Unit 10

Sequences and series

Introduction

This unit introduces the concept of a *sequence*, which is the mathematical name for a list of numbers arranged in a particular order. The list can stop at a particular number, or it can contain infinitely many numbers. Here are some of the sequences discussed in the unit:

$1,\ 3,\ 5,\ 7,\ \dots$

$7.15,\ 10.50,\ 13.85,\ 17.20,\ \dots,\ 245.00$

$1000,\ 950,\ 900,\ 850,\ \dots,\ 0$

$1000,\ 1050,\ 1102.50,\ 1157.63,\ 1215.51$

$2000,\ 1400,\ 980,\ 686,\ 480.2,\ \dots$

$1,\ 2,\ 6,\ 24,\ 120,\ \dots$

$\frac{1}{4},\ -\frac{1}{8},\ \frac{1}{16},\ -\frac{1}{32},\ \dots$

$1,\ 7,\ 21,\ 35,\ 35,\ 21,\ 7,\ 1.$

Here the three dots … stand for 'and so on', to indicate that the list continues. If the dots are followed by a number, then that number terminates the sequence. Otherwise, the sequence continues indefinitely.

Several of the sequences above arise from practical situations, such as estimating the height of a bouncing ball on successive bounces. Each of them has a definite mathematical pattern, and you may like to try to spot the pattern now in each case. (However, do not spend longer than a few minutes on this. The patterns may become harder to spot as you move down the list.) You'll see later what pattern is associated with each sequence.

In Section 1 you'll meet two different ways, known as *closed forms* and *recurrence systems*, to specify sequences using formulas. A closed form is a formula for each *term* (individual number) in a sequence, whereas a recurrence system indicates what the first term is and how each subsequent term is related to the previous term in the sequence.

In Section 2 you'll study two special types of sequence, known as *arithmetic* and *geometric* sequences, which include the majority of the examples above.

Section 3 explores how sequences can be visualised, and how they behave in the long term, that is, when a large number of terms are considered.

In some cases, it's possible to calculate the sum of the terms of a sequence, even if there are infinitely many of them. An expression that's obtained by adding the terms of a sequence, such as $1 + 3 + 5 + 7 + 9$, is known as a *series*. Series are the subject of Section 4.

In Section 5 the focus moves from sequences to the *binomial theorem*. This important result enables you to multiply out expansions such as $(1 + x)^4$, $(a + b)^7$ and $(2y - 3)^5$ quickly.

1 What is a sequence?

In this section you'll meet some notation used for sequences, and two different ways to specify sequences using formulas.

1.1 Sequence notation

We frequently see lists of numbers arranged in order. For example, a list of numbers can be used to represent a quantity that's varying over time, such as the midday temperature (in °C) at a particular location, recorded each day for a week:

$$13,\ 12,\ 10,\ 10,\ 10,\ 9,\ 8;$$

or the amount of money (in £) in a savings account on each 1 January over a five-year period:

$$1000.00,\ 1050.00,\ 1102.50,\ 1157.63,\ 1215.51.$$

In mathematics, a list of numbers is called a **sequence**, and each number in the list is called a **term** of the sequence.

A sequence that has a finite number of terms (and hence has a last term as well as a first term) is called a **finite** sequence, whereas a sequence that has an infinite number of terms is called an **infinite** sequence.

For example,

$$13,\ 12,\ 10,\ 10,\ 10,\ 9,\ 8$$

is a finite sequence, whose first term is 13 and whose last term is 8. On the other hand, the sequence of odd natural numbers,

$$1,\ 3,\ 5,\ 7,\ 9,\ \dots,$$

goes on forever, with first term 1 but no last term, so this is an infinite sequence. For each term in the sequence there is always a next term, an odd number 2 greater than the one before, so the sequence has no end.

Many sequences have a structure, or pattern, that allows us to give a concise description of the sequence and helps us to understand its behaviour. For example, the sequence $1, 3, 5, 7, 9, \dots$ has a simple mathematical structure; each term is exactly 2 more than the previous term:

$$1 + 2 = 3, \quad 3 + 2 = 5, \quad 5 + 2 = 7, \quad 7 + 2 = 9, \quad \dots.$$

The savings account sequence 1000, 1050, …, 1215.51 also has a simple structure, though this is less obvious. For this sequence, you can check that, to three significant figures:

$$\frac{1050}{1000} = 1.05, \quad \frac{1102.50}{1050} = 1.05, \quad \frac{1157.63}{1102.50} = 1.05, \quad \frac{1215.51}{1157.63} = 1.05,$$

so each term is 1.05 times the previous term (approximately, at least).

This pattern arises because the savings account pays compound interest of 5% per annum, and there have been no withdrawals. We consider this sequence again in Section 2.

The last two fractions above are not *exactly* equal to 1.05 because the numbers in the sequence have been rounded to two decimal places, that is, to the nearest penny. Without this rounding the sequence would be

$$1000.00,\ 1050.00,\ 1102.50,\ 1157.625,\ 1215.506\,25,$$

for which each term is exactly 1.05 times the previous term.

There's no reason to suppose that the temperature sequence above has any simple mathematical structure, since it is obtained by making independent measurements each day. In this unit, we'll investigate various types of sequence that *do* have simple underlying mathematical structures. Such sequences arise in various real-world situations and have many applications.

First, however, here's some notation associated with sequences. In elementary algebra, we use letters such as $a, b, c, x, y, z, A, B, C$, and so on, to represent variables. With a sequence, we represent the terms by using one particular letter with an attached subscript; this subscript is an integer that indicates which term of the sequence is referred to. Thus the sequence

$$a_1,\ a_2,\ a_3,\ \dots,\ a_{10}$$

has 10 terms, the first being a_1 and the last being a_{10}. This notation is called **subscript notation** (or **suffix notation**). The terms above are read as: a-one (or a-sub-one), a-two (or a-sub-two), and so on. (You've seen subscripts used previously, for coordinates in Unit 2, and for vector components in Unit 5.)

Sometimes it is possible to choose an appropriate letter for a sequence. For example, you might use t for the temperature sequence. Since the first term is 13, you write $t_1 = 13$, and so on, giving

$$t_1 = 13, \quad t_2 = 12, \quad t_3 = 10, \quad t_4 = 10, \quad t_5 = 10, \quad t_6 = 9, \quad t_7 = 8.$$

You can use either upper- or lower-case letters to represent sequences. The use of appropriate letters can be helpful, especially when dealing with several sequences, but it is not necessary and indeed not always possible.

If you want to refer to a general term of a sequence, rather than a particular term, then you use a letter for the subscript as well, as follows:

a_n denotes the term of the sequence with subscript n.

Here n represents a natural number in the appropriate range; for example, for the sequence $a_1, a_2, \dots, a_{10}$, the range of n is $1, 2, \dots, 10$. In mathematics, the letter n often represents a natural number or, more generally, an integer; that is, one of the numbers $\dots, -2, -1, 0, 1, 2, \dots$. Other letters commonly used to represent integers are i, j, k, l, m, p and q.

The notation a_n denotes an individual term of a sequence – the term with subscript n. You can represent a whole sequence by using notation such as the following:

$(a_n)_{n=1}^{17}$ denotes the finite sequence $a_1,\ a_2,\ a_3,\ \dots,\ a_{17}$

$(a_n)_{n=1}^{\infty}$ denotes the infinite sequence $a_1,\ a_2,\ a_3,\ \dots$.

Because infinite sequences occur frequently, we use the abbreviated notation (a_n), with no subscript values, to mean $(a_n)_{n=1}^{\infty}$. That is,

(a_n) denotes the infinite sequence $a_1,\ a_2,\ a_3,\ \ldots.$

The variable n in the notation for a sequence is sometimes referred to as the **index variable** for the sequence. It's a **dummy variable**: you can change it to any other variable name you like without changing the meaning. For example,

$$(a_n)_{n=1}^{\infty}, \quad (a_i)_{i=1}^{\infty} \quad \text{and} \quad (a_k)_{k=1}^{\infty}$$

all denote the same sequence $a_1, a_2, a_3, \ldots.$ (The idea of a dummy variable featured also in Unit 8.)

Activity 1 *Using sequence notation*

Consider the infinite sequence

$(b_n)_{n=1}^{\infty}$ with terms $1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ \ldots.$

(a) Write down the values of b_1 and b_4.

(b) For which value of n is $b_n = 16$?

(c) Can you write down the value of b_0?

1.2 Closed forms for sequences

The notation a_n is useful when you want to specify a sequence by giving a formula for the terms. For example, suppose that you want to specify the infinite sequence of *perfect squares*:

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots.$$

If you choose to represent this sequence using the letter s (for square), then

$$s_1 = 1^2, \quad s_2 = 2^2, \quad s_3 = 3^2, \quad s_4 = 4^2, \quad \ldots.$$

For a general natural number n, we have $s_n = n^2$, which is a formula for the general term, called the ***n*th term**, of the sequence. To complete the specification of the sequence, you need to state the range of values of the subscript n. You do this using brackets, as follows:

$$s_n = n^2 \quad (n = 1, 2, 3, \ldots).$$

Note that it's important to include the range of values of n here. It tells you that for this particular sequence the first term corresponds to $n = 1$ and that the sequence continues indefinitely, rather than stopping after some number of terms.

A formula like $s_n = n^2$, for defining a sequence in terms of the subscript n, is called a **closed form** (or a **closed-form formula**). A closed form for a sequence enables you to calculate any term of the sequence directly, once you're given the value of n. Unfortunately, however, not all sequences have such a formula.

Closed form for a sequence

A **closed form** for a sequence is a formula that defines the general term a_n as an expression involving the subscript n. To specify a sequence using a closed form, two pieces of information are needed:

- the closed form
- the range of values for the subscript n.

Example 1 *Using closed forms for sequences*

For each of the sequences specified by the following closed forms and ranges of values of n, find the first four terms and the 10th term.

(a) $a_n = 2^n - n^2 \quad (n = 1, 2, 3, \ldots)$

(b) $b_n = 1/n^2 \quad (n = 1, 2, 3, \ldots)$

Solution

Substitute in turn $n = 1, 2, 3, 4, 10$ into the closed form for each sequence.

(a)
$$\begin{aligned} a_1 &= 2^1 - 1^2 = 1 \\ a_2 &= 2^2 - 2^2 = 0 \\ a_3 &= 2^3 - 3^2 = -1 \\ a_4 &= 2^4 - 4^2 = 0 \\ a_{10} &= 2^{10} - 10^2 = 924 \end{aligned}$$

(b)
$$\begin{aligned} b_1 &= 1/1^2 = 1 \\ b_2 &= 1/2^2 = 1/4 \\ b_3 &= 1/3^2 = 1/9 \\ b_4 &= 1/4^2 = 1/16 \\ b_{10} &= 1/10^2 = 1/100 \end{aligned}$$

Here are some examples for you to try.

Activity 2 *Using closed forms*

For each of the sequences specified by the following closed forms and ranges of values of n, find the first five terms and the 100th term.

(a) $a_n = 7n \quad (n = 1, 2, 3, \ldots)$

(b) $b_n = 1/n \quad (n = 1, 2, 3, \ldots)$

(c) $c_n = (-1)^{n+1} \quad (n = 1, 2, 3, \ldots)$

(d) $d_n = (-1)^n n \quad (n = 1, 2, 3, \ldots)$

(e) $e_n = (-2)^n \quad (n = 1, 2, 3, \ldots)$

Notice the device used in Activity 2(c)–(e) to specify a sequence whose terms alternate in sign (that is, are alternately positive and negative). The expression $(-1)^n$ has value -1 when n is odd and 1 when n is even, and the expression $(-1)^{n+1}$ has value 1 when n is odd and -1 when n is even. In part (e), $(-2)^n = (-1)^n 2^n$.

If you're given some terms of a particular sequence, then you may be able to spot a closed form for the sequence by recognising the pattern involved. It's helpful to consider questions such as the following.

- Is the nth term a constant multiple of n or of some fixed power of n?
- Is the nth term a constant multiple of the nth power of some fixed number?
- Do the terms alternate in sign?

Activity 3 *Spotting closed forms*

For each of the following sequences, given by the first four terms, try to spot a closed form for the sequence. Denote the nth term of the sequence by a_n in each case. Then use your closed form to find the 10th term in the sequence.

(Don't spend more than a short time attempting each part. Some of the closed forms are harder to spot than others.)

(a) $1,\ 2,\ 3,\ 4,\ \dots$ (b) $2,\ 4,\ 8,\ 16,\ \dots$ (c) $-1,\ 1,\ -1,\ 1,\ \dots$

(d) $1,\ 8,\ 27,\ 64,\ \dots$ (e) $6,\ -6,\ 6,\ -6,\ \dots$ (f) $\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \dots$

(g) $2,\ -4,\ 6,\ -8,\ \dots$

When you write down a closed form for a sequence, you should usually also write down the subscript range, as demonstrated in the solution to Activity 3, so that you have fully specified the sequence.

So far you've represented sequences using the subscript 1 for the first term. For example, you've seen that you can represent a sequence as $a_1, a_2, a_3, \dots$. This seems natural and easy to remember, but there are occasions when it is convenient to be flexible about how you represent the first term. Consider, for example, the sequence

$$1,\ 2,\ 4,\ 8,\ 16,\ \dots.$$

This is a sequence of powers of 2:

$$2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ \dots.$$

You can specify this sequence in closed form, using $a_n = 2^n$, but only if you start numbering the subscripts from 0 rather than from 1:

$$a_n = 2^n \quad (n = 0, 1, 2, \dots).$$

In this case, the simplicity of the formula $a_n = 2^n$ generally outweighs any inconvenience of starting with a_0. There is the possibility of confusion in having a_0 as the first term of a sequence, with a_1 as second term, and so on, but with practice this should not cause difficulties. If it were considered essential to have the first term with subscript 1, then you could specify the sequence as follows:

$$b_n = 2^{n-1} \quad (n = 1, 2, 3, \ldots).$$

Then, for example, $b_1 = 2^{1-1} = 2^0 = 1$.

Further flexibility about the subscript for the first term of a sequence is sometimes useful, as illustrated in the next activity.

Activity 4 *Using a subscript other than 1 for the first term*

(a) For each of the following closed forms and ranges of values of n, find the first three terms of the sequence specified.

(i) $a_n = 3^n \quad (n = 0, 1, 2, \ldots)$

(ii) $b_n = \dfrac{1}{n(n-1)} \quad (n = 2, 3, 4, \ldots)$

(iii) $c_n = \dfrac{1}{(n+1)n} \quad (n = 1, 2, 3, \ldots)$

(b) For each of the following sequences, given by the first four terms, try to spot a closed form for the sequence, using the notation specified. Then use your closed form to find the sixth term in the sequence. (Don't spend more than a short time attempting each part.)

(i) $1,\ 3,\ 9,\ 27,\ \ldots$ (general term d_n, first term d_0)

(ii) $\frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \ldots$ (general term e_n, first term e_5)

(iii) $\frac{1}{4},\ -\frac{1}{8},\ \frac{1}{16},\ -\frac{1}{32},\ \ldots$ (general term f_n, first term f_2)

The sequences in parts (a)(ii) and (a)(iii) of Activity 4 illustrate the fact that two sequences whose descriptions appear to be different at first sight can actually have exactly the same terms. This activity also highlights the importance of including the range of values of n in the definition of the sequence.

It's sometimes useful to convert a closed form for a sequence into a different closed form for the same sequence, which uses a different subscript for the first term. Here's an example.

Example 2 *Changing the subscripts used for a sequence*

Consider the sequence given by the closed form

$$a_n = n(n+2) \quad (n = 1, 2, 3, \dots).$$

For the same sequence, find a closed form

$$b_n = \dots \quad (n = 0, 1, 2, \dots).$$

Solution

The value of n corresponding to each term is reduced by 1 in moving from (a_n) to (b_n), so you need to replace each occurrence of n in the formula for the term by $n+1$, to leave the term unchanged.

The required closed form is

$$b_n = (n+1)(n+1+2) \quad (n = 0, 1, 2, \dots);$$

that is,

$$b_n = (n+1)(n+3) \quad (n = 0, 1, 2, \dots).$$

As a check, the closed form for (a_n) gives

$$a_1 = 1 \times 3, \quad a_2 = 2 \times 4, \quad a_3 = 3 \times 5, \quad \dots.$$

The closed form obtained for (b_n) gives

$$b_0 = 1 \times 3, \quad b_1 = 2 \times 4, \quad b_2 = 3 \times 5, \quad \dots,$$

which is the same sequence.

Activity 5 *Changing the subscripts used for sequences*

(a) For each of the sequences (a_n) given by the following closed forms, write down a closed form $b_n = \dots \quad (n = 0, 1, 2, \dots)$ that specifies the same sequence.

(i) $a_n = 2n \quad (n = 1, 2, 3, \dots)$

(ii) $a_n = 3^{n-1} \quad (n = 1, 2, 3, \dots)$

(iii) $a_n = 5 + n \quad (n = 1, 2, 3, \dots)$

(iv) $a_n = 4(n-1) \quad (n = 1, 2, 3, \dots)$

(b) For each of the sequences (a_n) given by the following closed forms, write down a closed form $b_n = \dots \quad (n = 1, 2, 3, \dots)$ that specifies the same sequence.

(i) $a_n = 0.4^n \quad (n = 0, 1, 2, \dots)$

(ii) $a_n = 5n \quad (n = 0, 1, 2, \dots)$

(iii) $a_n = \dfrac{1}{2^n} \quad (n = 0, 1, 2, \dots)$

(iv) $a_n = 2 + 3n \quad (n = 0, 1, 2, \dots)$

(c) For each of the sequences (a_n) given by the following closed forms, write down a closed form $b_n = \dots \quad (n = 2, 3, 4, \dots)$ that specifies the same sequence.

(i) $a_n = \dfrac{3^n}{n+1} \quad (n = 0, 1, 2, \dots)$

(ii) $a_n = \dfrac{1}{(n+1)(n+3)} \quad (n = 0, 1, 2 \dots)$

It's implicit in what we've done so far that where a simple pattern is evident from the first few terms of a sequence, that pattern is assumed to continue unchanged. For example, in Activity 3(d) you were asked to find a closed form for the sequence $1, 8, 27, 64, \dots$. The answer

$$a_n = n^3 \quad (n = 1, 2, 3, \dots)$$

was found by spotting that the first four terms are the cubes of $1, 2, 3, 4$, respectively, and assuming that this pattern continues. However, alternative closed forms can be written down that give the same first four terms; one of these is

$$b_n = n^3 + (n-1)(n-2)(n-3)(n-4) \quad (n = 1, 2, 3, \dots).$$

The terms of the sequence (b_n) differ from those of (a_n) from the fifth term onwards, so these are different sequences. In general, no finite number of terms can describe a sequence without ambiguity, whereas a sequence described by a closed form with a subscript range is unambiguously defined. Where sequences are described by giving the first few terms, we shall always assume that any simple pattern that is evident continues for the remainder of the sequence.

1.3 Recurrence relations for sequences

Some sequences have the property that each term (after the first) can be obtained *from the previous term* by using a formula. For example, consider the sequence with closed form $a_n = 7n$ $(n = 1, 2, 3, \dots)$. Its terms are

$$7,\ 14,\ 21,\ 28,\ 35,\ \dots.$$

Each term (after the first) of this sequence can be obtained from the previous term by adding 7:

$$a_2 = a_1 + 7, \quad a_3 = a_2 + 7, \quad a_4 = a_3 + 7, \quad \dots.$$

If a term (after the first) has subscript n, then the previous term has subscript $n-1$, so you can write

$$a_n = a_{n-1} + 7 \quad (n = 2, 3, 4, \dots).$$

Note that the range of values of n here begins with 2 rather than 1, because 2 is the first value of n for which the equation $a_n = a_{n-1} + 7$ applies.

In a similar way, the sequence with closed form $b_n = 2^n$ $(n = 0, 1, 2, \ldots)$, whose terms are

$$1,\ 2,\ 4,\ 8,\ 16,\ \ldots,$$

has the property that each term (after the first) is always twice the previous term:

$$b_1 = 2b_0, \quad b_2 = 2b_1, \quad b_3 = 2b_2, \quad \ldots.$$

Hence you can write

$$b_n = 2b_{n-1} \quad (n = 1, 2, 3, \ldots).$$

A formula like $a_n = a_{n-1} + 7$ or $b_n = 2b_{n-1}$, which allows each term of a sequence (after the first) to be obtained from the previous term, is called a **recurrence relation**. It's more specifically a **first-order** recurrence relation because it involves only the immediately preceding term of the sequence. (A second-order recurrence relation would involve the preceding two terms, and so on.) In this module, the words 'recurrence relation' always refer to a first-order recurrence relation, unless otherwise stated.

If you have a recurrence relation for a sequence, and you also know the first term of the sequence, then in principle you can determine any term of the sequence by starting from the first term and repeatedly applying the recurrence relation. For example, from the recurrence relation

$$x_n = x_{n-1}^2 \quad (n = 2, 3, 4, \ldots),$$

and the first term $x_1 = 2$, you can successively calculate

$$\begin{aligned} x_1 &= 2, \\ x_2 &= x_1^2 = 2^2 = 4, \\ x_3 &= x_2^2 = 4^2 = 16, \\ x_4 &= x_3^2 = 16^2 = 256, \end{aligned}$$

and so on. Here the expression x_1^2, for example, means $(x_1)^2$, the square of x_1.

Notice that if you keep the same recurrence relation but change the first term, x_1, then you obtain a different sequence; for example, with $x_1 = 1$, the recurrence relation above gives $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, ...

Taken together, the specification of a first term, a recurrence relation and the range of values of n for which the recurrence relation applies is called a **recurrence system**, and the resulting sequence is called a **recurrence sequence**. We display the three parts of a recurrence system as follows:

$$x_1 = 2, \qquad x_n = x_{n-1}^2 \quad (n = 2, 3, 4, \ldots);$$

the first term of the sequence is on the left, and the recurrence relation and the range of values of n are on the right. If the first term of the sequence is labelled not as x_1 but as x_0, say, then the range of values of n has to begin with 1 rather than 2, as follows:

$$x_0 = 2, \qquad x_n = x_{n-1}^2 \quad (n = 1, 2, 3, \ldots).$$

Recurrence system for a sequence

A **recurrence relation** for a sequence is an equation that defines each term other than the first as an expression involving the previous term. To specify a sequence using a **recurrence system**, three pieces of information are needed:

- the value of the first term
- the recurrence relation
- the range of values for the subscript n.

Example 3 *Using recurrence systems for sequences*

For each of the following recurrence systems, find the first five terms of the sequence specified.

(a) $a_1 = 1, \qquad a_n = 5a_{n-1} - 2 \quad (n = 2, 3, 4, \ldots)$

(b) $b_1 = 2, \qquad b_n = \dfrac{1}{b_{n-1}} \quad (n = 2, 3, 4, \ldots)$

(c) $c_0 = 1, \qquad c_n = nc_{n-1} \quad (n = 1, 2, 3, \ldots)$

Solution

Apply the recurrence relation for each value of n in turn.

(a) $a_1 = 1$
$a_2 = 5 \times a_1 - 2 = 3$
$a_3 = 5 \times a_2 - 2 = 13$
$a_4 = 5 \times a_3 - 2 = 63$
$a_5 = 5 \times a_4 - 2 = 313$

(b) $b_1 = 2$
$b_2 = 1/b_1 = \frac{1}{2}$
$b_3 = 1/b_2 = 2$
$b_4 = 1/b_3 = \frac{1}{2}$
$b_5 = 1/b_4 = 2$

(c) $c_0 = 1$
$c_1 = 1 \times c_0 = 1$
$c_2 = 2 \times c_1 = 2$
$c_3 = 3 \times c_2 = 6$
$c_4 = 4 \times c_3 = 24$

Activity 6 *Using recurrence systems*

For each of the following recurrence systems, find the first five terms of the sequence specified.

(a) $a_1 = 0, \qquad a_n = 2a_{n-1} + 1 \quad (n = 2, 3, 4, \dots)$

(b) $b_1 = 1, \qquad b_n = b_{n-1}^2 - 1 \quad (n = 2, 3, 4, \dots)$

(c) $c_0 = 2, \qquad c_n = \frac{1}{2}(c_{n-1} + 2/c_{n-1}) \quad (n = 1, 2, 3, \dots)$

In part (c), round the terms to six decimal places when writing them down, but maintain full calculator precision when calculating each term.

The sequence in Activity 6(c) has the property that it very rapidly gives good approximations to $\sqrt{2}$, as you can check by finding $\sqrt{2}$ on your calculator. A proof of this property is outside the scope of this module, but it illustrates an important use for recurrence systems, namely, that sequences defined by using recurrence systems can sometimes be used to calculate approximations to certain irrational numbers to as many decimal places as may be required.

The next box summarises what you have seen in this section.

Three ways to specify a sequence

A sequence (a_n) can be specified by giving one of the following.

- The values of the first few terms, if we assume that any simple pattern that is apparent continues.
- A closed form,

  $$a_n = \text{expression in } n,$$

  and a subscript range. This permits the value of a_n to be found directly for any value of n in the subscript range.
- A recurrence system, consisting of the value of the first term, a recurrence relation,

  $$a_n = \text{expression involving } a_{n-1},$$

  and a range of values of the subscript n. This permits the value of a_n to be found from the value of a_{n-1} for any value of n in the subscript range.

It's often (though not always) convenient to take the first subscript of a sequence to be 1, so to avoid constant repetition we adopt the following convention.

Convention

The first term of a sequence (a_n) is taken to be a_1 unless indicated otherwise.

In the next section, you'll study two particular types of recurrence system, and investigate whether closed forms can be found for the corresponding sequences. Many sequences defined by recurrence systems do have closed forms, but not all.

2 Arithmetic and geometric sequences

In this section, we consider two types of sequence that occur frequently in practice.

2.1 Arithmetic sequences

We begin with two sequences that arise in different ways but are of a similar mathematical type.

First, consider the finite sequence

$$7.15,\ 10.50,\ 13.85,\ 17.20,\ \ldots,\ 245.00.$$

This sequence could represent the heights in metres above ground level of successive floors in a very tall building, from the first floor upwards. (The terms are similar to the heights in metres of the habitable floors of the Shard, in London.) We'll call this sequence (h_n), so $h_1 = 7.15$, $h_2 = 10.50$, $h_3 = 13.85$, and so on.

The Shard, with St Paul's Cathedral on the left

To get from any term in this sequence to the next, we *add* the same number each time:

$$7.15 + 3.35 = 10.50,$$

$$10.50 + 3.35 = 13.85,$$

$$13.85 + 3.35 = 17.20,$$

and so on. The number 3.35 occurs here because it is the height difference in metres between successive floors. Thus this sequence can be defined by the recurrence system

$$h_1 = 7.15, \qquad h_n = h_{n-1} + 3.35 \quad (n = 2, 3, 4, \ldots, 72).$$

The last value in the range of n here is 72 because, as will be confirmed later in this subsection, the last term in the sequence is h_{72}. (The 72nd floor of the Shard is about 245 metres above ground level. The total height of the building is 310 metres.)

Next, consider the finite sequence

$$1000,\ 950,\ 900,\ 850,\ \ldots,\ 0.$$

This sequence could represent the volume, measured in litres on successive Saturdays, of oil in a tank supplying a boiler that uses 50 litres of oil per week. We'll call this sequence (v_n), so $v_1 = 1000$, $v_2 = 950$, $v_3 = 900$, and so on.

Once again, to get from any term in this sequence to the next, we *add* the same number each time:

$$1000 + (-50) = 950,$$

$$950 + (-50) = 900,$$

$$900 + (-50) = 850,$$

and so on. The negative number -50 occurs here because each week the volume in the tank is *reduced* by 50 litres. Thus this sequence can be defined by the recurrence system

$$v_1 = 1000, \qquad v_n = v_{n-1} - 50 \quad (n = 2, 3, 4, \ldots, 21).$$

The last value in the range of n here is 21 because 1000 litres at 50 litres per week lasts 20 weeks, so the last term in the sequence is $v_{21} = 0$.

Any sequence with the structure demonstrated above – the addition of a fixed number to obtain the next term – is called an **arithmetic sequence**, or alternatively an **arithmetic progression**. (In this context, 'arithmetic' is pronounced with emphasis on the syllable 'met'.) Thus a general arithmetic sequence is given by the recurrence system

$$x_1 = a, \qquad x_n = x_{n-1} + d \quad (n = 2, 3, 4, \ldots),$$

where a is the first term and d is the number that's added to each term to give the next term. That is, d is the difference $x_n - x_{n-1}$ between any pair of successive terms, usually called the **common difference** of the sequence. Choosing the values of the first term a and the common difference d determines a particular arithmetic sequence; we call a and d the **parameters** of the arithmetic sequence. For example, the floor heights sequence has parameters $a = 7.15$ and $d = 3.35$, and the oil volumes sequence has parameters $a = 1000$ and $d = -50$.

An arithmetic sequence (x_n) can be finite, as for the floor heights sequence and the oil volumes sequence, or infinite. Also, the first term can be x_0 rather than x_1, in which case the sequence is given by

$$x_0 = a, \qquad x_n = x_{n-1} + d \quad (n = 1, 2, 3, \ldots).$$

Activity 7 *Recognising arithmetic sequences*

Which of the following recurrence systems define arithmetic sequences? For each arithmetic sequence, write down the values of the first term a and common difference d.

(a) $x_1 = -1, \qquad x_n = x_{n-1} + 1 \quad (n = 2, 3, 4, \ldots, 100)$

(b) $y_1 = 2, \qquad y_n = -y_{n-1} + 1 \quad (n = 2, 3, 4, \ldots)$

(c) $z_0 = 1, \qquad z_n = z_{n-1} - 1.5 \quad (n = 1, 2, 3, \ldots)$

Suppose now that you have been given the first few terms of an arithmetic sequence (x_n). How can you find its parameters a and d? Well, a is just the first term, and d is the difference $x_n - x_{n-1}$ between any two successive terms x_n and x_{n-1} in the sequence, as illustrated in the following example.

Example 4 *Finding parameters of arithmetic sequences*

(a) For the infinite arithmetic sequence (x_n) whose first four terms are

$$100, \ 95, \ 90, \ 85,$$

find the values of the first term a and common difference d, and write down the corresponding recurrence system. Calculate also the next two terms of the sequence.

(b) Repeat part (a) for the infinite arithmetic sequence (y_n) whose first four terms are

$$\tfrac{1}{4}, \ \tfrac{1}{2}, \ \tfrac{3}{4}, \ 1.$$

Solution

By the convention stated on page 15, take the first term of each sequence to have subscript 1.

Remember that to specify a recurrence system, you have to write down three things: the first term, the recurrence relation and the subscript range.

(a) The first term is $a = 100$, and the common difference is $d = 95 - 100 = -5$. So a suitable recurrence system is

$$x_1 = 100, \qquad x_n = x_{n-1} - 5 \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$x_5 = x_4 - 5 = 85 - 5 = 80,$$
$$x_6 = x_5 - 5 = 80 - 5 = 75.$$

(b) The first term is $a = \frac{1}{4}$, and the common difference is $d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. So a suitable recurrence system is

$$y_1 = \tfrac{1}{4}, \qquad y_n = y_{n-1} + \tfrac{1}{4} \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$y_5 = y_4 + \tfrac{1}{4} = 1 + \tfrac{1}{4} = \tfrac{5}{4},$$
$$y_6 = y_5 + \tfrac{1}{4} = \tfrac{5}{4} + \tfrac{1}{4} = \tfrac{3}{2}.$$

Activity 8 *Finding parameters of arithmetic sequences*

(a) For the infinite arithmetic sequence (x_n) whose first four terms are

$$1,\ 4,\ 7,\ 10,$$

find the values of the first term a and common difference d, and write down the corresponding recurrence system. Calculate also the next two terms of the sequence.

(b) Repeat part (a) for the infinite arithmetic sequence (y_n) whose first four terms are

$$2.1,\ 3.2,\ 4.3,\ 5.4.$$

(c) Repeat part (a) for the finite arithmetic sequence (z_n) with eleven terms, whose first four terms are

$$1,\ 0.9,\ 0.8,\ 0.7.$$

A special type of arithmetic sequence arises when the common difference d is zero. In this case, each term of the sequence is equal to the first term. Such a sequence is called a **constant sequence**. For example,

$$3,\ 3,\ 3,\ \ldots$$

is a constant sequence.

Finite arithmetic sequences

As you've seen, the method of finding the values of the parameters a and d is the same for any arithmetic sequence, whether it's finite or infinite. For an infinite sequence, the recurrence system has the form

$$x_1 = a, \qquad x_n = x_{n-1} + d \quad (n = 2, 3, 4, \ldots),$$

whereas for a finite sequence, with N terms say, the recurrence system has the form

$$x_1 = a, \qquad x_n = x_{n-1} + d \quad (n = 2, 3, 4, \ldots, N).$$

Here the final number in the range of values of n is N because the final term x_N is obtained by applying the recurrence relation with $n = N$.

If you have the first few terms and the last term of a particular finite arithmetic sequence, and you want to find a recurrence system that specifies the sequence, then you need to find not only the values of the parameters a and d, but also the subscript of the last term. To see how to do this, consider the floor heights sequence,

$$7.15,\ 10.50,\ 13.85,\ 17.20,\ \ldots,\ 245.00.$$

To find how many terms there are in the sequence, you need to work out how many times the common difference 3.35 has been added to the first term 7.15 to produce the last term 245.00. The total amount added is 245.00 − 7.15, so the number of times that 3.35 has been added is

$$\frac{245.00 - 7.15}{3.35} = 71.$$

So there are 71 terms of the sequence after the first term, and hence there are 72 terms altogether. The corresponding recurrence system for the floor heights sequence is therefore

$$h_1 = 7.15, \qquad h_n = h_{n-1} + 3.35 \quad (n = 2, 3, 4, \dots, 72),$$

as stated earlier. In general, the number of terms of a finite non-constant arithmetic sequence is given by

$$\text{number of terms} = \frac{\text{last term} - \text{first term}}{\text{common difference}} + 1.$$

That is, if a finite arithmetic sequence has N terms, with first term x_1, last term x_N and common difference d, then

$$N = \frac{x_N - x_1}{d} + 1. \tag{1}$$

You'll meet this equation in a slightly different form in the next subsection.

Activity 9 *Finding the number of terms in a finite arithmetic sequence*

(a) Find the number of terms in the finite arithmetic sequence

$$1000,\ 970,\ 940,\ 910,\ \dots,\ 10.$$

(b) Hence write down a recurrence system for this sequence, denoting the nth term by x_n.

2.2 Closed forms for arithmetic sequences

Arithmetic sequences have a particularly simple form: to get from one term to the next, you add the same number each time. This pattern allows you to obtain closed forms for such sequences, which makes them easier to handle mathematically.

To see how to do this, consider the sequence

$$5,\ 8,\ 11,\ 14,\ \dots,\ 38.$$

This is a finite arithmetic sequence with parameters $a = 5$, $d = 3$ and 12 terms, which can be described by the recurrence system

$$b_1 = 5, \qquad b_n = b_{n-1} + 3 \quad (n = 2, 3, 4, \dots, 12).$$

So $b_1 = 5$, $b_2 = 8$, and so on, up to $b_{12} = 38$. The way in which the terms of this sequence are obtained from the recurrence relation can be pictured as follows.

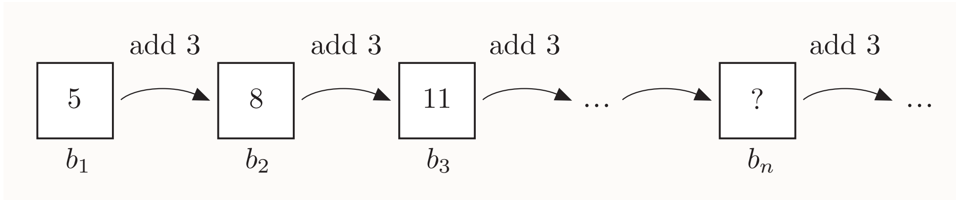


Figure 1 Obtaining the terms of an arithmetic sequence

Starting from $b_1 = 5$,

to obtain b_2, we add 3,

to obtain b_3, we add 3 twice,

to obtain b_4, we add 3 three times,

and so on. In each case the number of added 3s is one fewer than the subscript on the left. Thus, to obtain the general term b_n, you have to add 3 exactly $n - 1$ times; that is, you add $3(n - 1)$. This gives the value of the general term as

$$b_n = 5 + 3(n - 1) \quad (n = 1, 2, 3, \ldots, 12),$$

which can be simplified to

$$b_n = 3n + 2 \quad (n = 1, 2, 3, \ldots, 12).$$

(Notice that the first value in this range of n is 1, whereas the first value in the range of the recurrence relation was 2.)

For example, using this closed form we find that $b_4 = 3 \times 4 + 2 = 14$, as expected.

The reasoning above can be applied to a *general* arithmetic sequence, to obtain a formula for the nth term. Consider the arithmetic sequence given by the recurrence system

$$x_1 = a, \qquad x_n = x_{n-1} + d \quad (n = 2, 3, 4, \ldots).$$

Figure 2 shows how each successive term is obtained from the term before.

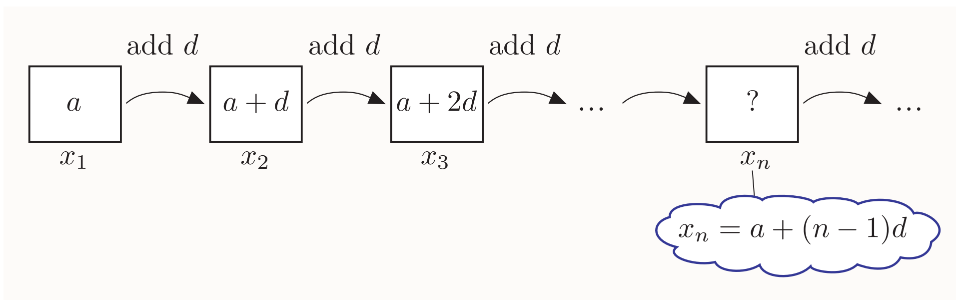


Figure 2 Obtaining the terms of a general arithmetic sequence

To obtain a general term x_n, you start with $x_1 = a$ and add d exactly $n - 1$ times, so $x_n = a + (n - 1)d$.

Closed form for an arithmetic sequence

The arithmetic sequence with recurrence system

$$x_1 = a, \qquad x_n = x_{n-1} + d \quad (n = 2, 3, 4, \dots)$$

has the closed form and subscript range

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \dots).$$

If you have a finite arithmetic sequence, with N terms say, then it has the closed form stated in the box above, but a finite subscript range:

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \dots, N).$$

It's easy to check that the formula in the box above gives the correct answers for the first few terms. For example, when $n = 1$ and $n = 2$, the formula gives the correct values $x_1 = a$ and $x_2 = a + d$.

You can simplify the expression $a + (n-1)d$ in the closed form for an arithmetic sequence when a and d have particular values. For example,

$$5 + 3(n-1) = 3n + 2.$$

The next example and activity are about applying the closed form in the box above to particular sequences.

Example 5 *Finding a closed form for an arithmetic sequence*

Find a closed form for the arithmetic sequence $5, 9, 13, 17, \dots$, given by the recurrence system

$$x_1 = 5, \qquad x_n = x_{n-1} + 4 \quad (n = 2, 3, 4, \dots).$$

Check that your answer gives the correct value for the fourth term, and also calculate the 10th term of the sequence.

Solution

Apply the closed-form formula $x_n = a + (n-1)d$ $(n = 1, 2, 3, \dots)$.

Since $a = 5$ and $d = 4$, the closed form is

$$\begin{aligned} x_n &= 5 + 4(n-1) \\ &= 4n + 1 \quad (n = 1, 2, 3, \dots). \end{aligned}$$

This gives $x_4 = 4 \times 4 + 1 = 17$, as expected, and also

$$x_{10} = 4 \times 10 + 1 = 41.$$

Activity 10 *Finding a closed form for an arithmetic sequence*

(a) Find a closed form for the arithmetic sequence $1, 4, 7, 10, \dots$, given by the recurrence system

$$x_1 = 1, \qquad x_n = x_{n-1} + 3 \quad (n = 2, 3, 4, \dots).$$

Check that your answer gives the correct value for the fourth term, and also calculate the 10th term of the sequence.

(b) Repeat part (a) for the arithmetic sequence $2.1, 3.2, 4.3, 5.4, \dots$, given by the recurrence system

$$y_1 = 2.1, \qquad y_n = y_{n-1} + 1.1 \quad (n = 2, 3, 4, \dots).$$

(c) Repeat part (a) for the arithmetic sequence $1, 0.9, 0.8, 0.7, \dots, 0$, given by the recurrence system

$$z_1 = 1, \qquad z_n = z_{n-1} - 0.1 \quad (n = 2, 3, 4, \dots, 11).$$

(You were asked to write down the recurrence systems for these sequences in Activity 8.)

Note that if a finite arithmetic sequence has last term x_N, then the closed form that you've met gives

$$x_N = a + (N-1)d,$$

which can be written as

$$x_N = x_1 + (N-1)d.$$

This is a rearrangement of equation (1) on page 19, so the two formulas are really just saying the same thing.

Finally, note that an arithmetic sequence with first term a and common difference d has an alternative closed form that's sometimes useful. The closed form that you've met,

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \dots),$$

holds when the subscript n takes values starting from 1. If instead you choose to have the subscript n start from 0, then the closed form is

$$y_n = a + nd \quad (n = 0, 1, 2, \dots).$$

2.3 Geometric sequences

Next we investigate a different type of sequence. We begin once again with two sequences from real-world contexts.

First, consider the savings account sequence,

$$1000.00,\ 1050.00,\ 1102.50,\ 1157.63,\ 1215.51.$$

This represents the amount of money (in £) in a savings account on each successive 1 January over a five-year period. As pointed out in Subsection 1.1, the terms of this sequence after the third term have been rounded to two decimal places, since they are amounts of money and hence need to be expressed to the nearest penny. The corresponding sequence with unrounded values is

$$1000,\ 1050,\ 1102.5,\ 1157.625,\ 1215.506\,25.$$

We shall call this unrounded sequence (s_n), so $s_1 = 1000$, $s_2 = 1050$, $s_3 = 1102.50$, and so on.

To get from any term in this sequence to the next, we *multiply* by the same number each time:

$$1.05 \times 1000 = 1050,$$

$$1.05 \times 1050 = 1102.50,$$

$$1.05 \times 1102.50 = 1157.625,$$

and so on. The number 1.05 occurs here because the interest added at the end of each year is 0.05 times (that is, 5% of) the amount in the account at the *beginning* of the year. Thus this sequence can be defined by the recurrence system

$$s_1 = 1000, \qquad s_n = 1.05s_{n-1} \quad (n = 2, 3, 4, 5).$$

Here the range of values of n stops at 5, since the last term in the sequence is s_5. Note that, where rounding is required, you should carry out the complete calculation for each term using exact arithmetic first and round only at the end, rather than rounding at each application of the recurrence relation.

Next, consider the sequence

$$2000,\ 1400,\ 980,\ 686,\ 480.2,\ \ldots.$$

This could represent the heights, measured in millimetres, of successive bounces of a ball that is assumed to rebound to 70% of the height from which it falls. We shall call this sequence (h_n), so $h_1 = 2000$, $h_2 = 1400$, $h_3 = 980$, and so on.

Once again, to get from any term in this sequence to the next, we *multiply* by the same number each time:

$$0.7 \times 2000 = 1400,$$

$$0.7 \times 1400 = 980,$$

$$0.7 \times 980 = 686,$$

and so on. The number 0.7 occurs here because each successive height is 70% of the previous one. Thus this sequence can be defined by the recurrence system

$$h_1 = 2000, \qquad h_n = 0.7h_{n-1} \quad (n = 2, 3, 4, \ldots),$$

where we have assumed (unrealistically) that the ball will bounce infinitely many times.

Geometric sequins

Any sequence with the structure demonstrated above – multiplication by a fixed number to obtain the next term – is called a **geometric sequence**, or alternatively a **geometric progression**. Thus a general geometric sequence is given by the recurrence system

$$x_1 = a, \qquad x_n = rx_{n-1} \quad (n = 2, 3, 4, \ldots),$$

where a is the first term and r is the number by which you multiply each term to obtain the next term. That is, r is the constant ratio x_n/x_{n-1} of any two successive terms, often called the **common ratio** of the sequence. Choosing the values of the first term a and common ratio r determines a particular geometric sequence; we call a and r the **parameters** of the geometric sequence. For example, the savings account sequence (s_n) has parameters $a = 1000$ and $r = 1.05$, and the bouncing ball sequence (h_n) has parameters $a = 2000$ and $r = 0.7$.

A geometric sequence (x_n) can be finite, as for the savings account sequence, or infinite, as for the bouncing ball sequence. The first term can be x_0 rather than x_1, in which case the sequence is given by the recurrence system

$$x_0 = a, \qquad x_n = rx_{n-1} \quad (n = 1, 2, 3, \ldots).$$

Notice that when $r = 1$ we obtain the constant sequence $a, a, a, \ldots$, so constant sequences are not only a special type of arithmetic sequence, as you saw earlier, but also a special type of geometric sequence.

Activity 11 *Recognising geometric sequences*

Which of the following recurrence systems define geometric sequences? For each geometric sequence, write down the values of the first term a and common ratio r.

(a) $x_1 = -1, \qquad x_n = 3x_{n-1} \quad (n = 2, 3, 4, \ldots)$

(b) $y_0 = 1, \qquad y_n = -0.9y_{n-1} \quad (n = 1, 2, 3, \ldots)$

(c) $z_1 = 2, \qquad z_n = -z_{n-1} + 1 \quad (n = 2, 3, 4, \ldots)$

Suppose now that you know the first few terms of a geometric sequence (x_n) and you want to find its parameters. As with arithmetic sequences, the parameter a is just the first term. The other parameter, r, is the ratio of any pair of successive terms.

The next example and activity are about finding the parameters for some geometric sequences, and so obtaining recurrence systems for these sequences.

Example 6 *Finding parameters of geometric sequences*

(a) For the infinite geometric sequence (x_n) whose first four terms are

$$2,\ -3,\ 4.5,\ -6.75,$$

find the values of the first term a and common ratio r, and write down the corresponding recurrence system. Hence calculate the next two terms of the sequence, to three decimal places.

(b) Repeat part (a) for the infinite geometric sequence (y_n) whose first four terms are

$$100,\ 99,\ 98.01,\ 97.0299,$$

calculating the next two terms to three significant figures.

Solution

Using the convention stated on page 15, take the first term of each sequence to have subscript 1.

Remember that to specify a recurrence system, you have to write down three things: the first term, the recurrence relation and the subscript range.

(a) The first term is $a = 2$, and the common ratio is $r = (-3)/2 = -1.5$. So the recurrence system is

$$x_1 = 2, \qquad x_n = -1.5x_{n-1} \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$x_5 = -1.5x_4 = -1.5 \times (-6.75) = 10.125$$
$$x_6 = -1.5x_5 = -1.5 \times 10.125 = -15.1875 = -15.188 \text{ (to 3 d.p.)}.$$

(b) The first term is $a = 100$, and the common ratio is $r = 99/100 = 0.99$. So the recurrence system is

$$y_1 = 100, \quad y_n = 0.99y_{n-1} \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$\begin{aligned} y_5 &= 0.99y_4 \\ &= 0.99 \times 97.0299 \\ &= 96.059\,601 = 96.1 \text{ (to 3 s.f.)}, \end{aligned}$$

$$\begin{aligned} y_6 &= 0.99y_5 \\ &= 0.99 \times 96.059\,601 \\ &= 95.099\,004\ldots = 95.1 \text{ (to 3 s.f.)}. \end{aligned}$$

Activity 12 *Finding parameters of geometric sequences*

(a) For the infinite geometric sequence (x_n) whose first four terms are

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},$$

find the values of the first term a and common ratio r, and write down the corresponding recurrence system. Calculate also the next two terms of the sequence.

(b) Repeat part (a) for the infinite geometric sequence (y_n) whose first four terms are

$$4.2,\ 7.14,\ 12.138,\ 20.6346.$$

Give the next two terms to three decimal places.

(c) Repeat part (a) for the infinite geometric sequence (z_n) whose first four terms are

$$2,\ -2,\ 2,\ -2.$$

Finite geometric sequences

The method of finding values for the parameters a and r is the same for any geometric sequence, whether it is finite or infinite. However, for a finite sequence you may need to find how many terms there are, as you did for finite arithmetic sequences.

For example, suppose that the savings account sequence is changed by lengthening the period over which the account balance accumulates, giving this sequence of annual balances:

$$1000.00,\ 1050.00,\ 1102.50,\ 1157.63,\ \ldots,\ 2078.93.$$

Here, as before, the amounts are rounded, after their calculation, to two decimal places (that is, to the nearest penny, since the amounts are measured in £). As before, the interest rate is 5% per year, and hence each term (before rounding) is 1.05 times the previous term. How can you find the number of years for which the money is kept in the account? In other words, how many terms are there in the sequence?

Apart from the rounding that takes place, this is a finite geometric sequence with first term 1000, last term 2078.93 and common ratio 1.05. In order to determine how many terms there are in the sequence, N say, you need to find how many times $s_1 = 1000$ has to be multiplied by 1.05 in order to obtain $s_N = 2078.93$. Since progressing from s_1 to s_N involves $N - 1$ multiplications by 1.05, we have

$$1000 \times 1.05^{N-1} = 2078.93.$$

This is an exponential equation of a type that you saw how to solve in Subsection 4.4 of Unit 3. Dividing through by 1000 and taking the natural logarithm of both sides gives

$$1.05^{N-1} = \frac{2078.93}{1000} = 2.078\,93$$
$$(N-1)\ln 1.05 = \ln 2.078\,93$$
$$N = 1 + \frac{\ln 2.078\,93}{\ln 1.05} = 16.0000 \quad \text{(to 4 d.p.).}$$

(Recall that the value of s_N has been rounded to two decimal places, so it is not surprising that the outcome for N is not exactly an integer.)

Hence there are 16 terms in the sequence, and the corresponding recurrence system is

$$s_1 = 1000, \qquad s_n = 1.05 s_{n-1} \quad (n = 2, 3, 4, \dots, 16).$$

The same argument can be applied in the general case. If the finite geometric sequence $x_1, x_2, x_3, \dots, x_N$ has common ratio r, then you obtain x_N from x_1 by multiplying by r exactly $N-1$ times. That is,

$$x_N = x_1 r^{N-1}.$$

So you can find the number of terms, N, in the sequence by solving the following equation for N:

$$r^{N-1} = \frac{x_N}{x_1}.$$

You can always do this by using logarithms, as above (except in the simple special cases where $x_1 = 0$, $r = 0$ or $r = \pm 1$). However, care is needed if $r < 0$. In this case, you can find N by solving the equation $|r|^{N-1} = |x_N/x_1|$ by using logarithms.

Here is an example for you to try.

Activity 13 *Finding the number of terms in a finite geometric sequence*

(a) Find the number of terms in the finite geometric sequence (z_n) whose terms are

$$7,\ 35,\ 175,\ 875,\ \dots,\ 2\,734\,375.$$

(b) Hence write down a recurrence system for this sequence.

The situation posed above for the savings account, in which $s_N = 2078.93$ is known but N is unknown, is somewhat artificial. A more realistic question is: how many years will it take for an initial sum of £1000, placed in a savings account paying 5% interest per year, to double in value? In other words, what is the smallest value of N for which $s_N \geq 2000$?

This question can be answered with a similar approach to that above, by solving the equation

$$1000 \times 1.05^{N-1} = 2000.$$

The solution is given by

$$N = 1 + \frac{\ln 2}{\ln 1.05} = 15.2067 \quad \text{(to 4 d.p.)}.$$

This shows that when $N = 15$ the account balance has not yet reached twice its initial value, whereas when $N = 16$ it has more than doubled.

2.4 Closed forms for geometric sequences

Geometric sequences, like arithmetic sequences, have a particularly simple form: to get from one term to the next, you multiply by the same number each time. This pattern allows you to obtain a closed form for such sequences.

To see how to do this, consider the bouncing ball sequence,

$$2000,\ 1400,\ 980,\ 686,\ 480.2,\ \ldots .$$

This is a geometric sequence with parameters $a = 2000$ and $r = 0.7$, which can be defined as

$$h_1 = 2000, \qquad h_n = 0.7h_{n-1} \quad (n = 2, 3, 4, \ldots).$$

The way in which the terms of this sequence are obtained from the recurrence relation can be pictured as follows.

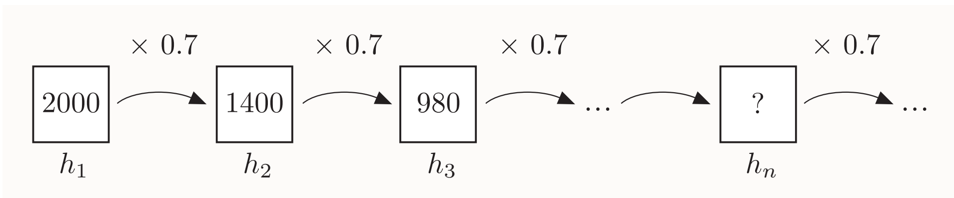


Figure 3 Obtaining the terms of the bouncing ball sequence

Starting from $h_1 = 2000$,

to obtain h_2, you multiply by 0.7,

to obtain h_3, you multiply by 0.7 twice, i.e. by 0.7^2,

to obtain h_4, you multiply by 0.7 three times, i.e. by 0.7^3,

and so on. To obtain the general term h_n, you have to multiply by 0.7 exactly $n - 1$ times; that is, you multiply by 0.7^{n-1}. This gives the value of the general term as $2000 \times 0.7^{n-1}$, so the closed form is

$$h_n = 2000 \times 0.7^{n-1} \quad (n = 1, 2, 3, \ldots).$$

For example, using this closed form you find that $h_4 = 2000 \times 0.7^3 = 686$, as expected.

The reasoning above can be applied to a *general* geometric sequence to obtain a formula for the nth term. Consider the geometric sequence given by the recurrence system

$$x_1 = a, \qquad x_n = rx_{n-1} \quad (n = 2, 3, 4, \ldots).$$

Figure 4 shows how each successive term is obtained from the term before.

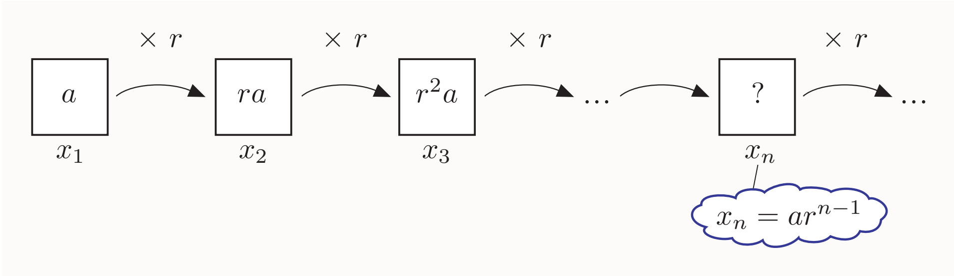


Figure 4 Obtaining the terms of a general geometric sequence

To obtain a general term x_n, you start with $x_1 = a$ and multiply by r exactly $n - 1$ times, so $x_n = ar^{n-1}$.

Closed form for a geometric sequence

The geometric sequence with recurrence system

$$x_1 = a, \qquad x_n = rx_{n-1} \quad (n = 2, 3, 4, \dots)$$

has the closed form and subscript range

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \dots).$$

If you have a finite geometric sequence, with N terms say, then it has the closed form stated in the box above, but a finite subscript range:

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \dots, N).$$

Once again, you can check that this formula gives the correct answers for the first few terms. For example, in the cases $n = 1$ and $n = 2$, the formula gives the correct values $x_1 = a$ and $x_2 = ar$. Also, if the ratio is $r = 1$, then you obtain the constant sequence $x_n = a$ $(n = 1, 2, 3, \dots)$, as expected.

The next example and activity are about applying the closed form in the box above to particular sequences.

Example 7 *Finding a closed form for a geometric sequence*

Find a closed form for the geometric sequence $2, 6, 18, 54, \dots$, given by the recurrence system

$$x_1 = 2, \qquad x_n = 3x_{n-1} \quad (n = 2, 3, 4, \dots).$$

Check that your answer gives the correct value for the fourth term, and also calculate the 10th term of the sequence.

Solution

Apply the closed-form formula $x_n = ar^{n-1}$ $(n = 1, 2, 3, \ldots)$.

Since $a = 2$ and $r = 3$, the closed form is

$$x_n = 2 \times 3^{n-1} \quad (n = 1, 2, 3, \ldots).$$

This gives $x_4 = 2 \times 3^{4-1} = 54$, as expected, and also

$$x_{10} = 2 \times 3^{10-1} = 2 \times 3^9 = 39\,366.$$

Activity 14 *Finding a closed form for a geometric sequence*

(a) Find a closed form for the geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, given by the recurrence system

$$x_1 = 1, \qquad x_n = \tfrac{1}{2}x_{n-1} \quad (n = 2, 3, 4, \ldots).$$

Check that your answer gives the correct value for the fourth term, and also calculate the 10th term of the sequence (correct to four significant figures).

(b) Repeat part (a) for the geometric sequence 4.2, 7.14, 12.138, 20.6346, ..., given by the recurrence system

$$y_1 = 4.2, \qquad y_n = 1.7y_{n-1} \quad (n = 2, 3, 4, \ldots).$$

(c) Repeat part (a) for the geometric sequence 2, −2, 2, −2, ..., given by the recurrence system

$$z_1 = 2, \qquad z_n = -z_{n-1} \quad (n = 2, 3, 4, \ldots).$$

(You were asked to write down the recurrence systems for these sequences in Activity 12.)

Finally, note that the geometric sequence with first term a and common ratio r has an alternative closed form that's sometimes useful. The closed form that you've met,

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \ldots),$$

holds when the subscript n takes values starting from 1. If instead you choose to have the subscript n start from 0, then the closed form is

$$y_n = ar^n \quad (n = 0, 1, 2, \ldots).$$

3 Graphs and long-term behaviour

In this section you'll see how the information contained in a sequence can be plotted on a graph. You'll then look at what can be said about the behaviour of a sequence after a large number of terms.

3.1 Graphs of sequences

In Unit 3 you were introduced to the idea of a function. You can think of any sequence as a function whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$. For example, you can think of the sequence

$$1,\ 4,\ 7,\ 10,\ 13,\ \ldots$$

as the function for which the input 1 gives the output 1, the input 2 gives the output 4, the input 3 gives the output 7, and so on, as shown in the mapping diagram in Figure 5.

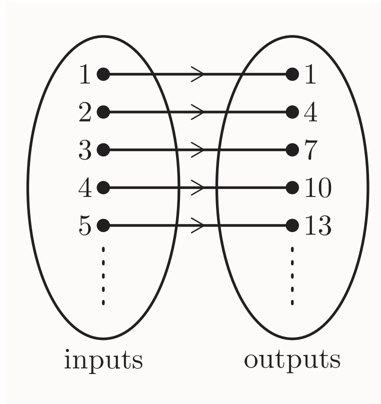


Figure 5 A sequence viewed as a function

In general, the sequence $(x_n)_{n=1}^{\infty}$ defines a function for which each input number n gives the output x_n, as shown in Figure 6. If you want the subscript of the first term of the sequence to be 0 rather than 1, then you should take the domain of the function to be $\{0, 1, 2, \ldots\}$ rather than $\{1, 2, 3, \ldots\}$. You can make similar adjustments for other possible ranges of subscripts.

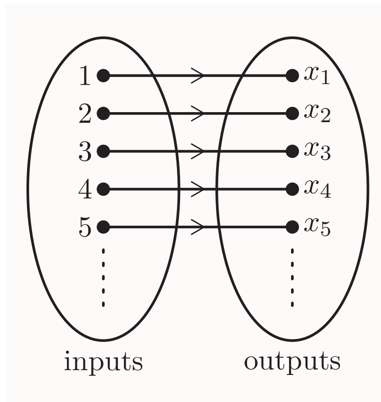


Figure 6 A general sequence viewed as a function

Functions can be represented by graphs, and this is in particular true of sequences. Each term x_n corresponds to a point (n, x_n) on the graph. The next example demonstrates how to plot a graph for a sequence.

Example 8 *Plotting the graph of an arithmetic sequence*

Plot a graph for the first six terms of the arithmetic sequence given by the closed form

$$x_n = 3n - 2 \quad (n = 1, 2, 3, \ldots).$$

Solution

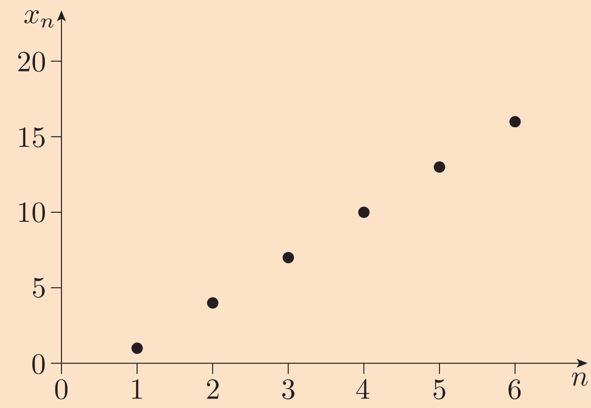
The sequence is $1, 4, 7, 10, 13, 16, \ldots$.

Plot the points (n, x_n) for $n = 1, 2, 3, \ldots, 6$.

The points to be plotted are

$$(1, 1),\ (2, 4),\ (3, 7),\ (4, 10),\ (5, 13),\ (6, 16).$$

The graph for the first six terms is as follows.



Activity 15 *Plotting the graph of an arithmetic sequence*

Plot a graph for the first six terms of the arithmetic sequence given by the recurrence system

$$z_1 = 1, \qquad z_n = z_{n-1} - 0.1 \quad (n = 2, 3, 4, \dots).$$

Notice that, in each of Example 8 and Activity 15, the graph of the sequence consists of points that lie on a straight line. This happens for every arithmetic sequence. To see why, consider the arithmetic sequence in Example 8, which is given by

$$x_n = 3n - 2 \quad (n = 1, 2, 3, \dots).$$

This equation defines x_n as a *linear* function of n, as introduced in Subsection 1.6 of Unit 3. So it is the equation of a straight line, with n and x_n in place of the usual variables x and y, respectively. You can read off the gradient, 3, in the usual way. (You can also read off the vertical intercept, but it has no relevance here since 0 is not in the domain of the function.) However, the graph of the arithmetic sequence consists only of those isolated points on the line that have first coordinate $n = 1, 2, 3, \dots$, rather than the whole straight line. This graph and the straight line on which it lies are shown in Figure 7.

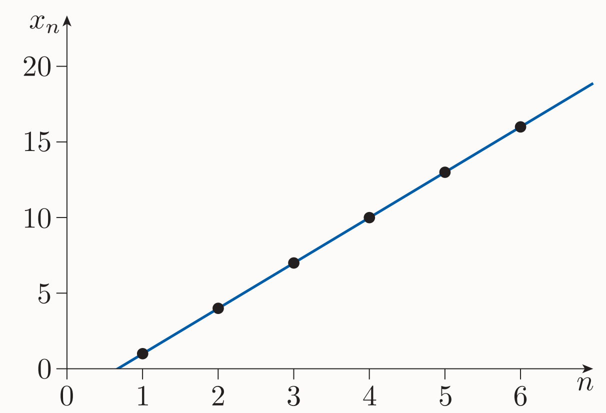


Figure 7 The graph of an arithmetic sequence lying on a straight line

In general, the arithmetic sequence with first term a and common difference d has the closed form

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \ldots),$$

which can be rearranged as

$$x_n = dn + (a-d) \quad (n = 1, 2, 3, \ldots).$$

So its graph consists of points that lie on a straight line with gradient d.

Note that in the particular case $d = 0$, the straight line has gradient zero and so is horizontal. Correspondingly, the arithmetic sequence is a constant sequence, $a, a, a, \ldots$.

We turn next to the graphs of geometric sequences.

Example 9 *Plotting the graph of a geometric sequence*

Plot a graph for the first six terms of the geometric sequence

$$x_n = \left(\tfrac{1}{2}\right)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

Solution

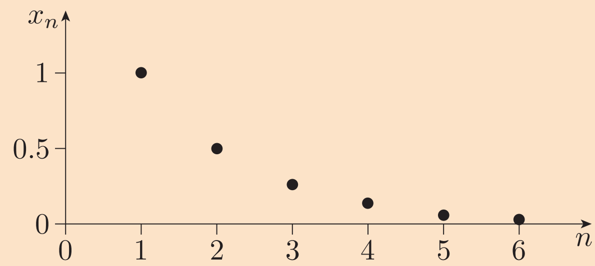
The sequence is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$.

Plot the points (n, x_n) for $n = 1, 2, 3, \ldots, 6$.

The points to be plotted are

$$(1,1),\ (2, \tfrac{1}{2}),\ (3, \tfrac{1}{4}),\ (4, \tfrac{1}{8}),\ (5, \tfrac{1}{16}),\ (6, \tfrac{1}{32}).$$

The graph showing the first six terms is as follows.



In each part of the next activity, your first task is to calculate the terms for which the corresponding points are to be plotted.

Activity 16 *Plotting the graphs of geometric sequences*

(a) Plot a graph for the first six terms of the sequence given by the recurrence system

$$y_1 = 4.2, \qquad y_n = 1.7y_{n-1} \quad (n = 2, 3, 4, \ldots).$$

(b) Repeat part (a) for the first five terms of the sequence given by the closed form

$$c_n = (-1)^{n+1} \quad (n = 1, 2, 3, \ldots).$$

In Example 9 the common ratio of the geometric sequence is $r = \frac{1}{2}$ and the graph is decreasing, whereas in Activity 16(a) the common ratio is $r = 1.7$ and the graph is increasing. In each case, the graph consists of points that lie on the graph of an exponential growth or decay function. You saw in Subsection 4.6 of Unit 3 that an *exponential growth* or *decay function* is a function of the form $f(x) = ae^{kx}$, where a and k are non-zero constants. Such a function can also be written in the form $f(x) = ab^x$, where a and b are constants with $a \neq 0$, $b > 0$ and $b \neq 1$. If $b > 1$, then f is an exponential growth function, whereas if $0 < b < 1$, then f is an exponential decay function.

For example, consider the sequence from Example 9. It has the closed form

$$x_n = \left(\tfrac{1}{2}\right)^{n-1} \quad (n = 1, 2, 3, \ldots),$$

which can be rearranged as $x_n = 2\left(\frac{1}{2}\right)^n$, and this is the formula for an exponential decay function, with n and x_n in place of the usual variables x and y, respectively. (Comparing this equation with the general form of an exponential decay function, from above, we have $a = 2$ and $b = \frac{1}{2}$.)

However, the graph of the geometric sequence consists only of those isolated points on the exponential decay curve that have first coordinate $n = 1, 2, 3, \ldots$, rather than the whole curve. The graph and the exponential decay curve on which it lies are shown in Figure 8.

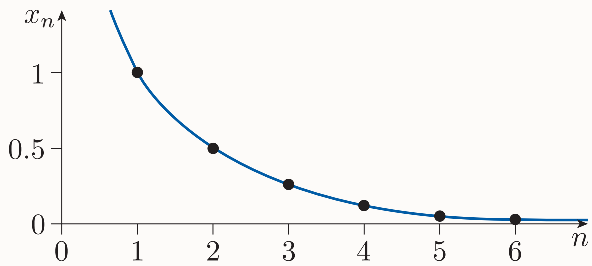


Figure 8 The graph of a geometric sequence lying on the graph of an exponential decay function

As another example, consider the sequence (y_n) from Activity 16(a). It has the closed form $y_n = 4.2 \times 1.7^{n-1}$, which can also be written as $y_n = (4.2/1.7) \times 1.7^n$; that is, $y_n = \frac{42}{17} \times 1.7^n$. So the points of this graph lie on the graph of the exponential growth function $f(x) = ab^x$ with $a = \frac{42}{17}$ and $b = 1.7$.

In general, the geometric sequence with first term a and common ratio r has the closed form

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \ldots),$$

which can be rearranged as

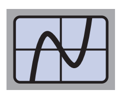
$$x_n = \left(\frac{a}{r}\right) r^n \quad (n = 1, 2, 3, \ldots),$$

provided that $r \neq 0$. If $r > 0$ and $r \neq 1$, then the graph of this geometric sequence consists of points that lie on the graph of an exponential growth or decay function.

Plotting graphs of sequences with a computer

When you plot the graphs of sequences by hand, you can usually plot only a small number of points. In the next activity you can find out how to use a computer to plot graphs of sequences showing many points. This work will prepare you for studying the long-term behaviour of sequences in the next subsection.

Activity 17 *Plotting graphs of sequences with a computer*



Work through Subsection 11.1 of the *Computer algebra guide*.

3.2 Long-term behaviour of sequences

A graph of a sequence can show information about the sequence only for a limited number of terms. We now investigate what can be said about the **long-term behaviour** of infinite sequences, that is, how each sequence will develop as more and more terms are considered. To start with, here's some terminology that's useful for describing the long-term behaviour of sequences.

Terminology for long-term behaviour

First, we use the words *increasing* and *decreasing* for sequences in much the same way as was introduced for functions in Subsection 1.4 of Unit 3. A sequence (x_n) is **increasing** if $x_{n-1} < x_n$ for each pair of sucessive terms x_{n-1} and x_n, and **decreasing** if $x_{n-1} > x_n$ for each pair of successive terms x_{n-1} and x_n. The graphs in Figure 9 illustrate how this terminology applies. The sequences in (e) and (f) are neither increasing nor decreasing.

Next, suppose that all the terms of a sequence (x_n) lie within some interval $[-A, A]$, where A is a fixed positive number, as illustrated in Figure 9(e), for example. Then we say that the sequence (x_n) is **bounded**. If there is no fixed value of A, however large, for which all the terms of the sequence (x_n) lie within the interval $[-A, A]$, then we say that the sequence (x_n) is **unbounded**, and also that the terms of the sequence become **arbitrarily large**. Again, the graphs in Figure 9 illustrate how this terminology applies.

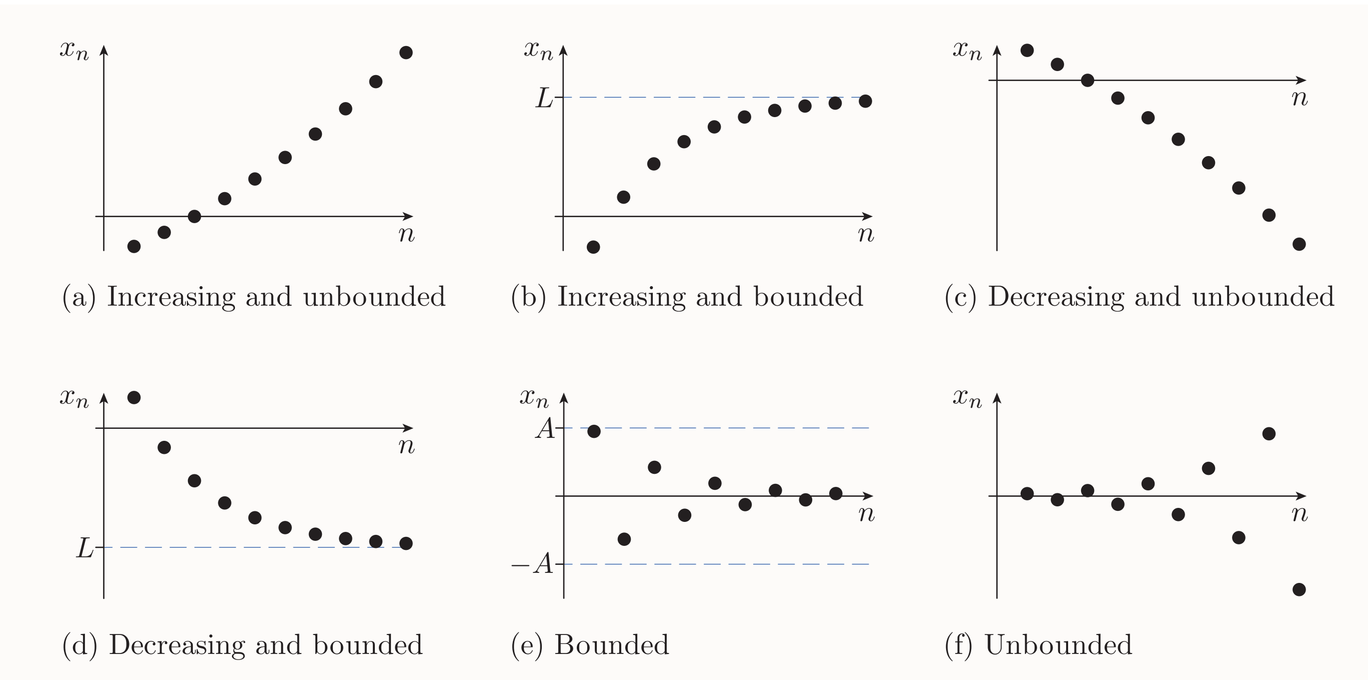


Figure 9 Sequences with various long-term behaviours

If the terms of a sequence (x_n) approach 0 more and more closely, in such a way that they eventually lie within any interval $[-h, h]$, no matter how small the positive number h is taken to be, then we say that the terms of the sequence (x_n) become **arbitrarily small**. More formally, we say that x_n **tends to** 0 **as** n **tends to infinity**, and we write

$$x_n \to 0 \text{ as } n \to \infty. \tag{2}$$

Three sequences with this property are illustrated in Figure 10.

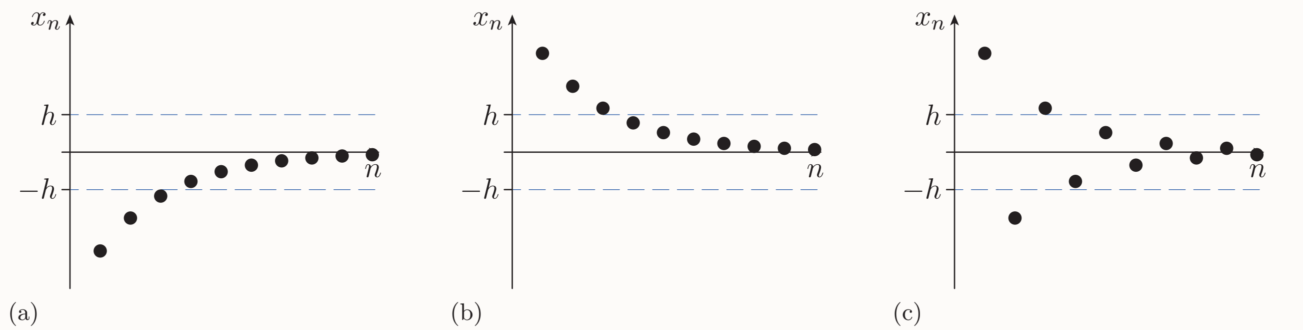


Figure 10 Sequences that tend to 0

More generally, suppose that the terms of a sequence (x_n) approach a particular number L more and more closely, so they eventually lie within any interval $[L-h, L+h]$, no matter how small the positive number h is taken to be. Then we say that x_n **tends to** L **as** n **tends to infinity**, and we write

$$x_n \to L \text{ as } n \to \infty. \tag{3}$$

Three sequences with this property are illustrated in Figure 11.

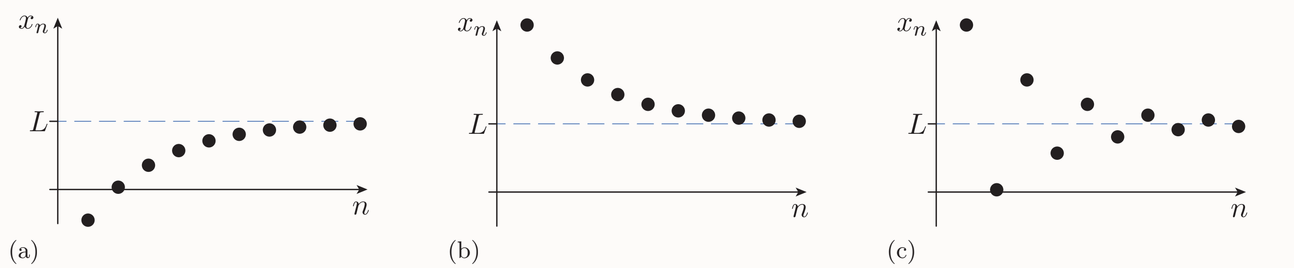


Figure 11 Sequences that tend to L

We also say in such a case that the **limit** of the sequence (x_n) is L, and that the sequence (x_n) **converges** or is **convergent** to the limit L. An alternative way to write statement (3) is

$$\lim_{n\to\infty} x_n = L,$$

which is read as 'the limit as n tends to infinity of x-sub-n is L'. In particular, statement (2) above can be written as

$$\lim_{n\to\infty} x_n = 0.$$

You saw a similar notation for other types of limits in Subsection 1.4 of Unit 6, and in Subsection 1.2 of Unit 8.

We also use arrows to denote 'tends to' in some cases where the sequence is unbounded. If a sequence (x_n) has the property that, whatever positive number A you take, no matter how large, the terms of (x_n) eventually lie in the interval $[A, \infty)$, then we say that x_n **tends to infinity as** n **tends to infinity**, and we write

$$x_n \to \infty \text{ as } n \to \infty.$$

Similarly, if a sequence (x_n) has the property that, whatever positive number A you take, the terms of (x_n) eventually lie in the interval $(-\infty, -A]$, then we say that x_n **tends to minus infinity as** n **tends to infinity**, and we write

$$x_n \to -\infty \text{ as } n \to \infty.$$

These definitions are illustrated in Figure 12.

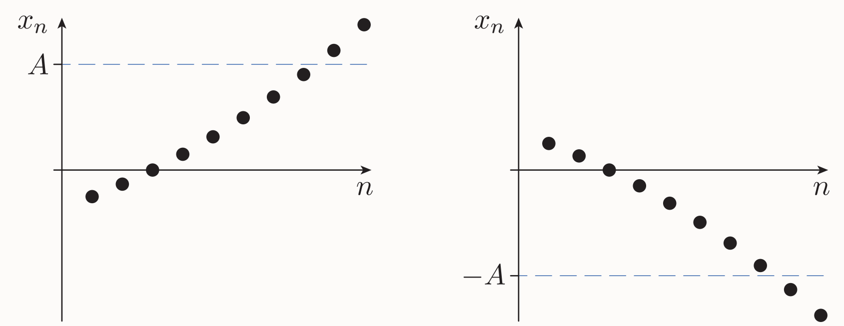


Figure 12 Sequences that tend to infinity or minus infinity

Example 10 *Using the notation for long-term behaviour*

Assuming that the patterns of behaviour suggested by the graphs in Figure 9 (on page 36) continue unchanged as n increases, complete the statement

$$x_n \to \dots \text{ as } n \to \infty,$$

for each of the sequences (x_n) shown in Figure 9(a) and (d).

Solution

The graph in Figure 9(a) shows that, for this sequence,

$$x_n \to \infty \text{ as } n \to \infty.$$

The graph in Figure 9(d) shows that, for this sequence,

$$x_n \to L \text{ as } n \to \infty \qquad \left(\text{or } \lim_{n\to\infty} x_n = L\right).$$

Activity 18 *Using the notation for long-term behaviour*

Assuming that the patterns of behaviour suggested by the graphs in Figure 9 continue unchanged as n increases, complete the statement

$$x_n \to \ldots \text{ as } n \to \infty,$$

for each of the sequences (x_n) shown in Figure 9(b), (c) and (e).

Long-term behaviour of arithmetic sequences

You've seen that the closed form of an arithmetic sequence (x_n) with first term a and common difference d is

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \ldots).$$

This closed form can be rearranged as

$$x_n = (a-d) + nd \quad (n = 1, 2, 3, \ldots),$$

which is the same as

$$x_n = b + nd \quad (n = 1, 2, 3, \ldots),$$

where $b = a - d$. This last formula is a little simpler than the first one since it involves n rather than $n - 1$. So, when studying the long-term behaviour of arithmetic sequences, we'll consider sequences with the closed form $x_n = b + nd$, where b and d are constants, with $d \neq 0$, and where the range of values of n is $1, 2, 3, \ldots$. These are arithmetic sequences with first term $b + d$ and common difference d.

Since the points that form the graph of an arithmetic sequence lie on a straight line, the long-term behaviour of arithmetic sequences is straightforward to describe.

Long-term behaviour of arithmetic sequences

Suppose that (x_n) is an arithmetic sequence with common difference d.

- If $d > 0$, then (x_n) is increasing and $x_n \to \infty$ as $n \to \infty$.
- If $d < 0$, then (x_n) is decreasing and $x_n \to -\infty$ as $n \to \infty$.
- If $d = 0$, then (x_n) is constant.

The three cases are illustrated in Figure 13.

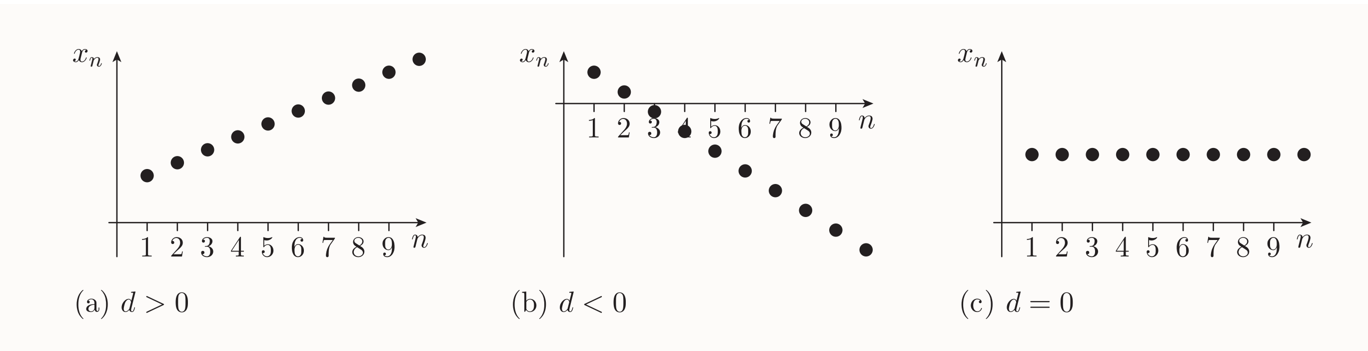


Figure 13 Graphs of arithmetic sequences

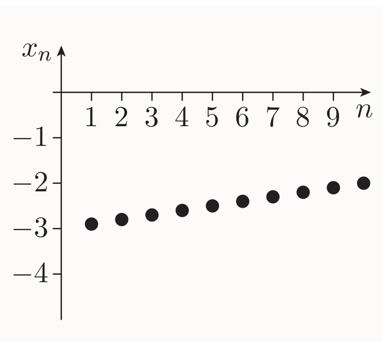


Figure 14 The graph of the sequence in Example 11

Example 11 *Finding the long-term behaviour of an arithmetic sequence*

Describe the long-term behaviour of the sequence (x_n) given by

$$x_n = \tfrac{1}{10}n - 3 \quad (n = 1, 2, 3, \ldots).$$

Solution

Recognise that (x_n) is an arithmetic sequence, find the common difference and use the facts in the box above.

The sequence (x_n) with closed form $x_n = \frac{1}{10}n - 3$ is arithmetic, with common difference $d = \frac{1}{10}$.

Since $d > 0$, the sequence (x_n) is increasing and $x_n \to \infty$ as $n \to \infty$.

The graph of the sequence in Example 11 is shown in Figure 14.

Activity 19 *Finding the long-term behaviour of an arithmetic sequence*

Describe the long-term behaviour of the sequence (x_n) given by

$$x_n = 3 - \tfrac{4}{5}n \quad (n = 1, 2, 3, \dots).$$

Long-term behaviour of geometric sequences

You've seen that the closed form of a geometric sequence (x_n) with first term a and common ratio r is

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \dots).$$

This closed form can be rearranged as

$$x_n = \left(\frac{a}{r}\right) r^n \quad (n = 1, 2, 3, \dots),$$

which is the same as

$$x_n = cr^n \quad (n = 1, 2, 3, \dots),$$

where $c = a/r$. This last formula is a little simpler than the original formula since it involves n rather than $n - 1$. So, when studying the long-term behaviour of geometric sequences, we'll consider sequences with closed form cr^n, where c and r are constants, with $r \neq 0$, and where the range of values of n is $1, 2, 3, \dots$. These are geometric sequences with first term cr and common ratio r.

The long-term behaviour of geometric sequences is much more varied than that of arithmetic sequences. However, you can determine the long-term behaviour of any particular geometric sequence if you know the long-term behaviour of sequences of the form (r^n), for all the different possible values of r. So let's start by looking at that. (The notation (r^n) means the sequence (x_n), where $x_n = r^n$ for $n = 1, 2, 3, \dots$, as you'd expect.)

First, there are three special cases.

If $r = 0$, then (r^n) is the sequence $0, 0, 0, 0, \dots$.
If $r = 1$, then (r^n) is the sequence $1, 1, 1, 1, \dots$.
If $r = -1$, then (r^n) is the sequence $-1, 1, -1, 1, \dots$.

So the sequences for $r = 0$ and $r = 1$ are constant sequences, and the sequence for $r = -1$ alternates indefinitely between -1 and 1.

Let's now look at the cases where r is *positive*, but not equal to 1. You can see what happens in these cases by using facts about exponential functions that you saw in Subsection 4.1 of Unit 3. Recall that the graph of the exponential function $f(x) = b^x$, where b is positive but not equal to 1, is

- increasing if $b > 1$, becoming steeper as x increases;
- decreasing if $0 < b < 1$, becoming less steep as x increases.

These two cases are shown in Figure 15.

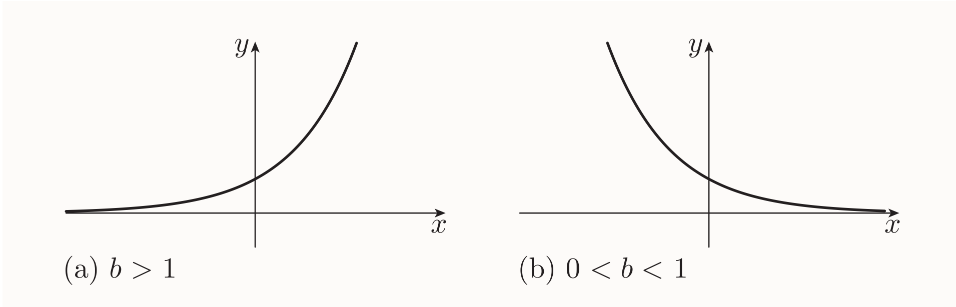


Figure 15 Graphs of $y = b^x$

Remember also from Unit 3 that the x-axis is an asymptote of the graph of the exponential function $f(x) = b^x$. When $0 < b < 1$, this means that the graph gets closer and closer to the positive x-axis as x increases, as illustrated in Figure 15(b), so the value of b^x approaches 0 more and more closely as x increases.

The only difference between the sequence (r^n) and the exponential function $y = r^x$ is that the domain of (r^n) consists of all the natural numbers, whereas the domain of $y = r^x$ consists of all the real numbers. Hence the graph of the sequence (r^n) is made up of isolated points, all lying on the graph of the function $y = r^x$, which is a continuous curve. So the long-term behaviour of the sequence (r^n), in the cases where r is positive but not equal to 1, is as follows.

- If $r > 1$, then (r^n) is increasing and $r^n \to \infty$ as $n \to \infty$.
- If $0 < r < 1$, then (r^n) is decreasing and $r^n \to 0$ as $n \to \infty$.

These two cases are illustrated in Figure 16.

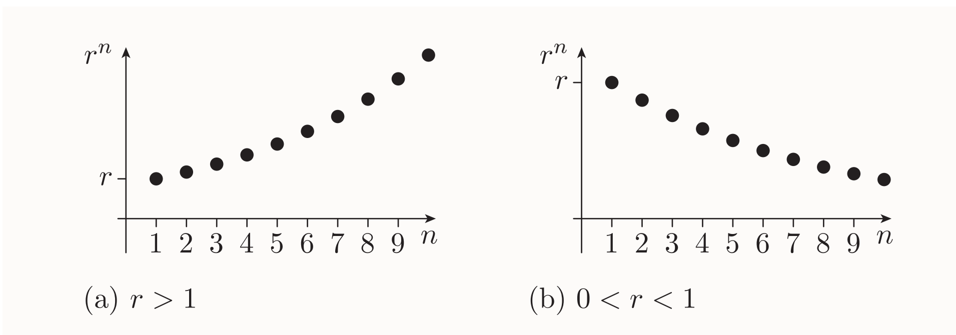


Figure 16 Graphs of sequences of the form (r^n)

For example,

- the sequence (2^n) is increasing and $2^n \to \infty$ as $n \to \infty$
- the sequence (0.5^n) is decreasing and $0.5^n \to 0$ as $n \to \infty$,

as illustrated in Figure 17.

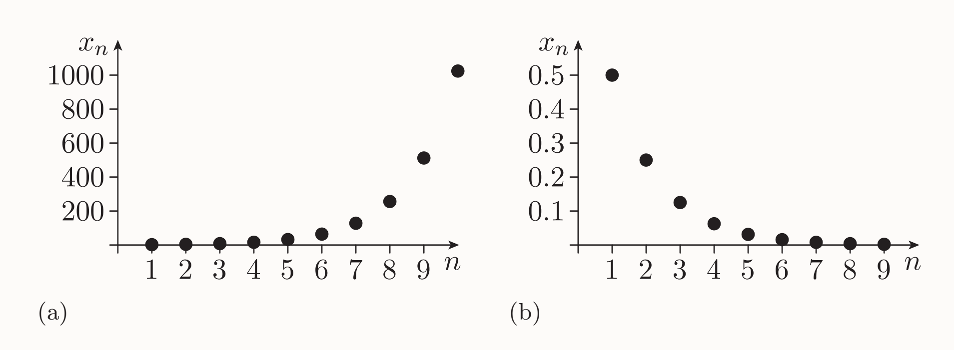


Figure 17 The graphs of the sequences (a) (2^n) (b) (0.5^n)

Now let's consider the long-term behaviour of sequences of the form (r^n) where r is *negative* but not equal to -1.

For example, consider the sequence $((-2)^n)$. The nth term of this sequence, $(-2)^n$, can also be written as $(-1)^n \times 2^n$, so the terms of the sequence have the same magnitude as the terms of the sequence (2^n), but alternate in sign, as illustrated in Figure 18(a). Hence the sequence $((-2)^n)$ is neither increasing nor decreasing, but is unbounded.

Similarly, the terms of the sequence $((-0.5)^n)$ have the same magnitude as the terms of the sequence (0.5^n), but alternate in sign, as illustrated in Figure 18(b). So the sequence $((-0.5)^n)$ is neither increasing nor decreasing, but $(-0.5)^n \to 0$ as $n \to \infty$.

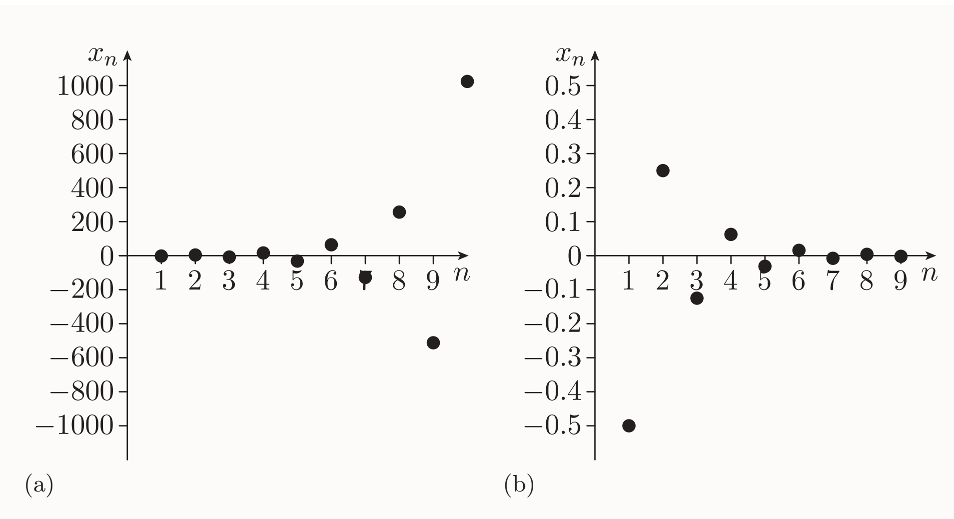


Figure 18 The graphs of the sequences (a) $((-2)^n)$ (b) $((-0.5)^n)$

In general, the long-term behaviour of the sequence (r^n), where r is negative but not equal to -1, is as follows.

- If $r < -1$, then r^n alternates between positive and negative values, and (r^n) is unbounded.
- If $-1 < r < 0$, then r^n alternates between positive and negative values, and $r^n \to 0$ as $n \to \infty$.

Here's a summary of the facts that you've seen about the behaviour of the sequence (r^n), for all possible values of r. All the cases except $r = 0$ are illustrated in Figure 19.

Long-term behaviour of the sequence (r^n)

Value of r	Behaviour of (r^n)
$r > 1$	Increasing, $r^n \to \infty$ as $n \to \infty$
$r = 1$	Constant: $1, 1, 1, \ldots$
$0 < r < 1$	Decreasing, $r^n \to 0$ as $n \to \infty$
$r = 0$	Constant: $0, 0, 0, \ldots$
$-1 < r < 0$	Alternates in sign, $r^n \to 0$ as $n \to \infty$
$r = -1$	Alternates between -1 and 1
$r < -1$	Alternates in sign, unbounded

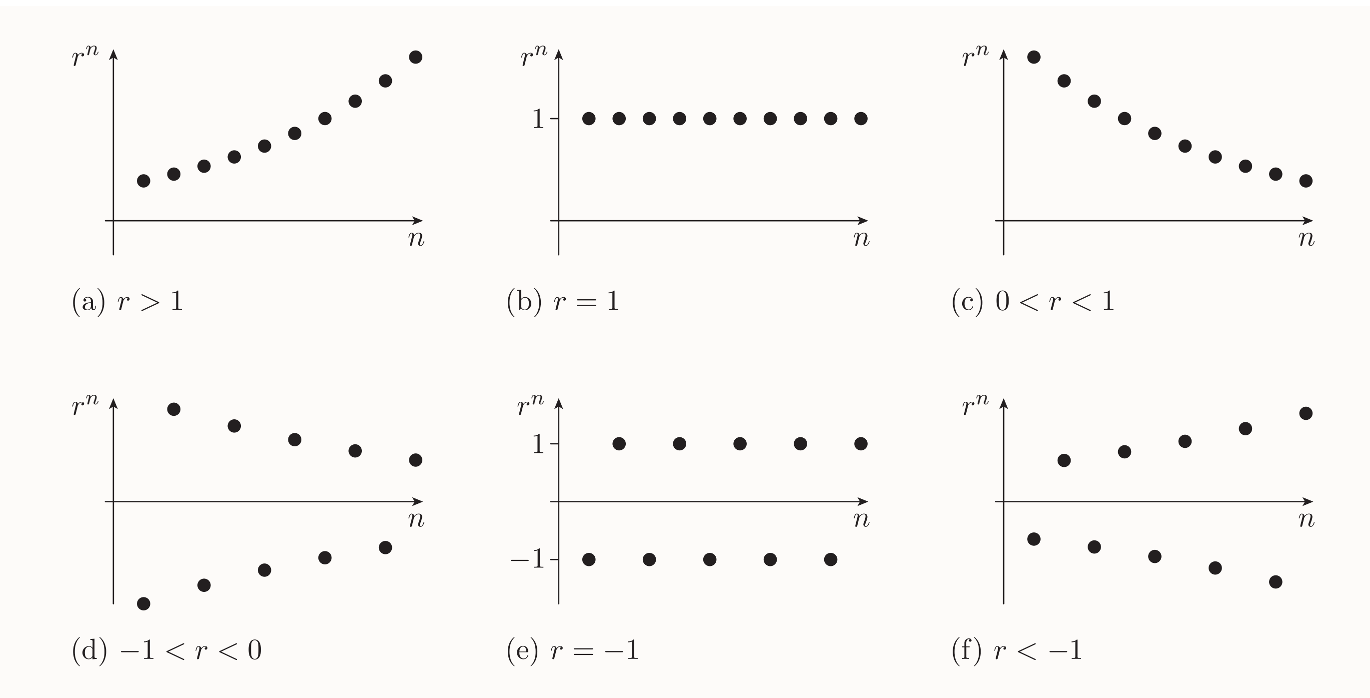


Figure 19 Sequences of the form (r^n) for all non-zero values of r

Now remember that any geometric sequence has the form (cr^n). You can work out the long-term behaviour of any sequence of this form by thinking about the long-term behaviour of the sequence (r^n) and using the fact that the terms of (cr^n) are obtained by multiplying the terms of (r^n) by c.

For example, since the sequence (2^n) is increasing and tends to infinity, as illustrated in Figure 20(a), it follows that the sequence $\left(\frac{2}{3} \times 2^n\right)$ is also increasing and tends to infinity, as illustrated in Figure 20(b). It also follows that the sequence $\left(-\frac{2}{3} \times 2^n\right)$ is decreasing and tends to minus infinity, as illustrated in Figure 20(c).

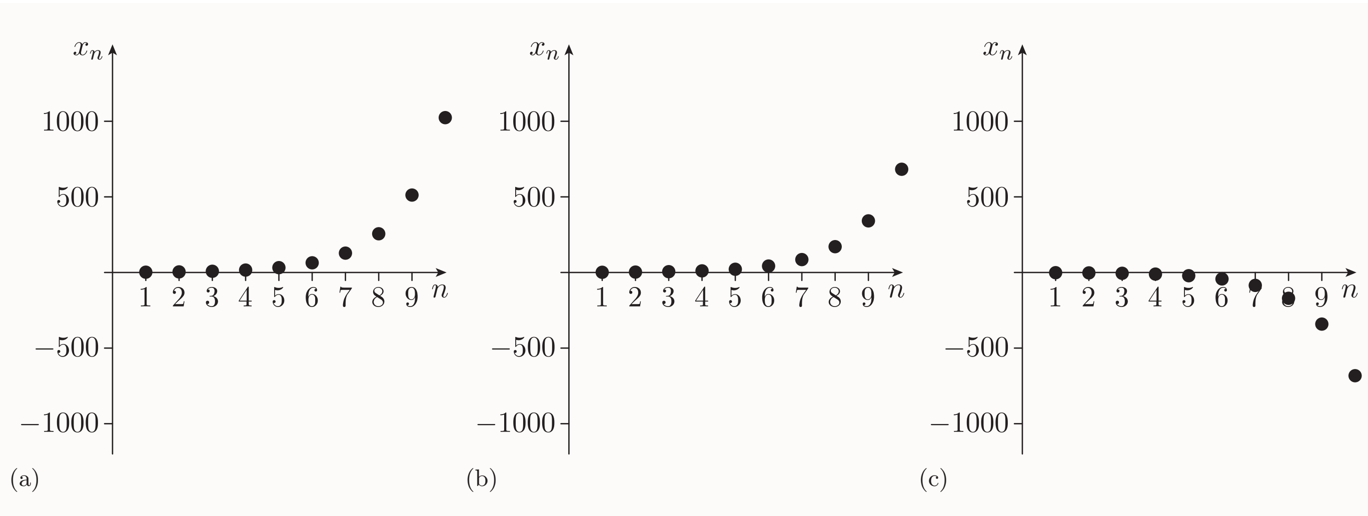


Figure 20 The sequences (a) (2^n) (b) $\left(\frac{2}{3} \times 2^n\right)$ (c) $\left(-\frac{2}{3} \times 2^n\right)$

When you're working out conclusions like these, it's helpful to use the ideas about scalings of graphs that you met in Unit 3. For example, it follows from what you saw there that the graph of the sequence $\left(\frac{2}{3} \times 2^n\right)$ is obtained from the graph of the sequence (2^n) by scaling it vertically by a factor of $\frac{2}{3}$. This squashes the graph vertically. Similarly, the graph of the sequence $\left(-\frac{2}{3} \times 2^n\right)$ is obtained from the graph of the sequence (2^n) by scaling it vertically by a factor of $-\frac{2}{3}$. This squashes the graph vertically, and reflects it in the horizontal axis.

The following box summarises some useful facts about multiplying the terms of a general sequence by a constant. However, when you need to use these facts you may find it easier just to think about the graph of the sequence, as in the paragraph above, rather than apply the facts directly. Try to check the facts in the box by thinking about the graphs of sequences in this way.

Multiplying each term of a sequence by a constant

Suppose that (x_n) is an infinite sequence and c is a constant.

$$\text{If } c \neq 0 \text{ and } (x_n) \left\{\begin{array}{l}\text{is constant}\\ \text{alternates in sign}\\ \text{is bounded}\\ \text{is unbounded}\\ \text{tends to } 0\end{array}\right\}, \text{ then so is/does } (cx_n).$$

$$\text{If } c > 0 \text{ and } (x_n) \left\{\begin{array}{l}\text{is increasing}\\ \text{is decreasing}\\ \text{tends to } \infty\\ \text{tends to } -\infty\end{array}\right\}, \text{ then so is/does } (cx_n).$$

$$\text{If } c < 0 \text{ and } (x_n) \left\{\begin{array}{l}\text{is increasing}\\ \text{is decreasing}\\ \text{tends to } \infty\\ \text{tends to } -\infty\end{array}\right\}, \text{ then } (cx_n) \left\{\begin{array}{l}\text{is decreasing}\\ \text{is increasing}\\ \text{tends to } -\infty\\ \text{tends to } \infty\end{array}\right\}.$$

You might find the tutorial clip for the example below particularly helpful.

Example 12 *Finding the long-term behaviour of geometric sequences*

Describe the long-term behaviour of each of the sequences given by the following closed forms.

(a) $x_n = -20 \times 0.7^n \quad (n = 1, 2, 3, \dots)$

(b) $y_n = \frac{1}{5} \times 1.5^n \quad (n = 1, 2, 3, \dots)$

(c) $z_n = 2(-1.1)^n \quad (n = 1, 2, 3, \dots)$

Solution

Use the facts in the box on page 44 and in the box above, and think about the graphs of the sequences involved.

(a) Since $0 < 0.7 < 1$, the sequence (0.7^n) is decreasing and $0.7^n \to 0$ as $n \to \infty$.

To obtain (x_n) we multiply each term by the negative constant -20. Hence (x_n) is increasing and $x_n \to 0$ as $n \to \infty$.

(b) Since $1.5 > 1$, the sequence (1.5^n) is increasing and $1.5^n \to \infty$ as $n \to \infty$.

To obtain (y_n) we multiply each term by the positive constant $\frac{1}{5}$. Hence (y_n) is increasing and $y_n \to \infty$ as $n \to \infty$.

(c) Since $-1.1 < -1$, the sequence $((-1.1)^n)$ alternates in sign and is unbounded.

To obtain (z_n) we multiply each term by the non-zero constant 2. Hence (z_n) also alternates in sign and is unbounded.

The graphs of the sequences in Example 12 are shown in Figure 21. You can see that the long-term behaviour of the sequences appears to be as determined in the solution to Example 12.

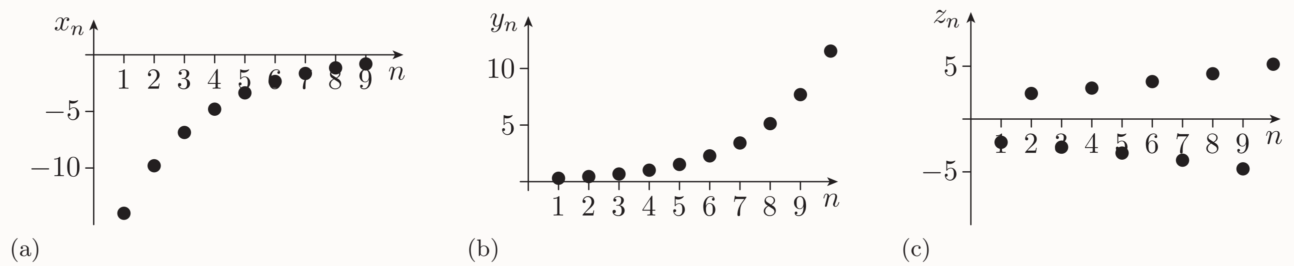


Figure 21 The graphs of the sequences in Example 12

Activity 20 *Finding the long-term behaviour of geometric sequences*

Describe the long-term behaviour of each of the sequences given by the following closed forms.

(a) $x_n = -\frac{1}{3} \times 1.2^n \quad (n = 1, 2, 3, \dots)$

(b) $y_n = 5(-0.9)^n \quad (n = 1, 2, 3, \dots)$

(c) $z_n = -\frac{1}{10}(-2.5)^n \quad (n = 1, 2, 3, \dots)$

You can check your answers to this question by using the computer algebra system to plot the graphs of the sequences.

Long-term behaviour of further sequences

You can find the long-term behaviour of slightly more complicated sequences by using the following facts.

Suppose that a, c and L are constants.

- If $x_n \to L$ as $n \to \infty$, then $x_n + a \to L + a$ as $n \to \infty$.
- If $x_n \to \infty$ as $n \to \infty$, then $x_n + a \to \infty$ as $n \to \infty$.
- If $x_n \to -\infty$ as $n \to \infty$, then $x_n + a \to -\infty$ as $n \to \infty$.
- If $x_n \to L$ as $n \to \infty$, then $cx_n \to cL$ as $n \to \infty$.

To understand why these facts hold, you can again use the ideas about scalings and translations of graphs that you met in Unit 3. For example, since the sequence (0.5^n) has limit 0, as illustrated in Figure 22(a), it follows that the sequence $(0.5^n + 0.1)$ has limit 0.1, as illustrated in Figure 22(b).

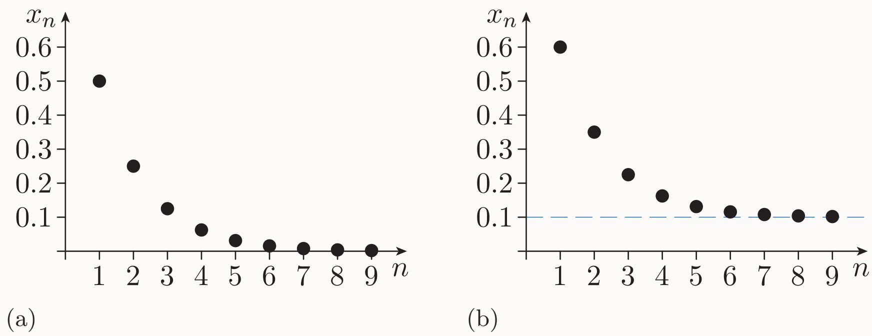


Figure 22 The sequences (a) (0.5^n) (b) $(0.5^n + 0.1)$

The following example shows how you can use the facts above, together with the facts that you met earlier in this subsection.

Example 13 *Finding the long-term behaviour of more sequences*

Describe the long-term behaviour of each of the sequences with the following closed forms.

(a) $x_n = -30 \times 0.9^n + 80 \quad (n = 1, 2, 3, \dots)$

(b) $y_n = 5 \times 2^n - 7 \quad (n = 1, 2, 3, \dots)$

Solution

Deal first with the term that involves r^n.

(a) Since $0 < 0.9 < 1$, the sequence (0.9^n) is decreasing and has limit 0. Hence the sequence (-30×0.9^n) is increasing and also has limit 0. Adding 80 to each term gives another increasing sequence, with limit 80. Hence the sequence (x_n) is increasing and $x_n \to 80$ as $n \to \infty$.

(b) Since $2 > 1$, the sequence (2^n) is increasing, and $2^n \to \infty$ as $n \to \infty$. Since the constant 5 is positive, the sequence (5×2^n) is also increasing and $5 \times 2^n \to \infty$ as $n \to \infty$. Subtracting 7 from each term gives another increasing sequence that tends to infinity. Hence the sequence (y_n) is increasing and $y_n \to \infty$ as $n \to \infty$.

The graphs of the sequences in Example 13 are shown in Figure 23.

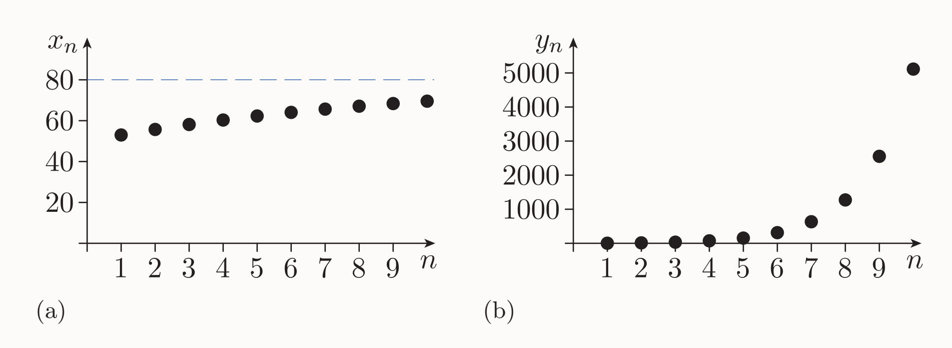


Figure 23 The graphs of the sequences in Example 13

Activity 21 *Finding the long-term behaviour of more sequences*

Describe the long-term behaviour of each of the sequences with the following closed forms. (These sequences are closely related to the sequences in Activity 20.)

(a) $a_n = 17 - \frac{1}{3} \times 1.2^n \quad (n = 1, 2, 3, \ldots)$

(b) $b_n = 5(-0.9)^n + 45 \quad (n = 1, 2, 3, \ldots)$

You can check your answers to this question by using the computer algebra system to plot the graphs of the sequences.

4 Series

In this section you'll investigate the sums of terms of sequences, and meet a useful notation for such sums.

4.1 Summing finite series

It's sometimes useful or interesting to add up consecutive terms of a sequence. For example, consider the following sums:

$$2 + 2^2 + 2^3 + 2^4 + 2^5,$$

$$(3 \times 4) + (3 \times 5) + (3 \times 6) + \cdots + (3 \times 20).$$

The first expression here is the sum of the first five terms of the geometric sequence (2^n), and the second expression is the sum of the terms, from the fourth to the twentieth, of the arithmetic sequence $(3n)$. Expressions like these are called *series*. That is, a **series** is an expression obtained by adding consecutive terms of a sequence. (The singular and plural forms of the word 'series' are the same.)

The series above are *finite* series, since they have only a finite number of terms. By contrast, an *infinite* series has an infinite number of terms. We'll consider finite series in this subsection and infinite series in the next.

Series are important in calculus, as you'll see in the next unit, and are also used frequently in statistics. They can be of interest too in their own right. For example, look at the series below, whose terms come from the sequence of odd integers:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16.$$

An obvious question is whether this pattern of adding up consecutive odd integers to obtain square numbers continues. You'll be able to answer that question later in the subsection.

The number that you obtain when you add up all the terms of a series is called the **sum** of the series, and the process of finding this sum is called **summing** the series, or **evaluating** the series. For example, the sum of the series $1 + 3 + 5 + 7$ is 16. You may think that this terminology is a little strange, because a series is *already* a sum, but it's standard terminology, and is convenient in practice.

In the rest of this subsection we'll look at summing some particular types of finite series.

Finite arithmetic series

Let's start by looking at sums of finite *arithmetic* series. As you'd expect, an **arithmetic series** is one whose terms come from an arithmetic sequence. Here's an example:

$$5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 + 38 + 41 + 44.$$

The terms of this series are the terms of an arithmetic sequence with first term 5 and common difference 3. One way to find the sum of this series is simply to add up all 14 terms. However, there's another and more illuminating way of finding the sum. First reverse the order of the terms of the series, and then write the result under the original series, with the terms aligned:

$$\begin{array}{l} 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 + 38 + 41 + 44 \\ 44 + 41 + 38 + 35 + 32 + 29 + 26 + 23 + 20 + 17 + 14 + 11 + 8 + 5. \end{array}$$

Then add the two copies of the series together, starting by adding each term to the one below. This gives

$$\begin{aligned} &49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 + 49 \\ &= 14 \times 49 = 686. \end{aligned}$$

This number is twice the sum of the original series (since we added two copies of the series together). So the sum of the original series is

$$\tfrac{1}{2} \times 686 = 343.$$

The reason for all the 49s above is that each term in the original series is 3 *more* than the one before, whereas each term in the reverse series is 3 *less* than the one before. When you add each term to the one below, these increases and decreases cancel out, and you obtain a new series all of whose terms are equal to its first term. The first term of this new series is the sum of the first and last terms of the original series, that is, $5 + 44 = 49$.

You can see that you could use the same approach to find the sum of any finite arithmetic series. So, in general, the sum of any finite arithmetic series is

$$\tfrac{1}{2} \times (\text{number of terms}) \times (\text{first term} + \text{last term}). \qquad (4)$$

If you don't know the number of terms then you can work it out by using the equation

$$\text{number of terms} = \frac{\text{last term} - \text{first term}}{\text{common difference}} + 1,$$

which you saw on page 19.

Example 14 *Finding the sum of an arithmetic series*

Find the sum of the arithmetic series

$$517 + 527 + 537 + \cdots + 1007.$$

Solution

First find the number of terms.

The first term is 517, the last term is 1007, and the common difference is $527 - 517 = 10$, so the number of terms is

$$\frac{1007 - 517}{10} + 1 = 50.$$

Hence the sum is

$$\begin{aligned}
&\tfrac{1}{2} \times (\text{number of terms}) \times (\text{first term} + \text{last term}) \\
&= \tfrac{1}{2} \times 50 \times (517 + 1007) \\
&= 25 \times 1524 \\
&= 38\,100.
\end{aligned}$$

Activity 22 *Finding the sum of an arithmetic series*

Find the sum of the arithmetic series

$$6 + 13 + 20 + \cdots + 90.$$

As you saw in Subsection 2.2, the nth term of the arithmetic sequence (x_n) with first term a and common difference d is given by

$$x_n = a + (n - 1)d.$$

It follows from this formula and expression (4) that the sum of an arithmetic series with n terms is

$$\begin{aligned}&\tfrac{1}{2} \times (\text{number of terms}) \times (\text{first term} + \text{last term})\\ &= \tfrac{1}{2}n(x_1 + x_n)\\ &= \tfrac{1}{2}n(a + a + (n-1)d)\\ &= \tfrac{1}{2}n\left(2a + (n-1)d\right).\end{aligned}$$

This useful formula is summarised in the box below.

Sum of a finite arithmetic series

The arithmetic series with first term a, common difference d and n terms has sum

$$\tfrac{1}{2} \times (\text{number of terms}) \times (\text{first term} + \text{last term});$$

that is,

$$\tfrac{1}{2}n\left(2a + (n-1)d\right). \qquad (5)$$

Activity 23 *Finding the sums of more arithmetic series*

Use formula (5) to find the sums of the following arithmetic series.

(a) $1 + 2 + 3 + \cdots + 100$

(b) $12 + 15 + 18 + \cdots + 60$

(c) $1 + 3 + 5 + \cdots + 19$

As you'll see shortly, the result of Activity 23(a) can be generalised to a useful formula for the sum of the first n natural numbers. Also, you can generalise the result of Activity 23(c) to prove that adding up consecutive odd numbers, starting at 1, always gives a square number – see the solution to the activity.

Finite geometric series

Now let's look at sums of *geometric* series: a **geometric series** is a series whose terms come from a geometric sequence. Again there's a particular approach that can be used to evaluate the sum of such a series. We'll apply it to find a general formula for the sum of a finite geometric series, after first considering a particular example.

Suppose that you want to evaluate the sum of the finite geometric series

$$1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11}.$$

One way to find this sum is simply to evaluate all of the 12 terms and then add them together. But there is a quicker and more illuminating way of finding this sum. Suppose that the sum is s. First write down the expressions for s and for $2s$ on successive lines (here $2s$ was chosen because 2 is the common ratio of the series):

$$\begin{aligned} s &= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11}, \\ 2s &= \phantom{1 + {}} 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12}. \end{aligned}$$

The terms in the sums s and $2s$ have been aligned to emphasise that they're the same, except for the appearance of 1 in the expression for s, and 2^{12} in the expression for $2s$. Now subtract the first of these equations from the second. All the common terms cancel on the right-hand side, so

$$s = 2^{12} - 1 = 4096 - 1 = 4095.$$

You can use the same approach to find a general formula for the sum of a finite geometric series.

Consider the finite geometric series with first term a, common ratio r and n terms, and suppose that its sum is s; that is,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \tag{6}$$

Then multiply the series by the common ratio r, and write it down below the original series, with like terms aligned. This gives

$$\begin{aligned} s &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}, \\ rs &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \end{aligned}$$

If you now subtract the bottom series from the top series, then most of the terms cancel each other out, and you obtain

$$s - rs = a - ar^n.$$

Taking out the common factor on each side gives

$$(1 - r)s = a(1 - r^n).$$

So, if $r \neq 1$, then

$$s = \frac{a(1 - r^n)}{1 - r}.$$

If $r = 1$, then it follows directly from equation (6) that

$$s = \underbrace{a + a + \cdots + a}_{n \text{ terms}} = na.$$

Sum of a finite geometric series

The geometric series with first term a, common ratio $r \neq 1$ and n terms has sum

$$\frac{a(1 - r^n)}{1 - r}. \tag{7}$$

As you saw earlier, the common ratio of a geometric sequence can be positive or negative, and when it's negative the terms of the sequence alternate in sign. Expression (7) still applies when the common ratio is negative.

Activity 24 *Finding the sums of geometric series*

(a) Find the sum of the first 10 powers of 3, that is,

$$3+3^2+3^3+\cdots+3^{10}.$$

(b) Find the value of the sum

$$1-\tfrac{1}{3}+\left(\tfrac{1}{3}\right)^2-\left(\tfrac{1}{3}\right)^3+\cdots+\left(\tfrac{1}{3}\right)^8,$$

giving your answer correct to six decimal places.

Other finite series

We now briefly consider some formulas for the sums of other standard finite series.

First, consider the sum of the first n natural numbers:

$$1+2+3+\cdots+n.$$

This is a finite arithmetic series with first term 1, last term n and n terms, so its sum is

$$\tfrac{1}{2}\times(\text{number of terms})\times(\text{first term}+\text{last term})=\tfrac{1}{2}n(1+n).$$

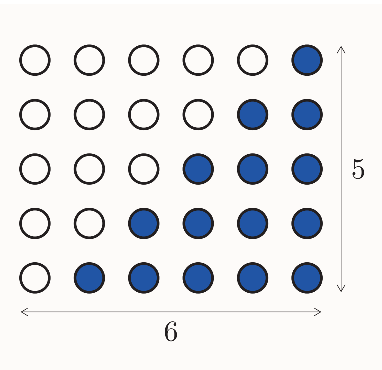


Figure 24 A geometric interpretation of $1+2+3+4+5=\frac{1}{2}\times5\times6$

Sum of the first n natural numbers

$$1+2+3+\cdots+n=\tfrac{1}{2}n(n+1)$$

This result has a geometric interpretation, which is illustrated (for the case $n=5$) in Figure 24. The number of shaded dots, which is $1+2+3+4+5$, is equal to half of the total number of dots, which is $\frac{1}{2}\times5\times6$.

Carl Friedrich Gauss (1777–1855)

It is said that the great German mathematician Carl Friedrich Gauss, at the age of ten, was asked along with the rest of his school class to find the sum $1+2+3+\cdots+100$. The teacher intended this to be a lengthy task. However, Gauss came up with the correct answer almost immediately, by applying the approach used earlier to derive expression (4) on page 51.

There are also formulas for the sums of the first n square numbers and the first n cube numbers, as stated in the following box. The proofs of these formulas are beyond the scope of this module, but you'll see the proofs if you go on to study MST125 *Essential mathematics 2*.

Sum of the first n square or cube numbers

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

Activity 25 *Using the formulas for the sums of consecutive natural numbers, squares and cubes*

(a) Use the formulas above to find the sums of the following finite series.

(i) $1 + 2 + 3 + \cdots + 30$

(ii) $1^2 + 2^2 + 3^2 + \cdots + 10^2$

(iii) $1^2 + 2^2 + 3^2 + \cdots + 30^2$

(iv) $1^3 + 2^3 + 3^3 + \cdots + 30^3$

(b) (i) Use your answers to parts (a)(ii) and (a)(iii) to find the sum of the series

$$11^2 + 12^2 + 13^2 + \cdots + 30^2.$$

(ii) Use a similar method to find the sum of the series

$$20^3 + 21^3 + 22^3 + \cdots + 40^3.$$

4.2 Summing infinite series

In this subsection we'll look at **infinite series**, which are expressions such as

$$a_1 + a_2 + a_3 + \cdots,$$

where (a_n) is a sequence. So that you can see what it means to add up all the terms in an infinite series, let's start by considering the infinite series

$$\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 + \cdots.$$

This is an infinite *geometric* series – its terms are those of the infinite geometric sequence with first term $\frac{1}{2}$ and common ratio $\frac{1}{2}$. Consider what happens as you add up more and more of the terms of this series:

$$\begin{aligned}
&\tfrac{1}{2} = \tfrac{1}{2}\\
&\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}\\
&\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8}\\
&\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16}\\
&\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} = \tfrac{31}{32}\\
&\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^6 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} = \tfrac{63}{64}\\
&\quad\vdots
\end{aligned}$$

As you add on more and more terms, the totals get closer and closer to 1. In fact, they get arbitrarily close to 1, but never reach 1, because each time you add on a new term, the difference between the total and 1 halves. You can think of this as meaning that, if you add on *infinitely many* terms, then the total is *exactly* 1. This is illustrated in Figure 25. So in this sense the sum of the infinite series above is 1.

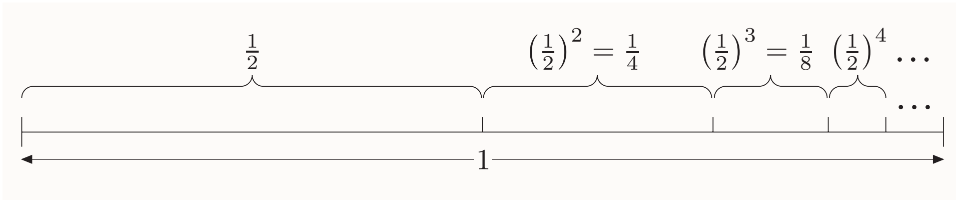


Figure 25 The sum of the infinite series $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$ is 1

One of Zeno's paradoxes

The result illustrated in Figure 25 is related to one of the famous paradoxes attributed to the Greek mathematician Zeno of Elea (*c.* 490–430 BC). It is known as the *dichotomy paradox*, and also as the *race course paradox*. One statement of it is based on the following:

> That which is in locomotion must arrive at the half-way stage before it arrives at the goal.
>
> (Aristotle, *Physics* VI:9, trans. R.P. Hardie and R.K. Gaye (2009), New York, Digireads.com.)

Before covering a fixed distance, half the distance must be covered, but before that, a quarter of the distance must be covered. Before that, an eighth must be covered, and so on. As this involves an infinite number of tasks, it is impossible to achieve.

Many infinite series don't have sums. For example, consider the infinite geometric series with first term 1 and common ratio 2:

$$1 + 2 + 2^2 + 2^3 + \cdots .$$

Consider what happens as you add up more and more of the terms of this series:

$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 2^2 = 1 + 2 + 4 = 7$$
$$1 + 2 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$$
$$1 + 2 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$$
$$\vdots$$

With this series, as you add on more and more terms, the total just keeps getting larger and larger, without getting closer and closer to any particular number. So this infinite series doesn't have a sum.

The ideas illustrated above can be applied to any infinite series, to determine whether it has a sum, and to find the value of its sum if it has one. We make the following definitions.

Consider any infinite series

$$a_1 + a_2 + a_3 + \cdots .$$

Let

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned}$$

and so on. The numbers $s_1, s_2, s_3, \ldots$ are called the **partial sums** of the series, and the infinite sequence (s_n) that they form is called the **sequence of partial sums** of the series.

If the sequence of partial sums of an infinite series converges to a limit, say s, then we call s the **sum** of the infinite series. On the other hand, if the sequence of partial sums doesn't converge, then the infinite series doesn't have a sum.

For example, you saw at the beginning of this subsection that the infinite series

$$\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 + \cdots$$

has the sequence of partial sums

$$\tfrac{1}{2},\ \tfrac{3}{4},\ \tfrac{7}{8},\ \tfrac{15}{16},\ \tfrac{31}{32},\ \ldots .$$

Since this sequence converges to 1, the infinite series has sum 1, as you saw.

Similarly, you saw that the infinite series

$$1 + 2 + 2^2 + 2^3 + \cdots$$

has the sequence of partial sums

$$1,\ 3,\ 7,\ 15,\ 31,\ \ldots .$$

Since this sequence doesn't converge, the infinite series has no sum.

It's often quite difficult to determine whether the sequence of partial sums of a particular infinite series converges, and if it does converge, what the limit is. However, it's fairly straightforward to deal with infinite arithmetic and geometric series. We'll consider each of these two types of infinite series in turn, starting with geometric series.

Infinite geometric series

Every infinite geometric series has the form

$$a + ar + ar^2 + ar^3 + \cdots,$$

where a is the first term and r is the common ratio. A useful way to determine whether such a series has a sum is to use the fact that each of its partial sums is the sum of a *finite* geometric series, and to apply the formula that you met in the previous subsection for such a sum.

To illustrate this approach, consider once more the infinite geometric series with first term $\frac{1}{2}$ and common ratio $\frac{1}{2}$:

$$\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 + \cdots.$$

Its partial sums are

$$\begin{aligned}
s_1 &= \tfrac{1}{2}\\
s_2 &= \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2\\
s_3 &= \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3\\
&\ \vdots
\end{aligned}$$

In general, the nth term of its sequence of partial sums is

$$s_n = \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^n.$$

This is the sum of a finite geometric series with first term $a = \frac{1}{2}$, common ratio $r = \frac{1}{2}$ and n terms, so by formula (7) on page 53, we have

$$s_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{\frac{1}{2}} = 1 - \left(\tfrac{1}{2}\right)^n.$$

Since $\left(\frac{1}{2}\right)^n \to 0$ as $n \to \infty$, it follows that

$$s_n = 1 - \left(\tfrac{1}{2}\right)^n \to 1 - 0 = 1 \quad \text{as } n \to \infty.$$

So the sequence of partial sums of this series converges to 1, and hence the series has sum 1. This confirms the result found earlier.

You can obtain a general result about the sums of infinite geometric series by applying the same approach. Consider the infinite geometric series with first term $a \neq 0$ and common ratio r:

$$a + ar + ar^2 + ar^3 + \cdots.$$

Its partial sums are

$$\begin{aligned}
s_1 &= a\\
s_2 &= a + ar\\
s_3 &= a + ar + ar^2\\
s_4 &= a + ar + ar^2 + ar^3\\
&\ \vdots
\end{aligned}$$

The nth term of its sequence of partial sums is

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

This is the sum of a finite geometric series with first term a, common ratio r and n terms. So by formula (7) on page 53, if $r \neq 1$, then

$$s_n = \frac{a(1-r^n)}{1-r}.$$

If $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$, as you saw on page 44, and hence

$$s_n = \frac{a(1-r^n)}{1-r} \to \frac{a(1-0)}{1-r} = \frac{a}{1-r} \quad \text{as } n \to \infty.$$

So if $-1 < r < 1$, then the series has the sum $a/(1-r)$.

If $r < -1$ or $r > 1$, then r^n is unbounded, as you also saw on page 44, and it follows that $s_n = a(1-r^n)/(1-r)$ is also unbounded. So if $r < -1$ or $r > 1$, then the series doesn't have a sum.

The only other possible values of r are $r = -1$ and $r = 1$. When $r = -1$, the series is

$$a - a + a - a + a - a + \cdots .$$

The sequence of partial sums of this series is $a, 0, a, 0, a, \ldots$, which doesn't converge (since $a \neq 0$), so the series doesn't have a sum. When $r = 1$, the series is

$$a + a + a + a + a + \cdots .$$

The sequence of partial sums of this series is $a, 2a, 3a, 4a, \ldots$, which doesn't converge (since $a \neq 0$), so again the series doesn't have a sum.

In summary, the following facts hold.

Sum of an infinite geometric series

The infinite geometric series with first term $a \neq 0$ and common ratio r has

$$\text{sum } \frac{a}{1-r}, \quad \text{if } -1 < r < 1;$$

$$\text{no sum}, \quad \text{if } r \leq -1 \text{ or } r \geq 1.$$

The only infinite geometric series with first term $a = 0$ is the series $0 + 0 + 0 + \cdots$, which has sum 0.

Example 15 *Summing infinite geometric series*

For each of the following infinite geometric series, determine whether or not it has a sum, and find the value of the sum if it exists.

(a) $\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots$

(b) $\frac{5}{2} - \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^3 - \left(\frac{5}{2}\right)^4 + \cdots$

Solution

(a) This is an infinite geometric series with first term $a = \frac{2}{3}$ and common ratio $r = \frac{2}{3}$. Since $-1 < r < 1$, the series has a sum, namely

$$\frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

(b) This is an infinite geometric series with first term $a = \frac{5}{2}$ and common ratio $r = -\frac{5}{2}$. Since $r < -1$, the series doesn't have a sum.

Here are some examples for you to try.

Activity 26 *Summing infinite geometric series*

For each of the following infinite geometric series, determine whether or not it has a sum, and find the value of the sum if it exists.

(a) $\frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots$

(b) $1 - \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 + \cdots$

(c) $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots$

(d) $\frac{1}{4} - \frac{1}{4}\left(\frac{2}{3}\right) + \frac{1}{4}\left(\frac{2}{3}\right)^2 - \frac{1}{4}\left(\frac{2}{3}\right)^3 + \cdots$

In fact, you've been dealing with the sums of infinite geometric series since Unit 1 of this module, because any recurring decimal can be thought of as the sum of a terminating decimal (possibly 0) plus an infinite geometric series. For example,

$$0.333\,333\ldots = 3\left(\tfrac{1}{10}\right) + 3\left(\tfrac{1}{10}\right)^2 + 3\left(\tfrac{1}{10}\right)^3 + \cdots,$$

$$5.178\,178\,178\,178\ldots = 5 + 178\left(\tfrac{1}{1000}\right) + 178\left(\tfrac{1}{1000}\right)^2 + 178\left(\tfrac{1}{1000}\right)^3 + \cdots.$$

It follows that one way to find a fraction equivalent to a given recurring decimal is to use the formula $a/(1-r)$ for the sum of a geometric series.

However, there's a neater method, which is equivalent to using this formula. It's illustrated in the next example.

Example 16 *Finding a fraction equivalent to a recurring decimal*

Find a fraction equivalent to $0.123\,123\,123\ldots$.

Solution

Put $s = 0.123\,123\,123\ldots$.

The repeating group, '123', is 3 digits long, so multiply s by 10^3.

Then

$$1000s = 123.123\,123\,123\ldots = 123 + s.$$

Hence $999s = 123$, so

$$s = \frac{123}{999} = \frac{41}{333}.$$

To find a fraction equivalent to a recurring decimal that has one or more non-zero digits before the recurring part, you can apply the method of Example 16 to find a fraction equivalent to the recurring part, and then add this to the number formed by the other digits. For example, by the solution to Example 16,

$$1.123\,123\,123\ldots = 1 + \frac{41}{333} = \frac{374}{333}.$$

Activity 27 *Finding fractions equivalent to recurring decimals*

Find a fraction equivalent to each of the following numbers.

(a) $0.454\,545\ldots$ (b) $3.729\,729\,729\ldots$

We'll now look at what happens when you try to sum an infinite *arithmetic* series.

Infinite arithmetic series

Every infinite arithmetic series has the form

$$a + (a + d) + (a + 2d) + \cdots,$$

where a is the first term and d is the common difference.

The situation for infinite arithmetic series is much simpler than for infinite geometric series. In summary, the only infinite arithmetic series that has a sum is the infinite series with first term 0 and common difference 0, that is, the series $0+0+0+\cdots$, which has sum 0. You can see this by using an approach similar to the one used earlier for infinite geometric series. You use the fact that each partial sum of an infinite arithmetic series is the sum of a *finite* arithmetic series, and apply the formula for the sum of a finite arithmetic series that you met in Subsection 4.1, as follows.

Consider the infinite arithmetic series with first term a and common difference d:

$$a+(a+d)+(a+2d)+\cdots.$$

The nth term of its sequence of partial sums is

$$s_n = a+(a+d)+(a+2d)+\cdots+(a+(n-1)d).$$

This is the sum of a finite arithmetic series with first term a, common difference d and n terms. So, by formula (5) on page 52, we have

$$s_n = \tfrac{1}{2}n(2a+(n-1)d).$$

If $d \neq 0$, then the expression on the right-hand side of this equation is a quadratic expression in n, so s_n tends to infinity or minus infinity. If $d=0$, then $s_n = na$, so again s_n tends to infinity or minus infinity, as long as $a \neq 0$. Therefore, provided that a and d are not both zero, the series doesn't have a sum.

4.3 Sigma notation

There's a useful notation for writing series concisely. It can be used for both finite and infinite series, but we'll begin by looking at how it's used for finite series.

For any sequence (x_n), the sum

$$x_p + x_{p+1} + \cdots + x_q$$

(that is, the sum of the terms from the pth term to the qth term) is denoted by

$$\sum_{n=p}^{q} x_n. \qquad (8)$$

(This is read as 'the sum from n equals p to q of x-sub-n.) This notation is called **sigma notation** or **summation notation** – the symbol $\sum$ is the upper-case Greek letter sigma. The numbers p and q, which tell you the terms to start and finish at, respectively, are called the **lower** and **upper limits** of the summation, respectively, and the variable n is called an **index variable**. For example, with this notation we can write

$$2+2^2+2^3+2^4+2^5 = \sum_{n=1}^{5} 2^n,$$

and

$$(3\times 4)+(3\times 5)+(3\times 6)+\cdots+(3\times 20)=\sum_{n=4}^{20} 3n.$$

The index variable n in sigma notation is a dummy variable – you can use any other variable name in its place. For example,

$$2+2^2+2^3+2^4+2^5=\sum_{n=1}^{5} 2^n=\sum_{j=1}^{5} 2^j=\sum_{k=1}^{5} 2^k.$$

In this module we'll usually use n or k for the index variable in sigma notation. Sometimes for a finite series it's natural to use n to denote the upper limit, and in that situation we'll use k for the index variable.

When sigma notation is in a line of text, it's sometimes written with the limits to the right of, instead of below and above, the symbol $\sum$. For example, expression (8) can be written as $\sum_{n=p}^{q} x_n$.

Example 17 *Converting from and to sigma notation*

(a) Write each of the following sums without sigma notation, giving the first three terms and the last.

(i) $\displaystyle\sum_{n=1}^{7} n$ (ii) $\displaystyle\sum_{n=4}^{19} n^2$ (iii) $\displaystyle\sum_{n=1}^{17} (n+3)$

(b) Write each of the following sums in sigma notation.

(i) $3^3+4^3+5^3+\cdots+9^3$

(ii) $1+2+3+\cdots+n$

(iii) The finite geometric series $a+ar+ar^2+\cdots+ar^{n-1}$.

Solution

(a) In each case, put n equal in turn to each of the integers from the lower to the upper limit of the sum, and add the resulting terms.

The sums are as follows.

(i) $$\sum_{n=1}^{7} n=1+2+3+\cdots+7$$

(ii) $$\sum_{n=4}^{19} n^2=4^2+5^2+6^2+\cdots+19^2$$

(iii) $$\sum_{n=1}^{17} (n+3)=4+5+6+\cdots+20$$

(b) In each case, find an expression for a typical term of the sum, then identify the lower and upper limits.

(i) The sum is $3^3 + 4^3 + 5^3 + \cdots + 9^3$.

The terms are of the form n^3, going from $n = 3$ to $n = 9$.

It can be written as $\displaystyle\sum_{n=3}^{9} n^3$.

(ii) The sum is $1 + 2 + 3 + \cdots + n$.

The sum already contains the variable name n, so we have to use a different letter for the index variable, say k. The terms are of the form k, going from $k = 1$ to $k = n$.

It can be written as $\displaystyle\sum_{k=1}^{n} k$.

(iii) The sum is $a + ar + ar^2 + \cdots + ar^{n-1}$.

Again we have to use a letter other than n for the index variable, say k. The terms are of the form ar^{k-1}, going from $k = 1$ to $k = n$. (Recall that $ar^0 = a \times 1 = a$ and that $ar^1 = ar$.)

It can be written as $\displaystyle\sum_{k=1}^{n} ar^{k-1}$.

This sum can also be written as $\displaystyle\sum_{k=0}^{n-1} ar^k$.

Activity 28 *Converting from and to sigma notation*

(a) Write each of the following sums without sigma notation, giving the first three terms and the last.

(i) $\displaystyle\sum_{n=5}^{20} n^4$ (ii) $\displaystyle\sum_{n=4}^{19} (n+1)^4$ (iii) $\displaystyle\sum_{n=1}^{6} (2n-1)$

(b) Write each of the following sums in sigma notation.

(i) $1 + 2 + 3 + \cdots + 150$

(ii) $5^2 + 6^2 + 7^2 + \cdots + 13^2$

(iii) $2 + 2^2 + 2^3 + \cdots + 2^{12}$

You might have noticed that the series in Activity 28(a)(i) and (ii) are the same, even though they're written differently in sigma notation. The same applies to the two alternative expressions given in the solution to

Example 17(b)(iii). In general, any series can be written in many different ways using sigma notation, even if you use the same index variable.

The formulas that you met earlier for the sums of finite arithmetic and geometric series can be stated concisely using sigma notation, as in the box below. We use k for the index variable since the upper limit is n in each series. In fact we'll use k as the index variable in the rest of this subsection, since the upper limit is sometimes n.

Sums of finite arithmetic and geometric series (in sigma notation)

$$\sum_{k=1}^{n}(a+(k-1)d) = \tfrac{1}{2}n(2a+(n-1)d)$$

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$$

Similarly, the formulas for the sum of the first n natural numbers, the sum of the first n square numbers and the sum of the first n cube numbers can be stated concisely using sigma notation, as in the box below. The box also includes, at the beginning, the simple formula for adding up n copies of the number 1, which is sometimes useful when you're working with series in sigma notation, as you'll see shortly.

Sums of standard finite series (in sigma notation)

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^3 = \tfrac{1}{4}n^2(n+1)^2$$

Activity 29 *Using formulas in sigma notation*

Use the formulas in the box above to find the sums of the following series.

(a) $\displaystyle\sum_{k=1}^{24} k$ (b) $\displaystyle\sum_{k=1}^{24} k^2$

Sigma notation allows you to work more easily with complicated finite series, and hence find their sums efficiently. To do this, you need to become familiar with a few rules for manipulating finite series in sigma notation. These are stated in the following box, and explained after the box.

Rules for manipulating finite series in sigma notation

$$\sum_{k=p}^{q} cx_k = c\sum_{k=p}^{q} x_k \quad \text{(where } c \text{ is a constant)}$$

$$\sum_{k=p}^{q}(x_k + y_k) = \sum_{k=p}^{q} x_k + \sum_{k=p}^{q} y_k$$

$$\sum_{k=p}^{q} x_k = \sum_{k=1}^{q} x_k - \sum_{k=1}^{p-1} x_k \quad \text{(where } 1 < p \le q\text{)}$$

The second rule in the box also holds if you replace the plus signs by minus signs:

$$\sum_{k=p}^{q}(x_k - y_k) = \sum_{k=p}^{q} x_k - \sum_{k=p}^{q} y_k.$$

To see why these rules hold, you can translate them from sigma notation into the usual, longer notation for sums. When you do this, the first rule becomes

$$cx_p + cx_{p+1} + \cdots + cx_q = c(x_p + x_{p+1} + \cdots + x_q).$$

This is just the usual rule for multiplying out brackets.

The second rule becomes

$$\begin{aligned}&(x_p + y_p) + (x_{p+1} + y_{p+1}) + \cdots + (x_q + y_q)\\ &\quad = (x_p + x_{p+1} + \cdots + x_q) + (y_p + y_{p+1} + \cdots + y_q).\end{aligned}$$

This rule holds simply because you can add numbers in any order. A similar argument shows that the version with minus signs holds. (Alternatively, you can deduce the version with minus signs by combining the first two rules in the box, taking $c = -1$.)

Finally, the third rule becomes

$$\begin{aligned}&x_p + x_{p+1} + \cdots + x_q\\ &\quad = (x_1 + x_2 + \cdots + x_q) - (x_1 + x_2 + \cdots + x_{p-1})\end{aligned}$$

This rule says that if you split a series into two parts, then the sum of the second part is equal to the sum of the whole series minus the sum of the first part. You were asked to use this fact in Activity 25(b) on page 55.

The next example illustrates how you can use the rules for manipulating finite series, together with some of the standard formulas for the sums of finite series, to find the sums of some other finite series.

Example 18 *Using series manipulations to find the sums of finite series*

Find the sums of the following finite series.

(a) $\displaystyle\sum_{k=1}^{25}(k^2+2k)$ (b) $\displaystyle\sum_{k=50}^{100}(9k-4)$

Solution

(a) Use the rules for manipulating series to express the series in terms of simpler series.

$$\begin{aligned}\sum_{k=1}^{25}(k^2+2k) &= \sum_{k=1}^{25}k^2+\sum_{k=1}^{25}2k\\ &= \sum_{k=1}^{25}k^2+2\sum_{k=1}^{25}k\end{aligned}$$

Use the formulas for the sums of standard series.

$$\begin{aligned}&= \tfrac{1}{6}(25)(25+1)(2\times 25+1)+2\times\tfrac{1}{2}(25)(25+1)\\ &= \tfrac{1}{6}\times 25\times 26\times 51+25\times 26\\ &= 5525+650\\ &= 6175\end{aligned}$$

(b) The lower limit isn't 1, so start by applying the third rule for manipulating series.

$$\sum_{k=50}^{100}(9k-4)=\sum_{k=1}^{100}(9k-4)-\sum_{k=1}^{49}(9k-4)$$

Now use the other rules to express each of the series on the right-hand side in terms of simpler series.

Now

$$\begin{aligned}\sum_{k=1}^{100}(9k-4) &= \sum_{k=1}^{100}9k-\sum_{k=1}^{100}4\\ &= 9\sum_{k=1}^{100}k-4\sum_{k=1}^{100}1\\ &= 9\times\tfrac{1}{2}\times 100\times(100+1)-4\times 100\\ &= 45\,450-400\\ &= 45\,050.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{k=1}^{49}(9k-4) &= \sum_{k=1}^{49}9k - \sum_{k=1}^{49}4\\ &= 9\sum_{k=1}^{49}k - 4\sum_{k=1}^{49}1\\ &= 9\times\tfrac{1}{2}\times 49\times(49+1) - 4\times 49\\ &= 11\,025 - 196\\ &= 10\,829.\end{aligned}$$

Hence

$$\sum_{k=50}^{100}(9k-4) = 45\,050 - 10\,829 = 34\,221.$$

Activity 30 *Using series manipulations to find the sums of finite series*

Find the sums of the following finite series.

(a) $\displaystyle\sum_{k=1}^{30}(2k^3-k)$ (b) $\displaystyle\sum_{k=1}^{40}(\tfrac{1}{4}k^2-1)$ (c) $\displaystyle\sum_{k=65}^{125}(6k+7)$

Sigma notation for infinite series

To write an infinite series in sigma notation, you write the symbol ∞ in place of the upper limit. For example,

$$\tfrac{1}{2}+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^3+\left(\tfrac{1}{2}\right)^4+\cdots=\sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n.$$

Activity 31 *Using sigma notation for infinite series*

Write the following infinite series in sigma notation.

(a) $\frac{1}{5}+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\cdots$

(b) $\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\cdots$

(c) $1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^2+\cdots$

The formula that you met earlier (on page 59) for the sum of an infinite geometric series can be stated concisely using sigma notation, as follows.

Sum of an infinite geometric series (in sigma notation)

If $-1 < r < 1$, then

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

The rules for manipulating finite series in sigma notation, given in the box on page 66, apply also for infinite series once the upper limit q is replaced everywhere by ∞, provided that each series involved has a sum. Thus, for example, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\left(\tfrac{1}{2}\right)^n + \left(\tfrac{1}{3}\right)^n\right) &= \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\tfrac{1}{3}\right)^n \\ &= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{1}{3}}{1-\frac{1}{3}} \\ &= 1 + \frac{1}{2} = \frac{3}{2}.\end{aligned}$$

4.4 Summing series using a computer

In the following activity you can learn how to use the module computer algebra system to work with series.

Activity 32 *Summing series on a computer*

Work through Subsection 11.2 of the *Computer algebra guide*.

5 The binomial theorem

In this final section you'll meet an important result, called the *binomial theorem*, which will help you to multiply out expressions such as

$$(1+x)^4, \quad (a+b)^7 \quad \text{and} \quad (2y-3)^5.$$

5.1 Expanding powers of binomials

In Subsection 2.3 of Unit 1, you saw how to expand squared brackets, such as

$$(x-5)^2, \quad (x+1)^2 \quad \text{and} \quad (2p-3q)^2. \tag{9}$$

For example,

$$\begin{aligned}(x-5)^2 &= (x-5)(x-5)\\ &= x^2-5x-5x+25\\ &= x^2-10x+25.\end{aligned}$$

Each of expressions (9) is of the form

$$(a+b)^2,$$

where a and b represent terms. For example, in the first expression $a = x$ and $b = -5$, and in the third one $a = 2p$ and $b = -3q$.

You saw in Unit 1 that the following general formula holds for multiplying out squared brackets:

$$(a+b)^2 = a^2 + 2ab + b^2. \tag{10}$$

So the square of the sum of two terms is equal to the square of the first term, plus twice the product of the two terms, plus the square of the second term.

The next example reminds you how equation (10) is applied.

Example 19 *Using the formula to expand squared brackets*

Use formula (10) to expand the expression

$$(2x-1)^2.$$

Solution

Substitute $a = 2x$ and $b = -1$ into the formula for $(a+b)^2$.

$$\begin{aligned}(2x-1)^2 &= (2x)^2 + 2(2x)(-1) + (-1)^2\\ &= 4x^2 - 4x + 1\end{aligned}$$

In Unit 1 you also met the formula

$$(a-b)^2 = a^2 - 2ab + b^2, \tag{11}$$

but in fact formula (10) above is enough alone. For example, you saw in Example 19 that you can expand $(2x-1)^2$ by using formula (10) with $a = 2x$ and $b = -1$, as an alternative to using formula (11) with $a = 2x$ and $b = 1$.

Activity 33 *Using the formula to expand squared brackets*

Use formula (10) to expand each of the following squared brackets.

(a) $(c+5)^2$ (b) $(1-3x)^2$ (c) $(p^2-q^2)^2$

In Activity 33 you probably found that using formula (10) to expand the squared brackets is only a little quicker than just writing the expression as two pairs of brackets multiplied together and expanding them in the usual way. However, sometimes you have to expand *cubed* brackets, of the form $(a+b)^3$, or similar brackets raised to an even higher power. It's much quicker to use a formula to expand expressions like these. In this subsection you'll see a general formula for expanding any expression of the form

$$(a+b)^n,$$

where n is a natural number and a and b are terms. This formula is known as the *binomial theorem*. A **binomial** is an expression that is the sum of two terms, so $(a+b)^n$ is a *power of a binomial*.

The word 'binomial' is derived from a Latin word meaning 'having two names', and is related to the word 'polynomial'.

You've seen and used a formula for $(a+b)^2$, so let's now find a formula for $(a+b)^3$. The easiest way to do this is to take the formula for $(a+b)^2$, and multiply it by $a+b$:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a(a^2+2ab+b^2)+b(a^2+2ab+b^2)\\ &= a^3+2a^2b+\ \ ab^2\\ &\qquad +\ \ a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

Notice that in the working above, the expression after the fourth equals sign has been written over two lines, with the like terms aligned in columns. This makes it easy to collect the like terms. The working shows that the formula for $(a+b)^3$ is

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3. \qquad (12)$$

Activity 34 *Using the formula to expand cubed brackets*

Use formula (12) to expand the following cubed brackets.

(a) $(5+p)^3$ (b) $(1-2x)^3$ (c) $(2x+3y)^3$

So that you can begin to see what happens for higher powers of binomials, let's work out one more formula, for $(a+b)^4$. We take the formula for $(a+b)^3$ and multiply it by $a+b$:

$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)^3\\ &= (a+b)(a^3+3a^2b+3ab^2+b^3)\\ &= a(a^3+3a^2b+3ab^2+b^3)+b(a^3+3a^2b+3ab^2+b^3)\\ &= a^4+3a^3b+3a^2b^2+ab^3\\ &\qquad+a^3b+3a^2b^2+3ab^3+b^4\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4.\end{aligned}$$

So the formula for $(a+b)^4$ is

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

You could use the same method to find formulas for $(a+b)^5$, $(a+b)^6$ and so on: you just keep multiplying by $a+b$. The formulas that are obtained by doing this are shown below, along with the three formulas found above. The 'formulas' for $(a+b)^0$ and $(a+b)^1$ are also shown, as it's helpful to consider these as part of the general pattern. Remember that any number raised to the power 0 is 1. The right-hand sides of the formulas have been aligned at their centres, to make it easier to see a particular pattern in them, which will be described shortly.

$$\begin{array}{lc}(a+b)^0 = & 1\\ (a+b)^1 = & a+b\\ (a+b)^2 = & a^2+2ab+b^2\\ (a+b)^3 = & a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 = & a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ (a+b)^5 = & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\ (a+b)^6 = & a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\end{array}$$

There are two useful things to observe about the formulas above.

First, notice that in each formula on the right-hand side, the powers in each term add up to the power of the brackets on the left-hand side. For example, look at the formula for $(a+b)^4$, and at the second term on the right-hand side, which is $4a^3b$. In this term, a has power 3 and b has power 1, and $3+1=4$, which is the power of the brackets in $(a+b)^4$. You can check that the powers add up to 4 in all the other terms in the formula for $(a+b)^4$.

You can see why this property holds for all the formulas if you look at, and think about, the working that produced the formulas for $(a+b)^2$, $(a+b)^3$ and $(a+b)^4$. Each new formula is produced by multiplying the formula before by $a+b$. When this is done, each term of the old formula is multiplied by a to give terms of the new formula, and separately multiplied by b to give further terms of the new formula (before the like terms are collected). This raises the sum of the powers in each term by 1. So, for example, since the sum of the powers in each term in the formula for

$(a+b)^2$ is 2, it follows that the sum of the powers in each term in the formula for $(a+b)^3$ is 3, and so on.

This property means that the terms on the right-hand side of each formula can be arranged in a standard order, as is done above. In the first term, a is raised to the same power as the brackets on the left-hand side. In each subsequent term, the power of a is decreased by 1 and the power of b is increased by 1 until, in the last term, b is raised to the same power as the brackets.

The second thing to notice about the formulas is that each coefficient on the right-hand side is the sum of the two adjacent coefficients in the line above. For example, the coefficient of a^4b^2 in the formula for $(a+b)^6$ is 15, which is the sum of the two adjacent coefficients 5 and 10 in the formula for $(a+b)^5$, as shown below.

$$\begin{array}{rcl}
\vdots & & \vdots \\
(a+b)^5 = & & a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 = & & a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\
\vdots & & \vdots
\end{array}$$

To see why this always happens, again look at and think about the working for $(a+b)^3$ and $(a+b)^4$, at the steps where the like terms are aligned. You can see that each coefficient in each new formula is obtained by adding two adjacent coefficients from the formula before.

This means that you can find the coefficients for any of the formulas from the triangular array below, which is known as **Pascal's triangle**. The array has 1s down each edge, and each of its other numbers is the sum of the two adjacent numbers in the line above. It can be continued indefinitely.

$$\begin{array}{lccccccccccccc}
(a+b)^0 & & & & & & & 1 & & & & & & \\
(a+b)^1 & & & & & & 1 & & 1 & & & & & \\
(a+b)^2 & & & & & 1 & & 2 & & 1 & & & & \\
(a+b)^3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
(a+b)^4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
(a+b)^5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
(a+b)^6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\vdots & & & & & & & \vdots & & & & & &
\end{array}$$

Notice that the array is symmetrical – the numbers in each row are the same whether you read them from left to right or from right to left.

Pascal's triangle contains many interesting sequences. For example, look at the diagonal lines of numbers. The first diagonal is $1, 1, 1, 1, 1, \ldots$, and the second diagonal contains the natural numbers $1, 2, 3, 4, 5, \ldots$. The third diagonal contains the numbers $1, 3, 6, 10, 15, \ldots$, which are known as the **triangular numbers**, because they correspond to triangular patterns of dots, as shown in Figure 26. There are many other less obvious patterns in the triangle.

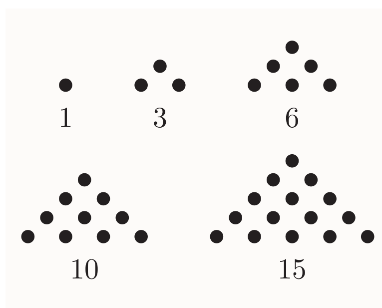


Figure 26 Triangular numbers

Activity 35 *Using Pascal's triangle to find a formula for* $(a+b)^7$

(a) Calculate the row of Pascal's triangle that gives the coefficients for the formula for $(a+b)^7$.

(b) Hence write down the formula for $(a+b)^7$.

Activity 36 *Expanding an expression of the form* $(a+b)^n$

For each of the following expressions, use one of the formulas that you have seen in this subsection to expand the brackets.

(a) $(3x+2)^5$ (b) $(x-4y)^4$

$$\begin{array}{ccccccc} & & & {}^0C_0 & & & \\ & & {}^1C_0 & & {}^1C_1 & & \\ & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & \\ {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 \\ & & & \vdots & & & \end{array}$$

Figure 27 Notation for binomial coefficients

The numbers in Pascal's triangle are called **binomial coefficients**. They can be denoted as follows. If the rows of Pascal's triangle are numbered as row 0, row 1, row 2 and so on, and the coefficients within each row are numbered from left to right as coefficient 0, coefficient 1, coefficient 2 and so on, then coefficient k in row n is denoted by ${}^nC_k$, as illustrated in Figure 27. For example,

$${}^3C_0 = 1, \quad {}^3C_1 = 3, \quad {}^3C_2 = 3 \quad \text{and} \quad {}^3C_3 = 1.$$

The letter C in this notation doesn't stand for 'coefficient', as you might expect, but rather for 'choose' or 'combination' – the reason for this is mentioned shortly. The notation ${}^nC_k$ is read as 'n choose k' or just as 'n C k'.

A common alternative notation for the binomial coefficients is

$$\binom{n}{k},$$

which is again read as 'n choose k'. (This notation has no connection with the column notation for two-dimensional vectors that you met in Unit 5, though it looks the same.)

With the notation ${}^nC_k$, the general formula for $(a+b)^n$ can be written as follows.

The binomial theorem

For any natural number n,

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + \cdots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n,$$

where ${}^nC_k$ is coefficient k in row n of Pascal's triangle (where the rows, and the coefficients within each row, are numbered 0, 1, 2, ...).

The right-hand side of the equation in the binomial theorem is known as the **binomial expansion** of $(a+b)^n$.

A good way to remember the terms in the equation in the binomial theorem is to use the fact that the powers of a and b on the right-hand side of the equation follow the pattern described earlier. In the first term a is raised to the same power as the brackets on the left-hand side, and in each subsequent term the power of a is decreased by 1 and the power of b is increased by 1, until in the last term b is raised to the same power as the brackets. To complete the expansion, you just need to include the coefficients of the terms, which are the binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$.

Intriguingly, the binomial coefficients also occur in a seemingly unrelated area of mathematics, concerned with finding answers to questions such as 'How many different sets of six lottery numbers can be chosen from 49 possible numbers?' This is the context which led to ${}^nC_k$ being read as 'n choose k'. You'll learn about it if you go on to study the module MST125 *Essential mathematics 2*.

Blaise Pascal (1623–1662)

Pascal's triangle is named after the French mathematician and philosopher Blaise Pascal. He was far from the first person to study this array of numbers, but his work on it in his *Traité du Triangle Arithmétique* was influential. Research on binomial coefficients was also carried out at about the same time by John Wallis (1616–1703) and then by Isaac Newton (1642–1727), who discovered that the binomial theorem can be generalised to negative and fractional powers. You'll learn about this in Unit 11.

Pascal contributed to many other areas of mathematics in his short life. He worked on conic sections and projective geometry, and, together with Pierre de Fermat, laid the foundations for the theory of probability.

Pascal's triangle was studied centuries earlier by the Chinese mathematician Yanghui and the Persian astronomer and poet Omar Khayyám, and is known as the Yanghui triangle in China (see Figure 28).

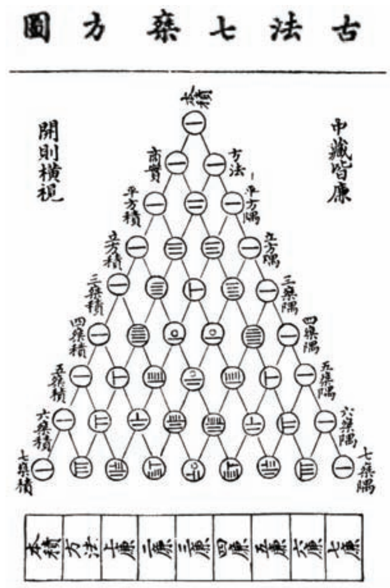


Figure 28 The Yanghui triangle (Pascal's triangle), from a publication of Zhu Shijie, AD 1303

5.2 A formula for binomial coefficients

Pascal's triangle is a convenient way to obtain the binomial coefficients for fairly small values of n. However, it would be tedious to obtain a particular coefficient in the expansion of $(a+b)^{12}$, say, by using Pascal's triangle. It is therefore desirable to have a closed-form formula for ${}^nC_k$, and you'll meet such a formula in this subsection. First you need to learn about *factorials*, which occur in the formula for ${}^nC_k$.

For any natural number n, the product of all the natural numbers up to and including n is called the **factorial** of $\boldsymbol{n}$, and denoted by $n!$ (this notation is read as 'n factorial' or 'factorial n'). So

$$n! = 1 \times 2 \times 3 \times \cdots \times n.$$

For example,

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.$$

We also define

$$0! = 1,$$

because this interpretation of $0!$ works well with formulas that involve factorials, as you'll see shortly.

The values of $n!$ increase rapidly as n increases. The first few values are shown in Table 1.

Table 1 Values of $n!$

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5040	40 320	362 880	3 628 800

Many calculators can evaluate $n!$, often for values of n up to 69. (The value of 69! is about 1.7×10^{98}, whereas 70! is about 1.2×10^{100}. Typically, calculators don't carry out calculations that involve numbers greater in magnitude than 10^{100}.)

Notice that

$$1! = 1 \times 0!, \quad 2! = 2 \times 1!, \quad 3! = 3 \times 2!, \quad \text{and so on.}$$

In general, $n!$ can be expressed in terms of $(n-1)!$ as follows:

$$n! = n(n-1)! \quad (n = 1, 2, 3, \ldots).$$

So the sequence of factorials, $1, 1, 2, 6, 24, 120, \ldots$, is generated by the recurrence system

$$c_0 = 1, \qquad c_n = nc_{n-1} \quad (n = 1, 2, 3, \ldots).$$

You met this recurrence system, and its first few terms, in Example 3(c) on page 13.

There's a formula for the binomial coefficients that can be stated concisely in terms of factorials, as below. You'll see a justification of this formula at the end of this section.

If n and k are integers with $0 \leq k \leq n$, then

$$^{n}C_k = \frac{n!}{k!\,(n-k)!}. \qquad (13)$$

For example, this formula gives

$$^{5}C_2 = \frac{5!}{2!\,3!} = \frac{120}{2\times 6} = 10,$$

which accords with the value of $^{5}C_2$ in Pascal's triangle.

When you use the formula above to evaluate binomial coefficients by hand, it's generally best *not* to evaluate the three factorials in the formula and then do the division. There's usually an easier and quicker way to proceed. To see this, consider the following calculation of $^{7}C_3$ using the formula, which is written out in full:

$$^{7}C_3 = \frac{7!}{3!\,4!} = \frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{(3\times 2\times 1)\times(4\times 3\times 2\times 1)}.$$

The two occurrences of $4\times 3\times 2\times 1$ on the top and bottom cancel out, so

$$^{7}C_3 = \frac{7\times 6\times 5}{3\times 2\times 1}.$$

We can now do some more cancelling, and hence obtain the value $^{7}C_3 = 35$.

In general, whenever you use formula (13) to work out a value of $^{n}C_k$, where k is not equal to either 0 or n, the factorial $(n-k)!$ in the denominator cancels with the 'tail' of the factorial $n!$ in the numerator. The numerator then contains the product of all the integers from n down to the integer that's one larger than $n-k$, and the denominator contains just $k!$. So, when k is not equal to 0 or n, formula (13) can be restated in the following form, which is not as neat but usually easier to apply. In fact this form of the formula also applies when $k=n$, but in this case it's just as easy to use the original form.

If n and k are integers with $0 < k \le n$, then

$$^{n}C_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.$$

This formula has k factors on the bottom and also k factors on the top. So, to apply it, you start with the integer n on the top and the integer k on the bottom, and for each of these integers you keep multiplying by the next integer down until you have k factors in total.

This is illustrated in the next example, but first here's one more fact to keep in mind when you're evaluating binomial coefficients. Remember that the binomial coefficients in each row of Pascal's triangle are the same no matter whether you read them from left to right or right to left. In other words, we have the following general fact.

$$^{n}C_k = {}^{n}C_{n-k}$$

So, for example, if you want to evaluate $^{10}C_7$ using the method described above, then it's better to evaluate $^{10}C_{10-7}$, that is, $^{10}C_3$, instead. You'll get the same answer, but the working will be easier, because you'll have only 3 factors on each of the top and bottom of the fraction, instead of 7.

Example 20 *Evaluating binomial coefficients using the formula*

Evaluate the following binomial coefficients without using a calculator.

(a) ${}^{9}C_{4}$ (b) ${}^{17}C_{15}$ (c) ${}^{4}C_{4}$

Solution

(a) Write down a fraction with 9 on the top and 4 on the bottom. For each of these integers, keep multiplying by the next integer down until you have 4 factors.

$$ {}^{9}C_{4} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} $$

Do all the cancelling that you can do, and hence evaluate the fraction.

$$ = 9 \times 2 \times 7 = 126 $$

(b) Use the fact that ${}^{17}C_{15} = {}^{17}C_{17-15}$, then proceed as in part (a).

$$ {}^{17}C_{15} = {}^{17}C_{2} = \frac{17 \times 16}{2 \times 1} = 17 \times 8 = 136 $$

(c) Since $n = k$ here, use the original form of formula (13), rather than the alternative form. Remember that $0! = 1$.

$$ {}^{4}C_{4} = \frac{4!}{4!\,0!} = \frac{4!}{4! \times 1} = 1 $$

Activity 37 *Evaluating binomial coefficients using the formula*

Evaluate each of the following without using a calculator.

(a) ${}^{8}C_{4}$ (b) ${}^{12}C_{10}$ (c) ${}^{5}C_{5}$ (d) ${}^{5}C_{0}$ (e) ${}^{21}C_{20}$

Another way to evaluate binomial coefficients is simply to use your calculator. (You should find that ${}^{n}C_{r}$ is a function for one of the buttons.)

Activity 38 *Evaluating binomial coefficients using a calculator*

Evaluate each of the following by using a calculator.

(a) ${}^{25}C_{19}$ (b) ${}^{32}C_{17}$

The binomial coefficients were known to early Islamic mathematicians. For example, al-Karajī, who flourished early in the 11th century, provided a table of binomial coefficients in a text (now lost but known through a copy in *The Brilliant in Algebra* by al-Samaw'al (1125–1174)), while the formula for them was given by al-Kāshī (d. 1429) in his book *The Calculator's Key.* A proof that this formula is correct was given by Pascal.

5.3 Working with the binomial theorem

In Subsection 5.1 you saw a statement of the binomial theorem, as follows: For any natural number n,

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + \cdots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n,$$

where ${}^nC_k$ is coefficient k in row n of Pascal's triangle (where the rows, and the coefficients within each row, are numbered $0, 1, 2, \ldots$).

In Subsection 5.2 it was stated that the binomial coefficients are given by

$${}^nC_k = \frac{n!}{k!\,(n-k)!}. \qquad (13)$$

Hence the binomial theorem can now be restated, without direct reference to Pascal's triangle, as follows.

The binomial theorem (restated)

For any natural number n,

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + \cdots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n,$$

where

$${}^nC_k = \frac{n!}{k!\,(n-k)!} \quad (k = 0, 1, 2, \ldots, n).$$

The binomial theorem can also be written in sigma notation, as follows.

The binomial theorem (sigma notation)

For any natural number n,

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k}b^k = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, a^{n-k}b^k.$$

It's useful to be familiar with the first few binomial coefficients in the expansion of $(a+b)^n$. The formula for ${}^nC_r$ gives

$${}^nC_0 = 1, \quad {}^nC_1 = n, \quad {}^nC_2 = \frac{n(n-1)}{2!}, \quad {}^nC_3 = \frac{n(n-1)(n-2)}{3!}.$$

Example 21 *Using the binomial theorem*

Use the binomial theorem to find the first four terms in the expansion of each of the following expressions.

(a) $(x+y)^8$ (b) $(2-x)^7$

Solution

(a) In the binomial theorem, put $n=8$, $a=x$ and $b=y$.

The first four terms in the binomial expansion of $(x+y)^8$ are

$$\begin{aligned}&x^8 + {}^8C_1x^7y + {}^8C_2x^6y^2 + {}^8C_3x^5y^3\\ &= x^8 + 8x^7y + \frac{8\times 7}{2!}x^6y^2 + \frac{8\times 7\times 6}{3!}x^5y^3\\ &= x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3.\end{aligned}$$

(b) In the binomial theorem, put $n=7$, $a=2$ and $b=-x$.

The first four terms in the binomial expansion of $(2-x)^7$ are

$$\begin{aligned}&2^7 + {}^7C_1\times 2^6(-x) + {}^7C_2\times 2^5(-x)^2 + {}^7C_3\times 2^4(-x)^3\\ &= 2^7 - 7\times 2^6x + \frac{7\times 6}{2!}\times 2^5x^2 - \frac{7\times 6\times 5}{3!}\times 2^4x^3\\ &= 128 - 448x + 672x^2 - 560x^3.\end{aligned}$$

Activity 39 *Using the binomial theorem*

Use the binomial theorem to find the first four terms in the expansion of each of the following expressions.

(a) $(1+x)^{10}$ (b) $\left(2+\frac{1}{3}x\right)^{10}$

Sometimes it's useful to find a particular term in a binomial expansion. The next example demonstrates how to do this.

Example 22 *Finding a particular term in a binomial expansion*

Find the coefficient of $x^{12}y^9$ in the expansion of

$$\left(2x-\tfrac{1}{2}y\right)^{21}.$$

Solution

By the binomial theorem, each term in the expansion is of the form

$$^{21}C_k(2x)^{21-k}\left(-\tfrac{1}{2}y\right)^k = {}^{21}C_k \times 2^{21-k}x^{21-k}\left(-\tfrac{1}{2}\right)^k y^k$$

Write the powers of x and y at the end, and simplify the powers of 2.

$$\begin{aligned} &= {}^{21}C_k(-1)^k 2^{21-k}2^{-k}x^{21-k}y^k \\ &= {}^{21}C_k(-1)^k 2^{21-2k}x^{21-k}y^k. \end{aligned}$$

The term in $x^{12}y^9$ is obtained when $k = 9$. Hence the coefficient of $x^{12}y^9$ is

$$^{21}C_9(-1)^9 2^{21-2\times 9} = -{}^{21}C_9 \times 2^3$$

Use a calculator to evaluate $^{21}C_9$.

$$\begin{aligned} &= -293\,930 \times 8 \\ &= -2351\,440. \end{aligned}$$

Activity 40 *Finding particular terms in binomial expansions*

(a) Find the coefficient of a^6b^5 in the expansion of $(a+b)^{11}$.

(b) Find the coefficient of c^5d^{15} in the expansion of $(3c-d)^{20}$.

Here's a slightly more complicated example.

Example 23 *Finding a particular term in another binomial expansion*

Find the coefficient of p^4 in the expansion of

$$\left(p^2 + \frac{1}{3p}\right)^{17}.$$

Give your answer as a fraction in its lowest terms.

Solution

By the binomial theorem, each term in the expansion is of the form

$$^{17}C_k\left(p^2\right)^{17-k}\left(\frac{1}{3p}\right)^k = {}^{17}C_k \times p^{2(17-k)}\left(\frac{1}{3}\right)^k\left(\frac{1}{p}\right)^k$$

Simplify the powers of p.

$$= {}^{17}C_k \left(\frac{1}{3^k}\right) p^{34-2k} p^{-k}$$

$$= {}^{17}C_k \left(\frac{1}{3^k}\right) p^{34-3k}.$$

For the term in p^4, we need

$$34 - 3k = 4,$$

which gives $30 = 3k$; that is, $k = 10$. Hence the coefficient of p^4 is

$$^{17}C_{10} \left(\frac{1}{3^{10}}\right) = \frac{19\,448}{59\,049}.$$

Activity 41 *Finding particular terms in more binomial expansions*

(a) Find the constant term in the expansion of

$$\left(x - \frac{1}{x}\right)^{12}.$$

(b) Consider the expansion of

$$\left(h - \frac{3}{2h^2}\right)^{15}.$$

(i) Find the coefficient of h^3.

(ii) Find the coefficient of h^{-12}.

(iii) Show that there is no term in h^2.

An important special case of the binomial theorem, which occurs frequently, is obtained by taking $a = 1$ and $b = x$. This gives the following expansion.

For each natural number n,

$$\begin{aligned}(1+x)^n &= 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_k x^k + \cdots + x^n \\ &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.\end{aligned}$$

Activity 42 *Applying a special case of the binomial theorem*

Use the form of the binomial theorem in the box above to evaluate the first six terms of the expansion of $(1-x)^{17}$.

Various interesting results can be obtained by choosing particular values for a and b in the statement of the binomial theorem. For example, if $a = b = 1$, we have that

$$1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_{n-1} + 1 = 2^n.$$

In sigma notation, this is

$$\sum_{k=0}^{n} {}^nC_k = 2^n.$$

Hence, for each natural number n, the finite series formed from the sequence of binomial coefficients ${}^nC_k$, $k = 0, 1, 2, \ldots, n$, has sum 2^n. In other words, the nth row of Pascal's triangle has sum 2^n.

A proof of the formula for the binomial coefficients

We can prove that the formula

$$ {}^nC_k = \frac{n!}{k!\,(n-k)!} \tag{14}$$

is correct by proving the following two facts.

1. Formula (14) gives the correct result for each of the 'edge values' of Pascal's triangle – that is, for $n = 0, 1, 2, \ldots$,

   $${}^nC_0 = 1 \text{ and } {}^nC_n = 1.$$

2. Formula (14) has the key property of Pascal's triangle described on page 73 – that is, each binomial coefficient (apart from the edge values) in each row is the sum of the two adjacent coefficients in the row above.

Fact 1 holds because, for $n = 0, 1, 2, \ldots$, formula (14) gives

$${}^nC_0 = \frac{n!}{n!\,0!} = 1 \quad \text{and} \quad {}^nC_n = \frac{n!}{0!\,n!} = 1.$$

Proving fact 2 takes more work. Consider any binomial coefficient ${}^nC_k$ that isn't an edge value (so $k \neq 0$ and $k \neq n$) in row n of Pascal's triangle. The two adjacent coefficients in the row above (row $n-1$) are

$${}^{n-1}C_{k-1} \quad \text{and} \quad {}^{n-1}C_k.$$

Fact 2 can therefore be expressed algebraically as the equation

$${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k, \tag{15}$$

for $n = 2, 3, 4, \ldots$ and $k = 1, 2, 3, \ldots, n-1$. We now verify that this equation holds.

The formula

$$^{n}C_k = \frac{n!}{k!\,(n-k)!}$$

gives

$$^{n-1}C_k = \frac{(n-1)!}{k!\,((n-1)-k)!} = \frac{(n-1)!}{k!\,(n-k-1)!}$$

and

$$^{n-1}C_{k-1} = \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} = \frac{(n-1)!}{(k-1)!\,(n-k)!}.$$

Hence the expression on the right-hand side of equation (15) is

$$^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}.$$

Multiplying the top and bottom of the first fraction by k, and the top and bottom of the second fraction by $n-k$, gives

$$\begin{aligned}
^{n-1}C_{k-1} + {}^{n-1}C_k &= \frac{k\,(n-1)!}{k!\,(n-k)!} + \frac{(n-k)\,(n-1)!}{k!\,(n-k)!}\\
&= \frac{(k+n-k)\,(n-1)!}{k!\,(n-k)!}\\
&= \frac{n(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = {}^{n}C_k,
\end{aligned}$$

which is the left-hand side of equation (15). This completes the proof.

Learning outcomes

After studying this unit, you should be able to:

- understand and use standard terminology and notation for sequences
- find the parameters of an arithmetic or geometric sequence from the first few terms, and hence write down a recurrence system for the sequence
- find a closed form for an arithmetic or geometric sequence, given the first few terms of the sequence or a recurrence system for it
- draw the graph of the first few terms of a sequence
- determine the long-term behaviour of certain simple sequences
- find the sum of a finite arithmetic or geometric series
- where possible, find the sum of an infinite geometric series
- understand the link between Pascal's triangle and binomial expansions
- find terms in the binomial expansion of an expression of the form $(a+b)^n$, by applying the binomial theorem and evaluating binomial coefficients.

Solutions to activities

Solution to Activity 1

(a) The first term of the sequence

$$(b_n)_{n=1}^{\infty} \text{ with terms } 1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ \dots$$

is the term with subscript 1; that is, $b_1 = 1$. The fourth term is $b_4 = 10$.

(b) Counting along the terms of the sequence, 16 is the 6th term. Hence if $b_n = 16$, then $n = 6$.

(c) No, b_0 is not defined, because 0 is not included in the range of values $n = 1, 2, 3, \dots$.

Solution to Activity 2

(a) $a_1 = 7$, $a_2 = 14$, $a_3 = 21$, $a_4 = 28$, $a_5 = 35$, $a_{100} = 700$.

(b) $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{3}$, $b_4 = \frac{1}{4}$, $b_5 = \frac{1}{5}$, $b_{100} = \frac{1}{100}$.

(You may have converted these simple fractions to decimals, but there is no need to do this in such cases.)

(c) $c_1 = (-1)^{1+1} = (-1)^2 = 1$,
$c_2 = (-1)^{2+1} = (-1)^3 = -1$,
$c_3 = 1$, $c_4 = -1$, $c_5 = 1$, $c_{100} = -1$.

(d) $d_1 = (-1)^1 \times 1 = -1$,
$d_2 = (-1)^2 \times 2 = 2$,
$d_3 = -3$, $d_4 = 4$, $d_5 = -5$, $d_{100} = 100$.

(e) $e_1 = (-2)^1 = -2$,
$e_2 = (-2)^2 = 4$,
$e_3 = -8$, $e_4 = 16$, $e_5 = -32$,
$e_{100} = (-2)^{100} = 1.27 \times 10^{30}$ (to 3 s.f.).

Solution to Activity 3

(a) We have $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$. A closed form is

$$a_n = n \quad (n = 1, 2, 3, \dots),$$

and the 10th term is $a_{10} = 10$.

(b) We have $a_1 = 2 = 2^1$, $a_2 = 2^2$, $a_3 = 2^3$, $a_4 = 2^4$. A closed form is

$$a_n = 2^n \quad (n = 1, 2, 3, \dots).$$

The 10th term is $a_{10} = 2^{10} = 1024$.

(c) We have $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $a_4 = 1$. A closed form is

$$a_n = (-1)^n \quad (n = 1, 2, 3, \dots),$$

and the 10th term is $a_{10} = (-1)^{10} = 1$.

(d) We have $a_1 = 1^3$, $a_2 = 2^3$, $a_3 = 3^3$, $a_4 = 4^3$. A closed form is

$$a_n = n^3 \quad (n = 1, 2, 3, \dots),$$

and the 10th term is $a_{10} = 10^3 = 1000$.

(e) We have $a_1 = 6 \times 1$, $a_2 = 6 \times (-1)$, $a_3 = 6 \times 1$, $a_4 = 6 \times (-1)$. A closed form is

$$a_n = 6 \times (-1)^{n+1} \quad (n = 1, 2, 3, \dots).$$

Since $(-1)^{n+1} = (-1)^n(-1)^1 = -(-1)^n$, this closed form can also be written as

$$a_n = -6 \times (-1)^n \quad (n = 1, 2, 3, \dots).$$

The 10th term is $a_{10} = 6 \times (-1)^{11} = -6$.

(f) We have $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{4}$, $a_4 = \frac{4}{5}$. A closed form is

$$a_n = \frac{n}{n+1} \quad (n = 1, 2, 3, \dots).$$

Since $n = (n+1) - 1$, this closed form can also be written as

$$a_n = 1 - \frac{1}{n+1} \quad (n = 1, 2, 3, \dots).$$

The 10th term is $a_{10} = \frac{10}{11}$.

(g) We have $a_1 = 2 \times 1$, $a_2 = -2 \times 2$, $a_3 = 2 \times 3$, $a_4 = -2 \times 4$. A closed form is

$$a_n = 2n(-1)^{n+1} \quad (n = 1, 2, 3, \dots).$$

Since $(-1)^{n+1} = (-1)^n(-1)^1 = -(-1)^n$, this closed form can also be written as

$$a_n = -2n(-1)^n \quad (n = 1, 2, 3, \dots).$$

The 10th term is $a_{10} = 2 \times 10 \times (-1)^{11} = -20$.

Solution to Activity 4

(a) (i) $a_0 = 3^0 = 1$, $a_1 = 3^1 = 3$, $a_2 = 3^2 = 9$.

(ii) $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{6}$, $b_4 = \frac{1}{12}$.

(iii) $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{6}$, $c_3 = \frac{1}{12}$.

(b) (i) We have $d_0 = 3^0$, $d_1 = 3^1$, $d_2 = 3^2$, $d_3 = 3^3$. A closed form is

$$d_n = 3^n \quad (n = 0, 1, 2, \dots),$$

and the sixth term is $d_5 = 3^5 = 243$.

(ii) We have $e_5 = \frac{1}{5}$, $e_6 = \frac{1}{6}$, $e_7 = \frac{1}{7}$, $e_8 = \frac{1}{8}$. A closed form is

$$e_n = \frac{1}{n} \quad (n = 5, 6, 7, \dots),$$

and the sixth term is $e_{10} = \frac{1}{10}$.

(iii) We have $f_2 = \left(-\frac{1}{2}\right)^2$, $f_3 = \left(-\frac{1}{2}\right)^3$, $f_4 = \left(-\frac{1}{2}\right)^4$, $f_5 = \left(-\frac{1}{2}\right)^5$. A closed form is

$$f_n = \left(-\tfrac{1}{2}\right)^n \quad (n = 2, 3, 4, \ldots),$$

and the sixth term is $f_7 = \left(-\frac{1}{2}\right)^7 = -\frac{1}{128}$.

Solution to Activity 5

(a) (i) $b_n = 2(n+1) \quad (n = 0, 1, 2, \ldots)$

(ii) $b_n = 3^n \quad (n = 0, 1, 2, \ldots)$

(iii) $b_n = 6 + n \quad (n = 0, 1, 2, \ldots)$

(iv) $b_n = 4n \quad (n = 0, 1, 2, \ldots)$

(b) (i) $b_n = 0.4^{n-1} \quad (n = 1, 2, 3, \ldots)$

(ii) $b_n = 5(n-1) \quad (n = 1, 2, 3, \ldots)$

(iii) $b_n = \dfrac{1}{2^{n-1}} \quad (n = 1, 2, 3, \ldots)$

(iv) $b_n = 2 + 3(n-1) \quad (n = 1, 2, 3, \ldots)$, which simplifies to

$$b_n = -1 + 3n \quad (n = 1, 2, 3, \ldots).$$

(c) (i) $b_n = \dfrac{3^{n-2}}{n-1} \quad (n = 2, 3, 4, \ldots)$

(ii) $b_n = \dfrac{1}{(n-1)(n+1)} \quad (n = 2, 3, 4, \ldots)$

Solution to Activity 6

(a) $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $a_4 = 7$, $a_5 = 15$.

(b) $b_1 = 1$, $b_2 = 0$, $b_3 = -1$, $b_4 = 0$, $b_5 = -1$.

(c) $c_0 = 2$, $c_1 = 1.5$,
$c_2 = 17/12 = 1.416\,667$ (to 6 d.p.),
$c_3 = 577/408 = 1.414\,216$ (to 6 d.p.),
$c_4 = 1.414\,214$ (to 6 d.p.).

Solution to Activity 7

(a) The sequence (x_n) is arithmetic, with parameters $a = -1$ and $d = 1$.

(b) The sequence (y_n) is not arithmetic. The term y_{n-1} on the right of the recurrence relation is multiplied by -1.

(c) The sequence (z_n) is arithmetic, with parameters $a = 1$ and $d = -1.5$.

Solution to Activity 8

(a) The first term is $a = 1$, and the common difference is $d = 4 - 1 = 3$. So the recurrence system is

$$x_1 = 1, \qquad x_n = x_{n-1} + 3 \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$x_5 = x_4 + 3 = 10 + 3 = 13,$$
$$x_6 = x_5 + 3 = 13 + 3 = 16.$$

(b) The first term is $a = 2.1$, and the common difference is $d = 3.2 - 2.1 = 1.1$. So the recurrence system is

$$y_1 = 2.1, \qquad y_n = y_{n-1} + 1.1 \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$y_5 = y_4 + 1.1 = 5.4 + 1.1 = 6.5,$$
$$y_6 = y_5 + 1.1 = 6.5 + 1.1 = 7.6.$$

(c) The first term is $a = 1$, and the common difference is $d = 0.9 - 1 = -0.1$. So the recurrence system is

$$z_1 = 1, \qquad z_n = z_{n-1} - 0.1 \quad (n = 2, 3, 4, \ldots, 11).$$

The next two terms are

$$z_5 = z_4 - 0.1 = 0.7 - 0.1 = 0.6,$$
$$z_6 = z_5 - 0.1 = 0.6 - 0.1 = 0.5.$$

Solution to Activity 9

(a) The first term is 1000, the last term is 10 and the common difference is $d = 970 - 1000 = -30$. Hence the number of terms is

$$\frac{10 - 1000}{-30} + 1 = 34,$$

so there are 34 terms in the sequence.

(b) The corresponding recurrence system is

$$x_1 = 1000, \quad x_n = x_{n-1} - 30 \quad (n = 2, 3, 4, \ldots, 34).$$

Solution to Activity 10

(a) Since $a = 1$ and $d = 3$, the closed form is

$$\begin{aligned} x_n &= 1 + 3(n-1) \\ &= 3n - 2 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

This gives $x_4 = 3 \times 4 - 2 = 10$, as expected, and also

$$x_{10} = 3 \times 10 - 2 = 28.$$

(b) Since $a = 2.1$ and $d = 1.1$, the closed form is

$$\begin{aligned} y_n &= 2.1 + 1.1(n-1) \\ &= 1.1n + 1 \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

This gives $y_4 = 1.1 \times 4 + 1 = 5.4$, as expected, and also

$$y_{10} = 1.1 \times 10 + 1 = 12.$$

(c) Since $a = 1$ and $d = -0.1$, the closed form is

$$\begin{aligned} z_n &= 1 - 0.1(n-1) \\ &= 1.1 - 0.1n \quad (n = 1, 2, 3, \ldots, 11). \end{aligned}$$

This gives $z_4 = 1.1 - 0.1 \times 4 = 0.7$, as expected, and also

$$z_{10} = 1.1 - 0.1 \times 10 = 0.1.$$

Solution to Activity 11

(a) The sequence (x_n) is geometric, with parameters $a = -1$ and $r = 3$.

(b) The sequence (y_n) is geometric, with parameters $a = 1$ and $r = -0.9$.

(c) The sequence (z_n) is not geometric, because the expression on the right of the recurrence relation contains the term $+1$.

Solution to Activity 12

(a) The first term is $a = 1$, and the common ratio is $r = \frac{1}{2}/1 = \frac{1}{2}$. So the recurrence system is

$$x_1 = 1, \qquad x_n = \tfrac{1}{2}x_{n-1} \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$\begin{aligned} x_5 &= \tfrac{1}{2}x_4 = \tfrac{1}{2} \times \tfrac{1}{8} = \tfrac{1}{16}, \\ x_6 &= \tfrac{1}{2}x_5 = \tfrac{1}{2} \times \tfrac{1}{16} = \tfrac{1}{32}. \end{aligned}$$

(b) The first term is $a = 4.2$, and the common ratio is $r = 7.14/4.2 = 1.7$. So the recurrence system is

$$y_1 = 4.2, \qquad y_n = 1.7y_{n-1} \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$\begin{aligned} y_5 = 1.7y_4 &= 1.7 \times 20.6346 \\ &= 35.078\,82 \\ &= 35.079 \text{ (to 3 d.p.)}, \end{aligned}$$

$$\begin{aligned} y_6 = 1.7y_5 &= 1.7 \times 35.078\,82 \\ &= 59.633\,994 \\ &= 59.634 \text{ (to 3 d.p.)}. \end{aligned}$$

(c) The first term is $a = 2$, and the common ratio is $r = (-2)/2 = -1$. So the recurrence system is

$$z_1 = 2, \qquad z_n = -z_{n-1} \quad (n = 2, 3, 4, \ldots).$$

The next two terms are

$$\begin{aligned} z_5 &= -z_4 = -(-2) = 2, \\ z_6 &= -z_5 = -2. \end{aligned}$$

Solution to Activity 13

(a) Suppose that the sequence has N terms, with first term $z_1 = 7$. Then the last term is $z_N = 2734\,375$. The common ratio is $r = z_2/z_1 = 5$.
Now N is given by

$$r^{N-1} = \frac{z_N}{z_1};$$

that is,

$$5^{N-1} = \frac{2734\,375}{7} = 390\,625.$$

Taking the natural logarithm of both sides of this equation gives

$$(N-1)\ln 5 = \ln(390\,625),$$

from which

$$N = 1 + \frac{\ln(390\,625)}{\ln 5} = 9.$$

Hence there are nine terms in the sequence.

(b) The corresponding recurrence system is

$$z_1 = 7, \qquad z_n = 5z_{n-1} \quad (n = 2, 3, 4, \ldots, 9).$$

Solution to Activity 14

(a) Since $a = 1$ and $r = \frac{1}{2}$, the closed form is

$$\begin{aligned} x_n &= 1 \times \left(\tfrac{1}{2}\right)^{n-1} \\ &= \left(\tfrac{1}{2}\right)^{n-1} \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

This gives $x_4 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, as expected, and also

$$\begin{aligned} x_{10} = \left(\tfrac{1}{2}\right)^9 &= \tfrac{1}{512} \\ &= 1.953 \times 10^{-3} \text{ (to 4 s.f.)}. \end{aligned}$$

(b) Since $a = 4.2$ and $r = 1.7$, the closed form is

$$y_n = 4.2 \times 1.7^{n-1} \quad (n = 1, 2, 3, \ldots).$$

This gives $y_4 = 4.2 \times 1.7^3 = 20.6346$, as expected, and also

$$y_{10} = 4.2 \times 1.7^9 = 498.1 \text{ (to 4 s.f.)}.$$

(c) Since $a = 2$ and $r = -1$, the closed form is

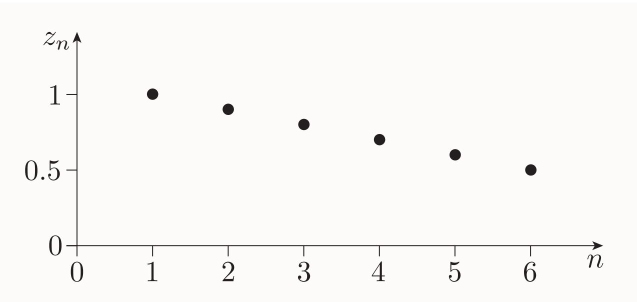
$$z_n = 2(-1)^{n-1} \quad (n = 1, 2, 3, \ldots).$$

This gives $z_4 = 2(-1)^3 = -2$, as expected, and also

$$z_{10} = 2(-1)^9 = -2.$$

Solution to Activity 15

The first point to be plotted is $(1, z_1) = (1, 1)$. The second point is $(2, z_2) = (2, 0.9)$. The subsequent points are $(3, 0.8)$, $(4, 0.7)$, $(5, 0.6)$ and $(6, 0.5)$. The graph is as follows.

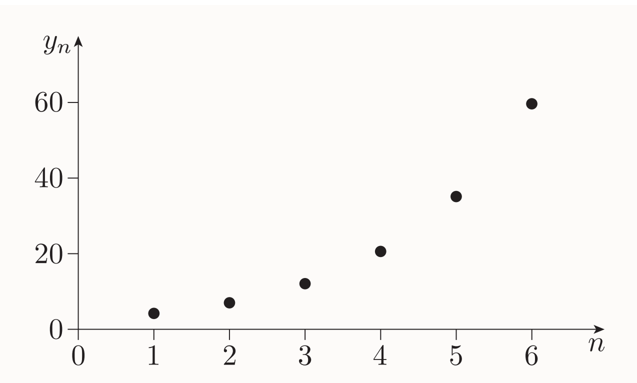


Solution to Activity 16

(a) The first point to be plotted is $(1, y_1) = (1, 4.2)$. The second point is

$$(2, y_2) = (2, 4.2 \times 1.7) \approx (2, 7.1).$$

The subsequent points (to 1 d.p.) are $(3, 12.1)$, $(4, 20.6)$, $(5, 35.1)$ and $(6, 59.6)$. The graph is as follows.



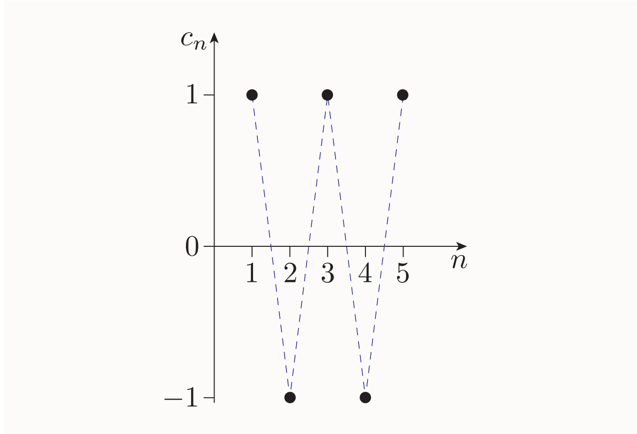
(b) The first point to be plotted is

$$(1, c_1) = (1, (-1)^2) = (1, 1).$$

The second point is

$$(2, c_2) = (2, (-1)^3) = (2, -1).$$

The subsequent points are $(3, 1)$, $(4, -1)$ and $(5, 1)$. The graph is as follows.



The dashed lines shown above are not part of the graph of the sequence, but draw attention to the fact that the terms of this sequence alternate in sign.

Solution to Activity 18

The graph in Figure 9(b) shows that, for this sequence,

$$x_n \to L \text{ as } n \to \infty \qquad \left(\text{or } \lim_{n\to\infty} x_n = L\right).$$

The graph in Figure 9(c) shows that, for this sequence,

$$x_n \to -\infty \text{ as } n \to \infty.$$

The graph in Figure 9(e) shows that, for this sequence,

$$x_n \to 0 \text{ as } n \to \infty \qquad \left(\text{or } \lim_{n\to\infty} x_n = 0\right).$$

Solution to Activity 19

The sequence (x_n) with closed form $x_n = 3 - \frac{4}{5}n$ is arithmetic, with common difference $d = -\frac{4}{5}$. Since $-\frac{4}{5} < 0$, the sequence (x_n) is decreasing and $x_n \to -\infty$ as $n \to \infty$.

Solution to Activity 20

(a) Since $r = 1.2 > 1$, the sequence (1.2^n) is increasing and $1.2^n \to \infty$ as $n \to \infty$. To obtain (x_n) we multiply each term by the negative constant $-\frac{1}{3}$. Hence (x_n) is decreasing and $x_n \to -\infty$ as $n \to \infty$.

(b) Since $-1 < -0.9 < 0$, the sequence $((-0.9)^n)$ alternates in sign and $(-0.9)^n \to 0$ as $n \to \infty$. To obtain (y_n) we multiply each term by the non-zero constant 5. Hence (y_n) alternates in sign and $y_n \to 0$ as $n \to \infty$.

(c) Since $-2.5 < -1$, the sequence $((-2.5)^n)$ alternates in sign and is unbounded. To obtain (z_n) we multiply each term by the negative constant $-\frac{1}{10}$. Hence (z_n) also alternates in sign and is unbounded.

(Details of how to use the CAS to plot graphs of these sequences are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

Solution to Activity 21

(a) From Activity 20(a), the sequence with terms $y_n = -\frac{1}{3} \times 1.2^n$ is decreasing and $y_n \to -\infty$ as $n \to \infty$. Since $a_n = y_n + 17$ for each n, it follows that the sequence (a_n) is also decreasing and $a_n \to -\infty$ as $n \to \infty$.

(b) From Activity 20(b), the sequence with terms $z_n = 5(-0.9)^n$ has limit 0 and its terms are of alternating sign. Since $b_n = z_n + 45$ for each n, it follows that $b_n \to 45$ as $n \to \infty$; that is,

$$\lim_{n\to\infty} b_n = 45.$$

Also, the terms of (b_n) alternate either side of 45.

(Details of how to use the CAS to plot graphs of these sequences are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

Solution to Activity 22

The sequence is $6, 13, 20, \dots, 90$. The first term is 6, the last term is 90 and the common difference is $d = 13 - 6 = 7$, so the number of terms is

$$\frac{90-6}{7} + 1 = 13.$$

Hence the sum is

$$\begin{aligned}\tfrac{1}{2} \times 13 \times (6+90) &= 13 \times 48\\ &= 624.\end{aligned}$$

Solution to Activity 23

Expression (5) is used in each case.

(a) Here $a = 1$, $d = 1$ and $n = 100$. The sum is

$$\begin{aligned}\tfrac{1}{2} \times 100 \times (2 \times 1 + 99 \times 1) &= 50 \times 101\\ &= 5050.\end{aligned}$$

(b) Here $a = 12$, $d = 3$ and

$$n = \frac{60-12}{3} + 1 = 17.$$

The sum is

$$\begin{aligned}\tfrac{1}{2} \times 17 \times (2 \times 12 + 16 \times 3) &= \tfrac{1}{2} \times 17 \times 72\\ &= 612.\end{aligned}$$

(c) Here $a = 1$, $d = 2$ and

$$n = \frac{19-1}{2} + 1 = 10.$$

The sum is

$$\begin{aligned}\tfrac{1}{2} \times 10 \times (2 \times 1 + 9 \times 2) &= 5 \times 20\\ &= 100.\end{aligned}$$

Each of these answers could also have been obtained using expression (4).

(Notice that the result of part (c) is another example of a pattern observed near the start of this subsection, namely that adding up consecutive odd numbers, starting at 1, always seems to give a square number. Here's how you can prove that this pattern holds in general.

The sum of the first n odd numbers forms an arithmetic series with first term $a = 1$, common difference $d = 2$ and n terms. By the formula in the box above this activity, this series has sum

$$\begin{aligned}\tfrac{1}{2}n(2 \times 1 + (n-1) \times 2) &= \tfrac{1}{2}n(2 + 2n - 2)\\ &= \tfrac{1}{2}n \times 2n\\ &= n^2.)\end{aligned}$$

Solution to Activity 24

(a) The numbers added form a geometric sequence whose 10 terms can be expressed as

$$3^n \quad (n = 1, 2, \dots, 10).$$

The first term is $a = 3$ and the common ratio is $r = 3$. Using expression (7) with $n = 10$, the sum of these terms is

$$\begin{aligned}\frac{3\left(1-3^{10}\right)}{1-3} &= \tfrac{3}{2}(59\,049 - 1)\\ &= 88\,572.\end{aligned}$$

(b) The numbers added form a geometric sequence with first term 1 and common ratio $-\frac{1}{3}$, and with 9 terms. (Be careful when working out the number of terms of a geometric sequence that has first term 1.) From expression (7) the sum is

$$\begin{aligned}\frac{1-\left(-\frac{1}{3}\right)^9}{1-\left(-\frac{1}{3}\right)} &= \tfrac{3}{4}\left(1 + \left(\tfrac{1}{3}\right)^9\right)\\ &= 0.750\,038 \quad \text{(to 6 d.p.)}.\end{aligned}$$

Solution to Activity 25

(a) (i) The formula for the first n natural numbers is

$$1+2+3+\cdots+n=\tfrac{1}{2}n(n+1).$$

Here $n=30$, and so

$$1+2+3+\cdots+30=\tfrac{1}{2}\times 30\times 31 = 465.$$

(ii) The formula for the squares of the first n natural numbers is

$$1^2+2^2+3^2+\cdots+n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Here $n=10$, and so

$$1^2+2^2+3^2+\cdots+10^2 = \tfrac{1}{6}\times 10\times 11\times 21 = 385.$$

(iii) We use the same formula as in part (a)(ii). Here $n=30$, and so

$$1^2+2^2+3^2+\cdots+30^2 = \tfrac{1}{6}\times 30\times 31\times 61 = 9455.$$

(iv) The formula for the cubes of the first n natural numbers is

$$1^3+2^3+3^3+\cdots+n^3=\tfrac{1}{4}n^2(n+1)^2.$$

Here $n=30$, and so

$$1^3+2^3+3^3+\cdots+30^3=\tfrac{1}{4}\times 30^2\times 31^2 = 216\,225.$$

(b) (i) Note that

$$11^2+12^2+13^2+\cdots+30^2 = (1^2+2^2+3^2+\cdots+30^2) - (1^2+2^2+3^2+\cdots+10^2).$$

It follows from the results of parts (a)(ii) and (iii) that

$$11^2+12^2+13^2+\cdots+30^2 = 9455-385 = 9070.$$

(ii) Note that

$$20^3+21^3+22^3+\cdots+40^3 = (1^3+2^3+3^3+\cdots+40^3) - (1^3+2^3+3^3+\cdots+19^3).$$

Applying the formula from part (a)(iv), we have (with $n=19$)

$$1^3+2^3+3^3+\cdots+19^3 = \tfrac{1}{4}\times 19^2\times 20^2 = 36\,100$$

and (with $n=40$)

$$1^3+2^3+3^3+\cdots+40^3 = \tfrac{1}{4}\times 40^2\times 41^2 = 672\,400.$$

It follows that

$$20^3+21^3+22^3+\cdots+40^3 = 672\,400-36\,100 = 636\,300.$$

Solution to Activity 26

(a) This is an infinite geometric series with first term $a=\frac{3}{4}$ and common ratio $r=\frac{3}{4}$. Since $-1<r<1$, the series has a sum, namely

$$\frac{a}{1-r}=\frac{\frac{3}{4}}{1-\frac{3}{4}}=\frac{\frac{3}{4}}{\frac{1}{4}}=3.$$

(b) This is an infinite geometric series with first term $a=1$ and common ratio $r=-\frac{1}{2}$. Since $-1<r<1$, the series has a sum, namely

$$\frac{a}{1-r}=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{1}{\frac{3}{2}}=\frac{2}{3}.$$

(c) This is an infinite geometric series with first term $a=1$ and common ratio $r=-2$. Since $r<-1$, the series doesn't have a sum.

(d) This is an infinite geometric series, with first term $a=\frac{1}{4}$ and common ratio $r=-\frac{2}{3}$. Since $-1<r<1$, the series has a sum, namely

$$\frac{a}{1-r}=\frac{\frac{1}{4}}{1-\left(-\frac{2}{3}\right)}=\frac{\frac{1}{4}}{\frac{5}{3}}=\frac{3}{20}.$$

Solution to Activity 27

(a) Put $s=0.454\,545\ldots$. The repeating group, '45', is 2 digits long, so multiply s by 10^2, to obtain

$$100s=45.454\,545\ldots=45+s.$$

Hence we have $s=\frac{45}{99}=\frac{5}{11}$.

(b) Put $s = 0.729\,729\,729\ldots$. The repeating group, '729', is 3 digits long, so multiply s by 10^3, to obtain

$$1000s = 729.729\,729\,729\ldots = 729 + s.$$

Hence we have $s = \frac{729}{999} = \frac{27}{37}$. It follows that

$$3.729\,729\,729\ldots = 3 + \frac{27}{37} = \frac{138}{37}.$$

Solution to Activity 28

(a) The sums are as follows.

(i) $$\sum_{n=5}^{20} n^4 = 5^4 + 6^4 + 7^4 + \cdots + 20^4$$

(ii) $$\sum_{n=4}^{19} (n+1)^4 = 5^4 + 6^4 + 7^4 + \cdots + 20^4$$

(iii) $$\sum_{n=1}^{6} (2n-1) = 1 + 3 + 5 + \cdots + 11$$

(b) (i) The sum is

$$1 + 2 + 3 + \cdots + 150 = \sum_{n=1}^{150} n.$$

(ii) The sum is

$$5^2 + 6^2 + 7^2 + \cdots + 13^2 = \sum_{n=5}^{13} n^2.$$

(iii) The sum is

$$2 + 2^2 + 2^3 + \cdots + 2^{12} = \sum_{n=1}^{12} 2^n.$$

Solution to Activity 29

(a) $$\sum_{k=1}^{24} k = \tfrac{1}{2} \times 24 \times (24+1) = 300$$

(b) $$\begin{aligned}\sum_{k=1}^{24} k^2 &= \tfrac{1}{6} \times 24 \times (24+1) \times (2 \times 24 + 1)\\ &= \tfrac{1}{6} \times 24 \times 25 \times 49\\ &= 4900\end{aligned}$$

Solution to Activity 30

The solutions below apply the rules for manipulating finite series using sigma notation, in the box on page 66, and the formulas for the sums of standard finite series in the box on page 65.

(a) $$\begin{aligned}\sum_{k=1}^{30}(2k^3 - k) &= \sum_{k=1}^{30} 2k^3 - \sum_{k=1}^{30} k\\ &= 2\sum_{k=1}^{30} k^3 - \sum_{k=1}^{30} k\\ &= 2 \times \tfrac{1}{4}(30)^2(31)^2 - \tfrac{1}{2}(30)(31)\\ &= 432\,450 - 465\\ &= 431\,985\end{aligned}$$

(b) $$\begin{aligned}\sum_{k=1}^{40}(\tfrac{1}{4}k^2 - 1) &= \sum_{k=1}^{40} \tfrac{1}{4}k^2 - \sum_{k=1}^{40} 1\\ &= \tfrac{1}{4}\sum_{k=1}^{40} k^2 - \sum_{k=1}^{40} 1\\ &= \tfrac{1}{4} \times \tfrac{1}{6}(40)(41)(81) - 40\\ &= 5535 - 40\\ &= 5495\end{aligned}$$

(c) We have

$$\sum_{k=65}^{125}(6k+7) = \sum_{k=1}^{125}(6k+7) - \sum_{k=1}^{64}(6k+7).$$

Now

$$\begin{aligned}\sum_{k=1}^{125}(6k+7) &= \sum_{k=1}^{125} 6k + \sum_{k=1}^{125} 7\\ &= 6\sum_{k=1}^{125} k + 7\sum_{k=1}^{125} 1\\ &= 6 \times \tfrac{1}{2}(125)(126) + 7 \times 125\\ &= 47\,250 + 875\\ &= 48\,125.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{k=1}^{64}(6k+7) &= \sum_{k=1}^{64} 6k + \sum_{k=1}^{64} 7\\ &= 6\sum_{k=1}^{64} k + 7\sum_{k=1}^{64} 1\\ &= 6 \times \tfrac{1}{2}(64)(65) + 7 \times 64\\ &= 12\,480 + 448\\ &= 12\,928.\end{aligned}$$

Hence

$$\sum_{k=65}^{125}(6k+7) = 48\,125 - 12\,928 = 35\,197.$$

Solution to Activity 31

(a) $\frac{1}{5}+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\cdots=\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$

(b) $\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\cdots=\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$

(c) $1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^2+\cdots=\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2$

Solution to Activity 33

(a) $(c+5)^2=c^2+2c\times 5+5^2$
$\qquad\qquad =c^2+10c+25$

(b) $(1-3x)^2=1^2+2\times 1\times(-3x)+(-3x)^2$
$\qquad\qquad =1-6x+9x^2$

(c) $(p^2-q^2)^2=(p^2)^2+2p^2(-q^2)+(-q^2)^2$
$\qquad\qquad =p^4-2p^2q^2+q^4$

Solution to Activity 34

(a) By formula (12),

$$\begin{aligned}(5+p)^3&=5^3+3\times 5^2\times p+3\times 5\times p^2+p^3\\&=125+75p+15p^2+p^3.\end{aligned}$$

(b) By formula (12),

$$\begin{aligned}(1-2x)^3&=1^3+3\times 1^2\times(-2x)\\&\quad+3\times 1\times(-2x)^2+(-2x)^3\\&=1-6x+12x^2-8x^3.\end{aligned}$$

(c) By formula (12),

$$\begin{aligned}(2x+3y)^3&=(2x)^3+3\times(2x)^2\times 3y\\&\quad+3\times 2x\times(3y)^2+(3y)^3\\&=8x^3+36x^2y+54xy^2+27y^3.\end{aligned}$$

Solution to Activity 35

(a) The row is 1, 7, 21, 35, 35, 21, 7, 1.

(b) This gives the formula

$$\begin{aligned}(a+b)^7&=a^7+7a^6b+21a^5b^2+35a^4b^3\\&\quad+35a^3b^4+21a^2b^5+7ab^6+b^7.\end{aligned}$$

Solution to Activity 36

(a) We use the formula

$$\begin{aligned}&(a+b)^5\\&=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.\end{aligned}$$

Substituting $a=3x$ and $b=2$ gives

$$\begin{aligned}&(3x+2)^5\\&=(3x)^5+5(3x)^4\times 2+10(3x)^3\times 2^2\\&\quad+10(3x)^2\times 2^3+5(3x)\times 2^4+2^5\\&=243x^5+810x^4+1080x^3+720x^2+240x+32.\end{aligned}$$

(b) We use the formula

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

Substituting $a=x$ and $b=-4y$ gives

$$\begin{aligned}&(x-4y)^4\\&=x^4+4x^3(-4y)+6x^2(-4y)^2\\&\quad+4x(-4y)^3+(-4y)^4\\&=x^4-16x^3y+96x^2y^2-256xy^3+256y^4.\end{aligned}$$

Solution to Activity 37

(a) ${}^8C_4=\dfrac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1}=70$

(b) ${}^{12}C_{10}={}^{12}C_2=\dfrac{12\times 11}{2\times 1}=66$

(c) ${}^5C_5=\dfrac{5!}{5!0!}=1$

(d) ${}^5C_0=\dfrac{5!}{0!5!}=1$

(e) ${}^{21}C_{20}={}^{21}C_1=\dfrac{21}{1}=21$

Solution to Activity 38

(a) ${}^{25}C_{19}=177\,100$

(b) ${}^{32}C_{17}=565\,722\,720$

Solution to Activity 39

(a) We put $n=10$, $a=1$ and $b=x$ in the binomial theorem. The first four terms in the binomial expansion of $(1+x)^{10}$ are

$$\begin{aligned}&1^{10}+{}^{10}C_1\times 1^9x+{}^{10}C_2\times 1^8x^2+{}^{10}C_3\times 1^7x^3\\&=1+10x+\frac{10\times 9}{2!}x^2+\frac{10\times 9\times 8}{3!}x^3\\&=1+10x+45x^2+120x^3.\end{aligned}$$

(b) Here $n=10$ (as in part (a)), $a=2$ and $b=\frac{1}{3}x$. The binomial coefficients needed were obtained in part (a). The first four terms in the binomial expansion of $(2+\frac{1}{3}x)^{10}$ are

$$\begin{aligned}&2^{10}+10\times 2^9\left(\tfrac{1}{3}x\right)+45\times 2^8\left(\tfrac{1}{3}x\right)^2+120\times 2^7\left(\tfrac{1}{3}x\right)^3\\&=1024+\tfrac{5120}{3}x+1280x^2+\tfrac{5120}{9}x^3.\end{aligned}$$

Solution to Activity 40

(a) By the binomial theorem, each term in the expansion is of the form

$${}^{11}C_k a^{11-k} b^k.$$

The term in a^6b^5 is obtained when $k = 5$. Hence the coefficient of a^6b^5 is

$${}^{11}C_5 = \frac{11!}{6!\,5!} = \frac{11\times 10\times 9\times 8\times 7}{5\times 4\times 3\times 2\times 1} = 462.$$

(b) By the binomial theorem, each term in the expansion is of the form

$${}^{20}C_k(3c)^{20-k}(-d)^k = {}^{20}C_k(-1)^k 3^{20-k}c^{20-k}d^k.$$

The term in c^5d^{15} is obtained when $k = 15$. Hence the coefficient of c^5d^{15} is

$$\begin{aligned}{}^{20}C_{15}(-1)^{15}3^5 &= -{}^{20}C_{15}\times 3^5\\ &= -15\,504\times 243\\ &= -3767\,472.\end{aligned}$$

Solution to Activity 41

(a) The constant term arises when the power of x and the power of $-1/x$ in the expansion are the same, namely, 6. By the binomial theorem, this term is

$$\begin{aligned}{}^{12}C_6\times x^6\left(-\frac{1}{x}\right)^6 &= {}^{12}C_6\\ &= \frac{12\times 11\times 10\times 9\times 8\times 7}{6\times 5\times 4\times 3\times 2\times 1}\\ &= 924.\end{aligned}$$

(b) By the binomial theorem, each term in the expansion is of the form

$$\begin{aligned}&{}^{15}C_k\times h^{15-k}\left(-\frac{3}{2h^2}\right)^k\\ &= {}^{15}C_k\times h^{15-k}\left(-\tfrac{3}{2}\right)^k(h^{-2})^k\\ &= {}^{15}C_k\left(-\tfrac{3}{2}\right)^k h^{15-k}h^{-2k}\\ &= {}^{15}C_k\left(-\tfrac{3}{2}\right)^k h^{15-3k}.\end{aligned}$$

(i) For the term in h^3, we need

$$15-3k = 3,$$

which gives $12 = 3k$; that is, $k = 4$. Hence the coefficient of h^3 is

$${}^{15}C_4\left(-\tfrac{3}{2}\right)^4 = 1365\times\frac{81}{16} = \frac{110\,565}{16}.$$

(ii) For the term in h^{-12}, we need

$$15-3k = -12,$$

which gives $27 = 3k$; that is, $k = 9$. Hence the coefficient of h^{-12} is

$$\begin{aligned}{}^{15}C_9\left(-\tfrac{3}{2}\right)^9 &= -5005\times\frac{19\,683}{512}\\ &= -\frac{98\,513\,415}{512}.\end{aligned}$$

(iii) For a term in h^2, we need

$$15-3k = 2,$$

which gives $13 = 3k$. Since there is no integer value of k that satisfies this equation, there is no term in h^2.

Solution to Activity 42

We put $n = 17$ and replace x by $-x$ in the equation in the box. The first six terms of the expansion are

$$\begin{aligned}&1 + {}^{17}C_1(-x) + {}^{17}C_2(-x)^2 + {}^{17}C_3(-x)^3\\ &\quad + {}^{17}C_4(-x)^4 + {}^{17}C_5(-x)^5\\ &= 1 - 17x + \frac{17\times 16}{2!}x^2 - \frac{17\times 16\times 15}{3!}x^3\\ &\quad + \frac{17\times 16\times 15\times 14}{4!}x^4\\ &\quad - \frac{17\times 16\times 15\times 14\times 13}{5!}x^5\\ &= 1 - 17x + 136x^2 - 680x^3 + 2380x^4 - 6188x^5.\end{aligned}$$

Acknowledgements

Grateful acknowledgement is made to the following sources:

Page 15: Ton Zijlstra / www.flickr.com/photos/tonz/9679968537/. This file is licensed under the Creative Commons Attribution-Noncommercial-ShareAlike Licence http://creativecommons.org/licenses/by-nc-sa/3.0/

Every effort has been made to contact copyright holders. If any have been inadvertently overlooked the publishers will be pleased to make the necessary arrangements at the first opportunity.