MST2102306F1PV1

MST210

Module Examination 2023

Mathematical methods, models and modelling

Friday 9 June 2023

There are **three sections** in this examination.

In **Section 1** you should **submit answers to <u>all</u> 19 questions**. Each question is worth 2% of the total mark. A wholly incorrectly answered question will get zero marks. Answers to Section 1 should be submitted using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to <u>all</u> 6 questions**. Each question is worth 5% of the total mark.

In **Section 3** you should **attempt <u>both</u> questions**. Each question is worth 16% of the total mark.

For **Sections 2 and 3**:

Include all your working, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST210.

PLAGIARISM WARNING – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

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- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

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Section 1

You should ***attempt all questions****.* ***Each question is worth 2%.***

Question 1

Consider the following differential equation

$$\frac{dy}{dx} + xy^2 = x^2y^2$$

Select the option that correctly describes how the solution can or cannot be found.

A The separation of variables method only

B Either the integrating factor method or the separation of variables method

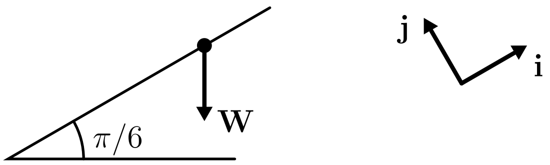
C Neither the integrating factor method nor the separation of variables method

D The integrating factor method only

Answer:

Question 2

Consider a particle on a slope inclined at an angle $\pi/6$ to the horizontal. Let $\mathbf{W}$ be the weight of the particle and choose unit vectors with $\mathbf{i}$ up the slope and $\mathbf{j}$ perpendicularly out of the slope, as shown in the diagram below.



Select the option that gives the force $\mathbf{W}$ expressed in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

A $\frac{1}{2}|\mathbf{W}|\mathbf{i} + \frac{\sqrt{3}}{2}|\mathbf{W}|\mathbf{j}$

B $\frac{\sqrt{3}}{2}|\mathbf{W}|\mathbf{i} - \frac{1}{2}|\mathbf{W}|\mathbf{j}$

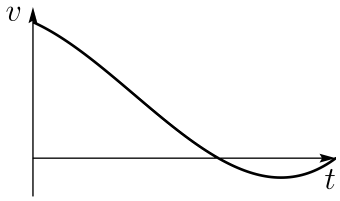
C $-\frac{\sqrt{3}}{2}|\mathbf{W}|\mathbf{i} + \frac{1}{2}|\mathbf{W}|\mathbf{j}$

D $-\frac{1}{2}|\mathbf{W}|\mathbf{i} - \frac{\sqrt{3}}{2}|\mathbf{W}|\mathbf{j}$

Answer:

Question 3

Consider the following graph that shows the x-component of the velocity v of a particle moving along the x-axis plotted versus time.



Select the true statement from the following options.

A During the time interval shown, the acceleration of the particle is always in same direction and the particle never returns to its starting position

B During the time interval shown, the acceleration of the particle is always in same direction, and the particle returns to its starting position

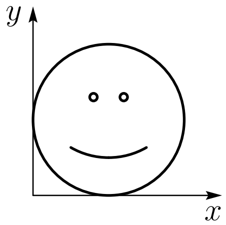
C During the time interval shown, the acceleration of the particle changes direction, and the particle never returns to its starting position

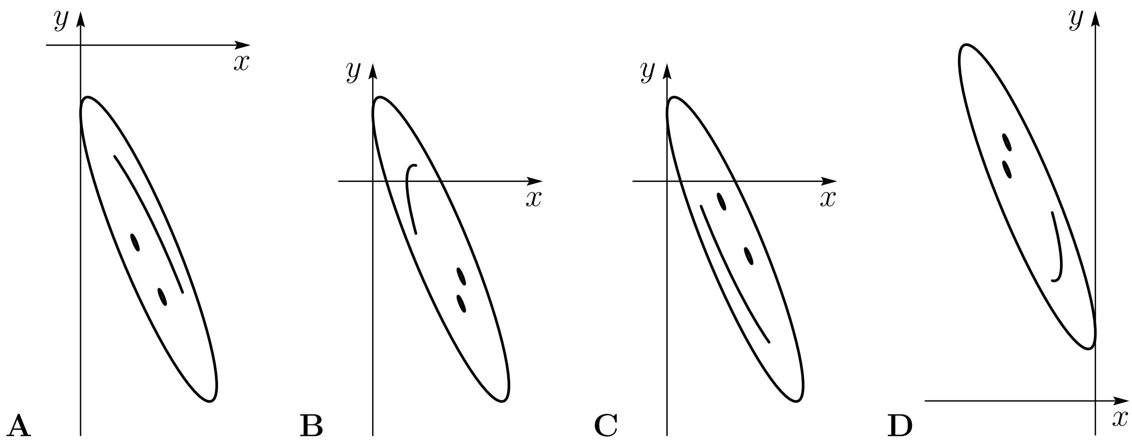
D During the time interval shown, the acceleration of the particle changes direction, and the particle returns to its starting position

Answer: ☐

Question 4

Select the option that shows the result of applying the linear transformation $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ to the following figure.





Answer: ☐

Question 5

The eigenvalues of a matrix $\mathbf{A}$ are $-\frac{1}{2}$, 2 and 3.

Enter the eigenvalue of largest magnitude of $-\mathbf{I} + (\mathbf{A}^{-1})^3$ in the box below.

eigenvalue of largest magnitude of $-\mathbf{I} + (\mathbf{A}^{-1})^3 =$ [ ]

Question 6

A system of differential equations has a general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-5t} + \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}.$$

Enter an expression in the box below that gives the value of α in the particular solution that satisfies $x = 4$ and $y = \frac{11}{2}$ when $t = 0$.

$\alpha =$ [ ]

Question 7

A function of two variables $f(x, y)$ has a stationary point with Hessian matrix

$$\mathbf{H} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}.$$

Select the option that describes this stationary point.

A The stationary point is a saddle point.

B The stationary point is a local maximum.

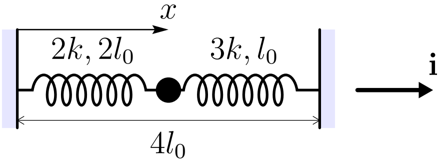
C The stationary point is a local minimum.

D The nature of the stationary point cannot be determined from the Hessian matrix.

Answer: [ ]

Question 8

A particle rests on a smooth surface and is attached by two horizontal model springs to two fixed walls a distance $4l_0$ apart. The left hand spring is of stiffness $2k$ and natural length $2l_0$ and the right hand spring is of stiffness $3k$ and natural length l_0. Let x be the distance of the particle from the left hand wall, as shown in the following diagram.

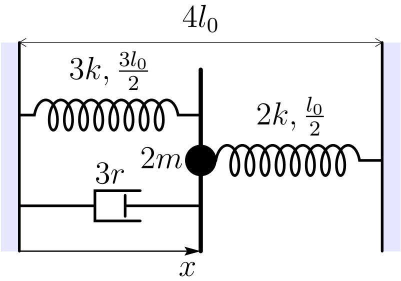


Enter in the box below the distance x from the left hand wall when the system is in equilibrium (use `l_0` to input l_0).

$x =$ [ ]

Question 9

A particle of mass $2m$ is attached to a fixed wall on its left by a spring with stiffness $3k$ and natural length $\frac{3l_0}{2}$ and a damper with damping constant $3r$. The particle is also attached to another fixed wall a distance $4l_0$ from the left hand wall by a spring of stiffness $2k$ and natural length $\frac{l_0}{2}$. Let x be the distance of the particle from the left-hand wall, as shown in the following diagram.



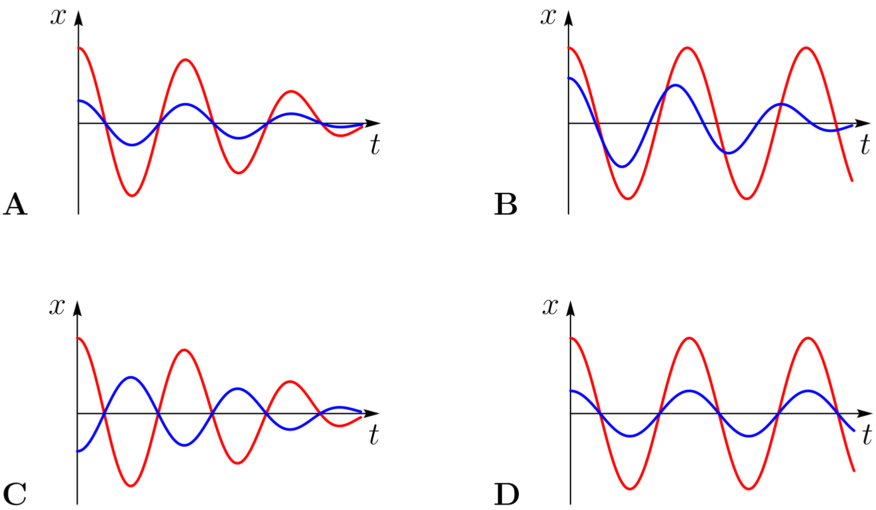
Enter an expression in the box below that gives the right hand side of the equation of motion of the particle (use `l_0` to enter l_0).

$2m\ddot{x} + 3r\dot{x} + 5kx =$ ☐

Question 10

A mechanical system with two particles constrained to move along a straight line is set in motion starting from rest. The x-coordinates of the particles along their line of motion are plotted against time in the graphs below, which show the resulting motion.

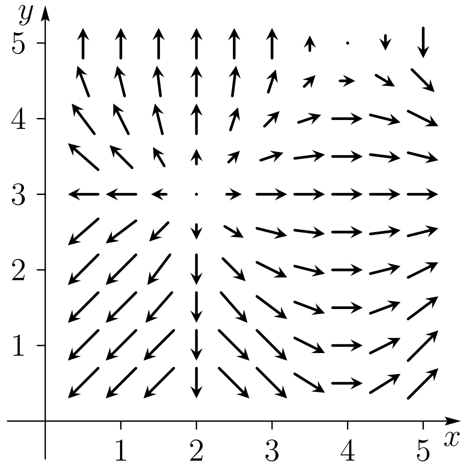
Select the option that corresponds to the particles moving in normal mode motion.



Answer: ☐

Question 11

Consider the autonomous system of non-linear equations $\dot{\mathbf{x}} = \mathbf{u}(x, y)$, where $\mathbf{u}$ is a vector field shown below.



This system has an equilibrium point at (2, 3). Assume that the behaviour of the vector field close to the equilibrium point is the same as the pattern of arrows shown.

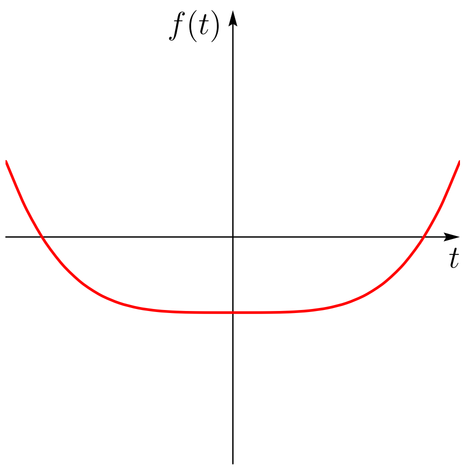
Select the option that could be the classification of the equilibrium point at (2, 3).

A Sink **B** Star source **C** Saddle **D** Centre

Answer: $\square$

Question 12

Consider the function with the graph shown below.



Select the option that best describes this function.

A The function is even.

B The function is both even and odd.

C The function is odd.

D The function is neither even nor odd.

Answer: $\square$

Question 13

Consider applying the method of separation of variables with $u(x,t) = X(x)\,T(t)$ to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2}.$$

Select the option that gives the resulting pair of ordinary differential equations (where μ is a non-zero separation constant).

A $X''(x) - X'(x) = \mu X(x), \quad \ddot{T}(t) = \mu T(t)$

B $X''(x) - X'(x) = \mu, \quad \ddot{T}(t) = \mu$

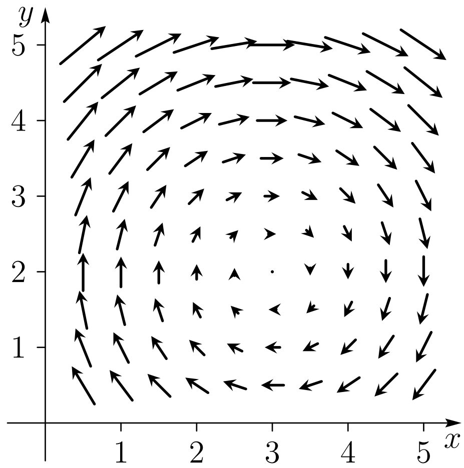
C $X''(x) = \mu, \quad \ddot{T}(t) - \dot{T}(t) = \mu$

D $X''(x) = \mu X(x), \quad \ddot{T}(t) - \dot{T}(t) = \mu T(t)$

Answer: $\square$

Question 14

Consider the vector field shown below.



Select the option that gives the equations that could correspond to this vector field.

A $(2-y)\,\mathbf{i} + (x-3)\,\mathbf{j}$

B $(y-2)\,\mathbf{i} + (3-x)\,\mathbf{j}$

C $(2-x)\,\mathbf{i} + (y-3)\,\mathbf{j}$

D $(x-2)\,\mathbf{i} + (3-y)\,\mathbf{j}$

Answer: $\square$

Question 15

A vector field is defined in spherical coordinates by

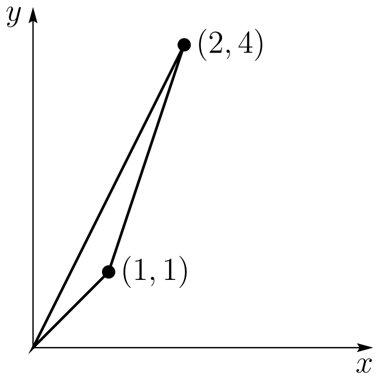
$$\mathbf{F}(r,\theta,\phi) = r\sin\theta\,\mathbf{e}_r + r^2\sin\theta\,\mathbf{e}_\theta + r\cos\theta\,\mathbf{e}_\phi.$$

Enter the expression that gives the divergence of $\mathbf{F}$ in the box below (write θ as `theta` and ϕ as `phi`).

$\boldsymbol{\nabla}\cdot\mathbf{F} =$ $\square$

Question 16

Consider the triangle with vertices $(0,\ 0)$, $(1,\ 1)$ and $(2,\ 4)$, as shown in the diagram below.



Select the option that gives an expression that evaluates to the area of this triangle.

A $\displaystyle\int_{y=0}^{y=4}\left(\int_{x=y}^{x=2y} 1\,dx\right)\,dy$

B $\displaystyle\int_{x=0}^{x=1}\left(\int_{y=x}^{y=2x} 1\,dy\right)\,dx + \int_{x=1}^{x=2}\left(\int_{y=3x-2}^{y=2x} 1\,dy\right)\,dx$

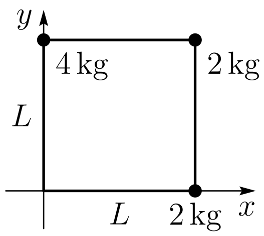
C $\displaystyle\int_{y=0}^{y=1}\left(\int_{x=y}^{x=2y} 1\,dx\right)\,dy + \int_{y=1}^{y=4}\left(\int_{x=3y-2}^{x=2y} 1\,dx\right)\,dy$

D $\displaystyle\int_{x=0}^{x=2}\left(\int_{y=x}^{y=2x} 1\,dy\right)\,dx$

Answer: ☐

Question 17

A light wire frame is formed as a square of side L lies in the positive quadrant of the plane with one vertex at the origin. Particles of mass 2 kg, 2 kg, and 4 kg are fixed to the other three corners in an anticlockwise order, as shown in the diagram.



Select the option that gives the coordinates of the centre of mass of the system.

A $(\frac{L}{4},\ \frac{3L}{4})$ **B** $(\frac{L}{2},\ \frac{3L}{4})$ **C** $(\frac{L}{4},\ \frac{L}{4})$ **D** $(\frac{L}{2},\ \frac{L}{4})$

Answer: ☐

Question 18

A particle of mass 2 kg is attached by a light inextensible string to a fixed point O. The particle moves in uniform circular motion in a horizontal plane with the string horizontal in a circle of radius 4 metres with speed $\frac{1}{3}\,\mathrm{m\,s^{-1}}$.

Enter an expression in the box below that gives the magnitude of the tension force due to the string (do not include units in the box below).

magnitude of the tension force = $\boxed{}$ N

Question 19

Consider a solid cube of side $3L$ and mass $2M$ that rotates about an axis along one of the edges of the cube.

Select the option that gives the moment of inertia of this solid about the axis described.

A $\frac{15}{2}ML^2$ **B** $12ML^2$ **C** $3ML^2$ **D** $9ML^2$

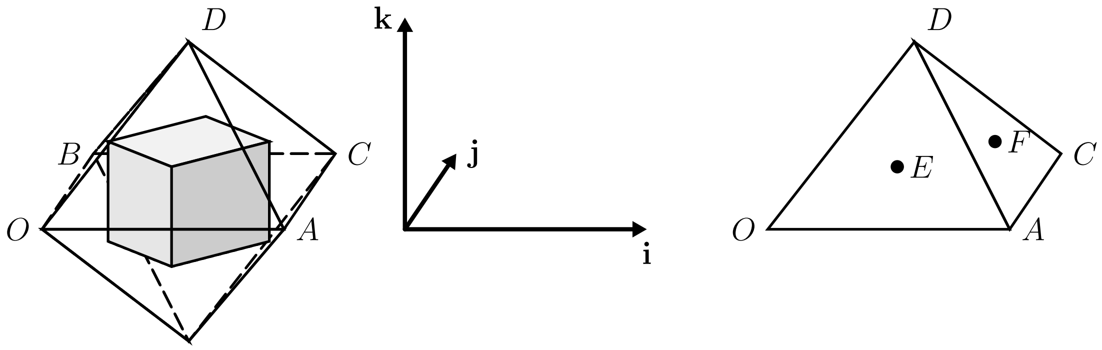
Answer: $\boxed{}$

Section 2

*You should **attempt all questions**. **Each question is worth 5%.***

Question 20

For every polyhedron there is a construction for a so called dual polyhedron by connecting together the centres of the faces. The diagram on the left below shows this construction for a regular octahedron that constructs a cube inside it.



Choose an origin O to be the front left vertex of the regular octahedron. Choose the $\mathbf{i}$ to be $\overrightarrow{OA}$ and choose $\mathbf{j}$ to be $\overrightarrow{OB}$. Choose $\mathbf{k}$ to be vertically upwards, so that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ forms a right-handed Cartesian coordinate system. In this coordinate system the vertex D has position vector, $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + a\mathbf{k}$, where a is a positive constant.

Let E be the centre of the face OAD and F be the centre of the face ACD, as shown in the right-hand diagram.

(a) The centre of a triangular face can be calculated as the average of the position vectors of the vertices. Show that the position vector of point E is

$$\overrightarrow{OE} = \tfrac{1}{2}\mathbf{i} + \tfrac{1}{6}\mathbf{j} + \tfrac{a}{3}\mathbf{k}. \qquad [2]$$

(b) Calculate the position vector $\overrightarrow{OF}$ of the point F. *[1]*

(c) Calculate the displacement vector $\overrightarrow{EF}$ and hence calculate the side length of the cube. *[2]*

Question 21

Consider the following simultaneous linear ordinary differential equations.

$$\frac{dx}{dt} = -11x + 7y, \quad \frac{dy}{dt} = -21x + 17y.$$

(a) Express the system of equations in matrix form. *[1]*

(b) Find the eigenvalues and eigenvectors of the matrix of coefficients. *[3]*

(c) Hence write down the general solution to this system. *[1]*

Question 22

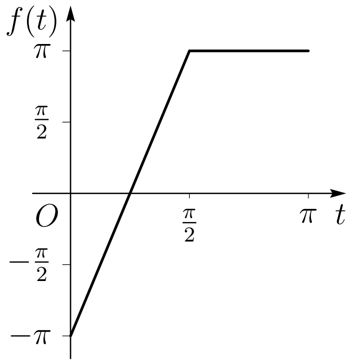
The function of three variables x, y and z given by

$$f(x, y, z) = \frac{x^2}{2} + 3x + 3yz + z^2 + (y - 5)^2,$$

has a stationary point at $(-3,\ -4,\ 6)$. Note that you are not required to verify this fact. Calculate the Hessian matrix of f at this stationary point and hence classify the stationary point. [5]

Question 23

Consider the function $f(t)$ defined on the interval $(0, \pi)$ that corresponds to the graph



The graph consists of a straight line segment from $(0, -\pi)$ to $(\frac{\pi}{2}, \pi)$ followed by another line segment from $(\frac{\pi}{2}, \pi)$ to (π, π).

(a) Sketch the even and odd extensions function $f(t)$ over the range $-3\pi \leq t \leq 3\pi$, clearly indicating each case. [3]

(b) The even extension of $f(t)$ has Fourier series $F(t)$ given by

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nt)$$

Calculate the constant term A_0. [2]

Question 24

A path C is defined by the parametric equations

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \leq t \leq 1,$$

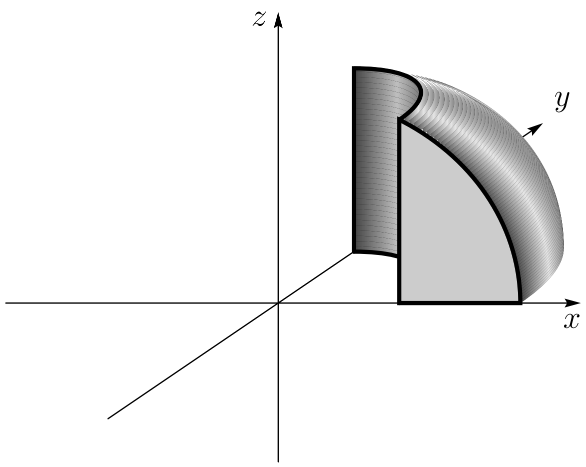
and a vector field is defined by the equation

$$\mathbf{F} = x^2\mathbf{i} + 4xy^2\mathbf{j}.$$

Calculate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$. [5]

Question 25

Use cylindrical coordinates to find the volume of the object represented in the following figure, which is the region where the following inequalities are satisfied: $x^2 + y^2 + z^2 \leq 4$, $x^2 + y^2 \geq 1$, $x \geq 0$, $y \geq 0$, and $z \geq 0$,.



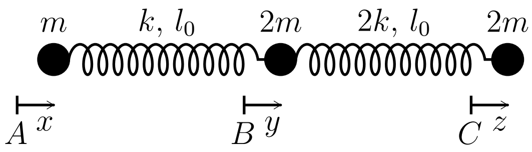
[5]

Section 3

You should ***attempt all questions****.* ***Each question is worth 16%.***

Question 26

Three particles of mass m, $2m$ and $2m$ are constrained to move along a frictionless horizontal groove. The first two particles are connected by a spring of stiffness k and natural length l_0. The last two particles are connected by a spring of stiffness $2k$ and natural length l_0. Initially the particles are in equilibrium at points A, B and C and we measure the displacements x, y and z of the three particles from these equilibrium points, as shown in the following diagram.



The system is displaced from equilibrium and we are concerned with modelling the subsequent motion.

(a) Draw a force diagram for each particle, showing all of the forces acting on the particles. *[2]*

(b) Determine the changes in spring forces from their equilibrium values in terms of the above variables and parameters. *[3]*

(c) Derive the equation of motion of this mechanical system in the form

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\mathbf{A}$ is a matrix that you should specify explicitly. *[3]*

(d) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ by calculating the corresponding eigenvalue. What is the physical significance of this eigenvector in terms of the motion of the particles? *[2]*

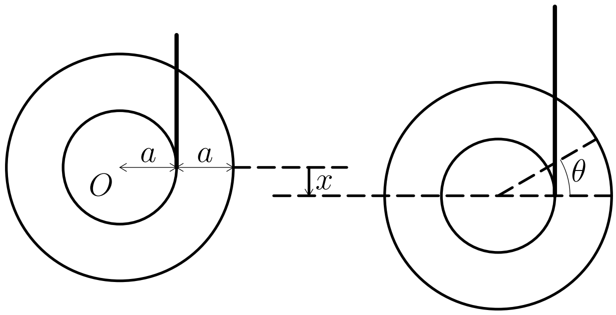
(e) Show that $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and write down the corresponding eigenvalue. *[2]*

(f) Use the trace formula to calculate a third eigenvalue of $\mathbf{A}$ and then calculate the corresponding eigenvector. *[2]*

(g) Hence write down the solution to the equation of motion for this system. *[2]*

Question 27

Consider a model of a spherical toy yo-yo that consists of a solid sphere of radius $2a$ and mass m. A thin groove of depth a is cut into the sphere and a model string is wrapped around this groove, as shown in the left-hand diagram below. Consider the downward motion of the yo-yo that occurs while the string unwraps without slipping from the groove.



The yo-yo is released from rest and falls a distance x as it rotates an angle θ, as shown in the right-hand diagram above. Assume that the top end of the string is fixed and that the effects of air resistance can be ignored.

Calculate the acceleration of the yo-yo in terms of g, the magnitude of the acceleration due to gravity. [*16*]

[END OF QUESTION PAPER]