equal to the total external torque acting on the system. This extends the torque law to an n-particle system, which we now refer to as just the torque law.

Torque law

Consider a system of n particles. Let $\mathbf{r}_i$ be the position vector (relative to a fixed origin O) of particle i, and let $\mathbf{F}_i$ be the total external force on particle i, for $i = 1, 2, \ldots, n$. The total external torque on the system about O is

$$\boldsymbol{\Gamma} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i. \tag{22}$$

The rate of change of the total angular momentum $\mathbf{L}$ of the system about O equals the total external torque acting on the system, that is,

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}. \tag{23}$$

In particular, when the total external torque about O is zero, the total angular momentum vector about O is conserved, that is, it is constant.

Exercise 10

Suppose that all the external forces on a system of particles are applied at the same fixed point X. What can you deduce about the angular momentum of the system?

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3 Rigid-body rotation about a fixed axis

In this section we model a **rigid body** as an n-particle system where all the inter-particle distances remain constant. To simplify matters, we will consider rigid bodies that are rotating about a fixed axis. For such a body, the rotational motion can be expressed in a very convenient way by using the moment of inertia. This is done in Subsection 3.1.

Rigid bodies were introduced in Unit 2.

Moments of inertia were introduced in Unit 17.

Unit 17 showed how to use multiple integrals to calculate the moments of inertia of complicated shapes. In this unit we model systems using simple geometric shapes, and we will give a table of moments of inertia for these in Subsection 3.1. This table covers only moments of inertia about an axis through the centre of mass of the body. You may want to find moments of inertia about other axes, and this is covered in Subsection 3.2. The kinetic energy of a rotating rigid body can also be expressed in terms of its moment of inertia, and we consider that in Subsection 3.3.

3.1 Angular momentum and moments of inertia

In this section we confine our attention to motion in which a rigid body is rotating about a fixed axis. For simplicity, we choose a frame of reference in which the z-axis is the axis of rotation.

Now, in order to apply the torque law $\dot{\mathbf{L}} = \mathbf{\Gamma}$ to a rotating rigid body, we need to find an expression for the angular momentum of the body (relative to a point on the axis of rotation). Suppose that the body is composed of particles A, B, C, and so on. The particles that lie on the axis of rotation, like C and D in Figure 14(a), do not move. On the other hand, particles like A and B do move, but their distances from the z-axis remain constant, so they travel in circles centred on a point on the z-axis, and parallel to the (x, y)-plane. The angular velocity of the body (and of each of its constituent particles) is $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, where θ is the angle through which the body is rotated about its axis of rotation, as shown in Figure 14(b).

We choose x-, y-, z-axes that are fixed in space, and an origin O that is fixed and on the axis of rotation. Here $\mathbf{k}$ is a unit vector in the z-direction.

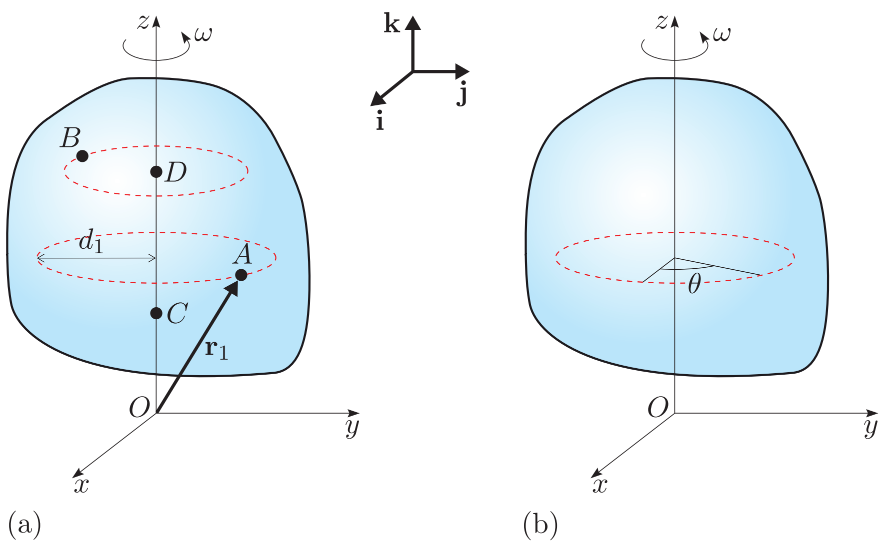

Figure 14 Rigid body rotating about the z-axis, showing (a) some individual particles, (b) the relationship with the angular displacement θ

First consider particle A, which has mass m_1 and position vector $\mathbf{r}_1$. From Unit 20, particle A has velocity $\dot{\mathbf{r}}_1 = \boldsymbol{\omega} \times \mathbf{r}_1$. Therefore its angular momentum is

$$\begin{aligned}\mathbf{L}_1 &= \mathbf{r}_1 \times m_1\dot{\mathbf{r}}_1\\ &= \mathbf{r}_1 \times m_1(\boldsymbol{\omega} \times \mathbf{r}_1)\\ &= \mathbf{r}_1 \times m_1\dot{\theta}(\mathbf{k} \times \mathbf{r}_1).\end{aligned} \tag{24}$$

If $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, we have

$$\begin{aligned}\mathbf{k} \times \mathbf{r}_1 &= \mathbf{k} \times (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})\\ &= x_1\mathbf{j} - y_1\mathbf{i}.\end{aligned}$$

Then, on substituting for $\mathbf{r}_1$ and $\mathbf{k} \times \mathbf{r}_1$ in equation (24), we obtain

$$\begin{aligned}\mathbf{L}_1 &= (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \times m_1\dot{\theta}(x_1\mathbf{j} - y_1\mathbf{i})\\ &= m_1\dot{\theta}(x_1^2\mathbf{k} + y_1^2\mathbf{k} - x_1z_1\mathbf{i} - y_1z_1\mathbf{j}).\end{aligned} \tag{25}$$

Let d_1 be the perpendicular distance of particle A from the axis of rotation. Then $d_1^2 = x_1^2 + y_1^2$, and equation (25) simplifies to

$$\mathbf{L}_1 = m_1\dot{\theta}(-x_1z_1\mathbf{i} - y_1z_1\mathbf{j} + d_1^2\mathbf{k}).$$

Suppose that the rigid body consists of n particles, where the ith particle (for $i = 1, 2, \ldots, n$) has mass m_i and position vector $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$. The total angular momentum $\mathbf{L}$ of the body will be the sum of the angular momenta of all these particles, that is,

$$\mathbf{L} = \dot{\theta}\sum_{i=1}^{n} m_i(-x_iz_i\mathbf{i} - y_iz_i\mathbf{j} + d_i^2\mathbf{k}), \tag{26}$$

where d_i is the perpendicular distance of the ith particle from the z-axis. This angular momentum has non-zero components perpendicular to the axis of rotation (in the $\mathbf{i}$- and $\mathbf{j}$-directions). However, in many applications it is only the component in the direction of the axis of rotation that is needed (the situation in Exercise 11 below is an exception). With our choice of axes, this component is in the z-direction, and is

$$L_z = \dot{\theta}\sum_{i=1}^{n} m_i d_i^2.$$

This result is very important. For any particular rigid body, the quantity $\sum_{i=1}^{n} m_i d_i^2 = I$ is a constant, since both m_i and d_i are constant for all i. It depends on the distribution of mass about the axis of rotation, and as you saw in Unit 17, it is called the moment of inertia. Hence for a rigid body spinning about a fixed axis, the component of the angular momentum in the direction of the axis can be conveniently expressed in terms of its moment of inertia.

We have established this result with the z-axis as the axis of rotation. However, for any rigid body rotating about a fixed axis, we could choose axes such that the z-axis is the axis of rotation, so there is no loss of generality in our argument.

Angular momentum of a rigid body rotating about a fixed axis

Suppose that a rigid body is rotating about a fixed axis with angular velocity $\boldsymbol{\omega}$. Let $\mathbf{L}$ be the angular momentum of the body about a point O on the axis, and let L_{axis} be the component of $\mathbf{L}$ in the direction of the axis. Then

$$L_{\text{axis}} = I\dot{\theta}, \tag{27}$$

where I is the moment of inertia of the body about the axis of rotation.

Note that the *magnitude* of the component of the angular momentum in the axis of rotation can be written as $|L_{\text{axis}}| = I\omega$, where ω is the angular speed.

Exercise 11

Throughout this exercise, torque and angular momentum are measured about the origin O; also, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are Cartesian unit vectors, with $\mathbf{i}$ and $\mathbf{j}$ in a horizontal plane and $\mathbf{k}$ pointing vertically upwards.

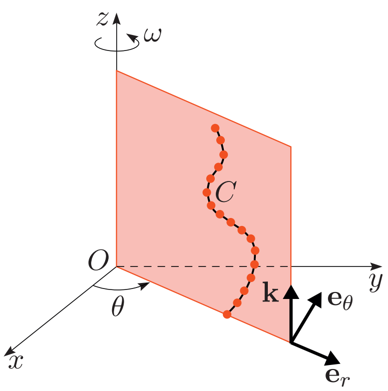

Figure 15 Rigid body modelled as n particles placed on a curve C embedded in a vertical plane rotating about the z-axis

(a) Consider a rigid body modelled as a system of n particles lying on a curve C embedded in a vertical plane as shown in Figure 15. The curve C does not cross the z-axis or extend below the (x, y)-plane, and is moving anticlockwise with constant angular speed ω about the z-axis.

(i) Show that the angular momentum of the system can be expressed as

$$\mathbf{L} = I\omega\mathbf{k} - A\omega\mathbf{e}_r, \tag{28}$$

where I is the moment of inertia of the body about the z-axis, $A > 0$ is a constant, and $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is the unit vector pointing horizontally from the z-axis within the vertical plane containing the rigid body, as shown in Figure 15. (You met this use of $\mathbf{e}_r$ in Unit 20.)

(*Hint*: Express the x- and y-coordinates of each of the particles in polar coordinates.)

(ii) Hence show that

$$\dot{\mathbf{L}} = -A\omega^2\mathbf{e}_\theta, \tag{29}$$

where the unit vector $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$ lies in the (x, y)-plane and is normal to $\mathbf{e}_r$, pointing in the direction of increasing θ. (You met this use of $\mathbf{e}_\theta$ in Unit 20.)

Deduce that the body must be subject to a non-zero torque in the tangential direction but pointing in the opposite direction to $\mathbf{e}_\theta$.

(b) Consider a skater whose centre of mass is moving with constant angular speed ω in a circle parallel to the (x, y)-plane.

(i) By analysing the motion of an equivalent particle located at the skater's centre of mass, deduce the direction of the total external force $\mathbf{F}$ on the skater.

(ii) Suppose that the skater tries to follow the circle while remaining perfectly vertical. Use equation (29) to show that this is impossible. What must the skater do if she is to follow such a circle? (Model the skater as a rigid system of particles lying on a curve, as discussed in part (a). Assume that the only forces on the skater are her weight and the force exerted by the ice on her skates.)

Suppose that $\mathbf{k}$ is a unit vector along the axis of rotation of a rigid body, so $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. If we look at just the $\mathbf{k}$-components, then the torque law and equation (27) yield

$$\begin{aligned}\boldsymbol{\Gamma} \cdot \mathbf{k} &= \dot{\mathbf{L}} \cdot \mathbf{k} \\ &= \frac{d}{dt}(\mathbf{L} \cdot \mathbf{k}) \\ &= \frac{d}{dt}(I\dot{\theta}) \\ &= I\ddot{\theta}.\end{aligned}$$

So in this situation the torque law leads to the following important and elegant result.

Equation of rotational motion

Suppose that a rigid body is rotating about a fixed axis, and that its angular displacement (from some fixed line normal to the axis) is θ. Then

$$I\ddot{\theta} = \Gamma_{\text{axis}}, \tag{30}$$

where Γ_{axis} is the component of the total external torque on the body in the direction of the axis of rotation.

The moments of inertia of a number of common regular geometric shapes are given in Table 1. In each case we assume that the rigid body is continuous and of uniform density, and we consider only objects of this type. The moments of inertia given in the table have been found by integration, as discussed in Unit 17. You may use these moments of inertia when doing the exercises in this unit.

The objects are therefore homogeneous rigid bodies, as defined in Unit 19.

Table 1 Moments of inertia of homogeneous rigid bodies

Object	Axis	Dimensions	Moment of inertia	Figure
Solid cylinder	Axis of cylinder	Radius R	$\frac{1}{2}MR^2$	16(a), axis AB
Solid cylinder	Normal to axis of cylinder	Radius R, length h	$\frac{1}{4}MR^2 + \frac{1}{12}Mh^2$	16(a), axis CD
Hollow cylinder	Axis of cylinder	Inner radius a, outer radius R	$\frac{1}{2}M(R^2 + a^2)$	16(b), axis AB
Hollow cylinder	Normal to axis of cylinder	Inner radius a, outer radius R, length h	$\frac{1}{4}M(R^2 + a^2) + \frac{1}{12}Mh^2$	16(b), axis CD
Solid rectangular cuboid	Normal to one pair of faces	Faces normal to axis have sides of lengths a, b	$\frac{1}{12}M(a^2 + b^2)$	16(c), axis AB
Thin straight rod	Normal to rod	Length h	$\frac{1}{12}Mh^2$	
Solid sphere	Through centre	Radius R	$\frac{2}{5}MR^2$	
Hollow sphere	Through centre	Inner radius a, outer radius R	$\frac{2}{5}M\dfrac{R^5 - a^5}{R^3 - a^3}$	
Thin spherical shell	Through centre	Radius R	$\frac{2}{3}MR^2$	

In each case, the mass of the object is M and the axis passes through its centre of mass G (see Figure 16).

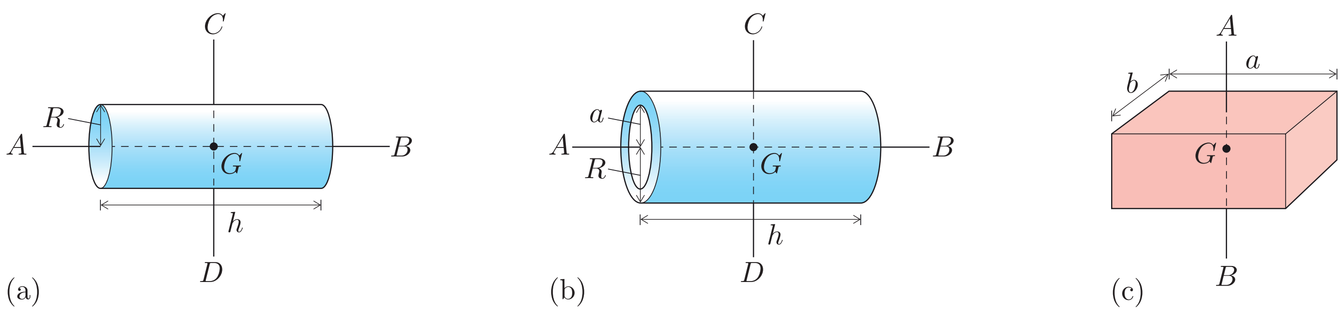

Figure 16 (a) Solid cylinder, (b) hollow cylinder, (c) solid rectangular cuboid

The moment of inertia of the 'thin straight rod' in Table 1 is calculated using the assumption that all its particles lie on a straight line. While this will never be absolutely true for a three-dimensional object, we sometimes use such limiting cases as convenient models. Similarly, a 'thin spherical shell' is an object whose particles are confined to the surface of a sphere. The moment of inertia of a thin spherical shell is, in fact, the limit of that for a hollow sphere as $a \to R$, but that limit is not obvious, so this case is given separately in the table.

The next example illustrates the use of Table 1 to find the moment of inertia of a 'thin disc', which is another example of a two-dimensional model of a three-dimensional object.

Example 5

Consider a uniform solid circular disc of mass M and radius R, and negligible thickness. What is its moment of inertia about an axis normal to the disc and through its centre?

Solution

The distribution of the mass of the disc relative to the specified axis is the same as for a solid cylinder rotating about its own axis. From Table 1, the moment of inertia in this case is $\frac{1}{2}MR^2$.

Exercise 12

Show that the moment of inertia about the axis of a thin cylindrical shell of mass M and radius R is MR^2.

To find the moment of inertia of a compound object formed from several simple shapes joined together, we find the moment of inertia of each component part separately, and then sum these. We use this approach in the next example.

Example 6

A playground roundabout can be modelled as a uniform solid circular disc of radius $1.2\,\mathrm{m}$ and mass $240\,\mathrm{kg}$. The disc is horizontal and rotates about a vertical axis through its centre, making 0.5 revolutions per second. A man of mass $80\,\mathrm{kg}$ is standing stationary with respect to the disc at a position $0.2\,\mathrm{m}$ from the centre O of the disc.

Suppose that the man moves towards the edge of the disc, stopping when he is $1\,\mathrm{m}$ away from the centre. Assume that the external forces on the roundabout are such that the component of the total external torque about O in the direction of the axis of rotation is zero. Assume also that the man can be modelled as a particle. At what rate will the disc be rotating when he is in his new position?

Solution

Since Γ_{axis}, the component of the total external torque about O in the direction of the axis of rotation, is zero, from equations (27) and (30) it follows that L_{axis}, the angular momentum about the axis of rotation through O, is conserved. To proceed, we need to know the moment of inertia of the combined 'man-plus-roundabout' system for each of the two positions of the man.

Now, the moment of inertia of the roundabout about the axis through O (in $\text{kg}\,\text{m}^2$) is

$$\tfrac{1}{2}MR^2 = \tfrac{1}{2} \times 240(1.2)^2 = 172.8.$$

If we model the man as a particle of mass m located at a distance r from O, then his moment of inertia is mr^2. So when he is 0.2 m from the centre of the disc, the moment of inertia of the combined system is

$$I_1 = 172.8 + 80(0.2)^2 = 176.$$

With the man 1 m from the centre of the disc, the moment of inertia of the combined system is

$$I_2 = 172.8 + 80(1)^2 = 252.8.$$

When the man is nearer to the centre of the disc, the angular speed ω_1 is 0.5 revolutions per second, that is, $\omega_1 = \pi$ (in $\text{rad}\,\text{s}^{-1}$). Suppose that the angular speed is ω_2 after the man has moved closer to the edge of the disc. Conservation of angular momentum gives

$$I_1\omega_1 = I_2\omega_2,$$

hence

$$\omega_2 = \frac{176}{252.8}\pi \simeq 2.19.$$

Therefore after the man has moved closer to the edge of the roundabout, the roundabout rotates more slowly, at about $2.19\,\text{rad}\,\text{s}^{-1}$, that is, at about 0.35 revolutions per second.

The following example and exercise consolidate ideas presented in this subsection.

Example 7

A bucket used to draw water from a well has mass m and is attached to a light inextensible rope that is wound round a heavy wheel (see Figure 17). The wheel is a uniform solid disc, with centre O, radius R and mass M. The distance from O to the surface of the water in the well is h. Suppose that the bucket is released from rest at a point X, level with O, and allowed to fall down the well, causing the wheel to rotate. Model the bucket as a particle and the rope as a model string, and assume that the force supporting the wheel acts at the point O and that there is no friction there.

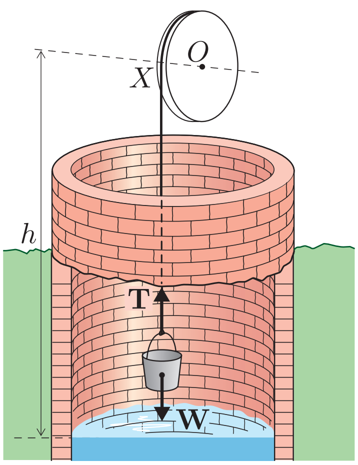

Figure 17 Bucket attached to a wheel centred at O, drawing water from a well

(a) If $\mathbf{T}$ denotes the tension in the rope, and z denotes the distance that the bucket has travelled down the well in time t, write down the equation of motion for the bucket in terms of m, $|\mathbf{T}|$ and z.

(b) What is the torque acting on the wheel about its centre?

(c) Suppose that the wheel has rotated through an angle θ while the bucket has been falling. What is the relationship between z and θ?

(d) Write down the equation of rotational motion for the wheel.

(e) Find an expression for the acceleration $\ddot{z}$ of the bucket by eliminating $|\mathbf{T}|$ and θ from the equations of motion obtained in parts (a) and (d), and by using the result from part (c).

(f) Hence find the time taken for the bucket to descend the well.

Solution

(a) Choose Cartesian unit vectors with $\mathbf{k}$ pointing vertically downwards, $\mathbf{j}$ in the direction of the axis of rotation of the wheel, and $\mathbf{i}$ horizontal, as shown in Figure 18. The origin O is at the centre of the wheel.

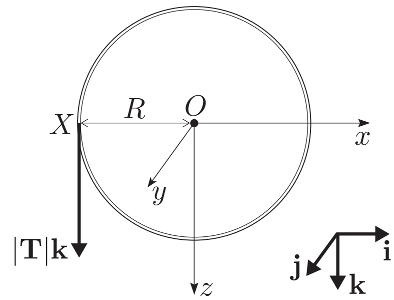

Figure 18 Force on the wheel

The forces acting on the bucket are the tension force in the string, $-|\mathbf{T}|\mathbf{k}$, and the force due to gravity, $\mathbf{W} = mg\mathbf{k}$. So by Newton's second law,

$$m\ddot{z}\mathbf{k} = mg\mathbf{k} - |\mathbf{T}|\mathbf{k}.$$

Resolving in the $\mathbf{k}$-direction gives the equation of motion:

$$m\ddot{z} = mg - |\mathbf{T}|. \tag{31}$$

(b) The force exerted on the wheel by the rope is $|\mathbf{T}|\mathbf{k}$, acting at the point X whose position relative to O is $-R\mathbf{i}$ (see Figure 18). Therefore the torque acting on the wheel about its centre is $(-R\mathbf{i}) \times |\mathbf{T}|\mathbf{k} = R|\mathbf{T}|\mathbf{j}$.

(c) If the bucket has fallen a distance z, then the quantity of rope that has unwound from the wheel as the bucket falls must also be of length z. As the bucket falls, the wheel turns through an angle θ anticlockwise, so the part of the perimeter of the wheel that has moved past the point X in Figure 19 has length $R\theta$. Hence $z = R\theta$.

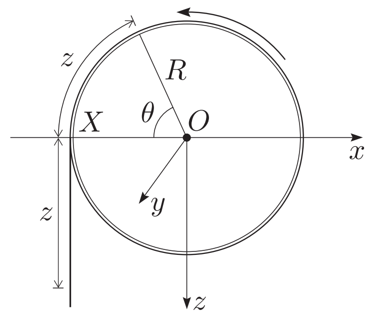

Figure 19 Rotation of the wheel

(d) The wheel is a disc turning about an axis through its centre and normal to its plane. It has moment of inertia $I = \frac{1}{2}MR^2$ (from Example 5) about the axis of rotation. Then from equation (30), the equation of rotational motion of the wheel (in the $\mathbf{j}$-direction) is

$$I\ddot{\theta} = \Gamma_{\text{axis}},$$

and from part (b), $\Gamma_{\text{axis}} = R|\mathbf{T}|$. So we obtain

$$\tfrac{1}{2}MR^2\ddot{\theta} = R|\mathbf{T}|,$$

or equivalently,

$$MR\ddot{\theta} = 2|\mathbf{T}|. \tag{32}$$

(e) From part (c) we have $z = R\theta$, and since R is constant, $\ddot{z} = R\ddot{\theta}$. On substituting in equation (32), we find

$$M\ddot{z} = 2|\mathbf{T}|. \tag{33}$$

Now from equation (31), $|\mathbf{T}| = mg - m\ddot{z}$. So substituting for $|\mathbf{T}|$ in equation (33) yields

$$M\ddot{z} = 2(mg - m\ddot{z}).$$

Hence

$$\ddot{z} = \frac{2mg}{M + 2m}. \tag{34}$$

(f) The expression for $\ddot{z}$ in equation (34) is a constant, so using the constant acceleration formula $z = z_0 + v_0 t + \frac{1}{2}a_0 t^2$ with $z = h$, $z_0 = 0$, $v_0 = 0$ and a_0 given by the right-hand side of equation (34), we have

$$h = \frac{mg}{M + 2m}\,t^2.$$

Rearranging this gives the time for the bucket to reach the surface of the water as $\sqrt{h(M + 2m)/mg}$.

Exercise 13

A roundabout of mass 250 kg is modelled as a uniform solid disc of radius 1.2 m, which turns in a horizontal plane about a vertical axis through its centre O. It is being pushed with a force $\mathbf{F}$, using a handle at X as in Figure 20. This force, which has constant magnitude 100 N, is applied at a distance of 1.5 m from O, and is horizontal and normal to OX. A force $\mathbf{R}$ resists the rotational motion. It has magnitude $c\omega$, where c is a constant and ω is the angular speed of the roundabout; its point of action is 0.1 m from O, and its direction is opposite to the velocity of that point on the roundabout. Assume throughout that $\dot{\theta} \geq 0$ (anticlockwise rotation), so $\omega = \dot{\theta}$.

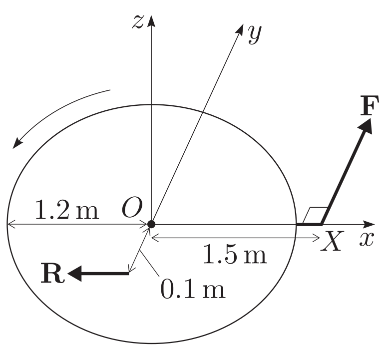

Figure 20 Roundabout rotating about O and pushed by force $\mathbf{F}$ with resistive force $\mathbf{R}$

(a) Obtain a differential equation in terms of ω for the rotational motion of the roundabout.

(b) If the pushing starts at time $t = 0$ with the roundabout at rest, find ω as a function of t.

(c) If $c = 10$, what is the maximum possible angular speed that the roundabout could reach according to this model? Do you think that this could be achieved in practice?

3.2 Parallel axes theorem

The moments of inertia of various objects about their centres of mass were given in Table 1. To find the moments of inertia about some other axis, we can use the *parallel axes theorem*.

Parallel axes theorem

Suppose that I_{AB} is the moment of inertia of a rigid body of mass M about a line AB (see Figure 21(a)). Let EF be a line through the centre of mass of the body that is parallel to AB, let the distance between the lines AB and EF be d, and let I_{EF} be the moment of inertia of the body about EF. Then

$$I_{AB} = I_{EF} + Md^2. \tag{35}$$

To see this, we follow the argument set out below.

Figure 21(a) shows a rigid body of mass M, with the centre of mass O taken as the origin. Suppose that we want to find the moment of inertia of the body about the line AB. As shown in Figure 21(b), choose as the z-axis a line EF through O that is parallel to AB. Let the perpendicular distance between these two parallel lines be d. Choose as the x-axis a line through O that intersects the line AB and is normal to AB.

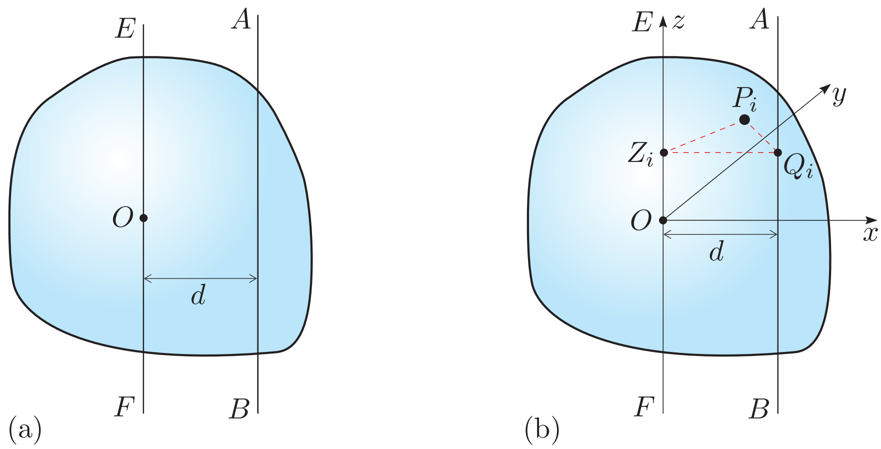

Figure 21 Rigid body, showing (a) two axes of rotation, EF and AB, and (b) the coordinate axes and a particular particle P_i

Suppose that the rigid body is composed of n particles P_i, where for $i = 1, 2, \ldots, n$ the ith particle has mass m_i and position (x_i, y_i, z_i). The distance of particle P_i from the z-axis is d_i (P_iZ_i in Figures 21(b) and 22), where $d_i^2 = x_i^2 + y_i^2$. Also, the particle's distance from the line AB is s_i (P_iQ_i in Figures 21(b) and 22), where

$$\begin{aligned} s_i^2 &= (d - x_i)^2 + y_i^2 \\ &= d^2 - 2x_i d + x_i^2 + y_i^2 \\ &= d^2 - 2x_i d + d_i^2. \end{aligned}$$

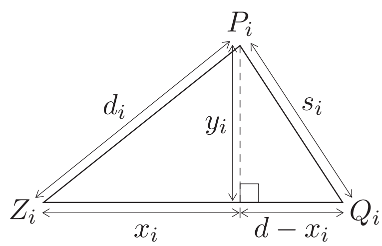

Figure 22 Part of a section through the rigid body in Figure 21(b), parallel to the (x, y)-plane

By definition (see Unit 17), the moment of inertia of the rigid body about the line AB is

$$\begin{aligned} I_{AB} &= \sum_{i=1}^{n} m_i s_i^2 \\ &= \sum_{i=1}^{n} m_i(d^2 - 2x_i d + d_i^2) \\ &= d^2 \sum_{i=1}^{n} m_i - 2d \sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n} m_i d_i^2. \end{aligned} \tag{36}$$

Now $\sum_{i=1}^{n} m_i = M$ and $\sum_{i=1}^{n} m_i d_i^2 = I_{EF}$, where I_{EF} is the moment of inertia of the object about the z-axis (the line EF). Also, from Unit 19, Subsection 2.1, $\left(\sum_{i=1}^{n} m_i x_i\right)/M$ is the x-coordinate of the centre of mass of the body. Since the origin was chosen at the centre of mass, we have $\sum_{i=1}^{n} m_i x_i = 0$. Hence equation (36) becomes

$$I_{AB} = I_{EF} + Md^2,$$

giving the parallel axes theorem.

Example 8

An ice skater is rotating about a fixed axis AB at an angular speed of $8\,\mathrm{rad\,s^{-1}}$. One of the skater's arms is modelled as a uniform solid cylinder of mass 5.5 kg, length 0.66 m and diameter 0.08 m. The cylinder is normal to the axis of rotation AB, and the end of the cylinder is 0.09 m from the axis (see Figure 23).

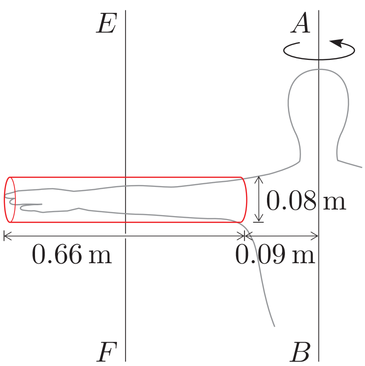

Figure 23 Skater with outstretched arm

Find the magnitude of the component of the angular momentum in the direction of the axis of rotation for this model of the arm.

Solution

From Table 1, the moment of inertia of the cylinder about an axis EF through its centre of mass and normal to the cylinder is

$$I_{EF} = \tfrac{1}{4}MR^2 + \tfrac{1}{12}Mh^2 = \tfrac{1}{4} \times 5.5(0.04)^2 + \tfrac{1}{12} \times 5.5(0.66)^2 \simeq 0.202$$

(in $\mathrm{kg\,m^2}$). The axis of rotation AB is $0.33 + 0.09 = 0.42$ (in m) from the centre of mass of the cylinder, so from the parallel axes theorem, the moment of inertia of the cylinder about the axis of rotation AB is

$$I_{AB} \simeq 0.202 + 5.5(0.42)^2 \simeq 1.172$$

(in $\mathrm{kg\,m^2}$). As the skater's angular speed is $\omega = 8$ (in $\mathrm{rad\,s^{-1}}$), it follows from equation (27) that for this model of the arm, the component of angular momentum in the direction of the axis of rotation has magnitude

$$|L_{\text{axis}}| = I_{AB}\,\omega \simeq 1.172 \times 8 = 9.376$$

(in $\mathrm{kg\,m^2\,s^{-1}}$).

Note that we could treat other parts of the body in a similar fashion and, in principle, sum up the moments of inertia of all the individual body parts to obtain the moment of inertia of the whole body.

Exercise 14

The mace used by the leader of a troupe of drum majorettes can be modelled as a sphere of radius r attached to the end of a uniform cylindrical rod of length d and radius R. The mass of the sphere is M, and the mass of the rod is m.

Find the moment of inertia of the mace about an axis AB through the end of the rod and normal to it (as shown in Figure 24).

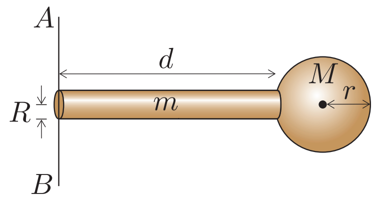

Figure 24 Mace rotating about AB

3.3 Kinetic energy of a rotating rigid body

You saw in Unit 17 how the kinetic energy of a rotating particle can be expressed in terms of its moment of inertia. Recall that the kinetic energy of a particle with mass m moving at speed v is given by $T = \frac{1}{2}mv^2$. If the particle is moving in a circle of radius d (the distance from the axis of rotation) with angular speed ω, then $v = d\omega$, so we have $T = \frac{1}{2}md^2\omega^2 = \frac{1}{2}I\omega^2$, where $I = md^2$ is the moment of inertia of the particle.

The kinetic energy of a rotating rigid body can be expressed in a similar way. Consider a rigid body rotating about a fixed axis with angular speed ω. The kinetic energy of the body is the sum of the kinetic energies of all its constituent particles. If the ith particle is a distance d_i from the axis of rotation, then its speed is $d_i\omega$ and its kinetic energy is $\frac{1}{2}m_i(d_i\omega)^2$. Hence the total kinetic energy of the body is

$$\sum_{i=1}^{n} \tfrac{1}{2}m_i(d_i\omega)^2 = \tfrac{1}{2}\left(\sum_{i=1}^{n} m_i d_i^2\right)\omega^2 = \tfrac{1}{2}I\omega^2,$$

where I is the body's moment of inertia about the axis of rotation.

Kinetic energy of a rigid body rotating about a fixed axis

Suppose that a rigid body is rotating with angular speed ω about a fixed axis. Let I be the moment of inertia of the body about the axis of rotation. Then the kinetic energy of the body is

$$T = \tfrac{1}{2}I\omega^2. \qquad (37)$$

Note that as long as rotation is about a fixed axis, the expression $\frac{1}{2}I\omega^2$ gives the *total* kinetic energy of the body.

Example 9

A planet is modelled as a uniform solid sphere of radius 6400 km and mass 6.0×10^{24} kg. It is turning on its axis once every 24 hours.

If the axis of rotation is fixed, what is the kinetic energy of the planet?

Solution

The planet's kinetic energy is $\frac{1}{2}I\omega^2$, where

$$\omega = \frac{2\pi}{24 \times 60^2}$$

(in $\mathrm{rad\,s^{-1}}$), and from Table 1,

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5} \times 6 \times 10^{24} \times (6.4 \times 10^6)^2$$

(in $\mathrm{kg\,m^2}$). Therefore the kinetic energy of the planet (in joules) is

$$\tfrac{1}{2} \times \tfrac{2}{5} \times 6 \times 10^{24} \times (6.4 \times 10^6)^2 \left(\frac{2\pi}{24 \times 60^2}\right)^2 \simeq 2.6 \times 10^{29}.$$

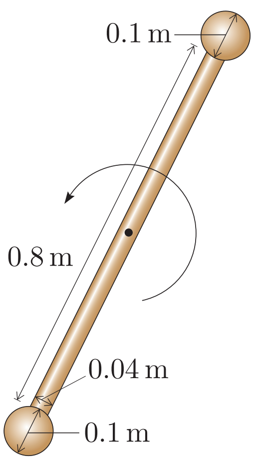

Figure 25 Drum majorette's baton rotating about an axis through its centre of mass

Exercise 15

A drum majorette's baton is modelled as a uniform cylindrical rod with spheres of equal mass at either end. The rod has mass 0.1 kg, length 0.8 m and diameter 0.04 m. Each sphere has diameter 0.1 m and mass 0.25 kg. The baton is rotated at 1 revolution per second about a fixed axis that is normal to the axis of the cylinder and through the centre of mass (see Figure 25).

Determine the kinetic energy of the baton.

Exercise 16

(a) In the situation considered in Example 6, determine the total kinetic energy of the system (man plus roundabout) when the man is stationary at 0.2 m from the centre of the roundabout, and then when he is stationary at 1 m from the centre. (Use the results found in the solution to Example 6, as needed.)

(b) In the reverse of the situation in Example 6, the man starts 1 m from the centre of the roundabout and moves inwards until he is 0.2 m from the centre. What happens to the kinetic energy of the system when he does this?

In a system such as that dealt with in Example 6 and Exercise 16, the angular momentum is constant because the total external torque is zero (both quantities are measured about the centre of the roundabout and refer to components in the direction of the axis of rotation). However, the internal forces can change the kinetic energy of the system. This means that the kinetic energy is *not* necessarily constant, as you saw in Exercise 16(b), where the effort put in by the man in moving across the roundabout increased the total kinetic energy of that system.

Exercise 17

A rigid body of mass M is rotating about a fixed axis with angular speed ω. Suppose that the centre of mass is following a circle of radius R at speed v, and that I_G is the moment of inertia of the body about an axis parallel to the axis of rotation and through the centre of mass.

Show that the kinetic energy of the body is

$$\tfrac{1}{2}I_G\omega^2 + \tfrac{1}{2}Mv^2.$$

Exercise 18

A diver begins a dive at rest and vertical in the handstand position. She starts to rotate from this position with negligible angular speed. She lets go of the diving board when her body makes an angle α with the vertical (where $0 < \alpha \leq \frac{\pi}{2}$), and she then starts the 'in-flight' part of the dive. Let ω be the angular speed of the diver at the moment of letting go of the board. Assume that while she is in contact with the board at O, the diver is a rigid body able to rotate about an axis through O, normal to the plane of Figure 26.

Use the principle of conservation of mechanical energy to obtain ω in terms of α and the following parameters: l, the distance from the diver's hands (in the handstand position) to her centre of mass; m, the mass of the diver; I_G, the moment of inertia of the diver about an axis through her centre of mass when her body is straight.

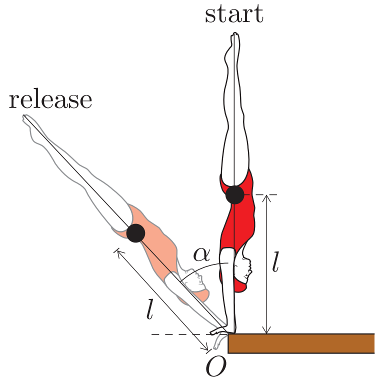

Figure 26 Diver beginning a dive in the handstand position

Exercise 19

Consider the diver from Exercise 18 when she is part way through her dive (as shown in Figure 27). Suppose that she has rotated through an angle θ ($\theta < \alpha$) in time t, and in that position the force exerted on her hands by the board has component vectors $R_1\mathbf{i}$ and $R_2\mathbf{j}$, as shown in the figure.

(a) With the origin and axes shown in Figure 27, let $x\mathbf{i} + y\mathbf{j}$ be the position of the diver's centre of mass. Express x and y in terms of θ. By differentiating, show that

$$\ddot{x} = l\dot{\theta}^2 \sin\theta - l\ddot{\theta}\cos\theta, \quad \ddot{y} = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta. \tag{38}$$

(b) Let I_O be the diver's moment of inertia about an axis through O. Give the equation of rotational motion for the diver in terms of θ, I_O, m and l.

(c) Obtain another equation for the rotational motion by applying the principle of conservation of mechanical energy to the diver. Verify that differentiation of this equation leads to the equation of rotational motion that you found in part (b).

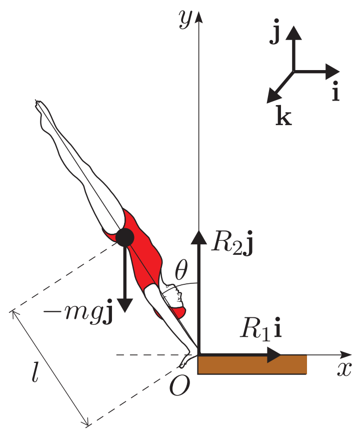

Figure 27 Diver part way through her dive

(d) (i) Write down Newton's second law for the motion of the diver's centre of mass, and obtain two equations of motion.

(ii) Use equations (38) to substitute for $\ddot{x}$ and $\ddot{y}$ in these equations of motion.

(e) (i) Use the results of parts (b) and (c) to substitute for $\dot{\theta}^2$ and $\ddot{\theta}$ in your equation for R_1 in part (d)(ii). Hence show that

$$R_1 = mg\,\frac{ml^2}{I_O}\sin\theta\,(2 - 3\cos\theta). \tag{39}$$

(ii) Deduce that for angles θ between 0 and $\pi/2$, the horizontal component R_1 of the force exerted by the diving board is zero when $\theta = 0$ or when $\theta = \arccos\frac{2}{3}$.

(f) Estimate I_O by modelling the diver as a uniform thin straight rod of length $2l$. Then estimate, as a multiple of the magnitude of her weight, the magnitude of the horizontal force that the diver would need to exert on the board to be able to remain in contact with it until $\theta = \frac{\pi}{2}$.

(g) At what point is the diver likely to lose contact with the board? Assume that there is nothing on the diving board on which the diver can grip, so she cannot pull on the board but can only push on it.

4 Rotation about a moving axis

The motion of an extended body may be much more complicated than that considered in Section 3, where we confined our attention to a *rigid* body rotating about a *fixed* axis. In this section we look at a more general situation where the axis of rotation is moving. The general case is considered in Subsection 4.1, and here we derive the result used in Section 1 that the motion can be decomposed into the motion of the centre of mass together with the rotation about the centre of mass. In Subsection 4.2, we consider the motion of a *rigid* body when the body is rotating about an axis whose *direction is fixed*, though the centre of mass may be moving. In Subsection 4.3, we look at the motion of rolling objects, such as cans rolling down slopes.

We will not consider the motion of a rigid body when the direction of the axis of rotation varies.

4.1 Torque law relative to the centre of mass

In Unit 19 we showed how Newton's second law of motion can be extended to an n-particle system. You saw there that the centre of mass of the system moves as if it is a single particle with the same mass as the whole system, and with all the external forces applied to this particle. The motion of the centre of mass is often referred to as the **linear motion** of the system, to distinguish it from the rotational motion of the system.

That is not to suggest that the centre of mass will always travel in a straight line!

It is convenient to analyse complicated motion in two parts: the linear motion, and the motion relative to the centre of mass. Now, to apply the torque law as stated in Section 2, we need to work relative to a *fixed* origin. However, as you will now see, we can still use the torque law when these quantities are calculated relative to a point that is *moving*, as long as *that point is at the centre of mass.*

To demonstrate this, we need some notation. Throughout this subsection, we will consider an extended body modelled as a system of n particles, with total mass M and centre of mass at a position $\mathbf{R}$ relative to some fixed point O. The ith particle of the system (for $i = 1, 2, \ldots, n$) has mass m_i and position relative to O given by

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^{\mathrm{rel}}, \tag{40}$$

where $\mathbf{r}_i^{\mathrm{rel}}$ denotes the position of the particle relative to the centre of mass. In the following discussion it is also useful to obtain a relationship between the velocities of the particles relative to the different origins by differentiating equation (40) to give

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}_i^{\mathrm{rel}}. \tag{41}$$

There is an additional equation involving $\mathbf{r}_i^{\mathrm{rel}}$ that results from the fact that $\mathbf{R}$ is the centre of mass of the system. From Unit 19, we have

This is where we use the fact that $\mathbf{R}$ is the centre of mass of the system.

$$\begin{aligned} M\mathbf{R} = \sum_{i=1}^{n} m_i \mathbf{r}_i &= \sum_{i=1}^{n} m_i \left(\mathbf{R} + \mathbf{r}_i^{\mathrm{rel}}\right) \\ &= \sum_{i=1}^{n} m_i \mathbf{R} + \sum_{i=1}^{n} m_i \mathbf{r}_i^{\mathrm{rel}} \\ &= M\mathbf{R} + \sum_{i=1}^{n} m_i \mathbf{r}_i^{\mathrm{rel}}. \end{aligned}$$

Thus, as one would expect,

$$\sum_{i=1}^{n} m_i \mathbf{r}_i^{\mathrm{rel}} = \mathbf{0}. \tag{42}$$

This equation states that if position vectors are taken relative to the centre of mass, then the centre of mass is at the origin.

This takes us to the following important definitions.

Torque and angular momentum relative to the centre of mass

If $\mathbf{r}_i^{\mathrm{rel}}$ denotes the position of the ith particle relative to the centre of mass, and $\mathbf{F}_i$ denotes the total external force on the ith particle, then the total external torque on the system relative to the centre of mass is defined by

$$\boldsymbol{\Gamma}^{\mathrm{rel}} = \sum_{i=1}^{n} \mathbf{r}_i^{\mathrm{rel}} \times \mathbf{F}_i, \tag{43}$$

and the total angular momentum relative to the centre of mass is defined as

$$\mathbf{L}^{\mathrm{rel}} = \sum_{i=1}^{n} \mathbf{r}_i^{\mathrm{rel}} \times m_i \dot{\mathbf{r}}_i^{\mathrm{rel}}. \tag{44}$$

We aim to relate $\boldsymbol{\Gamma}^{\text{rel}}$ and $\mathbf{L}^{\text{rel}}$ to the corresponding quantities calculated relative to the fixed origin O, and start by looking at the total angular momentum relative to O, which is defined as

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i.$$

Substituting for $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$, using equations (40) and (41), gives

$$\mathbf{L} = \sum_{i=1}^{n} (\mathbf{R} + \mathbf{r}_i^{\text{rel}}) \times m_i (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^{\text{rel}}).$$

Expanding the brackets gives

$$\mathbf{L} = \sum_{i=1}^{n} \left(\mathbf{R} \times m_i \dot{\mathbf{R}} + \mathbf{R} \times m_i \dot{\mathbf{r}}_i^{\text{rel}} + \mathbf{r}_i^{\text{rel}} \times m_i \dot{\mathbf{R}} + \mathbf{r}_i^{\text{rel}} \times m_i \dot{\mathbf{r}}_i^{\text{rel}} \right).$$

The last term can be recognised as the total angular momentum relative to the centre of mass (equation (44)), so we have

$$\mathbf{L} = \sum_{i=1}^{n} \left(\mathbf{R} \times m_i \dot{\mathbf{R}} + \mathbf{R} \times m_i \dot{\mathbf{r}}_i^{\text{rel}} + \mathbf{r}_i^{\text{rel}} \times m_i \dot{\mathbf{R}} \right) + \mathbf{L}^{\text{rel}}.$$

As $\mathbf{R}$ is independent of i, this expression can be written as

$$\mathbf{L} = \mathbf{R} \times \left(\sum_{i=1}^{n} m_i \right) \dot{\mathbf{R}} + \mathbf{R} \times \left(\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^{\text{rel}} \right) + \left(\sum_{i=1}^{n} m_i \mathbf{r}_i^{\text{rel}} \right) \times \dot{\mathbf{R}} + \mathbf{L}^{\text{rel}}.$$

The first bracketed term above is the total mass M of the system. The third bracketed term is zero by equation (42), and the second bracketed term is zero by differentiating equation (42). So we arrive at the result

$$\mathbf{L} = \mathbf{R} \times M \dot{\mathbf{R}} + \mathbf{L}^{\text{rel}}.$$

The next exercise asks you to derive a similar relationship that holds between the torque relative to the centre of mass and the torque relative to O.

Exercise 20

Starting from the definition of $\boldsymbol{\Gamma}$, the total external torque about O, and using equations (40) and (43), show that

$$\boldsymbol{\Gamma} = \mathbf{R} \times \mathbf{F} + \boldsymbol{\Gamma}^{\text{rel}},$$

where $\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i$ is the total external force on the system.

The two results derived above are worth re-stating formally.

Decomposition theorems

Let $\mathbf{R}$ be the position vector of the centre of mass of a body relative to some fixed point O, let M be the total mass of the body, and let $\mathbf{F}$ be the total external force on the body. If the total angular momentum of the body relative to the centre of mass is $\mathbf{L}^{\text{rel}}$, then the total angular momentum of the body relative to O is given by

$$\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{L}^{\text{rel}}. \tag{45}$$

If $\mathbf{\Gamma}^{\text{rel}}$ is the total external torque on the body relative to the centre of mass, then the total external torque on the body about O is given by

$$\mathbf{\Gamma} = \mathbf{R} \times \mathbf{F} + \mathbf{\Gamma}^{\text{rel}}. \tag{46}$$

This states that both the total angular momentum and the total external torque of an n-particle system can be decomposed into the corresponding quantity for an equivalent particle located at the centre of mass plus the corresponding quantity for the rotational motion relative to the centre of mass.

Now we move on to derive the central result of this subsection, and one of the key results of the unit, which is the *torque law relative to the centre of mass*. To derive this result, we start by differentiating equation (45) to obtain

$$\dot{\mathbf{L}} = \frac{d}{dt}\big(\mathbf{R} \times M\dot{\mathbf{R}}\big) + \dot{\mathbf{L}}^{\text{rel}}.$$

Using the torque law relative to a fixed origin O gives $\dot{\mathbf{L}} = \mathbf{\Gamma}$, so

$$\mathbf{\Gamma} = \frac{d}{dt}\big(\mathbf{R} \times M\dot{\mathbf{R}}\big) + \dot{\mathbf{L}}^{\text{rel}}.$$

Using the product rule for differentiating the cross product gives

See Unit 20.

$$\mathbf{\Gamma} = \dot{\mathbf{R}} \times M\dot{\mathbf{R}} + \mathbf{R} \times M\ddot{\mathbf{R}} + \dot{\mathbf{L}}^{\text{rel}}.$$

The first term on the right-hand side is zero, since $\dot{\mathbf{R}} \times \dot{\mathbf{R}} = \mathbf{0}$, so

$$\mathbf{\Gamma} - \mathbf{R} \times M\ddot{\mathbf{R}} = \dot{\mathbf{L}}^{\text{rel}}.$$

By Newton's second law for n-particle systems, we have $\mathbf{F} = M\ddot{\mathbf{R}}$, hence

The result $\mathbf{F} = M\ddot{\mathbf{R}}$ was established in Unit 19, Subsection 2.1.

$$\mathbf{\Gamma} - \mathbf{R} \times \mathbf{F} = \dot{\mathbf{L}}^{\text{rel}}.$$

Now we use equation (46) to obtain the desired relationship between the torque and angular momentum relative to the centre of mass:

$$\mathbf{\Gamma}^{\text{rel}} = \dot{\mathbf{L}}^{\text{rel}}.$$

Torque law relative to the centre of mass

The total external torque on an extended body relative to its centre of mass is equal to the rate of change of the total angular momentum relative to the centre of mass. So if $\mathbf{\Gamma}^{\text{rel}}$ is the total external torque on the body relative to the centre of mass, and $\mathbf{L}^{\text{rel}}$ is the total angular momentum of the body relative to the centre of mass, as defined in equations (43) and (44), then

$$\mathbf{\Gamma}^{\text{rel}} = \dot{\mathbf{L}}^{\text{rel}}. \tag{47}$$

Remember that we are modelling the extended body as a system of n particles.

This result shows that we can extend the torque law of Section 2 by considering torques and angular momentum *relative to the centre of mass*. Therefore to model rotational motion, we can work in a frame of reference where the centre of mass is taken as the origin and thought of as fixed for the purpose of treating the relative motion. We can then deal with motion of the centre of mass separately; this can be done by applying Newton's second law to an equivalent particle located at the centre of mass, that is, $\mathbf{F} = M\ddot{\mathbf{R}}$.

We can use the torque law relative to the centre of mass to justify an assumption that we made in Section 1, that the angular momentum relative to the centre of mass is conserved for projectiles in flight.

Conservation of angular momentum: special case

Suppose that each particle in a system of n particles is subject to an external force of the form $cm_i\mathbf{k}$, where c is a constant, m_i is the mass of the ith particle, and $\mathbf{k}$ is a fixed vector, and that there are no other external forces on the system. Then $\boldsymbol{\Gamma}^{\text{rel}} = \mathbf{0}$, so $\mathbf{L}^{\text{rel}}$ is constant.

The next exercise asks you to establish this result.

Exercise 21

(a) Use equation (43) and the torque law relative to the centre of mass to establish the boxed result above.

(b) Show that the boxed result applies to a body (such as a diver or gymnast) in flight and subject only to gravity.

Exercise 22

Suppose that all the external forces acting on a system of particles are directed towards the origin. Show that $\boldsymbol{\Gamma}^{\text{rel}} = -\mathbf{R} \times \mathbf{F}$.

We end this subsection with another decomposition theorem, this time for kinetic energy. This theorem is valuable if we wish to tackle a problem about rotational motion by using conservation of mechanical energy (rather than by using equations of motion). In terms of the vectors defined at the start of this subsection, the square of the speed of the ith particle is $\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$, so the total kinetic energy of the system is

Recall that $|\dot{\mathbf{r}}_i|^2 = \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$.

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i.$$

Using equation (41) gives

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^{\text{rel}}\right) \cdot \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}_i^{\text{rel}}\right).$$

Expanding the brackets gives

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i \big(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + 2\dot{\mathbf{R}} \cdot \dot{\mathbf{r}}_i^{\text{rel}} + \dot{\mathbf{r}}_i^{\text{rel}} \cdot \dot{\mathbf{r}}_i^{\text{rel}} \big).$$

Here we have used $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ in order to collect terms.

Using the fact that $\dot{\mathbf{R}}$ is independent of i allows us to rearrange to

$$T = \tfrac{1}{2} \left(\sum_{i=1}^{n} m_i \right) \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \dot{\mathbf{R}} \cdot \left(\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^{\text{rel}} \right) + \tfrac{1}{2} \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^{\text{rel}} \cdot \dot{\mathbf{r}}_i^{\text{rel}}.$$

The first bracketed term in this equation is the total mass M of the system. By differentiating equation (42) with respect to t, we can show that the second bracketed term is zero. So the equation reduces to

$$T = \tfrac{1}{2} M \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \tfrac{1}{2} \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^{\text{rel}} \cdot \dot{\mathbf{r}}_i^{\text{rel}}. \qquad (48)$$

This result can be stated in words as follows.

Kinetic energy decomposition theorem

The kinetic energy of an extended body is equal to the kinetic energy of an equivalent particle that has the velocity of the body's centre of mass, plus the sum of the kinetic energies due to the motion, relative to the centre of mass, of all the body's constituent particles.

The term 'equivalent particle' is used to mean a particle of the same mass as the body located at its centre of mass.

4.2 Rigid body rotating with fixed orientation

Consider a rigid body that is in motion, rotating about an axis that may itself move but remains pointing in the same direction. A cylindrical can rolling down a slope, with its axis pointing in the same horizontal direction throughout, provides an example of such motion. In this situation, the centre of mass of the rigid body may be moving, but the motion *relative to the centre of mass* is of the type considered in Section 3. The position of each particle in the rigid body, relative to the centre of mass, is constrained in the same way that the position of each particle was in our discussion in Subsection 3.1. This means that arguments similar to those in Section 3 can be used to deduce expressions, in terms of the moment of inertia, for the angular momentum and kinetic energy of the rigid body, relative to its centre of mass.

To illustrate these points, take a rigid body of mass M that is rotating about an axis of fixed orientation through its centre of mass with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, where $\mathbf{k}$ is a fixed unit vector and θ is the usual anticlockwise angular displacement relative to the axis. Let I be the body's moment of inertia about that axis. Then the $\mathbf{k}$-component of the angular momentum of the body relative to the centre of mass is $I\dot{\theta}$, while the kinetic energy of the body relative to the centre of mass is $\frac{1}{2}I\omega^2$, where $\omega = |\boldsymbol{\omega}| = |\dot{\theta}|$ is the angular speed. Combining these results with the torque law relative to the centre of mass (equation (47)) and the kinetic energy decomposition theorem (equation (48)) leads to the following.

Rigid body rotating with fixed orientation

A rigid body of mass M is rotating about an axis of fixed orientation through its centre of mass, with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, where $\mathbf{k}$ is a fixed unit vector and θ is the anticlockwise angular displacement relative to the axis of rotation. Let I be the moment of inertia of the body about the axis of rotation.

The $\mathbf{k}$-component $L^{\text{rel}}_{\text{axis}}$ $(= \mathbf{L}^{\text{rel}} \cdot \mathbf{k})$ of the angular momentum of the body relative to the centre of mass is given by

$$L^{\text{rel}}_{\text{axis}} = I\dot{\theta}. \tag{49}$$

The **equation of relative rotational motion** of the body is

$$\Gamma^{\text{rel}}_{\text{axis}} = I\ddot{\theta}, \tag{50}$$

where $\Gamma^{\text{rel}}_{\text{axis}}$ $(= \boldsymbol{\Gamma}^{\text{rel}} \cdot \mathbf{k})$ is the $\mathbf{k}$-component of the total external torque relative to the centre of mass.

The kinetic energy T of the body is the sum of the kinetic energy of an equivalent particle at the centre of mass and the rotational kinetic energy relative to the centre of mass, that is,

$$T = \tfrac{1}{2}M\left|\dot{\mathbf{R}}\right|^2 + \tfrac{1}{2}I\omega^2, \tag{51}$$

where $\mathbf{R}$ is the position vector of the centre of mass and ω $(= |\dot{\theta}|)$ is the angular speed.

Exercise 23

You considered an identical baton in Exercise 15. Use any results from the solution to that exercise that you find useful.

A drum majorette's baton is modelled as a uniform cylindrical rod with spheres of equal mass at each end. The rod has mass 0.1 kg, length 0.8 m and diameter 0.04 m. Each sphere has diameter 0.1 m and mass 0.25 kg. The baton has been thrown upwards and is rotating at 1 revolution per second about a horizontal axis through its centre of mass and normal to the axis of the cylinder. Its centre of mass, which was initially at O, is moving vertically upwards at $5\,\text{m}\,\text{s}^{-1}$.

(a) Find the kinetic energy of the baton at the instant described.

(b) Find the magnitude of the $\mathbf{k}$-component of the angular momentum of the baton, relative to O, where $\mathbf{k}$ is a unit vector in the direction of the axis of rotation.

Recall the Highland sport of tossing the caber. The process of tossing the caber can be divided into five phases, as shown in Figure 7. In the following exercise, we consider an early part of the toss, that is, the transition between phases (i) and (ii).

Exercise 24

(a) During a caber-tossing competition, a competitor runs forward holding the caber (carrying the end X shown in Figure 28) and stops suddenly. While the competitor is running, the caber is vertical, and the whole caber has the same forward speed v. When the competitor stops, a large force is exerted at X for a very short time, with the effect that the end X of the caber becomes stationary. Model the caber as a uniform thin rod of length $2l$ and mass m, and assume that the motion of the caber is confined to two dimensions, in the (x, y)-plane in Figure 28.

(i) What will be the angular speed of the caber just after the competitor stops?

(ii) What will be the kinetic energy of the caber just after the competitor stops?

(b) Suppose that after stopping, the competitor holds the end X of the caber stationary while the caber falls forward under gravity. Assuming that resistive forces are negligible, estimate the angular speed of the caber when it makes an angle θ with the vertical.

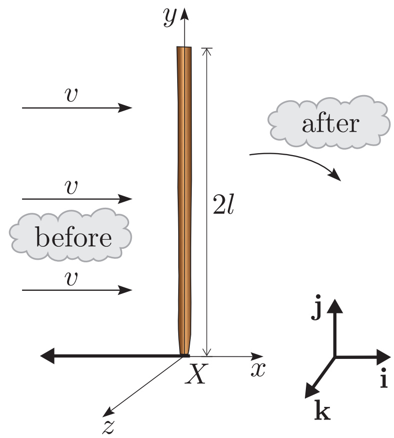

Figure 28 The instant of change between phases (i) and (ii) of Figure 7

We now consider a much later part of the motion of the caber: the start of phase (v) (see Figure 7) or the moment at which it strikes the ground (see Figure 29). The aim of the toss is to ensure that the caber finishes lying on the ground with the end X, which was originally being held by the competitor, now furthest away from him. Even if the caber strikes the ground as shown in the figure, before it has rotated sufficiently for X to have moved to the right of Y, it may maintain sufficient rotation after impact for X to swing past Y, and for the caber to fall with X pointing away from the competitor. How might we model the effect of the caber hitting the ground on its rotational motion?

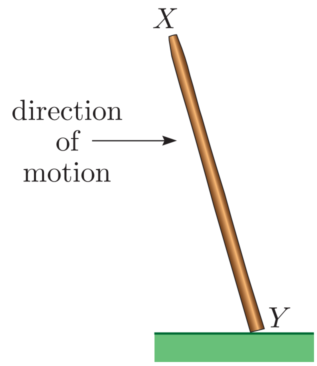

Figure 29 Instant where the caber strikes the ground at the start of phase (v)

When an object hits the ground, there is an **impact**, during which the object is subject to forces of great magnitude over a short period of time. These forces drastically change the motion of the object. To model the impact, we assume that after hitting the ground, the end Y of the caber is stationary, and that during impact, all the forces on the caber are acting at the point Y. Take an origin at the point Y, and consider the torque and angular momentum about that point. Since we are assuming that all the external forces act at Y during the impact, the torque about Y is zero. Then by the torque law, the *angular momentum about Y is conserved during the impact.*

We are assuming that during the impact, the force due to gravity is negligible compared with the forces acting at Y.

You are asked to develop this model further in the following exercise.

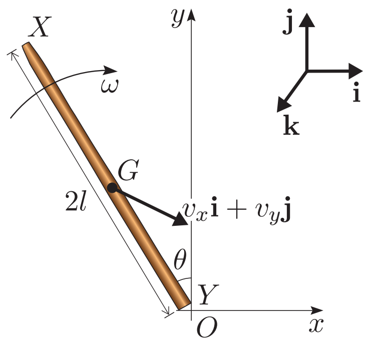

Figure 30 Coordinate set-up at the start of phase (v)

Exercise 25

A caber is in flight, and its end Y is about to strike the ground at O at an angle θ from the vertical. Use the axes shown in Figure 30, and assume that the motion of the caber is confined to the (x, y)-plane throughout. Just before the end Y comes in contact with the ground, the centre of mass G of the caber has velocity $v_x\mathbf{i} + v_y\mathbf{j}$, and the angular speed of the caber about G is ω, rotating clockwise. Model the caber as a uniform thin straight rod of length $2l$ and mass m.

(a) Use a decomposition theorem to find the **k**-component of the angular momentum of the caber about O just before it hits the ground.

(b) Let ω_1 be the angular speed of rotation of the caber about O just after it hits the ground. Show that

$$\omega_1 = \tfrac{1}{4}\omega + \frac{3}{4l}(v_x \cos\theta + v_y \sin\theta). \qquad (52)$$

(c) To rotate past the vertical, the caber must have sufficient kinetic energy after impact that it reaches the vertical with non-zero kinetic energy. Use this fact to show that in order for the caber to rotate past the vertical, we must have that

$$\omega_1^2 > \frac{3g(1 - \cos\theta)}{2l}. \qquad (53)$$

The values of θ, v_x, v_y and ω, as defined in Exercise 25, when the caber strikes the ground will be determined by the way that the competitor launches the caber. If we know these values and the length $2l$ of the caber, then we can calculate ω_1 from equation (52). Then condition (53) enables us to determine whether or not the caber will pass the vertical.

4.3 Rolling objects

It is interesting to compare how solid and hollow cylindrical cans roll down an inclined plane – a situation mentioned in the Introduction. We are now in a position to model that situation quantitatively. To do so, we will assume that there is no loss of mechanical energy when a cylinder rolls down a slope. This is an important assumption that will be fully justified later on.

Consider a cylinder of mass M and radius R rolling down a plane inclined at an angle α to the horizontal (Figure 31(a)). We will first look at the behaviour of a uniform *solid* cylinder. Its moment of inertia about an axis through its centre of mass is $\frac{1}{2}MR^2$ (from Table 1). The cylinder starts from rest at the origin O, and we want to find how long it will take to reach the point A, where the distance OA is l.

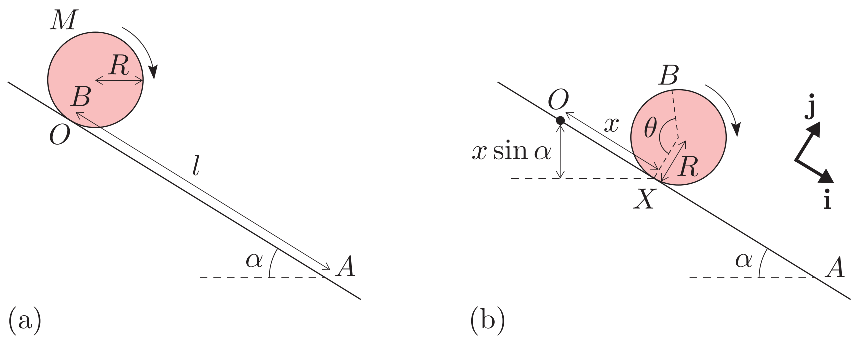

Figure 31 Cylinder rolling down an inclined plane: (a) initial position, (b) after it has rolled a distance x along the plane

Take the situation where the cylinder has rolled as far as X (Figure 31(b)), and choose Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ as shown in the figure. Let $OX = x$, and suppose that in rolling from O to X the cylinder has turned through an angle θ, without any slipping having occurred. Note that since the cylinder is rotating clockwise, θ becomes increasingly negative. The distance OX must be equal to the length of the circumference of the cylinder from B to X, where B is the point of contact between the cylinder and the slope at the outset. Thus taking 0 as the initial value of θ, so that its subsequent values are negative, we have

Note that, as usual, θ is measured positive in an anticlockwise sense.

$$x = -R\theta. \tag{54}$$

We refer to this equation as the **rolling condition**.

The rolling condition is often referred to as the *no slip condition*.

When the cylinder is at the point shown in Figure 31(b), its centre of mass has position $x\mathbf{i} + R\mathbf{j}$. Hence the velocity of the centre of mass is $\dot{x}\mathbf{i}$, since R is constant. The cylinder is rotating clockwise at an angular speed $|\dot{\theta}|$. Therefore by equation (51), the cylinder has kinetic energy

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\dot{\theta}^2. \tag{55}$$

From equation (54) we have $R|\dot{\theta}| = |\dot{x}|$, so

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{4}M\dot{x}^2 = \tfrac{3}{4}M\dot{x}^2.$$

Because the cylinder starts from rest at O, its initial kinetic energy is zero. In moving from O to X, the centre of mass of the cylinder descends a vertical distance $x\sin\alpha$. So the potential energy of the cylinder is reduced by $Mgx\sin\alpha$. Then, if mechanical energy is conserved, we have

$$\tfrac{3}{4}M\dot{x}^2 = Mgx\sin\alpha. \tag{56}$$

By differentiating each side of this equation with respect to t, we obtain

$$\tfrac{3}{2}M\dot{x}\ddot{x} = Mg\dot{x}\sin\alpha. \tag{57}$$

On dividing by $M\dot{x}$ and rearranging, we obtain

The cylinder is not stationary, so $\dot{x}$ is not zero.

$$\ddot{x} = \tfrac{2}{3}g\sin\alpha. \tag{58}$$

You saw this in Unit 3.

Now, if the cylinder were simply *sliding* down the slope (without friction) rather than rolling down, it would have acceleration $g\sin\alpha$. But from equation (58) we see that the rolling cylinder has a lower acceleration than if it were to slide down the slope. This is because in the case of the rolling object, some of its potential energy has been converted into kinetic energy of rotation, while for a sliding object all the kinetic energy is associated with the linear motion of the centre of mass.

Exercise 26

(a) Adapt the foregoing argument to obtain an expression for the acceleration of a hollow cylinder of mass M and radius R when, starting from rest, it is rolling down the same slope as considered above. (Model the hollow cylinder as a thin cylindrical shell, all of whose mass is at a distance R from its axis.)

(b) In a 'race' over a distance of 2 m down a slope angled at $\frac{\pi}{6}$ to the horizontal, how much faster will a solid cylinder travel than a hollow one? Do the masses of the cylinders matter?

We assume that air resistance is negligible.

We now consider the forces on a cylinder as it rolls down a slope without slipping. Since the cylinder starts from rest, it is gaining angular momentum about its centre of mass as it rolls, so there must be some force supplying a torque relative to the centre of mass. This cannot be the weight $\mathbf{W}$ – you saw in Exercise 21 that the force of gravity on a body gives a zero torque relative to the centre of mass. Another force on the cylinder is that from the slope: the normal reaction $\mathbf{N}$ of the slope acts through the centre of mass, so again it gives a zero torque about the centre of mass. That leaves only a force between the slope and the cylinder in the direction of the slope, which is supplied by friction. Therefore we must include friction $\mathbf{F}$ in our model if the model is to have any hope of predicting the motion that is observed.

If there were no friction between the slope and the cylinder, the cylinder would simply slide down the slope without rolling.

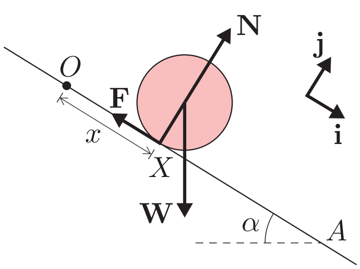

Figure 32 Forces on a rolling cylinder

Exercise 27

(a) Use Newton's second law (for the motion of an equivalent particle at the centre of mass) and the equation of relative rotational motion (equation (50)) to find an expression for the acceleration of a solid cylinder of mass M and radius R when it is rolling (without slipping) down a slope of angle α to the horizontal, as in Figure 32.

(b) The condition for the cylinder to roll without slipping is $|\mathbf{F}| \leq \mu|\mathbf{N}|$, where μ is the coefficient of static friction. Obtain a condition relating α and μ that must hold if only rolling is to occur.

In a system where there is friction, you might reasonably expect there to be a loss of mechanical energy. However, this is not necessarily the case. A cornering car needs a sideways frictional force between each tyre and the road to avoid skidding, as you saw in Unit 20, but as long as the car does not skid sideways, there is no loss of mechanical energy due to this force. The point of contact between the tyre and the road does not move in the direction of the frictional force, thus no mechanical energy is lost to friction. The situation is similar for the rolling cylinder, though this is perhaps more difficult to see. The point on the cylinder that is in contact with the slope at any instant does not move relative to the slope (if it did, the cylinder would skid and the rolling condition would not hold). Hence there is no loss of mechanical energy due to the frictional force. The truth of this assertion was demonstrated in Exercise 27(a), where you saw that the equation for the acceleration of the centre of mass obtained using Newton's second law, and making no assumption about the mechanical energy, is the same as equation (58) obtained earlier in this subsection using the assumption that mechanical energy is conserved.

That is, the frictional force $\mathbf{F}$ is normal to the velocity $\dot{\mathbf{r}}$, so $\mathbf{F} \cdot \dot{\mathbf{r}} = 0$. Therefore the work done by $\mathbf{F}$ is zero (see Unit 16).

Learning outcomes

After studying this unit, you should be able to:

- model the motion of an extended body as the motion of an equivalent particle at the centre of mass combined with the motion of the body relative to its centre of mass
- find the moments of inertia of rigid bodies of common geometrical shapes about axes of symmetry by reference to Table 1, and use the parallel axes theorem to find the moments of inertia of such bodies about other axes
- determine the angular momentum and kinetic energy of a rigid body rotating about an axis whose direction is fixed, using the appropriate decomposition theorem if necessary
- for an extended body subjected to an impact at a particular point, use conservation of angular momentum about that point to relate the motions of the body before and after the impact
- give the equation of rotational motion for a rigid body rotating about a fixed axis, or the equation of relative rotational motion for a rigid body rotating about an axis whose direction is fixed
- use the rolling condition to relate the translational and rotational motions of a body rolling across a plane surface without slipping.

Solutions to exercises

Solution to Exercise 1

(a) To determine the time of flight t, we need only consider the vertical motion of the centre of mass. The centre of mass starts with an upward velocity component (in $\mathrm{m\,s^{-1}}$) of $4\cos\frac{\pi}{6} = 2\sqrt{3}$, but is subject to a constant acceleration of magnitude g (taken to be $9.81\,\mathrm{m\,s^{-2}}$) downwards, and it descends from an initial height (above ground level) of $3 - 1.2\cos\frac{\pi}{3} = 2.4$ to a final height of 1.2 (both in m). Hence, from the constant acceleration equation,

$$1.2 = 2.4 + 2\sqrt{3}t - \tfrac{1}{2} \times 9.81t^2,$$

or

$$\tfrac{1}{2} \times 9.81t^2 - 2\sqrt{3}t - 1.2 = 0.$$

Solving for t, and rejecting the negative root, we obtain $t = 0.96$. So the gymnast is in flight for about 0.96 s.

(b) The gymnast rotates through $\frac{5\pi}{3}$ (from Example 2(b)) in approximately 0.96 s, and therefore has an angular speed of $5\pi/(3 \times 0.96) \simeq 5.45$ (in $\mathrm{rad\,s^{-1}}$). Suppose that his moment of inertia about an axis through his centre of mass is I_{T} in the tuck position, and I_{E} when fully extended. Just after leaving the bar, he is fully extended and has angular speed $\frac{10}{3}\,\mathrm{rad\,s^{-1}}$ (from Example 2(a)).

Now, angular momentum is given by $I\omega$, where I is the moment of inertia and ω is the angular speed. Since angular momentum is conserved, we have

$$I_{\mathrm{E}}\tfrac{10}{3} = I_{\mathrm{T}}5.45.$$

Hence $I_{\mathrm{T}}/I_{\mathrm{E}} = 10/(3 \times 5.45) \simeq 0.61$. So adopting a tuck position must reduce the moment of inertia by at least 40%, if it is to allow the gymnast to complete the dismount. (In practice, the reduction in the moment of inertia would need to be greater than this, since the gymnast will need time to get into and out of the tuck position.)

Solution to Exercise 2

The pairs of forces shown in parts (a), (b) and (c) of Figure 8 satisfy equation (2). Those in part (d) do not.

Solution to Exercise 3

The pair of forces that satisfy equation (2) but do not satisfy Newton's third law is in Figure 8(c).

(The forces in Figures 8(a) and 8(b) satisfy Newton's third law: they are opposite in direction and act in the same straight line. Figure 8(d) does not show forces satisfying Newton's third law: although they act in the same straight line, they are *not* in opposite directions.)

Solution to Exercise 4

Differentiating the given expression for $\mathbf{r}$, we have

$$\dot{\mathbf{r}} = -6\sin(2t)\mathbf{i} + 8\cos(2t)\mathbf{j}.$$

Then at $t = 0$, the expressions for $\mathbf{r}$ and $\dot{\mathbf{r}}$ reduce to

$$\mathbf{r} = 3\mathbf{i} + 5\mathbf{k},$$
$$\dot{\mathbf{r}} = 8\mathbf{j}.$$

So the angular momentum of the particle about O at $t = 0$ is

$$\mathbf{r} \times m\dot{\mathbf{r}} = (3\mathbf{i} + 5\mathbf{k}) \times 20(8\mathbf{j}) = 160(3\mathbf{k} - 5\mathbf{i}).$$

Solution to Exercise 5

(a) The angular speed ω of the particle is v/R, and the angular velocity $\boldsymbol{\omega}$ is $(v/R)\mathbf{k}$, using the right-hand grip rule (see Unit 20) for the direction.

(b) By definition, $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$. Now, using ω as defined in part (a), we have from the hint that

$$\mathbf{r} = R\big(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\big),$$

and therefore (since ω is constant)

$$\dot{\mathbf{r}} = R\omega\big(-\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}\big).$$

Then, substituting for $\mathbf{r}$ and $\dot{\mathbf{r}}$ in $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$, we find

$$\begin{aligned}\mathbf{L} &= R\big(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\big) \times mR\omega\big(-\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}\big)\\ &= mR^2\omega\big(\cos^2(\omega t)\mathbf{k} + \sin^2(\omega t)\mathbf{k}\big)\\ &= mR^2\omega\mathbf{k}.\end{aligned}$$

(c) For a single particle, the moment of inertia is mr^2 (from Unit 17), where r is the distance of the particle from the relevant axis. In this case $r = R$, so $I = mR^2$.

(d) With $\boldsymbol{\omega} = \omega\mathbf{k}$ and using the result from part (c), we have

$$I\boldsymbol{\omega} = mR^2\boldsymbol{\omega} = mR^2\omega\mathbf{k},$$

which is equal to $\mathbf{L}$ as given by the solution to part (b).

Solution to Exercise 6

(a) As $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, it follows that

$$\mathbf{r} \times \dot{\mathbf{r}} = (x\mathbf{i} + y\mathbf{j}) \times (\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) = x\dot{y}\mathbf{k} - y\dot{x}\mathbf{k} = (x\dot{y} - y\dot{x})\mathbf{k}.$$

(b) The angular momentum of the particle about O is given by

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = m(\mathbf{r} \times \dot{\mathbf{r}}) = m(x\dot{y} - y\dot{x})\mathbf{k},$$

from part (a). But

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Hence, using the product and chain rules,

$$\dot{x} = \dot{r}\cos\theta + r\,\frac{d}{dt}(\cos\theta) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$
$$\dot{y} = \dot{r}\sin\theta + r\,\frac{d}{dt}(\sin\theta) = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

Then

$$\begin{aligned} x\dot{y} - y\dot{x} &= r\cos\theta(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) - r\sin\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \\ &= r\dot{r}\cos\theta\sin\theta + r^2\dot{\theta}\cos^2\theta - r\dot{r}\cos\theta\sin\theta + r^2\dot{\theta}\sin^2\theta \\ &= r^2\dot{\theta}. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathbf{L} &= m(x\dot{y} - y\dot{x})\mathbf{k} \\ &= mr^2\dot{\theta}\mathbf{k}, \end{aligned}$$

as required.

Solution to Exercise 7

In Example 3(b)(i), we found

$$\dot{\mathbf{r}} = -R\omega\sin(\omega t)\mathbf{i} + R\omega\cos(\omega t)\mathbf{j}.$$

Differentiating this gives the acceleration

$$\ddot{\mathbf{r}} = -R\omega^2\cos(\omega t)\mathbf{i} - R\omega^2\sin(\omega t)\mathbf{j}.$$

The total force on the particle is $\mathbf{F} = m\ddot{\mathbf{r}}$. So, substituting for $\ddot{\mathbf{r}}$ as given above, we have

$$\mathbf{F} = -mR\omega^2\big(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\big).$$

Now, $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$, which, on substituting for $\mathbf{r}$ from equation (8) and for $\mathbf{F}$ as given above, becomes

$$\begin{aligned} \boldsymbol{\Gamma} &= \big(R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j} + h\mathbf{k}\big) \times (-mR\omega^2)\big(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\big) \\ &= -mR^2\omega^2\big(\cos(\omega t)\sin(\omega t)\mathbf{k} - \cos(\omega t)\sin(\omega t)\mathbf{k}\big) \\ &\quad - mhR\omega^2\big(\cos(\omega t)\mathbf{j} - \sin(\omega t)\mathbf{i}\big) \\ &= mhR\omega^2\big(\sin(\omega t)\mathbf{i} - \cos(\omega t)\mathbf{j}\big). \end{aligned}$$

Comparing this with equation (11), we see that we have obtained the same expression for $\boldsymbol{\Gamma}$.

Solution to Exercise 8

(a) From the solution to Exercise 5(b), the angular momentum, about O, of the particle at A is $m_1a^2\boldsymbol{\omega}$, while the angular momentum, about O, of the particle at B is $m_2b^2\boldsymbol{\omega}$. So the total angular momentum of the two-particle system modelling the roundabout is

$$m_1a^2\boldsymbol{\omega} + m_2b^2\boldsymbol{\omega} = I\boldsymbol{\omega},$$

where $I = m_1a^2 + m_2b^2$.

(b) Since the force at C is applied in a horizontal direction at right angles to OC, it exerts a torque about O of $cF\mathbf{k}$. There are other external forces on the system that need to be taken into account: the force at O exerted by the fixed spindle, and the weight of each of the two particles. The force from the spindle has zero torque about O. Each weight exerts a non-zero torque about O, but the condition $m_1 a = m_2 b$ ensures that the torques exerted by the two weights are equal and opposite, so their sum is zero. Hence the total external torque on the two-particle system is $cF\mathbf{k}$.

Then, as the total angular momentum of the system is $I\boldsymbol{\omega}$ (from part (a)), the torque law applied to the system consisting of the two particles at A and B gives

$$\frac{d}{dt}(I\boldsymbol{\omega}) = cF\mathbf{k}.$$

Because I is a constant and $\boldsymbol{\omega} = \omega\mathbf{k}$, this equation can be rewritten as

$$I\dot{\omega}\mathbf{k} = cF\mathbf{k},$$

which simplifies to

$$I\dot{\omega} = cF.$$

(c) From the previous equation,

$$\dot{\omega} = \frac{cF}{I}.$$

Now $c = 1.5$ and $F = 315$, and I can be calculated from

$$I = m_1a^2 + m_2b^2 = (45 \times 1^2) + (60 \times 0.75^2) = 78.75.$$

So, on substituting, we have (in $\mathrm{rad\,s^{-2}}$)

$$\dot{\omega} = \frac{1.5 \times 315}{78.75} = 6.$$

Rotations at 0.5 revolutions per second require an angular speed of $\pi\,\mathrm{rad\,s^{-1}}$. A force of magnitude $315\,\mathrm{N}$ increases the angular speed from 0 to $6\,\mathrm{rad\,s^{-1}}$ in one second, so to reach an angular speed of $\pi\,\mathrm{rad\,s^{-1}}$ starting from rest will take $\frac{\pi}{6}$ seconds, which is about $0.5\,\mathrm{s}$.

Solution to Exercise 9

The only force assumed to be acting on each star is the gravitational force due to the other star, which is an internal force. Hence the total external force acting on the two-particle system is zero, and consequently so is the external torque. It follows from the torque law that the angular momentum of the system must be conserved, so its magnitude L is constant.

Suppose that the distances of the stars from their common centre of mass O are d_1 and d_2, and that the angular velocity of each star about O is $\boldsymbol{\omega} = \omega\mathbf{k}$. (Since the two stars are rotating as though tied together by a rigid rod, their angular velocities must be the same.) Then from equation (7), the angular momentum of star 1 is $m_1d_1^2\omega\mathbf{k}$, and that of star 2 is $m_2d_2^2\omega\mathbf{k}$.

The total angular momentum of the system is

$$\mathbf{L} = m_1 d_1^2 \omega \mathbf{k} + m_2 d_2^2 \omega \mathbf{k} = (m_1 d_1^2 + m_2 d_2^2)\omega \mathbf{k}.$$

By an argument similar to that used in Example 4, this must be equal to

$$\frac{m_1 m_2}{m_1 + m_2}\, d^2 \omega \mathbf{k}.$$

Therefore

$$L = \frac{m_1 m_2}{m_1 + m_2}\, d^2 \omega,$$

and this can be rearranged to give

$$\omega = \frac{m_1 + m_2}{m_1 m_2 d^2}\, L.$$

Thus the period of rotation of the system is

$$\frac{2\pi}{\omega} = \frac{2\pi m_1 m_2 d^2}{(m_1 + m_2)L}.$$

Solution to Exercise 10

Apply the torque law about the point X. Since all the external forces are applied at X, the total external torque about X is zero. Hence the rate of change of angular momentum about X is zero, that is, the angular momentum of the system *about* X is constant.

Solution to Exercise 11

(a) (i) Start from equation (26), which can be expressed as

$$\mathbf{L} = I\dot{\theta}\mathbf{k} - \dot{\theta} \sum_{i=1}^{n} m_i z_i (x_i \mathbf{i} + y_i \mathbf{j}),$$

where the definition $I = \sum_{i=1}^{n} m_i d_i^2$ has been used. Now write $x_i = d_i \cos\theta$ and $y_i = d_i \sin\theta$, since d_i is the perpendicular distance of the ith particle from the z-axis. Note also that $\dot{\theta} = \omega$, since the body is rotating anticlockwise with angular speed ω. Hence

$$\begin{aligned}\mathbf{L} &= I\omega\mathbf{k} - \omega \sum_{i=1}^{n} m_i d_i z_i (\cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}) \\ &= I\omega\mathbf{k} - \omega \sum_{i=1}^{n} m_i d_i z_i \mathbf{e}_r \\ &= I\omega\mathbf{k} - A\omega\mathbf{e}_r,\end{aligned}$$

where

$$A = \sum_{i=1}^{n} m_i d_i z_i,$$

which is constant since m_i, d_i and z_i are constants. Moreover, we must have $A > 0$ since for all $1 \le i \le n$ we have $m_i > 0$, $d_i > 0$ and $z_i > 0$, except possibly for the bottom-most particle where z_i may be zero.

(ii) None of I, ω, $\mathbf{k}$ or A varies with time, but the vector $\mathbf{e}_r$ does (because its direction is changing). Hence differentiating equation (28) with respect to time yields $\dot{\mathbf{L}} = -A\omega\dot{\mathbf{e}}_r$. But from Unit 20 we have $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta = \omega\mathbf{e}_\theta$ (using the chain rule). Thus $\dot{\mathbf{L}} = -A\omega^2\mathbf{e}_\theta$, as required.

By the torque law, the total torque on the body is $\mathbf{\Gamma} = \dot{\mathbf{L}}$. Since A and ω^2 are positive constants, we see from equation (29) that $\dot{\mathbf{L}}$, and therefore $\mathbf{\Gamma}$, is a negative multiple of $\mathbf{e}_\theta$. Thus in order to keep the body rotating with constant angular speed, it must be subjected to a non-zero torque in a direction tangential to the motion but pointing in the opposite direction to $\mathbf{e}_\theta$.

(b) (i) Since the skater's centre of mass is following a circle at constant angular speed, its acceleration is towards the z-axis, that is, in the direction of $-\mathbf{e}_r$. Thus by Newton's second law, the total external force $\mathbf{F}$ on the skater must be in this direction.

(ii) Let $\mathbf{W}$ be the skater's weight, and let the force exerted on her skates by the ice have component forces $\mathbf{N}$ vertically upwards and $\mathbf{R}$ horizontally. So $\mathbf{F} = \mathbf{W} + \mathbf{N} + \mathbf{R}$. From part (b)(i), the vertical component of $\mathbf{F}$ is zero, so $\mathbf{W} + \mathbf{N} = \mathbf{0}$. If the skater is perfectly vertical, then $\mathbf{N}$ and $\mathbf{W}$ have exactly the same line of action, so the sum of their torques must also be zero. Since $\mathbf{F}$ must be in the direction of $-\mathbf{e}_r$, the component of $\mathbf{R}$ normal to $\mathbf{e}_r$ must be zero. Therefore if O is the centre of the circle followed by the skater's feet, the line of action of $\mathbf{R}$ passes through O, and hence $\mathbf{R}$ has zero torque about O. This means that if the skater is vertical, the total torque about O will be zero. But part (a) showed that a system of this kind must be subject to a non-zero torque in the tangential direction! So it is impossible for the skater to follow a circle while remaining vertical.

To follow a circle, the skater needs to adopt a position in which there is a non-zero torque in the tangential direction. To do this, she must lean towards the centre of the circle. This will mean that $\mathbf{W}$ and $\mathbf{N}$ do not have the same line of action, so they provide a non-zero torque. However, note that if the skater leaned in the opposite direction, away from the centre of the circle while following the circle, there would still be a non-zero torque but in the opposite tangential direction to that predicted in part (a), so this motion would be impossible.

Solution to Exercise 12

The required moment of inertia can be obtained from that of a hollow cylinder (inner radius a, outer radius R) about its axis, as given in Table 1, by taking the limit as $a \to R$, in which case

$$I = \tfrac{1}{2}M(R^2 + a^2) \to MR^2.$$

Solution to Exercise 13

(a) With unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the directions of the x-, y-, z-axes shown in Figure 20, we have $\mathbf{F} = 100\mathbf{j}$ and $\mathbf{R} = -c\omega\mathbf{i}$. So the total torque about O is

$$\mathbf{\Gamma} = 1.5\mathbf{i} \times \mathbf{F} + (-0.1\mathbf{j}) \times \mathbf{R} = 150\mathbf{k} - 0.1c\omega\mathbf{k}.$$

Thus the z-component of the total torque on the roundabout is $150 - 0.1c\omega$.

The moment of inertia of the roundabout (from Example 5) is $\frac{1}{2}(250)(1.2)^2 = 180$, so from equation (30), the equation of rotational motion for the roundabout is

$$180\dot{\omega} = 150 - 0.1c\omega.$$

(b) Employing methods from Unit 1 (either separation of variables or the integrating factor method can be used in this case), we can obtain the general solution of the differential equation derived in part (a), which is

$$\omega = Ae^{-ct/1800} + \frac{1500}{c},$$

where A is a constant. Since the roundabout is at rest at the outset, we have $\omega = 0$ at $t = 0$, so

$$0 = A + \frac{1500}{c},$$

thus

$$A = -\frac{1500}{c}.$$

The particular solution satisfying this initial condition is therefore

$$\omega = \frac{1500}{c}\left(1 - e^{-ct/1800}\right).$$

(c) If pushing continues indefinitely, the final expression for ω implies that the rotational speed will increase to almost $1500/c$ as the exponential term becomes negligible, and it will then become steady. For $c = 10$, this maximum angular speed is $150\,\mathrm{rad\,s^{-1}}$. To follow the roundabout at that angular speed, the pusher would need to be travelling at a speed of $1.5(150) = 225\,\mathrm{m\,s^{-1}}$. This is impossible for a human pusher to achieve.

Solution to Exercise 14

The moment of inertia I_{M} of the mace about AB is the sum of the moment of inertia I_{S} of the sphere about AB and the moment of inertia I_{R} of the rod about AB. Using Table 1, in conjunction with the parallel axes theorem, we have

$$I_{\mathrm{S}} = \tfrac{2}{5}Mr^2 + M(d+r)^2,$$
$$I_{\mathrm{R}} = \tfrac{1}{4}mR^2 + \tfrac{1}{12}md^2 + m\left(\tfrac{1}{2}d\right)^2 = \tfrac{1}{4}mR^2 + \tfrac{1}{3}md^2,$$

so

$$I_{\mathrm{M}} = I_{\mathrm{S}} + I_{\mathrm{R}} = \tfrac{2}{5}Mr^2 + M(d+r)^2 + \tfrac{1}{4}mR^2 + \tfrac{1}{3}md^2.$$

Solution to Exercise 15

The kinetic energy of the baton is given by $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the baton about the axis of rotation, and ω is the angular speed. Now, the moment of inertia of the baton is the sum of the moments of inertia of the cylindrical rod and the two spheres.

The cylinder is rotating about its centre of mass, and its moment of inertia (from Table 1) is

$$\tfrac{1}{4}(0.1)(0.02)^2 + \tfrac{1}{12}(0.1)(0.8)^2 \simeq 0.005\,34$$

(in $\mathrm{kg\,m^2}$).

Each sphere has its centre of mass $0.45\,\mathrm{m}$ from the centre of the baton. To find the moment of inertia of a sphere about the axis of rotation, we find the moment of inertia about its centre of mass from Table 1 and use the parallel axes theorem, to obtain

$$\tfrac{2}{5}(0.25)(0.05)^2 + 0.25(0.45)^2 \simeq 0.0509$$

(in $\mathrm{kg\,m^2}$).

So the moment of inertia of the baton about the axis of rotation is

$$I = 0.005\,34 + 2(0.0509) \simeq 0.1071$$

(in $\mathrm{kg\,m^2}$). The angular speed ω of the baton is $2\pi\,\mathrm{rad\,s^{-1}}$. Hence the kinetic energy of the baton (in joules) is

$$\begin{aligned}\tfrac{1}{2}I\omega^2 &= \tfrac{1}{2}(0.1071)(2\pi)^2\\ &\simeq 2.11.\end{aligned}$$

Solution to Exercise 16

(a) As you saw in Example 6, when the man is $0.2\,\mathrm{m}$ from the centre of the roundabout, the angular speed of the roundabout is $\omega_1 = \pi$ (in $\mathrm{rad\,s^{-1}}$). The moment of inertia of the combined system under these circumstances is $I_1 = 176$ (in $\mathrm{kg\,m^2}$). So the kinetic energy of the system (in joules) is

$$\begin{aligned}\tfrac{1}{2}I_1\omega_1^2 &= \tfrac{1}{2}\times 176\pi^2\\ &\simeq 869.\end{aligned}$$

Similarly, when the man is $1\,\mathrm{m}$ from the centre, the angular speed is $\omega_2 \simeq 2.19$ (in $\mathrm{rad\,s^{-1}}$). The moment of inertia of the combined system is $I_2 = 252.8$ (in $\mathrm{kg\,m^2}$). So the kinetic energy of the system (in joules) is

$$\begin{aligned}\tfrac{1}{2}I_2\omega_2^2 &\simeq \tfrac{1}{2}\times 252.8(2.19)^2\\ &\simeq 606.\end{aligned}$$

(b) When the man moves inwards, the kinetic energy of the system *increases*, from about 606 joules to about 869 joules.

Solution to Exercise 17

The kinetic energy of the body is given by $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the body about the axis of rotation. By the parallel axes theorem,

$$I = I_G + MR^2.$$

The centre of mass is following a circle of radius R at angular speed ω, so $R\omega = v$, and hence the kinetic energy of the body is

$$\begin{aligned}\tfrac{1}{2}I\omega^2 &= \tfrac{1}{2}(I_G + MR^2)\omega^2\\ &= \tfrac{1}{2}I_G\omega^2 + \tfrac{1}{2}M(R\omega)^2\\ &= \tfrac{1}{2}I_G\omega^2 + \tfrac{1}{2}Mv^2.\end{aligned}$$

Solution to Exercise 18

The diver's centre of mass starts at a height l above the board. In the position at which the diver lets go of the board, the vertical displacement (above the board) of her centre of mass is $l\cos\alpha$, so the potential energy has decreased by

$$mgl(1 - \cos\alpha).$$

Let I_O be the moment of inertia of the diver about O. Then the kinetic energy at the time of letting go of the board is $\frac{1}{2}I_O\omega^2$.

The kinetic energy at the outset is zero, as the diver is at rest, so by the conservation of mechanical energy, we have

$$\tfrac{1}{2}I_O\omega^2 = mgl(1 - \cos\alpha).$$

The parallel axes theorem gives

$$I_O = I_G + ml^2.$$

Hence

$$\omega = \sqrt{\frac{2mgl(1 - \cos\alpha)}{I_G + ml^2}}.$$

Solution to Exercise 19

(a) From Figure 27, we have

$$x = -l\sin\theta, \quad y = l\cos\theta.$$

Differentiating with respect to t, the chain rule is used to obtain

$$\dot{x} = -l\dot{\theta}\cos\theta, \quad \dot{y} = -l\dot{\theta}\sin\theta.$$

Differentiating again with respect to t (using both the product and chain rules) gives

$$\begin{aligned}\ddot{x} &= l\dot{\theta}^2\sin\theta - l\ddot{\theta}\cos\theta,\\ \ddot{y} &= -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta.\end{aligned}$$

(b) The diver is rotating about a fixed axis through O. The component of the total external torque about O in the direction of the axis of rotation (the $\mathbf{k}$-direction in Figure 27) is $mgl\sin\theta$ since $\mathbf{\Gamma} = (-l\sin\theta\,\mathbf{i} + l\cos\theta\,\mathbf{j}) \times (-mg\,\mathbf{j}) = mgl\sin\theta\,\mathbf{k}$. So from equation (30), the equation of rotational motion about O for the diver is

$$I_O\ddot{\theta} = mgl\sin\theta.$$

(c) As in Exercise 18, the principle of conservation of mechanical energy gives

$$\tfrac{1}{2}I_O\dot{\theta}^2 = mgl(1-\cos\theta).$$

Differentiating this equation with respect to t and using the chain rule gives

$$\tfrac{1}{2}I_O(2\dot{\theta}\ddot{\theta}) = mgl\dot{\theta}\sin\theta,$$

which simplifies to (since $\dot{\theta} \neq 0$)

$$I_O\ddot{\theta} = mgl\sin\theta,$$

the equation of rotational motion derived in part (b).

(d) (i) By applying Newton's second law, we obtain

$$R_1\mathbf{i} + R_2\mathbf{j} - mg\mathbf{j} = m\ddot{\mathbf{r}} = m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}).$$

Resolving in the x- and y-directions gives

$$R_1 = m\ddot{x},$$
$$R_2 - mg = m\ddot{y}.$$

(ii) Substituting from equation (38) for $\ddot{x}$ and $\ddot{y}$ in the results obtained in part (d)(i) gives

$$R_1 = ml(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta),$$
$$R_2 = mg + ml(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta).$$

(e) (i) From parts (b) and (c), we have

$$I_O\ddot{\theta} = mgl\sin\theta,$$
$$\tfrac{1}{2}I_O\dot{\theta}^2 = mgl(1-\cos\theta).$$

Then, from the results of part (d)(ii), we obtain

$$\begin{aligned} R_1 &= ml(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) \\ &= ml\left(\frac{2mgl}{I_O}(1-\cos\theta)\sin\theta - \frac{mgl}{I_O}\sin\theta\cos\theta\right) \\ &= mg\,\frac{ml^2}{I_O}\sin\theta\,(2-3\cos\theta). \end{aligned}$$

(ii) From this equation we have $R_1 = 0$ when $\sin\theta = 0$, with the only solution in the given range $0 \leq \theta \leq \frac{\pi}{2}$ being $\theta = 0$, or when $2 - 3\cos\theta = 0$, that is, when $3\cos\theta = 2$, with the only solution in the given range being $\theta = \arccos\frac{2}{3}$ (which is 0.8411, or about 48°).

(f) From Table 1, the moment of inertia of a thin rod of length $2l$ and mass m about an axis through its centre of mass is $\frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2$. So for this model of the diver, using the parallel axes theorem, $I_O = \frac{1}{3}ml^2 + ml^2 = \frac{4}{3}ml^2$.

Then from equation (39), we obtain

$$R_1 = \tfrac{3}{4}mg\sin\theta\,(2 - 3\cos\theta).$$

At $\theta = \frac{\pi}{2}$, the right-hand side of this equation is $\frac{3}{2}mg$. Therefore if the diver were still in contact with the board when $\theta = \frac{\pi}{2}$, she would need to be exerting a horizontal force on the board equal in magnitude to R_1 (and opposite in direction), that is, about 50% greater than the magnitude of her weight.

(g) Equation (39) shows that $R_1 < 0$ for small positive values of θ, since $\sin\theta \geq 0$ for $0 < \theta < \pi$, and for $\theta \simeq 0$, $\cos\theta \simeq 1$ so $2 - 3\cos\theta \simeq -1$. The board is 'pushing' (horizontally) on the diver's hands, and by Newton's third law, this corresponds to the diver pushing on the board. However, for $\theta > \arccos\frac{2}{3}$, we have $R_1 > 0$. We are assuming that the diver cannot pull on the board to provide a force in this direction, so the diver is unlikely to be able to rotate past $\arccos\frac{2}{3}$. The diver is therefore likely to lose contact with the board when θ is approximately $\arccos\frac{2}{3}$ (i.e. about 48°).

Solution to Exercise 20

Start from the definition of the total external torque relative to the fixed origin O, namely

$$\boldsymbol{\Gamma} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i.$$

Now use equation (40) to substitute for $\mathbf{r}_i$, to yield

$$\boldsymbol{\Gamma} = \sum_{i=1}^{n} (\mathbf{R} + \mathbf{r}_i^{\text{rel}}) \times \mathbf{F}_i.$$

Expanding the bracket gives

$$\boldsymbol{\Gamma} = \sum_{i=1}^{n} \mathbf{R} \times \mathbf{F}_i + \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i.$$

The first term on the right-hand side simplifies because $\mathbf{R}$ is independent of i, while the second term is by definition the total external torque relative to the centre of mass (equation (43)), so

$$\boldsymbol{\Gamma} = \mathbf{R} \times \left(\sum_{i=1}^{n} \mathbf{F}_i\right) + \boldsymbol{\Gamma}^{\text{rel}} = \mathbf{R} \times \mathbf{F} + \boldsymbol{\Gamma}^{\text{rel}},$$

as required.

Solution to Exercise 21

(a) The total external force on the ith particle is $\mathbf{F}_i = cm_i\mathbf{k}$. Now, from equation (43),

$$\boldsymbol{\Gamma}^{\text{rel}} = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times cm_i\mathbf{k} = \left(\sum_{i=1}^{n} m_i\mathbf{r}_i^{\text{rel}}\right) \times c\mathbf{k}.$$

But from equation (42), $\sum_{i=1}^{n} m_i\mathbf{r}_i^{\text{rel}} = \mathbf{0}$, so

$$\boldsymbol{\Gamma}^{\text{rel}} = \mathbf{0} \times c\mathbf{k} = \mathbf{0}.$$

Then from the torque law relative to the centre of mass (equation (47)), $\mathbf{L}^{\text{rel}}$ is constant.

(b) If we model the body in flight as a system of particles, then each particle of mass m_i is subject only to a force $\mathbf{F}_i = m_i g\mathbf{k}$, where $\mathbf{k}$ is a unit vector pointing vertically downwards. So the only external force on each particle has the form $cm_i\mathbf{k}$, where $c = g$ is a constant, hence the boxed result applies, that is, the angular momentum of the body relative to the centre of mass is conserved.

Solution to Exercise 22

If the external force on the ith particle is acting towards the origin, it must act along the line of the position vector $\mathbf{r}_i$ of the particle, so $\mathbf{F}_i = c_i\mathbf{r}_i$, for some scalar c_i. The external torque on this particle about the origin is

$$\boldsymbol{\Gamma}_i = \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r}_i \times c_i\mathbf{r}_i = \mathbf{0}.$$

Then, summing over all particles, $\boldsymbol{\Gamma} = \mathbf{0}$. Hence from the torque decomposition theorem (equation (46)),

$$\mathbf{R} \times \mathbf{F} + \boldsymbol{\Gamma}^{\text{rel}} = \boldsymbol{\Gamma} = \mathbf{0}.$$

Thus $\boldsymbol{\Gamma}^{\text{rel}} = -\mathbf{R} \times \mathbf{F}$, as required.

Solution to Exercise 23

(a) The model of the baton is identical to that used in Exercise 15, and the angular speed (about the centre of mass) is the same as there. So the kinetic energy due to the rotation of the baton is 2.11 J. The total mass of the baton is 0.6 kg, and the speed of the centre of mass is $5\,\text{m}\,\text{s}^{-1}$. Hence the kinetic energy (in joules) of an equivalent particle at the centre of mass is

$$\tfrac{1}{2}(0.6)5^2 = 7.5.$$

Therefore by equation (51), the total kinetic energy of the baton (in joules) is $7.5 + 2.11 = 9.61$.

(b) An equivalent particle at the centre of mass of the baton would be moving vertically upwards. Since the centre of mass is vertically above O, this means that the position vector $\mathbf{R}$ of the centre of mass and the velocity $\dot{\mathbf{R}}$ of the centre of mass have the same direction.

Hence $\mathbf{R} \times \dot{\mathbf{R}} = \mathbf{0}$. So the angular momentum of an equivalent particle at the centre of mass is

$$\mathbf{R} \times M\dot{\mathbf{R}} = M(\mathbf{R} \times \dot{\mathbf{R}}) = \mathbf{0}.$$

Therefore by the angular momentum decomposition theorem (equation (45)), the angular momentum of the baton about O is equal to its angular momentum relative to the centre of mass. By equation (49), the $\mathbf{k}$-component of this is $I\omega$, where $I = 0.1071$ (from Exercise 15) and $\omega = 2\pi$, so

$$I\omega = 0.1071(2\pi) \simeq 0.6729.$$

So the angular momentum of the baton, relative to O, has a $\mathbf{k}$-component with magnitude approximately $0.67\,\mathrm{kg\,m^2\,s^{-1}}$.

Solution to Exercise 24

(a) (i) During the brief period while the competitor is in the act of stopping, the large force exerted at X has zero torque about X. We would be justified in regarding the effect of the torque exerted by the weight of the caber during this short period as negligible, but since the caber is vertically above X, the line of action of its weight passes through X, thus will have zero torque about X anyway. Hence while the competitor is in the act of stopping, angular momentum about X is constant. Thus the angular momentum of the caber about X is the same just before and just after the competitor stops.

Before the competitor stops, the whole caber is moving forward in the $\mathbf{i}$-direction at speed v. We now use the angular momentum decomposition theorem, equation (45), to find the angular momentum of the caber about X. The angular momentum relative to the centre of mass is zero, since the caber is stationary relative to the centre of mass. The angular momentum (about X) of an equivalent particle of mass m and velocity $v\mathbf{i}$ at the centre of mass is

$$l\mathbf{j} \times mv\mathbf{i} = -mvl\mathbf{k}.$$

Therefore the total angular momentum of the caber about X is $-mvl\mathbf{k}$.

Just after the competitor stops, the caber will be rotating about an axis through X (in the negative $\mathbf{k}$-direction). Suppose that its angular speed is ω, that is, its angular velocity is $\boldsymbol{\omega} = -\omega\mathbf{k}$, and its moment of inertia about the z-axis through X is I. Now, angular momentum about X is the same just before and just after the competitor stops, so equating $\mathbf{k}$-components gives

$$-mvl = -I\omega,$$

thus

$$\omega = mvl/I.$$

Using Table 1 and the parallel axes theorem, we have

$$I = ml^2 + \tfrac{1}{12}m(2l)^2 = \tfrac{4}{3}ml^2.$$

Hence on substituting this into $\omega = mvl/I$, we obtain $\omega = \frac{3}{4}v/l$.

So just after the competitor stops, the caber will be rotating forwards (clockwise) at an angular speed of $\frac{3}{4}v/l$.

(ii) The kinetic energy of the caber just after the competitor stops is

$$\tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(\tfrac{4}{3}ml^2)\left(\frac{3v}{4l}\right)^2 = \tfrac{3}{8}mv^2.$$

(b) Suppose that the caber has angular speed ω_1 about X when its angle with the vertical is θ. Since we assume that resistive forces are negligible, we can apply the conservation of mechanical energy, which implies that

$$\tfrac{1}{2}I\omega^2 + mgl = \tfrac{1}{2}I\omega_1^2 + mgl\cos\theta.$$

Rearranging gives

$$\omega_1^2 = \omega^2 + \frac{2mgl}{I}(1-\cos\theta),$$

which, on substituting for $\omega = 3v/(4l)$ and I from part (a)(i), yields

$$\omega_1^2 = \frac{9v^2}{16l^2} + \frac{3g}{2l}(1-\cos\theta).$$

So when the caber makes an angle θ with the vertical, its angular speed is

$$\omega_1 = \sqrt{\frac{9v^2}{16l^2} + \frac{3g}{2l}(1-\cos\theta)}.$$

Solution to Exercise 25

(a) From the angular momentum decomposition theorem, the angular momentum $\mathbf{L}$ of the caber about O just prior to impact is the sum of the angular momentum $\mathbf{L}_G$ of an equivalent particle at the centre of mass, and the angular momentum $\mathbf{L}^{\text{rel}}$ relative to the centre of mass. Now

$$\mathbf{L}_G = \mathbf{R} \times m\dot{\mathbf{R}},$$

where $\mathbf{R} = (-l\sin\theta)\mathbf{i} + (l\cos\theta)\mathbf{j}$ and $\dot{\mathbf{R}} = v_x\mathbf{i} + v_y\mathbf{j}$. So

$$\begin{aligned}
\mathbf{L}_G &= \big((-l\sin\theta)\mathbf{i} + (l\cos\theta)\mathbf{j}\big) \times m(v_x\mathbf{i} + v_y\mathbf{j}) \\
&= ml\big(-(v_y\sin\theta)\mathbf{k} - (v_x\cos\theta)\mathbf{k}\big) \\
&= -ml(v_x\cos\theta + v_y\sin\theta)\mathbf{k}.
\end{aligned}$$

From Table 1, the moment of inertia of the caber (modelled as a thin rod of length $2l$) about an axis through its centre of mass is $I = \frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2$. Now, since the angular speed is ω and the caber is rotating clockwise, the angular velocity is $\boldsymbol{\omega} = -\omega\mathbf{k}$ (i.e. $\dot{\theta} = -\omega$). So using equation (49), the $\mathbf{k}$-component of the angular momentum of the caber relative to the centre of mass is $L^{\text{rel}}_{\text{axis}} = I(-\omega) = -\frac{1}{3}ml^2\omega$.

The **k**-component of the angular momentum $\mathbf{L}$ just before the caber hits the ground is the sum of the **k**-component of $\mathbf{L}^{\text{rel}}$ and the **k**-component of $\mathbf{L}_G$, and thus is

$$-ml\left(\tfrac{1}{3}l\omega + v_x\cos\theta + v_y\sin\theta\right).$$

(b) From the parallel axes theorem, the moment of inertia of the caber about an axis through O is

$$\tfrac{1}{3}ml^2 + ml^2 = \tfrac{4}{3}ml^2.$$

Just after impact we have $\boldsymbol{\omega} = -\omega_1\mathbf{k}$ since the caber is continuing to rotate clockwise but with angular speed ω_1. Assuming that the angular momentum about O is unchanged by the impact, we have, on equating **k**-components of the angular momentum just before impact (from part (a)) and just after impact,

$$-ml\left(\tfrac{1}{3}l\omega + v_x\cos\theta + v_y\sin\theta\right) = -\tfrac{4}{3}ml^2\omega_1.$$

Hence

$$\begin{aligned}\omega_1 &= \frac{3}{4l}\left(\tfrac{1}{3}l\omega + v_x\cos\theta + v_y\sin\theta\right)\\ &= \tfrac{1}{4}\omega + \frac{3}{4l}(v_x\cos\theta + v_y\sin\theta).\end{aligned}$$

(c) After impact, the caber is rotating about its end O at an angular speed ω_1, so its kinetic energy is

$$\tfrac{1}{2}I\omega_1^2 = \tfrac{1}{2}\left(\tfrac{4}{3}ml^2\right)\omega_1^2 = \tfrac{2}{3}ml^2\omega_1^2.$$

To reach the vertical, its gain in potential energy (which equals loss in kinetic energy) must be

$$mgl(1-\cos\theta).$$

To pass the vertical, the caber must reach the vertical while still retaining some kinetic energy, so we need

$$\tfrac{2}{3}ml^2\omega_1^2 - mgl(1-\cos\theta) > 0,$$

that is (as $m > 0$ and $l > 0$),

$$\omega_1^2 > \frac{3g(1-\cos\theta)}{2l}.$$

Solution to Exercise 26

This moment of inertia was obtained in Exercise 12.

(a) For a cylindrical shell with all its mass at a distance R from its axis, the moment of inertia about the centre of mass is MR^2. Then the total kinetic energy of the cylinder is

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}(MR^2)\dot{\theta}^2 = M\dot{x}^2$$

(compare with equation (55)).

The assumption of conservation of mechanical energy then gives

$$M\dot{x}^2 = Mgx\sin\alpha$$

(compare with equation (56)). Differentiating with respect to time, we have

$$2M\dot{x}\ddot{x} = Mg\dot{x}\sin\alpha$$

(compare with equation (57)), so

$$\ddot{x} = \tfrac{1}{2}g\sin\alpha$$

(compare with equation (58)).

(We see that a hollow cylinder has a smaller acceleration than a solid one. A higher proportion of the potential energy lost goes into rotational kinetic energy in the case of a hollow cylinder.)

(b) From the final expressions for $\ddot{x}$ for solid and hollow cylinders, we can see that the acceleration is independent of the mass in each case. With $\alpha = \frac{\pi}{6}$, $\sin\alpha = \frac{1}{2}$, and substituting into the expressions for $\ddot{x}$, we obtain

$$\ddot{x}_{\text{solid}} = \tfrac{1}{3}g, \quad \ddot{x}_{\text{hollow}} = \tfrac{1}{4}g.$$

To find the time τ to move 2 m down the slope at constant acceleration a_0, starting from rest, we can use the formula $x = x_0 + v_0t + \frac{1}{2}a_0t^2$ from Unit 3 (with $x_0 = v_0 = 0$) to obtain $\frac{1}{2}a_0\tau^2 = 2$. So $\tau = 2/\sqrt{a_0}$, and substituting for a_0 from the above expressions for $\ddot{x}_{\text{solid}}$ and $\ddot{x}_{\text{hollow}}$, the times (in seconds) are

$$\tau_{\text{solid}} = 1.11, \quad \tau_{\text{hollow}} = 1.28.$$

Therefore the solid cylinder is 0.17 s faster.

Solution to Exercise 27

(a) The centre of mass has acceleration $\ddot{x}\mathbf{i}$, where x is displacement down the slope. So Newton's second law applied to an equivalent particle at the centre of mass gives

$$\begin{aligned} M\ddot{x}\mathbf{i} &= \mathbf{W} + \mathbf{N} + \mathbf{F} \\ &= Mg(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j}) + |\mathbf{N}|\mathbf{j} - |\mathbf{F}|\mathbf{i}. \end{aligned}$$

Resolving in the **i**- and **j**-directions, we obtain

$$\begin{aligned} M\ddot{x} &= Mg\sin\alpha - |\mathbf{F}|, \\ 0 &= -Mg\cos\alpha + |\mathbf{N}|. \end{aligned}$$

The rotational acceleration of the cylinder about its centre of mass is $\ddot{\theta}$, where $x = -R\theta$ (from equation (54)), so $\ddot{x} = -R\ddot{\theta}$. The moment of inertia of a solid cylinder of mass M about an axis in the **k**-direction through its centre of mass is $I = \frac{1}{2}MR^2$ (see Table 1). The total external torque relative to the centre of mass is given by

$$\mathbf{\Gamma}^{\text{rel}} = (-R\mathbf{j})) \times (-|\mathbf{F}|\mathbf{i}) = -R|\mathbf{F}|\mathbf{k},$$

so

$$\Gamma^{\text{rel}}_{\text{axis}} = -R|\mathbf{F}|.$$

Hence the equation of relative rotational motion, $\Gamma^{\text{rel}}_{\text{axis}} = I\ddot{\theta}$, implies that

$$-R|\mathbf{F}| = \tfrac{1}{2}MR^2\ddot{\theta} = -\tfrac{1}{2}MR\ddot{x}.$$

So we have

$$|\mathbf{F}| = \tfrac{1}{2}M\ddot{x}.$$

Substituting this into $M\ddot{x} = Mg\sin\alpha - |\mathbf{F}|$ gives

$$Mg\sin\alpha - \tfrac{1}{2}M\ddot{x} = M\ddot{x},$$

hence

$$g\sin\alpha = \tfrac{3}{2}\ddot{x},$$

which can be rearranged as

$$\ddot{x} = \tfrac{2}{3}g\sin\alpha.$$

This is the same as equation (58), which was obtained under the assumption of conservation of mechanical energy.

(b) From part (a) we have

$$|\mathbf{N}| = Mg\cos\alpha$$

and

$$|\mathbf{F}| = \tfrac{1}{2}M\ddot{x} = \tfrac{1}{2}M\left(\tfrac{2}{3}g\sin\alpha\right) = \tfrac{1}{3}Mg\sin\alpha.$$

So

$$\frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{Mg\sin\alpha}{3Mg\cos\alpha} = \tfrac{1}{3}\tan\alpha,$$

or equivalently,

$$|\mathbf{F}| = \tfrac{1}{3}\tan\alpha\,|\mathbf{N}|.$$

The condition that the cylinder rolls without slipping is $|\mathbf{F}| \le \mu|\mathbf{N}|$, that is, $\tfrac{1}{3}\tan\alpha\,|\mathbf{N}| \le \mu|\mathbf{N}|$, so

$$\tan\alpha \le 3\mu.$$