Unit 16

Further vector calculus

Introduction

In Unit 15 you saw that the spatial variations of a scalar field can be described by the gradient vector field, which can be expressed in terms of the three partial derivatives of the scalar field. Vector fields are more complicated than scalar fields. For a given Cartesian coordinate system there are nine partial derivatives associated with a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ at each field point. In the first two sections of this unit we will introduce two new fields that are combinations of these partial derivatives and that represent important physical properties of the vector field $\mathbf{F}$. One of these fields, a scalar field called the *divergence* of $\mathbf{F}$, is the subject of Section 1. The other, a vector field called the *curl* of $\mathbf{F}$, is the subject of Section 2.

These partial derivatives are $\partial F_1/\partial x$, $\partial F_1/\partial y$, $\partial F_1/\partial z$, $\partial F_2/\partial x$, $\partial F_2/\partial y$, $\partial F_2/\partial z$, $\partial F_3/\partial x$, $\partial F_3/\partial y$, $\partial F_3/\partial z$.

In Section 3 we measure the work done by a force in moving a particle from one point to another in a vector field, by introducing the scalar line integral. Finally, Section 4 develops links between gradient, curl and the scalar line integral.

This unit will also build on the concepts of kinetic energy and potential energy, introduced in Unit 9.

1 Divergence of a vector field

We will approach divergence in two different, but equivalent, ways. In Subsections 1.1 and 1.2 we define divergence mathematically in terms of partial derivatives, using the vector differential operator $\boldsymbol{\nabla}$, and give you practice in calculating the divergence of given vector fields. Subsection 1.3 discusses the problem of modelling the flow of heat energy in a uranium fuel rod, and predicting the temperature field in the rod. This enables us to develop a physical interpretation of divergence as the rate of heat outflow per unit volume at each point in the rod – one of many examples of divergence in the physical world.

1.1 Defining divergence

The gradient $\mathbf{grad}\, f$ of a scalar field f can be expressed as $\boldsymbol{\nabla} f$, where $\boldsymbol{\nabla}$ (del) is the vector differential operator

You saw the del operator, denoted $\boldsymbol{\nabla}$, in Unit 15, Subsection 3.4.

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

We now form the dot product of $\boldsymbol{\nabla}$ with an arbitrary vector field $\mathbf{F}(x, y, z)$, treating $\boldsymbol{\nabla}$ as if it is a vector.

We write the order of symbols in the dot product as $\nabla \cdot \mathbf{F}$, so

The order is important here. We do *not* write $\mathbf{F} \cdot \nabla$, since this would result in a differential operator looking for something to operate on.

$$\nabla \cdot \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}),$$

where we have written the vector field $\mathbf{F}$ in terms of its Cartesian component fields $F_1(x, y, z)$, etc. We can evaluate this expression by multiplying the brackets, letting the differential operators act on the scalar component functions, and using the rule for the dot products of the unit vectors. For example, multiplying the first term in each bracket yields $(\partial F_1/\partial x)\,\mathbf{i} \cdot \mathbf{i} = \partial F_1/\partial x$. Terms with dot products of mutually perpendicular unit vectors evaluate to zero, so we are left with the sum of the three terms $\partial F_1/\partial x$, $\partial F_2/\partial y$ and $\partial F_3/\partial z$. This combination of partial derivatives is defined as the *divergence* of the vector field $\mathbf{F}$, or $\operatorname{div} \mathbf{F}$ for short. You can see that $\operatorname{div} \mathbf{F}$ is a scalar quantity that can be evaluated at each point in the domain of $\mathbf{F}$. Thus $\operatorname{div} \mathbf{F}$ is a scalar field.

Divergence

The **divergence** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ in **Cartesian coordinates** is a scalar field given by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \qquad (1)$$

$\operatorname{div} \mathbf{F}$ and $\nabla \cdot \mathbf{F}$ are alternative notations for the divergence of $\mathbf{F}$. Note that we do not embolden 'div' in $\operatorname{div} \mathbf{F}$ since $\operatorname{div} \mathbf{F}$ is a scalar.

A physical interpretation of divergence will be given in Subsection 1.3. Meanwhile we show how the divergence of a given vector field can be calculated at specified points.

1.2 Calculating divergence

To calculate the divergence of a given vector field $\mathbf{F}$, we need to identify its components F_1, F_2 and F_3, work out the partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$ and $\partial F_3/\partial z$, and add them as in equation (1).

Example 1

(a) Determine the divergence of each of the following vector fields:

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k},$$
$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
$$\mathbf{V}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 5\mathbf{k}.$$

(b) Evaluate $\operatorname{div} \mathbf{F}$ at the origin and at the point $(1, 2, 3)$.

Solution

(a) The components of $\mathbf{F}$ are $F_1 = x^2$, $F_2 = y^2$, $F_3 = z^2$. Hence

$$\operatorname{div} \mathbf{F} = 2x + 2y + 2z = 2(x + y + z).$$

Similarly, $r_1 = x$, $r_2 = y$, $r_3 = z$. Hence

$$\operatorname{div}\mathbf{r} = 1 + 1 + 1 = 3.$$

Also, $V_1 = -y$, $V_2 = x$, $V_3 = 5$. Hence

$$\operatorname{div}\mathbf{V} = 0 + 0 + 0 = 0.$$

(b) The value of $\operatorname{div}\mathbf{F}$ at $(0,0,0)$ is 0, and its value at $(1,2,3)$ is $2(1+2+3) = 12$.

Exercise 1

Determine the scalar field $\operatorname{div}\mathbf{F}$, where $\mathbf{F}(x,y,z) = xy\mathbf{i} - yz\mathbf{k}$, and evaluate $\operatorname{div}\mathbf{F}$ at the point $(3,-1,2)$.

Exercise 2

Find $\boldsymbol{\nabla}\cdot\mathbf{F}$, where $\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, and hence evaluate $\boldsymbol{\nabla}\cdot\mathbf{F}(1,2,3)$.

Exercise 3

Consider the three-dimensional vector field $\mathbf{F} = \widehat{\mathbf{r}}/|\mathbf{r}|^n$ $(\mathbf{r} \neq \mathbf{0})$, where $\mathbf{r}$ is a position vector, $\widehat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$, and n is a positive integer.

Express the field in Cartesian form, and show that $\mathbf{F}$ has zero divergence everywhere only when $n = 2$.

When vector fields are given in cylindrical or spherical coordinates, it is not always convenient to use the Cartesian expression for divergence. The cylindrical and spherical coordinate expressions for divergence are quite complicated, and we will derive only the expression for cylindrical coordinates.

First, recall that the gradient vector operator in cylindrical coordinates is given by

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z \frac{\partial}{\partial z}.$$

See Unit 15, Subsection 4.1.

Recall that in cylindrical coordinates, we write

$$\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z.$$

To compute the divergence $\boldsymbol{\nabla}\cdot\mathbf{F}$, we must therefore calculate

$$\left(\mathbf{e}_\rho \frac{\partial}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z \frac{\partial}{\partial z}\right)\cdot\left(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z\right).$$

Now $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ depend on ϕ. Indeed, from the relationships

Note that $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ do not depend on ρ or z.

$$\begin{pmatrix}\mathbf{e}_\rho\\ \mathbf{e}_\phi\\ \mathbf{e}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix},$$

we see, for example, that

$$\mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}.$$

Therefore

$$\frac{\partial \mathbf{e}_\rho}{\partial \phi} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} = \mathbf{e}_\phi.$$

Exercise 4

Find a similar expression for $\dfrac{\partial \mathbf{e}_\phi}{\partial \phi}$.

Now, returning to the calculation of $\boldsymbol{\nabla}\cdot\mathbf{F}$, we proceed term by term as follows:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &= \mathbf{e}_\rho\cdot\frac{\partial}{\partial\rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) + \frac{1}{\rho}\mathbf{e}_\phi\cdot\frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &\quad + \mathbf{e}_z\cdot\frac{\partial}{\partial z}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z).\end{aligned}$$

Since none of $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ or $\mathbf{e}_z$ depends on ρ, we have

$$\frac{\partial}{\partial\rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{\partial F_\rho}{\partial\rho}\mathbf{e}_\rho + \frac{\partial F_\phi}{\partial\rho}\mathbf{e}_\phi + \frac{\partial F_z}{\partial\rho}\mathbf{e}_z,$$

so

$\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ form a right-handed set of unit vectors, thus are mutually perpendicular.

$$\mathbf{e}_\rho\cdot\frac{\partial}{\partial\rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{\partial F_\rho}{\partial\rho}.$$

Next, using the results $\partial\mathbf{e}_\rho/\partial\phi = \mathbf{e}_\phi$, $\partial\mathbf{e}_\phi/\partial\phi = -\mathbf{e}_\rho$ and $\partial\mathbf{e}_z/\partial\phi = \mathbf{0}$, we have

$$\begin{aligned}&\frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &= \frac{\partial F_\rho}{\partial\phi}\mathbf{e}_\rho + F_\rho\frac{\partial\mathbf{e}_\rho}{\partial\phi} + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi + F_\phi\frac{\partial\mathbf{e}_\phi}{\partial\phi} + \frac{\partial F_z}{\partial\phi}\mathbf{e}_z\\ &= \frac{\partial F_\rho}{\partial\phi}\mathbf{e}_\rho + F_\rho\mathbf{e}_\phi + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi - F_\phi\mathbf{e}_\rho + \frac{\partial F_z}{\partial\phi}\mathbf{e}_z.\end{aligned}$$

So

$$\frac{1}{\rho}\mathbf{e}_\phi\cdot\frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}.$$

Finally, since none of $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ depends on z, the contribution from the z-component of $\boldsymbol{\nabla}$ is

$$\mathbf{e}_z\cdot\frac{\partial}{\partial z}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{\partial F_z}{\partial z}.$$

So we have finally arrived at the formula for $\operatorname{div}\mathbf{F}$ in cylindrical coordinates:

$$\operatorname{div}\mathbf{F} = \frac{\partial F_\rho}{\partial\rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}.$$

Divergence in cylindrical and spherical coordinates

The **divergence** of a vector field $\mathbf{F}(\rho,\theta,z) = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z$ is given in **cylindrical coordinates** by

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_\rho}{\partial\rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}. \qquad (2)$$

For $\mathbf{F}(r,\theta,\phi) = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ in **spherical coordinates**, the divergence is given by

$$\operatorname{div}\mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}\left(\frac{\partial F_\theta}{\partial\theta} + 2F_r\right) + \frac{1}{r\sin\theta}\left(\frac{\partial F_\phi}{\partial\phi} + F_\theta\cos\theta\right). \qquad (3)$$

This follows from a similar but more complicated derivation that we do not consider here.

Example 2

Consider the vector field $\mathbf{F} = \widehat{\mathbf{r}}/|\mathbf{r}|^n$ ($\mathbf{r} \neq \mathbf{0}$), where $\mathbf{r}$ is a position vector, $\widehat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$, and n is a positive integer.

This is the same field as in Exercise 3.

Express $\mathbf{F}$ in spherical coordinates, and hence confirm that $\operatorname{div}\mathbf{F} = 0$ everywhere only when $n = 2$.

Solution

The spherical form of the field is $\mathbf{F}(r,\theta,\phi) = \mathbf{e}_r/r^n$ ($r > 0$), so the field has components $F_r = 1/r^n$ and $F_\theta = F_\phi = 0$. We can use the spherical expression for $\operatorname{div}\mathbf{F}$ given in equation (3) to obtain

Recall that $r = |\mathbf{r}|$ and $\mathbf{e}_r = \widehat{\mathbf{r}}$.

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial(1/r^n)}{\partial r} + \frac{2}{r}\,(1/r^n)\\ &= (-n)\,\frac{1}{r^{n+1}} + \frac{2}{r^{n+1}}\\ &= \frac{2-n}{r^{n+1}} \quad (r > 0),\end{aligned}$$

which is zero only when $n = 2$.

You can see the advantage of using spherical coordinates in Example 2. (Compare the solution with that for Exercise 3.)

Exercise 5

Express the position vector $\mathbf{r}$ in spherical coordinates, and hence determine $\operatorname{div}\mathbf{r}$ using the spherical expression for divergence as given by equation (3).

1.3 Divergence in physical laws

Our definition of div **F** as a sum of partial derivatives gives little clue to the physical meaning of divergence. In this subsection we describe one physical meaning of divergence by considering the flow of heat energy in a uranium fuel rod. In the steady state, the rate of flow of heat energy out of any small region of the rod must be equal to the rate at which heat is generated by nuclear fission in that small region. We will show that this heat–energy balance can be expressed by the equation

Our model assumes that there are no other heat losses or gains.

$$\operatorname{div}\mathbf{J} = S,$$

where div **J** represents the *rate of heat outflow per unit volume* at a point in the rod, and S is the *rate at which heat is generated per unit volume at that point*. (The scalar field S is sometimes referred to as the *heat source density*.) This equation represents the *law of energy conservation* for heat flow in the rod.

See Unit 15, Subsection 2.3.

You saw in Unit 15 that the heat flow rate in a heat conductor may be represented as a vector field, **J**, where the direction is that of the heat flow, and the magnitude is given by

$$|\mathbf{J}| = \lim_{A\to 0} \frac{\text{heat flow rate across surface area } A}{A}.$$

For a uranium fuel rod in a nuclear reactor (see Figure 1), heat is generated throughout the body of the rod by nuclear fission of uranium atoms. The heat generated in the rod flows by conduction through the rod and out into the surrounding coolant, a flowing gas or liquid that takes the heat away to drive turbines. In the steady state, the coolant establishes a uniform, relatively cool, constant temperature Θ_a on the outside surface of the rod (where the radius of the rod is a). An important question in reactor design is: how does the temperature distribution Θ depend on the heat source and on the constant surface temperature Θ_a established by the coolant? In preparation for this, we now derive the equation $\operatorname{div}\mathbf{J} = S$.

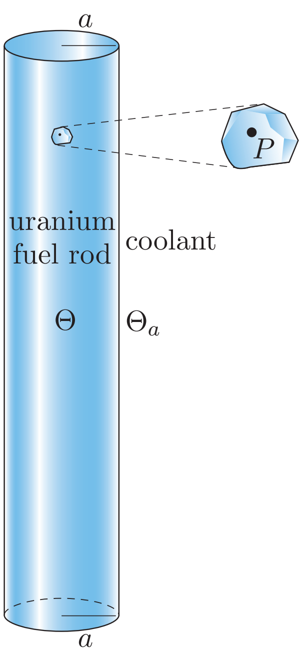

Figure 1 Uranium fuel rod in a nuclear reactor

When the reactor has been in operation for some time, a steady state is reached where the temperature field in the rod stays constant in time and the net outflow of heat from any region of the rod is balanced by the total heat production by nuclear fission in that region. Let us consider this energy balance more closely. If P is a point in a small region of volume V inside the rod, and q is the rate at which heat is generated by nuclear fission in the region, then the quantity

$$S = \lim_{V\to 0} \frac{q}{V}$$

is the rate (in $\mathrm{W\,m^{-3}}$) at which heat is generated per unit volume at P. The rate S of heat generation may vary from point to point in the rod, and is a scalar field.

In Figure 2, the point P has position vector $\mathbf{r}$ and lies inside a small region R, with volume V, entirely within the rod. If the region R is small enough, then the field S is effectively constant throughout the small region and is approximately equal to $S(\mathbf{r})$. So the net rate at which heat is generated in R is approximately $S(\mathbf{r})\,V$ (in watts). In the steady state, this rate of heat generation is balanced by the net rate of outflow of heat from the region R into neighbouring regions of the rod. Thus we have

$$\text{net outflow rate from small region containing } P \simeq S(\mathbf{r})\,V.$$

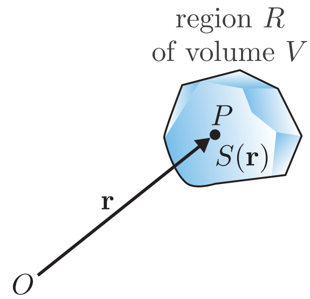

Figure 2 Small region containing point P inside the fuel rod

Dividing both sides by V, we obtain

$$\frac{\text{net outflow rate from small region containing } P}{V} \simeq S(\mathbf{r}). \tag{4}$$

Let us take the limit as the region R becomes smaller and eventually shrinks onto the point P, that is, as $V \to 0$. Then while the right-hand side of approximation (4) remains constant, the left-hand side becomes the *net outflow rate per unit volume at P*, and the approximation becomes an equality. Thus we have

$$\text{net outflow rate per unit volume at } P = S(\mathbf{r}). \tag{5}$$

We can evaluate the net outflow rate per unit volume on the left-hand side of this equation by considering a region R in the shape of a small cube centred on P, as shown in Figure 3. We will determine the net rate of outflow across the three pairs of parallel faces of the cube, and then take the limit as $V \to 0$ to give the net outflow rate per unit volume. Consider first the two parallel faces normal to the x-direction and separated from one another by a distance L, the side length of the cube. Let the left-hand face be in the plane $x = X$, with its midpoint at (X, Y, Z). If $\mathbf{J} = J_1\mathbf{i} + J_2\mathbf{j} + J_3\mathbf{k}$ is the heat flow vector field (in $\mathrm{W\,m^{-2}}$), only the component of $\mathbf{J}$ normal to this face, in the direction of negative x, will contribute to the outflow rate across this face. (The y- and z-components of $\mathbf{J}$ represent flow in the plane of the face.) If the cube is small enough, we can ignore any variation of $\mathbf{J}$ on the face and take its value $\mathbf{J}(X, Y, Z)$ at the centre of the face. The unit vector in the negative x-direction is $-\mathbf{i}$, so the outflow rate per unit area (in $\mathrm{W\,m^{-2}}$) across this face is $-\mathbf{J} \cdot \mathbf{i} = -J_1$, and the total outflow rate (in watts) across the area L^2 of the face is approximately $-J_1(X, Y, Z)\,L^2$. Similarly, the total outflow rate across the parallel face in the plane $x = X + L$ is $J_1(X + L, Y, Z)\,L^2$, since only the component of $\mathbf{J}$ in the positive x-direction contributes. The net outflow rate across the two faces is therefore approximately

$$[-J_1(X, Y, Z) + J_1(X + L, Y, Z)]\,L^2.$$

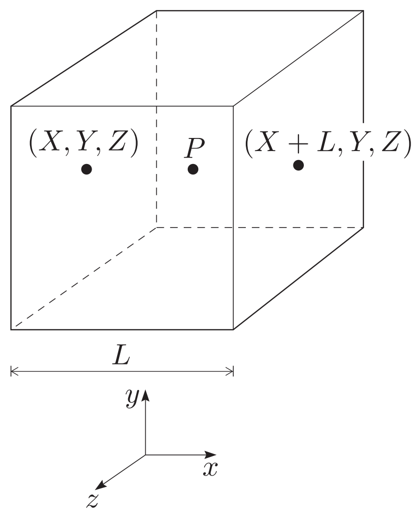

Figure 3 Cube of side length L centred at point P

(In Figure 3, the Cartesian coordinate system is oriented so that the coordinate planes are parallel to the plane faces of the cube. We use an orientation of x, y and z that is different from that used generally in this unit.)

Dividing by the volume $V = L^3$ and taking the limit as $L \to 0$ (equivalent to $V \to 0$) gives the contribution from these two parallel faces to the net rate of outflow per unit volume as

The units of outflow rate per unit volume are therefore $\mathrm{W\,m^{-3}}$.

$$\lim_{L\to 0} \frac{J_1(X+L,Y,Z) - J_1(X,Y,Z)}{L}.$$

You should recognise this limit from Unit 7. It is the partial derivative of J_1 with respect to x at the point (X,Y,Z). In other words, it is the partial derivative $\partial J_1(X,Y,Z)/\partial x$. In the same way, you can see that the contributions to the net rate of outflow per unit volume from the other two pairs of parallel faces of the cube are $\partial J_2(X,Y,Z)/\partial y$ and $\partial J_3(X,Y,Z)/\partial z$. The net rate of heat outflow per unit volume at the point P is the sum of these three partial derivatives, namely,

Taking the limit as $L \to 0$ in all three directions places (X,Y,Z) at the point P.

$$\frac{\partial J_1}{\partial x}(X,Y,Z) + \frac{\partial J_2}{\partial y}(X,Y,Z) + \frac{\partial J_3}{\partial z}(X,Y,Z).$$

You will recognise this expression as the *divergence* of $\mathbf{J}$.

Hence at each point P in the rod, we have

$$\text{net outflow rate per unit volume} = \operatorname{div}\mathbf{J},$$

so from equation (5), energy balance requires

$$\operatorname{div}\mathbf{J} = S \tag{6}$$

Note that this equation is dimensionally correct.

to be satisfied at each point P in the rod.

The equation $\operatorname{div}\mathbf{J} = S$ is a first-order partial differential equation for the vector field $\mathbf{J} = \mathbf{J}(x,y,z)$. It may be written as

$$\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} = S, \tag{7}$$

where J_1, J_2, J_3 and S are functions of x, y and z.

You have seen (in Unit 15, Subsection 3.3) that we can describe the heat flow vector $\mathbf{J}$ by Fourier's law as

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta, \tag{8}$$

where Θ is the temperature field and κ is the thermal conductivity. If κ is constant throughout the fuel rod, then the divergence of equation (8), divided by $-\kappa$, can be written as

$$-\frac{1}{\kappa}\operatorname{div}\mathbf{J} = \operatorname{div}(\mathbf{grad}\,\Theta) = \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2},$$

which, from equation (6), gives us the second-order partial differential equation

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2} = -\frac{S}{\kappa}. \tag{9}$$

This equation for the temperature distribution Θ is called *Poisson's equation*.

Knowing S and κ, and the appropriate boundary conditions, equation (9) can be solved for the temperature field Θ.

Example 3

Steady-state conductive heat flow is governed by the partial differential equation

$$\operatorname{div}\mathbf{J} = S,$$

where $\mathbf{J}$ (in $\mathrm{W\,m^{-2}}$) is the heat flow per unit area, and the scalar field S (in $\mathrm{W\,m^{-3}}$) is the heat generated per unit volume. Nuclear fission of uranium atoms in a long cylindrical fuel rod of radius $0.02\,\mathrm{m}$ generates heat at a constant rate S uniformly throughout the rod. When the axis of the rod is along the z-axis, the solution of the equation describing steady-state heat flow in the rod is known to be of the form

The description of the rod as 'long' implies that we can ignore any anomalies in the heat flow or heat generation at the ends of the rod, i.e. we ignore any 'end effects'.

$$\mathbf{J}(x,y,z) = A(x\mathbf{i} + y\mathbf{j}),$$

where A is a constant.

(a) Determine A when $S = 4 \times 10^6$.

(b) Hence determine the magnitude and direction of the vector $\mathbf{J}$ at:

  (i) points on the z-axis

  (ii) a point T with Cartesian coordinates $(0.005, 0, 0)$

  (iii) a point Q with Cartesian coordinates $(0.015, 0, 0)$.

Solution

(a) We find that

$$\operatorname{div}\mathbf{J} = \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} = \frac{\partial(Ax)}{\partial x} + \frac{\partial(Ay)}{\partial y} = A + A = 2A.$$

Now from equation (6), we have $2A = S$, where S is given as $4 \times 10^6\,\mathrm{W\,m^{-3}}$, so $A = 2 \times 10^6$.

(b) (i) On the z-axis, $x = y = 0$, so $\mathbf{J} = \mathbf{0}$. The magnitude is zero and there is no direction associated with the zero vector.

  (ii) At point T, $\mathbf{J}(0.005, 0, 0) = 2 \times 10^6 \times (0.005\mathbf{i}) = 10^4\mathbf{i}$, which is a vector pointing in the positive x-direction, that is, radially outwards, of magnitude $10^4\,\mathrm{W\,m^{-2}} = 10\,\mathrm{kW\,m^{-2}}$.

  (iii) Similarly, at Q, $\mathbf{J} = 2 \times 10^6 \times (0.015\mathbf{i})$, which is a vector directed in the positive x-direction of magnitude $3 \times 10^4\,\mathrm{W\,m^{-2}} = 30\,\mathrm{kW\,m^{-2}}$.

Figure 4 depicts the heat flow vector $\mathbf{J}$ at the points T and Q described in Example 3. The vector field $\mathbf{J}(x, y, z)$ can be expressed in cylindrical coordinates as $\mathbf{J}(\rho, \phi, z) = A\rho\,\mathbf{e}_\rho$, so $\mathbf{J}$ points radially outwards from and perpendicular to the z-axis everywhere, and $|\mathbf{J}|$ is the same at all points on a cylindrical surface of radius ρ.

You will be asked to confirm this expression for $\mathbf{J}$ in cylindrical coordinates in Exercise 7.

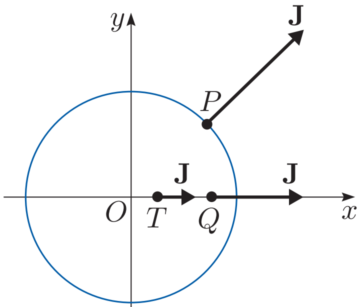

Figure 4 Cross-section of the fuel rod used in Example 3 and Exercise 6

Exercise 6

This exercise extends Example 3. Determine the magnitude of the heat flow field $\mathbf{J} = A(x\mathbf{i} + y\mathbf{j})$ at any point P on the outer cylindrical surface of the rod (see Figure 4). Hence deduce the total rate of heat flow out of a one-metre length of the rod. Determine also the total rate of heat generated in the one-metre length.

Exercise 7

Express the vector field $\mathbf{J}(x, y, z) = A(x\mathbf{i} + y\mathbf{j})$, given in Example 3, in cylindrical coordinates, and hence determine $\operatorname{div}\mathbf{J}$ using the cylindrical expression for divergence (equation (2)).

Now let us return to the general problem of the fuel rod. Recall that we had arrived at the two equations

$$\operatorname{div}\mathbf{J} = S, \tag{10}$$

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta. \tag{11}$$

The shape of the fuel rod makes it particularly appropriate to use cylindrical coordinates, and armed with the formula for div in cylindrical coordinates, we can make rapid progress. We take the z-axis along the centre of the rod so that the symmetry of the situation allows us to make further simplifications. The flow rate vector $\mathbf{J} = J_\rho\mathbf{e}_\rho + J_\phi\mathbf{e}_\phi + J_z\mathbf{e}_z$ cannot depend on the angle ϕ, because we have assumed that the material of the rod is uniform. Hence $J_\phi = 0$. Similarly, if we assume that the rod is very long in comparison with its thickness, then we can neglect the effects of the ends of the rod. In these circumstances $\mathbf{J}$ will not depend on z either, and $J_z = 0$. Thus the formula for $\operatorname{div}\mathbf{J}$ in cylindrical coordinates reduces to

$$\operatorname{div}\mathbf{J} = \frac{\partial J_\rho}{\partial \rho} + \frac{1}{\rho}J_\rho,$$

and equation (10) becomes

$$\frac{\partial J_\rho}{\partial \rho} + \frac{1}{\rho}J_\rho = S. \tag{12}$$

Now equation (11) gives

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta = -\kappa\left(\mathbf{e}_\rho\frac{\partial\Theta}{\partial\rho} + \frac{1}{\rho}\,\mathbf{e}_\phi\frac{\partial\Theta}{\partial\phi} + \mathbf{e}_z\frac{\partial\Theta}{\partial z}\right),$$

so $J_\rho = -\kappa\,\partial\Theta/\partial\rho$. Since κ is constant, equation (12) leads to the following second-order differential equation for Θ:

$$\frac{\partial^2\Theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Theta}{\partial\rho} = -\frac{S}{\kappa}.$$

In fact, because Θ does not depend on ϕ or z, this partial differential equation can be written as a second-order *ordinary* differential equation:

$$\frac{d^2\Theta}{d\rho^2} + \frac{1}{\rho}\frac{d\Theta}{d\rho} = -\frac{S}{\kappa}.$$

However, it is not a constant-coefficient equation, so the methods of Unit 1 are not directly applicable. Fortunately, we can reduce the equation to a pair of first-order equations, each of which can be solved. Replacing $d\Theta/d\rho$ by u, we have

$$\frac{du}{d\rho} + \frac{1}{\rho}u = -\frac{S}{\kappa}, \quad \frac{d\Theta}{d\rho} = u.$$

The first of these is a linear differential equation, with integrating factor

$$\exp\left(\int \frac{1}{\rho}\,d\rho\right) = \exp(\ln\rho) = \rho.$$

Multiplying through by this factor gives

$$\frac{d}{d\rho}(u\rho) = -\frac{S}{\kappa}\rho,$$

so

$$u\rho = -\tfrac{1}{2}\frac{S}{\kappa}\rho^2 + C,$$

where C is an arbitrary constant. If we assume that Θ has a maximum at the centre of the rod (which seems eminently reasonable!), then we must have

$$\frac{d\Theta}{d\rho} = u = 0 \quad \text{when} \quad \rho = 0,$$

so $C = 0$. Thus

$$u = \frac{d\Theta}{d\rho} = -\tfrac{1}{2}\frac{S}{\kappa}\rho,$$

and

$$\Theta = -\tfrac{1}{4}\frac{S}{\kappa}\rho^2 + D,$$

where D is another arbitrary constant. Finally, since the temperature on the edge of the rod (when $\rho = a$) is known to be Θ_a, we have

$$\Theta_a = \Theta(a) = -\tfrac{1}{4}\frac{S}{\kappa}a^2 + D,$$

thus

$$D = \Theta_a + \tfrac{1}{4}\frac{S}{\kappa}a^2.$$

Therefore

$$\Theta = \Theta_a + \tfrac{1}{4}\frac{S}{\kappa}a^2 - \tfrac{1}{4}\frac{S}{\kappa}\rho^2 = \Theta_a + \frac{S}{4\kappa}(a^2 - \rho^2)$$

is the solution to the problem.

Exercise 8

(a) Suppose that a laser beam transports energy through air as a cylindrical beam of light, that is, in a uniform parallel beam, at the rate of $5000\,\mathrm{W\,m^{-2}}$. Taking the z-axis to be in the beam direction, along the axis of the cylinder, we can write the energy flow field as $\mathbf{N}(x, y, z) = 5000\mathbf{k}$, where $\mathbf{k}$ is a unit vector in the direction of the positive z-axis. Determine $\operatorname{div}\mathbf{N}$.

(b) When the laser beam shines through fog, it becomes weaker and its intensity falls off exponentially with distance. If the laser is placed so that the beam originates at $z = 0$, then the energy flow is given by $\mathbf{N}(x, y, z) = 5000e^{-\alpha z}\mathbf{k}$, where α is a positive constant (the absorption coefficient of the fog). Determine $\operatorname{div}\mathbf{N}$.

You have seen that for steady-state heat flow in a uranium rod with heat flow field $\mathbf{J}$, $\operatorname{div}\mathbf{J}$ is equal to the heat source density at each point. In fact, for *any* steady-state heat flow field $\mathbf{J}$, $\operatorname{div}\mathbf{J}$ is equal to the heat source density at each point. A similar result applies to the laser beam in Exercise 8, where $\operatorname{div}\mathbf{N}$ is equal to the source density of light energy in the beam. In Exercise 8(a) you found $\operatorname{div}\mathbf{N} = 0$. This result represents the fact that there are no light sources or sinks in the air through which the laser beam passes. On the other hand, when the beam passes through fog, which absorbs light and so attenuates the beam (hence the factor $e^{-\alpha z}$), the fog acts as a sink of light. The presence of this sink of light is represented by the fact that $\operatorname{div}\mathbf{N}$ is negative.

A vector field $\mathbf{F}$ for which $\operatorname{div}\mathbf{F} = 0$ everywhere is called **solenoidal**.

When a source density S is negative, the source is commonly called a *sink* (cf. the terminology for stability in Unit 12).

Exercise 9

Consider the vector field

$$\mathbf{f}(\rho, \phi, z) = \frac{\mathbf{e}_\rho}{\rho^n},$$

where $\rho > 0$ and n is a positive integer. For what value(s) of n is $\operatorname{div}\mathbf{f} = 0$ everywhere?

We have interpreted divergence in terms of net outflow rate from a point. Not all vector fields are associated with flow of material or energy. For example, there is no flow associated with a static magnetic field or an electrostatic field. However, the divergences of such fields can still have physical significance.

For example, the divergence of any magnetic field $\mathbf{B}$ is always zero. This law,

$$\operatorname{div}\mathbf{B} = 0, \tag{13}$$

This law is one of four equations called *Maxwell's equations*. It can be read as saying that the magnetic field $\mathbf{B}$ is a solenoidal vector field.

is true for all possible magnetic fields and is an expression of the fact that there are no sources or sinks in a magnetic field.

Exercise 10

Use equation (13) to decide which of the following vector fields could be magnetic fields:

$$\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + xy\mathbf{k},$$
$$\mathbf{G}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k},$$
$$\mathbf{H}(x,y,z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}.$$

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2 Curl of a vector field

You have seen that the divergence of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a scalar field given by the sum of the three partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$ and $\partial F_3/\partial z$. This is one way of differentiating a vector field, resulting in a scalar. We now introduce the *curl* of a vector field $\mathbf{F}$, written $\mathbf{curl}\,\mathbf{F}$. This new field is a vector field constructed from the other six partial derivatives of F_1, F_2 and F_3 with respect to Cartesian coordinates x, y, z. You can think of these partial derivatives as 'sideways' derivatives. For example, $\partial F_1/\partial y$ describes how the x-component of $\mathbf{F}$ changes with small displacements in the y-direction. Subsections 2.1 and 2.2 introduce the curl of a vector field and show how to calculate it. Subsection 2.3 provides a physical interpretation of curl.

Note that 'curl' is emboldened in $\mathbf{curl}\,\mathbf{F}$ since it is a vector field; in handwritten work it is denoted using a straight or wavy underline.

2.1 Defining curl

In this subsection we consider the mathematical definition of $\mathbf{curl}\,\mathbf{F}$ using the vector differential operator $\boldsymbol{\nabla}$. You have seen that for any scalar field f and for any vector field $\mathbf{F}$, we have

$$\mathbf{grad}\, f = \boldsymbol{\nabla} f,$$

$\mathbf{grad}\, f$ is a vector field.

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F}.$$

$\operatorname{div}\mathbf{F}$ is a scalar field.

We now define the vector field $\mathbf{curl}\,\mathbf{F}$ to be the cross product, $\boldsymbol{\nabla}\times\mathbf{F}$, of $\boldsymbol{\nabla} = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

Curl

The **curl** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ in **Cartesian coordinates** is

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \qquad (14)\end{aligned}$$

The determinant form of the cross product was given in Unit 4, Subsection 2.4.

The name 'curl' comes from the fact that the vector field $\mathbf{curl}\,\mathbf{F}$ represents the 'local rotation' in the field $\mathbf{F}$. Local rotation will be discussed in Subsection 2.3.

When the vector field $\mathbf{F}(x, y)$ is confined to the (x, y)-plane, F_3 is zero, and F_1 and F_2 do not depend on z. Then you can see from equation (14) that only the z-component of $\mathbf{curl}\,\mathbf{F}$ can be non-zero, and we have the following simplified expression.

Curl of a two-dimensional vector field

The **curl of a two-dimensional vector field** $\mathbf{F}(x, y)$ is

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \qquad (15)$$

This means that the curl of a two-dimensional vector field must either be the zero vector or be directed at right angles to the plane of the field.

Examples of both possibilities are given in Example 4.

2.2 Calculating curl

To calculate the curl of a given vector field, first identify the component fields, then evaluate the relevant partial derivatives, and substitute these partial derivatives into the appropriate expression for curl in equation (14) or equation (15).

Example 4

(a) Find the curl of each of the following vector fields.

(i) $\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j}$

(ii) $\mathbf{g}(x, y) = -y\mathbf{i} + x\mathbf{j}$

(iii) $\mathbf{F}(x, y, z) = (xy + z^2)\mathbf{i} + x^2\mathbf{j} + (xz - 2)\mathbf{k}$

(b) (i) Find $\mathbf{curl}\,\mathbf{F}(1,-1,3)$, where $\mathbf{F}$ is the vector field specified in part (a)(iii).

(ii) Find points where $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, where $\mathbf{F}$ is the vector field specified in part (a)(iii).

Solution

(a) (i) The components of $\mathbf{f}$ are $f_1 = x$ and $f_2 = y$, so

$$\frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_2}{\partial x} = 0.$$

The vector field $f(x, y)$ is confined to the (x, y)-plane, so we can use the simplified version. Substituting in equation (15), we find $\mathbf{curl}\,\mathbf{f} = \mathbf{0}$.

We write the bold zero vector here since curl is a *vector* field.

(ii) The components of $\mathbf{g}$ are $g_1 = -y$ and $g_2 = x$, so

$$\frac{\partial g_1}{\partial y} = -1, \quad \frac{\partial g_2}{\partial x} = 1.$$

Substituting in equation (15), we obtain

$$\mathbf{curl}\,\mathbf{g} = \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)\mathbf{k} = (1 - (-1))\mathbf{k} = 2\mathbf{k}.$$

(iii) We have $F_1 = xy + z^2$, $F_2 = x^2$, $F_3 = xz - 2$, so

$$\frac{\partial F_1}{\partial y} = x, \quad \frac{\partial F_1}{\partial z} = 2z,$$
$$\frac{\partial F_2}{\partial x} = 2x, \quad \frac{\partial F_2}{\partial z} = 0,$$
$$\frac{\partial F_3}{\partial x} = z, \quad \frac{\partial F_3}{\partial y} = 0.$$

Substituting in equation (14), we obtain

$$\mathbf{curl}\,\mathbf{F} = (0 - 0)\mathbf{i} + (2z - z)\mathbf{j} + (2x - x)\mathbf{k} = z\mathbf{j} + x\mathbf{k}.$$

(b) (i) At the point $(1, -1, 3)$, we have $\mathbf{curl}\,\mathbf{F} = 3\mathbf{j} + \mathbf{k}$.

(ii) When $x = 0$ and $z = 0$, and for any value of y, $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, that is, $\mathbf{curl}\,\mathbf{F}$ is zero everywhere on the y-axis.

Exercise 11

(a) Find $\mathbf{curl}\,\mathbf{F}$, where $\mathbf{F}(x, y, z) = x(y - z)\mathbf{i} + 3x^2\mathbf{j} + yz\mathbf{k}$.

(b) Determine $\mathbf{curl}\,\mathbf{F}$ at:

(i) the origin

(ii) the point $(1, 2, 3)$

(iii) the point that is 5 units from the origin on the positive z-axis.

Exercise 12

Consider $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\widehat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$. Find $\mathbf{curl}\,\mathbf{F}$ in each of the following cases.

(a) $\mathbf{F} = \mathbf{r}$

(b) $\mathbf{F} = f(r)\,\widehat{\mathbf{r}}$, where f is a function of r only

(c) $\mathbf{F} = (y^2 + 2z)\mathbf{i} + (xy + 6z)\mathbf{j} + (z^2 + 2xz + y)\mathbf{k}$

(d) $\mathbf{F} = \mathbf{grad}\,f$, where f is any differentiable scalar field

Exercise 13

Suppose that two vector fields, $\mathbf{A}$ and $\mathbf{B}$, are related by $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$. Show that provided that $\mathbf{A}$ is sufficiently smooth, this implies that $\operatorname{div}\mathbf{B} = 0$ everywhere, that is, $\mathbf{B}$ is a solenoidal field.

If a vector field $\mathbf{F}$ has $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere, then it is said to be an **irrotational field**. Irrotational force fields are important in mechanics. If a force field acts on a particle such that the total mechanical energy of the particle, that is, potential energy + kinetic energy, is conserved, then the force field will be irrotational. More will be said about this in Section 4.

Exercise 14

Identify irrotational fields from those considered in Example 4 and Exercise 12.

Exercise 15

Show that any vector field of the form $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$, where f is a function of x only and g is a function of y only, is irrotational.

2.3 Curl and local rotation

Whereas the divergence of a vector field represents the outflow rate from a point, the curl of a vector field represents rotation at a point.

We are assuming that the flow has settled down so that there are no changes in time. It is steady flow.

Consider the two-dimensional velocity field on the surface of a river. Let the surface be in the (x, y)-plane with the coordinate axes fixed relative to the river bank. Then the water surface flows through fixed points (x, y) with velocity $\mathbf{v}(x, y)$. This surface velocity field can be revealed by watching a floating object such as a small leaf. Then the velocity $\mathbf{v}(x, y)$ is the velocity of the leaf as it passes through the fixed point (x, y).

If you watch a small leaf floating down a river, you may notice two types of motion. First, the leaf floats downstream following a vector field line of the velocity field $\mathbf{v}(x, y)$. Then the leaf may rotate in the plane of the surface. The rate of rotation may be quite fast for an object floating near the bank of a fast flowing river. However, near midstream, the rate of rotation is likely to be slow or zero, even when the surface velocity is fast.

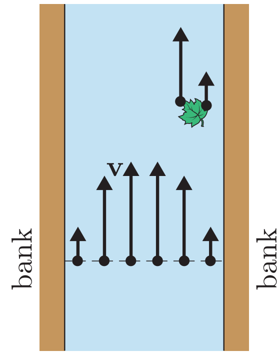

Figure 5 Velocity field on the surface of a river

The rotation of a floating body is easily explained. It happens whenever the velocity, and hence the drag, is greater on one side of the object than on the other. There is then a torque on the object, thus it rotates. Figure 5 shows a typical distribution of surface velocities $\mathbf{v}$ on a line across a river. You can see that the velocity magnitude increases quite rapidly as you move out from a river bank, but varies hardly at all near midstream. Hence the difference in drag on the near and far sides of a small floating object may be quite large near a bank and very small near midstream.

A rotation about the z-axis is described by a vector $\boldsymbol{\omega} = \omega\mathbf{k}$, called the **angular velocity**. The magnitude $|\boldsymbol{\omega}| = \omega$ describes the angular speed, that is, the rate at which angle (such as the angle of orientation of the floating leaf on the river) changes in time during the rotation (measured in radians per second). The direction of the vector $\boldsymbol{\omega}$ is at right angles to the plane of rotation in the sense given by the right-hand grip rule. In the case of the rotating leaf on a river, the direction of the rotation is at right angles to the river surface, that is, parallel to the z-axis.

Rotation about an axis will be discussed in more detail in Unit 20.

See Unit 2 for the right-hand grip rule.

Perhaps you can now begin to see a relationship between the rotation of a small floating object, as described by a vector $\boldsymbol{\omega}$, and the curl of the surface velocity field $\mathbf{curl}\,\mathbf{v}$. The curl of any two-dimensional vector field on a surface is directed at right angles to the surface. Hence $\boldsymbol{\omega}$ and $\mathbf{curl}\,\mathbf{v}$ are both vectors directed at right angles to the water surface. Furthermore, $\mathbf{curl}\,\mathbf{v}$ is built up from the partial derivatives $\partial v_1/\partial y$ and $\partial v_2/\partial x$, which represent variation perpendicular to the directions of the corresponding component vectors $v_1\mathbf{i}$ and $v_2\mathbf{j}$, and so correspond to rotational motion. Furthermore, the angular speed $|\boldsymbol{\omega}|$ is largest where the *downstream* surface velocity varies most rapidly with distance *across* the river. In fact, the rotation of a small floating body can be modelled quite well by

$$\boldsymbol{\omega} = A\,\mathbf{curl}\,\mathbf{v},$$

where A is a positive constant that depends on the size and shape of the body, the nature of its surface and the properties of the fluid.

The above relationship applies also in three dimensions and is the basis of experimental techniques for exploring the curl of velocity fields in fluids by using a neutral density probe, that is, a small object that can float at a point beneath the surface of a fluid without sinking or rising. The magnitude and direction of the rotation of the floating probe provide an indication of the curl of the velocity at that point.

In the next example we consider the surface velocity field on a model river, calculate the curl of the velocity, and confirm that it describes the way in which a small floating object would rotate.

Example 5

The surface water velocity on a straight uniform river can often be modelled by a two-dimensional vector field of the form

$$\mathbf{v}(x, y) = Cx(d - x)\mathbf{j} \quad (0 \leq x \leq d),$$

where the y-axis is along one bank and points downstream, d is the width of the river, and C is a positive constant (see Figure 6).

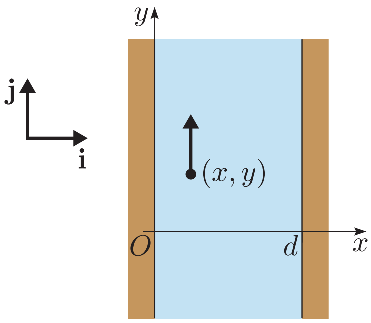

Figure 6 River with surface velocity $\mathbf{v}(x, y)$ and banks running along $x = 0$ and $x = d$

(a) What is the surface velocity at the river banks and at midstream? How does the magnitude of the surface velocity change as you move out from one bank towards midstream?

(b) Find the vector field $\mathbf{curl\,v}$. State the magnitude and direction of $\mathbf{curl\,v}$ at the river banks and at midstream.

(c) Explain why the answers to part (b) are consistent with the idea that $\mathbf{curl\,v}$ describes the rate and sense of rotation of a small leaf floating on the river surface.

Solution

(a) The river banks are the lines $x = 0$ (the y-axis) and $x = d$. We find that $\mathbf{v}(0, y) = \mathbf{v}(d, y) = \mathbf{0}$, so the river velocity is the zero vector at the banks. At midstream, $x = \frac{1}{2}d$ and $\mathbf{v}(\frac{1}{2}d, y) = \frac{1}{4}Cd^2\mathbf{j}$. Hence the velocity at midstream is directed downstream (that is, up the page in Figure 6) and is of magnitude $\frac{1}{4}Cd^2$. The surface speed increases as you move out from either bank towards midstream.

(b) The components of $\mathbf{v}$ are $v_1 = 0$ and $v_2 = Cx(d - x)$. Hence, using equation (15), we have $\mathbf{curl\,v} = (Cd - 2Cx)\mathbf{k} = C(d - 2x)\mathbf{k}$. The value of $\mathbf{curl\,v}$ at the bank $x = 0$ is $Cd\mathbf{k}$. The value of $\mathbf{curl\,v}$ at the bank $x = d$ is $-Cd\mathbf{k}$. The value of $\mathbf{curl\,v}$ at midstream ($x = \frac{1}{2}d$) is $\mathbf{0}$.

(c) We have found that at $x = 0$, $\mathbf{curl\,v}$ points in the positive z-direction (vertically upwards).

This corresponds to an anticlockwise rotation by the right-hand grip rule of Unit 2; this will be explained in greater detail in Unit 20.

A small floating leaf near $x = 0$ would rotate anticlockwise as it floats downstream, since its outer edge is pulled faster than the inner edge by the flow.

Similarly, a leaf near the bank $x = d$ would rotate clockwise. This is in accordance with $\mathbf{curl\,v}$ being a vector pointing in the negative z-direction at $x = d$.

At midstream, the leaf would float without rotating, since both sides of the leaf are pulled equally by the flow. This corresponds to $\mathbf{curl\,v} = \mathbf{0}$ at midstream.

The rotation in the velocity field $\mathbf{v}$ described by $\mathbf{curl}\,\mathbf{v}$ is the **local rotation** in the field, as measured by the rotation of a small floating object. Local rotation should be distinguished from **bulk rotation**, which is simply the movement of material around a circular path.

There is local rotation, but no bulk rotation, in Example 5, since the flow is parallel to the y-direction everywhere. There is bulk rotation of water in the velocity fields $\mathbf{u}$ and $\mathbf{v}$ that model the surface velocity of water in an emptying bathtub in Exercise 16 below, but you will find that there is local rotation in only one of them.

Exercise 16

The velocity field on the surface of an emptying bathtub can be modelled by two functions, the first describing the vigorously swirling vortex of radius a in a central region, and the second describing the more gently rotating fluid outside the vortex region. These functions are

$$\mathbf{u}(x,y) = \omega(-y\mathbf{i} + x\mathbf{j}) \quad \left(\sqrt{x^2+y^2} \leq a\right),$$
$$\mathbf{v}(x,y) = \frac{\omega a^2(-y\mathbf{i} + x\mathbf{j})}{x^2+y^2} \quad \left(\sqrt{x^2+y^2} \geq a\right),$$

Note that $\mathbf{u} = \mathbf{v}$ when $(x^2+y^2)^{1/2} = a$.

where ω is a positive constant and the water surface is assumed to lie in the (x,y)-plane.

Find $\mathbf{curl}\,\mathbf{u}$ and $\mathbf{curl}\,\mathbf{v}$.

Exercise 17

Express the two vector fields $\mathbf{u}$ and $\mathbf{v}$ of Exercise 16 in cylindrical coordinates. What shape are the vector field lines of $\mathbf{u}$ and $\mathbf{v}$?

Both velocity fields $\mathbf{u}$ and $\mathbf{v}$ in Exercise 16 have a bulk rotation of fluid around a central point, but field $\mathbf{v}$ has $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$, showing that there is no local rotation in the field $\mathbf{v}$. A small floating object in the region of the field $\mathbf{v}$ would float around a circular field line, but would not rotate, that is, it would maintain the same orientation in space.

If you try this experiment next time you have a bath, you will probably see some rotation. This is hardly surprising, since the field $\mathbf{v}$ is a simple model that assumes that the flow is circular, whereas in reality the flow *spirals into* the plughole.

In Exercise 17 you found that

$$\mathbf{u}(\rho,\phi,z) = \omega\rho\,\mathbf{e}_\phi \quad (\rho \leq a),$$
$$\mathbf{v}(\rho,\phi,z) = \frac{\omega a^2}{\rho}\,\mathbf{e}_\phi \quad (\rho \geq a).$$

Hence the fields $\mathbf{u}$ and $\mathbf{v}$ have simple forms when expressed in cylindrical coordinates. When we are given a vector field in cylindrical coordinates, it is usually easier to calculate the curl using the cylindrical form of curl rather than converting into Cartesian coordinates.

Exercise 18

The Earth rotates about its north–south axis at the rate of 2π radians per 24 hours. This rotation has angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$, where the unit vector $\mathbf{k}$ points from south to north, and $\omega = |\boldsymbol{\omega}|$ is the angular speed. The velocity of a point inside the Earth or on the Earth's surface is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad (|\mathbf{r}| \leq R),$$

where $\mathbf{r}$ is the position vector of the point measured from an origin O at the Earth's centre, and R is the Earth's radius.

(a) Express the vector field $\mathbf{v}$ in Cartesian coordinates, taking the z-axis to be directed from south to north along the axis of rotation.

(b) Find $\mathbf{curl}\,\mathbf{v}$ and determine its magnitude.

(c) Comment on the relationship between the magnitude and direction of $\mathbf{curl}\,\mathbf{v}$, and the rate and sense of rotation of your home.

Curl in cylindrical and spherical coordinates

We will derive the formula for curl only in cylindrical coordinates. (The method for spherical coordinates is similar, but the calculations are more complicated.)

First recall, from Subsection 4.1 of Unit 15, that in cylindrical coordinates the gradient vector is given by

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}.$$

To compute $\mathbf{curl}\,\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F}$, we must therefore calculate

$$\left(\mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \times (F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z),$$

and since this is a complicated expression, it is sensible to proceed term by term.

The cross product of any vector with itself is $\mathbf{0}$, and

$$\mathbf{e}_\rho \times \mathbf{e}_\phi = \mathbf{e}_z,$$
$$\mathbf{e}_\phi \times \mathbf{e}_z = \mathbf{e}_\rho,$$
$$\mathbf{e}_z \times \mathbf{e}_\rho = \mathbf{e}_\phi.$$

Since $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ do not depend on ρ, the first term is

$$\begin{aligned}\mathbf{e}_\rho \frac{\partial}{\partial \rho} \times (F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) &= \mathbf{e}_\rho \times \frac{\partial}{\partial \rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &= \mathbf{e}_\rho \times \left(\frac{\partial F_\rho}{\partial \rho}\mathbf{e}_\rho + \frac{\partial F_\phi}{\partial \rho}\mathbf{e}_\phi + \frac{\partial F_z}{\partial \rho}\mathbf{e}_z\right)\\ &= \frac{\partial F_\phi}{\partial \rho}\mathbf{e}_z - \frac{\partial F_z}{\partial \rho}\mathbf{e}_\phi.\end{aligned}$$

Next, using the results $\partial\mathbf{e}_\rho/\partial\phi = \mathbf{e}_\phi$, $\partial\mathbf{e}_\phi/\partial\phi = -\mathbf{e}_\rho$ and $\partial\mathbf{e}_z/\partial\phi = \mathbf{0}$, we have

$$\begin{aligned}
&\mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} \times (F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) \\
&= \frac{1}{\rho}\mathbf{e}_\phi \times \frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) \\
&= \frac{1}{\rho}\mathbf{e}_\phi \times \left(\frac{\partial F_\rho}{\partial\phi}\mathbf{e}_\rho + F_\rho\mathbf{e}_\phi + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi - F_\phi\mathbf{e}_\rho + \frac{\partial F_z}{\partial\phi}\mathbf{e}_z\right) \\
&= -\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi}\mathbf{e}_z + \frac{1}{\rho}F_\phi\,\mathbf{e}_z + \frac{1}{\rho}\frac{\partial F_z}{\partial\phi}\mathbf{e}_\rho.
\end{aligned}$$

Finally, since $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ do not depend on z,

$$\begin{aligned}
\mathbf{e}_z \frac{\partial}{\partial z} \times (F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) &= \mathbf{e}_z \times \frac{\partial}{\partial z}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) \\
&= \mathbf{e}_z \times \left(\frac{\partial F_\rho}{\partial z}\mathbf{e}_\rho + \frac{\partial F_\phi}{\partial z}\mathbf{e}_\phi + \frac{\partial F_z}{\partial z}\mathbf{e}_z\right) \\
&= \frac{\partial F_\rho}{\partial z}\mathbf{e}_\phi - \frac{\partial F_\phi}{\partial z}\mathbf{e}_\rho.
\end{aligned}$$

Hence the formula, which we summarise below. A more complicated derivation provides a corresponding formula in spherical coordinates.

Curl in cylindrical and spherical coordinates

The **curl** of a vector field $\mathbf{F} = \mathbf{F}(\rho, \phi, z) = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z$ in **cylindrical coordinates** is

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} = &\left(\frac{1}{\rho}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\mathbf{e}_\phi \\
&+ \left(\frac{\partial F_\phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi} + \frac{1}{\rho}F_\phi\right)\mathbf{e}_z. \qquad (16)
\end{aligned}$$

The curl of a vector field $\mathbf{F}(r, \theta, \phi) = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ in **spherical coordinates** is

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} = &\left(\frac{1}{r}\frac{\partial F_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\phi} + \frac{\cot\theta}{r}F_\phi\right)\mathbf{e}_r \\
&+ \left(-\frac{\partial F_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}F_\phi\right)\mathbf{e}_\theta \\
&+ \left(\frac{\partial F_\theta}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{r}F_\theta\right)\mathbf{e}_\phi. \qquad (17)
\end{aligned}$$

As you can see, the full cylindrical expression for curl is quite complicated. However, for vector fields where there is no z-component and no variation in the z-direction, we have the following.

Curl of a two-dimensional vector field

The **curl** of a two-dimensional vector field $\mathbf{F} = \mathbf{F}(\rho, \phi) = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi$ in **cylindrical coordinates** is

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial F_\phi}{\partial \rho} + \frac{F_\phi}{\rho} - \frac{1}{\rho} \frac{\partial F_\rho}{\partial \phi} \right) \mathbf{e}_z. \tag{18}$$

Example 6

Use equation (18) to calculate $\mathbf{curl}\,\mathbf{u}$ and $\mathbf{curl}\,\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are the velocity fields in cylindrical form that you found in Exercise 17.

Solution

For the field $\mathbf{u}$, we have $u_\rho = 0$, $u_\phi = \omega\rho$. The only non-zero partial derivative is $\partial u_\phi / \partial \rho = \omega$. Hence

$$\mathbf{curl}\,\mathbf{u} = \left(\omega + \frac{\omega\rho}{\rho} \right) \mathbf{e}_z = 2\omega \mathbf{e}_z = 2\omega \mathbf{k}.$$

Similarly, we have $v_\rho = 0$, $v_\phi = \omega a^2/\rho$, so the only non-zero partial derivative for $\mathbf{v}$ is $\partial v_\phi / \partial \rho = -\omega a^2/\rho^2$. Hence

$$\mathbf{curl}\,\mathbf{v} = \left(-\frac{\omega a^2}{\rho^2} + \frac{\omega a^2}{\rho^2} \right) \mathbf{e}_z = \mathbf{0}.$$

These results agree with those of Exercise 16.

Exercise 19

Consider the vector field $\mathbf{F}$ expressed in cylindrical coordinates as

$$\mathbf{F}(\rho, \phi) = f(\rho)\, \mathbf{e}_\rho + \rho^n g(\phi)\, \mathbf{e}_\phi, \quad \rho > 0,$$

where $f(\rho)$ is a function of ρ only, $g(\phi)$ is a function of ϕ only that is in general non-zero, and n is an integer.

Show that $\mathbf{F}$ is irrotational if and only if $n = -1$.

Exercise 20

Consider the vector field $\mathbf{F} = f(r)\, \widehat{\mathbf{r}}$, where $r = |\mathbf{r}|$, $\widehat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (the position vector), and $f(r)$ is a function of r only.

Use the expression for curl in spherical coordinates to show that $\mathbf{F}$ is irrotational.

Exercise 21

Consider the vector field $\mathbf{A}$ expressed in spherical coordinates as

$$\mathbf{A}(r,\theta,\phi) = \frac{\psi(r,\theta)}{r\sin\theta}\,\mathbf{e}_\phi, \quad r>0,\ 0<\theta<\pi,$$

where $\psi(r,\theta)$ does not depend on ϕ, and let a second vector field, $\mathbf{F}$, be given by $\mathbf{F} = \widehat{\mathbf{r}}/r^2 = \mathbf{e}_r/r^2$ (in spherical coordinates).

Show that $\mathbf{F} = \mathbf{curl}\,\mathbf{A}$ provided that the function $\psi(r,\theta)$ satisfies

$$\frac{\partial\psi}{\partial r} = 0, \quad \frac{\partial\psi}{\partial\theta} = \sin\theta.$$

Hence deduce that the family of functions that satisfy these requirements is given by $\psi(r,\theta) = c - \cos\theta$, for some arbitrary constant c.

3 The scalar line integral

Scalar line integrals occur whenever we sum scalar values along a line or along a curve in space. An important example of a scalar line integral is the *work done by a force*. Subsection 3.1 defines the work done by a force acting on a particle moving along the x-axis. This definition is then generalised to motion along a curve in space. In Subsection 3.2 we show that the work done by the force is a *scalar line integral* of the force vector along the curve, and we also show how scalar line integrals are evaluated. In Subsection 3.3 we show how the length of a curve can be expressed as a scalar line integral.

We often refer to scalar line integrals as just *line integrals*.

3.1 Work done by forces

We begin with a one-dimensional problem in which a particle is acted on by a single force $\mathbf{F}$. Suppose that a constant force $F\mathbf{i}$ accelerates a particle of mass m in the direction of the positive x-axis. If $a_0\mathbf{i}$ is the constant acceleration of the particle, $x_0\mathbf{i}$ is its initial position, and v_0 is its initial speed, then the position $x\mathbf{i}$ and velocity $v\mathbf{i}$ of the particle are related by the constant acceleration equation

You met problems of this sort in Unit 3. If F is constant, then so is the acceleration $a_0\mathbf{i}$, and $F = ma_0$.

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

See Subsection 1.2 of Unit 3.

Multiplying by $\frac{1}{2}m$ and rearranging gives the energy conservation equation

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = ma_0(x-x_0) = F(x-x_0). \qquad (19)$$

In Unit 9 an equation of this type was interpreted as stating that the gain (or loss) of kinetic energy equals the loss (or gain) of potential energy. Here we are interested in two other ways of interpreting equation (19).

First, since both the force and the displacement of the particle are directed along the x-axis, the right-hand side is just the dot product of $\mathbf{F} = F\mathbf{i}$ and $(x - x_0)\mathbf{i}$, so

$$ma_0(x - x_0) = F(x - x_0) = \mathbf{F} \cdot (x - x_0)\mathbf{i}.$$

Second, since F is a constant and the particle is moving in a straight line from x_0 to a general position x, we can write

$$F(x - x_0) = \int_{x_0}^{x} F\,dx, \tag{20}$$

so the right-hand side of equation (19) is just the integral of F, along the x-axis, from x_0 to x. We call this integral the *work done* by $\mathbf{F}$ in moving the particle along the straight line from x_0 to x.

Here $h = x - x_0$, $a_0 = g$ and $v_0 = 0$ in equation (19).

We ignore all other forces such as air resistance acting on the stone.

You can see from this that the work done by the force is equal to the *gain* in kinetic energy of the particle. For a stone falling a distance h from rest, the constant force of gravity, $F_{\text{g}}\mathbf{i} = mg\mathbf{i}$, does work $F_{\text{g}}h = mgh$, which is equal to the *loss* of gravitational potential energy. The work done by the force is equal to the amount of gravitational potential energy that becomes kinetic energy.

We now generalise the work integral (20) to motion that is no longer confined to the x-axis but is either two-dimensional, along a path in a plane, or in three-dimensional space. In general, the force $\mathbf{F}$ will act at points $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in space, and we use the notation $\mathbf{F}(\mathbf{r})$ instead of $\mathbf{F}(x, y, z)$. We calculate the work done by any force field $\mathbf{F}(\mathbf{r})$, which depends at most on the position vector $\mathbf{r}$. Thus we exclude, for example, forces that depend on velocity, such as friction, or forces that depend explicitly on time as well as position.

We start with Newton's second law, $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$, for a particle of mass m. We take the dot product of both sides of the equation with the velocity $\dot{\mathbf{r}} = d\mathbf{r}/dt$ to give $m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \mathbf{F}(\mathbf{r}) \cdot \dot{\mathbf{r}}$. Since $d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/dt = 2\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, this is equivalent to $\frac{1}{2}m\,d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/dt = \mathbf{F}(\mathbf{r}) \cdot \dot{\mathbf{r}}$. We then integrate both sides of the equation with respect to time over the interval from t_0 to t_1. Thus, with $\mathbf{v}_0 = \dot{\mathbf{r}}(t_0)$ and $\mathbf{v}_1 = \dot{\mathbf{r}}(t_1)$, we have

$$\begin{aligned}\int_{t_0}^{t_1} \left(\tfrac{1}{2}m\,\frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt}\right) dt &= \tfrac{1}{2}m\dot{\mathbf{r}}(t_1) \cdot \dot{\mathbf{r}}(t_1) - \tfrac{1}{2}m\dot{\mathbf{r}}(t_0) \cdot \dot{\mathbf{r}}(t_0) \\ &= \tfrac{1}{2}m\mathbf{v}_1 \cdot \mathbf{v}_1 - \tfrac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0.\end{aligned}$$

Since it is based on Newton's second law, equation (21) is valid only when $\mathbf{F}$ is the resultant force on the particle. However, when there is more than one force acting simultaneously, we can still define the *work done by each force separately* by an integral of the form of equation (22).

Hence we obtain a generalised form of equation (19),

$$\tfrac{1}{2}m\mathbf{v}_1 \cdot \mathbf{v}_1 - \tfrac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0 = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt, \tag{21}$$

where the left-hand side is the change of kinetic energy, and the definite integral

$$W = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt \tag{22}$$

is the **work done** by the force $\mathbf{F}$ on the particle when it moves along a path described by the position vector $\mathbf{r}(t)$, in the interval from t_0 to t_1.

Consider again the case of a particle moving in the positive direction of the x-axis, so that $\mathbf{r} = x\mathbf{i}$ and $\dot{\mathbf{r}} = \dot{x}\mathbf{i}$, but now we allow the magnitude of $\mathbf{F}$ to vary, so $\mathbf{F} = F(x)\,\mathbf{i}$. Suppose that the particle is at $x = a$ at time t_0, and at $x = b$ at time t_1, as shown in Figure 7. Since the force $\mathbf{F}$ depends at most on position x, and not on other variables such as velocity, or explicitly on time, the work done by the force, from equation (22), is

The case of a varying force accelerating a particle along the x-axis was considered in Section 3 of Unit 9.

$$\begin{aligned} W = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt &= \int_{t_0}^{t_1} F(x)\,\mathbf{i} \cdot \frac{dx}{dt}\mathbf{i}\,dt \\ &= \int_{t_0}^{t_1} F(x)\,\frac{dx}{dt}\,dt \\ &= \int_a^b F(x)\,dx. \qquad (23) \end{aligned}$$

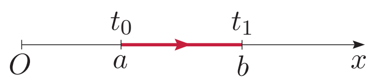

Figure 7 Path of a particle moving in one dimension

Example 7

A particle moves along the x-axis from $x = 2$ to $x = 5$ under the action of a force $F(x)\,\mathbf{i}$ given by

$$F(x) = -100(x - 1).$$

Determine the work done by the force, and the change in the kinetic energy of the particle. (Assume that the initial kinetic energy of the particle is sufficiently high for the motion to be possible.)

We use SI units.

Solution

To calculate the work done, we can use equation (23), obtaining

$$W = \int_a^b F(x)\,dx = -100\int_2^5 (x-1)\,dx = -100\left[\tfrac{1}{2}x^2 - x\right]_2^5 = -750.$$

Hence the work done by the force is $-750\,\mathrm{J}$ as the particle moves from $x = 2$ to $x = 5$.

The SI units of work (work done) are the same as those of energy (joules).

Assuming that $F(x)\,\mathbf{i}$ is the resultant force acting on the particle, the change in the particle's kinetic energy is $-750\,\mathrm{J}$. Note that the kinetic energy is reduced because the force always acts in the opposite direction to the direction of motion.

Exercise 22

The force on a particle of mass m moving on an interval of the x-axis is

$$F(x)\,\mathbf{i} = -\frac{A}{x^2}\,\mathbf{i} \quad (x \neq 0),$$

where A is a positive constant.

Determine the work done by the force when the particle moves from $x = 3$ to $x = 1$, and hence find an expression for the speed v of the particle when it is at $x = 1$, given that its speed at $x = 3$ was u.

Before we give the details of a general method for evaluating integrals like that in equation (22), for the work done by a force, let us first consider two special cases when the integrand, and hence the work done, is zero.

- $\mathbf{F} = \mathbf{0}$, which makes sense since if the force is zero, it can do no work.
- $\mathbf{F}$ and $d\mathbf{r}/dt$ are perpendicular, so $\mathbf{F} \cdot d\mathbf{r}/dt = 0$. Since the velocity vector $d\mathbf{r}/dt$ points along the tangent to the curve at each point, only the component of the force in this direction contributes to the work done. Hence no work is done if the only force is perpendicular to the direction of motion.

See Unit 20.

An example of the second case is provided by a particle moving at constant speed along a circular path. In this case the force is directed along the radius, towards the centre; it therefore has zero component along the tangent to the circle at any point, and hence does no work. Other examples of situations where forces always act at right angles to the path of a moving particle, and hence do no work, are the tension force in the string of a simple pendulum and the normal reaction on an object moving on an inclined plane.

The parameter t (for time) appears in equation (22) because it is often a convenient way of parametrising the path of the particle. We could equally well describe the path in terms of another parameter, such as the distance s from the point where $\mathbf{r} = \mathbf{r}(t_0)$. For this reason we normally write equation (22) symbolically in parameter-free form as

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \qquad (24)$$

where C is the path of the particle. However, when we need to evaluate equation (24), we express the path in terms of a suitable **parameter**, usually t (which may or may not represent time), so that it once again takes the form of equation (22).

You considered t as a parameter for the position vector $\mathbf{r}$ in Unit 3.

To evaluate equation (22) with the dot product in the integrand expressed in Cartesian form, we need to know the functions $\mathbf{F}(\mathbf{r})$ and $d\mathbf{r}/dt$. Now for $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, we have

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

and we can express the components of $\mathbf{F}(\mathbf{r})$ at any point on C as functions of t. For example,

$$F_1(\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \mathbf{i} = F_1(x, y, z) = F_1(x(t), y(t), z(t)) = F_1(t),$$

so

Notice that in employing a parameter, we also abuse notation by reducing a function of three coordinates $\mathbf{F}(\mathbf{r})$ to a function of one parameter $\mathbf{F}(t)$.

$$\mathbf{F}(t) = F_1(t)\,\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k} \quad \text{on } C.$$

Thus the integrand in equation (22) is

$$\mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} = F_1(t)\,\frac{dx}{dt} + F_2(t)\,\frac{dy}{dt} + F_3(t)\,\frac{dz}{dt}.$$

Work done by a force

The work done by a force $\mathbf{F}(\mathbf{r})$, given in Cartesian coordinates, in moving a particle along a path C given by $\mathbf{r} = \mathbf{r}(t)$ from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$ is

$$\begin{aligned} W &= \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= \int_{t_0}^{t_1} \left(F_1(t)\,\frac{dx}{dt} + F_2(t)\,\frac{dy}{dt} + F_3(t)\,\frac{dz}{dt} \right) dt. \qquad (25) \end{aligned}$$

Thus we have expressed the work done by a force as an ordinary definite integral that we can try to evaluate by standard integration techniques.

Example 8

Find the work done by the force

$$\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$$

when it moves a particle from the point $(0,0,0)$ to the point $(3,1,1)$ along the curve C given by $\mathbf{r} = 3t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, from $t_0 = 0$ to $t_1 = 1$.

Solution

The components of $\mathbf{r}$ in terms of t, that is, the parametric equations for C, are

$$x = 3t, \quad y = t^2, \quad z = t^3, \quad \text{where } t \text{ goes from } t_0 = 0 \text{ to } t_1 = 1,$$

and differentiating these equations yields

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 3t^2.$$

The components of $\mathbf{F}$ in terms of t are

$$F_1 = yz = t^5, \quad F_2 = zx = 3t^4, \quad F_3 = xy = 3t^3.$$

Substituting into equation (25) gives

$$\begin{aligned} W = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt} \right) dt \\ &= \int_0^1 \big((t^5)(3) + (3t^4)(2t) + (3t^3)(3t^2) \big)\, dt \\ &= \int_0^1 18t^5\, dt \\ &= 18\left[\tfrac{1}{6}t^6\right]_0^1 \\ &= 3. \end{aligned}$$

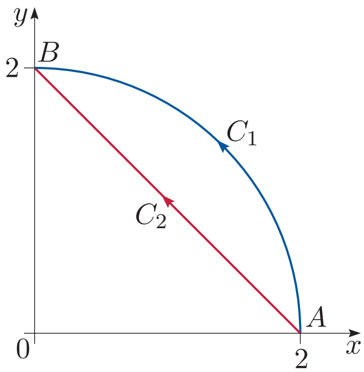

Figure 8 Two different paths from A to B

Exercise 23

Find the work done by the force

$$\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$$

as it moves a particle from the point $(2, 0)$ to the point $(0, 2)$ along the quarter-circle C_1 shown in Figure 8. The quarter-circle has centre at the origin and radius 2, and can be parametrised by the equations

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 0, \quad \text{where } t \text{ goes from } t_0 = 0 \text{ to } t_1 = \tfrac{\pi}{2}.$$

Sometimes a curve in the (x, y)-plane is specified by an equation $y = f(x)$. We can then effectively use x as the parameter t in equation (25).

Example 9

The force $\mathbf{F}$ is the same as that in Exercise 23.

Find the work done by the vector field $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ acting along the straight line C_2, specified by the equation $y = 2 - x$, from the point $(2, 0)$ to the point $(0, 2)$ (see Figure 8).

Solution

We put $x = t$, and the parametric equations for C_2 are

$$x = t, \quad y = 2 - t, \quad z = 0,$$

where t runs from $t_0 = 2$ to $t_1 = 0$ (see Figure 8). Then we have

$$F_1 = 2x + y = 2 + t, \quad F_2 = -x = -t,$$
$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -1.$$

Substituting into equation (25) yields

$$\begin{aligned} W = \int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_2^0 \big((2 + t) + (-t)(-1)\big)\, dt \\ &= \int_2^0 (2t + 2)\, dt \\ &= \big[t^2 + 2t\big]_2^0 \\ &= -8. \end{aligned}$$

Exercise 24

Find the work done by the force $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ acting along the parabolic segment C specified by the equation $y = 1 - x^2$, from the point $(1, 0)$ to the point $(0, 1)$ (see Figure 9).

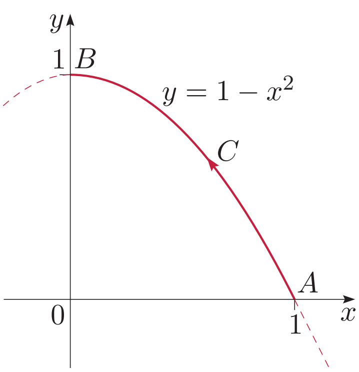

Figure 9

Of course, there are many paths joining any two given points.

Exercise 25

Find two different paths between the points $(0,0,0)$ and $(1,1,1)$, giving parametrisations for each.

A parametrisation of a particular curve is not unique, nor does the work done along a particular path depend on the parametrisation.

Exercise 26

The straight line C_2 in Example 9 can be parametrised by the equations

$$x = 2(1-t), \quad y = 2t, \quad z = 0, \quad \text{for } t \text{ from 0 to 1.}$$

Find the work done by the force $\mathbf{F} = (2x+y)\mathbf{i} - x\mathbf{j}$ using this parametrisation, and compare your answer with that found in Example 9.

3.2 Scalar line integrals

You have seen that the work integral in equation (22) may be written as

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \tag{26}$$

This is equation (24).

where C specifies the path (or directed curve) along which the particle moves, and $\mathbf{r}$ is the position vector of points on C. You can think of the dot product between the force $\mathbf{F}(\mathbf{r})$ and the symbol $d\mathbf{r}$ in equation (26) as representing the fact that we are integrating only the component of the force parallel to the tangent to the curve in the direction of motion. This is illustrated in Figure 10, which shows a very short segment PQ of the curve C. The displacement $\delta\mathbf{r}_i = \mathbf{r}_Q - \mathbf{r}_P$ is nearly parallel to the tangents to the path along this segment, and the work done by the force along PQ is approximately the dot product $\mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{r}_i$, where $\mathbf{F}(\mathbf{r}_i)$ is the force vector at some point $\mathbf{r}_i$ on PQ. If we divide up C into N such short segments, then adding the approximations for the work done along all N segments, we obtain an estimate for the work done by $\mathbf{F}$ as it acts along the whole of C, that is,

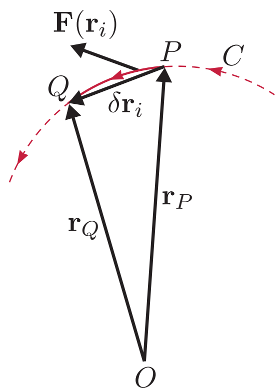

Figure 10 Small segment of a path of a particle subject to a force $\mathbf{F}(\mathbf{r})$

$$W \simeq \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{r}_i.$$

The approximation becomes exact in the limit as $N \to \infty$ and each $|\delta\mathbf{r}_i| \to 0$, to give

$$W = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{r}_i = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$

The work integral, equation (22) or equation (26), is an example of a *scalar line integral* of a vector field. We can form the scalar line integral of any vector field $\mathbf{F}(\mathbf{r})$ along a path C.

Note that t may be any parameter, not necessarily time.

Scalar line integral

The **scalar line integral** of a vector field $\mathbf{F}(\mathbf{r})$ along a path C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$, is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{27}$$

The physical interpretation of the scalar line integral will depend on the nature of the particular vector field $\mathbf{F}(\mathbf{r})$ being considered. We will consider only the case of force fields, where the scalar line integral represents the work done by the force.

We indicate the direction by an arrow in diagrams, e.g. Figures 8 and 9.

The path C in a scalar line integral is a directed curve (with starting point A and endpoint B). It is often convenient to denote the scalar line integral by writing it as

$$\int_{AB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r},$$

where the order AB indicates the direction along the path.

We could choose to traverse the *same* path but in the opposite sense, starting at B and ending at A. This would reverse the direction of each displacement $\delta\mathbf{r}_i$ in Figure 10 and therefore change the sign of the scalar line integral. Thus we have

$$\int_{BA} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -\int_{AB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \tag{28}$$

Closed curves are also referred to as *loops*.

Line integrals can be evaluated along segments of **open curves** where the starting point A and the endpoint B are distinct points, as in the examples above, or around **closed curves**. The straight line and quarter-circle of Figure 8 are examples of open curves. Suppose now that we reverse the direction of the straight-line segment in Figure 8. Then the reversed line segment together with the quarter-circle make the closed curve shown in Figure 11. We can traverse this closed curve in an anticlockwise sense, $APBQA$, by starting at point A, moving along the quarter-circle via P to B, and returning along the straight line BA via Q. The line integral of a vector field $\mathbf{F}$ for one complete anticlockwise traversal of this loop is the sum of the two line integrals. We can write this as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} + \int_{BQA} \mathbf{F} \cdot d\mathbf{r},$$

where the circle on the integral sign indicates that the path C is closed. The sense of each of the two line integrals on the right-hand side of the above equation, that is, the direction along the path, is indicated by the order of the letters APB and BQA.

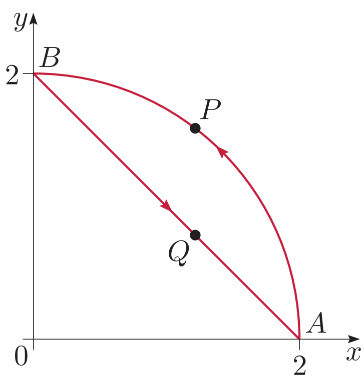

Figure 11 Path consisting of a closed curve $APBQA$ obtained from the paths in Figure 8 by combining path C_1 with the reverse of path C_2

Example 10

Evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ and C is the closed curve $APBQA$ shown in Figure 11.

Solution

We sum the line integrals along the two segments APB and BQA of the loop. The line integral of $\mathbf{F}$ along the quarter-circle C_1 $(= APB)$ was evaluated in the solution to Exercise 23, where you obtained

$$\int_{APB} \mathbf{F} \cdot d\mathbf{r} = -4 - 2\pi.$$

The line integral along the straight line C_2 $(= AQB)$ was evaluated in Example 9, where we obtained

$$\int_{AQB} \mathbf{F} \cdot d\mathbf{r} = -8.$$

But we want the line integral along BQA, that is, along the same straight-line segment but in the opposite direction. Using equation (28), we just change the sign, to obtain

$$\int_{BQA} \mathbf{F} \cdot d\mathbf{r} = 8.$$

Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} + \int_{BQA} \mathbf{F} \cdot d\mathbf{r} = -4 - 2\pi + 8 = 4 - 2\pi.$$

Exercise 27

Evaluate the scalar line integral of the vector field $\mathbf{G} = x^2\mathbf{i} + y\mathbf{j}$ along each of the paths C_1 and C_2 specified in Figure 8. (Use the parametrisations of Exercise 23 and Example 9, respectively.) Evaluate also the scalar line integral of $\mathbf{G}$ around the closed curve $APBQA$ in Figure 11.

Exercise 28

Evaluate the scalar line integral of $\mathbf{F} = 2x\mathbf{i} + (xz - 2)\mathbf{j} + xy\mathbf{k}$ along the path C from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ defined by the parametrisation

$$x = t, \quad y = t^2, \quad z = t^3 \quad (0 \leq t \leq 1).$$

Example 11

A force $\mathbf{F}$ acts along a path C of length L. The tangential component F_t of $\mathbf{F}$ is constant along C. Show that the work done by $\mathbf{F}$ is $F_t L$.

Solution

The component of $\mathbf{F}$ normal to C does no work since it is perpendicular to the tangent to C. So if the parameter l measures the length along C from $l_0 = 0$ to $l_1 = L$, and $F_t = \mathbf{F} \cdot d\mathbf{r}/dl$, then the work done, W, is given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{l_0}^{l_1} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dl} \right) dl = \int_{l_0}^{l_1} F_t \, dl = \int_0^L F_t \, dl = F_t L.$$

Exercise 29

Evaluate the scalar line integral of each of the two vector fields $\mathbf{u}$ and $\mathbf{v}$ given below, around the closed circular path C of radius a centred on the origin and defined by the parametric equations

$$x = a\cos t, \quad y = a\sin t, \quad \text{where } t \text{ goes from } 0 \text{ to } 2\pi.$$

(a) $\mathbf{u}(x,y) = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$

(b) $\mathbf{v}(x,y) = \omega(-y\mathbf{i} + x\mathbf{j})$, where ω is a positive constant

You may have spotted that the line integrals in Exercise 29 can be evaluated without explicitly carrying out an integration, since in each case the tangential component of the vector field is constant everywhere on the circle. This is obvious in part (a), since $x\mathbf{i} + y\mathbf{j} = \mathbf{r}$ is the position vector, which points radially outwards everywhere, and is therefore always perpendicular to the tangent to any circle centred on the origin. Hence the tangential component of $\mathbf{u}$ is zero everywhere on the circle, and the line integral is zero. It is not so obvious in part (b), but you can see that $\mathbf{r} \cdot \mathbf{v} = 0$, so the field $\mathbf{v}$ is at right angles to the position vector and is therefore directed tangentially to the path. Also, the field $\mathbf{v}$ has a constant magnitude, $\omega((-y)^2 + x^2)^{1/2} = \omega a$, on the circle. Thus the line integral of $\mathbf{v}$ for one complete traversal of the circle is $\omega a \times 2\pi a = 2\omega\pi a^2$. When evaluating line integrals, you can sometimes spot that the tangential component of the vector field is constant and, as this example shows, obtain the value directly using the result of Example 11.

The field lines of $\mathbf{v}$ are circles centred on the origin.

You might also note that on the circle of radius a we have $\mathbf{v} = \omega a \mathbf{e}_\theta$, where $\mathbf{e}_\theta$ is the unit vector in the direction of increasing θ in polar coordinates.

Exercise 30

Evaluate the scalar line integral of the vector field $\mathbf{v}$ defined in Exercise 29 along the x-axis from $x = 1$ to $x = -1$.

Exercise 31

Determine each of the following line integrals, which you can do without explicitly carrying out an integration.

(a) The line integral of $\mathbf{F} = z^2\mathbf{j}$ along the x-axis from $x = 1$ to $x = 2$

(b) The line integral of $\mathbf{F} = 5\mathbf{k}$ along the z-axis from $z = 0$ to $z = 6$

(c) The line integral of $\mathbf{F} = r^2\mathbf{e}_\theta$ on a semicircle in the (x, y)-plane centred on the origin and of radius 3

$\mathbf{F}$ is given here in polar coordinates.

3.3 The length of a curve

Consider the scalar line integral of the vector field $\mathbf{F}$ along the curve C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$, defined by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt}\, dt,$$

where t represents time. The expression $\mathbf{r}(t)$ represents the point on the curve corresponding to time t, and $d\mathbf{r}/dt$ represents the velocity of the point as it moves along the curve. Now consider what happens when we choose the vector function

$$\mathbf{F}(t) = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} = \widehat{\dot{\mathbf{r}}(t)},$$

which represents a unit vector in the direction of the velocity vector $\dot{\mathbf{r}}(t)$. In this case

$$\mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \cdot \dot{\mathbf{r}}(t) = |\dot{\mathbf{r}}(t)|,$$

and the scalar line integral becomes

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} |\dot{\mathbf{r}}(t)|\, dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad (29)$$

This gives the length of the curve C between the two points $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$. Indeed, one can see this intuitively by considering what happens to the point that is currently at $\mathbf{r}(t)$ during a small interval of time δt. During this time, the point will move along the curve by an approximate distance $|\dot{\mathbf{r}}(t)|\, \delta t$, so the length of the curve is approximated by the sum

$$\sum_i |\dot{\mathbf{r}}_i(t)|\, \delta t_i.$$

Allowing the intervals δt_i to shrink to zero in the usual way, we can replace the sum by an integral that is precisely equation (29). Let us check this in an example.

Example 12

Use the scalar line integral equation (29) and a suitable parametrisation to find the length of a semicircle of radius 1.

Solution

Take the centre of the circle to be at the origin, and consider the semicircle as lying in the upper half-plane ($y \geq 0$). Then a suitable parametrisation is given by $x = \cos t$, $y = \sin t$ ($0 \leq t \leq \pi$). Thus $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, and $\dot{\mathbf{r}}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$.

Therefore

$$|\dot{\mathbf{r}}(t)| = 1,$$

and the integral in equation (29) is

$$\int_0^{\pi} 1\,dt = \pi,$$

which we know to be the correct answer.

Exercise 32

Find the length of the curve given by

$$x = t, \quad y = -\ln(\cos t) \quad (0 \leq t \leq \pi/4).$$

4 Linking line integrals, curl and gradient

In this section a connection is established between the properties of scalar line integrals of a vector field and some of the properties of scalar and vector fields that you met in Unit 15 and earlier in this unit, in particular the curl of the vector field. Subsection 4.2 shows how the properties of scalar line integrals are used for classifying vector fields as *conservative* or *non-conservative*.

4.1 Line integrals and curl

In this subsection we make important links between line integrals, the curl of a vector field, potential functions and the gradient of a scalar field. The curl of a vector field was defined in Section 2 in terms of partial derivatives. In fact, it is also possible to define the curl of a vector field in terms of line integrals. We do this below, and in the next subsection you will see that it leads to the *curl test* for conservative fields.

Figure 12 shows a closed curve C in the (x, y)-plane enclosing a point P. Let the area enclosed by the curve be A. Consider the line integral of an arbitrary vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ for one complete anticlockwise traversal of the curve, then divide the value of the line integral by the area inside the curve to obtain the quotient

$$Q = \frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

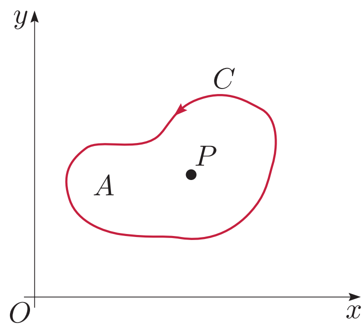

Figure 12 Closed curve with area A enclosing point P

In general, the value of Q might be expected to depend on the size and shape of the loop enclosing P. But suppose that we consider a sequence of loops of smaller and smaller area A, all enclosing P. We then find that the limit of Q for $A \to 0$ is independent of the shapes of the loops and of how the limit is approached, and depends only on the location of the point P, and of course the vector field $\mathbf{F}$. Furthermore – and here is a surprise – the limit turns out to be something familiar:

$$\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r}\right) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},$$

where the partial derivatives are evaluated at P. (Note that F_3 does not appear here since $\mathbf{k}$ is perpendicular to the plane containing the loop C.) You will recognise the right-hand side as the z-component of $\mathbf{curl}\,\mathbf{F}$ at P. In other words, we have

$$\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r}\right) = \mathbf{k} \cdot \mathbf{curl}\,\mathbf{F}. \tag{30}$$

By considering loops in the (y, z)-plane and in the (z, x)-plane, we can obtain similar relationships for the x- and y-components of $\mathbf{curl}\,\mathbf{F}$.

These results are special cases of Stokes' theorem relating the curl of a vector field to the line integral of the field around a closed curve.

We will not prove these results for the general case, but ask you to confirm that equation (30) is true in a particular case.

Exercise 33

Consider the vector field $\mathbf{v} = \omega(-y\mathbf{i} + x\mathbf{j})$, where ω is a positive constant.

(a) Evaluate

$$\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{v} \cdot d\mathbf{r}\right)$$

for anticlockwise traversal of the circle C of radius a centred on the origin.

The line integral is evaluated in the solution to Exercise 29(b).

(b) Evaluate $\mathbf{curl}\,\mathbf{v}$ and $\mathbf{k} \cdot \mathbf{curl}\,\mathbf{v}$.

From equation (30), and its counterparts for the x- and y-components, we can see that if a vector field $\mathbf{F}$ has the property that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C in the domain of $\mathbf{F}$, then $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere, that is, $\mathbf{F}$ is irrotational. However, the converse is not true *unless* the domain of $\mathbf{F}$ is simply-connected, a property that is defined as follows. A **simply-connected region** is a region where *any* closed curve contained in it can be continuously shrunk to a point without leaving the region.

Roughly speaking, a simply-connected region in two dimensions is one that contains no holes, and in three dimensions contains no holes that pass all the way though the region. For example, the (x, y)-plane is simply-connected, but the (x, y)-plane *excluding* the origin is *not*, for in this case it is not possible to continuously shrink a loop that encloses the origin to a point. Three-dimensional space $\mathbb{R}^3$ is simply-connected, as is $\mathbb{R}^3$ excluding the origin, but $\mathbb{R}^3$ excluding the z-axis is *not* simply-connected as loops that enclose the z-axis cannot be shrunk to a point.

Now, it can be shown that if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ at all points in a vector field on a *simply-connected domain*, then $\oint_C \mathbf{F} \cdot d\mathbf{r}$ must be zero for any closed curve C within the domain, and vice versa. You will see later how to make use of this idea, but first we consider how to link line integrals with energy conservation, and the idea of a line integral being independent of the path along which it is evaluated. Suppose that a particle can move along paths in the two- or three-dimensional domain of a force field $\mathbf{F}(\mathbf{r})$. In general, a particle may move from one fixed point A to another fixed point B via an infinite number of different paths, so we might ask: does the work done by the force depend on the particular path taken from A to B? You will see that if the work done by the force is independent of the particular path, then we can define a potential energy function and mechanical energy is conserved.

In Example 9, we found that the line integral of the vector field $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ on the straight line C_2, from the point $(2, 0)$ to the point $(0, 2)$, has the value -8; but in Exercise 23 you found that the line integral of the *same* vector field along the quarter-circle C_1 between the *same* two points $(2, 0)$ and $(0, 2)$ is $-4 - 2\pi$. In other words, the value of the line integral of the vector field between the two points depends on the actual path taken between the two points. The line integral of $\mathbf{F}$ is **path-dependent**. This is not surprising, since different paths between two fixed points will sample different vectors.

In Exercise 27, on the other hand, you found that the line integrals of the vector field $\mathbf{G} = x^2\mathbf{i} + y\mathbf{j}$ between the same two points have the same values for each of the two paths C_1 and C_2. In fact, the line integral of $\mathbf{G}$ between *any* two fixed points has the same value for *any* path between those two points. The line integral of $\mathbf{G}$ is **path-independent** and depends only on the starting point and endpoint of the path.

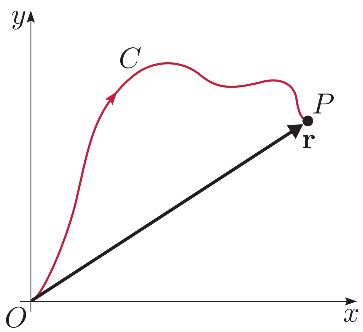

Figure 13 Path C joining O to P with position vector $\mathbf{r}$

We now turn to an idea that you met in Unit 15, that of the gradient vector. Suppose that the line integral of a vector field $\mathbf{F}(\mathbf{r})$ is path-independent. Given an origin O, we can define a scalar function $U(\mathbf{r})$ at every point P with position vector $\mathbf{r}$ in the domain of $\mathbf{F}$, by putting

$$U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r},$$

where OP is *any* path C from O to P (see Figure 13) and the minus sign is a conventional choice consistent with U being a potential energy when $\mathbf{F}$ is a force field. Then, for any two points A and B in the field,

$$\int_{AB} \mathbf{F} \cdot d\mathbf{r} = \int_{OB} \mathbf{F} \cdot d\mathbf{r} - \int_{OA} \mathbf{F} \cdot d\mathbf{r} = -\big(U(\mathbf{r}_B) - U(\mathbf{r}_A)\big), \qquad (31)$$

where $U(\mathbf{r}_A)$ and $U(\mathbf{r}_B)$ denote the values of U at the points A and B, respectively. The scalar function $U(\mathbf{r})$ is a scalar field called a **potential field** or simply a **potential** of the vector field $\mathbf{F}$.

When the vector field $\mathbf{F}$ is a force acting on a particle, the potential field U is the potential energy of the particle. Note that only *differences* of potential are defined by equation (31). As in one dimension, we can choose the value of the potential field to be zero at any convenient point (the *datum*). In this case, the datum is the origin O.

The potential energy of a particle was discussed in Unit 9.

Example 13

A potential of the two-dimensional field $\mathbf{F} = 2x\mathbf{i} - y\mathbf{j}$ is

$$U(\mathbf{r}) = U(x, y) = -(x^2 - \tfrac{1}{2}y^2).$$

Determine the line integral of $\mathbf{F}$ along a path from the origin to the point $(-3, 7)$.

Solution

Using equation (31), the line integral of $\mathbf{F}$ is given by

$$-\big(U(-3,7) - U(0,0)\big) = -\big(-((-3)^2 - \tfrac{1}{2} \times 7^2) - 0\big) = -\tfrac{31}{2}.$$

We can show that any scalar field U is a potential of the vector field $-\mathbf{grad}\, U$, by evaluating the line integral $\int_C (-\mathbf{grad}\, U) \cdot d\mathbf{r}$. We use the Cartesian form of $\mathbf{grad}\, U$,

$$\mathbf{grad}\, U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k},$$

in equation (25). Thus

$$\int_C (-\mathbf{grad}\, U) \cdot d\mathbf{r} = -\int_{t_0}^{t_1} \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} \right) dt.$$

The integrand is the expression for dU/dt as given by the chain rule. So we have

The chain rule was discussed in Unit 7.

$$\int_C (-\mathbf{grad}\, U) \cdot d\mathbf{r} = -\int_{t_0}^{t_1} \frac{dU}{dt}\, dt = -\big(U(t_1) - U(t_0)\big).$$

Notice again the abuse of notation. Since $\mathbf{r} = \mathbf{r}(t)$, $U(\mathbf{r})$ becomes $U(t)$.

Comparing this result with equation (31), you can see that the scalar field U is a potential of the vector field $\mathbf{F} = -\mathbf{grad}\, U$. If we now choose a datum at the origin, we have

$$\mathbf{F}(\mathbf{r}) = -\mathbf{grad}\, U \quad \text{and} \quad U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r}, \tag{32}$$

where $\mathbf{r}$ is the position vector of P, any point in space. These statements generalise, to two and three dimensions, the one-dimensional relationships

$$F(x) = -\frac{dU}{dx} \quad \text{and} \quad U(a) = -\int_0^a F(x)\, dx,$$

See Unit 9.

where a is any point on the x-axis.

Exercise 34

Given that the line integral of the field $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ is path-independent, find a potential function $U(x, y, z)$ for $\mathbf{F}$.

The approach taken in the solution to Exercise 34 is summarised in the following procedure.

Procedure 1 Determining the potential function from the line integral of a force

Suppose that we are given the force $\mathbf{F}(\mathbf{r})$ whose line integral is path-independent. To determine the potential function $U(\mathbf{r})$, with the datum set at the origin, carry out the following steps.

1. Take C to be the direct path from $(0, 0, 0)$ to the general point (a, b, c) parametrised by

$$\mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \quad (0 \le t \le 1).$$

2. With this choice of parametrisation, calculate the scalar line integral

$$U(a, b, c) = -\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt,$$

and write $U(\mathbf{r}) = U(x, y, z)$.

In Subsection 3.1 we considered the one-dimensional case of a force $F\mathbf{i}$ accelerating a particle along the x-axis, and showed that the work done by the force is equal to the change of kinetic energy of the particle. By combining equations (23) and (31), we can see that when F depends on position x, we are able to express the work done by the force as minus the change of potential energy. Furthermore, equation (21) shows that the work done equals the gain in kinetic energy. Thus for any force $F(x)\,\mathbf{i}$ that acts parallel to the x-axis and depends only on position x, the total mechanical energy (i.e. kinetic energy plus potential energy) is conserved.

Compare the discussion in Unit 9, Section 4.

We now have several links between line integrals of a force $\mathbf{F}$, path independence, the curl of $\mathbf{F}$, a potential for $\mathbf{F}$, and conservation of mechanical energy. In the next subsection we tie all these ideas together with various definitions of a conservative vector field.

4.2 Conservative vector fields

We classify vector fields according to whether their line integrals are path-dependent or path-independent. Those vector fields for which *all* the line integrals between *all* pairs of points are path-independent are called **conservative fields**. If there is at least one pair of points for which the line integrals of the vector field are path-dependent, then the vector field is *non-conservative*.

Conservative fields: Property (a)

The line integral of a conservative vector field along a path between any two fixed points A and B depends only on the starting point A and the endpoint B of the path. It is independent of the actual path taken between the two points.

Note that in one dimension, where the particle is restricted to the x-axis, the only possibility for varying the path between two fixed points $x = a$ and $x = b$ is when the particle overshoots into the regions $x > b$ or $x < a$ (see Figure 7). But then the particle always has to reverse back through these regions to reach $x = a$ or $x = b$, so the net contribution to the line integral from these forward and reverse motions vanishes. So all one-dimensional vector fields $F(x)\,\mathbf{i}$ that depend only on position x are conservative.

It follows from Property (a) that if you are asked to evaluate the line integral of a conservative field between any two given points, then you are free to choose the path that produces the simplest possible evaluation.

Example 14

Given that the vector field

$$\mathbf{F} = 2x\mathbf{i} - y\mathbf{j}$$

This field was considered in Example 13.

is a conservative field, evaluate the line integral of $\mathbf{F}$ from the origin to the point $(2, 0)$.

Solution

We are free to choose the path for a conservative field. The simplest path to take is the segment of the x-axis from $x = 0$ to $x = 2$. This path can be parametrised by

$$x = t, \quad y = 0 \quad (0 \le t \le 2),$$

so $dx/dt = 1$ and $dy/dt = 0$. The components of $\mathbf{F}$ on the path are $F_1 = 2x = 2t$ and $F_2 = -y = 0$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 2t\,dt = 4.$$

The line integral of a conservative field around any closed curve is zero. To see this, consider the loop in Figure 14. The two points A and B divide the loop into two segments, an upper segment APB and a lower segment AQB. We know that the line integral of any conservative field $\mathbf{F}$ from A to B has the same value for both segments, that is,

$$\int_{APB} \mathbf{F} \cdot d\mathbf{r} = \int_{AQB} \mathbf{F} \cdot d\mathbf{r}. \tag{33}$$

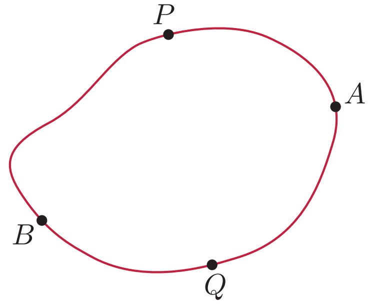

Figure 14 Closed path

Now the line integral for one complete anticlockwise traversal of the loop, starting and ending at A, consists of the line integral along APB plus that along BQA, that is,

$$\oint_{APBQA} \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} + \int_{BQA} \mathbf{F} \cdot d\mathbf{r}.$$

But we know from equation (28) that

$$\int_{BQA} \mathbf{F} \cdot d\mathbf{r} = -\int_{AQB} \mathbf{F} \cdot d\mathbf{r}.$$

Hence

$$\oint_{APBQA} \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} - \int_{AQB} \mathbf{F} \cdot d\mathbf{r} = 0,$$

by equation (33). A similar argument applies to any other loop in the domain of $\mathbf{F}$.

This gives us another important property of a conservative field.

Conservative fields: Property (b)

The line integral of any conservative field $\mathbf{F}$ around *any closed curve* C is zero, that is,

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0. \qquad (34)$$

Note that the line integral of a conservative field must be zero for *all possible loops* in its domain. If just one loop has a non-zero line integral, that is sufficient to make the field non-conservative.

Exercise 35

State which of the two vector fields $\mathbf{u}$ and $\mathbf{v}$ in Exercise 29 is definitely non-conservative.

How can we tell whether or not a given vector field is conservative? To use Property (a), we should need to know that all the line integrals between all possible pairs of fixed points are path-independent. It is obviously not practicable to evaluate all possible line integrals. Alternatively, we could think of using Property (b), but this would seem to require us to evaluate the line integrals around all possible loops! What we need is a simple test for a conservative field. What about the result involving curl that we illustrated via equation (30)? Let us see how this relates to conservative fields. We know that all line integrals around closed loops are zero if $\mathbf{F}$ is a conservative field. Hence from equation (30) and the corresponding x- and y-components, $\mathbf{curl}\,\mathbf{F}$ must be zero for any conservative field.

Conservative fields: Property (c)

If $\mathbf{F}$ is a conservative field, then $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere.

Property (c) holds for all conservative fields, but it does not provide us with a sufficient test for conservative fields. Perhaps there are vector fields $\mathbf{F}$ with $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ that are non-conservative? However, it can be shown that the converse of Property (c) is also true *provided* that the domain of $\mathbf{F}$ is *simply-connected*. Hence we have the following 'curl test' to determine if a field is conservative.

Curl test for a conservative field

If $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere, then $\mathbf{F}$ is a conservative field provided that the domain of $\mathbf{F}$ is simply-connected.

Thus it turns out that a conservative field is always irrotational, but conversely, an irrotational field is not necessarily conservative unless its domain is simply-connected. We will not prove this converse statement, but we will show in an example (in Exercise 40) where the curl test fails due to the domain being non-simply-connected. For all other cases in this module, you may assume that the curl test is a sufficient test for a conservative field.

We can use the curl test to show that if $\mathbf{F} = -\mathbf{grad}\,U$ for some scalar field U, then $\mathbf{F}$ is a conservative field. This follows from Exercise 12(d), where you found that $\mathbf{curl}\,(\mathbf{grad}\,f) \equiv \mathbf{0}$ for any differentiable scalar field f.

Conservative fields: Property (d)

If a vector field is a gradient, then it is conservative. Conversely, any conservative vector field can be expressed as a gradient of a scalar field.

Using equations (32), we can express this last property in terms of potentials.

Conservative fields: Property (e)

For a conservative field $\mathbf{F}$ and fixed origin O (the datum), there exists a unique potential field defined by

$$U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r}$$

with $\mathbf{F} = -\mathbf{grad}\,U$, where $\overrightarrow{OP} = \mathbf{r}$.

For a different datum, the potential field will differ from U by a constant.

Property (e) tells us that we can determine line integrals of a conservative field simply by finding its corresponding potential field and using equation (31), as we did in Example 13. What is more, Property (e) combined with equation (21) shows that mechanical energy, being the sum of kinetic and potential energies, is conserved when a particle is acted on by a conservative force field.

Exercise 36

(a) Use the curl test to determine which of the following vector fields are conservative:

$$\begin{aligned}&\mathbf{h} = x\mathbf{i} + y\mathbf{j},\\ &\mathbf{u} = \omega(-y\mathbf{i} + x\mathbf{j}), \quad \text{where } \omega \text{ is a positive constant},\\ &\mathbf{F} = 2x\mathbf{i} + (xz - 2)\mathbf{j} + xy\mathbf{k}.\end{aligned}$$

(b) Show that $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$ for $\mathbf{G} = kr^n\,\widehat{\mathbf{r}}$, where k and n are non-zero constants, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}|$ and $\widehat{\mathbf{r}} = \mathbf{r}/r$ $(r \neq 0)$.

The vector field $\mathbf{G} = kr^n\,\widehat{\mathbf{r}}$ (for any n or k) in Exercise 36(b) is directed radially towards or away from the origin depending on the sign of the constant k. Such fields are called **central fields**. In this case, the magnitude of the central field depends only on the distance from the origin r, that is, it is spherically symmetric, and a general expression for a force of this type is given by $\mathbf{F} = f(r)\,\widehat{\mathbf{r}}$ (as in Exercise 20).

The domain of $\mathbf{G}$ is simply-connected, even for $n < 0$, since although $\mathbf{G}$ cannot be defined at the origin when $n < 0$, we know that the space $\mathbb{R}^3$ excluding the origin is still simply-connected. You have found that $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$, so it follows that $\mathbf{G}$ is conservative for all n. Indeed, since we know from Exercise 20 that $\mathbf{F}$ is irrotational, it follows that all spherically symmetric central fields $\mathbf{F}$ are conservative. For example, the force of gravity near a large spherical object such as a planet or a star is always directed towards the centre of the sphere, with magnitude depending only on the distance to the centre of the sphere. Thus the force of gravity is a spherically symmetric central field and is therefore conservative. We can use the law of conservation of mechanical energy for conservative forces for any problem involving gravitational forces only – for example, the motion of planets around the Sun or the motion of artificial Earth satellites.

We can express the law of conservation of mechanical energy as

$$\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 = U(\mathbf{r}_A) - U(\mathbf{r}_B), \tag{35}$$

This equation is a combination of equations (31) and (21), and we have used $v_A^2 = \mathbf{v}_A \cdot \mathbf{v}_A$ and $v_B^2 = \mathbf{v}_B \cdot \mathbf{v}_B$.

where the left-hand side is the change in kinetic energy when the body moves from A to B, with position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, and the right-hand side is minus the change in potential energy. Equation (35) can be rewritten as

$$\tfrac{1}{2}mv_A^2 + U(\mathbf{r}_A) = \tfrac{1}{2}mv_B^2 + U(\mathbf{r}_B),$$

which tells us that the total mechanical energy,

$$E = \tfrac{1}{2}mv^2 + U(\mathbf{r}),$$

is a constant throughout the motion, provided that the force is conservative. This generalises the law of conservation of energy, given in Unit 9, Section 4, to motion in more than one dimension.

Exercise 37

The gravitational potential energy field of a body of mass m in the vicinity of the Earth is given by

$$U(\mathbf{r}) = mgR\left(1 - \frac{R}{|\mathbf{r}|}\right) \quad (|\mathbf{r}| \geq R),$$

where the position vector $\mathbf{r}$ of the body is measured from the centre of the Earth, g is the magnitude of the acceleration due to gravity at the Earth's surface, R is the Earth's radius, and the datum is on the Earth's surface. Derive an expression for the Earth's gravitational force $\mathbf{F}(\mathbf{r})$ on the body.

Example 15

An Earth satellite moves in an elliptical orbit with its nearest and furthest points from the Earth's surface being $\frac{1}{2}R$ and $2R$, where R is the Earth's radius. If its slowest speed is u, find its fastest speed. Assume that the only force acting on the satellite is the Earth's gravity.

Solution

Gravity is a conservative field, so mechanical energy is conserved throughout the motion of the satellite. The potential energy function U is given in Exercise 37. We can use the law of conservation of mechanical energy, equation (35), with A the furthest point reached by the satellite, $|\mathbf{r}_A| = 3R$, and B the nearest point, $|\mathbf{r}_B| = \frac{3}{2}R$, measured from the centre of the Earth. Then the speed at A is given as $v_A = u$, and we need to find v_B, the speed at B. Thus

$$\begin{aligned}
\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mu^2 &= U(\mathbf{r}_A) - U(\mathbf{r}_B) \\
&= mgR\left(\left(1 - \frac{R}{3R}\right) - \left(1 - \frac{2R}{3R}\right)\right) \\
&= \tfrac{1}{3}mgR.
\end{aligned}$$

Hence the fastest speed is

$$v_B = \sqrt{u^2 + 2gR/3}.$$

All spherically symmetric central fields are conservative, but not all conservative fields are central fields. Consider, for example, the two-dimensional vector field $\mathbf{F}(\mathbf{r}) = -2px\mathbf{i} - py\mathbf{j}$, where p is a constant. The vector $\mathbf{F}(\mathbf{r})$ at any point is not, in general, parallel to the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ at that point, so the field is not a central field. However, this vector field is conservative, as you can confirm in the next exercise.

Exercise 38

The force acting on a particle is given by

$$\mathbf{F} = -2px\mathbf{i} - py\mathbf{j},$$

where p is a constant.

(a) Show that the force is a conservative field.

(b) Confirm that $U(x, y) = p(x^2 + \frac{1}{2}y^2)$ is a potential energy function for $\mathbf{F}$, and hence find the work done by the force, in terms of p, when the particle moves from $(5, 0)$ to $(0, 5)$.

Finally, it is useful to gather together the properties of conservative fields.

Properties of conservative fields

Let $\mathbf{F}$ be a conservative vector field.

(a) All line integrals of $\mathbf{F}$ between any two fixed points in the domain are path-independent.

(b) The line integrals of $\mathbf{F}$ around all closed curves in the domain are zero.

(c) The curl of $\mathbf{F}$ is zero everywhere in the domain.

(d) All gradient fields are conservative, and a conservative vector field can be expressed as the gradient of a scalar field.

(e) For a fixed origin O (the datum), there exists a unique potential field defined by

$$U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r}$$

with $\mathbf{F} = -\mathbf{grad}\, U$, where $\overrightarrow{OP} = \mathbf{r}$.

We have taken statement (a) as the definition of a conservative field, but statements (b) and (e) are entirely equivalent to statement (a) and may alternatively be taken as the definition. Statement (c) is also equivalent to each of statements (a), (b) and (e) *provided* that the domain of $\mathbf{F}$ is simply-connected.

Exercise 39

Show that any vector field of the form $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$, where f is a function of x only, and g is a function of y only, is a conservative field. Can a vector field of the form $\mathbf{H} = g(y)\,\mathbf{i} + f(x)\,\mathbf{j}$ be conservative?

Exercise 40

Consider the two-dimensional vector field

$$\mathbf{B}(x,y) = \frac{k(-y\mathbf{i} + x\mathbf{j})}{x^2+y^2} \quad (x^2+y^2>0),$$

where k is a constant.

(a) Show that

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = 2\pi k,$$

where C is a circle of radius a with centre at the origin.

(b) Show that $\mathbf{curl}\,\mathbf{B} = \mathbf{0}$.

Learning outcomes

After studying this unit, you should be able to:

- calculate the divergence of a vector field
- calculate the curl of a vector field
- appreciate the use of gradient, divergence and curl to model simple laws and problems involving heat flow, fluid flow and local rotation in a vector field
- evaluate scalar line integrals of vector fields
- solve simple problems in one, two and three dimensions involving changes of kinetic energy and potential energy, and the work done by a force
- use the curl test for identifying conservative fields.

Solutions to exercises

Solution to Exercise 1

The components of $\mathbf{F}$ are

$$F_1 = xy, \quad F_2 = 0, \quad F_3 = -yz.$$

The relevant partial derivatives are

$$\frac{\partial F_1}{\partial x} = y, \quad \frac{\partial F_2}{\partial y} = 0, \quad \frac{\partial F_3}{\partial z} = -y.$$

Hence $\operatorname{div}\mathbf{F} = y + 0 + (-y) = 0$.

The scalar field $\operatorname{div}\mathbf{F}$ is zero everywhere, so $\operatorname{div}\mathbf{F}(3,-1,2) = 0$.

Solution to Exercise 2

The components of $\mathbf{F}$ are $F_1 = xy$, $F_2 = yz$ and $F_3 = zx$, so

$$\frac{\partial F_1}{\partial x} = y, \quad \frac{\partial F_2}{\partial y} = z, \quad \frac{\partial F_3}{\partial z} = x.$$

Thus $\boldsymbol{\nabla}\cdot\mathbf{F} = y + z + x$ and $\boldsymbol{\nabla}\cdot\mathbf{F}(1,2,3) = 2 + 3 + 1 = 6$.

Solution to Exercise 3

The position vector in Cartesian coordinates is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so

$$\begin{aligned}\widehat{\mathbf{r}} &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}},\\ \mathbf{F}(x,y,z) &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{(n+1)/2}}.\end{aligned}$$

The x-component is

$$F_1 = \frac{x}{(x^2+y^2+z^2)^{(n+1)/2}},$$

and

$$\frac{\partial F_1}{\partial x} = \frac{(x^2+y^2+z^2)^{(n+1)/2} - x^2(n+1)(x^2+y^2+z^2)^{(n-1)/2}}{(x^2+y^2+z^2)^{n+1}}.$$

The component F_2 and the partial derivative $\partial F_2/\partial y$ are obtained from F_1 and $\partial F_1/\partial x$ by interchanging x and y. Similarly, F_3 and $\partial F_3/\partial z$ are obtained by interchanging x and z. When this is done, we obtain

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{3(x^2+y^2+z^2)^{(n+1)/2} - (x^2+y^2+z^2)(n+1)(x^2+y^2+z^2)^{(n-1)/2}}{(x^2+y^2+z^2)^{n+1}}\\ &= \frac{(3-(n+1))(x^2+y^2+z^2)^{(n+1)/2}}{(x^2+y^2+z^2)^{n+1}}\\ &= \frac{2-n}{(x^2+y^2+z^2)^{(n+1)/2}}.\end{aligned}$$

Hence $\operatorname{div}\mathbf{F} = 0$ everywhere only when $n = 2$.

Solution to Exercise 4

We have

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j},$$

so

$$\frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\cos\phi\,\mathbf{i} - \sin\phi\,\mathbf{j} = -\mathbf{e}_\rho.$$

Solution to Exercise 5

In spherical coordinates, $\mathbf{r} = r\mathbf{e}_r$. Hence, using equation (3),

$$\operatorname{div}\mathbf{r} = \frac{\partial r}{\partial r} + \frac{2r}{r} = 3.$$

This should be compared to the result given in Example 1(a) for $\operatorname{div}\mathbf{r}$, which was obtained using Cartesian coordinates.

Solution to Exercise 6

The heat flow vector field $\mathbf{J}$ has cylindrical symmetry. Anywhere on the outer surface, $\mathbf{J}$ is directed radially outwards and has magnitude

$$|\mathbf{J}| = A\sqrt{x^2+y^2} = 2\times 10^6 \times 0.02 = 4\times 10^4.$$

The total surface area (in m^2) of a one-metre length of rod is

$$2\pi \times 0.02 \times 1 = 4\pi \times 10^{-2}.$$

Hence the total outward heat flow rate from a one-metre length is given (in watts) by

$$(4\times 10^4)\times(4\pi\times 10^{-2}) = 1600\pi,$$

that is, the heat flow rate is about 5 kW.

This is also the rate of heat generated in the one-metre length, since the heat flow is steady. Alternatively,

$$\begin{aligned} S \times \text{volume} &= (4\times 10^6)\times \pi(0.02)^2 \times 1 \\ &= 1600\pi. \end{aligned}$$

Solution to Exercise 7

In cylindrical coordinates, $\rho = \sqrt{x^2+y^2}$ and $\mathbf{e}_\rho = (x\mathbf{i}+y\mathbf{j})/\sqrt{x^2+y^2}$. Hence $\mathbf{J}(\rho,\phi,z) = A\rho\,\mathbf{e}_\rho$, and the cylindrical components of $\mathbf{J}$ are $J_\rho = A\rho$, $J_\phi = J_z = 0$. The only non-zero partial derivative is $\partial J_\rho/\partial\rho = A$. Hence, using equation (2), we have

$$\operatorname{div}\mathbf{J} = A + \frac{1}{\rho}A\rho = 2A,$$

which is in agreement with the result of Example 3.

Solution to Exercise 8

(a) The components of $\mathbf{N}$ are (in $\mathrm{W\,m^{-2}}$)

$$N_1 = N_2 = 0, \quad N_3 = 5000,$$

so

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_2}{\partial y} = \frac{\partial N_3}{\partial z} = 0.$$

Thus $\operatorname{div}\mathbf{N} = 0$.

(b) The only non-zero component of $\mathbf{N}$ is $N_3 = 5000e^{-\alpha z}$, so

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_2}{\partial y} = 0, \quad \frac{\partial N_3}{\partial z} = 5000(-\alpha)e^{-\alpha z}.$$

Hence $\operatorname{div}\mathbf{N} = -5000\alpha e^{-\alpha z}$.

Solution to Exercise 9

The only non-zero component of $\mathbf{f}$ is $f_\rho = \rho^{-n}$, so $\partial f_\rho/\partial\rho = -n\rho^{-(n+1)}$. From equation (2), we obtain

$$\operatorname{div}\mathbf{f} = \frac{-n}{\rho^{n+1}} + \frac{1}{\rho}\frac{1}{\rho^n} = \frac{1-n}{\rho^{n+1}}.$$

This is zero everywhere only when $n = 1$.

Solution to Exercise 10

We have

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= 0 + 0 + 0 = 0,\\
\operatorname{div}\mathbf{G} &= 2x + 2y + 2z,\\
\operatorname{div}\mathbf{H} &= 1 + 1 - 2 = 0.
\end{aligned}$$

So $\mathbf{F}$ and $\mathbf{H}$ could be magnetic fields, but $\mathbf{G}$ could not, since the divergence of a magnetic field is always zero.

Solution to Exercise 11

(a) The components of $\mathbf{F}(x,y,z) = x(y-z)\mathbf{i} + 3x^2\mathbf{j} + yz\mathbf{k}$ are $F_1 = x(y-z)$, $F_2 = 3x^2$ and $F_3 = yz$.

The partial derivatives required for substitution in equation (14) are

$$\begin{aligned}
&\frac{\partial F_1}{\partial y} = x, \quad \frac{\partial F_1}{\partial z} = -x,\\
&\frac{\partial F_2}{\partial x} = 6x, \quad \frac{\partial F_2}{\partial z} = 0,\\
&\frac{\partial F_3}{\partial x} = 0, \quad \frac{\partial F_3}{\partial y} = z.
\end{aligned}$$

Hence, using equation (14),

$$\mathbf{curl}\,\mathbf{F} = z\mathbf{i} - x\mathbf{j} + 5x\mathbf{k}.$$

(b) (i) At the origin, $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$.

(ii) At $(1, 2, 3)$, $\mathbf{curl}\,\mathbf{F} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

(iii) At $(0, 0, 5)$, $\mathbf{curl}\,\mathbf{F} = 5\mathbf{i}$.

Solution to Exercise 12

(a) The components of $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ are

$$F_1 = x, \quad F_2 = y, \quad F_3 = z,$$

so

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial z} = \frac{\partial F_2}{\partial x} = \frac{\partial F_3}{\partial y} = \frac{\partial F_3}{\partial x} = 0.$$

Hence, substituting into equation (14),

$$\mathbf{curl}\,\mathbf{F} = \mathbf{0}.$$

(b) The components of $\mathbf{F} = f(r)\,\widehat{\mathbf{r}} = (f(r)/r)\mathbf{r}$ are

$$F_1 = f(r)\,\frac{x}{r}, \quad F_2 = f(r)\,\frac{y}{r}, \quad F_3 = f(r)\,\frac{z}{r}.$$

The x-component of $\mathbf{curl}\,\mathbf{F}$ (using the chain rule) is

$$\begin{aligned}
\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= z\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial y} - y\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial z} \\
&= z\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{y}{r} - y\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{z}{r} \\
&= 0.
\end{aligned}$$

Note that we have used $\partial r/\partial y = y(x^2 + y^2 + z^2)^{-1/2} = y/r$ and $\partial r/\partial z = z/r$. Similarly, the y- and z-components of $\mathbf{curl}\,\mathbf{F}$ are also zero, so

$$\mathbf{curl}\,(f(r)\,\widehat{\mathbf{r}}) = \mathbf{0}$$

for all functions f of r only.

(c) We have

$$\mathbf{F} = (y^2 + 2z)\mathbf{i} + (xy + 6z)\mathbf{j} + (z^2 + 2xz + y)\mathbf{k},$$

so the x-, y- and z-components of $\mathbf{curl}\,\mathbf{F}$ are

$$\begin{aligned}
\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= 1 - 6 = -5, \\
\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= 2 - 2z = 2(1 - z), \\
\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= y - 2y = -y,
\end{aligned}$$

thus

$$\mathbf{curl}\,\mathbf{F} = -5\mathbf{i} + 2(1 - z)\mathbf{j} - y\mathbf{k}.$$

(d) Since

$$\mathbf{F} = \mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

we have

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y} = 0.$$

Similarly, the y- and z-components are zero. Hence

$$\mathbf{curl}\,(\mathbf{grad}\, f) \equiv \mathbf{0}.$$

Solution to Exercise 13

With $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, using equation (14) to determine $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$, we find

$$B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \quad B_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \quad B_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

Hence

$$\begin{aligned}
\operatorname{div} \mathbf{B} &= \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \\
&= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y} \\
&= 0,
\end{aligned}$$

where we have used the mixed derivative theorem for second-order partial derivatives (Unit 7, Subsection 2.2), which assumes that $\mathbf{A}$ (and therefore A_1, A_2 and A_3) is sufficiently smooth.

Solution to Exercise 14

Any vector field having zero curl everywhere is an irrotational field. The fields in parts (a), (b) and (d) of Exercise 12 and part (a)(i) of Example 4 are irrotational.

Solution to Exercise 15

The field $\mathbf{F}$ is confined to the (x, y)-plane, so

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x))\right)\mathbf{k} = \mathbf{0}.$$

Solution to Exercise 16

Both $\mathbf{u}$ and $\mathbf{v}$ are two-dimensional fields in the (x, y)-plane.

For the field $\mathbf{u}$,

$$u_1 = -\omega y \quad \text{and} \quad u_2 = \omega x,$$

so

$$\frac{\partial u_2}{\partial x} = \omega, \quad \frac{\partial u_1}{\partial y} = -\omega.$$

Hence

$$\mathbf{curl}\,\mathbf{u} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\mathbf{k} = 2\omega\mathbf{k}.$$

For the field $\mathbf{v}$,

$$v_1 = -\frac{\omega a^2 y}{x^2+y^2} \quad \text{and} \quad v_2 = \frac{\omega a^2 x}{x^2+y^2},$$

so

$$\frac{\partial v_2}{\partial x} = \frac{(x^2+y^2)\omega a^2 - \omega a^2 x(2x)}{(x^2+y^2)^2} = \frac{\omega a^2(y^2-x^2)}{(x^2+y^2)^2},$$

$$\frac{\partial v_1}{\partial y} = \frac{(x^2+y^2)(-\omega a^2) + \omega a^2 y(2y)}{(x^2+y^2)^2} = \frac{\omega a^2(y^2-x^2)}{(x^2+y^2)^2}.$$

Hence

$$\mathbf{curl}\,\mathbf{v} = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Note that at the boundary of the central region, $\mathbf{curl}\,\mathbf{u} \neq \mathbf{curl}\,\mathbf{v}$.

Solution to Exercise 17

In cylindrical coordinates we have

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

Hence

$$\mathbf{u}(\rho,\phi,z) = -\rho\omega\sin\phi\,\mathbf{i} + \rho\omega\cos\phi\,\mathbf{j} = \omega\rho\,\mathbf{e}_\phi \quad (\rho \le a),$$

and since $x^2+y^2=\rho^2$,

$$\mathbf{v}(\rho,\phi,z) = \frac{a^2}{\rho^2}\,\omega\rho\,\mathbf{e}_\phi = \frac{\omega a^2}{\rho}\,\mathbf{e}_\phi \quad (\rho \ge a).$$

The vector field lines of $\mathbf{u}$ and $\mathbf{v}$ are circles.

Solution to Exercise 18

(a) Using a Cartesian coordinate system with the z-axis lying on the axis of the Earth's rotation and in the direction of the angular velocity, we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence

$$\mathbf{v}(x,y,z) = \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \omega(x\mathbf{j} - y\mathbf{i}).$$

(b) Now $v_1 = -\omega y$, $v_2 = \omega x$ and $v_3 = 0$, so using equation (15) we have

$$\mathbf{curl}\,\mathbf{v} = \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right)\mathbf{k} = 2\omega\mathbf{k} = 2\boldsymbol{\omega}.$$

Hence

$$|\mathbf{curl}\,\mathbf{v}| = 2\omega = 2 \times \frac{2\pi}{24\times 60\times 60} = 1.454\times 10^{-4}.$$

(c) Every object fixed on the Earth (including your home) rotates with angular velocity $\boldsymbol{\omega}$. So from part (b), the rate of rotation (in $\mathrm{rad\,s^{-1}}$) is

$$|\boldsymbol{\omega}| = \tfrac{1}{2}|\mathbf{curl}\,\mathbf{v}| = 7.272\times 10^{-5}.$$

Solution to Exercise 19

Note that $\mathbf{F}$ has no z-component and no z-dependence in the other components, so it is a two-dimensional vector field. Hence we can use equation (18) to determine $\mathbf{curl}\,\mathbf{F}$, from which, with $F_\rho = f(\rho)$ and $F_\phi = \rho^n g(\phi)$, we have

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \left(\frac{\partial F_\phi}{\partial \rho} + \frac{F_\phi}{\rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right)\mathbf{e}_z \\ &= \left(n\rho^{n-1}g(\phi) + \frac{\rho^n g(\phi)}{\rho} + 0\right)\mathbf{e}_z \\ &= (n+1)\rho^{n-1}g(\phi)\,\mathbf{e}_z.\end{aligned}$$

Since in general $g(\phi) \neq 0$, we have $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational) only when $n = -1$.

Solution to Exercise 20

In spherical coordinates, $\mathbf{F} = f(r)\,\mathbf{e}_r$ (since $\widehat{\mathbf{r}} = \mathbf{e}_r$), so $F_r = f(r)$, $F_\theta = F_\phi = 0$ and, since F_r does not depend on θ and ϕ, $\partial F_r/\partial\theta = \partial F_r/\partial\phi = 0$. Thus, from equation (17), it immediately follows that $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is irrotational.

Solution to Exercise 21

We have $A_r = A_\theta = 0$ and $A_\phi = \psi(r,\theta)/(r\sin\theta)$. Writing $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ and using equation (17) for $\mathbf{F} = \mathbf{curl}\,\mathbf{A}$, for each spherical component in $\mathbf{F}$ we find

$$\begin{aligned}F_r &= \frac{1}{r}\frac{\partial A_\phi}{\partial\theta} + \frac{\cot\theta}{r}A_\phi \\ &= \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\psi}{\sin\theta}\right) + \frac{\psi\cot\theta}{r^2\sin\theta} \\ &= \frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta} - \psi\cos\theta + \psi\cos\theta\right) = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad (36)\\ F_\theta &= -\left(\frac{\partial A_\phi}{\partial r} + \frac{1}{r}A_\phi\right) \\ &= -\frac{1}{\sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{\psi}{r}\right) + \frac{\psi}{r^2}\right) \\ &= -\frac{1}{r^2\sin\theta}\left(r\frac{\partial\psi}{\partial r} - \psi + \psi\right) = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \qquad (37)\\ F_\phi &= 0.\end{aligned}$$

Since $\mathbf{F} = \mathbf{e}_r/r^2$, we require $F_r = 1/r^2$, which implies from equation (36) that $\partial\psi/\partial\theta = \sin\theta$. Also, we require $F_\theta = 0$, which implies from equation (37) that $\partial\psi/\partial r = 0$. Thus we have the required partial derivatives

$$\frac{\partial\psi}{\partial r} = 0, \quad \frac{\partial\psi}{\partial\theta} = \sin\theta.$$

The first of these partial derivatives implies that $\psi(r,\theta)$ is a function of θ only, which we write as $\psi(r,\theta) = f(\theta)$. The second partial derivative then implies that $f'(\theta) = \sin\theta$, which is integrated to give $f(\theta) = -\cos\theta + c$, where c is an arbitrary constant. Hence the required expression for $\psi(r,\theta)$.

Note that using the result found in Exercise 13, since $\mathbf{F} = \mathbf{curl}\,\mathbf{A}$, it immediately follows that $\operatorname{div}\mathbf{F} = 0$ everywhere (in the domain of $\mathbf{A}$), which is consistent with the findings of Exercise 3 and Example 2 (for $n = 2$).

Solution to Exercise 22

The work done is

$$W = -A\int_3^1 \frac{1}{x^2}\,dx = -A\left[-\frac{1}{x}\right]_3^1 = A\left(1-\tfrac{1}{3}\right) = \tfrac{2}{3}A.$$

The change in kinetic energy is $\frac{2}{3}A$, so $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{2}{3}A$ and

$$v = \left(\tfrac{4}{3}(A/m) + u^2\right)^{1/2}.$$

Solution to Exercise 23

We have $dx/dt = -2\sin t$, $dy/dt = 2\cos t$, $F_1 = 2x + y = 4\cos t + 2\sin t$, $F_2 = -x = -2\cos t$. Hence

$$\begin{aligned}\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/2}\big((4\cos t + 2\sin t)(-2\sin t) + (-2\cos t)2\cos t\big)\,dt\\ &= \int_0^{\pi/2}\left(-8\sin t\cos t - 4(\sin^2 t + \cos^2)\right)dt\\ &= \int_0^{\pi/2}(-4\sin(2t) - 4)\,dt\\ &= \left[2\cos(2t) - 4t\right]_0^{\pi/2} = -4 - 2\pi.\end{aligned}$$

Note that for the third equality, we used the trigonometric identities $2\sin t\cos t = \sin 2t$ and $\sin^2 t + \cos^2 t = 1$.

Solution to Exercise 24

Put $x = t$. Then $y = 1 - t^2$, with t going from 1 to 0. The components of the vector field $\mathbf{F}$ are $F_1 = x^2 = t^2$ and $F_2 = xy = t(1-t^2)$. The derivatives are $dx/dt = 1$ and $dy/dt = -2t$. Hence

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_1^0\left(t^2 - 2t^2(1-t^2)\right)dt\\ &= \int_1^0 (2t^4 - t^2)\,dt\\ &= \left[\tfrac{2}{5}t^5 - \tfrac{1}{3}t^3\right]_1^0 = -\left(\frac{6-5}{15}\right) = -\tfrac{1}{15}.\end{aligned}$$

Solution to Exercise 25

There are infinitely many paths between the points $(0, 0, 0)$ and $(1, 1, 1)$, so we choose two examples. The first moves from the origin along the x-axis to $(1, 0, 0)$, then parallel to the y-axis to $(1, 1, 0)$, and finally parallel to the z-axis to $(1, 1, 1)$. We can parametrise this as

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 1, \\ \mathbf{i} + (t-1)\mathbf{j}, & 1 \leq t \leq 2, \\ \mathbf{i} + \mathbf{j} + (t-2)\mathbf{k}, & 2 \leq t \leq 3. \end{cases}$$

Another alternative is the straight line joining the two points, which can be parametrised much more easily, as

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \quad (0 \leq t \leq 1).$$

Solution to Exercise 26

With $x = 2(1-t)$, $y = 2t$ and $z = 0$, we have $dx/dt = -2$, $dy/dt = 2$, $dz/dt = 0$, and $F_1 = 2x + y = 4 - 2t$, $F_2 = -x = -2(1-t)$. Hence

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \big((4-2t)(-2) - 2(1-t)2\big)\, dt \\ &= \int_0^1 (8t - 12)\, dt = \big[4t^2 - 12t\big]_0^1 = -8, \end{aligned}$$

which is the same value as that found in Example 9.

Solution to Exercise 27

In Exercise 23, C_1 is parametrised by $x = 2\cos t$, $y = 2\sin t$. Thus $dx/dt = -2\sin t$, $dy/dt = 2\cos t$. Also, $G_1 = x^2 = 4\cos^2 t$, $G_2 = y = 2\sin t$. Hence

$$\begin{aligned} \int_{C_1} \mathbf{G} \cdot d\mathbf{r} &= \int_0^{\pi/2} \big((4\cos^2 t)(-2\sin t) + 2\sin t(2\cos t)\big)\, dt \\ &= \int_0^{\pi/2} \big(-8\cos^2 t \sin t + 2\sin(2t)\big)\, dt \\ &= \big[\tfrac{8}{3}\cos^3 t - \cos(2t)\big]_0^{\pi/2} = -\tfrac{2}{3}. \end{aligned}$$

In Example 9, C_2 is parametrised by $x = t$, $y = 2 - t$. Thus $dx/dt = 1$, $dy/dt = -1$. Also, $G_1 = x^2 = t^2$, $G_2 = y = 2 - t$. Hence

$$\int_{C_2} \mathbf{G} \cdot d\mathbf{r} = \int_2^0 \big(t^2 + (2-t)(-1)\big)\, dt = \big[\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 - 2t\big]_2^0 = -\tfrac{2}{3}.$$

The closed curve $APBQA$ is the path C_1 plus the reverse of path C_2. Hence the scalar line integral of $\mathbf{G}$ along the closed curve $APBQA$ is $-\frac{2}{3} - (-\frac{2}{3}) = 0$.

Solution to Exercise 28

We have

$$x = t, \quad y = t^2, \quad z = t^3, \quad t_0 = 0, \quad t_1 = 1,$$
$$F_1 = 2t, \quad F_2 = t^4 - 2, \quad F_3 = t^3,$$
$$dx/dt = 1, \quad dy/dt = 2t, \quad dz/dt = 3t^2,$$

so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \big(2t + (t^4 - 2)(2t) + t^3(3t^2)\big)\, dt \\ &= \int_0^1 (5t^5 - 2t)\, dt \\ &= -\tfrac{1}{6}.\end{aligned}$$

Solution to Exercise 29

(a) $\displaystyle\oint_C \mathbf{u} \cdot d\mathbf{r} = \int_0^{2\pi} \big((a\cos t/a^2)(-a\sin t) + (a\sin t/a^2)(a\cos t)\big)\, dt = 0.$

(b) $$\begin{aligned}\oint_C \mathbf{v} \cdot d\mathbf{r} &= \omega \int_0^{2\pi} \big((-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)\big)\, dt \\ &= \omega a^2 \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\ &= \omega a^2 \int_0^{2\pi} 1\, dt \\ &= 2\omega\pi a^2.\end{aligned}$$

Solution to Exercise 30

The x-axis is the line $y = 0$, so the vector field $\mathbf{v}$ on the x-axis is $\mathbf{v}(x, 0) = \omega x\mathbf{j}$, which has no component along the x-axis. Thus the tangential component of $\mathbf{v}$ along the line is zero. Hence the scalar line integral along the line segment on the x-axis is zero.

Solution to Exercise 31

(a) In these examples, the component of the vector field $\mathbf{F}$ in the direction tangential to the curve is constant.

Here the curve C is a segment of the x-axis, on which $\mathbf{i} \cdot \mathbf{F} = 0$, so the line integral is zero.

(b) Here the curve C is a segment of the z-axis, on which $\mathbf{k} \cdot \mathbf{F} = 5$, a constant, so the line integral is $5 \times (6 - 0) = 30$.

(c) The tangential component of $\mathbf{F}$ on the semicircle is $\mathbf{e}_\theta \cdot \mathbf{F}(3, \theta) = 3^2 = 9$, a constant, so the line integral is $9 \times 3\pi = 27\pi$.

Solution to Exercise 32

We have

$$\mathbf{r}(t) = t\,\mathbf{i} - \ln(\cos t)\,\mathbf{j},$$

so

$$\dot{\mathbf{r}}(t) = \mathbf{i} + \tan t\,\mathbf{j}.$$

Thus the required length is given by

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1+\tan^2 t}\,dt &= \int_0^{\pi/4} \sec t\,dt \\ &= \big[\ln(\sec t + \tan t)\big]_0^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln(1) \\ &= \ln(\sqrt{2}+1).\end{aligned}$$

In practice, although equation (29) is a very useful formula, the actual computations can involve complicated integrals.

Solution to Exercise 33

(a) The line integral was found in the solution to Exercise 29(b) to have the value $2\omega\pi a^2$. The area of the circle is πa^2, so the required limit is $2\omega\pi a^2/\pi a^2 = 2\omega$.

(b) From equation (14), $\mathbf{curl}\,\mathbf{v} = (\partial(\omega x)/\partial x - \partial(-\omega y)/\partial y)\mathbf{k} = 2\omega\mathbf{k}$, and $\mathbf{k}\cdot\mathbf{curl}\,\mathbf{v} = 2\omega$.

Solution to Exercise 34

We have $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$. Since the line integral is path-independent, take the direct path from $(0,0,0)$ to a general point (a,b,c) parametrised by

$$\mathbf{r}(t) = ta\mathbf{i} + tb\mathbf{j} + tc\mathbf{k} \quad (0 \le t \le 1),$$

so

$$\frac{d\mathbf{r}}{dt} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 (2atbtct\mathbf{i} + (at)^2ct\mathbf{j} + (at)^2bt\mathbf{k})\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_0^1 (2a^2bc + a^2bc + a^2bc)t^3\,dt \\ &= 4a^2bc\left[\tfrac{1}{4}t^4\right]_0^1 \\ &= a^2bc \\ &= U(0,0,0) - U(a,b,c).\end{aligned}$$

Setting the datum for the potential energy function at the origin, so that $U(0,0,0) = 0$, we can deduce that a potential function for $\mathbf{F}$ is

$$U(x,y,z) = -x^2yz.$$

Solution to Exercise 35

The field $\mathbf{v}$ is non-conservative, because its line integral around the closed loop C is non-zero.

We cannot yet say from the solution to Exercise 29 that the field $\mathbf{u}$ is conservative, since there may be loops on which the line integral of $\mathbf{u}$ is non-zero. (In fact, $\mathbf{u}$ *is* conservative.)

Solution to Exercise 36

(a) $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ everywhere, $\mathbf{curl}\,\mathbf{u} = 2\omega\mathbf{k}$, and

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz-2)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(2x)}{\partial z} - \frac{\partial(xy)}{\partial x}\right)\mathbf{j} \\ &\quad + \left(\frac{\partial(xz-2)}{\partial x} - \frac{\partial(2x)}{\partial y}\right)\mathbf{k} \\ &= -y\mathbf{j} + z\mathbf{k}.\end{aligned}$$

Thus field $\mathbf{h}$ is conservative, since $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ everywhere, and the other two fields are non-conservative, since $\mathbf{curl}\,\mathbf{u} \neq \mathbf{0}$ and, in general, $\mathbf{curl}\,\mathbf{F} \neq \mathbf{0}$.

(b) This is just a special case of Exercise 20, where it was shown that a vector field of this type is irrotational. In this case $f(r)$ (in the notation of Exercise 20) is now given by $f(r) = kr^n$, and just as in Exercise 20, the simplest way to determine $\mathbf{curl}\,\mathbf{G}$ is to use the expression for curl in spherical coordinates after noting that $\widehat{\mathbf{r}} = \mathbf{e}_r$. Since $G_r = kr^n$ and $G_\theta = G_\phi = 0$, we have $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$.

Alternatively, we can use Cartesian coordinates as follows. Since $\mathbf{G} = kr^{n-1}\mathbf{r}$, we have $G_1 = kr^{n-1}x$, $G_2 = kr^{n-1}y$ and $G_3 = kr^{n-1}z$. The $\mathbf{i}$-component of $\mathbf{curl}\,\mathbf{G}$ is then

$$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = k(n-1)(zr^{n-2}y/r - yr^{n-2}z/r) = 0,$$

where we have used $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$. Similarly, the $\mathbf{j}$- and $\mathbf{k}$-components are zero, so $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$.

Note that it is much simpler to use spherical coordinates in this case, where we have been able to exploit the spherical symmetry of the field.

Solution to Exercise 37

Express $U(\mathbf{r})$ in Cartesian or spherical coordinates, and use $\mathbf{F} = -\mathbf{grad}\,U$ (the coordinate forms of $\mathbf{grad}\,U$ are given in Unit 15). It is easiest to use spherical coordinates, since U depends only on the distance $r = |\mathbf{r}|$. (It is a spherically symmetric scalar field.) Thus we have

$$U(r,\theta,\phi) = mgR\left(1 - \frac{R}{r}\right)$$

and

$$\mathbf{grad}\,U = \frac{\partial U}{\partial r}\,\mathbf{e}_r = \frac{mgR^2}{r^2}\,\mathbf{e}_r.$$

Thus the gravitational force in spherical coordinates is

$$\mathbf{F}(r,\theta,\phi) = -\mathbf{grad}\,U = -\frac{mgR^2}{r^2}\,\mathbf{e}_r,$$

or in coordinate-free form,

$$\mathbf{F}(\mathbf{r}) = -\frac{mgR^2}{|\mathbf{r}|^2}\,\widehat{\mathbf{r}}.$$

Note that on the surface of the Earth, $\mathbf{F} = -mg\,\widehat{\mathbf{r}}$, as required.

Solution to Exercise 38

(a) The field $\mathbf{F}$ is confined to the (x,y)-plane, so

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} = \left(\frac{\partial(-py)}{\partial x} - \frac{\partial(-2px)}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Hence $\mathbf{F}$ is conservative.

(b) We have

$$\mathbf{grad}\,U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} = 2px\mathbf{i} + py\mathbf{j}.$$

Hence $\mathbf{F} = -\mathbf{grad}\,U$, so U is a potential energy function for $\mathbf{F}$.

The work done by the force is the line integral of $\mathbf{F}$ for any path from $(5,0)$ to $(0,5)$. From equation (31), this line integral is simply the difference between potential energies,

$$-(U(0,5) - U(5,0)) = -\tfrac{25}{2}p + 25p = \tfrac{25}{2}p.$$

Solution to Exercise 39

Use the curl test. We have $\mathbf{curl}\,\mathbf{F} = [\partial g(y)/\partial x - \partial f(x)/\partial y]\mathbf{k}$, where $\partial g/\partial x = \partial f/\partial y = 0$. Hence $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, so $\mathbf{F}$ is conservative.

We have $\mathbf{curl}\,\mathbf{H} = [\partial f(x)/\partial x - \partial g(y)/\partial y]\mathbf{k} = (f'(x) - g'(y))\mathbf{k}$. This can be zero, making $\mathbf{H}$ conservative, only if $f'(x) = g'(y)$, and since the left-hand side of this equation is a function of x only, and the right-hand side is a function of y only, both sides must equal the same constant, m say. Thus for $\mathbf{H}$ to be conservative, we must have $f(x) = mx + c_1$ and $g(y) = my + c_2$, for constants m, c_1 and c_2. Otherwise the curl is non-zero and $\mathbf{H}$ is non-conservative.

Solution to Exercise 40

(a) The circle C can be parametrised by $x = a\cos t$, $y = a\sin t$. Hence $B_1 = -(k/a)\sin t$, $B_2 = (k/a)\cos t$, $dx/dt = -a\sin t$, $dy/dt = a\cos t$. So

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} k\,dt = 2\pi k.$$

Alternatively, the field can be expressed in polar coordinates as

$$\mathbf{B}(r,\theta) = \frac{k}{r}\,\mathbf{e}_\theta \quad (r > 0).$$

Hence the field is directed tangentially, and the tangential component of $\mathbf{B}$ on the circle of radius $r = a$ is k/a, which is constant. Hence the line integral around the circle is $(k/a)2\pi a = 2\pi k$.

(b) It is easiest to use the cylindrical form of curl given in Subsection 2.3. We can write $\mathbf{B}(\rho,\phi) = k\mathbf{e}_\phi/\rho$, so we have $B_\phi = k/\rho$ and $B_\rho = B_z = 0$. Thus, using equation (18), since the vector field is two-dimensional, we have

$$\mathbf{curl}\,\mathbf{B} = (-k/\rho^2 + k/\rho^2)\mathbf{e}_z = \mathbf{0}.$$

So part (a) shows that the vector field $\mathbf{B}$ is not a conservative field, since it has a non-zero line integral for the circle, but this part shows that $\mathbf{curl}\,\mathbf{B}$ is zero. This failure of the curl test occurs because the domain of $\mathbf{B}$ is not simply-connected. The origin, where $x^2 + y^2 = 0$, is excluded. Whenever there is a hole in the domain of a two-dimensional vector field, there is a possibility that the curl test will fail. You may take it that the curl test can be applied to all other vector fields that we ask you to work with in this module.