Unit 4

Matrices and determinants

Introduction

In this unit we examine some of the properties and applications of matrices, where an $m \times n$ **matrix** is a rectangular array of *elements* with m rows and n columns. For example, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a 2×3 matrix.

In this module the elements of matrices are real numbers. Other words for *element* are *component* and *entry*.

Among their many applications, matrices are useful in describing electrical circuits, in the analysis of forces, and in writing down the equations of motion for a system of particles; and they form an essential component of an engineer's or scientist's toolkit. Matrices have a role to play whenever we need to manipulate arrays of numbers; in applications, m and n may be very large, so do not be misled by the fact that the discussion concentrates on small matrices.

In applied mathematics, one common problem is to solve a system of equations involving unknown constants, that is, to determine values of the constants that satisfy the equations. Matrices can be used to store details of such a problem. For example, a system of equations such as

$$\begin{cases} 2x + 3y = 8, \\ 4x - 5y = -6, \end{cases} \tag{1}$$

We sometimes use a curly bracket as shown here to emphasise that we are dealing with a *system* of equations.

contains three relevant pieces of information:

- the numbers on the left-hand sides of the equations, which can be stored in the 2×2 matrix
  $$\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$$
- the constants to be determined, which can be stored in the 2×1 matrix
  $$\begin{pmatrix} x \\ y \end{pmatrix}$$
- the numbers on the right-hand sides of the equations, which can be stored in the 2×1 matrix
  $$\begin{pmatrix} 8 \\ -6 \end{pmatrix}.$$

A matrix with one column is often referred to as a *vector*, or sometimes as a *column vector*. You met vectors in Unit 2.

With this notation, the essential information in these equations can be written in the form

$$\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}. \tag{2}$$

If we put

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ -6 \end{pmatrix},$$

then equation (2) can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{3}$$

As with vectors, in handwritten work, matrices are denoted with a straight or wavy underline, e.g. $\underline{A}$ or $\utilde{A}$.

For the moment you may regard equation (2) as merely a convenient shorthand for the original system of equations, but later we will see that it is compatible with matrix multiplication. Generally, the matrix $\mathbf{A}$ will be an $n \times n$ matrix, while $\mathbf{x}$ and $\mathbf{b}$ are vectors containing n elements, where n may be large. In this unit we will be concerned with the solutions of such systems, which are known as systems of *linear* equations, and the problem of finding a solution can be expressed as one of finding the vector $\mathbf{x}$ that satisfies equation (3).

The term *linear* comes from the fact that each of the equations can be represented graphically by a straight line.

There is a graphical interpretation of the system of equations (1). Each equation represents a straight line, as you can see by rearranging the equations as $y = -\frac{2}{3}x + \frac{8}{3}$ and $y = \frac{4}{5}x + \frac{6}{5}$. The solution of this system of equations thus lies at the point of intersection of the graphs of the straight lines, as illustrated in Figure 1. For this pair of equations, there is just one solution, namely $x = 1$, $y = 2$.

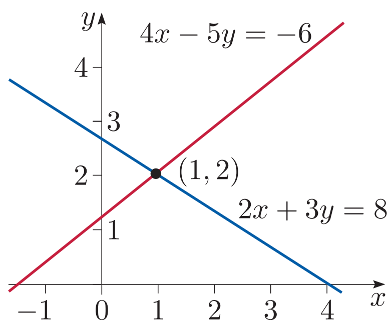


Figure 1 The intersection of two straight lines

For two equations in two unknowns it is easy to draw graphs or to manipulate the equations to determine the solution. It would be much more difficult if we were faced with a problem involving 100 equations in 100 unknowns. Then we would need a systematic method for working with matrices to obtain the solution, because this would enable us to program a computer to solve systems of linear equations.

Matrices and matrix operations are revised in Section 2.

You may already have met a matrix method for solving a system of two linear equations in two unknowns using the inverse of a matrix. Although this method works well for 2×2 matrices, it is not very efficient for large systems of equations compared with other methods.

In Section 1 you will be introduced to a matrix method for solving large systems of linear equations – the Gaussian elimination method – and you will see the conditions required for the method to work. In Section 2 we review some of the properties of matrices that make them so useful, and introduce the determinant of a matrix (a concept that is important for the discussion of eigenvalues in the next unit). In Section 3 we investigate the problem of determining a polynomial of degree n that passes through $n + 1$ data points. Section 4 shows that in certain situations, numerical errors can accumulate and render a solution unreliable.

1 Simultaneous linear equations

The purpose of this section is to outline an efficient systematic method for obtaining the solution of a system of linear equations, known as *Gaussian elimination*. In Subsection 1.1 we introduce the method by manipulating equations, then in Subsection 1.2 we do the same calculations using matrices. To complete this section we look at the types of solution that can arise when solving such systems of linear equations. We illustrate the method by solving systems of three equations in three unknowns.

Gaussian elimination is named after the great mathematician Carl Friedrich Gauss (1777–1855).

1.1 Manipulating equations

We begin with an example where the solution of a system of three equations in three unknowns can be found fairly easily.

Example 1

Find the solution of the system of equations

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, &\quad E_1\\ 10x_2 - 3x_3 &= 34, &\quad E_2\\ 2x_3 &= 4. &\quad E_3 \end{aligned}$$

For ease of reference, we label the equations E_1, E_2 and E_3. The unknowns in our equations will always be written in the order x_1, x_2, x_3.

Solution

Starting with E_3, we obtain $x_3 = 2$. Substituting this value into E_2, we obtain $10x_2 - (3 \times 2) = 34$, so $x_2 = 4$. Substituting the values for x_2 and x_3 into E_1, we obtain $x_1 - (4 \times 4) + (2 \times 2) = -9$, so $x_1 = 3$. Hence the solution is

$$x_1 = 3, \quad x_2 = 4, \quad x_3 = 2.$$

The system of equations in the above example is easy to solve because the equations are in **upper triangular form**, that is, the first non-zero coefficient in E_i is the coefficient of x_i. For a system of equations of the form

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, &\quad E_1\\ 3x_1 - 2x_2 + 3x_3 &= 7, &\quad E_2\\ 8x_1 - 2x_2 + 9x_3 &= 34, &\quad E_3 \end{aligned}$$

the objective of the first stage of the Gaussian elimination process is to manipulate the equations so that they are in upper triangular form. The second stage of the process is to solve the system of equations in upper triangular form using **back substitution**: starting with the last equation, we work *back* to the first equation, *substituting* the known values in order to determine the next value, as demonstrated in Example 1.

Stage 1: elimination

Stage 2: back substitution

The key property that we will use is that we may add and subtract multiples of the equations without affecting the desired solution. For example, $x_1 = 3$, $x_2 = 4$, $x_3 = 2$ satisfies both of the equations

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, \qquad E_1 \\ 3x_1 - 2x_2 + 3x_3 &= 7. \qquad E_2 \end{aligned}$$

Suppose that we form E_{2a} by subtracting 3 times E_1 from E_2 ($E_{2a} = E_2 - 3E_1$):

$$3x_1 - 2x_2 + 3x_3 - 3(x_1 - 4x_2 + 2x_3) = 7 - 3(-9),$$

giving

$$10x_2 - 3x_3 = 34 \qquad E_{2a}.$$

Then $x_1 = 3$, $x_2 = 4$, $x_3 = 2$ also satisfies E_{2a}.

More generally, we could form an equation E_{2a} by writing $E_{2a} = pE_1 + qE_2$ for any numbers p and q. Then E_{2a} is said to be a *linear combination* of E_1 and E_2, and again $x_1 = 3$, $x_2 = 4$, $x_3 = 2$ satisfies E_{2a}.

An *algorithm* is a procedure or set of rules to be used in a calculation.

Our strategy in the elimination stage is to form linear combinations of equations to reduce the system to upper triangular form. The Gaussian elimination method uses a particular algorithm in which linear combinations of the form $E_{2a} = E_2 - mE_1$ are used, as you will see in the following discussion and examples.

Example 2

Use the Gaussian elimination method to reduce the following simultaneous equations (introduced above) to upper triangular form, and hence deduce their solution:

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, \qquad E_1 \\ 3x_1 - 2x_2 + 3x_3 &= 7, \qquad E_2 \\ 8x_1 - 2x_2 + 9x_3 &= 34. \qquad E_3 \end{aligned}$$

Solution

Stage 1: elimination.

Each step of the Gaussian elimination algorithm is indicated in the margin below.

We begin with the elimination stage, where we eliminate x_1 (in Stage 1(a)) and then x_2 (in Stage 1(b)).

Stage 1(a) To eliminate x_1, first we subtract a multiple of E_1 from E_2 to obtain a new equation with no term in x_1. Subtracting 3 times E_1 from E_2 gives

$E_{2a} = E_2 - 3E_1$

$$10x_2 - 3x_3 = 34. \qquad E_{2a}$$

Now we subtract a multiple of E_1 from E_3 to obtain a new equation with no term in x_1. Subtracting 8 times E_1 from E_3 gives

$E_{3a} = E_3 - 8E_1$

$$30x_2 - 7x_3 = 106. \qquad E_{3a}$$

So on completion of Stage 1(a), the equations have been reduced to

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, && E_1\\ 10x_2 - 3x_3 &= 34, && E_{2a}\\ 30x_2 - 7x_3 &= 106. && E_{3a}\end{aligned}$$

Stage 1(b) Next we eliminate x_2. We see that E_{2a} and E_{3a} are two equations in two unknowns. So we subtract a multiple of E_{2a} from E_{3a} to obtain an equation E_{3b} with no term in x_2. Subtracting 3 times E_{2a} from E_{3a} gives

$$2x_3 = 4. \qquad E_{3b}$$

$E_{3b} = E_{3a} - 3E_{2a}$

At this point, the elimination process is finished. We have brought our equations into upper triangular form as

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, && E_1\\ 10x_2 - 3x_3 &= 34, && E_{2a}\\ 2x_3 &= 4. && E_{3b}\end{aligned}$$

Stage 2: back substitution.

This system was solved in Example 1, using back substitution, to give the solution

$$x_1 = 3, \quad x_2 = 4, \quad x_3 = 2.$$

Checking that the solution satisfies the original system of equations, we have

LHS of $E_1 = x_1 - 4x_2 + 2x_3 = (1 \times 3) - (4 \times 4) + (2 \times 2) = -9 = \text{RHS}$,
LHS of $E_2 = 3x_1 - 2x_2 + 3x_3 = (3 \times 3) - (2 \times 4) + (3 \times 2) = 7 = \text{RHS}$,
LHS of $E_3 = 8x_1 - 2x_2 + 9x_3 = (8 \times 3) - (2 \times 4) + (9 \times 2) = 34 = \text{RHS}$.

The process of solving simultaneous equations described in Examples 1 and 2 is called the **Gaussian elimination method**. The method provides a systematic approach to the problem of solving simultaneous equations that should cope with the rather larger sets of equations that can occur in real-life applications. In practice, hand calculation will almost certainly involve fractions, and computer calculations will involve numeric approximations to these fractions and so will introduce rounding errors. We have avoided such problems in Example 2 and in the next exercise.

We do not give a formal procedure for the Gaussian elimination method at this point, since we give a matrix formulation shortly.

Exercise 1

Solve the following simultaneous equations using the Gaussian elimination method:

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, && E_1\\ 5x_1 + 2x_2 + 2x_3 &= 20, && E_2\\ 4x_1 - 2x_2 - 3x_3 &= 15. && E_3\end{aligned}$$

1.2 Manipulating matrices

The Gaussian elimination method relies on manipulating equations. In this subsection you will see that it can be formulated efficiently in terms of matrices.

When we use the Gaussian elimination method to solve the equations in Example 2, the new equation E_{2a} is obtained by subtracting a multiple of E_1 from E_2. The actions are performed on the numbers multiplying x_1, x_2 and x_3 (the coefficients of x_1, x_2 and x_3). For instance, the operation on the coefficients of x_2 in the equation $E_{2a} = E_2 - 3E_1$, namely $-2x_2 - 3 \times (-4x_2) = 10x_2$, could be recorded as $-2 - 3 \times (-4) = 10$, provided that we remember that the operation is associated with x_2.

Thus during the elimination stage, we need to record just the coefficients of x_1, x_2 and x_3, and the right-hand side of each equation, rather than the whole system of equations each time. We record the coefficients in E_1, E_2 and E_3 in a **coefficient matrix A**, the unknown constants in a vector $\mathbf{x}$, and the right-hand sides in a **right-hand-side vector b** as

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 2 \\ 3 & -2 & 3 \\ 8 & -2 & 9 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -9 \\ 7 \\ 34 \end{pmatrix}.$$

The problem, in terms of matrices, is to determine the vector $\mathbf{x}$ that satisfies the equation $\mathbf{Ax} = \mathbf{b}$, which can be written as

$$\underbrace{\begin{pmatrix} 1 & -4 & 2 \\ 3 & -2 & 3 \\ 8 & -2 & 9 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} -9 \\ 7 \\ 34 \end{pmatrix}}_{\mathbf{b}}.$$

For computing purposes it is sufficient just to record the information in the **augmented matrix**

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & -4 & 2 & -9 \\ 3 & -2 & 3 & 7 \\ 8 & -2 & 9 & 34 \end{array}\right).$$

Alternatively, we could say that the first three columns of $\mathbf{A}|\mathbf{b}$ represent, respectively, the coefficients of x_1, x_2 and x_3, and the column after the bar represents the right-hand sides of the equations.

The first row of $\mathbf{A}|\mathbf{b}$ contains the coefficients of x_1, x_2 and x_3, and the right-hand side of E_1; the second row contains similar information about E_2; and the third row contains similar information about E_3.

Exercise 2

Write the following systems of equations in augmented matrix form.

(a) $\begin{cases} 3x_1 - 5x_2 = 8 \\ 4x_1 + 7x_2 = 11 \end{cases}$ (b) $\begin{cases} x_1 + 2x_2 \phantom{{}+x_3} = 3 \\ 2x_1 - x_2 + x_3 = 1 \\ \phantom{2x_1 - {}} x_2 - x_3 = -1 \end{cases}$

Once the information has been written in augmented matrix form, the stages in the Gaussian elimination process are equivalent to manipulating the rows of the matrix, as we demonstrate in the next example.

Example 3

Solve the simultaneous equations of Example 2 using the matrix form of the Gaussian elimination method, that is,

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & -4 & 2 & -9 \\ 3 & -2 & 3 & 7 \\ 8 & -2 & 9 & 34 \end{array}\right).$$

Solution

We use the matrix representing these equations throughout the elimination procedure. For brevity, we denote the first row of the matrix, namely $\left(\begin{array}{ccc|c} 1 & -4 & 2 & -9 \end{array}\right)$, by $\mathbf{R}_1$, and so on.

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & -4 & 2 & -9 \\ 3 & -2 & 3 & 7 \\ 8 & -2 & 9 & 34 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Notice that $\mathbf{R}_i$ encapsulates all the information about the equation E_i.

Stage 1: elimination.

Each part of the elimination stage in Example 2 has an equivalent part in matrix form.

Stage 1(a) First we eliminate x_1, as before. Equation E_{2a} in Example 2 was found by subtracting $3E_1$ from E_2. Arithmetically, this is the same operation as subtracting $3\mathbf{R}_1$ from $\mathbf{R}_2$, and it is useful to record this operation. The way in which the new rows are formed is recorded on the left of the matrix.

Adding or subtracting a multiple of one row to/from another is called a **row operation**.

It is essential to keep a written record of the derivation of each row if you wish to check your working, as illustrated in the example.

$$\begin{array}{l} \\ \mathbf{R}_2 - 3\mathbf{R}_1 \\ \mathbf{R}_3 - 8\mathbf{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & -4 & 2 & -9 \\ 0 & 10 & -3 & 34 \\ 0 & 30 & -7 & 106 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) Now we eliminate x_2, as before.

$$\begin{array}{l} \\ \\ \mathbf{R}_{3a} - 3\mathbf{R}_{2a} \end{array} \left(\begin{array}{ccc|c} 1 & -4 & 2 & -9 \\ 0 & 10 & -3 & 34 \\ 0 & 0 & 2 & 4 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

A triangle of zeros appears in the matrix at the end of the elimination stage. This shows us that each equation has one fewer unknown than the previous one, which is what we need in order to do the back substitution.

The final coefficient matrix is known as an **upper triangular matrix** because the only non-zero elements that it contains are on or above the **leading diagonal**, that is, the diagonal from top left to bottom right, here containing the numbers 1, 10 and 2.

The leading diagonal is sometimes called the *main diagonal*.

Stage 2: back substitution.

Before carrying out the back substitution stage, we write the final matrix as a system of equations:

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= -9, \qquad & E_1 \\ 10x_2 - 3x_3 &= 34, \qquad & E_{2a} \\ 2x_3 &= 4. \qquad & E_{3b} \end{aligned}$$

This is exactly the same as in Example 2. The solution $x_1 = 3$, $x_2 = 4$, $x_3 = 2$ is then found using back substitution as before.

The objective of Stage 1 of the Gaussian elimination method is to reduce the matrix $\mathbf{A}$ to an upper triangular matrix, $\mathbf{U}$ say. At the same time, the right-hand-side vector $\mathbf{b}$ is transformed into a new right-hand-side vector, $\mathbf{c}$ say, where in this example,

$$\mathbf{U} = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 10 & -3 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -9 \\ 34 \\ 4 \end{pmatrix}.$$

The Gaussian elimination method in Procedure 1 underpins the algorithm used by many computer algebra packages.

This procedure does not always work. We examine cases where it breaks down in the next subsection.

Procedure 1 Gaussian elimination method

To solve a system of n linear equations in n unknowns, with coefficient matrix $\mathbf{A}$ and right-hand-side vector $\mathbf{b}$, carry out the following steps.

1. Write down the augmented matrix $\mathbf{A}|\mathbf{b}$ with rows $\mathbf{R}_1, \ldots, \mathbf{R}_n$.

Stage 1: elimination

Stage 1(a)

2. Subtract multiples of $\mathbf{R}_1$ from $\mathbf{R}_2, \mathbf{R}_3, \ldots, \mathbf{R}_n$ to reduce the elements below the leading diagonal in the first column to zero.

Stage 1(b)

   In the new matrix obtained, subtract multiples of $\mathbf{R}_2$ from $\mathbf{R}_3, \mathbf{R}_4, \ldots, \mathbf{R}_n$ to reduce the elements below the leading diagonal in the second column to zero.

Stage 1(c), ...

   Continue this process until $\mathbf{A}|\mathbf{b}$ is reduced to $\mathbf{U}|\mathbf{c}$, where $\mathbf{U}$ is an upper triangular matrix.

Stage 2: back substitution

3. Solve the system of equations with coefficient matrix $\mathbf{U}$ and right-hand-side vector $\mathbf{c}$ by back substitution.

The steps of the elimination stage of Example 3 are

$$\begin{aligned} \mathbf{R}_{2\text{a}} &= \mathbf{R}_2 - 3\mathbf{R}_1, \\ \mathbf{R}_{3\text{a}} &= \mathbf{R}_3 - 8\mathbf{R}_1, \\ \mathbf{R}_{3\text{b}} &= \mathbf{R}_{3\text{a}} - 3\mathbf{R}_{2\text{a}}. \end{aligned}$$

The numbers 3, 8 and 3 are called *multipliers*. In general, to obtain, for example, an equation $E_{2\text{a}}$ without a term in x_1 from

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots &= d_1, && E_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots &= d_2, && E_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots &= d_3, && E_3 \end{aligned}$$

where $a_{11} \neq 0$, we subtract $(a_{21}/a_{11})E_1$ from E_2. The number a_{21}/a_{11} is the **multiplier**.

In forming a multiplier, we divide by a number, a_{11} in the above generalisation. The number by which we divide is referred to as a **pivot** or **pivot element**, and the row in which it lies is the **pivot row**.

Looking again at Example 3: in Stage 1(a) the multipliers are $3 = 3/1 = a_{21}/a_{11}$ and $8 = 8/1 = a_{31}/a_{11}$, and the pivot is $a_{11} = 1$; in Stage 1(b) the multiplier is $3 = 30/10 = a_{32}/a_{22}$, and the pivot is $a_{22} = 10$. In general, the kth pivot is the number in the denominator of the multipliers in Stage $1(k)$ of the elimination stage. At the end of the elimination stage, the pivots comprise the elements of the leading diagonal of $\mathbf{U}$.

Example 4

Use the matrix form of the Gaussian elimination method to solve the following simultaneous equations:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 1, \qquad && E_1 \\ 5x_1 + x_2 + 2x_3 &= 6, && E_2 \\ 4x_1 - 2x_2 - 3x_3 &= 3. && E_3 \end{aligned}$$

Solution

The augmented matrix representing these equations is as follows.

The row operations used to solve this example are highlighted in the margin below.

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 5 & 1 & 2 & 6 \\ 4 & -2 & -3 & 3 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

The pivot is a_{11}.

$$\begin{array}{r} \\ \mathbf{R}_2 - \frac{5}{3}\mathbf{R}_1 \\ \mathbf{R}_3 - \frac{4}{3}\mathbf{R}_1 \end{array} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 0 & -\frac{2}{3} & \frac{11}{3} & \frac{13}{3} \\ 0 & -\frac{10}{3} & -\frac{5}{3} & \frac{5}{3} \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

$\mathbf{R}_{2a} = \mathbf{R}_2 - (a_{21}/a_{11})\mathbf{R}_1$

$\mathbf{R}_{3a} = \mathbf{R}_3 - (a_{31}/a_{11})\mathbf{R}_1$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

The pivot is a_{22}.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3a} - \frac{(-10/3)}{(-2/3)}\mathbf{R}_{2a} \end{array} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 0 & -\frac{2}{3} & \frac{11}{3} & \frac{13}{3} \\ 0 & 0 & -20 & -20 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

$\mathbf{R}_{3b} = \mathbf{R}_{3a} - (a_{32}/a_{22})\mathbf{R}_{2a}$, so here $\mathbf{R}_{3b} = \mathbf{R}_{3a} - 5\mathbf{R}_{2a}$.

Stage 2 The equations represented by the new matrix are

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 1, \qquad && E_1 \\ -\tfrac{2}{3}x_2 + \tfrac{11}{3}x_3 &= \tfrac{13}{3}, && E_{2a} \\ -20x_3 &= -20. && E_{3b} \end{aligned}$$

From E_{3b}, we have $x_3 = 1$. From E_{2a}, we have $-\frac{2}{3}x_2 + \frac{11}{3} = \frac{13}{3}$, so $x_2 = -1$. From E_1, we have $3x_1 - 1 - 1 = 1$, so $x_1 = 1$. Hence the solution is

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1.$$

You can test the validity of this solution by showing that E_2 and E_3 are satisfied by these values. Such a final step is always a worthwhile check on your working.

Exercise 3

Use the matrix form of the Gaussian elimination method to solve the following simultaneous equations:

$$\begin{aligned} x_1 + 2x_2 &= 4, && E_1 \\ 3x_1 - x_2 &= 5. && E_2 \end{aligned}$$

Exercise 4

Use the matrix form of the Gaussian elimination method to solve the following simultaneous equations:

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, && E_1 \\ 5x_1 + 2x_2 + 2x_3 &= 20, && E_2 \\ 4x_1 - 2x_2 - 3x_3 &= 15. && E_3 \end{aligned}$$

1.3 Special cases

The previous examples and exercises may have led you to believe that Procedure 1 will always be successful, but this is not the case. The procedure will fail if at any stage of the calculation a pivot is zero. We will see that sometimes it is possible to overcome this difficulty, but this is not so in every case. In the following example we point out some difficulties, and indicate whether they can be overcome.

Example 5

Consider the following four systems of linear equations. Try to solve them using the matrix form of the Gaussian elimination method as given in Procedure 1. In each case the method breaks down. Suggest, if possible, a method of overcoming the difficulty, and hence determine the solution if one exists.

(a) $$\begin{cases} \phantom{x_1 -{}} 10x_2 - 3x_3 = 34 \\ x_1 - 4x_2 + 2x_3 = -9 \\ \phantom{x_1 - 4x_2 +{}} 2x_3 = 4 \end{cases}$$

(b) $$\begin{cases} x_1 + 10x_2 - 3x_3 = 8 \\ x_1 + 10x_2 + 2x_3 = 13 \\ x_1 + 4x_2 + 2x_3 = 7 \end{cases}$$

(c) $$\begin{cases} x_1 + 4x_2 - 3x_3 = 2 \\ x_1 + 2x_2 + 2x_3 = 5 \\ 2x_1 + 2x_2 + 9x_3 = 7 \end{cases}$$

(d) $$\begin{cases} x_1 + 4x_2 - 3x_3 = 2 \\ x_1 + 2x_2 + 2x_3 = 5 \\ 2x_1 + 2x_2 + 9x_3 = 13 \end{cases}$$

Solution

(a) Since the first pivot a_{11} is zero, there is no multiple of the first row that we can subtract from the second row to eliminate the term in x_1. However, interchanging two equations does not change the solution of a system of equations. Hence interchanging the first two equations gives a system of equations in upper triangular form, from which we can determine the solution

This may appear to be a trivial difficulty, but if we hope to devise a procedure that could be implemented on a computer, the process must allow for every contingency.

$$x_1 = 3, \quad x_2 = 4, \quad x_3 = 2.$$

(b) We begin by writing down the augmented matrix.

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 10 & -3 & 8 \\ 1 & 10 & 2 & 13 \\ 1 & 4 & 2 & 7 \end{array}\right) \end{array} \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2 - \mathbf{R}_1 \\ \mathbf{R}_3 - \mathbf{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 10 & -3 & 8 \\ 0 & 0 & 5 & 5 \\ 0 & -6 & 5 & -1 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We now want to reduce the element in column 2 that is below the leading diagonal to zero. The difficulty here is that we cannot subtract a multiple of $\mathbf{R}_{2a}$ from $\mathbf{R}_{3a}$ to eliminate the coefficient of x_2, since the pivot a_{22} is zero.

We can overcome this difficulty by interchanging $\mathbf{R}_{2a}$ and $\mathbf{R}_{3a}$.

$$\begin{array}{r} \\ \\ \mathbf{R}_{2a} \leftrightarrow \mathbf{R}_{3a} \end{array} \left(\begin{array}{ccc|c} 1 & 10 & -3 & 8 \\ 0 & -6 & 5 & -1 \\ 0 & 0 & 5 & 5 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2b} \\ \mathbf{R}_{3b} \end{array}$$

The notation $\mathbf{R}_{2a} \leftrightarrow \mathbf{R}_{3a}$ indicates that we have interchanged $\mathbf{R}_{2a}$ with $\mathbf{R}_{3a}$. Such interchanges are also called row operations.

This is now in upper triangular form, and we can proceed to Stage 2 to determine the solution using back substitution. We find that the solution is

$$x_1 = x_2 = x_3 = 1.$$

(c) We begin by writing down the augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 1 & 2 & 2 & 5 \\ 2 & 2 & 9 & 7 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2 - \mathbf{R}_1 \\ \mathbf{R}_3 - 2\mathbf{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 0 & -2 & 5 & 3 \\ 0 & -6 & 15 & 3 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3a}-3\mathbf{R}_{2a} \end{array} \left(\begin{array}{rrr|r} 1 & 4 & -3 & 2 \\ 0 & -2 & 5 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2 We now try to solve the system of equations represented by the above matrix by back substitution. The coefficient matrix is in upper triangular form, but if we write out the system of equations as

$$\begin{aligned} x_1 + 4x_2 - 3x_3 &= 2, \\ -2x_2 + 5x_3 &= 3, \\ 0x_3 &= -6, \end{aligned}$$

we see that the last equation has no solution since no value of x_3 can give $0x_3 = -6$. Hence the system of equations has no solution.

(d) We begin by writing down the augmented matrix.

$$\left(\begin{array}{rrr|r} 1 & 4 & -3 & 2 \\ 1 & 2 & 2 & 5 \\ 2 & 2 & 9 & 13 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

We will refer back to these steps in Example 6.

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2-\mathbf{R}_1 \\ \mathbf{R}_3-2\mathbf{R}_1 \end{array} \left(\begin{array}{rrr|r} 1 & 4 & -3 & 2 \\ 0 & -2 & 5 & 3 \\ 0 & -6 & 15 & 9 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3a}-3\mathbf{R}_{2a} \end{array} \left(\begin{array}{rrr|r} 1 & 4 & -3 & 2 \\ 0 & -2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2 We now try to solve the system of equations represented by the above matrix by back substitution. The coefficient matrix is in upper triangular form, but if we write out the system of equations as

Systems of equations of this kind occur in Unit 5, so you should take careful note of the method of solution used in this example.

$$\begin{aligned} x_1 + 4x_2 - 3x_3 &= 2, \\ -2x_2 + 5x_3 &= 3, \\ 0x_3 &= 0, \end{aligned}$$

we see that *any* value of x_3 gives $0x_3 = 0$. If we let $x_3 = k$, where k is an arbitrary number, then proceeding with the back substitution gives $-2x_2 + 5k = 3$, so $x_2 = -\frac{3}{2} + \frac{5}{2}k$. The first equation gives $x_1 + (-6 + 10k) - 3k = 2$, so $x_1 = 8 - 7k$. So there is an infinite number of solutions of the form

$$x_1 = 8 - 7k, \quad x_2 = -\tfrac{3}{2} + \tfrac{5}{2}k, \quad x_3 = k.$$

The *general solution* can be written as $(8-7k \quad -\frac{3}{2}+\frac{5}{2}k \quad k)^T$, where k is an arbitrary number. You should verify that this solution satisfies the original equations.

Here we have used the *transpose* notation, which you met in Unit 2. More will be said about transposes of matrices in Subsection 2.1.

In Example 5 parts (a) and (b), we were able to overcome the difficulty of a zero pivot by making an **essential row interchange**. In general, whenever one of the pivots is zero, we interchange that row of the augmented matrix with the first available row below it that would lead to a non-zero pivot, effectively reordering the original system of equations. The difficulties in (a) and (b) are thus easily overcome, but those occurring in (c) and (d) are more fundamental. In (c) there was a zero pivot that could not be avoided by interchanging rows, and we were left with an **inconsistent system of equations** for which there is no solution. The final example (d) illustrates the case where a pivot is zero and the system of equations has an **infinite number of solutions**. It is these last two cases in particular that we explore in the next subsection.

1.4 No solutions and an infinite number of solutions

We begin by looking at the general system of two linear equations in two unknowns, given by

$$\begin{aligned} ax + by &= e, \qquad && E_1 \\ cx + dy &= f. \qquad && E_2 \end{aligned}$$

All the points satisfying E_1 lie on one straight line, while all the points satisfying E_2 lie on another. The solution of the system of linear equations can be described graphically as the coordinates of the point of intersection of these two lines. However, if we draw two lines at random in a plane, there are three situations that can arise, as illustrated in Figure 2.

- Figure 2(a) illustrates the typical case, where the two lines are not parallel and there is a *unique solution* of the system of linear equations, corresponding to the point of intersection of the two lines.
- Figure 2(b) illustrates the special case where the two lines are parallel (thus do not intersect) and the corresponding system of linear equations has *no solution*.
- Figure 2(c) illustrates the very special case where the two lines coincide, so any point on the line satisfies both equations and we have an *infinite number of solutions*.

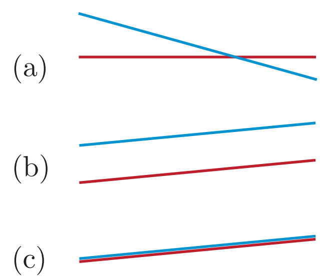


Figure 2 Points of intersection of two straight lines

Exercise 5

For each of the following pairs of linear equations, sketch their graphs and hence determine the number of solutions.

(a) $\begin{cases} x + y = 4 \\ 2x + 2y = 5 \end{cases}$ (b) $\begin{cases} x + y = 4 \\ 2x - 3y = 8 \end{cases}$

(c) $\begin{cases} x + y = 4 \\ 2x + 2y = 8 \end{cases}$ (d) $\begin{cases} y = 4 \\ 2x + 2y = 5 \end{cases}$

Exercise 6

For the system of equations

$$ax + by = e,$$
$$cx + dy = f,$$

what conditions ensure that the two lines are parallel or coincident?

A linear equation involving three unknowns x, y and z of the form

$$ax + by + cz = d,$$

where a, b, c and d are constants, can be represented graphically as a plane in three-dimensional space. (Further explanation of this graphical representation will be given in Unit 7.) For a system of three equations in three unknowns, a graphical interpretation gives rise to three types of solution, as illustrated in Figure 3 where none of the planes are parallel or coincident. (Two non-parallel or non-coincident planes intersect along a line called the *line of intersection*.)

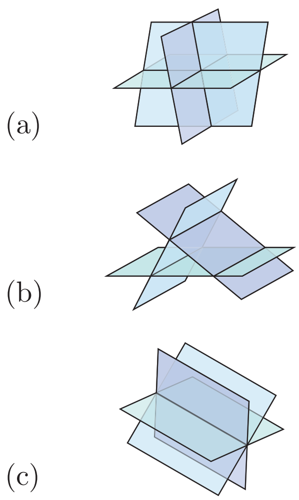


Figure 3 Points of intersection of three planes

- Figure 3(a) illustrates the situation when the three planes intersect at a single point, so there is a *unique solution* at the point of intersection.
- Figure 3(b) illustrates the situation when the three planes form a tent shape, having no point of intersection and hence *no solution* (the three lines where a pair of planes meet are parallel). A variant of this case occurs when two (or even all three) of the planes are parallel, thus cannot meet at all.
- Figure 3(c) illustrates the situation when the three planes have (at least) a common line of intersection and hence an *infinite number of solutions*. A variant of this case occurs when at least two of the planes are coincident.

The next example gives an algebraic interpretation of one of these three types of solution. For this example, the following definition is required.

Linear combination

A **linear combination** of rows $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ is a row $\mathbf{R}$ such that $\mathbf{R} = q_1\mathbf{R}_1 + q_2\mathbf{R}_2 + \cdots + q_n\mathbf{R}_n$, where the q_i $(i = 1, 2, \ldots, n)$ are numbers.

A **non-trivial linear combination** is one where at least one q_i is non-zero.

Example 6

Consider the system of linear equations in Example 5(d), where the corresponding augmented matrix is as follows.

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 1 & 2 & 2 & 5 \\ 2 & 2 & 9 & 13 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Show that there is a non-trivial linear combination of the rows of this matrix that is equal to a row of zeros.

Solution

We use the results of the elimination process in Example 5(d), noting that $\mathbf{R}_{3\mathrm{b}}$ is equal to a row of zeros, that is, $\mathbf{R}_{3\mathrm{b}} = \mathbf{0}$. From this, we see that

We use $\mathbf{0}$ to denote a row of zeros (or 'zero row').

$$\begin{aligned} \mathbf{R}_{3\mathrm{b}} &= \mathbf{R}_{3\mathrm{a}} - 3\mathbf{R}_{2\mathrm{a}} \\ &= (\mathbf{R}_3 - 2\mathbf{R}_1) - 3(\mathbf{R}_2 - \mathbf{R}_1) = \mathbf{R}_1 - 3\mathbf{R}_2 + \mathbf{R}_3 = \mathbf{0}. \end{aligned}$$

Hence there is a non-trivial linear combination of the rows that is equal to the zero row.

In Example 6 we saw that $\mathbf{R}_{3\mathrm{b}} = \mathbf{R}_1 - 3\mathbf{R}_2 + \mathbf{R}_3$, which means that $\mathbf{R}_{3\mathrm{b}}$ is a linear combination of the rows of $\mathbf{A}|\mathbf{b}$. However, in this example, we have something more: such a linear combination produces a row of zeros (and the corresponding equations have an infinite number of solutions, as we found in Example 5(d)). When such a relationship exists between the rows of a matrix, we say that the rows are *linearly dependent*.

This is the case where the three planes, corresponding to the three equations, meet in a line (see Figure 3(c)).

Linear dependence and linear independence

The rows $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_n$ of a matrix are **linearly dependent** if a non-trivial linear combination of these rows is equal to the zero row, that is,

$$q_1\mathbf{R}_1 + q_2\mathbf{R}_2 + \cdots + q_n\mathbf{R}_n = \mathbf{0}, \tag{4}$$

where the numbers $q_1, q_2, \dots, q_n$ are not all zero.

Rows that are not linearly dependent are **linearly independent**.

If the rows of the matrix $\mathbf{A}$ are linearly independent, then the Gaussian elimination method works and produces a unique solution of the system of equations $\mathbf{Ax} = \mathbf{b}$. However, if the rows of $\mathbf{A}$ are linearly dependent, then the corresponding system of linear equations may have an infinite number of solutions, or no solution. The elimination process will reveal which is the case.

Exercise 7

In the following systems of linear equations, determine which has a single solution, which has an infinite number of solutions, and which has no solution. Where the equations have a unique solution, find it. Where the equations have an infinite number of solutions, find the general solution and a non-trivial linear combination of the rows of $\mathbf{A}|\mathbf{b}$ that gives a row of zeros.

(a) $\begin{cases} x_1 - 2x_2 + 5x_3 = 7 \\ x_1 + 3x_2 - 4x_3 = 20 \\ x_1 + 18x_2 - 31x_3 = 40 \end{cases}$ (b) $\begin{cases} x_1 - 2x_2 + 5x_3 = 6 \\ x_1 + 3x_2 - 4x_3 = 7 \\ 2x_1 + 6x_2 - 12x_3 = 12 \end{cases}$

(c) $\begin{cases} x_1 - 4x_2 + x_3 = 14 \\ 5x_1 - x_2 - x_3 = 2 \\ 6x_1 + 14x_2 - 6x_3 = -52 \end{cases}$

2 Properties of matrices

Matrices were first used in 1858 by the English mathematician Arthur Cayley.

In this section we review the algebraic properties of matrices, and show how solving the matrix equation $\mathbf{Ax} = \mathbf{b}$ can be interpreted as finding the vector $\mathbf{x}$ that is mapped to the vector $\mathbf{b}$ by the transformation defined by $\mathbf{A}$. Then we investigate a related number, called the *determinant* of the matrix, that can be used to decide whether a given system of linear equations has a unique solution. Finally, we look at some applications of determinants.

2.1 Algebra of matrices

You met these notations for column vectors in Unit 2.

The element a_{ij} is in the ith row and the jth column.

A matrix of **order** or **size** $m \times n$ is a rectangular array of elements (usually real numbers) with m rows and n columns. If $m = n$, the matrix is a **square matrix**, an example of which is the 2×2 matrix $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$. If $m = 1$, the matrix can be regarded as a **row vector**, an example of which is the 1×3 matrix $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$. If $n = 1$, the matrix can be regarded as a **column vector**, an example of which is the 2×1 matrix $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$, which we often write in text as $(5 \quad 7)^T$. The general $m \times n$ matrix $\mathbf{A}$ can be written as (a_{ij}) to denote the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are **equal** if they have the same order and $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. If all the elements of a matrix are zero, the matrix is the **zero matrix**, denoted by $\mathbf{0}$. (Strictly speaking, we should write the $m \times n$ zero matrix as $\mathbf{0}_{mn}$, but the size of the zero matrix will always be clear from the context.)

Addition and scalar multiplication of $m \times n$ matrices

If $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are matrices of the same order, we can form the **sum**

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij}) \quad \text{(componentwise addition).} \tag{5}$$

Note that matrix addition is commutative, $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, and associative, $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

For any matrix $\mathbf{A}$, and $\mathbf{0}$ of the same order,

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A} \quad \text{(the components are unaltered).} \tag{6}$$

The **scalar multiple** of a matrix $\mathbf{A} = (a_{ij})$ by a number k is given by

$$k\mathbf{A} = (ka_{ij}) \quad \text{(multiply each component by } k\text{).} \tag{7}$$

The **negative** of the matrix $\mathbf{A} = (a_{ij})$ is $-\mathbf{A} = (-a_{ij})$, so

$$\mathbf{A} + (-\mathbf{A}) = -\mathbf{A} + \mathbf{A} = \mathbf{0}. \tag{8}$$

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, $\mathbf{A} - \mathbf{B}$ is given by

$$\mathbf{A} + (-\mathbf{B}). \tag{9}$$

In this context, the number k is sometimes referred to as a *scalar* in order to distinguish it from a matrix. The same idea was used in Unit 2 in the context of vectors.

Exercise 8

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 7 \\ 4 & -5 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & 1 & -6 \\ -5 & 3 & 3 \end{pmatrix}$.

(a) Calculate $3\mathbf{A}$ and $-\mathbf{A}$.

(b) Calculate $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{C}$.

(c) Use the results of part (b) to verify that $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Exercise 9

Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 7 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 0 \\ 4 & -5 \end{pmatrix}$.

(a) Calculate $\mathbf{A} - \mathbf{B}$ and $\mathbf{B} - \mathbf{A}$.

(b) Verify that $\mathbf{B} - \mathbf{A} = -(\mathbf{A} - \mathbf{B})$.

Exercise 10

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 & 7 \\ 4 & -5 & 0 \end{pmatrix}$.

(a) Calculate $2\mathbf{A} - 5\mathbf{B}$.

(b) Verify that $3(\mathbf{A} + \mathbf{B}) = 3\mathbf{A} + 3\mathbf{B}$.

The multiplication of matrices is more complicated. We illustrate the method by forming the product of two 2×2 matrices.

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. To form the product $\mathbf{AB} = \mathbf{C}$, we define c_{ij} using the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$, so that

$$\begin{aligned}
c_{11} &= a_{11}b_{11} + a_{12}b_{21} = (1 \times 5) + (2 \times 7) = 19, \\
c_{12} &= a_{11}b_{12} + a_{12}b_{22} = (1 \times 6) + (2 \times 8) = 22, \\
c_{21} &= a_{21}b_{11} + a_{22}b_{21} = (3 \times 5) + (4 \times 7) = 43, \\
c_{22} &= a_{21}b_{12} + a_{22}b_{22} = (3 \times 6) + (4 \times 8) = 50.
\end{aligned}$$

Thus

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

Similarly, if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 5 & 3 \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{AB} &= \begin{pmatrix} (1 \times 2) + (2 \times (-1)) + (3 \times 5) & (1 \times 4) + (2 \times (-2)) + (3 \times 3) \\ (4 \times 2) + (5 \times (-1)) + (6 \times 5) & (4 \times 4) + (5 \times (-2)) + (6 \times 3) \end{pmatrix} \\
&= \begin{pmatrix} 15 & 9 \\ 33 & 24 \end{pmatrix}.
\end{aligned}$$

The above procedure can be carried out only when the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

Matrix multiplication

The **product** of an $m \times p$ matrix $\mathbf{A}$ and a $p \times n$ matrix $\mathbf{B}$ is the $m \times n$ matrix $\mathbf{C} = \mathbf{AB}$, where c_{ij} is formed using the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$, so that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}. \tag{10}$$

Exercise 11

Calculate the following matrix products, where they exist.

(a) $\begin{pmatrix}1 & 2\\ 2 & -3\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}$ (b) $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1 & 6\\ 0 & 2\end{pmatrix}$ (c) $\begin{pmatrix}1\\ 2\end{pmatrix}\begin{pmatrix}3 & 0 & -4\end{pmatrix}$

(d) $\begin{pmatrix}3 & 1 & 2\\ 0 & 5 & 1\end{pmatrix}\begin{pmatrix}-2 & 0 & 1\\ 1 & 3 & 0\\ 4 & 1 & -1\end{pmatrix}$ (e) $\begin{pmatrix}2\\ 4\\ -1\end{pmatrix}\begin{pmatrix}3 & 2\\ 4 & -1\end{pmatrix}$

Exercise 12

Calculate $\mathbf{Ax}$ when

$$\mathbf{A} = \begin{pmatrix}3 & -1 & 5\\ 6 & 4 & 7\\ 2 & -3 & 0\end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix},$$

and hence show that the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = (2 \quad 5 \quad 6)^T$, is equivalent to the system of equations

$$\begin{aligned} 3x_1 - x_2 + 5x_3 &= 2,\\ 6x_1 + 4x_2 + 7x_3 &= 5,\\ 2x_1 - 3x_2 \phantom{{}+7x_3} &= 6. \end{aligned}$$

In Section 1 we referred to $\mathbf{Ax} = \mathbf{b}$ as a convenient representation of a system of equations. This is consistent with the interpretation of $\mathbf{Ax}$ as the matrix product of $\mathbf{A}$ with $\mathbf{x}$, as we show here.

Earlier, you saw that addition of matrices is commutative and associative. We now give the rules of matrix multiplication.

Rules of matrix multiplication

For any matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of appropriate sizes, matrix multiplication is **associative**, that is,

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}), \tag{11}$$

and **distributive** over matrix addition, that is,

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}. \tag{12}$$

The phrase 'of appropriate sizes' means that all the matrix sums and products can be formed.

In general, matrix multiplication is *not commutative*, so $\mathbf{AB}$ may not be equal to $\mathbf{BA}$. But multiplication of numbers *is* commutative: $ab = ba$ for any numbers a and b. So this is a significant difference between the algebra of matrices and the algebra of numbers.

Exercise 13

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$.

Verify each of the following statements.

(a) $\mathbf{AB} \neq \mathbf{BA}$ (b) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (c) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

For a square matrix $\mathbf{A}$, we define **powers** of the matrix in the obvious way: $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA}$, and so on.

$\mathbf{A}^T$ is read as 'A transpose'.

If we denote $\mathbf{A}$ by (a_{ij}) and $\mathbf{A}^T$ by (a^T_{ij}), then $a^T_{ij} = a_{ji}$.

An operation that we can apply to any matrix $\mathbf{A}$ is to form its **transpose** $\mathbf{A}^T$ by interchanging its rows and columns. Thus the rows of $\mathbf{A}^T$ are the columns of $\mathbf{A}$, and the columns of $\mathbf{A}^T$ are the rows of $\mathbf{A}$, taken in the same order. If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix. Examples of transposes are

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 7 \\ -6 & 1 \\ 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 & 0 \\ 7 & 1 & 4 \end{pmatrix}.$$

Rules for transposes of matrices

For any matrix $\mathbf{A}$,

$$(\mathbf{A}^T)^T = \mathbf{A}. \tag{13}$$

For any matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size,

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T. \tag{14}$$

If $\mathbf{A}$ is an $m \times p$ matrix and $\mathbf{B}$ is a $p \times n$ matrix, then

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T. \tag{15}$$

Notice the change in order of the terms involving $\mathbf{A}$ and $\mathbf{B}$.

Remember that a vector is simply a matrix with one column.

Notice in passing that the dot product of two vectors can be written using matrix multiplication: if $\mathbf{a} = (a_1 \quad a_2 \quad a_3)^T$ and $\mathbf{b} = (b_1 \quad b_2 \quad b_3)^T$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \mathbf{a}^T\mathbf{b}.$$

This fact will turn out to be extremely useful when we come to discuss vector calculus later in the module.

A square matrix $\mathbf{A}$ is **symmetric** if $\mathbf{A} = \mathbf{A}^T$. Symmetric here refers to symmetry about the leading diagonal. The matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

are examples of symmetric matrices.

Exercise 14

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$.

(a) Write down $\mathbf{A}^T$, $\mathbf{B}^T$ and $\mathbf{C}^T$.

(b) Verify that $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) Verify that $(\mathbf{AC})^T = \mathbf{C}^T\mathbf{A}^T$.

(d) Explain why $\mathbf{C}$ is not a symmetric matrix.

(e) Is $\mathbf{CC}^T$ a symmetric matrix?

Exercise 15

Given the symmetric matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

show that $\mathbf{AB}$ is not symmetric. Verify that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Of particular importance in the solution of simultaneous linear equations are the triangular matrices. An **upper triangular matrix** is a square matrix in which each element below the leading diagonal is 0. A **lower triangular matrix** is a square matrix in which each element above the leading diagonal is 0. A **diagonal matrix** is a square matrix where all the elements off the leading diagonal are 0. A matrix that is upper triangular, lower triangular or both (i.e. diagonal) is sometimes referred to simply as a **triangular matrix**. For example, the matrix on the left below is upper triangular, the one in the middle is lower triangular, and the one on the right is diagonal:

You met examples of an upper triangular matrix $\mathbf{U}$ in Section 1.

Note that the transpose of an upper triangular matrix is a lower triangular matrix, and vice versa.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Exercise 16

For each of the following matrices, state whether it is upper triangular, lower triangular, diagonal, or none of these.

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Earlier, you met the $m \times n$ zero matrix $\mathbf{0}$, which has the property that $\mathbf{A} + \mathbf{0} = \mathbf{A}$ for each $m \times n$ matrix $\mathbf{A}$. The analogue for matrix multiplication is the $n \times n$ **identity matrix I**, which is a diagonal matrix where each diagonal element is 1. For example, the 3×3 identity matrix is

Strictly speaking, we should write the $n \times n$ identity matrix as $\mathbf{I}_n$, but the size of any identity matrix will be clear from the context.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{I}$ is the $n \times n$ identity matrix, then

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

If there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$, and we write $\mathbf{B} = \mathbf{A}^{-1}$. Only square matrices can have inverses. A matrix that has an inverse is called **invertible** (and a matrix that does not have an inverse is called **non-invertible**!).

$\mathbf{A}^{-1}$ is read as 'A inverse'.

An invertible matrix is sometimes called *non-singular*, while a non-invertible matrix is called *singular*.

Exercise 17

For each of the following pairs of matrices $\mathbf{A}$ and $\mathbf{B}$, calculate $\mathbf{AB}$ and deduce that $\mathbf{B} = \mathbf{A}^{-1}$.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 7 & 8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{8}{9} & -\frac{1}{9} \\ -\frac{7}{9} & \frac{2}{9} \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$

Finding the inverse of an invertible matrix

There is a way to compute the inverse of an invertible square matrix using row operations similar to those used for Gaussian elimination. An example will make the method clear.

Example 7

Find the inverse of the invertible matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 \\ 3 & 5 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Solution

We form the augmented 3×6 matrix

$$\left(\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

consisting of $\mathbf{A}$ together with the identity matrix. Then we perform row operations in order to reduce the left-hand matrix to the identity, as follows.

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{l} \\ \mathbf{R}_2 - \frac{3}{2}\mathbf{R}_1 \\ \mathbf{R}_3 - \frac{1}{2}\mathbf{R}_1 \end{array} \left(\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & \frac{5}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{l} \\ \\ \mathbf{R}_{3a} - \frac{1}{2}\mathbf{R}_{2a} \end{array} \left(\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & \frac{5}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & 1 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Note that the left-hand matrix is now in upper triangular form.

Stage 2(a) We adjust the element at the bottom of column 3 to one.

$$\begin{array}{l} \\ \\ 4\mathbf{R}_{3b} \end{array} \left(\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & \frac{5}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 4 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3c} \end{array}$$

Stage 2(b) We reduce the elements above the leading diagonal in column 3 to zero.

$$\begin{array}{l} \mathbf{R}_1 + \mathbf{R}_{3c} \\ \mathbf{R}_{2a} - \frac{5}{2}\mathbf{R}_{3c} \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 2 & 2 & 0 & 2 & -2 & 4 \\ 0 & 2 & 0 & -4 & 6 & -10 \\ 0 & 0 & 1 & 1 & -2 & 4 \end{array}\right) \begin{array}{l} \mathbf{R}_{1a} \\ \mathbf{R}_{2b} \\ \mathbf{R}_{3c} \end{array}$$

Stage 2(c) We adjust the element on the leading diagonal in column 2 to one.

$$\begin{array}{l} \\ \frac{1}{2}\mathbf{R}_{2b} \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 2 & 2 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & -2 & 3 & -5 \\ 0 & 0 & 1 & 1 & -2 & 4 \end{array}\right) \begin{array}{l} \mathbf{R}_{1a} \\ \mathbf{R}_{2c} \\ \mathbf{R}_{3c} \end{array}$$

Stage 2(d) We reduce the element at the top of column 2 to zero.

$$\begin{array}{l} \mathbf{R}_{1a} - 2\mathbf{R}_{2c} \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 6 & -8 & 14 \\ 0 & 1 & 0 & -2 & 3 & -5 \\ 0 & 0 & 1 & 1 & -2 & 4 \end{array}\right) \begin{array}{l} \mathbf{R}_{1b} \\ \mathbf{R}_{2c} \\ \mathbf{R}_{3c} \end{array}$$

Stage 2(e) We adjust the element at the top of column 1 to one.

$$\begin{array}{l} \frac{1}{2}\mathbf{R}_{1b} \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & -2 & 3 & -5 \\ 0 & 0 & 1 & 1 & -2 & 4 \end{array}\right) \begin{array}{l} \mathbf{R}_{1c} \\ \mathbf{R}_{2c} \\ \mathbf{R}_{3c} \end{array}$$

The resulting matrix on the right-hand side is the required inverse,

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -4 & 7 \\ -2 & 3 & -5 \\ 1 & -2 & 4 \end{pmatrix},$$

as you can readily check by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

This technique extends to larger square matrices, but because it is rather inefficient, it is not widely used.

Procedure 2 Finding the inverse of an invertible square matrix

To find the inverse of an invertible square matrix $\mathbf{A}$, carry out the following steps.

1. Form the augmented matrix $\mathbf{A}|\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{A}$.
2. Use row operations to reduce the left-hand side to the identity matrix $\mathbf{I}$.
3. The matrix on the right-hand side is the inverse of $\mathbf{A}$.

Row interchanges will be necessary if one or more of the pivots is zero.

Exercise 18

Use Procedure 2 to find the inverse of a general 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (\text{where } ad - bc \neq 0).$$

Note that you will have to treat $a = 0$ as a special case.

The existence (or otherwise) of the inverse of a given square matrix $\mathbf{A}$ depends solely on the value of a single number called the *determinant* of $\mathbf{A}$, written $\det \mathbf{A}$. For a 2×2 matrix, Exercise 18 yields the following result (which saves us working through Procedure 2).

It is sometimes convenient to write $\det(\mathbf{A})$ rather than $\det \mathbf{A}$. We study determinants in Subsection 2.3.

Inverse of an invertible 2×2 matrix

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the **determinant** of $\mathbf{A}$ is $\det \mathbf{A} = ad - bc$.

If $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{16}$$

So to find the inverse of a 2×2 matrix, interchange the diagonal elements, take the negatives of the other two elements, and divide each resulting element by the determinant. You may like to check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

You will see shortly that it is possible to define $\det \mathbf{A}$ for all square matrices, and the following result holds (although we do not prove it).

Condition for invertibility of a matrix $\mathbf{A}$

A matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.

This is equivalent to saying that a matrix $\mathbf{A}$ is non-invertible if and only if $\det \mathbf{A} = 0$.

Exercise 19

For each of the following 2×2 matrices $\mathbf{A}$, calculate $\det \mathbf{A}$ and determine $\mathbf{A}^{-1}$, if it exists.

(a) $\mathbf{A} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$

Properties of invertible matrices

- The inverse of an invertible matrix $\mathbf{A}$ is unique (i.e. if $\mathbf{AB} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$, then $\mathbf{B} = \mathbf{C} = \mathbf{A}^{-1}$).
- If $\mathbf{AB} = \mathbf{I}$, then $\mathbf{BA} = \mathbf{I}$, so $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, and hence the inverse of $\mathbf{A}^{-1}$ is $\mathbf{A}$.
- The rows of a square matrix are linearly independent if and only if the matrix is invertible.
- If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices of the same size, then $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

We do not prove these properties here.

Notice the change in order of the terms involving $\mathbf{A}$ and $\mathbf{B}$.

Exercise 20

(a) Show that if $\mathbf{A}$ and $\mathbf{B}$ are any two square matrices of the same size, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Now let $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$.

Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

(c) For $\mathbf{A}, \mathbf{B}$ as in part (b), verify that $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

(d) For $\mathbf{A}, \mathbf{B}$ as in part (b), verify that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

2.2 Linear transformations of the plane

Linear transformations of the plane provide examples of a use of matrices.

You will see further examples in Unit 5.

The transformation is called *linear* because straight lines are mapped to straight lines.

A **linear transformation** of the plane is a function that maps a two-dimensional vector $(x \quad y)^T$ to the image vector $(ax + by \quad cx + dy)^T$, where a, b, c and d are real numbers.

We can represent any such linear transformation by the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, since $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. The image of any given vector can then be calculated. For example, the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ maps $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to $\mathbf{A}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

Exercise 21

Consider the linear transformation that maps a vector $(x \quad y)^T$ to the image vector $(x + 2y \quad 3x + 4y)^T$.

(a) Write down the matrix $\mathbf{A}$ for this linear transformation.

(b) Use the matrix $\mathbf{A}$ to find the image of each of the following vectors.

(i) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For any linear transformation with matrix $\mathbf{A}$, the images of the Cartesian unit vectors $\mathbf{i} = (1 \quad 0)^T$ and $\mathbf{j} = (0 \quad 1)^T$ are the columns of $\mathbf{A}$. To see this, consider the general linear transformation that maps each vector $(x \quad y)^T$ to the image vector $(a_{11}x + a_{12}y \quad a_{21}x + a_{22}y)^T$. Then

$$\text{the image of } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is } \begin{pmatrix} a_{11} \times 1 + a_{12} \times 0 \\ a_{21} \times 1 + a_{22} \times 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix},$$

and

$$\text{the image of } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is } \begin{pmatrix} a_{11} \times 0 + a_{12} \times 1 \\ a_{21} \times 0 + a_{22} \times 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}.$$

These images are the columns of the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

An interesting offshoot of these ideas concerns the *determinant* of the matrix of a linear transformation.

Consider the general linear transformation with matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The image of $(1 \quad 0)^T$ is $(a \quad c)^T$, and the image of $(0 \quad 1)^T$ is $(b \quad d)^T$. It can be shown that the unit square (with vertices at the points $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$) is mapped to the parallelogram defined by these image vectors, as shown in Figure 4.

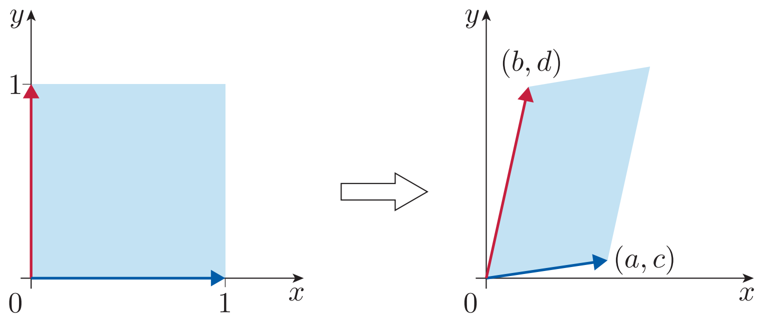


Figure 4 The unit square maps to a parallelogram

From Unit 2, we know that the area of this parallelogram is the magnitude of the cross product of the two position vectors $(a \quad c)^T$ and $(b \quad d)^T$, which is $|(a\mathbf{i} + c\mathbf{j}) \times (b\mathbf{i} + d\mathbf{j})| = |ad - bc|$. The determinant of $\mathbf{A}$ is $\det \mathbf{A} = ad - bc$. So the area of the parallelogram is the magnitude of $\det \mathbf{A}$, denoted $|\det \mathbf{A}|$. But the parallelogram is the image of the unit square defined by the two vectors $(1 \quad 0)^T$ and $(0 \quad 1)^T$, so $|\det \mathbf{A}|$ is the area of the image of the unit square. Accordingly, the larger $|\det \mathbf{A}|$, the larger the images of shapes under the transformation.

Exercise 22

For each of the following matrices, calculate $\det \mathbf{A}$ and compare your answer with the area of the parallelogram defined by the images of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & -6 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

We can also link these ideas with systems of linear equations. For example, the system of linear equations

$$\begin{cases} x_1 + x_2 = 0, \\ x_1 + 2x_2 = 1, \end{cases} \tag{17}$$

can be written in matrix form as $\mathbf{A}\mathbf{x} = \mathbf{b}$, given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{18}$$

Solving equations (17) is equivalent to finding the vector $\mathbf{x}$ that is mapped to the vector $\mathbf{b}$ by the linear transformation with matrix $\mathbf{A}$ as shown in equation (18). This is the 'inverse process' of our earlier work (e.g. in Exercise 21), where we were given $\mathbf{x}$ and asked to find the image vector $\mathbf{b} = \mathbf{A}\mathbf{x}$. This suggests that we might consider using the inverse matrix as a way of solving such systems of linear equations.

For the matrix $\mathbf{A}$ in equation (18),

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Here $\det \mathbf{A} = 1$.

We wish to multiply the equation $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$, but we must be careful with the order of the multiplication. Multiplying both sides of the equation *on the left* by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}.$$

Here we use associativity: $\mathbf{A}^{-1}(\mathbf{Ax}) = (\mathbf{A}^{-1}\mathbf{A})\mathbf{x}$.

Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, we have

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b},$$

so $\mathbf{x}$ is the image of $\mathbf{b}$ under transformation by the inverse matrix $\mathbf{A}^{-1}$. Therefore

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}.\end{aligned}$$

Thus $x_1 = -1$, $x_2 = 1$ is the solution of this system of linear equations.

Exercise 23

Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, calculate $\det \mathbf{A}$ and $\mathbf{A}^{-1}$. Hence write down the solution of the system of equations

$$\begin{aligned} x + 2y &= 1, \\ -x + y &= -1. \end{aligned}$$

This matrix approach to solving a system of linear equations can be used whenever the matrix $\mathbf{A}^{-1}$ exists, that is, whenever $\mathbf{A}$ is invertible. However, except for 2×2 matrices, the inverse matrix is usually tedious to calculate, and it is more efficient to use the Gaussian elimination method to solve the system of equations.

We come now to a result that is important with respect to the material in the next unit. Suppose that $\mathbf{b} = \mathbf{0}$, so that we are looking for a solution to $\mathbf{Ax} = \mathbf{0}$. What can we say about $\mathbf{x}$? If $\mathbf{A}$ has an inverse, then multiplying both sides of $\mathbf{Ax} = \mathbf{0}$ on the left by $\mathbf{A}^{-1}$ gives $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{0}$, so $\mathbf{x} = \mathbf{0}$. Therefore, if $\mathbf{Ax} = \mathbf{0}$ is to have a non-zero solution $\mathbf{x}$, then $\mathbf{A}$ cannot have an inverse, that is, it must be non-invertible. So we have the following result.

Non-invertible square matrix

If $\mathbf{A}$ is a square matrix and $\mathbf{Ax} = \mathbf{0}$ with $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{A}$ is non-invertible and therefore $\det \mathbf{A} = 0$.

2.3 Determinants

In this subsection we summarise the main properties of determinants of 2×2 matrices and extend the ideas to $n \times n$ matrices.

Properties of 2×2 determinants

We frequently use the 'vertical line' notation for determinants:

$$\text{if } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } \det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Earlier in this section you saw that if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible. In the following exercise we investigate some further properties of 2×2 determinants.

Exercise 24

Calculate the determinants of the following matrices, which are related to the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In parts (b), (c), (e), (f), (g) and (h), compare your answer with $\det \mathbf{A}$.

We will refer back to the results of this exercise later.

(a) $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ (lower triangular, upper triangular and diagonal matrices)

(b) $\begin{pmatrix} c & d \\ a & b \end{pmatrix}$ and $\begin{pmatrix} b & a \\ d & c \end{pmatrix}$ (formed by interchanging rows or columns of $\mathbf{A}$)

(c) $\mathbf{A}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

(d) $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ and $\begin{pmatrix} a & a \\ c & c \end{pmatrix}$ (matrices with linearly dependent rows or columns)

(e) $\begin{pmatrix} ka & kb \\ c & d \end{pmatrix}$ and $\begin{pmatrix} ka & b \\ kc & d \end{pmatrix}$ (formed by multiplying a row or column of $\mathbf{A}$ by a scalar k)

(f) $k\mathbf{A} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

(g) $\begin{pmatrix} a & b \\ c - ma & d - mb \end{pmatrix}$ and $\begin{pmatrix} a - mc & b - md \\ c & d \end{pmatrix}$ (formed by subtracting a multiple of one row of $\mathbf{A}$ from the other row)

(h) $\mathbf{A}^{-1} = \begin{pmatrix} \dfrac{d}{ad - bc} & -\dfrac{b}{ad - bc} \\ -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{pmatrix}$

Introducing 3×3 and $n \times n$ determinants

Just as the magnitude of the 2×2 determinant

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

represents the area of the parallelogram defined by the vectors $(a_1 \quad a_2)^T$ and $(b_1 \quad b_2)^T$, so we can define a 3×3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

whose magnitude represents the volume of the parallelepiped defined by the vectors $\mathbf{a} = (a_1 \quad a_2 \quad a_3)^T$, $\mathbf{b} = (b_1 \quad b_2 \quad b_3)^T$ and $\mathbf{c} = (c_1 \quad c_2 \quad c_3)^T$, as shown in Figure 5. (We will look at this further in Subsection 2.4.)

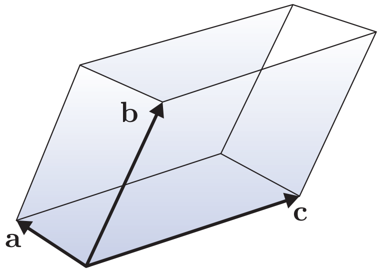


Figure 5 A parallelepiped defined by the vectors **a**, **b** and **c**

The **determinant** of the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

is given by

$$\det \mathbf{A} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, \qquad (19)$$

or equivalently,

$$\det \mathbf{A} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1. \qquad (20)$$

This is sometimes known as 'expanding the determinant by the top row'. Notice the minus sign before the second term.

As before, we frequently use 'vertical line' notation. For example,

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & -1 \\ 2 & 5 & 6 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 5 & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 11 - 40 + 52 = 23.$$

Exercise 25

Evaluate the following determinants.

(a) $\begin{vmatrix} 4 & 1 & 0 \\ 0 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$

As a way to understand how to expand 3×3 determinants, first consider the matrix of signs

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix},$$

which we will call the **sign matrix**. To evaluate the determinant of the 3×3 matrix $\mathbf{A}$ in the above definition, select a row or column. The determinant is then given by a sum of three terms, one for each element in the selected row or column. Each term consists of the element itself multiplied by the determinant of the 2×2 matrix formed by removing the row and column passing through the element, and this is then multiplied by the corresponding sign in the above sign matrix.

Note that in the definition of the determinant in equation (19), this procedure was carried out by selecting the top row. If the middle row is chosen for the expansion, then $\det \mathbf{A}$ is given by

$$\det \mathbf{A} = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix},$$

Note how the signs of the terms correspond to those of the middle row of the sign matrix.

and if the third column is selected, then $\det \mathbf{A}$ is given by

$$\det \mathbf{A} = a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

Note how the signs of the terms correspond to those of the third column of the sign matrix.

and so on. Any row or column could be selected for the expansion.

You can see from this procedure that the sign of the determinant will change if two adjacent rows or two adjacent columns are interchanged.

You can also see that

$$\det(\mathbf{A}^T) = \det \mathbf{A},$$

a result that extends to $n \times n$ matrices (though we do not prove this). This is a consequence of the fact that to evaluate $\det \mathbf{A}$ using the above procedure, we can expand using columns in place of rows, as illustrated in Example 8 below.

In fact, the procedure makes it easy to evaluate any 3×3 determinant in which one row or column is particularly simple, such as when a row or column contains mainly zeros.

Example 8

Evaluate the determinant of each of the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 8 & 9 & -6 \\ 1 & 0 & 0 \\ 32 & -7 & 14 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 8 & 9 & 0 \\ 13 & -4 & 2 \\ -6 & 2 & 0 \end{pmatrix}$

Solution

(a) We choose to expand $\det \mathbf{A}$ by the second row, as this contains two zeros:

$$\begin{aligned} \begin{vmatrix} 8 & 9 & -6 \\ 1 & 0 & 0 \\ 32 & -7 & 14 \end{vmatrix} &= -1 \begin{vmatrix} 9 & -6 \\ -7 & 14 \end{vmatrix} + 0 \begin{vmatrix} 8 & -6 \\ 32 & 14 \end{vmatrix} - 0 \begin{vmatrix} 8 & 9 \\ 32 & -7 \end{vmatrix} \\ &= -1(126 - 42) \\ &= -84. \end{aligned}$$

(b) We choose to expand $\det \mathbf{B}$ by the third column, as this contains two zeros:

$$\begin{vmatrix} 8 & 9 & 0 \\ 13 & -4 & 2 \\ -6 & 2 & 0 \end{vmatrix} = 0\begin{vmatrix} 13 & -4 \\ -6 & 2 \end{vmatrix} - 2\begin{vmatrix} 8 & 9 \\ -6 & 2 \end{vmatrix} + 0\begin{vmatrix} 8 & 9 \\ 13 & -4 \end{vmatrix}$$
$$= -2(16+54)$$
$$= -140.$$

The armoury of techniques for evaluating determinants can be expanded by noting some general rules.

The general $n \times n$ determinant is defined below.

You may find it helpful to compare these general rules with the results for 2×2 matrices in Exercise 24. The letters (a) to (h) here relate to the relevant parts of that exercise.

The multiple can be negative or positive, so rule (g) covers subtracting a multiple of one row from another row.

Rules for $n \times n$ determinants

(a) If $\mathbf{A}$ is a diagonal, upper triangular or lower triangular matrix, then $\det \mathbf{A}$ is the product of the diagonal elements.

(b) Interchanging any two rows or any two columns of $\mathbf{A}$ changes the sign of $\det \mathbf{A}$.

(c) $\det(\mathbf{A}^T) = \det \mathbf{A}$.

(d) If the rows or columns of $\mathbf{A}$ are linearly dependent, then $\det \mathbf{A} = 0$; otherwise, $\det \mathbf{A} \neq 0$.

(e) Multiplying any row or any column of $\mathbf{A}$ by a scalar k multiplies $\det \mathbf{A}$ by k.

(f) For any number k, $\det(k\mathbf{A}) = k^n \det \mathbf{A}$.

(g) Adding a multiple of one row of $\mathbf{A}$ to another row does not change $\det \mathbf{A}$.

(h) The matrix $\mathbf{A}$ is non-invertible if and only if $\det \mathbf{A} = 0$. If $\det \mathbf{A} \neq 0$, then $\det(\mathbf{A}^{-1}) = 1/\det \mathbf{A}$.

(i) For any two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, we have

$$\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}.$$

Note that since $\det(\mathbf{A}^T) = \det \mathbf{A}$, we can also deduce that $\det \mathbf{A} = 0$ if and only if the *columns* of $\mathbf{A}$ are linearly dependent.

Rule (d) tells us that $\det \mathbf{A} = 0$ if and only if the rows of $\mathbf{A}$ are linearly dependent. Hence any system of linear equations with coefficient matrix $\mathbf{A}$ has a *unique* solution if and only if $\det \mathbf{A} \neq 0$. If $\det \mathbf{A} = 0$, then the system has an infinite number of solutions or no solution.

Exercise 26

(a) Calculate $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 6 \end{vmatrix}$.

(b) Use the result of part (a) and the above rules to *write down* the values of the following determinants.

(i) $\begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 6 \\ 0 & 2 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 6 \end{vmatrix}$ (iii) $\begin{vmatrix} 2 & 3 & 3 \\ 3 & 3 & 6 \\ 0 & 6 & 1 \end{vmatrix}$

(c) Let $\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{pmatrix}$. Calculate $\det \mathbf{A}$.

Subtract twice the second row of $\mathbf{A}$ from the third row to obtain an upper triangular matrix $\mathbf{U}$. Calculate $\det \mathbf{U}$ using the definition of a 3×3 determinant, and compare this with the product of the diagonal elements of $\mathbf{U}$.

It is also possible to define larger determinants. To do this, we proceed one step at a time, defining an $n \times n$ determinant in terms of $(n-1) \times (n-1)$ determinants in a way analogous to the definition of a 3×3 determinant. For example, to define a 4×4 determinant, we have the sign matrix

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix},$$

and proceed by selecting a row or column. The determinant is now given as a sum of four terms, one for each element in the selected row or column. Again, each term consists of the element itself but now multiplied by the determinant of the 3×3 matrix formed by removing the row and column passing through the element, and this is then multiplied by the corresponding sign in the 4×4 sign matrix. So, for example, by selecting the top row, we have

Note how the signs of the terms correspond to those of the top row of the sign matrix.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}.$$

Except in special cases, the calculation of large determinants like this can be very tedious. However, rule (g) for determinants provides the clue to a simpler method. Procedure 1 (Gaussian elimination) applied to a matrix $\mathbf{A}$ consists of a sequence of row operations where a multiple of one row is subtracted from another. Each such operation does not change the value of the determinant. Thus we can deduce that $\det \mathbf{A} = \det \mathbf{U}$, where $\mathbf{U}$ is the upper triangular matrix obtained at the end of the elimination stage. Since the determinant of an upper triangular matrix is the product of the diagonal elements, Procedure 1 also provides an efficient way of calculating determinants of any size.

If we need to make essential row interchanges to avoid a zero pivot, we note that each row interchange will change the sign of the determinant (see rule (b)).

Exercise 27

In Example 3 we applied the Gaussian elimination method to the matrix
$\mathbf{A} = \begin{pmatrix} 1 & -4 & 2 \\ 3 & -2 & 3 \\ 8 & -2 & 9 \end{pmatrix}$ to obtain $\mathbf{U} = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 10 & -3 \\ 0 & 0 & 2 \end{pmatrix}$.

Calculate $\det \mathbf{A}$ and $\det \mathbf{U}$, and hence show that $\det \mathbf{A} = \det \mathbf{U}$.

Exercise 28

Evaluate $\begin{vmatrix} 1 & -2 & 0 & 3 \\ 4 & -5 & 0 & 6 \\ -2 & -1 & 10 & 4 \\ 5 & -7 & 0 & 9 \end{vmatrix}$.

2.4 Some applications

We conclude this section by illustrating how 3×3 determinants can be used to represent areas and volumes, and products of vectors. We start by revisiting the scalar triple product.

Scalar triple products and the other topics in this subsection were mentioned in Unit 2.

Scalar triple product

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. The volume of the parallelepiped with sides defined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (see Figure 5) is given by the magnitude of the scalar triple product $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$, which can also be written as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. The cross product of vectors $\mathbf{b}$ and $\mathbf{c}$ is

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= (b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k} \\ &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k}, \end{aligned} \qquad (21)$$

In Unit 2, the scalar triple product was written as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. But because any of the faces of the parallelepiped can be thought of as the base, we can begin this expression with the cross product of *any two* of the defining vectors and not alter the *magnitude* of the scalar triple product.

hence

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}. \end{aligned} \qquad (22)$$

Exercise 29

Use equation (22) and the result of Exercise 26(a) to find the scalar triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ when

$$\mathbf{a} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}, \quad \mathbf{b} = 2\mathbf{j}+\mathbf{k} \quad \text{and} \quad \mathbf{c} = 3\mathbf{i}+\mathbf{j}+6\mathbf{k}.$$

Exercise 30

Use equation (22) to find the volume of the parallelepiped defined by the vectors

$$\mathbf{a} = \mathbf{i}+\mathbf{k}, \quad \mathbf{b} = \mathbf{i}+2\mathbf{j} \quad \text{and} \quad \mathbf{c} = \mathbf{j}+3\mathbf{k}.$$

Cross product

The similarity between formula (20) for a 3×3 determinant and the component form of the cross product of two vectors in equation (21) gives another way of remembering the formula for the cross product. We have seen above that if $\mathbf{b} = (b_1 \quad b_2 \quad b_3)^T$ and $\mathbf{c} = (c_1 \quad c_2 \quad c_3)^T$, then

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}\mathbf{k}.$$

This expression can be remembered more easily if we write it as a 3×3 determinant:

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}.$$

Exercise 31

Express $\mathbf{b}\times\mathbf{c}$ as a determinant, and calculate its value, in the following cases.

(a) $\mathbf{b} = 3\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ and $\mathbf{c} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$

(b) $\mathbf{b} = (1 \quad 2 \quad 3)^T$ and $\mathbf{c} = (6 \quad 5 \quad 4)^T$

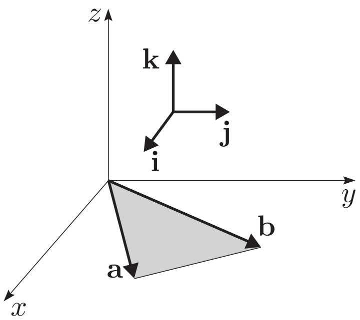


Figure 6 Triangle in the plane defined by the vectors **a** and **b**

As in the solution to Exercise 30, we use det for determinant here, rather than | |, to avoid confusion with the modulus function, which is also present.

Area of a triangle in the plane

Consider the triangle defined by the origin and the points with position vectors $\mathbf{a} = (a_1 \quad a_2 \quad 0)^T$ and $\mathbf{b} = (b_1 \quad b_2 \quad 0)^T$, as shown in Figure 6. Its area A is given by

$$\begin{aligned} A &= \tfrac{1}{2}\,|\mathbf{a}\times\mathbf{b}| \\ &= \tfrac{1}{2}\left|\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{pmatrix}\right| \\ &= \tfrac{1}{2}\left|\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\mathbf{k}\right| \\ &= \tfrac{1}{2}\left|\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right|. \end{aligned}$$

This formula agrees with our earlier interpretation of the magnitude of the determinant as the area of the parallelogram defined by **a** and **b**.

We can extend this result to give a formula containing determinants for the area of a triangle whose vertices have position vectors $\mathbf{a} = (a_1 \quad a_2 \quad 0)^T$, $\mathbf{b} = (b_1 \quad b_2 \quad 0)^T$ and $\mathbf{c} = (c_1 \quad c_2 \quad 0)^T$. Two sides of the triangle are given by the vectors $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}-\mathbf{c}$, so we can find the area as

Recall that $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ and $-\mathbf{c}\times\mathbf{b} = \mathbf{b}\times\mathbf{c}$ (see Unit 2).

$$\begin{aligned} A &= \tfrac{1}{2}\,|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})| \\ &= \tfrac{1}{2}\,|(\mathbf{a}\times\mathbf{b}) - (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c})| \\ &= \tfrac{1}{2}\left|\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{pmatrix} - \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ c_1 & c_2 & 0 \end{pmatrix} + \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}\right| \\ &= \tfrac{1}{2}\big|(a_1b_2 - a_2b_1)\mathbf{k} - (a_1c_2 - a_2c_1)\mathbf{k} + (b_1c_2 - b_2c_1)\mathbf{k}\big| \\ &= \tfrac{1}{2}\big|(a_1b_2 - a_2b_1) - (a_1c_2 - a_2c_1) + (b_1c_2 - b_2c_1)\big| \\ &= \tfrac{1}{2}\left|\det\begin{pmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}\right|. \end{aligned} \tag{23}$$

Exercise 32

Use equation (23) to find the area of the triangle whose vertices are

$$\mathbf{a} = 3\mathbf{i}+\mathbf{j}, \quad \mathbf{b} = \mathbf{i}+2\mathbf{j} \quad \text{and} \quad \mathbf{c} = 2\mathbf{i}-\mathbf{j}$$

(see Figure 7).

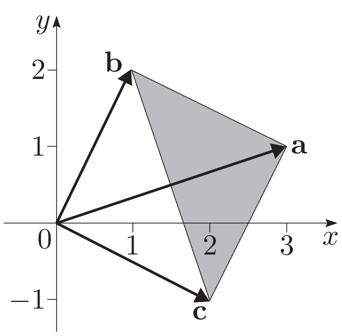


Figure 7

3 Polynomial interpolation

In Section 1 we used matrix notation to describe the solution of systems of linear equations by the Gaussian elimination method. The ubiquitous nature of matrices is partially explained by the fact that such systems of equations arise in many areas of applied mathematics, numerical analysis and statistics. In this section we look at an application that involves solving systems of linear equations: *polynomial interpolation*.

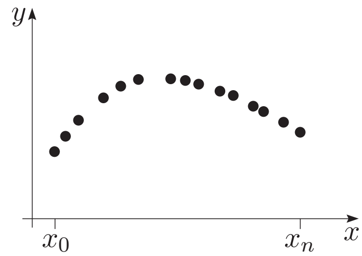


Figure 8 A set of data points

In this application, we are given a set of $n+1$ data points (x_i, y_i), $i = 0, 1, \ldots, n$, where $x_0 < x_1 < \ldots < x_n$, as shown in Figure 8. We determine the polynomial $y(x) = a_0 + a_1 x + \cdots + a_n x^n$ such that $y(x_i) = y_i$, $i = 0, 1, \ldots, n$. This polynomial, defined on the interval $x_0 \leq x \leq x_n$, is called the **interpolating polynomial**. The graph of such a function passes through each data point.

We are often given a table of values showing the variation of one variable with another. For example, in Example 5 of Unit 1 we met the initial-value problem

$$\frac{dy}{dx} = x + y, \quad y(0) = 1. \tag{24}$$

Using Euler's method with step size $h = 0.2$, we obtained the approximate values for the solution shown in Table 1.

Table 1

i	x_i	Y_i
0	0	1
1	0.2	1.2
2	0.4	1.48
3	0.6	1.856
4	0.8	2.347 2
5	1.0	2.976 64

Suppose that we wish to approximate the solution at $x = 0.47$. One way of doing this is to construct a polynomial through some or all of the data values and then use this interpolating polynomial to approximate the solution at $x = 0.47$. There are many ways of constructing interpolating polynomials, but we present just one method here. We start with a straight-line approximation through the two data points closest to $x = 0.47$.

Example 9

Find the equation of the straight line through $(0.4, 1.48)$ and $(0.6, 1.856)$, and use it to approximate the solution of system (24) at $x = 0.47$.

Solution

Suppose that the line is $y = a_0 + a_1 x$. Since, from Table 1, $y = 1.48$ when $x = 0.4$, and $y = 1.856$ when $x = 0.6$, we obtain the system of equations

$$a_0 + 0.4a_1 = 1.48,$$
$$a_0 + 0.6a_1 = 1.856.$$

The solution of these equations is $a_0 = 0.728$, $a_1 = 1.88$. Hence the equation of the straight line is

$$y = 0.728 + 1.88x.$$

When $x = 0.47$, we have $y = 0.728 + 1.88 \times 0.47 = 1.6116$.

In general, if we require the line through (x_0, y_0) and (x_1, y_1), we have (as in Example 9)

$$a_0 + a_1 x_0 = y_0,$$
$$a_0 + a_1 x_1 = y_1,$$

which can be written in matrix form as

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix};$$

that is,

$$\mathbf{Xa} = \mathbf{y},$$

where the matrix $\mathbf{X}$ contains the given values x_i, and the vector $\mathbf{y}$ contains the given values y_i $(i = 1, 2)$. We wish to determine the vector $\mathbf{a}$, and this could be done, for example, by using Gaussian elimination.

Exercise 33

Consider the data in Table 1. Find the straight line through the points $(0.6, 1.856)$ and $(0.8, 2.3472)$, and use it to find an approximation to the value of y at $x = 0.65$ and at $x = 0.47$.

We now have two approximations for $y(0.47)$, which do not agree, even to one decimal place. However, in Example 9 the value of $x = 0.47$ lies within the domain $0.4 \le x \le 0.6$ of the interpolating polynomial, and we have *interpolated* a straight line to obtain the approximation. In Exercise 33 the value of $x = 0.47$ lies outside the domain $0.6 \le x \le 0.8$ of the interpolating polynomial, and we have *extrapolated* a straight line to obtain the approximation. **Extrapolation** is, in general, less accurate than interpolation, and we should use the result for the approximate value of $y(0.47)$ from Example 9 rather than that from Exercise 33.

In general, given $n + 1$ data points, we can determine the interpolating polynomial of degree n of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

in a similar way, so we can fit a straight line through two points, a quadratic through three points, a cubic through four points, and so on.

Example 10

Find the interpolating quadratic polynomial for the three data points $(0.4, 1.48)$, $(0.6, 1.856)$ and $(0.8, 2.3472)$, and use it to find an approximation to the value of y at $x = 0.47$.

Solution

The three data points give rise to the system of linear equations

$$a_0 + 0.4a_1 + (0.4)^2 a_2 = 1.48,$$
$$a_0 + 0.6a_1 + (0.6)^2 a_2 = 1.856,$$
$$a_0 + 0.8a_1 + (0.8)^2 a_2 = 2.3472,$$

that is,

$$\underbrace{\begin{pmatrix} 1 & 0.4 & 0.16 \\ 1 & 0.6 & 0.36 \\ 1 & 0.8 & 0.64 \end{pmatrix}}_{\mathbf{X}} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}}_{\mathbf{a}} = \underbrace{\begin{pmatrix} 1.48 \\ 1.856 \\ 2.3472 \end{pmatrix}}_{\mathbf{y}}.$$

Using the Gaussian elimination method to solve these equations, we find $a_0 = 1.0736$, $a_1 = 0.44$, $a_2 = 1.44$. Hence the interpolating quadratic polynomial is

$$y = 1.0736 + 0.44x + 1.44x^2.$$

So $y(0.47) = 1.598\,496$.

This value for $y(0.47)$ is fairly close to the solution found in Example 9.

Procedure 3 Polynomial interpolation

To determine the interpolating polynomial of degree n

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{25}$$

through $n + 1$ data points (x_i, y_i), $i = 0, 1, \ldots, n$, proceed as follows.

Solve, for the coefficients $a_0, a_1, \ldots, a_n$, the system of equations $\mathbf{Xa} = \mathbf{y}$ given by

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}. \tag{26}$$

There are a number of questions that should be asked at this stage.

- *Will we always obtain a unique polynomial?*

  The answer is yes, provided that all the x_i values are different.

- *How accurate is any estimate obtained from an interpolating polynomial?*

  This depends on the accuracy of the data. For accurate data, it is often sufficient to obtain interpolating polynomials of increasing degree and then to look for consistency in the estimates. Estimates for values of x close to the data points are likely to be more accurate than those that are further away from the data points. Interpolation is, in general, more accurate than extrapolation.

Table 1 provides only an approximate solution to the differential equation at the specified values of x, and this illustrates a case where the initial data do not represent values of the 'true' function. The accuracy of the interpolated values is then limited by the accuracy of the values of y.

- *What degree polynomial should we use?*

  This again depends on the accuracy of the data. In theory, if the data are very accurate, then we can use polynomials of high degree. In practice, high-degree interpolating polynomials often oscillate rapidly, which may cause difficulties. A sensible strategy is to start with a low-degree polynomial and increase the degree while looking for an appropriate level of consistency in the estimates.

- *Which points should we use?*

  The best strategy, when using an interpolating polynomial of degree n, is to select the $n + 1$ points that are closest to the value of x for which you want to estimate the value of the underlying function. Unfortunately, if you need estimates at several different points, this might involve calculating several different interpolating polynomials of degree n, each based on a different subset of $n + 1$ points selected from the data points. A sensible compromise might be to use a different interpolating polynomial of degree n for each subinterval $x_i \leq x \leq x_{i+1}$, based on the $n + 1$ data points closest to this subinterval.

Exercise 34

Determine the quadratic polynomial that passes through the data points $(0.2, 1.2)$, $(0.4, 1.48)$ and $(0.6, 1.856)$ from Table 1. Use it to estimate the solution of the initial-value problem $dy/dx = x + y$, $y(0) = 1$ at $x = 0.47$. Is this estimate likely to be more accurate than that found in Example 10?

4 Ill-conditioning

In this section we briefly examine a difficulty that may occur when we attempt to find the numerical solution to a given problem. This arises because some problems are inherently unstable in the sense that very small changes to the input data (due perhaps to experimental errors or rounding errors) may dramatically alter the output numerical values. Such problems are said to be **ill-conditioned**.

Such changes 'perturb' the data.

Problems that are not ill-conditioned are said to be **well-conditioned**.

We use examples to help us to define what we mean by ill-conditioning for a system of linear equations $\mathbf{Ax} = \mathbf{b}$. A proper analysis of ill-conditioning for such a system would include a discussion of the effect of small changes to the coefficient matrix $\mathbf{A}$, but to simplify the theory, we discuss only the effect of small changes to the right-hand-side vector $\mathbf{b}$, and we assume that $\mathbf{A}$ is exact.

In earlier modules, we introduced the idea of *absolute error*: for real numbers, the error in an estimate $\overline{x}$ of the exact value x is $\overline{x} - x$, and the absolute error is $|\overline{x} - x|$. We need to extend this idea to vectors. There are a number of ways of doing this; we do it by defining the **norm of a vector** $\mathbf{x} = (x_1 \quad x_2 \quad \ldots \quad x_n)^T$ as the magnitude of the element of largest magnitude in $\mathbf{x}$. Thus the norm of $\mathbf{x}$, using the notation $\|\mathbf{x}\|$, is

See the Handbook.

Note that $\overline{x}$ here denotes the estimate of x, *not* the complex conjugate.

$$\|\mathbf{x}\| = \max_{i=1,\ldots,n} |x_i|.$$

For example, if $\mathbf{x} = (2 \quad -3 \quad 1)^T$, then $\|\mathbf{x}\| = \max\{2, 3, 1\} = 3$.

Suppose that we have two vectors, $\mathbf{x}$ and $\overline{\mathbf{x}}$, where $\overline{\mathbf{x}}$ is an estimate of the exact vector $\mathbf{x}$. The **change** in $\mathbf{x}$ is $\delta\mathbf{x} = \overline{\mathbf{x}} - \mathbf{x}$, and the **absolute change** is $\|\delta\mathbf{x}\| = \|\overline{\mathbf{x}} - \mathbf{x}\|$.

We prefer to discuss changes here rather than errors, since we are making small changes to the data to see the effect on the solution.

Example 11

Suppose that $\mathbf{x} = (2 \quad -3 \quad 1)^T$ and that $\overline{\mathbf{x}} = (2.02 \quad -3.11 \quad 1.03)^T$ is an approximation to $\mathbf{x}$. Compute the change and the absolute change in $\mathbf{x}$.

Solution

The change is $\delta\mathbf{x} = \overline{\mathbf{x}} - \mathbf{x} = (0.02 \quad -0.11 \quad 0.03)^T$.

The absolute change is $\|\delta\mathbf{x}\| = \max\{0.02, 0.11, 0.03\} = 0.11$.

Exercise 35

Determine the absolute change in the approximation $\overline{\mathbf{x}} = (3.04 \quad 2.03 \quad 0.95)^T$ to the exact vector $\mathbf{x} = (3 \quad 2 \quad 1)^T$.

In discussing ill-conditioning, we are interested in the solution $\mathbf{x}$ of the equation

$$\mathbf{Ax} = \mathbf{b},$$

and, in particular, in how small changes in $\mathbf{b}$ give rise to changes in $\mathbf{x}$. The solution may be written as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, which we can regard as a linear transformation of $\mathbf{b}$ to $\mathbf{x}$. If we allow each element of $\mathbf{b}$ to change by the small amount $\pm\varepsilon$, forming the vector $\overline{\mathbf{b}} = \mathbf{b} + \delta\mathbf{b}$, then the solution that we obtain will be $\overline{\mathbf{x}} = \mathbf{A}^{-1}(\mathbf{b} + \delta\mathbf{b})$, where the change is

We assume that $\mathbf{A}$ is invertible.

$$\delta\mathbf{x} = \overline{\mathbf{x}} - \mathbf{x} = \mathbf{A}^{-1}(\mathbf{b} + \delta\mathbf{b}) - \mathbf{A}^{-1}\mathbf{b} = \mathbf{A}^{-1}\delta\mathbf{b}. \tag{27}$$

If $\|\delta\mathbf{x}\|$ is large compared to $\|\delta\mathbf{b}\|$, then we know that the system of equations is ill-conditioned.

Example 12

Consider the equation

$$\begin{pmatrix} 1 & -2 \\ 1 & -1.9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ -0.8 \end{pmatrix}. \tag{28}$$

If the values on the right-hand side are changed by ± 0.1, how big a change in the solution might arise?

Solution

We have

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & -1.9 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ -0.8 \end{pmatrix}, \quad \text{so} \quad \mathbf{A}^{-1} = \begin{pmatrix} -19 & 20 \\ -10 & 10 \end{pmatrix}.$$

Thus the solution of equation (28) is

$$\mathbf{x} = \begin{pmatrix} -19 & 20 \\ -10 & 10 \end{pmatrix} \begin{pmatrix} -1 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Now taking $\varepsilon = 0.1$, the possible values for $\overline{\mathbf{b}}$ are

$$\begin{pmatrix} -1.1 \\ -0.9 \end{pmatrix}, \quad \begin{pmatrix} -1.1 \\ -0.7 \end{pmatrix}, \quad \begin{pmatrix} -0.9 \\ -0.9 \end{pmatrix}, \quad \begin{pmatrix} -0.9 \\ -0.7 \end{pmatrix},$$

with corresponding values for $\delta\mathbf{b}$

$$\begin{pmatrix} -0.1 \\ -0.1 \end{pmatrix}, \quad \begin{pmatrix} -0.1 \\ 0.1 \end{pmatrix}, \quad \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}, \quad \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}.$$

Note that $\|\delta\mathbf{b}\| = 0.1$ for each of these vectors.

Applying $\mathbf{A}^{-1}$ to each $\delta\mathbf{b}$ yields, respectively, the following values of $\delta\mathbf{x}$:

$$\begin{pmatrix} -0.1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3.9 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -3.9 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}.$$

Looking at the norm of $\delta\mathbf{x}$ in each case, we see that the middle two vectors have

$$\|\delta\mathbf{x}\| = 3.9,$$

which is 39 times bigger than the norm of $\delta\mathbf{b}$. Thus a change of magnitude 0.1 in the elements of $\mathbf{b}$ may cause a change of magnitude 3.9 in the elements of the solution.

A geometric interpretation will help to make this clearer. If, in Example 12, we change the elements of $\mathbf{b}$ by ± 0.1, then the changed vector $\overline{\mathbf{b}}$ lies at a corner of the square with centre $\mathbf{b} = (-1 \quad -0.8)^T$ shown in Figure 9.

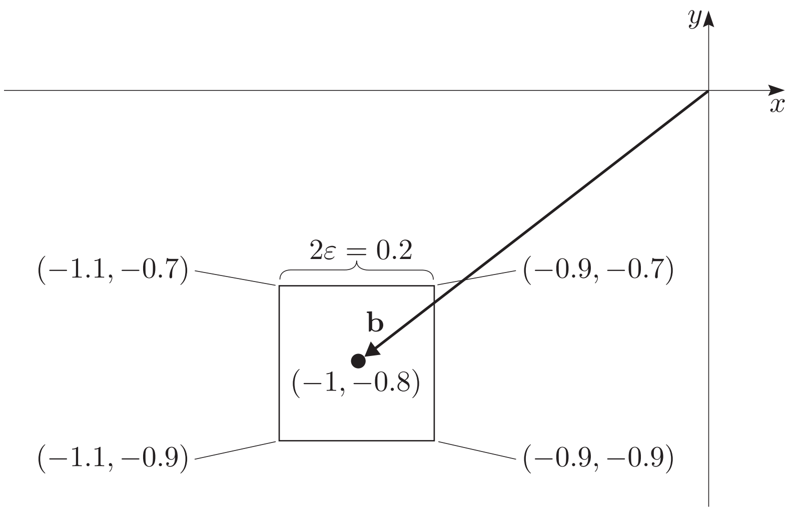


Figure 9 Changing elements of vector $\mathbf{b}$ to $\overline{\mathbf{b}}$, which lies at a corner of the square (not to scale)

Such a point $\overline{\mathbf{b}}$ is mapped to a point $\overline{\mathbf{x}}$ under the linear transformation represented by $\mathbf{A}^{-1}$, that is, $\overline{\mathbf{x}} = \mathbf{A}^{-1}\overline{\mathbf{b}}$, so the point $\overline{\mathbf{x}}$ must lie at a vertex of the parallelogram shown in Figure 10. (You saw in Subsection 2.2 how linear transformations map squares to parallelograms.)

Vector positions for the vertices of the parallelogram are $\mathbf{b} + \delta\mathbf{b}$, namely $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 3.9 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$.

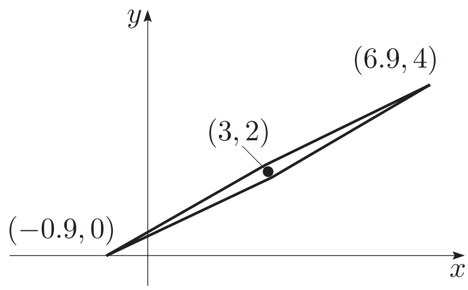


Figure 10 Transformation of the square in Figure 9

The greatest change in $\mathbf{x}$ occurs when $\overline{\mathbf{x}}$ is at a vertex furthest from (the exact solution) $\mathbf{x} = (3 \quad 2)^T$ in Figure 10, in other words either at $\overline{\mathbf{x}} = (6.9 \quad 4)^T$ or at $\overline{\mathbf{x}} = (-0.9 \quad 0)^T$.

The vertex $\overline{\mathbf{x}} = (6.9 \quad 4)^T$ arises from choosing $\overline{\mathbf{b}} = (-1.1 \quad -0.7)^T$, and $\overline{\mathbf{x}} = (-0.9 \quad 0)^T$ arises from choosing $\overline{\mathbf{b}} = (-0.9 \quad -0.9)^T$.

In either case, $\|\delta\mathbf{x}\| = \|\overline{\mathbf{x}} - \mathbf{x}\| = 3.9$, as we have seen. So here we have a situation in which a numerical change of 0.1 in the elements of $\mathbf{b}$ has caused a change of 3.9 in an element of the solution. We would certainly regard such a system of equations as ill-conditioned. It is the ratio of $\|\delta\mathbf{x}\|$ to $\|\delta\mathbf{b}\|$ that is relevant here. In this instance, we have found a point $\overline{\mathbf{b}}$ and its image $\overline{\mathbf{x}}$ for which this ratio is $3.9/0.1 = 39$ (a rather large number). Once we have found one instance of $\overline{\mathbf{b}}$ for which the ratio is large, we say that the system of equations is ill-conditioned.

We define the **absolute condition number** k_{a} for the problem of solving $\mathbf{Ax} = \mathbf{b}$, when $\mathbf{b}$ is subject to small changes of up to ε in magnitude, to be the largest possible value of the ratio of $\|\delta\mathbf{x}\|$ to $\|\delta\mathbf{b}\|$, that is,

$$k_{\mathrm{a}} = \max_{\|\delta\mathbf{b}\|\leq\varepsilon} \frac{\|\delta\mathbf{x}\|}{\|\delta\mathbf{b}\|}.$$

We do not prove this here.

Because $\mathbf{A}^{-1}$ represents a linear transformation, this largest value occurs when the perturbed vector $\overline{\mathbf{b}}$ lies at a corner of the square of side 2ε centred on $\mathbf{b}$. In the example above, we have shown that $k_{\mathrm{a}} = 39$, which tells us that the system of equations is *absolutely ill-conditioned*.

We will discuss methods of determining k_{a} and specific criteria for absolute ill-conditioning shortly.

The cause of the ill-conditioning can be deduced by a re-examination of equation (28). Solving this equation corresponds to finding the point of intersection of the two lines $x - 2y = -1$ and $x - 1.9y = -0.8$. These lines are almost parallel, so a small change in $\mathbf{b}$ can give rise to a large change in the solution.

These ideas can be applied to many problems other than those involving systems of linear equations.

Criteria for absolute ill-conditioning

Suppose that small changes are made in the data for a problem. The problem is **absolutely ill-conditioned** if it is possible for the absolute change in the solution to be significantly larger than the absolute change in the data.

Alternatively, we can think of small errors or uncertainties in the data giving rise to significantly larger errors or uncertainties in the solution.

Usually, the interpretation of *significantly larger* is dependent on the context. However, for the sake of clarity and certainty, we adopt the following module convention. A problem is judged to be:

- absolutely well-conditioned if the absolute condition number k_{a} for the problem is less than 5
- neither absolutely well-conditioned nor absolutely ill-conditioned if k_{a} is greater than 5 but less than 10
- absolutely ill-conditioned if k_{a} is greater than 10.

Different numbers might be more appropriate for large systems of equations, but 5 and 10 are suitable choices for systems of two or three linear equations.

For very large systems of equations, we may try to detect ill-conditioning by making small changes in the data. If, for the changes that we try, the changes in the solution remain small, then we can say only that we have found *no evidence of ill-conditioning* and that the problem *may* be well-conditioned. For small systems of equations, however, where it is feasible to compute the inverse matrix $\mathbf{A}^{-1}$, we can give a much better way of detecting ill-conditioning or well-conditioning.

Changes that we have not tried might cause significantly larger changes in the solution.

From equation (27) we have

$$\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{b}.$$

Hence any change in the right-hand-side vector $\mathbf{b}$ will be multiplied by $\mathbf{A}^{-1}$ to give the change in the solution.

To see how this works, we return to the linear problem in Example 12:

$$\begin{pmatrix} 1 & -2 \\ 1 & -1.9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ -0.8 \end{pmatrix},$$

where

$$\mathbf{A}^{-1} = \begin{pmatrix} -19 & 20 \\ -10 & 10 \end{pmatrix}.$$

The argument will be clearer if we let $\delta\mathbf{b} = (\varepsilon_1 \quad \varepsilon_2)^T$, where $\varepsilon_1 = \pm\varepsilon$ and $\varepsilon_2 = \pm\varepsilon$. Then we have

$$\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{b} = \begin{pmatrix} -19 & 20 \\ -10 & 10 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} -19\varepsilon_1 + 20\varepsilon_2 \\ -10\varepsilon_1 + 10\varepsilon_2 \end{pmatrix}.$$

We can see that the largest element of $\delta\mathbf{x}$ takes its maximum value when $\varepsilon_1 = -\varepsilon$ and $\varepsilon_2 = \varepsilon$, giving $\delta\mathbf{x} = (39\varepsilon \quad 20\varepsilon)^T$ and $\|\delta\mathbf{x}\| = 39\varepsilon$.

Notice that the sign of ε_i is taken to be the same as that of the element of $\mathbf{A}$ that multiplies it so as to give the largest possible result for $\|\delta\mathbf{x}\|$.

Now $\|\delta\mathbf{b}\| = \varepsilon$, therefore $k_\mathrm{a} = 39$, as we observed above. It is no coincidence that this is also the maximum row sum (i.e. the sum of the elements in a row) of the magnitudes of the elements of $\mathbf{A}^{-1}$.

This example illustrates the following result (which we do not prove here).

Absolute condition number of an invertible $n \times n$ matrix

The absolute condition number for small changes to the right-hand-side vector $\mathbf{b}$ in the solution of $\mathbf{Ax} = \mathbf{b}$ is given by the maximum row sum of the magnitudes of the elements of $\mathbf{A}^{-1}$, that is,

$$k_\mathrm{a} = \max_i\{|c_{i1}| + |c_{i2}| + \cdots + |c_{in}|\}, \tag{29}$$

where the c_{ij} are the elements of the matrix $\mathbf{C} = \mathbf{A}^{-1}$.

Exercise 36

(a) Determine the absolute condition number for small changes to the right-hand-side vector $\mathbf{b}$ in the solution of $\mathbf{Ax} = \mathbf{b}$ when $\mathbf{A}$ and its inverse are given by

$$\mathbf{A} = \begin{pmatrix} 6 & 4 & 3 \\ 7.5 & 6 & 5 \\ 10 & 7.5 & 6 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} 6 & 6 & -8 \\ -20 & -24 & 30 \\ 15 & 20 & -24 \end{pmatrix}.$$

(b) What form does $\delta\mathbf{b} = (\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3)^T$ take, where $\varepsilon_1 = \pm\varepsilon$, $\varepsilon_2 = \pm\varepsilon$ and $\varepsilon_3 = \pm\varepsilon$, so that $\|\delta\mathbf{x}\|$ takes its maximum value? Confirm that the largest value of $\|\delta\mathbf{x}\|/\|\delta\mathbf{b}\|$ equals the absolute condition number.

In order to determine the conditioning of a problem, we have had to do what we had hoped to avoid: calculate the inverse of the matrix $\mathbf{A}$. However, this may be the price that we have to pay if we are worried that our problem may be sensitive to small changes in the data.

Note also that we have not discussed the effect of changes to the elements in $\mathbf{A}$.

The cure for ill-conditioning is fairly drastic. We can abandon the current equations and try to find some more data – or even abandon the model.

Exercise 37

For each of the following examples, determine the absolute condition number and comment on the conditioning of the problem.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1.4 & 1 \\ 2 & 1.4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3.7 \\ 5.2 \end{pmatrix}$

Learning outcomes

After studying this unit, you should be able to:

- understand how a system of linear equations can be represented using matrices
- solve 2×2 and 3×3 systems of linear equations by the Gaussian elimination method, using essential row interchanges where necessary
- add, subtract and multiply matrices of suitable sizes, and multiply a matrix by a scalar
- understand the terms transpose of a matrix, symmetric matrix, diagonal matrix, upper triangular matrix, lower triangular matrix, zero matrix, identity matrix, inverse matrix, invertible matrix and non-invertible matrix
- understand that a matrix can be used to represent a linear transformation, and know what this means geometrically for a 2×2 matrix
- find the inverse of a 2×2 matrix
- evaluate the determinant of an $n \times n$ matrix, by hand when $n = 2$ or 3, and using the Gaussian elimination method for $n > 3$
- use the determinant of a matrix to evaluate cross products, areas and volumes
- find the interpolating polynomial of degree n passing through $n + 1$ data points, by hand for $n \leq 3$
- understand what is meant by absolute ill-conditioning for a system of linear equations, and know how to determine whether a particular problem is absolutely ill-conditioned.

Solutions to exercises

Solution to Exercise 1

Stage 1(a) We eliminate x_1 using $E_2 - 5E_1$, which gives

$$-3x_2 + 7x_3 = 10, \qquad E_{2a}$$

followed by $E_3 - 4E_1$, which gives

$$-6x_2 + x_3 = 7. \qquad E_{3a}$$

Stage 1(b) We eliminate x_2 using $E_{3a} - 2E_{2a}$, which gives

$$-13x_3 = -13. \qquad E_{3b}$$

Stage 2 The solution is obtained by back substitution. From E_{3b}, we find $x_3 = 1$. Substituting this into E_{2a} gives $-3x_2 + 7 = 10$, hence $x_2 = -1$. From E_1, $x_1 - 1 - 1 = 2$, hence $x_1 = 4$. So the solution is

$$x_1 = 4, \quad x_2 = -1, \quad x_3 = 1.$$

Solution to Exercise 2

(a) $\mathbf{A}|\mathbf{b} = \left(\begin{array}{cc|c} 3 & -5 & 8 \\ 4 & 7 & 11 \end{array}\right)$

(b) $\mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$

Solution to Exercise 3

The augmented matrix representing these equations is as follows.

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -1 & 5 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \end{array}$$

Stage 1 We reduce to zero the element below the leading diagonal.

$$\begin{array}{r} \\ \mathbf{R}_2 - 3\mathbf{R}_1 \end{array} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -7 & -7 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \end{array}$$

Stage 2 The equations represented by the new matrix are

$$\begin{aligned} x_1 + 2x_2 &= 4, \qquad && E_1 \\ -\,7x_2 &= -7. && E_{2a} \end{aligned}$$

From E_{2a}, we find $x_2 = 1$. Substituting this into E_1, we obtain $x_1 + 2 = 4$. Hence $x_1 = 2$, giving the solution

$$x_1 = 2, \quad x_2 = 1.$$

Solution to Exercise 4

The augmented matrix representing these equations is as follows.

$$\left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 5 & 2 & 2 & 20 \\ 4 & -2 & -3 & 15 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2 - 5\mathbf{R}_1 \\ \mathbf{R}_3 - 4\mathbf{R}_1 \end{array} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & -3 & 7 & 10 \\ 0 & -6 & 1 & 7 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3a} - 2\mathbf{R}_{2a} \end{array} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & -3 & 7 & 10 \\ 0 & 0 & -13 & -13 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2 The equations represented by the new matrix are

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, && E_1 \\ - 3x_2 + 7x_3 &= 10, && E_{2a} \\ - 13x_3 &= -13. && E_{3b} \end{aligned}$$

From E_{3b}, we have $x_3 = 1$. From E_{2a}, we have $-3x_2 + 7 = 10$, so $x_2 = -1$. From E_1, we have $x_1 - 1 - 1 = 2$, so $x_1 = 4$. Hence the solution is

$$x_1 = 4, \quad x_2 = -1, \quad x_3 = 1,$$

as we saw in the solution to Exercise 1. As a useful check, note via direct substitution that this solution satisfies E_2 and E_3.

Solution to Exercise 5

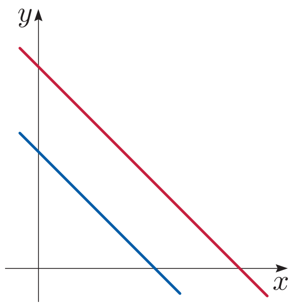
The graphs of the four pairs of equations are sketched below.

(a) The two lines are parallel, so there is no solution.

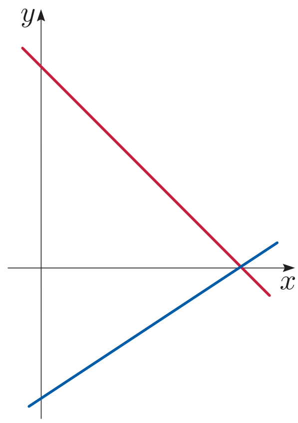
(b) There is one solution, at the intersection of the two lines.

(c) The two lines coincide, so there is an infinite number of solutions, consisting of all the points that lie on the line.

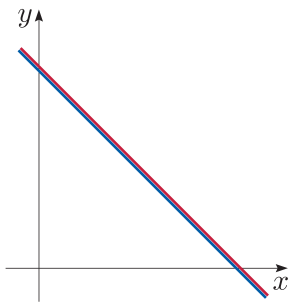
(d) There is one solution, at the intersection of the two lines.



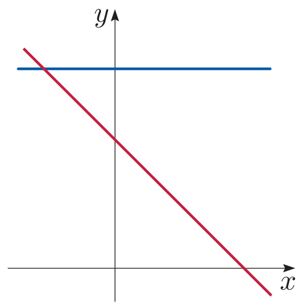
(a)



(b)



(c)



(d)

Solution to Exercise 6

For the two lines to be parallel or coincident, they must have the same slope.

If $b = 0$, the first equation becomes $ax = e$ and the line has infinite slope. Thus we must also have $d = 0$, and the second equation becomes $cx = f$. For a non-trivial solution (i.e. $a \neq 0$ and $c \neq 0$), these lines are coincident if $e/a = f/c$. The same argument applies if we start from $d = 0$.

The remaining case is when $b \neq 0$ and $d \neq 0$. Then the system of equations can be written as

$$y = -\frac{a}{b}x + \frac{e}{b},$$
$$y = -\frac{c}{d}x + \frac{f}{d}.$$

The first line has slope $-a/b$, while the second line has slope $-c/d$. Hence the two lines are parallel if

$$\frac{a}{b} = \frac{c}{d}.$$

The lines are coincident when $e/b = f/d$ also.

Solution to Exercise 7

(a) First, we write down the augmented matrix.

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{rrr|r} 1 & -2 & 5 & 7 \\ 1 & 3 & -4 & 20 \\ 1 & 18 & -31 & 40 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2 - \mathbf{R}_1 \\ \mathbf{R}_3 - \mathbf{R}_1 \end{array} \left(\begin{array}{rrr|r} 1 & -2 & 5 & 7 \\ 0 & 5 & -9 & 13 \\ 0 & 20 & -36 & 33 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3a} - 4\mathbf{R}_{2a} \end{array} \left(\begin{array}{rrr|r} 1 & -2 & 5 & 7 \\ 0 & 5 & -9 & 13 \\ 0 & 0 & 0 & -19 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2 We try to solve the equations represented by the rows of the above matrix.

Since the rows of $\mathbf{A}$ are linearly dependent but the rows of $\mathbf{A}|\mathbf{b}$ are not, there is no solution.

(b) First, we write down the augmented matrix.

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{rrr|r} 1 & -2 & 5 & 6 \\ 1 & 3 & -4 & 7 \\ 2 & 6 & -12 & 12 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2 - \mathbf{R}_1 \\ \mathbf{R}_3 - 2\mathbf{R}_1 \end{array} \left(\begin{array}{rrr|r} 1 & -2 & 5 & 6 \\ 0 & 5 & -9 & 1 \\ 0 & 10 & -22 & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3a} - 2\mathbf{R}_{2a} \end{array} \left(\begin{array}{rrr|r} 1 & -2 & 5 & 6 \\ 0 & 5 & -9 & 1 \\ 0 & 0 & -4 & -2 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3b} \end{array}$$

Stage 2 We solve the equations by back substitution.

Since the rows of $\mathbf{A}$ are linearly independent, there is a unique solution. Back substitution gives

$$x_1 = 5.7, \quad x_2 = 1.1, \quad x_3 = 0.5.$$

(c) First, we write down the augmented matrix.

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{ccc|c} 1 & -4 & 1 & 14 \\ 5 & -1 & -1 & 2 \\ 6 & 14 & -6 & -52 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{array}$$

Stage 1(a) We reduce the elements below the leading diagonal in column 1 to zero.

$$\begin{array}{r} \\ \mathbf{R}_2 - 5\mathbf{R}_1 \\ \mathbf{R}_3 - 6\mathbf{R}_1 \end{array} \left(\begin{array}{ccc|c} 1 & -4 & 1 & 14 \\ 0 & 19 & -6 & -68 \\ 0 & 38 & -12 & -136 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2\mathrm{a}} \\ \mathbf{R}_{3\mathrm{a}} \end{array}$$

Stage 1(b) We reduce the element below the leading diagonal in column 2 to zero.

$$\begin{array}{r} \\ \\ \mathbf{R}_{3\mathrm{a}} - 2\mathbf{R}_{2\mathrm{a}} \end{array} \left(\begin{array}{ccc|c} 1 & -4 & 1 & 14 \\ 0 & 19 & -6 & -68 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2\mathrm{a}} \\ \mathbf{R}_{3\mathrm{b}} \end{array}$$

Stage 2 We solve the equations by back substitution.

Since the rows of $\mathbf{A}|\mathbf{b}$ are linearly dependent, there is an infinite number of solutions. Putting $x_3 = k$, back substitution gives the set of solutions

$$x_1 = (5k - 6)/19, \quad x_2 = (6k - 68)/19, \quad x_3 = k.$$

Here

$$\begin{aligned} \mathbf{R}_{3\mathrm{b}} &= \mathbf{R}_{3\mathrm{a}} - 2\mathbf{R}_{2\mathrm{a}} \\ &= (\mathbf{R}_3 - 6\mathbf{R}_1) - 2(\mathbf{R}_2 - 5\mathbf{R}_1) \\ &= \mathbf{R}_3 - 2\mathbf{R}_2 + 4\mathbf{R}_1 = \mathbf{0}, \end{aligned}$$

which is the required non-trivial linear combination of the rows of $\mathbf{A}|\mathbf{b}$ that gives a row of zeros.

Solution to Exercise 8

(a) $3\mathbf{A} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}, \quad -\mathbf{A} = \begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{pmatrix}.$

(b) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1+1 & 2+3 & 3+7 \\ 4+4 & 5+(-5) & 6+0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 10 \\ 8 & 0 & 6 \end{pmatrix},$

$\mathbf{B} + \mathbf{C} = \begin{pmatrix} 1+(-2) & 3+1 & 7+(-6) \\ 4+(-5) & (-5)+3 & 0+3 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 1 \\ -1 & -2 & 3 \end{pmatrix}.$

(c) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \begin{pmatrix} 2 & 5 & 10 \\ 8 & 0 & 6 \end{pmatrix} + \begin{pmatrix} -2 & 1 & -6 \\ -5 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 4 \\ 3 & 3 & 9 \end{pmatrix},$

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} -1 & 4 & 1 \\ -1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 4 \\ 3 & 3 & 9 \end{pmatrix}.$

Thus $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Solution to Exercise 9

(a)
$$\begin{aligned}\mathbf{A}-\mathbf{B} &= \begin{pmatrix}1 & -2\\ 7 & -3\end{pmatrix} - \begin{pmatrix}-3 & 0\\ 4 & -5\end{pmatrix}\\ &= \begin{pmatrix}1-(-3) & -2-0\\ 7-4 & -3-(-5)\end{pmatrix} = \begin{pmatrix}4 & -2\\ 3 & 2\end{pmatrix},\\ \mathbf{B}-\mathbf{A} &= \begin{pmatrix}-3 & 0\\ 4 & -5\end{pmatrix} - \begin{pmatrix}1 & -2\\ 7 & -3\end{pmatrix}\\ &= \begin{pmatrix}-3-1 & 0-(-2)\\ 4-7 & -5-(-3)\end{pmatrix} = \begin{pmatrix}-4 & 2\\ -3 & -2\end{pmatrix}.\end{aligned}$$

(b) From part (a) we have $\mathbf{B}-\mathbf{A} = -(\mathbf{A}-\mathbf{B})$.

Solution to Exercise 10

(a) $2\mathbf{A}-5\mathbf{B} = \begin{pmatrix}2 & 4 & 6\\ 8 & 10 & 12\end{pmatrix} - \begin{pmatrix}5 & 15 & 35\\ 20 & -25 & 0\end{pmatrix} = \begin{pmatrix}-3 & -11 & -29\\ -12 & 35 & 12\end{pmatrix}.$

(b) $3(\mathbf{A}+\mathbf{B}) = 3\begin{pmatrix}2 & 5 & 10\\ 8 & 0 & 6\end{pmatrix} = \begin{pmatrix}6 & 15 & 30\\ 24 & 0 & 18\end{pmatrix},$

$3\mathbf{A}+3\mathbf{B} = \begin{pmatrix}3 & 6 & 9\\ 12 & 15 & 18\end{pmatrix} + \begin{pmatrix}3 & 9 & 21\\ 12 & -15 & 0\end{pmatrix} = \begin{pmatrix}6 & 15 & 30\\ 24 & 0 & 18\end{pmatrix}.$

Thus $3(\mathbf{A}+\mathbf{B}) = 3\mathbf{A}+3\mathbf{B}$.

Solution to Exercise 11

(a) $\begin{pmatrix}1 & 2\\ 2 & -3\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix} = \begin{pmatrix}5 & 1\\ -4 & 9\end{pmatrix}.$

(b) $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1 & 6\\ 0 & 2\end{pmatrix} = \begin{pmatrix}2 & 14\end{pmatrix}.$

(c) $\begin{pmatrix}1\\ 2\end{pmatrix}\begin{pmatrix}3 & 0 & -4\end{pmatrix} = \begin{pmatrix}3 & 0 & -4\\ 6 & 0 & -8\end{pmatrix}.$

(d) $\begin{pmatrix}3 & 1 & 2\\ 0 & 5 & 1\end{pmatrix}\begin{pmatrix}-2 & 0 & 1\\ 1 & 3 & 0\\ 4 & 1 & -1\end{pmatrix} = \begin{pmatrix}3 & 5 & 1\\ 9 & 16 & -1\end{pmatrix}.$

(e) The product does not exist, because the left-hand matrix has only 1 column, whereas the right-hand matrix has 2 rows.

Solution to Exercise 12

$$\mathbf{Ax} = \begin{pmatrix}3 & -1 & 5\\ 6 & 4 & 7\\ 2 & -3 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}3x_1 - x_2 + 5x_3\\ 6x_1 + 4x_2 + 7x_3\\ 2x_1 - 3x_2\end{pmatrix}.$$

The two matrices $\mathbf{Ax}$ and $\mathbf{b}$ are equal only if

$$\begin{aligned}3x_1 - x_2 + 5x_3 &= 2,\\ 6x_1 + 4x_2 + 7x_3 &= 5,\\ 2x_1 - 3x_2 &= 6.\end{aligned}$$

Hence the equation $\mathbf{Ax} = \mathbf{b}$ is equivalent to the system of equations.

Solution to Exercise 13

(a) $$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 14 \end{pmatrix},$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 9 \\ 5 & 4 \end{pmatrix}.$$

Thus $\mathbf{AB} \neq \mathbf{BA}$.

(b) $$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} 3 & 5 \\ 7 & 14 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 25 \\ 28 & 70 \end{pmatrix},$$

$$\begin{aligned} \mathbf{A}(\mathbf{BC}) &= \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \left(\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 6 & 20 \\ 5 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 25 \\ 28 & 70 \end{pmatrix}. \end{aligned}$$

Thus $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

(c) $$\begin{aligned} \mathbf{A}(\mathbf{B}+\mathbf{C}) &= \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \left(\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 15 & 24 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \mathbf{AB}+\mathbf{AC} &= \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 \\ 7 & 14 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 8 & 10 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 15 & 24 \end{pmatrix}. \end{aligned}$$

Thus $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.

Solution to Exercise 14

(a) $\mathbf{A}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, $\mathbf{B}^T = \begin{pmatrix} 2 & -1 & 3 \\ 5 & -4 & 1 \end{pmatrix}$, $\mathbf{C}^T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

(b) $$(\mathbf{A}+\mathbf{B})^T = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 5 \\ -1 & -4 \\ 3 & 1 \end{pmatrix} \right)^T = \begin{pmatrix} 3 & 7 \\ 2 & 0 \\ 8 & 7 \end{pmatrix}^T = \begin{pmatrix} 3 & 2 & 8 \\ 7 & 0 & 7 \end{pmatrix},$$

$$\mathbf{A}^T + \mathbf{B}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 3 \\ 5 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 8 \\ 7 & 0 & 7 \end{pmatrix}.$$

Thus $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) $$(\mathbf{AC})^T = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \right)^T = \begin{pmatrix} 5 & 6 \\ 11 & 12 \\ 17 & 18 \end{pmatrix}^T = \begin{pmatrix} 5 & 11 & 17 \\ 6 & 12 & 18 \end{pmatrix},$$

$$\mathbf{C}^T\mathbf{A}^T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 11 & 17 \\ 6 & 12 & 18 \end{pmatrix}.$$

Thus $(\mathbf{AC})^T = \mathbf{C}^T\mathbf{A}^T$.

(d) From the solution to part (a), we can immediately see that $\mathbf{C}^T \neq \mathbf{C}$ and therefore $\mathbf{C}$ is not symmetric.

(e) Let $\mathbf{D} = \mathbf{C}\mathbf{C}^T$. Then, using the rules for transposes of matrices, we have

$$\mathbf{D}^T = (\mathbf{C}\mathbf{C}^T)^T = (\mathbf{C}^T)^T\mathbf{C}^T = \mathbf{C}\mathbf{C}^T = \mathbf{D},$$

so $\mathbf{D} = \mathbf{C}\mathbf{C}^T$ is a symmetric matrix.

This is true for *any* matrix $\mathbf{C}$, but we can verify that this result holds for the $\mathbf{C}$ given here since

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix},$$

which is clearly symmetric.

Solution to Exercise 15

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 \\ 5 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix},$$

so we see that $\mathbf{AB}$ is not symmetric.

$$(\mathbf{AB})^T = \begin{pmatrix} 3 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix},$$

$$\mathbf{B}^T\mathbf{A}^T = \mathbf{BA} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix},$$

so $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Solution to Exercise 16

(a) The matrix is upper triangular.

(b) The matrix is neither upper triangular nor lower triangular, thus cannot be diagonal.

(c) The matrix is upper triangular and lower triangular, and hence is diagonal.

Solution to Exercise 17

(a) $$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} \frac{8}{9} & -\frac{1}{9} \\ -\frac{7}{9} & \frac{2}{9} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Since $\mathbf{AB} = \mathbf{I}$, it follows that $\mathbf{B} = \mathbf{A}^{-1}$.

(b) $$\mathbf{AB} = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Since $\mathbf{AB} = \mathbf{I}$, it follows that $\mathbf{B} = \mathbf{A}^{-1}$.

Solution to Exercise 18

We form the 4×2 augmented matrix.

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \end{array}$$

First, assume $a \neq 0$. Going through the stages of Procedure 2 in turn (as in Example 7), we obtain the following.

$$\mathbf{R}_2 - \tfrac{c}{a}\mathbf{R}_1 \quad \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2\mathrm{a}} \end{array}$$

$$\tfrac{a}{ad-bc}\mathbf{R}_{2\mathrm{a}} \quad \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2\mathrm{b}} \end{array}$$

$$d - \frac{c}{a}b = \frac{ad-bc}{a}$$

$$\mathbf{R}_1 - b\mathbf{R}_{2\mathrm{b}} \quad \left(\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \begin{array}{l} \mathbf{R}_{1\mathrm{a}} \\ \mathbf{R}_{2\mathrm{b}} \end{array}$$

$$\tfrac{1}{a}\mathbf{R}_{1\mathrm{a}} \quad \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \begin{array}{l} \mathbf{R}_{1\mathrm{b}} \\ \mathbf{R}_{2\mathrm{b}} \end{array}$$

Hence

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $a = 0$, we must start by interchanging $\mathbf{R}_1$ and $\mathbf{R}_2$.

$$\mathbf{R}_1 \leftrightarrow \mathbf{R}_2 \quad \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_{1\mathrm{a}} \\ \mathbf{R}_{2\mathrm{a}} \end{array}$$

Again, going through the stages of Procedure 2 in turn, noting that $bc \neq 0$ since $ad - bc \neq 0$, we obtain the following.

$$\tfrac{1}{b}\mathbf{R}_{2\mathrm{a}} \quad \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_{1\mathrm{a}} \\ \mathbf{R}_{2\mathrm{b}} \end{array}$$

$$\mathbf{R}_{1\mathrm{a}} - d\,\mathbf{R}_{2\mathrm{b}} \quad \left(\begin{array}{cc|cc} c & 0 & -\frac{d}{b} & 1 \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_{1\mathrm{b}} \\ \mathbf{R}_{2\mathrm{b}} \end{array}$$

$$\tfrac{1}{c}\mathbf{R}_{1\mathrm{b}} \quad \left(\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right) \begin{array}{l} \mathbf{R}_{1\mathrm{c}} \\ \mathbf{R}_{2\mathrm{b}} \end{array}$$

This gives the same inverse matrix as substituting $a = 0$ into the inverse matrix found earlier.

Solution to Exercise 19

(a) Since $\det \mathbf{A} = (7 \times 3) - (4 \times 5) = 1$,

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}.$$

(b) Since $\det \mathbf{A} = (6 \times 3) - (2 \times 9) = 0$, $\mathbf{A}^{-1}$ does not exist.

(c) Since $\det \mathbf{A} = (4 \times 3) - (2 \times 5) = 2$,

$$\mathbf{A}^{-1} = \tfrac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix}.$$

Solution to Exercise 20

(a) We have

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$

and

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I},$$

hence $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Since $\det \mathbf{A} = 2$, we have

$$\mathbf{A}^{-1} = \tfrac{1}{2}\begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{pmatrix}.$$

Since $\det \mathbf{B} = -2$, we have

$$\mathbf{B}^{-1} = \tfrac{1}{-2}\begin{pmatrix} -2 & -4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

(c)

$$(\mathbf{A}+\mathbf{B})^{-1} = \begin{pmatrix} 5 & 6 \\ 3 & 3 \end{pmatrix}^{-1} = \tfrac{1}{-3}\begin{pmatrix} 3 & -6 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -\frac{5}{3} \end{pmatrix},$$

$$\mathbf{A}^{-1}+\mathbf{B}^{-1} = \begin{pmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2} & 1 \\ -\frac{5}{2} & -\frac{1}{2} \end{pmatrix}.$$

Thus $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1}+\mathbf{B}^{-1}$.

(d)

$$\begin{aligned}(\mathbf{AB})^{-1} &= \left(\begin{pmatrix} 2 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} 4 & 4 \\ 7 & 6 \end{pmatrix}^{-1} = \tfrac{1}{-4}\begin{pmatrix} 6 & -4 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 1 \\ \frac{7}{4} & -1 \end{pmatrix},\end{aligned}$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} \frac{5}{2} & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 1 \\ \frac{7}{4} & -1 \end{pmatrix}.$$

Thus $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Solution to Exercise 21

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(b) (i) $\mathbf{A}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$.

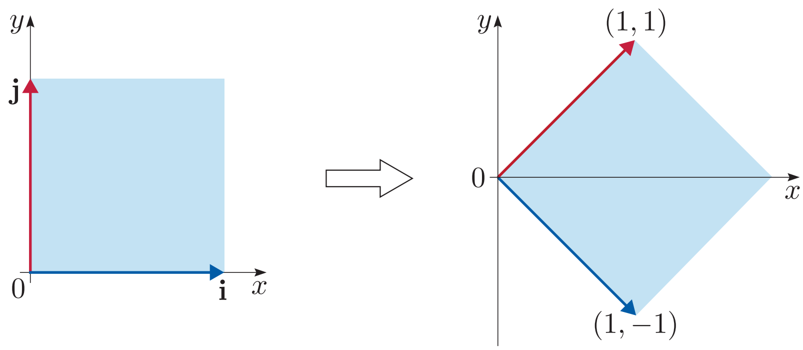
(ii) $\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(iii) $\mathbf{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Solution to Exercise 22

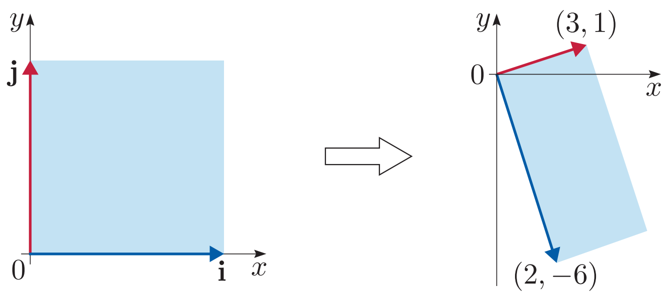
(a) In each case, the area of the parallelogram is the magnitude of $\det \mathbf{A}$. (Note that in each case the parallelogram is a rectangle!)

In this case, $\det \mathbf{A} = 2$.



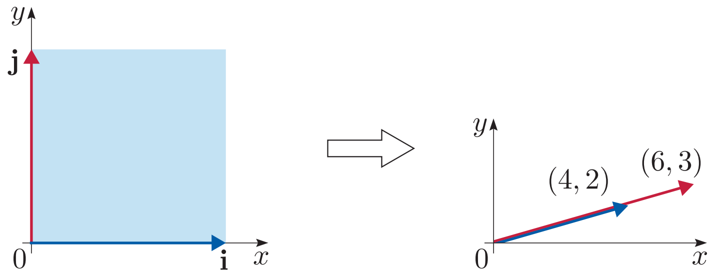
From the figure, the area of the parallelogram is the area of a square of side $\sqrt{1+1} = \sqrt{2}$. That is, the area is 2, agreeing with the value of $|\det \mathbf{A}|$.

(b) In this case, $\det \mathbf{A} = -20$.



From the figure, the area of the parallelogram is the area of a rectangle with sides $\sqrt{3^2+1^2} = \sqrt{10}$ and $\sqrt{2^2+(-6)^2} = \sqrt{40} = 2\sqrt{10}$. That is, the area is 20, agreeing with the value of $|\det \mathbf{A}|$.

(c) In this case, $\det \mathbf{A} = 0$.



From the figure, the area of the parallelogram must be 0, because the vectors defining it are parallel. This agrees with the value of $|\det \mathbf{A}|$.

Solution to Exercise 23

$\det \mathbf{A} = 1 - (-2) = 3$ and $\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$.

Putting $\mathbf{x} = (x \quad y)^T$ and $\mathbf{b} = (1 \quad -1)^T$, we need to solve $\mathbf{Ax} = \mathbf{b}$. Multiplying both sides on the left by $\mathbf{A}^{-1}$, we obtain $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$. This simplifies to

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so $x = 1$ and $y = 0$.

Solution to Exercise 24

(a) $\begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad.$

(b) $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = \begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad = -(ad - bc) = -\det \mathbf{A}.$

(c) $\det \mathbf{A}^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = \det \mathbf{A}.$

(d) $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & a \\ c & c \end{vmatrix} = 0.$

(e) $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k(ad - bc) = k \det \mathbf{A}.$

(f) $\det(k\mathbf{A}) = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2(ad - bc) = k^2 \det \mathbf{A}.$

(g) $\begin{vmatrix} a & b \\ c - ma & d - mb \end{vmatrix} = ad - amb - bc + bma = ad - bc = \det \mathbf{A},$

$\begin{vmatrix} a - mc & b - md \\ c & d \end{vmatrix} = ad - mcd - bc + mdc = ad - bc = \det \mathbf{A}.$

(h) From part (f) with $k = (ad - bc)^{-1}$, we have

$$\begin{aligned}\det \mathbf{A}^{-1} &= \begin{vmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{vmatrix} \\ &= \frac{1}{(ad-bc)^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} \\ &= \frac{1}{ad-bc} \\ &= \frac{1}{\det \mathbf{A}}.\end{aligned}$$

Solution to Exercise 25

(a)
$$\begin{aligned}\begin{vmatrix} 4 & 1 & 0 \\ 0 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix} &= 4\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} \\ &= (4 \times 5) - (1 \times 2) + 0 \\ &= 18.\end{aligned}$$

(b)
$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (1 \times (-3)) - (2 \times (-6)) + (3 \times (-3)) \\ &= -3 + 12 - 9 \\ &= 0.\end{aligned}$$

(c)
$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} &= 1\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 0 & 6 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} \\ &= (1 \times 24) - (2 \times 0) + (3 \times 0) \\ &= 24.\end{aligned}$$

Solution to Exercise 26

(a) Expanding by the second row, we have

$$\begin{aligned}\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 6 \end{vmatrix} &= -0\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ &= (2 \times 3) - (1 \times (-1)) \\ &= 6 + 1 \\ &= 7.\end{aligned}$$

(b) (i) We obtain -7 (interchanging two rows).

(ii) We obtain 7 (taking the transpose).

(iii) We obtain -21 (interchanging two rows, as in part (b)(i), then multiplying column 2 by 3).

(c) Expanding by the first column, we have

$$\det \mathbf{A} = 1\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix} - 0\begin{vmatrix}3 & 2\\ 2 & 3\end{vmatrix} + 0\begin{vmatrix}3 & 2\\ 1 & -1\end{vmatrix} = 5.$$

We obtain

$$\mathbf{U} = \begin{pmatrix}1 & 3 & 2\\ 0 & 1 & -1\\ 0 & 0 & 5\end{pmatrix},$$

then (expanding by the first column)

$$\det \mathbf{U} = 1\begin{vmatrix}1 & -1\\ 0 & 5\end{vmatrix} - 0\begin{vmatrix}3 & 2\\ 0 & 5\end{vmatrix} + 0\begin{vmatrix}3 & 2\\ 1 & -1\end{vmatrix} = 5.$$

The product of the diagonal elements of $\mathbf{U}$ is also $1 \times 1 \times 5 = 5$.

Note that $\det \mathbf{U} = \det \mathbf{A}$, which follows from rule (g) above.

Solution to Exercise 27

We have

$$\begin{aligned}\det \mathbf{A} = \begin{vmatrix}1 & -4 & 2\\ 3 & -2 & 3\\ 8 & -2 & 9\end{vmatrix} &= 1\begin{vmatrix}-2 & 3\\ -2 & 9\end{vmatrix} - (-4)\begin{vmatrix}3 & 3\\ 8 & 9\end{vmatrix} + 2\begin{vmatrix}3 & -2\\ 8 & -2\end{vmatrix}\\ &= -12 + 12 + 20 = 20\end{aligned}$$

and

$$\det \mathbf{U} = \begin{vmatrix}1 & -4 & 2\\ 0 & 10 & -3\\ 0 & 0 & 2\end{vmatrix} = 1 \times 10 \times 2 = 20.$$

Thus $\det \mathbf{A} = \det \mathbf{U}$.

Solution to Exercise 28

By selecting the third column in the expansion of the determinant, we have

$$\begin{vmatrix}1 & -2 & 0 & 3\\ 4 & -5 & 0 & 6\\ -2 & -1 & 10 & 4\\ 5 & -7 & 0 & 9\end{vmatrix} = 10\begin{vmatrix}1 & -2 & 3\\ 4 & -5 & 6\\ 5 & -7 & 9\end{vmatrix} = 10\begin{vmatrix}1 & -2 & 3\\ 5 & -7 & 9\\ 5 & -7 & 9\end{vmatrix} = 0,$$

where the second equality followed by adding Row 1 to Row 2, and using the rule that such an operation leaves the determinant unchanged. The final equality to zero follows from rule (d) about two linearly dependent rows.

Solution to Exercise 29

Using equation (22) and the result of Exercise 26(a),

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix}2 & 1 & 3\\ 0 & 2 & 1\\ 3 & 1 & 6\end{vmatrix} = 7.$$

Solution to Exercise 30

The volume is

$$\left|\det\begin{pmatrix}1&0&1\\1&2&0\\0&1&3\end{pmatrix}\right| = \left|1\times\det\begin{pmatrix}2&0\\1&3\end{pmatrix} - 0\times\det\begin{pmatrix}1&0\\0&3\end{pmatrix} + 1\times\det\begin{pmatrix}1&2\\0&1\end{pmatrix}\right|$$
$$= \left|(1\times 6) - 0 + (1\times 1)\right| = 7.$$

Solution to Exercise 31

(a) $$\mathbf{b}\times\mathbf{c} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&-4\\1&-1&3\end{vmatrix}$$
$$= \begin{vmatrix}2&-4\\-1&3\end{vmatrix}\mathbf{i} - \begin{vmatrix}3&-4\\1&3\end{vmatrix}\mathbf{j} + \begin{vmatrix}3&2\\1&-1\end{vmatrix}\mathbf{k}$$
$$= 2\mathbf{i} - 13\mathbf{j} - 5\mathbf{k}.$$

(b) $$\mathbf{b}\times\mathbf{c} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\6&5&4\end{vmatrix}$$
$$= \begin{vmatrix}2&3\\5&4\end{vmatrix}\mathbf{i} - \begin{vmatrix}1&3\\6&4\end{vmatrix}\mathbf{j} + \begin{vmatrix}1&2\\6&5\end{vmatrix}\mathbf{k}$$
$$= -7\mathbf{i} + 14\mathbf{j} - 7\mathbf{k}.$$

Solution to Exercise 32

The area is

$$\tfrac{1}{2}\left|\det\begin{pmatrix}1&1&1\\3&1&2\\1&2&-1\end{pmatrix}\right| = \tfrac{1}{2}\left|1\times\det\begin{pmatrix}1&2\\2&-1\end{pmatrix} - 1\times\det\begin{pmatrix}3&2\\1&-1\end{pmatrix}\right.$$
$$\left. + 1\times\det\begin{pmatrix}3&1\\1&2\end{pmatrix}\right|$$
$$= \tfrac{1}{2}|-5+5+5| = 2.5.$$

Solution to Exercise 33

Suppose that the line is $y = a_0 + a_1x$, so the interpolation equations are

$$a_0 + 0.6a_1 = 1.856,$$
$$a_0 + 0.8a_1 = 2.3472.$$

The solution of this pair of equations is $a_0 = 0.3824$ and $a_1 = 2.456$, so the equation of the interpolating straight line is

$$y = 0.3824 + 2.456x.$$

Thus $y(0.65) = 1.9788$ and $y(0.47) = 1.536\,72$.

Solution to Exercise 34

Suppose that the quadratic is $y = a_0 + a_1x + a_2x^2$, so the interpolation equations are

$$a_0 + 0.2a_1 + (0.2)^2a_2 = 1.2,$$
$$a_0 + 0.4a_1 + (0.4)^2a_2 = 1.48,$$
$$a_0 + 0.6a_1 + (0.6)^2a_2 = 1.856.$$

The augmented matrix form of these equations is

$$\mathbf{X}|\mathbf{y} = \left(\begin{array}{ccc|c} 1 & 0.2 & 0.04 & 1.2 \\ 1 & 0.4 & 0.16 & 1.48 \\ 1 & 0.6 & 0.36 & 1.856 \end{array}\right).$$

After Stage 1 of the Gaussian elimination method, we have

$$\mathbf{U}|\mathbf{c} = \left(\begin{array}{ccc|c} 1 & 0.2 & 0.04 & 1.2 \\ 0 & 0.2 & 0.12 & 0.28 \\ 0 & 0 & 0.08 & 0.096 \end{array}\right).$$

Back substitution gives $\mathbf{a} = (1.016 \quad 0.68 \quad 1.2)^T$, so the equation of the interpolating quadratic is

$$y = 1.016 + 0.68x + 1.2x^2.$$

This gives $y(0.47) = 1.600\,68$. This estimate is very close to the value $1.598\,496$ obtained in Example 10. However, since the three points closest to $x = 0.47$ are 0.2, 0.4 and 0.6, the value $1.600\,68$ would be regarded as the most reliable of the four estimates that we have computed, in Examples 9 and 10, and Exercises 33 and 34.

Solution to Exercise 35

The change in $\mathbf{x}$ is

$$\delta\mathbf{x} = \overline{\mathbf{x}} - \mathbf{x} = (0.04 \quad 0.03 \quad -0.05)^T,$$

so the absolute change is

$$\|\delta\mathbf{x}\| = 0.05.$$

Solution to Exercise 36

(a) The row sums of the magnitudes of the elements of $\mathbf{A}^{-1}$ are 20, 74 and 59. Hence the absolute condition number is $k_a = 74$.

(b) Now,

$$\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{b} = \begin{pmatrix} 6 & 6 & -8 \\ -20 & -24 & 30 \\ 15 & 20 & -24 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} 6\varepsilon_1 + 6\varepsilon_2 - 8\varepsilon_3 \\ -20\varepsilon_1 - 24\varepsilon_2 + 30\varepsilon_3 \\ 15\varepsilon_1 + 20\varepsilon_2 - 24\varepsilon_3 \end{pmatrix}.$$

From this we can see that the largest element of $\delta\mathbf{x}$ takes its maximum value when $\varepsilon_1 = -\varepsilon$, $\varepsilon_2 = -\varepsilon$ and $\varepsilon_3 = \varepsilon$, so that $\delta\mathbf{b}$ takes the form

$$\delta\mathbf{b} = (-\varepsilon \quad -\varepsilon \quad \varepsilon)^T.$$

This gives $\delta\mathbf{x} = (-20\varepsilon \quad 74\varepsilon \quad -59\varepsilon)^T$, so $\|\delta\mathbf{x}\| = 74\varepsilon$. Since $\|\delta\mathbf{b}\| = \varepsilon$, the maximum value of $\|\delta\mathbf{x}\|/\|\delta\mathbf{b}\|$ is 74, as required.

Solution to Exercise 37

(a) The inverse matrix is

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

The problem is absolutely well-conditioned, since $k_a = 1$.

(The solution is

$$\mathbf{x} = \mathbf{A}^{-1}\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

though you were not asked to find it.)

(b) The inverse matrix is

$$\mathbf{A}^{-1} = \begin{pmatrix} -35 & 25 \\ 50 & -35 \end{pmatrix}.$$

The problem is absolutely ill-conditioned, since $k_a = 85$.

(The solution is

$$\mathbf{x} = \mathbf{A}^{-1}\begin{pmatrix} 3.7 \\ 5.2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 3 \end{pmatrix},$$

though you were not asked to find it.)