Unit 18

Reviewing the model

Introduction

In Unit 8, mathematical modelling was introduced via a set of key stages as follows.

1 **Specify the purpose of the model:**
define the problem;
decide which aspects of the problem to investigate.

2 **Create the model:**
state assumptions;
choose variables and parameters;
formulate mathematical relationships.

3 **Do the mathematics:**
solve equations;
draw graphs;
derive results.

4 **Interpret the results:**
collect relevant data;
describe the mathematical solution in words;
decide what results to compare with reality.

5 **Evaluate the model:**
test the model by comparing its predictions with reality;
criticise the model.

The diagram in Figure 1 was introduced to emphasise these key stages.

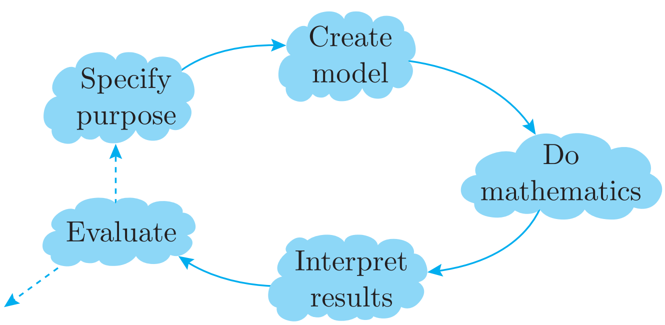


Figure 1 Mathematical modelling process

In Unit 8 the emphasis was on creating a first model, by making sufficient simplifying assumptions so that the essential features of the model are considered, without getting lost in a myriad of practical details. In this unit you will have the opportunity to improve your modelling skills and to see how models might be improved by assessing their deficiencies, before going round the modelling cycle again, taking steps to rectify some of those deficiencies. It should be noted that most complex models start out as very simple models, with more features added stage by stage to overcome the various shortcomings of the current model.

It should be stated at the outset that in modelling there is no right model. Any model that follows on from the assumptions and simplifications made is a useful model. Other models may give better predictions, but a simple model that includes the essential details may be much better than a complex model that tries to cover too much. When you start writing modelling reports, you should produce a fairly simple model and suggest how this might be revised. In writing a report, the key thing is to follow the modelling process, for example, listing the assumptions and simplifications, and showing how this establishes the model. In this module, this is more important than the model that you derive.

In Section 1 we introduce the method of dimensional analysis as a way of creating relationships between variables that are dimensionally consistent. In Section 2 we look at the sensitivity of the solution to variations in the parameters as a method of estimating the accuracy of the solution. We also revisit the tin can model created in Unit 8, to discuss its limitations with a view to trying to improve the model. In Section 3 we return to the problem of pollution in the Great Lakes to give examples of how the very simple model developed in Unit 8 can be gradually made more sophisticated by revising some of the simplifying assumptions that were made. Finally, Section 4 presents a new case study as a demonstration of the way in which a modelling investigation can be written up.

1 Method of dimensional analysis

In the final section of Unit 8, we used the base dimensions of physical quantities to check the consistency of equations and to find the dimensions of parameters in equations. However, the fact that an equation must be dimensionally consistent can also be used to *suggest* or check for a possible form of the relationship between the physical quantities involved in modelling a situation, once the key variables and parameters have been identified. This process is called the method of **dimensional analysis**.

We will illustrate the method by considering the example of a projectile, which we met in Unit 3, Section 5. Our aim is to find the form of the formula for the range of a projectile thrown on horizontal ground in terms of the parameters involved in the situation. If we assume that the projectile is thrown from ground level and neglect any effects due to air resistance, then the principal quantities involved in the situation are those listed in Table 1.

In this case we already know the formula, so this example illustrates the general method in a situation with which we are already familiar.

Table 1

Physical quantity	Symbol	Dimensions
Range of projectile	R	L
Mass of projectile	m	M
Launch speed of projectile	u	$\mathrm{L\,T^{-1}}$
Launch angle of projectile	θ	1
Magnitude of acceleration due to gravity	g	$\mathrm{L\,T^{-2}}$

Our aim is to find the range R as a function of the four parameters. In other words, we are looking for a function f of four variables such that

$$R = f(m, u, \theta, g).$$

We begin by assuming that

$$R = k\, m^{\alpha}\, u^{\beta}\, \theta^{\gamma}\, g^{\delta}, \tag{1}$$

where k is a (dimensionless) real number, and the powers α, β, γ and δ are to be found.

For this equation to be dimensionally consistent, we must have

$$[R] = [k\, m^{\alpha}\, u^{\beta}\, \theta^{\gamma}\, g^{\delta}] = [k]\, [m]^{\alpha}\, [u]^{\beta}\, [\theta]^{\gamma}\, [g]^{\delta}.$$

As in Unit 8, square brackets denote 'the dimensions of'.

In terms of the base dimensions, we have

$$\mathrm{L} = \mathrm{M}^{\alpha}\, (\mathrm{L\,T^{-1}})^{\beta}\, (\mathrm{L\,T^{-2}})^{\delta} = \mathrm{M}^{\alpha}\, \mathrm{L}^{\beta+\delta}\, \mathrm{T}^{-\beta-2\delta},$$

as k and θ are dimensionless. Equating powers of M, L and T on both sides of this equation, we have

$$0 = \alpha, \quad 1 = \beta + \delta, \quad 0 = -\beta - 2\delta.$$

Solving these equations for α, β, γ and δ gives

$$\alpha = 0, \quad \beta = 2, \quad \delta = -1.$$

So the requirement that the equation must be dimensionally consistent has given us values of α, β and δ. Because the parameter θ is dimensionless, the method cannot give us a value for γ.

We have shown that

$$R = k\, u^2\, g^{-1}\, \theta^{\gamma}$$

is dimensionally consistent for any value of the power γ. We can take an additive combination of such expressions with different values of γ and k, which will still be dimensionally consistent. Hence the best we can deduce is that

$$R = k(\theta)\, u^2\, g^{-1} = k(\theta)\, \frac{u^2}{g}, \tag{2}$$

where k is now a function of θ. As the launch angle θ is dimensionless, the function $k(\theta)$ is also dimensionless, and equation (2) is dimensionally consistent.

Dimensional analysis can tell us nothing further about the form of the function $k(\theta)$. In order to find this function, we would have to use Newton's second law to determine it analytically, as we did in Unit 3, finding that

$$k(\theta) = \sin 2\theta,$$

or we would have to use a suitable experiment to find it numerically.

In the following exercise, you will find the form of the relationship for the period of a pendulum. From Unit 11, we already know this relationship for *small* oscillations of the pendulum, where we can use the approximation $\sin\theta \simeq \theta$ throughout the oscillations, but here we will consider the case of *large* oscillations.

Exercise 1

In this exercise we neglect air resistance.

A pendulum is oscillating in a vertical plane. Use dimensional analysis to find a possible form for the expression for the period τ of the pendulum in terms of the mass m of the bob, the length l of the pendulum stem, the angular amplitude Φ of the oscillations, and g, the magnitude of the acceleration due to gravity.

In Exercise 1, we derived the result

$$\tau = k(\Phi)\sqrt{\frac{l}{g}} \tag{3}$$

for the period of a pendulum. You may have been disappointed in this result, in that we have not been able to determine the function $k(\Phi)$. But in fact the result is very powerful. First, it shows that the period of oscillation of the pendulum is independent of the mass of the bob – a fact that we can easily verify by experiment in situations where air resistance is small. Second, it implies that for a given angular amplitude, the period is proportional to $\sqrt{l/g}$. So if we can determine the period of oscillation for one known length of pendulum stem for a particular angular amplitude, say $\Phi = \pi/4$, then we can deduce the period of oscillation of any pendulum, no matter what its length is, for this amplitude, even if the pendulum is on Mars!

Let us now return to our initial assumption that the relationship between the parameters involves their powers, such as in equation (1). You might think that the relationship could involve transcendental functions such as exp, ln, sin, cos, tan, and so on. However, as we remarked in Unit 8, the argument of any such function must be dimensionless. In our previous example of the simple pendulum, if we ignore air resistance, there is no dimensionless combination of the parameters m, l and g, so the only possible transcendental function involved is in the term $k(\Phi)$.

So far we have discussed situations where there is only one possible combination of the parameters with the required dimensions. We will now discuss a situation where this is not the case.

We consider again our investigation of the range of a projectile, but include air resistance. Now the important parameters are R, m, u, θ, g (as before) and air resistance F. (Of course, air resistance is a function of speed at time t, which is not a parameter, but we can define F to be the magnitude of the air resistance at a typical speed, such as the launch speed u. The key thing here is that F will have the dimensions of force.) These parameters are listed in Table 2, together with their dimensions.

We have been rather vague about defining F. However, the second-order differential equation derived from Newton's second law can be solved numerically to determine the range, and this will depend uniquely on the initial conditions such as the launch speed and launch angle.

Table 2

Physical quantity	Symbol	Dimensions
Range of projectile	R	L
Mass of projectile	m	M
Launch speed	u	$\mathrm{L\,T^{-1}}$
Launch angle	θ	1
Acceleration due to gravity	g	$\mathrm{L\,T^{-2}}$
Typical air resistance	F	$\mathrm{M\,L\,T^{-2}}$

Again, we will assume a relationship between these parameters of the form

$$R = k\, m^\alpha\, u^\beta\, \theta^\gamma\, g^\delta\, F^\varepsilon, \tag{4}$$

where k is a dimensionless number. For this equation to be dimensionally consistent, we must have

$$\begin{aligned}[R] &= [k\, m^\alpha\, u^\beta\, \theta^\gamma\, g^\delta\, F^\varepsilon] \\ &= [k]\, [m]^\alpha\, [u]^\beta\, [\theta]^\gamma\, [g]^\delta\, [F]^\varepsilon.\end{aligned}$$

In terms of the base dimensions M, L and T, we have

$$\begin{aligned}\mathrm{L} &= \mathrm{M}^\alpha\, (\mathrm{L\,T^{-1}})^\beta\, (\mathrm{L\,T^{-2}})^\delta\, (\mathrm{M\,L\,T^{-2}})^\varepsilon \\ &= \mathrm{M}^{\alpha+\varepsilon}\, \mathrm{L}^{\beta+\delta+\varepsilon}\, \mathrm{T}^{-\beta-2\delta-2\varepsilon},\end{aligned}$$

as k and θ are dimensionless. Equating powers of M, L and T on both sides of this equation, we have

$$\alpha + \varepsilon = 0, \quad \beta + \delta + \varepsilon = 1, \quad -\beta - 2\delta - 2\varepsilon = 0.$$

We have three equations involving the four unknowns α, β, δ and ε. As before, the constant γ remains indeterminate. By adding twice the second of these three equations to the third, we deduce that $\beta = 2$. But we still have two equations, namely

$$\alpha + \varepsilon = 0, \quad \delta + \varepsilon = -1,$$

in the three unknowns α, δ and ε.

To proceed, we choose to write two of these unknowns in terms of the third. If we let $\varepsilon = a$, where a is an unknown parameter, then we have $\alpha = -a$ and $\delta = -1 - a$. Equation (4) becomes

$$\begin{aligned}R &= k\, m^{-a}\, u^2\, \theta^\gamma\, g^{-1-a}\, F^a \\ &= k\, \frac{u^2}{g}\, \theta^\gamma \left(\frac{F}{mg}\right)^a.\end{aligned}$$

Each choice of values of the parameters γ and a gives a possible expression for the range R, as does a linear combination of these possibilities. More generally, we can write

In the absence of air resistance, the range is given by
$$R = \frac{u^2 \sin 2\theta}{g},$$
which is of this form, as we saw earlier.

$$R = \frac{u^2}{g}\, f\left(\theta, \frac{F}{mg}\right), \tag{5}$$

where f is a function of two variables. The two quantities θ and F/mg are dimensionless.

You may be wondering whether we would have arrived at a different result if, when solving the equations $\alpha + \varepsilon = 0$, $\delta + \varepsilon = -1$, we had made a choice other than $\varepsilon = a$ for the unknown parameter. Suppose that we let $\delta = b$ instead. Then the solution of the equations is $\alpha = 1 + b$ and $\varepsilon = -1 - b$. So equation (4) becomes

$$\begin{aligned} R &= k\, m^{1+b}\, u^2\, \theta^\gamma\, g^b\, F^{-1-b} \\ &= k\, \frac{mu^2}{F}\, \theta^\gamma \left(\frac{mg}{F}\right)^b, \end{aligned}$$

and we deduce that the general expression for the range is

$$R = \frac{mu^2}{F}\, h\left(\theta, \frac{mg}{F}\right),$$

where h is a function of two variables.

At first sight this appears different from equation (5). However,

$$\begin{aligned} \frac{mu^2}{F}\, h\left(\theta, \frac{mg}{F}\right) &= \frac{u^2}{g}\, \frac{mg}{F}\, h\left(\theta, \frac{mg}{F}\right) \\ &= \frac{u^2}{g}\, f\left(\theta, \frac{F}{mg}\right), \end{aligned}$$

where

$$f(x, y) = \frac{1}{y}\, h\left(x, \frac{1}{y}\right).$$

(Similarly, if we had made a third possible choice, and let $\alpha = c$, we would again have arrived at an expression for the range R that is equivalent to equation (5).)

Dimensionless combinations of physical quantities, such as F/mg in equation (5), are called **dimensionless groups**. These are particularly important in situations where we cannot find an analytical solution to our model. For example, in the case of the range of a projectile with air resistance, equation (5) means that we have reduced the problem to finding, either numerically or by experiment, an unknown function of two dimensionless groups, namely θ and F/mg, so we do not have to consider the effect of varying every individual parameter involved in the model.

The above discussion of the range of a projectile, taking into account the effects of air resistance, illustrates the method of dimensional analysis, which is summarised in the following box.

Method of dimensional analysis

To find dimensionally consistent relationships, carry out the following steps.

1. List the parameters $y, x_1, x_2, \ldots, x_n$ that represent the important features involved in the situation being modelled, and determine their dimensions.
2. Assume a relationship involving the powers of these parameters, namely

   $$y = k\, x_1^{\alpha}\, x_2^{\beta}\, x_3^{\gamma} \cdots x_n^{\nu},$$

   where k is a dimensionless constant.
3. Use the principle of dimensional consistency to write

   $$[y] = [x_1]^{\alpha}\, [x_2]^{\beta}\, [x_3]^{\gamma} \cdots [x_n]^{\nu},$$

   and equate the powers of M, L and T on both sides of this equation.
4. Solve the four simultaneous equations obtained in Step 3 for the powers $\alpha, \beta, \ldots, \nu$. This solution will usually involve unknown parameters.
5. Substitute for the powers in the expression in Step 2, and rewrite it in terms of dimensionless groups. Hence write down a general expression for y, which will usually involve an unknown function of these dimensionless groups.

In practice, if you omit some of the key parameters, you may end up with a set of equations for the powers where there are, for example, more equations than variables, and these equations may well have no solution.

Exercise 2

Use dimensional analysis to find an expression for the frequency f of vibration of a guitar string, assuming that it depends on only the mass m of the string, the length l of the string, and the tension T_{eq} of the string.

This problem was considered in Unit 14.

2 Evaluating the model

Having used the techniques outlined in Unit 8, and in Section 1, to formulate mathematical models, in this section we will focus on their evaluation. We will describe three ways in which a model can be evaluated. The first, which you saw in Unit 8, is to use the model to predict the outcome of several events, which can then be compared with real data, if they are available. For example, in Unit 8, the tin can problem was based on the assumption that a manufacturer of cans would wish to minimise the amount of material used to construct a cylindrical can to hold a specified volume of food. After making a number of additional simplifying assumptions, we constructed a model that predicted that the height of the can would be equal to its diameter. Checking with a range of tin cans used to store food, you will find that rarely does the height of the can equal its diameter (so that, sideways on, the can would be square-shaped), so something has gone wrong.

If the predictions are accurate, then the model can be used with confidence. If they are accurate only for specific ranges of the parameters, then these limitations can be noted. If they are inaccurate for a range of the parameters in which the model is required to make predictions, then we need to look at the assumptions that underpin the model to see if one or two of them can be changed to try to improve the accuracy of the model. This second process is called *criticising the model*.

There is a third check that we need to carry out on our model, to see how sensitive it is to variations in the parameters of the model. We may have made many assumptions and simplifications in order to arrive at a mathematical model. If a small change in one of the parameters causes a significantly larger change in the prediction, then we have a problem, since it is likely that neither the model nor the data used to interpret the results will be particularly accurate, and such sensitivities may undermine the usefulness of the model to make predictions. We start this section by looking at the sensitivity of models.

2.1 Sensitivity analysis

The aim of this subsection is to determine the sensitivity of the output value, obtained from a calculation, to the values of the parameters used in that calculation. You will need an understanding of partial derivatives, as described in Unit 7, to follow the analysis in this subsection. We will also look at empirical methods of assessing sensitivity, by carrying out numerical experiments on the model.

This approach is very similar to the approach to ill-conditioning in Unit 4.

A modelling process usually results in an equation that gives the value of the variable, or variables, of interest in terms of the model parameters. Figure 2 shows this in diagrammatic form.

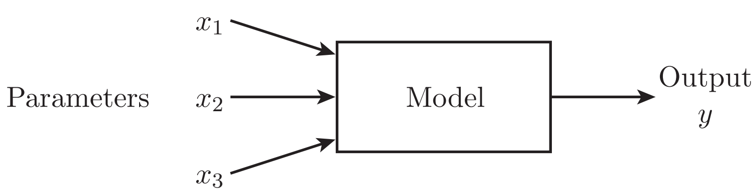


Figure 2 Diagram showing the inputs and output for a model

In most situations, the values of the parameters are not known exactly, and they will have some error margin associated with them, as shown in Figure 3.

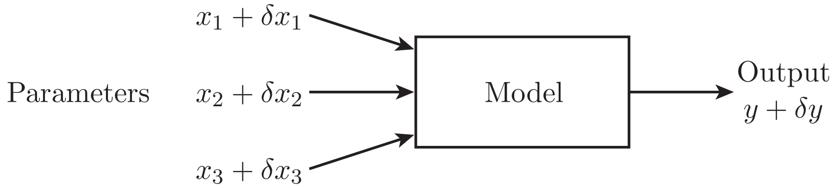


Figure 3 Diagram showing the inputs and output, with errors, for a model

Sensitivity analysis is used to quantify the effect that parameter values have on the output of the model. Errors in some of the parameters may have more effect on the final value than errors in other parameters. The output is sensitive to a particular parameter if a small change in that parameter has a significant effect on the output. To examine this, we need some definitions.

The **absolute sensitivity** of a solution value y to a parameter value x is defined as $\delta y/\delta x$, where δy is the change in y caused by a small change δx in the value of the parameter x.

The **relative sensitivity** of a solution value y to a parameter value x is defined as

$$\frac{\delta y/y}{\delta x/x} = \frac{x}{y}\frac{\delta y}{\delta x}, \tag{6}$$

where $\delta y/y$ is the relative (or proportionate) change in y for a small relative change $\delta x/x$ in the value of the parameter x.

Where the dimensions, or units, of x and y are different, it is better to consider the relative sensitivity, since a change in the unit of measurement will affect the value of the absolute sensitivity, but not the relative sensitivity, which is dimensionless.

There are two ways of determining the sensitivity of the final answer to changes in the parameter values. One way is an empirical approach, achieved by changing the value of a parameter by a small amount, then considering its effect on the final solution. The other method is analytical, based on finding the rate of change of the variable of interest with respect to each of the parameters.

In the empirical approach, only one parameter value at a time is changed; if more than one parameter value is changed, then the effects of one change may be masked by another.

For the analytical approach, where $y = f(x_1, x_2, \dots, x_n)$, the absolute sensitivity of y with respect to x_i for infinitesimally small changes in x_i is $\partial y/\partial x_i$; the relative sensitivity is

$$\frac{x_i}{y}\frac{\partial y}{\partial x_i}. \tag{7}$$

We consider both approaches in the following example.

Example 1

We want to estimate the depth of a well by measuring the time that a pebble takes to hit the surface of the water when dropped into the well. A simple model, neglecting air resistance, gives

This model was derived in Unit 3.

$$h = ut + \tfrac{1}{2}gt^2,$$

where h is the depth of the well, u is the initial speed of the pebble, g is the value of the acceleration due to gravity, and t is the time for the pebble to drop.

Although the pebble may be dropped from rest, so that $u = 0$, u is included so that the effect of a non-zero value can be considered.

In one measurement, the pebble was dropped from rest and a splashing sound was heard 2.8 s later. The estimate for the depth of the well is

$$h = \tfrac{1}{2} \times 9.81 \times 2.8^2 = 38.455.$$

Although the accuracy quoted is far in excess of what is needed, when performing a sensitivity analysis it is necessary to use greater precision, since the changes to the parameter values will be small.

So the well is about 38.5 m deep. The question is, how accurate is this estimate of the depth? Is it unduly sensitive to the measurement of the time taken? How accurate should the value of g be?

Solution

Since the values that are used to estimate the depth are in units different to those for the depth, it is better to consider the *relative* change in the values; otherwise a change of units would affect the calculation of the sensitivity.

- First, try the empirical approach. Change the value of the time taken to fall, then consider the effect that this has on the estimate of the depth. On increasing the measured time from 2.8 to 2.83, a relative change of about 1%, the estimate of the depth changes from 38.455 to 39.284, an increase of approximately 0.8 m. This is equivalent to a relative change of $(39.284 - 38.455)/38.455 \simeq 0.02$ or 2%. So the small relative change in the time for the pebble to drop, when $t = 2.8$, has magnified the relative change in the depth by a factor of 2.

  Now, instead, change the value of g to 9.80, a relative change of $|(9.80 - 9.81)/9.81| \simeq 0.001$, which is 0.1%. The resulting value for the depth is now 38.416, a decrease of approximately 0.3 m. The relative change in the value for the depth of the well is $|(38.416 - 38.455)/38.455| \simeq 0.001$ or 0.1%. The relative change in depth is approximately the same as the relative change in the value of g when $g = 9.81$.

- Now consider the analytical approach. Think of the expression for h as a function of the two variables t and g:

  $$h(t, g) = ut + \tfrac{1}{2}gt^2.$$

  Taylor approximations for functions of two variables were developed in Unit 7.

  The first-order Taylor approximation around the nominal values $t = t_0$ and $g = g_0$ is

  $$h(t_0 + \delta t, g_0 + \delta g) \simeq h(t_0, g_0) + \delta t\,\frac{\partial h}{\partial t}(t_0, g_0) + \delta g\,\frac{\partial h}{\partial g}(t_0, g_0),$$

  where δt is the small change in the value of t, and δg is the small change in the value of g.

Thus the change in the depth δh is given by

$$\delta h = h(t_0 + \delta t, g_0 + \delta g) - h(t_0, g_0) \simeq \delta t \, \frac{\partial h}{\partial t}(t_0, g_0) + \delta g \, \frac{\partial h}{\partial g}(t_0, g_0).$$

For this problem, the partial derivatives are

$$\frac{\partial h}{\partial t}(t, g) = u + gt \quad \text{and} \quad \frac{\partial h}{\partial g}(t, g) = \tfrac{1}{2}t^2,$$

so we have, at $t = t_0$ and $g = g_0$,

$$\delta h \simeq (u + g_0 t_0)\, \delta t + \tfrac{1}{2} t_0^2 \, \delta g.$$

When $u = 0$, $g_0 = 9.81$ and $t_0 = 2.8$, we have $\partial h/\partial t = u + g_0 t_0 = 27.5$ and $\partial h/\partial g = \frac{1}{2}t_0^2 = 3.92$. Thus the absolute change in the depth of the well for small changes in t and g is

$$\delta h \simeq 27.5\, \delta t + 3.92\, \delta g.$$

As we said earlier, these results are not particularly useful since h, t and g have different units, and a change in the units of measurement will affect the results. We can, however, say that with the units that we have chosen, the model is absolutely sensitive to changes in t.

We describe the module convention for absolute sensitivity at the end of this example.

A more important measure is the effect of relative changes. The relative error in the value of the depth of the well is $\delta h/h_0$, where $h_0 = ut_0 + \frac{1}{2}g_0 t_0^2$. The Taylor approximation can be rearranged to give the relative error in h as

$$\begin{aligned}
\frac{\delta h}{h_0} &\simeq \frac{\frac{\partial h}{\partial t}(t_0, g_0)}{h_0}\, \delta t + \frac{\frac{\partial h}{\partial g}(t_0, g_0)}{h_0}\, \delta g \\
&= \frac{t_0}{h_0} \frac{\partial h}{\partial t}(t_0, g_0) \left(\frac{\delta t}{t_0}\right) + \frac{g_0}{h_0} \frac{\partial h}{\partial g}(t_0, g_0) \left(\frac{\delta g}{g_0}\right),
\end{aligned}$$

which gives the relative error in the depth of the well h in terms of the relative errors in t and g.

For our model, where

$$h = ut + \tfrac{1}{2}gt^2, \quad \frac{\partial h}{\partial t}(t, g) = u + gt \quad \text{and} \quad \frac{\partial h}{\partial g}(t, g) = \tfrac{1}{2}t^2,$$

we have, at $t = t_0$ and $g = g_0$,

$$\begin{aligned}
\frac{\delta h}{h_0} &\simeq \frac{t_0(u + g_0 t_0)}{ut_0 + \frac{1}{2}g_0 t_0^2} \left(\frac{\delta t}{t_0}\right) + \frac{g_0 \frac{1}{2} t_0^2}{ut_0 + \frac{1}{2}g_0 t_0^2} \left(\frac{\delta g}{g_0}\right) \\
&= \frac{(u + g_0 t_0)}{u + \frac{1}{2}g_0 t_0} \left(\frac{\delta t}{t_0}\right) + \frac{\frac{1}{2}g_0 t_0}{u + \frac{1}{2}g_0 t_0} \left(\frac{\delta g}{g_0}\right).
\end{aligned}$$

If we assume that the pebble is dropped from rest, so that $u = 0$, then considerable further simplification occurs and we have

$$\frac{\delta h}{h_0} \simeq 2\left(\frac{\delta t}{t_0}\right) + \left(\frac{\delta g}{g_0}\right)$$

as we predicted in the earlier empirical approach. However, the analytical approach tells us that these results hold for a range of values for t_0 and g_0, provided that the pebble is dropped from rest.

We conclude that a small relative change in the time is magnified by a factor of 2 in the relative change in the depth of the well, while a small relative change in the value of g gives a similar relative change in the depth of the well. Hence the predicted depth is less sensitive to relative changes in g than to relative changes in t.

Example 1 illustrates that for absolute and relative sensitivity, the empirical approach to the effect of a change in the parameters $x_1, x_2, \dots, x_n$ on the value of $y(x_1, x_2, \dots, x_n)$ can be done numerically by simply changing one of the parameters by a small amount and assessing its effect on the output, while the analytic approach is based on a first-order Taylor series approximation

$$\begin{aligned}\delta y &= y(x_1 + \delta x_1, x_2 + \delta x_2, \dots, x_n + \delta x_n) - y(x_1, x_2, \dots, x_n)\\ &\simeq \delta x_1 \frac{\partial y}{\partial x_1} + \delta x_2 \frac{\partial y}{\partial x_2} + \cdots + \delta x_n \frac{\partial y}{\partial x_n}.\end{aligned}$$

We could use this to consider a change in one variable at a time as

$$\begin{aligned}\delta y_i &= y(x_1, x_2, \dots, x_i + \delta x_i, \dots, x_n) - y(x_1, x_2, \dots, x_i, \dots, x_n)\\ &\simeq \delta x_i \frac{\partial y}{\partial x_i},\end{aligned}$$

or to consider the effect of changing several of the variables, if we so wished. Similarly, the corresponding relative changes are given by

$$\frac{\delta y}{y} \simeq \frac{\delta x_1}{x_1}\left(\frac{x_1}{y}\frac{\partial y}{\partial x_1}\right) + \frac{\delta x_1}{x_2}\left(\frac{x_2}{y}\frac{\partial y}{\partial x_2}\right) + \cdots + \frac{\delta x_n}{x_n}\left(\frac{x_n}{y}\frac{\partial y}{\partial x_n}\right).$$

This equation relates the relative change in y, namely $\delta y/y$, to the relative change in each variable x_j, namely $\delta x_j/x_j$. So the coefficient of $\delta x_j/x_j$ must be the relative sensitivity, as given in equation (7).

If we simply wish to consider the effect of a relative change in the parameter x_j, with all the other parameters fixed, then we have

$$\frac{\delta y}{y} \simeq \frac{\delta x_j}{x_j}\left(\frac{x_j}{y}\frac{\partial y}{\partial x_j}\right),$$

and this formula will enable us to analyse the sensitivity of the output y for a range of values for the parameters, not just a particular value. Hence this approach is more thorough than the empirical approach, which gives results only for particular values of the parameters.

Conditions for absolute and relative sensitivity

Suppose that small absolute or relative changes are made in the parameters of a model. The model is absolutely (relatively) sensitive if it is possible for the absolute (relative) change in the solution to be significantly larger than the absolute (relative) change in one or more of the parameters.

Usually the interpretation of *significantly larger* is dependent on the context. However, for the sake of clarity and certainty, we adopt the module convention. A model is judged to be:

- absolutely (relatively) insensitive with respect to a particular parameter if a small absolute (relative) change in the parameter is magnified by a factor less than or equal to 5
- neither absolutely (relatively) sensitive nor absolutely (relatively) insensitive with respect to a particular parameter if a small absolute (relative) change in the parameter is magnified by a factor of between 5 and 10
- absolutely (relatively) sensitive with respect to a particular parameter if a small absolute (relative) change in the parameter is magnified by a factor greater than or equal to 10.

Exercise 3

Consider the following problem of mortgage repayment. Let C_n be the capital sum owing in year n, let i be the annual interest rate (assumed constant), and let R be the constant sum repaid each year, where C_n and R are measured in pounds. The capital sum owing the next year is the capital sum owing in the previous year plus the interest on this capital sum less the repayment. Expressed mathematically, this becomes

$$C_{n+1} = C_n + iC_n - R.$$

This is an iterative formula whose explicit solution is

$$C_n = \left(C_0 - \frac{R}{i}\right)(1+i)^n + \frac{R}{i},$$

where C_0 is the capital sum originally owed.

The solution can be checked by substituting values of $0, 1, 2, \ldots$ for n.

Take an example where the initial mortgage is £100 000, and £8000 is paid every year, with an interest rate of 5%. Using the defined symbols, this gives $R = 8000$, $i = 0.05$ and $C_0 = 100\,000$, and the capital sum owing is given by

$$C_n = \left(100\,000 - \frac{8000}{0.05}\right)(1.05)^n + \frac{8000}{0.05} = 160\,000 - 60\,000(1.05)^n.$$

So the sum still outstanding 15 years after the start of the mortgage will be

$$\begin{aligned} C_{15} = 160\,000 - 60\,000 \times 1.05^{15} &= 160\,000 - 60\,000 \times 2.078\,928 \\ &= 35\,264.31. \end{aligned}$$

How sensitive is the amount owing after 15 years to changes in the values of the following parameters? Use the empirical approach to determine the sensitivity of the solution to changes in the values of both parameters.

(a) The annual payment

(b) The interest rate

In Exercise 3 we considered the effect of increasing the annual repayment. As the number of years increases, the amount owing decreases more rapidly as more is paid off each year, which also reduces the amount of interest on the remaining mortgage. Consequently the extra payment can have a significant effect on the time taken to repay the mortgage. The model becomes increasingly sensitive to the values of the parameters as n increases and the amount owing decreases. For example, suppose that the model is used to predict how long it will take to pay off the mortgage. For the given values of the parameters, the model suggests that the mortgage will be repaid some time in the second month of the 21st year. If the repayment is increased by £100 per year, then the mortgage will be repaid in the ninth month of the 20th year, some five months earlier. Similarly, if the interest rate is reduced by 0.1% then the mortgage will be repaid in the tenth month of the 20th year, four months earlier, assuming that the repayments are continued at the same rate.

Exercise 4

Consider the trajectory of a particle launched from a height h, with speed u, at an angle θ to the horizontal. If we assume that the only force acting on the particle is the weight of the particle, then the trajectory is given by

See Unit 3, Subsection 5.2 for a derivation of this model.

$$y = h + x\tan\theta - x^2\frac{g}{2u^2}\sec^2\theta.$$

We are interested in how far the particle travels (i.e. the value of x) before it hits the ground (i.e. when $y = 0$). The analytical approach is to be used to investigate the relative sensitivity of the model.

(a) Derive a formula that can be used to determine the relative sensitivity of this model to changes in the launch speed u.

(b) If $h = 2$, $u = 20$, $\theta = \pi/4$ and $g = 9.81$, then it can be shown that the particle hits the ground when $x = 42.68$. Use the formula found in part (a) to estimate how much further the particle travels before it hits the ground if u is increased to 21.

2.2 Comparing predictions with reality and criticising the model

In Unit 8 we developed a model for the manufacture of tin cans in which we wished to determine the height h and radius r of a cylindrical can that would minimise the amount of metal used for a given volume V of food. The following simplifying assumptions were made.

We have modified the wording slightly here.

Assumption 1: The cans are manufactured from sheet metal of a uniform thickness.

Assumption 2: Any overlapping metal needed to form joins is ignored.

Assumption 3: Any leftover metal can be recycled.

Assumption 4: The costs of manufacture can be ignored.

We thus had to minimise the surface area A given by the cylindrical portion of the can plus two circular pieces, that is,

$$A = 2\pi rh + 2\pi r^2,$$

such that the volume $V = \pi r^2 h$ is fixed. Substituting $h = V/(\pi r^2)$ into the equation for A leads to the problem of determining the value of r that minimises

$$A = \frac{2V}{r} + 2\pi r^2.$$

Differentiating with respect to r to find the maximum and minimum values of A gives

$$\frac{dA}{dr} = -\frac{2V}{r^2} + 4\pi r = 0,$$

This is a minimum since

$$\frac{d^2A}{dr^2} = \frac{4V}{r^3} + 4\pi > 0.$$

from which we deduced that $r = (V/(2\pi))^{1/3}$ and $h = 2r$, and the minimum area of metal needed is

$$A = 2\pi rh + 2\pi r^2 = 4\pi r^2 + 2\pi r^2 = 6\pi r^2 = 6\pi \left(\frac{V}{2\pi}\right)^{2/3} = 3\sqrt[3]{2V^2\pi}.$$

We concluded that the height of the can should be the same as its diameter.

It was commented in Unit 8 that this rarely matches reality, with most cans being taller than they are wide. So what has gone wrong? It is tempting to conclude that manufacturers have no knowledge of mathematics and hence are not finding the optimal solution, but this might well be wrong. Having found a solution that does not match reality, we need to revisit our modelling process and re-examine what has been done. For example, is the real problem to minimise the amount of metal used? The sides of the can are often used to advertise the product. Is this important? Studies have suggested that certain shapes are more pleasing to the eye, and such products may fly off the shelves more rapidly than cans with a square-shaped profile. These may well be issues that the manufacturer takes into consideration.

Before we look at the assumptions, it is helpful to consider the sensitivity of the area A of metal used to changes in r while keeping the volume fixed. At the minimum, $dA/dr = 0$, so the model is relatively insensitive to changes in r.

We can also express A in terms of h as

$$A = 2\sqrt{\pi V h} + \frac{2V}{h},$$

and minimise the area by solving $dA/dh = 0$ for h. Hence we can deduce that at the minimum, the model is also insensitive to changes in h.

Away from the optimal value, suppose that we set $h_1 = 4r_1$ so the height of the can is twice the width. The volume $V = \pi r_1^2 h_1 = 4\pi r_1^3$ is still fixed, so $r_1 = (V/(4\pi))^{1/3}$. The output from the model is the area

$$\begin{aligned} A_1 &= 2\pi r_1 h_1 + 2\pi r_1^2 \\ &= 8\pi r_1^2 + 2\pi r_1^2 = 10\pi r_1^2 = 10\pi \left(\frac{V}{4\pi}\right)^{2/3} = 5\sqrt[3]{V^2\pi/2}. \end{aligned}$$

The relative change in the area of metal needed is

$$\frac{A_1 - A}{A} = \frac{(5\sqrt[3]{1/2} - 3\sqrt[3]{2})\sqrt[3]{V^2\pi}}{3\sqrt[3]{2}\sqrt[3]{V^2\pi}}$$
$$= \frac{5\sqrt[3]{1/2} - 3\sqrt[3]{2}}{3\sqrt[3]{2}} = \frac{5}{3\sqrt[3]{4}} - 1 = 0.050.$$

This is a rather remarkable result since it tells us that having a can that is twice as high as it is wide requires only an extra 5% of material to manufacture. This might well explain why manufacturers may use different criteria than minimising the amount of material.

Exercise 5

(a) How much extra material is needed for the tin can if $h_2 = r_2$ so that the height is half the width of the can? What can you conclude about the sensitivity of the amount of material A_2 needed for $r \leq h \leq 4r$?

(b) For the cases $h_1 = 4r_1$ and $h_2 = r_2$, what is the change in the amount of advertising space $W = 2\pi rh$ on the wall of the can?

However, let us stay with the idea that we are trying in some way to minimise the amount of material, or perhaps the total cost of manufacture, and have a look at the simplifying assumptions that we made.

Assumption 1 looks fairly reasonable, although it could be noted that some cans are sometimes *ribbed* to give added strength to the sides. It is also possible to manufacture cans where the top and bottom are made of a different thickness than the sides. Would this enable savings to be made?

Assumption 2 could certainly be modified to take into account the overlap for the circular pieces and in the vertical wall of the can.

Assumption 3 could be modified. Maybe all components of the can are made from rectangular pieces of metal, with leftover pieces discarded or scrapped.

Assumption 4 could be reviewed. The cost of manufacture includes the cost of the materials plus the cost of soldering the joins. Should the soldering costs be included?

The examination of the simplifying assumptions has led to at least four different ways of modifying the first model. You are asked to consider two of these in the following exercises.

Exercise 6

For the tin can problem, suppose that Assumption 3 is modified to

Assumption 3: Any leftover metal is discarded.

In this case the parts of the can are cut from rectangular pieces of metal.

In particular, the top and bottom of the can are cut from squares of side $2r$, with the leftover bits discarded. We will assume that these are the only bits that will be discarded.

(a) What is the new model for the area of material used?

(b) How much material is needed for the optimal-sized can, and what is the ratio of the height to the diameter of the can?

Exercise 7

For the tin can problem, consider a modification to Assumption 2:

Assumption 2: An overlap of width b is included in the revised model for the circular pieces and the vertical seam.

(a) What is the new model for the area of material used?

(b) What is the equation to be solved to find the radius that minimises (or maximises) the area of material?

Exercise 8

Consider the problem of estimating the depth of a well in Example 1. The model for estimating the height h of the well is

$$h = ut + \tfrac{1}{2}gt^2,$$

where u is the initial velocity of a pebble, and t is the time that it takes to drop to the bottom of the well. It is based on the following assumptions.

Assumption 1: The pebble is treated as a particle.

Assumption 2: All forces other than the gravitational force are ignored.

Assumption 3: The sound of the pebble hitting the water is heard instantaneously.

For a particular well, a pebble of diameter 1 cm was dropped from rest and the sound was heard 2.8 s later. This suggested that the well was 38.5 m deep. A subsequent measurement of the well gave the depth as 34 m.

(a) Discuss each of the assumptions in terms of the discrepancy in the depth of the well.

(b) How would the first model change if a linear model for air resistance was assumed?

(c) How would the first model change if the time for the sound to travel up the well was incorporated into the model? What difference would this make to the estimated depth of the well?

The speed of sound is $343\,\mathrm{m\,s^{-1}}$.

In the next section we return to the Great Lakes model, developed in Unit 8, to see how the assumptions that underpin it can be reassessed in order to develop an improved model.

3 A return visit to the Great Lakes

The information for this revised case study comes from two main sources:

The Great Lakes: An Environmental Atlas and Resource Book, available online at www.epa.gov/grtlakes/atlas/intro.html (accessed 23 November 2014);

R.V. Thomann and J.A. Mueller (1987) *Principles of Surface Water Quality Modeling and Control*, Harper & Row.

This section evaluates and revises the model developed in Section 1 of Unit 8, for pollution in the Great Lakes of North America. It is worth revisiting that section to remind yourself of the basic model and of the assumptions and simplifications that led to it. These are fundamental to a critical evaluation.

Subsection 3.1 evaluates the success of the first model by comparing the model's predictions against data on contamination levels in the Great Lakes. Subsection 3.2 proposes some revisions to the first model in an attempt to overcome some of its deficiencies. In Subsection 3.3 we briefly discuss the techniques that were used, in order to highlight the key activities involved.

3.1 Evaluation of the model

◀ Evaluate ▶

To see if the model is realistic, consider its purpose. The problem is to predict how long it will take for the level of pollution in a lake to reduce to a target level if all sources of pollution are eliminated. It is intended to use the model to investigate pollution levels in any one of the Great Lakes, although the model could be used for any polluted lake.

The usual way of evaluating a model is to compare the predictions of the model with real data. These data can come either from your own experiments or from some published source. In the current case, collecting data from your own experiments is not feasible. However, there are useful published data, based on research studies conducted for the US Environmental Protection Agency and the US National Oceanographic and Atmospheric Administration by the University of Minnesota and the University of Wisconsin. One such set of data is shown in Table 3.

PCB stands for polychlorinated biphenyl. PCBs have been used in the manufacture of transformers, capacitors and other electrical equipment.

Table 3 Total PCB concentrations in Lake Superior

Year	PCB concentration ($10^{-9}\,\mathrm{kg\,m^{-3}}$)
1978	1.73
1979	4.04
1980	1.13
1983	0.80
1986	0.56
1988	0.33
1990	0.32
1992	0.18

The model considered pollution in general, but applies equally well to any single pollutant. The set of data in Table 3 is based on one particular type of pollutant, PCBs, which were a major source of pollution in the Great Lakes in the 1960s. The manufacture and importation of PCBs was prohibited in the USA in 1975. It is useful to consider this pollutant in the model since it is fairly stable, that is, it does not react with other chemicals, is not broken down by sunlight, and so on.

Exercise 9

Consider the discussion above.

(a) Why do you think we need to consider a stable pollutant?

(b) Is it relevant that the manufacture and importation of PCBs stopped in 1975?

(c) What are the advantages of using Lake Superior for the evaluation of the model?

A plot of the data in Table 3 reveals that there is a curious and unexpectedly high concentration in 1979. All of the other data points seem to lie close to a smooth curve. For the moment, put this anomaly aside, and concentrate on the remaining data points. From the data it looks as if it takes 14 years (from 1978 to 1992, the actual time range of the data) for the concentration to fall to approximately a tenth of its former value, which relates to the specific question posed in the original model. However, the simple model predicts that it will take 440 years for the pollution level in Lake Superior to drop by a factor of ten. Clearly there is an enormous quantitative discrepancy between the model and these data.

Are the data at fault, or is the model at fault? One might question the data, particularly on the basis of the unexpectedly high concentration given for 1979. Was this due to measurement error? Does this particular value indicate that the data may be unreliable? Does it really reflect the level of pollution in that year? Do PCBs actually satisfy the assumptions of the model, or is there some process, relevant to PCBs, that has been overlooked? These are the kinds of question that you should ask when using data to evaluate the success of a model.

Without going back over the data and checking how they were obtained, it is difficult to explain the anomalous value for 1979. It has been suggested that this may have been caused by unusual disturbances of the sediment (for example, by storms, earthquakes or high winds). The sediment, which may contain high doses of the pollutant, would be drawn into the water and raise the measured value, while the reversion back to a much lower level the following year could have been caused by the PCBs settling back into the sediment from which they were disturbed. Regardless of whether this anomaly can be explained, these data will not validate the quantitative predictions of the model derived in Unit 8.

The model is not supported by the data. However, it is difficult to decide how to revise the model on the basis of a mismatch over a single value, namely, the time that it takes for the pollution concentration to reach a given level. A set of values or an equation to validate might be a better way to assess the success of the model and to give ideas on how best to revise it.

The main equation in the simple model proposes a relationship between the pollutant concentration, at any time after all pollution has ceased, and the time:

$$c(t) = c(0)\, e^{-kt},$$

where $c(0)$ is the initial concentration of pollutant. Do the data bear any qualitative resemblance to this equation? Since it involves a negative exponential, taking the logarithm of both sides gives

$$\ln(c(t)) = \ln(c(0)) - kt,$$

which predicts a straight-line graph when $\ln(c(t))$ is plotted against t (in years).

As Figure 4 illustrates, except for the anomalous value for 1979, the data are reasonably well approximated by a straight line. This confirms that the simple model does have some good qualitative agreement with the data, in the sense that a negative exponential model approximates reasonably well the data for PCBs. The value of k is given by the negative of the slope in Figure 4, that is, by $4.8 \times 10^{-9}\,\mathrm{s}^{-1}$. The value of k derived from the model is $0.166 \times 10^{-9}\ \mathrm{s}^{-1}$, which differs substantially from the experimental value, by a factor of approximately 30.

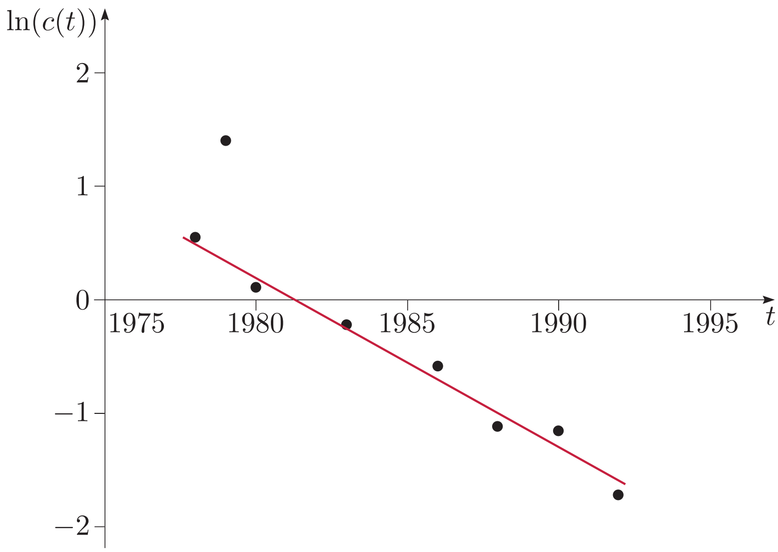


Figure 4 Plot of the data for Lake Superior

This is only one set of data, and more sets of data ought to be tried. Unfortunately, there are few published data on the actual concentrations of pollutants in the lakes (possibly because their measurement is time-consuming and costly). However, there are extensive data available on the contaminant concentration in fish and birds inhabiting the Great Lakes. It has been shown that over the long term, trends of contaminants (including PCBs) in such creatures have followed those in the water and thus provide a measure of the trends in the ecosystem of the Great Lakes. So it is reasonable to assume that if the simple model applies to the pollutant concentrations in the lakes, then it will apply to those in its inhabitants too.

Source: D. De Vault and R. Hesselberg (1994) ‘Contaminant trends in lake trout and walleye from the St Lawrence Great Lakes’, *Journal of Great Lakes Research*.

Table 4 gives the concentrations of PCBs in herring gull eggs collected around Lake Superior over a number of years. The concentrations are given in micrograms per gram (or equivalently, parts per million), rather than in kilograms per cubic metre, since this is a more common method of measuring concentrations in solids.

Table 4 PCB concentrations in herring gull eggs from Lake Superior

Year	PCB concentration ($\mu g\,g^{-1}$)
1974	65
1975	81
1977	59
1978	44
1979	60
1980	28
1981	36
1982	37
1983	23
1984	19

Year	PCB concentration ($\mu g\,g^{-1}$)
1985	21
1986	17
1987	14
1988	15
1989	17
1990	13
1991	15
1992	15
1993	16

To compare the concentration in herring gull eggs with the concentration in water, note that $1\,m^3$ of water weighs 1000 kg. Hence in 1978 the concentration of PCBs in the water (from Table 3) was 1.73×10^{-12} grams of PCB per gram of water. In the herring gull eggs that year, the concentration was 44×10^{-6} grams of PCB per gram of egg, which is a magnification of about 25 million.

The high concentrations here as compared with those in the lake are an example of *biomagnification*, in which small concentrations of pollutant at one point cause higher and higher concentrations when moving up the food chain.

Exercise 10

When evaluating the model, why should you disregard at least the first data point in Table 4?

Eliminating the first data point from Table 4 gives a value for k of 3.1×10^{-9}, which is similar to the value obtained with the first set of data. This confirms that while the negative exponential form of the relationship is approximately correct, the actual numerical values predicted by the model are in error.

The comparison being made here is valid provided that the magnification factor (from concentration in the water to concentration in gulls) and the time delay (referred to in the solution to Exercise 10) are both constant.

While the quantitative comparison (between the data and the model) of the time for the concentration of pollutant to drop to a tenth of its initial value gives no clue how to revise the model, the qualitative comparison gives some information on where to focus revision of the model – in the assumptions concerning the rate at which the pollutant dissipates.

3.2 Revisions to the model

This subsection investigates two possible revisions to the model as applied to PCBs. The first takes account of a process that affects the rate at which PCB concentration diminishes. The second revision tries to take into account the variation in concentration across the lake.

In considering revisions to the model, first review the modelling assumptions.

Exercise 11

Consider the assumptions made for the first model. Which of these assumptions do you think should be re-examined?

First revision: adding a process

◄ Create model ►

To explain the discrepancy between the prediction of the model and evidence of the data, the physics and chemistry of PCBs need examination, to determine the likeliest ways in which PCB concentrations can diminish other than by being flushed out of the lake by the flow of water. For any chemical in solution, there are four principal mechanisms that can cause its concentration to diminish:

The details of these processes would be important to a mathematical modeller investigating this problem, but you only need to understand that there are various additional processes that may affect the model.

- *volatilisation*, where the chemical is absorbed into and from the atmosphere at the surface of the lake
- *photolysis*, where the action of sunlight on water near the surface of the lake causes certain chemicals to degrade
- *hydrolysis*, where the chemical reacts with water, causing it to break down
- *biodegradation*, where the chemical is degraded by bacteria and other living organisms.

Now, PCBs are stable. This means that they are difficult to break down, which tends to rule out photolysis, hydrolysis and biodegradation as relevant processes. However, there is evidence that PCBs are volatile, so we consider this mechanism further. Inclusion of volatilisation amends Assumption (b) as follows:

(b) The pollutant does not biodegrade in the lake or decay through any other biological, chemical or physical process *except through volatilisation*.

An additional assumption is also needed, about the process of volatilisation.

Exercise 12

On what physical properties does the rate of volatilisation depend?

For a simple revision of the model, consider only the first two factors mentioned in the solution to Exercise 12. This results in the following additional assumption:

(h) The pollutant is lost from the lake to the atmosphere at a rate that is proportional to the total surface area of the lake and to the concentration of pollutant in the lake.

Suppose that the surface area of the lake is A (measured in square metres). The assumption is that the mass of the pollutant is lost at a rate that is proportional to both A and $c(t)$, where $c(t)$ is the pollutant concentration in the lake. Hence pollutant is lost at a rate proportional to $A\,c(t)$. Thus the differential equation that models the rate of loss of the mass of pollutant is

$$\frac{dm}{dt} = -k\,m(t) - pA\,c(t), \tag{8}$$

A detailed derivation of this differential equation would involve use of the input–output principle.

where p is the constant of proportionality, known as the *volatilisation rate*. Now, the model includes the equation $c(t) = m(t)/V$, which is still valid. So, by eliminating $c(t)$, equation (8) can be expressed entirely in terms of the mass m of pollutant, as

$$\frac{dm}{dt} = -k\,m(t) - \frac{pA}{V}\,m(t) = -\left(k + \frac{pA}{V}\right)m(t) = -\kappa\,m(t), \tag{9}$$

κ is the Greek letter kappa.

where $\kappa = k + pA/V$. This is essentially the same model as the first one, except that the *proportionate flow rate* k has now become a *proportionate decay rate* κ, and κ should be greater than k.

◀ Do mathematics ▶

The solution of equation (9) is

$$m(t) = m(0)\,e^{-\kappa t},$$

where $m(0)$ is the initial mass of pollutant. The corresponding equation for the concentration is

$$c(t) = c(0)\,e^{-\kappa t},$$

where $c(0)$ is the initial concentration of pollutant. The time T taken for the pollutant concentration to reduce to the target level $c(T) = c_{\text{target}}$ is given by

$$T = -\frac{1}{\kappa}\ln\left(\frac{c_{\text{target}}}{c(0)}\right).$$

The proportionate decay rate κ determines the value of T.

◀ Interpret results ▶

Now we use this model to predict what will happen to PCB concentrations in Lake Superior. It is a negative exponential decay model, therefore it will have the same desirable qualitative behaviour that was apparent in the first model. Since the proportionate decay rate κ is larger than the proportionate flow rate k, it is also likely to give better quantitative agreement than the first model.

R.V. Thomann and J.A. Mueller (1987) *Principles of Surface Water Quality Modeling and Control*, Harper & Row.

Thomann and Mueller suggest that volatilisation probably occurs at a rate of about 0.1 m (that is, $0.1\,\mathrm{m}^3$ per m^2) per day. So in SI units ($\mathrm{m\,s^{-1}}$), we have $p \simeq 0.1/(24 \times 60 \times 60) \simeq 1.16 \times 10^{-6}$. The surface area of Lake Superior is given as $A \simeq 8.21 \times 10^{10}\,\mathrm{m}^2$. Hence in SI units,

Note that pA/V, which was assumed to be zero in the first model, is about 50 times as big as k, and is clearly the more dominant process.

$$\begin{aligned}\kappa &= k + pA/V \\ &\simeq 0.166 \times 10^{-9} + 1.16 \times 10^{-6} \times 8.21 \times 10^{10}/(12.10 \times 10^{12}) \\ &\simeq 8.03 \times 10^{-9}.\end{aligned}$$

The estimated proportionate decay rate is about 50 times larger than the proportionate flow rate. It is clear that the volatilisation of PCBs has a much greater influence on the rate at which their concentration is reduced than does the flow of pollutant out of the lake. The revised time, in seconds, for the pollutant concentration to fall by a factor of ten is

$$T = \frac{\ln 10}{\kappa} \simeq 2.87 \times 10^8,$$

which is approximately 9.1 years.

◀ Evaluate ▶

The results of this revised model are much more encouraging. The values in Table 3 indicate that PCB concentrations should take about 14 years to reduce by a factor of ten, while the revised model predicts about 9 years (compared with the 440 years predicted by the original model). So from a model that grossly overestimates the time taken, the revised model underestimates it. It may be that for some purposes, the revised model is sufficient. If it is not, then look again for possible reasons for the discrepancy between the revised model's prediction and reality.

You saw how values of A, V and k for Lake Superior led to values for κ and for the decay time T (the time for pollutant concentration to fall by a factor of ten). Table 5 gives the corresponding results for κ and decay time for all of the lakes.

The values of k in Table 5 were obtained, from the water flow rate and volume of each lake, in Unit 8. The decay time is given (in seconds) by

$$T = (\ln 10)/k$$

in the original model, and by

$$T = (\ln 10)/\kappa$$

in the revised model.

Table 5 Proportionate flow and decay rates, and times to fall by a factor of ten

Lake	Volume V ($10^{12}\,\mathrm{m}^3$)	Surface area A ($10^{10}\,\mathrm{m}^2$)	Original model: Prop. flow rate k ($10^{-9}\,\mathrm{s}^{-1}$)	Original model: Decay time (years)	Revised model: Prop. decay rate κ ($10^{-9}\,\mathrm{s}^{-1}$)	Revised model: Decay time (years)
Superior	12.10	8.21	0.166	440	8.02	9.1
Michigan	4.92	5.78	0.319	229	13.92	5.2
Huron	3.54	5.96	1.441	51	20.93	3.5
Erie	0.48	2.57	12.292	6	74.26	1.0
Ontario	1.64	1.90	4.134	18	17.54	4.2

Possible reasons for the discrepancy between the data in Table 3 and the prediction of the revised model in Table 5, for Lake Superior, include the following.

- The volatilisation rate is given to only one significant figure, and this may cause a sizeable numerical error. A more accurate estimate for this rate is required to ensure reliable predictions.
- The volatilisation of PCBs from the lake to the atmosphere has been included, but the volatilisation of PCBs from the atmosphere to the lake has been ignored.
- Although PCBs were banned in 1975, there will be a residual amount in the catchment area that will continue to wash into the lake over time, so Assumption (a) may need to be revised.
- Some of the PCBs settle into the sediment at the bottom of the lake, and will be flushed out of this sediment at a much slower rate, so the early samples of water will have lower PCB concentrations than expected.
- The volatilisation rate is based on the assumption that the pollutant is uniformly distributed over the whole lake. However, it is known that the pollutant concentration varies considerably within the lake, so this assumption may have to be reconsidered.

The first item in this list is not a criticism of the model but a requirement for more accurate data. Each of the other criticisms could form the basis of a second revised model.

The process of improving a model is one of evaluating the model at each stage and then addressing the assumption that seems most likely to improve the model; it is an iterative process. It is best to change one feature at a time, even though that may affect more than one assumption.

In the second revision of the model below, the assumption of a uniform concentration of pollutant in the whole lake is revised, to take some account of the last criticism.

Second revision: segmenting the lake

◀ Create model ▶

To account for the variation in pollutant concentration across a lake, consider the technique of breaking down the region considered by the model into a number of smaller subregions or cells, where each cell has its own distinct properties. A model for the whole region is built up by considering what happens in each cell, and how it interacts with the other cells.

This technique is known as *segmentation* or *compartmentalization.*

The mathematical model for pollution in the Great Lakes used by the US Environmental Protection Agency considers what happens in each lake by segmenting it into a large number of smaller volumes. This segmentation is done by first constructing a horizontal grid on the lake. The horizontal segmentation takes into account, for example, regions of the lake where the pollutant concentration is much higher or lower than average. Then for each cell created by the horizontal segmentation, there is also vertical segmentation. This allows for different behaviour between the upper regions of the lake, where photolysis and volatilisation take place, and the lower regions, where the presence of sediment may have an effect.

In this way, each lake is divided up into hundreds of small cells, on which more precise data are available and for which more precise information may be required. Each cell has its own variables representing, for example, its pollutant concentration and the rates of flow to and from adjacent cells.

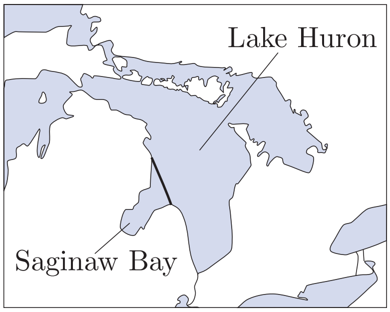


Figure 5 Saginaw Bay in Lake Huron

Models that use segmentation can be very complicated, and this is true of the models used to analyse levels of pollution in the Great Lakes. So to keep the analysis relatively simple, the revised model here is a simple segmentation of Lake Huron. In the south-west of Lake Huron is a large bay, called Saginaw Bay (see Figure 5). The Saginaw River flows into this bay, and was a major source of pollution for the whole lake. Suppose that a model of the concentration of PCBs in Lake Huron is required. It would be reasonable to consider Saginaw Bay separately from the rest of the lake, as shown in Figure 5, since the concentration of pollutant in this bay is likely to be significantly higher than that in the rest of the lake.

Two further assumptions are now needed, in addition to those made for the first revised model:

(i) Saginaw Bay and the rest of Lake Huron can be considered as two separate lakes.

Diffusion of pollutant is a transportation process that involves the movement of molecules of the pollutant relative to the water, rather than bodily movement of the water itself.

(j) The rate of diffusion of pollutant from Saginaw Bay into the rest of Lake Huron is proportional to the difference in concentrations between the two regions of the lake.

The first assumption here implies that the pollutant in Saginaw Bay is evenly dispersed throughout the bay, so that the concentration of pollutant in the bay varies only with time and not with position. The second assumption models the interaction between the two regions of the lake.

There will be approximately twice as many variables and parameters as before, with one set for each region of the lake.

The variables are:

t the time, in seconds, since all sources of pollution cease;

$m_B(t)$ the mass, in kilograms, of pollutant in the bay at time t;

$m_L(t)$ the mass, in kilograms, of pollutant in the rest of the lake at time t;

$c_B(t)$ the concentration, in kilograms per cubic metre, of pollutant in the bay at time t;

$c_L(t)$ the concentration, in kilograms per cubic metre, of pollutant in the rest of the lake at time t.

The parameters are:

V_B the volume, in cubic metres, of water in the bay;

V_L the volume, in cubic metres, of water in the rest of the lake;

r_B the water flow rate, in cubic metres per second, into and out of the bay;

r_L the water flow rate, in cubic metres per second, into and out of the rest of the lake;

A_{B} the surface area, in square metres, of the bay;
A_{L} the surface area, in square metres, of the rest of the lake;
p the volatilisation rate, in metres per second, for the pollutant;
E the bulk exchange rate, in cubic metres per second.

The meaning of E will become apparent shortly.

Starting with Saginaw Bay, consider what happens to the mass of pollutant in the bay over a time interval $[t, t+\delta t]$. According to Assumptions (a) and (g), there is no new pollution entering the lake, so there is no input of pollutant to the bay from external sources. The output (in kilograms per second) of pollutant from the bay takes three main forms.

- Pollutant flows from Saginaw Bay into the rest of Lake Huron at a rate $r_{\mathrm{B}}\, c_{\mathrm{B}}(t) = r_{\mathrm{B}}\, m_{\mathrm{B}}(t)/V_{\mathrm{B}}$.
- Pollutant is lost to the atmosphere, through volatilisation, at a rate $pA_{\mathrm{B}}\, c_{\mathrm{B}}(t) = pA_{\mathrm{B}}\, m_{\mathrm{B}}(t)/V_{\mathrm{B}}$.
- Pollutant diffuses from a region of higher concentration (the bay) to a region of lower concentration (the rest of the lake) at a rate $E(c_{\mathrm{B}}(t) - c_{\mathrm{L}}(t)) = E(m_{\mathrm{B}}(t)/V_{\mathrm{B}} - m_{\mathrm{L}}(t)/V_{\mathrm{L}})$, where the parameter E depends, for example, on the vertical cross-sectional area separating Saginaw Bay from the rest of Lake Huron.

This follows from Assumption (j). The parameter E is known as the *bulk exchange rate* for diffusion of pollutant across the boundary between the bay and the rest of the lake.

Thus the total output of pollutant from the bay during the time interval $[t, t+\delta t]$ is

$$\left(\frac{r_{\mathrm{B}}}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) + \frac{pA_{\mathrm{B}}}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) + \frac{E}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) - \frac{E}{V_{\mathrm{L}}}\, m_{\mathrm{L}}(t)\right) \delta t.$$

The accumulation of the mass of pollutant within the bay is

$$m_{\mathrm{B}}(t+\delta t) - m_{\mathrm{B}}(t).$$

Using the input–output principle then gives

$$\begin{aligned} &m_{\mathrm{B}}(t+\delta t) - m_{\mathrm{B}}(t) \\ &= -\left(\frac{r_{\mathrm{B}}}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) + \frac{pA_{\mathrm{B}}}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) + \frac{E}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) - \frac{E}{V_{\mathrm{L}}}\, m_{\mathrm{L}}(t)\right) \delta t. \end{aligned}$$

Dividing through by δt, then letting $\delta t \to 0$, leads to the differential equation

$$\frac{dm_{\mathrm{B}}}{dt} = -k_1\, m_{\mathrm{B}}(t) + k_2\, m_{\mathrm{L}}(t), \tag{10}$$

where $k_1 = r_{\mathrm{B}}/V_{\mathrm{B}} + pA_{\mathrm{B}}/V_{\mathrm{B}} + E/V_{\mathrm{B}}$ and $k_2 = E/V_{\mathrm{L}}$.

Consider now the rest of Lake Huron. The input (in kilograms per second) of pollutant to this part of the lake comes from two main sources:

- the flow of polluted water from Saginaw Bay, at a rate $r_{\mathrm{B}}\, m_{\mathrm{B}}(t)/V_{\mathrm{B}}$
- diffusion across the boundary between the two parts of the lake, at a rate $E(m_{\mathrm{B}}(t)/V_{\mathrm{B}} - m_{\mathrm{L}}(t)/V_{\mathrm{L}})$.

This ignores any input of pollutant to Lake Huron from the upstream lakes, in keeping with Assumption (g). This assumption may not be justified for Lake Huron, but it keeps the model simple.

Hence the input of pollutant to the rest of Lake Huron in the time interval $[t, t+\delta t]$ is

$$\left(\frac{r_{\mathrm{B}}}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) + \frac{E}{V_{\mathrm{B}}}\, m_{\mathrm{B}}(t) - \frac{E}{V_{\mathrm{L}}}\, m_{\mathrm{L}}(t)\right) \delta t.$$

During this same time interval, the output of pollutant from this part of the lake takes two main forms.

- Pollutant flows from Lake Huron to Lake Erie at a rate $r_L\, c_L(t) = r_L\, m_L(t)/V_L$.
- Pollutant is lost to the atmosphere, through volatilisation, at a rate $pA_L\, c_L(t) = pA_L\, m_L(t)/V_L$.

Thus the total output of pollutant from the rest of the lake in the time interval $[t, t+\delta t]$ is

$$\left(\frac{r_L}{V_L}\, m_L(t) + \frac{pA_L}{V_L}\, m_L(t)\right)\delta t.$$

The accumulation of the mass of pollutant within the rest of the lake is

$$m_L(t+\delta t) - m_L(t).$$

Using the input–output principle applied to the rest of Lake Huron then leads to the differential equation

$$\begin{aligned}\frac{dm_L}{dt} &= \frac{r_B}{V_B}\, m_B(t) + \frac{E}{V_B}\, m_B(t) - \frac{E}{V_L}\, m_L(t) - \frac{r_L}{V_L}\, m_L(t) - \frac{pA_L}{V_L}\, m_L(t)\\ &= k_3\, m_B(t) - k_4\, m_L(t),\end{aligned} \tag{11}$$

where $k_3 = r_B/V_B + E/V_B$ and $k_4 = E/V_L + r_L/V_L + pA_L/V_L$.

Techniques for solving systems of linear differential equations can be found in Unit 6.

The revised model has led to equations (10) and (11). These are a pair of linear differential equations that describe the behaviour of pollutant in the two regions of the lake.

Exercise 13

It is worthwhile estimating the contribution from each term that arose in applying the input–output principle. Table 6 gives the relevant data on parameter values for the two regions of the lake.

Table 6 Parameter values for the second revised model

Parameter	Value
V_B	$25 \times 10^9\,\mathrm{m}^3$
A_B	$4.2 \times 10^9\,\mathrm{m}^2$
r_B	$153\,\mathrm{m}^3\,\mathrm{s}^{-1}$
V_L	$3.24 \times 10^{12}\,\mathrm{m}^3$
A_L	$57 \times 10^9\,\mathrm{m}^2$
r_L	$4967\,\mathrm{m}^3\,\mathrm{s}^{-1}$
p	$1.16 \times 10^{-6}\,\mathrm{m}\,\mathrm{s}^{-1}$
E	$11\,000\,\mathrm{m}^3\,\mathrm{s}^{-1}$

(a) Suppose that the concentration of pollutant in the rest of the lake is half that in Saginaw Bay. Compare the rates of pollutant output from the bay for each of the three forms of pollutant transfer: water flow, volatilisation and diffusion. Hence decide which of these mechanisms is the most significant in removing pollutant from the bay.

(b) How will the rate of pollutant output for each of these three forms change if the concentration of pollutant in the rest of the lake is less than half that in the bay?

(c) What is the main mechanism for the removal of pollutant from the rest of the lake?

3.3 Review

In this section you have seen how data were used to evaluate a simple model for the behaviour of pollution in the Great Lakes. The data were obtained from published sources, but it was not possible to check their reliability. There was good qualitative agreement with the model, but rather poor quantitative agreement.

Prior to making revisions, the assumptions made for the first model were evaluated. The first revision was based on the inclusion of an additional process, to try to explain the reason for the substantial quantitative differences between the published data and the predictions of the model. This revision was very effective, since it demonstrated that volatilisation is likely to have a much more substantial effect on the removal of PCBs than does the flow of water out of the lake.

The second revision was based on a segmentation of the lake, or division into compartments, to take into account the non-uniformity of the concentration of pollutant. This technique is widely applicable in mathematical modelling. Exercise 13 shows that diffusion, as well as volatilisation, is more significant than water flow in removing pollutant from Saginaw Bay. In fact, the only form of pollutant output considered in the original model turns out to be the least important in the revised models.

The original model, based on reasonable simplifying assumptions, made a start in the problem-solving process. The evaluation of this model revealed its deficiencies, and as a result other forms of output were considered to improve the model. The development that was undertaken here emphasises the role of the first simple model in initialising the modelling process.

4 Sample modelling report

The marginal notes give more information about what is required in each section.

We round off this unit by looking at and commenting on a report on a modelling activity based on bungee jumping, starting with the statement of the problem that is to be investigated. Remember that it is the modelling process rather than the model that is important in writing up the report. The report given here is intended to be not a model report, but simply an attempt to outline the approach to writing a modelling report.

Try not to worry about the *details* of the sample report. It is provided to show you the sort of thing that you might produce, not to teach you any new mathematics.

In order to follow this report, you need one extra piece of physics that is not taught in this module. From Unit 9, you know that the magnitude of the force exerted by a spring is proportional to the extension, and that this constant of proportionality k is called the stiffness of the spring. Here we need to include something that links in the cross-sectional area of the elastic material used in making the bungee, and the required model is that the magnitude of the force F is given by

$$F = \frac{EAe}{L},$$

The difference between a spring and a bungee is that the cross-sectional area of the bungee changes when it is stretched.

where A is the original (unstretched) cross-sectional area of the elastic material, L is the natural length of the bungee, e is the extension, and E is the *Young's modulus* (sometimes called the modulus of elasticity) for the elastic material. We can deduce that the stiffness is $k = AE/L$.

Problem statement: Bungee jumping

The extreme sport of bungee jumping involves a participant diving off a high platform, below which is clear space, while attached by a safety harness to an elastic cord. The cord eventually brings the jumper momentarily to rest before he begins to move upwards. It is paramount that those who organise such jumps should ensure the safety of the jumpers, which means among other things that if there is a solid surface such as the ground beneath the jump, then the jumper should not collide with it. (With water beneath, as in several famous jump locations, this safety issue is less crucial, but typically the jumper will wish to remain above the water at all times.) However, within this concern for safety, the jumper will wish to experience the greatest possible thrill.

Develop a mathematical model that will enable you to advise the organisers of a bungee jump on how to ensure the safety or dryness of participants, while not diminishing the element of excitement more than is necessary.

Title: Bungee jumping

Author: Model Student, 30 September 2014

Specify the purpose of the model

Definition of the problem

In the sport of bungee jumping, each participant (jumper) drops into space from the top of a high tower or platform. The jumper is tied to one end of an elastic cord (called a bungee), the other end of which is secured to the top of the tower. The safety and wellbeing of the jumpers are most important. For a jump over water, the jumper may not wish to be soaked in the process. Another important aim is that the force magnitudes experienced by the jumper should not be too high, since this also can cause injury or significant discomfort.

In many cases, the problem specified may be rather vague. This gives you scope to choose which aspects of the problem you think are important. You should outline, in a couple of sentences, the approach that you have adopted in your report.

This report considers what advice should be provided to those who organise bungee jumps over water, to ensure that each jumper remains dry during the jump and is not subject at any stage to excessive forces. In particular, the aim is to specify the characteristics of the bungee so as to maximise the distance that a jumper falls without entering the water beneath, while keeping the maximum force encountered to an acceptable limit.

Aspects of the problem to be investigated

The most important elements of a bungee jump are the mass of the jumper, and the length, cross-sectional area and elastic properties of the bungee. It will be shown that these factors determine both the forces encountered and the distance fallen. Consequently, to achieve a given distance of fall in safety, the length and elastic properties of the bungee can be selected for a jumper of given mass. See also Appendix 3.

Create the model

Outline of the approach in the first model

The situation will be simplified by ignoring dissipative effects (like friction or air resistance). This enables the principle of conservation of mechanical energy to be applied. The constituent quantities that will feature in applying energy conservation are the kinetic energy and gravitational potential energy of the jumper, and the elastic potential energy of the bungee. The sum of these quantities will be constant throughout the jump. For a chosen total distance of fall, the energy conservation equation provides a relationship between the mass of the jumper on the one hand and the unstretched length and stiffness of the bungee on the other. The stiffness is in turn related to other physical properties of the bungee.

You need to specify the approach that you will take in the first model. This should be an outline of how you will tackle the model, giving the reader some idea of what you are attempting to do. It should not be mathematical, but you may need to use some mathematical jargon occasionally.

Assumptions

It should be possible to justify each major step in the formulation of mathematical relationships in your model, by appealing to one or more of the assumptions. On the other hand, an assumption that is unnecessary for the derivation of your model is superfluous, and should be omitted.

In your first model, it is best to simplify the problem as much as you can, without losing any of its most essential features.

1. The jumper is a particle (so his size and shape and any rotations are ignored).
2. The motion is entirely in the vertical direction and hence in one dimension.
3. The initial speed of the jumper (on leaving the platform) is zero.
4. The mass of the bungee is ignored throughout.
5. Until the bungee is extended (beyond its unstretched length), it exerts no force on the jumper.
6. Once the bungee is extended, it acts as a model spring (satisfying Hooke's law) with natural length equal to its unstretched length.
7. The bungee is made of uniform material, which behaves according to a standard model for elastic materials.
8. The physical properties of the bungee are unaffected by the local air temperature, exposure to sunshine or other weather conditions.
9. The distance between the take-off platform and the water surface beneath is constant in time.
10. Air resistance is ignored.
11. The force exerted on the jumper by the bungee should not be too great.
12. The bungee will at no point be in danger of breaking.

It is helpful to number the assumptions for reference later in the report.

Definition of variables and parameters

You may decide to list all the variables and parameters together in this section, or you may decide to give just the main ones here and to introduce others where they are needed in developing the model. In either case, make sure that each variable or parameter is clearly and unambiguously defined, and that the units or dimensions are given if appropriate.

The report is much clearer if all the variables and parameters are defined in a table.

Symbol	Description	Units
m	Mass of the jumper	kg
x	Distance of jumper below platform	m
H	Distance from jumping platform to lowest point of motion, at surface of water	m
L	Unstretched length of bungee	m
e	Extension of bungee when stretched	m
A	Cross-sectional area of unstretched bungee	m^2
F	Magnitude of force applied to end of bungee	N
E	Modulus of elasticity (Young's modulus) for the bungee material	$\mathrm{N\,m^{-2}}$
k	Stiffness of bungee	$\mathrm{N\,m^{-1}}$
g	Magnitude of acceleration due to gravity	$\mathrm{m\,s^{-2}}$
λg	Maximum acceleration permitted for the jumper	$\mathrm{m\,s^{-2}}$
p	Constant ($p = mg/(AE)$)	–

Formulation of mathematical relationships

By Assumptions 5, 6 and 10, there is no energy lost during the motion, hence the law of conservation of mechanical energy can be applied. By Assumptions 1 and 4, the system to be considered is a single particle of mass m, together with a model spring when in extension. By Assumption 2, the motion is in one (vertical) dimension, starting at $x = 0$ and reaching its lowest point at $x = H$. A diagram of the situation is given in Figure 1.

In formulating each of the mathematical relationships in your model, you need to explain to the reader, in words, what the relationship is and how it arises from your assumptions. This is probably the most difficult part of the report, so make sure that you re-read what you have written with a critical eye to ensure that it is clear. If possible, ask someone else to read this part of the report, and ask him or her whether your description of the formulation is easy to follow. You may wish to check the dimensional consistency of any mathematical relationships.

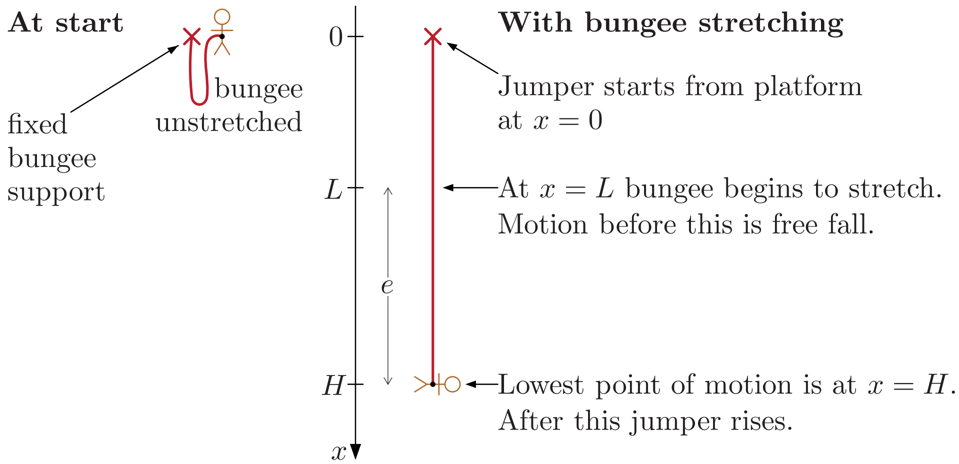


Figure 1 Positions and distances in bungee jumping

By Assumption 9, H is constant. Let $x = H$ be the datum for gravitational potential energy.

At $x = 0$ the jumper has zero kinetic energy (since he starts with speed zero, by Assumption 3) and gravitational potential energy mgH (by the choice of datum). The bungee, being massless (by Assumption 4) and unstretched, has zero potential energy.

At $x = H$ the jumper has zero kinetic energy (since he comes momentarily to rest at this point) and gravitational potential energy zero (by the choice of datum). The bungee (by Assumptions 6 and 8) has spring potential energy $\frac{1}{2}ke^2$, where the stiffness k is constant and $e = H - L$.

Equating the sums of potential and kinetic energies at $x = 0$ and at $x = H$, we have

$$mgH = \tfrac{1}{2}ke^2 = \tfrac{1}{2}k(H - L)^2. \qquad (1)$$

It is helpful to number your equations and relations as you develop the solution.

By Assumptions 6 and 8, we can apply Hooke's law to obtain the magnitude of the total upwards force at the bottom of the jump as

$$F = ke - mg. \qquad (2)$$

Using the standard elasticity model referred to in Assumption 7, we note that the stiffness of the spring k is inversely proportional to the length of the bungee. We have

$$k = \frac{AE}{L}, \tag{3}$$

where A is the cross-sectional area of the bungee, and E is the Young's modulus for the material of the bungee.

By Assumption 11, the acceleration exerted on the jumper should be no more than the maximum multiple of g permitted. From equation (2) this occurs at the bottom of the jump, so

$$\frac{F}{m} \leq \lambda g. \tag{4}$$

Equations (1)–(3) and inequality (4) constitute the first model. In addition, by Assumption 12, F will not exceed (or even come close to) the breaking strength of the bungee.

Do the mathematics

Solution of the equations

Explain to the reader, in words, what mathematical methods you use and how this will solve the problem. Write out the solution. A computer algebra package may be useful here.

With $p = mg/(AE)$, we have, from equations (1) and (3),

$$\frac{mgH}{k} = \frac{mgHL}{AE} = pHL = \tfrac{1}{2}(H-L)^2,$$

so

$$L^2 - 2(1+p)LH + H^2 = 0.$$

We can solve this for L (with $H > L$, since the stretched length of the bungee is greater than the unstretched length), in terms of the mass of the jumper m and the height H, as

$$L = \left((1+p) - \sqrt{p^2+2p}\right) H. \tag{5}$$

The negative square root is taken here because $L < H$. This is dimensionally consistent provided that p is dimensionless. We have

$$[p] = [mg/(AE)] = \frac{[m][g]}{[A][E]} = \frac{\mathrm{M}\,(\mathrm{L\,T^{-2}})}{(\mathrm{L}^2)\,(\mathrm{M\,L^{-1}\,T^{-2}})} = 1,$$

as required.

The alternative of solving equation (1) for the extension e in terms of L (choosing the positive square root to ensure that $e > 0$) gives

$$e = pL\left(1 + \sqrt{1 + \frac{2}{p}}\right). \tag{6}$$

From equations (2), (3) and (6), the maximum upwards acceleration experienced by the jumper due to the bungee is

$$\frac{F}{m} = \frac{ke}{m} - g = \left(\frac{e}{Lp} - 1\right) g = \left(\sqrt{1 + \frac{2}{p}}\right) g.$$

From inequality (4), the condition for maximum acceleration magnitude can therefore be written as

$$\sqrt{1+\frac{2}{p}} \leq \lambda, \quad \text{where } \lambda > 1,$$

and this can be interpreted as a limit on the weight of the jumper, since using $p = mg/(AE)$ we can deduce that

$$mg \geq \frac{2AE}{\lambda^2 - 1}. \tag{7}$$

This means that to avoid excessive acceleration at the bottom of the jump, the weight of the jumper must be sufficiently large. We can also deduce that for a given mass of jumper, the cross-sectional area of the bungee must satisfy

$$A \leq \frac{mg(\lambda^2 - 1)}{2E}. \tag{8}$$

Graphs

Various graphs may be drawn based on the relationships above – two examples are shown below.

A computer algebra package (as used here) may help in presenting graphs of the solution for sample values of the parameters.

Figure 2 is a graph of the unstretched bungee length L (as a proportion of H) against the parameter $p = mg/(AE)$ $(0.2 < p < 0.4)$, which can be interpreted as a graph of the unstretched length of the bungee against the mass of the jumper, when A is fixed, using equation (5).

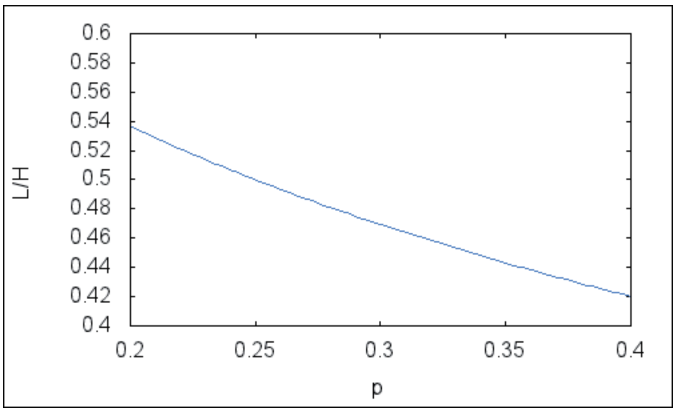


Figure 2 Graph of the unstretched length L of the bungee as a proportion of the height H, against p

An expected feature here is that the unstretched length L of the bungee decreases as the mass increases.

Figure 3 gives the maximum extension e of the bungee (as a proportion of L) against the parameter $p = mg/(AE)$ $(0.2 < p < 0.4)$, using equation (6).

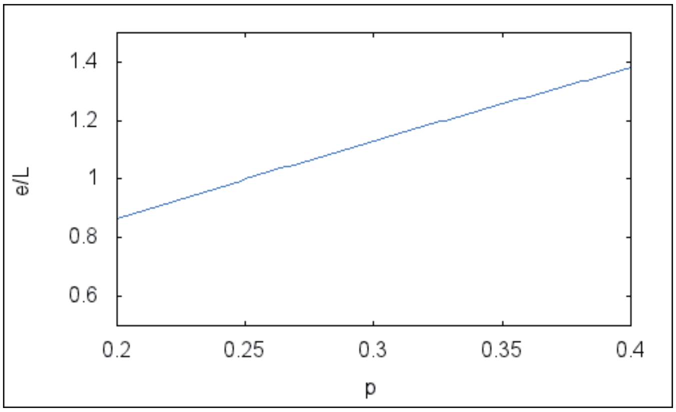


Figure 3 *Graph of the maximum extension e of the bungee as a proportion of the unstretched length L, against p*

Results

Give the mathematical solution to the problem in algebraic form.

As the maximum permitted acceleration λg to be exerted by the bungee occurs at the lowest point of motion, to obtain the maximum thrill of the jump, the inequality (8) becomes an equation. In this case, we have

$$A = \frac{mg(\lambda^2 - 1)}{2E} \quad (\lambda > 1). \tag{9}$$

Also, using equation (5) we have an expression for the length of the bungee, given the weight of the jumper, as

$$L = \left(\left(1 + \frac{mg}{AE}\right) - \sqrt{\left(\frac{mg}{AE}\right)^2 + 2\frac{mg}{AE}}\right) H. \tag{10}$$

Equations (9) and (10) constitute the main results from the model: for a jumper of mass m, a drop of total distance H and a maximum acceleration exerted of λg, these two equations give the appropriate values to be selected for the unstretched bungee cross-sectional area A and length L. Values are needed for the parameters g and E.

Interpret the results

Data

You should present the data that you have collected, giving references to any sources of data that you have used (though we do not do so here). `Internet sources` have been suppressed in this sample report, but you should give any internet sources that you have used, together with the date when you referenced each source.

Quantity	Value	Source or justification
g	$9.8\,\mathrm{m\,s^{-2}}$	Standard value
E	$1\text{–}5 \times 10^6\,\mathrm{N\,m^{-2}}$	For natural rubber, `internet source`

There is a wide variation in the values for Young's modulus of natural rubber to be found in various sources, and some are well outside the range quoted above. We will take $E = 1 \times 10^6\,\mathrm{N\,m^{-2}}$.

The masses of jumpers are assumed typically to vary between 40 kg and 120 kg, though lighter and heavier jumpers are also accommodated on some jumps.

The allowable maximum acceleration of the bungee jumper is often laid down in legislation. For example, `internet source` specifies that the maximum G-force allowable on a jumper should be $4\frac{1}{2}g$ for a waist-and-chest harness, and $3\frac{1}{2}g$ for an ankle harness. Other jurisdictions lay down restrictions in terms of cord extension. The Hong Kong Code of Practice for Bungee Jumping states a condition that is equivalent to $2.7 \leq \lambda \leq 3.3$.

Values of H also vary widely. A famous example is the first permanent commercial bungee jump site at Kawarau Bridge, Queenstown, New Zealand, where the drop distance is 43 m. However, the model should also apply to miniature bungee jumps, such as that considered later.

Note that the extension of the bungee, at which it is likely to break, is of the order of 750–850% (according to `internet source`). The extensions estimated by the current model are well outside this range, so should be completely safe, as required by Assumption 12.

If you have done your own experiments to collect data, then the descriptions of those experiments should appear in an appendix at the end of the report. (See Appendices 1 and 2 here.)

Interpretation of results

Using the given data for g and E, equation (9) becomes

$$A = 4.9 \times 10^{-6} m(\lambda^2 - 1) \quad (\lambda > 1),$$

whereas

$$p = \frac{mg}{AE} = \frac{2}{\lambda^2 - 1}.$$

Selecting as input the reasonable values $H = 43$ and $\lambda = 3$ gives $p = 0.25$, so

$$L = \left((1+p) - \sqrt{p^2 + 2p}\right) H = 0.5H = 21.5$$

and

$$A = 39.2 \times 10^{-6} m.$$

Hence for a person of mass 80 kg jumping from Kawarau Bridge, the bungee should have length 21.5 m and cross-sectional area $A = 3.14 \times 10^{-3}$ or $31.4\,\mathrm{cm}^2$ (i.e. a diameter of about 6 cm). For these values of H and λ, the graph of A against m is shown in Figure 4.

You should, if possible, give both a qualitative and quantitative interpretation of your solution. The qualitative interpretation might include how the solution varies with different parameters and whether this agrees with common sense, or it might examine whether the solution behaves as expected in limiting cases, or in simpler situations. Choose an appropriate level of accuracy for your numerical results.

Be careful to present your solution in such a way that it addresses the purpose of the mathematical model.

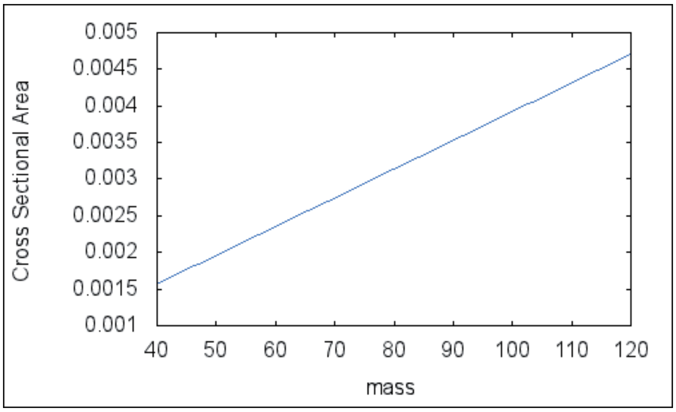


Figure 4 *Graph of bungee cross-sectional area against jumper mass (for* $H = 43$, $\lambda = 3$*)*

Clearly (from either this graph or equation (9)), the cross-sectional area should be chosen in proportion to the mass of the jumper. This conclusion applies for any values of H and λ. Provided that this rule is adhered to, each jumper, whatever their mass, should fall the same distance and experience the same maximum acceleration at the bottom of the jump. The length L of the bungee required to achieve this is given by equation (5), and once λ has been selected, L should be a fixed proportion of the drop distance H.

To specify a precise cross-sectional area for an unstretched bungee does not appear to be very practical advice. However, bungee cords are typically made up of a large number of individual rubber threads whose cross-sectional area is very small (for example, '22 gauge' at 0.643 mm diameter is common). Achieving a particular cross-sectional area is then a matter of aggregating the correct number of rubber threads. In practice there are four or five set numbers of threads that may be selected, each appropriate for a certain range of jumper mass, so there will be some (acceptable) variation in jump performance, for a given bungee, between jumpers at either end of the corresponding mass range. The assessment of 'worst case' performance must be carried out with respect to the heaviest mass that each type of bungee will be expected to carry.

As regards the sensitivity of the results to small changes in the parameters, it is clear from the proportionality of A and m that a 10% change in the value of m will lead to a 10% change in the value of A. Similarly, a 10% change in the value of H will lead to a 10% change in the value of L. The relative sensitivity is therefore 1 in each case.

It can be shown that the unstretched length L of the bungee can be expressed in terms of λ as

$$L = \frac{(\lambda - 1)H}{\lambda + 1}.$$

Thus the relative sensitivities of A and L to changes in λ are given respectively by

$$\frac{\lambda}{A}\frac{\partial A}{\partial \lambda} = \frac{2\lambda^2}{\lambda^2 - 1} \quad \text{and} \quad \frac{\lambda}{L}\frac{\partial L}{\partial \lambda} = -\frac{2\lambda}{\lambda^2 - 1}.$$

Both variables are relatively insensitive provided that λ is not close to 1. The typical data for λ quoted earlier show that values close to 1 do not occur in practice.

The conclusion is that neither the unstretched bungee cross-sectional area nor the length is unduly sensitive to changes in the parameter values of the model.

Choice of results to compare with reality

There is no shortage of relevant data available on internet sites, but it is difficult to find any site that provides a complete and consistent set of data to validate this model. The premises that underlie the data provided are often not made explicit. In what follows, some data from the internet are supplied, for comparison with certain aspects of the model, but the chief validation is based on an experiment. The experiment seeks to check the validity of equation (6), which predicts the maximum extension to be expected for given bungee length, bungee stiffness and jumper mass. The details are given in Appendix 1.

Which of your results can be tested against reality? These tests do not have to be directly related to the purpose of the model, and there may be ways of testing the reliability of your model by using it to make predictions in simple situations, or for only parts of the model.

Evaluate the model

Comparison with reality

The experimental results reported in Appendix 1 (for which the data and graphs are shown in Appendix 2) indicate some deficiencies in the model. One notable feature is that Hooke's law does not seem to apply at all closely over the range of masses considered. A comparison of the predicted and measured values for e as m is varied is shown in Figure 5. This indicates that the measured values are quite close numerically to the values predicted by the model. However, the shapes of the two graphs are noticeably different.

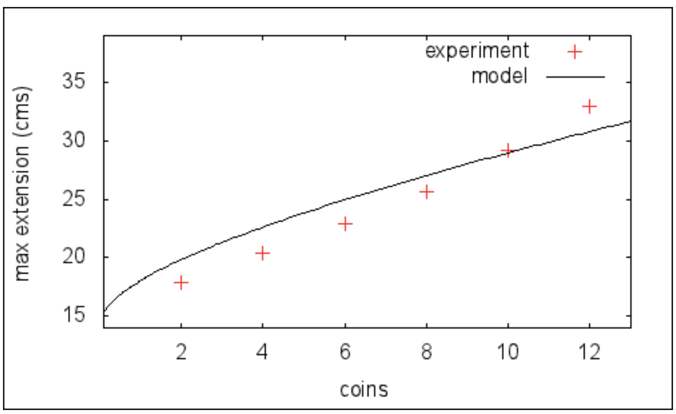


Figure 5 Comparison of predicted and experimental values for maximum extension

You should try to compare some of the predictions of your model with real data, in order to test the model's accuracy and reliability. Again, you should give references to any sources of data, and if you have carried out any experiments, then you should present your results here. The descriptions of the experiments can be put in an appendix at the end of the report. How reliable are your results? Are any of the simplifications that you made likely to have led to an answer that is too large or too small?

If you feel that there are no appropriate tests for your model, then you should explain why.

The data from internet sites given in Appendix 3 confirm that Hooke's law does not hold at all closely for bungee cords. However, there is considerable support here for the prediction that the cross-sectional area of a bungee (or equivalently, the number of threads from which it is made) should be approximately proportional to the mass of the jumper.

Criticism of the model

Criticising your work is a healthy practice, provided that it is done in moderation and constructively. You should praise the positive aspects of your model as well as criticising the negative ones. If the model is deficient, either qualitatively or quantitatively, then you should reconsider the assumptions that you made earlier. Decide whether each assumption is valid, based on the outcome of evaluation. Try to identify the key assumption(s) that, if changed, would lead to improvement in the predictions of the model. Make sure that the discussion is relevant.

In some respects the predictions of the model match both common sense and available data. This applies in particular to the outcome that the cross-sectional area of the bungee should be proportional to the mass of the jumper. However, the experimental results and data from the internet both indicate that the extension of a bungee cord is not proportional to the force applied at its end, leading to significant inaccuracies in the model's predictions.

Consider in turn each of the assumptions at the start of the report.

1. Since the jumper could be up to 2 m in length, the assumption that he is a particle might need revision to avoid some part of the jumper entering the water.
2. There will be minor departures from the vertical in the motion, but these should hardly affect the predictions of the model.
3. The jumper might project himself with a non-zero velocity, but this need not be a major factor for revision. It could be incorporated if desired, however.
4. The mass of the bungee is, in fact, quite significant (as one of the sources quoted in Appendix 3 indicates), and its weight provides an additional downward force on the jumper in the initial stage of the jump. This could be taken into account in a revised model.
5. Assuming that the bungee exerts no force on the jumper is reasonable as regards the elastic properties of the bungee, but its weight has an effect, as discussed above.
6. The experiment, and other available data, indicate that the elastic material does *not* behave like a model spring, and a revised model might attempt to capture more closely the behaviour actually observed. In fact, there are 'viscoelastic' energy losses that cause the bungee to heat up.
7. It is reasonable to assume that the bungee material is uniform, but the standard model referred to appears not to apply at all closely, except for very small suspended masses.
8. The assumption that the physical properties of the bungee are unaffected by weather etc. is not too bad, though the behaviour of rubber does alter significantly with temperature, and over-exposure to sunlight causes a deterioration in performance.
9. It is reasonable to assume a constant height between platform and water. If there were slight seasonal variations, then the model could take these into account.

10. Air resistance has an effect, but in practice the principal energy losses are within the bungee itself, as remarked above.
11. It is sensible to limit G-forces for safety, and this is in accordance with regulations for bungee jumping.
12. It is essential that the bungee should not break, but this hardly needed to feature in the model.

The chief candidates for revision, in the extent to which they affect the model's predictions, seem to be Assumption 4, on the one hand, and Assumptions 6 and 7 on the other; the latter pair are linked to each other. We will consider here the effect of removing Assumption 4, and hence taking the mass of the bungee into account. This can be a significant proportion of the mass of the jumper.

Revise the model

Description of the revision

Of the original assumptions, Assumption 4 is now ignored, while the other assumptions remain in force. By Assumption 7, the mass of the bungee (denoted by M, in kg) is distributed uniformly. The law of conservation of mechanical energy can still be applied, since there are no dissipative forces. Taking into account the mass of the bungee increases the loss of gravitational potential energy that takes place during the jump. Since this is translated into spring potential energy, the jumper can be expected to fall further than in the first model, unless the unstretched bungee length is reduced to take this into account. At both $x = 0$ and $x = H$ the system is momentarily at rest, so the kinetic energy of the bungee need not be considered.

Your best policy for revising and improving your model is to relax one (and no more) of the simplifying assumptions that you made and subsequently evaluated. In writing this section, you need to make clear which assumption has been changed. You do not need to solve the revised model, but you do need to follow through the changes to the formulation of your first model that occur as a consequence of the new assumption.

You should make a qualitative assessment of the effects of the change to the model that result.

Note that using your first model again with a new set of values for the parameters does not constitute a revision of the model. The revision needs to be based on a change in the assumptions that underpin the first model.

After changing an assumption, you will probably need to modify the work that you did in creating the first model.

The revised model

At $x = 0$ the jumper has gravitational potential energy mgH as before. The bungee hangs down from the attachment at $x = 0$ to approximately $x = \frac{1}{2}L$, then rises back up to the jumper at $x = 0$ again, so its centre of mass can be taken to be at $x = \frac{1}{4}L$. Hence the potential energy of jumper and bungee together at the start is

$$mgH + Mg\left(H - \tfrac{1}{4}L\right).$$

When the jumper reaches $x = H$ (the bottom of the jump), the centre of mass of the bungee has moved to $x = \frac{1}{2}H$, hence its gravitational potential energy is $\frac{1}{2}MgH$. As before, the bungee then has spring potential energy $\frac{1}{2}ke^2$. Hence conservation of mechanical energy leads to the equation

$$mgH + Mg\left(H - \tfrac{1}{4}L\right) = \tfrac{1}{2}MgH + \tfrac{1}{2}ke^2,$$

or

$$\left(mH + \tfrac{1}{4}M(2H - L)\right)g = \tfrac{1}{2}ke^2 = \tfrac{1}{2}k(H - L)^2.$$

The other equations from the first model remain unchanged. The consequence of having $M > 0$ is that for a given drop distance H, the extension e will be greater than with $M = 0$. So either the cross-sectional area A of the bungee must increase or its unstretched length L must decrease, to avoid accidents.

Conclusions

This is your opportunity to sum up the work that you have done, to say whether you have obtained a satisfactory solution to the problem that you set out to solve, and to indicate how you might further improve your model.

One prediction from the model – that the cross-sectional area or number of threads in the bungee should be proportional to the mass of the jumper – is supported both by common sense and by the available data.

The model also predicts that as the mass increases, for a fixed cross-sectional area of bungee, the length of the unstretched bungee should decrease. The formula (10) for bungee length in terms of drop distance should be treated with suspicion, since it is based on a linear model for the deformation of rubber that does not apply at all closely. The mass of the bungee is also a significant factor, but as indicated above, the model could be revised to take this into account.

Any general advice provided on the basis of the model should be accompanied by appropriate warnings as to its limitations. A practical approach at the actual jump site would appear to be most important. This could involve the dropping of a sandbag of known mass, to gauge the appropriate length of bungee. The result could then be scaled (in terms of cross-sectional area) to match the mass of each jumper.

Appendix 1: The experiment and summary of outcomes

You should use appendices to describe any experiments that you have conducted to collect data for the interpretation or evaluation of your model. You might also use an appendix to present data from referenced sources, if you need to adapt them for use with your model.

An experiment was conducted to check the validity of equation (6) from the model, expressed in the form

$$e = \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2Lk}{mg}}\right). \tag{11}$$

In the first part of the experiment, masses were hung from a rubber band, to provide data from which the stiffness k could be estimated. Then the maximum extensions obtained by dropping various masses were measured.

Part 1 of the experiment

Note that it made sense to give measurements in centimetres here, rather than in the SI units of metres.

The experimental set-up is shown in Figure 6. The masses used were £1 coins, each of mass 9.5 g. The range of masses was extended using a bag of ground almonds weighing 11 times as much as a £1 coin. The mass of the plastic bag containing the coins was about $1\frac{1}{2}$ g, and this mass was neglected.

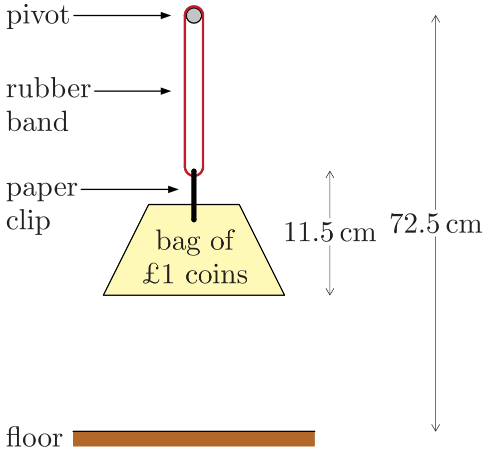


Figure 6 Set-up for both parts of the experiment

While it was not possible to obtain a direct measurement for the unstretched length of the rubber band, the data obtained for small suspended masses indicated that the appropriate value to take was $L = 14.6$ (in cm). The detailed data obtained for extensions are shown in Appendix 2.

From the data on extension e against suspended mass m, an average value for e/m was obtained, leading to an estimate for stiffness of $k = 22.2$. (Other approaches to fitting the data would lead to different values for k.)

However, the graphs indicate that except for small masses within the range considered, Hooke's law does not hold for this situation. Another notable feature was that the extensions measured when coins were being added progressively were smaller than the extensions when coins were being removed one by one. (This behaviour of natural rubber, known as 'hysteresis', is confirmed by various internet sources; see, for example, 'Hysteresis and rubber bands' at `internet source`.) To match the initial circumstances of a bungee jump, it seems sensible to consider only the data obtained when masses are being added and the rubber band is being initially stretched, and only those values for e/m were used in evaluating k.

Part 2 of the experiment

With values established for L and k from Part 1, equation (11) provides a prediction of what maximum extension e is to be expected for a given mass m attached via the rubber band and dropped from the pivot point. In this part of the experiment, the raw data obtained were measurements upwards from floor level, obtained using a pile of paperback books and judging whether the bag containing the dropped mass did or did not touch the top book. Hence the accuracy of these measurements was not as good as in Part 1. These data were then transformed appropriately into data for e. Again, the detailed outcomes are shown in Appendix 2.

It can be seen from the graph of maximum extension against dropped mass that while the quantitative results are in the right general area, the functional behaviour predicted seems to be at some odds with that observed. In particular, the second derivatives of the theoretical and empirical functions have opposite signs. This may well be due to the approximations involved in using a single value for the stiffness k instead of modelling the static extension behaviour more accurately.

Appendix 2: Details of experimental results

The results of the experiments outlined in Appendix 1 will be analysed using a computer algebra system. We begin by inputting the data for the length of the elastic band as coins are added to the bag, as `load1`, and as the coins are removed from the bag as `unload1`. The experiment is repeated to obtain `load2` and `unload2`. The average values for two sets of data are used to specify `load` and `unload`.

```
(%i1)  load1: [[1,14.9],[2,15.2],[3,15.5],[4,15.8],[5,16.1],
[6,16.4],[7,16.9],[8,17.3],[9,17.8],[10,18.3],[11,18.9],
[12,19.4]]$
```

```
(%i2)  unload1: [[1,15.1],[2,15.5],[3,15.8],[4,16.1],[5,16.5],
[6,16.9],[7,17.3],[8,17.7],[9,18.2],[10,18.6],[11,19.2],
[12,19.7]]$
```

```
(%i3)  load2: [[1,14.9],[2,15.2],[3,15.5],[4,15.8],[5,16.1],
[6,16.4],[7,16.8],[8,17.2],[9,17.7],[10,18.2],[11,18.8],
[12,19.4]]$
```

```
(%i4)  unload2: [[1,15.1],[2,15.4],[3,15.8],[4,16.2],[5,16.5],
[6,16.9],[7,17.3],[8,17.8],[9,18.2],[10,18.8],[11,19.2],
[12,19.7]]$
```

```
(%i5)  load: (load1+load2)/2$
```

```
(%i6)  unload: (unload1+unload2)/2$
```

The data in `load` can be used to predict the stiffness k for the elastic band by finding a least squares fit (simple linear regression) to the data.

```
(%i7)  res: simple_linear_regression(load);
```

$$(\%o7)\quad \begin{pmatrix} \text{SIMPLE LINEAR REGRESSION} \\ \text{model} = 0.40297202797203\,\text{x} + 14.23484848484848 \\ \text{correlation} = 0.99095108077158 \\ \text{v\_estimation} = 0.042602855477855 \\ \text{b\_conf\_int} = [0.36451341780587, 0.44143063813818] \\ \text{hypotheses} = \text{H0} : \text{b} = 0, \text{H1} : \text{b} \neq 0 \\ \text{statistic} = 23.34659584269099 \\ \text{distribution} = [\text{student\_t}, 10] \\ \text{p\_value} = 4.7062287400478908\,10^{-10} \end{pmatrix}$$

```
(%i8) bestfit: take_inference(model,res);
```

(%o8) $0.40297202797203\,x + 14.23484848484848$

The package calculates the best fit as

$$y = 0.40297202797203x + 14.23484848484848,$$

where y is the extension in centimetres and x is the number of coins. Now the stiffness k measured in newtons per metre satisfies

$$k \times \text{extension in metres} = \text{mass in kilograms} \times g,$$

so

$$k = \frac{\text{mass in kilograms} \times g}{\text{extension in metres}}.$$

The slope of the graph, suitably converted to the correct units, gives an estimate for k as

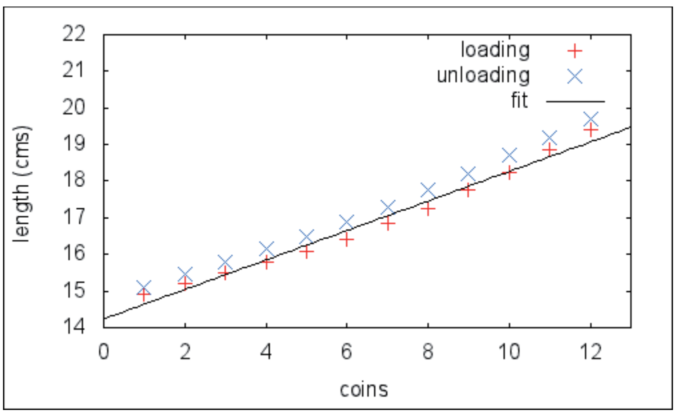
$$k = \frac{0.0095 \times 9.81 \times 100}{0.40297202797203} = 23.13,$$

so the estimate for the stiffness is $23.13\,\mathrm{N\,m^{-1}}$.

The experimental data and the least squares fit are compared in the following figure.

```
(%i9) wxplot2d([[discrete,load], [discrete,unload], bestfit],
[x, 0,13], [xlabel, "coins"], [y, 14,22],
[ylabel, "length (cms)"], [style, points,points,lines],
[point_type, plus,times,minus], [color, red,blue,black],
[legend,"loading","unloading","fit"])$
```

(%t9)



The fit looks pretty reasonable, but we need to see if the relationship still holds for large extensions. We thus continue the experiment by adding more coins in `load3`, `unload3`, `load4` and `unload4`.

```
(%i10) load3: [[11,18.8],[12,19.5],[13,20.0],[14,20.7],
[15,21.4],[16,22.1],[17,22.9],[18,23.9],[19,24.7],[20,25.7],
[21,26.7],[22,27.8],[23,28.8]]$
```

```
(%i11) unload3: [[11,20.5],[12,21.1],[13,21.8],[14,22.5],
[15,23.2],[16,23.8],[17,24.5],[18,25.4],[19,26.2],[20,27.0],
[21,27.8],[22,28.6],[23,29.4]]$
```

```
(%i12) load4: [[11,18.8],[12,19.6],[13,20.1],[14,20.8],
[15,21.4],[16,22.2],[17,23.0],[18,24.1],[19,25.0],[20,26.0],
[21,27.0],[22,28.0],[23,29.0]]$
```

```
(%i13) unload4: [[11,19.9],[12,20.3],[13,21.6],[14,22.3],
[15,23.1],[16,24.0],[17,24.8],[18,25.5],[19,26.3],[20,27.0],
[21,27.9],[22,28.6],[23,29.5]]$
```

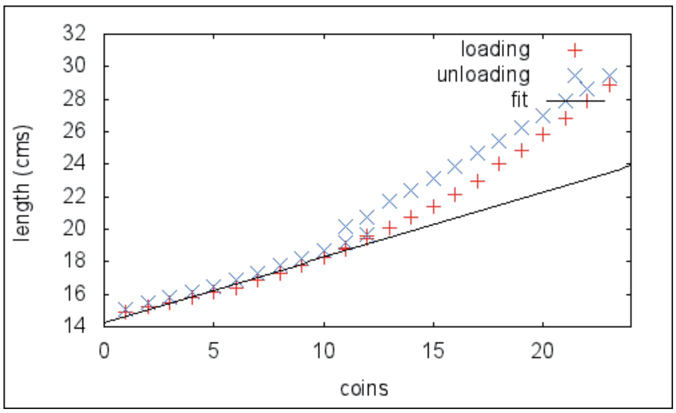
The data are again averaged and used with the original data to draw a graph, which shows that the linear fit is very unsatisfactory for the rubber band, where the cross-sectional area of the rubber band reduces as the band extends.

```
(%i14) allload: append(load,(load3+load4)/2)$
```

```
(%i15) allunload: append(unload,(unload3+unload4)/2)$
```

```
(%i16) wxplot2d([[discrete,allload],[discrete,allunload],bestfit],
[x, 0,24],[xlabel, "coins"],[y, 14,32], [ylabel, "length (cms)"],
[style, points,points,lines],[point_type, plus,times],
[color, red,blue,black],[legend, "loading","unloading","fit"])$
```

(%t16)



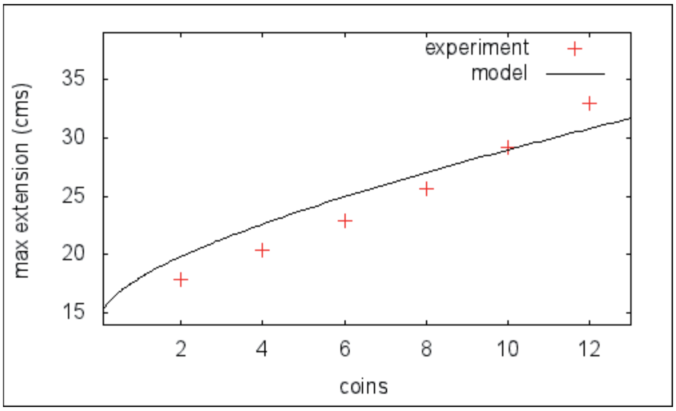
Part 2 of the experiment is to estimate the extension when the bag of coins is dropped. The results of two runs of the experiment are recorded in `drop1` and `drop2`, and averaged in `drop`. This is used to compare the results with the predicted results, using the model established in the following equation for the extension:

$$e = \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2kL}{mg}}\right).$$

```
(%i17) drop1: [[2,18.0],[4,20.5],[6,23.0],[8,25.5],
[10,28.0],[12,32.5]]$
```

```
(%i18) drop2: [[2,17.8],[4,20.2],[6,22.7],[8,25.8],
[10,30.4],[12,33.5]]$
(%i19) drop: (drop1+drop2)/2$
(%i20) extension: 14.235+100*0.0095*x*9.81/23.13
*(1+sqrt(2*0.14235*23.13/(0.0095*x*9.81)))$
(%i21) wxplot2d([[discrete,drop],extension], [x, 0.1,13],
[xlabel, "coins"], [y, 14,39], [ylabel, "max extension (cms)"],
[style, points,lines], [point_type, plus,times],
[color, red,black], [legend, "experiment","model"])$
```

(%t21)



This graph is also reproduced in the main body of the report (Figure 5).

Appendix 3: Further data obtained from the internet

From one internet website

This site is at: `internet source`

Bungee type	Number of threads	Mass range (kg)
Green	960	40–61
Red	1200	62–84
Blue	1440	85–98
Black	1600	99–118

The threads are 22 gauge. Extension achieved is 325–360%. Cord retirement is recommended after 250 jumps. Tensile break occurs at an extension of 650–700%.

From a second internet website

This site is at: `internet source`

Bungee type	Diameter (mm)	Number of threads	Mass range (kg)	Mass of cord ($kg\,m^{-1}$)
Yellow	37.5	920	45–62	1.10
Blue	41.0	1120	62–79	1.22
Red	46.0	1320	79–96	1.47
Black	48.0	1520	96–113	1.55

Extension of 350% is allowed in each case. Direct ultraviolet rays and ozone are the main culprits for shortening the lifespan of rubbers. It is recommended to use and discard cords within 150 days of the purchase date, and the number of jumps should not exceed 800.

Learning outcomes

After studying this unit, you should be able to:

- appreciate the role of dimensional analysis in creating relationships between the variables and parameters of a model
- understand the need to examine the sensitivity of a model to absolute and relative changes in its parameters, and be able to test the sensitivity of a model by analytic means in simple cases, and empirically in others
- appreciate the role of the assumptions of a model in thinking about possible revisions
- appreciate that there are a number of possible revisions to a model
- use a variety of techniques to revise a simple model
- appreciate that a complex model may be built up from a simple model by gradually including more and more features
- write a short report outlining the key stages of the modelling process associated with a particular problem.

Solutions to exercises

Solution to Exercise 1

The dimensions of the quantities assumed to be involved are listed in the table below.

Physical quantity	Symbol	Dimensions
Period	τ	T
Mass of bob	m	M
Length of pendulum stem	l	L
Angular amplitude of oscillations	Φ	1
Magnitude of acceleration due to gravity	g	$\mathrm{L\,T^{-2}}$

We wish to find an expression for the period τ in terms of the other quantities. We assume that this expression takes the form

$$\tau = k\,m^\alpha\,l^\beta\,\Phi^\gamma\,g^\delta,$$

where k is a dimensionless constant. For this equation to be dimensionally consistent, we must have

$$[\tau] = [k]\,[m]^\alpha\,[l]^\beta\,[\Phi]^\gamma\,[g]^\delta.$$

Hence

$$\mathrm{T} = \mathrm{M}^\alpha\,\mathrm{L}^\beta\,(\mathrm{L\,T^{-2}})^\delta = \mathrm{M}^\alpha\,\mathrm{L}^{\beta+\delta}\,\mathrm{T}^{-2\delta}.$$

Equating powers of M, L and T on both sides of this equation leads to

$$0 = \alpha, \quad 0 = \beta+\delta, \quad 1 = -2\delta.$$

Since $\alpha = 0$, any formula based only on the physical quantities in the table will not involve the mass of the bob.

Solving these equations gives

$$\alpha = 0, \quad \beta = \tfrac{1}{2}, \quad \delta = -\tfrac{1}{2}.$$

Hence

$$\tau = k\,l^{1/2}\,g^{-1/2}\,\Phi^\gamma.$$

Since additive combinations of this relationship with different values of k and γ are also dimensionally consistent, we conclude that

$$\tau = k(\Phi)\,l^{1/2}\,g^{-1/2} = k(\Phi)\sqrt{\frac{l}{g}},$$

where $k(\Phi)$ is an (undetermined) dimensionless function of Φ.

Solution to Exercise 2

The dimensions of the quantities involved in the situation are listed below.

Physical quantity	Symbol	Dimensions
Frequency of vibration of string	f	T^{-1}
Mass of string	m	M
Length of string	l	L
Tension of string	T_{eq}	$\mathrm{M\,L\,T}^{-2}$

We wish to find an expression for the frequency f in terms of the other quantities. We assume that this expression takes the form

$$f = k\,m^{\alpha}\,l^{\beta}\,T_{\mathrm{eq}}^{\gamma},$$

where k is a dimensionless constant. For this equation to be dimensionally consistent, we must have

$$[f] = [k]\,[m]^{\alpha}\,[l]^{\beta}\,[T_{\mathrm{eq}}]^{\gamma},$$

or

$$\mathrm{T}^{-1} = \mathrm{M}^{\alpha}\,\mathrm{L}^{\beta}\,(\mathrm{M\,L\,T}^{-2})^{\gamma} = \mathrm{M}^{\alpha+\gamma}\,\mathrm{L}^{\beta+\gamma}\,\mathrm{T}^{-2\gamma}.$$

Equating powers of M, L and T on both sides of this equation leads to

$$0 = \alpha + \gamma, \quad 0 = \beta + \gamma, \quad -1 = -2\gamma.$$

The solution of these equations is

$$\alpha = -\tfrac{1}{2}, \quad \beta = -\tfrac{1}{2}, \quad \gamma = \tfrac{1}{2}.$$

The relationship derived in Unit 14 was
$$f = \frac{1}{2}\sqrt{\frac{T_{\mathrm{eq}}}{ml}}.$$

Hence

$$\begin{aligned} f &= k\,m^{-1/2}\,l^{-1/2}\,T_{\mathrm{eq}}^{1/2} \\ &= k\sqrt{\frac{T_{\mathrm{eq}}}{ml}}, \end{aligned}$$

where k is a dimensionless constant.

Solution to Exercise 3

(a) Suppose that we increase the annual payment from £8000 to £8100. The capital sum owing becomes

$$\begin{aligned} C_n &= \left(100\,000 - \frac{8100}{0.05}\right)(1.05)^n + \frac{8100}{0.05} \\ &= 162\,000 - 62\,000(1.05)^n. \end{aligned}$$

After 15 years, the sum outstanding is $C_{15} = 33\,106.45$, which is a reduction $\delta C_{15} = -2157.86$. This is significantly larger than the £100 increase in the annual repayment. However, this comparison makes no sense, not least because the annual payment is made 15 times in the course of 15 years. The relative sensitivity is more useful, where a change of $(100/8000) \times 100$, that is, 1.25%, results in a change of

$(2157.86/35\,264.31) \times 100$, that is, a 6.12% reduction in the sum owing after 15 years. By the module definition of relative sensitivity, when $C_0 = 100\,000$, $R = 8000$ and $i = 0.05$, the sum owing after 15 years is relatively insensitive to changes in the annual payment, since the magnification is $6.12/1.25 = 4.90$ to two decimal places, and this is less than 5.

(b) Suppose that we change the interest rate from 5% to 4.9%, a decrease of $(0.001/0.05) \times 100$, that is, 2%. The capital sum owing becomes

$$\begin{aligned} C_n &= \left(100\,000 - \frac{8000}{0.049}\right)(1.049)^n + \frac{8000}{0.049} \\ &= 163\,265.31 - 63\,265.31(1.049)^n. \end{aligned}$$

After 15 years, the sum outstanding is $C_{15} = 33\,607.71$, which is a reduction $\delta C_{15} = -1656.60$, that is, a 4.70% reduction in the sum owing after 15 years. Again, by the module definition of relative sensitivity, when $C_0 = 100\,000$, $R = 8000$ and $i = 0.05$, the sum owing after 15 years is not relatively sensitive to changes in the interest rate, since the magnification is $4.70/2 = 2.35$ to two decimal places, and this is less than 5.

Solution to Exercise 4

(a) We are interested in the positive value of x when $y = 0$. The change in x for a change in u is given by $\partial x/\partial u$. Differentiating partially with respect to u keeping all the other parameters fixed gives

$$0 = 0 + \frac{\partial x}{\partial u}\tan\theta - 2x\frac{\partial x}{\partial u}\frac{g}{2u^2}\sec^2\theta + x^2\frac{g}{u^3}\sec^2\theta.$$

Collecting up the terms in $\partial x/\partial u$, we have

$$\frac{\partial x}{\partial u}\left(x\frac{g}{u^2}\sec^2\theta - \tan\theta\right) = x^2\frac{g}{u^3}\sec^2\theta.$$

For relative sensitivity, we calculate expression (7):

$$\frac{u}{x}\frac{\partial x}{\partial u} = \frac{xg\sec^2\theta}{xg\sec^2\theta - u^2\tan\theta} = \frac{1}{1 - u^2\sin\theta\cos\theta/(xg)}.$$

For a given set of data, we can use this formula to assess the relative sensitivity of the output x to changes in the launch speed u.

(b) Using the given values we have

$$\frac{u}{x}\frac{\partial x}{\partial u} = \frac{1}{1 - 200/(42.68 \times 9.81)} \simeq 1.9145.$$

$u^2\sin\theta\cos\theta = 20^2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$
$= 200.$

Thus a 5% increase in the launch speed results in a 1.9% increase in the distance, which is approximately 0.81 m.

Since the ratio of the relative change in u to the relative change in x is approximately $1.91/5 = 0.38$, which is small, we can conclude that the model, for this set of data, is relatively insensitive to changes in the launch speed.

Solution to Exercise 5

(a) If $h_2 = r_2$, then $V = \pi r_2^2 h_2 = \pi r_2^3$, giving $r_2 = (V/\pi)^{1/3}$, and $A_2 = 2\pi r_2 h_2 + 2\pi r_2^2 = 4\pi r_2^2 = 4\pi(V/\pi)^{2/3} = 4\sqrt[3]{V^2\pi}$. The relative increase in the amount of material needed is

$$\begin{aligned}\frac{A_2 - A}{A} &= \frac{(4 - 3\sqrt[3]{2})\sqrt[3]{V^2\pi}}{3\sqrt[3]{2}\sqrt[3]{V^2\pi}}\\ &= \frac{4 - 3\sqrt[3]{2}}{3\sqrt[3]{2}} = \frac{4}{3\sqrt[3]{2}} - 1 = 0.058.\end{aligned}$$

Thus an extra 5.8% of material is needed for a can where the diameter is twice the height. The amount of material required appears to be pretty insensitive to changes in the height, for a given volume, for $r \leq h \leq 4r$.

(b) For the optimal solution, when $h = 2r$ and $r = (V/(2\pi))^{1/3}$, the area of the wall of the can is

$$W = 2\pi rh = 4\pi r^2 = 4\pi\left(\frac{V}{2\pi}\right)^{2/3} = 2\sqrt[3]{2V^2\pi}.$$

When the height is twice the diameter, $h_1 = 4r_1$ and $r_1 = (V/(4\pi))^{1/3}$, so the area of the wall of the can is

$$W_1 = 2\pi r_1 h_1 = 8\pi r_1^2 = 8\pi\left(\frac{V}{4\pi}\right)^{2/3} = 2\sqrt[3]{4V^2\pi}.$$

The relative change in this area is

$$\frac{W_1 - W}{W} = \frac{2\sqrt[3]{4} - 2\sqrt[3]{2}}{2\sqrt[3]{2}} = \sqrt[3]{2} - 1 = 0.260.$$

Thus the increase in advertising area when the height is twice the width is 26%.

When the height is half the diameter, $h_2 = r_2$ and $r_2 = (V/\pi)^{1/3}$, so the area of the wall of the can is

$$W_2 = 2\pi r_2 h_2 = 2\pi r_2^2 = 2\pi\left(\frac{V}{\pi}\right)^{2/3} = 2\sqrt[3]{V^2\pi}.$$

The relative change in this area is

$$\frac{W_2 - W}{W} = \frac{2 - 2\sqrt[3]{2}}{2\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}} - 1 = -0.206.$$

Thus the decrease in advertising area when the height is half the width is 21%.

These results support the idea that increasing the height gives more advertising space, which may explain why cans tend to have height greater than width.

Solution to Exercise 6

(a) The area of material used is

$$A_3 = 2\pi rh + 8r^2,$$

where the term $8r^2$ is the area of two squares of side $2r$ of material used to make the circular end pieces, with the excess discarded. The volume is still $V = \pi r^2 h$. Thus $h = V/(\pi r^2)$ and the area of material to be minimised is

$$A_3 = \frac{2V}{r} + 8r^2.$$

(b) Differentiating with respect to r to find the maximum and minimum values of A_3 gives

$$\frac{dA_3}{dr} = -\frac{2V}{r^2} + 16r = 0,$$

from which we can deduce that $r = V^{1/3}/2$. This gives a minimum (rather than a maximum) since $d^2A/dr^2 > 0$. We can also deduce that $h = V/(\pi r^2) = 4V^{1/3}/\pi$, so $h/r = 8/\pi = 2.55$, which is larger than the ratio $h/r = 2$ for our first model.

The area of material used is

$$A_3 = \frac{2V}{V^{1/3}/2} + 8(V^{2/3}/4) = 6\sqrt[3]{V^2}.$$

The relative increase in the amount of material used is

$$\begin{aligned}\frac{A_3 - A}{A} &= \frac{(6 - 3\sqrt[3]{2\pi})\sqrt[3]{V^2}}{3\sqrt[3]{2\pi}\sqrt[3]{V^2}}\\ &= \frac{6 - 3\sqrt[3]{2\pi}}{3\sqrt[3]{2\pi}} = 2/\sqrt[3]{2\pi} - 1 = 0.084.\end{aligned}$$

Thus discarding the leftover pieces requires an extra 8.4% of material to manufacture the cans. It is clearly more important to recycle the leftover pieces than to worry about the precise ratio of h to r.

Solution to Exercise 7

(a) The radius of the circular pieces is now $r + b$, and the length of material for the walls of the can is now $2\pi r + b$. The area of material required is

$$A_4 = (2\pi r + b)h + 2\pi(r + b)^2,$$

where $h = V/(\pi r^2)$.

(b) Replacing h in the equation for A_4 gives

$$A_4 = \frac{(2\pi r + b)V}{\pi r^2} + 2\pi(r + b)^2.$$

Differentiating with respect to r to find the maximum and/or minimum values of A_4 gives

$$\frac{dA_4}{dr} = \frac{2\pi V}{\pi r^2} - \frac{2V(2\pi r + b)}{\pi r^3} + 4\pi(r+b) = 0.$$

Multiplying through by $r^3/(2\pi)$ and collecting up the terms in powers of r gives

$$2r^4 + 2r^3 b - \frac{rV}{\pi} - \frac{Vb}{\pi^2} = 0.$$

This is a quartic equation in r with potentially four roots, which will depend on the size of the overlap b. Only one of these roots will be the value of r for a practical solution that minimises the area.

Solution to Exercise 8

(a) Assumption 1 is reasonable, since the pebble will be very small. Assumption 2 could be changed to incorporate air resistance. This would slow the pebble down and hence reduce the estimate for the depth of the well. Assumption 3 could be varied to include the time taken for the sound to travel back up the well. This would also reduce the estimate for the depth of the well.

(b) Starting with Newton's second law, with the x-axis pointing down the well, the equation of motion is

$$m\ddot{x}\mathbf{i} = mg\mathbf{i} - c_1 D\dot{x}\mathbf{i},$$

where D is the diameter of the pebble, and $c_1 = 1.7 \times 10^{-4}$ for $D|\dot{x}\mathbf{i}| < 10^{-5}$ is a constant, given in Unit 3. Writing $k = c_1 D/m$, the equation to be solved is

$$\ddot{x} + k\dot{x} = g.$$

This linear equation can be solved using the techniques for second-order differential equations given in Unit 1, to give the general solution

This exact problem was solved in Example 7(a) in Unit 3 and in Example 1 in Unit 10.

$$x = A + Be^{-kt} + gt/k.$$

The particular solution with $x(0) = \dot{x}(0) = 0$ is

$$x = \frac{g}{k^2}\left(e^{-kt} - 1\right) + \frac{gt}{k}.$$

For a pebble of diameter 1 cm ($D = 0.01$) and mass 0.01 kg we have $k = 1.7 \times 10^{-4}$. For a time of 2.8 s for the pebble to reach the bottom of the well, the model estimates the depth of the well as 38.45 m, which is almost identical to the estimate without air resistance.

It should be noted that the linear model of air resistance is not suitable for this problem since $D|\dot{x}\mathbf{i}| > 10^{-5}$ for a considerable part of the motion. The quadratic model applied to this problem gives the estimate 37.5 m.

(c) The time for the pebble to drop down the well, from rest, is

$$T_1 = \sqrt{2h/g}.$$

The time taken for the sound to travel up again is $T_2 = h/c$, where c is the speed of sound. Thus the total time T is

$$T = T_1 + T_2 = \sqrt{2h/g} + h/c.$$

Taking the h/c over to the other side and squaring both sides gives

$$\left(T - \frac{h}{c}\right)^2 = \frac{2h}{g}.$$

Hence to find h given T we need to solve the quadratic equation

$$\frac{1}{c^2}h^2 - \left(\frac{2T}{c} + \frac{2}{g}\right)h + T^2 = 0.$$

With $c = 343$, $T = 2.8$ and $g = 9.81$, the quadratic is

$$8.5 \times 10^{-6}h^2 - 0.22020h + 7.84 = 0,$$

and this has roots

$$h = \frac{0.22020 \pm \sqrt{0.22020^2 - 4 \times 8.5 \times 10^{-6} \times 7.84}}{1.7 \times 10^{-5}}.$$

Thus the two estimates for the depth of the well are $h = 35.65$ and the spurious solution $h = 25\,870$ (which was introduced when $T - h/c$ was negative when we squared both sides). Taking into account the time for the sound to travel back up the well has reduced the estimate of the depth from 38.5 m to 35.7 m. So there is evidence that this has made a significant improvement to the model, at least for wells around 35 m deep.

Solution to Exercise 9

(a) The model is based on the assumption that the pollutant does not biodegrade or decay. The model would not be valid if the pollutant were unstable, in that it might be broken down by other elements in the water or by sunlight, for example.

(b) A significant reduction in the input of PCBs after this time would be expected, although there may be some residual pollutants in the catchment area of the lake that may take a number of years before they are washed into the lake. The model is based on the assumption that all sources of new pollution cease. This assumption may be reasonable if the input of PCBs to the lake has, as a result of the ban, reduced to insignificant levels.

(c) As one of the two lakes (Lake Michigan is the other) that is not downstream of any other lake in the system, there would be no input of pollution to Lake Superior from any of the other lakes.

Solution to Exercise 10

The model assumes no new pollution, and PCBs were banned in 1975. Any data from before this are therefore inappropriate for comparison purposes. Since there is a time delay in the ecosystem as pollution works its way up the food chain, it is likely that the level of pollution in herring gull eggs reflects the pollution level in the lake at an earlier time. It may thus be wise to also disregard the data point for 1975.

Solution to Exercise 11

Any of the assumptions, listed in Unit 8, could be challenged.

Assumption (a) is that all sources of pollution have been removed. In reality there may continue to be new pollution for several years.

Assumption (b) is that the pollutant does not biodegrade or decay. In reality all substances biodegrade or decay, although some do so more quickly than others.

Assumption (c) is that the pollutant is evenly dispersed. In reality some areas of the lake, close to the sources of pollution, will have a higher concentration of pollutant than the rest of the lake.

Assumption (d) is that water flows into and out of the lake at a constant rate. In reality there will be seasonal effects, with more water entering the lake during rainy periods, and less water entering during dry periods.

Assumption (e) is that all other water gains and losses can be ignored. In reality drinking water is extracted, water leaves the lake by evaporation, and so on.

Assumption (f) is that the volume of water in the lake is constant. In reality there will be seasonal effects, with more water in the lake during rainy periods, and less water during dry periods.

Assumption (g) is that negligible pollution flows into downstream lakes from upstream lakes (this is not relevant for Lake Superior and Lake Michigan). This appears to be unlikely.

Since the rate at which PCB concentration diminishes in Lake Superior is much faster than that predicted by the model, the likeliest candidate for re-examination is Assumption (b).

Solution to Exercise 12

The rate of volatilisation will depend on the surface area of the lake and on the concentration of pollutant near to the surface of the lake. It will also depend on the strength of the wind and on the concentration of the pollutant already in the atmosphere. It may also depend on temperature.

Solution to Exercise 13

(a) Let $c_{\mathrm{B}} = c$ be the concentration of pollutant in the bay, so $c_{\mathrm{L}} = \frac{1}{2}c$ is the concentration of pollutant in the rest of the lake. The rate of output of pollutant from the bay (in $\mathrm{kg\,s^{-1}}$) by each of the three mechanisms is as follows.

Water flow: $r_{\mathrm{B}}c_{\mathrm{B}} = 153c$.
Volatilisation: $pA_{\mathrm{B}}c_{\mathrm{B}} = 4872c$.
Diffusion: $E(c_{\mathrm{B}} - c_{\mathrm{L}}) = 5500c$.

Hence volatilisation and diffusion are the most important mechanisms for removing pollutant from the bay. The amount of pollutant removal by water flow is very small by comparison.

(b) The relative importance of the three mechanisms will not change greatly if the concentration of pollutant in the rest of the lake is less than half that in the bay (assuming that the concentration in the bay is still given by $c_{\mathrm{B}} = c$). Even if $c_{\mathrm{L}}/c_{\mathrm{B}}$ is very small, the loss of pollutant from the bay by diffusion will at most double. (If c_{B} and c_{L} are nearly equal, then the loss of pollutant by diffusion may be small, but since the bay is in practice significantly more polluted than the rest of the lake, this scenario is unlikely to occur.)

(c) Two mechanisms, water flow and volatilisation, are involved here. The rate of output of pollutant from the rest of the lake (in $\mathrm{kg\,s^{-1}}$) by each mechanism is as follows, for a pollutant concentration c_{L}.

Water flow: $r_{\mathrm{L}}c_{\mathrm{L}} = 4967c_{\mathrm{L}}$.
Volatilisation: $pA_{\mathrm{L}}c_{\mathrm{L}} = 66\,120c_{\mathrm{L}}$.

Hence volatilisation is the most important mechanism for pollutant output from the lake. (This is in line with a conclusion drawn from the first revised model.)