Unit 9

Oscillations and energy

Introduction

All around you there are mechanical systems that vibrate or oscillate (the two words can be used interchangeably). Each day you probably experience oscillations in a wide variety of forms: the buzzing of an alarm clock, the vibrations of an electric hair dryer or razor, the sideways movements of a train or boat, and so on. Vibrations vary in scale from the small motions of atoms within solid objects, to the swaying of bridges and tall buildings; and from the irregular bending of a tree in the wind, to the extremely regular oscillations involved in any device that is designed to measure accurately the passage of time.

A common feature of many of these vibrations or oscillations is *periodicity*, where the vibration exhibits a pattern of movement or displacement that repeats itself over and over again as time progresses. Not all oscillations repeat themselves exactly. They may also become more or less pronounced if the system is *forced* or *damped*, or the oscillations may even become chaotic. In this unit, however, attention will be directed towards systems for which the effects of damping or forcing are of secondary importance. We model such a system by assuming that such complicating effects may be ignored completely. As a result, the behaviour predicted by the model is periodic and, in particular, *sinusoidal* (having a pattern of motion in which the displacement can be represented by a sine or cosine function of time).

Forced and damped oscillations are considered in Unit 10.

Sinusoidal oscillations arise in a wide variety of electrical, thermal and mechanical systems, but we concentrate here on the last of these. In this context, the sinusoidal nature of the oscillations can often be seen as a consequence of a further modelling assumption: namely, that the force acting on a moving object in order to bring it back towards its equilibrium position is proportional to its displacement from that position. This 'force law' was first put forward by Robert Hooke (Figure 1) in 1678.

This force is often referred to as a *restoring force*.

Robert Hooke

Robert Hooke was a contemporary of Isaac Newton. There was a dispute between the two men as to which had first proposed an inverse square law to describe the effects of gravitation.

Figure 1 Robert Hooke (1635–1703)

As an aid to visualising a restoring force, we introduce the concept of a *model spring* to embody it. The sinusoidal behaviour that arises from a model spring is known as *simple harmonic motion*, and is the simplest effective model for an oscillating system.

A model spring is a hypothetical object that can be either stretched or compressed from its 'natural length', and for which the force exerted depends on its length. This is in contrast to the *model string* introduced in Unit 2. A model string is inelastic, and hence has a fixed length, though it is capable of sustaining a range of pulling forces. The model spring, on the other hand, can provide pushing as well as pulling forces.

Figure 2 shows, in diagrammatic form, some of the systems that we will discuss. In each case, part of the system moves up and down, or to and fro, about an average position. We will attempt to explain these oscillations by applying the laws of Newtonian mechanics.

The system in Figure 2(a) is a weight hanging from an elastic string. This is a simplified representation of a system that might be modelled using a model spring. Figure 2(b) shows an object connected to a spring, which is attached to a fixed point at its other end. This might be used as a model for a spring-loaded buffer, which reverses the motion of a vehicle making contact with it. Alternatively, it could model the release mechanism in a pinball machine while in contact with the ball. Figure 2(c) represents a floating pontoon, bobbing up and down in water.

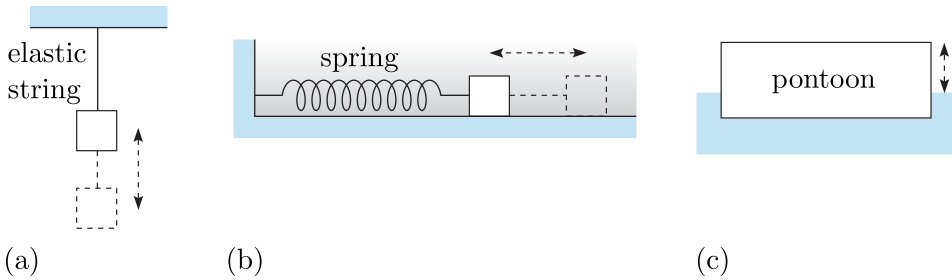


Figure 2 Examples of oscillating systems

Section 1 describes a simple experiment involving an oscillating system. It also introduces *Hooke's law* as a model for the force exerted by a spring, and goes on to consider how this law applies in various situations where no movement takes place.

Section 2 uses Newton's second law to model the oscillations of the simplest oscillating system, which consists of a single particle attached to a single horizontal spring. Variations on extended systems, such as vertical springs and systems with two springs, are also analysed.

Unit 11 takes up this story where more than one particle is involved.

Energy is an important concept in physics. What we mean by mechanical energy will be formally defined in Section 3, but for now we will use the everyday notion. It is a relatively new concept when compared to the ideas of mass and force. The concept of energy that can be converted from one form to another came to the fore when James Joule (see Figure 3) did experiments converting mechanical energy into heat in 1843.

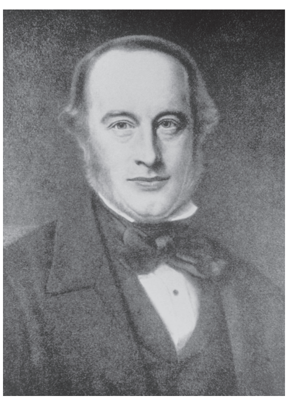


Figure 3 James Joule (1818–1889)

James Joule

James Joule was an amateur physicist who did many experiments over a forty-year period that established laws about conversion of many types of energy, including mechanical energy, heat and electrical energy. It is for this reason that the SI unit of energy is named after him.

Joule found that energy can be neither created nor destroyed: it can merely be converted from one form to another. A particular instance of this fact will be given in Section 4 as the *law of conservation of mechanical energy*.

In Section 4 you will see that the use of conservation of mechanical energy to solve mechanics problems is entirely equivalent to using Newton's second law of motion. In certain cases, it is possible to derive the law of conservation of mechanical energy from Newton's second law, and vice versa. So in a sense, no new problems can be solved using conservation of energy – every problem could equally well be solved by appealing to Newton's second law.

So why should one learn about energy? There are many answers to this. One answer is that in some situations it is easier to solve problems by considering energy, as you will see in this unit.

A second reason to introduce energy is because it is a physical concept that is important in many fields other than mechanics. By introducing other forms of energy, such as electromagnetic, chemical, heat, and so on, conservation of energy can be extended beyond the field of mechanics, and becomes one of the fundamental principles of physics. In this module, however, we concentrate on mechanical energy only.

A third reason to introduce energy is because it leads the way from Newtonian mechanics to Lagrangian and Hamiltonian mechanics. These other methods for solving mechanics problems have richer mathematical structures than Newtonian mechanics, making the generalisation to relativistic and quantum mechanics easier.

Lagrangian and Hamiltonian mechanics are Level 3 applied mathematics topics.

In Section 4 the law of conservation of mechanical energy is applied to systems involving gravity and springs. For this first look at energy, the discussion will mostly be limited to one-dimensional problems.

Two- and three-dimensional problems are studied in Unit 16.

1 Characteristics of a model spring

In this section the important concept of a *model spring* is introduced. Subsection 1.2 shows how the idealised concept of a model spring leads to a simple type of force law. In Subsection 1.3 this model is applied to some statics problems.

The behaviour of a simple mechanical system that can be set up at home is used first to obtain a feel for the concept of a model spring, before it is defined mathematically.

1.1 An experiment

Figure 4 Experimental arrangement

The mechanical system to be considered is illustrated in Figure 4. It is constructed by clamping a metal ruler of length 30 cm to a heavy table so that the ruler is horizontal. A hanger is attached to the free end of the ruler, and 'particles' of known mass (metal nuts) are hung from the hanger. This may not seem like a spring, but it has all the characteristics of a spring – applying a force to the end of the ruler causes a displacement. An accurate measuring device is used to measure the displacement d of the top of the end of the ruler from the horizontal.

A member of the module team carried out the experiment described below. Six of the nuts used as 'particles' have total mass 0.055 kg, so we assume that each nut has a mass of about 9.2×10^{-3} kg. (For machine-produced simple articles, it is a good assumption that each nut has the same mass.)

With no nuts on the hanger, the displacement was 0.22 mm. We call this value of d the zero displacement, and denote it by the symbol d_0, where the suffix indicates that there are no nuts hanging. Similarly, d_n represents the displacement when there are n nuts hanging.

Table 1 shows the displacements when 1 to 6 nuts were hung from the end of the ruler. It also shows the magnitudes of the weights corresponding to n nuts, to three decimal places. These were found (to two decimal places) by multiplying the total mass of the nuts by $g = 9.81\,\mathrm{m\,s^{-2}}$.

Table 1 Experimental data for displacement of ruler and number of nuts

Number of nuts, n	0	1	2	3	4	5	6
Displacement of end, d_n (millimetres)	0.22	1.15	2.07	2.99	3.85	4.78	5.68
Magnitude of weight of nuts, $w(d_n)$ (newtons)	0	0.09	0.18	0.27	0.36	0.45	0.54

Figure 5 gives a graph of the some of the results in Table 1. We work in SI units, so the millimetres used in the table have been converted to metres for use in the graph.

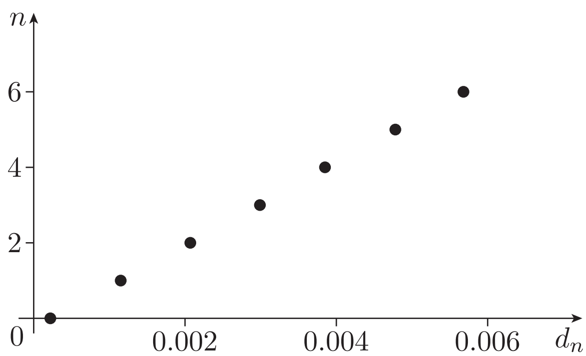


Figure 5 Plot of number of nuts n against displacement d_n (with d_n in metres)

Figure 5 shows that the number of nuts n and the displacement d_n are related in an approximately linear way when up to six nuts are hanging from the end of the ruler.

Figure 6 shows the magnitude of the weight of the nuts plotted against the displacement as well as the best straight line that goes through these points. Whereas Figure 5 considered discrete values of displacement, Figure 6 shows continuous values, hence d_n is replaced by d.

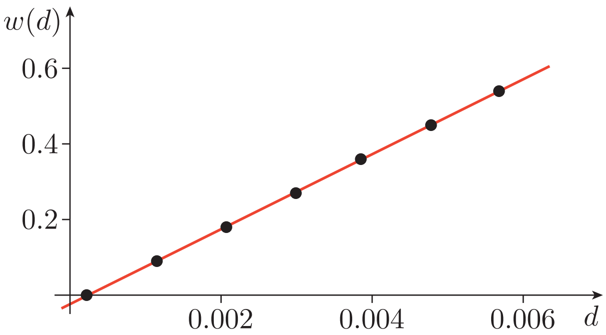


Figure 6 Plot of magnitude of weight against displacement, plus the best straight line through the points

A straight line that fits the experimental data points well is given by

$$w(d) \simeq -0.024 + 99.044d \quad \text{to 3 d.p.} \tag{1}$$

Later reference will be made to this equation, which is a relationship between the displacement of the ruler and the magnitude of the vertical force applied to it. Figure 7 shows the forces acting on the hanger. Since the hanger with n nuts is in equilibrium, its weight $\mathbf{W}$ must be exactly counteracted by the upward force provided by the ruler on the hanger. The force provided by the ruler is therefore $\mathbf{F} = -\mathbf{W}$, and the magnitude of this force is $|-\mathbf{W}| = |\mathbf{W}|$. Hence equation (1) tells us that the displacement of the ruler is related almost linearly to the magnitude of the force applied to the ruler. Another way of putting this is that as nuts are added to the hanger, the *increase* in the displacement of the ruler is approximately proportional to the *increase* in the magnitude of the force on it (or vice versa).

Figure 7 Forces acting on the hanger

From now on this unit considers only model springs, not real springs. The word 'spring' suggests a helical coil made of metal wire, and in representing a model spring diagrammatically, it is a stylised form of this physical realisation that is portrayed. However, a 'model spring' is in fact, as it says, nothing more than a model for a particular simple, linear, variation of a force with displacement. While it may be used as an idealised representation of a real coil spring, it can also be made to stand for part of a mechanical system that contains no spring at all, but behaves in the appropriate way, as in the experiment described above. In our idealised model, the model spring has zero mass.

The concept of a model spring may be compared with the concepts of a model string and a model pulley, which were introduced in Unit 2.

Note that the relationship between the length of the spring and the magnitude of the force in a model spring is represented by a straight line, and a straight line has two parameters – the slope (or gradient) and the intercept. For a spring, these two parameters are called the spring *stiffness* and the *natural length* of the spring, usually denoted by the symbols k and l_0, respectively. These parameters are often given next to a spring in a diagram, as shown in Figure 8. They will be explained further in the next subsection.

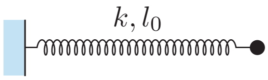


Figure 8 Spring representation

Figure 8 shows the diagrammatic convention that this module adopts. The bar on the left indicates a fixed and immovable wall to which the left-hand end of the spring is attached. The spring itself is coiled, and usually an end is attached to a particle, which is represented by a black filled circle; if no particle is attached, then the free end of the spring often ends in an open circle.

Model spring

A **model spring** is characterised by two positive constants, its **natural length** l_0 and its **stiffness** k. It has zero mass.

Exercise 1

Use equation (1) to estimate answers to the following questions.

(a) If the displacement is 2.5 mm, what is the magnitude of the weight on the ruler?

(b) If the magnitude of the weight on the hanger is 0.67 N, what will be the displacement?

(c) If the magnitude of the weight on the hanger is 60 N, what will be the displacement?

(d) If the displacement is −2.5 mm, what is the magnitude of the weight on the ruler?

1.2 Force law for a model spring

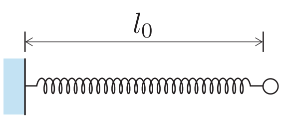


Figure 9 Free spring

Suppose that Figure 9 represents a model spring that is in its undisturbed state. One end is attached to a fixed object such as a wall, and the other end is free to move on a frictionless horizontal surface. The spring has no weight, and there is no normal reaction on it from the surface; there are no horizontal forces. So although the free end of the spring is free to move, it does not do so because the resultant force is zero. The length of the spring under these circumstances is its **natural length**, which we denote by l_0. If its length is made longer or shorter than l_0, then the spring will always try to return to its natural length.

Suppose now that the spring is *extended* (see Figure 10), so that its length is greater than its natural length; such a spring is said to be *in extension*, or alternatively *in tension*. You probably know from experience that if you hold the free end of an extended spring, then you will be pulled towards the fixed end. The magnitude of this pulling force is called the **tension** in the spring (just as, in Unit 2, we refer to the tension in a taut string). The more the spring is extended, the more it pulls; in other words, the tension in the spring increases with its extension.

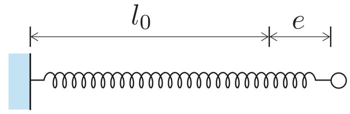


Figure 10 Spring extended by e

Now suppose that the spring is *compressed* (see Figure 11), so that its length is less than its natural length; such a spring is said to be *in compression*. In this case, if you hold the free end of the spring, then the spring will push you away from its fixed end. Again, the more the spring is compressed, the more it pushes; the magnitude of the pushing force increases with compression.

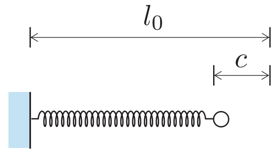


Figure 11 Spring compressed by c

A *model spring* is a special model of a real spring. This model is based on the assumption that *the magnitude of the force exerted by the spring is proportional to the amount of its deformation* (extension or compression). The constant of proportionality k, which relates the force magnitude to the deformation for a model spring, is known as the **stiffness** of the spring. The SI units for stiffness are newtons per metre ($\mathrm{N\,m^{-1}}$).

Force law

A model spring exerts a force on any object attached to either of its ends, as follows.

- When the length of the spring equals l_0, the magnitude of the force exerted by the spring is zero.
- When the spring is extended by an amount e (where $e > 0$), so that its length is $l_0 + e$, the force is directed along the axis of the spring towards the centre of the spring and has magnitude ke (stiffness × extension).
- When the spring is compressed by an amount c (where $c > 0$), so that its length is $l_0 - c$, the force is directed along the axis of the spring away from the centre of the spring and has magnitude kc (stiffness × compression).

The force law for a model spring is often known as **Hooke's law**, in honour of Robert Hooke. In 1678 Hooke declared, in *De Potentia Restitutiva*:

> It is very evident that the Rule or Law of Nature in every springing body is, that the force or power thereof to restore itself to its natural position is always proportionate to the Distance or space it is removed therefrom.

Actually, this rather overstates the case. The 'model spring' is only a model. Real springs exert forces that have a non-linear dependence on extension or compression and that may also depend on other factors such as the velocities of the ends or the mass of the spring. Also, as pointed out earlier, the concept of natural length may not carry over to an actual physical length associated with a real spring. Nevertheless, many real springs conform fairly closely to the idealised behaviour of a model spring for certain ranges of extension and compression. We will therefore use model springs to represent all the springs discussed in this unit. A bonus of this model is that it leads to a differential equation that can be solved. However, you should remember that this is only an *approximation* when comparing predictions with the real world. As in Exercise 1(c), if the modelling results in a value that is outside the validity of the model, then it must be revised.

Exercise 2

A model spring has natural length 0.3 m and stiffness $200\,\mathrm{N\,m^{-1}}$. It is attached to a fixed bracket at one end, while the other end is attached to an object (lying on a frictionless horizontal surface) whose distance l from the fixed bracket is variable (see Figure 12).

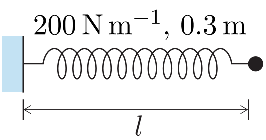


Figure 12

Determine the magnitude and direction, relative to the centre of the spring, of the force on the object when $l = 0.35$ and when $l = 0.2$.

So far we have discussed a spring with one end attached to a fixed point and considered the force at the free end of the spring. However, if you take a spring with an end in each hand and extend it, you will be made aware that there are pulling forces exerted by the spring at *both* ends.

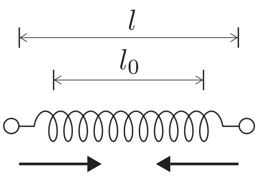


Figure 13 Spring extended at both ends

For a model spring in tension, these two forces have equal magnitudes and are both directed towards the centre of the spring (see Figure 13). The magnitude of each of these forces is given, as before, by stiffness $\times$ extension. If we denote the length of the extended spring by l, and its natural length by l_0, then its extension is $e = l - l_0$ and so, by the force law for a model spring, the tension in the spring is equal to $k(l - l_0)$.

If the same spring is compressed by being pushed towards its centre at both ends, then its length l is less than its natural length l_0 by the amount $c = l_0 - l$, and the magnitude of the force provided by the spring is equal to $k(l_0 - l)$.

We need to look at two different cases whenever we consider the force that may be exerted by a model spring on a particle or object attached to an end of the spring, depending on whether the spring is extended or compressed. However, it is possible to write a single *vector* formula for the force, which will apply correctly whatever the configuration of the spring.

The force exerted by a model spring of length l, natural length l_0 and stiffness k, on any object attached to one of its ends, is:

- directed from the object towards the spring, with magnitude $k(l - l_0)$, if the spring is extended from its natural length ($l > l_0$)
- directed away from the spring towards the object, with magnitude $k(l_0 - l)$, if the spring is compressed ($l < l_0$).

Suppose that we are concerned with the forces on an object attached to one end of a spring, and that $\widehat{\mathbf{s}}$ is a unit vector in the direction from the object towards the spring (see Figure 14). The force $\mathbf{H}$ that the spring exerts on the object can be written in terms of $\widehat{\mathbf{s}}$ as follows.

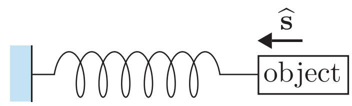


Figure 14 Direction of $\widehat{\mathbf{s}}$

- If the spring is extended ($l > l_0$), then it provides a force in the same direction as $\widehat{\mathbf{s}}$, namely $\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}$.
- If the spring is compressed ($l < l_0$), then it provides a force in the opposite direction to $\widehat{\mathbf{s}}$, namely $\mathbf{H} = k(l_0 - l)(-\widehat{\mathbf{s}})$.

Since $-k(l - l_0) = k(l_0 - l)$, we actually have the same algebraic relationship in each case. It follows that this vector equation provides a complete description of the model spring force law, whether the spring is extended or compressed, and this is a consequence of there being a linear relationship between the component of force and the change in length of the spring.

The vector equation also gives the correct outcome of zero force when the spring has its natural length ($l = l_0$).

Vector form of Hooke's law

Suppose that a model spring has natural length l_0 and stiffness k. Then on an object attached to a particular end, the spring exerts a force

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}, \tag{2}$$

where l is the length of the spring and $\widehat{\mathbf{s}}$ is a unit vector in the direction from the end where the object is attached towards the centre of the spring.

Exercise 3

Use the vector form of Hooke's law to answer once more the questions posed in Exercise 2. The diagram of the spring is repeated as Figure 15.

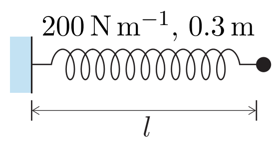


Figure 15

1.3 Model springs at rest

The chief purpose behind introducing the concept of a model spring is to provide a simple mathematical model that describes the oscillatory behaviour of certain systems, such as those shown in the Introduction. However, before considering systems in motion, it is instructive to look at some static configurations that feature model springs.

The approach to analysing these situations is similar to that which you studied in Unit 2, with the addition of Hooke's law to describe the force provided by a model spring.

See Procedure 2 in Unit 2.

Example 1

A particle of mass 5 kg is suspended from a model spring that is attached at its top end to a fixed point. The natural length of the spring is 0.3 m. The length of the spring is 0.5 m when the particle is suspended from it.

What is the stiffness k of the spring?

Solution

◀ Draw picture ▶

◀ Choose axes ▶

◀ Draw force diagram(s) ▶

Figure 16 shows the spring and the force diagram. Take the $\mathbf{i}$-direction to be vertically downwards, as shown.

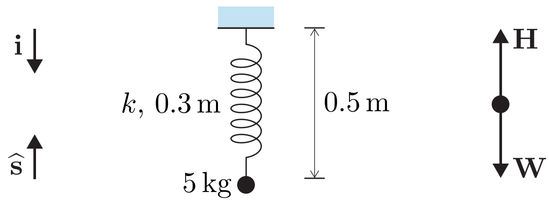


Figure 16 Mechanical system and force diagram

The mass of the particle is 5 kg, so its weight is $\mathbf{W} = 5g\mathbf{i}$.

The spring force $\mathbf{H}$ will depend on the values of k, l, l_0 and $\widehat{\mathbf{s}}$. We are given that $l = 0.5$ and $l_0 = 0.3$, and k is unknown. The unit vector $\widehat{\mathbf{s}}$ in the direction from the particle towards the centre of the spring points upwards, so $\widehat{\mathbf{s}} = -\mathbf{i}$.

◀ Apply law(s) ▶

By Hooke's law (equation (2)), the model spring provides a force

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}} = k(0.5 - 0.3)(-\mathbf{i}) = -0.2k\mathbf{i}.$$

The equilibrium condition for particles was introduced in Unit 2.

Since the particle is in equilibrium, we have

$$\mathbf{W} + \mathbf{H} = \mathbf{0}, \quad \text{that is,} \quad 5g\mathbf{i} - 0.2k\mathbf{i} = \mathbf{0}.$$

◀ Solve equation(s) ▶

Resolving in the $\mathbf{i}$-direction gives

$$5g - 0.2k = 0,$$

so

$$k = 25g \simeq 245 \quad \text{to 3 s.f., using } g = 9.81\,\mathrm{m\,s^{-2}}.$$

◀ Interpret solution ▶

We conclude that the stiffness of the spring is $245\,\mathrm{N\,m^{-1}}$.

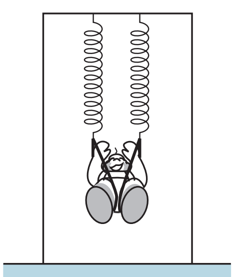


Figure 17 A baby bouncer

Exercise 4

A 'baby bouncer' (see Figure 17) consists of a seat and straps, whose combined mass is 0.5 kg, suspended from two identical springs that are attached to a door lintel. When a baby of mass 7.5 kg is strapped into the seat, the seat descends by 3 cm.

What is the stiffness of each of the springs?

The procedure to find the force due to a spring can be made more systematic.

Procedure 1 Finding the force on a particle due to a spring

1. Determine the length l of the spring; this is best done by drawing a diagram to show the various lengths in the system. Note that l may be in terms of unknown displacements such as x.
2. Use the given natural length l_0 of the spring to find the extension $l - l_0$.
3. From the diagram, determine $\widehat{\mathbf{s}}$, the unit vector from the particle towards the centre of the spring.
4. The spring tension force is $\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}$, where k is the stiffness of the spring.

If there is more than one spring, then consider a tabular approach (see Example 2).

Note that l and $\widehat{\mathbf{s}}$ are deduced from the position of the spring in relation to the particle, or the point at which the force due to the spring is required; k and l_0 are characteristics of the spring. The converse, finding the length of a spring when the spring tension force is known, is also possible using an approach like Procedure 1.

Example 2

An object of mass m is attached to two model springs, whose other ends are attached to fixed points that are $0.9\,\mathrm{m}$ apart. The object can be considered as a particle of negligible size. The left-hand spring has stiffness $40\,\mathrm{N\,m^{-1}}$ and natural length $0.3\,\mathrm{m}$, while the right-hand spring has stiffness $60\,\mathrm{N\,m^{-1}}$ and natural length $0.2\,\mathrm{m}$. The object is supported by a table and is free to move on the table without friction. This set-up is shown in Figure 18.

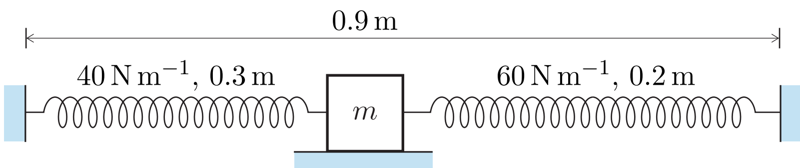


Figure 18

Determine the equilibrium position of the object, and find the tension in the springs.

Solution

All the forces acting on a particle should be drawn on a force diagram even if it is anticipated that some forces may not play a part in the subsequent derivation. The application of the equilibrium law or Newton's second law sorts out what is required.

The forces acting on the particle are shown in Figure 19, where $\mathbf{W}$ is the weight, $\mathbf{N}$ is the normal reaction, and $\mathbf{H}_1$ and $\mathbf{H}_2$ are the forces exerted by the left-hand and right-hand springs, respectively.

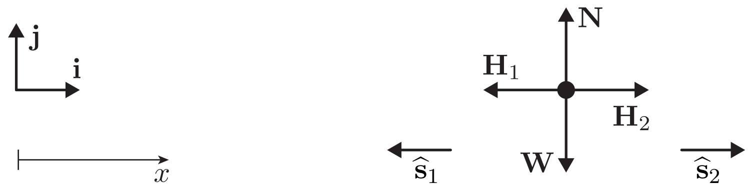


Figure 19 Axis and force diagram

Take $\mathbf{i}$ to be horizontal and positive to the right, and take $\mathbf{j}$ to be positive vertically upwards. Hence $\mathbf{W} = mg(-\mathbf{j})$ and $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$.

For each spring, the force exerted is given by Hooke's law as

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}},$$

where $\widehat{\mathbf{s}}$ is a unit vector directed from the particle towards the centre of the spring. The x-axis is chosen to point from left to right, in the same direction as $\mathbf{i}$, with origin at the left-hand fixed point, as shown in Figure 19. Then the lengths of the two springs are x (left spring) and $0.9 - x$ (right spring).

Choosing x to be in the contrary direction to $\mathbf{i}$ is not recommended; similarly for y and $\mathbf{j}$.

For problems where there is more than one spring, you might like to use a tabular approach to find the spring forces $\mathbf{H}$, and identify the values of k, l, l_0 and $\widehat{\mathbf{s}}$ either from the problem statement or from the diagram that accompanies it, as is done in Table 2.

Here $\widehat{\mathbf{s}}_1 = -\mathbf{i}$ since the direction from the particle towards the spring is to the left, that is, in the $-\mathbf{i}$-direction. Similarly, $\widehat{\mathbf{s}}_2 = \mathbf{i}$ since the direction from the particle towards the spring is to the right.

Table 2 Tabular approach to finding spring forces

Spring	l	l_0	$l - l_0$	k	$\widehat{\mathbf{s}}$	$\mathbf{H}$
Left	x	0.3	$x - 0.3$	40	$-\mathbf{i}$	$40(x - 0.3)(-\mathbf{i})$
Right	$0.9 - x$	0.2	$0.7 - x$	60	$\mathbf{i}$	$60(0.7 - x)\mathbf{i}$

From Table 2,

$$\mathbf{H}_1 = 40(x - 0.3)(-\mathbf{i}) = 40(0.3 - x)\mathbf{i},$$
$$\mathbf{H}_2 = 60(0.7 - x)\mathbf{i}.$$

Since the particle is held in equilibrium,

$$\begin{aligned}\mathbf{W} + \mathbf{N} + \mathbf{H}_1 + \mathbf{H}_2 &= mg(-\mathbf{j}) + |\mathbf{N}|\,\mathbf{j} + 40(0.3 - x)\mathbf{i} + 60(0.7 - x)\mathbf{i} \\ &= \mathbf{0}.\end{aligned}$$

Resolving in the **i**-direction gives

$$60(0.7 - x) + 40(0.3 - x) = 0.$$

The solution of this equation is $x = 0.54$, so the particle is 0.54 m from the left-hand fixed point (and hence 0.36 m from the right-hand fixed point).

Resolving in the **j**-direction gives $|\mathbf{N}| - mg = 0$, but this is of no consequence in this solution as it just tells us that the weight is balanced by the normal reaction.

The magnitudes of the spring forces can be found by substituting $x = 0.54$ into the above expressions for the forces to obtain

$$\mathbf{H}_1 = 40(0.3 - 0.54)\mathbf{i} = -9.6\mathbf{i},$$
$$\mathbf{H}_2 = 60(0.7 - 0.54)\mathbf{i} = 9.6\mathbf{i}.$$

So, as expected, since the particle is not moving, the forces exerted by the springs are of equal magnitude and in opposite directions (which provides a useful quick check). The tension in the springs is the magnitude of the above forces, that is, 9.6 N.

Exercise 5

Consider a modified version of the problem stated in Example 2, in which the distance between the two fixed ends is reduced to 0.3 m, but all other aspects of the problem remain unchanged (see Figure 20).

Solve this modified problem (finding, at the end, the magnitude of the forces exerted by the springs).

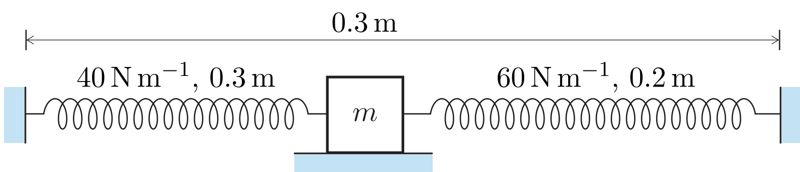


Figure 20

Example 3

A model spring hangs vertically from a fixed point, with a particle of mass m suspended from its lower end. The particle is in equilibrium.

Express the length l of the spring in terms of its stiffness k and natural length l_0, the mass m, and the magnitude of the acceleration due to gravity g.

Solution

Choose the **i**-direction to be vertically downwards, as shown in Figure 21, which also shows the force diagram.

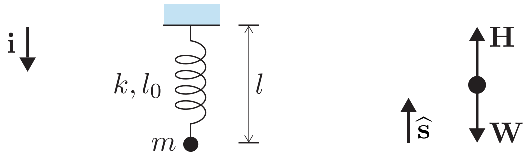


Figure 21 Set-up and force diagram

The weight of the particle is $\mathbf{W} = mg\mathbf{i}$, and the force exerted on it by the spring is

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}} = k(l - l_0)(-\mathbf{i}).$$

Since $\mathbf{H} + \mathbf{W} = \mathbf{0}$ (because the particle is in equilibrium), we have

$$-k(l - l_0) + mg = 0,$$

that is,

$$l = l_0 + \frac{mg}{k}.$$

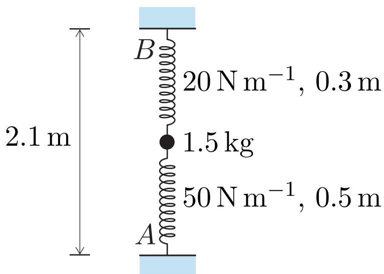


Figure 22

Exercise 6

An object of mass 1.5 kg is attached to the floor at A by a model spring of stiffness $50\,\mathrm{N\,m^{-1}}$ and natural length 0.5 m. It is also attached to the ceiling at B, a point 2.1 m vertically above A, by a model spring of stiffness $20\,\mathrm{N\,m^{-1}}$ and natural length 0.3 m (see Figure 22).

When the object is in equilibrium, what is the height of the object above the floor?

Exercise 7

A battery of length 5 cm is put into a battery holder. The holder has an internal length of 5.5 cm, and has a spring at each end to hold the battery in place (see Figure 23). The left-hand spring has stiffness $30\,\mathrm{N\,m^{-1}}$ and natural length 0.4 cm, while the right-hand spring has stiffness $10\,\mathrm{N\,m^{-1}}$ and natural length 0.3 cm.

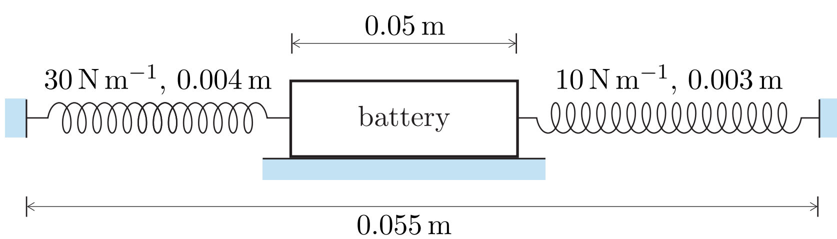


Figure 23 Schematic arrangement of the battery holder (not to scale)

What is the magnitude of the spring forces acting on the battery, assuming that there are no friction forces?

2 Oscillating systems

In Unit 3 you saw how to analyse the one-dimensional motion of a particle under the influence of forces such as gravity and air resistance. Here we are again concerned solely with one-dimensional motion, but now we concentrate on how a particle behaves when connected to the end of a model spring. In this unit friction and air resistance will be considered to be negligible, and the only forces that will be considered will be gravity, normal reactions and spring forces. There is no fundamental difference in approach; it is just that the force due to a spring depends on the length of the spring, and this results in a different outcome.

2.1 Setting up an equation of motion

As before, the analysis is based on Newton's second law, which states that if a particle has mass m and experiences a resultant force $\mathbf{F}$, then its acceleration $\mathbf{a}$ is given by

$$\mathbf{F} = m\mathbf{a}.$$

You saw in Unit 3 that the acceleration $\mathbf{a}(t)$ of a particle at time t is defined as the derivative of its velocity $\mathbf{v}(t)$, which in turn is defined as the derivative of its position vector $\mathbf{r}(t)$. Using Newton's 'dot' notation for derivatives with respect to time, we therefore have $\mathbf{a} = \ddot{\mathbf{r}}$, so Newton's second law may also be expressed as

Recall that $\dot{\mathbf{r}}$ means $d\mathbf{r}/dt$ and $\ddot{\mathbf{r}}$ means $d\dot{\mathbf{r}}/dt = d^2\mathbf{r}/dt^2$. Each dot stands for '$d/dt$'.

$$m\ddot{\mathbf{r}} = \mathbf{F}. \tag{3}$$

Example 4 illustrates the type of approach that is required where a model spring is involved. As indicated in the margin alongside the solution, it is still appropriate to adopt here the procedural steps for applying Newton's second law that were used in Unit 3. The last two stages of this procedure, solving the differential equation and interpreting the solution, will be dealt with in the following subsection.

See Procedure 1 of Unit 3 for the listing of these steps.

Example 4

A spring is attached to a fixed wall at one end. The other end of the spring is attached to a glider that moves along a track, such that the motion of the glider is friction-free. Model the glider as a particle of mass m and the spring as a model spring of natural length l_0 and stiffness k.

As mentioned above, air resistance is deemed to be negligible.

Find the equation of motion for the particle when its position x is measured in the direction away from the wall, with the origin at the fixed end.

(This model system is similar to that of Figure 2(b) in the Introduction, and as mentioned there, it might provide a first model for a spring-loaded buffer.)

Solution

◀ Draw picture ▶
◀ Choose axes ▶
◀ State assumptions ▶

Figure 24 sketches the physical situation, including the x-axis and unit vectors $\mathbf{i}$ and $\mathbf{j}$. We model the glider as a particle of mass m and the spring as a model spring. All friction and air resistance forces are ignored.

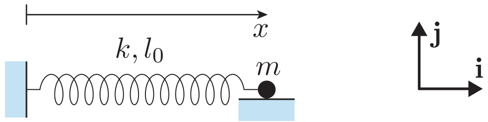


Figure 24 Simple oscillating system

The unit vector $\mathbf{i}$ is directed away from the fixed end. The length of the spring is given by x, in the same direction as $\mathbf{i}$.

◀ Draw force diagram ▶

In addition to its weight $\mathbf{W}$ and the normal reaction $\mathbf{N}$, the only force acting on the particle is that provided by the model spring, but how should this be portrayed on a force diagram? As you saw in Subsection 1.2, this force will pull the particle if the spring is extended and push the particle if the spring is compressed, so we can expect the force exerted by the spring on the particle to act in the direction of $\mathbf{i}$ at some times and in the direction of $-\mathbf{i}$ at others. We are therefore unable to make a single 'correct' choice for the direction of this force $\mathbf{H}$, but *either choice will do* for the purposes of the diagram. (The force diagram shown in Figure 25 corresponds to the situation when the spring is extended, and this is the usual convention.)

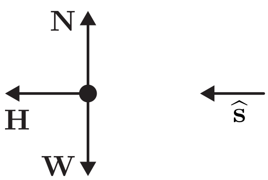


Figure 25 Force diagram

◀ Apply Newton's 2nd law ▶

Applying Newton's second law to the particle gives

$$m\ddot{\mathbf{r}} = \mathbf{H} + \mathbf{W} + \mathbf{N}. \tag{4}$$

To proceed further we need to model all the forces. The force $\mathbf{H}$ exerted by the model spring on the particle is described by Hooke's law, that is,

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}},$$

where l is the length of the spring and $\widehat{\mathbf{s}}$ is a unit vector in the direction from the end where the particle is attached towards the centre of the spring.

For the current situation, $l = x$ and $\widehat{\mathbf{s}} = -\mathbf{i}$, so the force in the spring that is exerted on the particle is

$$\mathbf{H} = k(x - l_0)(-\mathbf{i}).$$

The other two forces are $\mathbf{W} = mg(-\mathbf{j})$ and $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$. Substituting into equation (4) gives

$$m\ddot{\mathbf{r}} = \mathbf{H} + \mathbf{W} + \mathbf{N} = -k(x - l_0)\mathbf{i} - mg\mathbf{j} + |\mathbf{N}|\,\mathbf{j}. \tag{5}$$

Since the motion is only in the x-direction, the position vector of the particle is $\mathbf{r}(t) = x(t)\mathbf{i}$, so $\dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i}$ and $\ddot{\mathbf{r}}(t) = \ddot{x}(t)\mathbf{i}$. Hence resolving equation (5) in the **i**-direction gives $m\ddot{x} = -k(x - l_0)$, that is,

$$m\ddot{x} + kx = kl_0. \tag{6}$$

This is the equation of motion.

Equation (6) is a second-order inhomogeneous linear constant-coefficient differential equation. This is exactly the type of differential equation that was studied in Unit 1 (this is no coincidence), and it can be solved using the methods described in that unit. This will be done in the next subsection, but to conclude this subsection, we ask you to look at the derivation of another equation of motion.

As in Unit 1, we collect all the terms involving the unknown function (x in this case) and its derivatives onto the left-hand side, and any remaining terms onto the right-hand side.

Exercise 8

Figure 26 shows the same spring and mass as described in Example 4, but now it is arranged so that it moves vertically, hanging from a fixed support at the top. Find the equation of motion for the particle in this case.

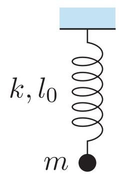


Figure 26 Vertical spring and mass

Exercise 9

What are the similarities, and the differences, between the equation of motion derived in Exercise 8 and equation (6) derived in Example 4 for a system that moves horizontally?

2.2 Solving the equation of motion

In the previous subsection we derived

$$m\ddot{x} + kx = kl_0. \tag{7}$$

as the equation of motion for a particle attached to one end of a horizontal model spring, whose other end is fixed. The solution of this equation will be found in two parts: first the complementary function, and then the particular integral.

We now complete Procedure 1 of Unit 3 for the problem stated in Example 4, by solving the differential equation and interpreting the solution.

The method for solving the associated homogeneous equation

$$m\ddot{x} + kx = 0 \tag{8}$$

to obtain the complementary function was described in Unit 1, but quite a lot can be deduced simply by looking at the form of the equation.

Equation (8), written as $\ddot{x} = (-k/m)x$, must be satisfied by a function $x(t)$ whose second derivative $\ddot{x}(t)$ is a negative constant multiple $(-k/m)$ of the function $x(t)$ itself. What functions satisfy this specification? There are at least two that come to mind, namely the sine and cosine functions. Hence functions of the form $x(t) = B\cos\omega t$ and $x(t) = C\sin\omega t$ (where B, C and ω are constants) are contenders to be solutions of equation (8). Since we are dealing with a homogeneous linear differential equation, it follows from the principle of superposition that the sum of two such functions is also a contender.

See Unit 1, Theorem 2.

Exercise 10

Show by substitution that $x(t) = B\cos\omega t + C\sin\omega t$ is a solution of the differential equation $m\ddot{x} + kx = 0$, where $\omega^2 = k/m$.

Exercise 11

By using the auxiliary equation, find the general solution of the differential equation $m\ddot{x} + kx = 0$.

This method of solution is described in Unit 1.

Exercises 10 and 11 confirm that our reasoning and the methods of Unit 1 give the same solution.

Exercise 12

Find the general solutions of the equations of motion found in Example 4 and Exercise 8, namely

$$m\ddot{x} + kx = kl_0,$$

$$m\ddot{x} + kx = kl_0 + mg.$$

Exercise 13

The equilibrium length of a horizontal spring is its natural length, l_0. Example 3 found that the equilibrium length of a hanging spring is $l_0 + mg/k$.

Compare each of these values with the value of the constant in the appropriate general solution found in Exercise 12.

Exercise 13 shows that the constant term in the solution of the differential equation for a horizontal or hanging spring is exactly the same as the equilibrium position of the particle. From now on this will be denoted by x_{eq}. In many systems that oscillate, the complementary function, which is associated with the oscillatory behaviour of the system, is of more consequence than the equilibrium position, which is associated with the particular integral.

Using the results of Exercises 12 and 13, the general solution of

$$m\ddot{x} + kx = kx_{\text{eq}}$$

is

$$x(t) = B\cos\omega t + C\sin\omega t + x_{\text{eq}}, \tag{9}$$

where $\omega = \sqrt{k/m}$, the constant term x_{eq} is the equilibrium position of the particle, and B and C are arbitrary constants. The values of these arbitrary constants can be found if initial conditions are provided for the function $x(t)$, usually $x(0)$ and $\dot{x}(0)$.

Motion of the type described by equation (9), arising from a differential equation of the form $\ddot{x} + \omega^2 x = 0$, is known as **simple harmonic motion** (sometimes abbreviated to **SHM**).

It is sometimes more convenient to replace $B\cos\omega t + C\sin\omega t$ with $A\cos(\omega t + \phi)$, which is directly equivalent. To see this, use a compound-angle formula to write

$$A\cos(\omega t + \phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi).$$

Equating terms in $\cos\omega t$ and $\sin\omega t$ gives $B = A\cos\phi$ and $C = -A\sin\phi$. Then

$$B^2 + C^2 = A^2\cos^2\phi + (-A)^2\sin^2\phi = A^2(\cos^2\phi + \sin^2\phi) = A^2.$$

So

$$A = \sqrt{B^2 + C^2}, \quad \cos\phi = \frac{B}{A}, \quad \sin\phi = -\frac{C}{A}. \tag{10}$$

We take the positive root for A so that $A > 0$. The value of ϕ that satisfies the equations for $\cos\phi$ and $\sin\phi$ lies in the interval $[-\pi, \pi]$.

The equivalent form of equation (9) is

$$x(t) = B\cos\omega t + C\sin\omega t + x_{\text{eq}} = A\cos(\omega t + \phi) + x_{\text{eq}}. \tag{11}$$

Note that the addition of a cosine term and a sine term with the same argument leads to a cosine (or sine) term that is shifted.

Exercise 14

Given that

$$x(t) = 10\cos\left(7t + \arctan\left(\tfrac{3}{4}\right)\right),$$

find constants B, C and ω in the equivalent form

$$x(t) = B\cos\omega t + C\sin\omega t.$$

When applying initial conditions, it is usually easier to use the form $B\cos\omega t + C\sin\omega t$, whereas the expression $A\cos(\omega t + \phi)$ gives a better feel for the features of motion.

The quantity ϕ, which is called the **phase angle** (or **phase**) of the motion, has the interpretation that $A\cos\phi$ is the particle's position at time $t = 0$. Also, if ϕ is positive, then $-\phi/\omega$ gives the last time, prior to $t = 0$, at which the position x reaches its maximum value (see Figure 27). If ϕ is negative, then $-\phi/\omega$ is the first time after $t = 0$ at which the position x reaches its maximum value. The graph of $A\cos(\omega t + \phi)$ may be obtained from the graph of $A\cos\omega t$ by a horizontal shift of ϕ/ω to the left.

The parameters in the general solution of a spring–mass system have special terminology, and at this point it is useful to define the terms used for describing oscillations. First, consider a plot of the solution

$$x(t) = A\cos(\omega t + \phi) + x_{\text{eq}} = A\cos\left(\omega\left(t + \frac{\phi}{\omega}\right)\right) + x_{\text{eq}}.$$

$A > 0$ and $-\pi < \phi \leq \pi$.

The form of the solution is a sinusoid added to a constant; the cosine has amplitude A and is displaced by an amount $t = \phi/\omega$ to the left. The period is the distance between successive peaks, and represents the time taken for the oscillation to repeat itself. These features are shown in Figure 27. (The graph has been drawn for $\phi > 0$.)

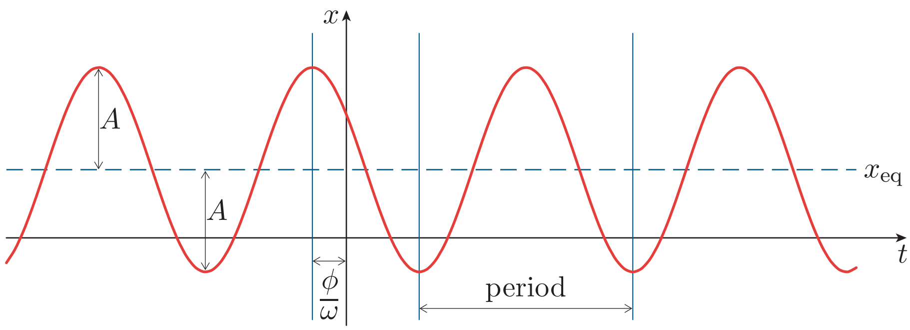


Figure 27 Solution for a general oscillating system

As the angle ωt is measured in radians, the SI units of the angular frequency ω are radians per second (rad s^{-1}).

The variable ω is called the **angular frequency** of the motion. Simple harmonic motion is periodic with a **period** τ given by $\tau = 2\pi/\omega$, that is, the motion repeats itself every $\tau = 2\pi/\omega$ seconds.

The SI unit for frequency is the *hertz* (Hz), where 1 hertz is 1 cycle per second. Hence $1\,\text{Hz} = 1\,\text{s}^{-1}$.

Since τ is the time for one cycle, the number f of cycles per unit time is equal to $1/\tau$. This is called the **frequency** of the motion and is given by

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi},$$

so the angular frequency ω (in rad s^{-1}) is equal to 2π times the frequency f (in Hz).

We now demonstrate how particular values may be found for the arbitrary constants and when specific initial conditions are given.

Example 5

In a model of a vibrating system, the position $x(t)$, with origin at zero, of a particle at time t is given by

$$x(t) = B\cos 10t + C\sin 10t,$$

where B and C are constants. The particle is set in motion at time $t = 0$, when its position is $-0.2\,\text{m}$ and its speed is $1\,\text{m}\,\text{s}^{-1}$ in the direction of the positive x-axis (both measured relative to the given x-axis and origin).

(a) Find the particular position function for the particle that corresponds to these initial conditions. Where is the particle after 0.5 seconds?

(b) What are the angular frequency, frequency and period of the oscillations?

(c) Write the solution in the form $A\cos(\omega t + \phi)$, and hence state the amplitude and phase angle to three decimal places. Sketch the graph of $x(t)$ against t for $0 \leq t \leq 2$.

Solution

(a) The initial conditions may be written as

$$x(0) = -0.2, \quad \dot{x}(0) = 1.$$

Since $x(t) = B\cos 10t + C\sin 10t$, we have

$$\dot{x}(t) = -10B\sin 10t + 10C\cos 10t.$$

Substituting $t = 0$ into these equations and using the given initial conditions, we have

$$x(0) = B = -0.2, \quad \dot{x}(0) = 10C = 1.$$

Hence $B = -0.2$ and $C = 0.1$, giving the particular position function

$$x(t) = -0.2\cos 10t + 0.1\sin 10t.$$

After 0.5 seconds, the particle is at

$$x(0.5) = -0.2\cos 5 + 0.1\sin 5 \simeq -0.15 \quad \text{to 2 d.p.},$$

that is, $0.15\,\text{m}$ along the x-axis in the negative direction from the origin.

(b) The angular frequency is 10, so the frequency is $10/2\pi = 5/\pi$, and the period is $2\pi/10 = \pi/5$.

(c) Using equations (10), we have

$$A = \sqrt{B^2 + C^2} = \sqrt{(-0.2)^2 + 0.1^2} = \sqrt{0.05}$$

and

$$\cos\phi = \frac{B}{A} = -\frac{0.2}{\sqrt{0.05}}, \quad \sin\phi = -\frac{C}{A} = -\frac{0.1}{\sqrt{0.05}}.$$

Thus

$$\tan\phi = \frac{0.1}{0.2} = \tfrac{1}{2},$$

and since both $\sin\phi$ and $\cos\phi$ are negative, this gives $\phi = -\pi + \arctan\left(\tfrac{1}{2}\right)$. Hence

$$x(t) = \sqrt{0.05}\cos\left(10t - \pi + \arctan\left(\tfrac{1}{2}\right)\right).$$

The amplitude A is $\sqrt{0.05} \simeq 0.224$ and the phase angle ϕ is $-\pi + 0.464 \simeq -2.678$, both to three decimal places, so $\phi/\omega = -2.678/10 \simeq -0.27$.

A sketch of x against t is shown in Figure 28.

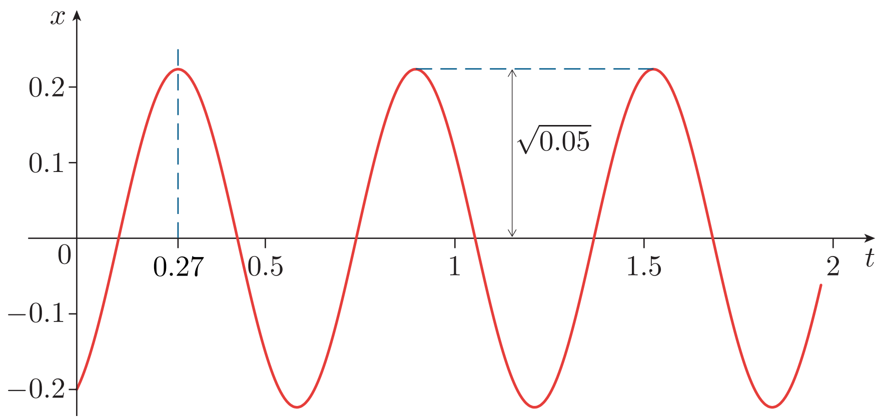


Figure 28

Exercise 15

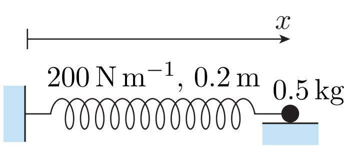


Figure 29 Oscillating system

For a particular case of the system shown in Figure 24, and reproduced here as Figure 29, the model spring's stiffness k is $200\,\mathrm{N\,m^{-1}}$, the natural length l_0 is $0.2\,\mathrm{m}$, and the particle's mass m is $0.5\,\mathrm{kg}$. The system is set in motion when $t = 0$, at which time the position is $0.3\,\mathrm{m}$ from the left-hand end and the speed is $2\,\mathrm{m\,s^{-1}}$ to the right.

The solution of the equation of motion is

$$x(t) = B\cos\omega t + C\sin\omega t + l_0,$$

where $\omega = \sqrt{k/m}$, and B and C are constants.

(a) Find the angular frequency ω and the period τ of the motion.

(b) Find the frequency f of the motion.

(c) Find the position function for the particle at time t.

(d) Find the minimum distance of the particle from the left-hand end.

You saw in Exercise 14 that the equivalence of the expressions for $x(t)$ in equations (11) is obtained by showing how the constants B and C in the first form are related to the constants A and ϕ in the second form. It is often helpful to draw a diagram such as the figure shown in the solution to Exercise 14.

Exercise 16

In Exercise 15 you showed that the particular oscillation described there is represented by the equation

$$x(t) = 0.1\cos 20t + 0.1\sin 20t + 0.2.$$

(a) Find the phase angle of this oscillation.

(b) Sketch the graph of x against t.

(c) Determine the times when the spring takes its natural length.

We summarise the results about simple harmonic motion in the following box.

Simple harmonic motion

The simple harmonic motion equation

$$\ddot{x} + \omega^2 x = \omega^2 x_{\text{eq}} \tag{12}$$

This equation is equivalent to $m\ddot{x} + kx = kx_{\text{eq}}$, where $\omega^2 = k/m$.

has a general solution that can be written in the form

$$x(t) = B\cos\omega t + C\sin\omega t + x_{\text{eq}}, \tag{13}$$

where B and C are arbitrary constants, or alternatively as

$$x(t) = A\cos(\omega t + \phi) + x_{\text{eq}}, \tag{14}$$

where

$$A = \sqrt{B^2 + C^2}, \quad \cos\phi = B/A, \quad \sin\phi = -C/A, \tag{15}$$

with $A > 0$ and $-\pi < \phi \leq \pi$.

The following names are given to quantities that are characteristic of simple harmonic motion:

- the *angular frequency* of the oscillations is ω
- the *period* (time for one complete cycle) is $\tau = 2\pi/\omega$
- the *frequency* (number of cycles per second) is $f = 1/\tau = \omega/(2\pi)$
- the *amplitude* of the oscillations is A
- the *phase angle* of the oscillations is ϕ
- the equilibrium position about which the particle oscillates is x_{eq}.

We now extend the experiment that was described in Subsection 1.1.

We can consider the system of the ruler and weights as a particle on the end of a model spring, as shown in Figure 30.

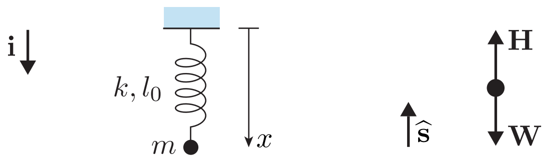


Figure 30 Model for the ruler experiment

Equation (1) of Subsection 1.1 can be rewritten in the vector form of Hooke's law, using x instead of d (the displacement of the end of the ruler). The spring force is in the upward direction, so

$$\mathbf{H} = -99.044\left(x - \frac{0.024}{99.044}\right)\mathbf{i} = -99.044(x - 0.000\,24)\,\mathbf{i},$$

which gives the spring stiffness as about $99\,\mathrm{N\,m^{-1}}$.

To test the mathematical model for oscillations, a mass of 0.03 kg was firmly attached to the ruler, then the ruler was deflected downwards by about a centimetre and released. The resulting oscillations were measured by a recording device, and some of them are shown in Figure 31, where t is the time in seconds and x is the distance of the end of the ruler below a fixed point.

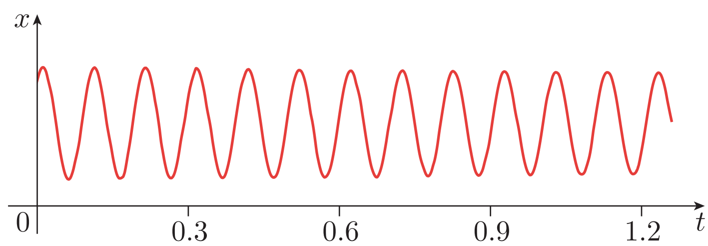


Figure 31 Measurements of an oscillating ruler

Example 6

Consider the oscillations of the end of the ruler as shown in Figure 31.

(a) Is the motion periodic?

(b) From the graph, estimate the period of the motion.

(c) Compare the measured period with the period obtained from considering the stiffness of the model spring and the mass of the particle on the end.

Solution

(a) The motion does look periodic.

(b) There are thirteen peaks in all, so twelve oscillations, and the time interval for these twelve oscillations is approximately 1.25 seconds, so the period is $1.25/12 \simeq 0.104$ seconds to three significant figures.

(c) For a mass m and spring stiffness k, the period is given by $\tau = 2\pi\sqrt{m/k}$. From above, the spring stiffness was estimated to be $99\,\mathrm{N\,m^{-1}}$, and the mass is $0.03\,\mathrm{kg}$. Therefore an estimate of the period is $2\pi\sqrt{0.03/99} \simeq 0.109$ seconds, which compares well with the measured value.

This simple model of a ruler that is oscillating with a mass on its end gives a very good comparison between the observed period and that deduced from considering the equivalent spring stiffness of the ruler. One may detect a slight decrease in the amplitude of the oscillations as time goes on, and this will be the subject of the next unit.

This section concludes with some exercises involving springs and masses in different set-ups.

Note that a useful check when setting up equations of motion for oscillating systems is that all the individual terms in x and $\ddot{x}$ when taken onto one side have the same sign; this confirms that each spring does what is required of it to restore the system to equilibrium.

Exercise 17

A block of mass m lying on a frictionless surface oscillates horizontally and is attached to two springs: the one on the left has stiffness k and natural length $2l_0$; the one on the right has stiffness $3k$ and natural length l_0 (see Figure 32). The other ends of the springs are attached to fixed points aligned horizontally, a distance $5l_0$ apart. Assume that the block may be modelled as a particle.

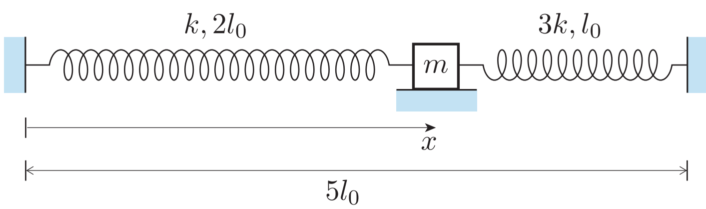


Figure 32

(a) Show that the distance x of the block from the left-hand end at time t during the oscillations satisfies the differential equation

$$m\ddot{x} + 4kx = 14kl_0.$$

(b) Hence find the distance from the left-hand end at which the block could remain in equilibrium.

(c) Find the period τ of the block's oscillations.

(d) Suppose that the block is initially (at $t = 0$) released from rest halfway between the two ends. Find an expression for $x(t)$.

(e) Does the block hit one of the ends in its subsequent motion?

Exercise 18

A block P of mass m is attached to three springs whose other ends are attached to fixed points Q, R and S. The point R is a distance $\frac{1}{2}l_0$ below Q, and the point S is a distance $4l_0$ below Q. The parameters of the three springs are given in Figure 33, which also illustrates the arrangement of the springs and the block.

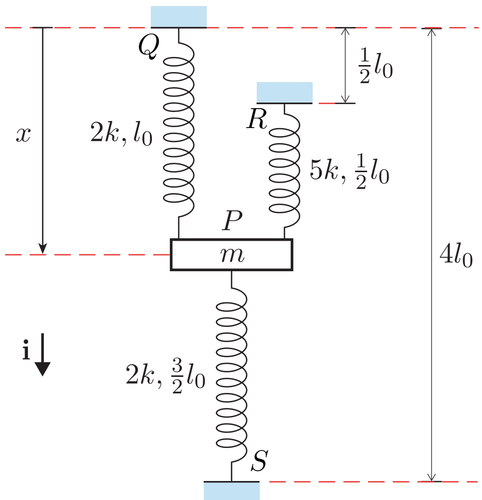


Figure 33

Model the block as a particle and the springs as model springs. Take the origin at Q, with the vertical distance of P from Q being x, so that the x-axis is as shown, and $\mathbf{i}$ is pointing downwards.

(a) Draw a force diagram indicating all the forces acting on the particle.

(b) Find the spring force for each spring.

(c) Derive the equation of motion of the particle.

(d) Find the position of equilibrium for the particle.

(e) Find the general solution of the differential equation that you found in part (c).

(f) The particle is initially released from rest at a distance $\frac{4}{3}l_0$ below Q. Determine the solution of the differential equation that satisfies these initial conditions.

(g) Write down the period and the amplitude of the oscillations of the particle during its subsequent motion.

(h) Draw a sketch of the graph of x against t for $t \geq 0$, clearly indicating the amplitude, period, starting position and average position of the block.

Figure 34 shows the uniform cross-section of a rectangular pontoon that floats on water. The position x of the upper surface of the pontoon is measured upwards from the water level as shown in Figure 34, which also shows the positive **i**-direction. The pontoon has height h, mass m and uniform horizontal cross-sectional area A; ρ is the density of water, and g is the magnitude of the acceleration due to gravity.

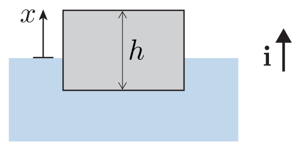


Figure 34 Pontoon on the water

Archimedes' principle states that if an object is wholly or partly immersed in a liquid, then the resulting buoyancy force on the object is directed vertically upwards and is equal in magnitude to the weight of liquid displaced by the object.

Figure 35 Forces acting on the pontoon

There are two forces acting on the pontoon: $\mathbf{W} = mg(-\mathbf{i})$, the weight of the pontoon, and $\mathbf{B}$, the buoyancy force (see Figure 35). In order to use Archimedes' principle to find the buoyancy force, first we need to find the volume of water displaced, which is $A(h-x)$. We then multiply this by ρg to obtain the magnitude of the buoyancy force; since the force is upwards,

$$\mathbf{B} = A\rho g(h-x)\mathbf{i}. \tag{16}$$

Note the similarity of the buoyancy force to the spring force $\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}}$.

Applying Newton's second law gives

$$m\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{B} = -mg\mathbf{i} + A\rho g(h-x)\mathbf{i}.$$

Resolving in the **i**-direction and taking the x terms onto the left-hand side gives

$$m\ddot{x} + A\rho g x = A\rho g h - mg. \tag{17}$$

According to this model, the 'bobbing up and down' of the pontoon is therefore a simple harmonic motion.

Exercise 19

(a) What is the range of x for which model (16) for the buoyancy force is applicable?

(b) Show that for a pontoon of given dimensions, this model predicts that the period τ of the oscillations is proportional to $\sqrt{m}$, and hence increases with the mass of the pontoon. Do you think that there is any limit to the period that could be attained, for a suitably massive pontoon?

---

3 Introduction to energy

In this section an alternative approach to solving a certain class of mechanics problems is introduced. This approach is based on the concept of *mechanical energy*. The application of this concept to solving mechanics problems is based on the constancy or *conservation* of mechanical energy – we discuss this in Section 4.

Here we introduce a method for solving mechanics problems for systems where the forces *depend only on the position* of the particle. Such forces are very common. For example, the force acting on a steel pin due to the attraction of a magnet depends on the distance between the pin and the magnet. In fact, the magnitude of this force increases as the pin is brought closer to the magnet. On the other hand, a dog tethered to a tree by a length of elastic rope feels a restraining force, once the elastic is taut, whose magnitude increases as the dog moves away from the tree. In either case, we can consider a one-dimensional situation and define the **i**-direction to be the direction of the force, and then we have

$$\mathbf{F} = F\mathbf{i} = F(x)\,\mathbf{i},$$

where we have written $F(x)$ in the last expression to emphasise that the **i**-component of the force depends on x alone. We assume that the form of the function $F(x)$ is known, and use this knowledge to answer questions about the motion. It is possible here for $F(x)$ to be a constant function, so in saying that 'F depends only on position', we really mean that F depends *at most* on position, that is, it does not depend directly on other variables of the motion, such as velocity, acceleration or time.

For example, the weight and tension force in a spring are forces that depend on position only. It is tempting to think that friction is a constant, in the same way that weight is, but this is not so. Friction **F** opposes the motion in the direction opposite to the direction in which the particle is travelling, thus can be written in terms of the velocity **v** of the particle as

$$\mathbf{F} = -|\mathbf{F}|\,\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Air resistance and friction are examples of dissipative forces that cause a loss of mechanical energy to other forms of energy such as heat or sound.

3.1 Forms of energy

Consider a particle of mass m acted on by a total force $\mathbf{F}(x)$ that is dependent on x alone. Choose **i** to be a unit vector in the direction of the force, so $\mathbf{F}(x) = F(x)\,\mathbf{i}$ (where the distinction between scalar and vector notation is important, because $F(x)$ can be positive or negative).

It is natural to begin by writing down Newton's second law,

$$\mathbf{F} = m\mathbf{a},$$

where as usual **a** is the acceleration of the particle. The motion is one-dimensional and we have $\mathbf{a} = \ddot{x}\mathbf{i}$, so this equation can be resolved in the **i**-direction to obtain

$$F(x) = m\ddot{x}. \tag{18}$$

This is a differential equation in which the independent variable is t. You have seen in Unit 3 equations (9) how $\ddot{x}$ can be transformed into a derivative with x as the independent variable, so we can write

$$\ddot{x} = v\,\frac{dv}{dx}, \tag{19}$$

where $v = \dot{x}$ is the component of velocity in the **i**-direction. A derivative with respect to t has been transformed into a derivative with respect to x. Using this transformation on equation (18), and integrating both sides with respect to x from the initial position with $x = x_0$, $v = v_0$ to a general position x, v gives

$$\int_{x_0}^{x} F(x)\,dx = m\int_{x_0}^{x} v\,\frac{dv}{dx}\,dx = m\int_{v_0}^{v} v\,dv = m\left[\tfrac{1}{2}v^2\right]_{v_0}^{v}, \tag{20}$$

where the limits for the right-hand integral have been changed to those of the variable being integrated.

Let $U(x) = -\int F(x)\,dx$, and substitute this into equation (20) to give

The reason for the negative sign will be explained soon.

$$-\big[U(x)\big]_{x_0}^{x} = m\left[\tfrac{1}{2}v^2\right]_{v_0}^{v},$$

which on rearrangement becomes

$$U(x_0) + \tfrac{1}{2}mv_0^2 = U(x) + \tfrac{1}{2}mv^2. \tag{21}$$

This equation has a useful symmetry: both sides of the equation involve the expression $\frac{1}{2}mv^2 + U(x)$; the only difference is that the right-hand side involves v and x at a general position x, and the left-hand side involves the initial values of those variables.

Each of these quantities, $\frac{1}{2}mv^2$ and $U(x)$, has a special significance, and we will consider them individually.

Kinetic energy

The term $\frac{1}{2}mv^2$ is dependent on speed (note that the sign of v is immaterial since it is squared – the direction of motion is not important), and it is given the name *kinetic energy*.

Kinetic energy

The **kinetic energy** T of a particle of mass m moving with component of velocity v (where $v = \dot{x}$) is given by

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{x}^2. \tag{22}$$

It is conventional to use T to denote the kinetic energy because E is usually reserved for the total energy, as you will see later.

The word *kinetic* in the term *kinetic energy* emphasises that the energy is due to the motion of the object.

The SI unit for energy is the *joule* (J). So a particle of mass 4 kg moving with speed $5\,\mathrm{m\,s^{-1}}$ has kinetic energy $\frac{1}{2} \times 4 \times 5^2 = 50$ joules.

$1\,\mathrm{J} = 1\,\mathrm{kg\,m^2\,s^{-2}}$.

Potential energy

Now consider the term $U(x)$ in equation (21). We call this term the potential energy *function* to emphasise that it is dependent on the position of the particle.

It is conventional to use U to denote the potential energy.

Potential energy

For a particle under the influence of a force $\mathbf{F} = F(x)\,\mathbf{i}$ that is either constant or depends only on the position $\mathbf{r} = x\mathbf{i}$, the **potential energy function** $U(x)$ is

$$U(x) = -\int F(x)\,dx, \tag{23}$$

or equivalently,

$$F(x) = -\frac{dU}{dx}. \tag{24}$$

By definition, $U(x)$ will incorporate a constant of integration, so it will not be unique. However, note that it occurs on both sides of equation (21), so the same constant of integration will occur on both sides.

Exercise 20

Check that potential energy has the same dimensions as kinetic energy.

Total energy

Consider again equation (21), repeated here:

$$U(x_0) + \tfrac{1}{2}mv_0^2 = U(x) + \tfrac{1}{2}mv^2.$$

The left-hand side of this equation is a constant, which we call E, the *total mechanical energy* of the particle.

Total mechanical energy

The **total mechanical energy** of a particle is given by the sum of its kinetic and potential energies, that is,

$$E = \tfrac{1}{2}mv^2 + U(x). \tag{25}$$

If there is more than one force, then the total potential energy $U(x)$ is the sum of the potential energy functions $U_1(x), U_2(x), \ldots, U_n(x)$ due to each of the n forces.

The potential energy function is defined in equation (23) as an indefinite integral, so it is defined only to within a constant of integration. In practice, however, the value of this constant is unimportant, because it can be absorbed into the constant E appearing in equation (25). The constant of integration in the definition of the potential energy function can therefore be chosen to have any convenient value. In other words, we can choose the value of the potential energy function $U(x)$ to be zero at any convenient point. This point is called the **datum** of the potential energy function. Thus if $x = x_0$ is the chosen datum, then $U(x_0) = 0$.

You saw a similar situation in Unit 1, where the integrating factor for a first-order linear differential equation was defined in terms of an indefinite integral.

3.2 Calculating potential energy functions

For the remainder of this section, we ask you to concentrate on the process of finding the potential energy function $U(x)$ for given forces.

Exercise 21

Using an x-axis pointing vertically upwards (see Figure 36) and equation (23), verify that the potential energy function for the force due to gravity acting on a particle of mass m is

$$U(x) = mgx,$$

where the datum O for this function is taken to be the origin $x = 0$.

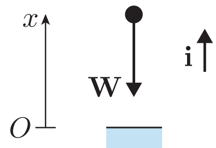


Figure 36

Exercise 22

Consider the motion of a particle of mass m along a straight frictionless track that is at an angle θ to the horizontal. Using an x-axis pointing down the slope, verify that the potential energy function for the total force acting on the particle is

$$U(x) = -mgx\sin\theta,$$

where the datum is taken to be the origin $x = 0$.

In Exercise 21, you showed that for a suitable choice of datum ($x = 0$), the potential energy function is $U(x) = mgx$ if the x-axis is directed vertically upwards. In Exercise 22, you showed that for a suitable choice of datum ($x = 0$), the potential energy function is $U(x) = -mgx\sin\theta$ if the x-axis is directed down the slope. It can be shown that it is always the case that for positions above the datum the potential energy function is positive, and for those below the datum the potential energy function is negative.

If the x-axis in Exercise 21 were pointing in the opposite direction, that is, vertically downwards (see Figure 37), then the potential energy function would be $U(x) = -mgx$. Now if the particle is above the datum, then the coordinate x is negative, so $U(x)$ is again positive. Conversely, if the particle is below the datum, then x is positive, so $U(x)$ is negative. These signs are the same as for the potential energy function with an upwards pointing x-axis. This leads to the following memorable result for the potential energy function due to gravity, henceforth called the *gravitational potential energy*.

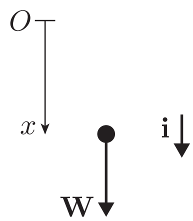


Figure 37 Reversing the direction of the x-axis

Gravitational potential energy

The **gravitational potential energy** of a particle of mass m is given by

$$U = mg \times \text{height above datum}, \tag{26}$$

where the height is measured vertically from some chosen datum, and g is the magnitude of the acceleration due to gravity.

The result stated for gravitational potential energy applies even when the direction of motion of the particle is not vertical. For example, in Exercise 22 you showed that for a particle moving on a slope at an angle θ to the horizontal, with the x-axis pointing down the slope, the potential energy function is $U(x) = -mgx\sin\theta$. Since the vertical height of the particle above the datum is $-x\sin\theta$, we again have $U(x) = mg \times \text{height}$.

Exercise 23

In equation (16) the buoyancy force was derived to be

$$\mathbf{B} = A\rho g(h - x)\mathbf{i},$$

where x is the height of the top of a pontoon above the water level, h is the height of the pontoon, and $A\rho g$ is a constant.

Find the potential energy function for the buoyancy force, taking the datum to be the level when the pontoon is just out of the water, that is, $x = h$.

Exercise 24

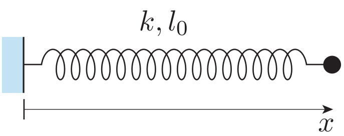


Figure 38

Figure 38 shows a model spring of stiffness k and natural length l_0 whose total length is x.

Verify that the potential energy function for the force exerted by the spring is

$$U(x) = \tfrac{1}{2}k(x - l_0)^2,$$

where the datum is taken to be the point of zero deformation, $x = l_0$.

As before, 'deformation' refers to either extension or compression from the natural length, as appropriate.

For the model spring force in Exercise 24, you showed the following, which we state for future reference.

Potential energy stored in a spring

The potential energy stored in a model spring is

$$U = \tfrac{1}{2} \times \text{stiffness} \times (\text{deformation})^2, \tag{27}$$

where the datum is chosen to be at the natural length of the spring.

You could use a different datum for the potential energy of a spring, but this is the most natural choice.

We now investigate what happens to the potential energy function when the force acting on a particle is the sum of two forces.

Exercise 25

Find the potential energy function for each of the following forces (where a and b are constants). State, in each case, the datum that you use.

(a) $F(x)\,\mathbf{i} = -ax^2\mathbf{i}$

(b) $F(x)\,\mathbf{i} = bx^{-2}\mathbf{i}$ $(x > 0)$

Exercise 26

(a) Suppose that the potential energy functions $U_1(x)$ and $U_2(x)$ correspond to the forces $F_1(x)\,\mathbf{i}$ and $F_2(x)\,\mathbf{i}$, respectively. Show that a potential energy function corresponding to the force $F(x)\,\mathbf{i} = F_1(x)\,\mathbf{i} + F_2(x)\,\mathbf{i}$ is

$$U(x) = U_1(x) + U_2(x).$$

(b) Using the result of part (a) and your answers to Exercise 25, write down a potential energy function for the force

$$F(x)\,\mathbf{i} = (-ax^2 + bx^{-2})\mathbf{i} \quad (x > 0),$$

where a and b are constants.

Exercise 26 raises the question of what occurs when two potential energy functions that have different datum points are added together. In this case, it is possible to set up a fresh datum at any desired point for the combined potential energy function, by adding an appropriate constant to the expression for $U(x)$, since the potential energy is defined in terms of an indefinite integral (equation (23)). Specifying a datum amounts to pinning down a value for the arbitrary constant in the indefinite integral, but while this may sometimes be convenient, it is not essential to do it. Indeed, it is not necessary for the potential energy to take the value zero at any point.

Note that $U(x) + C$ is also a potential energy function, for any constant C.

4 Energy conservation

In the previous section we concentrated mainly on the relationship between the potential energy function $U(x)$ and the force component $F(x)$. Now we look at how to use energy conservation to solve problems.

The fact that the total mechanical energy E is constant throughout the motion of a particle that is subject to forces that depend only on position means that the particle *does not lose or gain mechanical energy*.

Law of conservation of mechanical energy

If the force on a particle depends only on the particle's position, then the total mechanical energy of the particle is constant. In other words,

$$E = \tfrac{1}{2}mv^2 + U(x), \tag{28}$$

is a constant.

To see the reasonableness of this, consider the familiar example of an object falling vertically downwards under gravity, ignoring air resistance. As the object speeds up (and gains kinetic energy), it loses height (and loses potential energy). On the other hand, if the object is rising (gaining potential energy), then it is also slowing down (losing kinetic energy). So conservation of mechanical energy says that a gain in kinetic energy must be accompanied by a loss of potential energy, and vice versa.

Energy conservation is useful in solving problems that involve only the speed and position of objects, rather than, say, the position of the object at specific times. This is illustrated in the following example and exercise.

Example 7

A stone of mass m is thrown vertically upwards from ground level with initial speed $15\,\mathrm{m\,s^{-1}}$. Assume that gravity is the only force acting on the stone.

(a) Find the speed of the stone when its height is 5 m.

(b) Find the maximum height of the stone.

Solution

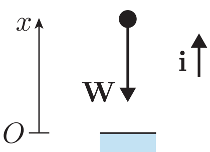


Figure 39 Stone falling under gravity

(a) Choose an x-axis pointing upwards, with the origin (and datum) at the point of projection (see Figure 39). The only force acting on the stone is the weight due to gravity, which is constant, so equation (25) applies. Using equation (26), the potential energy is mgx, thus the total energy is given by

$$E = \tfrac{1}{2}mv^2 + mgx.$$

Initially, $v = 15$ when $x = 0$, so

$$E = \tfrac{1}{2}m \times 15^2 + m \times 9.81 \times 0 = 112.5m.$$

Hence throughout the motion, since energy is conserved, we have

$$\tfrac{1}{2}v^2 + gx = 112.5. \tag{29}$$

Notice that the m terms have cancelled, so this equation applies to a stone of any mass; all stones will follow the same trajectory.

When $x = 5$, this gives

$$\tfrac{1}{2}v^2 + 9.81 \times 5 = 112.5,$$

hence

$$v^2 = 2 \times (112.5 - 9.81 \times 5) = 126.9.$$

So the speed of the stone at height $x = 5$ is

$$|v| = \sqrt{126.9} \simeq 11.3 \quad \text{to 3 s.f.}$$

This speed of about $11.3\,\mathrm{m\,s^{-1}}$ is the same whether the stone is moving up or down. The sign of the velocity will depend on the direction of motion.

(b) At maximum height, we have $v = 0$. Substituting $v = 0$ into equation (29) produces

$$gx = 112.5$$

or

$$x = 112.5/9.81 \simeq 11.5 \quad \text{to 3 s.f.},$$

so the maximum height attained by the stone is approximately 11.5 m.

Exercise 27

(a) A marble, initially at rest, is dropped from the Clifton Suspension Bridge and falls into the River Avon, 77 m below (see Figure 40). Assuming that the only force acting on the marble is the force of gravity, use energy conservation to estimate the speed of the marble just before it hits the water. How far has it fallen when its speed reaches $20\,\mathrm{m\,s^{-1}}$?

(Choose a downward-pointing x-axis with origin at the point of release, and take this origin as the datum of potential energy. Take care with the sign of the potential energy function.)

(b) Why can the law of conservation of mechanical energy not be used if air resistance is taken into account in the situation of part (a)?

You considered this situation in Unit 3.

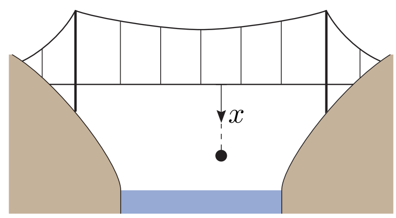


Figure 40 A marble dropped from the Clifton Suspension Bridge

Models involving friction or air resistance do not exhibit conservation of mechanical energy. In fact, energy is conserved, but the mechanical energy lost by a particle is converted to another form of energy such as heat or sound.

4.1 General method for solving problems using energy principles

Example 7 is typical of many problems that can be solved by using the law of conservation of mechanical energy. There are two reasons why this problem is particularly suitable. First, it involves a force, namely gravity, that depends only on position (in this case, of course, the force is a constant function of position). This is crucial, because the law of conservation of mechanical energy applies only to such forces. Second, the problem asks for a relationship between position and speed, which is ideal because the conservation law immediately gives the desired relationship.

Procedure 2 Applying conservation of energy

This procedure may be applied to a mechanics problem involving the one-dimensional motion of a particle in which:

- the total force depends only on the position of the particle (or is constant)
- the question to be answered refers to a relationship between the position and the speed of the particle.

Carry out the following steps.

◀ Draw picture ▶

1. Drawing a picture is always the first step in solving a mechanics problem.

◀ State assumptions ▶

2. State any assumptions used, such as ignoring friction or air resistance.

◀ Choose datum ▶

3. The next step is to choose a datum. For gravitational potential energy it is usual to choose the origin. For a spring, the datum is usually chosen to be the point of zero deformation. (For other forces this step could be deferred until the potential energy function is known; choose the datum to simplify the expression.)

◀ Find potential energy ▶

4. If gravity or model springs are present, simply add together the corresponding potential energy functions as given in equations (26) and (27). Otherwise, identify the x-component $F(x)$ of the total force acting on the particle, then apply the potential energy function definition in equation (23).

◀ Find E ▶

5. Use equation (28) and the initial conditions to calculate the value of the constant E, the total mechanical energy of the system.

◀ Find x or v ▶

6. Solve the resulting version of equation (28), either for x (at a specified value of v) or for v (at a specified value of x).

◀ Interpret solution ▶

7. Interpret the solution in terms of the original problem.

The steps outlined in Procedure 2 are used in the following example.

Example 8

A toy manufacturer is testing a trampoline and wants to estimate the largest mass of child that can safely use it. An experiment is devised in which a sandbag of mass m is dropped from a height h, and the maximum deformation d of the trampoline is measured.

(a) Model the trampoline as a model spring of stiffness k. Find an equation relating the given parameters and g. You may assume that no energy is lost in the impact.

A better approach would be to collect more data so that a better estimate could be made and the model could be tested.

(b) It is observed that when a 20 kg sandbag is dropped from a height of 1 m above the trampoline, the maximum deformation of the trampoline is 25 cm. Use these data to calculate the stiffness of the model spring.

(c) If the maximum safe deformation is 30 cm, estimate the maximum mass that may be dropped on the trampoline from a height of 1 m.

Solution

(a) Figure 41 shows pictures of the sandbag at the top and bottom of its motion. As shown, we choose an x-axis pointing upwards, with origin at the height of the undeformed trampoline. ◀ Draw picture ▶

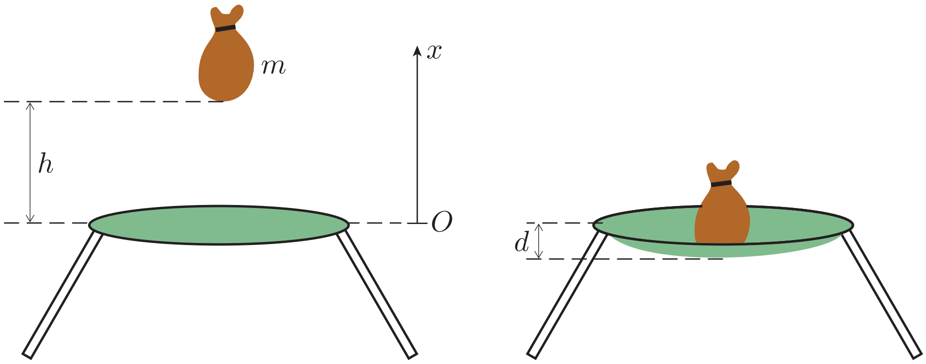


Figure 41 A sandbag being dropped onto a trampoline

We model the sandbag as a particle, which is subject only to the gravitational force in flight. We model the trampoline as a model spring. ◀ State assumptions ▶

Take the origin as the datum for the gravitational potential energy. Then the potential energy function due to the weight is ◀ Choose datum ▶ ◀ Find potential energy ▶

$$U_{\text{grav}}(x) = mgx.$$

Take the origin as the datum for the spring potential energy function (as this corresponds to the point of zero deformation of the spring). The trampoline exerts a force only when the sandbag is in contact with it, so the potential energy is stored in the spring only if $x < 0$. So the potential energy stored in the spring is

$$U_{\text{spring}}(x) = \begin{cases} 0, & x \geq 0, \\ \frac{1}{2}kx^2, & x < 0. \end{cases}$$

To find the total potential energy, we add together the two contributions:

$$U(x) = U_{\text{spring}}(x) + U_{\text{grav}}(x) = \begin{cases} mgx, & x \geq 0, \\ \frac{1}{2}kx^2 + mgx, & x < 0. \end{cases}$$

Initially, the sandbag is at rest ($v = 0$) at a height $x = h > 0$ above the trampoline, so using equation (28) gives ◀ Find E ▶

$$E = \tfrac{1}{2}m0^2 + mgh = mgh.$$

At the bottom of the motion, the sandbag is again at rest, at the point $x = -d$ (where $d > 0$ is the maximum deformation of the trampoline). At this point we have ◀ Find x or v ▶

$$E = \tfrac{1}{2}m0^2 + U(-d) = \tfrac{1}{2}kd^2 - mgd.$$

This gives the desired relationship

$$mgh = \tfrac{1}{2}kd^2 - mgd. \tag{30}$$

◀ Interpret solution ▶

(b) To calculate k, rearrange equation (30) to make k the subject, then substitute in the known values:

$$k = \frac{2mg(h+d)}{d^2} = \frac{2 \times 20 \times 9.81 \times (1+0.25)}{0.25^2} = 7848.$$

So the spring has stiffness approximately $8000\,\mathrm{N\,m^{-1}}$.

(c) To calculate the mass that corresponds to the maximum deformation of the trampoline, rearrange equation (30) to make m the subject, then substitute in the known values:

$$m = \frac{kd^2}{2g(h+d)} = \frac{7848 \times 0.3^2}{2 \times 9.81 \times (1+0.3)} \simeq 27.7 \quad \text{to 3 s.f.}$$

So the maximum safe mass is approximately 27 kg.

Note that here we round *down* rather than to the nearest integer (because 28 kg would be *above* the safe maximum mass).

Now try applying the law of conservation of energy yourself by attempting the following exercises.

Exercise 28

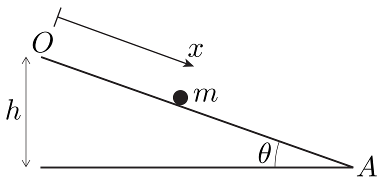


Figure 42 A mass moving down a slope

A particle of mass m moves without friction down a slope inclined at an angle θ to the horizontal (see Figure 42). The particle starts from rest at the point O, and slides down to the point A, which is a vertical distance h below O. The displacement of the particle from O, measured down the slope, is denoted by x. The point A is to be taken as the datum for the potential energy function.

(a) Show that the total mechanical energy of the particle can be written as

$$E = \tfrac{1}{2}mv^2 + mg(h - x\sin\theta),$$

where v is the speed of the particle.

(b) Obtain, in terms of h, an expression for the speed of the particle at A, and comment on your result.

(c) Apply the result from part (b) to the specific case of a crate of empty bottles that slides 2 m down a smooth ramp set at an angle of $\pi/6$ radians to the horizontal, starting from rest.

This problem was considered in Example 5 of Unit 3.

Exercise 29

A particle of mass 0.5 kg moves along a horizontal straight frictionless track. The particle is attached to an end of a model spring of stiffness $2\,\mathrm{N\,m^{-1}}$, the other end of the spring being fixed (see Figure 43).

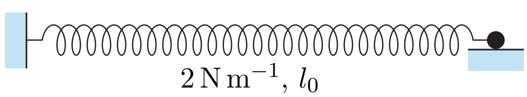


Figure 43

Initially, the extension of the spring is 2 m and the particle's speed is $3\,\mathrm{m\,s^{-1}}$, directed towards the centre of the spring. What is the particle's maximum speed during the subsequent motion, and at what positions is it momentarily at rest? (Assume that the natural length l_0 of the model spring is greater than 2.5 m.)

Exercise 30

A particle of mass m is attached to the lower end of a model spring of natural length l_0 and stiffness $k = 10mg/l_0$, where g is the magnitude of the acceleration due to gravity. The spring is hung vertically, with its upper end fixed (see Figure 44).

The particle is pulled vertically downwards until the spring's length is $\frac{5}{4}l_0$ and then released from rest. By using the law of conservation of energy, find the speed of the particle when the spring is of length l_0 in the subsequent motion.

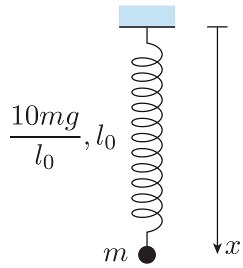


Figure 44

4.2 Energy in oscillating systems

As you have seen in the above example and exercises, the law of conservation of mechanical energy,

$$E = \tfrac{1}{2}mv^2 + U(x),$$

provides a convenient method for finding the relationship between the velocity $v\mathbf{i}$ and the position $x\mathbf{i}$ whenever the force acting on a particle is a function of position. Since the term $\frac{1}{2}mv^2$ can never be negative, the same must be true of the quantity $E - U(x)$, which means that motion is possible only in regions for which $E - U(x) \geq 0$. Also, the particle will be momentarily at rest ($v = 0$) at points for which $E - U(x) = 0$. At such points, called **turning points**, the particle changes its direction of motion; this is illustrated in Figure 45. A particle moving under the action of a force that gives rise to the illustrated potential energy function $U(x)$, and with total mechanical energy E, will oscillate backwards and forwards between the two turning points A and B, changing its direction of motion each time it reaches one of these points.

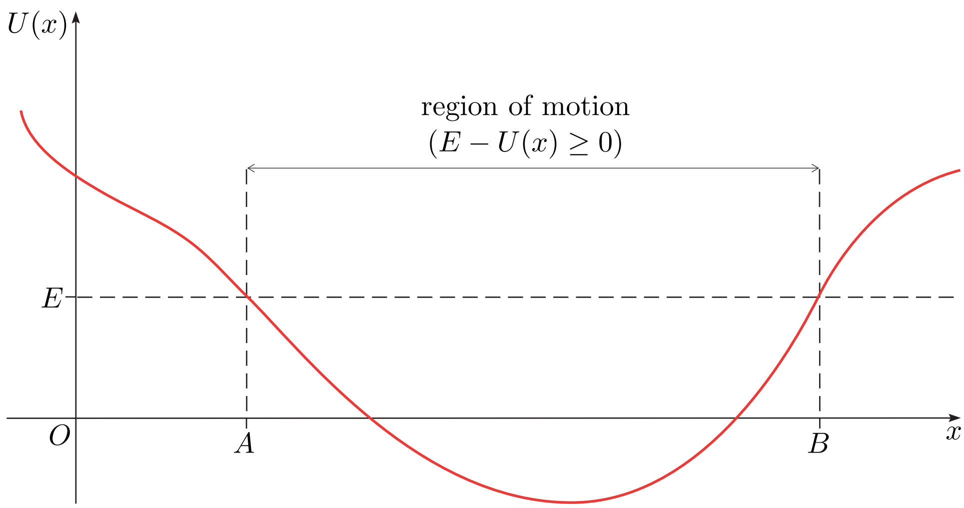


Figure 45 Turning points

Exercise 31

Throughout this question, all quantities are measured in the appropriate SI units.

A particle of mass $m = 2$ moves under the influence of a force that has x-component $F(x) = 2 - 2x$.

(a) Find a potential energy function $U(x)$ for this force.

(b) If the particle is released from rest at $x = -1$, find the total mechanical energy of the particle in the subsequent motion.

(c) Use the law of conservation of mechanical energy to find an expression for the velocity $v\mathbf{i}$ of the particle at position $x\mathbf{i}$.

(d) Find the region of motion of the particle, and its speed at the midpoint of this region.

We now show how energy methods can be used to find solutions for oscillating systems. The process is the reverse of that used in Section 3 to introduce the concept of energy. This may seem like a circular argument, but it is sometimes easier to solve problems with oscillations using energy methods.

Start with the equation for total energy,

$$E = \tfrac{1}{2}mv^2 + U(x).$$

Differentiate this with respect to t to obtain

$$\frac{dE}{dt} = \tfrac{1}{2}m\,\frac{d}{dt}\big(v^2\big) + \frac{d}{dt}(U(x)).$$

Use the chain rule on the derivatives on the right-hand side:

$$\frac{dE}{dt} = \tfrac{1}{2}m\,\frac{d}{dv}\big(v^2\big)\,\frac{dv}{dt} + \frac{d}{dx}(U(x))\,\frac{dx}{dt} = \tfrac{1}{2}m2v\,\frac{dv}{dt} + U'(x)\,v.$$

If the system is such that energy is conserved (no dissipative forces), then E is a constant, so $dE/dt = 0$. Divide by v to obtain

We may assume that $v \neq 0$ in general.

$$0 = m\,\frac{dv}{dt} + U'(x)$$

or

$$0 = m\ddot{x} + U'(x). \tag{31}$$

One consequence of equation (31) is that it makes it easy to find positions of equilibrium. (The system is in equilibrium if the particle remains at rest.) At equilibrium $\ddot{x} = 0$, so the solution of $U'(x) = 0$ will give the values of x at which a system is in equilibrium.

Exercise 32

Find the positions of equilibrium for the potential energy function $U(x) = x^3 - 3x$.

The plot of the potential energy function in Figure 46 shows that the positions of equilibrium found in Exercise 32 occur at a local maximum or local minimum of $U(x)$. The nature of the equilibrium at each of these points is different.

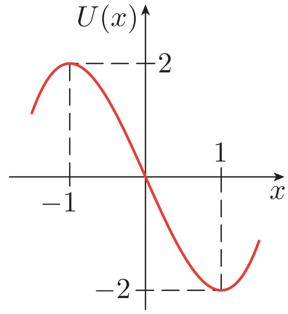


Figure 46 Positions of equilibrium

Consider an equilibrium point $x_{\rm eq}$ of $U(x)$ so that $U'(x_{\rm eq}) = 0$. Expand $U(x)$ as a Taylor series about $x_{\rm eq}$ to give

$$U(x) = U(x_{\rm eq}) + U'(x_{\rm eq})(x - x_{\rm eq}) + \frac{1}{2!}U''(x_{\rm eq})(x - x_{\rm eq})^2 + \frac{1}{3!}U'''(x_{\rm eq})(x - x_{\rm eq})^3 + \cdots .$$

Since $U'(x_{\rm eq}) = 0$, the Taylor series simplifies to

$$U(x) = U(x_{\rm eq}) + \frac{1}{2!}U''(x_{\rm eq})(x - x_{\rm eq})^2 + \frac{1}{3!}U'''(x_{\rm eq})(x - x_{\rm eq})^3 + \cdots .$$

Now we make the approximation that $x - x_{\rm eq}$ is small, so the second term of the above series will dominate the remainder if $U''(x_{\rm eq}) \neq 0$; if $U''(x_{\rm eq}) = 0$, then the quadratic term disappears and the following discussion does not hold.

The case where $U''(x_{\rm eq}) = 0$ is not considered further.

So the potential energy is approximately a quadratic:

$$U(x) \simeq U(x_{\rm eq}) + \tfrac{1}{2}U''(x_{\rm eq})\,(x - x_{\rm eq})^2.$$

Since we require $U'(x)$ in equation (31), we differentiate to obtain

$$U'(x) \simeq U''(x_{\rm eq})\,(x - x_{\rm eq}).$$

From the above arguments it is sensible to construct a model of the behaviour of the small oscillations where the potential energy is exactly quadratic, and then

$$U'(x) = U''(x_{\rm eq})\,(x - x_{\rm eq}).$$

Using this in equation (31) gives

$$0 = m\ddot{x} + U''(x_{\rm eq})\,(x - x_{\rm eq}),$$

which on rearrangement becomes

$$m\ddot{x} + U''(x_{\rm eq})\,x = U''(x_{\rm eq})\,x_{\rm eq}.$$

If $U''(x_{\rm eq}) > 0$, then the motion is simple harmonic around $x_{\rm eq}$, so the motion is simple harmonic with angular frequency $\omega = \sqrt{U''(x_{\rm eq})/m}$. You will appreciate from this general argument that this kind of motion occurs frequently. Hence we give a mechanical system of this kind a name: it is called a **harmonic oscillator**.

Note that for the angular frequency to be real, we must have $U''(x_{\rm eq}) > 0$, which *is* the case if $U(x)$ has a local minimum at $x_{\rm eq}$.

If $U''(x_{\rm eq}) < 0$, then the solution is

$$x = Ae^{\lambda t} + Be^{-\lambda t} + x_{\rm eq}$$

where A and B are arbitrary constants, and $\lambda = \sqrt{U''(x_{\rm eq})/m}$. Unless $A = 0$, which is extremely unlikely in practice, x will increase exponentially.

An example of the two types of equilibrium is provided by a ball in a concave bowl, which will oscillate about the bottom point, and a ball on top of a convex bowl, where any slight departure from equilibrium leads to catastrophic departure from equilibrium.

Equilibrium for a potential function

For a system with potential energy function $U(x)$, positions of equilibrium x_{eq} occur whenever $U'(x_{\mathrm{eq}}) = 0$.

If $U''(x_{\mathrm{eq}}) > 0$, then for small oscillations the motion is simple harmonic with angular frequency $\omega = \sqrt{U''(x_{\mathrm{eq}})/m}$.

If $U''(x_{\mathrm{eq}}) < 0$, then the equilibrium point is unstable.

Of course, when $\mathbf{F}$ is a non-zero constant force, $U'(x) = -F(x)$ is never zero, and there are no equilibrium points.

For the potential energy function defined in Exercise 32, the equilibrium point at $x = 1$ (a minimum) will have oscillations around it, and the one at $x = -1$ (a maximum) will not. For this case $U''(-1) = -6$ and the equation of motion is $m\ddot{x} - 6x = -6x_{\mathrm{eq}}$, for which the general solution is $x = Be^{\sqrt{6/m}} + Ce^{-\sqrt{6/m}} + x_{\mathrm{eq}}$, showing that the point is unstable.

Exercise 33

All quantities here are measured in the appropriate SI units.

A particle of mass $m = 3$ moves along a straight line and experiences a force that repels it from the origin at position $x = 0$. This force has x-component $F(x) = 2/x^2$ for $x > 0$. Initially, the particle is moving towards the origin, being at position $x\mathbf{i} = 10\mathbf{i}$ with velocity $v\mathbf{i} = -\mathbf{i}$.

(a) Find a potential energy function for this force for $x > 0$, stating the datum that you use.

(b) Find the total mechanical energy of the particle.

(c) What is the closest point to the origin that is reached by the particle?

(d) Sketch the graph of the potential energy function, and indicate the region of motion. Describe the motion of the particle.

(e) Use your graph to determine whether there are any equilibrium points for this potential energy function.

Exercise 34

This is not a realistic model, since the truck would bounce off such buffers.

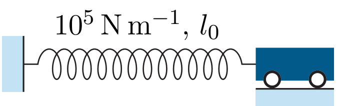


Figure 47

The system in Figure 47 can be used to model the motion of a railway truck while in contact with buffers. The mass of the truck is 2000 kg, and the stiffness of the (model) buffer spring is $10^5\,\mathrm{N\,m^{-1}}$. If the truck comes to rest at a position where the spring is compressed by 0.1 m, find the speed of the truck when it first came into contact with the buffers. (You should neglect friction, and assume that the spring has its natural length when the truck first touches the buffers.)

Exercise 35

This question analyses the same system of springs and a mass that was used in Exercise 18, but now using energy methods.

A block P of mass m is attached to three springs whose other ends are attached to fixed points Q, R and S. The point R is a distance $\frac{1}{2}l_0$ below Q, and the point S is a distance $4l_0$ below Q. The parameters of the three springs are given in Figure 48, which also illustrates the arrangement of the springs and the block.

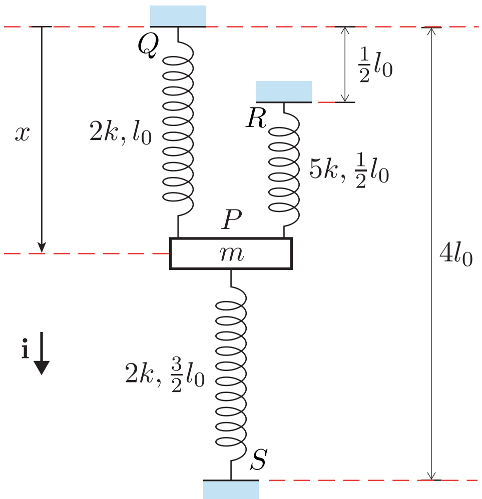


Figure 48

Model the block as a particle and the springs as model springs. Take the origin at Q, with the vertical distance of P from Q being x, so that the x-axis is as shown, and $\mathbf{i}$ is pointing downwards.

(a) Choose a datum and write down the gravitational potential energy of particle P at a general point of its motion.

(b) Write down the kinetic energy of particle P at a general point of its motion.

(c) Determine the potential energy stored in each spring at a general point of its motion.

(d) Write down an equation representing the total mechanical energy for the system at a general point of its motion. Use equation (31) to verify that your answer is equivalent to the equation of motion derived in Exercise 18(c).

Exercise 36

This question concerns a system with potential energy function given by

$$U(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2$$

whose graph is sketched in Figure 49.

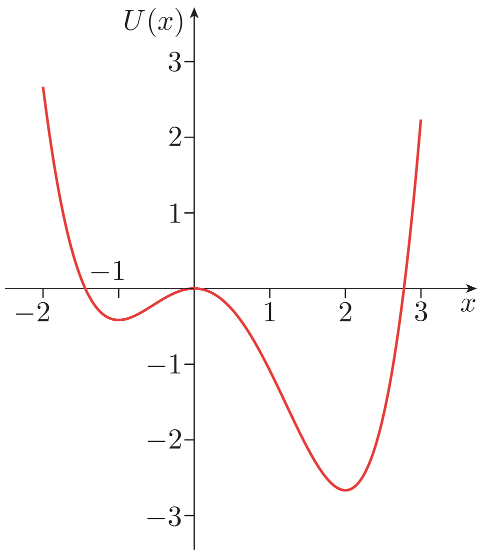


Figure 49

(a) Write down, in the form $F(x)\,\mathbf{i}$, the force that gives rise to this potential function.

(b) The total energy of the system is a constant E. For each of the following values of E, state, with a reason, a range (or ranges) accurate to one decimal place of x-values (if any) that could represent a motion of the system.

- $E = -3$
- $E = -2$
- $E = -0.25$
- $E = 1$

(c) Find the equilibrium points for this potential energy function, and state when motion around the equilibrium point will be oscillatory.

Learning outcomes

After studying this unit, you should be able to:

- apply the force law (Hooke's law) for a model spring in statics and dynamics problems
- find the equilibrium position of a particle that is acted on by model springs and gravity
- derive an equation of motion for a system involving model springs and gravity
- understand the basic features of simple harmonic motion, and use the terminology associated with it
- find the amplitude, phase angle, angular frequency and period of a system undergoing simple harmonic motion
- understand and explain the concepts of kinetic energy, potential energy and total mechanical energy
- find the kinetic energy of a particle
- find the gravitational potential energy of a particle
- find the energy stored in a spring
- apply, in appropriate circumstances, the law of conservation of mechanical energy to solve simple problems in mechanics
- identify the turning points and region of motion for a particle with given potential energy and total mechanical energy.

Solutions to exercises

Solution to Exercise 1

(a) If $d = 0.0025$, then equation (1) gives

$$w(d) = -0.024 + 99.044 \times 0.0025 = 0.224 \quad \text{to 3 d.p.}$$

The magnitude of the weight on the ruler is approximately 0.22 N when the displacement is 2.5 mm.

(b) If $w(d) = 0.67$, then equation (1) gives

$$0.67 = -0.024 + 99.044 \times d.$$

Solving for d, we obtain

$$d = \frac{0.67 + 0.024}{99.044} \simeq 0.007 \quad \text{to 3 d.p.,}$$

so the displacement is 7 mm.

(c) If $w(d) = 60$, then equation (1) gives

$$60 = -0.024 + 99.044 \times d.$$

Solving for d, we obtain

$$d = \frac{60 + 0.024}{99.044} \simeq 0.606 \quad \text{to 3 d.p.,}$$

so the mathematical model predicts a displacement of 0.61 m to two decimal places. (In reality this is impossible, since the ruler itself is only 0.3 m long. Any solution of the mathematical model must be reviewed in the light of its validity to represent the actual situation.)

(d) If $d = -0.0025$, then equation (1) gives

$$w(d) = -0.024 + 99.044 \times (-0.0025) = -0.272 \quad \text{to 3 d.p.}$$

The magnitude of the weight on the ruler is about 0.27 N when the displacement is -2.5 mm; however, the negative sign indicates that the force applied to the ruler is upwards. In this particular case the ruler can be bent upwards as well as downwards, so it is practically possible; in other situations it may not be possible to have a negative displacement.

Solution to Exercise 2

When $l = 0.35$, the spring is extended by an amount $e = 0.05$, so the magnitude of the spring force is $0.05 \times 200 = 10$ (newtons). Since the spring is in tension, the force on the object at the end is directed towards the centre of the spring.

When $l = 0.2$, the spring is compressed by an amount $c = 0.1$, so the magnitude of the spring force is $0.1 \times 200 = 20$ (newtons). Since the spring is in compression, the force on the object at the end is directed away from the centre of the spring.

Solution to Exercise 3

In each case we have $k = 200$ and $l_0 = 0.3$. Take $\widehat{\mathbf{s}}$ to be a unit vector in the direction from the object towards the centre of the spring.

With $l = 0.35$, the force acting on the object is

$$\mathbf{H} = 200(0.35 - 0.3)\,\widehat{\mathbf{s}} = 10\,\widehat{\mathbf{s}},$$

with magnitude 10 N and direction given by $\widehat{\mathbf{s}}$, that is, towards the centre of the spring.

When $l = 0.2$, the force acting on the object is

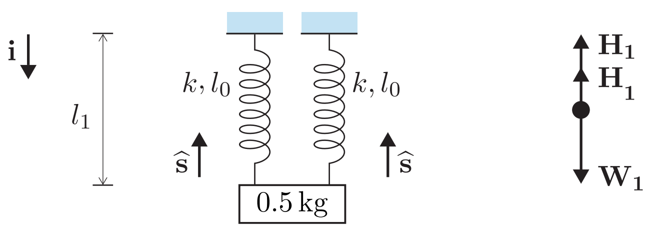
$$\mathbf{H} = 200(0.2 - 0.3)\,\widehat{\mathbf{s}} = -20\,\widehat{\mathbf{s}} = 20(-\widehat{\mathbf{s}}),$$

with magnitude 20 N and direction given by $-\widehat{\mathbf{s}}$, that is, away from the centre of the spring.

These results agree with what we found in Exercise 2.

Solution to Exercise 4

Let the length of each spring be l_1 when there is no baby in the baby bouncer. Assume that the two springs may be modelled satisfactorily by model springs, each with the same stiffness k and natural length l_0. Choose the $\mathbf{i}$-direction to be vertically downwards. Thus $\widehat{\mathbf{s}}$ for this problem is $-\mathbf{i}$ for both springs. The situation and force diagram are shown below.



Each spring exerts a force

$$\mathbf{H}_1 = k(l_1 - l_0)\,\widehat{\mathbf{s}} = k(l_1 - l_0)(-\mathbf{i}).$$

Since the two springs are identical, and they are both vertical, the spring force in each can be represented by the same vector.

The weight of the seat plus straps is $\mathbf{W}_1 = 0.5g\mathbf{i}$.

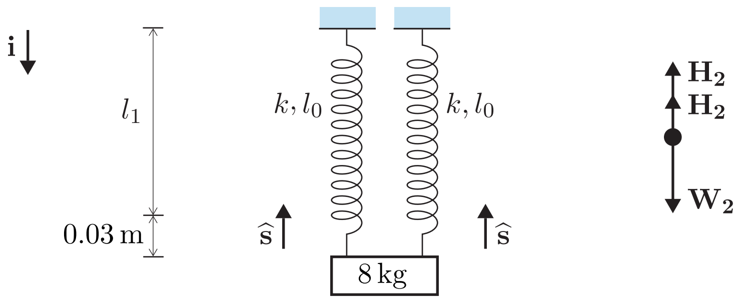
The system is in equilibrium, so

$$\mathbf{W}_1 + 2\mathbf{H}_1 = \mathbf{0}.$$

Resolving in the $\mathbf{i}$-direction gives

$$2k(l_1 - l_0) - 0.5g = 0.$$

With the baby strapped into the seat, the length of each spring becomes $l_1 + 0.03$ (in metres). The new situation and force diagram are shown below.



Now each spring exerts a force

$$\mathbf{H}_2 = k(l_1 + 0.03 - l_0)(-\mathbf{i}),$$

while the total weight of the baby, seat and straps is $\mathbf{W}_2 = 8g\mathbf{i}$.

The system is again in equilibrium, so

$$\mathbf{W}_2 + 2\mathbf{H}_2 = \mathbf{0}.$$

Resolving in the **i**-direction gives

$$2k(l_1 + 0.03 - l_0) - 8g = 0.$$

We have $2k(l_1 - l_0) = 0.5g$ from the earlier calculation, and subtracting this from the equation above gives $0.06k = 7.5g$. Hence $k = 125g \simeq 1200$, and the stiffness of each spring is $1200\,\mathrm{N\,m^{-1}}$.

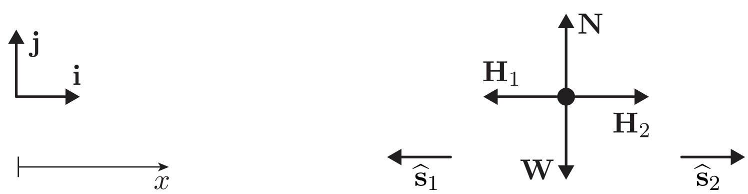
Solution to Exercise 5

As before, the object is modelled as a particle. In Example 2, the distance between the fixed points was greater than the sum of the natural lengths of the two springs, hence in equilibrium each spring was extended. Here the distance between the fixed points is less than the sum of the natural lengths of the two springs, hence in equilibrium each spring is compressed. However, this does not affect the approach to the solution. (This shows the usefulness of the vector formulation for the force in a spring, be it in extension or compression.)

◀ Choose axes ▶

◀ Draw force diagram(s) ▶

We again choose the x-axis to point from left to right, with origin at the left-hand fixed point. The unit vectors will be the same as before, as shown in the diagram below.



Strictly speaking, as we know that both springs will be in compression, we should have $\mathbf{H}_1$ and $\mathbf{H}_2$ in the opposite directions. We address this issue at the end of the solution.

The lengths of the two springs are x (left spring) and $0.3 - x$ (right spring), and this will alter the spring forces. As can be seen from the diagram above, everything else is the same as in the solution to Example 2.

◀ Apply law(s) ▶

Spring	l	l_0	$l - l_0$	k	$\widehat{\mathbf{s}}$	$\mathbf{H}$
Left	x	0.3	$x - 0.3$	40	$-\mathbf{i}$	$40(x - 0.3)(-\mathbf{i})$
Right	$0.3 - x$	0.2	$0.1 - x$	60	$\mathbf{i}$	$60(0.1 - x)\mathbf{i}$

Thus we have

$$\mathbf{H}_1 = 40(x - 0.3)(-\mathbf{i}) = 40(0.3 - x)\mathbf{i},$$
$$\mathbf{H}_2 = 60(0.1 - x)\,\mathbf{i}.$$

Since the particle is held in equilibrium by these two forces plus the weight $\mathbf{W}$ of the particle and the normal reaction $\mathbf{N}$, we have

$$\begin{aligned}\mathbf{W} + \mathbf{N} + \mathbf{H}_1 + \mathbf{H}_2 &= mg(-\mathbf{j}) + |\mathbf{N}|\,\mathbf{j} + 40(0.3 - x)\mathbf{i} + 60(0.1 - x)\mathbf{i}\\ &= \mathbf{0}.\end{aligned}$$

Resolving in the $\mathbf{i}$-direction gives

◀ Solve equation(s) ▶

$$60(0.1 - x) + 40(0.3 - x) = 0.$$

The solution of this equation is $x = 0.18$, so the particle is 0.18 m from the left-hand fixed point (and hence 0.12 m from the right-hand fixed point). Using this value of x in the equation for $\mathbf{H}_1$ gives

$$\mathbf{H}_1 = 40(0.3 - 0.18)\mathbf{i} = 4.8\mathbf{i},$$

that is, the left-hand spring force has a magnitude of 4.8 N that acts to the right, so it is pushing the particle away – it is a compressive force. The spring force in the right-hand spring will be compressive as well, pushing the particle to the left. Also, as expected,

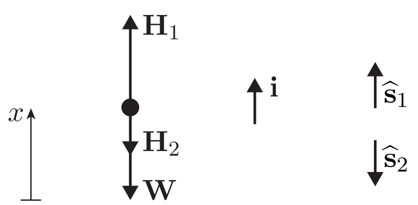
$$\mathbf{H}_2 = 60(0.1 - 0.18)\mathbf{i} = -4.8\mathbf{i}.$$

Note that the spring force $\mathbf{H}_1$ was shown as a tension, in the direction to the left, in the force diagram; but this analysis has shown it to be compressive and acting to the right. However, this does not affect the analysis.

Solution to Exercise 6

Model the object as a particle. The force diagram is shown below, where $\mathbf{W}$ is the weight of the particle, $\mathbf{H}_1$ is the spring force in the upper spring, and $\mathbf{H}_2$ is the spring force in the lower spring.

◀ Draw force diagram(s) ▶



◀ Choose axes ▶

Choose the x-axis to point vertically upwards, with origin at A, so $\mathbf{i}$ is also upwards. Then the weight of the particle is

$$\mathbf{W} = 1.5g(-\mathbf{i}).$$

◀ Apply law(s) ▶

The springs have lengths x (lower spring) and $2.1 - x$ (upper spring), and the forces exerted by them are shown in the table below.

Spring	l	l_0	$l - l_0$	k	$\widehat{\mathbf{s}}$	$\mathbf{H}$
Upper	$2.1 - x$	0.3	$1.8 - x$	20	$\mathbf{i}$	$20(1.8 - x)\mathbf{i}$
Lower	x	0.5	$x - 0.5$	50	$-\mathbf{i}$	$50(x - 0.5)(-\mathbf{i})$

So

$$\mathbf{H}_1 = 20(1.8 - x)\mathbf{i},$$
$$\mathbf{H}_2 = 50(x - 0.5)(-\mathbf{i}) = 50(0.5 - x)\mathbf{i}.$$

In equilibrium,

$$\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W} = \mathbf{0}.$$

◀ Solve equation(s) ▶

Resolving in the $\mathbf{i}$-direction gives

$$20(1.8 - x) + 50(0.5 - x) - 1.5g = 0,$$

so $70x = 61 - 1.5g$. The solution of this equation is $x \simeq 0.66$.

◀ Interpret solution ▶

So the object is about 0.66 m above the floor.

You may like to reformulate the solution to the problem using the length y measured downwards from point B (and $\mathbf{i}$ now pointing downwards as well). Determine the value of y and hence confirm the value for x given above.

Solution to Exercise 7

Once the battery has been placed in its holder, the sum of the spring lengths is $(5.5 - 5) = 0.5$ cm, which in SI units is 0.005 m.

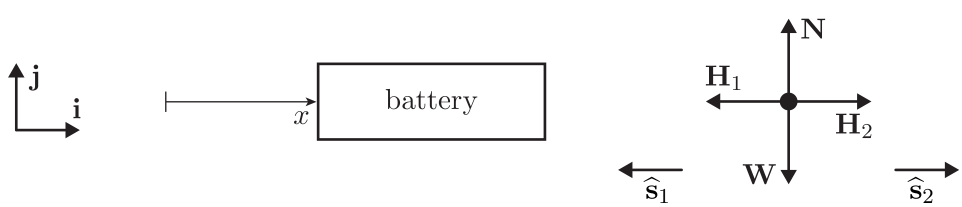
Let $\mathbf{W}$ be the weight of the battery, let $\mathbf{N}$ be the normal reaction from the battery holder acting on the battery, let $\mathbf{H}_1$ be the spring force in the left-hand spring, and let $\mathbf{H}_2$ be the spring force in the right-hand spring.

◀ Choose axes ▶

Take the x-axis, and the $\mathbf{i}$-direction, to be from left to right, with the origin for x at the left-hand end of the battery holder; x is used as the distance from the left-hand end of the battery holder to the left-hand end of the battery.

◀ Draw force diagram(s) ▶

The axis, unit vectors and force diagram are shown below.



The spring at the left-hand side of the battery has length x. The length of the right-hand spring is $0.055 - 0.05 - x$ (allowing for the length of the battery). From the battery to the centre of the left-hand spring is from right to left, so $\widehat{\mathbf{s}}_1 = -\mathbf{i}$ (for the left-hand spring); we look in the opposite direction for the right-hand spring, so $\widehat{\mathbf{s}}_2 = \mathbf{i}$.

The spring forces are derived in the table below.

◀ Apply law(s) ▶

Spring	l	l_0	$l - l_0$	k	$\widehat{\mathbf{s}}$	$\mathbf{H}$
Left	x	0.004	$x - 0.004$	30	$-\mathbf{i}$	$30(x - 0.004)(-\mathbf{i})$
Right	$0.005 - x$	0.003	$0.002 - x$	10	$\mathbf{i}$	$10(0.002 - x)\,\mathbf{i}$

So

$$\mathbf{H}_1 = 30(x - 0.004)(-\mathbf{i}) = 30(0.004 - x)\mathbf{i},$$
$$\mathbf{H}_2 = 10(0.002 - x)\mathbf{i}.$$

Since the battery is held in equilibrium by these two forces plus the weight $\mathbf{W} = mg(-\mathbf{j})$ of the battery and the normal reaction $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$ on it (both acting vertically), we have

$$\mathbf{W} + \mathbf{N} + \mathbf{H}_1 + \mathbf{H}_2 = \mathbf{0}.$$

Resolving in the (horizontal) $\mathbf{i}$-direction gives

◀ Solve equation(s) ▶

$$30(0.004 - x) + 10(0.002 - x) = 0,$$

which gives

$$0.14 - 40x,$$

with solution $x = 0.0035$ (metres). Resolving in the $\mathbf{i}$-direction also tells us that the magnitudes of the (compressive) spring forces are equal. This magnitude is

$$\begin{aligned} |\mathbf{H}_1| &= |30(0.004 - 0.0035)| \\ &= 30 \times 0.0005 \\ &= 0.015, \end{aligned}$$

that is, 0.015 N.

Solution to Exercise 8

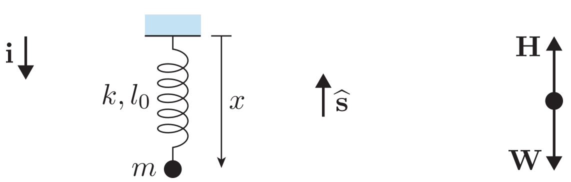
Define the length of the model spring to be x, with the origin at the support at the top. Define $\mathbf{i}$ to be in the direction of increasing x. (Only one unit vector is needed as all the forces are vertical.) We model the system as a particle of mass m and a model spring; air resistance forces are ignored. The set-up and force diagram are shown below.

◀ Draw picture ▶

◀ Choose axes ▶

◀ State assumptions ▶

◀ Draw force diagram ▶



There are two forces acting on the particle, the weight and that due to the spring. Using the unit vector given, $\mathbf{W} = mg\mathbf{i}$.

The force $\mathbf{H}$ exerted by the spring on the particle is given by equation (2), that is,

$$\mathbf{H} = k(l - l_0)\,\widehat{\mathbf{s}},$$

where l is the length of the spring and $\widehat{\mathbf{s}}$ is a unit vector directed from the particle towards the centre of the spring. Here $l = x$ and $\widehat{\mathbf{s}} = -\mathbf{i}$, which gives

$$\mathbf{H} = k(x - l_0)(-\mathbf{i}).$$

◀ Apply Newton's 2nd law ▶

Apply Newton's second law (3) to obtain

$$m\ddot{\mathbf{r}} = m\ddot{x}\mathbf{i} = -k(x - l_0)\mathbf{i} + mg\mathbf{i}.$$

Resolving in the $\mathbf{i}$-direction gives

$$m\ddot{x} = -k(x - l_0) + mg,$$

that is,

$$m\ddot{x} + kx = kl_0 + mg.$$

Solution to Exercise 9

The associated homogeneous differential equation, $m\ddot{x} + kx = 0$, is the same in both cases. However, the constant term on the right-hand side is different. So the complementary function will be the same, but the particular integral will be different.

Solution to Exercise 10

If $x(t) = B\cos\omega t + C\sin\omega t$, then

$$\dot{x}(t) = -B\omega\sin\omega t + C\omega\cos\omega t,$$
$$\ddot{x}(t) = -B\omega^2\cos\omega t - C\omega^2\sin\omega t = -\omega^2\,x(t).$$

Putting $\omega^2 = k/m$ and writing x for $x(t)$, we have

$$\ddot{x} = -kx/m,$$

that is,

$$m\ddot{x} + kx = 0.$$

Hence $x(t) = B\cos\omega t + C\sin\omega t$ is a solution of this differential equation.

Solution to Exercise 11

Write the differential equation in the form

$$\ddot{x} + \omega^2 x = 0,$$

where $\omega^2 = k/m$. Then the auxiliary equation is

$$\lambda^2 + \omega^2 = 0,$$

with solutions $\lambda = 0 \pm i\omega$.

So the general solution is

$$\begin{aligned} x(t) &= e^0(B\cos\omega t + C\sin\omega t) \\ &= B\cos\omega t + C\sin\omega t, \end{aligned}$$

where B and C are arbitrary constants.

Solution to Exercise 12

Both differential equations have a constant on the right-hand side, so consider the solution of

$$m\ddot{x} + kx = K,$$

where $K = kl_0$ for Example 4 and $K = kl_0 + mg$ for Exercise 8.

Based on the methods in Unit 1, since the function on the right-hand side is a constant, try a solution of the form $x_\text{p} = c$, where c is a constant. Then $\dot{x}_\text{p} = \ddot{x}_\text{p} = 0$, and substituting this into the differential equation gives

$$m \times 0 + kc = K, \quad \text{thus} \quad c = \frac{K}{k}.$$

The general solution is the sum of the complementary function and the particular integral. So for Example 4, the general solution is

$$x(t) = B\cos\omega t + C\sin\omega t + l_0,$$

and for Exercise 8, the general solution is

$$x(t) = B\cos\omega t + C\sin\omega t + l_0 + \frac{mg}{k},$$

where B and C are arbitrary constants.

Solution to Exercise 13

For both the horizontal spring and the vertical spring, the equilibrium length is the same as the value of the constant.

Each general solution is a sinusoid plus a constant term. The sinusoid represents the oscillatory motion, and the constant term gives the position about which the oscillations take place. It is sometimes called the average position of the oscillations.

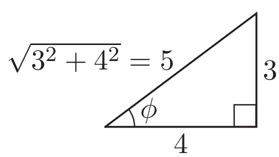
Solution to Exercise 14

The coefficient of t is 7, so $\omega = 7$.

Using a compound-angle formula gives

$$\cos\big(7t + \arctan\left(\tfrac{3}{4}\right)\big) = \cos 7t \cos\big(\arctan\left(\tfrac{3}{4}\right)\big) - \sin 7t \sin\big(\arctan\left(\tfrac{3}{4}\right)\big).$$

From the figure in the margin, it can be seen that if $\arctan\left(\frac{3}{4}\right) = \phi$, then $\cos\phi = \frac{4}{5}$ and $\sin\phi = \frac{3}{5}$, so



$$10\cos\big(7t + \arctan\left(\tfrac{3}{4}\right)\big) = 10\cos 7t \times \tfrac{4}{5} - 10\sin 7t \times \tfrac{3}{5}.$$

Thus $B = 8$ and $C = -6$, and we have the solution

$$x(t) = 8\cos\omega t - 6\sin\omega t.$$

Solution to Exercise 15

(a) The angular frequency is

$$\omega = \sqrt{k/m} = \sqrt{200/0.5} = 20,$$

that is, $20\,\mathrm{rad\,s^{-1}}$.

The period is

$$\tau = 2\pi/\omega = 2\pi/20 \simeq 0.314 \quad \text{to 3 d.p.,}$$

that is, about 0.31 s.

(b) The frequency is

$$f = \omega/(2\pi) = 20/(2\pi) \simeq 3.18 \quad \text{to 2 d.p.,}$$

that is, about 3.18 Hz.

(c) With $\omega = 20$ and $l_0 = 0.2$, we have

$$x(t) = B\cos 20t + C\sin 20t + 0.2,$$

whose derivative is

$$\dot{x}(t) = -20B\sin 20t + 20C\cos 20t.$$

Now $x = 0.3$ and $\dot{x} = 2$ when $t = 0$, so

$$0.3 = B + 0.2, \quad 2 = 20C.$$

Hence $B = 0.1$ and $C = 0.1$. Thus the position function for the particle is

$$x(t) = 0.1\cos 20t + 0.1\sin 20t + 0.2.$$

(d) The amplitude of the oscillation is $\sqrt{0.1^2 + 0.1^2} = 0.141$ to three decimal places. So the minimum distance is $0.2 - 0.141 = 0.059$ m, or approximately 6 cm.

Solution to Exercise 16

(a) Using equations (10), we have $A = \sqrt{0.1^2 + 0.1^2} = 0.1\sqrt{2}$ and

$$\cos\phi = 0.1/A, \quad \sin\phi = -0.1/A.$$

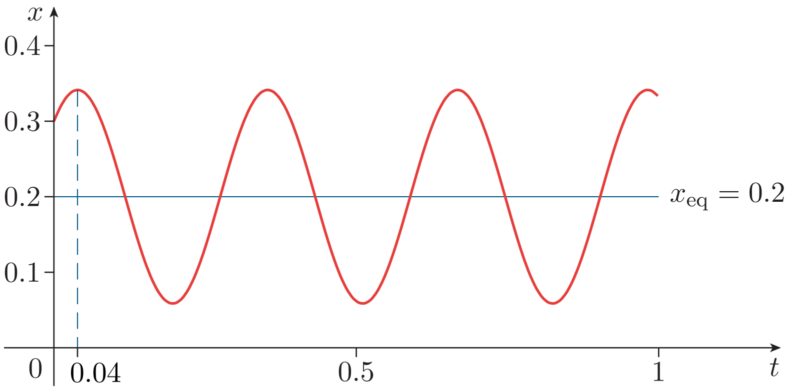
Since $\cos\phi > 0$ and $\sin\phi < 0$, the angle ϕ lies in the fourth quadrant (where $-\frac{\pi}{2} < \phi < 0$), so

$$\phi = \arcsin\left(\frac{-0.1}{0.1\sqrt{2}}\right) = -\frac{\pi}{4}.$$

(b) Using the result of part (a), the oscillation may be described by the equation

$$x(t) = 0.1\sqrt{2}\cos\left(20t - \tfrac{\pi}{4}\right) + 0.2.$$

This is sketched below (where $\phi/\omega = -\frac{\pi}{4}/20 \simeq 0.04$).



(c) The spring takes its natural length when $x = l_0$. From part (b), this occurs when

$$0.2 = 0.1\sqrt{2}\cos\left(20t - \tfrac{\pi}{4}\right) + 0.2,$$

that is, when $20t - \pi/4 = (2n+1)\pi/2$, with n any integer. Thus the times are given by

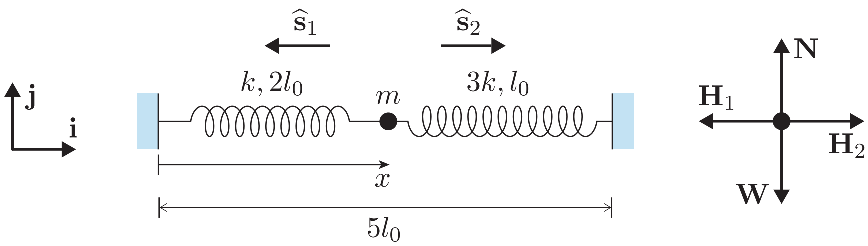
$$t = \frac{1}{20}\left(\frac{\pi}{4} + \frac{(2n+1)\pi}{2}\right) = \frac{\pi}{20}\left(\tfrac{3}{4} + n\right), \quad n \in \mathbb{Z}.$$

Solution to Exercise 17

(a) A picture of the set-up is shown below, along with unit vectors in the horizontal and vertical directions.

◀ Draw picture ▶

◀ Choose axes ▶



We model the block as a particle of mass m and the springs as model springs. All friction and air resistance forces are ignored.

◀ State assumptions ▶

The force diagram is drawn (in the picture above) for the case where both springs are extended. There are four forces acting on the particle: the weight $\mathbf{W}$ of the particle, the normal reaction $\mathbf{N}$ from the table, the spring force $\mathbf{H}_1$ from the left-hand spring, and the spring force $\mathbf{H}_2$ from the right-hand spring.

◀ Draw force diagram ▶

The spring forces are derived in the following table (where the headings refer to general notation and the rows contain the parameters for this particular set-up).

◀ Apply Newton's 2nd law ▶

Spring	l	l_0	$l - l_0$	k	$\widehat{\mathbf{s}}$	$\mathbf{H}$
Left	x	$2l_0$	$x - 2l_0$	k	$-\mathbf{i}$	$k(x - 2l_0)(-\mathbf{i})$
Right	$5l_0 - x$	l_0	$4l_0 - x$	$3k$	$\mathbf{i}$	$3k(4l_0 - x)\mathbf{i}$

So

$$\mathbf{H}_1 = k(2l_0 - x)\mathbf{i},$$
$$\mathbf{H}_2 = 3k(4l_0 - x)\mathbf{i}.$$

In addition, $\mathbf{W} = mg(-\mathbf{j})$ and $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$.

From Newton's second law, we have

$$\begin{aligned} m\ddot{x}\mathbf{i} &= \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W} + \mathbf{N} \\ &= k(2l_0 - x)\mathbf{i} + 3k(4l_0 - x)\mathbf{i} - mg\mathbf{j} + |\mathbf{N}|\,\mathbf{j}. \end{aligned}$$

Resolving in the **i**-direction and rearranging, we obtain the equation

$$m\ddot{x} + 4kx = 14kl_0,$$

as required.

◀ Solve differential equation ▶

(b) In equilibrium, with $x = x_{\text{eq}}$ say, we have $\dot{x} = 0$ and $\ddot{x} = 0$. Substituting into the equation of motion gives

$$4kx_{\text{eq}} = 14kl_0,$$

so the equilibrium distance from the left-hand end is

$$x_{\text{eq}} = \tfrac{7}{2}l_0.$$

(c) The general solution of the equation of motion is

$$x(t) = B\cos\omega t + C\sin\omega t + x_{\text{eq}},$$

where B and C are constants, and the angular frequency ω is $\sqrt{4k/m}$. So the period of the oscillations is

$$\tau = \frac{2\pi}{\omega} = \pi\sqrt{\frac{m}{k}}.$$

(d) The initial conditions are $x(0) = \frac{5}{2}l_0$ (the halfway point) and $\dot{x}(0) = 0$. So we have

$$\tfrac{5}{2}l_0 = B + \tfrac{7}{2}l_0, \quad 0 = C\omega.$$

Hence $B = -l_0$ and $C = 0$, giving the particular solution

$$x(t) = -l_0\cos\left(2\sqrt{\frac{k}{m}}\,t\right) + \tfrac{7}{2}l_0.$$

(e) The amplitude of the motion is $A = \sqrt{(-l_0)^2 + 0^2} = l_0$.

Note that one of the extrema is the point at which the particle was released from rest.

The minimum distance is $x_{\text{eq}} - A = \frac{7}{2}l_0 - l_0 = \frac{5}{2}l_0$.

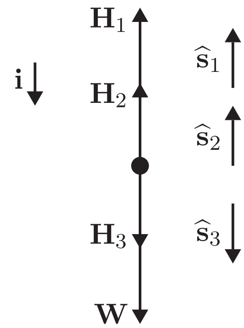
The maximum distance is $x_{\text{eq}} + A = \frac{7}{2}l_0 + l_0 = \frac{9}{2}l_0$.

The range of motion is within the fixed ends at $x = 0$ and $x = 5l_0$, so the particle never hits the ends.

Solution to Exercise 18

(a) Since air resistance can be ignored, there are just four forces acting, namely the weight $\mathbf{W}$, and three spring forces: $\mathbf{H}_1$ from spring QP, $\mathbf{H}_2$ from spring RP, and $\mathbf{H}_3$ from spring SP. The block is modelled as a particle, so the forces act at the same point.

The force diagram in the margin also shows the unit vectors $\widehat{\mathbf{s}}_1$, $\widehat{\mathbf{s}}_2$ and $\widehat{\mathbf{s}}_3$ associated with each spring.



(b) The spring forces can be obtained using a table (where once again the headings refer to general notation and the rows contain the parameters for this particular set-up).

Spring	l	l_0	$l-l_0$	k	$\widehat{\mathbf{s}}$	$\mathbf{H}$
QP	x	l_0	$x-l_0$	$2k$	$-\mathbf{i}$	$2k(x-l_0)(-\mathbf{i})$
RP	$x-\frac{1}{2}l_0$	$\frac{1}{2}l_0$	$x-l_0$	$5k$	$-\mathbf{i}$	$5k(x-l_0)(-\mathbf{i})$
SP	$4l_0-x$	$\frac{3}{2}l_0$	$\frac{5}{2}l_0-x$	$2k$	$\mathbf{i}$	$2k\left(\frac{5}{2}l_0-x\right)\mathbf{i}$

So

$$\mathbf{H}_1 = 2k(l_0-x)\mathbf{i},$$
$$\mathbf{H}_2 = 5k(l_0-x)\mathbf{i},$$
$$\mathbf{H}_3 = 2k\left(\tfrac{5}{2}l_0-x\right)\mathbf{i}.$$

(c) Applying Newton's second law, the equation of motion is

$$m\ddot{x}\mathbf{i} = \mathbf{W}+\mathbf{H}_1+\mathbf{H}_2+\mathbf{H}_3.$$

The weight is given by $\mathbf{W}=mg\mathbf{i}$, so using the results from part (b), we have

$$m\ddot{x}\mathbf{i} = mg\mathbf{i} + 2k(l_0-x)\mathbf{i} + 5k(l_0-x)\mathbf{i} + 2k\left(\tfrac{5}{2}l_0-x\right)\mathbf{i}.$$

Resolving in the $\mathbf{i}$-direction, the equation of motion is

$$m\ddot{x}+9kx = mg+12kl_0.$$

(d) In equilibrium, $x=x_{\text{eq}}$ and the acceleration is zero, so

$$9kx_{\text{eq}} = mg+12kl_0.$$

Thus the equilibrium position is given by

$$x_{\text{eq}} = \tfrac{4}{3}l_0 + \frac{mg}{9k}.$$

(e) Starting with the equation of motion from part (c), the associated homogeneous differential equation is

$$\ddot{x}+\frac{9k}{m}x = 0.$$

Take $\omega^2 = 9k/m$, so that the complementary function is

$$x_c = A\cos\omega t + B\sin\omega t,$$

where A and B are constants, that is,

$$x_c = A\cos\left(3\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(3\sqrt{\frac{k}{m}}\,t\right).$$

The particular integral x_p is the equilibrium position, thus the general solution is

$$\begin{aligned} x(t) &= x_c + x_p \\ &= A\cos\left(3\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(3\sqrt{\frac{k}{m}}\,t\right) + \frac{mg}{9k} + \tfrac{4}{3}l_0. \end{aligned}$$

(f) If the particle is released from rest ($\dot{x}(0) = 0$) at time $t = 0$ at a position $\frac{4}{3}l_0$ below Q, then there are two equations to solve.

Using the initial condition on the displacement gives

$$\tfrac{4}{3}l_0 = A + \frac{mg}{9k} + \tfrac{4}{3}l_0,$$

so

$$A = -\frac{mg}{9k}.$$

After differentiating the general solution to obtain

$$\dot{x}(t) = -3A\sqrt{\frac{k}{m}}\sin\left(3\sqrt{\frac{k}{m}}\,t\right) + 3B\sqrt{\frac{k}{m}}\cos\left(3\sqrt{\frac{k}{m}}\,t\right),$$

insertion of the initial condition for the speed gives

$$0 = 3B\sqrt{\frac{k}{m}},$$

so $B = 0$.

Thus the equation of motion in this case is

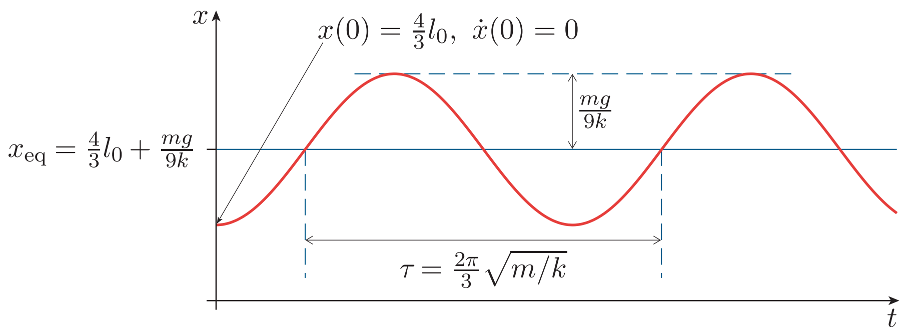
$$x(t) = \tfrac{4}{3}l_0 + \frac{mg}{9k}\left(1 - \cos\left(3\sqrt{\frac{k}{m}}\,t\right)\right).$$

(g) From the form of the solution, the frequency is $\omega = 3\sqrt{k/m}$, so the period of the oscillations is

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{3}\sqrt{\frac{m}{k}}.$$

The amplitude is $mg/9k$.

(h) A sketch graph is shown below.



Solution to Exercise 19

(a) Once the pontoon is out of the water, the buoyancy force is zero, so the upper limit for x is h. When the pontoon is fully submerged, the buoyancy force is constant, and not dependent on x, so the lower limit for x is 0. The expression for the buoyancy force is valid for $0 < x < h$.

(b) The given equation of motion (17) is equivalent to

$$\ddot{x} + \omega^2 x = \omega^2 x_{eq},$$

where $\omega^2 = A\rho g/m$ and $x_{eq} = h - m/(\rho A)$. The period τ is therefore given by

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{A\rho g}} = C\sqrt{m},$$

where $C = 2\pi/\sqrt{A\rho g}$ is constant if A is fixed. Thus τ is proportional to $\sqrt{m}$.

If the mass m is increased progressively, for given dimensions of the pontoon, then eventually the pontoon will sink. The quoted equation of motion then no longer applies.

From part (a), for the pontoon to float we must have $x_{eq} > 0$, which occurs if $m < Ah\rho$. This gives an upper limit $2\pi\sqrt{h/g}$ for the period of oscillations.

Solution to Exercise 20

Using the method of dimensions from Unit 8,

$$[T(x)] = \left[\tfrac{1}{2}\right][m]\left[v^2\right] = 1 \times \mathrm{M} \times \left(\mathrm{L\,T^{-1}}\right)^2 = \mathrm{M\,L^2\,T^{-2}}$$

Take care to distinguish T denoting kinetic energy from T used as the dimensions of time.

and

$$[U(x)] = \left[-\int F(x)\,dx\right] = [F(x)]\,[x] = \mathrm{M\,L\,T^{-2}} \times \mathrm{L} = \mathrm{M\,L^2\,T^{-2}}.$$

So, not surprisingly, kinetic energy and potential energy have the same dimensions.

Solution to Exercise 21

With the given x-axis, the force due to gravity has $\mathbf{i}$-component $F = -mg$. Using the definition of the potential energy function (23), we have

$$U(x) = -\int F(x)\,dx = \int mg\,dx = mgx + C,$$

where C is a constant. The given datum is the origin, so $U(0) = 0$, thus $C = 0$, and we obtain

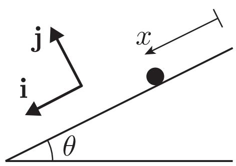
$$U(x) = mgx.$$

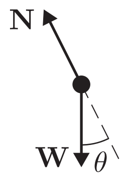
This is a justification for the negative sign in the definition of $U(x)$; the energy is more positive the higher the particle is above the datum.

So for the force due to gravity, the potential energy function is mg times the height above the chosen datum.

Solution to Exercise 22

Choose the unit vectors $\mathbf{i}$ and $\mathbf{j}$ as shown below.





Then the forces acting on the particle are the normal reaction $\mathbf{N}$ (in the $\mathbf{j}$-direction) and the weight

$$\mathbf{W} = mg\,(\sin\theta\,\mathbf{i} + \cos\theta\,(-\mathbf{j})) = mg\sin\theta\,\mathbf{i} - mg\cos\theta\,\mathbf{j}.$$

The motion (and hence the resultant force) is along the slope. The total force $F(x)\,\mathbf{i}$ acting on the particle along the slope is given by

$$F(x) = (\mathbf{N} + \mathbf{W}) \cdot \mathbf{i} = mg\sin\theta.$$

Using the definition of the potential energy function (23), we have

$$U(x) = -\int F(x)\,dx = \int -mg\sin\theta\,dx = -mgx\sin\theta + C,$$

where C is a constant. The given datum is the origin, so $U(0) = 0$, thus $C = 0$, and we obtain

$$U(x) = -mgx\sin\theta.$$

Since $x\sin\theta$ is the vertical height of the particle as measured *below* the datum, the potential energy function is again mg times the height above the chosen datum (since this is now negative, because the x-axis is down the slope).

Solution to Exercise 23

The $\mathbf{i}$-component of the buoyancy force is $A\rho g(h - x)$, so

Exercise 19(a) shows that the range of validity for the equation for $\mathbf{B}$ is $0 < x < h$.

$$U(x) = -\int A\rho g(h - x)\,dx = -A\rho g\left(hx - \tfrac{1}{2}x^2\right) + C,$$

where C is a constant. Using the given datum, $U(h) = 0$, so

$$0 = -A\rho g\left(h^2 - \tfrac{1}{2}h^2\right) + C,$$

therefore

$$C = \tfrac{1}{2}A\rho g h^2.$$

Thus

$$U(x) = -A\rho g\left(hx - \tfrac{1}{2}x^2 + \tfrac{1}{2}A\rho g h^2\right) = \tfrac{1}{2}A\rho g(h - x)^2.$$

This makes sense because when the pontoon is fully submerged ($x = 0$), the potential energy is a maximum. This can be demonstrated by holding a light ball (such as a table tennis ball) underwater; on being released, it shoots upwards. Potential energy is converted to kinetic energy.

Solution to Exercise 24

With the given choice of origin, the force due to the spring is

$$\mathbf{H} = k(x - l_0)(-\mathbf{i}) = -k(x - l_0)\mathbf{i},$$

whose x-component is $F(x) = -k(x - l_0)$. Hence the potential energy function is

$$U(x) = -\int F(x)\,dx = \int k(x - l_0)\,dx = \tfrac{1}{2}k(x - l_0)^2 + C,$$

where C is a constant. The given datum is $x = l_0$, so $U(0) = 0$, thus $C = 0$, and we have

$$U(x) = \tfrac{1}{2}k(x - l_0)^2.$$

So for the force due to the spring, the potential energy function is

$$\tfrac{1}{2} \times \text{stiffness} \times (\text{deformation})^2.$$

Solution to Exercise 25

(a) For $F(x) = -ax^2$, we have

$$U(x) = -\int F(x)\,dx = \int ax^2\,dx = \tfrac{1}{3}ax^3 + C,$$

where C is a constant. Choosing $x = 0$ as the datum, so $U(0) = 0$, gives $C = 0$. Hence the potential energy function is

$$U(x) = \tfrac{1}{3}ax^3.$$

(b) For $F(x) = bx^{-2}$ $(x > 0)$, we have

$$U(x) = -\int F(x)\,dx = -\int bx^{-2}\,dx = bx^{-1} + C,$$

It is not possible to use $x = 0$ as the datum, since $U(x)$ is not defined at $x = 0$.

where C is a constant. Choosing $x = \infty$ as the datum, so $U(\infty) = 0$, gives $C = 0$. Hence the potential energy function is

$$U(x) = bx^{-1} \quad (x > 0).$$

Choosing a different datum, say x_0 such that $U(x_0) = 0$, leads to

The motion of a comet can be represented by this model.

$$U(x) = \frac{b}{x} - \frac{b}{x_0} \quad (x > 0).$$

Solution to Exercise 26

(a) From the definition of potential energy, if $U(x)$ is a potential energy function corresponding to the force $F(x)\,\mathbf{i}$, then

$$\begin{aligned} U(x) &= -\int F(x)\,dx \\ &= -\int \big[F_1(x) + F_2(x)\big]\,dx \\ &= -\int F_1(x)\,dx - \int F_2(x)\,dx \\ &= U_1(x) + U_2(x). \end{aligned}$$

(b) If $F_1(x) = -ax^2$ and $F_2(x) = bx^{-2}$ $(x > 0)$, then from Exercise 25, we have

$$U_1(x) = \tfrac{1}{3}ax^3, \quad U_2(x) = bx^{-1} \ (x > 0).$$

Now the given force is $F(x)\,\mathbf{i} = F_1(x)\,\mathbf{i} + F_2(x)\,\mathbf{i}$, so its potential energy function is

$$\begin{aligned} U(x) &= U_1(x) + U_2(x) \\ &= \tfrac{1}{3}ax^3 + bx^{-1} \quad (x > 0). \end{aligned}$$

(Your answer could differ from the above by a constant if you chose a different datum for either F_1 or F_2; see the text that follows this exercise.)

Solution to Exercise 27

(a) Using the origin as the datum with a downward-pointing x-axis, the height above the datum is $-x$. So using equation (26), the potential energy is $mg(-x)$, thus the total energy is

$$E = \tfrac{1}{2}mv^2 - mgx.$$

The initial condition is $v = 0$ when $x = 0$, so $E = 0$. Hence we have

$$0 = \tfrac{1}{2}v^2 - gx, \quad \text{or} \quad v^2 = 2gx,$$

throughout the motion since energy is conserved. So when $x = 77$, we find that

$$|v| = \sqrt{2 \times 9.81 \times 77} \simeq 38.9 \quad \text{to 3 s.f.}$$

Hence the speed of the marble just before it hits the water is $38.9\,\mathrm{m\,s^{-1}}$.

(The same answer was obtained by direct application of Newton's second law for constant acceleration in Unit 3, Example 4.)

When $v = 20$, we have

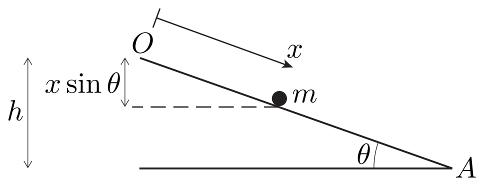
$$x = 20^2/(2 \times 9.81) \simeq 20.4 \quad \text{to 3 s.f.}$$

So the marble has fallen $20.4\,\mathrm{m}$ when its speed reaches $20\,\mathrm{m\,s^{-1}}$.

(b) The force of air resistance is a function of velocity. Hence the total force acting on the marble is not dependent on position alone, and the law of conservation of mechanical energy is not applicable.

Solution to Exercise 28

(a) First, we draw a picture. ◀ Draw picture ▶



We assume that the particle travels in a straight line down a straight frictionless slope, subject only to the gravitational force and the normal reaction. Air resistance is ignored. ◀ State assumptions ▶

The point A is chosen as the datum, at $x = h/\sin\theta$. ◀ Choose datum ▶

The component of the force down the slope is $mg\sin\theta$, so the potential energy function is ◀ Find potential energy ▶

$$\begin{aligned} U(x) &= -\int_{h/\sin\theta}^{x} F(x)\,dx \\ &= -\int_{h/\sin\theta}^{x} mg\sin\theta\,dx \\ &= mgh - mgx\sin\theta. \end{aligned}$$

So the total mechanical energy of the particle is

$$E = \tfrac{1}{2}mv^2 + mg(h - x\sin\theta).$$

(b) At O we have $x = 0$ and $v = 0$, so $E = mgh$ and ◀ Find E ▶

$$\tfrac{1}{2}mv^2 + mg(h - x\sin\theta) = mgh$$

throughout the motion.

At A we have $x\sin\theta = h$, so $\frac{1}{2}mv^2 = mgh$. Since $v > 0$, this gives ◀ Find x or v ▶

$$v = \sqrt{2gh}.$$

The speed of the particle at A is therefore $\sqrt{2gh}$. ◀ Interpret solution ▶

This result depends on the height h, but not on the angle θ at which the slope is inclined. For a less steep slope, the acceleration of the particle will be less, but then the particle has further to travel down the slope in order to descend a given vertical distance h. The result depends on the assumption of no friction between the particle and the surface, but does not depend on the mass of the particle.

(c) For the given data, we obtain a value at A of

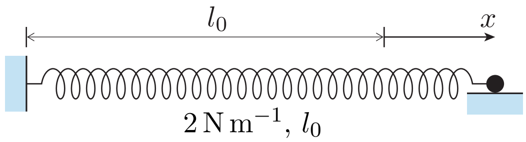
$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2 \sin \tfrac{\pi}{6}} \simeq 4.4,$$

so the crate of empty bottles reaches the bottom of the slope with speed $4.4\,\mathrm{m\,s^{-1}}$. (This agrees with the answer obtained in Example 5 of Unit 3.)

Solution to Exercise 29

◀ Draw picture ▶

First, we draw a picture.



◀ State assumptions ▶

We assume that the track is straight, level and frictionless, that the spring is a model spring, and that air resistance forces are ignored.

◀ Choose datum ▶

Take the datum at $x = 0$ (the point of zero deformation).

◀ Find potential energy ▶

For the force exerted by a spring with deformation x, the potential energy function is $U(x) = \frac{1}{2}kx^2$ (from equation (27)), so the law of conservation of mechanical energy gives

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E.$$

◀ Find E ▶

Since $m = 0.5$ and $k = 2$, this becomes

$$\tfrac{1}{4}v^2 + x^2 = E.$$

Initially, $v = -3$ when $x = 2$, so

$$E = \tfrac{9}{4} + 4 = \tfrac{25}{4}.$$

◀ Find x or v ▶

Hence we have

$$v^2 + 4x^2 = 25.$$

From this equation, $|v|$ will take its maximum value when $4x^2$ takes its minimum value, which is zero. So the maximum speed is $|v| = \sqrt{25} = 5$.

The particle is at rest when $v = 0$, that is, at $x = \pm\frac{5}{2}$.

◀ Interpret solution ▶

Summarising, the maximum speed of the particle is $5\,\mathrm{m\,s^{-1}}$, and it is momentarily at rest at $\pm\frac{5}{2}$ m from the equilibrium position.

Solution to Exercise 30

We assume that the spring is a model spring and that the particle only moves up and down, subject only to the spring force and the gravitational force. Air resistance is ignored. ◀ State assumptions ▶

We choose the fixed top of the spring as the datum for gravitational potential energy. We choose the point of zero deformation as the datum for the spring potential energy. ◀ Choose datum ▶

Let x be the length of the spring, measured downwards from the datum. The potential energy function is the sum of two contributions, one due to gravity and one due to the spring. The height above the datum is $-x$, so the gravitational potential energy is $-mgx$. The extension of the spring is $x - l_0$, so the spring contributes a term $\frac{1}{2}k(x - l_0)^2$ to the potential energy. So the potential energy of the particle is ◀ Find potential energy ▶

$$U(x) = \tfrac{1}{2}k(x - l_0)^2 - mgx.$$

Hence by the law of conservation of energy,

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\frac{10mg}{l_0}(x - l_0)^2 - mgx,$$

where we have substituted in the given value of k.

Initially, the particle is at rest at a depth of $\frac{5}{4}l_0$, so ◀ Find E ▶

$$\begin{aligned} E &= \tfrac{1}{2}m0^2 + \tfrac{1}{2}\frac{10mg}{l_0}(\tfrac{5}{4}l_0 - l_0)^2 - mg\tfrac{5}{4}l_0 \\ &= 0 + \tfrac{5}{16}mgl_0 - \tfrac{5}{4}mgl_0 \\ &= -\tfrac{15}{16}mgl_0. \end{aligned}$$

We wish to find $|v|$ when the spring has its natural length. Substituting $x = l_0$ into the equation for E gives ◀ Find x or v ▶

$$-\tfrac{15}{16}mgl_0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\frac{10mg}{l_0}(l_0 - l_0)^2 - mgl_0.$$

This leads to $|v| = \sqrt{gl_0/8}$.

The speed of the particle when the spring is of length l_0 is $\sqrt{gl_0/8}$. ◀ Interpret solution ▶

Solution to Exercise 31

(a) We choose the origin as the datum (you may have made a different choice). The potential energy function is then

$$U(x) = -\int F(x)\,dx = -\int (2 - 2x)\,dx = -2x + x^2.$$

(b) The total mechanical energy of the particle of mass $m = 2$ is

$$E = \tfrac{1}{2}mv^2 + U(x) = v^2 - 2x + x^2.$$

Now $v = 0$ when $x = -1$, so

$$E = 0 + 2 + 1 = 3.$$

(If you used a different datum for the potential energy, so that $U(x) = -2x + x^2 + C$, for C a constant, then your answer for the total mechanical energy will be $E = 3 + C$.)

(c) By the law of conservation of mechanical energy,

$$3 = v^2 - 2x + x^2,$$

so

$$v = \pm\sqrt{3 + 2x - x^2}.$$

(d) The region of motion is given by

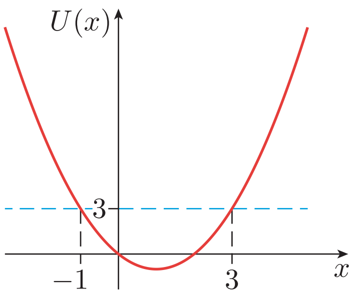
$$3 + 2x - x^2 \geq 0,$$

which factorises to give

$$(1 + x)(3 - x) \geq 0,$$

that is,

$$-1 \leq x \leq 3.$$



At the midpoint (corresponding to the minimum value for $U(x)$ – see the sketch in the margin) $x = 1$, the particle's speed is

$$|v| = \sqrt{3 + 2 - 1} = 2.$$

Solution to Exercise 32

Differentiating gives $U'(x) = 3x^2 - 3$, so $U'(x) = 0$ when $3x^2 - 3 = 0$, which has solutions $x = \pm 1$. These are the positions of equilibrium.

Solution to Exercise 33

(a) We choose $x = \infty$ as the datum. The potential energy function is then

$$U(x) = -\int F(x)\,dx = -\int \frac{2}{x^2}\,dx = \frac{2}{x}.$$

(If you chose a different datum, then your answer should be $U(x) = 2x^{-1} + C$, where C is a constant.)

(b) The law of conservation of mechanical energy is

$$\tfrac{1}{2}mv^2 + U(x) = E,$$

which gives

$$\tfrac{3}{2}v^2 + \frac{2}{x} = E.$$

Initially, $v = -1$ when $x = 10$, which leads to

$$E = \tfrac{17}{10} \quad \text{(in joules)}.$$

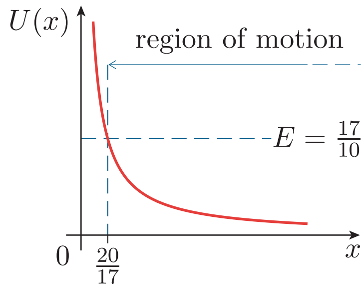
(If you used a different datum for the potential energy function, as above, then you should have found $E = \frac{17}{10} + C$.)

(c) At the point of closest approach, the particle must be stationary, so $v = 0$. Then from the law of conservation of mechanical energy, the corresponding value of x is given by

$$\tfrac{3}{2} \times 0^2 + \frac{2}{x} = \tfrac{17}{10}.$$

Hence the point of closest approach is $x = \frac{20}{17}$, that is, about 1.18 m.

(d) The region of motion is $x \geq \frac{20}{17}$, and a sketch of $U(x)$ is shown in the margin.



The particle is initially at $x = 10$, moving in the direction of decreasing x. It comes instantaneously to rest at $x = \frac{20}{17}$, and changes its direction of motion. It then continues to move in the direction of increasing x. For large values of x, its velocity will approach $v = \sqrt{17/15}$ (again using the law of conservation of mechanical energy), that is, about $1.06\,\mathrm{m\,s^{-1}}$.

(e) The graph shows no extrema, so there are no equilibrium points for this potential energy function.

Solution to Exercise 34

When the buffer spring is at maximum compression, the spring potential energy (using the natural length as the datum) is

$$\tfrac{1}{2} \times \text{stiffness} \times (\text{deformation})^2 = \tfrac{1}{2} \times 10^5 \times (0.1)^2 = 500,$$

and the kinetic energy of the truck is zero. So the total mechanical energy of the system is 500 J.

When the truck first comes into contact with the buffers, the spring potential energy is zero and the kinetic energy of the truck is $\frac{1}{2} \times 2000v^2$, where v is the speed of the truck. Using conservation of energy, we have

$$1000v^2 = 500,$$

so $|v| = 1/\sqrt{2}$, that is, the speed of the truck when it first hits the buffers is about $0.71\,\mathrm{m\,s^{-1}}$ to two significant figures.

Solution to Exercise 35

(a) Take Q as a datum for gravitational potential energy; this choice is the easiest for subsequent working.

The position of the particle above the datum is $-x$, so the gravitational potential energy is

$$U_1 = -mgx.$$

(b) The kinetic energy is given by

$$T_1 = \tfrac{1}{2}m\dot{x}^2.$$

(c) The potential energy stored in a stretched (or compressed) spring is given by $\frac{1}{2}kd^2$, where d is the deformation (extension or compression) relative to the natural length of the spring, and k is the stiffness. A tabular approach similar to that used for Exercise 18, and repeating many of the entries, is useful. We use the natural length of each spring as the datum point for the energy of the spring.

Spring	Deformation	Stiffness	Potential energy
QP	$x - l_0$	$2k$	$U_2 = \frac{1}{2}(2k)(x - l_0)^2$
RP	$x - l_0$	$5k$	$U_3 = \frac{1}{2}(5k)(x - l_0)^2$
SP	$\frac{5}{2}l_0 - x$	$2k$	$U_4 = \frac{1}{2}(2k)\left(\frac{5}{2}l_0 - x\right)^2$

So the potential energy stored in each spring is

$$U_2 = k(x - l_0)^2,$$
$$U_3 = \tfrac{5}{2}k(x - l_0)^2,$$
$$U_4 = k\left(\tfrac{5}{2}l_0 - x\right)^2.$$

(d) The total mechanical energy in the system at a general point is the sum of the various energies listed above, that is,

$$\begin{aligned} E &= T_1 + U_1 + U_2 + U_3 + U_4 \\ &= \tfrac{1}{2}m\dot{x}^2 - mgx + k(x - l_0)^2 + \tfrac{5}{2}k(x - l_0)^2 + k\left(\tfrac{5}{2}l_0 - x\right)^2. \end{aligned}$$

This is a constant since this system conserves mechanical energy.

Writing

$$U(x) = -mgx + k(x - l_0)^2 + \tfrac{5}{2}k(x - l_0)^2 + k\left(\tfrac{5}{2}l_0 - x\right)^2$$

gives

$$\begin{aligned} U'(x) &= -mg + 2k(x - l_0) + 5k(x - l_0) - 2k\left(\tfrac{5}{2}l_0 - x\right) \\ &= -mg + 9kx - 12kl_0, \end{aligned}$$

so from equation (31) we have

$$0 = m\ddot{x} - mg + 9kx - 12kl_0,$$

which can be rearranged as

$$m\ddot{x} + 9kx = mg + 12kl_0,$$

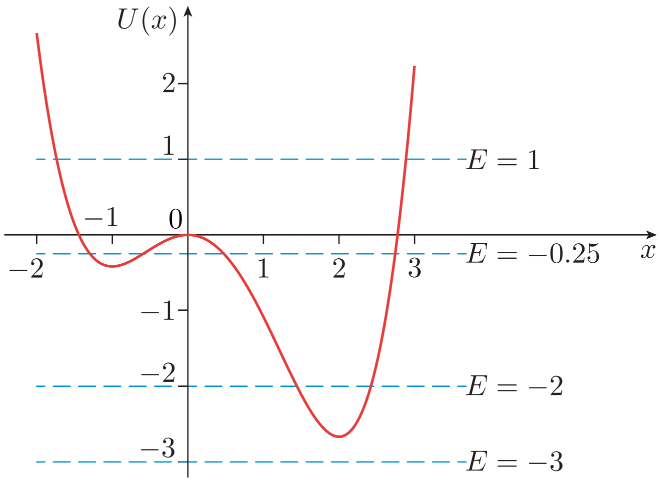
in agreement with the solution to Exercise 18(c).

Solution to Exercise 36

(a) The force is given by $F(x)\,\mathbf{i}$ where

$$F(x) = -\frac{dU}{dx} = -x^3 + x^2 + 2x.$$

(b) In each case the system is constrained by the fact that $E \geq U$. The graph below shows the potential energy function and the four values for E.



- From the graph it is apparent that the inequality $U(x) \leq -3$ has no solution (i.e. the minimum of $U(x)$ is greater than -3).
- For $E = -2$ the graph gives an approximate range

  $$1.4 < x < 2.4$$

  as a range for possible motion.
- It is clear from the graph that there are two possible ranges of motion when $E = -0.25$. The graph shows two disjoint ranges of x where $E \geq U$, namely

  $$-1.3 < x < -0.6 \quad \text{and} \quad 0.5 < x < 2.7.$$
- The range of possible values for $E = 1$ is

  $$-1.7 < x < 2.9.$$

(c) We have

$$U'(x) = x^3 - x^2 - 2x = x(x-2)(x+1),$$

so there are three equilibrium points, -1, 0 and 2. It is clear from the graph that -1 and 2 are minima, thus oscillations will take place around these two equilibrium points.

Acknowledgements

Grateful acknowledgement is made to the following sources:

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