Unit 13

Fourier series

Introduction

In Unit 7 you saw that many functions can be approximated by a Taylor polynomial

$$f(x) \simeq f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2!}f''(x_0)\,(x - x_0)^2 + \cdots \\ + \frac{1}{n!}f^{(n)}(x_0)\,(x - x_0)^n.$$

It is often the case that a small number of terms gives a useful approximation, and it is tempting to ask whether the approximation may be made exact by taking an *infinite* number of terms – in other words, is

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)\,(x - x_0)^n$$

true? This is indeed true for sufficiently smooth functions, but it is not necessarily true for all functions. An example for which the above formula is true is when $f(x) = \exp(x)$ and $x_0 = 0$. In this case we have $f^{(n)}(x_0) = e^0 = 1$, so

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n,$$

for any real number x.

In this unit we are primarily concerned not with polynomial functions (which, if not constant, become numerically very large as $x \to \pm\infty$), but with *periodic* functions, such as $\sin x$ and $\cos x$. A great deal of beautiful mathematics has arisen from the analysis that we will describe, and there are also important practical benefits.

One example arises directly from Unit 10, where you studied differential equations modelling forced and damped oscillations. You saw how they could be used to predict the response of various mechanical and electrical systems. In particular, you saw how they responded to a sinusoidal forcing term like $\cos t$, with graph as in Figure 1.

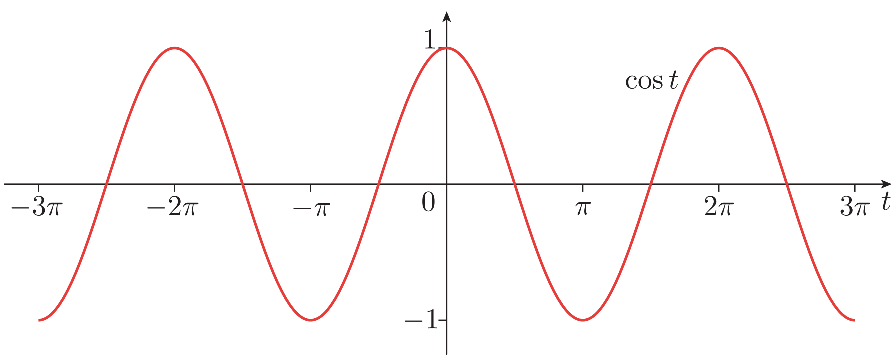


Figure 1 The sinusoidal function $\cos t$

Such forcing terms occur frequently in applications; for instance, they model the force acting on the suspensions of cars travelling on bumpy roads and the effect of radio signals acting on electrical circuits. The resulting model leads to a differential equation of the form

$$m\ddot{x} + r\dot{x} + kx = P\cos(\Omega t),$$

with steady-state solution

$$x(t) = PM\cos(\Omega t + \phi).$$

The amplitude magnification M and the phase angle ϕ are rather complicated functions of the forcing frequency Ω, which we need not consider here.

However, a forcing term, though periodic, may be more complicated than a purely sinusoidal function. Figures 2, 3 and 4 depict periodic functions $b(t)$, $g(t)$ and $h(t)$ that are reasonably easy to visualise and describe.

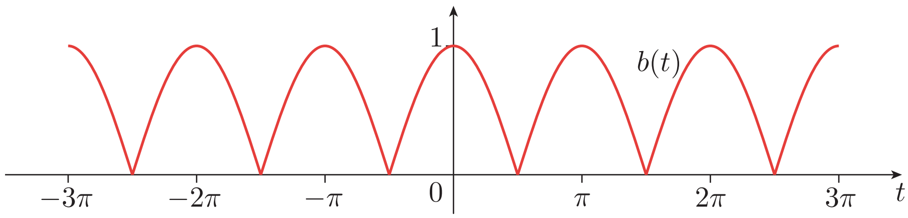


Figure 2 The periodic function $b(t)$, which resembles a series of bumps

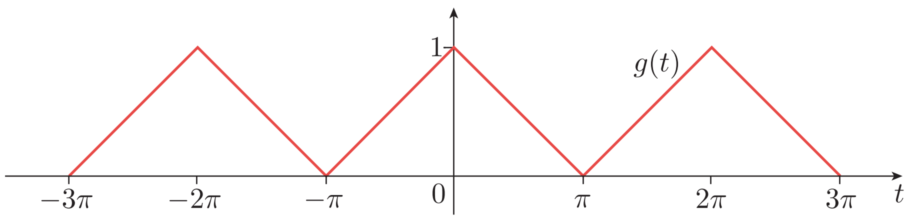


Figure 3 The periodic function $g(t)$, also known as a *sawtooth function*

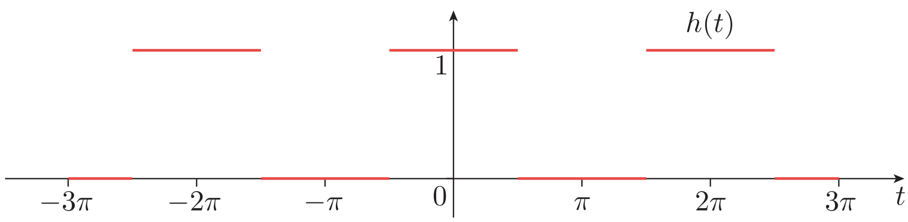


Figure 4 The periodic function $h(t)$, also known as a *square-wave function*

The graph in Figure 2 is similar to that of $\cos t$, except that all the values are positive. The function $b(t)$ is described by

$$b(t) = |\cos t|.$$

This graph differs from that of the cosine function, which turns smoothly. Here the direction changes abruptly every time the graph reaches the t-axis. However, $b(t)$ is still continuous in that there are no sudden jumps in the function value as t increases smoothly; it is possible to draw the graph without taking the pen off the paper.

The graph of $g(t)$ in Figure 3 is also continuous, but with abrupt changes of direction at both the highest and lowest values. Between the points where the direction changes, the graph is a straight line. The whole graph looks rather like the blade of a saw, so $g(t)$ is known as a **sawtooth function**. You will meet this function again in Unit 14, where it will be used to model the initial displacement of a plucked guitar string.

The graph in Figure 4 shows a function that takes the value 1 whenever t lies in the interval $[(2k - \frac{1}{2})\pi, (2k + \frac{1}{2})\pi]$ for some integer k, and the value 0 otherwise, that is,

$$h(t) = \begin{cases} 1 & \text{for } (2k - \frac{1}{2})\pi \le t \le (2k + \frac{1}{2})\pi \quad (k \in \mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

This graph differs quite radically from those of $\cos t$, $b(t)$ and $g(t)$. Here there are abrupt jumps in the function value itself (rather than merely abrupt changes of direction) at the points $t = (k + \frac{1}{2})\pi$. The function $h(t)$ is discontinuous, while $\cos t$, $b(t)$ and $g(t)$ are continuous. The function $h(t)$ is known as a **square-wave function**.

At present we have no way of predicting the response of a mechanical or electrical system to a forcing function like $b(t)$, $g(t)$ or $h(t)$ (although such systems are extremely common). Fourier series provide the answer. The differential equation

$$m\ddot{x} + r\dot{x} + kx = P\cos(\Omega t)$$

is *linear*. Therefore the principle of superposition tells us that if

See Unit 1.

$$x = P_1 M_1 \cos(\Omega_1 t + \phi_1)$$

is a solution of

$$m\ddot{x} + r\dot{x} + kx = P_1 \cos(\Omega_1 t),$$

and

$$x = P_2 M_2 \cos(\Omega_2 t + \phi_2)$$

is a solution of

$$m\ddot{x} + r\dot{x} + kx = P_2 \cos(\Omega_2 t),$$

then

$$x = P_1 M_1 \cos(\Omega_1 t + \phi_1) + P_2 M_2 \cos(\Omega_2 t + \phi_2)$$

is a solution of

$$m\ddot{x} + r\dot{x} + kx = P_1 \cos(\Omega_1 t) + P_2 \cos(\Omega_2 t).$$

If we could express $b(t)$ as a linear combination of cosine functions, then we would be able to apply this idea to find a solution of

$$m\ddot{x} + r\dot{x} + kx = b(t)$$

(and similarly for $g(t)$ and $h(t)$). This is precisely what we will do in this unit, except that the linear combinations will involve an *infinite* number of terms.

As you will see, the functions $b(t)$, $g(t)$ and $h(t)$ introduced above correspond, respectively, to the infinite sums

$$B(t) = \tfrac{2}{\pi}\left(1 + \tfrac{2}{1\times 3}\cos 2t - \tfrac{2}{3\times 5}\cos 4t + \tfrac{2}{5\times 7}\cos 6t - \dots\right), \tag{1}$$

$$G(t) = \tfrac{1}{2} + \tfrac{4}{\pi^2}\cos t + \tfrac{4}{9\pi^2}\cos 3t + \tfrac{4}{25\pi^2}\cos 5t + \tfrac{4}{49\pi^2}\cos 7t + \cdots, \tag{2}$$

$$H(t) = \tfrac{1}{2} + \tfrac{2}{\pi}\cos t - \tfrac{2}{3\pi}\cos 3t + \tfrac{2}{5\pi}\cos 5t - \tfrac{2}{7\pi}\cos 7t + \cdots. \tag{3}$$

We will use the convention that a function named with an upper-case letter (such as $B(t)$) corresponds to a given function with a lower-case letter (such as $b(t)$). Later in the unit the precise nature of this correspondence will be stated, and it will turn out that for almost all values of t these two functions will have equal values.

Figure 5 The French mathematician, physicist and historian Joseph Fourier (1768–1830)

Infinite sums like $B(t)$, $G(t)$ and $H(t)$ are called **Fourier series** (named after Joseph Fourier; see Figure 5). Successive terms in each sum are functions that belong to a family of sinusoidal functions whose frequencies are related. The sums are infinite in the sense that they do not stop after a finite number of terms, though in practice we take only as many terms as are needed to make the result as accurate as we require.

Fourier series are not just of interest in the analysis and application of damped forced oscillations, but are widely applicable and are of fundamental theoretical importance. Whenever a system exhibits variation at a range of frequencies, it is sensible to see if this variation can be explained by some combination of sinusoidal terms.

In Unit 14 you will look at transverse vibrations of guitar strings and at the conduction of heat along metal rods. These effects will be modelled by differential equations involving the partial derivatives that you met in Unit 7. In the case of the vibrating systems, there are sinusoidal solutions with a range of frequencies corresponding to the normal modes of Unit 11. These can be combined to find particular solutions. However, it is not so obvious that solutions to the heat-conduction problem can also be found as sums of sinusoidal terms. This was one of Fourier's many great discoveries.

Fourier realised that by considering series of sinusoidal functions, he could approximate most periodic functions. It is these series that we introduce and explore in this unit. You have already studied the mathematics that you need. Here all we have to do is draw it together to obtain powerful results.

In Section 1 we introduce Fourier series. This first involves a discussion of families of periodic functions and their periods, frequencies and fundamental intervals, and of even and odd functions. Section 2 is devoted to the task of calculating the Fourier series for a particular function, and in doing so we establish the principles for finding *any* Fourier series. Section 3 extends the results in Section 2 to find general formulas for the Fourier series for both even and odd functions with any given period. In Section 4 we extend these formulas to deal with functions that are neither even nor odd. Finally, we look at the problem of extending a function defined on an interval so that the extended function is periodic and has desirable properties. It is this final technique that will be used in the next unit to solve partial differential equations.

1 Introducing Fourier series

In this section we ask what kinds of functions can be expressed as Fourier series. You will see that a requirement is that the function is periodic. This section investigates a family of periodic functions, then looks at general even and odd functions. For a periodic function, we introduce the notions of *period*, *angular frequency* and *fundamental interval*.

1.1 Families of cosine functions

Suppose that we *define* the function $G(t)$ by the following Fourier series (an infinite series of cosine terms):

$$G(t) = \frac{1}{2} + \frac{4}{\pi^2}\cos t + \frac{4}{9\pi^2}\cos 3t + \frac{4}{25\pi^2}\cos 5t + \frac{4}{49\pi^2}\cos 7t + \cdots. \qquad (4)$$

We now investigate some properties of $G(t)$, *without* assuming any connection with the sawtooth function $g(t)$.

The individual cosine terms in the sum are periodic, so it is not surprising to find that the sum $G(t)$ is also periodic.

Exercise 1

Let $G(t)$ be defined as in equation (4). Find $G(t + 2\pi)$ in terms of $G(t)$.

You saw in Exercise 1 that the function $G(t)$ defined by equation (4) is periodic with period 2π, just like the cosine function $\cos t$. But what of the individual terms in the sum? Apart from the constant term, they are multiples of

$$\cos t, \quad \cos 3t, \quad \cos 5t, \quad \cos 7t, \quad \ldots. \qquad (5)$$

Exercise 2

What are the angular frequencies and periods of the functions in sequence (5)?

(Recall that in the expression $\cos(\omega t)$, the constant ω is called the **angular frequency**. The angular frequency is related to the period using the fact that the cosine function will repeat when its argument increases by 2π, so $\omega\tau = 2\pi$, where τ is the period, that is, $\tau = 2\pi/\omega$.)

From Exercise 2 you can see that we have a family of cosine functions where all the angular frequencies are integer multiples of the smallest angular frequency 1, and whose periods are integer fractions of the *fundamental period* 2π. We have seen that this is the period of the function $G(t)$, since all the component functions will have repeated after this time – some having repeated several times.

More generally (as you may recall from Unit 9), any function $f(t)$ is said to be **periodic** if it repeats regularly, that is, if there is some positive value λ such that for all t, $f(t+\lambda) = f(t)$. In this case, it is also true that for all t, $f(t+2\lambda) = f(t+\lambda) = f(t)$, so 2λ could be taken as a period for $f(t)$ instead of λ, and in general, $n\lambda$ could be taken as the period (where n is any positive integer). The **fundamental period** of a periodic function is the *smallest* possible (positive) value for the period.

In applications, the fundamental period is far more important than the other periods. For this reason, 'fundamental period' is usually shortened to simply *period*. For example, the fundamental period of a pendulum (the time that it takes to swing to and fro) is usually called *the* period of the pendulum. We will occasionally use this shorthand when there is no risk of confusion. If we talk about *the period* of a function, then we mean its fundamental period.

In this unit, time is the independent variable of functions, and τ is used to denote the period of a periodic function.

Example 1

Let $f(t) = \cos 4t + 3\cos 6t$. What are the angular frequencies and corresponding periods of the component functions? What is the period of the function $f(t)$?

Solution

The angular frequencies of the component functions are 4 and 6. Their corresponding periods are $\frac{\pi}{2}$ and $\frac{\pi}{3}$, respectively. Hence the period of the combined function $f(t)$ is $\tau = \pi$ (as this is the shortest time that is an integer multiple of both $\frac{\pi}{2}$ and $\frac{\pi}{3}$). After this time, the first cosine term will have completed two cycles, while the other cosine term will have completed three.

Example 1 suggests that the period of a sum of sinusoidal terms is the least integer common multiple of the periods of the component functions. This period gives the first time after which *all* the component functions have repeated.

Exercise 3

Let $f(t) = 2\cos \pi t + 3\cos \frac{3\pi}{2}t - \cos 2\pi t$. What are the angular frequencies and corresponding periods of the component functions? What is the period of the function $f(t)$?

A Fourier series is an infinite sum of sinusoidal terms each of which is periodic, so, as has been exemplified above, a Fourier series is also periodic. Hence a sensible restriction on a function that is to be described by a Fourier series is that it should itself be periodic. However, as you have seen above, if you are interested in obtaining Fourier series for a function with period τ, then you must consider not only sinusoidal functions with period τ in the infinite sum, but also sinusoidal functions with fractional periods

$$\frac{\tau}{2}, \frac{\tau}{3}, \frac{\tau}{4}, \frac{\tau}{5}, \ldots,$$

since functions with these periods also repeat after time τ. Corresponding to the periods $\tau, \frac{\tau}{2}, \frac{\tau}{3}, \frac{\tau}{4}, \frac{\tau}{5}, \ldots$ are the angular frequencies

$$\frac{2\pi}{\tau}, \frac{4\pi}{\tau}, \frac{6\pi}{\tau}, \frac{8\pi}{\tau}, \frac{10\pi}{\tau}, \ldots.$$

So, for example, for a Fourier series of cosine functions, you must consider the family of functions

$$C_n(t) = \cos\left(\frac{2n\pi t}{\tau}\right), \quad \text{where } n \text{ is a positive integer.} \tag{6}$$

Since these functions repeat after a time τ, we do not need to draw their graphs for all values of t. We can restrict our attention to any interval of length τ. We will choose the interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$ as it has the correct length and is centred on the origin. In general, any interval whose length is the fundamental period can be chosen as the **fundamental interval** for functions of that period. The graph of the function $C_1(t)$ is shown in Figure 6.

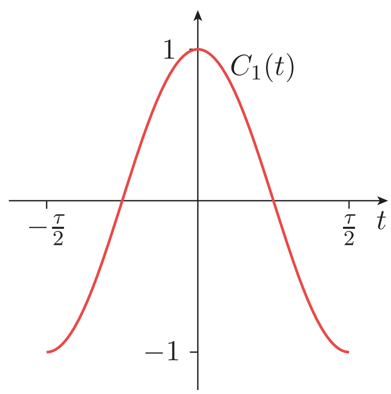


Figure 6 The function $C_1(t)$ plotted over its fundamental interval

Exercise 4

Sketch the graphs of the functions $C_2(t)$ and $C_3(t)$ on the fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$.

What happens if you try to define $C_0(t)$ using formula (6)?

We have now obtained a family of cosine functions that repeat after a time τ, including the constant function $C_0(t) = 1$. (Whatever period τ is chosen, it is trivially true that a constant function has the same value after that period.) You will see in Section 2 how this family can be used to obtain Fourier series. But first, in Subsection 1.2, we need to digress slightly to discuss even and odd functions as these ideas are used to simplify later calculations.

1.2 Even and odd functions

The previous subsection dealt solely with cosine functions, but sine functions are also periodic, so why have we not used them? In fact, there is a distinguishing feature that separates these two families of functions. We will investigate this difference as it leads to a simplification of calculations later in the unit.

Figure 7 shows the graphs of a cosine function and a sine function, namely

$$C_1(t) = \cos\left(\frac{2\pi t}{\tau}\right), \quad S_1(t) = \sin\left(\frac{2\pi t}{\tau}\right).$$

They both have period τ and hence the same fundamental interval.

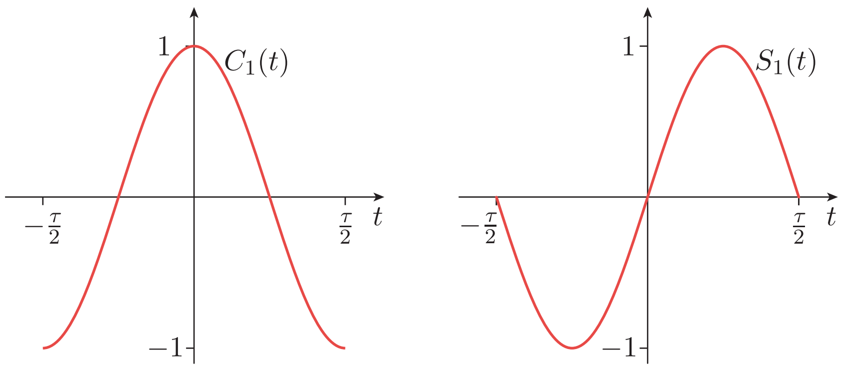


Figure 7 The functions $C_1(t)$ and $S_1(t)$ compared on their fundamental interval

Both graphs exhibit symmetry. You can see that the graph of the cosine function $C_1(t)$ takes the same values at corresponding points on either side of the vertical axis. We say that the function is *even*. By contrast, in the graph of the sine function $S_1(t)$, the values at corresponding points on either side of the vertical axis have the same magnitude but opposite signs. We say that the function is *odd*.

A function need not be either even or odd, as you will see below.

The function $f(t)$ is an **even function** if

$$f(-t) = f(t) \quad \text{for all values of } t;$$

it is an **odd function** if

$$f(-t) = -f(t) \quad \text{for all values of } t.$$

Example 2

Suppose that the function $f(t)$ is defined by $f(t) = t^2$. Is this function even, odd, or neither even nor odd?

Solution

Since $f(-t) = (-t)^2 = t^2 = f(t)$ for all t, the function is even.

Exercise 5

Suppose that the function $g(t)$ is defined by $g(t) = t^3$. Is this function even, odd, or neither even nor odd?

Exercise 6

If $f(t)$ and $g(t)$ are both odd functions, show that the function $k(t)$ defined by

$$k(t) = f(t) + g(t)$$

is also an odd function.

The graphs of the functions $f(t)$ and $g(t)$ defined in Example 2 and Exercise 5, and shown in Figure 8, should make the definitions clearer.

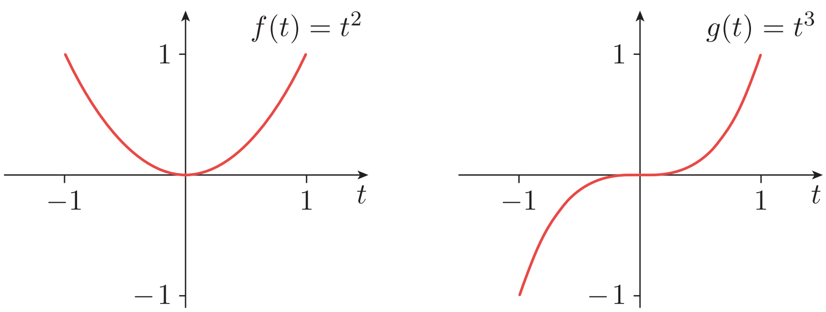


Figure 8 Graphs of the even function $f(t) = t^2$ and the odd function $g(t) = t^3$

For the even function $f(t)$, the same values appear on either side of the vertical axis, so the graph has reflection symmetry about this line. For the odd function $g(t)$, the values on either side of the vertical axis have opposite signs, so the graph has rotational symmetry through the angle π about the origin.

Generalising from the function $f(t) = t^2$, we can state that polynomial functions where *all* the powers are even are themselves even functions. Similarly, polynomial functions where *all* the powers are odd are themselves odd functions. Indeed, this is the origin of the terms 'even function' and 'odd function'.

Exercise 7

(a) Is the function $C_0(t)$, given by $C_0(t) = 1$ for $-\tau/2 \leq t \leq \tau/2$, even, odd or neither?

(b) Is the function $h(t)$, defined by $h(t) = t^2 + t^3$ for all t, even, odd or neither?

The way that even and odd functions combine is similar to the way that positive and negative numbers combine. That is:

- the sum of two even functions (positive numbers) is even (positive)
- the sum of two odd functions (negative numbers) is odd (negative)
- the sum of an even function (positive number) and an odd function (negative number) is neither even nor odd (positive, negative or zero)
- the product of two even functions (positive numbers) is even (positive)
- the product of two odd functions (negative numbers) is even (positive)
- the product of an even function (positive number) and an odd function (negative number) is odd (negative).

In the next example and exercise, we use the first two properties in the list above to demonstrate the last two properties.

Example 3

If $f(t) = t^3 + 2t^5$ and $g(t) = t - t^3$, show that the function defined by $h(t) = f(t)\,g(t)$ is an even function.

Solution

Calculating explicitly,

$$\begin{aligned} h(t) = f(t)\,g(t) &= (t^3 + 2t^5)(t - t^3) \\ &= t^4 - t^6 + 2t^6 - 2t^8 \\ &= t^4 + t^6 - 2t^8. \end{aligned}$$

This is a polynomial where all the powers are even, therefore $h(t)$ is an even function.

Alternatively, since both $f(t)$ and $g(t)$ are odd functions, we know by definition that

$$f(-t) = -f(t), \quad g(-t) = -g(t).$$

Hence

$$h(-t) = f(-t)\,g(-t) = (-f(t))(-g(t)) = f(t)\,g(t) = h(t),$$

so $h(t)$ is an even function.

Exercise 8

If $f(t) = t^3 + 2t^5$ and $g(t) = 3t^2 - t^4$, show that the function defined by $h(t) = f(t)\,g(t)$ is an odd function.

For much of this subsection we have been concerned with even and odd functions that are not periodic. If a function $f(t)$ *is* periodic, of period $2a$, then an advantage of choosing a fundamental interval $[-a, a]$ centred on the origin is that we can tell whether $f(t)$ is even, odd or neither by seeing whether it is even, odd or neither on $[-a, a]$.

Even and odd periodic functions

Let $f(t)$ be periodic of period $2a$. Then $f(t)$ is even provided that it is even over the interval $[-a, a]$. Similarly, $f(t)$ is odd provided that it is odd over the interval $[-a, a]$.

Now we investigate the properties of odd and even functions that will simplify later calculations, namely what happens when these functions are integrated over the fundamental interval $[-a, a]$.

For an odd function $f(t)$, the integral is zero because the integral for positive t-values is exactly cancelled by the integral for negative t-values. This is illustrated in Figure 9 and is proved by the following argument. Calling the integral I and splitting it into two halves gives

$$I = \int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt.$$

Now use the rule that changing the order of integration changes the sign of the integral to obtain

$$I = -\int_{0}^{-a} f(t)\,dt + \int_{0}^{a} f(t)\,dt.$$

We can use the substitution $u = -t$ to yield

$$I = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(t)\,dt.$$

Finally, use the fact that f is odd, so $f(-u) = -f(u)$, to get

$$I = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(t)\,dt = 0,$$

where the final equality follows because the two integrals are the same integral written with different integration variables.

If f is an even function, then the argument above can be applied up until the penultimate step, then instead of terms cancelling, the result will be twice the integral from 0 to a.

These results are worth remembering as they can save a lot of effort when evaluating integrals of even and odd functions.

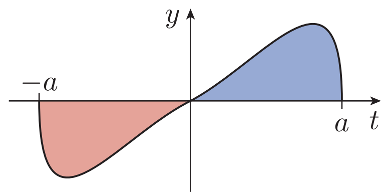


Figure 9 An odd function with area above the horizontal axis shaded blue and area below shaded red. By symmetry, the two areas are equal in size, so the integral is zero (as the red area is counted as negative).

Integrals of even and odd periodic functions

$$\int_{-a}^{a} g(t)\,dt = 2\int_{0}^{a} g(t)\,dt \quad \text{if } g \text{ is an even function.} \tag{7}$$

$$\int_{-a}^{a} f(t)\,dt = 0 \quad \text{if } f \text{ is an odd function.} \tag{8}$$

From Figure 7, you can see that the cosine function is even, but the sine function is odd. The suggestion is that to approximate an even function (such as the sawtooth function or the square-wave function) as a sum of sinusoidal terms, we ought to ensure that the approximating function is even. The only way to do this is to ensure that only cosine functions appear in the sum, as you will see in the next section.

2 Fourier series for even functions with period 2π

In this section you will see how to obtain the Fourier series for the sawtooth and square-wave functions, and in doing this you will obtain general formulas that can be applied to find the Fourier series for any even function with period 2π. We start with this special case as it is the simplest. However, even though it is a special case, you will see later that the arguments used to find the Fourier series also apply in the general case.

2.1 A series of approximations

In Section 1 you saw that there are certain useful things that can be said about the series $G(t)$ in equation (4), namely that it is periodic of period 2π and that it is an even function. But this does not tell us why this particular series corresponds to the sawtooth function $g(t)$ described in the Introduction. In this section we derive the Fourier series for the sawtooth function $g(t)$ by a mathematical argument, and we ask you to do the same for the square-wave function.

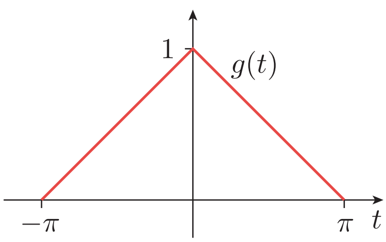


Figure 10 Graph of the sawtooth function $g(t)$

The graph of $g(t)$ (see Figure 10) coincides with the line $g(t) = 1 + t/\pi$ in the range $-\pi \le t < 0$ and with the line $g(t) = 1 - t/\pi$ in the range $0 \le t \le \pi$. The interval $[-\pi, \pi]$ is a fundamental interval for $g(t)$, on which it is defined as

$$g(t) = \begin{cases} \dfrac{1}{\pi}t + 1 & \text{for } -\pi \le t < 0, \\ -\dfrac{1}{\pi}t + 1 & \text{for } 0 \le t \le \pi. \end{cases} \tag{9}$$

Exercise 9

What are the period and angular frequency of the sawtooth function $g(t)$?

It seems reasonable that an even periodic function will have a Fourier series consisting of only even sinusoidal terms with the same period. This is indeed the case, as we will justify later. Here, we proceed to use a family of even sinusoidal functions, namely the family of cosine functions $\{C_0(t), C_1(t), C_2(t), \ldots\}$ that you met earlier:

$$C_n(t) = \cos nt \quad (n = 0, 1, 2, \ldots).$$

We consider a series

$$\begin{aligned} G(t) &= A_0\,C_0(t) + A_1\,C_1(t) + A_2\,C_2(t) + A_3\,C_3(t) + \cdots \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos nt, \end{aligned} \tag{10}$$

We usually write the sum in this way with the constant term singled out because this form is easier to use in calculations.

and ask how the **Fourier coefficients** A_n can be chosen so that as we add successive terms of the series to obtain the approximations

$$\begin{aligned} G_0(t) &= A_0, \\ G_1(t) &= A_0 + A_1 \cos nt, \\ G_2(t) &= A_0 + A_1 \cos nt + A_2 \cos(2nt), \end{aligned}$$

and so on, the values approach $g(t)$ for any chosen value of t.

So the central problem is: how can A_n be chosen so that the approximation to $g(t)$ given by $G_n(t)$ gets better as n increases? We can also consider what happens when infinitely many terms are added to the approximation, so that we obtain the function $G(t)$. Can we choose the coefficients A_n so that $G(t) = g(t)$ for all t?

The argument that we use is quite general, in that it works for *any* even periodic function with fundamental interval $[-\pi, \pi]$. Thus for the remainder of this section, we will use the notation $f(t)$ to refer to a general such function, and

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nt \tag{11}$$

for the corresponding series whose coefficients $A_0, A_1, A_2, \ldots$ we are trying to find. We will then apply the general argument to the sawtooth function (in Examples 4–6), and ask you to apply it to other functions, including the square-wave function (in Exercises 11, 13 and 14).

We begin by deriving the first of the approximations listed above, namely $G_0(t)$.

2.2 A first approximation

The easiest coefficient to find is A_0. The technique is based on the observation that all the functions $C_n(t)$, some of which are illustrated in Figure 6 and the solution to Exercise 4, oscillate, and the positive contributions to an integral over the fundamental interval exactly cancel out the negative contributions. This means that if you integrate the cosine functions over the fundamental interval $[-\pi, \pi]$, then you obtain 0; that is,

$$\int_{-\pi}^{\pi} \cos nt\, dt = 0 \quad (n = 1, 2, 3, \ldots). \tag{12}$$

Exercise 10

Verify that the integral in formula (12) is zero (for each $n = 1, 2, 3, \ldots$).

In this module we assume the validity of term-by-term integration and differentiation of infinite series. The process is valid in all the practical cases that concern us.

If we integrate both sides of equation (11) term by term over the fundamental interval $[-\pi, \pi]$, then we find that

$$\begin{aligned}\int_{-\pi}^{\pi} F(t)\, dt &= \int_{-\pi}^{\pi} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos nt \right) dt \\ &= \int_{-\pi}^{\pi} A_0\, dt + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos nt\, dt.\end{aligned}$$

Now all the terms involving integrals of cosine functions vanish, by formula (12), leaving us with

$$\int_{-\pi}^{\pi} F(t)\, dt = \int_{-\pi}^{\pi} A_0\, dt = 2\pi A_0 \quad (\text{since } A_0 \text{ is a constant}).$$

Hence

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)\, dt.$$

We do not know the coefficients of $F(t)$, but our aim is to ensure that $F(t) = f(t)$. So to determine A_0, we replace $F(t)$ by $f(t)$ in the above equation to obtain the following result.

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt. \tag{13}$$

You can think of A_0 as the average value taken by the function $f(t)$.

Example 4

Find the value of A_0 when the general function $f(t)$ is replaced by the sawtooth function $g(t)$ given by equation (9).

Solution

Equation (13) becomes

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,dt.$$

As the function $g(t)$ is even, equation (7) applies and the integral is equal to twice the integral over the positive t-values:

$$A_0 = \frac{1}{\pi}\int_{0}^{\pi} g(t)\,dt.$$

Using this fact simplifies the calculation as $g(t)$ is defined piecewise, with different formulas for positive and negative values; here we need to consider only one of the formulas.

The function $g(t)$ is given by equation (9), so the above integral is

$$\frac{1}{\pi}\int_0^{\pi} g(t)\,dt = \frac{1}{\pi}\int_0^{\pi}\left(-\frac{1}{\pi}t+1\right)dt = \frac{1}{\pi}\left[-\frac{1}{2\pi}t^2+t\right]_0^{\pi} = \frac{1}{2}.$$

Thus for the case of the sawtooth function,

$$A_0 = \tfrac{1}{2}.$$

The argument used to simplify the first step of the calculation in Example 4 is a general argument that applies whenever the integrand is an even function, and it could also be applied to equation (13) to obtain a simplified formula. But we will not do this here because, as we will see later, equation (13) is the formula that applies in general (not just to even functions).

Figure 11 shows graphs of the original function $g(t)$ and the first approximation $G_0(t) = A_0 = \frac{1}{2}$. You can see from the figure that $A_0 = \frac{1}{2}$ is the average value of the function $g(t)$.

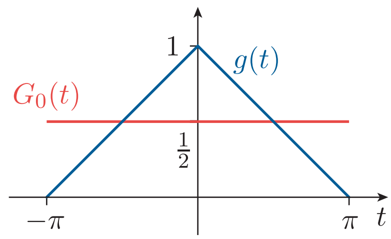


Figure 11 Approximation $G_0(t)$ compared to $g(t)$

Exercise 11

Suppose that the general function $f(t)$ is now replaced by the square-wave function $h(t)$ of the Introduction, which can be defined on the fundamental interval $[-\pi, \pi]$ by

$$h(t) = \begin{cases} 1 & \text{for } -\frac{\pi}{2} \le t \le \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of the Fourier coefficient A_0.

Hint: As the function $h(t)$ is zero over some of the fundamental interval,

$$\int_{-\pi}^{\pi} h(t)\,dt = \int_{-\pi/2}^{\pi/2} h(t)\,dt.$$

Integration enabled us to eliminate the coefficients A_n $(n > 0)$ and hence to find the coefficient A_0. The next subsection looks at how we can find the coefficient A_1 and hence find a better approximation to $g(t)$.

2.3 A second approximation

To find the next coefficient, A_1, we must somehow eliminate A_0 and all the other coefficients. The technique is to multiply both sides of equation (11) by the term $\cos t$ to give

$$F(t)\cos t = A_0\cos t + \sum_{n=1}^{\infty} A_n \cos nt\, \cos t. \tag{14}$$

If we integrate both sides of equation (14) term by term over the fundamental interval $[-\pi, \pi]$, then we find that

$$\int_{-\pi}^{\pi} F(t)\cos t\, dt = \int_{-\pi}^{\pi} A_0 \cos t\, dt + \sum_{n=1}^{\infty}\left(A_n \int_{-\pi}^{\pi} \cos nt \cos t\, dt\right). \tag{15}$$

Now we evaluate each of these integrals separately.

Exercise 12

(a) Show that

$$\int_{-\pi}^{\pi} A_0 \cos t\, dt = 0.$$

(b) Use the trigonometric identity

$$\cos\alpha\cos\beta = \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta)$$

to show that

$$\int_{-\pi}^{\pi} \cos 2t \cos t\, dt = 0.$$

(c) More generally, use the identity given in part (b) to show that

$$\int_{-\pi}^{\pi} \cos nt \cos t\, dt = 0 \quad \text{when } n \text{ is an integer and } n > 1.$$

(d) Use the trigonometric identity

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

to evaluate the integral

$$\int_{-\pi}^{\pi} \cos^2 t\, dt.$$

Exercise 12 shows that most of the integrals on the right-hand side of equation (15) evaluate to zero. The only remaining term involves the coefficient A_1, and we are left with

$$\int_{-\pi}^{\pi} F(t)\cos t\, dt = A_1 \int_{-\pi}^{\pi} \cos^2 t\, dt = A_1\pi.$$

Hence the coefficient A_1 is given by

$$A_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos t\, dt.$$

We are trying to choose the coefficients so that $F(t) = f(t)$, so we replace $F(t)$ by $f(t)$ and obtain the following result.

$$A_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t\,dt. \tag{16}$$

Before applying this formula to calculate A_1 for the sawtooth function, we state two useful integrals that often arise when calculating Fourier series.

Two useful integrals

For a a non-zero constant and C a constant,

$$\int t\sin(at)\,dt = \frac{1}{a^2}\big(\sin(at) - at\cos(at)\big) + C, \tag{17}$$

$$\int t\cos(at)\,dt = \frac{1}{a^2}\big(\cos(at) + at\sin(at)\big) + C. \tag{18}$$

Both of these integrals are easy to derive using integration by parts, but it is quicker to state the standard result. One of these integrals will be used in the calculation of A_1 for the sawtooth function in the next example.

Example 5

Returning to the sawtooth function $g(t)$ defined by equation (9), find the value of the coefficient A_1.

Solution

The coefficient is given by

$$A_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos t\,dt.$$

As $g(t)$ is even and $\cos t$ is even, the product $g(t)\cos t$ is even, and we may make use of equation (7) to simplify the calculation by evaluating twice the integral over the positive t-values:

$$A_1 = \frac{2}{\pi}\int_{0}^{\pi} g(t)\cos t\,dt.$$

Substituting the definition of $g(t)$ from equation (9) gives

$$\begin{aligned} A_1 &= \frac{2}{\pi}\int_{0}^{\pi} \left(-\frac{1}{\pi}t + 1\right)\cos t\,dt \\ &= -\frac{2}{\pi^2}\int_{0}^{\pi} t\cos t\,dt + \frac{2}{\pi}\int_{0}^{\pi}\cos t\,dt. \end{aligned}$$

The first integral is one of the two useful integrals (equation (18) with $a = 1$) that were stated immediately preceding this example, so performing the integrations yields

$$\begin{aligned} A_1 &= -\frac{2}{\pi^2}\left[\cos t + t\sin t\right]_0^\pi + \frac{2}{\pi}\left[\sin t\right]_0^\pi \\ &= -\frac{2}{\pi^2}(-1-1) = \frac{4}{\pi^2}. \end{aligned}$$

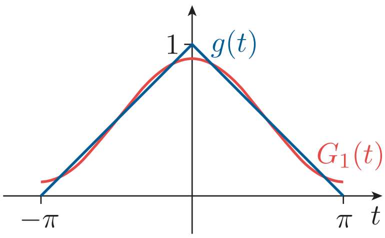


Figure 12 Approximation $G_1(t)$ compared to $g(t)$

So our second approximation to the function $g(t)$ (our first non-constant approximation) is

$$G_1(t) = \frac{1}{2} + \frac{4}{\pi^2}\cos t.$$

The graph of this approximation is compared with the graph of $g(t)$ in Figure 12. Already $G_1(t)$ is quite a reasonable approximation to $g(t)$.

Exercise 13

Suppose that $f(t)$ in equation (16) is replaced by the square-wave function $h(t)$ of the Introduction, defined in Exercise 11. Find the value of the coefficient A_1.

2.4 Better approximations

The coefficient A_n can be found in a similar way to the way in which we found A_1, by multiplying by an appropriate sinusoidal function and integrating. In order to do this, we need to evaluate many integrals, but like those in Exercise 12, most vanish. Generalising the results of that exercise, we find the following.

Trigonometric integrals over the interval $[-\pi, \pi]$

For any positive integers m and n,

$$\int_{-\pi}^{\pi} \cos mt\, dt = 0, \tag{19}$$

$$\int_{-\pi}^{\pi} \cos mt \cos nt\, dt = 0 \quad (m \neq n), \tag{20}$$

$$\int_{-\pi}^{\pi} \cos^2 mt\, dt = \pi. \tag{21}$$

These results mean that if we multiply both sides of equation (11) by $\cos mt$ and integrate over the fundamental interval, then all of the coefficients except A_m disappear. To see this, multiply both sides of equation (11) by $\cos mt$, to obtain

$$F(t)\cos mt = A_0 \cos mt + \sum_{n=1}^{\infty} A_n \cos mt\, \cos nt.$$

Then integration gives

$$\int_{-\pi}^{\pi} F(t)\cos mt\, dt = \int_{-\pi}^{\pi} A_0 \cos mt\, dt + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos mt \cos nt\, dt,$$

and using formulas (19) and (20), we find that all of the terms of the above infinite sum are zero except when $n = m$, so we get

$$\int_{-\pi}^{\pi} F(t)\cos mt\, dt = A_m \int_{-\pi}^{\pi} \cos^2 mt\, dt.$$

Finally, formula (21) gives the value of the right-hand integral as π, so

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos mt\, dt.$$

This key result is worth remembering, so we re-state it in the form with the original function that we are trying to approximate, $f(t)$, instead of $F(t)$, and the usual index n instead of m.

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \quad (n > 0). \tag{22}$$

Example 6

Returning once again to the sawtooth function $g(t)$ as defined by equation (9), find the values of the coefficients A_2 and A_3.

Solution

Substituting $g(t)$ into equation (22) gives

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\, dt.$$

Now we use the fact that the integrand is even, since it is the product of two even functions, $g(t)$ and $\cos nt$, to obtain

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_{0}^{\pi} \left(-\frac{1}{\pi}t + 1\right)\cos nt\, dt \\ &= -\frac{2}{\pi^2}\int_{0}^{\pi} t\cos nt\, dt + \frac{2}{\pi}\int_{0}^{\pi} \cos nt\, dt, \end{aligned}$$

where we have substituted for $g(t)$ using the definition in equation (9).

This integral can be evaluated by recognising the first integral as one of the two useful integrals (equation (18) with $a = n$), to give

$$A_n = -\frac{2}{\pi^2}\left[\frac{1}{n^2}(\cos nt + nt\sin nt)\right]_0^\pi + \frac{2}{\pi}\left[\frac{1}{n}\sin nt\right]_0^\pi$$
$$= -\frac{2}{\pi^2}\left(\frac{1}{n^2}((-1)^n - 1)\right),$$

since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, so

$$A_n = \frac{2}{n^2\pi^2}(1-(-1)^n).$$

If $n = 2$, then $(-1)^n = 1$, so $A_2 = 0$.

If $n = 3$, then $(-1)^n = -1$, so $A_3 = 4/(9\pi^2)$.

From Example 6 we have $A_2 = 0$, so the approximation $G_2(t)$ is equal to $G_1(t)$ and hence is no better as an approximation to $g(t)$. However, from Examples 4, 5 and 6, we can derive a better approximation to $g(t)$ by writing

$$G_3(t) = A_0 + A_1\cos t + A_2\cos 2t + A_3\cos 3t$$
$$= \frac{1}{2} + \frac{4}{\pi^2}\left(\cos t + \frac{1}{9}\cos 3t\right).$$

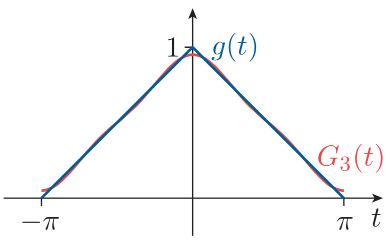


Figure 13 Approximation $G_3(t)$ compared to $g(t)$

The graph of $G_3(t)$ is shown in Figure 13. This shows a further improvement in the accuracy of the approximation, to the extent that on this scale it is hard to see a difference between the graphs of $G_3(t)$ and $g(t)$.

Exercise 14

Suppose that $f(t)$ in formula (22) is replaced by the square-wave function $h(t)$, defined in Exercise 11. Find the values of the coefficients A_2 and A_3.

Examples 4, 5 and 6 can be generalised to find all the coefficients in the Fourier series for the sawtooth function $g(t)$. We find that

$$A_0 = \frac{1}{2},\quad A_1 = \frac{4}{\pi^2},\quad A_2 = 0,\quad A_3 = \frac{4}{\pi^2}\times\frac{1}{9},\quad A_4 = 0,$$
$$A_5 = \frac{4}{\pi^2}\times\frac{1}{25},\quad A_6 = 0,\quad A_7 = \frac{4}{\pi^2}\times\frac{1}{49},\quad A_8 = 0,\quad \ldots,$$

and a clear pattern has appeared. Hence the Fourier series for the sawtooth function $g(t)$ is

$$G(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \frac{1}{49}\cos 7t + \cdots\right),$$

confirming equation (2).

The terms of this Fourier series can be written more compactly using the sigma notation for summations. The angular frequencies of the terms form a pattern of successive odd numbers. The coefficient of each term in the brackets is the reciprocal of the square of the angular frequency. Recall that if $n = 1, 2, 3, \ldots$, then $2n = 2, 4, 6, \ldots$ runs through the even numbers and $2n - 1 = 1, 3, 5, \ldots$ runs through the odd numbers. With these observations, the Fourier series can be written as

$$G(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)t.$$

It is sometimes more convenient to write Fourier series in so-called **closed form** like this, using the summation symbol $\sum$. Now try doing this yourself.

Exercise 15

You have found, in Exercises 11, 13 and 14, the Fourier coefficients A_0, A_1, A_2 and A_3 for the square-wave function $h(t)$ defined in Exercise 11. In fact, as indicated in equation (3), its Fourier series is

$$H(t) = \frac{1}{2} + \frac{2}{\pi} \cos t - \frac{2}{3\pi} \cos 3t + \frac{2}{5\pi} \cos 5t - \frac{2}{7\pi} \cos 7t + \cdots .$$

Write down this series in closed form.

The following exercises ask you to apply the formulas derived in this section to find Fourier series for other even periodic functions.

Exercise 16

Find the Fourier series for the even periodic function $f(t)$ defined on the fundamental interval $[-\pi, \pi]$ by

$$f(t) = t^2.$$

(*Hint*: You may find the following integral obtained by integration by parts useful:

$$\int t^2 \cos nt \, dt = \frac{1}{n^3} \big((n^2t^2 - 2) \sin nt + 2nt \cos nt\big) + C,$$

where C is a constant.)

Exercise 17

A variant of the sawtooth function can be defined on the fundamental interval $[-\pi, \pi]$ by

$$w(t) = |t|.$$

Recall that the absolute value function is defined by

$$|t| = \begin{cases} t & \text{for } t \geq 0, \\ -t & \text{for } t < 0. \end{cases}$$

Find the Fourier series for this function, and write down the approximation $W_5(t)$.

2.5 Convergence

You have just tackled a substantial piece of work, and this has involved finding Fourier series for several even functions. Having found them, you need to take stock of what you have done. You have seen that just a few terms of the Fourier series for the sawtooth function $g(t)$ give a very good approximation. However, the first few terms of the Fourier series for the square-wave function $h(t)$ do not give a particularly good approximation, as Figure 14 illustrates.

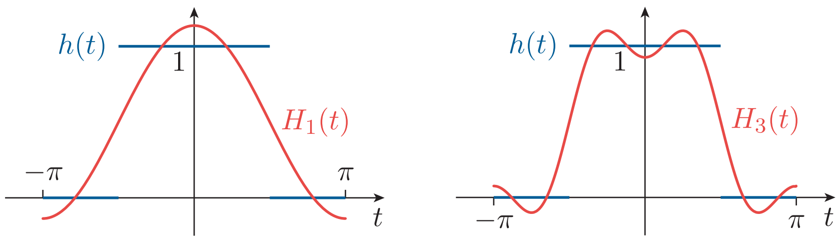


Figure 14 The first two approximations (red lines) to the square-wave function (blue line) by its Fourier series

The situation improves only slowly for the square-wave function. Plotting the graphs for the sums as far as the $\cos 7t$, $\cos 11t$ and $\cos 21t$ terms, better approximations to $h(t)$ are obtained, as expected (see Figure 15).

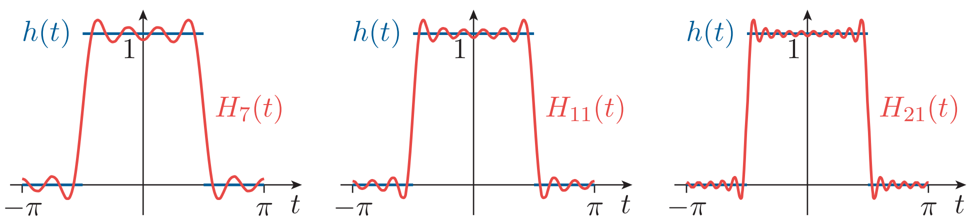


Figure 15 More approximations to the square-wave function by its Fourier series

However, even $H_{21}(t)$ does not approximate $h(t)$ as well as $G_3(t)$ does $g(t)$ in Figure 13. This is because of the discontinuities in $h(t)$. We cannot reasonably expect the sum of continuous sinusoidal functions to provide a good approximation to a discontinuous function. From the graphs, you can see that the approximations to $h(t)$ are worse near the discontinuities, that is, near the points where the value of $h(t)$ jumps from 0 to 1 and back again. Nevertheless, even for a discontinuous function such as $h(t)$, we can, remarkably, approximate reasonably well using Fourier series. At a discontinuity, the Fourier series takes the average value of the function at either side of the discontinuity. This is formally stated in the following theorem (which we do not prove) that guarantees the nature of the Fourier series for a wide class of functions at points in the fundamental interval.

Theorem 1 Pointwise convergence

If, on the interval $[-\pi, \pi]$, the function f has a continuous derivative except at a finite number of points, then at each point $x_0 \in [-\pi, \pi]$, the Fourier series for f converges to

$$\tfrac{1}{2}\left(f(x_0^+) + f(x_0^-)\right).$$

Here $f(x_0^+)$ is the limit of $f(x)$ as x approaches x_0 from above, and $f(x_0^-)$ is the limit of $f(x)$ as x approaches x_0 from below.

It is worth remarking that if f is continuous at x_0, then $f(x_0^-) = f(x_0^+) = f(x_0)$, and in this case the theorem states that the Fourier series converges to $f(x_0)$.

The following example shows how Theorem 1 is used to determine the values to which Fourier series converge.

Example 7

Consider the function

$$f(t) = \begin{cases} -1 & \text{for } -1 \le t < 0, \\ t & \text{for } 0 \le t < 1, \end{cases}$$

$$f(t+2) = f(t),$$

and its corresponding Fourier series $F(t)$.

(a) Calculate $f(-1)$, $f(\frac{1}{2})$ and $f(2)$.

(b) Calculate $F(-1)$, $F(\frac{1}{2})$ and $F(2)$.

(c) Compare the values obtained in parts (a) and (b).

Solution

(a) The first step in solving problems such as this is to draw a sketch graph, such as the one shown in Figure 16.

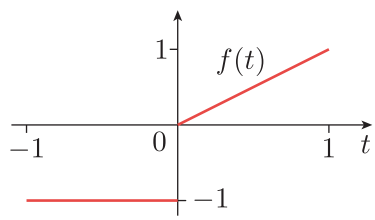


Figure 16 Sketch graph of $f(t)$ on its fundamental interval

Using the sketch as a guide, we can calculate

$$f(-1) = -1,$$
$$f(\tfrac{1}{2}) = \tfrac{1}{2},$$
$$f(2) = f(0) = 0.$$

(b) The pointwise convergence theorem gives

$$F(-1) = \frac{f(-1^+) + f(-1^-)}{2}.$$

Approaching $t = -1$ from above is included in the fundamental interval of f, so we can say that $f(-1^+) = -1$.

Approaching $t = -1$ from below is not in the fundamental interval, so we use the periodicity of f to say that the value is the same as approaching $t = 1$ from below, that is, $f(-1^-) = f(1^-)$. So from the sketch we have $f(1^-) = 1$.

Substituting these values into the formula gives

$$F(-1) = \frac{(-1) + 1}{2} = 0.$$

At $t = \frac{1}{2}$ the function $f(t)$ is continuous, so the Fourier series will be equal to the given function here, hence

$$F(\tfrac{1}{2}) = f(\tfrac{1}{2}) = \tfrac{1}{2}.$$

As F is also periodic with period 2, we have $F(2) = F(0)$. So

$$F(2) = F(0) = \frac{f(0^+) + f(0^-)}{2}.$$

Now, using the sketch of the function as a guide, we obtain that approaching $t = 0$ from above gives $f(0^+) = 0$ and approaching $t = 0$ from below gives $f(0^-) = -1$.

Substituting these values into the formula gives

$$F(2) = \frac{0 + (-1)}{2} = -\tfrac{1}{2}.$$

(c) We have $f(\frac{1}{2}) = F(\frac{1}{2})$, which is to be expected as f is continuous at $t = \frac{1}{2}$.

If f is not continuous at a point, then the Fourier series does not necessarily converge to the value of the function. This is the case for the other two points considered here, where we have found that $f(-1) \neq F(-1)$ and $f(2) \neq F(2)$. Both of these points are points of discontinuity of f.

Now try the following exercise in applying the pointwise convergence theorem.

Exercise 18

Consider the function

$$f(t) = \frac{|t| + t}{2} \quad \text{for } -1 \leq t < 1,$$
$$f(t+2) = f(t),$$

and its corresponding Fourier series $F(t)$. Calculate the values $F(-1)$ and $F(0)$.

3 Fourier series for even and odd periodic functions

In the previous section we concentrated on finding the Fourier series for two particular even periodic functions with period $\tau = 2\pi$ and hence fundamental interval $[-\pi, \pi]$. In this section we extend the technique to periodic functions that are either even or odd and have any fixed period τ.

3.1 Fourier series for even functions with period τ

We will now examine a generalisation of the function $w(t)$ described in Exercise 17 that has period τ, where τ is a specific positive number. The graph of $w(t)$ repeats along the t-axis and looks like a sawtooth function (see Figure 17).

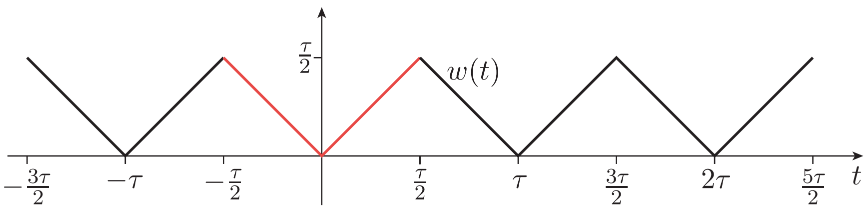


Figure 17 Graph of the function $w(t)$ over four periods, with the fundamental interval highlighted in red

If the function is defined to have period τ (Exercise 17 considered the special case $\tau = 2\pi$), then a fundamental interval is $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$, on which $w(t)$ is shown as the red curve in Figure 17. On this interval the function is given by $w(t) = |t|$ or

$$w(t) = \begin{cases} -t & \text{for } -\frac{\tau}{2} \leq t < 0, \\ t & \text{for } 0 \leq t < \frac{\tau}{2}. \end{cases} \tag{23}$$

You saw in Subsection 1.1 that when we are trying to find a Fourier series for a function with period τ, we must also consider functions with the shorter periods

$$\frac{\tau}{2}, \frac{\tau}{3}, \frac{\tau}{4}, \frac{\tau}{5}, \ldots .$$

Corresponding to the fundamental period τ and to these shorter periods are the angular frequencies

$$\frac{2\pi}{\tau}, \frac{4\pi}{\tau}, \frac{6\pi}{\tau}, \frac{8\pi}{\tau}, \frac{10\pi}{\tau}, \ldots .$$

Hence we consider the family of even functions

$$C_n(t) = \cos\left(\frac{2n\pi t}{\tau}\right) \quad (n = 1, 2, 3, \ldots).$$

Since we are dealing with even functions, we also include the constant function

$$C_0(t) = 1.$$

As in Section 2, the argument used here is a general one. Thus for the remainder of this subsection, we use the symbol $f(t)$ to refer to a general even periodic function with fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$. As before, we assume that we can choose the coefficients in an infinite sum of these functions $C_n(t)$ in such a way that we can approximate the original function as accurately as required. We write this (as before) as

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right). \tag{24}$$

To find the coefficients in this sum, we need to evaluate integrals of the cosine functions that generalise equations (19)–(21) used in the previous section. The integrals in which we are interested are for functions defined over the fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$, which are obtained from the former integrals by the substitution $u = \tau t/(2\pi)$.

Trigonometric integrals over the interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$

For any positive integers m and n,

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2m\pi t}{\tau}\right) dt = 0, \tag{25}$$

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2m\pi t}{\tau}\right) \cos\left(\frac{2n\pi t}{\tau}\right) dt = 0 \quad (m \neq n), \tag{26}$$

$$\int_{-\tau/2}^{\tau/2} \cos^2\left(\frac{2m\pi t}{\tau}\right) dt = \frac{\tau}{2}. \tag{27}$$

To find the coefficients in Fourier series (24), we proceed as before. We first multiply both sides of equation (24) by a chosen cosine function. Then we integrate, and all but one of the coefficients become zero. We are left with a formula for that remaining coefficient.

First, to find the constant A_0, we integrate both sides of equation (24) over the fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$ to obtain

$$\int_{-\tau/2}^{\tau/2} F(t)\, dt = \int_{-\tau/2}^{\tau/2} A_0\, dt + \sum_{n=1}^{\infty} A_n \int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right) dt.$$

Using formula (25), all the integrals in the infinite sum become zero, leaving

$$\int_{-\tau/2}^{\tau/2} F(t)\, dt = \int_{-\tau/2}^{\tau/2} A_0\, dt = \tau A_0,$$

so

$$A_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} F(t)\, dt.$$

As in Section 2, we now use the fact that we wish to choose the coefficients so that $F(t) = f(t)$. Thus we put $F(t) = f(t)$ to give the following result.

$$A_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\, dt. \tag{28}$$

The constant A_0 can again be thought of as the average value of the function $f(t)$ on the fundamental interval.

The derivation of a formula for the coefficient A_n for functions with period τ is very similar to the derivation of equation (22) for functions with period 2π. We will not repeat the argument; we simply state the result.

$$A_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos\left(\frac{2n\pi t}{\tau}\right) dt. \tag{29}$$

Now apply these results to do the following exercise.

Exercise 19

For the sawtooth function $w(t)$ defined by equation (23), find the coefficients A_0 and A_n.

Substituting the coefficients that you have found in Exercise 19 into equation (24) gives the Fourier series $W(t)$ corresponding to the sawtooth function $w(t)$:

$$\begin{aligned} W(t) &= \frac{\tau}{4} - \frac{\tau}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{2n\pi t}{\tau}\right) \\ &= \frac{\tau}{4} - \frac{\tau}{\pi^2} \left(2\cos\left(\frac{2\pi t}{\tau}\right) + \frac{2}{9}\cos\left(\frac{6\pi t}{\tau}\right)\right. \\ &\qquad\qquad \left. + \frac{2}{25}\cos\left(\frac{10\pi t}{\tau}\right) + \cdots\right). \end{aligned} \tag{30}$$

Since $w(t)$ is very similar in form to the sawtooth function that we investigated in Section 2, it should come as no surprise that the successive approximations to $w(t)$ generated by series (30) converge rapidly to $w(t)$. Figure 18 compares the graph of $w(t)$ with the graphs of the approximations

$$W_0(t) = \frac{\tau}{4} \quad \text{and} \quad W_1(t) = \frac{\tau}{4} - \frac{2\tau}{\pi^2} \cos\left(\frac{2\pi t}{\tau}\right).$$

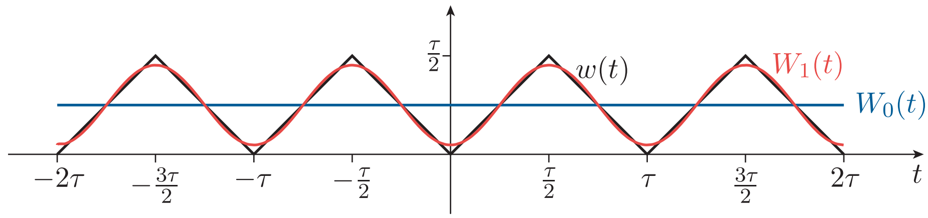


Figure 18 The sawtooth function $w(t)$ compared to the first two Fourier series approximations

3.2 Fourier series for odd functions with period τ

We have so far concentrated on even periodic functions, but it is equally straightforward to deal with odd periodic functions. As an example of an odd periodic function, consider the function $v(t)$ whose graph is shown in Figure 19. You can think of $v(t)$ as representing another type of sawtooth function, or as a broken surface made up of successive ramps and steps.

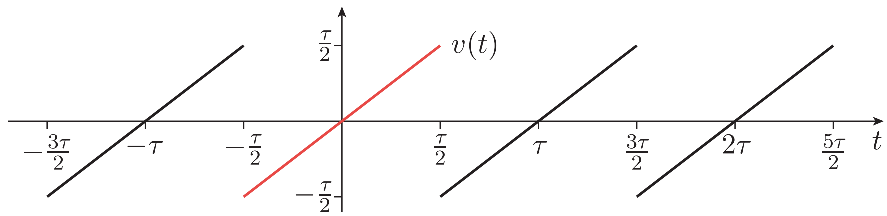


Figure 19 Graph of the function $v(t)$ with the fundamental interval highlighted in red

The function $v(t)$ is defined to be equal to t within the fundamental interval $-\tau/2 < t < \tau/2$, but what values should be assigned to the ends of the fundamental interval, that is, at $t = -\tau/2$ and $t = \tau/2$? Note that if the function has period τ, then $v(t) = v(t + \tau)$, so with $t = -\tau/2$ we have $v(-\tau/2) = v(\tau + (-\tau/2)) = v(\tau/2)$. For our purposes it makes no difference how $v(t)$ is defined on these points because we are only interested in integrals over the fundamental interval of $v(t)$, and changing values at a single point does not affect the integrals. As we are free to choose, we conventionally pick the value at the left of the interval and define $v(t)$ as

$$v(t) = t \quad \text{for } -\tau/2 \leq t < \tau/2, \tag{31}$$
$$v(t + \tau) = v(t).$$

The second line of this definition repeats the definition on the fundamental interval to make $v(t)$ have period τ.

The function $v(t)$ as defined above is essentially an odd function as it differs from an odd function only at the endpoints (an odd periodic function is zero at the endpoints). For our purposes this is close enough to being an odd function as changing a function at a single point does not change the value of an integral of the function.

In order to approximate the odd function $v(t)$, we need odd trigonometric functions with period τ. As usual, we must consider functions with periods

$$\tau,\ \frac{\tau}{2},\ \frac{\tau}{3},\ \frac{\tau}{4},\ \frac{\tau}{5},\ \ldots .$$

Corresponding to these periods are the angular frequencies

$$\frac{2\pi}{\tau},\ \frac{4\pi}{\tau},\ \frac{6\pi}{\tau},\ \frac{8\pi}{\tau},\ \frac{10\pi}{\tau},\ \ldots .$$

So we must consider the family of sine functions

$$S_n(t) = \sin\left(\frac{2n\pi t}{\tau}\right) \quad (n = 1, 2, 3, \ldots),$$

which is a family of odd functions.

In contrast to Subsection 3.1, where we considered the cosine series, we do not bother with the function $S_0(t) = \sin 0 = 0$, since any multiple of this function is zero.

Example 8

Sketch the graphs of the functions $S_1(t)$, $S_2(t)$, $S_3(t)$ and $S_4(t)$ on the fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$.

Solution

Sketches of the four functions are shown in Figure 20.

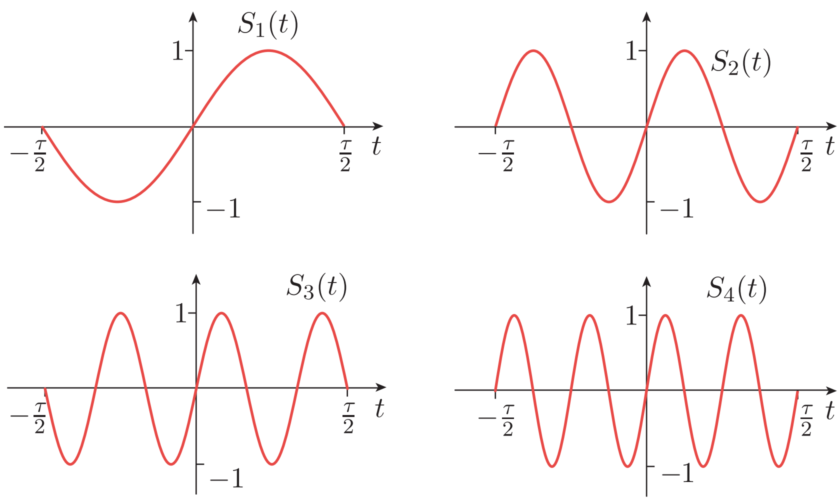


Figure 20 The first four members of a family of sine functions

As before, the argument used to find the coefficients in a Fourier series for an odd function is a general one. Thus for the remainder of this subsection we use the function $f(t)$ to refer to a general odd periodic function with fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$. Then we assume that we can choose the coefficients in an infinite sum of the functions $S_n(t)$ in such a way that we can approximate the original function $f(t)$ as accurately as required.

We write the sum as

$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right), \tag{32}$$

where the coefficients $B_1, B_2, B_3, \ldots$ are constants depending on the particular function $f(t)$. We will refer to equation (32) as the Fourier series for the odd function $f(t)$.

To find the coefficients B_n, we multiply both sides of equation (32) by the function $\sin(2m\pi t/\tau)$ and integrate over the fundamental interval to give

$$\begin{aligned}&\int_{-\tau/2}^{\tau/2} F(t)\sin\left(\frac{2m\pi t}{\tau}\right)dt\\ &= \sum_{n=1}^{\infty} B_n \int_{-\tau/2}^{\tau/2} \sin\left(\frac{2m\pi t}{\tau}\right)\sin\left(\frac{2n\pi t}{\tau}\right)dt.\end{aligned} \tag{33}$$

To simplify equation (33), we need evaluated integrals of sine functions analogous to those for the cosine functions in Subsection 3.1.

Trigonometric integrals over the interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$

For any positive integers m and n,

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2m\pi t}{\tau}\right)dt = 0, \tag{34}$$

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2m\pi t}{\tau}\right)\sin\left(\frac{2n\pi t}{\tau}\right)dt = 0 \quad (m \neq n), \tag{35}$$

$$\int_{-\tau/2}^{\tau/2} \sin^2\left(\frac{2m\pi t}{\tau}\right)dt = \frac{\tau}{2}. \tag{36}$$

Using formula (35), all the integrals on the right-hand side of equation (33) become zero except when $n = m$. That is,

$$\int_{-\tau/2}^{\tau/2} F(t)\sin\left(\frac{2m\pi t}{\tau}\right)dt = B_m \int_{-\tau/2}^{\tau/2} \sin^2\left(\frac{2m\pi t}{\tau}\right)dt.$$

Using formula (36), the right-hand side reduces to $B_m\tau/2$, and we can make B_m the subject (and as before replace $F(t)$ by the function $f(t)$ and rewrite the index as n instead of m).

$$B_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt. \tag{37}$$

Example 9

(a) Find the coefficients B_1, B_2 and B_3 for the function $v(t)$ defined in equation (31).

(b) Sketch the graph of

$$V_3(t) = B_1 \sin\left(\frac{2\pi t}{\tau}\right) + B_2 \sin\left(\frac{4\pi t}{\tau}\right) + B_3 \sin\left(\frac{6\pi t}{\tau}\right)$$

on the interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$, and compare it with the graph of $v(t)$.

Solution

(a) The coefficients B_n are given by formula (37) with $f(t) = t$:

$$B_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} t \sin\left(\frac{2n\pi t}{\tau}\right) dt.$$

As the function $f(t) = t$ is odd and the sine term is odd, the product is even, so we can use equation (7) to rewrite the integral as twice the sum over the positive t-values:

$$B_n = \frac{4}{\tau} \int_{0}^{\tau/2} t \sin\left(\frac{2n\pi t}{\tau}\right) dt.$$

This integral is one of the two useful integrals (equation (17) with $a = 2n\pi/\tau$), so

$$\begin{aligned}
B_n &= \frac{4}{\tau} \left[\frac{\tau^2}{4n^2\pi^2} \left(\sin\left(\frac{2n\pi t}{\tau}\right) - \frac{2n\pi t}{\tau} \cos\left(\frac{2n\pi t}{\tau}\right)\right)\right]_0^{\tau/2} \\
&= \frac{\tau}{n^2\pi^2} \left[\sin\left(\frac{2n\pi t}{\tau}\right) - \frac{2n\pi t}{\tau} \cos\left(\frac{2n\pi t}{\tau}\right)\right]_0^{\tau/2} \\
&= \frac{\tau}{n^2\pi^2}(-n\pi \cos n\pi) \\
&= -\frac{\tau(-1)^n}{n\pi}.
\end{aligned}$$

So $B_1 = \tau/\pi$, $B_2 = -\tau/(2\pi)$ and $B_3 = \tau/(3\pi)$.

(b) Substituting the coefficients into the given equation and simplifying gives

$$V_3(t) = \frac{\tau}{\pi} \left[\sin\left(\frac{2\pi t}{\tau}\right) - \frac{1}{2} \sin\left(\frac{4\pi t}{\tau}\right) + \frac{1}{3} \sin\left(\frac{6\pi t}{\tau}\right)\right].$$

The graphs of $v(t)$ and $V_3(t)$ on $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$ are shown in Figure 21.

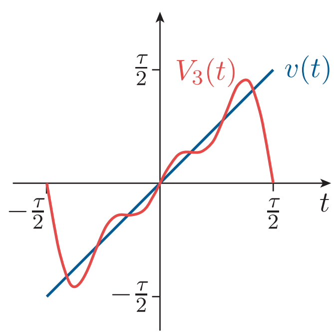


Figure 21 Graphs of $v(t)$ and $V_3(t)$

The Fourier series for the ramp function $v(t)$ is thus

$$V(t) = -\frac{\tau}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{2n\pi t}{\tau}\right).$$

Our sketch in Figure 21 of the sum of the first three terms gives an approximation to the function, but it is not as good as the corresponding approximation to the sawtooth function that you met in the previous subsection. As in the case of the square-wave function studied in Subsection 2.5, this is due to the discontinuities in the original function. At these points, the Fourier series takes the average value in the middle of the jump. Here that value is 0. In the case of a continuous function, the sizes of the coefficients in the Fourier series generally have an n^2 factor in the denominator and so decrease quite rapidly (such as in equation (30)). By contrast, here and for the square-wave function, where there are discontinuities, the sizes of the coefficients of the Fourier series have only a factor n in the denominator and so the coefficients decrease more slowly. This is a general phenomenon: functions with discontinuities converge more slowly than smooth functions.

Now try to find a Fourier series for an odd function yourself by working through the following exercise.

Exercise 20

The periodic function $f(t)$ with period τ is defined on the interval $[-\frac{\tau}{2}, \frac{\tau}{2}]$ by

$$f(t) = \begin{cases} -1 & \text{for } -\frac{\tau}{2} < t \leq 0, \\ 1 & \text{for } 0 < t < \frac{\tau}{2}. \end{cases}$$

Find the first three non-zero terms of the Fourier series for this function.

4 Fourier series for any periodic function

In the previous section you saw how to find Fourier series for even and odd periodic functions. Unfortunately, not all periodic functions are even or odd. However, the next exercise shows that any function is a sum of an even function and an odd function, so you would expect to be able to approximate functions that are neither even nor odd with a Fourier series involving both sine and cosine terms.

Exercise 21

Consider a general function $f(x)$.

(a) Show that the function $g(x)$ defined by

$$g(x) = \frac{f(x) + f(-x)}{2}$$

is even.

(b) Show that the function $h(x)$ defined by

$$h(x) = \frac{f(x) - f(-x)}{2}$$

is odd.

(c) Show that the function $f(x)$ can be written as the sum

$$f(x) = g(x) + h(x),$$

where $g(x)$ and $h(x)$ are as defined above.

Exercise 21 shows that one way of finding the Fourier series for a general function f is to find Fourier series for the functions g and h as defined in the exercise, and then add them. However, we can find the Fourier series for a general function more directly, as you will see in Subsection 4.1.

The modelling of a real problem may involve a function $f(t)$ that is defined only on some interval. We can choose the interval to be of the form $\left[0, \frac{\tau}{2}\right]$. Then we can extend the definition of the function to the interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$ by choosing the function to be either even or odd on this interval. From there, we can extend the definition of the function to all the real numbers as a periodic function. You will see how to do this in Subsection 4.2.

4.1 Fourier series for periodic functions

Suppose that you have a periodic function $f(t)$ with period τ and fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$. In general, the function will be neither even nor odd. However, it can always be written as a sum of an even function and an odd function, so it should have a Fourier series involving both cosine and sine terms. That is, we can try to represent $f(t)$ as the general Fourier series

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right).$$

As in Subsections 2.4, 3.1 and 3.2, the basic technique is to multiply by cosine or sine terms and integrate over the fundamental interval. Hence we need the following formula that gives the integral when cosine and sine terms are multiplied together.

Trigonometric integrals over the interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$

For any pair of integers m and n,

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2m\pi t}{\tau}\right) \cos\left(\frac{2n\pi t}{\tau}\right) dt = 0. \tag{38}$$

Using this result, no new terms appear when we form our products, so we arrive at the same formulas as before, which are summarised as follows.

Procedure 1 Fourier series for periodic functions

For a periodic function $f(t)$, with period τ and fundamental interval $\left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$, the Fourier series

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right)$$

is found by using the formulas

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt,$$

$$A_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos\left(\frac{2n\pi t}{\tau}\right) dt \quad (n = 1, 2, \ldots),$$

$$B_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right) dt \quad (n = 1, 2, \ldots).$$

It should be noted that the integrals for determining A_0, A_n and B_n are all over intervals that are symmetric about the origin, which means that the results for integrals of even and odd functions (equations (7) and (8)) will apply. In particular, if f is even then B_n is zero, and if f is odd then A_0 and A_n are zero.

The Fourier coefficients of a given function are unique. So if a function is itself a sum of sine and cosine functions, then it is its own Fourier series. For example, the Fourier series for $\frac{1}{2}\sin 3t$ is $\frac{1}{2}\sin 3t$ – that is, $B_3 = \frac{1}{2}$ and all other Fourier coefficients are zero.

Example 10

The periodic function $f(t)$ is defined by

$$f(t) = e^t \quad \text{for } -1 \leq t \leq 1,$$

on the fundamental interval $[-1, 1]$. Find its Fourier series.

Solution

The function $f(t)$ has the graph shown in Figure 22.

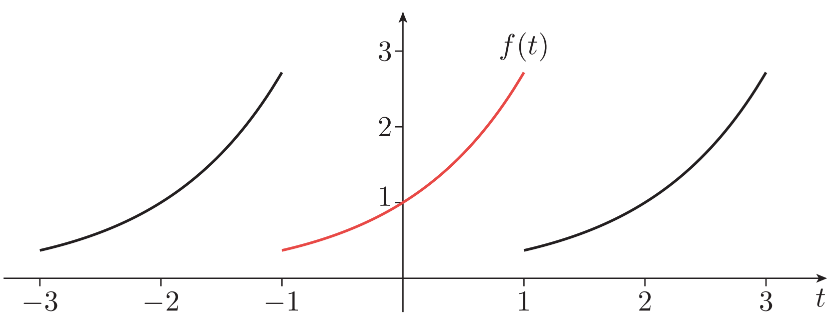


Figure 22 Graph of the function $f(t)$, with the fundamental interval highlighted in red

This function is clearly neither even nor odd. Using Procedure 1, with period $\tau = 2$, we first obtain

$$A_0 = \tfrac{1}{2}\int_{-1}^{1} e^t\,dt = \tfrac{1}{2}(e - e^{-1})$$

and

$$A_n = \int_{-1}^{1} e^t \cos(n\pi t)\,dt.$$

Now use integration by parts (integrating e^t and differentiating $\cos(n\pi t)$) to get

$$\begin{aligned} A_n &= \left[e^t\cos(n\pi t)\right]_{-1}^{1} - \int_{-1}^{1} e^t(-n\pi\sin(n\pi t))\,dt \\ &= e\cos n\pi - e^{-1}\cos n\pi + n\pi\int_{-1}^{1} e^t\sin(n\pi t)\,dt. \qquad (39) \end{aligned}$$

The integral on the right-hand side of this equation can also be integrated by parts (again integrating e^t and differentiating $\sin(n\pi t)$), so

$$A_n = (e - e^{-1})\cos n\pi + n\pi\left(\left[e^t\sin(n\pi t)\right]_{-1}^{1} - \int_{-1}^{1} e^t n\pi\cos(n\pi t)\,dt\right).$$

This can be simplified by using $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ to yield

$$A_n = (e - e^{-1})(-1)^n - n^2\pi^2\int_{-1}^{1} e^t\cos(n\pi t)\,dt.$$

The integral on the right-hand side is the integral that we started with for A_n, so we have

$$A_n = (e - e^{-1})(-1)^n - n^2\pi^2 A_n.$$

Solving this for A_n gives the required coefficient:

$$A_n = \frac{(e - e^{-1})(-1)^n}{1 + n^2\pi^2}.$$

To find the coefficient B_n, we need to evaluate

$$B_n = \int_{-1}^{1} e^t\sin(n\pi t)\,dt,$$

but this is easier as this integral has already appeared in equation (39) during the computation of A_n. Substituting for A_n in equation (39) gives

$$\frac{(e - e^{-1})(-1)^n}{1 + n^2\pi^2} = (e - e^{-1})(-1)^n + n\pi B_n.$$

Rearranging this equation and taking out common factors gives

$$B_n = \frac{(e - e^{-1})(-1)^n}{n\pi}\left(\frac{1}{1 + n^2\pi^2} - 1\right).$$

Simplifying then gives

$$B_n = -\frac{n\pi(e - e^{-1})(-1)^n}{1 + n^2\pi^2}.$$

Every coefficient in the Fourier series involves the term $e - e^{-1}$. The series is therefore conveniently written as

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right)$$
$$= (e - e^{-1})\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2}\big(\cos(n\pi t) - n\pi\sin(n\pi t)\big)\right).$$

The graph of the function $f(t)$ and the graph of $F_8(t)$, the sum of the constant and the first eight cosine terms and first eight sine terms in the Fourier series, are shown in Figure 23.

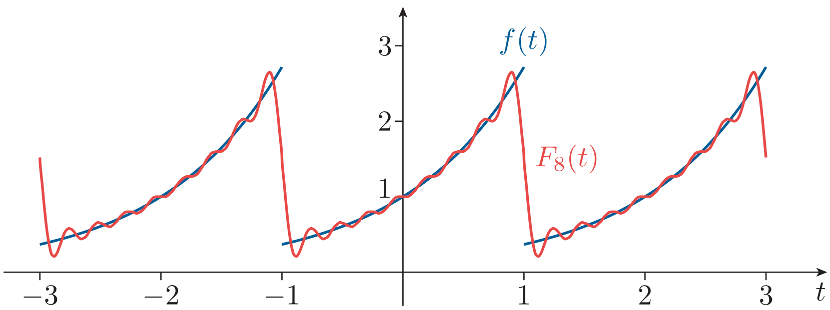


Figure 23 Graph of the function $f(t)$ together with the approximation $F_8(t)$

There is a fairly good approximation to the function, except at the endpoints. At each endpoint, the Fourier series takes the average value, $\frac{1}{2}(e + e^{-1})$, of the function either side of the endpoint.

The following exercises ask you to apply Procedure 1 to find Fourier series.

Exercise 22

Suppose that the periodic function $f(t)$ is defined on the fundamental interval $[-1, 1]$ by

$$f(t) = \begin{cases} 1 & \text{for } -1 \le t < 0, \\ t & \text{for } 0 \le t \le 1. \end{cases}$$

Find the coefficients of its Fourier series.

Exercise 23

(a) Find a fundamental interval for, and hence the period of, the function $b(t)$ defined in the Introduction as

$$b(t) = |\cos t|.$$

(b) State whether $b(t)$ is even, odd, or neither even nor odd, and find the Fourier series for this function.

(*Hint*: Use the trigonometric identity

$$\cos t \cos 2nt = \tfrac{1}{2}\big(\cos(2n-1)t + \cos(2n+1)t\big),$$

which is based on a more general identity given in the Handbook.)

4.2 Functions defined on an interval

So far, you have seen how to calculate the Fourier series for any *periodic* function. However, this is not the whole story. It is also possible to calculate the Fourier series for (almost) *any* function that is defined on a finite interval. This idea will be particularly useful in the next unit.

Suppose that a function $f(t)$ is defined within the finite interval $0 \le t \le T$, such as the curve shown in Figure 24. Furthermore, suppose that we do not care about what happens to the function outside this interval.

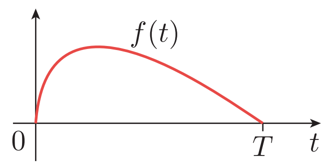


Figure 24 A function $f(t)$ defined on a finite interval $0 \le t \le T$

Then we can always define another function $f_{\text{ext}}(t)$ to be equal to $f(t)$ on the interval $0 \le t \le T$, and to be periodic with fundamental period T everywhere else. This function is called a **periodic extension** of $f(t)$ and is written as

$$f_{\text{ext}}(t) = f(t) \quad \text{for } 0 \le t < T,$$
$$f_{\text{ext}}(t+T) = f_{\text{ext}}(t).$$

The graph of $f_{\text{ext}}(t)$ consists of copies of $f(t)$ shifted by T and by all positive and negative integer multiples of T, as shown in Figure 25.

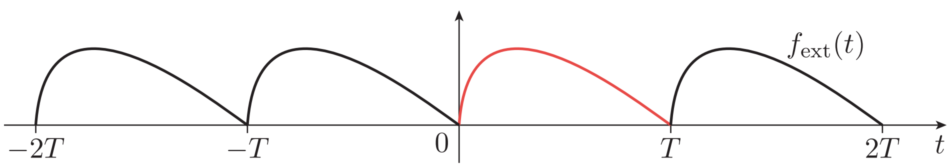


Figure 25 A periodic extension $f_{\text{ext}}(t)$ of $f(t)$

The periodic extension $f_{\text{ext}}(t)$ is a periodic function of fundamental period T, and we can find its Fourier series in the usual way. The resulting Fourier series will be equal to $f_{\text{ext}}(t)$ everywhere, and is equal to $f(t)$ for $0 \le t \le T$. So this Fourier series represents the non-periodic function $f(t)$ inside its domain of definition, $0 \le t \le T$. The periodic extension shown in Figure 25 is neither even nor odd, so the Fourier series contains both sine and cosine terms.

With a little preparation, we can use $f(t)$ to construct periodic functions that are either even or odd, before extending over all t. This is generally a sensible thing to do because the resulting Fourier series will be simpler. The definition of the even extension is straightforward.

Even periodic extension

Consider a function $f(t)$ defined over a finite domain $0 \le t \le T$.

The **even periodic extension** of $f(t)$ is given by

$$f_{\text{even}}(t) = \begin{cases} f(t) & \text{for } 0 \le t < T, \\ f(-t) & \text{for } -T \le t < 0, \end{cases}$$

$$f_{\text{even}}(t + 2T) = f_{\text{even}}(t).$$

An example of this extension is shown in Figure 26.

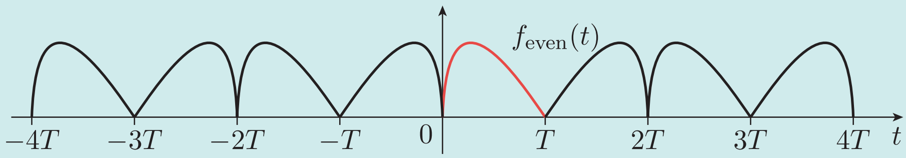


Figure 26 The even periodic extension $f_{\text{even}}(t)$ of $f(t)$

The definition of the odd periodic extension needs a little more care so that the resulting function is both periodic and odd. The reason for this is that any function that is both periodic and odd is zero at the origin and at the endpoints of a fundamental interval centred on the origin.

To show that any odd function $f(t)$ has the value zero at the origin, let a be the value at the origin, that is, $a = f(0)$. Then since f is odd, we have $f(-0) = -f(0)$, thus $a = -a$, that is, $2a = 0$ and so $a = 0$. To show that any odd function with period $2T$ is zero at the right-hand endpoint, let $a = f(T)$. As f is periodic with period $2T$, we must have $f(-T) = f(2T - T) = f(T) = a$. As f is odd, we must have $f(-T) = -f(T)$, so this again leads to the equation $a = -a$, and as before $a = 0$. These results are built into the definition of the odd extension of a function in the following definition.

Odd periodic extension

Consider a function $f(t)$ defined over a finite domain $0 \le t \le T$.

The **odd periodic extension** of $f(t)$ is given by

$$f_{\text{odd}}(t) = \begin{cases} f(t) & \text{for } 0 < t < T, \\ -f(-t) & \text{for } -T < t < 0, \\ 0 & \text{for } t = 0 \text{ or } t = T, \end{cases}$$

$$f_{\text{odd}}(t + 2T) = f_{\text{odd}}(t).$$

An example of this extension is shown in Figure 27.

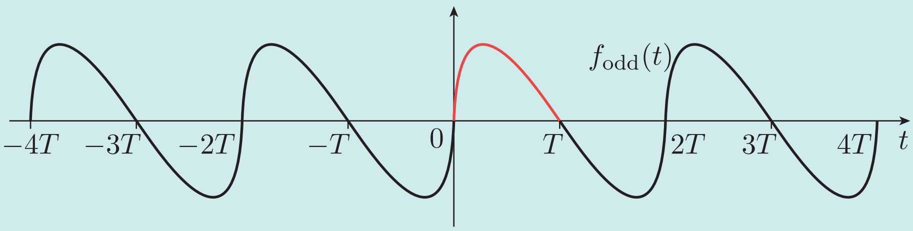


Figure 27 The odd periodic extension $f_{\mathrm{odd}}(t)$ of $f(t)$

In general, both even and odd extensions have fundamental period $\tau = 2T$, but in exceptional cases the even periodic extension may have a smaller fundamental period (see Exercise 25 for an example with fundamental period $\tau = T$).

The next example shows how to use these definitions to define even and odd extensions.

Example 11

Find and sketch the even and odd periodic extensions of the function

$$f(t) = t \quad \text{for } 0 \le t \le 1.$$

Solution

The even periodic extension is given by

$$f_{\mathrm{even}}(t) = \begin{cases} t & \text{for } 0 \le t < 1, \\ -t & \text{for } -1 \le t < 0, \end{cases}$$

$$f_{\mathrm{even}}(t+2) = f_{\mathrm{even}}(t).$$

This function is sketched in Figure 28, with the original function shown in red.

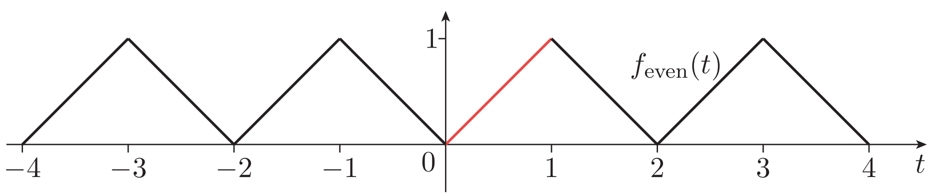


Figure 28 The given function is shown in red and the even periodic extension is shown in black

The odd periodic extension is given by

$$f_{\mathrm{odd}}(t) = \begin{cases} t & \text{for } 0 < t < 1, \\ t & \text{for } -1 < t < 0, \\ 0 & \text{for } t = 0 \text{ or } t = 1 \end{cases}$$

$$f_{\mathrm{odd}}(t+2) = f_{\mathrm{odd}}(t).$$

This function is sketched in Figure 29, with the original function shown in red.

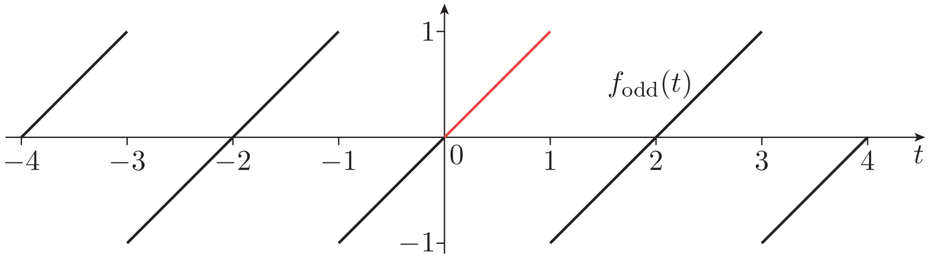


Figure 29 The given function is shown in red and the odd periodic extension is shown in black

In this particular case, both extensions can be expressed in alternative forms. The even periodic extension is

$$f_{\text{even}}(t) = |t| \quad \text{for } -1 \le t < 1,$$
$$f_{\text{even}}(t+2) = f_{\text{even}}(t),$$

and the odd periodic extension is

$$f_{\text{odd}}(t) = \begin{cases} t & \text{for } -1 < t < 1, \\ 0 & \text{for } t = 1, \end{cases}$$
$$f_{\text{odd}}(t+2) = f_{\text{odd}}(t).$$

Exercise 24

Consider the function shown in Figure 30 and defined by

$$q(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 3. \end{cases}$$

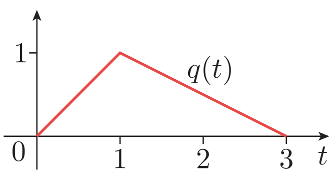


Figure 30

Define the even and odd periodic extensions of $q(t)$, simplifying the formulas if possible. State the fundamental periods, and sketch each extension over a range of three periods.

The following example illustrates how a function defined on a finite interval can be represented by a Fourier series. This result will be used in the next unit.

Example 12

Consider the function defined on the finite interval $0 \le t \le T$ by

$$f(t) = \begin{cases} t/T & \text{for } 0 \le t < T/2, \\ (T-t)/T & \text{for } T/2 \le t \le T, \end{cases}$$

where T is a positive constant. Express $f(t)$ as a Fourier series that involves only sine terms.

Solution

Because we are looking for a Fourier series that involves only sine terms, we need to consider the *odd* periodic extension of $f(t)$, denoted by $f_{\text{odd}}(t)$. This is sketched in Figure 31.

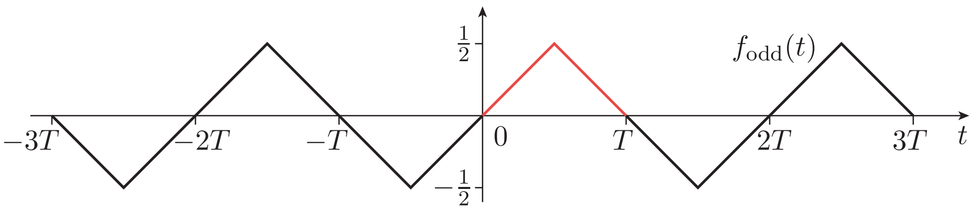


Figure 31 The odd extension of $f(t)$

The function $f_{\text{odd}}(t)$ is odd and has period $\tau = 2T$, so its Fourier series (from equation (32)) takes the form

$$F_{\text{odd}}(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{T}\right),$$

where the Fourier coefficients B_n are given by

$$B_n = \frac{1}{T}\int_{-T}^{T} f_{\text{odd}}(t) \sin\left(\frac{n\pi t}{T}\right) dt.$$

The integrand is even as it is the product of two odd functions, so the integral can be written as twice the integral over positive values. Also, $f_{\text{odd}}(t) = f(t)$ on this interval, so

$$B_n = \frac{2}{T}\int_{0}^{T} f(t) \sin\left(\frac{n\pi t}{T}\right) dt.$$

Using the given piecewise definition of $f(t)$, we obtain

$$B_n = \frac{2}{T^2}\int_{0}^{T/2} t \sin\left(\frac{n\pi t}{T}\right) dt + \frac{2}{T^2}\int_{T/2}^{T} (T-t) \sin\left(\frac{n\pi t}{T}\right) dt.$$

Expanding the second integral gives

$$B_n = \frac{2}{T^2}\int_{0}^{T/2} t \sin\left(\frac{n\pi t}{T}\right) dt + \frac{2}{T}\int_{T/2}^{T} \sin\left(\frac{n\pi t}{T}\right) dt - \frac{2}{T^2}\int_{T/2}^{T} t \sin\left(\frac{n\pi t}{T}\right) dt.$$

The first and third integrals are of the form of one of the two useful integrals (equation (17) with $a = n\pi/T$), so we get

$$\begin{aligned} B_n = {} & \frac{2}{T^2}\left[\frac{T^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi t}{T}\right) - \frac{n\pi t}{T}\cos\left(\frac{n\pi t}{T}\right)\right)\right]_0^{T/2} \\ & + \frac{2}{T}\left[-\frac{T}{n\pi}\cos\left(\frac{n\pi t}{T}\right)\right]_{T/2}^{T} \\ & - \frac{2}{T^2}\left[\frac{T^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi t}{T}\right) - \frac{n\pi t}{T}\cos\left(\frac{n\pi t}{T}\right)\right)\right]_{T/2}^{T}. \end{aligned}$$

Simplifying,

$$B_n = \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) - \frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\left(-n\pi\cos n\pi - \left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right)\right).$$

There is a lot of cancellation of terms, and the expression simplifies to

$$B_n = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad (n = 1, 2, 3, \ldots).$$

So

$$F_{\text{odd}}(t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi t}{T}\right).$$

Since $f(t)$ and $f_{\text{odd}}(t)$ coincide on the interval $0 \le t \le T$, this is the required sine Fourier series $F_{\text{odd}}(t)$ for the odd extension of $f(t)$.

For $n = 1, 2, 3, 4, 5, 6, 7$, the values of $\sin(n\pi/2)$ are $1, 0, -1, 0, 1, 0, -1$, so the first few terms in the Fourier series are

$$F_{\text{odd}}(t) = \frac{4}{\pi^2}\left(\sin(\pi t) - \frac{1}{3^2}\sin(3\pi t) + \frac{1}{5^2}\sin(5\pi t) - \frac{1}{7^2}\sin(7\pi t) + \cdots\right).$$

Exercise 25

Consider the same function $f(t)$ as that discussed in Example 12. Within its domain of definition, $0 \le t \le T$, represent this function by a Fourier series that involves only constant and cosine functions.

To represent the original function $f(t)$ in Example 12 by a Fourier series, we can use the odd periodic extension, obtaining a series that contains only sine terms (as in Example 12), or we can use the even periodic extension, obtaining a series that contains only constant and cosine terms (as in Exercise 25).

Sometimes one choice is better than the other. In general, if we want to approximate a function by a truncated Fourier series, it is better to use a periodic extension that is continuous, rather than discontinuous. This is because, as pointed out earlier, the Fourier series for a continuous function converges more rapidly than that for a discontinuous function. So for the function discussed in Example 11 we would use the even periodic extension to obtain the Fourier series.

However, in the next unit we will use Fourier series to solve partial differential equations, and in that case our choice of an even or odd periodic extension is generally dictated by other factors, namely the boundary conditions.

Exercise 25 is an exceptional case in which the even periodic extension has fundamental period $\tau = T$ rather than $\tau = 2T$. This simplifies the calculations because we need integrals only over the range from 0 to $T/2$. If we were to treat the function in Exercise 25 as having period $2T$, then the usual formula would eventually give the *same* Fourier series, although the calculations would be longer. In general, making the mistake of using a non-fundamental period rather than the fundamental period will always give the same final Fourier series, but at the expense of more labour.

Exercise 26

Consider the function

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} \le t < \pi. \end{cases}$$

(a) Define the even periodic extension, simplifying your answer as much possible. Sketch this function over $-3\pi \le t \le 3\pi$, and state its fundamental period.

(b) Find the Fourier series for the even periodic extension.

(c) Define the odd periodic extension. Sketch this function over $-3\pi \le t \le 3\pi$, and state its fundamental period.

Learning outcomes

After studying this unit, you should be able to:

- understand the terms frequency, period and fundamental interval, and obtain them for a periodic function
- understand the terms even and odd as applied to functions, and test a function to see if it is either
- find the Fourier series for a periodic function
- compare the graph of a function with the graph of a sum of terms in the Fourier series, and comment on the closeness of the approximation to the function
- understand how to modify a function defined on an interval to give an even or odd periodic extension.

Solutions to exercises

Solution to Exercise 1

We have

$$\begin{aligned}
G(t+2\pi) &= \frac{1}{2} + \frac{4}{\pi^2}\Big(\cos(t+2\pi) + \frac{1}{9}\cos 3(t+2\pi) \\
&\qquad + \frac{1}{25}\cos 5(t+2\pi) + \frac{1}{49}\cos 7(t+2\pi) + \cdots\Big) \\
&= \frac{1}{2} + \frac{4}{\pi^2}\Big(\cos(t+2\pi) + \frac{1}{9}\cos(3t+6\pi) \\
&\qquad + \frac{1}{25}\cos(5t+10\pi) + \frac{1}{49}\cos(7t+14\pi) + \cdots\Big) \\
&= \frac{1}{2} + \frac{4}{\pi^2}\Big(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \frac{1}{49}\cos 7t + \cdots\Big) \\
&= G(t).
\end{aligned}$$

Note that $G(t)$ would also be unchanged by adding any integer multiple of 2π to its argument. These results hold because adding 2π any number of times to a cosine argument does not change the value of the cosine.

Solution to Exercise 2

The angular frequencies are $1,\ 3,\ 5,\ 7,\ \ldots,$

and the periods are $2\pi,\ \frac{2\pi}{3},\ \frac{2\pi}{5},\ \frac{2\pi}{7},\ \ldots.$

Solution to Exercise 3

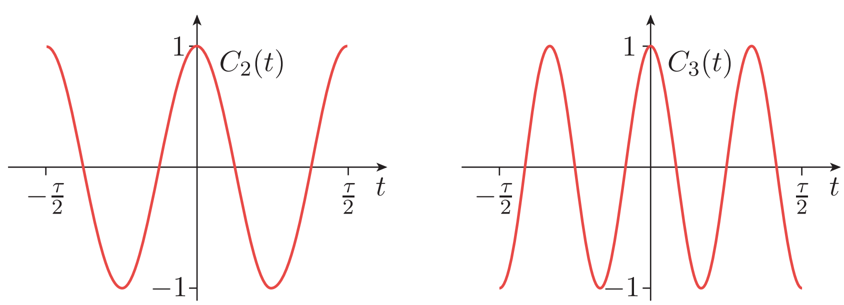
The angular frequencies of the component functions are $\pi,\ \frac{3\pi}{2},\ 2\pi,$

and the corresponding periods are $2,\ \frac{4}{3},\ 1.$

The least common multiple of these periods is 4, so the period of the function $f(t)$ is $\tau = 4$.

Solution to Exercise 4

The two graphs are as follows.



When $n = 0$, formula (6) gives

$$C_0(t) = \cos\left(\frac{2 \times 0\pi t}{\tau}\right) = \cos 0 = 1.$$

This is a constant function, which is periodic.

Solution to Exercise 5

Since $g(-t) = (-t)^3 = -t^3 = -g(t)$ for al t, the function is odd.

Solution to Exercise 6

Since $f(t)$ and $g(t)$ are both odd functions,

$$f(-t) = -f(t), \quad g(-t) = -g(t).$$

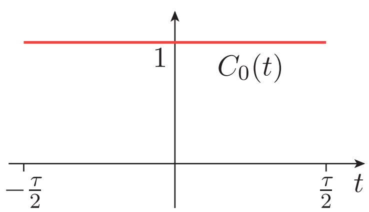
Hence

$$k(-t) = f(-t) + g(-t) = -f(t) + (-g(t)) = -(f(t) + g(t)) = -k(t),$$

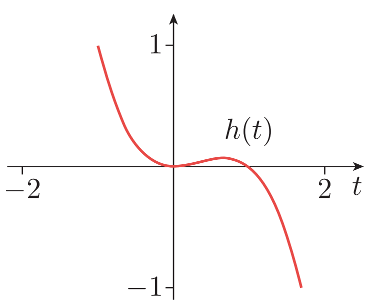
so $k(t)$ is an odd function.

Solution to Exercise 7

(a) The graph of the function $C_0(t)$ is shown below. As the graph is symmetrical about the vertical axis, the function must be even.



(b) Here $h(-t) = (-t)^2 + (-t)^3 = t^2 - t^3$. In general, this is equal to neither $h(t)$ nor $-h(t)$, so the function is neither even nor odd. This can also be seen by the asymmetrical nature of the curve in the diagram below.



Solution to Exercise 8

We have

$$\begin{aligned} h(t) = f(t)\,g(t) &= (t^3 + 2t^5)(3t^2 - t^4) \\ &= 3t^5 - t^7 + 6t^7 - 2t^9 \\ &= 3t^5 + 5t^7 - 2t^9. \end{aligned}$$

All the powers of t are odd, so $h(t)$ is an odd function.

Alternatively, since $f(t)$ and $g(t)$ are odd and even functions, respectively, we know by definition that

$$f(-t) = -f(t), \quad g(-t) = g(t).$$

Hence

$$h(-t) = f(-t)\,g(-t) = -f(t)\,g(t) = -h(t),$$

so $h(t)$ is an odd function.

Solution to Exercise 9

From the graph, the values of the function repeat after an interval of length 2π. Hence the period is $\tau = 2\pi$. The angular frequency ω satisfies $\tau = 2\pi/\omega$, so here $\omega = 1$.

Solution to Exercise 10

For $n = 1, 2, 3, \ldots$,

$$\int_{-\pi}^{\pi} \cos nt\, dt = \left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi} = 0,$$

since $\sin n\pi = 0$ and $\sin(-n\pi) = 0$.

Solution to Exercise 11

Applying equation (13) gives

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)\, dt.$$

Using the hint, this simplifies to

$$A_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\, dt,$$

where we have used the fact that $h(t) = 1$ for $-\pi/2 \le t \le \pi/2$. Performing the integration gives

$$A_0 = \frac{1}{2\pi}\big[t\big]_{-\pi/2}^{\pi/2} = \tfrac{1}{2}.$$

Solution to Exercise 12

(a) As A_0 is a constant, it can be taken outside the integral to obtain

$$\int_{-\pi}^{\pi} A_0 \cos t\, dt = A_0 \int_{-\pi}^{\pi} \cos t\, dt = A_0 \big[\sin t\big]_{-\pi}^{\pi} = 0.$$

(b) Using the given trigonometric identity with $\alpha = 2t$ and $\beta = t$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \cos 2t \cos t\, dt &= \int_{-\pi}^{\pi} \left(\tfrac{1}{2}\cos 3t + \tfrac{1}{2}\cos t\right) dt \\ &= \left[\tfrac{1}{6}\sin 3t + \tfrac{1}{2}\sin t\right]_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

(c) Again using the trigonometric identity, this time with $\alpha = nt$ and $\beta = t$, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nt \cos t\, dt &= \int_{-\pi}^{\pi} \left(\tfrac{1}{2}\cos(n+1)t + \tfrac{1}{2}\cos(n-1)t\right) dt \\ &= \left[\frac{\sin(n+1)t}{2(n+1)} + \frac{\sin(n-1)t}{2(n-1)}\right]_{-\pi}^{\pi} \qquad (\text{as } n > 1) \\ &= 0.\end{aligned}$$

(d) Rearranging the given trigonometric identity to make $\cos^2 \alpha$ the subject, and using $\alpha = t$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 t\, dt &= \int_{-\pi}^{\pi} \tfrac{1}{2}(\cos 2t + 1)\, dt \\ &= \left[\tfrac{1}{4}\sin 2t + \tfrac{1}{2}t\right]_{-\pi}^{\pi} \\ &= \pi.\end{aligned}$$

Solution to Exercise 13

Substitute the definition of $h(t)$ from Exercise 11 into equation (16) to obtain

$$A_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\cos t\, dt.$$

Now use the definition of $h(t)$ to get (as $h(t) = 0$ outside the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$)

$$A_1 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \times \cos t\, dt.$$

Performing the integration gives

$$A_1 = \frac{1}{\pi}\big[\sin t\big]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\big(1 - (-1)\big) = \frac{2}{\pi}.$$

Solution to Exercise 14

We could obtain a general formula for A_n, as in Example 6, and then substitute $n = 2$ and $n = 3$ into that. Alternatively, put $n = 2$ into equation (22) and obtain

$$A_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\cos 2t\,dt.$$

Now use the definition of $h(t)$, which is zero outside $-\pi/2 \le t \le \pi/2$ and equal to 1 on this interval, to yield

$$A_2 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \times \cos 2t\,dt.$$

Performing the integration gives

$$A_2 = \frac{1}{\pi}\left[\tfrac{1}{2}\sin 2t\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left(\tfrac{1}{2}\sin\pi - \tfrac{1}{2}\sin(-\pi)\right) = 0.$$

Substituting $n = 3$ into equation (22) gives

$$A_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\cos 3t\,dt.$$

Now use the definition of $h(t)$ again to yield

$$A_3 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \times \cos 3t\,dt.$$

Performing the integration gives

$$\begin{aligned} A_3 = \frac{1}{\pi}\left[\tfrac{1}{3}\sin 3t\right]_{-\pi/2}^{\pi/2} &= \frac{1}{3\pi}\left(\sin\tfrac{3\pi}{2} - \sin\left(-\tfrac{3\pi}{2}\right)\right) \\ &= \frac{1}{3\pi}\big((-1) - 1\big) = -\frac{2}{3\pi}. \end{aligned}$$

Solution to Exercise 15

The constant term does not fit in the general pattern, so we can leave this term outside the summation. All of the other terms have a factor 2 in the numerator and a factor π in the denominator, so we can factorise the expression as

$$H(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right).$$

The sum in the brackets is a sum over all odd numbers. As $n = 1, 2, 3, \ldots$, the expression $2n - 1$ evaluates to $1, 3, 5, \ldots$, so this is the expression to be used in the angular frequency and the denominator. The terms also change sign, so we need a factor $(-1)^n$ in the coefficients. The first sign is positive, and $(-1)^n$ is negative for $n = 1$, so we need an extra minus sign (either by writing this as $(-1)^{n+1}$ or by taking a minus sign out of the bracket as below).

Putting all this together gives the closed form as

$$H(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos(2n-1)t.$$

Solution to Exercise 16

We have

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{2\pi}\left[\tfrac{1}{3}t^3\right]_{-\pi}^{\pi} = \tfrac{1}{3}\pi^2,$$

and for $n = 1, 2, 3, \ldots$, we use the hint to obtain

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt \\ &= \frac{1}{\pi}\left[\frac{1}{n^3}\left((n^2t^2 - 2)\sin nt + 2nt\cos nt\right)\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \times \frac{1}{n^3}\left(2n\pi(-1)^n - (-2n\pi(-1)^n)\right), \end{aligned}$$

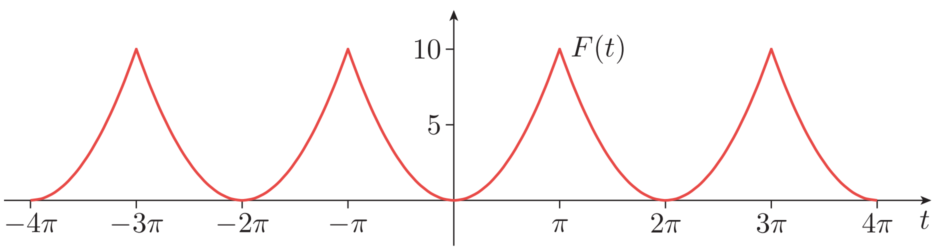
where we have used $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ as n is an integer. This simplifies to

$$A_n = \frac{4(-1)^n}{n^2}.$$

The Fourier series is therefore

$$\begin{aligned} F(t) &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nt \\ &= \frac{\pi^2}{3} - 4\cos t + \cos 2t - \frac{4}{9}\cos 3t + \frac{1}{4}\cos 4t + \cdots. \end{aligned}$$

(The graph of $F(t)$ matches the expected graph, which consists of a parabola on the fundamental interval repeated indefinitely to both the left and the right – see below.)



Solution to Exercise 17

The function $w(t)$ is even, and $w(t) = t$ on the interval $[0, \pi]$. Therefore

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi} t\,dt = \frac{1}{\pi}\left[\tfrac{1}{2}t^2\right]_0^{\pi} = \frac{\pi}{2}.$$

For the other coefficients we must evaluate

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos nt\,dt.$$

As $w(t)$ is even and equal to t for t positive, this simplifies to

$$A_n = \frac{2}{\pi}\int_{0}^{\pi} t\cos nt\,dt.$$

This is one of the useful integrals (equation (18) with $a = n$), so

$$A_n = \frac{2}{\pi} \times \frac{1}{n^2}\left[\cos nt + nt \sin nt\right]_0^{\pi}.$$

Evaluating this expression (and using $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$), we get

$$A_n = \frac{2}{n^2\pi}\left((-1)^n - 1\right).$$

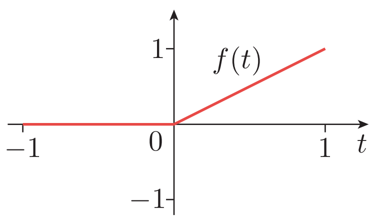
Thus the Fourier series is

$$W(t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nt.$$

So the requested approximation is

$$W_5(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos t - \frac{4}{9\pi}\cos 3t - \frac{4}{25\pi}\cos 5t.$$

Solution to Exercise 18



The easiest way to proceed is to draw a graph of the given function on the fundamental interval, such as in the figure in the margin. The pointwise convergence theorem gives

$$F(-1) = \frac{f(-1^+) + f(-1^-)}{2}.$$

Using the graph as a guide, we see that the given function is discontinuous at the point $t = -1$ and that approaching the point from the right gives $f(-1^+) = 0$. Approaching $t = -1$ from the left is outside the fundamental interval, so we add one period to see that this has the same value as approaching $t = 1$ from the left, that is, $f(-1^-) = f(1^-) = 1$. So

$$F(-1) = \frac{0+1}{2} = \frac{1}{2}.$$

The given function $f(t)$ is continuous at $t = 0$, so $F(t) = f(t)$ at $t = 0$, that is,

$$F(0) = f(0) = 0.$$

(You may observe that the slope of the curve changes abruptly at $t = 0$, so the derivative of $f(t)$ is discontinuous at $t = 0$. This property is independent of the fact that $f(t)$ is continuous at $t = 0$.)

Solution to Exercise 19

Using equation (28) gives

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} w(t)\, dt.$$

As $w(t)$ is even, this integral can be simplified and evaluated as

$$A_0 = \frac{2}{\tau}\int_0^{\tau/2} t\, dt = \frac{2}{\tau}\left[\tfrac{1}{2}t^2\right]_0^{\tau/2} = \tfrac{1}{4}\tau.$$

Similarly, we can compute A_n by using equation (29):

$$A_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} w(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt.$$

Proceeding as before, we can simplify the integral by using the fact that the integrand $w(t)\cos(2n\pi t/\tau)$ is even, to get

$$\begin{aligned}A_n &= \frac{4}{\tau}\int_{0}^{\tau/2} w(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt\\ &= \frac{4}{\tau}\int_{0}^{\tau/2} t\cos\left(\frac{2n\pi t}{\tau}\right)dt.\end{aligned}$$

This integral is again of the form of one of the two useful integrals (equation (18) with $a = 2n\pi/\tau$), so

$$A_n = \frac{4}{\tau}\times\frac{\tau^2}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi t}{\tau}\right) + \frac{2n\pi t}{\tau}\sin\left(\frac{2n\pi t}{\tau}\right)\right]_0^{\tau/2}.$$

Evaluating this expression (and using $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$) gives

$$\begin{aligned}A_n &= \frac{\tau}{n^2\pi^2}\big((\cos n\pi + n\pi\sin n\pi) - (1-0)\big)\\ &= \frac{\tau((-1)^n - 1)}{n^2\pi^2}.\end{aligned}$$

(So we have

$$\begin{aligned}&A_0 = \frac{\tau}{4},\quad A_1 = -\frac{2\tau}{\pi^2},\quad A_2 = 0,\quad A_3 = -\frac{2\tau}{9\pi^2},\quad A_4 = 0,\\ &A_5 = -\frac{2\tau}{25\pi^2},\quad A_6 = 0,\quad A_7 = -\frac{2\tau}{49\pi^2},\quad A_8 = 0,\quad \ldots .)\end{aligned}$$

Solution to Exercise 20

The function is odd, so its Fourier series involves only sine terms:

$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right),$$

where the coefficients are given by

$$B_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt.$$

As $f(t)$ is odd, the integrand is even (as the product of two odd functions). In addition, $f(t)$ is equal to 1 for positive t, so the integral simplifies to

$$B_n = \frac{4}{\tau}\int_{0}^{\tau/2} \sin\left(\frac{2n\pi t}{\tau}\right)dt.$$

Evaluating the integral gives

$$\begin{aligned}B_n = \frac{4}{\tau}\left[-\frac{\tau}{2n\pi}\cos\left(\frac{2n\pi t}{\tau}\right)\right]_0^{\tau/2} &= \frac{2\big(-\cos n\pi - (-1)\big)}{n\pi},\\ &= \frac{2(1-(-1)^n)}{n\pi}.\end{aligned}$$

Evaluating each coefficient in turn yields

$$\frac{4}{\pi},\ 0,\ \frac{4}{3\pi},\ 0,\ \frac{4}{5\pi},\ 0,\ \ldots,$$

so the first three non-zero terms of the Fourier series for $f(t)$ are

$$F(t) = \frac{4}{\pi}\left(\sin\left(\frac{2\pi t}{\tau}\right) + \frac{1}{3}\sin\left(\frac{6\pi t}{\tau}\right) + \frac{1}{5}\sin\left(\frac{10\pi t}{\tau}\right) + \cdots\right).$$

(Note that since $B_n = 0$ for even n, the Fourier series could be written as a sum over odd terms,

$$F(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{2(2n-1)\pi t}{\tau}\right),$$

but this was not required in this exercise.)

Solution to Exercise 21

(a) Using the definition of $g(x)$, we have

$$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = g(x).$$

Hence $g(x)$ is an even function.

(b) Using the definition of $h(x)$, we have

$$h(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -h(x).$$

Hence $h(x)$ is an odd function.

(c) Using the definitions of $g(x)$ and $h(x)$, we have

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x).$$

Solution to Exercise 22

Using Procedure 1, we obtain

$$\begin{aligned}
A_0 &= \tfrac{1}{2}\int_{-1}^{0} 1\,dt + \tfrac{1}{2}\int_{0}^{1} t\,dt \\
&= \tfrac{1}{2}\big[t\big]_{-1}^{0} + \tfrac{1}{2}\left[\tfrac{1}{2}t^2\right]_0^1 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4},
\end{aligned}$$

Using equation (18) with $a = n\pi$.

$$\begin{aligned}
A_n &= \int_{-1}^{0}\cos(n\pi t)\,dt + \int_{0}^{1} t\cos(n\pi t)\,dt \\
&= \frac{1}{n\pi}\big[\sin(n\pi t)\big]_{-1}^{0} + \left[\frac{1}{n^2\pi^2}\left(\cos(n\pi t) + n\pi t\sin(n\pi t)\right)\right]_0^1 \\
&= \frac{1}{n^2\pi^2}\left((-1)^n - 1\right),
\end{aligned}$$

$$\begin{aligned}
B_n &= \int_{-1}^{0} \sin(n\pi t)\,dt + \int_{0}^{1} t\sin(n\pi t)\,dt \\
&= -\frac{1}{n\pi}\big[\cos(n\pi t)\big]_{-1}^{0} + \left[\frac{1}{n^2\pi^2}\big(\sin(n\pi t) - n\pi t\cos(n\pi t)\big)\right]_0^1 \\
&= -\frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n^2\pi^2}\,(n\pi\cos n\pi) \\
&= -\frac{1}{n\pi} + \frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi} \\
&= -\frac{1}{n\pi}.
\end{aligned}$$

Using equation (17) with $a = n\pi$.

Solution to Exercise 23

(a) It is clear from the graph of $b(t)$ (Figure 2) that $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is a fundamental interval, so $b(t)$ has period π.

(b) The function $b(t)$ is even, so its Fourier series involves only the constant and cosine terms, hence

$$B(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos 2nt.$$

Starting with the constant term, we need to calculate

$$A_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t\,dt.$$

Evaluating the integral gives

$$A_0 = \frac{1}{\pi}\big[\sin t\big]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\big(1-(-1)\big) = \frac{2}{\pi}.$$

Similarly, the coefficient A_n can be calculated from

$$A_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t\cos 2nt\,dt.$$

As the integrand is even, this integral simplifies to

$$A_n = \frac{4}{\pi}\int_{0}^{\pi/2} \cos t\cos 2nt\,dt.$$

Using the hint, we write this as

$$A_n = \frac{4}{\pi}\int_{0}^{\pi/2} \tfrac{1}{2}\big(\cos(2n-1)t + \cos(2n+1)t\big)\,dt.$$

Evaluating the integral gives

$$\begin{aligned}
A_n &= \frac{2}{\pi}\left[\frac{\sin(2n-1)t}{2n-1} + \frac{\sin(2n+1)t}{2n+1}\right]_0^{\pi/2} \\
&= \frac{2}{\pi}\left(\frac{\sin(n\pi - \frac{\pi}{2})}{2n-1} + \frac{\sin(n\pi + \frac{\pi}{2})}{2n+1}\right).
\end{aligned}$$

Now use the addition formulas for sine to get

$$A_n = \frac{2}{\pi}\left(\frac{\sin n\pi \cos\frac{\pi}{2} - \cos n\pi \sin\frac{\pi}{2}}{2n-1} + \frac{\sin n\pi \cos\frac{\pi}{2} + \cos n\pi \sin\frac{\pi}{2}}{2n+1}\right).$$

Using $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ and $\sin\frac{\pi}{2} = 1$, this simplifies to

$$\begin{aligned} A_n &= \frac{2}{\pi}\left(\frac{-(-1)^n}{2n-1} + \frac{(-1)^n}{2n+1}\right) \\ &= \frac{2(-1)^n}{\pi}\left(-\frac{1}{2n-1} + \frac{1}{2n+1}\right) \\ &= \frac{2(-1)^n}{\pi} \times \frac{-(2n+1)+(2n-1)}{(2n-1)(2n+1)} \\ &= -\frac{4(-1)^n}{\pi(4n^2-1)}. \end{aligned}$$

So the Fourier series is

$$\begin{aligned} B(t) &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} \cos 2nt \\ &= \frac{2}{\pi}\left(1 + \frac{2}{3}\cos 2t - \frac{2}{15}\cos 4t + \frac{2}{35}\cos 6t - \frac{2}{63}\cos 8t + \cdots\right). \end{aligned}$$

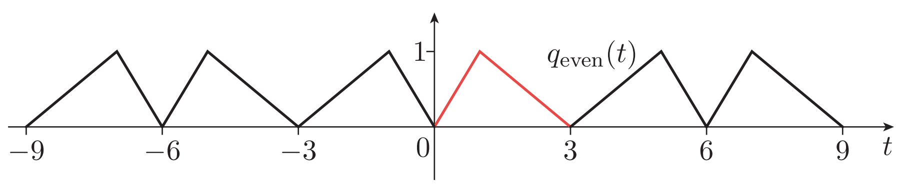
Solution to Exercise 24

Using the definition of the even periodic extension, we have

$$q_{\text{even}}(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 3, \\ -t & \text{for } -1 \le t < 0, \\ \frac{3}{2} + \frac{1}{2}t & \text{for } -3 < t < -1, \end{cases}$$

$$q_{\text{even}}(t+6) = q_{\text{even}}(t).$$

This function has fundamental period $\tau = 6$, and its formula cannot be made much simpler. Its graph is sketched below.

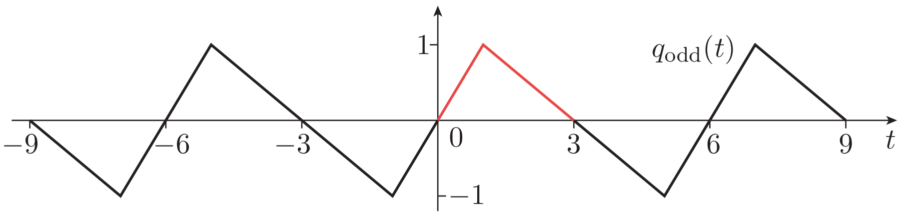


The odd periodic extension is given by

$$q_{\text{odd}}(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 3, \\ t & \text{for } -1 \le t < 0, \\ -\frac{3}{2} - \frac{1}{2}t & \text{for } -3 < t < -1, \end{cases}$$

$$q_{\text{odd}}(t+6) = q_{\text{odd}}(t).$$

This function has fundamental period $\tau = 6$, and its graph is sketched below.



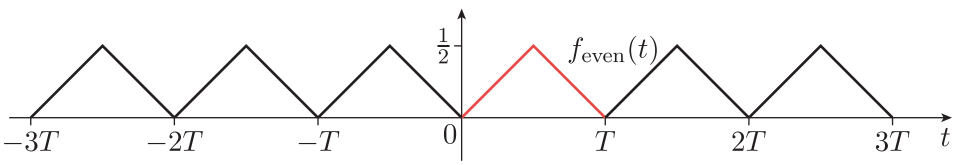
By examining this graph, we see that the formula can be simplified to

$$q_{\text{odd}}(t) = \begin{cases} t & \text{for } -1 \le t < 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 \le t < 5, \end{cases}$$

$$q_{\text{odd}}(t+6) = q_{\text{odd}}(t).$$

Solution to Exercise 25

Because we are looking for a Fourier series that involves only constant and cosine terms, we need to consider the *even* periodic extension of $f(t)$, denoted by $f_{\text{even}}(t)$. This is sketched in the figure below.



The function $f_{\text{even}}(t)$ is even and has period $\tau = T$, so its Fourier series takes the form

$$F_{\text{even}}(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{T}\right),$$

where the Fourier coefficients A_0 and A_n are given by

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f_{\text{even}}(t)\,dt,$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f_{\text{even}}(t)\cos\left(\frac{2n\pi t}{T}\right)dt \quad (n = 1, 2, 3, \ldots).$$

But both of the integrands are even, and on the interval $0 \le t \le T/2$ we have $f_{\text{even}}(t) = f(t) = t/T$, so the integrals simplify to

$$A_0 = \frac{2}{T}\int_0^{T/2} \frac{t}{T}\,dt,$$

$$A_n = \frac{4}{T}\int_0^{T/2} \frac{t}{T}\cos\left(\frac{2n\pi t}{T}\right)dt \quad (n = 1, 2, 3, \ldots).$$

Evaluating the first integral gives

$$A_0 = \frac{2}{T^2}\int_0^{T/2} t\,dt = \frac{2}{T^2}\left[\tfrac{1}{2}t^2\right]_0^{T/2} = \tfrac{1}{4}.$$

Using equation (18) with $a = 2n\pi/T$, we get

$$\begin{aligned} A_n &= \frac{4}{T^2}\left(\frac{T}{2n\pi}\right)^2\left[\cos\left(\frac{2n\pi t}{T}\right) + \frac{2n\pi t}{T}\sin\left(\frac{2n\pi t}{T}\right)\right]_0^{T/2} \\ &= \frac{(-1)^n - 1}{n^2\pi^2}. \end{aligned}$$

So

$$F_{\text{even}}(t) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\left(\frac{2n\pi t}{T}\right).$$

Since $f(t)$ and $f_{\text{even}}(t)$ coincide on the interval $0 \le t \le T$, this is the required cosine Fourier series $F_{\text{even}}(t)$ for $f(t)$.

The first few terms in this Fourier series are

$$F_{\text{even}}(t) = \frac{1}{4} - \frac{2}{\pi^2}\left(\cos\left(\frac{2\pi t}{T}\right) + \frac{1}{3^2}\cos\left(\frac{6\pi t}{T}\right) + \frac{1}{5^2}\cos\left(\frac{10\pi t}{T}\right) + \cdots\right).$$

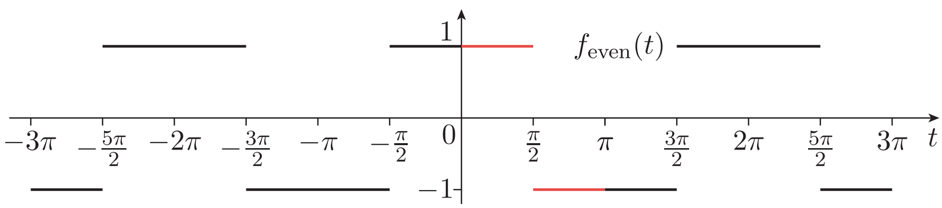
Solution to Exercise 26

(a) The even periodic extension is given by

$$f_{\text{even}}(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} < t \le \pi, \\ -1 & \text{for } -\pi < t < -\frac{\pi}{2}, \\ 1 & \text{for } -\frac{\pi}{2} \le t < 0, \end{cases}$$

$$f_{\text{even}}(t + 2\pi) = f_{\text{even}}(t),$$

and is drawn below.



The fundamental period of this even extension is $\tau = 2\pi$, and its formula can be simplified to

$$f_{\text{even}}(t) = \begin{cases} 1 & \text{for } -\frac{1}{2}\pi \le t < \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} \le t < \frac{3\pi}{2}, \end{cases}$$

$$f_{\text{even}}(t + 2\pi) = f_{\text{even}}(t).$$

(b) Because the function $f_{\text{even}}(t)$ is even and has period $\tau = 2\pi$, its Fourier series takes the form

$$F_{\text{even}}(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nt.$$

The coefficient A_0 is given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{\text{even}}(t)\,dt \\ &= \frac{1}{2\pi}\left(-\int_{-\pi}^{-\pi/2} 1\,dt + \int_{-\pi/2}^{\pi/2} 1\,dt - \int_{\pi/2}^{\pi} 1\,dt\right) = 0. \end{aligned}$$

This result is not unexpected, as the graph shows clearly that the average value of the function is zero.

As the integrand is even, the coefficients A_n are given by

$$\begin{aligned} A_n &= \frac{4}{2\pi}\int_0^{\pi} f_{\text{even}}(t)\cos nt\,dt \\ &= \frac{2}{\pi}\left(\int_0^{\pi/2}\cos nt\,dt - \int_{\pi/2}^{\pi}\cos nt\,dt\right) \\ &= \frac{2}{n\pi}\left(\big[\sin nt\big]_0^{\pi/2} - \big[\sin nt\big]_{\pi/2}^{\pi}\right) \\ &= \frac{4}{n\pi}\sin\frac{n\pi}{2}. \end{aligned}$$

Hence

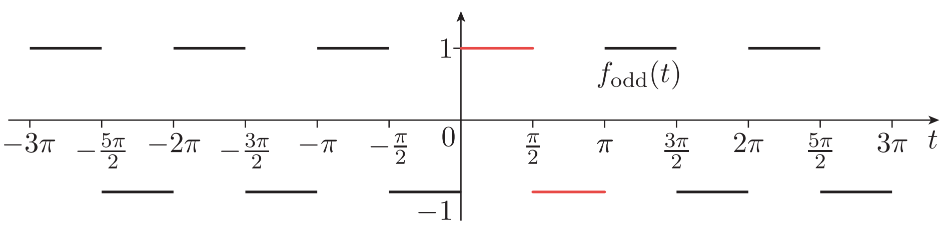
$$F_{\text{even}}(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos nt.$$

(c) The odd periodic extension is defined by

$$f_{\text{odd}}(t) = \begin{cases} 1 & \text{for } 0 < t \le \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} < t < \pi, \\ 1 & \text{for } -\pi < t < -\frac{\pi}{2}, \\ -1 & \text{for } -\frac{\pi}{2} \le t < 0, \\ 0 & \text{for } t = 0 \text{ or } t = \pi, \end{cases}$$

$$f_{\text{odd}}(t + 2\pi) = f_{\text{odd}}(t),$$

and is drawn below.



This has fundamental period $\tau = \pi$.