TMA03

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Question 1

Differentiate the following functions, simplifying your answers as far as possible.

a) $f(x) = \cos(x) \ln(\frac{1}{2}x)$

Solution:
Let $h(x) = \cos(x)$, $u = \frac{1}{2}x$, and $g(u) = \ln(u)$. Then by the product rule:

$$\begin{aligned} f'(x) &= h(x)g'(u) + g(u)h'(x) \\ &= \cos(x)g'(u) + g(u)(-\sin x) \end{aligned}$$

By the chain rule, $g'(u) = \frac{1}{u} \times \frac{1}{2} = \frac{1}{x}$. Substituting $g'(u) = \frac{1}{x}$ and $g(u) = \ln(\frac{1}{2}x)$ gives

$$\begin{aligned} f'(x) &= \cos(x) \times \frac{1}{x} + \ln\left(\frac{1}{2}\right)(-\sin x) \\ &= \frac{\cos x}{x} - \ln\left(\frac{x}{2}\right)\sin x \end{aligned}$$

b) $g(y) = \frac{y^{1/3}}{e^{2y+1}}$

Solution:
Let $h(y) = y^{1/3}$, $f(y) = e^{2y+1}$, and $f'(y) = 2e^{2y+1}$ (by the chain rule). Then by the quotient rule:

$$g'(y) = \frac{f(y)h'(y) - h(y)f'(y)}{(f(y))^2}$$

$$= \frac{e^{2y+1}(\frac{1}{3}y^{-\frac{2}{3}}) - y^{\frac{1}{3}}2e^{2y+1})}{(e^{2y+1})(e^{2y+1})}$$

$$= \frac{\frac{2}{3}y^{-\frac{1}{3}} - 2y^{\frac{1}{3}}}{e^{2y+1}}$$

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c) $h(z) = \sin(-z^2 + 2z)$

Solution:
Let $u = -z^2 + 2z$, then by the chain rule:

$$\begin{aligned} h'(z) &= \cos(u) \times (-2z + 2) \\ &= -(2z - 2)\cos(z^2 - 2z) \end{aligned}$$

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d) $k(t) = (\cos(t)e^{7t} + 1)^5$

Solution:
Let $f(t) = 7t$, $g(t) = e^{7t}$, and $h(t) = \cos(t)e^{7t} + 1$. then $f'(t) = 7$, $g'(t) = 7e^{7t}$ (by the chain rule), and $h'(t) = \cos(t)e^{7t} - e^{7t}\sin(t)$ (by the product rule).
Then by the chain rule:

$$k'(t) = 5(e^{7t}\cos(t) + 1)^4(7e^7t\cos(t) - e^{7t}\sin(t))$$

Question 2

a) Write down an equation for the total length of the framing material in terms of the height h and width w. Use this equation to express h in terms of w.

Solution:
The total length of the window frame L is the sum of the outside lengths of the rectangle and semicircle that form it ($L_{Rectangle}$ and $L_{Semicircle}$, respectively) and is given by

$$\begin{aligned} L &= L_{Rectangle} + L_{Semicircle} \\ &= 2h + w + \frac{\pi w}{4} \end{aligned}$$

Substituting $L = 4$ gives

$$4 = 2h + w + \frac{\pi w}{4}$$

$$2h = 4 - w - \frac{\pi w}{4}$$

$$h = 2 - \frac{w}{2} - \frac{\pi w}{8}$$

So the equation $h = 2 - \frac{w}{2} - \frac{\pi w}{8}$ expresses h in terms of w.

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b) Show that the total area A, in terms of w, is

$$A = 2w - \left(\frac{\pi + 4}{8}\right) w^2$$

Solution:
The total area of the window frame A is the sum of the area of the rectangle and semicircle that form it ($A_{Rectangle}$ and $A_{Semicircle}$, respectively) and is given by

$$\begin{aligned} A &= A_{Rectangle} + A_{Semicircle} \\ &= hw + \frac{\pi w^2}{4} \end{aligned}$$

Substituting $h = 2 - \frac{w}{2} - \frac{\pi w}{8}$ as derived in part a of the question, and simplifying gives

$$A = \left(2 - \frac{w}{2} - \frac{\pi w}{8}\right)w + \frac{\pi w^2}{4}$$

$$= 2w - \frac{w^2}{2} - \frac{\pi w^2}{8} + \frac{\pi w^2}{4}$$

$$= 2w - \frac{4w^2 - \pi w^2 + 2\pi w^2}{8}$$

$$= 2w - \left(\frac{\pi + 4}{8}\right)w^2$$

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c) Use differentiation to determine the width w that gives the maximum area A of the window, and show that this is a maximum. What is the maximum area of the window?

Solution:
The equation for the area A in terms of w is a quadratic equation with a negative coefficient on the w^2 term. This means it has a single local maximum at the vertex of its graph (i.e. the first derivative test isn't required). Solving the quadratic equation to find the possible values of w when $A = 0$ gives the endpoints of the interval for which the area can be positive:

$$0 = 2w - \left(\frac{\pi + 4}{8}\right)w^2$$

$$2w = \left(\frac{\pi + 4}{8}\right)w^2$$

$$2 = \left(\frac{\pi + 4}{8}\right)w$$

$$w = 0 \text{ and } w = \frac{16}{\pi + 4}$$

Therefore $w \in (0, \frac{16}{\pi+4})$ when $A > 0$.

Differentiating A with respect to w and solving when $\frac{dA}{dw} = 0$ finds the stationary point (i.e. the vertex) of the equation for A:

$$\frac{dA}{dx} = 2 - \frac{(\pi + 4)w}{4}$$

$$0 = 2 - \frac{(\pi + 4)w}{4}$$

$$= 8 - (\pi + 4)w$$
$$w = \frac{8}{\pi + 4}$$

Calculating the area of the window at the two endpoints of the interval of w and at the vertex gives

$$A_{Lower} = 2 \times 0 - \left(\frac{\pi + 4}{8}\right) \times 0^2$$

$$= 0$$

$$A_{Vertex} = 2 \times \frac{8}{(\pi + 4)} - \left(\frac{\pi + 4}{8}\right) \times \left(\frac{8}{\pi + 4}\right)^2$$

$$= \frac{8}{\pi + 4}$$

$$A_{Upper} = 2 \times \frac{16}{\pi + 4} - \left(\frac{\pi + 4}{8}\right) \times \left(\frac{16}{\pi + 4}\right)^2$$

$$= 0$$

Therefore the width w that gives the maximum area of the window is $\frac{8}{\pi+4}$ meters, which gives a window area of $\frac{8}{\pi+4}$ meters^2.

Question 3

Find the indefinite integral of the function

$$f(x) = \frac{1}{2}\big(2 + \sqrt{x(x+1)}\big)\big(-2 + \sqrt{x(x+1)}\big)$$

Solution:
Simplifying $f(x)$ gives

$$\begin{aligned} f(x) &= \frac{1}{2}\big(2 + \sqrt{x(x+1)}\big)\big(-2 + \sqrt{x(x+1)}\big) \\ &= \frac{1}{2}\big(2 + \sqrt{x^2+x}\big)\big(-2 + \sqrt{x^2+x}\big) \\ &= \frac{1}{2}(x^2 + x - 4) \\ &= \frac{x^2}{2} + \frac{x}{2} - 2 \end{aligned}$$

In this form, the indefinite integral of $f(x)$ is easier to find:

$$\int f(x)dx = \frac{x^3}{6} + \frac{x^2}{4} - 2x + c$$

where c is the constant of integration.

Question 4

Use integration by substitution to find the indefinite integral in part (a) and to evaluate the definite integral in part (b).

a)

$$\int \frac{\sin(x)\cos(x) - e^x}{\cos^2(x) + 2e^x} dx$$

Solution:
Let $u = -\frac{\cos^2(x)}{2} - e^x$ and $\frac{du}{dx} = \sin(x)\cos(x) - e^x$, then integrating by substitution gives

$$\int \frac{\sin(x)\cos(x) - e^x}{\cos^2(x) + 2e^x} dx = \int -\frac{1}{2u} \times \frac{du}{dx} dx$$

$$= \int -\frac{1}{2u} du$$

$$= -\frac{1}{2} \ln |u| + c$$

$$= -\frac{1}{2} \ln(\cos^2(x) + 2e^x) + c$$

where c is the constant of integration.

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b)

$$\int_0^2 (2t-4)^3(t+4)dt$$

Solution:
Let $f(t) = (2t-4)^3(t+4)$, then expanding the brackets and simplifying gives

$$\begin{aligned} f(t) &= (2t-4)^3(t+4) \\ &= 8t^4 - 16t^3 - 96t^2 + 320t - 256 \\ &= 8(t^4 - 2t^3 - 12t^2 + 40t - 32) \end{aligned}$$

Let $F(t) = 8(\frac{1}{5}t^5 - \frac{1}{2}t^4 - 4t^3 + 20t^2 + 32t)$, an antiderivative of $f(t)$, then by the fundamental theorem of calculus:

$$\int_0^2 f(t)dt = [F(t)]_0^2$$

$$\int_0^2 (2t-4)^3(t+4)dt = \left[8\left(\frac{1}{5}t^5 - \frac{1}{2}t^4 - 4t^3 + 20t^2 + 32t\right)\right]_0^2$$

$$= -\frac{704}{5}$$

Question 5

a) Explain why the graph of f lies on or above the x-axis for all values of θ in the interval $[4\pi, 6\pi]$.

Solution:
When $\theta = 4\pi$:

$$\begin{aligned} f(\theta) &= (4\pi + 3)\sin(4\pi/2) \\ &= (4\pi + 3) \times 0 \\ &= 0 \end{aligned}$$

and when $\theta = 6\pi$:

$$\begin{aligned} f(\theta) &= (6\pi + 3)\sin(6\pi/2) \\ &= (6\pi + 3) \times 0 \\ &= 0 \end{aligned}$$

So to determine if $f(\theta) = 0$ when $\theta \in [4\pi, 6\pi]$, I start by finding the first derivative of $f(\theta)$ to identify any stationary points in this interval.

Let $h(\theta) = \theta + 3$ and $g(\theta) = \sin(\theta/2)$, then $h'(\theta) = 1$ and $g'(\theta) = \frac{1}{2}\cos(\theta/2)$. Then using the product rule:

$$\begin{aligned} f'(\theta) &= h(\theta)g'(\theta) + g(\theta)h'(\theta) \\ &= (\theta + 3)\frac{1}{2}\cos(\theta/2) + \sin(\theta/2) \times 1 \\ &= \frac{\theta\cos(\theta/2) + 3\cos(\theta/2)}{2} + \sin(\theta/2) \\ &= \frac{\cos(\theta/2)(\theta + 3)}{2} + \sin(\theta/2) \end{aligned}$$

To identify any stationary points, I substitute $f'(\theta) = 0$ and solve for θ:

$$
\begin{aligned}
0 &= \frac{\cos(\theta/2)(\theta+3)}{2} + \sin(\theta/2) \\
\\
&= \theta\cos(\theta/2) + 3\cos(\theta/2) + 2\sin(\theta/2) \\
\theta\cos(\theta/2) &= -3\cos(\theta/2) - 2\sin(\theta/2) \\
\theta &= -\frac{3\cos(\theta/2) + 2\sin(\theta/2)}{\cos(\theta/2)}
\end{aligned}
$$

As there is only one solution to $f'(\theta)(\theta \in [4\pi, 6\pi])$, there is only a single stationary point of $f(\theta)$ in this interval. To determine if the stationary point is a local maximum or minimum, I apply the first derivative test. As $f(\theta)$ is a continuous function with a single stationary point when $\theta \in [4\pi, 6\pi]$, I evaluate $f'(\theta)$ at $\theta = 4.5\pi$ and $\theta = 5.5\pi$:

$$
\begin{aligned}
f'(4.5\pi) &= \frac{\cos(4.5\pi/2)(4.5\pi+3)}{2} + \sin(4.5\pi/2) \\
&= 6.766...
\end{aligned}
$$

$$
\begin{aligned}
f'(5.5\pi) &= \frac{\cos(5.5\pi/2)(5.5\pi+3)}{2} + \sin(3.5\pi/2) \\
&= -6.462...
\end{aligned}
$$

As the first derivative is positive to the left and negative to the right of the stationary point, the stationary point is a local maximum. Therefore, given that

$f(\theta) = 0$ when $\theta = 4\pi$ and $\theta = 6\pi$, and that there is a single local maximum between these points, the function lies on or above the x-axis in this interval.

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b) Write down an expression, involving a definite integral, that gives the area between the graph of f and the θ-axis, from $\theta = 4\pi$ to $\theta = 6\pi$.

Solution:
Let $F(\theta)$ be an antiderivative of $f(\theta)$, then the definite integral of $f(\theta)$ on the interval $\theta \in [4\pi, 6\pi]$ is given by the expression

$$
\int_{4\pi}^{6\pi} f(\theta)d\theta = \big[F(\theta)\big]_{4\pi}^{6\pi}
$$

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c) Use integration by parts to find the area described in part (b), giving both the exact answer and an approximation to three decimal places.

Solution:
Let $h(\theta) = \theta + 3$, $g(\theta) = \sin(\theta/2)$, and $G(\theta) = 2\cos(\theta/2)$, where $G(\theta)$ is an antiderivative of $g(\theta)$ (found using integration by substitution).
Using integration by parts, the indefinite integral of $f(\theta)$ is

$$\int (\theta+3)\sin(\theta/2)d\theta = h(\theta)G(\theta) - \int h'(\theta)G(\theta)d\theta$$

$$= (\theta+3)(-2\cos(\theta/2)) + \int 2\cos(\theta/2)$$

$$= -2\theta\cos(\theta/2) - 6\cos(\theta/2) + 4\sin(\theta/2)$$

The definite integral of $f(\theta)$ from $\theta = 4\pi$ to $\theta = 6\pi$ is found using the fundamental theorem of calculus:

$$\begin{aligned}\int_{4\pi}^{6\pi} (\theta+3)\sin(\theta/2)d\theta &= \big[-2\theta\cos(\theta/2) - 6\cos(\theta/2) + 4\sin(\theta/2)\big]_{4\pi}^{6\pi} \\ &= 12\pi + 6 - (-8\pi - 6) \\ &= 20\pi + 12 \\ &= 74.8318...\end{aligned}$$

So the area between the graph of $f(\theta)$ and the θ-axis from $\theta = 4\pi$ to $\theta = 6\pi$ is exactly $20\pi + 12$ or approximately 74.832 (2 d.p.).

Question 6

a) Use integration by substitution to find the indefinite integral

$$\int (x^2 + 2)(x + 1)^{42} dx$$

Solution:
Let $u = x + 1$, then $\frac{du}{dx} = 1$, and $x^2 = (u - 1)^2$. Substituting these values into the expression for the indefinite integral gives

$$\begin{aligned}
\int (x^2 + 2)(x + 1)^{42} dx &= \int \big((u - 1)^2 + 2\big)u^{42} \frac{du}{dx} dx \\
&= \int (u^2 - 2u + 3)u^{42} du \\
&= \int u^{44} - 2u^{43} + 3u^{42} du \\
&= \frac{1}{45}u^{45} - \frac{1}{22}u^{44} + \frac{3}{43}u^{43} + c \\
&= \frac{(x + 1)^{45}}{45} - \frac{(x + 1)^{44}}{22} + \frac{3(x + 1)^{43}}{43} + c
\end{aligned}$$

where c is the constant of integration.

Note: I really struggled with this question and would appreciate some more practice on similar questions if you can point me to any.

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b) Use integration by parts for find the indefinite integral

$$\int x^2 e^{-3x} dx$$

Solution:
Let $f(x) = x^2$ and $g(x) = e^{-3x}$, then $f'(x) = 2x$, and $G(x) = -\frac{e^{-3x}}{3}$, where $G(x)$ is an antiderivative of $g(x)$. Substituting these values and substituting by parts gives:

$$
\begin{aligned}
\int x^2 e^{-3x} dx &= f(x)G(x) - \int f'(x)G(x)dx \\
&= x^2\left(-\frac{e^{-3x}}{3}\right) - \int 2x\left(-\frac{e^{-3x}}{3}\right)dx \\
&= -\frac{x^2 e^{-3x}}{3} - 2x\left(\frac{e^{-3x}}{9}\right) - \int 2\left(\frac{e^{-3x}}{9}\right) \\
&= -\frac{x^2 e^{-3x}}{3} - \frac{2xe^{-3x}}{9} - \frac{2e^{-3x}}{27} + c \\
&= -\frac{e^{-3x}(9x^2 + 6x + 2)}{27} + c
\end{aligned}
$$

where c is the constant of differentiation.

Question 7

This question is about the function:

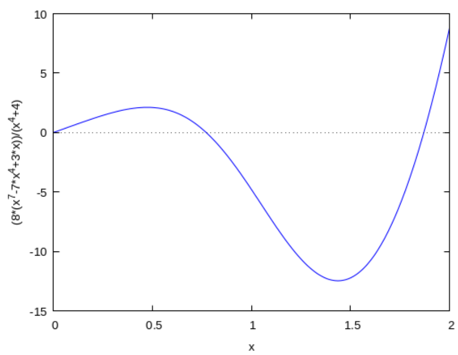
(%i1) `f(x) := (8·(x^7 − 7·x^4 + 3·x)) / (x^4 + 4);`

(%o1) $f(x) := \dfrac{8\,(x^7 - 7\,x^4 + 3\,x)}{x^4 + 4}$

a) Plot the graph of f, choosing ranges of values on the x- and y-axes to make its two stationary points clearly visible.

(%i2) `wxplot2d(f(x), [x, 0, 2], [y, −15, 10]);`

(%t2)



(%o2)

b) Find the derivative of f.

(%i3) `f_prime(x):= ''(diff(f(x), x, 1));`

(%o3) $f_{prime}(x) := \dfrac{8\,(7\,x^6 - 28\,x^3 + 3)}{x^4 + 4} - \dfrac{32\,x^3\,(x^7 - 7\,x^4 + 3\,x)}{(x^4 + 4)^2}$

c) Calculate the x- and y-coordinates of the local maximum of f, giving your answers to three decimal places.

(%i4) `loc_max_x: find_root(f_prime(x), x, 0.4, 0.6);`

(%o4) 0.4731664816180412

(%i5) `loc_max_y: f(loc_max_x);`

(%o5) 2.121284107033947

Therefore, the x- and y-coordinates of the local maximum of f are 0.473 and 2.121, respectively (both to 3 d.p.)

d) Calculate the value of x where the graph first crosses the x-axis to the right of x = 0. Find the area enclosed by the graph of f and the x-axis, between x = 0 and this value, giving your answer to three decimal places.

(%i6) `first_cross: find_root(f(x), x, 0.1, 1.5);`

(%o6) 0.7711708363212311

(%i7) `float(integrate(f(x), x, 0, first_cross));`

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(%o7) 1.035379653886037

The area enclosed by the graph of f and the x-axis between x=0 and where the graph next crosses the x-axis is 1.035 (to 3 d.p.)

Question 8

a) Write a brief explanation of how you will prepare for your exam.

In preparation for the exam, I will start by creating a revision timetable that will set out what topics I aim to revise on each day. I will study at my desk in the space I reserve for studying, in evenings and weekends. I will read through the notes I've taken throughout the module, then attempt the practice quiz on each topic, noting questions I struggle with and contacting my tutor for support. In particular, I foresee requiring more time on trigonometry, vectors, and integration.

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b) i) Approximately how long should you spend on answering a question that is worth 2% of the total marks? Approximately how long should you spend on answering a question that is worth 4% of the total marks?

Solution:
If we consider the 8 questions each worth 4% to be equivalent to 16 questions each worth 2%, then the total number of questions would be $34 + 16 = 50$. 180 minutes divided by 50 questions gives 3.4 minutes per question. Therefore, I should spend approximately 3.6 minutes per question worth 2%, and 7.2 minutes per question worth 4%.

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ii) List any other factors that you may need to take into account when estimating how long you should spend on each question.

Solution:
Questions on topics I am weaker with will require more time. Also some topics, such as integration, typically require more steps in their calculations.

Question 9

Please find the answers to the mini examination paper inline over the next three pages.

MST124 Mini examination paper

This paper has **TWO** sections. You should attempt **ALL** questions in each section.

Section A has 6 questions, each worth 2 marks.
Section B has 2 question, each worth 4 marks.

Each question in Section A is multiple-choice, with **ONE** correct answer from five options. Answer each question by making a mark within the circle to the left-hand side of one of the options given. No marks will be deducted for incorrectly answered questions.

For both questions in Section B, write your answers in the boxes provided. Do not include any working; full marks will be given for a completely correct answer only. No marks will be deducted for incorrectly answered questions.

SECTION A

Question 1

Which of the following is equivalent to $(2-x)(3x+1)-3(1-x^2)$?

○ **A** $-x-1$

○ **B** -1

○ **C** $6x-3$

⊗ **D** $5x-1$

○ **E** $-2x^2-5x-1$

Question 2

Which of the following is equivalent to the inequality $\dfrac{1}{x+2} > x-1$?

○ **A** $-x^2-x-3>0$

○ **B** $\dfrac{-x^2-x+3}{x+2}<0$

○ **C** $-x^2-x-3<0$

⊗ **D** $\dfrac{-x^2-x+3}{x+2}>0$

○ **E** $\dfrac{-x^2-x-1}{x+2}>0$

Question 3

How many values of θ in radians between $-\pi/2$ and $3\pi/2$ satisfy $\cos(\theta) = -1/2$?

- ◯ **A** 1
- ⊗ **B** 2
- ◯ **C** 3
- ◯ **D** 4
- ◯ **E** 5

Question 4

What is the derivative of $e^x \cos(x)$?

- ◯ **A** $e^x \cos(x) + e^x \sin(x)$
- ◯ **B** $-e^x \sin(x)$
- ⊗ **C** $e^x \cos(x) - e^x \sin(x)$
- ◯ **D** $\dfrac{e^x \cos(x) + e^x \sin(x)}{\cos^2(x)}$
- ◯ **E** $e^x - \sin(x)$

Question 5

On which interval is the function $f(x) = -x^2 + 3x - 2$ increasing?

- ⊗ **A** $(-\infty, 3/2)$
- ◯ **B** $(1, 2)$
- ◯ **C** $[1, 2]$
- ◯ **D** $(3/2, \infty)$
- ◯ E $[1, 3/2)$

Question 6

Which of the following is equal to $\int_0^1 \left(\sqrt{x} - 2x\right) \mathrm{d}x$?

- ◯ **A** $-\frac{2}{3}$
- ◯ **B** $\frac{1}{2}$
- ◯ **C** $\frac{1}{3}$
- ◯ **D** $\frac{2}{3}$
- ⊗ **E** $-\frac{1}{3}$

SECTION B

Question 7

A function f has derivative $f'(x) = 2x^2 + x$. What are the stationary points of f, and their natures?

The x-coordinate of one stationary point is at $x =$ $\boxed{0}$.

It is a $\boxed{\text{local minimum}}$ (options: local maximum, local minimum or horizontal point of inflection).

The x-coordinate of the other stationary point is at $x =$ $\boxed{-\frac{1}{2}}$.

It is a $\boxed{\text{local maximum}}$ (options: local maximum, local minimum or horizontal point of inflection).

Question 8

The velocity v (in metres per second) of an object in terms of the time t (in seconds) is given by

$$v = 4t^2 + 1.$$

The displacement of the object at time $t = 1$ is $3\,\text{m}$. What is the displacement of the object (in metres) at time $t = 3$?

The displacement is $\boxed{0.67}$ metres (to 2 d.p.) at $t = 3$.

[END OF QUESTION PAPER]