Unit 6

Systems of linear differential equations

Introduction

You have already met several types of linear differential equation in this module. For example, in Unit 1 you met linear first-order differential equations of the form

$$\frac{dx}{dt} + g(t)\,x = h(t),$$

where $x(t)$ is a function of the independent variable t, and $g(t)$ and $h(t)$ are given functions of t. Also in Unit 1 you met linear constant-coefficient second-order differential equations of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \quad a \neq 0,$$

where a, b and c are constants, and $f(t)$ is a given function of the independent variable t.

However, the homogeneous case (i.e. setting $f(t) = 0$) of this second equation can be expressed as a pair of coupled linear first-order differential equations:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -(cx + by)/a,\end{aligned}$$

Recall that $\dot{x}$ means dx/dt and $\dot{y}$ means dy/dt.

where $y(t)$ is a another function of t. Systems of this type, that is, of linear differential equations relating two or more functions and their derivatives, are the subject of this unit. Another simple example of such a system is

$$\begin{aligned}\dot{x} &= 3x + 2y,\\ \dot{y} &= x + 4y,\end{aligned}$$

where $x(t)$ and $y(t)$ are both functions of the independent variable t. You can think of the vector $(\dot{x} \quad \dot{y})^T = (\dot{x}(t) \quad \dot{y}(t))^T$ as the velocity at time t of a particle at position $(x, y) = (x(t), y(t))$. If we are given a system and an initial condition for the position of the particle at time $t = 0$, then solving the differential equations will give us the position $(x(t), y(t))$ of the particle at any subsequent time t.

In this unit we develop techniques for solving such systems of linear differential equations with constant coefficients. We will see that such systems can be written in matrix form, and that we can solve them by calculating the eigenvalues and eigenvectors of the resulting coefficient matrix.

We also examine systems of equations involving second derivatives, such as

$$\begin{aligned}\ddot{x} &= x + 4y,\\ \ddot{y} &= x - 2y,\end{aligned}$$

where you can think of the vector $(\ddot{x} \quad \ddot{y})^T = (\ddot{x}(t) \quad \ddot{y}(t))^T$ as the acceleration at time t of a particle at position $(x, y) = (x(t), y(t))$.

In Section 1 we show how various situations can be modelled by a system of linear differential equations. In Section 2 we show how such a system can be written in matrix form, and use eigenvalues and eigenvectors to solve it when the equations are homogeneous with constant coefficients. This discussion spills over into Section 3, where we discuss the inhomogeneous case, and into Section 4, where we show how similar techniques can be used to solve certain systems of *second-order* differential equations.

Although many of the problems that we consider in this unit may appear to be rather restricted in scope, the type of system discussed here arises surprisingly often in practice. In particular, systems of linear constant-coefficient differential equations occur frequently in modelling situations, especially when we need to make simplifying assumptions about the situations involved. They also play an important role in understanding certain properties of the solutions of particular *non-linear* systems, as we will see in Unit 12.

1 Systems of differential equations as models

In this short motivational section, you will see how systems of linear differential equations arise in the process of modelling three rather different types of situation.

We do not give here full details of the modelling process that is required in each case, since the aim is to give a fairly rapid impression of where systems of differential equations might occur in practice. You should not spend too much time dwelling on the details.

1.1 Conflicts

Frederick William Lanchester

The model described in this subsection was first published by F.W. Lanchester (Figure 1) in 1914.

Lanchester was an English polymath and an engineer of outstanding ability. In addition to his mathematical work on conflicts, he also made significant contributions to aeronautics and automotive engineering, and played an important pioneering role in the early development of the British car industry, founding the Lanchester Engine Company in 1899.

Figure 1 Frederick William Lanchester (1868–1946)

Consider a situation in which two different groups are in direct competition for survival. In a military context, the individual members of these groups might be humans (soldiers, say) or they might be tanks, ships or aircraft. In the absence of any external means of stopping the conflict, a battle unfolds by a process of attrition, in which individual members of the two groups are in one way or another rendered inactive (in the case of humans, killed or severely wounded). The battle terminates when one side or the other has lost all of its active members.

What factors affect who will 'win' such a conflict? Other things being equal, we would expect a larger group to prevail over a smaller one, so the size of each group is important. However, it is often the case that one side is more effective per member than the other. Militarily, this effectiveness is determined by the choice and design of the weaponry used, and a recognised measure of this effectiveness is the abhorrent term *kill rate*, that is, the rate at which single members of one group can, on average, render members of the other group inactive.

For two groups of equal initial size, the more effective group will, on average, win a battle. But what will occur when group X is numerically larger than group Y but has inferior weaponry? We will describe a simple model that is capable of providing a first answer to this question.

The model is a continuous one: it approximates the actual situation, in which the active group size at any time is an integer, by assuming that the group size is capable of continuous variation. This is a very reasonable approximation if each group has a large number of members.

Let the active sizes of groups X and Y be denoted by x and y, respectively. These sizes vary with time t, so $x(t)$ represents the active size of group X at time t, and similarly for $y(t)$. Suppose that the constant kill rates of the two groups are α for group X, and β for group Y, where α and β are both positive.

We suppose that the rate of reduction of each group is proportional to the size of the other, so

$$\frac{dx}{dt} = -\beta y \quad \text{and} \quad \frac{dy}{dt} = -\alpha x.$$

This pair of equations can be written alternatively as

$$\begin{cases} \dot{x} = -\beta y, \\ \dot{y} = -\alpha x, \end{cases} \tag{1}$$

which is a system of two first-order differential equations. Neither of these equations is soluble directly by the methods of Unit 1, because they are *coupled*; that is, the equation that features the derivative of one of the variables, x say, also includes the other variable y on the right-hand side, and vice versa. In order to solve the first equation for x, we need to know explicitly what the function $y(t)$ is, and similarly for the second equation.

You will appreciate in Section 2 that this system of equations may be solved in terms of the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix},$$

which arises from expressing equations (1) in the matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Indeed, this approach is the central theme of this unit, and although there are also other methods of solution, these will not be discussed further in this unit. However, as a digression, it is worth noting that without solving the system of differential equations, it is possible to deduce an interesting conclusion about such conflicts. Suppose that we multiply both sides of the first equation of (1) by αx, then both sides of the second equation by βy, and finally subtract the resulting equations. This produces

$$\alpha x \dot{x} - \beta y \dot{y} = 0,$$

which, by the chain rule, can also be expressed as

$$\tfrac{1}{2}\alpha \frac{d}{dt}(x^2) - \tfrac{1}{2}\beta \frac{d}{dt}(y^2) = 0.$$

Integration with respect to time then gives

$$\alpha x^2 - \beta y^2 = c,$$

where c is a constant. If the initial sizes of the two groups (at time $t = 0$) are respectively x_0 and y_0, then the value of c is $\alpha x_0^2 - \beta y_0^2$, and we have

$$\alpha x^2 - \beta y^2 = \alpha x_0^2 - \beta y_0^2 \qquad (2)$$

throughout the conflict.

This relationship allows us to predict when two sides are equally matched in a conflict. Neither side wins if both have their size reduced to zero at the same time, that is, if $x = 0$ when $y = 0$. In this case, we must have $\alpha x^2 - \beta y^2 = 0$ throughout the conflict, so at the start $\alpha x_0^2 = \beta y_0^2$. This reasoning led Lanchester to define the *fighting strength* of a force as 'the square of its size multiplied by the kill rate of its individual units'.

Lanchester applied this model to the situation at the Battle of Trafalgar (1805), and was able to demonstrate why Nelson's tactic of splitting the opposition fleet into two parts might have been expected to succeed, even though Nelson had a smaller total number of ships.

According to this definition, a force that outnumbers an adversarial force by only half as much again is more than twice as strong, assuming equal effectiveness on both sides. For any potential conflict, the initial fighting strengths of the two sides can be estimated, and according to equation (2), any difference in these strengths remains constant throughout the conflict. At the end, the size and therefore the strength of the loser is zero, so this difference is equal to the remaining strength of the winning side. From this, the number of survivors can be predicted.

This model is of course very simple, and will not be applicable in exact form for any particular conflict. Nevertheless, the rule of thumb that it provides is an informative one, and is taken into account by, for example, military strategists.

We again stress that the method leading to equation (2) will not be used further in this unit.

1.2 Two-compartment drug model

The administration of clinical drugs is a complex task. The doctor must ensure that the concentration of the drug in the patient's blood remains between certain upper and lower limits. One model that has been developed to assist in understanding the process is given by the following differential equation, in which $c(t)$ represents the concentration of drug in the patient's blood (as a function of time):

We will not go into detail on how the concentration $c(t)$ is defined. A number of choices are possible.

$$\frac{dc}{dt} = \frac{q}{V} - \lambda c.$$

Here the drug enters the bloodstream at a constant rate q, V is the 'apparent volume of distribution', and λ is a constant (which may be interpreted as the proportionate rate at which the kidneys excrete the drug). You are not expected to understand how this equation is derived.

Such a constant rate q of drug input to the bloodstream may be achieved by intravenous infusion, but this requires the patient to be connected to certain apparatus and is consequently inconvenient. A modern alternative is to use slow-release capsules that are taken orally. These capsules gradually dissolve within the stomach and by so doing raise the drug concentration there. The drug reaches the bloodstream from the stomach by a process that may be represented as passing through a membrane that separates the stomach 'compartment' from the bloodstream 'compartment'. Later, the drug is excreted from the kidneys, as before. The whole process is indicated diagrammatically in Figure 2.

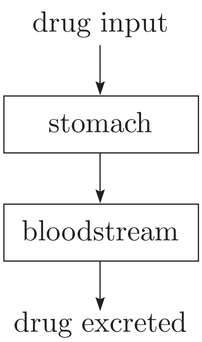

Figure 2 Schematic flow diagram showing how a drug passes through a patient

We will now indicate briefly how to model such a two-compartment situation. We make the following assumptions.

(a) The slow-release capsule has the effect of providing a constant rate of input of the drug to the stomach, until the capsule is completely dissolved.

(b) The concentrations of drug within the stomach compartment and within the bloodstream compartment are each uniform at any instant of time.

(c) The rate at which the drug passes from the stomach to the bloodstream is proportional to the difference in concentrations between them.

(d) The rate of excretion from the bloodstream via the kidneys is (as in the earlier model) proportional to the concentration in the bloodstream.

Assumption (c) is based on *Fick's law*, which is an empirical result stating that 'the amount of material passing through a membrane is proportional to the difference in concentration between the two sides of the membrane'.

Suppose that the drug concentrations in the stomach and bloodstream at time t are denoted by $x(t)$ and $y(t)$, respectively, and consider the period after the slow-release capsule has been swallowed but before it has completely dissolved.

The drug concentration in the stomach will be raised at a constant rate, k_1 say, by Assumption (a), but simultaneously lowered at a rate $k_2(x-y)$, where k_2 is a constant, by Assumption (c). Hence we have

$$\frac{dx}{dt} = k_1 - k_2(x-y).$$

Depending on how the concentrations are defined, we might expect that $k_2 = k_3$ if all the drug leaving the stomach enters the bloodstream, but this is not one of our assumptions.

The drug concentration in the blood will be raised at a rate $k_3(x-y)$, by Assumption (c), but also lowered at a rate k_4y, by Assumption (d) (where k_3 and k_4 are constants). This gives

$$\frac{dy}{dt} = k_3(x-y) - k_4y.$$

The drug concentrations in the two compartments are therefore governed within the model by the pair of differential equations

$$\begin{cases} \dot{x} = -k_2x + \qquad k_2y + k_1, \\ \dot{y} = \quad k_3x - (k_3+k_4)y\,, \end{cases} \tag{3}$$

which can be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -k_2 & k_2 \\ k_3 & -(k_3+k_4) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_1 \\ 0 \end{pmatrix}.$$

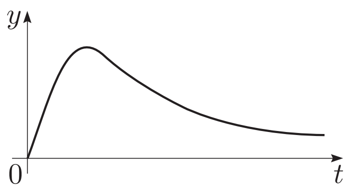

Figure 3 Plot of the drug concentration in the bloodstream, $y(t)$, against time t

Figure 3 sketches the type of behaviour that this model predicts for the drug concentration $y(t)$ in the bloodstream. It is characterised by an absorption phase followed by a steady decline as the drug is eliminated. The graph for the drug concentration $x(t)$ in the stomach is of similar shape. The drug concentrations do not approach zero for large values of t so long as k_1 is non-zero. In reality, a slow-release capsule will eventually dissolve completely, and then the patient must swallow another capsule in order to prevent the drug concentration from falling below some predetermined level.

The eigenvalues and eigenvectors of the matrix of coefficients

$$\begin{pmatrix} -k_2 & k_2 \\ k_3 & -(k_3+k_4) \end{pmatrix}$$

You will study methods of solution in the following two sections.

provide a starting point for solving the system of equations (3). However, the form of this system differs from the one derived in the previous subsection. The presence of the term k_1 on the right-hand side of the first equation, which is non-zero and does not depend on x or y, makes this system *inhomogeneous*, whereas system (1) in Subsection 1.1 is *homogeneous*, since it has no such term present. You will see shortly that, as with ordinary differential equations, the solution of an inhomogeneous system is related to, but slightly more complicated than, that of a homogeneous system.

However, system (3) becomes homogeneous when modelling the system after the slow-release capsule dissolves completely and is not replaced. This corresponds to putting $k_1 = 0$.

1.3 Motion of a ball-bearing in a bowl

The previous examples each led to a system of two first-order linear differential equations, homogeneous in the case of the conflict model, and inhomogeneous for the two-compartment drug model.

We now look at an example where the system that arises involves *second-order* linear differential equations.

When a ball-bearing (that is, a small metallic ball) is placed in a bowl and set in motion, it tends to roll along the surface of the bowl for some considerable time before coming to rest. We will now indicate how this motion might be modelled for a generalised 'bowl'.

To simplify matters, we model the ball-bearing as a particle (so that the rolling aspect of the motion is ignored), and assume that the surface of the bowl is frictionless and that there is no air resistance. To describe the surface, suppose that a three-dimensional Cartesian coordinate system is chosen with the (x, y)-plane horizontal, the z-axis vertical, and the lowest point of the bowl's surface at the origin, $(0, 0, 0)$. We assume further that in the vicinity of the origin, the surface may be described by an equation of the form

$$z = \tfrac{1}{2}ax^2 + bxy + \tfrac{1}{2}cy^2, \tag{4}$$

where the constants a, b, c satisfy the conditions $ac - b^2 > 0$ and $a > 0$. The first condition here ensures that the surface under consideration is a *paraboloid*, for which any vertical cross-section through the origin is a parabola, while the second condition means that these cross-sectional parabolas are all concave upwards rather than downwards. The surface is sketched in Figure 4.

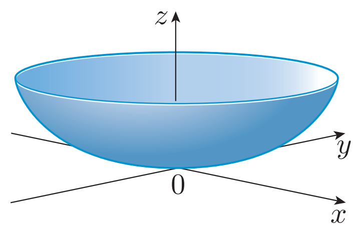

Figure 4 Surface of a bowl, inside which a ball-bearing is set in motion

It may seem restrictive to specify the particular form of surface given by equation (4). However, it turns out that many other functions $z = f(x, y)$ with a minimum value $z = 0$ at $(0, 0)$ can be approximated satisfactorily by a function of the form (4) near to the origin.

You will be able to appreciate this point more fully after studying the topic of two-variable Taylor polynomial approximations in Unit 7.

Since the surface is assumed to be frictionless and there is no air resistance, the only forces that act on the particle are its weight $\mathbf{W} = -mg\mathbf{k}$ (where m is its mass and g is the acceleration due to gravity) and the normal reaction $\mathbf{N}$ from the surface. In order to describe the latter, we need to be able to write down a vector that is normal to the surface given by equation (4) at any point (x, y, z) on the surface. One such vector, pointing inwards, is

$$-(ax + by)\mathbf{i} - (bx + cy)\mathbf{j} + \mathbf{k}.$$

We ask you to take this on trust.

Hence the normal reaction is

$$\mathbf{N} = C(-(ax + by)\mathbf{i} - (bx + cy)\mathbf{j} + \mathbf{k}),$$

where C is some positive quantity. Newton's second law then gives

Newton's second law is considered in Unit 3.

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{N} + \mathbf{W} \\ &= -C(ax + by)\mathbf{i} - C(bx + cy)\mathbf{j} + (C - mg)\mathbf{k}, \end{aligned}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of a point on the surface, relative to the origin. Resolving in the $\mathbf{i}$-, $\mathbf{j}$- and $\mathbf{k}$-directions, we obtain

$$\begin{aligned} m\ddot{x} &= -C(ax + by), \\ m\ddot{y} &= -C(bx + cy), \\ m\ddot{z} &= C - mg. \end{aligned}$$

On eliminating the quantity C between these equations, and dividing through by m, we have

$$\ddot{x} = -(ax+by)(g+\ddot{z}) \quad \text{and} \quad \ddot{y} = -(bx+cy)(g+\ddot{z}).$$

For motions that do not move too far from the lowest point of the bowl, the vertical component of acceleration, $\ddot{z}$, will be small in magnitude compared with g, so to a good approximation the horizontal motion of the particle is governed by the pair of equations

$$\ddot{x} = -g(ax+by),$$
$$\ddot{y} = -g(bx+cy).$$

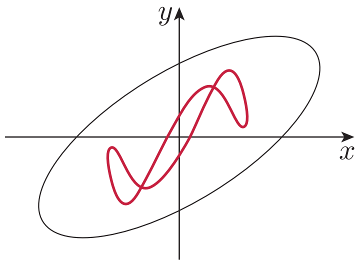

Figure 5 The projection of the ball-bearing's motion in the (x,y)-plane, showing an example of a path that might be observed

This motion is what would be observed if you looked down onto the surface from some distance above it, as indicated in Figure 5.

For example, if the surface is given by

$$z = 0.25x^2 - 0.4xy + 0.25y^2,$$

then the corresponding equations for horizontal motion (taking $g \simeq 10\,\mathrm{m\,s^{-2}}$) are

$$\begin{cases} \ddot{x} = -5x + 4y, \\ \ddot{y} = 4x - 5y. \end{cases} \tag{5}$$

These differential equations are linear, constant-coefficient, homogeneous and second-order. You will see them solved, and the possible motions interpreted, in Subsection 4.2. As you will see, their solutions can be expressed in terms of the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix},$$

which arises from expressing the pair of equations (5) in matrix form as

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

2 First-order homogeneous systems

In this unit we discuss systems of equations involving either two functions of time $x(t)$ and $y(t)$, or three functions of time $x(t)$, $y(t)$ and $z(t)$ (which we abbreviate to x, y and z). For larger systems it is often convenient to denote the functions by subscripts, for example $x_1, x_2, \ldots, x_n$, but we do not use this notation here because we wish to use subscripts later for another purpose. Throughout, we write

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

as appropriate.

In Unit 4 you saw that any system of linear equations can be written in matrix form. For example, the equations

$$\begin{aligned} 3x + 2y &= 5, \\ x + 4y &= 5, \end{aligned}$$

can be written in matrix form as

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix},$$

that is, as $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

In a similar way, we can write systems of linear differential equations in matrix form. To see what is involved, consider the system

$$\begin{cases} \dot{x} = 3x + 2y + 5t, \\ \dot{y} = x + 4y + 5, \end{cases} \tag{6}$$

Here x and y are functions of t.

which can be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5t \\ 5 \end{pmatrix},$$

that is, as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} 5t \\ 5 \end{pmatrix}.$$

Here $\mathbf{h}$ is generally a function of t, but $\mathbf{A}$ is constant.

Note that $\dot{\mathbf{x}} = (\dot{x} \quad \dot{y})^T$.

We can similarly represent systems of three, or more, linear differential equations in matrix form. For example, the system

$$\begin{aligned} \dot{x} &= 3x + 2y + 2z + e^t, \\ \dot{y} &= 2x + 2y + 2e^t, \\ \dot{z} &= 2x + 4z, \end{aligned}$$

can be written in matrix form as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, where

Note that here $\dot{\mathbf{x}} = (\dot{x} \quad \dot{y} \quad \dot{z})^T$.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} e^t \\ 2e^t \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^t. \tag{7}$$

Homogeneous and inhomogeneous equations

A matrix equation of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ is said to be **homogeneous** if $\mathbf{h} = \mathbf{0}$, and **inhomogeneous** otherwise.

Note that in an inhomogeneous equation, some, *but not all*, of the components of $\mathbf{h}$ may be 0.

For example, the matrix equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is homogeneous, whereas the matrix equations represented by (6) and (7) are inhomogeneous.

Exercise 1

Write each of the following systems in matrix form, and classify it as homogeneous or inhomogeneous.

(a) $\begin{cases} \dot{x} = 2x + y + 1 \\ \dot{y} = x \qquad\quad - 2 \end{cases}$ (b) $\begin{cases} \dot{x} = y \\ \dot{y} = t \end{cases}$ (c) $\begin{cases} \dot{x} = 5x \\ \dot{y} = x + 2y + z \\ \dot{z} = x + y + 2z \end{cases}$

In this section we present an algebraic method for solving homogeneous systems of linear first-order differential equations with constant coefficients. In Section 3 we show how this method can be adapted to inhomogeneous systems.

2.1 Eigenvalue method

Our intention is to develop a method for finding the general solution of any homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, however large, but we start with a 2×2 matrix $\mathbf{A}$. We begin by solving a system of differential equations using a non-matrix method, then we look at the same example again, but this time the emphasis is on the links to matrices.

Suppose that we wish to find a solution of the system

$$\dot{x} = 3x + 2y, \tag{8}$$

$$\dot{y} = x + 4y. \tag{9}$$

Equation (9) gives $x = \dot{y} - 4y$, so $\dot{x} = \ddot{y} - 4\dot{y}$. Substituting into equation (8), we obtain

$$\ddot{y} - 4\dot{y} = 3(\dot{y} - 4y) + 2y,$$

which simplifies to give

$$\ddot{y} - 7\dot{y} + 10y = 0.$$

Using the methods of Unit 1, we see that the auxiliary equation is

$$\lambda^2 - 7\lambda + 10 = 0,$$

with roots $\lambda = 5$ and $\lambda = 2$, from which we have $y = \alpha e^{5t} + \beta e^{2t}$ (for arbitrary constants α and β). Having found y, we can substitute for y and $\dot{y}$ in equation (9) and obtain $x = \alpha e^{5t} - 2\beta e^{2t}$, so the solution is

$$x = \alpha e^{5t} - 2\beta e^{2t} \quad \text{and} \quad y = \alpha e^{5t} + \beta e^{2t}. \tag{10}$$

Thus we have found the general solution of our original system of two equations, and this solution contains two arbitrary constants, as expected (one for each derivative).

General and particular solutions

The **general solution** of a system of n linear constant-coefficient first-order differential equations is a collection of all possible solutions of the system of equations.

A **particular solution** of a system of n linear constant-coefficient first-order differential equations is a solution containing no arbitrary constants and satisfying given conditions.

Compare the definitions for the single equation cases in Unit 1, Subsection 1.1.

Usually, the general solution of a system of n first-order differential equations contains n arbitrary constants.

The above method does not extend very well to larger systems and will no longer be used in what follows. However, notice that the above general solution is a linear combination of exponential terms of the form $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ (where in the above case $\lambda_1 = 5$ and $\lambda_2 = 2$). This suggests an alternative approach to solving such systems of equations, and this will turn out to be the method forming the subject of this unit. In the above case we have a solution $x = e^{5t}$, $y = e^{5t}$ corresponding to choosing $\alpha = 1$ and $\beta = 0$, and another solution $x = -2e^{2t}$, $y = e^{2t}$ corresponding to choosing $\alpha = 0$ and $\beta = 1$. Thus the general solution is a linear combination of much simpler solutions, and this suggests that we might be able to solve such systems by looking for these simpler solutions. So the idea is to search for solutions of the form $x = Ce^{\lambda t}$, $y = De^{\lambda t}$, and then form linear combinations of such solutions in order to find the general solution. There is only one minor problem – we have to convince ourselves that a linear combination of two solutions is itself a solution.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the matrix equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$; then $\dot{\mathbf{x}}_1 = \mathbf{A}\mathbf{x}_1$ and $\dot{\mathbf{x}}_2 = \mathbf{A}\mathbf{x}_2$. If α and β are arbitrary constants, then

$$\frac{d}{dt}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha\dot{\mathbf{x}}_1 + \beta\dot{\mathbf{x}}_2 = \alpha\mathbf{A}\mathbf{x}_1 + \beta\mathbf{A}\mathbf{x}_2 = \mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2),$$

so $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is indeed a solution. This is a particular case of a more general result known as the *principle of superposition* that you will meet later (in Theorem 2).

You met the principle of superposition in the case of second-order differential equations in Unit 1.

We apply the above technique in the following example.

Example 1

(a) For the following pair of equations, find a solution of the form $x = Ce^{\lambda t}$, $y = De^{\lambda t}$, and hence find the general solution:

$$\begin{cases} \dot{x} = 3x + 2y, \\ \dot{y} = x + 4y. \end{cases} \tag{11}$$

(b) Find the particular solution for which $x(0) = 1$ and $y(0) = 4$.

Solution

(a) We investigate possible solutions of the form

$$x = Ce^{\lambda t}, \quad y = De^{\lambda t},$$

where C and D are constants. Since $\dot{x} = C\lambda e^{\lambda t}$ and $\dot{y} = D\lambda e^{\lambda t}$, substituting the expressions for x and y into equations (11) gives

$$\begin{cases} C\lambda e^{\lambda t} = 3Ce^{\lambda t} + 2De^{\lambda t}, \\ D\lambda e^{\lambda t} = Ce^{\lambda t} + 4De^{\lambda t}. \end{cases}$$

$e^{\lambda t}$ can never be zero.

Cancelling the $e^{\lambda t}$ terms, we obtain the simultaneous linear equations

$$\begin{cases} C\lambda = 3C + 2D, \\ D\lambda = C + 4D, \end{cases} \tag{12}$$

Note that equations (12) can be expressed in matrix form:
$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \lambda \begin{pmatrix} C \\ D \end{pmatrix}.$$

which can be rearranged to give

$$\begin{cases} (3-\lambda)C + 2D = 0, \\ C + (4-\lambda)D = 0. \end{cases} \tag{13}$$

See the summary of results for non-invertible matrices in Unit 5.

This system of linear equations has non-zero solutions for C and D only if the determinant of the coefficient matrix is 0, that is, if

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 0.$$

Expanding this determinant gives $(3-\lambda)(4-\lambda) - 2 = 0$, which simplifies to $\lambda^2 - 7\lambda + 10 = 0$, and we deduce that $\lambda = 5$ or $\lambda = 2$. We now substitute these values of λ in turn into equations (13).

- For $\lambda = 5$, equations (13) become

  $$\begin{aligned} -2C + 2D &= 0, \\ C - D &= 0, \end{aligned}$$

  You will see later that choosing any non-zero value will work.

  which reduce to the single equation $C = D$. Choosing $C = D = 1$ provides us with the solution $x = e^{5t}$, $y = e^{5t}$.

- For $\lambda = 2$, equations (13) become

  $$\begin{aligned} C + 2D &= 0, \\ C + 2D &= 0, \end{aligned}$$

  which reduce to the single equation $C = -2D$. Choosing $C = -2$ and $D = 1$ provides us with another solution, $x = -2e^{2t}$, $y = e^{2t}$.

In vector form, these solutions are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}.$$

Linear independence of vectors is defined in Unit 5.

(Notice that these solution vectors are linearly independent.)

Thus the general solution can be written in the form

Here we are using the principle of superposition.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}. \tag{14}$$

The fact that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent ensures that this expression contains *two* arbitrary constants (in other words, one term cannot be absorbed into the other). Note that this general solution is identical to that expressed in equations (10), found using the earlier method.

We do not prove that this is the case.

(b) We seek the solution for which $x = 1$ and $y = 4$ when $t = 0$. Substituting these values into equation (14) gives the simultaneous linear equations

$$\begin{aligned} 1 &= \alpha - 2\beta, \\ 4 &= \alpha + \beta. \end{aligned}$$

Solving these equations gives $\alpha = 3$, $\beta = 1$, so the required particular solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{5t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}.$$

You may have noticed that the method in Example 1(a) is similar to that used for calculating eigenvalues and eigenvectors in Unit 5. This similarity is not coincidental. Indeed, if we write the system of differential equations (11) in matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, that is,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

then the numbers $\lambda = 5$ and $\lambda = 2$ arising in the general solution (14) turn out to be the *eigenvalues* of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix},$$

and the vectors $(1 \quad 1)^T$ and $(-2 \quad 1)^T$ appearing in the general solution are corresponding *eigenvectors*.

You may like to check that

$$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{A}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

To see why this happens, consider a general system of homogeneous linear constant-coefficient first-order differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, of any size. Suppose that there is a solution of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is a constant column vector. Then $\dot{\mathbf{x}} = \mathbf{v}\lambda e^{\lambda t}$, and the system of differential equations becomes $\mathbf{v}\lambda e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}$. Dividing the latter equation by $e^{\lambda t}$ (which is never 0) and rearranging, we have

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, and λ is the corresponding eigenvalue.

Conversely, we have the following result.

Theorem 1

If λ is an eigenvalue of the matrix $\mathbf{A}$ corresponding to an eigenvector $\mathbf{v}$, then $\mathbf{x} = \mathbf{v}e^{\lambda t}$ is a solution of the system of differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

This theorem holds because if $\mathbf{x} = \mathbf{v}e^{\lambda t}$, then $\dot{\mathbf{x}} = \lambda\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{x}$.

Example 2

A particle moves in the (x, y)-plane in such a way that its position (x, y) at any time t satisfies the simultaneous differential equations

$$\dot{x} = x + 4y,$$
$$\dot{y} = x - 2y.$$

Find the position (x, y) at time t if $x(0) = 2$ and $y(0) = 3$.

Solution

Note that the differential equations can be written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where the matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}.$$

These eigenvectors and eigenvalues were found in Unit 5, Exercise 9(a).

The eigenvectors of $\mathbf{A}$ are $(4 \quad 1)^T$ with corresponding eigenvalue 2, and $(1 \quad -1)^T$ with corresponding eigenvalue -3. The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t},$$

where α and β are arbitrary constants.

Since $x(0) = 2$ and $y(0) = 3$, we have, on putting $t = 0$,

$$2 = 4\alpha + \beta,$$
$$3 = \alpha - \beta.$$

Solving these equations gives $\alpha = 1$, $\beta = -2$, so the required particular solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} e^{-3t}.$$

In the above example the particle starts at the point $(2, 3)$ when $t = 0$, and follows a certain path as t increases. The ultimate direction of this path is easy to determine because e^{-3t} is small when t is large, so we have $(x \quad y)^T \simeq (4 \quad 1)^T e^{2t}$, that is, $x \simeq 4e^{2t}$, $y \simeq e^{2t}$, so $x \simeq 4y$ and $y \simeq 0.25x$. Thus the solution approaches the line $y = 0.25x$ as t increases.

Exercise 2

Use the above method to solve the system of differential equations

$$\dot{x} = 5x + 2y,$$
$$\dot{y} = 2x + 5y,$$

given that $x = 4$ and $y = 0$ when $t = 0$.

(The eigenvalues of $\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ are 7 and 3, corresponding to eigenvectors $(1 \quad 1)^T$ and $(1 \quad -1)^T$, respectively.)

The above method works equally well for larger systems. Consider the following example of a system of three differential equations.

Example 3

Find the general solution of the system of differential equations

$$\begin{aligned} \dot{x} &= 3x + 2y + 2z, \\ \dot{y} &= 2x + 2y, \\ \dot{z} &= 2x \qquad\;\; + 4z. \end{aligned}$$

Solution

The matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

The eigenvectors of $\mathbf{A}$ are $(2 \quad 1 \quad 2)^T$, $(1 \quad 2 \quad -2)^T$ and $(-2 \quad 2 \quad 1)^T$, corresponding to the eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$ and $\lambda_3 = 0$, respectively. The general solution is therefore

These eigenvectors and eigenvalues were found in Unit 5, Example 8.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{6t} + \beta \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} e^{3t} + \gamma \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix},$$

Note that the last term on the right-hand side corresponds to the term in $e^{\lambda_3 t} = e^{0t} = 1$.

where α, β and γ are arbitrary constants.

Exercise 3

A particle moves in three-dimensional space in such a way that its position (x, y, z) at any time t satisfies the simultaneous differential equations

$$\begin{aligned} \dot{x} &= 5x, \\ \dot{y} &= \;\; x + 2y + \;\; z, \\ \dot{z} &= \;\; x + \;\; y + 2z. \end{aligned}$$

Find the position (x, y, z) at time t if $x(0) = 4$, $y(0) = 6$ and $z(0) = 0$.

(The eigenvectors of $\begin{pmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ are $(2 \quad 1 \quad 1)^T$, $(0 \quad 1 \quad 1)^T$ and $(0 \quad 1 \quad -1)^T$, corresponding to eigenvalues 5, 3 and 1, respectively.)

The above method can be used to solve any system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ of linear constant-coefficient first-order differential equations for which the matrix $\mathbf{A}$ has distinct real eigenvalues. We summarise the procedure as follows.

Procedure 1 Solving a system of equations when there are distinct real eigenvalues

To solve a system of linear constant-coefficient first-order differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix with distinct real eigenvalues, carry out the following steps.

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and a corresponding set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
2. Write down the general solution in the form
$$\mathbf{x} = C_1\mathbf{v}_1e^{\lambda_1 t} + C_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + C_n\mathbf{v}_ne^{\lambda_n t}, \tag{15}$$
where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

In the next subsection we investigate what happens when the eigenvalues of the matrix $\mathbf{A}$ are not distinct, or when they are complex numbers.

2.2 Two variations

This result is not obvious; we ask you to take it on trust.

The above method relies on the fact that eigenvectors corresponding to distinct eigenvalues are linearly independent, which ensures that the solution given in equation (15) contains n arbitrary constants hence is the required general solution. If the eigenvalues are not distinct, then we may not be able to write down n linearly independent eigenvectors, so when we attempt to construct the solution for $\mathbf{x}$ given in equation (15), we find that it contains too few constants. In the following example we are able to find a sufficient number of eigenvectors in spite of the fact that an eigenvalue is repeated.

Example 4

Find the general solution of the system of differential equations

$$\begin{aligned}\dot{x} &= 5x \phantom{{}+2y} + 3z,\\ \dot{y} &= 3x + 2y + 3z,\\ \dot{z} &= -6x \phantom{{}+2y} - 4z.\end{aligned}$$

Solution

The matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 3 \\ 3 & 2 & 3 \\ -6 & 0 & -4 \end{pmatrix}.$$

Using the techniques of Unit 5, we can calculate the eigenvalues of $\mathbf{A}$. They are $\lambda = -1$ and $\lambda = 2$ (repeated), corresponding to eigenvectors $(1 \quad 1 \quad -2)^T$ and $(0 \quad 1 \quad 0)^T$, respectively.

You need to understand what goes wrong before we show you how to put it right, so suppose that we try to follow Procedure 1, and let

$$\mathbf{x} = C_1(1 \quad 1 \quad -2)^T e^{-t} + C_2(0 \quad 1 \quad 0)^T e^{2t}.$$

This is certainly a solution, but it is not the general solution because it contains only two arbitrary constants and we require three.

The answer to our difficulty lies in our method of calculating the eigenvectors when the eigenvalue is repeated. The eigenvector equations are

$$\begin{aligned}(5-\lambda)x \quad\quad\quad\quad &+ \quad\quad 3z = 0,\\ 3x + (2-\lambda)y &+ \quad\quad 3z = 0,\\ -6x \quad\quad\quad\quad &+ (-4-\lambda)z = 0.\end{aligned}$$

For $\lambda = 2$, the eigenvector equations become

$$3x + 3z = 0, \quad 3x + 3z = 0 \quad \text{and} \quad -6x - 6z = 0.$$

All three equations give $z = -x$, but there is no restriction on y. It follows that any vector of the form $(k \quad l \quad -k)^T$ is an eigenvector corresponding to $\lambda = 2$, where k and l are any real numbers, not both zero. (In particular, the vector $(0 \quad 1 \quad 0)^T$ mentioned above is an eigenvector corresponding to $\lambda = 2$.)

Remember that we need to find three linearly independent eigenvectors, and it might appear that we have found only two, namely $(1 \quad 1 \quad -2)^T$ corresponding to $\lambda = -1$ and $(k \quad l \quad -k)^T$ corresponding to $\lambda = 2$. However, the numbers k and l are arbitrary, so there are infinitely many vectors in the second category; our task is to choose two that are linearly independent. If we write

This approach to finding eigenvectors corresponding to repeated eigenvalues was touched on in Unit 5, Exercise 31.

$$(k \quad l \quad -k)^T = k(1 \quad 0 \quad -1)^T + l(0 \quad 1 \quad 0)^T,$$

then we see at once that $(1 \quad 0 \quad -1)^T$ and $(0 \quad 1 \quad 0)^T$ are suitable candidates (corresponding to $k = 1$, $l = 0$ and $k = 0$, $l = 1$, respectively, but of course other combinations are also possible).

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} e^{-t} + \left(C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) e^{2t},$$

where C_1, C_2 and C_3 are arbitrary constants, is a solution, by Theorem 1 and the principle of superposition. The fact that the three eigenvectors are linearly independent ensures that there are three arbitrary constants, and hence that this is the general solution.

Exercise 4

A particle moves in three-dimensional space in such a way that its position (x, y, z) at any time t satisfies the simultaneous differential equations

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= y, \\ \dot{z} &= x. \end{aligned}$$

Find the position (x, y, z) at time t if $x(0) = 7$, $y(0) = 5$ and $z(0) = 1$.

(The eigenvalues of $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ are $\lambda = -1$, corresponding to an eigenvector $(1 \quad 0 \quad -1)^T$, and $\lambda = 1$ (repeated), corresponding to an eigenvector $(k \quad l \quad k)^T$.)

In Example 4 we were able to determine the general solution even though there were only two distinct eigenvalues. This is because we were able to find three linearly independent eigenvectors. We now consider a situation in which there are too few linearly independent eigenvectors.

Suppose that we try to apply the above method to find the general solution of the system of differential equations

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= x + y. \end{aligned}$$

The matrix of coefficients is

Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are simply the diagonal entries.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and the eigenvalues are $\lambda = 1$ (repeated). We now substitute this eigenvalue into the eigenvector equations

$$\begin{aligned} (1-\lambda)x \qquad\qquad &= 0, \\ x + (1-\lambda)y &= 0. \end{aligned}$$

For $\lambda = 1$, the eigenvector equations become $0 = 0$ and $x = 0$. The first equation tells us nothing; the second equation gives $x = 0$, but imposes no restriction on y. It follows that $(0 \quad k)^T$ is an eigenvector corresponding to $\lambda = 1$, for any non-zero value of k. Thus choosing $k = 1$, we have the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t.$$

But this is clearly not the general solution, since it has only one arbitrary constant instead of two; the procedure has failed.

Before extending our matrix procedure to cover the above case, it is illuminating to solve the given system of differential equations directly.

The first equation, $\dot{x} = x$, has solution $x = De^t$, where D is an arbitrary constant. Substituting this solution into the second equation gives $\dot{y} = y + De^t$, which can be solved by the integrating factor method of Unit 1 to give $y = Ce^t + Dte^t$. Thus the solution for which we are searching is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{C\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{eigenvector}} e^t + \underbrace{D\begin{pmatrix} 1 \\ t \end{pmatrix}}_{\text{notice the } t \text{ here}} e^t,$$

and this is the general solution since it contains two arbitrary constants.

There are *two* arbitrary constants since $(0 \quad 1)^T$ and $(1 \quad t)^T$ are linearly independent.

The above solution provides a clue to a general method. The origins of the first term on the right-hand side are clear – it corresponds to an eigenvector – but let us look more closely at the second term on the right-hand side. We have

$$D\begin{pmatrix} 1 \\ t \end{pmatrix} e^t = D\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{eigenvector}} te^t + D\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t = D\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t.$$

This suggests that if we wish to solve the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ for which $\mathbf{A}$ has a repeated eigenvalue λ, giving rise to too few eigenvectors, it may be helpful to search for solutions of the form

$$\mathbf{x} = (\mathbf{v}t + \mathbf{b})e^{\lambda t},$$

where $\mathbf{v}$ is an eigenvector corresponding to λ, and $\mathbf{b}$ is to be determined.

Let us suppose that our system has a solution of this form, and examine the consequences. Substituting this proposed solution into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ gives

$$\lambda(\mathbf{v}t + \mathbf{b})e^{\lambda t} + \mathbf{v}e^{\lambda t} = \mathbf{A}(\mathbf{v}t + \mathbf{b})e^{\lambda t},$$

or, on dividing by $e^{\lambda t}$ and rearranging,

$$\mathbf{A}(\mathbf{v}t + \mathbf{b}) - \lambda(\mathbf{v}t + \mathbf{b}) = \mathbf{v},$$

and a further rearrangement leads to

$$(\mathbf{A}\mathbf{v} - \lambda\mathbf{v})t + (\mathbf{A}\mathbf{b} - \lambda\mathbf{b}) = \mathbf{v}.$$

The first term disappears because $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ (so $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$), and we are left with $(\mathbf{A} - \lambda\mathbf{I})\mathbf{b} = \mathbf{v}$. If this equation can be solved for $\mathbf{b}$, then we will have found a solution of the required form.

For example, in the above case we had $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\lambda = 1$, so

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus if $\mathbf{b} = (b_1 \quad b_2)^T$, the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{b} = \mathbf{v}$ becomes

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Any solution of this equation for b_1 and b_2 is acceptable. In fact, this equation gives just $b_1 = 1$ (with no condition on b_2), and we can take $\mathbf{b}$ to be $(1 \quad 0)^T$, giving the solution

The value of b_2 is irrelevant, since it is absorbed into the arbitrary constant.

$$\left(\begin{pmatrix}0\\1\end{pmatrix}t+\begin{pmatrix}1\\0\end{pmatrix}\right)e^t=\begin{pmatrix}1\\t\end{pmatrix}e^t.$$

Now we have two linearly independent solutions of our system of differential equations, namely $(0 \quad 1)^T e^t$ and $(1 \quad t)^T e^t$, so the general solution is

$$\begin{pmatrix}x\\y\end{pmatrix}=C\underbrace{\begin{pmatrix}0\\1\end{pmatrix}e^t}_{\substack{\text{solution}\\ \mathbf{v}e^{\lambda t}}}+D\underbrace{\begin{pmatrix}1\\t\end{pmatrix}e^t}_{\substack{\text{solution}\\ (\mathbf{v}t+\mathbf{b})e^{\lambda t}}},$$

as we found before.

Procedure 2 Solving a system of equations when there are repeated real eigenvalues

To solve a system of linear constant-coefficient first-order differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix with some repeated real eigenvalues (where any eigenvalue is repeated at most once), carry out the following steps.

This procedure covers only the case where an eigenvalue is repeated once. It could be extended to cover the case where an eigenvalue is repeated several times, but we choose not to generalise.

1. For the non-repeated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, write down the set S of solutions of the form $\mathbf{v}_i e^{\lambda_i t}$.
2. Examine the eigenvector equations corresponding to each repeated eigenvalue, and attempt to construct two linearly independent eigenvectors to add two solutions to S.

   If this fails, then for each repeated eigenvalue λ_i that gives rise to only one eigenvector $\mathbf{v}_i$, construct a solution $(\mathbf{v}_i t + \mathbf{b}_i)e^{\lambda_i t}$ for which $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{b}_i = \mathbf{v}_i$, and add this and the solution $\mathbf{v}_i e^{\lambda_i t}$ to the set S.
3. If S contains n linearly independent solutions, then the general solution of the system of differential equations is an arbitrary linear combination of the solutions in S.

This arbitrary linear combination will contain n arbitrary constants.

Exercise 5

A particle moves in the plane in such a way that its position (x, y) at any time t satisfies the simultaneous differential equations

$$\begin{aligned}\dot{x} &= 2x + 3y,\\ \dot{y} &= 2y.\end{aligned}$$

Find the position (x, y) of the particle at time t if its position is $(4, 3)$ at time $t = 0$.

So far, all our examples and exercises have involved *real* eigenvalues. We now investigate what happens when the characteristic equation has at least one complex root (giving a *complex eigenvalue*). In fact, since the arguments leading to Procedure 1 did not rely on the eigenvalues being real, Procedure 1 applies *whenever* the eigenvalues are distinct – it does not matter whether they are real or complex. However, using Procedure 1 for complex eigenvalues leads to a complex-valued solution involving complex arbitrary constants. For a system of differential equations with real coefficients, we would generally want real-valued solutions. So here we see how to adapt Procedure 1 to obtain a real-valued solution when some of the eigenvalues are complex.

For matrices that are real (ours always are), complex roots occur as complex conjugate pairs $a \pm bi$.

We begin with an example that is simple enough for us to be able to apply a direct method in order to find the general solution. We then solve the system again by applying a matrix method based on Procedure 1, which has the advantage that it can be extended to larger systems of equations.

Example 5

Solve the system of differential equations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases} \tag{16}$$

Solution

If we differentiate the first equation, giving $\ddot{x} = \dot{y}$, and substitute for $\dot{y}$ in the second equation, we obtain $\ddot{x} = -x$, so $\ddot{x} + x = 0$. This second-order differential equation has auxiliary equation $\lambda^2 + 1 = 0$, so the general solution is $x = C\cos t + D\sin t$. Therefore $y = \dot{x} = -C\sin t + D\cos t$. Thus the general solution of equations (16) is

See Unit 1.

$$\begin{pmatrix} x \\ y \end{pmatrix} = C\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + D\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Note that if we want *real* solutions, then C and D must be real.

Now we will obtain the same general solution using a matrix method based on Procedure 1.

The matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 + 1 = 0$, giving the complex eigenvalues $\lambda = i$ and $\lambda = -i$. The eigenvector equations are

$$-\lambda x + y = 0 \quad \text{and} \quad -x - \lambda y = 0.$$

- For $\lambda = i$, the eigenvector equations become

  $$-ix + y = 0 \quad \text{and} \quad -x - iy = 0,$$

  which reduce to the single equation $y = ix$. It follows that $(1 \quad i)^T$ is an eigenvector corresponding to $\lambda = i$.

Note that the first equation is i times the second equation.

- For $\lambda = -i$, the eigenvector equations become

  $$ix + y = 0 \quad \text{and} \quad -x + iy = 0,$$

  which reduce to the single equation $y = -ix$. It follows that $(1 \quad -i)^T$ is an eigenvector corresponding to $\lambda = -i$.

Here, the second equation is i times the first equation.

Note that λ and $\mathbf{v}$ occur in complex conjugate pairs.

(Recall that the complex conjugate of a vector $\mathbf{v}$ is the vector $\overline{\mathbf{v}}$ whose elements are the complex conjugates of the respective elements of $\mathbf{v}$.)

Now since Procedure 1 works for complex as well as real eigenvalues, the general solution of the given system of differential equations can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + D \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it},$$

where C and D are arbitrary *complex* constants.

If we are interested only in *real-valued* solutions, then we need to rewrite the above solution in such a way as to eliminate the terms involving i. We do this by using Euler's formula, which gives

You saw Euler's formula used in a similar way in Unit 1.

$$e^{it} = \cos t + i \sin t \quad \text{and} \quad e^{-it} = \cos t - i \sin t$$

(using $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$). Then

$$\begin{aligned} x &= C(\cos t + i \sin t) + D(\cos t - i \sin t) \\ &= (C + D) \cos t + (Ci - Di) \sin t, \\ y &= Ci(\cos t + i \sin t) - Di(\cos t - i \sin t) \\ &= (Ci - Di) \cos t - (C + D) \sin t. \end{aligned}$$

Writing $\alpha = C + D$ and $\beta = Ci - Di$, we have

Since C and D are arbitrary complex constants, so are α and β. Note also that α and β will be real if C and D are complex conjugates.

$$x = \alpha \cos t + \beta \sin t \quad \text{and} \quad y = \beta \cos t - \alpha \sin t.$$

Thus we can write the general solution of equations (16) as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \beta \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

where α and β are arbitrary complex constants. For real-valued solutions we must use only real values for α and β.

In Example 5, notice that

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

so

$$\operatorname{Re}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} \right) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \operatorname{Im}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} \right) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

and these are the expressions that appear in our final solution.

In general, complex eigenvalues of a real matrix $\mathbf{A}$ occur in complex conjugate pairs λ and $\overline{\lambda}$, with corresponding complex conjugate eigenvectors $\mathbf{v}$ and $\overline{\mathbf{v}}$. These give rise to two complex solutions, $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda} t}$, which contribute

$$C\mathbf{v}e^{\lambda t} + D\overline{\mathbf{v}}e^{\overline{\lambda} t}$$

to the general solution (where C and D are arbitrary complex constants). To obtain a real-valued solution, this expression can be rewritten in the form

$$\alpha\,\mathrm{Re}(\mathbf{v}e^{\lambda t}) + \beta\,\mathrm{Im}(\mathbf{v}e^{\lambda t}),$$

where α and β are arbitrary real constants.

The components of $\mathrm{Re}(\mathbf{v}e^{\lambda t})$ and $\mathrm{Im}(\mathbf{v}e^{\lambda t})$ are sinusoidal functions.

Procedure 3 Solving a system of equations when there are complex eigenvalues

To obtain a real-valued solution of a system of linear constant-coefficient first-order differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix with distinct eigenvalues, some of which are complex (occurring in complex conjugate pairs λ and $\overline{\lambda}$, with corresponding complex conjugate eigenvectors $\mathbf{v}$ and $\overline{\mathbf{v}}$), carry out the following steps.

If any of the real eigenvalues are repeated, this procedure will need to be adapted as in Procedure 2.

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and a corresponding set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
2. Write down the general solution in the form
$$\mathbf{x} = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + C_n\mathbf{v}_n e^{\lambda_n t}, \tag{17}$$
where $C_1, C_2, \ldots, C_n$ are arbitrary complex constants.
3. Replace the terms $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ appearing in the general solution with $\mathrm{Re}(\mathbf{v}e^{\lambda t})$ and $\mathrm{Im}(\mathbf{v}e^{\lambda t})$.

The general solution will then be real-valued for real $C_1, C_2, \ldots, C_n$.

Example 6

Solve the system of differential equations

$$\dot{x} = 3x - y,$$
$$\dot{y} = 2x + y,$$

given that $x = 3$ and $y = 1$ when $t = 0$.

Solution

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$.

The eigenvalues of $\mathbf{A}$ are $\lambda = 2 + i$ and $\overline{\lambda} = 2 - i$, corresponding to eigenvectors $\mathbf{v} = (1 \quad 1 - i)^T$ and $\overline{\mathbf{v}} = (1 \quad 1 + i)^T$, respectively. So the general solution can be written as

$$\mathbf{x} = C\mathbf{v}e^{\lambda t} + D\overline{\mathbf{v}}e^{\overline{\lambda}t} = C\begin{pmatrix} 1 \\ 1 - i \end{pmatrix} e^{(2+i)t} + D\begin{pmatrix} 1 \\ 1 + i \end{pmatrix} e^{(2-i)t},$$

where C and D are arbitrary complex constants.

To obtain a real-valued solution, we follow Procedure 3 and write

$$\begin{aligned}
\mathbf{v}e^{\lambda t} &= \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(2+i)t} \\
&= e^{2t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{it} \\
&= e^{2t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} (\cos t + i \sin t) \\
&= e^{2t} \begin{pmatrix} \cos t + i \sin t \\ (1-i)(\cos t + i \sin t) \end{pmatrix} \\
&= e^{2t} \begin{pmatrix} \cos t + i \sin t \\ (\cos t + \sin t) + i(\sin t - \cos t) \end{pmatrix} \\
&= \underbrace{e^{2t} \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix}}_{\text{real part}} + i \underbrace{e^{2t} \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}}_{\text{imaginary part}}.
\end{aligned}$$

The real-valued general solution of the given system of equations is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\, e^{2t} \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + \beta\, e^{2t} \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}, \tag{18}$$

where α and β are arbitrary real constants.

In order to find the required particular solution, we substitute $x = 3$, $y = 1$ and $t = 0$ into equation (18), to obtain

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

so $3 = \alpha$ and $1 = \alpha - \beta$, giving $\beta = 2$, and the solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{2t}(3\cos t + 2\sin t) \\ e^{2t}(\cos t + 5\sin t) \end{pmatrix}.$$

We will ask you to solve systems of linear constant-coefficient first-order differential equations by hand only for 2×2 and 3×3 coefficient matrices, but you may well encounter larger systems when you use a computer algebra package.

Exercise 6

Using the eigenvalues and eigenvectors given, find the real-valued solution of each of the following systems of differential equations, given that $x = y = 1$ when $t = 0$.

(a) $\begin{cases} \dot{x} = 2x + 3y \\ \dot{y} = 2x + y \end{cases}$

(The matrix $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ has eigenvalues 4 and -1, corresponding to eigenvectors $(3 \quad 2)^T$ and $(1 \quad -1)^T$, respectively.)

(b) $\begin{cases} \dot{x} = -3x - 2y \\ \dot{y} = 4x + y \end{cases}$

(The matrix $\begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}$ has eigenvalues $-1+2i$ and $-1-2i$, corresponding to eigenvectors $(1 \quad -1-i)^T$ and $(1 \quad -1+i)^T$, respectively.)

Exercise 7

Write down the general solution of each of the following systems of equations.

(a) $\begin{cases} \dot{x} = x \phantom{{}+2y} - z \\ \dot{y} = x + 2y + z \\ \dot{z} = 2x + 2y + 3z \end{cases}$

(The matrix $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ has eigenvalues 1, 2 and 3, corresponding to eigenvectors $(1 \quad -1 \quad 0)^T$, $(-2 \quad 1 \quad 2)^T$ and $(1 \quad -1 \quad -2)^T$, respectively.)

(b) $\begin{cases} \dot{x} = 5x - 6y - 6z \\ \dot{y} = -x + 4y + 2z \\ \dot{z} = 3x - 6y - 4z \end{cases}$

(The matrix $\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$ has eigenvalues 1 and 2 (repeated), with corresponding eigenvectors $(3 \quad -1 \quad 3)^T$ and $(2l+2k \quad l \quad k)^T$, respectively, where k and l are arbitrary real numbers, not both zero.)

Exercise 8

Find the general real-valued solution of the system of equations

$$\begin{aligned} \dot{x} &= x \phantom{{}+y} + z, \\ \dot{y} &= \phantom{-x+{}} y\,, \\ \dot{z} &= -x + y\,. \end{aligned}$$

Find the solution for which $x = y = 1$ and $z = 2$ when $t = 0$.

(The matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ has eigenvalues 1, $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\overline{\lambda} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, corresponding to the eigenvectors $(1 \quad 1 \quad 0)^T$, $\mathbf{v} = (1 \quad 0 \quad -\frac{1}{2} + \frac{\sqrt{3}}{2}i)^T$ and $\overline{\mathbf{v}} = (1 \quad 0 \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i)^T$, respectively.)

3 First-order inhomogeneous systems

In the previous section you saw how to solve a system of differential equations of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a given constant-coefficient matrix. We now extend our discussion to systems of the form

Here we write $\mathbf{h}(t)$ to emphasise that $\mathbf{h}$ is a function of t. Henceforth we will abbreviate this to $\mathbf{h}$.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}(t),$$

where $\mathbf{h}(t)$ is a given function of t. Our method involves finding a 'particular integral' for the system, and mirrors the approach taken for inhomogeneous second-order differential equations in Unit 1.

3.1 A basic result

In Unit 1 we discussed inhomogeneous differential equations such as

See Unit 1, Example 18.

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x} + 18x + 18. \tag{19}$$

To solve such an equation, we proceed as follows.

- We first find the *complementary function* of the corresponding homogeneous equation

  $$\frac{d^2y}{dx^2} + 9y = 0,$$

  which is

  $$y_{\mathrm{c}} = C_1 \cos 3x + C_2 \sin 3x,$$

  where C_1 and C_2 are arbitrary constants.
- We then find a *particular integral* of the inhomogeneous equation (19):

  $$y_{\mathrm{p}} = \tfrac{1}{9}e^{3x} + 2x + 2.$$

The general solution y of the original equation is then obtained by adding these two functions (using the principle of superposition) to give

$$y = y_{\mathrm{c}} + y_{\mathrm{p}} = (C_1 \cos 3x + C_2 \sin 3x) + \left(\tfrac{1}{9}e^{3x} + 2x + 2\right). \tag{20}$$

A similar situation holds for *systems* of linear first-order differential equations. For example, in order to find the general solution of the inhomogeneous system

$$\begin{cases} \dot{x} = 3x + 2y + 4e^{3t}, \\ \dot{y} = x + 4y - e^{3t}, \end{cases} \tag{21}$$

which in matrix form becomes

This is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, where $\mathbf{h} = (4e^{3t} \quad -e^{3t})^T$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4e^{3t} \\ -e^{3t} \end{pmatrix},$$

we first find the general solution of the corresponding homogeneous system

$$\begin{cases} \dot{x} = 3x + 2y, \\ \dot{y} = x + 4y, \end{cases}$$

which is the *complementary function*

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t},$$

where α and β are arbitrary constants (see Example 1(a)).

We next find a particular solution, or *particular integral*, of the original system (21), namely

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{3t},$$

as we will show in Subsection 3.2 (Example 9). The general solution of the original system (21) is then obtained by adding these two:

We use the term *particular integral* rather than particular solution. The latter is more appropriately used for the solution of system (21) that satisfies given initial or boundary conditions.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{3t}.$$

To describe the above expression as 'the general solution' is perhaps premature, because it is not immediately obvious that it is even a solution. In order to establish that this is the case, we may use the following general result.

Theorem 2 Principle of superposition

If $\mathbf{x}_1$ is a solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_1$ and $\mathbf{x}_2$ is a solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_2$, then $p\mathbf{x}_1 + q\mathbf{x}_2$ is a solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + p\mathbf{h}_1 + q\mathbf{h}_2$, where p and q are constants.

This result is easy to prove, for we have

$$\begin{aligned} \frac{d}{dt}(p\mathbf{x}_1 + q\mathbf{x}_2) &= p\dot{\mathbf{x}}_1 + q\dot{\mathbf{x}}_2 \\ &= p(\mathbf{A}\mathbf{x}_1 + \mathbf{h}_1) + q(\mathbf{A}\mathbf{x}_2 + \mathbf{h}_2) \\ &= \mathbf{A}(p\mathbf{x}_1 + q\mathbf{x}_2) + p\mathbf{h}_1 + q\mathbf{h}_2. \end{aligned}$$

The particular case that is relevant here corresponds to choosing $\mathbf{h}_1 = \mathbf{0}$ and $\mathbf{h}_2 = \mathbf{h}$, say, and putting $p = q = 1$, $\mathbf{x}_1 = \mathbf{x}_c$, the complementary function, and $\mathbf{x}_2 = \mathbf{x}_p$, a particular integral. Then the above result gives rise to the following theorem.

Theorem 3

If $\mathbf{x}_c$ is the complementary function of the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and $\mathbf{x}_p$ is a particular integral of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, then $\mathbf{x}_c + \mathbf{x}_p$ is the general solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$.

Exercise 9

Write down the general solution of the system

$$\dot{x} = 3x + 2y + t,$$
$$\dot{y} = x + 4y + 7t,$$

given that a particular integral is

$$x_{\mathrm{p}} = t + \tfrac{4}{5}, \quad y_{\mathrm{p}} = -2t - \tfrac{7}{10}.$$

(The matrix $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ has eigenvalues 5 and 2, corresponding to eigenvectors $(1 \quad 1)^T$ and $(-2 \quad 1)^T$, respectively.)

At this stage it is natural to ask how we were able to find the above particular integral (although it is easy to verify that it *is* a solution to the given system, by direct substitution). Before we address that question in detail, we should emphasise the importance of the principle of superposition. Consider the following example, paying particular attention to the form of $\mathbf{h}$, which is made up of both exponential and linear terms.

Example 7

Here $\mathbf{h} = \begin{pmatrix} 4e^{3t} + 2t \\ -e^{3t} + 14t \end{pmatrix}$.

Find the general solution of the system

$$\dot{x} = 3x + 2y + 4e^{3t} + 2t,$$
$$\dot{y} = x + 4y - e^{3t} + 14t.$$

Solution

Choosing $\mathbf{h}_1 = (4e^{3t} \quad -e^{3t})^T$, we see from the results derived for equations (21) that $\mathbf{x}_1 = (3 \quad -2)^T e^{3t}$ is a solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_1$. Also, choosing $\mathbf{h}_2 = (t \quad 7t)^T$ gives $\mathbf{x}_2 = (t + \frac{4}{5} \quad -2t - \frac{7}{10})^T$ as a solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_2$, from Exercise 9. Thus from the principle of superposition, $\mathbf{x}_1 + 2\mathbf{x}_2$ is a particular integral of the given system written as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_1 + 2\mathbf{h}_2$. Hence by Theorem 3, using the complementary function in Exercise 9, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} \right) + \left(\begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{3t} + \begin{pmatrix} 2t + \frac{8}{5} \\ -4t - \frac{7}{5} \end{pmatrix} \right).$$

Example 7 illustrates a general technique, the principle of which is to break down the term $\mathbf{h}$ into a sum of manageable components.

3.2 Finding particular integrals

We now show you how to find a particular integral $\mathbf{x}_p$ in some special cases. We consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ in the situations where $\mathbf{h}$ is a vector whose components are:

- polynomial functions
- exponential functions.

Our treatment will be similar to that of Unit 1, where we found particular integrals for linear second-order differential equations using the method of undetermined coefficients. As in that unit, a number of exceptional cases arise, where our methods need to be slightly modified.

To illustrate the ideas involved, we consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}, \quad \text{where } \mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}. \tag{22}$$

The first stage in solving any inhomogeneous system is to find the complementary function, that is, the solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, which you saw in Subsection 3.1 is

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t}. \tag{23}$$

To this complementary function we add a particular integral that depends on the form of $\mathbf{h}$. We now look at examples of the above two forms for $\mathbf{h}$, and derive a particular integral in each case.

Example 8

Find the general solution of the system

$$\begin{aligned} \dot{x} &= 3x + 2y + t, \\ \dot{y} &= x + 4y + 7t. \end{aligned}$$

Solution

The complementary function is given in equation (23).

We note that $\mathbf{h} = (t \quad 7t)^T$ consists entirely of linear functions, so it seems natural to seek a particular integral of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} at + b \\ ct + d \end{pmatrix},$$

You may have been tempted to use a simpler trial solution, of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} at \\ ct \end{pmatrix}$. Unfortunately, this does not work – try it and see! You may recall something similar in Unit 1.

where a, b, c and d are constants that we need to determine. We find them by substituting $x = at + b$, $y = ct + d$ into the original system. This gives the simultaneous equations

$$\begin{aligned} a &= 3(at + b) + 2(ct + d) + t, \\ c &= (at + b) + 4(ct + d) + 7t, \end{aligned}$$

which give, on rearranging,

$$\begin{cases} (3a + 2c + 1)t + (3b + 2d - a) = 0, \\ (a + 4c + 7)t + (b + 4d - c) = 0. \end{cases} \tag{24}$$

These equations hold for *all* values of t, which means that each of the bracketed terms must be zero.

Equating the coefficients of t to zero in equations (24) gives

$$\begin{aligned} 3a + 2c + 1 &= 0, \\ a + 4c + 7 &= 0, \end{aligned}$$

which have the solution

$$a = 1, \quad c = -2.$$

Equating the constant terms to zero in equations (24), and putting $a = 1$, $c = -2$, gives the equations

$$\begin{aligned} 3b + 2d - 1 &= 0, \\ b + 4d + 2 &= 0, \end{aligned}$$

which have the solution

$$b = \tfrac{4}{5}, \quad d = -\tfrac{7}{10}.$$

Thus the required particular integral is

$$\begin{pmatrix} x_{\text{p}} \\ y_{\text{p}} \end{pmatrix} = \begin{pmatrix} t + \frac{4}{5} \\ -2t - \frac{7}{10} \end{pmatrix},$$

and the general solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x_{\text{c}} \\ y_{\text{c}} \end{pmatrix} + \begin{pmatrix} x_{\text{p}} \\ y_{\text{p}} \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} t + \frac{4}{5} \\ -2t - \frac{7}{10} \end{pmatrix}. \end{aligned}$$

Exercise 10

Find the general solution of the system

$$\begin{aligned} \dot{x} &= x + 4y - t + 2, \\ \dot{y} &= x - 2y + 5t. \end{aligned}$$

(*Hint*: For the complementary function, see Example 2.)

Example 9

Find the general solution of the system

$$\begin{aligned} \dot{x} &= 3x + 2y + 4e^{3t}, \\ \dot{y} &= x + 4y - e^{3t}. \end{aligned}$$

Solution

The complementary function is given in equation (23).

We note that $\mathbf{h} = (4e^{3t} \quad -e^{3t})^T$ consists entirely of exponentials, so it seems natural to seek a particular integral of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ae^{3t} \\ be^{3t} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{3t},$$

This method fails when $\mathbf{h}$ involves e^{5t} or e^{2t}, which occur in the complementary function. We deal with such examples in the 'Exceptional cases' subsection below. You may recall similar failures in Unit 1.

where a and b are constants that we need to determine. We find them by substituting $x = ae^{3t}$, $y = be^{3t}$ into the original system. This gives the simultaneous equations

$$\begin{aligned} 3ae^{3t} &= 3ae^{3t} + 2be^{3t} + 4e^{3t}, \\ 3be^{3t} &= ae^{3t} + 4be^{3t} - e^{3t}, \end{aligned}$$

or, on dividing by e^{3t},

$$\begin{aligned} 3a &= 3a + 2b + 4, \\ 3b &= a + 4b - 1. \end{aligned}$$

Rearranging these equations gives

$$\begin{aligned} 2b &= -4, \\ a + b &= 1, \end{aligned}$$

which have the solution

$$a = 3, \quad b = -2.$$

Thus the required particular integral is

$$\begin{pmatrix} x_{\text{p}} \\ y_{\text{p}} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -2e^{3t} \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{3t},$$

and the general solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x_{\text{c}} \\ y_{\text{c}} \end{pmatrix} + \begin{pmatrix} x_{\text{p}} \\ y_{\text{p}} \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{3t}. \end{aligned}$$

Exercise 11

Find the general solution of the system

$$\begin{aligned} \dot{x} &= x + 4y + 4e^{-t}, \\ \dot{y} &= x - 2y + 5e^{-t}. \end{aligned}$$

(*Hint*: For the complementary function, see Exercise 10.)

Exercise 12

Solve the system of differential equations

$$\dot{x} = 2x + 3y + e^{2t},$$
$$\dot{y} = 2x + y + 4e^{2t},$$

subject to the initial conditions $x(0) = \frac{5}{6}$, $y(0) = \frac{2}{3}$.

(The matrix $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ has eigenvectors $(1 \quad -1)^T$ and $(3 \quad 2)^T$, corresponding to eigenvalues -1 and 4, respectively.)

Recall that the technique used in Unit 1, when determining the particular integral of an inhomogeneous differential equation, was to look for a solution of the same form as the right-hand side. Thus if $\mathbf{h}$ contains sinusoidal terms of the form $\cos\omega t$ and/or $\sin\omega t$, the particular integral will be of the form

$$\begin{pmatrix} x_{\text{p}} \\ y_{\text{p}} \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \cos\omega t + \begin{pmatrix} b \\ d \end{pmatrix} \sin\omega t.$$

There is a more efficient way of treating this case, by exploiting the relationship between the sinusoidal functions and complex exponentials (given by Euler's formula). One then proceeds as with the case for real exponential terms in $\mathbf{h}$, only now with complex-valued undetermined coefficients.

This will generally lead to *four* equations in the undetermined coefficients a, b, c and d. Therefore subsequent calculations tend to be rather long-winded, and for this reason the case when $\mathbf{h}$ consists of sinusoidal functions will not be pursued further in this module.

Procedure 4 Finding a particular integral

To find a particular integral $\mathbf{x}_{\text{p}} = (x_{\text{p}} \quad y_{\text{p}})^T$ for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, carry out the following steps.

1. When the elements of $\mathbf{h}$ are polynomials of degree less than or equal to k, choose x_{p} and y_{p} to be polynomials of degree k.

   When the elements of $\mathbf{h}$ are multiples of the same exponential function, choose x_{p} and y_{p} to be multiples of this exponential function.
2. To determine the coefficients in x_{p} and y_{p}, substitute into the system of differential equations, and equate coefficients.

Exercise 13

Using the information from Exercise 12, find the general solution of the system of differential equations

$$\dot{x} = 2x + 3y + t,$$
$$\dot{y} = 2x + y + 4e^{-2t}.$$

(*Hint*: Use the principle of superposition.)

Exceptional cases

In Unit 1 we discussed the differential equation

$$\frac{d^2y}{dx^2} - 4y = 2e^{2x}.$$

From Unit 1, Exercise 18(c), the complementary function is $y_c = \alpha e^{-2x} + \beta e^{2x}$, where α and β are arbitrary constants. For the particular integral, it would be natural to try $y = ke^{2x}$, where k is a constant to be determined. However, as you saw in Unit 1, Example 16, this fails since e^{2x} is already included in the complementary function. So we insert an extra factor x, and try a particular integral of the form $y = kxe^{2x}$.

The situation is similar for systems of linear differential equations in cases where the usual trial solution is part of the complementary function, as we show in the following example.

Example 10

Find the general solution of the system

$$\begin{cases} \dot{x} = 3x + 2y + 6e^{2t}, \\ \dot{y} = x + 4y + 3e^{2t}. \end{cases} \tag{25}$$

Solution

From equation (23), the complementary function is

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t},$$

where α and β are arbitrary constants.

Since the complementary function includes an e^{2t} term, a particular integral (suggested by $\mathbf{h} = (6e^{2t} \quad 3e^{2t})^T$) of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 e^{2t} \\ a_2 e^{2t} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{2t}$$

will not work. That is, if we substitute this into equations (25), we obtain

$$2a_1e^{2t} = 3a_1e^{2t} + 2a_2e^{2t} + 6e^{2t},$$
$$2a_2e^{2t} = a_1e^{2t} + 4a_2e^{2t} + 3e^{2t},$$

or, on dividing by e^{2t} and rearranging,

$$a_1 + 2a_2 = -6,$$
$$a_1 + 2a_2 = -3.$$

These equations clearly have no solution, so the method fails.

However, there is a method that will succeed. First write $\mathbf{v}_1 = (1 \quad 1)^T$, $\mathbf{v}_2 = (-2 \quad 1)^T$, $\lambda_1 = 5$, $\lambda_2 = 2$, so that $\mathbf{v}_1$, $\mathbf{v}_2$ are eigenvectors corresponding respectively to the eigenvalues λ_1, λ_2, and write $\mathbf{h} = \mathbf{k}e^{\lambda_2 t}$, where $\mathbf{k} = (6 \quad 3)^T$.

Recall from Unit 5, Section 1 that any n-dimensional vector $\mathbf{v}$ can be written as a linear combination of n linearly independent n-dimensional vectors.

Now $\mathbf{k}$ can be written as a linear combination of the linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{k} = p\mathbf{v}_1 + q\mathbf{v}_2.$$

In this case, $(6 \quad 3)^T = p(1 \quad 1)^T + q(-2 \quad 1)^T$, so $p = 4$ and $q = -1$.

Next we look for a trial solution of the form

$$\mathbf{x} = (a\mathbf{v}_1 + b\mathbf{v}_2 t)e^{\lambda_2 t}.$$

Substituting this and $\mathbf{h} = \mathbf{k}e^{\lambda_2 t} = p\mathbf{v}_1 e^{\lambda_2 t} + q\mathbf{v}_2 e^{\lambda_2 t}$ into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$ gives

$$b\mathbf{v}_2 e^{\lambda_2 t} + \lambda_2(a\mathbf{v}_1 + b\mathbf{v}_2 t)e^{\lambda_2 t} = \mathbf{A}(a\mathbf{v}_1 + b\mathbf{v}_2 t)e^{\lambda_2 t} + p\mathbf{v}_1 e^{\lambda_2 t} + q\mathbf{v}_2 e^{\lambda_2 t}.$$

Dividing by $e^{\lambda_2 t}$ and using the fact that $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, we obtain

$$a(\lambda_2 - \lambda_1)\mathbf{v}_1 + b\mathbf{v}_2 = p\mathbf{v}_1 + q\mathbf{v}_2.$$

Equating coefficients is appropriate as $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Also, a is well defined since $\lambda_1 \neq \lambda_2$.

Equating the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_2$ gives

$$a = p/(\lambda_2 - \lambda_1) \quad \text{and} \quad b = q.$$

We know $\lambda_1 = 5$, $\lambda_2 = 2$, $p = 4$ and $q = -1$, so $a = -\frac{4}{3}$ and $b = -1$, and a particular integral is

$$\mathbf{x}_\mathrm{p} = \left(-\tfrac{4}{3}\begin{pmatrix}1\\1\end{pmatrix} - 1\begin{pmatrix}-2\\1\end{pmatrix}t\right)e^{2t} = \begin{pmatrix}2t - \frac{4}{3}\\ -t - \frac{4}{3}\end{pmatrix}e^{2t}.$$

So the general solution of equations (25) is

$$\mathbf{x} = \alpha\begin{pmatrix}1\\1\end{pmatrix}e^{5t} + \beta\begin{pmatrix}-2\\1\end{pmatrix}e^{2t} + \begin{pmatrix}2t - \frac{4}{3}\\ -t - \frac{4}{3}\end{pmatrix}e^{2t}.$$

The method used in Example 10 works in general, provided that the eigenvalues are distinct.

Procedure 5 Finding a particular integral – special case

To find a particular integral $\mathbf{x}_\mathrm{p} = (x_\mathrm{p} \quad y_\mathrm{p})^T$ for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}$, where $\mathbf{A}$ has distinct real eigenvalues λ_1 and λ_2, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, and $\mathbf{h} = \mathbf{k}e^{\lambda_2 t}$, first determine p and q such that

$$\mathbf{k} = p\mathbf{v}_1 + q\mathbf{v}_2.$$

Then a particular integral has the form

$$\mathbf{x}_\mathrm{p} = (a\mathbf{v}_1 + b\mathbf{v}_2 t)e^{\lambda_2 t}, \tag{26}$$

where $a = p/(\lambda_2 - \lambda_1)$ and $b = q$.

A similar procedure can be followed if $\mathbf{h}$ contains polynomial components, but we do not go into the details here.

Exercise 14

Using the information from Exercise 12, find the general solution of the system of differential equations

$$\dot{x} = 2x + 3y + e^{4t},$$
$$\dot{y} = 2x + y + 2e^{4t}.$$

4 Second-order systems

In this section we show how the methods introduced earlier in this unit can be adapted to finding the solutions of certain systems of homogeneous second-order differential equations. We then consider a particular case in which the solutions are all sinusoidal. Such cases arise often in connection with oscillating mechanical systems or electrical circuits.

4.1 Homogeneous second-order systems

We now turn our attention to systems of linear constant-coefficient second-order differential equations of the form $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. You have seen that the general solution of a first-order system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ of n equations can be written as a linear combination of n linearly independent solutions involving n arbitrary constants. In a similar way, it can be shown that the general solution of a second-order system $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ of n equations can be written as a linear combination of $2n$ linearly independent solutions involving $2n$ arbitrary constants. The following example will show you what is involved in the solution of such systems; the treatment is similar to that of Example 1.

We do not prove this.

Example 11

Find the general solution of the system of differential equations

$$\begin{cases} \ddot{x} = x + 4y, \\ \ddot{y} = x - 2y. \end{cases} \tag{27}$$

Solution

In order to find the general solution of this pair of equations, it is sufficient to find four linearly independent solutions and write down an arbitrary linear combination of them.

We begin by attempting to find solutions of the form

$$x = Ce^{\mu t}, \quad y = De^{\mu t},$$

where C and D are constants.

The reason for using μ rather than λ will become apparent as we proceed.

Since $\ddot{x} = C\mu^2 e^{\mu t}$ and $\ddot{y} = D\mu^2 e^{\mu t}$, we have, on substituting the expressions for x and y into equations (27),

$$\begin{aligned} C\mu^2 e^{\mu t} &= Ce^{\mu t} + 4De^{\mu t}, \\ D\mu^2 e^{\mu t} &= Ce^{\mu t} - 2De^{\mu t}. \end{aligned}$$

Dividing through by $e^{\mu t}$, we obtain

$$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \mu^2 \begin{pmatrix} C \\ D \end{pmatrix},$$

so μ^2 is an eigenvalue of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$. However, from Example 2 we know that the eigenvalues of $\mathbf{A}$ are 2 and -3, so $\mu = \pm\sqrt{2}$ or $\mu = \pm\sqrt{3}i$.

The verification can be done by direct substitution.

The eigenvalue 2 corresponds to an eigenvector $(4 \quad 1)^T$, and it is easy to verify that the values $\mu = \pm\sqrt{2}$ provide us with two linearly independent solutions of equations (27), namely

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{\sqrt{2}t} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-\sqrt{2}t}.$$

The eigenvalue -3 corresponds to an eigenvector $(1 \quad -1)^T$, and choosing $\mu = \sqrt{3}i$ gives a further solution

Here we use Euler's formula to write down the exponential in terms of sinusoids.

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\sqrt{3}it} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (\cos\sqrt{3}t + i\sin\sqrt{3}t);$$

we can verify that the real and imaginary parts of the expression on the right-hand side are both solutions of equations (27). (Choosing $\mu = -\sqrt{3}i$ leads to the same two linearly independent solutions.)

Thus we have found two more linearly independent solutions of equations (27), namely

$$\mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\sqrt{3}t \quad \text{and} \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\sqrt{3}t.$$

We do not prove this.

Using a version of the principle of superposition applicable to second-order systems, we can now take linear combinations of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ to find further solutions. The expression

$$\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + C_3\mathbf{x}_3 + C_4\mathbf{x}_4,$$

Note that there are four arbitrary constants, as expected.

where C_1, C_2, C_3, C_4 are arbitrary constants, is the general solution of equations (27).

Comparing the above solution with that of Example 2, we notice many similarities. The main difference is that λ is replaced by μ^2, giving rise to four values for μ, instead of two for λ. Consequently, we obtain a general solution with four arbitrary constants.

This discussion mirrors the corresponding discussion in Section 2.

In general, consider a system of differential equations of the form $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. If we try an exponential solution of the form $\mathbf{x} = \mathbf{v}e^{\mu t}$, where $\mathbf{v}$ is a constant column vector, then $\ddot{\mathbf{x}} = \mathbf{v}\mu^2 e^{\mu t}$, and the system of differential equations becomes $\mathbf{v}\mu^2 e^{\mu t} = \mathbf{A}\mathbf{v}e^{\mu t}$.

Dividing this equation by $e^{\mu t}$ and rearranging, we have

$$\mathbf{A}\mathbf{v} = \mu^2\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, and μ^2 is the corresponding eigenvalue.

Theorem 4

If μ^2 is an eigenvalue of the matrix $\mathbf{A}$ corresponding to an eigenvector $\mathbf{v}$, then $\mathbf{x} = \mathbf{v}e^{\mu t}$ is a solution of the system of differential equations $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

If $\mathbf{x} = \mathbf{v}e^{\mu t}$, then $\dot{\mathbf{x}} = \mu\mathbf{v}e^{\mu t}$ and $\ddot{\mathbf{x}} = \mu^2\mathbf{v}e^{\mu t} = \mathbf{A}\mathbf{v}e^{\mu t} = \mathbf{A}\mathbf{x}$.

Example 12

Find the general solution of the system of differential equations

$$\ddot{x} = 3x + 2y,$$
$$\ddot{y} = x + 4y.$$

Solution

The matrix of coefficients is

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

From Example 1, the eigenvectors of $\mathbf{A}$ are $(1 \quad 1)^T$ and $(-2 \quad 1)^T$, corresponding to the eigenvalues $\lambda = 5$ and $\lambda = 2$, respectively.

Using the notation of Example 11, it follows that μ has the values $\sqrt{5}$, $-\sqrt{5}$, $\sqrt{2}$ and $-\sqrt{2}$, and that the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\sqrt{5}t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\sqrt{5}t} + C_3\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{\sqrt{2}t} + C_4\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-\sqrt{2}t}.$$

Exercise 15

Find the general solution of the system of differential equations

$$\ddot{x} = 5x + 2y,$$
$$\ddot{y} = 2x + 5y.$$

(The matrix $\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ has eigenvectors $(1 \quad 1)^T$ and $(1 \quad -1)^T$, corresponding to eigenvalues 7 and 3, respectively.)

The above ideas can be formalised in the following procedure.

Complex eigenvalues and repeated real eigenvalues are not discussed here, but they can be dealt with in a fashion similar to that for the first-order case.

Procedure 6 Solving a second-order homogeneous linear system

To solve a system $\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix with n distinct real eigenvalues, carry out the following steps.

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$, and a corresponding set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
2. Each *positive* eigenvalue, μ^2 say, corresponding to an eigenvector $\mathbf{v}$, gives rise to two linearly independent solutions
$$\mathbf{v}e^{\mu t} \quad \text{and} \quad \mathbf{v}e^{-\mu t}.$$
Each *negative* eigenvalue, $-\omega^2$ say, corresponding to an eigenvector $\mathbf{v}$, gives rise to two linearly independent solutions
$$\mathbf{v}\cos\omega t \quad \text{and} \quad \mathbf{v}\sin\omega t.$$
A *zero* eigenvalue corresponding to an eigenvector $\mathbf{v}$ gives rise to two linearly independent solutions
$$\mathbf{v} \quad \text{and} \quad \mathbf{v}t.$$
3. The general solution is then an arbitrary linear combination of the $2n$ linearly independent solutions found in step 2, involving $2n$ arbitrary constants.

We do not show this here, but you can verify it in any particular case (see Example 13 below). It is analogous to the case of a single second-order differential equation with both roots of the auxiliary equation equal to zero.

We illustrate this procedure in the following example.

Example 13

Find the general solution of the system of differential equations

$$\begin{aligned}\ddot{x} &= 3x + 2y + 2z,\\ \ddot{y} &= 2x + 2y,\\ \ddot{z} &= 2x \phantom{{}+2y} + 4z.\end{aligned}$$

Solution

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}$.

The eigenvectors of $\mathbf{A}$ are $(2 \;\; 1 \;\; 2)^T$, corresponding to the eigenvalue $\lambda = 6$, $(1 \;\; 2 \;\; -2)^T$, corresponding to the eigenvalue $\lambda = 3$, and $(-2 \;\; 2 \;\; 1)^T$, corresponding to the eigenvalue $\lambda = 0$. It follows from Procedure 6 that the general solution of the above system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}(C_1 e^{\sqrt{6}t} + C_2 e^{-\sqrt{6}t}) + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}(C_3 e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t}) + \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}(C_5 + C_6 t).$$

You may like to verify that $(-2 \;\; 2 \;\; 1)^T$ and $(-2 \;\; 2 \;\; 1)^T t$ are both solutions of the system.

Exercise 16

Find the general solution of the system of equations

$$\begin{aligned}\ddot{x} &= x + 4y,\\ \ddot{y} &= x + y.\end{aligned}$$

Exercise 17

Find the general solution of the system of differential equations

$$\begin{aligned}\ddot{x} &= 2x + y - z,\\ \ddot{y} &= - 3y + 2z,\\ \ddot{z} &= 4z.\end{aligned}$$

(The matrix $\begin{pmatrix} 2 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$ has eigenvalues 2, -3 and 4, corresponding to eigenvectors $(1 \quad 0 \quad 0)^T$, $(1 \quad -5 \quad 0)^T$ and $(-5 \quad 4 \quad 14)^T$, respectively.)

4.2 Simple harmonic motion

Simple harmonic motion is an often observed phenomenon. It arises, for example, if a quantity satisfies a second-order differential equation of the form

$$\ddot{x} = -\omega^2 x,$$

where ω is a constant. In this case solutions are of the form

$$x = \alpha \cos \omega t + \beta \sin \omega t,$$

where α and β are arbitrary constants.

In Subsection 1.3 we developed a model for the horizontal motion of a ball-bearing within a bowl of specified shape. The resulting equations were

$$\begin{cases} \ddot{x} = -5x + 4y, \\ \ddot{y} = 4x - 5y, \end{cases} \tag{28}$$

where $x(t)$ and $y(t)$ are the coordinates of the ball-bearing at time t. These second-order differential equations may be expressed in matrix form as

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $\begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, corresponding to eigenvalues -1 and -9, respectively.

It follows from Procedure 6 that the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (C_1 \cos t + C_2 \sin t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (C_3 \cos 3t + C_4 \sin 3t),$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Let us now consider the paths that a ball-bearing takes for given initial conditions. The following list gives just four of the many possibilities.

The initial conditions are chosen so that the ball-bearing is kept sufficiently close to the lowest point of the bowl (i.e. the origin). Recall that this was the assumption underlying the linear approximation leading to equations (28).

(a) $x(0) = 0.1, \quad y(0) = 0.1, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0.$

(b) $x(0) = 0.1, \quad y(0) = -0.1, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0.$

(c) $x(0) = 0.1, \quad y(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0.$

(d) $x(0) = 0.1, \quad y(0) = 0.2, \quad \dot{x}(0) = -0.1, \quad \dot{y}(0) = 0.1.$

In case (a) we find $C_1 = 0.1$ and $C_2 = C_3 = C_4 = 0$, so the solution is

$$\begin{pmatrix} x & y \end{pmatrix}^T = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}^T \cos t,$$

and the ball-bearing performs simple harmonic motion in the direction of the vector $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$, that is, along the line $y = x$, with angular frequency 1 (as shown in Figure 6(a)).

In case (b) we find $C_3 = 0.1$ and $C_1 = C_2 = C_4 = 0$, so the solution is

$$\begin{pmatrix} x & y \end{pmatrix}^T = \begin{pmatrix} 0.1 & -0.1 \end{pmatrix}^T \cos 3t,$$

and the ball-bearing performs simple harmonic motion in the direction of the vector $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$, that is, along the line $y = -x$, with angular frequency 3 (as shown in Figure 6(b)).

Note that cases (a) and (b) consist of motion with a *single* angular frequency – these are the *normal modes* of the system, which will be the subject of Unit 11.

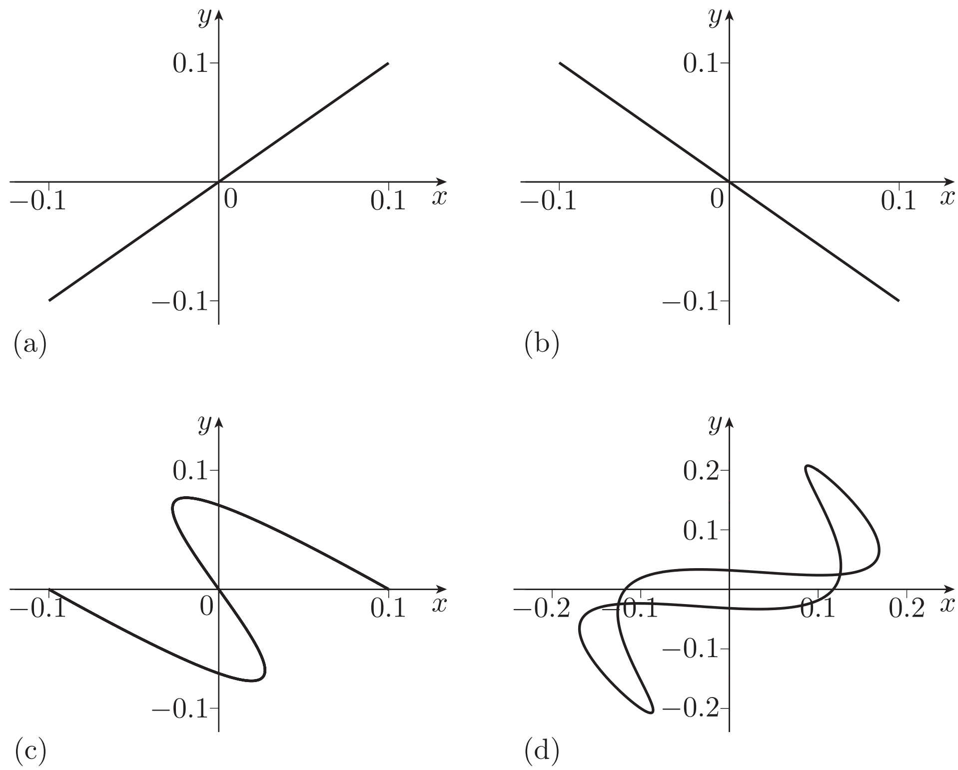

Figure 6 Paths of the ball-bearing for initial conditions (a), (b), (c) and (d)

In case (c) we find $C_1 = C_3 = 0.05$ and $C_2 = C_4 = 0$, so the solution is

$$(x \quad y)^T = 0.05\,(1 \quad 1)^T \cos t + 0.05\,(1 \quad -1)^T \cos 3t,$$

which is a combination of the previous motions (as shown in Figure 6(c)).

In case (d) we find $C_1 = 0.15$, $C_3 = -0.05$, $C_2 = 0$ and $C_4 = -\frac{1}{30}$, so the solution is

$$(x \quad y)^T = (1 \quad 1)^T(0.15\cos t) + (1 \quad -1)^T(-0.05\,\cos 3t - \tfrac{1}{30}\sin 3t)$$

(as shown in Figure 6(d)).

Cases (c) and (d) consist of superpositions of the two normal modes.

Note that the coefficients in equations (28) have been contrived so that the resulting angular frequencies ω are rational numbers, and this implies periodic motion for the ball-bearing. In general, this need not be so – irrational ω could give non-periodic motion. More accurately, the motion is in general *non-periodic* if the *ratio* of the two angular frequencies (normal mode frequencies) is *irrational*; otherwise, the motion is always periodic.

Exercise 18

An object moves in a plane so that its coordinates at time t satisfy the equations

$$\begin{aligned} \ddot{x} &= -\tfrac{25}{7}x + \tfrac{6}{7}y, \\ \ddot{y} &= \tfrac{9}{7}x - \tfrac{10}{7}y. \end{aligned}$$

Find two directions in which the object can describe simple harmonic motion along a straight line. Give the angular frequencies of such motions.

Learning outcomes

After studying this unit, you should be able to:

- understand and use the terminology associated with systems of linear constant-coefficient differential equations
- obtain the general solution of a homogeneous system of two or three first-order differential equations, by applying knowledge of the eigenvalues and eigenvectors of the coefficient matrix
- obtain a particular solution of an inhomogeneous system of two first-order differential equations in certain simple cases, by using a trial solution
- understand the role of the principle of superposition in determining a particular integral of an inhomogeneous linear system of differential equations
- obtain the general solution of an inhomogeneous system of two or three first-order differential equations, by combining its complementary function and a particular integral
- apply given initial conditions to obtain the solution of an initial-value problem that features a system of two first-order differential equations
- obtain the general solution of a homogeneous system of two or three second-order equations, by applying knowledge of the eigenvalues and eigenvectors of the coefficient matrix
- appreciate how systems of linear constant-coefficient differential equations arise in mathematical models of the real world.

Solutions to exercises

Solution to Exercise 1

(a) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; inhomogeneous.

(b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}$; inhomogeneous.

(c) $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; homogeneous.

Solution to Exercise 2

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$.

We are given that the eigenvectors of $\mathbf{A}$ are $(1 \quad 1)^T$ with corresponding eigenvalue $\lambda = 7$, and $(1 \quad -1)^T$ with corresponding eigenvalue $\lambda = 3$. The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}.$$

Since $x = 4$ and $y = 0$ when $t = 0$, we have

$$4 = \alpha + \beta,$$
$$0 = \alpha - \beta.$$

Thus $\alpha = 2$ and $\beta = 2$, so

$$\begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{7t} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} e^{3t}.$$

Solution to Exercise 3

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

We are given that the eigenvectors of $\mathbf{A}$ are $(2 \quad 1 \quad 1)^T$, $(0 \quad 1 \quad 1)^T$ and $(0 \quad 1 \quad -1)^T$, corresponding to the eigenvalues 5, 3 and 1, respectively.

The general solution is therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{3t} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{t}.$$

Since $x = 4$, $y = 6$ and $z = 0$ when $t = 0$, we have

$$4 = 2\alpha,$$
$$6 = \alpha + \beta + \gamma,$$
$$0 = \alpha + \beta - \gamma.$$

Thus $\alpha = 2$, $\beta = 1$ and $\gamma = 3$, so

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= 2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{5t} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{3t} + 3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{t} \\ &= \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} e^{5t} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} e^{t}.\end{aligned}$$

Solution to Exercise 4

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

We are given that the eigenvectors are $(1 \quad 0 \quad -1)^T$ and $(k \quad l \quad k)^T$, corresponding to the eigenvalues -1 and 1 (repeated), respectively. But

$$(k \quad l \quad k)^T = k(1 \quad 0 \quad 1)^T + l(0 \quad 1 \quad 0)^T,$$

so the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + \beta\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + \gamma\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}.$$

Since $x = 7$, $y = 5$ and $z = 1$ when $t = 0$, we have

$$7 = \alpha + \beta, \quad 5 = \gamma \quad \text{and} \quad 1 = -\alpha + \beta.$$

Thus $\alpha = 3$, $\beta = 4$ and $\gamma = 5$, so

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= 3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + 4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} \\ &= \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} e^{-t} + \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix} e^{t}.\end{aligned}$$

Solution to Exercise 5

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, and the eigenvalues are $\lambda = 2$ (repeated).

The eigenvector equations are

$$(2 - \lambda)x + 3y = 0 \quad \text{and} \quad (2 - \lambda)y = 0,$$

which reduce, when $\lambda = 2$, to

$$3y = 0 \quad \text{and} \quad 0 = 0,$$

so $y = 0$. It follows that an eigenvector corresponding to $\lambda = 2$ is $(1 \quad 0)^T$, and that one solution is $(1 \quad 0)^T e^{2t}$.

To find the general solution, we solve the equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{b}=\mathbf{v}$, where

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-2\mathbf{I}=\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix},\quad \mathbf{v}=\begin{pmatrix}1\\ 0\end{pmatrix}\quad\text{and}\quad \mathbf{b}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}.$$

This gives

$$\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix},$$

which reduces to $3b_2=1$ with no condition on b_1, so $b_2=\frac{1}{3}$ and we can take $b_1=0$, giving a solution

$$\left(\begin{pmatrix}1\\ 0\end{pmatrix}t+\begin{pmatrix}0\\ \frac{1}{3}\end{pmatrix}\right)e^{2t}=\begin{pmatrix}t\\ \frac{1}{3}\end{pmatrix}e^{2t}.$$

Thus the general solution is

$$\begin{pmatrix}x\\ y\end{pmatrix}=\alpha\begin{pmatrix}1\\ 0\end{pmatrix}e^{2t}+\beta\begin{pmatrix}t\\ \frac{1}{3}\end{pmatrix}e^{2t}.$$

Since $x=4$ and $y=3$ when $t=0$, we have

$$4=\alpha\quad\text{and}\quad 3=\tfrac{1}{3}\beta,$$

giving $\alpha=4$ and $\beta=9$. Thus the required solution is

$$\begin{pmatrix}x\\ y\end{pmatrix}=4\begin{pmatrix}1\\ 0\end{pmatrix}e^{2t}+9\begin{pmatrix}t\\ \frac{1}{3}\end{pmatrix}e^{2t}=\begin{pmatrix}9t+4\\ 3\end{pmatrix}e^{2t}.$$

Solution to Exercise 6

(a) The matrix of coefficients is $\mathbf{A}=\begin{pmatrix}2 & 3\\ 2 & 1\end{pmatrix}$.

Using the given eigenvalues and eigenvectors, we obtain the general solution

$$\begin{pmatrix}x\\ y\end{pmatrix}=\alpha\begin{pmatrix}3\\ 2\end{pmatrix}e^{4t}+\beta\begin{pmatrix}1\\ -1\end{pmatrix}e^{-t}.$$

Since $x=y=1$ when $t=0$, we have

$$1=3\alpha+\beta\quad\text{and}\quad 1=2\alpha-\beta,$$

giving $\alpha=\frac{2}{5}$ and $\beta=-\frac{1}{5}$. The required solution is therefore

$$\begin{aligned}\begin{pmatrix}x\\ y\end{pmatrix}&=\tfrac{2}{5}\begin{pmatrix}3\\ 2\end{pmatrix}e^{4t}-\tfrac{1}{5}\begin{pmatrix}1\\ -1\end{pmatrix}e^{-t}\\ &=\begin{pmatrix}\frac{6}{5}\\ \frac{4}{5}\end{pmatrix}e^{4t}+\begin{pmatrix}-\frac{1}{5}\\ \frac{1}{5}\end{pmatrix}e^{-t}.\end{aligned}$$

(b) The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}$.

Using the given eigenvalues and eigenvectors, we obtain the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(-1+2i)t} + D \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(-1-2i)t}.$$

Now

$$\begin{aligned} \begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(-1+2i)t} &= \begin{pmatrix} \cos 2t + i \sin 2t \\ (-1-i)(\cos 2t + i \sin 2t) \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} \cos 2t + i \sin 2t \\ (\sin 2t - \cos 2t) - i(\sin 2t + \cos 2t) \end{pmatrix} e^{-t}. \end{aligned}$$

So we have

$$\operatorname{Re}\left(\begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(-1+2i)t} \right) = \begin{pmatrix} \cos 2t \\ \sin 2t - \cos 2t \end{pmatrix} e^{-t},$$

$$\operatorname{Im}\left(\begin{pmatrix} 1 \\ -1-i \end{pmatrix} e^{(-1+2i)t} \right) = \begin{pmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix} e^{-t},$$

and the general real-valued solution can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} \cos 2t \\ \sin 2t - \cos 2t \end{pmatrix} e^{-t} + D \begin{pmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix} e^{-t}.$$

Since $x = y = 1$ when $t = 0$, we have

$$1 = C \quad \text{and} \quad 1 = -C - D,$$

so $C = 1$, $D = -2$, and the required particular solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2t - 2 \sin 2t \\ 3 \sin 2t + \cos 2t \end{pmatrix} e^{-t}.$$

Solution to Exercise 7

(a) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{t} + \beta \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} e^{2t} + \gamma \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} e^{3t}.$$

(b) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} e^{t} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} e^{2t} + \gamma \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

Solution to Exercise 8

Using the given eigenvalues and eigenvectors in Procedure 3, we have

$$\begin{aligned}
\mathbf{v}e^{\lambda t} &= \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \end{pmatrix} e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)t} \\
&= e^{t/2}\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \end{pmatrix}\left(\cos\left(\tfrac{\sqrt{3}}{2}t\right)+i\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) \\
&= e^{t/2}\begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right)+i\sin\left(\frac{\sqrt{3}}{2}t\right) \\ 0 \\ \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+i\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \end{pmatrix} \\
&= \underbrace{e^{t/2}\begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ 0 \\ -\frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}}_{\text{real part}} \\
&\quad + i\,\underbrace{e^{t/2}\begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ 0 \\ \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}}_{\text{imaginary part}}.
\end{aligned}$$

Thus the general real-valued solution is

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= C_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^t + C_2\begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ 0 \\ -\frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}e^{t/2} \\
&\quad + C_3\begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ 0 \\ \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}e^{t/2},
\end{aligned}$$

where C_1, C_2 and C_3 are real constants.

Putting $x=y=1$ and $z=2$ when $t=0$, we have

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix},$$

so $C_1+C_2=1$, $C_1=1$ and $-\frac{1}{2}C_2+\frac{\sqrt{3}}{2}C_3=2$, which give $C_2=0$ and $C_3=\frac{4}{\sqrt{3}}$.

Thus the required solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t + \frac{4}{\sqrt{3}} \begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ 0 \\ \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} e^{t/2}.$$

Solution to Exercise 9

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \beta \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} t + \frac{4}{5} \\ -2t - \frac{7}{10} \end{pmatrix}.$$

Solution to Exercise 10

From Example 2, the complementary function is

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.$$

For a particular integral, we try

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} at + b \\ ct + d \end{pmatrix},$$

where a, b, c, d are constants to be determined.

Substituting $x = at + b$, $y = ct + d$ into the differential equations gives

$$\begin{aligned} a &= (at + b) + 4(ct + d) - t + 2, \\ c &= (at + b) - 2(ct + d) + 5t, \end{aligned}$$

which become

$$\begin{aligned} &(a + 4c - 1)t + (b + 4d + 2 - a) = 0, \\ &(a - 2c + 5)t + (b - 2d - c) = 0. \end{aligned}$$

Equating the coefficients of t to zero gives

$$\begin{aligned} a + 4c - 1 &= 0, \\ a - 2c + 5 &= 0, \end{aligned}$$

which have the solution $a = -3$, $c = 1$.

Equating the constant terms to zero, and putting $a = -3$, $c = 1$, gives

$$\begin{aligned} b + 4d + 5 &= 0, \\ b - 2d - 1 &= 0, \end{aligned}$$

which have the solution $b = -1$, $d = -1$.

Thus the required particular integral is

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} -3t - 1 \\ t - 1 \end{pmatrix},$$

and the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -3t - 1 \\ t - 1 \end{pmatrix}.$$

Solution to Exercise 11

From Exercise 10, the complementary function is

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.$$

For a particular integral, we try

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-t},$$

where a and b are constants to be determined.

Substituting $x = ae^{-t}$, $y = be^{-t}$ into the differential equations gives

$$\begin{aligned} -ae^{-t} &= ae^{-t} + 4be^{-t} + 4e^{-t}, \\ -be^{-t} &= ae^{-t} - 2be^{-t} + 5e^{-t}, \end{aligned}$$

which, on dividing by e^{-t} and rearranging, become

$$\begin{aligned} 2a + 4b &= -4, \\ a - b &= -5. \end{aligned}$$

These equations have the solution $a = -4$, $b = 1$.

Thus the required particular integral is

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{-t},$$

and the general solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} \\ &= \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{-t}. \end{aligned}$$

Solution to Exercise 12

The complementary function is

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

We try a particular integral of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{2t}.$$

Then

$$\begin{aligned} 2ae^{2t} &= 2ae^{2t} + 3be^{2t} + e^{2t}, \\ 2be^{2t} &= 2ae^{2t} + be^{2t} + 4e^{2t}, \end{aligned}$$

which give $3b + 1 = 0$ and $b - 2a = 4$, so $b = -\frac{1}{3}$ and $a = -\frac{13}{6}$. The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} - \tfrac{1}{6} \begin{pmatrix} 13 \\ 2 \end{pmatrix} e^{2t}.$$

Putting $t = 0$, we obtain

$$\tfrac{5}{6} = \alpha + 3\beta - \tfrac{13}{6}, \quad \tfrac{2}{3} = -\alpha + 2\beta - \tfrac{1}{3},$$

so $\alpha + 3\beta = 3$ and $-\alpha + 2\beta = 1$, which give $\alpha = \frac{3}{5}$ and $\beta = \frac{4}{5}$.

The required solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{3}{5}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \tfrac{4}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} - \tfrac{1}{6}\begin{pmatrix} 13 \\ 2 \end{pmatrix} e^{2t}.$$

Solution to Exercise 13

These equations are of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_1 + 4\mathbf{h}_2,$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{h}_1 = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}.$$

Choosing a particular integral

$$\mathbf{x}_1 = (at + b \quad ct + d)^T$$

with $\mathbf{h}_1 = (t \quad 0)^T$, and substituting into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_1$, we obtain

$$\begin{aligned} a &= 2(at + b) + 3(ct + d) + t, \\ c &= 2(at + b) + (ct + d), \end{aligned}$$

so

$$\begin{alignedat}{5} 2a & & &+ 3c & & &= -1&, \\ 2a & & &+ c & & &= 0&, \\ -a &+ 2b & & & &+ 3d &= 0&, \\ & 2b &&- c &&+ d &= 0&, \end{alignedat}$$

which have the solution $a = \frac{1}{4}$, $b = -\frac{7}{16}$, $c = -\frac{1}{2}$, $d = \frac{3}{8}$. Hence a particular integral of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_1$ is given by

$$\mathbf{x}_1 = \tfrac{1}{16}\begin{pmatrix} 4t - 7 \\ -8t + 6 \end{pmatrix}.$$

For $\mathbf{h}_2 = (0 \quad e^{-2t})^T$, we try a particular integral

$$\mathbf{x}_2 = (a \quad b)^T e^{-2t},$$

which is substituted into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_2$, followed by dividing through by e^{-2t}, to obtain

$$\begin{aligned} -2a &= 2a + 3b, \\ -2b &= 2a + b + 1, \end{aligned}$$

which simplify to

$$\begin{aligned} 4a + 3b &= 0, \\ 2a + 3b &= -1, \end{aligned}$$

giving the solution $a = \frac{1}{2}$ and $b = -\frac{2}{3}$.

Hence a particular integral of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{h}_2$ is given by

$$\mathbf{x}_2 = \tfrac{1}{6}\begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-2t}.$$

Using the complementary function from Exercise 12 and the principle of superposition, the required general solution is

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + \mathbf{x}_1 + 4\mathbf{x}_2 \\ &= \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + \tfrac{1}{16}\begin{pmatrix} 4t-7 \\ -8t+6 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-2t}.\end{aligned}$$

Solution to Exercise 14

These equations are of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{k}e^{4t}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{k} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

From Exercise 12, $\lambda_1 = -1$ and $\lambda_2 = 4$ are the eigenvalues corresponding to eigenvectors $\mathbf{v}_1 = (1 \quad -1)^T$ and $\mathbf{v}_2 = (3 \quad 2)^T$, respectively.

We need to use Procedure 5 because $e^{4t} = e^{\lambda_2 t}$. We have $\mathbf{k} = p\mathbf{v}_1 + q\mathbf{v}_2$ for some numbers p and q, that is, $(1 \quad 2)^T = p(1 \quad -1)^T + q(3 \quad 2)^T$, which gives $p = -\frac{4}{5}$ and $q = \frac{3}{5}$. A particular integral is of the form $\mathbf{x}_\text{p} = (a\mathbf{v}_1 + b\mathbf{v}_2 t)e^{4t}$, where $a = p/(\lambda_2 - \lambda_1)$ and $b = q$. Substituting our values for p and q into these expressions gives $a = -\frac{4}{5}/5 = -\frac{4}{25}$ and $b = \frac{3}{5}$, hence a particular integral is

$$\mathbf{x}_\text{p} = \left(-\tfrac{4}{25}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \tfrac{3}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} t\right) e^{4t}.$$

The required general solution is therefore

$$\mathbf{x} = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + \tfrac{1}{25}\begin{pmatrix} 45t-4 \\ 30t+4 \end{pmatrix} e^{4t}.$$

Solution to Exercise 15

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$, and we are given that the eigenvectors are $(1 \quad 1)^T$ corresponding to the eigenvalue $\lambda = 7$, and $(1 \quad -1)^T$ corresponding to the eigenvalue $\lambda = 3$.

It follows that the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\sqrt{7}t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\sqrt{7}t} + C_3\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\sqrt{3}t} + C_4\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\sqrt{3}t}.$$

Solution to Exercise 16

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$, so the eigenvalues are $\lambda = 3$ and $\lambda = -1$. Solving the eigenvector equations, we obtain eigenvectors $(2 \quad 1)^T$ and $(-2 \quad 1)^T$, respectively. Using Procedure 6, we then have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \left(C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t} \right) + \begin{pmatrix} -2 \\ 1 \end{pmatrix} (C_3 \cos t + C_4 \sin t).$$

Solution to Exercise 17

Using the given eigenvalues and eigenvectors, we obtain the general solution

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{\sqrt{2}t} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-\sqrt{2}t} \\ &\quad + C_3 \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \cos(\sqrt{3}t) + C_4 \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \sin(\sqrt{3}t) \\ &\quad + C_5 \begin{pmatrix} -5 \\ 4 \\ 14 \end{pmatrix} e^{2t} + C_6 \begin{pmatrix} -5 \\ 4 \\ 14 \end{pmatrix} e^{-2t}. \end{aligned}$$

Solution to Exercise 18

The matrix of coefficients is $\mathbf{A} = \begin{pmatrix} -\frac{25}{7} & \frac{6}{7} \\ \frac{9}{7} & -\frac{10}{7} \end{pmatrix}$.

The characteristic equation of $\mathbf{A}$ is $\lambda^2 + 5\lambda + 4 = 0$, so the eigenvalues are $\lambda = -4$ and $\lambda = -1$.

The eigenvector equations are

$$\begin{aligned} (-\tfrac{25}{7} - \lambda)x + \qquad\qquad \tfrac{6}{7}y &= 0, \\ \tfrac{9}{7}x + (-\tfrac{10}{7} - \lambda)y &= 0. \end{aligned}$$

- For $\lambda = -4$, the eigenvector equations become

  $$\begin{aligned} \tfrac{3}{7}x + \tfrac{6}{7}y &= 0, \\ \tfrac{9}{7}x + \tfrac{18}{7}y &= 0, \end{aligned}$$

  which reduce to the equation $-2y = x$, so a corresponding eigenvector is $(-2 \quad 1)^T$.

- For $\lambda = -1$, the eigenvector equations become

  $$\begin{aligned} -\tfrac{18}{7}x + \tfrac{6}{7}y &= 0, \\ \tfrac{9}{7}x - \tfrac{3}{7}y &= 0, \end{aligned}$$

  which reduce to the equation $y = 3x$, so a corresponding eigenvector is $(1 \quad 3)^T$.

The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} (C_1 \cos 2t + C_2 \sin 2t) + \begin{pmatrix} 1 \\ 3 \end{pmatrix} (C_3 \cos t + C_4 \sin t).$$

The object describes simple harmonic motion along the eigenvectors, so there are two possibilities for simple harmonic motion along a straight line:

- motion in the direction of the vector $(-2 \quad 1)^T$ (i.e. the line $x + 2y = 0$) with angular frequency 2 (corresponding to $C_3 = C_4 = 0$)
- motion in the direction of the vector $(1 \quad 3)^T$ (i.e. the line $y = 3x$) with angular frequency 1 (corresponding to $C_1 = C_2 = 0$).

Acknowledgements

Grateful acknowledgement is made to the following source:

Figure 1: © British Motor Industry Heritage Trust.

Every effort has been made to contact copyright holders. If any have been inadvertently overlooked, the publishers will be pleased to make the necessary arrangements at the first opportunity.