Unit 7

Differentiation methods and integration

Introduction

The first half of this unit continues with the topic of Unit 6, *differential calculus*. You'll meet the formulas for the derivatives of five standard functions, namely sin, cos, tan, exp and ln. You'll also learn some further ways of differentiating complicated functions by combining the formulas for the derivatives of standard functions.

By the end of this part of the unit you should have acquired skills in differentiating a wide variety of functions. You'll need these skills in the later parts of the module, and in any further modules that you study that involve calculus. So it's important that you become proficient in them. The only way to do that is to practise, so you should aim to do all the activities in this unit, and as many further activities in the practice quizzes and exercise booklets as you have time for, until you feel confident with your skills.

In the first half of the unit you'll also learn how to use your computer to find and work with derivatives, and you'll see some further applications of differentiation.

In the second half of the unit you'll start to learn about *integral calculus*, which can be thought of as the reverse of differential calculus. You've seen that in differential calculus you start off knowing the values taken by a continuously changing quantity throughout a period of change, and you use this information to find the values taken by the rate of change of the quantity throughout the same period. In integral calculus, you do the opposite: you start off knowing the values taken by the rate of change of a quantity throughout a period of change, and you use this information to find the values taken by the quantity throughout the same period. This process is called *integration*. The topic of integral calculus continues in Unit 8.

1 Derivatives of further standard functions

In this section you'll meet the formulas for the derivatives of the five standard functions sin, cos, tan, exp and ln.

1.1 Derivatives of sin, cos and tan

In this first subsection we'll obtain formulas for the derivatives of the trigonometric functions $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$.

When you're working with these functions, it's essential to remember that the input variable x is the size of an angle *measured in radians*. This is indicated by the formulas for the functions, because x represents a number, and when the size of an angle is given just as a number, with no units, then the units are assumed to be radians. You met this convention in Unit 4. For example, $\sin 10$ denotes the sine of 10 radians, whereas $\sin 10^\circ$ denotes the sine of 10 degrees.

When we're dealing with the derivatives of trigonometric functions, we always measure angles in radians. This is because it makes the formulas for the derivatives of trigonometric functions simpler. If you'd like to know more about this, then, once you've completed Sections 1 and 2 of this unit, read the document *Why do we use radians?* on the module website.

Derivative of sin

Our first aim in this subsection is to obtain a formula for the derivative of the sine function, $f(x) = \sin x$.

As you know, the derivative of a function is itself a function, and therefore the derivative has a graph. So, to get a rough idea of what the derivative of the sine function might be like, let's try to roughly sketch its graph, by estimating the gradient of the graph of the sine function at various points.

Figure 1 shows the graph of the sine function, drawn on axes with equal scales, and some tangents to the graph.

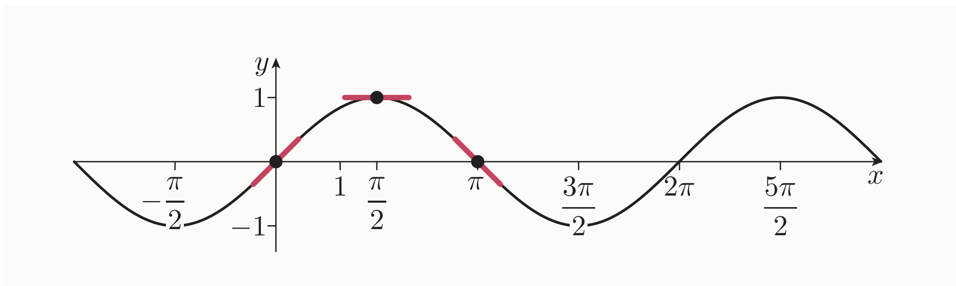


Figure 1 The graph of $f(x) = \sin x$

When $x = 0$, the gradient of the graph of the sine function seems to be about the same as the gradient of the line $y = x$; that is, it seems to be about 1. As x increases, the graph gradually gets less steep – that is, its gradient gradually decreases, until $x = \pi/2$, when the gradient seems to be zero. The gradient decreases slowly when x is only a little larger than 0, but decreases more rapidly as x gets closer to $\pi/2$.

In other words, for x between 0 and $\pi/2$ the value of the derivative seems to decrease from 1 to zero, slowly at first, but then more rapidly. So its graph must look something like the sketch in Figure 2.

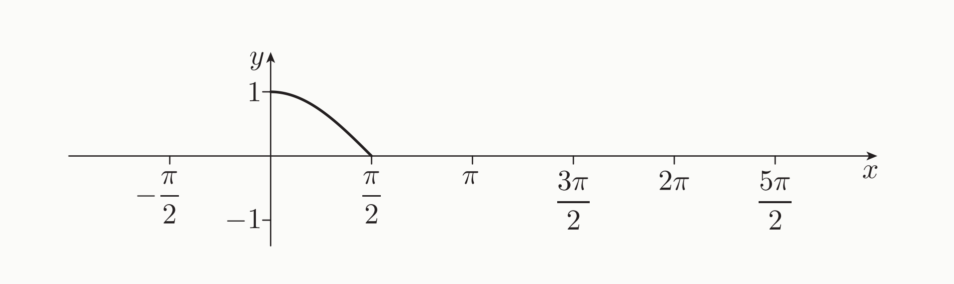


Figure 2 A sketch of the graph of the derivative of $f(x) = \sin x$, for x between 0 and $\pi/2$

Now look at the graph of the sine function for x between $\pi/2$ and π. The gradient starts at zero, then it becomes negative. At first the graph is not very steep – that is, the gradient has small negative values – but it becomes steeper and steeper – that is, the gradient takes negative values of greater and greater magnitude. Eventually, when $x = \pi$, the gradient seems to be about the same as the gradient of the line $y = -x$; that is, it seems to be about -1. The gradient decreases fairly rapidly when x is only a little larger than $\pi/2$, but decreases more slowly as x gets closer to π.

So, for x between $\pi/2$ and π, the value of the derivative seems to decrease from 0 to -1, fairly rapidly at first, but then more slowly. So the graph of the derivative must look something like the sketch in Figure 3.

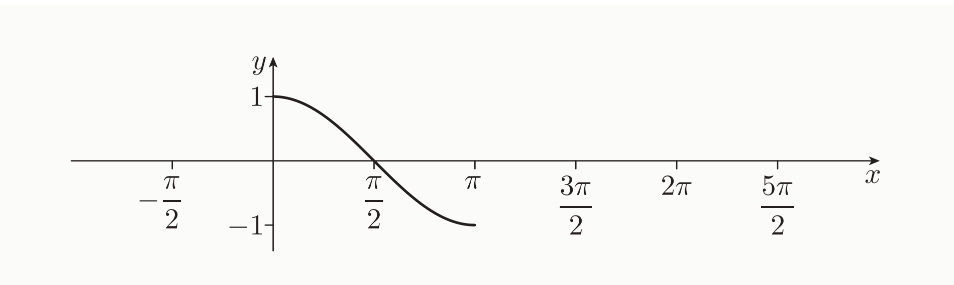


Figure 3 A sketch of the graph of the derivative of $f(x) = \sin x$, for x between 0 and π

In the next activity you're asked to extend the sketch in Figure 3 to cover more values of x.

Activity 1 *Extending the sketch of the graph of the derivative of sin*

By considering the gradients of the graph of the sine function (shown in Figure 1) for x between π and $3\pi/2$, and then for x between $3\pi/2$ and 2π, work out what the graph of the derivative of the sine function must look like for these values of x. Hence extend the sketch of the derivative of the sine function in Figure 3 to cover all values of x between 0 and 2π.

In Activity 1 you should have worked out that the graph of the derivative of the sine function for x between 0 and 2π must look something like the sketch in Figure 4.

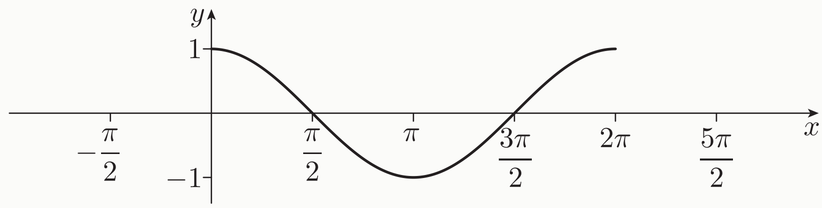


Figure 4 A sketch of the graph of the derivative of $f(x) = \sin x$, for x between 0 and 2π

Since the graph of the sine function repeats every 2π units, its gradients also repeat every 2π units. So you can now extend the sketch of the graph of the derivative to cover any interval of values of x that you wish. For example, Figure 5 shows a sketch of the graph for values of x between $-\pi$ and 3π.

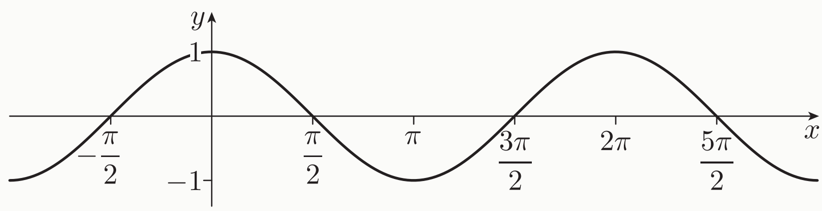


Figure 5 A sketch of the graph of the derivative of $f(x) = \sin x$, for x between $-\pi$ and 3π

The graph in Figure 5 should look familiar to you – it looks very like the graph of the cosine function! So it seems possible that the derivative of the sine function is the cosine function – in other words, that

$$\text{if } f(x) = \sin x, \text{ then } f'(x) = \cos x.$$

This is indeed true. It can be confirmed by using differentiation from first principles, but this is quite tricky to do. If you'd like to see some details about how it's done, then have a look at the document *The derivative of the sine function* on the module website.

Derivative of cos

Let's now look at obtaining a formula for the derivative of the cosine function, $f(x) = \cos x$. You could get a rough idea of what this derivative must be like by using a similar procedure to the one that you've just seen for the sine function. Figures 6 and 7 show the graph of $f(x) = \cos x$, and the sketch of its derivative that's obtained from this procedure.

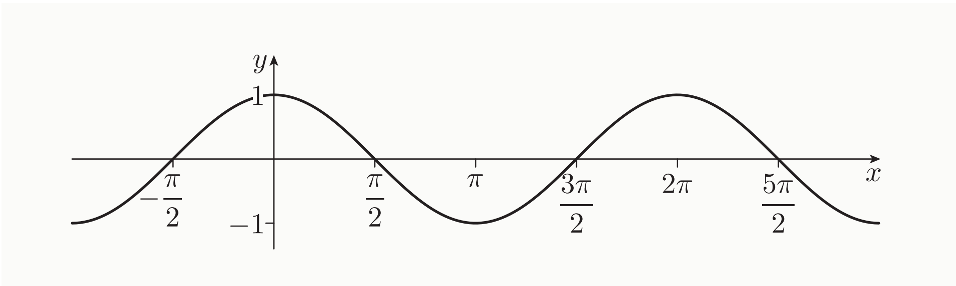


Figure 6 The graph of $f(x) = \cos x$

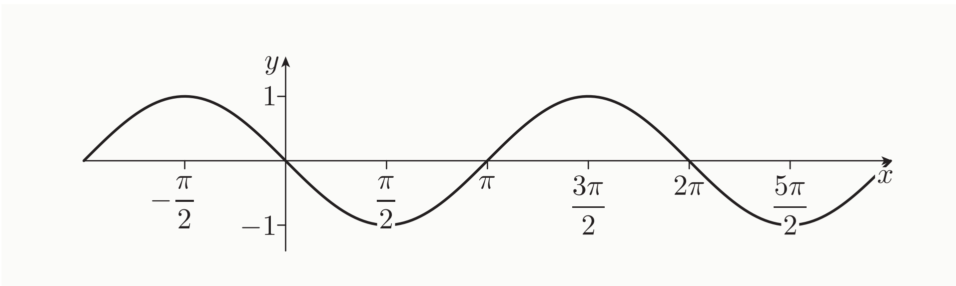


Figure 7 A sketch of the graph of the derivative of $f(x) = \cos x$

The graph in Figure 7 looks like the graph of the sine function, reflected in the x-axis. That is, it looks like the graph of $y = -\sin x$. So it seems possible that the derivative of the cosine function is the negative of the sine function – in other words, that

$$\text{if } f(x) = \cos x, \text{ then } f'(x) = -\sin x.$$

Again, this is indeed true. One way to confirm it is to use differentiation from first principles, but, as with the sine function, this is a tricky process.

A simpler way to confirm it is to deduce it from the fact that the derivative of the sine function is the cosine function. As you know, you can obtain the graph of the cosine function by translating the graph of the sine function by $\pi/2$ to the left. It follows that you can obtain the graph of the *derivative* of the cosine function by translating the graph of the *derivative* of the sine function by $\pi/2$ to the left. (This is because the two derivatives give the gradients of the two graphs.)

The derivative of the sine function is the cosine function, whose graph is shown in Figure 8. Translating this graph by $\pi/2$ to the left gives the graph in Figure 9. Because of the symmetries of the graphs of sine and cosine, this graph is the graph of $y = -\sin x$. So the derivative of $f(x) = \cos x$ is indeed $f'(x) = -\sin x$.

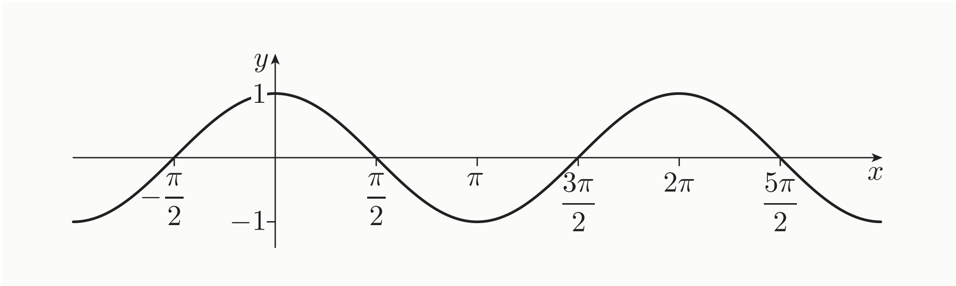


Figure 8 The graph of $y = \cos x$

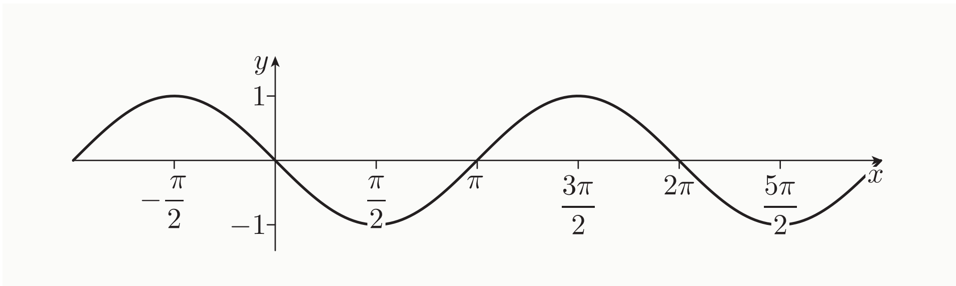


Figure 9 The graph of $y = \cos x$ translated by $\pi/2$ to the left

Derivative of tan

The final derivative of a trigonometric function that you'll meet in this subsection is the derivative of the tangent function. In fact,

$$\text{if } f(x) = \tan x, \text{ then } f'(x) = \sec^2 x.$$

Remember that $\sec^2 x$ means $(\sec x)^2$, and also that $\sec x$ means $1/(\cos x)$. So another way to write the derivative of $f(x) = \tan x$ is

$$f'(x) = \frac{1}{(\cos x)^2}.$$

However, we usually use the form $f'(x) = \sec^2 x$, for conciseness.

This formula for the derivative of the tangent function can be worked out by using the fact that $\tan x = (\sin x)/(\cos x)$, together with the formulas for the derivatives of $f(x) = \sin x$ and $f(x) = \cos x$. The process involves a rule for combining derivatives that you'll meet later in the unit, and you'll see the justification of the formula there.

For now, you might like to compare the graphs of $y = \tan x$ and $y = \sec^2 x$, shown in Figure 10, and convince yourself that the graph of $y = \sec^2 x$ does seem, at least roughly, to give the gradients of the graph of $y = \tan x$.

In particular, notice that the gradients of the graph of $y = \tan x$ are always positive and that the graph of $y = \tan x$ is the least steep whenever x is a multiple of π.

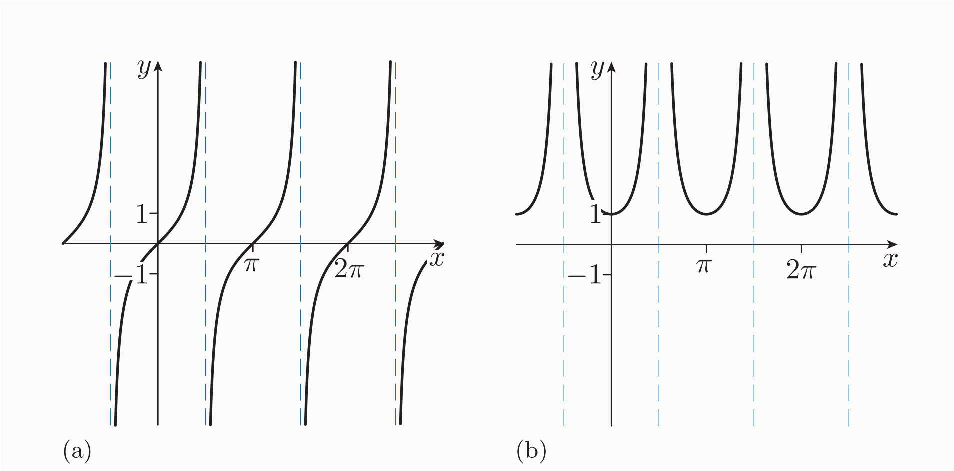


Figure 10 The graphs of (a) $y = \tan x$ (b) $y = \sec^2 x$

Here's a summary of the three derivatives that you've met in this subsection.

Derivatives of sin, cos and tan

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$$

In the next activity you're asked to use these formulas, together with the sum and constant multiple rules, to find the derivatives of some more functions. In this activity, and in general, remember that if a function is specified using function notation, then you should usually use Lagrange notation for its derivative, whereas if it's specified by an equation that expresses one variable in terms of another, then you should usually use Leibniz notation.

Activity 2 *Differentiating functions involving sin, cos and tan*

Write down the derivatives of the following functions.

(a) $f(x) = \sin x + \cos x$ (b) $g(u) = u^2 - \cos u$ (c) $P = 6\tan\theta$

(d) $r = -2(1 + \sin\phi)$

1.2 Derivatives of exp and ln

In this subsection you'll meet the formulas for the derivatives of the exponential function, $f(x) = e^x$, and its inverse function, the natural logarithm function $f(x) = \ln x$. Remember that e is a special irrational constant, whose value is about 2.718.

Derivative of exp

Let's look first at the exponential function, $f(x) = e^x$. Its graph is shown in Figure 11.

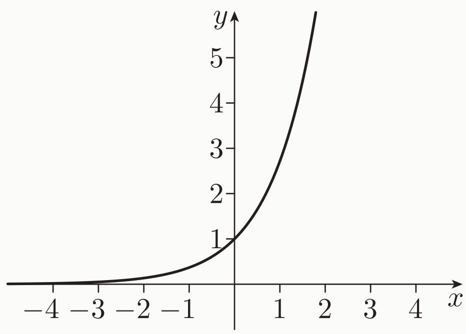


Figure 11 The graph of $y = e^x$

You saw in Unit 3 that the constant e has the special property that the graph of $f(x) = e^x$ has gradient exactly 1 at the point with x-coordinate 0, as shown in Figure 12. So you already know one value of $f'(x)$ for this function f: you know that $f'(0) = 1$.

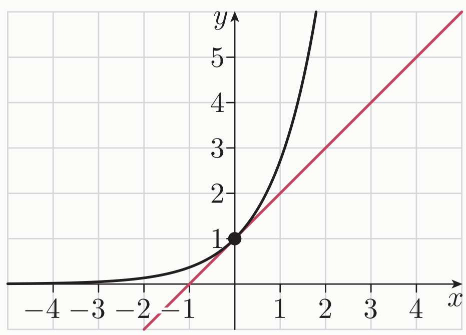


Figure 12 The graph of $y = e^x$ has gradient 1 when $x = 0$

You can find the general formula for the derivative $f'(x)$ of the function $f(x) = e^x$ by using differentiation from first principles. The difference quotient for $f(x) = e^x$ is

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h}.$$

As usual, to find a formula for $f'(x)$, you have to work out what happens to the value of the difference quotient as h gets closer and closer to zero. It's not immediately obvious how to do this, but surprisingly you can do it by using the fact that you already know what happens in the particular case when $x = 0$, since you know that $f'(0) = 1$.

When $x = 0$, the difference quotient is

$$\frac{e^{0+h} - e^0}{h} = \frac{e^h - 1}{h}.$$

So you already know that, as h gets closer and closer to zero, the value of this expression gets closer and closer to 1.

Now look again at the difference quotient for a general value of x. You can rearrange it as follows:

$$\frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = e^x \left(\frac{e^h - 1}{h} \right).$$

The expression in the brackets is the difference quotient in the case when $x = 0$. So as h gets closer and closer to zero, the expression in the brackets gets closer and closer to 1, and hence the value of the whole expression gets closer and closer to e^x. That is, the formula for the derivative is

$$f'(x) = e^x.$$

So the derivative of the exponential function is the exponential function!

To see that this makes sense, look at the graph of the exponential function in Figure 11. As x increases, the gradient of the graph increases, and it increases more and more rapidly. So as x increases, the value of the derivative of the exponential function increases, and it increases more and more rapidly. The exponential function itself has the property that as x increases, the value of the function increases, and it increases more and more rapidly.

The exponential function, and its constant multiples such as the function $f(x) = 3e^x$, are the only functions that are equal to their own derivatives (except, of course, for functions obtained by restricting the domains of these functions). It's this property that makes the constant e such a special number.

Derivative of ln

Now let's look at the natural logarithm function, $f(x) = \ln x$. Since this function is the inverse of the exponential function $f(x) = e^x$, its graph is the reflection of the graph of $f(x) = e^x$ in the line $y = x$, as shown in Figure 13.

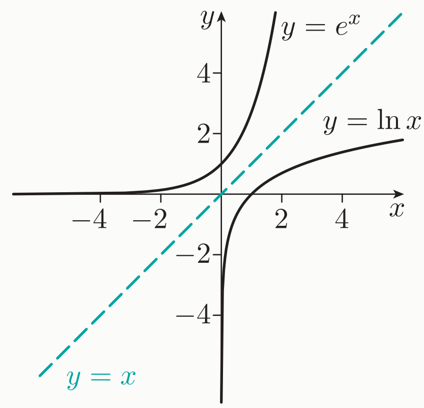


Figure 13 The graphs of $y = \ln x$ and $y = e^x$

So the gradients of the two graphs are related, and hence you can use the formula for the derivative of $f(x) = e^x$ to work out a formula for the derivative of $f(x) = \ln x$. If you'd like to see this done, then have a look at the document *The derivative of the natural logarithm function* on the module website.

However, a quicker way to work out the derivative of the natural logarithm function is to use a general rule, called the *inverse function rule*, that you'll meet later in the unit. This rule tells you how to use the formula for the derivative of *any* invertible function to work out a formula for the derivative of its inverse function. When you meet this rule, you'll see that the formula for the derivative of $f(x) = \ln x$ turns out to be

$$f'(x) = \frac{1}{x}.$$

For now, you might like to compare the graphs of $y = \ln x$ and $y = 1/x$, shown in Figure 14, and convince yourself that the function $y = 1/x$ does seem to give the gradients of the graph of the function $y = \ln x$. Only the right-hand half of the graph of $y = 1/x$ is shown. The left-hand half doesn't give gradients of the graph of $f(x) = \ln x$, because $\ln x$ is defined only for $x > 0$.

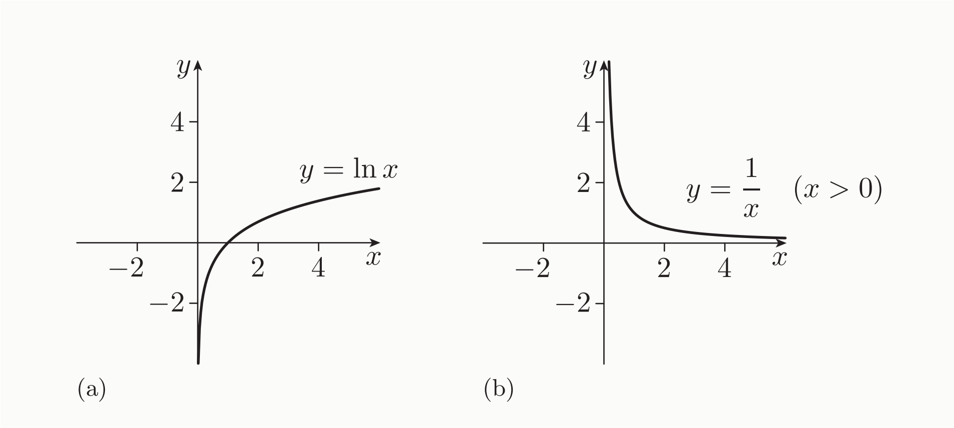


Figure 14 The graphs of (a) $y = \ln x$ (b) $y = 1/x$ $(x > 0)$

The two derivatives that you've met in this subsection are summarised in the box below.

Derivatives of exp and ln

$$\frac{\mathrm{d}}{\mathrm{d}x}\,(e^x) = e^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\,(\ln x) = \frac{1}{x}$$

In the next activity you're asked to use these formulas, together with the sum and constant multiple rules and the formulas for the derivatives of trigonometric functions, to find the derivatives of some more functions.

Activity 3 *Differentiating functions involving exp and log*

Find the derivatives of the following functions.

(a) $f(x) = e^x + \ln x$ (b) $h(r) = r - \cos r - 3\ln r$

(c) $v = \dfrac{1}{t} + \ln t$ (d) $w = 5 - 3e^u$ (e) $k = 4(\ln v - \tan v)$

The formulas for the derivatives of standard functions that you've met in this section are all included in the *Handbook*. However, it's worth memorising the formulas for the derivatives of sin, cos, exp and ln, at least, as they occur frequently.

2 Finding derivatives of more complicated functions

In this section you'll learn some more ways in which you can combine formulas for the derivatives of functions to obtain formulas for the derivatives of other, more complicated functions.

2.1 Product rule

Suppose that you know the formulas for the derivatives of two functions, and you want to know the formula for the derivative of their product. (Remember that the *product* of two functions is obtained by 'multiplying them together'.) For example, you already know the formulas for the derivatives of the functions $f(x) = x^2$ and $g(x) = \sin x$, but suppose that you want to know the formula for the derivative of the function $k(x) = x^2 \sin x$.

You've seen that to obtain a formula for the derivative of the *sum* of two functions, you just add the formulas for the derivatives of the two individual functions – this is the *sum rule*. So you might hope that to obtain a formula for the derivative of the *product* of two functions, you just multiply the formulas for the derivatives of the two individual functions.

Unfortunately, this doesn't work! For example, consider the two power functions $f(x) = x^2$ and $g(x) = x^3$. The derivatives of x^2 and x^3 are $2x$ and $3x^2$, respectively, and multiplying these two derivatives gives $6x^3$. However, the product of the functions $f(x) = x^2$ and $g(x) = x^3$ is the function $k(x) = x^5$, and you already know that the derivative of x^5 is $5x^4$. So multiplying the formulas for the derivatives of the two individual functions doesn't give the right answer for the derivative of their product.

In fact, to find a formula for the derivative of the product of two functions, you need to use the rule in the box below. It's given in Lagrange notation, but you'll see it in Leibniz notation shortly.

Product rule (Lagrange notation)

If $k(x) = f(x)g(x)$, then

$$k'(x) = f(x)g'(x) + g(x)f'(x),$$

for all values of x at which both f and g are differentiable.

The product rule can be proved by using the idea of differentiation from first principles, and you'll see this done at the end of this subsection.

Your main task in this subsection is to learn how to use the product rule. The next example demonstrates how it can be used to differentiate the function $k(x) = x^2 \sin x$, which was mentioned at the beginning of this section.

Example 1 *Using the product rule*

Differentiate the function

$$k(x) = x^2 \sin x.$$

Solution

Identify two functions f and g such that $k(x) = f(x)g(x)$.

Here $k(x) = f(x)g(x)$ where $f(x) = x^2$ and $g(x) = \sin x$.

Find their derivatives.

Then $f'(x) = 2x$ and $g'(x) = \cos x$.

Use the product rule.

By the product rule,

$$\begin{aligned} k'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^2 \cos x + (\sin x) \times 2x \end{aligned}$$

Simplify the answer.

$$= x^2 \cos x + 2x \sin x.$$

An alternative way to write the final answer in Example 1 is as $k'(x) = x(x \cos x + 2 \sin x)$. Either form is acceptable.

Here are some similar examples for you to try.

Activity 4 *Using the product rule*

Use the product rule to differentiate the following functions.

(a) $k(x) = x^3 \tan x$ (b) $k(x) = xe^x$

Activity 5 *Using the product rule again*

Check that if you find the derivative of the function $k(x) = x^5$ by writing $x^5 = x^2 \times x^3$ and using the product rule, then you obtain the same answer as you obtain by differentiating it directly.

The product rule can be written in Leibniz notation as follows.

Product rule (Leibniz notation)

If $y = uv$, where u and v are functions of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x},$$

for all values of x at which both u and v are differentiable.

(The phrase 'u and v are differentiable' in the box above is a condensed way of saying that if we write $u = f(x)$ and $v = g(x)$ then f and g are differentiable at x.)

However it's written, in essence the product rule tells you the following:

$$\begin{pmatrix}\text{derivative}\\ \text{of product}\\ \text{function}\end{pmatrix} = \begin{pmatrix}\text{first}\\ \text{function}\end{pmatrix} \times \begin{pmatrix}\text{derivative}\\ \text{of second}\\ \text{function}\end{pmatrix} + \begin{pmatrix}\text{second}\\ \text{function}\end{pmatrix} \times \begin{pmatrix}\text{derivative}\\ \text{of first}\\ \text{function}\end{pmatrix}.$$

For most people, the best way to remember and use it is as the abbreviated informal version below.

Product rule (informal)

$$\begin{pmatrix}\text{derivative}\\ \text{of product}\end{pmatrix} = (\text{first}) \times \begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix} + (\text{second}) \times \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix}.$$

When you use this informal version of the product rule, you can usually skip the steps of writing down the first and second functions and their derivatives, and just go straight to applying the rule. To do this, you recite the informal version of the rule in your head: as you think 'first', you write down the formula for the first function, then as you think 'derivative of second', you differentiate the second function in your head and write down the result, and so on. This is illustrated in the next example.

Example 2 *Using the product rule efficiently*

Differentiate the function

$$k(x) = (2x + 1)\ln x.$$

Solution

Identify the first and second functions: here first $= 2x + 1$ and second $= \ln x$. Use the rule

$$\begin{pmatrix}\text{derivative}\\ \text{of product}\end{pmatrix} = (\text{first}) \times \begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix} + (\text{second}) \times \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix}.$$

By the product rule,

$$k'(x) = (2x+1) \times \frac{1}{x} + (\ln x) \times 2$$

Simplify the answer.

$$\begin{aligned} &= \frac{2x+1}{x} + 2\ln x \\ &= \frac{2x+1}{x} + \frac{2x\ln x}{x} \\ &= \frac{2x+1+2x\ln x}{x}. \end{aligned}$$

There are various ways to write the final answer in Example 2. For example, the following are all acceptable:

$$2 + \frac{1}{x} + 2\ln x, \quad \frac{1}{x} + 2(1+\ln x) \quad \text{and} \quad \frac{2x(1+\ln x)+1}{x}.$$

You can practise using the informal version of the product rule in the next activity. Alternatively, you might prefer to continue using the formal version for a little longer, and move to the informal version when you feel more confident with it. You can use the exercises on the product rule in the Unit 7 practice quiz and exercise booklet for further practice.

Whichever version of the product rule you use, remember to simplify your final answers, where possible. You should always do this when you differentiate functions.

Activity 6 *Using the product rule efficiently*

Use the product rule to differentiate the following functions.

(a) $k(x) = (3x^2 + 2x + 1)e^x$ (b) $k(x) = \sin x \cos x$

(c) $k(z) = z \sin z$ (d) $v = (2t^2 - 1)\cos t$ (e) $m = (u^2 + 3)\ln u$

(f) $y = \sqrt{x} \sin x$

Hint: In part (f), convert the square root symbol to index notation before you apply the product rule.

Activity 7 *Using the product rule to find a gradient*

Find $\mathrm{d}y/\mathrm{d}x$ when $y = x^2e^x$. What is the gradient of the graph of $y = x^2e^x$ at the point with x-coordinate -1?

To finish this subsection, here's a proof of the product rule, using differentiation from first principles. It uses the Lagrange notation form of the product rule, which is repeated below.

Product rule (Lagrange notation)

If $k(x) = f(x)g(x)$, then

$$k'(x) = f(x)g'(x) + g(x)f'(x),$$

for all values of x at which both f and g are differentiable.

A proof of the product rule

Suppose that f and g are functions, and that the function k is given by $k(x) = f(x)g(x)$. Let x denote any value at which both f and g are differentiable. To determine whether $k'(x)$ exists, and find it if it does exist, you have to consider what happens to the difference quotient for k at x, which is

$$\frac{k(x+h) - k(x)}{h}$$

(where h can be either positive or negative, but not zero), as h gets closer and closer to zero. Since $k(x) = f(x)g(x)$, this difference quotient is equal to

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

You can rearrange this expression so that it includes the difference quotients for f and g at x. To do this, you first subtract and add $f(x)g(x+h)$ in the numerator. Then the expression becomes

$$\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h},$$

which is equal to

$$\left(\frac{f(x+h) - f(x)}{h}\right) g(x+h) + f(x)\left(\frac{g(x+h) - g(x)}{h}\right).$$

The expression in the first pair of large brackets is the difference quotient for f at x, and the expression in the second pair of large brackets is the difference quotient for g at x. So as h gets closer and closer to zero, the values of these two expressions get closer and closer to $f'(x)$ and $g'(x)$, respectively. Also, the value of $g(x+h)$ gets closer and closer to $g(x)$.

Hence the whole expression gets closer and closer to $f'(x)g(x) + f(x)g'(x)$; that is, $f(x)g'(x) + g(x)f'(x)$. So

$$k'(x) = f(x)g'(x) + g(x)f'(x),$$

which is the product rule.

2.2 Quotient rule

Suppose now that you know the formulas for the derivatives of two functions, and you want to know the formula for the derivative of their quotient. (Remember that a *quotient* of two functions is obtained by 'dividing one by the other'.) For example, you already know the formulas for the derivatives of the functions $f(x) = x^2$ and $g(x) = \sin x$, but suppose that you want to know the formula for the derivative of the function $k(x) = x^2/\sin x$.

You can find the derivative of the quotient of two functions by using the rule in the box below. It's given in Lagrange notation, but you'll see a version in Leibniz notation shortly.

Quotient rule (Lagrange notation)

If $k(x) = f(x)/g(x)$, then

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2},$$

for all values of x at which both f and g are differentiable and $g(x) \neq 0$.

You'll see a justification of the quotient rule in Subsection 2.5. Here's an example that illustrates its use.

Example 3 *Using the quotient rule*

Differentiate the function

$$k(x) = \frac{x^2}{\sin x}.$$

Solution

Identify two functions f and g such that $k(x) = f(x)/g(x)$.

Here $k(x) = f(x)/g(x)$ where $f(x) = x^2$ and $g(x) = \sin x$.

Find their derivatives.

Then $f'(x) = 2x$ and $g'(x) = \cos x$.

Use the quotient rule.

By the quotient rule,

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{(\sin x) \times 2x - x^2 \cos x}{(\sin x)^2}$$

Simplify the answer.

$$= \frac{2x \sin x - x^2 \cos x}{\sin^2 x}.$$

There are various ways to write the final answer in Example 3. For example, the following are all acceptable:

$$\frac{x(2\sin x - x\cos x)}{\sin^2 x}, \quad \frac{2x}{\sin x} - \frac{x^2 \cos x}{\sin^2 x}, \quad \frac{2x - x^2 \cot x}{\sin x}.$$

Here are two similar examples for you to try.

Activity 8 *Using the quotient rule*

Use the quotient rule to differentiate the following functions.

(a) $k(x) = \dfrac{e^x}{x}$ (b) $k(x) = \dfrac{x^3}{2x+1}$

The quotient rule can be written in Leibniz notation as follows.

Quotient rule (Leibniz notation)

If $y = u/v$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

for all values of x at which both u and v are differentiable and $v \neq 0$.

For most people the best way to remember and use the quotient rule is as the informal version below.

Quotient rule (informal)

$$\begin{pmatrix}\text{derivative}\\ \text{of quotient}\end{pmatrix} = \frac{(\text{bottom}) \times \begin{pmatrix}\text{derivative}\\ \text{of top}\end{pmatrix} - (\text{top}) \times \begin{pmatrix}\text{derivative}\\ \text{of bottom}\end{pmatrix}}{(\text{bottom})^2}.$$

You can use this informal version of the rule in a similar way to the informal version of the product rule. To differentiate a quotient of two functions by using the quotient rule, you recite the informal version of the rule in your head: as you think 'bottom', you write down the formula for the bottom function, then as you think 'derivative of top' you differentiate the top function in your head and write down the result, and so on. (Make sure that you remember that the quotient rule contains a minus sign rather than a plus sign.) This is illustrated in the next example.

Example 4 *Using the quotient rule efficiently*

Differentiate the function

$$k(x) = \frac{2x+1}{\ln x}.$$

Solution

Identify the top and bottom functions: here top $= 2x + 1$ and bottom $= \ln x$. Use the rule

$$\begin{pmatrix}\text{derivative}\\ \text{of quotient}\end{pmatrix} = \frac{(\text{bottom}) \times \begin{pmatrix}\text{derivative}\\ \text{of top}\end{pmatrix} - (\text{top}) \times \begin{pmatrix}\text{derivative}\\ \text{of bottom}\end{pmatrix}}{(\text{bottom})^2}.$$

By the quotient rule,

$$k'(x) = \frac{(\ln x) \times 2 - (2x+1) \times \dfrac{1}{x}}{(\ln x)^2}$$

Simplify the answer. To clear the fraction in the numerator, multiply both the numerator and the denominator by x.

$$= \frac{2x \ln x - (2x+1) \times \dfrac{1}{x} \times x}{x(\ln x)^2}$$

$$= \frac{2x \ln x - 2x - 1}{x(\ln x)^2}.$$

You can practise using the informal version of the quotient rule in the next activity. Alternatively, as with the product rule, you might prefer to continue using the formal version for a little longer, and move to the informal version when you feel more confident with it.

Activity 9 *Using the quotient rule*

Use the quotient rule to differentiate the following functions.

(a) $f(x) = \dfrac{e^x}{x^3}$ (b) $g(u) = \dfrac{u-1}{e^u}$ (c) $z = \dfrac{r+1}{r-1}$

(d) $m = \dfrac{\theta}{\cos\theta}$ (e) $p(t) = \dfrac{\ln t}{t^2}$

In Subsection 1.1 you met the fact that the derivative of $\tan x$ is $\sec^2 x$, but this result wasn't justified there. It can be confirmed by using the quotient rule, as shown in the next example. (Note, however, that this justification is incomplete, as you haven't yet seen a proof of the quotient rule!)

Example 5 *Using the quotient rule to find the derivative of tan*

Show that the derivative of $\tan x$ is $\sec^2 x$.

Solution

Use the fact that $\tan x = (\sin x)/(\cos x)$, and apply the quotient rule.

By the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\end{aligned}$$

Simplify this expression by using the facts that $\cos^2 x + \sin^2 x = 1$ and $1/\cos x = \sec x$.

$$\begin{aligned}&= \frac{1}{\cos^2 x}\\ &= \sec^2 x,\end{aligned}$$

as required.

You can also use the quotient rule to find the derivatives of the trigonometric functions cosec, sec and cot. These derivatives are given in the box below, and you're asked to confirm them in the next activity.

Derivatives of cosec, sec and cot

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Activity 10 *Finding the derivatives of further trigonometric functions*

Use the quotient rule to confirm the following formulas for derivatives.

(a) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ (b) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(c) $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Hint: Use the facts that $\cot x = (\cos x)/(\sin x)$, $\sec x = 1/(\cos x)$ and $\operatorname{cosec} x = 1/(\sin x)$.

The derivatives of cosec, sec and cot can now be included in the list of derivatives of standard functions that you have available to work with. The *Handbook* includes a table of all the standard derivatives given in this module.

2.3 Chain rule

Now that you know the product rule and the quotient rule, you can differentiate a much wider range of functions than you could before. However, there are still some fairly simple functions that you can't differentiate using any of the rules that you've met so far.

For example, consider the function $k(x) = \sin(x^2)$. This function isn't a sum, a product or a quotient of functions that you can differentiate. However, it's still made up of functions that you can differentiate, just combined in a different way. To work out $k(x)$ for a particular value of x, you have to first square x, then find the sine of the result. So the function k is a *composite* of the functions $f(x) = x^2$ and $g(x) = \sin x$, as shown in Figure 15. You met the idea of a composite of two functions in Subsection 3.2 of Unit 3.

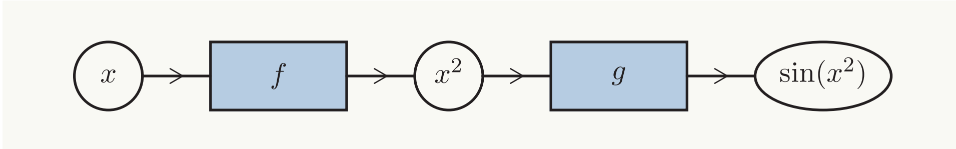


Figure 15 A composite of the functions $f(x) = x^2$ and $g(x) = \sin x$

The functions $f(x) = x^2$ and $g(x) = \sin x$ are functions that you can differentiate.

In this subsection you'll meet another rule for differentiation, called the *chain rule*, which allows you to differentiate functions that are composites of functions that you can already differentiate. You'll see towards the end of the subsection why it's called the chain rule. It's also sometimes called the *composite rule* or the *function of a function rule*.

The first step in learning to use the chain rule is to learn to recognise when a function is a composite of two functions, and how to decompose such a function into its constituent functions. (*Decomposing* is the opposite of *composing*.) To see how to do this, let's consider again the function

$$k(x) = \sin(x^2).$$

While you're learning about decomposing functions, it's easier to work with functions expressed using two variables rather than function notation, so let's first express this function as

$$y = \sin(x^2).$$

To recognise that it's a composite of two functions, you observe that it's of the form

$$y = \sin(\text{something}),$$

where the 'something' is an expression that contains the input variable x.

You can then decompose it into its two constituent functions.
A convenient way to do this is to set the 'something' equal to an extra variable, say u. This gives

$$y = \sin(u) \quad \text{where} \quad u = x^2.$$

These two equations specify the two constituent functions.

In general, suppose that you have an equation that specifies y as a function of x. If the equation is of the form

$$y = \text{a function of 'something'},$$

where the 'something' is a function of the input variable x, then the original function is a composite function. To decompose it into its two constituent functions, you set the 'something' equal to u. In other words, you write y as a function of u, where u is a function of x.

Here are some more examples that illustrate how to do this.

Example 6 *Decomposing composite functions*

For each of the following equations, write y as a function of u, where u is a function of x.

(a) $y = \cos(\ln x)$ (b) $y = e^{2x}$ (c) $y = \sin^2 x$

Solution

(a) $y = \cos(\text{something})$, so set the something $= u$.

$y = \cos u$ where $u = \ln x$.

(b) $y = e^{\text{something}}$, so set the something $= u$.

$y = e^u$ where $u = 2x$.

(c) Remember that $\sin^2 x$ means $(\sin x)^2$. Therefore $y = \text{something}^2$, so set the something $= u$.

$y = u^2$ where $u = \sin x$.

Here are some examples for you to try.

Activity 11 *Decomposing composite functions*

For each of the following equations, write y as a function of u, where u is a function of x.

(a) $y = \ln(x^4)$ (b) $y = \sin(x^2 - 1)$ (c) $y = \cos(e^x)$

(d) $y = \cos^3 x$ (e) $y = e^{\tan x}$ (f) $y = \sqrt{\ln x}$

(g) $y = (x + \sqrt{x})^9$ (h) $y = \sqrt[3]{x^2 - x - 1}$ (i) $y = e^{\sqrt{x}}$

The chain rule, which allows you to differentiate composite functions, is given below. It's given in Leibniz notation, because it's easier to remember in this form, but you'll see it in Lagrange notation later in this subsection.

Chain rule (Leibniz notation)

If y is a function of u, where u is a function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}x},$$

for all values of x where y as a function of u, and u as a function of x, are differentiable.

As you can see, in Leibniz notation the chain rule looks like an obvious statement about fractions, which makes it easy to remember. You should keep in mind, though, that this is *not* what it is! The pieces of notation $\mathrm{d}y/\mathrm{d}x$, $\mathrm{d}y/\mathrm{d}u$ and $\mathrm{d}u/\mathrm{d}x$ represent derivatives, not fractions.

As with the other rules for combining derivatives, the chain rule can be proved by using the idea of differentiation from first principles. Unfortunately, this is quite tricky to do, and the proof isn't included in this module.

However, here's an informal way to see that the chain rule makes sense. Suppose that y is a function of u, where u is a function of x, as in the statement of the chain rule. Suppose that for a particular value of x, the value of $\mathrm{d}u/\mathrm{d}x$ is 2, and for the value of u that corresponds to this particular value of x, the value of $\mathrm{d}y/\mathrm{d}u$ is 3. Since y is increasing at the rate of 3 units for every unit that u increases, and u is increasing at the rate of 2 units for every unit that x increases, you'd expect y to be increasing at the rate of $3 \times 2 = 6$ units for every unit that x increases. This is what the chain rule tells you.

The next example shows you how to use the chain rule.

Example 7 *Using the chain rule*

Differentiate the function $y = \sin(x^2)$.

Solution

Recognise that the given function is a function of 'something', where the 'something' involves the input variable x. Hence decompose the given function.

Here $y = \sin u$ where $u = x^2$.

Find the derivatives of the constituent functions.

This gives

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x.$$

Apply the chain rule, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

By the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos u) \times 2x$$

Express the answer in terms of x, by substituting for u in terms of x. Here we have $u = x^2$.

$$= (\cos(x^2)) \times 2x$$

Simplify the answer.

$$= 2x\cos(x^2).$$

Notice that when you apply the chain rule as illustrated in Example 7, you first obtain an answer that's expressed in terms of both u and x. You then have to substitute for u, using the equation for u in terms of x that you wrote down when you decomposed the function, to obtain a final answer in terms of x.

Activity 12 *Using the chain rule*

Differentiate the following functions.

(a) $y = e^{5x}$ (b) $y = \sin(2x^2 - 3)$ (c) $y = \sin\left(\dfrac{x}{4}\right)$

(d) $y = \cos(e^x)$ (e) $y = \ln(\sin x)$ (f) $y = \sin^3 x$

Hint: In part (f), remember that $\sin^3 x$ means $(\sin x)^3$.

You've now seen how to apply the chain rule by first introducing an extra variable, usually u, which you then remove again to obtain the final answer. Once you're familiar with using the chain rule, you can often apply it without introducing this extra variable at all.

To see how to do this, consider again the equation $y = \sin(x^2)$. Think of the expression that you would have written as u, that is, the expression that was referred to as 'something' earlier, as one 'big variable':

$$y = \sin(x^2)$$

To find $\mathrm{d}y/\mathrm{d}x$, first differentiate with respect to this big variable; that is, with respect to the 'something'. This gives

$$\cos(x^2)$$

By the chain rule, to obtain the final answer, you now need to multiply by the derivative of the something with respect to x. This gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x^2) \times 2x.$$

This is the final answer, but, as before, you should simplify it slightly to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\cos(x^2).$$

Here are two more examples. Notice that in the second of these examples the function is expressed using function notation rather than using two variables. This makes no difference to how you apply the ideas above.

Example 8 *Using the chain rule without introducing an extra variable*

Differentiate the following functions.

(a) $y = \ln(2x-1)$ (b) $k(x) = \cos^4 x$

Solution

(a) Think of the 'something', $2x-1$, as a 'big variable'. Differentiate with respect to this variable. Then multiply by the derivative of $2x-1$ with respect to x.

By the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2x-1} \times 2$$

Simplify the answer.

$$= \frac{2}{2x-1}.$$

(b) The function is $k(x) = (\cos x)^4$, so think of the 'something', $\cos x$, as a 'big variable'. Differentiate with respect to this variable. Then multiply by the derivative of $\cos x$ with respect to x.

By the chain rule,

$$k'(x) = (4\cos^3 x)(-\sin x)$$

Simplify the answer.

$$= -4\sin x \cos^3 x.$$

Try this quicker way of using the chain rule in the following activity.

Activity 13 *Using the chain rule without introducing an extra variable*

Differentiate the following functions.

(a) $f(x) = \cos(4x)$ (b) $f(x) = e^{2x+1}$ (c) $r(x) = \cos^5 x$

(d) $g(x) = \ln(x^2 + x + 1)$ (e) $v(t) = \sin(e^t)$ (f) $v = \ln(r^5)$

(g) $r = \tan^2\theta$ (h) $R = e^{\tan q}$ (i) $A = (1+p^2)^9$

You should try to get used to applying the chain rule in the quick way demonstrated in Example 8, as you'll find this skill useful in calculus. However, sometimes, if a function that you want to differentiate using the chain rule isn't straightforward, then you might find it easier to revert to introducing an extra variable. It's fine to do that if you find it helpful.

Sometimes you can make it easier to apply the chain rule without introducing an extra variable if you start by rearranging the formula for the function. This applies in particular to composites that involve power functions. Here's an example.

Example 9 *Applying the chain rule to a composite involving a power function*

Differentiate the function

$$f(x) = \frac{1}{\sqrt{\sin x}}.$$

Solution

The function f is of the form $f(x) = 1/\sqrt{\text{something}}$, where the 'something' is $\sin x$. So use the chain rule. Start by rearranging the formula for f into the form 'something^n'.

The function is

$$f(x) = \frac{1}{\sqrt{\sin x}};$$

that is,

$$f(x) = (\sin x)^{-1/2}.$$

Apply the chain rule.

By the chain rule,

$$\begin{aligned} f'(x) &= -\tfrac{1}{2}(\sin x)^{-3/2}(\cos x) \\ &= -\frac{\cos x}{2\sin^{3/2} x}. \end{aligned}$$

With practice, you may be able to do the sort of initial rearranging needed in Example 9 in your head.

Activity 14 *Applying the chain rule to composites involving power functions*

Differentiate the following functions.

(a) $f(x) = \sqrt{\ln x}$ (b) $p(w) = \dfrac{1}{(w^3+7)^4}$ (c) $f(x) = \cos(\sqrt{x})$

(d) $f(x) = \dfrac{1}{\sqrt{e^x}}$ (e) $a(t) = \dfrac{1}{(2-3t-3t^2)^{1/3}}$ (f) $C(p) = e^{1/p^2}$


Gerald was never very good at using the chain rule.

Now here's an explanation of where the name 'chain rule' comes from. Consider a function that's a composite of *three* functions. Suppose that

y is a function of u,
u is a function of v,
v is a function of x.

You can find the derivative of y with respect to x by applying the chain rule twice:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}$$

so, using the second equation to substitute in the first equation, you obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}.$$

As you can see, this gives a 'chain' of derivatives. You can keep adding 'links in the chain' indefinitely. One of the examples in Subsection 2.5, Example 12, illustrates how you can apply the chain rule twice in this way.

Finally in this subsection, let's translate the chain rule into Lagrange notation. You can do this by thinking about the way of applying it without introducing an extra variable. Suppose that f and g are functions, and you want to differentiate the expression $g(f(x))$ with respect to x. The chain rule says that you can do this by differentiating it with respect to the 'something' $f(x)$, then multiplying the result by the derivative of $f(x)$ with respect to x. This gives the version of the chain rule below.

Chain rule (Lagrange notation)

If $k(x) = g(f(x))$, where f and g are functions, then

$$k'(x) = g'(f(x))f'(x)$$

for all values of x such that f is differentiable at x and g is differentiable at $f(x)$.

The first use of the chain rule seems to be by Gottfried Wilhelm Leibniz in a memoir of 1676, in which he calculated (with mistakes!) the derivative of $\sqrt{a+bz+cz^2}$. Further examples of the use of the rule appear in the first calculus textbook, written by Guillaume de l'Hôpital and published in 1696, but nowhere is the rule actually stated. (There is more about l'Hôpital's book in Subsection 3.4.) It was not until the 19th century that the chain rule was commonly stated explicitly, and at that time it was generally described as a 'rule for differentiating a function of a function'. The term chain rule seems to have originated as a translation of the German word *Kettenregel*, which was used in German calculus books in the early 20th century.

2.4 Differentiating functions of linear expressions

This subsection is about a type of function that occurs frequently, and which you can differentiate by using the chain rule.

Consider the functions

$$k(x) = \sin(3x) \quad \text{and} \quad h(x) = e^{-2x}.$$

Each of them is a composite function, and in each of them the 'something' is a constant multiple of the input variable x; the 'somethings' are $3x$ and $-2x$, respectively. It's worth noticing, and remembering, what happens when you use the chain rule to differentiate functions like these. For example,

$$\text{if } k(x) = \sin(3x), \text{ then } k'(x) = \cos(3x) \times 3 = 3\cos(3x),$$
$$\text{if } h(x) = e^{-2x}, \text{ then } h'(x) = e^{-2x} \times (-2) = -2e^{-2x}.$$

You can see that to differentiate a composite function in which the 'something' is ax, where a is a constant, you differentiate with respect to ax, and then multiply by a.

It's worth remembering this special case of the chain rule, which can be stated as below.

Derivative of a function of the form $k(x) = f(ax)$

If $k(x) = f(ax)$, where a is a non-zero constant, then

$$k'(x) = af'(ax),$$

for all values of x such that f is differentiable at ax.

You can use this rule to quickly differentiate any function of the type discussed above. For example, it tells you that

$$\text{if } k(x) = \sin(5x), \text{ then } k'(x) = 5\cos(5x).$$

Try using the rule above in the next activity.

Activity 15 *Differentiating functions of the form $k(x) = f(ax)$*

Differentiate the following functions.

(a) $k(x) = \cos(7x)$ (b) $r(\theta) = \sin\left(\dfrac{\theta}{2}\right)$ (c) $g(u) = e^{3u/2}$

(d) $h(x) = e^{-x}$ (e) $p(\alpha) = \cos(-8\alpha)$ (f) $w(x) = \ln(7x)$

(g) $f(t) = \sin\left(\dfrac{-2t}{3}\right)$ (h) $k(\phi) = \tan(3\phi)$

In fact you can slightly extend the special case of the chain rule above. Notice that, by the chain rule,

$$\text{if } k(x) = \sin(3x-2), \text{ then } k'(x) = \cos(3x-2) \times 3 = 3\cos(3x-2),$$
$$\text{if } h(x) = e^{-2x+5}, \text{ then } h'(x) = e^{-2x+5} \times (-2) = -2e^{-2x+5}.$$

You can see that, in general, to differentiate a composite function in which the 'something' is $ax+b$, where a and b are constants, you differentiate with respect to $ax+b$, and then multiply by a. This special case of the chain rule is summarised below.

Derivative of a function of the form $k(x) = f(ax+b)$

If $k(x) = f(ax+b)$, where a and b are constants with a non-zero, then

$$k'(x) = af'(ax+b),$$

for all values of x such that f is differentiable at $ax+b$.

For example, this rule tells you that

$$\text{if } k(x) = \sin(5x-8), \text{ then } k'(x) = 5\cos(5x-8).$$

Try using it in the next activity.

Activity 16 *Differentiating functions of the form $k(x) = f(ax+b)$*

Differentiate the following functions.

(a) $k(x) = \cos(7x+4)$ (b) $h(x) = e^{-x-3}$ (c) $r(\theta) = \sin\left(\dfrac{\theta-1}{2}\right)$

(d) $s(\theta) = \sin\left(\dfrac{2-\theta}{3}\right)$

Remember that an expression of the form $ax + b$, where a and b are constants, is called a *linear* expression in x. So in this subsection you've seen how to quickly differentiate functions of linear expressions.

2.5 Using the differentiation rules together

You can differentiate a wide range of functions by combining the various differentiation rules that you've met. For example, you can use more than one differentiation rule, or more than one application of the same rule. This subsection starts with three examples that show you the kinds of things that you can do.

When you're combining differentiation rules in this way, it can be useful to use the notation $\mathrm{d}/\mathrm{d}x$ to indicate the derivative of a function that appears as part of your working, then actually do the differentiation in the next line. This is illustrated in the examples.

Example 10 *Combining the differentiation rules*

Find the derivative of

$$f(x) = x^2 \sin(x^3).$$

Solution

The function is of the form something × something. So apply the product rule. The second 'something', $\sin(x^3)$, isn't straightforward to differentiate, so use the notation $\mathrm{d}/\mathrm{d}x$ to indicate its derivative.

$$f'(x) = x^2 \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x^3)) + (\sin(x^3))(2x) \quad \text{(by the product rule)}$$

Now differentiate $\sin(x^3)$. It's of the form sin(something), so apply the chain rule.

$$= x^2(\cos(x^3))(3x^2) + (\sin(x^3))(2x) \quad \text{(by the chain rule)}$$

Simplify the answer.

$$= 3x^4 \cos(x^3) + 2x \sin(x^3).$$

Example 11 *Combining the differentiation rules again*

Find the derivative of

$$y = \left(\frac{x^2-2}{x^2+1}\right)^4.$$

Solution

The function is of the form (something)^4. So apply the chain rule. The 'something' isn't straightforward to differentiate, so use the notation $\mathrm{d}/\mathrm{d}x$ to indicate its derivative.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4\left(\frac{x^2-2}{x^2+1}\right)^3 \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-2}{x^2+1}\right) \quad \text{(by the chain rule)}$$

Now differentiate $(x^2-2)/(x^2+1)$. It's of the form something/something, so apply the quotient rule.

$$= 4\left(\frac{x^2-2}{x^2+1}\right)^3 \times \left(\frac{(x^2+1)(2x)-(x^2-2)(2x)}{(x^2+1)^2}\right)$$

(by the quotient rule)

Simplify the answer.

$$\begin{aligned} &= 4\left(\frac{x^2-2}{x^2+1}\right)^3 \times \frac{(2x)(x^2+1-x^2+2)}{(x^2+1)^2} \\ &= 4\left(\frac{x^2-2}{x^2+1}\right)^3 \times \frac{6x}{(x^2+1)^2} \\ &= \frac{24x(x^2-2)^3}{(x^2+1)^5}. \end{aligned}$$

Example 12 *Combining the differentiation rules yet again*

Find the derivative of

$$g(t) = \cos(e^{2t+1}).$$

Solution

The function is of the form cos(something). So apply the chain rule. The 'something' isn't straightforward to differentiate, so use the notation $\mathrm{d}/\mathrm{d}t$ to indicate its derivative.

$$g'(t) = -\sin(e^{2t+1}) \times \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2t+1}\right) \quad \text{(by the chain rule)}$$

Now differentiate e^{2t+1}. It's of the form $e^{\text{something}}$. So apply the chain rule again.

$$= -\sin(e^{2t+1}) \times e^{2t+1} \times 2 \quad \text{(by the chain rule again)}$$

Simplify the answer.

$$= -2e^{2t+1}\sin(e^{2t+1}).$$

You can practise combining the differentiation rules in the next activity. If you find it tricky to work out which rules you need to apply, then the checklist below might help.

Checklist for differentiating a function

1. Is it a standard function (is its derivative given in the *Handbook*)?
2. Can you use the constant multiple rule and/or sum rule?
3. Can you rewrite it to make it easier to differentiate? For example, multiplying out brackets may help.
4. Is it of the form $f(ax)$ or $f(ax+b)$, where a and b are constants and f is a function? If so, use the rule for differentiating a function of a linear expression.
5. Can you use the product rule (is it of the form $f(x) = \text{something} \times \text{something}$)?
6. Can you use the quotient rule (is it of the form $f(x) = \text{something}/\text{something}$)?
7. Can you use the chain rule (is it of the form $f(x) = \text{a function of something}$)?

When you use a differentiation rule, you usually have to find the derivatives of simpler functions. You can apply the checklist above to each of these simpler functions in turn.

Note also that when you need to apply more than one rule to differentiate a function, the first rule that you should apply is the one that applies to the overall structure of the rule of the function. For example, as you saw in Example 10, the overall structure of the function $f(x) = x^2\sin(x^3)$ is that it's a product, since $x^2\sin(x^3)$ is the product of x^2 and $\sin(x^3)$. So, to differentiate this function, you start by applying the product rule, ignoring for the moment the fact that one of the two functions in the product is itself a composite. After you've done that, you can deal with the composite.

Activity 17 *Combining the differentiation rules*

Differentiate the following functions.

(a) $g(x) = (x^2 + 1)(x^3 + 1)$ (b) $f(x) = \dfrac{x}{(x-2)^3}$

(c) $g(x) = \cos(x \ln x)$ (d) $h(x) = e^{x/2} \sin(3x)$ (e) $q(u) = \dfrac{\cos(4u)}{e^{3u}}$

(f) $f(x) = e^{\sin(5x)}$ (g) $h(x) = \cos^3(x^2)$ (h) $z(x) = \dfrac{e^{-2x}}{5}$

(i) $G(x) = 3 \sin x \cos x + 2 \sin(3x)$ (j) $p(t) = te^t \sin t$

(k) $r = \sin(2\theta) \cos \theta$ (l) $h = ze^{3z}$ (m) $v = \dfrac{\sin \sqrt{u}}{\sqrt{u}}$

Sometimes you have to differentiate functions that involve constants represented by letters. You can do this in the usual way, treating the constants like any other numbers.

Activity 18 *Differentiating functions involving constants*

Differentiate the following functions.

(a) $g(x) = e^{-kx}$, where k is a constant.

(b) $f(x) = x \cos(\pi x)$.

(c) $h = \sin^2(a\theta)$, where a is a constant.

(d) $y = \cos(3x) + c$, where c is a constant.

Here's another technique that's occasionally useful when you want to differentiate a complicated function. Whenever you have a function that you can differentiate using the quotient rule, an alternative way to differentiate it is to rearrange its rule to express it as a *product* of two functions, and then apply the product rule.

In the next activity you're asked to differentiate a function in this way. (You were asked to differentiate the same function using the quotient rule in Activity 9(e).)

Activity 19 *Differentiating a quotient by using the product rule*

Differentiate the function $p(t) = \dfrac{\ln t}{t^2}$ by writing it as $p(t) = t^{-2} \ln t$ and using the product rule.

In fact, the quotient rule is really just a combination of the product rule and a special case of the chain rule. Here's a proof of it, as promised in Subsection 2.2.

A proof of the quotient rule

Suppose that the function k has rule $k(x) = f(x)/g(x)$. This rule can be written as

$$k(x) = \frac{1}{g(x)} \times f(x);$$

that is,

$$k(x) = (g(x))^{-1} f(x).$$

So, for each value of x at which both f and g are differentiable and $g(x) \neq 0$,

$$\begin{aligned} k'(x) &= (g(x))^{-1} f'(x) + f(x) \frac{\mathrm{d}}{\mathrm{d}x}\left((g(x))^{-1}\right) \quad \text{(by the product rule)} \\ &= (g(x))^{-1} f'(x) + f(x)\left(-(g(x))^{-2}\right) g'(x) \quad \text{(by the chain rule)} \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}, \end{aligned}$$

which is the quotient rule.

A proof of the formula for the derivative of a power function

Finally in this subsection, here's a proof of the general formula for the derivative of a power function, as promised in Unit 6. The formula states that, for any constant n,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}.$$

The proof given here is valid only for $x > 0$, as you'll see. However, for a power function $y = x^n$ that's defined when $x < 0$, you can use the fact that the formula holds for $x > 0$, together with the symmetry of the graph of $y = x^n$, to deduce that the formula also holds for $x < 0$; the details are not given here. To see that the formula also holds when $x = 0$, you can use the fact that at $x = 0$ the gradient of the graph of $y = x^n$ is 0.

To prove the formula for $x > 0$, you can use the fact that $x = e^{\ln x}$ for such values of x to write the expression x^n in the following form:

$$x^n = (e^{\ln x})^n = e^{n \ln x}.$$

This gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(x^n) &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{n \ln x}) \\
&= (e^{n \ln x}) \frac{\mathrm{d}}{\mathrm{d}x}(n \ln x) \quad \text{(by the chain rule)} \\
&= (e^{n \ln x}) \times n \times \frac{\mathrm{d}}{\mathrm{d}x}(\ln x) \quad \text{(by the constant multiple rule)} \\
&= (e^{n \ln x}) \times n \times \frac{1}{x} \\
&= x^n \times n \times \frac{1}{x} \\
&= nx^{n-1},
\end{aligned}$$

which is the formula stated above.

Note that the two proofs that you've just seen, namely the proof of the quotient rule and the proof of the formula for the derivative of a power function, both depend on the chain rule. Since this module doesn't include a proof of the chain rule, strictly the proofs of these two results are incomplete.

3 More differentiation

In this section you'll see how to use differentiation to solve a type of problem that occurs frequently in applications of mathematics, how to use differentiation on a computer, and how to find the derivative of the inverse of a function if you know the derivative of the original function.

3.1 Optimisation problems

In mathematics, and its applications, it's often helpful to solve problems that involve identifying the best possible option from a choice of suitable possibilities. Problems of this type are known as **optimisation** problems.

For example, the manufacturer of an electronic device might want to identify the best price that it should charge for its device, if its goal is to achieve the maximum profit. Increasing the price might increase the profit, but increasing it further might decrease the profit, by causing fewer of the devices to be sold. This problem is an example of a **maximisation** problem, because it involves identifying the circumstances under which the *maximum* value of a quantity is obtained. Similarly, some optimisation problems are **minimisation** problems, which involve identifying the circumstances under which the *minimum* value of a quantity is obtained.

You can often solve optimisation problems by modelling them mathematically and then applying calculus. In particular, you can often use the strategy below, which you met in Unit 6, or an adaptation of it.

Strategy:
To find the greatest or least value of a function on an interval of the form $[a, b]$

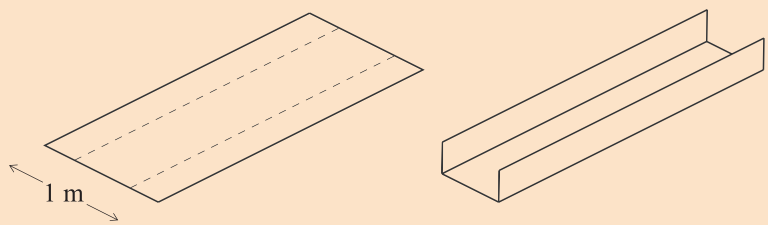
(This strategy is valid when the function is continuous on the interval, and differentiable at all values in the interval except possibly the endpoints.)

1. Find the stationary points of the function.
2. Find the values of the function at any stationary points inside the interval, and at the endpoints of the interval.
3. Find the greatest or least of the function values found.

The next example illustrates how you can use this strategy to solve an optimisation problem. You might like to try guessing the answer to the problem posed in the example, and then read on to see whether you're right.

Example 13 *Solving an optimisation problem*

A rectangular sheet of metal, of width 1 m, is to be bent along two straight lines parallel to the sides of the sheet, to form an open channel with a rectangular cross-section, as shown below. How far from the sides of the sheet should the metal be bent to achieve a channel of maximum volume?

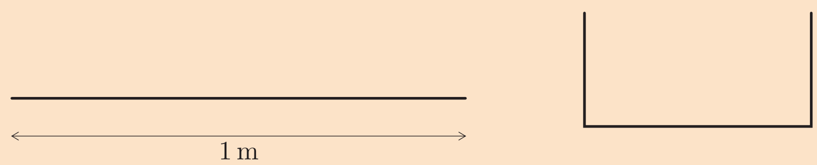


Solution

Simplify the problem if possible. Draw a diagram.

The volume of the channel is the area of its cross-section times its (fixed) length. So to achieve a channel of maximum volume, we have to achieve the maximum area of the cross-section.

The diagrams below show the cross-sections of the metal sheet and the channel. We have to determine how far from the side the metal sheet should be bent to achieve the maximum area of the cross-section.

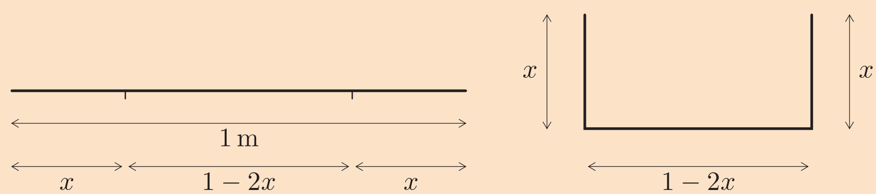


Identify the quantity that you can change, and represent it by a variable, noting the possible values that it can take. Identify the quantity to be maximised, and represent it by a variable too.

Let the distance from the side of the metal to where it is bent be x (in m). The value of x must be between 0 and $\frac{1}{2}$.

Let the area of the cross-section be A (in m^2).

Find a formula for the variable to be maximised (the dependent variable) in terms of the variable that you can change (the independent variable). Start by annotating the diagram.



The area of the cross-section is given by

$$A = x(1 - 2x).$$

Formulate the problem in terms of the variables.

We have to find the value of x, between 0 and $\frac{1}{2}$, that gives the maximum value of A.

Use calculus to solve the problem. You can use the strategy in the box given before this example.

The variable A is a polynomial function of x, so it is continuous on its whole domain and differentiable at every value of x. So the maximum value of A must occur either at an endpoint of the interval $[0, \frac{1}{2}]$, or at a stationary point.

Find the stationary points.

The formula for A is

$$\begin{aligned} A &= x(1-2x) \\ &= x - 2x^2, \end{aligned}$$

so

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 1 - 4x.$$

At a stationary point, $\mathrm{d}A/\mathrm{d}x = 0$, which gives

$$1 - 4x = 0;$$

that is,

$$x = \tfrac{1}{4}.$$

So the only stationary point occurs when $x = \frac{1}{4}$.

Consider the values of the function at any stationary points inside the interval, and at the endpoints of the interval.

The endpoints of the interval are 0 and $\frac{1}{2}$, so the maximum value of A occurs when $x = 0$, $x = \frac{1}{4}$ or $x = \frac{1}{2}$.

When $x = 0$, $A = 0$, and similarly when $x = \frac{1}{2}$, $A = 0$.

When $x = \frac{1}{4}$,

$$A = \tfrac{1}{4} \times (1 - 2 \times \tfrac{1}{4}) = \tfrac{1}{4} \times \tfrac{1}{2} = \tfrac{1}{8} = 0.125.$$

Find the greatest of the function values found.

The greatest of the three values of A found is 0.125, which is achieved when $x = \frac{1}{4}$.

So the maximum value of A is achieved when $x = \frac{1}{4}$.

State a conclusion that answers the original problem.

To achieve a channel of maximum volume, the metal should be bent at a distance of 25 cm from each side.

In fact, in Example 13 it isn't necessary to actually work out the values of A when $x = 0$, $x = \frac{1}{4}$ and $x = \frac{1}{2}$. When $x = 0$ the height of the channel is zero, and when $x = \frac{1}{2}$ its width is zero, so in both these cases the area A will be zero. When $x = \frac{1}{4}$, the height and the width of the channel are both positive, so the area A will also be positive. So you can see immediately that, of these three values of x, the value $x = \frac{1}{4}$ will give the greatest value of A.

The solution to Example 13 shows that the channel has maximum volume when it is twice as wide as it is high.

When you're solving a maximisation or minimisation problem, it can be helpful to sketch a graph of the function involved, or plot it using a computer. This will give you a rough idea of how the quantity that you want to maximise or minimise changes as the quantity on which it depends changes. For example, Figure 16 shows a graph of the equation $A = x(1 - 2x)$ from Example 13. You can see that, as x increases, the value of A first increases and then decreases, as you'd expect, and the maximum value of A seems to occur when x is about 0.25, which accords with the answer found in Example 13.

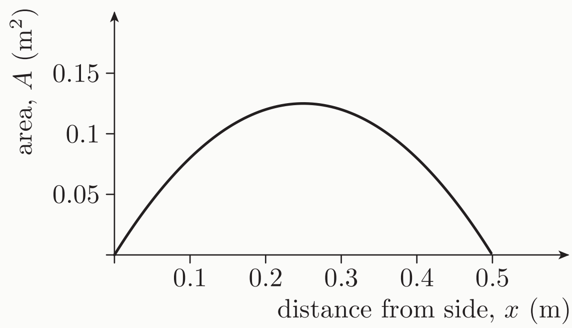


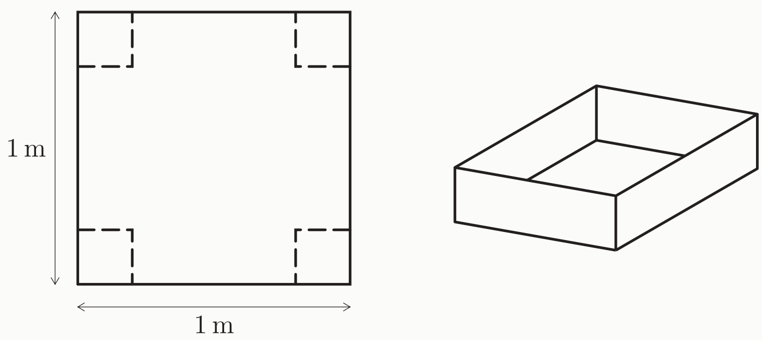
Figure 16 The graph of $A = x(1 - 2x)$ for $0 \le x \le \frac{1}{2}$

Notice that in fact you could have solved the problem in Example 13 without using calculus, because the function involved turned out to be a quadratic function. You could have found the value of x that gives the maximum value of A by using a method other than calculus to find the vertex of the parabola that's the graph of the function. You saw some methods for doing this in Unit 2. However, once you're familiar with calculus, it's usually easier to use it.

Each of the next two activities poses an optimisation problem for you to solve by using calculus. The first activity involves maximising the volume of a box, and the second one involves maximising the area of the cross-section of an open channel, in a similar way to Example 13. In both activities, you might like to try guessing the answers to the problems posed before you start work on the solutions.

Activity 20 *Solving an optimisation problem*

Suppose that you have a sheet of cardboard that measures 1 m by 1 m. You intend to cut a square from each of the four corners and then fold the resulting shape to form a box (without a top), as shown. What should the side length of the squares be, to give a box of maximum volume?



In the next activity you should use the fact that the area of a triangle with sides of lengths a and b and included angle θ (as illustrated in Figure 17) is given by the formula $A = \frac{1}{2}ab\sin\theta$. You met this formula in Unit 4.

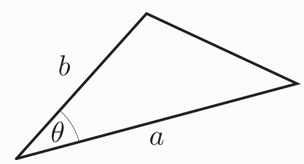
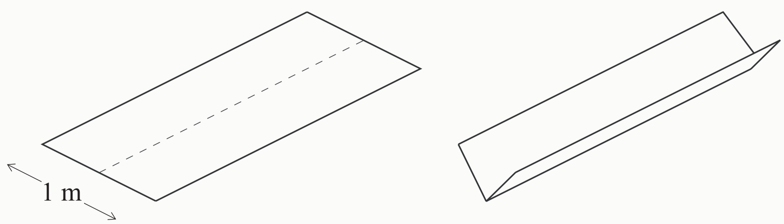


Figure 17 A triangle with sides of lengths a and b and included angle θ

Activity 21 *Solving another optimisation problem*

Consider again the problem of the open channel in Example 13. Suppose that instead of bending the metal along two straight lines, you intend to bend it along one straight line in the middle, to form an open channel with a v-shaped cross-section, as shown below. What should be the angle between the two sides of the v-shape, to give a channel with maximum volume?



Here's a summary of the main steps that you can use to solve an optimisation problem. Of course, you should combine these steps with other problem-solving techniques as appropriate, such as simplifying the problem if possible, drawing a diagram and drawing a graph.

Strategy:
To solve an optimisation problem

1. Identify the quantity that you can change, and represent it by a variable, noting the possible values that it can take. Identify the quantity to be maximised or minimised, and represent it by a variable. These variables are the independent and dependent variables, respectively.
2. Find a formula for the dependent variable in terms of the independent variable.
3. Use the techniques of differential calculus to find the value of the independent variable that gives the maximum/minimum value of the dependent variable.
4. Interpret your answer in the context of the problem.

The optimisation techniques that you've met in this subsection are useful in the design of everyday objects. For example, suppose that you want to design the shape of a food tin so that it has a certain volume but uses the minimum amount of metal. You can use calculus to determine the optimal shape of the tin. If you simplify the problem by assuming that the tin is simply a hollow cylinder, as shown in Figure 18, then it turns out that the optimal shape is for the diameter of the cylinder to be equal to its height. (You might like to try to prove this.) Food tins tend to have proportions close to this, though to optimise accurately the shape of a food tin you would need to take into account the extra metal needed for the joins, and any differences in the thicknesses of metal needed for the sides, base and top.

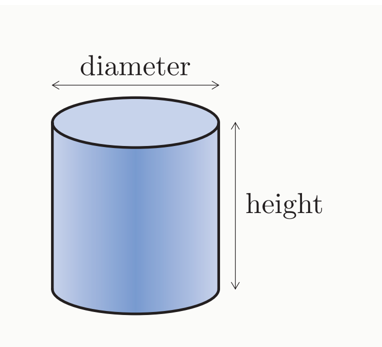


Figure 18 A cylinder

3.2 Differentiation using a computer

Sometimes it's convenient to use a computer to find the derivatives of functions, and to work with these derivatives. The next activity shows you how to use the module computer algebra system to do this.

Activity 22 *Using a computer for differentiation*

Work through Section 8 of the *Computer algebra guide*.

To illustrate how computers can be used to solve calculus problems, in the rest of this subsection we'll use a computer to solve the following minimisation problem.

> A man is in a forest, 1 km away from the nearest point on a straight road that passes through the forest, as shown in Figure 19. He wants to get to a point 2 km further along the road, as quickly as possible, by first running in a straight line through the forest and then running along the road. He can run at 8 kilometres per hour through the forest, and at 16 kilometres per hour along the road. At what point along the road should he aim to join it?

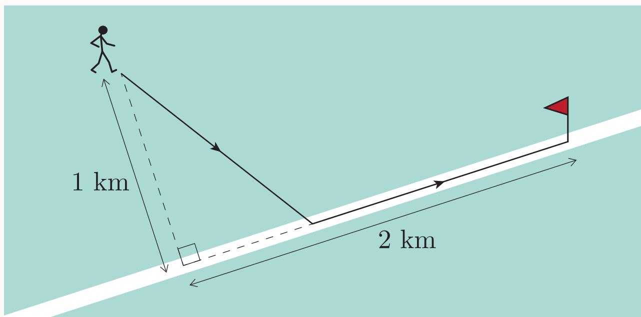


Figure 19 The position of a man in a forest and his path to his destination

Before we can use a computer, we need to carry out the first two steps of the strategy given at the end of the last subsection, to convert the problem from a word problem into a problem expressed in terms of variables and a function.

To do this, let x be the distance in kilometres between the point where the man joins the road and the point on the road that is closest to his initial position, as shown in Figure 20. Then x can take any value in the interval $[0, 2]$. Let T be the total time in hours taken by the man to reach his destination.

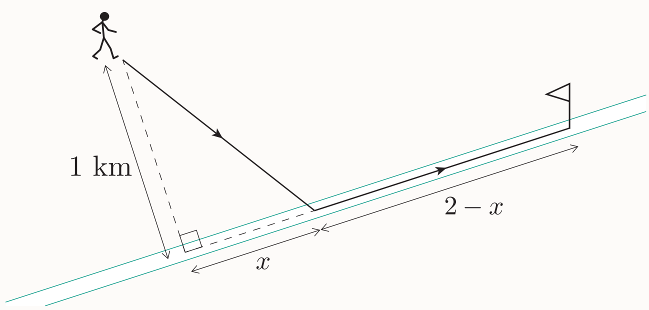


Figure 20 An annotated diagram of the man's position and path

We now have to express T in terms of x. By Pythagoras' theorem, the distance in kilometres that the man runs through the forest is

$$\sqrt{1^2 + x^2}.$$

The distance in kilometres that he runs along the road is simply

$$2 - x.$$

For each of the two parts of the man's run, the time that he takes is equal to the distance that he runs divided by the speed at which he runs. Hence the total time T that he takes to reach his destination is given by

$$T = \frac{\sqrt{1 + x^2}}{8} + \frac{2 - x}{16}.$$

To solve the problem, we now need to use calculus to find the value of x in the interval $[0, 2]$ that gives the minimum value of T. That is, we need to carry out the final step of the strategy.

You could do that without using a computer, and you might like to try this, if you want a challenge. However, in the next activity you're asked to do it by using a computer. This is quicker and easier, and it will give you practice in using a computer to solve problems like this. A by-hand solution is provided at the end of the solution to the activity.

Activity 23 *Solving an optimisation problem using a computer*

Solve the problem discussed above, by following the steps below. Carry out parts (a), (b), (d) and (e) using the module computer algebra system.

(a) Define a function f as follows:

$$f(x) = \frac{\sqrt{1 + x^2}}{8} + \frac{2 - x}{16}.$$

(b) Plot the graph of f, for values of x in the interval $[0, 2]$ at least.

(c) Use your plot to estimate roughly the value of x in $[0, 2]$ that gives the minimum value of $f(x)$.

(d) Find the derivative f' of f.

(e) Solve the equation $f'(x) = 0$.

(f) Hence state, to the nearest 10 metres, how far along the road the man should join it.

3.3 Derivatives of inverse functions

In this subsection you'll meet a rule, known as the *inverse function rule*, which you can use to work out a formula for the derivative of an inverse function when you know a formula for the derivative of the original function.

The inverse function rule can be derived from the chain rule, as follows. We'll use Leibniz notation, since the inverse function rule is usually easier to work with in Leibniz notation.

Suppose that y is an invertible function of x. Then x is also a function of y. As you know, the derivative of x with respect to x is 1, and we can write this fact as

$$\frac{\mathrm{d}x}{\mathrm{d}x} = 1.$$

However, by the chain rule,

$$\frac{\mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Therefore

$$\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

Hence, for values of x for which $\mathrm{d}x/\mathrm{d}y$ exists and is non-zero,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}.$$

So we have the following rule.

Inverse function rule (Leibniz notation)

If y is an invertible function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y},$$

for all values of x such that $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ exists and is non-zero.

You can see that, as with the chain rule, the Leibniz form of the inverse function rule looks like an obvious fact about fractions. As with the chain rule, you should keep in mind that that's not what it is.

Here's an informal way to see that the inverse function rule makes sense. Suppose that y is an invertible function of x, which means that x is also a function of y. Suppose that, for a particular value of x and its corresponding value of y, the value of x is increasing at the rate of 2 units for every unit that y increases, as illustrated in Figure 21. Then, at these values of x and y, you'd expect the value of y to be increasing at the rate of $\frac{1}{2}$ unit for every unit that x increases. This is what the inverse function rule tells you.

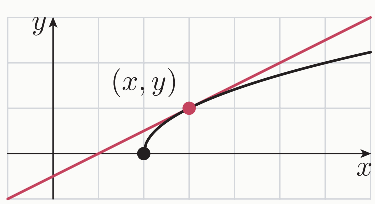


Figure 21 A point (x, y) on the graph of a function

You'll see the inverse function rule in Lagrange notation later in this subsection. First let's look at some examples of how you can use it.

Using the inverse function rule

In the next example, the inverse function rule in Leibniz notation is used to work out that the derivative of $\ln x$ is $1/x$. This is a fact that you met in Subsection 1.2.

Example 14 *Using the inverse function rule*

Use the inverse function rule, and the derivative of the exponential function, to differentiate $y = \ln x$.

Solution

Express x in terms of y.

We have $y = \ln x$, so

$$x = e^y.$$

Differentiate with respect to y.

Therefore

$$\frac{dx}{dy} = e^y.$$

Use the inverse function rule.

By the inverse function rule,

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Use the relationship between x and y to express the derivative in terms of x.

Hence, since $x = e^y$,

$$\frac{dy}{dx} = \frac{1}{x}.$$

As illustrated in Example 14, when you use the inverse function rule, you obtain an expression for the derivative that's expressed in terms of y rather than x. You then have to use the relationship between x and y to express the derivative in terms of x.

In the next example, the inverse function rule is used to find the derivative of the function $f(x) = \sin^{-1}(x)$. You saw in Unit 4 that this function has domain $[-1, 1]$. Its graph is shown in Figure 22.

Example 15 *Using the inverse function rule again*

Use the inverse function rule, and the derivative of the sine function, to differentiate $y = \sin^{-1} x$.

Solution

Express x in terms of y.

We have $y = \sin^{-1} x$, so

$$x = \sin y.$$

Differentiate x with respect to y.

Therefore

$$\frac{dx}{dy} = \cos y.$$

Use the inverse function rule.

By the inverse function rule,

$$\frac{dy}{dx} = \frac{1}{\cos y},$$

provided that $\cos y \neq 0$.

Use the relationship between x and y to express the derivative in terms of x. In this case the relationship is given by $x = \sin y$ (or $y = \sin^{-1} x$), and you need to express $\cos y$ in terms of x. You could write $\cos y = \cos(\sin^{-1} x)$, but you can obtain a simpler expression by using the identity $\sin^2 y + \cos^2 y = 1$, as follows.

The identity $\cos^2 y + \sin^2 y = 1$ gives

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

The $+$ sign applies here, because y takes values only in the interval $[-\pi/2, \pi/2]$ (since $y = \sin^{-1}(x)$) and so $\cos y$ is always non-negative. Hence

$$\cos y = \sqrt{1 - x^2}.$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}},$$

provided that $x \neq \pm 1$.

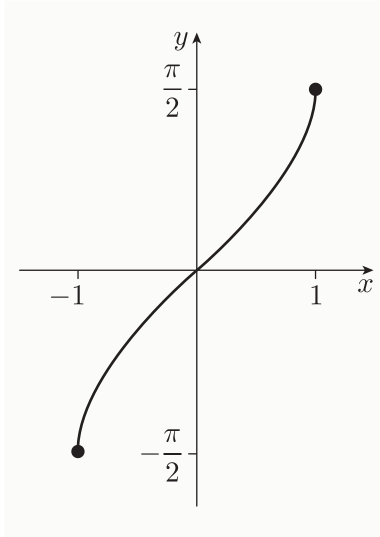


Figure 22 The graph of $y = \sin^{-1} x$

Notice that the formula for the derivative of $\sin^{-1} x$ found in Example 15 doesn't apply at the endpoints of the domain of this function, that is, when $x = -1$ and $x = 1$. This is because the function $f(x) = \sin^{-1} x$ isn't right-differentiable at $x = -1$ and isn't left-differentiable at $x = 1$, as you can see, roughly, from the shape of its graph, in Figure 22. The 'one-sided tangents' at these values of x are vertical.

Here's another point to note about Example 15. Strictly, it involves the sine function with its domain restricted to the interval $[-\pi/2, \pi/2]$, rather than the sine function with its usual domain, $\mathbb{R}$. As you saw in Unit 4, the sine function with its usual domain doesn't have an inverse function.

In the next activity you're asked to find the derivatives of the functions $f(x) = \cos^{-1} x$ and $f(x) = \tan^{-1} x$. You saw in Unit 4 that these functions have domains $[-1, 1]$ and $\mathbb{R}$, respectively. Their graphs are shown in Figures 23 and 24, respectively.

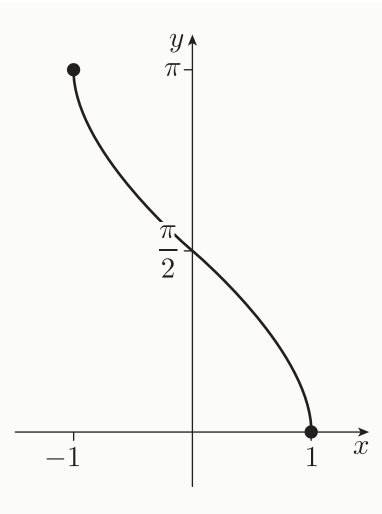


Figure 23 The graph of $y = \cos^{-1} x$

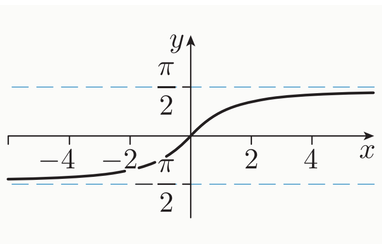


Figure 24 The graph of $y = \tan^{-1} x$

Activity 24 *Using the inverse function rule*

Use the inverse function rule, and the derivatives of the cosine and tangent functions, to differentiate the following functions.

(a) $y = \cos^{-1} x$ (b) $y = \tan^{-1} x$

Hint: To express the derivatives in terms of x, use the identity $\sin^2 y + \cos^2 y = 1$ in part (a), and the identity $\tan^2 y + 1 = \sec^2 y$ in part (b).

The derivatives found in Example 15 and Activity 24 are repeated below, and can now be included in the list of derivatives of standard functions that you have available to work with.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

In the next activity you're asked to combine these standard derivatives with some of the rules for differentiation that you've seen, to find the derivatives of some more functions.

Activity 25 *Finding derivatives of functions involving inverse trigonometric functions*

Differentiate the following functions.

(a) $f(x) = \sin^{-1}(3x)$ (b) $f(x) = e^x \cos^{-1} x$

The inverse function rule in Lagrange notation

Let's now translate the inverse function rule into Lagrange notation. To do this, we set $y = f^{-1}(x)$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (f^{-1})'(x),$$

where $(f^{-1})'$ denotes the derivative of f^{-1}, as you'd expect.

Also, the equation $y = f^{-1}(x)$ is equivalent to the equation $x = f(y)$, so

$$\frac{\mathrm{d}x}{\mathrm{d}y} = f'(y) = f'(f^{-1}(x)).$$

This gives the following form of the inverse function rule.

Inverse function rule (Lagrange notation)

If f is a function with inverse function f^{-1}, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

for all values of x such that $f'(f^{-1}(x))$ exists and is non-zero.

This is a formula for the derivative of f^{-1} in terms of the derivative of f, but it may look rather complicated and difficult to understand. Here's a helpful way to understand what it tells you, and why it holds.

First, it's important to realise the following fact. If two straight lines (drawn on axes with equal scales) are reflections of each other in the line $y = x$, then their gradients are reciprocals of each other. This is because if you consider any pair of points on one line, and their reflections on the other line, then the values of the run and the rise are interchanged for the two pairs of points, as illustrated in Figure 25(a). The lines in this diagram have gradients 2 and $\frac{1}{2}$.

Another fact, which you met in Unit 3 and will be useful here, is that if two points (plotted on axes with equal scales) are reflections of each other in the line $y = x$, then they're obtained from each other by interchanging x- and y-coordinates. For example, Figure 25(b) shows the point $(3, 1)$ and its reflection $(1, 3)$ in the line $y = x$.

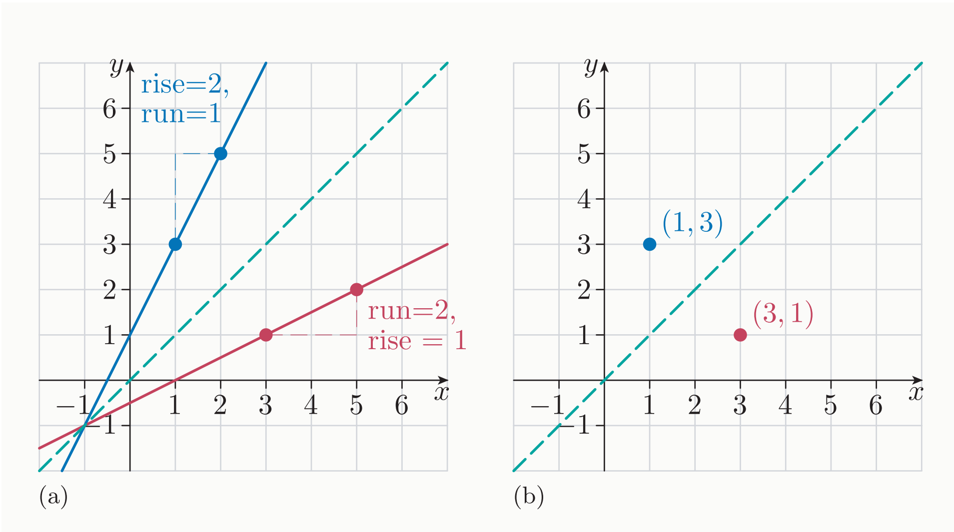


Figure 25 (a) Two straight lines that are reflections of each other in the line $y = x$ (b) Two points that are reflections of each other in the line $y = x$

Now remember that the graphs of a function and its inverse function are reflections of each other in the line $y = x$, when they're drawn on axes with equal scales. For example, Figure 26 shows the graphs of a function f and its inverse function f^{-1}. (The function is in fact $f(x) = x^2 + 2$ $(x \geq 0)$, with inverse function $f^{-1}(x) = \sqrt{x - 2}$.) The two tangents shown are reflections of each other in the line $y = x$, so their gradients are reciprocals of each other. The gradient of the red tangent is the gradient of the graph of f^{-1} at the point $(3, 1)$, which is $(f^{-1})'(3)$. The gradient of the blue tangent is the gradient of the graph of f at the point $(1, 3)$, which is $f'(1)$. Hence

$$(f^{-1})'(3) = \frac{1}{f'(1)}.$$

This is a particular instance of the inverse function rule, with $x = 3$ and $f^{-1}(x) = f^{-1}(3) = 1$.

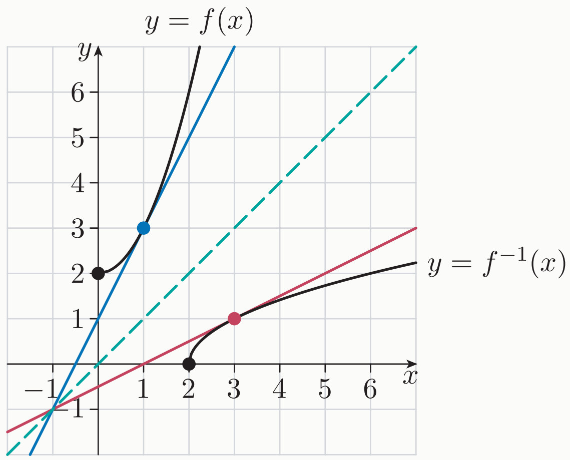


Figure 26 The graphs of a function f and its inverse function f^{-1}, and their tangents at $(1, 3)$ and $(3, 1)$, respectively

In general, consider the graphs of any invertible function f and its inverse function f^{-1}, as illustrated in Figure 27. Consider two (non-vertical) tangents that are reflections of each other in the line $y = x$, as illustrated. Their gradients are reciprocals of each other. The gradient of the red tangent is the gradient of the graph of f^{-1} at the point $(x, f^{-1}(x))$, which is $(f^{-1})'(x)$. The gradient of the blue tangent is the gradient of the graph of f at the point $(f^{-1}(x), x)$, which is $f'(f^{-1}(x))$. Hence

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This is the inverse function rule.

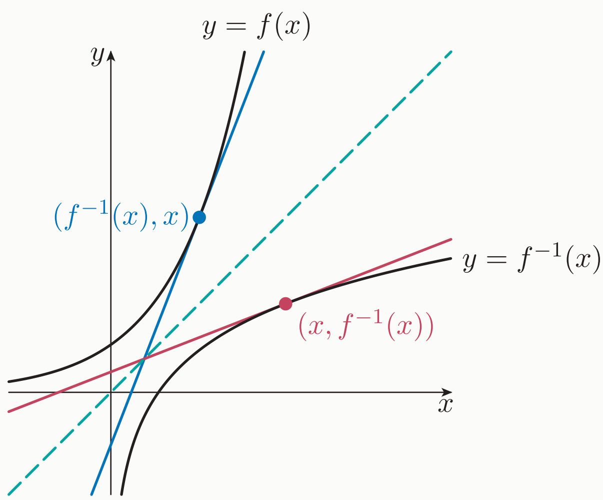
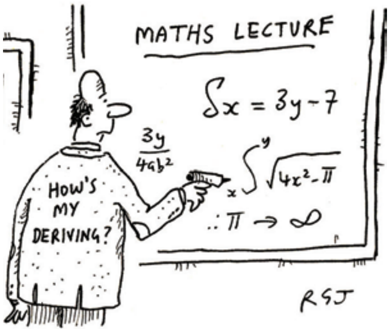


Figure 27 The graphs of a function f and its inverse function f^{-1}, and their tangents at $(f^{-1}(x), x)$ and $(x, f^{-1}(x))$, respectively

3.4 Table of standard derivatives



You've now reached the end of the material that introduces differential calculus in this module, though you'll use the ideas and methods again in Unit 11, *Taylor polynomials*. In the next section, you'll make a start on learning about integral calculus. To round off your introduction to differential calculus, here's a table of the standard derivatives that you've met. You should try to memorise at least the first six (those above the line in the middle of the table), as they're used frequently in mathematics. This table of standard derivatives is also included in the *Handbook*.

Standard derivatives

Function $f(x)$	Derivative $f'(x)$
a (constant)	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$

The first calculus textbook

In about 1690 the wealthy Marquis de l'Hôpital (1661–1704), a minor French nobleman, became interested in calculus as a result of journal articles by Gottfried Wilhelm Leibniz and the Swiss mathematicians Jacob and Johann Bernoulli. However, these articles contained little by way of explanation and so l'Hôpital paid Johann Bernoulli, then living in Paris, to tutor him in the new subject. When Bernoulli moved from Paris to Groningen to take up a professorship, they came to an arrangement whereby for a considerable monthly sum Bernoulli would send l'Hôpital material on calculus, including his new results, and would deny access to the material to anyone else. In effect, l'Hôpital bought the rights to Bernoulli's discoveries.

By 1696 l'Hôpital felt he understood enough of the mathematics to publish a book on it, *Analyse des Infiniment Petits* (*Analysis of Infinitely Small Quantities*). Although in the book l'Hôpital acknowledges his debt to Bernoulli, after l'Hôpital's death Bernoulli complained that l'Hôpital had won undeserved praise for the text. But because l'Hôpital was known to have been a competent mathematician and because Bernoulli was known to be a contentious egotist, the latter's complaints were generally dismissed. However, documents discovered in the 20th century have revealed that Bernoulli was in fact right to claim much of the book for himself.

Guillaume De l'Hôpital (1661–1704)

Johann Bernoulli (1667–1748)

ANALYSE
DES
INFINIMENT PETITS,
Pour l'intelligence des lignes courbes.

A PARIS,
DE L'IMPRIMERIE ROYALE.
M. DC. XCVI.

The title page of *Analyse des Infiniment Petits* (1696)

4 Reversing differentiation

In Unit 6 you learned that if you know the values taken by a continuously changing quantity throughout a period of change, then you can use differentiation to find the values taken by its rate of change throughout the same period. For example, you saw that if you have a formula for the displacement of a moving object in terms of time, then you can use differentiation to find a formula for its velocity in terms of time. You saw that the area of calculus that deals with this type of process is called *differential calculus*.

In the rest of this unit, and in Unit 8, we'll consider the opposite process. We'll consider situations where you know the values taken by the rate of change of a continuously changing quantity throughout a period of change, and you want to find the values taken by the quantity throughout the same period. For example, you might have a formula or a graph that tells you the values taken by the velocity of an object (such as the walking man discussed in Unit 6) over some time period, and you want to use this information to work out the values taken by the displacement of the object over the same time period. As mentioned earlier, the area of calculus that deals with this type of process is called *integral calculus*.

As you'll see, when you're working with integral calculus, it can be useful to consider whether a function is *continuous*. A **continuous** function is one whose graph has no discontinuities (that is, 'breaks') in it. In other words, informally, it's a function whose graph you can draw without taking your pen off the paper (though you might need an infinitely large piece of paper, as the graph might be infinitely long!). For example, the function $f(x) = 2x$ is a continuous function, as shown in Figure 28(a), but the function $f(x) = 1/x^2$ isn't, as its graph has a break at $x = 0$, as shown in Figure 28(b).

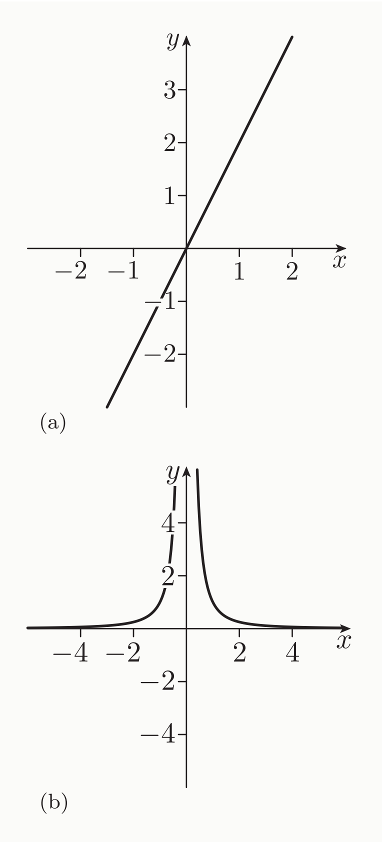


Figure 28 The graphs of
(a) $f(x) = 2x$
(b) $f(x) = 1/x^2$

4.1 Antiderivatives and indefinite integrals

In integral calculus you usually start with a function f, and you want to find another function, say F, whose derivative is f. Such a function F is called an **antiderivative** of the original function f. For example, an antiderivative of the function $f(x) = 2x$ is the function $F(x) = x^2$, because the derivative of x^2 is $2x$.

The process of finding an antiderivative of a function is called **antidifferentiation**, or, more commonly, **integration**. So integration is the reverse of differentiation.

The first thing to realise is that a function can have more than one antiderivative. To see this, try the following activity.

Activity 26 *Differentiating functions whose formulas are the same apart from a constant term*

Differentiate the following functions.

(a) $F(x) = x^2$ (b) $F(x) = x^2 + 3$ (c) $F(x) = x^2 - \frac{5}{7}$

You should have found that all three functions in Activity 26 have derivative $F'(x) = 2x$. In fact, you can see that any function of the form

$$F(x) = x^2 + c,$$

where c is a constant, has derivative $F'(x) = 2x$. This is because the derivative of a constant term is zero. So each function of the form $F(x) = x^2 + c$ is an antiderivative of the function $f(x) = 2x$.

Another way to think about the fact that all the functions of the form $F(x) = x^2 + c$ have the same derivative is to remember that adding a constant to the formula of a function has the effect of translating its graph vertically. So the graphs of all the functions of the form $F(x) = x^2 + c$ are vertical translations of each other, as illustrated in Figure 29(b). Translating a graph vertically doesn't change its gradient at each x-value, of course. In other words, it doesn't change the derivative of the function that the graph represents. So each of the functions in Figure 29(b) has the same derivative, namely $f(x) = 2x$, whose graph is shown in Figure 29(a).

Another thing to appreciate about all the functions of the form $F(x) = x^2 + c$, where c is a constant, is that these functions are the *only* antiderivatives of the function $f(x) = 2x$. That's because any antiderivative of $f(x) = 2x$ has the same gradient at each x-value as the function $F(x) = x^2$, and the functions of the form $F(x) = x^2 + c$ are the only functions with this property.

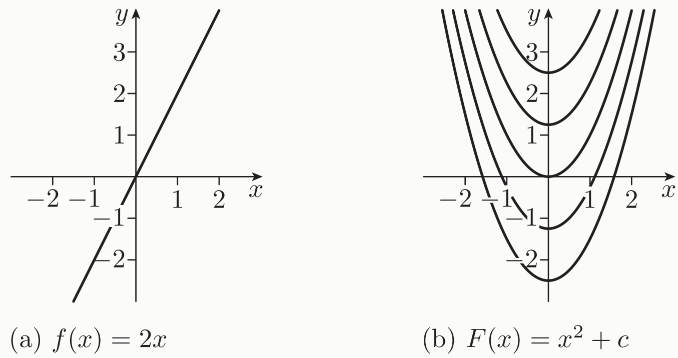


Figure 29 The graphs of (a) a function (b) some of its antiderivatives

So the formula $F(x) = x^2 + c$ describes the *complete family* of antiderivatives of the function $f(x) = 2x$. We call the general function $F(x) = x^2 + c$ the **indefinite integral** of the function $f(x) = 2x$. The word 'indefinite' refers to the fact that the constant c can take any value.

In general, consider any function f that has an antiderivative. You can see that you can add any constant that you like to the formula for the antiderivative, and you'll get the formula for another antiderivative of f. Equivalently, you can translate the graph of the antiderivative vertically by any amount, and you'll get the graph of another antiderivative of f.

These antiderivatives are the *only* antiderivatives of the function f, provided that f is a *continuous* function.

So, if f is a function that has an antiderivative, and f is continuous, then the general function

$$F(x) = (\text{formula for any particular antiderivative of } f) + c,$$

where c represents any constant, and whose domain is the same as the domain of f, describes the complete family of antiderivatives of f. We call this general function F the **indefinite integral** of the function f.

The constant c in an indefinite integral is called an **arbitrary constant**, or the **constant of integration**. It's usually denoted by c, but you can use any letter. Like the word 'indefinite' in 'indefinite integral', the word 'arbitrary' in 'arbitrary constant' refers to the fact that the constant c can take any value.

You don't need to be concerned about what happens when the function f *isn't* continuous, because normally when we're solving problems in integral calculus we work only with continuous functions. These are usually the only functions that we need for the sorts of calculations that we want to perform. However, if you'd like to know what the problem is with functions that aren't continuous, then read the explanation at the end of this subsection, when you reach it.

Before you go on, it's important to make sure that you fully understand the difference between an antiderivative and an indefinite integral of a function. The definitions that you've seen are summarised below.

Antiderivatives and indefinite integrals

Suppose that f is a function.

An **antiderivative** of f is any specific function whose derivative is f.

If f has an antiderivative, and f is continuous, then the **indefinite integral** of f is the *general* function obtained by adding an arbitrary constant c to the formula for an antiderivative of f. It describes the complete family of antiderivatives of f.

Example 16 *Understanding antiderivatives and indefinite integrals*

(a) Show that the function $F(x) = x^3$ is an antiderivative of the function $f(x) = 3x^2$.

(b) What is the indefinite integral of the function $f(x) = 3x^2$?

Solution

(a) Since

$$\frac{d}{dx}(x^3) = 3x^2,$$

it follows that $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.

(b) The indefinite integral of $f(x) = 3x^2$ is $F(x) = x^3 + c$.

Activity 27 *Understanding antiderivatives and indefinite integrals*

(a) Show that the function $F(x) = \frac{1}{2}\sin(2x)$ is an antiderivative of the function $f(x) = \cos(2x)$.

(b) What is the indefinite integral of the function $f(x) = \cos(2x)$?

(c) Write down an antiderivative of the function $f(x) = \cos(2x)$ other than the antiderivative in part (a).

Even though the idea of an indefinite integral really applies only to continuous functions, it's convenient in practice to state indefinite integrals of other functions. For example, consider again the function $f(x) = 1/x^2$, which isn't continuous. Its graph is repeated in Figure 30. An antiderivative of this function is the function $F(x) = -1/x$, as you can check by differentiating this function F. Even though f isn't continuous, we do still say

the function $f(x) = 1/x^2$ has indefinite integral $F(x) = -1/x + c$.

This is shorthand for

any continuous function f with rule $f(x) = 1/x^2$ has indefinite integral F with rule $F(x) = -1/x + c$.

For example, the continuous function $f(x) = 1/x^2$ $(x > 0)$, whose graph is shown in Figure 31, has indefinite integral $F(x) = -1/x + c$ $(x > 0)$.

We use this shorthand for any function that isn't continuous.

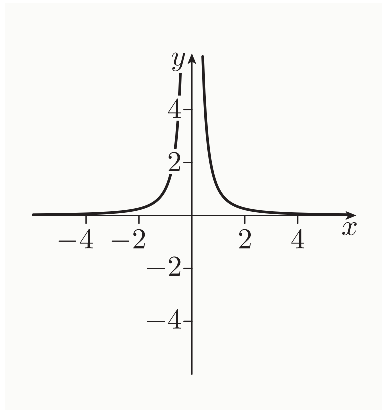


Figure 30 The graph of the function $f(x) = 1/x^2$

As you've seen, to find an indefinite integral of a function, you just find a formula for any antiderivative of the function, and add an arbitrary constant, usually c.

You can find antiderivatives of many functions by using what you already know about derivatives. In the next two subsections you'll practise finding antiderivatives, and hence indefinite integrals, of some simple functions in this way. Before you make a start on that, here are a few things that you need to know.

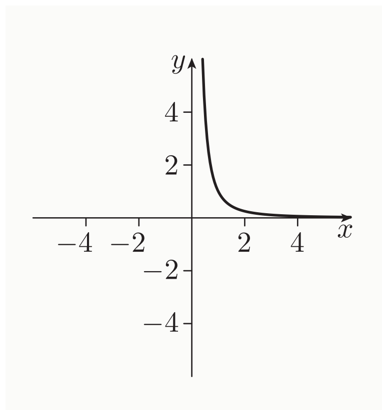


Figure 31 The graph of the function $f(x) = 1/x^2$ $(x > 0)$

First, note that when we're discussing antiderivatives and indefinite integrals, we use conventions similar to those for derivatives. For example, the terms 'antiderivative' and 'indefinite integral' can be applied directly to expressions that could appear on the right-hand side of the rule of a function, as well as to the function itself. For instance, rather than saying that

the indefinite integral of $f(x) = 2x$ is $F(x) = x^2 + c$,

we can simply say that

the indefinite integral of $2x$ is $x^2 + c$.

Also, as you'd expect, you can use any letters for functions and variables – you're not restricted to the standard ones, f and x.

When we're working with antiderivatives and indefinite integrals in the rest of this unit, we'll usually use a lower-case letter, such as f, to denote the function that we start with, and the corresponding upper-case letter, such as F, to denote any of its antiderivatives, or its indefinite integral. You'll meet a more adaptable, though more complicated, notation for indefinite integrals in Unit 8. It involves the *integral sign*, $\int$, which you may have seen elsewhere.

Functions that aren't continuous

If you'd like to know what the problem is with functions that aren't continuous, then read the discussion below. Otherwise just go on to Subsection 4.2.

A function f that has an antiderivative but which isn't continuous may have other antiderivatives as well as those obtained by vertically translating the graph of the original antiderivative. For example, consider the function $f(x) = 1/x^2$, whose graph is shown in Figure 32(a). One antiderivative of this function f is the function $F(x) = -1/x$, as mentioned above. The graph of this function F is shown in Figure 32(b). Every function obtained by vertically translating this graph is an antiderivative of f, but so is every function obtained by vertically translating each of the *two pieces* of the graph *individually*, since that doesn't change the gradient at each x-value. For example, Figure 32(c) shows an antiderivative of f obtained by translating the left-hand piece of the graph of $F(x) = -1/x$ down by 2 units, and the right-hand piece up by 1 unit.

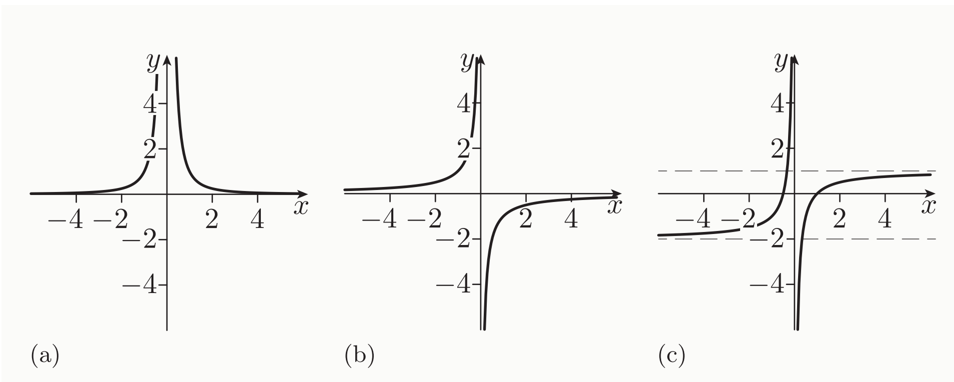


Figure 32 The graphs of (a) $f(x) = 1/x^2$ (b) the antiderivative $F(x) = -1/x$ of $f(x) = 1/x^2$ (c) another antiderivative of $f(x) = 1/x^2$

So if f is a function that has an antiderivative, but which isn't continuous, then the general function

$$F(x) = (\text{formula for any particular antiderivative of } f) + c,$$

where c represents any constant, and whose domain is the same as the domain of f, usually *doesn't* describe the complete family of antiderivatives of f, as stated earlier. As mentioned earlier, this isn't a problem, because normally when we're solving problems in integral calculus we work only with continuous functions.

4.2 Antiderivatives of power functions

In this section you'll see how to find antiderivatives, and hence indefinite integrals, of power functions.

You can find antiderivatives of power functions by using what you already know about derivatives of power functions. For example, suppose that you want to find an antiderivative of the power function $f(x) = x^6$. Consider what happens when you differentiate the power function whose exponent is 1 more than 6, namely $g(x) = x^7$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^7\right) = 7x^6.$$

This equation tells you that x^7 is 'nearly' an antiderivative of x^6. You can deduce from it that if you multiply x^7 by the constant $\frac{1}{7}$, then differentiating will give the answer that you want, x^6:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{7}x^7\right) = \frac{1}{7} \times 7x^6 = x^6.$$

So an antiderivative of $f(x) = x^6$ is $F(x) = \frac{1}{7}x^7$. Hence the indefinite integral of $f(x) = x^6$ is

$$F(x) = \tfrac{1}{7}x^7 + c.$$

You can find a general formula for an antiderivative of a power function by applying the thinking above to the general power function $f(x) = x^n$. Differentiating the power function $g(x) = x^{n+1}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{n+1}\right) = (n+1)x^n.$$

So if you multiply x^{n+1} by $1/(n+1)$, then differentiating will give the answer that you want, x^n:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{n+1}x^{n+1}\right) = \frac{1}{n+1} \times (n+1)x^n = x^n.$$

This equation gives a general formula for an antiderivative, and hence the indefinite integral, of a power function, which is stated below. The formula applies for every value of the exponent n, except $n = -1$. It doesn't apply for $n = -1$ because division by zero is undefined. (The case $n = -1$ is more complicated, and is dealt with in Subsection 6.1.)

Indefinite integral of a power function

For any number n except $n = -1$,

$$\text{the indefinite integral of } x^n \text{ is } \frac{1}{n+1}x^{n+1} + c.$$

Here's a helpful informal way to remember this formula.

Indefinite integral of a power function (informal)

To find an antiderivative of a power function (except the power function $f(x) = x^{-1}$), first increase the power by 1, then divide by the new power.

As usual, to obtain the indefinite integral, add the arbitrary constant c.

Example 17 *Finding indefinite integrals of power functions*

Find the indefinite integrals of the following functions.

(a) $f(x) = x^{10}$ (b) $g(x) = \dfrac{1}{x^5}$ (c) $h(x) = \sqrt{x}$

Solution

(a) To find an antiderivative, increase the power by 1, then divide by the new power. To obtain the indefinite integral, add the arbitrary constant c.

The indefinite integral is

$$F(x) = \frac{1}{11}x^{11} + c.$$

(b) First write the function in the form $g(x) = x^n$.

The function is $g(x) = x^{-5}$.

Proceed as in part (a).

The indefinite integral is

$$G(x) = \frac{1}{-4}x^{-4} + c$$

Simplify the answer.

$$= -\frac{1}{4x^4} + c.$$

(c) First write the function in the form $y = x^n$.

The function is $h(x) = x^{1/2}$.

Proceed as in part (a).

The indefinite integral is

$$H(x) = \frac{1}{3/2}x^{3/2} + c$$

Simplify the answer.

$$= \tfrac{2}{3}x^{3/2} + c.$$

Activity 28 *Finding indefinite integrals of power functions*

Find the indefinite integrals of the following functions.

(a) $f(x) = x^9$ (b) $p(u) = \dfrac{1}{u^3}$ (c) $f(x) = x^{2/3}$

(d) $f(x) = \sqrt[4]{x}$ (e) $g(t) = t^{-2/3}$ (f) $b(v) = \dfrac{1}{\sqrt{v}}$

Activity 29 *Finding different antiderivatives of a function*

Let f be the function with rule $f(x) = x^5$.

(a) Find the indefinite integral of f.

(b) Hence write down three different antiderivatives of f.

Notice that the formula for the indefinite integral of a power function tells you that the indefinite integral of the function

$$f(x) = 1 \quad \text{(which is the same as } f(x) = x^0\text{)}$$

is

$$F(x) = x + c.$$

(As in earlier units, we assume that $0^0 = 1$, so the formula works when $x = 0$.) Another way to obtain the indefinite integral above is, of course, to use the fact that the derivative of x is 1.

In Subsection 6.1 later in the unit, you'll see a formula for the indefinite integral of the only power function that isn't covered by the general formula that you've met in this subsection. This is the power function $f(x) = x^{-1}$, that is, the reciprocal function $f(x) = 1/x$.

4.3 Constant multiple rule and sum rule for antiderivatives

Just as with derivatives, once you have formulas for antiderivatives of some functions, you can combine them to obtain formulas for antiderivatives of further functions.

Unfortunately, though, it's generally more tricky to combine antiderivatives than it is to combine derivatives. For example, there's no straightforward rule that you can use to find an antiderivative of a product of two functions if you already know an antiderivative of each of the two individual functions. In other words, there's no 'product rule for antiderivatives'. Similarly, there's no 'quotient rule for antiderivatives' and no 'chain rule for antiderivatives'.

However, there is a constant multiple rule and a sum rule for antiderivatives, and they work in the same simple ways as the similar rules for derivatives, as follows.

Constant multiple rule for antiderivatives

If $F(x)$ is an antiderivative of $f(x)$, and k is a constant, then $kF(x)$ is an antiderivative of $kf(x)$.

Sum rule for antiderivatives

If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, then $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$.

(You might have noticed that the letter used for the constant in the statement above of the constant multiple rule for antiderivatives, namely k, is different from the letter used in Unit 6 for the constant in the constant multiple rule for derivatives, which was a. It doesn't matter which letter is used, of course, but in integral calculus it's sometimes convenient to reserve the letter a to denote something else, as you'll see in Unit 8.)

These rules follow directly from the similar rules for derivatives. To see why, first suppose that $F(x)$ is an antiderivative of $f(x)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = f(x).$$

It follows from the constant multiple rule for derivatives that, for any constant k,

$$\frac{\mathrm{d}}{\mathrm{d}x}(kF(x)) = kf(x).$$

This equation tells you that $kF(x)$ is an antiderivative of $kf(x)$, which is what the constant multiple rule for antiderivatives says.

Now suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = f(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}(G(x)) = g(x).$$

It follows from the sum rule for derivatives that

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x) + G(x)) = f(x) + g(x).$$

This equation tells you that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$, which is what the sum rule for antiderivatives says.

Remember that, as always when a constant multiple rule and a sum rule hold, the sum rule also holds if you replace the plus signs by minus signs. That is, if $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, then $F(x) - G(x)$ is an antiderivative of $f(x) - g(x)$.

Here's an example of how you can use the two rules above.

Example 18 *Using the constant multiple rule and the sum rule for antiderivatives*

Find the indefinite integrals of the following functions.

(a) $f(x) = 10x^4$ (b) $g(t) = \dfrac{6}{t^3} - 5\sqrt{t} + 1$

Solution

(a) Find an antiderivative by using the constant multiple rule, together with the rule for finding an antiderivative of a power function. To obtain the indefinite integral, add the arbitrary constant c.

The indefinite integral is

$$\begin{aligned} F(x) &= 10 \times \tfrac{1}{5}x^5 + c \\ &= 2x^5 + c. \end{aligned}$$

(b) First express the function as a sum of constant multiples of power functions.

The function is

$$g(t) = 6t^{-3} - 5t^{1/2} + 1.$$

By the sum rule, you can find an antiderivative by working term by term. Use the method of part (a) to find an antiderivative of each term. To obtain the indefinite integral, add the arbitrary constant c.

The indefinite integral is

$$\begin{aligned}G(t) &= 6 \times \frac{1}{-2}t^{-2} - 5 \times \frac{1}{3/2}t^{3/2} + t + c\\ &= -3t^{-2} - 5 \times \tfrac{2}{3}t^{3/2} + t + c\\ &= -\frac{3}{t^2} - \frac{10t^{3/2}}{3} + t + c.\end{aligned}$$

Notice in particular that it follows from the constant multiple rule for antiderivatives that, for any constant a, the indefinite integral of the function

$$f(x) = a \quad \text{(which is the same as } f(x) = a \times 1\text{)}$$

is

$$F(x) = ax + c.$$

Another way to see this is, of course, to use the fact that the derivative of ax is a.

This useful fact is summarised below.

Indefinite integral of a constant function

The indefinite integral of the constant a is $ax + c$.

You can practise using the constant multiple rule and the sum rule for antiderivatives in the activity below. In parts (d)–(g), you have to begin by using an algebraic technique, such as multiplying out brackets or expanding a fraction, to express the function as a sum of constant multiples of power functions.

Remember that you must include an arbitrary constant c in every indefinite integral that you find.

Activity 30 *Using the constant multiple rule and the sum rule for antiderivatives*

Find the indefinite integrals of the following functions.

(a) $f(x) = 8x^3$ (b) $f(x) = 7$ (c) $f(x) = \sqrt{x} - x^2$

(d) $f(x) = (x+2)(x-5)$ (e) $g(t) = (t+1)^2$

(f) $g(x) = \dfrac{x^2+1}{x^4}$ (g) $f(r) = \sqrt[3]{r}\left(10r^2 - \dfrac{1}{r^2}\right)$

You'll meet some more ways to combine antiderivatives in Unit 8.

5 Particular antiderivatives

Sometimes when you're working with a function, you need to find a *particular* antiderivative that it has, rather than *any* antiderivative, or its indefinite integral. For example, this is often what you need to do when the function is a mathematical model of a real-life situation. In this section, you'll learn how to find such particular antiderivatives, and see some examples of how you can use them.

5.1 Finding particular antiderivatives

You can work out which of the infinitely many antiderivatives of a function is the right one if you have some appropriate extra information. Usually the extra information that you have is the value taken by the antiderivative at some value of the input variable. Here's an example.

Example 19 *Finding a particular antiderivative*

Find the antiderivative F of the function $f(x) = x^2 + 5$ such that $F(3) = 20$.

Solution

Find the indefinite integral.

The indefinite integral is

$$F(x) = \tfrac{1}{3}x^3 + 5x + c.$$

Use the extra information to find the required value of the constant c.

Using the fact that $F(3) = 20$ gives

$$\tfrac{1}{3} \times 3^3 + 5 \times 3 + c = 20$$
$$9 + 15 + c = 20$$
$$c = -4.$$

Hence the required antiderivative is

$$F(x) = \tfrac{1}{3}x^3 + 5x - 4.$$

Here are some similar examples for you to try.

Activity 31 *Finding particular antiderivatives*

(a) Find the antiderivative F of the function $f(x) = x + 2$ such that $F(1) = \frac{9}{2}$.

(b) Find the antiderivative F of the function

$$f(x) = 1/x^3 \quad (x \in (0, \infty))$$

such that $F(3) = -\frac{1}{9}$.

5.2 Using integration to work with rates of change

As mentioned at the start of this section, it's often useful to find a particular antiderivative when you're working with a function that models a real-life situation. This is illustrated in the example below.

In this example, we'll consider again a situation that you met in Unit 6, namely the one where a man walks along a straight path, gradually slowing down as he walks. In Section 3.1 of Unit 6, we started with the equation for the man's displacement in terms of time, and we used differentiation to find an equation for his velocity (his rate of change of displacement) in terms of time. Here we'll do the opposite: we'll start with the equation for his velocity in terms of time, and we'll use integration to find an equation for his displacement in terms of time. Remember that integration is the opposite process to differentiation – it's the process of finding an antiderivative of a function.

Example 20 *Using integration to deduce displacement from velocity*

Suppose that a man walks along a straight path, and his velocity v (in kilometres per hour) at time t (in hours) after he begins his walk is given by the equation

$$v = 6 - \tfrac{5}{4}t.$$

Let s be his displacement in kilometres from his starting point.

(a) Find an equation that expresses s in terms of t.

(b) Hence find the man's displacement two hours after he began his walk.

Solution

(a) Write down the given equation for the man's velocity.

The man's velocity v at time t is given by

$$v = 6 - \tfrac{5}{4}t.$$

Use integration to find an equation for his displacement. It will contain an arbitrary constant, because at this stage we don't know which antiderivative is the right one.

Hence his displacement s at time t is given by

$$s = 6t - \tfrac{5}{4} \times \tfrac{1}{2}t^2 + c,$$

that is,

$$s = 6t - \tfrac{5}{8}t^2 + c,$$

where c is a constant.

Use extra information to find the value of the constant c.

When the man begins his walk, his displacement from his starting point is 0 km. That is, when $t = 0$, $s = 0$. Substituting these values into the equation above gives

$$0 = 6 \times 0 - \tfrac{5}{8} \times 0^2 + c,$$

that is,

$$c = 0.$$

Hence the required equation for s in terms of t is

$$s = 6t - \tfrac{5}{8}t^2.$$

(b) When $t = 2$,

$$s = 6 \times 2 - \tfrac{5}{8} \times 2^2 = 12 - \tfrac{5}{2} = \tfrac{19}{2} = 9.5.$$

So the man's displacement two hours after he began his walk is 9.5 km.

The calculation in Example 20 is the sort of calculation that integral calculus allows you to carry out: if you know the values taken by the rate of change of a quantity throughout a period of change, then you can use integration to find the values taken by the quantity throughout the same period.

Here's something else that's illustrated by Example 20. If the only information that you have about the motion of an object is an equation for its velocity in terms of time, then you *don't have enough information* to find an equation for its displacement in terms of time. The best that you can do is to find an equation for its displacement in terms of time that contains an arbitrary constant. To find a definite equation (that is, to choose the correct function from a family of functions whose graphs are vertical translations of each other), you need to use further information, which is usually a fact about the displacement of the object at some moment in time.

This is just what you'd expect: if all you know is an equation for the velocity of the object in terms of time, then you don't have enough information to determine its displacement in terms of time, because it could have started anywhere!

In this sort of situation you do, however, have enough information to find the *change in the displacement* of the object between any two points in time. You'll see some examples of this later in this subsection.

Here's an example similar to Example 20 for you to try.

Activity 32 *Using integration to deduce displacement from velocity*

Suppose that a marble is rolled in a straight line down a long slope, and its velocity v (in metres per second) at time t (in seconds) after it begins rolling is given by the equation

$$v = \tfrac{1}{10}t.$$

The marble starts rolling from a position that is 0.3 metres down the slope.

Let s be its displacement in metres from the top of the slope.

(a) Find an equation that expresses s in terms of t.

(Hint: First find an equation for s in terms of t that contains an arbitrary constant, then use the value of s when $t = 0$.)

(b) Hence find the displacement of the marble from the top of the slope 4 seconds after it begins rolling.

Let's now consider another example of motion in a straight line from Unit 6. You saw that if an object falls from rest under the influence of gravity, and the effects of air resistance are negligible, then its displacement s (in metres) at time t (in seconds) after it began falling is modelled by the equation

$$s = -4.9t^2.$$

Here the displacement of the object is measured from the point from which it starts to fall, and the positive direction along the line of motion is upwards.

You saw that if you differentiate this equation, to obtain an equation for the object's velocity in terms of time, and then differentiate again to obtain an equation for its acceleration in terms of time, then you obtain the fact that the object is moving with a constant acceleration of $-9.8\,\mathrm{m\,s^{-2}}$. The magnitude of this acceleration, $9.8\,\mathrm{m\,s^{-2}}$, is known as the *acceleration due to gravity*.

In the next example, we'll carry out this process in reverse. We'll start by assuming that the effect of gravity on any object falling from rest is to produce a constant acceleration of $-9.8\,\mathrm{m\,s^{-2}}$. We'll use this fact to write down an equation for the object's acceleration in terms of time, integrate it to obtain an equation for the object's velocity in terms of time, and then integrate again to obtain an equation for the object's displacement in terms of time. You'll see that the result is the equation stated above.

Example 21 *Using integration to deduce displacement from acceleration*

Assume that an object falling from rest has a constant acceleration of $-9.8\,\mathrm{m\,s^{-2}}$, where the positive direction is upwards.

Let s be the displacement in metres of the object from its initial position and let t be the time in seconds since it began falling.

Find an equation that expresses s in terms of t.

Solution

Define the variables that you intend to use.

Let v be the velocity of the object in $\mathrm{m\,s^{-1}}$ and let a be its acceleration in $\mathrm{m\,s^{-2}}$.

Write down the equation for the acceleration of the object.

The equation for a in terms of t is

$$a = -9.8.$$

The variable t doesn't appear on the right-hand side of this equation because the acceleration is constant.

Use integration to find an equation for v in terms of t, containing an arbitrary constant.

Hence the equation for v in terms of t is

$$v = -9.8t + c,$$

where c is a constant.

Use extra information to find the value of the constant c.

The object falls from rest, so when it begins falling its velocity is $0\,\mathrm{m\,s^{-1}}$. That is, when $t = 0$, $v = 0$. Substituting these values into the equation above gives

$$0 = -9.8 \times 0 + c, \quad \text{that is,} \quad c = 0.$$

So the equation for v in terms of t is

$$v = -9.8t.$$

Use integration again to find an equation for s in terms of t, containing another arbitrary constant. Don't use the letter c for the arbitrary constant, as it's been used already. Choose a different letter, such as b.

It follows that the equation for s in terms of t is

$$s = -9.8 \times \tfrac{1}{2}t^2 + b,$$

that is,

$$s = -4.9t^2 + b,$$

where b is a constant.

Use extra information to find the value of the constant b.

When the object begins falling, its displacement is 0 m. That is, when $t = 0$, $s = 0$. Substituting these values into the equation gives

$$0 = -4.9 \times 0^2 + b, \quad \text{that is,} \quad b = 0.$$

So the equation for s in terms of t is

$$s = -4.9t^2.$$

This is the required equation for the displacement of the object in terms of time.

The next activity is about the motion of a ball thrown vertically into the air. The ball is an example of a *vertical projectile*. A **projectile** is an object that's launched into the air by a force that ceases after launch, and a **vertical projectile** is a projectile that's launched vertically upwards.

Provided that the effects of air resistance are negligible, a vertical projectile has a constant downwards acceleration due to gravity of about $9.8\,\mathrm{m\,s^{-2}}$ at all times after its launch, just as an object falling from rest has. The only thing that makes its motion different from that of an object falling from rest is that at the start of its motion its velocity is not zero. In the next activity you're asked to use a process similar to that in Example 21 to find a formula for the displacement of a ball thrown vertically into the air, in terms of time.

Activity 33 *Using integration to deduce displacement from acceleration*

Suppose that a ball is thrown vertically upwards with initial speed $12\,\mathrm{m\,s^{-1}}$. Assume that its subsequent motion is modelled as having a constant acceleration of $-9.8\,\mathrm{m\,s^{-2}}$, where the positive direction along the line of motion is upwards.

Let the acceleration, velocity and displacement of the ball at time t (in seconds) after it was thrown be a (in $\mathrm{m\,s^{-2}}$), v (in $\mathrm{m\,s^{-1}}$) and s (in m), respectively, where displacement is measured from the point from which the ball was thrown.

(a) Find an equation that expresses v in terms of t.

(b) Hence find an equation that expresses s in terms of t.

(c) Use the formulas that you found in parts (a) and (b) to find the velocity and the displacement of the ball 1 second after it was thrown.

(d) Use the formula that you found in part (b) to determine how long it takes for the ball to fall back to the point from which it was thrown.

Hint: At the time when the ball has fallen back to this point, what is the value of s?

The next activity involves an economic model of a type that you met in Unit 6. You saw there that marginal cost is the derivative of total cost.

Activity 34 *Using integration to work with marginal cost and total cost*

A company that produces agricultural fertiliser has developed a model for its production costs. According to the model, if the amount of fertiliser that the company is currently producing each week is q (in tonnes), then the marginal weekly cost m (in £ per tonne) of producing extra fertiliser is given by the equation

$$m = 250 - 0.5q.$$

The model applies for values of q between 50 and 200.

Currently the company produces 80 tonnes of fertiliser each week, at a total weekly cost of £30 400.

Let the total weekly cost of producing q tonnes of fertiliser be t (in £).

(a) By using the fact that marginal cost is the derivative of total cost, find a formula for t in terms of q.

(b) Use the formula that you found in part (a) to find the total weekly cost of producing 160 tonnes of fertiliser.

(c) What is the cost per tonne (in other words, the unit cost, not the marginal cost) of producing the fertiliser if the company produces 80 tonnes of fertiliser each week? What is it if it produces 160 tonnes each week?

5.3 Changes in the values of antiderivatives

There's a fact about the antiderivatives of a continuous function that it's important to appreciate.

To understand this fact, consider again the function $f(x) = 2x$, and its antiderivatives, the functions of the form $F(x) = x^2 + c$, whose graphs are shown in Figure 33.

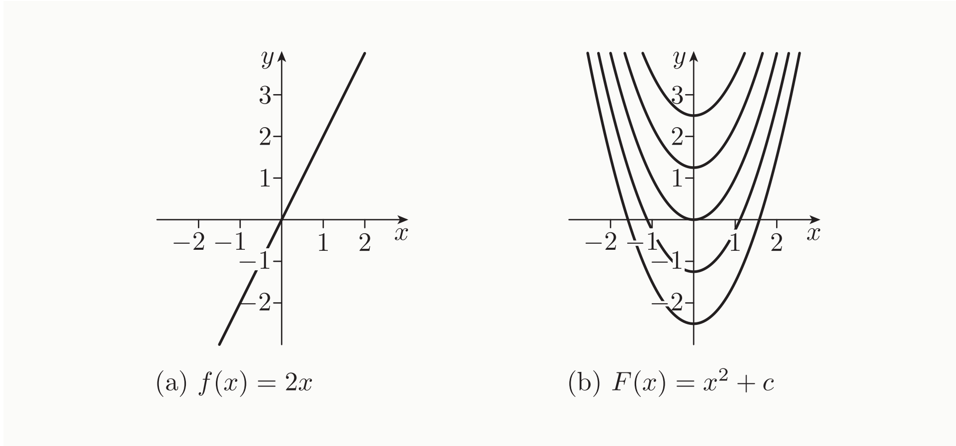


Figure 33 The graphs of a particular continuous function and some of its antiderivatives

Let's choose any two numbers in the domain of the function f, say 1 and 2, as shown in Figure 34(a), and consider the amounts by which the *antiderivatives* of f change as x changes from the first number to the second number. Figure 34(b) illustrates the amount of the change for each of two different antiderivatives of f. You can see that the amount of the change appears to be 3 units in each case.

The important fact to appreciate is that *the amount of the change is the same for all the antiderivatives of* f. That's because the graphs of all the antiderivatives are vertical translations of each other, and translating a graph vertically doesn't affect how much it changes from one value of x to another.

Another way to see that all the antiderivatives of the function $f(x) = 2x$ change by the same amount as x changes from $x = 1$ to $x = 2$ is to use algebra. Consider any antiderivative F of f. Then F is of the form $F(x) = x^2 + c$, where c is some constant. The value of F when $x = 1$ is

$$F(1) = 1^2 + c,$$

and the value of F when $x = 2$ is

$$F(2) = 2^2 + c.$$

So the change in the value of F as x changes from $x = 1$ to $x = 2$ is

$$\begin{aligned} F(2) - F(1) &= (2^2 + c) - (1^2 + c) \\ &= 4 + c - 1 - c \\ &= 3. \end{aligned}$$

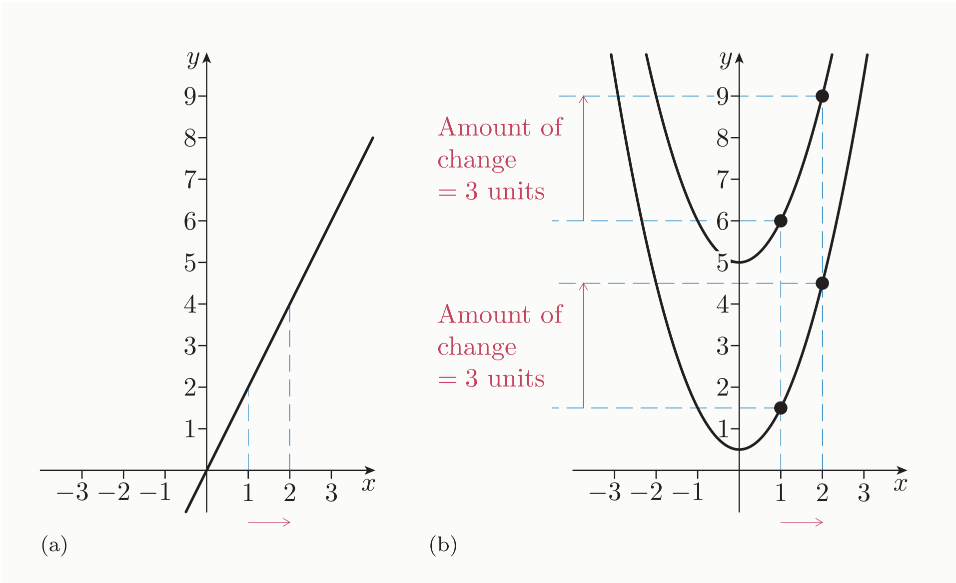


Figure 34 (a) The numbers 1 and 2 in the domain of $f(x) = 2x$ (b) The amounts that two different antiderivatives of $f(x) = 2x$ change as x changes from 1 to 2

You can see that no matter what the value of c is, the two occurrences of it will cancel out when you calculate $F(2) - F(1)$. Hence you'll always get the same answer for $F(2) - F(1)$, namely 3.

In general, we have the following important fact.

Suppose that f is a function that has an antiderivative, and f is continuous. If a and b are numbers in the domain of f, then all the antiderivatives of f change by the same amount as x changes from $x = a$ to $x = b$. In other words, the value of $F(b) - F(a)$ is the same for every antiderivative F of f.

So if you want to work out the amount by which an antiderivative of a continuous function f changes from $x = a$ to $x = b$, then you just need to find *any* antiderivative F of f, and calculate $F(b) - F(a)$.

You can use this method to solve problems in which you know the values taken by the rate of change of a quantity over some period of change, and you want to find the amount by which the quantity changes from one point in this period to another. Here's an example.

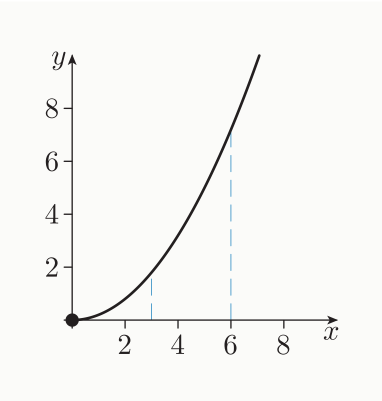


Figure 35 The graph of $y = \frac{1}{5}x^2$ $(x \geq 0)$

Example 22 *Finding a change in the value of an antiderivative*

Suppose that the rate of change of a quantity is given by the function $f(x) = \frac{1}{5}x^2$ $(x \geq 0)$, as shown in Figure 35. By what amount does the quantity change from $x = 3$ to $x = 6$?

Solution

Find any antiderivative F of f and calculate $F(6) - F(3)$.

An antiderivative of f is

$$F(x) = \tfrac{1}{5} \times \tfrac{1}{3}x^3 = \tfrac{1}{15}x^3.$$

So the change in the quantity from $x = 3$ to $x = 6$ is

$$\begin{aligned} F(6) - F(3) &= \left(\tfrac{1}{15} \times 6^3\right) - \left(\tfrac{1}{15} \times 3^3\right) \\ &= \tfrac{72}{5} - \tfrac{9}{5} \\ &= \tfrac{63}{5} = 12.6. \end{aligned}$$

Here's a similar activity for you to try.

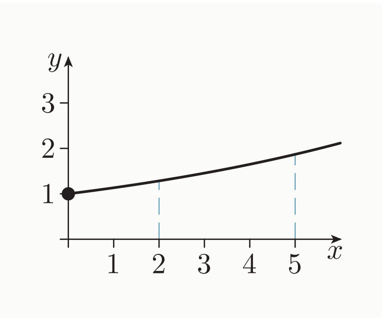


Figure 36 The graph of $f(x) = e^{x/8}$ $(x \geq 0)$

Activity 35 *Finding a change in the value of an antiderivative*

Suppose that the rate of change of a quantity is given by the function $f(x) = e^{x/8}$ $(x \geq 0)$, as shown in Figure 36. Find the amount by which the quantity changes from $x = 2$ to $x = 5$, to three significant figures.

In the next example and activity, the changing quantity is the displacement of a moving object, and hence the rate of change of the quantity is the velocity of the object.

The example is about a car whose velocity is increasing, as shown in Figure 37.

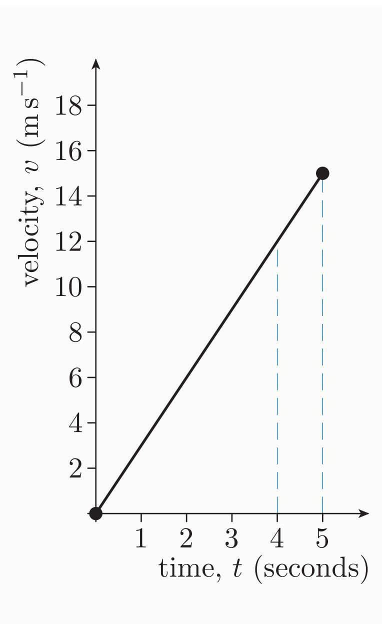


Figure 37 The graph of $v = 3t$ $(0 \leq t \leq 5)$

Example 23 *Finding a change in displacement*

Suppose that a car begins to move along a straight road, and its velocity v (in m s^{-1}) during the first five seconds of its journey is given by $v = f(t)$, where

$$f(t) = 3t.$$

Here t is the time in seconds since the car began moving. How far does the car travel from the end of the fourth second to the end of the fifth second?

Solution

Find any antiderivative F of f and calculate $F(5) - F(4)$.

An antiderivative of f is

$$F(t) = \tfrac{3}{2}t^2.$$

So the change in the displacement of the car from $t = 4$ to $t = 5$ is

$$\begin{aligned} F(5) - F(4) &= \left(\tfrac{3}{2} \times 5^2\right) - \left(\tfrac{3}{2} \times 4^2\right) \\ &= 37.5 - 24 \\ &= 13.5. \end{aligned}$$

That is, the car travels 13.5 m (in the positive direction) during the fifth second.

The activity below is about an object whose velocity is decreasing, and then becomes negative, as shown in Figure 38; that is, it starts to move in the opposite direction, as shown in Figure 39.

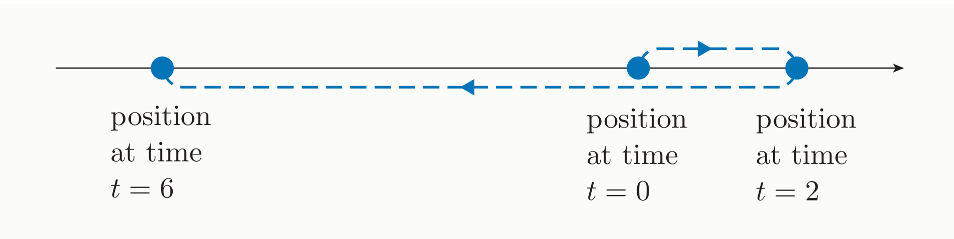


Figure 39 The object whose motion is given by Figure 38 is moving in the positive direction at time $t = 0$, then turns at time $t = 2$ to move in the negative direction

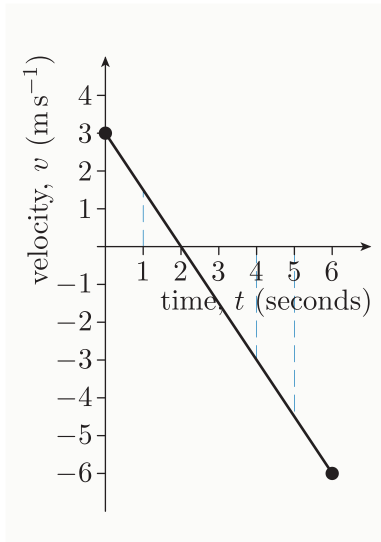


Figure 38 The graph of $v = 3 - \frac{3}{2}t$ $(0 \le t \le 6)$

Activity 36 *Finding changes in displacement*

Suppose that an object moves along a straight line, and its velocity v (in m s^{-1}) at time t (in seconds) is given by $v = f(t)$, where $f(t) = 3 - \frac{3}{2}t$ $(0 \le t \le 6)$, as shown in Figure 38.

(a) By what amount does the displacement of the object change from time $t = 0$ to time $t = 1$? What is the *distance* that the object travels in that time?

(b) Repeat part (a) for the times $t = 4$ and $t = 5$.

The ideas that you've met in this subsection will be important in Unit 8.

6 Antiderivatives of further standard functions

In this section you'll meet and use formulas for antiderivatives of some further standard functions. These formulas are all found by using formulas for derivatives that you met earlier in this unit.

6.1 An antiderivative of the reciprocal function

We'll begin by finding a formula for an antiderivative of the reciprocal function $f(x) = 1/x$, which is the only power function that isn't covered by the general formula for the indefinite integral of a power function that you met in Subsection 4.2.

Here's how you can find an antiderivative of this function, by using what you already know about differentiation.

Remember that you saw earlier in this unit that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}. \qquad (1)$$

Unfortunately this formula doesn't immediately give you a formula for an antiderivative of the function $f(x) = 1/x$, because the function $f(x) = 1/x$ has domain $(-\infty, 0) \cup (0, \infty)$, but the formula applies only when x is positive (because $\ln x$ is defined only when x is positive). However, the formula does tell you that an antiderivative of the function

$$g(x) = \frac{1}{x} \quad (x > 0)$$

is

$$G(x) = \ln x.$$

This fact is illustrated in Figure 40. The graph of $g(x) = 1/x$ $(x > 0)$, in Figure 40(a), gives the gradients of the graph of $G(x) = \ln x$, in Figure 40(b).

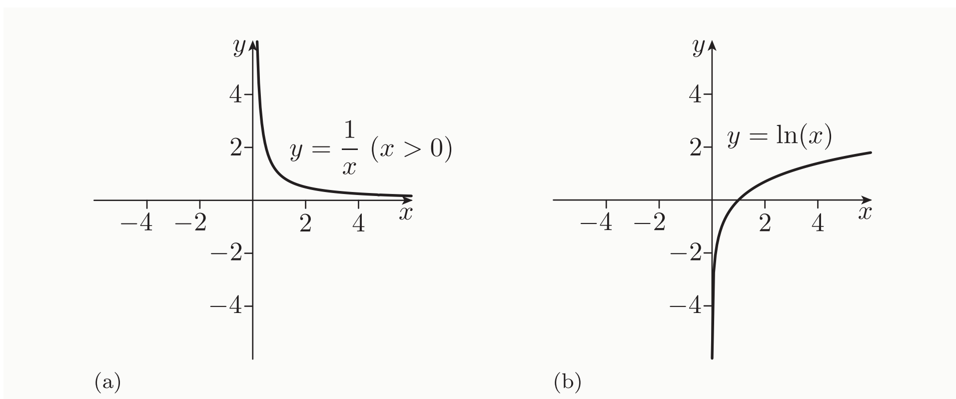


Figure 40 The graphs of (a) $g(x) = 1/x$ $(x > 0)$ and (b) $G(x) = \ln x$

To deal with the 'other half' of the function $f(x) = 1/x$, you can use the fact that equation (1), together with the chain rule, gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(-x)) = \frac{1}{-x} \times (-1) = \frac{1}{x}.$$

This formula applies for *negative* values of x, because $\ln(-x)$ is defined when $-x$ is positive, that is, when x is negative. So it tells you that an antiderivative of the function

$$h(x) = 1/x \quad (x < 0)$$

is

$$H(x) = \ln(-x).$$

This fact is illustrated in Figure 41. The graph of $h(x) = 1/x$ $(x < 0)$, in Figure 41(a), gives the gradients of the graph of $H(x) = \ln(-x)$, in Figure 41(b).

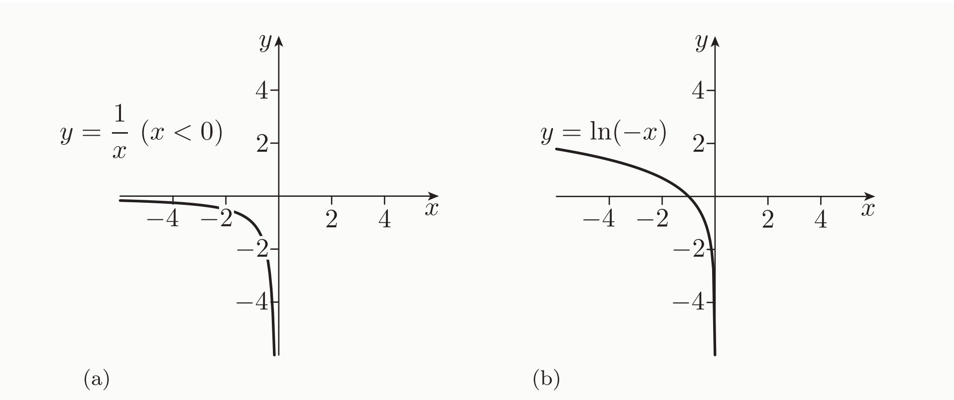


Figure 41 The graphs of (a) $h(x) = 1/x$ $(x < 0)$ and (b) $H(x) = \ln(-x)$

You can obtain an antiderivative for the function $f(x) = 1/x$, with its whole domain $(-\infty, 0) \cup (0, \infty)$, by putting together the antiderivatives found for the two 'halves' of f. The graph of this antiderivative is shown in Figure 42(b). The graph of $f(x) = 1/x$ is shown in Figure 42(a): it gives the gradients of the graph in Figure 42(b).

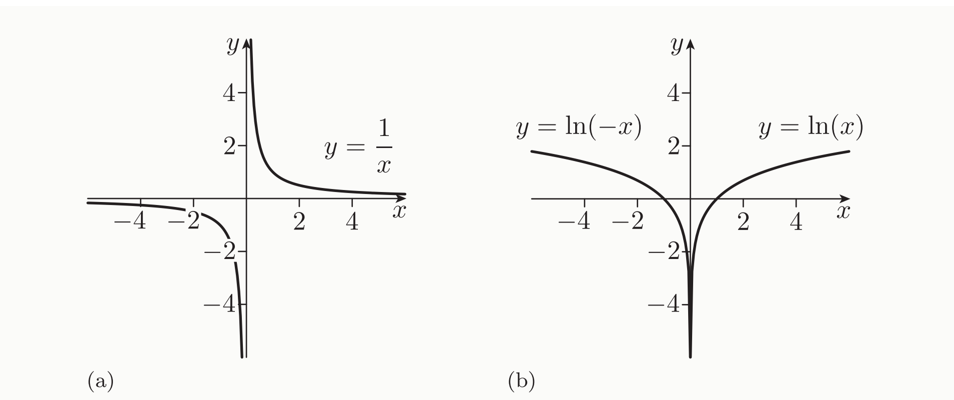


Figure 42 The graphs of (a) $f(x) = 1/x$ and (b) $F(x) = \ln x$ and $F(x) = \ln(-x)$

The formula for the antiderivative of $f(x) = 1/x$ found above can be written as

$$F(x) = \begin{cases} \ln(-x) & (x < 0) \\ \ln x & (x > 0). \end{cases} \tag{2}$$

However, there's a more concise way to write it, using the notation $|\ |$ for the modulus of a number. Notice that if x is a negative number, then

$$-x = |x|$$

(for example, $-(-3) = 3 = |-3|$). Also, if x is a positive number, then

$$x = |x|$$

(for example, $3 = |3|$). So formula (2) can be written concisely as

$$F(x) = \ln|x|.$$

Hence the indefinite integral of $f(x) = 1/x$ is

$$F(x) = \ln|x| + c.$$

This formula is sometimes useful, but often when you're working with the function $f(x) = 1/x$, the variable x takes only *positive* values. If this is the case, then you can just use the antiderivative of the function $f(x) = 1/x$ $(x > 0)$, which has the simpler formula

$$F(x) = \ln x + c.$$

The facts that you've seen so far in this section are summarised below.

Indefinite integral of the reciprocal function

The indefinite integral of $\dfrac{1}{x}$ is $\ln|x| + c$.

(If x takes only positive values, then the indefinite integral of $\dfrac{1}{x}$ is simply $\ln x + c$.)

In the next activity you're asked to use this formula, together with the constant multiple rule and the sum rule for antiderivatives, to find the indefinite integrals of some further functions.

Activity 37 *Finding indefinite integrals of more functions*

Find the indefinite integrals of the following functions.

(a) $f(x) = \dfrac{4}{x}$ (b) $f(x) = \dfrac{x+1}{x^2}$ (c) $f(x) = \dfrac{1}{2x}$ $(x > 0)$

6.2 Table of standard indefinite integrals

You can use all of the derivatives of standard functions that you met in Unit 6 and in the first three sections of this unit to deduce antiderivatives, and hence indefinite integrals, of further functions.

You saw in the last subsection that it isn't straightforward to use the fact that the derivative of $\ln x$ is $1/x$ to deduce an antiderivative of $1/x$, because the domain of the function $f(x) = 1/x$ contains numbers that aren't in the domain of the function $F(x) = \ln x$. Luckily there are no similar problems with any of the other derivatives of standard functions that you've met, so it's straightforward to use them to deduce antiderivatives.

For example, you can immediately deduce antiderivatives of the standard functions sin, cos and exp, as follows. Since

the derivative of $\sin x$ is $\cos x$,

it follows that

an antiderivative of $\cos x$ is $\sin x$.

Similarly, since

the derivative of $\cos x$ is $-\sin x$,

it follows that

an antiderivative of $-\sin x$ is $\cos x$,

and hence (by the constant multiple rule for antiderivatives)

an antiderivative of $\sin x$ is $-\cos x$.

Finally, since the derivative of e^x is e^x, it follows that an antiderivative of e^x is e^x.

The table in the box below contains all the indefinite integrals of standard functions that you've met so far, and all the further indefinite integrals that can be deduced from the table of standard derivatives in Subsection 3.4, in the way just described. You should try to memorise at least the ones above the line in the middle of the table. For the others, you should try to make sure that you can at least recognise the functions on the left as functions that can be integrated easily, even if you continue to refer to the table for their indefinite integrals. This table of standard indefinite integrals is also included in the *Handbook*.

Standard indefinite integrals

Function	Indefinite integral
a (constant)	$ax + c$
x^n $(n \neq -1)$	$\dfrac{1}{n+1}\,x^{n+1} + c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + c$ or $\ln x + c$, for $x > 0$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\operatorname{cosec}^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$ or $-\cos^{-1} x + c$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + c$

In the next activity you're asked to use these formulas, together with the constant multiple rule and the sum rule for antiderivatives, to find the indefinite integrals of some further functions. In some of the parts you have to start by using an algebraic technique to express the function in a form that you can integrate.

Activity 38 *Finding indefinite integrals of more functions*

Find the indefinite integrals of the following functions.

(a) $h(\theta) = 3\cos\theta + 4\sin\theta$ (b) $g(\phi) = 6 - 5\operatorname{cosec}^2\phi$ (c) $f(x) = \dfrac{7}{x}$

(d) $g(t) = \dfrac{1}{3t}$ (e) $f(x) = \dfrac{1}{\pi x}$ $(x > 0)$ (f) $f(x) = \dfrac{3}{1+x^2}$

(g) $h(t) = \dfrac{1}{4\sqrt{1-t^2}}$ (h) $p(x) = \dfrac{1}{5+5x^2}$ (i) $g(x) = \dfrac{1}{\sqrt{4-4x^2}}$

(j) $q(x) = 5(x-3)(2x-1)$ (k) $f(x) = \dfrac{x-2}{x}$ (l) $g(x) = \dfrac{x-2}{x^3}$

(m) $f(x) = 8e^x$ (n) $f(x) = e^{1+x}$ (o) $r(\phi) = -\operatorname{cosec}\phi\cot\phi$

(p) $r(\theta) = \sec\theta(\sec\theta + \tan\theta)$

You'll learn much more about integration in Unit 8. In particular you'll meet a surprising and useful alternative way to understand it. You'll also meet some further methods for combining antiderivatives of functions to obtain antiderivatives of more functions.

Learning outcomes

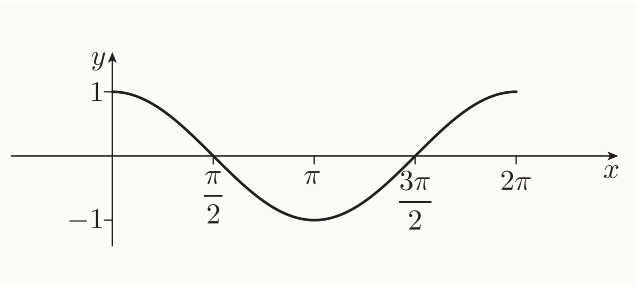
After studying this unit, you should be able to:

- remember the formulas for the derivatives of some standard functions
- differentiate a wide variety of functions, including products, quotients and composites
- use differentiation to solve simple optimisation problems
- find and use derivatives using a computer
- understand and use the relationship between the derivative of an invertible function and the derivative of its inverse function
- understand that integration reverses differentiation
- understand what's meant by an antiderivative and the indefinite integral of a function
- remember the formulas for the indefinite integrals of some standard functions
- use the constant multiple rule and the sum rule for antiderivatives
- use integration to work with the relationship between a changing quantity and its rate of change
- in particular, use integration to work with the relationships between displacement, velocity and acceleration.

Solutions to activities

Solution to Activity 1

The graph of the derivative of the sine function for x between 0 and 2π must look something like the sketch below.



(You can work this out as follows.

When $x = \pi$, the gradient of the graph of the sine function seems to be about the same as the gradient of the line $y = -x$; that is, it seems to be about -1. As x increases, the graph gradually gets less steep – that is, its gradient takes negative values of smaller and smaller magnitude, until $x = 3\pi/2$, when the gradient seems to be zero. The gradient changes slowly when x is only a little larger than π, but changes more rapidly as x gets closer to $3\pi/2$.

In other words, for x between π and $3\pi/2$ the value of the derivative seems to increase from -1 to zero, slowly at first, but then more rapidly.

When x increases from $3\pi/2$, the gradient starts at zero, then it becomes positive. At first the graph is not very steep – that is, the gradient has small positive values – but it becomes steeper and steeper – that is, the gradient takes larger and larger positive values. Eventually, when $x = 2\pi$, the gradient seems to be about the same as the gradient of the line $y = x$; that is, it seems to be about 1. The gradient increases fairly rapidly when x is only a little larger than $3\pi/2$, but increases more slowly as x gets closer to 2π.

So, for x between $3\pi/2$ and 2π, the value of the derivative seems to increase from 0 to 1, fairly rapidly at first, but then more slowly.)

Solution to Activity 2

(a) $f(x) = \sin x + \cos x$, so
$$f'(x) = \cos x - \sin x.$$

(b) $g(u) = u^2 - \cos u$, so
$$\begin{aligned} g'(u) &= 2u - (-\sin u) \\ &= 2u + \sin u. \end{aligned}$$

(c) $P = 6\tan\theta$, so
$$\frac{\mathrm{d}P}{\mathrm{d}\theta} = 6\sec^2\theta.$$

(d) $r = -2(1 + \sin\phi)$, so
$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\phi} &= -2(0 + \cos\phi) \\ &= -2\cos\phi. \end{aligned}$$

Solution to Activity 3

(a) $f(x) = e^x + \ln x$, so
$$f'(x) = e^x + \frac{1}{x}.$$

(b) $h(r) = r - \cos r - 3\ln r$, so
$$\begin{aligned} h'(r) &= 1 - (-\sin r) - 3 \times \frac{1}{r} \\ &= 1 + \sin r - \frac{3}{r}. \end{aligned}$$

(c) $v = \dfrac{1}{t} + \ln t$, so
$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{1}{t^2} + \frac{1}{t} \\ &= -\frac{1}{t^2} + \frac{t}{t^2} \\ &= \frac{t-1}{t^2}. \end{aligned}$$

(d) $w = 5 - 3e^u$, so
$$\frac{\mathrm{d}w}{\mathrm{d}u} = -3e^u.$$

(e) $k = 4(\ln v - \tan v)$, so
$$\frac{\mathrm{d}k}{\mathrm{d}v} = 4\left(\frac{1}{v} - \sec^2 v\right).$$

Solution to Activity 4

(a) The function is $k(x) = x^3\tan x$.
Let $f(x) = x^3$ and $g(x) = \tan x$.
Then $f'(x) = 3x^2$ and $g'(x) = \sec^2 x$.

By the product rule,

$$\begin{aligned} k'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^3 \sec^2 x + (\tan x) \times 3x^2 \\ &= x^3 \sec^2 x + 3x^2 \tan x \\ &= x^2(x \sec^2 x + 3\tan x). \end{aligned}$$

(b) The function is $k(x) = xe^x$.
Let $f(x) = x$ and $g(x) = e^x$.
Then $f'(x) = 1$ and $g'(x) = e^x$.
By the product rule,

$$\begin{aligned} k'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= xe^x + e^x \times 1 \\ &= xe^x + e^x \\ &= (x+1)e^x. \end{aligned}$$

Solution to Activity 5

The function is $k(x) = x^2 \times x^3$.
Let $f(x) = x^2$ and $g(x) = x^3$.
Then $f'(x) = 2x$ and $g'(x) = 3x^2$.
By the product rule,

$$\begin{aligned} k'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^2 \times 3x^2 + x^3 \times 2x \\ &= 3x^4 + 2x^4 \\ &= 5x^4. \end{aligned}$$

This is the same answer that you obtain by differentiating $k(x) = x^5$ directly, using the usual rule for the derivative of a power function.

Solution to Activity 6

(a) $k(x) = (3x^2 + 2x + 1)e^x$, so, by the product rule,

$$\begin{aligned} k'(x) &= (3x^2 + 2x + 1)e^x + e^x(6x + 2) \\ &= (3x^2 + 2x + 1 + 6x + 2)e^x \\ &= (3x^2 + 8x + 3)e^x. \end{aligned}$$

(b) $k(x) = \sin x \cos x$, so, by the product rule,

$$\begin{aligned} k'(x) &= (\sin x)(-\sin x) + (\cos x)(\cos x) \\ &= \cos^2 x - \sin^2 x. \end{aligned}$$

(c) $k(z) = z \sin z$, so, by the product rule,

$$\begin{aligned} k'(z) &= z\cos z + (\sin z) \times 1 \\ &= z\cos z + \sin z. \end{aligned}$$

(d) $v = (2t^2 - 1)\cos t$, so, by the product rule,

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= (2t^2 - 1)(-\sin t) + (\cos t) \times 4t \\ &= (1 - 2t^2)\sin t + 4t\cos t. \end{aligned}$$

(e) $m = (u^2 + 3)\ln u$, so, by the product rule,

$$\begin{aligned} \frac{\mathrm{d}m}{\mathrm{d}u} &= (u^2 + 3) \times \frac{1}{u} + (\ln u) \times 2u \\ &= \frac{u^2 + 3}{u} + 2u\ln u \\ &= \frac{u^2 + 3}{u} + \frac{2u^2 \ln u}{u} \\ &= \frac{u^2 + 3 + 2u^2 \ln u}{u} \\ &= \frac{u^2(1 + 2\ln u) + 3}{u}. \end{aligned}$$

(f) $y = \sqrt{x}\sin x = x^{1/2}\sin x$, so, by the product rule,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^{1/2}\cos x + (\sin x) \times \frac{1}{2}\,x^{-1/2} \\ &= \sqrt{x}\cos x + \frac{\sin x}{2\sqrt{x}} \\ &= \frac{2x\cos x + \sin x}{2\sqrt{x}}. \end{aligned}$$

Solution to Activity 7

If $y = x^2 e^x$, then, by the product rule,

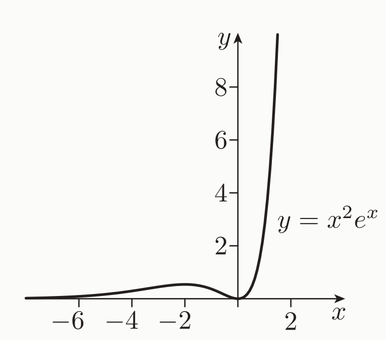
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 e^x + e^x \times 2x \\ &= x(x+2)e^x. \end{aligned}$$

Substituting $x = -1$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (-1)(-1+2)e^{-1} = -\frac{1}{e}.$$

So the gradient is $-1/e$, which is approximately -0.37.

(The graph of $y = x^2 e^x$ is shown below. You can see that the gradient at the point with x-coordinate -1 does seem to be approximately -0.37.)



Solution to Activity 8

(a) The function is $k(x) = \dfrac{e^x}{x}$.

Let $f(x) = e^x$ and $g(x) = x$.

Then $f'(x) = e^x$ and $g'(x) = 1$.

So, by the quotient rule,

$$\begin{aligned} k'(x) = &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{xe^x - e^x \times 1}{x^2} \\ &= \frac{(x-1)e^x}{x^2}. \end{aligned}$$

(b) The function is $k(x) = \dfrac{x^3}{2x+1}$.

Let $f(x) = x^3$ and $g(x) = 2x + 1$.

Then $f'(x) = 3x^2$ and $g'(x) = 2$.

So, by the quotient rule,

$$\begin{aligned} k'(x) = &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(2x+1) \times 3x^2 - x^3 \times 2}{(2x+1)^2} \\ &= \frac{(3(2x+1) - 2x)x^2}{(2x+1)^2} \\ &= \frac{(6x+3-2x)x^2}{(2x+1)^2} \\ &= \frac{(4x+3)x^2}{(2x+1)^2}. \end{aligned}$$

Solution to Activity 9

(a) $f(x) = \dfrac{e^x}{x^3}$, so, by the quotient rule,

$$\begin{aligned} f'(x) &= \frac{x^3e^x - e^x(3x^2)}{(x^3)^2} \\ &= \frac{x^2(x-3)e^x}{x^6} \\ &= \frac{(x-3)e^x}{x^4}. \end{aligned}$$

(b) $g(u) = \dfrac{u-1}{e^u}$, so, by the quotient rule,

$$\begin{aligned} g'(u) &= \frac{(e^u) \times 1 - (u-1)e^u}{(e^u)^2} \\ &= \frac{(1-u+1)e^u}{(e^u)^2} \\ &= \frac{2-u}{e^u}. \end{aligned}$$

(c) $z = \dfrac{r+1}{r-1}$, so, by the quotient rule,

$$\begin{aligned} \frac{dz}{dr} &= \frac{(r-1) \times 1 - (r+1) \times 1}{(r-1)^2} \\ &= \frac{r-1-r-1}{(r-1)^2} \\ &= -\frac{2}{(r-1)^2}. \end{aligned}$$

(d) $m = \dfrac{\theta}{\cos\theta}$, so, by the quotient rule,

$$\begin{aligned} \frac{dm}{d\theta} &= \frac{(\cos\theta) \times 1 - \theta(-\sin\theta)}{(\cos\theta)^2} \\ &= \frac{\cos\theta + \theta\sin\theta}{\cos^2\theta} \\ &= \frac{1 + \theta\tan\theta}{\cos\theta}. \end{aligned}$$

(The final expression here is obtained by dividing the top and bottom of the previous expression by $\cos\theta$. The previous expression would also be an acceptable final answer.)

(e) $p(t) = \dfrac{\ln t}{t^2}$, so, by the quotient rule,

$$\begin{aligned} p'(t) &= \frac{t^2 \times \frac{1}{t} - (\ln t) \times 2t}{(t^2)^2} \\ &= \frac{t - 2t\ln t}{t^4} \\ &= \frac{1 - 2\ln t}{t^3}. \end{aligned}$$

Solution to Activity 10

(a) By the quotient rule,

$$\begin{aligned} \frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\ &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\operatorname{cosec}^2 x. \end{aligned}$$

(b) By the quotient rule,

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= \frac{(\cos x)\times 0 - 1\times(-\sin x)}{(\cos x)^2}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)\\ &= \sec x \tan x.\end{aligned}$$

(c) By the quotient rule,

$$\begin{aligned}\frac{d}{dx}(\operatorname{cosec} x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ &= \frac{(\sin x)\times 0 - 1\times \cos x}{(\sin x)^2}\\ &= \frac{-\cos x}{\sin^2 x}\\ &= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)\\ &= -\operatorname{cosec} x \cot x.\end{aligned}$$

Solution to Activity 11

(a) $y = \ln(x^4)$
so $y = \ln u$ where $u = x^4$.

(b) $y = \sin(x^2 - 1)$
so $y = \sin u$ where $u = x^2 - 1$.

(c) $y = \cos(e^x)$
so $y = \cos u$ where $u = e^x$.

(d) $y = \cos^3 x$
so $y = u^3$ where $u = \cos x$.

(e) $y = e^{\tan x}$
so $y = e^u$ where $u = \tan x$.

(f) $y = \sqrt{\ln x}$
so $y = \sqrt{u}$ where $u = \ln x$.

(g) $y = (x + \sqrt{x})^9$
so $y = u^9$ where $u = x + \sqrt{x}$.

(h) $y = \sqrt[3]{x^2 - x - 1}$
so $y = \sqrt[3]{u}$ where $u = x^2 - x - 1$.

(i) $y = e^{\sqrt{x}}$
so $y = e^u$ where $u = \sqrt{x}$.

Solution to Activity 12

(a) $y = e^{5x}$, so

$$y = e^u \text{ where } u = 5x.$$

This gives

$$\frac{dy}{du} = e^u, \quad \frac{du}{dx} = 5.$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= e^u \times 5\\ &= e^{5x}\times 5\\ &= 5e^{5x}.\end{aligned}$$

(b) $y = \sin(2x^2 - 3)$, so

$$y = \sin u \text{ where } u = 2x^2 - 3.$$

This gives

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = 4x.$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= (\cos u)\times 4x\\ &= (\cos(2x^2 - 3))\times 4x\\ &= 4x\cos(2x^2 - 3).\end{aligned}$$

(c) $y = \sin\left(\frac{x}{4}\right)$, so

$$y = \sin u \text{ where } u = \frac{x}{4}.$$

This gives

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = \frac{1}{4}.$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= (\cos u)\times \frac{1}{4}\\ &= \frac{1}{4}\cos\left(\frac{x}{4}\right).\end{aligned}$$

(d) $y = \cos(e^x)$, so

$$y = \cos u \text{ where } u = e^x.$$

This gives

$$\frac{dy}{du} = -\sin u, \quad \frac{du}{dx} = e^x.$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= (-\sin u)\times e^x\\ &= (-\sin(e^x))\times e^x\\ &= -e^x\sin(e^x)\end{aligned}$$

(e) $y = \ln(\sin x)$, so

$$y = \ln u \text{ where } u = \sin x.$$

This gives

$$\frac{dy}{du} = \frac{1}{u},\qquad \frac{du}{dx} = \cos x.$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= \frac{1}{u}\times\cos x\\ &= \frac{1}{\sin x}\times\cos x\\ &= \frac{\cos x}{\sin x}\\ &= \cot x.\end{aligned}$$

(f) $y = \sin^3 x = (\sin x)^3$, so

$$y = u^3 \text{ where } u = \sin x.$$

This gives

$$\frac{dy}{du} = 3u^2,\qquad \frac{du}{dx} = \cos x.$$

By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= 3u^2\cos x\\ &= 3(\sin x)^2\cos x\\ &= 3\sin^2 x\cos x.\end{aligned}$$

Solution to Activity 13

(a) $f(x) = \cos(4x)$, so, by the chain rule,

$$\begin{aligned}f'(x) &= -\sin(4x)\times 4\\ &= -4\sin(4x).\end{aligned}$$

(b) $f(x) = e^{2x+1}$, so, by the chain rule,

$$\begin{aligned}f'(x) &= e^{2x+1}\times 2\\ &= 2e^{2x+1}.\end{aligned}$$

(c) $r(x) = \cos^5 x$, so, by the chain rule,

$$\begin{aligned}r'(x) &= (5\cos^4 x)(-\sin x)\\ &= -5\sin x\cos^4 x.\end{aligned}$$

(Remember that $\cos^5 x$ means $(\cos x)^5$.)

(d) $g(x) = \ln(x^2+x+1)$, so, by the chain rule,

$$\begin{aligned}g'(x) &= \frac{1}{x^2+x+1}\times(2x+1)\\ &= \frac{2x+1}{x^2+x+1}.\end{aligned}$$

(e) $v(t) = \sin(e^t)$, so, by the chain rule,

$$\begin{aligned}v'(t) &= \cos(e^t)\times e^t\\ &= e^t\cos(e^t).\end{aligned}$$

(f) $v = \ln(r^5)$, so, by the chain rule,

$$\begin{aligned}\frac{dv}{dr} &= \frac{1}{r^5}\times 5r^4\\ &= \frac{5}{r}.\end{aligned}$$

(Here's an alternative way to differentiate this function, without using the chain rule:

$v = \ln(r^5) = 5\ln r$, so, by the constant multiple rule,

$$\frac{dv}{dr} = 5\times\frac{1}{r} = \frac{5}{r}.)$$

(g) $r = \tan^2\theta$, so, by the chain rule,

$$\begin{aligned}\frac{dr}{d\theta} &= (2\tan\theta)\times\sec^2\theta\\ &= 2\tan\theta\sec^2\theta.\end{aligned}$$

(h) $R = e^{\tan q}$, so, by the chain rule,

$$\frac{dR}{dq} = e^{\tan q}\sec^2 q.$$

(i) $A = (1+p^2)^9$ so, by the chain rule,

$$\begin{aligned}\frac{dA}{dp} &= 9(1+p^2)^8\times 2p\\ &= 18p(1+p^2)^8.\end{aligned}$$

Solution to Activity 14

(a) $f(x) = \sqrt{\ln x} = (\ln x)^{1/2}$, so, by the chain rule,

$$\begin{aligned}f'(x) &= \tfrac{1}{2}(\ln x)^{-1/2}\times\frac{1}{x}\\ &= \frac{1}{2x\sqrt{\ln x}}.\end{aligned}$$

(b) $p(w) = \dfrac{1}{(w^3+7)^4} = (w^3+7)^{-4}$, so, by the chain rule,

$$\begin{aligned}p'(w) &= -4(w^3+7)^{-5}\times 3w^2\\ &= -\frac{12w^2}{(w^3+7)^5}.\end{aligned}$$

(c) $f(x) = \cos(\sqrt{x}) = \cos(x^{1/2})$, so, by the chain rule,
$$\begin{aligned} f'(x) &= -\sin(x^{1/2}) \times \tfrac{1}{2}x^{-1/2} \\ &= -\frac{\sin(\sqrt{x})}{2\sqrt{x}}. \end{aligned}$$

(d) $f(x) = \dfrac{1}{\sqrt{e^x}} = (e^x)^{-1/2} = e^{-x/2}$, so, by the chain rule,
$$\begin{aligned} f'(x) &= e^{-x/2} \times (-\tfrac{1}{2}) \\ &= -\tfrac{1}{2}e^{-x/2} \\ &= -\frac{1}{2\sqrt{e^x}}. \end{aligned}$$
(This solution uses the index laws $(a^m)^n = a^{mn}$ and $a^{m/n} = \sqrt[n]{a^m}$.)

(e) $a(t) = \dfrac{1}{(2-3t-3t^2)^{1/3}} = (2-3t-3t^2)^{-1/3}$, so, by the chain rule,
$$\begin{aligned} a'(t) &= -\tfrac{1}{3}(2-3t-3t^2)^{-4/3} \times (-3-6t) \\ &= \frac{1+2t}{(2-3t-3t^2)^{4/3}}. \end{aligned}$$

(f) $C(p) = e^{1/p^2} = e^{(p^{-2})}$, so, by the chain rule,
$$\begin{aligned} C'(p) &= e^{(p^{-2})} \times (-2p^{-3}) \\ &= -2p^{-3}e^{1/p^2} \\ &= -\frac{2e^{1/p^2}}{p^3}. \end{aligned}$$

Solution to Activity 15

(a) $k(x) = \cos(7x)$, so
$$k'(x) = -7\sin(7x).$$

(b) $r(\theta) = \sin\left(\dfrac{\theta}{2}\right)$, so
$$r'(\theta) = \tfrac{1}{2}\cos\left(\frac{\theta}{2}\right).$$

(c) $g(u) = e^{3u/2}$, so
$$g'(u) = \tfrac{3}{2}e^{3u/2}.$$

(d) $h(x) = e^{-x}$, so
$$h'(x) = -e^{-x}.$$

(e) $p(\alpha) = \cos(-8\alpha)$, so
$$\begin{aligned} p'(\alpha) &= -8(-\sin(-8\alpha)) \\ &= 8\sin(-8\alpha). \end{aligned}$$

(f) $w(x) = \ln(7x)$, so
$$w'(x) = \frac{7}{7x} = \frac{1}{x}.$$
(Here's an alternative way to differentiate this function:

$w(x) = \ln(7x) = \ln 7 + \ln x$, so
$$w'(x) = 0 + \frac{1}{x} = \frac{1}{x}.)$$

(g) $f(t) = \sin\left(\dfrac{-2t}{3}\right)$, so
$$f'(t) = -\frac{2}{3}\cos\left(\frac{-2t}{3}\right).$$

(h) $k(\phi) = \tan(3\phi)$, so
$$k'(\phi) = 3\sec^2(3\phi).$$

Solution to Activity 16

(a) $k(x) = \cos(7x+4)$, so
$$k'(x) = -7\sin(7x+4).$$

(b) $h(x) = e^{-x-3}$, so
$$h'(x) = -e^{-x-3}.$$

(c) $r(\theta) = \sin\left(\dfrac{\theta-1}{2}\right) = \sin\left(\dfrac{\theta}{2} - \dfrac{1}{2}\right)$, so
$$r'(\theta) = \frac{1}{2}\cos\left(\frac{\theta}{2} - \frac{1}{2}\right) = \frac{1}{2}\cos\left(\frac{\theta-1}{2}\right).$$

(d) $s(\theta) = \sin\left(\dfrac{2-\theta}{3}\right) = \sin\left(\dfrac{2}{3} - \dfrac{\theta}{3}\right)$, so
$$s'(\theta) = -\frac{1}{3}\cos\left(\frac{2}{3} - \frac{\theta}{3}\right) = -\frac{1}{3}\cos\left(\frac{2-\theta}{3}\right).$$

Solution to Activity 17

(a)
$$\begin{aligned} g(x) &= (x^2+1)(x^3+1) \\ &= x^5 + x^3 + x^2 + 1, \end{aligned}$$
so
$$\begin{aligned} g'(x) &= 5x^4 + 3x^2 + 2x \\ &= x(5x^3 + 3x + 2). \end{aligned}$$
(An alternative method is to use the product rule.)

(b) $f(x) = \dfrac{x}{(x-2)^3}$, so

$$f'(x) = \frac{(x-2)^3 \times 1 - x\dfrac{d}{dx}\left((x-2)^3\right)}{(x-2)^6}$$

(by the quotient rule)

$$= \frac{(x-2)^3 \times 1 - x \times 3(x-2)^2 \times 1}{(x-2)^6}$$

(by the chain rule)

$$= \frac{(x-2)^3 - 3x(x-2)^2}{(x-2)^6}$$

$$= \frac{(x-2) - 3x}{(x-2)^4}$$

$$= \frac{-2x-2}{(x-2)^4}$$

$$= -\frac{2(x+1)}{(x-2)^4}.$$

(c) $g(x) = \cos(x \ln x)$, so

$$g'(x) = -\sin(x\ln x)\frac{d}{dx}(x \ln x)$$

(by the chain rule)

$$= -\sin(x\ln x)\left(x \times \frac{1}{x} + (\ln x) \times 1\right)$$

(by the product rule)

$$= -\sin(x\ln x)(1 + \ln x).$$

(d) $h(x) = e^{x/2}\sin(3x)$, so

$$h'(x) = e^{x/2}\frac{d}{dx}(\sin(3x)) + \sin(3x)\frac{d}{dx}\left(e^{x/2}\right)$$

(by the product rule)

$$= e^{x/2} \times 3\cos(3x) + \sin(3x) \times \tfrac{1}{2}e^{x/2}$$

(by the chain rule)

$$= 3e^{x/2}\cos(3x) + \tfrac{1}{2}e^{x/2}\sin(3x)$$
$$= \tfrac{1}{2}e^{x/2}(6\cos(3x) + \sin(3x)).$$

(e) $q(u) = \dfrac{\cos(4u)}{e^{3u}}$, so

$$q'(u) = \frac{e^{3u}\dfrac{d}{dx}(\cos(4u)) - \cos(4u)\dfrac{d}{dx}\left(e^{3u}\right)}{(e^{3u})^2}$$

(by the quotient rule)

$$= \frac{e^{3u} \times (-4\sin(4u)) - \cos(4u) \times 3e^{3u}}{(e^{3u})^2}$$

(by the chain rule)

$$= \frac{-4e^{3u}\sin(4u) - 3e^{3u}\cos(4u)}{e^{6u}}$$

$$= \frac{-e^{3u}(4\sin(4u) + 3\cos(4u))}{e^{6u}}$$

$$= -\frac{4\sin(4u) + 3\cos(4u)}{e^{3u}}$$

(f) $f(x) = e^{\sin(5x)}$, so

$$f'(x) = e^{\sin(5x)} \times 5\cos(5x)$$

(by the chain rule)

$$= 5e^{\sin(5x)}\cos(5x).$$

(g) $h(x) = \cos^3(x^2)$, so

$$h'(x) = 3\cos^2(x^2)\frac{d}{dx}\left(\cos(x^2)\right)$$

(by the chain rule)

$$= 3\cos^2(x^2) \times (-\sin(x^2)) \times 2x$$

(by the chain rule again)

$$= -6x\cos^2(x^2)\sin(x^2).$$

(h) $z(x) = \dfrac{e^{-2x}}{5} = \tfrac{1}{5}e^{-2x}$, so

$$z'(x) = \tfrac{1}{5} \times (-2e^{-2x})$$
$$= -\frac{2}{5}e^{-2x}.$$

(i) $G(x) = 3\sin x\cos x + 2\sin(3x)$, so

$$G'(x) = 3((\sin x)(-\sin x) + (\cos x)(\cos x)) + 2(\cos(3x)) \times 3$$

(by the product rule)

$$= -3\sin^2 x + 3\cos^2 x + 6\cos(3x).$$

(You can simplify this answer further by using the trigonometric identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$:

$$-3\sin^2 x + 3\cos^2 x + 6\cos(3x)$$
$$= 3(-\sin^2 x + \cos^2 x) + 6\cos(3x)$$
$$= 3\cos(2x) + 6\cos(3x).)$$

(j) $p(t) = te^t \sin t = t(e^t \sin t)$, so

$$\begin{aligned} p'(t) &= t\,\frac{\mathrm{d}}{\mathrm{d}t}\,(e^t \sin t) + (e^t \sin t) \times 1 \\ &\qquad \text{(by the product rule)} \\ &= t(e^t \cos t + (\sin t)e^t) + e^t \sin t \\ &\qquad \text{(by the product rule again)} \\ &= te^t \cos t + te^t \sin t + e^t \sin t \\ &= e^t(t\cos t + t\sin t + \sin t). \end{aligned}$$

(Alternatively, you could start by writing $p(t) = (te^t)\sin t$, and again apply the product rule twice.)

(k) $r = \sin(2\theta)\cos\theta$, so

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \sin(2\theta)(-\sin\theta) + (\cos\theta) \times 2\cos(2\theta) \\ &\qquad \text{(by the product rule)} \\ &= 2\cos\theta\cos(2\theta) - \sin\theta\sin(2\theta). \end{aligned}$$

(l) $h = ze^{3z}$, so

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}z} &= z \times 3e^{3z} + e^{3z} \times 1 \\ &\qquad \text{(by the product rule)} \\ &= 3ze^{3z} + e^{3z} \\ &= (3z+1)e^{3z}. \end{aligned}$$

(m) $v = \dfrac{\sin\sqrt{u}}{\sqrt{u}} = \dfrac{\sin(u^{1/2})}{u^{1/2}}$, so

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{u^{1/2}\,\frac{\mathrm{d}}{\mathrm{d}u}\big(\sin(u^{1/2})\big) - \sin(u^{1/2}) \times \frac{1}{2}u^{-1/2}}{u} \\ &\qquad \text{(by the quotient rule)} \\ &= \frac{u^{1/2}\cos(u^{1/2}) \times \frac{1}{2}u^{-1/2} - \sin(u^{1/2}) \times \frac{1}{2}u^{-1/2}}{u} \\ &\qquad \text{(by the chain rule)} \\ &= \frac{\cos(u^{1/2}) - u^{-1/2}\sin(u^{1/2})}{2u} \\ &= \frac{u^{1/2}\cos(u^{1/2}) - \sin(u^{1/2})}{2u^{3/2}} \\ &= \frac{\sqrt{u}\cos\sqrt{u} - \sin\sqrt{u}}{2u\sqrt{u}}. \end{aligned}$$

Solution to Activity 18

(a) $g(x) = e^{-kx}$, where k is a constant, so

$$g'(x) = -ke^{-kx}.$$

(b) $f(x) = x\cos(\pi x)$, so, by the product rule,

$$\begin{aligned} f'(x) &= x \times (-\pi\sin(\pi x)) + (\cos(\pi x)) \times 1 \\ &= -\pi x\sin(\pi x) + \cos(\pi x) \\ &= \cos(\pi x) - \pi x\sin(\pi x). \end{aligned}$$

(c) $h = \sin^2(a\theta)$, where a is a constant, so, by the chain rule,

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}\theta} &= 2\sin(a\theta) \times a\cos(a\theta) \\ &= 2a\sin(a\theta)\cos(a\theta). \end{aligned}$$

(You can use the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to write this answer above in a neater form, namely $a\sin(2a\theta)$.)

(d) $y = \cos(3x) + c$, where c is a constant, so

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (-\sin(3x)) \times 3 + 0 \\ &= -3\sin(3x). \end{aligned}$$

Solution to Activity 19

The function is $p(t) = \dfrac{\ln t}{t^2} = t^{-2}\ln t$.

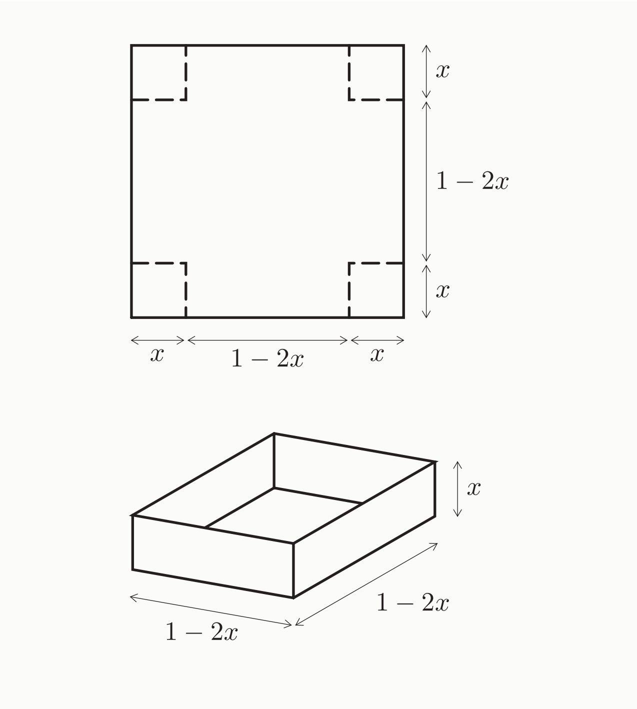
So, by the product rule,

$$\begin{aligned} p'(t) &= t^{-2} \times \frac{1}{t} + (\ln t)(-2t^{-3}) \\ &= \frac{1}{t^3} - \frac{2\ln t}{t^3} \\ &= \frac{1 - 2\ln t}{t^3}. \end{aligned}$$

Solution to Activity 20

Let the length of the sides of the squares be x (in m), as shown below. The value of x must be between 0 and $\frac{1}{2}$.

Let the volume of the box be V (in m^3).



Then the height of the box will be x, and the length and width of the box will each be $1-2x$. Hence

$$V = x(1-2x)^2.$$

We have to find the value of x, between 0 and $\frac{1}{2}$, that gives the maximum value of V.

V is a polynomial function of x, so it is continuous on its whole domain and differentiable at every value of x. So the maximum value of V must occur either at an endpoint of the interval $[0, \frac{1}{2}]$, or at a stationary point.

The formula for V is

$$\begin{aligned} V &= x(1-2x)^2 \\ &= x(1-4x+4x^2) \\ &= x-4x^2+4x^3, \end{aligned}$$

so

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 1-8x+12x^2.$$

At a stationary point, $\mathrm{d}V/\mathrm{d}x = 0$, which gives

$$1-8x+12x^2 = 0$$
$$12x^2-8x+1 = 0$$
$$(6x-1)(2x-1) = 0$$
$$x = \tfrac{1}{6} \text{ or } x = \tfrac{1}{2}.$$

So the stationary points occur when $x = \frac{1}{6}$ and $x = \frac{1}{2}$.

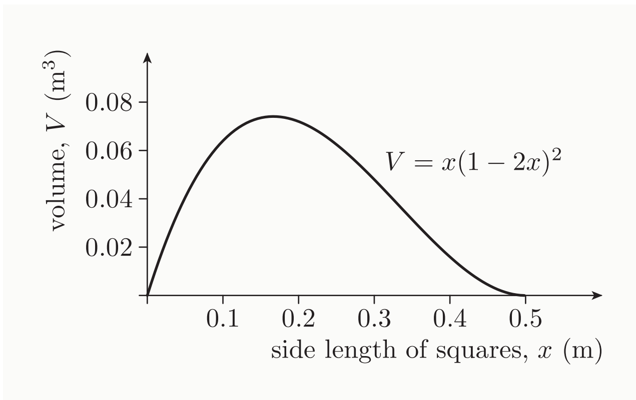
The endpoints of the interval are 0 and $\frac{1}{2}$, so the maximum value of V occurs when $x=0$, $x=\frac{1}{6}$ or $x=\frac{1}{2}$.

When $x=0$ the height of the box is zero, and when $x=\frac{1}{2}$ its length and width are zero, so in both these cases the volume V will be zero. When $x=\frac{1}{6}$, the length, width and height of the box are all positive, so the volume V will also be positive.

So the maximum value of V is achieved when $x=\frac{1}{6}$.

Hence the sides of the squares should be $\frac{1}{6}$ m; that is, about 16.7 cm.

(A graph of V against x, for x between 0 and $\frac{1}{2}$, is shown below. You can see that the maximum value of V does indeed seem to occur when x is approximately $\frac{1}{6} = 0.166\ldots$.)



(Note also that an alternative way to differentiate the formula for V, rather than first multiplying out the brackets, is to use a combination of the product rule and the chain rule, as follows:

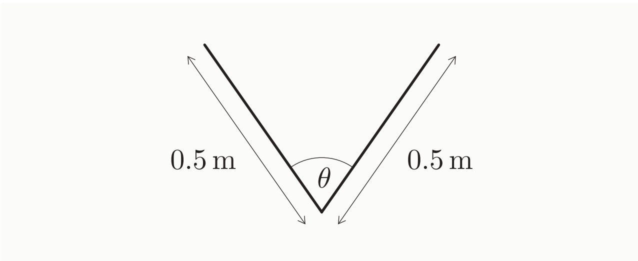
$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-2x)^2\right) \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\left((1-2x)^2\right) + (1-2x)^2 \times 1 \\ &= x \times 2(1-2x) \times (-2) + (1-2x)^2 \\ &= -4x(1-2x) + (1-2x)^2 \\ &= (1-2x)(-4x+1-2x) \\ &= (1-2x)(1-6x).) \end{aligned}$$

Solution to Activity 21

The volume of the channel is the area of its cross-section times its (fixed) length. So to achieve a channel of maximum volume, we have to achieve the maximum area of the cross-section.

Let the angle between the two sides of the v-shape be θ, as shown below. The value of θ must be between 0 and π.

Let the area of the cross-section of the channel be A (in m^2).



Each side of the v-shape has length $\frac{1}{2}$ m, so, by the formula $A = \frac{1}{2}ab\sin\theta$ for the area of a triangle,

$$A = \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\sin\theta = \tfrac{1}{8}\sin\theta.$$

We have to find the value of θ, between 0 and π, that gives the maximum value of A.

Since the sine function is continuous on its whole domain and differentiable everywhere, so is the variable A as a function of θ. So the maximum value of A must occur either at an endpoint of the interval $[0, \pi]$, or at a stationary point.

The formula stated above for A gives

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \tfrac{1}{8}\cos\theta.$$

At a stationary point, $\mathrm{d}A/\mathrm{d}\theta = 0$, which gives

$$\tfrac{1}{8}\cos\theta = 0$$
$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}$$

(since θ is in the interval $[0, \pi]$).

So the stationary point occurs when $\theta = \pi/2$.

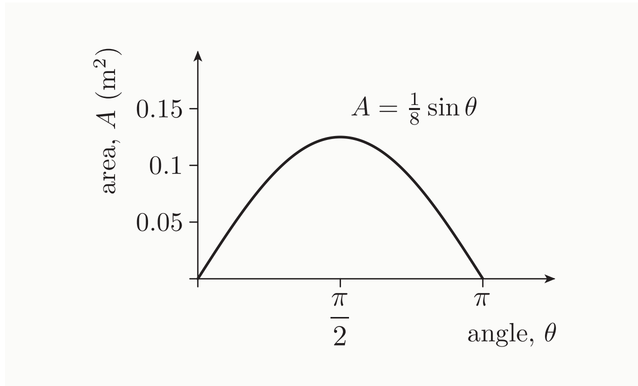
The endpoints of the interval are 0 and π, so the maximum value of A occurs when $\theta = 0$, $\theta = \pi/2$ or $\theta = \pi$.

When $\theta = 0$ the width of the channel is zero, and when $\theta = \pi$ its height is zero, so in both these cases the area A will be zero. When $\theta = \pi/2$, the width and height of the channel are both positive, so the area A will also be positive.

So the maximum value of A is achieved when $\theta = \pi/2$.

That is, to give a channel with maximum volume, the angle between the sides of the v-shape should be $\pi/2$, which is 90°.

(A graph of A against θ, for θ between 0 and π, is shown below. You can see that the maximum value of A does indeed seem to occur when θ is approximately $\pi/2$.)

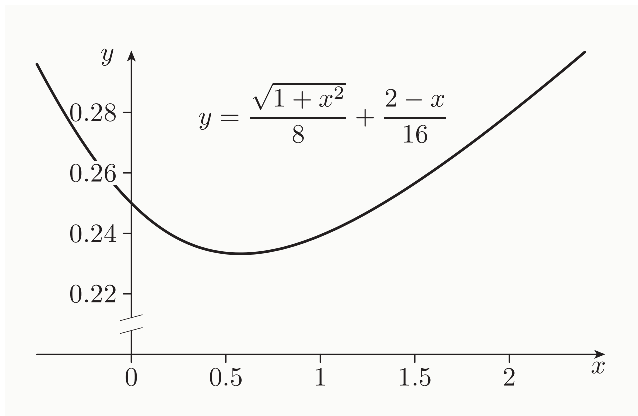


(Notice that in fact you could have answered this question without using calculus, because, from what you saw in Unit 4, the maximum value of $\sin\theta$ for values of θ between 0 and π occurs when $\theta = \pi/2$. It follows that the maximum value of $A = \frac{1}{8}\sin\theta$ for values of θ between 0 and π also occurs when $\theta = \pi/2$.)

Solution to Activity 23

(a) (You need to do this on a computer.)

(b)



(The symbol consisting of two short slant lines on the y-axis of this graph indicates that part of the axis has been omitted.)

(c) The value of x that gives the minimum value of $f(x)$ seems to be about 0.6.

(d) The derivative of f is

$$f'(x) = \frac{x}{8\sqrt{1+x^2}} - \frac{1}{16}.$$

(e) The solution of the equation $f'(x) = 0$ is $x = 0.577$ (to 3 s.f.).

(f) The man should join the road 0.58 kilometres (to the nearest 10 metres) along from the point that is closest to his initial position.

(Because the graph of f indicates that the minimum value of $f(x)$ occurs at a local minimum rather than at one of the endpoints of the interval $[0, 2]$, in the solution above we don't consider the values of $f(0)$ and $f(2)$.)

(Details of how to use the CAS for this question are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

By-hand solution

The problem in this question can be solved by hand as follows.

The function to be minimised has the rule

$$\begin{aligned} f(x) &= \frac{\sqrt{1+x^2}}{8} + \frac{2-x}{16} \\ &= \tfrac{1}{8}(1+x^2)^{1/2} + \tfrac{1}{16}(2-x). \end{aligned}$$

The variable x can take values between 0 and 2.

The function f is continuous on its domain $[0, 2]$, and differentiable at every value of x in this interval, except at the endpoints 0 and 2. (Remember that a function can't be differentiable at an endpoint of its domain.) (These facts can be deduced from the fact that f is a simple combination of functions that have these properties. You can learn more about making such deductions in later modules. For now, you can see from the graph of f that it appears to have these properties.)

Hence the minimum value of f on the interval $[0, 2]$ occurs at a stationary point of f or at an endpoint of the interval.

Differentiating f gives

$$\begin{aligned} f'(x) &= \tfrac{1}{8} \times \tfrac{1}{2}(1+x^2)^{-1/2} \times 2x + \tfrac{1}{16} \times (-1) \\ &= \frac{x}{8\sqrt{1+x^2}} - \frac{1}{16}. \end{aligned}$$

At a stationary point, $f'(x) = 0$, which gives

$$\begin{aligned} \frac{x}{8\sqrt{1+x^2}} - \frac{1}{16} &= 0 \\ \frac{x}{8\sqrt{1+x^2}} &= \frac{1}{16} \\ 16x &= 8\sqrt{1+x^2} \\ 2x &= \sqrt{1+x^2} \\ (2x)^2 &= 1+x^2 \\ 4x^2 &= 1+x^2 \\ 3x^2 &= 1 \\ x^2 &= \frac{1}{3} \\ x &= \frac{1}{\sqrt{3}}. \end{aligned}$$

(Here we have used the fact that $x \geq 0$.)

So the only stationary point of f is $x = 1/\sqrt{3}$. The value of f at this stationary point is

$$\begin{aligned} f\left(\frac{1}{\sqrt{3}}\right) &= \frac{\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2}}{8} + \frac{2-\left(\frac{1}{\sqrt{3}}\right)}{16} \\ &= \frac{\sqrt{1+\frac{1}{3}}}{8} + \frac{2\sqrt{3}-1}{16\sqrt{3}} \\ &= \frac{\sqrt{\frac{4}{3}}}{8} + \frac{2\sqrt{3}-1}{16\sqrt{3}} \\ &= \frac{2}{8\sqrt{3}} + \frac{2\sqrt{3}-1}{16\sqrt{3}} \\ &= \frac{2\sqrt{3}}{8\sqrt{3}\sqrt{3}} + \frac{2\sqrt{3}\sqrt{3}-\sqrt{3}}{16\sqrt{3}\sqrt{3}} \\ &= \frac{2\sqrt{3}}{24} + \frac{6-\sqrt{3}}{48} \\ &= \frac{4\sqrt{3}}{48} + \frac{6-\sqrt{3}}{48} \\ &= \frac{6+3\sqrt{3}}{48} \\ &= \frac{2+\sqrt{3}}{16} \approx 0.23. \end{aligned}$$

The values of f at the endpoints of the interval $[0, 2]$ are as follows:

$$\begin{aligned} f(0) &= \frac{\sqrt{1+0^2}}{8} + \frac{2-0}{16} \\ &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = 0.25 \end{aligned}$$

$$f(2) = \frac{\sqrt{1+2^2}}{8} + \frac{2-2}{16}$$
$$= \frac{\sqrt{5}}{8} + 0 = \frac{\sqrt{5}}{8} \approx 0.28.$$

So the minimum value of $f(x)$ is achieved when $x = 1/\sqrt{3} = 0.577$ (to 3 s.f.). Hence, as was found above using a computer, the man should join the road 0.58 kilometres (to the nearest 10 metres) along from the point that is closest to his initial position.

Solution to Activity 24

(a) We have $y = \cos^{-1} x$, so

$$x = \cos y.$$

Therefore

$$\frac{dx}{dy} = -\sin y.$$

By the inverse function rule,

$$\frac{dy}{dx} = -\frac{1}{\sin y},$$

provided that $\sin y \neq 0$.

The identity $\sin^2 y + \cos^2 y = 1$ gives

$$\sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-x^2}.$$

The $+$ sign applies here, because y takes values only in the interval $[0, \pi]$ (since $y = \cos^{-1}(x)$) and so $\sin y$ is always non-negative. Hence

$$\sin y = \sqrt{1-x^2}.$$

Therefore

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}},$$

provided that $x \neq \pm 1$.

(b) We have $y = \tan^{-1} x$, so

$$x = \tan y.$$

Therefore

$$\frac{dx}{dy} = \sec^2 y.$$

By the inverse function rule,

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

The identity $\tan^2 y + 1 = \sec^2 y$ gives

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

Therefore

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

Solution to Activity 25

(a) $f(x) = \sin^{-1}(3x)$, so, by the chain rule,

$$f'(x) = \frac{1}{\sqrt{1-(3x)^2}} \times 3$$
$$= \frac{3}{\sqrt{1-9x^2}}.$$

(b) $f(x) = e^x \cos^{-1} x$, so, by the product rule,

$$f'(x) = e^x \left(-\frac{1}{\sqrt{1-x^2}}\right) + (\cos^{-1} x)e^x$$
$$= \left(\cos^{-1} x - \frac{1}{\sqrt{1-x^2}}\right) e^x.$$

Solution to Activity 26

(a) $F(x) = x^2$, so $F'(x) = 2x$.

(b) $F(x) = x^2 + 3$, so $F'(x) = 2x$.

(c) $F(x) = x^2 - \frac{5}{7}$, so $F'(x) = 2x$.

Solution to Activity 27

(a) Since

$$\frac{d}{dx}\left(\tfrac{1}{2}\sin(2x)\right) = \tfrac{1}{2} \times 2\cos(2x) = \cos(2x),$$

it follows that $F(x) = \frac{1}{2}\sin(2x)$ is an antiderivative of the function $f(x) = \cos(2x)$.

(b) The indefinite integral of $f(x) = \cos(2x)$ is $F(x) = \frac{1}{2}\sin(2x) + c$.

(c) An antiderivative of $f(x) = \cos(2x)$, other than the antiderivative in part (a), is $F(x) = \frac{1}{2}\sin(2x) + 1$.

(You probably chose a different antiderivative. Any function obtained by setting the constant c in the indefinite integral $F(x) = \frac{1}{2}\sin(2x) + c$ equal to a particular number will do.)

Solution to Activity 28

(a) The indefinite integral of $f(x) = x^9$ is

$$F(x) = \tfrac{1}{10}x^{10} + c.$$

(b) The indefinite integral of $p(u) = \dfrac{1}{u^3} = u^{-3}$ is

$$P(u) = \frac{1}{-2}u^{-2} + c = -\frac{1}{2u^2} + c.$$

(c) The indefinite integral of $f(x) = x^{2/3}$ is

$$F(x) = \tfrac{3}{5}x^{5/3} + c.$$

(d) The indefinite integral of $f(x) = \sqrt[4]{x} = x^{1/4}$ is

$$F(x) = \tfrac{4}{5}x^{5/4} + c.$$

(e) The indefinite integral of $g(t) = t^{-2/3}$ is

$$G(t) = 3t^{1/3} + c.$$

(f) The indefinite integral of $b(v) = \dfrac{1}{\sqrt{v}} = v^{-1/2}$ is

$$B(v) = 2v^{1/2} + c = 2\sqrt{v} + c.$$

Solution to Activity 29

(a) The indefinite integral of $f(x) = x^5$ is

$$F(x) = \tfrac{1}{6}x^6 + c.$$

(b) Three different antiderivatives of f are

$$F(x) = \tfrac{1}{6}x^6,$$
$$F(x) = \tfrac{1}{6}x^6 + 1,$$
$$F(x) = \tfrac{1}{6}x^6 - 2.$$

(You probably chose different examples.)

Solution to Activity 30

(a) The indefinite integral of $f(x) = 8x^3$ is

$$\begin{aligned} F(x) &= 8 \times \tfrac{1}{4}x^4 + c \\ &= 2x^4 + c. \end{aligned}$$

(b) The indefinite integral of $f(x) = 7$ is

$$F(x) = 7x + c.$$

(c) The indefinite integral of $f(x) = \sqrt{x} - x^2 = x^{1/2} - x^2$ is

$$\begin{aligned} F(x) &= \tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3 + c \\ &= \tfrac{1}{3}(2x^{3/2} - x^3) + c. \end{aligned}$$

(d) We have

$$\begin{aligned} f(x) &= (x+2)(x-5) \\ &= x^2 - 3x - 10, \end{aligned}$$

which has indefinite integral

$$F(x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 - 10x + c.$$

(e) We have

$$\begin{aligned} g(t) &= (t+1)^2 \\ &= t^2 + 2t + 1, \end{aligned}$$

which has indefinite integral

$$G(t) = \tfrac{1}{3}t^3 + t^2 + t + c.$$

(f) We have

$$g(x) = \frac{x^2+1}{x^4} = \frac{1}{x^2} + \frac{1}{x^4} = x^{-2} + x^{-4},$$

which has indefinite integral

$$\begin{aligned} G(x) &= -x^{-1} - \tfrac{1}{3}x^{-3} + c \\ &= -\frac{1}{x} - \frac{1}{3x^3} + c. \end{aligned}$$

(g) We have

$$\begin{aligned} f(r) &= \sqrt[3]{r}\left(10r^2 - \frac{1}{r^2}\right) \\ &= r^{1/3}(10r^2 - r^{-2}) \\ &= 10r^{7/3} - r^{-5/3} \end{aligned}$$

which has indefinite integral

$$\begin{aligned} F(r) &= 10(\tfrac{3}{10}r^{10/3}) - (-\tfrac{3}{2}r^{-2/3}) + c \\ &= 3r^{10/3} + \tfrac{3}{2}r^{-2/3} + c \\ &= \tfrac{3}{2}r^{-2/3}(1 + 2r^4) + c. \end{aligned}$$

Solution to Activity 31

(a) The function is $f(x) = x + 2$. Its indefinite integral is

$$F(x) = \tfrac{1}{2}x^2 + 2x + c.$$

Using the fact that $F(1) = \frac{9}{2}$ gives

$$\begin{aligned} \tfrac{1}{2} \times 1^2 + 2 \times 1 + c &= \tfrac{9}{2} \\ \tfrac{5}{2} + c &= \tfrac{9}{2} \\ c &= \tfrac{4}{2} = 2. \end{aligned}$$

Hence the required antiderivative is

$$F(x) = \tfrac{1}{2}x^2 + 2x + 2.$$

(b) The function is $f(x) = 1/x^3 = x^{-3}$ with domain $(0, \infty)$. Its indefinite integral is

$$F(x) = \frac{1}{-2}x^{-2} + c = -\frac{1}{2x^2} + c.$$

Using the fact that $F(3) = -\frac{1}{9}$ gives

$$\begin{aligned} -\frac{1}{2 \times 3^2} + c &= -\frac{1}{9} \\ -\frac{1}{18} + c &= -\frac{1}{9} \\ c = -\frac{1}{9} + \frac{1}{18} &= -\frac{1}{18}. \end{aligned}$$

Hence the required antiderivative is

$$\begin{aligned} F(x) &= -\frac{1}{2x^2} - \frac{1}{18} \\ &= -\frac{9 + x^2}{18x^2} \quad (x \in (0, \infty)). \end{aligned}$$

Solution to Activity 32

(a) The given equation for v in terms of t is

$$v = \tfrac{1}{10}t.$$

Hence the equation for s in terms of t is

$$s = \tfrac{1}{10} \times \tfrac{1}{2}t^2 + c,$$

that is,

$$s = \tfrac{1}{20}t^2 + c,$$

where c is a constant.

The marble starts rolling from a position 0.3 metres down the slope, so when $t = 0$, $s = 0.3$.

Substituting these values of s and t into the equation above gives

$$0.3 = \tfrac{1}{20} \times 0^2 + c$$
$$c = 0.3.$$

So the equation for s in terms of t is

$$s = \tfrac{1}{20}t^2 + 0.3.$$

(b) Substituting $t = 4$ into the equation from part (a) gives

$$s = \tfrac{1}{20} \times 4^2 + 0.3 = 0.8 + 0.3 = 1.1.$$

That is, the displacement of the marble from the top of the slope after it has been rolling for 4 seconds is 1.1 metres.

Solution to Activity 33

(a) The equation for a in terms of t is

$$a = -9.8.$$

Integrating this equation gives the following equation for v in terms of t:

$$v = -9.8t + c,$$

where c is a constant.

At the start of the motion, the velocity of the ball is $12\,\mathrm{m\,s^{-1}}$. That is, when $t = 0$, $v = 12$.

Substituting these values into the equation for v above gives

$$12 = -9.8 \times 0 + c, \quad \text{that is,} \quad c = 12.$$

So the equation for v in terms of t is

$$v = 12 - 9.8t.$$

(b) Integrating the equation found in part (a) gives the following equation for s in terms of t:

$$s = 12t - 9.8 \times \tfrac{1}{2}t^2 + b,$$

that is,

$$s = 12t - 4.9t^2 + b,$$

where again b is a constant.

At the start of the motion, the displacement of the ball is 0 m. That is, when $t = 0$, $s = 0$.

Substituting these values into the equation for s above gives

$$0 = 12 \times 0 - 4.9 \times 0^2 + b, \quad \text{that is,} \quad b = 0.$$

So the equation for s in terms of t is

$$s = 12t - 4.9t^2.$$

(c) When $t = 1$,

$$v = 12 - 9.8 \times 1 = 12 - 9.8 = 2.2,$$

and

$$s = 12 \times 1 - 4.9 \times 1^2 = 12 - 4.9 = 7.1.$$

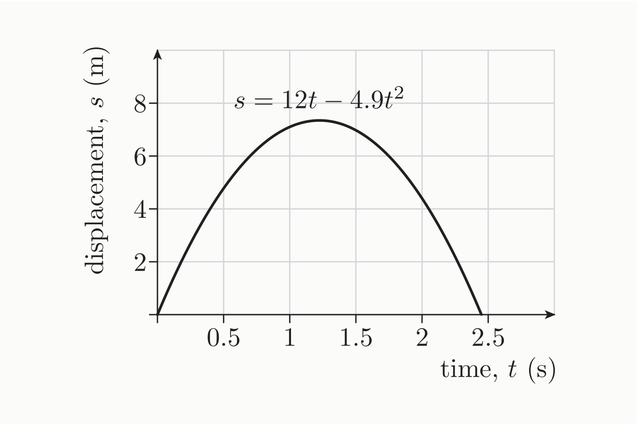
So the velocity and the displacement of the ball 1 second after it was thrown are $2.2\,\mathrm{m\,s^{-1}}$ and 7.1 m, respectively.

(d) When the ball has fallen back to the point from which it was thrown, $s = 0$. Substituting this value of s into the equation found in part (b) gives

$$0 = 12t - 4.9t^2$$
$$12t - 4.9t^2 = 0$$
$$t(12 - 4.9t) = 0$$
$$t = 0 \text{ or } t = 12/4.9 = 2.4 \text{ (to 1 d.p.)}.$$

So the ball takes about 2.4 seconds to fall back to the point from which it was thrown.

(The displacement–time graph for the ball is shown below. You can see that the ball first rises and then falls, as expected. You can also see that it reaches a height of about 7 m after 1 second, and that it returns to the point from which it was thrown after about 2.4 seconds, as you would expect from the answers found in parts (c) and (d).)



Solution to Activity 34

(a) The given equation for m in terms of q is

$$m = 250 - 0.5q.$$

Since marginal cost is the derivative of total cost, the equation for t in terms of q is

$$t = 250q - 0.5 \times \tfrac{1}{2}q^2 + c,$$

that is,

$$t = 250q - 0.25q^2 + c,$$

where c is a constant.

According to the information given in the question, when $q = 80$, $t = 30\,400$.

Substituting these values into the equation found for t above gives

$$30\,400 = 250 \times 80 - 0.25 \times 80^2 + c$$

$$30\,400 = 18\,400 + c$$

$$c = 12\,000.$$

Hence the equation for t in terms of q is

$$t = 250q - 0.25q^2 + 12\,000.$$

(b) The total weekly cost of producing 160 tonnes of fertiliser each week is

$$£(250 \times 160 - 0.25 \times 160^2 + 12\,000) = £45\,600.$$

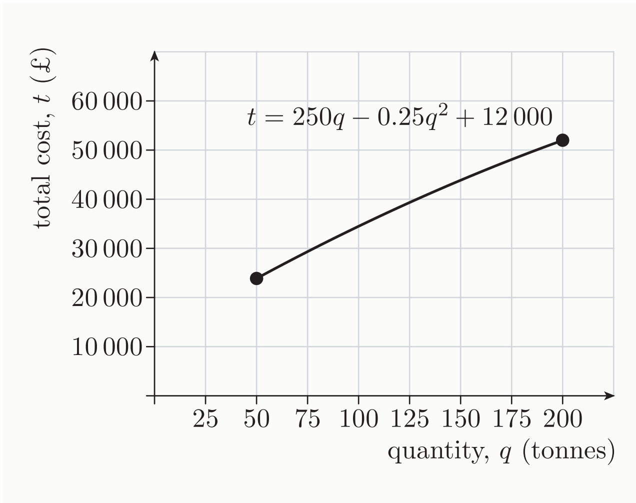
(c) The question states that the total weekly cost of producing 80 tonnes of fertiliser each week is £30 400. Hence the cost per tonne of producing 80 tonnes of fertiliser each week is

$$\frac{£30400}{80} = £380.$$

It was calculated in part (b) that the total weekly cost of producing 160 tonnes of fertiliser each week is £45 600. Hence the cost per tonne of producing 160 tonnes of fertiliser each week is

$$\frac{£45\,600}{160} = £285.$$

(The graph of t against q is shown below. You can see that the total weekly cost of producing 160 tonnes of fertiliser each week is about £45 000, as expected.)



Solution to Activity 35

An antiderivative of f is

$$F(x) = 8e^{x/8}.$$

So the change in the quantity from $x = 2$ to $x = 5$ is

$$\begin{aligned} F(5) - F(2) &= 8e^{5/8} - 8e^{2/8} \\ &= 4.67 \text{ (to 3 s.f.).} \end{aligned}$$

Solution to Activity 36

An antiderivative of f is

$$F(t) = 3t - \tfrac{3}{2} \times \tfrac{1}{2}t^2 = 3t - \tfrac{3}{4}t^2.$$

(a) The change in the displacement (in metres) of the object from $t = 0$ to $t = 1$ is

$$\begin{aligned} F(1) - F(0) &= (3 \times 1 - \tfrac{3}{4} \times 1^2) \\ &\quad - (3 \times 0 - \tfrac{3}{4} \times 0^2) \\ &= 2.25. \end{aligned}$$

That is, the object travels 2.25 m in that time (in the positive direction).

(b) The change in the displacement (in metres) of the object from $t = 4$ to $t = 5$ is

$$\begin{aligned} F(5) - F(4) &= (3 \times 5 - \tfrac{3}{4} \times 5^2) \\ &\quad - (3 \times 4 - \tfrac{3}{4} \times 4^2) \\ &= (15 - \tfrac{75}{4}) - (12 - 12) \\ &= -3.75. \end{aligned}$$

That is, the object travels 3.75 m in that time (in the negative direction).

Solution to Activity 37

(a) The function $f(x) = \dfrac{4}{x}$ has indefinite integral

$$F(x) = 4 \ln |x| + c.$$

(b) The function

$$f(x) = \frac{x+1}{x^2} = \frac{1}{x} + \frac{1}{x^2} = \frac{1}{x} + x^{-2}$$

has indefinite integral

$$F(x) = \ln |x| - x^{-1} + c = \ln |x| - \frac{1}{x} + c.$$

(c) The function

$$f(x) = \frac{1}{2x} \quad (x > 0)$$

has indefinite integral

$$F(x) = \tfrac{1}{2} \ln x + c.$$

(The answer $F(x) = \frac{1}{2} \ln |x| + c$ is also correct, but can be simplified, as x takes only positive values.)

Solution to Activity 38

(a) The indefinite integral of $h(\theta) = 3\cos\theta + 4\sin\theta$ is

$$\begin{aligned} H(\theta) &= 3\sin\theta + 4(-\cos\theta) + c \\ &= 3\sin\theta - 4\cos\theta + c. \end{aligned}$$

(b) The indefinite integral of $g(\phi) = 6 - 5\,\mathrm{cosec}^2\,\phi$ is

$$\begin{aligned} G(\phi) &= 6\phi - 5(-\cot\phi) + c \\ &= 6\phi + 5\cot\phi + c. \end{aligned}$$

(c) The indefinite integral of $f(x) = \dfrac{7}{x}$ is

$$F(x) = 7 \ln |x| + c.$$

(d) The function is

$$g(t) = \frac{1}{3t},$$

which can be written as

$$g(t) = \frac{1}{3} \times \frac{1}{t}.$$

Hence its indefinite integral is

$$G(t) = \frac{1}{3} \ln |t| + c.$$

(e) The function is

$$f(x) = \frac{1}{\pi x} \ (x > 0),$$

which can be written as

$$f(x) = \frac{1}{\pi} \times \frac{1}{x} \ (x > 0).$$

Hence its indefinite integral is

$$F(x) = \frac{1}{\pi} \ln x + c.$$

(f) The indefinite integral of $f(x) = \dfrac{3}{1+x^2}$ is

$$F(x) = 3\tan^{-1} x + c.$$

(g) The indefinite integral of $h(t) = \dfrac{1}{4\sqrt{1-t^2}}$ is

$$H(t) = \tfrac{1}{4} \sin^{-1} t + c.$$

(Or, alternatively, $H(t) = -\frac{1}{4} \cos^{-1} t + c$.)

(h) The function is

$$p(x) = \frac{1}{5 + 5x^2},$$

which can be written as

$$p(x) = \frac{1}{5} \times \frac{1}{1+x^2}.$$

Hence its indefinite integral is

$$P(x) = \tfrac{1}{5} \tan^{-1} x + c.$$

(i) The function is

$$g(x) = \frac{1}{\sqrt{4 - 4x^2}},$$

which can be written as

$$g(x) = \frac{1}{\sqrt{4}\sqrt{1-x^2}};$$

that is,

$$g(x) = \frac{1}{2\sqrt{1-x^2}}.$$

Hence its indefinite integral is

$$G(x) = \tfrac{1}{2} \sin^{-1} x + c.$$

(Or, alternatively, $G(x) = -\frac{1}{2} \cos^{-1} x + c$.)

(j) The function is

$$q(x) = 5(x-3)(2x-1),$$

which can be written as

$$q(x) = 5(2x^2 - 7x + 3).$$

Hence its indefinite integral is

$$\begin{aligned} Q(x) &= 5(2 \times \tfrac{1}{3}x^3 - 7 \times \tfrac{1}{2}x^2 + 3x) + c \\ &= \tfrac{10}{3}x^3 - \tfrac{35}{2}x^2 + 15x + c. \end{aligned}$$

(It could also be written as

$$Q(x) = \tfrac{5}{6}x(4x^2 - 21x + 18) + c,$$

for example.)

(k) The function is

$$f(x) = \frac{x-2}{x},$$

which can be written as

$$f(x) = 1 - \frac{2}{x}.$$

Hence its indefinite integral is

$$F(x) = x - 2\ln|x| + c.$$

(l) The function is

$$g(x) = \frac{x-2}{x^3},$$

which can be written as

$$g(x) = x^{-2} - 2x^{-3}.$$

Hence its indefinite integral is

$$\begin{aligned} G(x) &= -x^{-1} - 2 \times \frac{1}{-2}x^{-2} + c \\ &= -\frac{1}{x} + \frac{1}{x^2} + c \\ &= -\frac{x}{x^2} + \frac{1}{x^2} + c \\ &= \frac{1-x}{x^2} + c. \end{aligned}$$

(m) The indefinite integral of $f(x) = 8e^x$ is

$$F(x) = 8e^x + c.$$

(n) The function is

$$f(x) = e^{1+x},$$

which can be written as

$$f(x) = e \times e^x.$$

Hence its indefinite integral is

$$\begin{aligned} F(x) &= e \times e^x + c \\ &= e^{1+x} + c. \end{aligned}$$

(You can integrate the expression $e \times e^x$ by using the constant multiple rule for antiderivatives, since e is a constant.)

(o) The indefinite integral of $r(\phi) = -\operatorname{cosec}\phi \cot\phi$ is

$$\begin{aligned} R(\phi) &= -(-\operatorname{cosec}\phi) + c \\ &= \operatorname{cosec}\phi + c. \end{aligned}$$

(p) The function is

$$r(\theta) = \sec\theta(\sec\theta + \tan\theta)$$

which can be written as

$$r(\theta) = \sec^2\theta + \sec\theta\tan\theta.$$

Hence its indefinite integral is

$$R(\theta) = \tan\theta + \sec\theta + c.$$

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