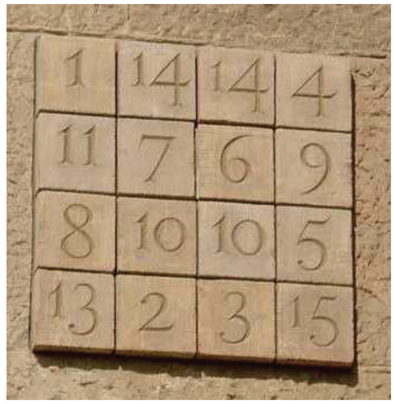
Unit 9

Matrices

Introduction

This unit introduces mathematical objects called *matrices* (the singular form is *matrix*). At their simplest, matrices are arrays of numbers used to hold information in an orderly way. In particular, they provide a useful way of storing data that can be organised in rows and columns. Everyday examples of such data include transport timetables and tables holding survey results.

In this unit you will discover that arrays of numbers can be thought of as 'multi-dimensional numbers'. For example, you can add, subtract and multiply them, just as you can add, subtract and multiply numbers. You've met a similar idea before: in Unit 5, you saw that column vectors are numerical entities made up of more than one number, and they can be added, subtracted and multiplied by scalars, for example.



A magic square (an example of an array of numbers) on the facade of the Sagrada Familia basilica in Barcelona

One advantage of arranging large amounts of numerical data in matrix form is precisely that a block of data can be treated as a single entity. Many problems that involve large amounts of numerical data are first reduced to matrix problems before being solved with a computer. So matrices are an important topic in mathematics, and find widespread use in mathematical modelling, as well as in computer graphics (where they are used to move and rotate objects), medical imaging, economic models, electrical networks, data encryption and many other applications.

As a simple practical example, consider a publisher that supplies its books to a number of bookshops. The publisher's sales department might collect the weekly order quantities for three of its bestselling books in a grid such as the one in Table 1.

Table 1 A publisher's order quantities for three books in a certain week

	Bestseller 1	Bestseller 2	Bestseller 3
Bookshop 1	10	25	12
Bookshop 2	5	10	5
Bookshop 3	7	5	0
Bookshop 4	10	8	10

The information in Table 1 can be stored more succinctly by omitting the row and column headings and enclosing the data in brackets, if it is understood that each row corresponds to a bookshop and each column corresponds to a book:

$$\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix}.$$

This is an example of a matrix. Information about order quantities can be collected every week for the life of the three books and stored in this form.

For example, the information for two consecutive weeks can be recorded in two separate matrices:

$$\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix}, \qquad \begin{pmatrix} 15 & 30 & 20 \\ 10 & 20 & 10 \\ 20 & 8 & 5 \\ 15 & 0 & 10 \end{pmatrix}.$$

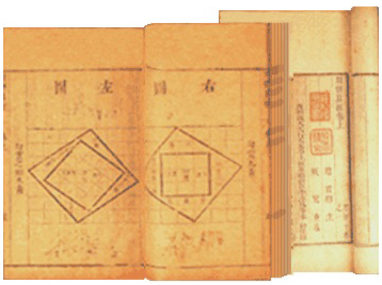
The order quantities for a fortnight can now be calculated by adding corresponding numbers in the two arrays, which gives

$$\begin{pmatrix} 10+15 & 25+30 & 12+20 \\ 5+10 & 10+20 & 5+10 \\ 7+20 & 5+8 & 0+5 \\ 10+15 & 8+0 & 10+10 \end{pmatrix}, \quad \text{that is,} \quad \begin{pmatrix} 25 & 55 & 32 \\ 15 & 30 & 15 \\ 27 & 13 & 5 \\ 25 & 8 & 20 \end{pmatrix}.$$

This calculation amounts to *adding* the two matrices, in a sense that will be made precise in Section 1.

You might wonder what the advantage is in carrying out calculations in this way. After all, the total number of orders for each bookshop can be worked out in a straightforward way without resorting to matrices. In a real-life situation, however, a large publisher will supply hundreds of bookshops with hundreds of books. Calculating order quantities and total charges requires careful accounting, and it is precisely in this sort of situation that matrices become a useful way to store data (especially electronically) and carry out calculations with them.

Matrices were first introduced in Western mathematics in their current form by the British mathematician Arthur Cayley (1821–1895). However, they appeared as early as 300–200 BC in the Chinese text *The nine chapters on the mathematical art*, where matrix methods are used to solve simultaneous equations. You'll meet a modern version of these methods in Section 4.



Pages from *The nine chapters on the mathematical art*

There are other useful operations that can be carried out when data are stored in matrix form, such as the multiplication of matrices by single numbers, and the multiplication of two matrices. You'll learn about these operations in Section 1, and see an application of matrix multiplication in Section 2. In Section 3 you'll learn about *matrix inverses*, and in Section 4 you'll meet a connection between matrices and systems of linear equations.

1 Matrices and matrix operations

In this section you'll meet matrices and standard matrix notation, and learn about matrix addition, multiplication of matrices by single numbers (known as *scalar multiplication* of matrices), matrix multiplication and matrix powers.

1.1 Matrix notation and terminology

A **matrix** (pronounced 'may-tricks') is a rectangular array of numbers, usually enclosed in brackets. Here are some examples:

$$\begin{pmatrix} 1 & 2 & -1 \\ 5 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{3} & 4 \\ 7 & -6 \\ 22 & 0 \end{pmatrix}.$$

Some texts use square brackets for matrices, like this:

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

In this module we'll always use round brackets.

We often use capital letters to denote matrices. For example, we might write

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 5 & 3 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \frac{1}{3} & 4 \\ 7 & -6 \\ 22 & 0 \end{pmatrix}.$$

First rule of consulting: never, *never* call this a "table".
We can bill four times as much when we call it a "matrix".

In the text in this module, the capital letters that denote matrices are in bold font, but you don't need to do anything special when you handwrite them. In particular, you don't need to underline them, unlike letters that denote vectors.

The numbers in a matrix are called its **elements** (or its **entries**, in some texts). For example, matrices **A** and **C** above each have six elements, and matrix **B** has four elements. In this module, the elements of matrices are always real numbers.

A horizontal line of numbers in a matrix is called a **row**, and a vertical line of numbers is called a **column**. For instance, in the matrix **A** above the first row is $\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$ and the second row is $\begin{pmatrix} 5 & 3 & 0 \end{pmatrix}$. Likewise, the first column is $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, the second column is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and the third column is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

The matrix **A** has two rows and three columns: we say that it is a 2×3 matrix. In general, a matrix with m rows and n columns is described as an $m \times n$ matrix, or a matrix of **size** $m \times n$. The first number in this notation is always the number of rows, and the second number is always the number of columns. The size $m \times n$ of a matrix is read as 'm by n'.

A matrix with the same number of rows as columns is called a **square matrix**. Thus, for example, matrix **B** above is square.

The column notation for vectors that you met in Unit 5 is a particular instance of matrix notation: two-dimensional column vectors are 2×1 matrices, and three-dimensional column vectors are 3×1 matrices. In general, the word **vector** is used to mean any matrix with a single column, even if it has more than 3 elements and hence has no geometric interpretation in the plane or three-dimensional space. For instance, the following matrices are vectors:

$$\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2} \\ -3 \\ \frac{1}{2} \\ 0 \\ 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \quad \text{where } w,\ x,\ y,\ z \text{ are variables.}$$

The elements of a vector are often called its **components**, as in Unit 5. A vector with n components is called an n-**dimensional** vector.

There is a useful notation for the elements of a matrix in terms of their row and column positions. The element in row i and column j of a matrix $\mathbf{A}$ is denoted by a_{ij}. Similarly, the element in row i and column j of a matrix $\mathbf{B}$ is denoted by b_{ij}, and so on.

For instance, if

$$\mathbf{A} = \begin{pmatrix} \frac{1}{3} & 4 \\ 7 & -6 \end{pmatrix},$$

then $a_{11} = \frac{1}{3}$, $a_{12} = 4$, $a_{21} = 7$ and $a_{22} = -6$.

This notation is used for matrices of any size. Thus, for instance, a general 3×4 matrix can be denoted by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}.$$

Activity 1 *Using matrix notation*

Write down the elements a_{33}, a_{34}, a_{43}, a_{14}, a_{44} and a_{42} of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 8 & 0 & -1 \\ 4 & 7 & 5 & -2 \\ 9 & -1 & 3 & -4 \\ 6 & -5 & -7 & -9 \end{pmatrix}.$$

What is the size of $\mathbf{A}$?

Two matrices are equal if they have the same numbers of rows and columns, and all corresponding elements are equal. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

but

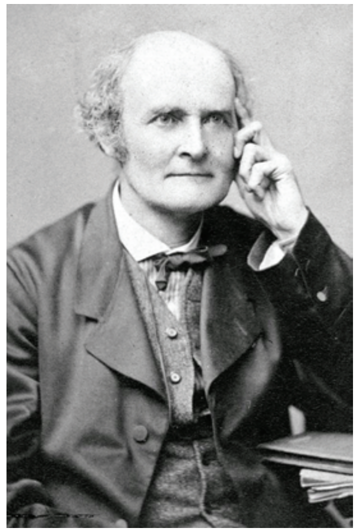
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

You can carry out operations on matrices in a similar way to numbers. For example, you can add, subtract and multiply them, as you'll see in this section. There's also a way to manipulate matrices that in certain circumstances plays the role of division, which you'll learn about in Section 3. While matrix addition is similar to addition of vectors, which you met in Unit 5, matrix multiplication is more surprising. We'll start by looking at matrix addition.

Arthur Cayley gave the rules for matrix operations in a paper presented to the Royal Society in 1858. The paper also contained the conditions under which a matrix has an inverse (you'll meet matrix inverses in Section 3). However, it was Cayley's friend and fellow mathematician James Joseph Sylvester who coined the term 'matrix', in 1850, and he also contributed to the theory of matrices.

Before being appointed to a mathematics professorship at the University of Cambridge, Cayley worked for 14 years as a lawyer in London. At the time, Sylvester was employed as an actuary. The two worked together at the courts of Lincoln's Inn, which gave them the opportunity to discuss mathematics.

Sylvester had a more varied mathematical career than Cayley, including professorships at University College London, Johns Hopkins University in the USA, and the University of Oxford. It is said that he tutored Florence Nightingale in mathematics, though firm evidence for this has yet to be found.



Arthur Cayley (1821–1895)

James Joseph Sylvester (1814–1897)

1.2 Matrix addition and subtraction

You can add two matrices only if they have the same size. The sum of two matrices is obtained by adding their corresponding elements. For example:

$$\begin{pmatrix} -2 & 4 \\ 1 & 5 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -2 & 10 \\ -6 & 4 \end{pmatrix} = \begin{pmatrix} -2+3 & 4+1 \\ 1+(-2) & 5+10 \\ 0+(-6) & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 15 \\ -6 & 1 \end{pmatrix}.$$

The rule for adding two matrices can be expressed formally as follows.

Matrix addition

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, then $\mathbf{A}+\mathbf{B}$ is the $m \times n$ matrix whose element in row i and column j is $a_{ij}+b_{ij}$.

Addition of two matrices of different sizes is not defined.

For instance, you cannot add a 3×3 matrix and a 3×5 matrix.

The rule for adding matrices extends to the sum of any number of matrices of the same size: you just add corresponding elements.

Activity 2 *Adding matrices*

For each of the pairs of matrices below, work out the matrix sum $\mathbf{A}+\mathbf{B}$.

(a) $\mathbf{A}=\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} 4 & 6 \\ 2 & -1 \end{pmatrix}$

(b) $\mathbf{A}=\begin{pmatrix} 2 & 0 & -4 \\ -1 & 6 & 0 \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(c) $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$

In Unit 5, you saw that vector addition shares many properties with the addition of numbers. This happens for matrix addition, too. For example, you can see from the definition of matrix addition that

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A},$$

for all matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size. As you saw with vectors, this property is expressed by saying that matrix addition is **commutative**.

You can also see from the definition of matrix addition that

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}),$$

for all matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the same size. In other words, when you add three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the same size, it doesn't matter whether you add $\mathbf{A}$ and $\mathbf{B}$ and then add $\mathbf{C}$ to the result, or add $\mathbf{B}$ and $\mathbf{C}$ and then add $\mathbf{A}$ to the result, as you get the same overall result either way. This property is expressed by saying that matrix addition is **associative**. It tells you that you can write the expression

$$\mathbf{A}+\mathbf{B}+\mathbf{C}$$

without ambiguity: it doesn't matter whether it means $(\mathbf{A}+\mathbf{B})+\mathbf{C}$ or $\mathbf{A}+(\mathbf{B}+\mathbf{C})$, as both are equal.

There are other respects in which matrix operations behave like number operations. You will discover more similarities between matrices and numbers as you meet other matrix operations.

You can also subtract matrices of the same size by subtracting their corresponding elements. For example,

$$\begin{pmatrix} 4 & 5 & 3 \\ 7 & -6 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 4 \\ 3 & 9 & 1 \end{pmatrix} = \begin{pmatrix} 4-1 & 5-3 & 3-4 \\ 7-3 & -6-9 & 0-1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 2 & -1 \\ 4 & -15 & -1 \end{pmatrix}.$$

Here is the formal definition of matrix subtraction.

Matrix subtraction

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, then $\mathbf{A} - \mathbf{B}$ is the $m \times n$ matrix whose element in row i and column j is $a_{ij} - b_{ij}$.

Subtraction of two matrices of different sizes is not defined.

Like the subtraction of numbers, matrix subtraction is neither commutative nor associative. For example, it is *not* true that $\mathbf{A} - \mathbf{B} = \mathbf{B} - \mathbf{A}$ for all matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, as illustrated by parts (a) and (b) of the next activity.

Activity 3 *Subtracting matrices*

Let $\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & 4 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$. Work out the following matrices.

(a) $\mathbf{A} - \mathbf{B}$ (b) $\mathbf{B} - \mathbf{A}$ (c) $\mathbf{B} - \mathbf{C}$

Since vectors are single-column matrices, they are added and subtracted by adding or subtracting corresponding components, as in the examples below.

$$\begin{pmatrix} 1 \\ -3 \\ 4 \\ \frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 7 \\ -2 \\ 1 \\ \frac{3}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} 1+7 \\ -3+(-2) \\ 4+1 \\ \frac{1}{2}+\frac{3}{2} \\ 0+(-1) \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10-(-2) \\ 3-3 \\ 1-2 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -1 \\ -2 \end{pmatrix}$$

In other words, the ideas of addition and subtraction of column vectors that you met in Unit 5 extend in the obvious way to vectors with more than three components.

A matrix each of whose elements is zero is called a **zero matrix**, and is usually denoted by $\mathbf{0}$. For example, the 2×2 zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Also, as you'd expect, if $\mathbf{A}$ is any matrix, then the matrix formed by changing each element of $\mathbf{A}$ to its negative is called the **negative** of $\mathbf{A}$, and denoted by $-\mathbf{A}$. For example,

$$-\begin{pmatrix} 0 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 4 \end{pmatrix}.$$

It follows from the definitions of matrix addition and subtraction that for any matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}),$$

as you'd expect.

1.3 Scalar multiplication of matrices

Any matrix can be added to itself, and a matrix sum $\mathbf{A} + \mathbf{A}$ can be written as $2\mathbf{A}$. Each element in $2\mathbf{A}$ is twice the corresponding element in $\mathbf{A}$, as in the following example:

$$\begin{pmatrix} 1 & 0 & 5 \\ -1 & -2 & 3 \\ 3 & \frac{1}{2} & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 5 \\ -1 & -2 & 3 \\ 3 & \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 10 \\ -2 & -4 & 6 \\ 6 & 1 & 2 \end{pmatrix}.$$

Similarly, adding $\mathbf{A}$ to itself n times, for any natural number n, gives a matrix in which each element is the corresponding element in $\mathbf{A}$ multiplied by n.

This idea can be generalised to any real number k, so $k\mathbf{A}$ is the matrix obtained by multiplying each element of $\mathbf{A}$ by k. For example,

$$\frac{1}{2}\begin{pmatrix} 4 & 3 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times 4 & \frac{1}{2} \times 3 \\ \frac{1}{2} \times 0 & \frac{1}{2} \times (-1) \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

The matrix $k\mathbf{A}$ is called the **scalar multiple** of the matrix $\mathbf{A}$ by the real number k, and the operation of forming $k\mathbf{A}$ is called **scalar multiplication**.

Scalar multiplication

If $\mathbf{A}$ is an $m \times n$ matrix and k is any real number, then $k\mathbf{A}$ is the matrix whose element in row i and column j is ka_{ij}.

The number k in a scalar multiple $k\mathbf{A}$ of a matrix $\mathbf{A}$ is sometimes referred to as the **scalar** k in this context, to emphasise that it's a number rather than a matrix.

Activity 4 *Multiplying matrices by scalars*

(a) Calculate the following products of scalars and matrices, simplifying your answers.

(i) $3\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ (ii) $\dfrac{1}{2}\begin{pmatrix} 4 & 6 \\ 2 & -1 \end{pmatrix}$ (iii) $\dfrac{2}{3}\begin{pmatrix} 6 & x \\ 3 & y \end{pmatrix}$

(b) Simplify each of the following matrices by writing it as a scalar multiple of a matrix with integer elements.

(i) $\begin{pmatrix} 4.5 & 3 \\ 2 & 1.5 \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{3}{2} \\ \frac{2}{3} \end{pmatrix}$ (iii) $\begin{pmatrix} 2x & 3x \\ 0 & 5x \end{pmatrix}$

Notice that it follows from the definition of scalar multiplication that, for any matrix $\mathbf{A}$,

$$(-1)\mathbf{A} = -\mathbf{A},$$

as you'd expect.

In the next activity you can practise the matrix operations that you've learned so far.

Activity 5 *Combining matrix addition, subtraction and scalar multiplication*

Let $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 4 & -2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & -1 \\ -2 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} \frac{3}{2} & 0 & -1 \\ -1 & 3 & 5 \end{pmatrix}$.

Where possible, evaluate $\mathbf{A} - \mathbf{C}$, $\mathbf{B} + 2\mathbf{A}$ and $\mathbf{C} - 2\mathbf{A}$.

Here's a summary of some useful properties of matrix addition and scalar multiplication of matrices. As you'd expect (since vectors are single-column matrices), these properties are the same as the properties of vector algebra that you met in Unit 5, but with vectors replaced by matrices.

Properties of matrix addition and scalar multiplication

The following properties hold for all matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for which the sums mentioned are defined, and for all scalars m and n.

1. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
2. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
3. $\mathbf{A}+\mathbf{0}=\mathbf{A}$
4. $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$
5. $m(\mathbf{A}+\mathbf{B})=m\mathbf{A}+m\mathbf{B}$
6. $(m+n)\mathbf{A}=m\mathbf{A}+n\mathbf{A}$
7. $m(n\mathbf{A})=(mn)\mathbf{A}$
8. $1\mathbf{A}=\mathbf{A}$

(In properties 3 and 4, $\mathbf{0}$ is the zero matrix of the same size as $\mathbf{A}$.)

1.4 Matrix multiplication

In Subsection 1.3 you learned how to multiply a number and a matrix. Under certain conditions, it is also possible to multiply two matrices. The way that this is done is less obvious than matrix addition or scalar multiplication.

Recall the example in the introduction to this unit, where a matrix was used to store the numbers of copies of three bestselling books that four bookshops ordered from a publisher in a certain week:

$$\begin{array}{r c}
 & \begin{array}{ccc}\text{Bestseller 1} & \text{Bestseller 2} & \text{Bestseller 3}\end{array} \\
\begin{array}{r}\text{Bookshop 1}\\ \text{Bookshop 2}\\ \text{Bookshop 3}\\ \text{Bookshop 4}\end{array} & \begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix}
\end{array}.$$

Now suppose that the publisher charges the bookshops £4 per copy for Bestseller 1, £7 per copy for Bestseller 2 and £9 per copy for Bestseller 3. According to the data in the matrix, Bookshop 1 ordered 10 copies of Bestseller 1, 25 copies of Bestseller 2 and 12 copies of Bestseller 3. Therefore the total amount, in £, that Bookshop 1 owes the publisher for that week's order is

$$10\times 4+25\times 7+12\times 9=323.$$

The expression on the left of this equation is obtained by multiplying each element in the first row of the matrix by the price of the corresponding book, and then adding the results.

If we write the prices (in £) for the three bestsellers as a three-dimensional vector, that is, as a 3×1 matrix, then we can picture this procedure as shown in Figure 1.

$$\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \quad \text{gives } 10 \times 4 + 25 \times 7 + 12 \times 9$$

Figure 1 The procedure for calculating the amount owed by Bookshop 1

If we want to calculate the amount owed by Bookshop 2, then we can carry out the same procedure with the second row of the matrix, and likewise for Bookshops 3 and 4 with the third and fourth rows, respectively. We can write the four amounts that we obtain as a four-dimensional vector. The procedure described above combines the two matrices

$$\begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix}$$

to give

$$\begin{pmatrix} 10 \times 4 + 25 \times 7 + 12 \times 9 \\ 5 \times 4 + 10 \times 7 + 5 \times 9 \\ 7 \times 4 + 5 \times 7 + 0 \times 9 \\ 10 \times 4 + 8 \times 7 + 10 \times 9 \end{pmatrix} = \begin{pmatrix} 323 \\ 135 \\ 63 \\ 186 \end{pmatrix}.$$

This calculation shows that the amounts owed by the four bookshops are £323, £135, £63 and £186, respectively.

This procedure for combining matrices comes up so often in applications that it is used to define matrix multiplication. We say that

$$\text{the } \textbf{product} \text{ of } \begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \text{ is } \begin{pmatrix} 323 \\ 135 \\ 63 \\ 186 \end{pmatrix}.$$

Now suppose that as well as calculating the amount owed by each of the four bookshops, the publisher also wants to calculate the profit made from sales to each individual bookshop. Suppose that the profits made on each copy of Bestsellers 1, 2 and 3 are £1, £2 and £2, respectively. You can use exactly the same procedure as above to calculate the profit made from the sales to each bookshop. You write the profits (in £) made on the three books as a three-dimensional vector, and form

$$\text{the product of} \quad \begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

which is

$$\begin{pmatrix} 10 \times 1 + 25 \times 2 + 12 \times 2 \\ 5 \times 1 + 10 \times 2 + 5 \times 2 \\ 7 \times 1 + 5 \times 2 + 0 \times 2 \\ 10 \times 1 + 8 \times 2 + 10 \times 2 \end{pmatrix} = \begin{pmatrix} 84 \\ 35 \\ 17 \\ 46 \end{pmatrix}.$$

So the profits made from sales to the four bookshops are £84, £35, £17 and £46, respectively.

In fact you can view the two matrix multiplications above – the one for the amounts owed, and the one for the profits – as a *single* matrix multiplication. To do this, you put the two vectors containing the prices and the profits together as a single matrix with two columns, and you put the two vectors that contain the results of the calculations together as a single matrix with two columns, and say that

$$\text{the product of} \quad \begin{pmatrix} 10 & 25 & 12 \\ 5 & 10 & 5 \\ 7 & 5 & 0 \\ 10 & 8 & 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 1 \\ 7 & 2 \\ 9 & 2 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 323 & 84 \\ 135 & 35 \\ 63 & 17 \\ 186 & 46 \end{pmatrix}. \qquad (1)$$

This calculation is an example of how matrices are multiplied in general. In Section 2 you'll meet another context that shows that it makes sense to multiply matrices in this way. Before that, you'll see a more detailed explanation of the procedure for multiplying matrices, and have a chance to practise it yourself.

Let's start by considering which pairs of matrices can be multiplied together. Remember that the element in the first row and first column of the product matrix in calculation (1) above was obtained by multiplying each element in the first *row* of the first matrix by the corresponding element in the first *column* of the second matrix, and adding the results:

$$10 \times 4 + 25 \times 7 + 12 \times 9 = 323.$$

It was possible to do this because *the number of columns in the first matrix is the same as the number of rows in the second matrix*, so each element in the first row of the first matrix *has* a corresponding element in the first column of the second matrix. You can multiply two matrices together only if this condition holds.

Thus, you can multiply together a 4×3 matrix and a 3×2 matrix. On the other hand, it is not possible to multiply together a 3×4 matrix and a 2×3 matrix. When it is not possible to multiply two matrices, we sometimes say that their product is **undefined**.

A convenient way to determine whether two matrices, of sizes $m \times n$ and $p \times q$, say, can be multiplied together is to write their sizes next to each other as follows:

$$m \times n \quad p \times q.$$

The matrices can be multiplied only if the two numbers in the middle are equal; that is, if $n = p$. If these numbers are equal, then the size of the product matrix is given by the remaining numbers. So the product of an $m \times n$ matrix and an $n \times q$ matrix is a matrix of size $m \times q$.

For example, a 4×3 matrix and a 3×2 matrix can be multiplied together, and the product has size 4×2. Here's a useful way to picture this fact:

$$4 \times \boxed{3} \quad \boxed{3} \times 2 \quad \text{gives} \quad 4 \times 2.$$

Now let's look at the procedure for multiplying two matrices step by step. It's demonstrated in the next example.

Example 1 *Multiplying matrices*

Let $\mathbf{A} = \begin{pmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 0 & -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \\ -1 & 4 \end{pmatrix}$.

Check that the product matrix $\mathbf{AB}$ can be formed, determine the size of $\mathbf{AB}$, and calculate it.

Solution

Write down the sizes of $\mathbf{A}$ and $\mathbf{B}$.

$\mathbf{A}$ has size 3×3 and $\mathbf{B}$ has size 3×2.

$$3 \times \boxed{3} \quad \boxed{3} \times 2$$

The numbers in the middle are equal, so the product $\mathbf{AB}$ can be formed. It has size 3×2.

To obtain the element in the *first row* and *first column* of $\mathbf{AB}$, multiply each element in the *first row* of $\mathbf{A}$ by the corresponding element in the *first column* of $\mathbf{B}$, and add the results.

$$\begin{pmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} (-2)\times 1 + 1 \times \frac{1}{2} + 3 \times (-1) & \\ & \\ & \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & \\ & \\ & \end{pmatrix}$$

To obtain the element in the *first row* and *second column* of $\mathbf{AB}$, apply the same procedure to the *first row* of $\mathbf{A}$ and the *second column* of $\mathbf{B}$.

$$\begin{pmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} & (-2)\times 0 + 1 \times 2 + 3 \times 4 \\ & \\ & \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & 14 \\ & \\ & \end{pmatrix}$$

To obtain the element in the *second row* and *first column* of $\mathbf{AB}$, apply the same procedure to the *second row* of $\mathbf{A}$ and the *first column* of $\mathbf{B}$.

$$\begin{pmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} & 14 \\ 2 \times 1 + 4 \times \frac{1}{2} + (-2) \times (-1) & \\ & \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & 14 \\ 6 & \\ & \end{pmatrix}$$

In general, to obtain the element in the ith row and jth column of $\mathbf{AB}$, multiply each element in the ith row of $\mathbf{A}$ by the corresponding element in the jth row of $\mathbf{B}$, and add the results.

$$\begin{pmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 2 \\ -1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & 14 \\ 6 & 2 \times 0 + 4 \times 2 + (-2) \times 4 \\ 0 \times 1 + (-1) \times \frac{1}{2} + 0 \times (-1) & 0 \times 0 + (-1) \times 2 + 0 \times 4 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & 14 \\ 6 & 0 \\ -\frac{1}{2} & -2 \end{pmatrix}$$

Here's a summary of the procedure for matrix multiplication.

Matrix multiplication

Let **A** and **B** be matrices. Then the product matrix **AB** can be formed only if the number of columns of **A** is equal to the number of rows of **B**.

If **A** has size $m \times n$ and **B** has size $n \times p$, then the product **AB** has size $m \times p$.

The element in row i and column j of the product matrix **AB** is obtained by multiplying each element in the ith row of **A** by the corresponding element in the jth column of **B** and adding the results.

In element notation, if c_{ij} denotes the element in the ith row and jth column of **AB**, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Figure 2 illustrates how row i of a matrix **A** and column j of a matrix **B** are combined to give the element in row i and column j of the product matrix **AB**.

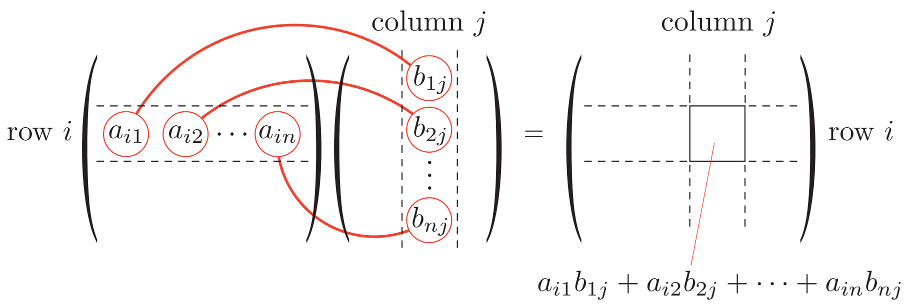


Figure 2 The element in row i and column j of a product matrix **AB** expressed in terms of elements from **A** and **B**

Matrix multiplication may seem quite complicated at first, but it is a very useful technique, and with practice you should become proficient at it.

Activity 6 *Multiplying matrices*

In each of parts (a)–(e) below, calculate the matrix product if it exists.

(a) $\begin{pmatrix} 2 & 9 \\ 7 & 1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} -2 & 3 & 1 \\ 4 & 0 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 9 \\ 7 & 1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 0 & 10 \\ 4 & 1 \\ -2 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}$

Matrix multiplication shares many of the properties of multiplication of numbers, as you'll see later in this subsection and later in the unit. However, there's an important difference between the properties for matrices and those for numbers, as you can find out in the next activity.

Activity 7 *Investigating matrix multiplication*

(a) Determine whether the products $\mathbf{AB}$ and $\mathbf{BA}$ are defined for the following pairs of matrices.

(i) $\mathbf{A} = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

(ii) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$

(b) For each of parts (a)(i) and (a)(ii), if the products $\mathbf{AB}$ and $\mathbf{BA}$ are both defined, evaluate them. What do you notice?

Activity 7 illustrates the important fact that *matrix multiplication is not commutative*. First, there are matrices $\mathbf{A}$ and $\mathbf{B}$ for which the product $\mathbf{AB}$ exists but the product $\mathbf{BA}$ is not defined. Second, even in cases where both the products $\mathbf{AB}$ and $\mathbf{BA}$ are defined, these two products can be *different matrices*.

In the next activity, you can investigate whether another property that holds for multiplication of numbers also holds for multiplication of matrices.

Activity 8 *Investigating matrix multiplication further*

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 3 & 0 \\ -3 & 4 & 1 \end{pmatrix}$.

(a) What are the sizes of the products $\mathbf{AB}$ and $\mathbf{BC}$? Calculate these matrices.

(b) Do the products $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ exist? If so, calculate them. What do you notice?

In Activity 8 you should have found that, for the particular three matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ in the activity, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. This does not of course tell you that this property holds for matrices in general. However, this property does hold in general: whenever $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are matrices such that the products $\mathbf{AB}$ and $\mathbf{BC}$ are defined, we have

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}).$$

In other words, matrix multiplication is *associative*. In this respect matrix multiplication behaves like multiplication of numbers.

In the next activity you can investigate how matrix multiplication interacts with multiplication by a scalar.

Activity 9 *Combining matrix multiplication and multiplication by a scalar*

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Show that

$$\mathbf{A}(2\mathbf{B}) = 2(\mathbf{AB}).$$

In Activity 9 you verified a particular instance of a general property of matrices: if the matrix product $\mathbf{AB}$ exists, then, for any scalar k,

$$\mathbf{A}(k\mathbf{B}) = (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}).$$

In the next activity you're asked to investigate how matrix multiplication interacts with matrix addition.

Activity 10 *Combining matrix addition and multiplication*

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

By calculating $\mathbf{A}(\mathbf{B} + \mathbf{C})$ and $\mathbf{AB} + \mathbf{AC}$, show that

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}.$$

In Activity 10 you verified a particular instance of another general property of matrices: if the matrix product $\mathbf{A}(\mathbf{B} + \mathbf{C})$ can be formed, then

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}.$$

It's also true that if the matrix product $(\mathbf{A} + \mathbf{B})\mathbf{C}$ can be formed, then

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

In other words, matrix multiplication is *distributive* over matrix addition.

So, for example, if you encounter a matrix expression of the form

$$\mathbf{AB} + \mathbf{AC},$$

then you can factorise it to give

$$\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C}).$$

Here's a summary of the properties of matrix multiplication that you've seen so far.

Some properties of matrix multiplication

The following properties hold for all matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for which the products and sums mentioned are defined.

- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$, for any scalar k
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

Remember also the following important fact.

Matrix multiplication is not commutative

- There are matrices $\mathbf{A}$, $\mathbf{B}$ such that the product $\mathbf{AB}$ exists but the product $\mathbf{BA}$ does not.
- Even when both products are defined, it can happen that $\mathbf{AB} \neq \mathbf{BA}$.

Matrix powers

Just as the square a^2 of a real number a is defined to be the product $a \times a$, so we define the **square** $\mathbf{A}^2$ of a matrix $\mathbf{A}$ to be the matrix product $\mathbf{AA}$, if it exists.

For example, if

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix},$$

then

$$\mathbf{A}^2 = \mathbf{AA} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix}.$$

In the next activity you're asked to investigate which matrices can be squared.

Activity 11 *Investigating which matrices can be squared*

(a) Let $\mathbf{M} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \end{pmatrix}$. Why is it not possible to form the product $\mathbf{MM}$?

(b) Let $\mathbf{N} = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$. Is the product $\mathbf{NN}$ defined? If, so, evaluate it.

(c) Describe which matrices can be squared.

In Activity 11 you saw that if $\mathbf{A}$ is a *square* matrix (a matrix with the same number of rows as columns), then its square $\mathbf{A}^2$ exists. The matrix $\mathbf{A}^2$ has the same size as the original matrix $\mathbf{A}$.

Be careful not to confuse the two different concepts *square matrix* and *square of a matrix*.

If $\mathbf{A}$ is a square matrix, then since $\mathbf{A}^2$ is a square matrix of the same size as $\mathbf{A}$, we can also form the products $(\mathbf{A}^2)\mathbf{A}$ and $\mathbf{A}(\mathbf{A}^2)$. Since matrix multiplication is associative, we know that

$$\mathbf{A}^2\mathbf{A} = (\mathbf{AA})\mathbf{A} = \mathbf{A}(\mathbf{AA}) = \mathbf{AA}^2.$$

Hence either of these products can be written unambiguously as $\mathbf{AAA}$. This matrix is called the **cube** of $\mathbf{A}$ and is denoted by $\mathbf{A}^3$. For example, for the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

discussed at the beginning of this subsection, we have

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 20 & 7 \\ 21 & 6 \end{pmatrix}.$$

In general, for any square matrix $\mathbf{A}$, the nth **power** of $\mathbf{A}$ is

$$\mathbf{A}^n = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{n\text{ times}}.$$

Activity 12 *Calculating matrix powers*

(a) Calculate the squares of the following matrices:

$$\begin{pmatrix} -1 & 4 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$. Calculate $\mathbf{A}^6$.

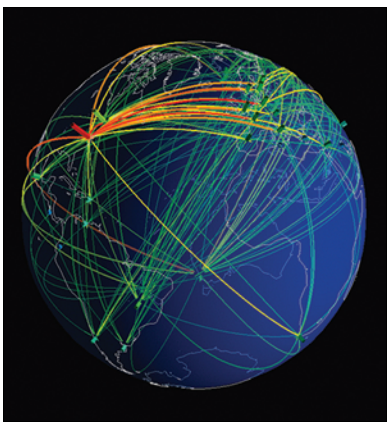
Hint: the cube $\mathbf{A}^3$ of this matrix $\mathbf{A}$ was calculated before the activity.

Matrix multiplication with a computer

Matrix multiplication can be time-consuming to carry out when the matrices involved are large. In the next activity you can find out how to use a computer to carry out matrix operations, including multiplication. This work will increase your familiarity with the way matrices behave, and help you distinguish between circumstances when it is worth using a computer to perform matrix calculations and when it is more efficient to do the calculations by hand.

Activity 13 *Working with matrices on a computer*

Work through Subsection 10.1 of the *Computer algebra guide*.



A network of world internet traffic

2 Matrices and networks

Matrices can be used to represent and analyse various flows in *networks*. A **network** is a collection of objects connected by links – either physical or abstract – such as cities connected by roads, telephone exchange points connected by telephone lines, or people connected by acquaintance. A foremost example is the internet, which is a network of communicating computers linked by a range of technologies. In this section you'll look at flows in networks of a particular type, and learn how to use matrices to work with them.

2.1 Networks

Figure 3 shows a simple example of a network of water pipes. Water can be poured into the system at position A or B, and it then flows down the pipes to come out at positions U, V and W. The pipes have different capacities: the three pipes that come out of position A are labelled with numbers that indicate the *proportions* of water that flow through these pipes when water is poured in at A, and the three pipes that come out of position B are labelled in the same way. For example, if a litre of water is poured in at position A, then 0.4 litres flows down the first pipe from A, 0.2 litres flows down the second pipe from A, and 0.4 litres flows down the third pipe from A.

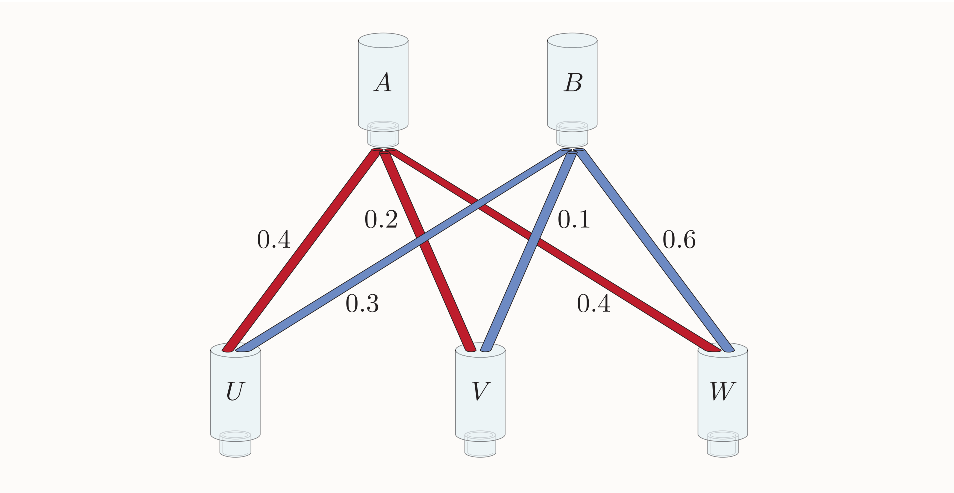


Figure 3 A network of water pipes

A network like this can be represented more simply as a **network diagram**, which is a mathematical representation of a physical network. The network diagram in Figure 4 represents the network in Figure 3.

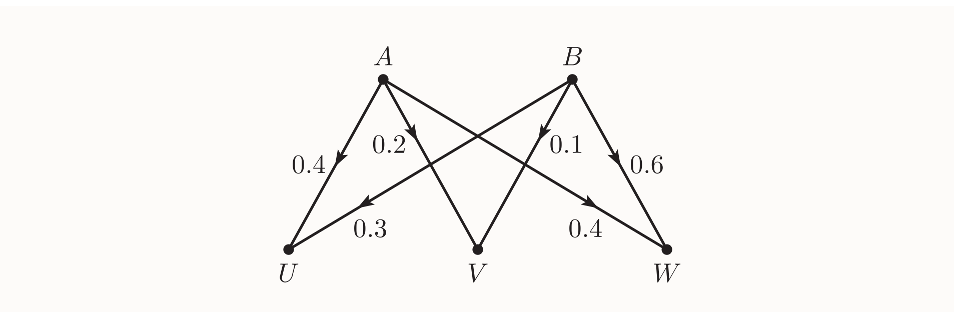
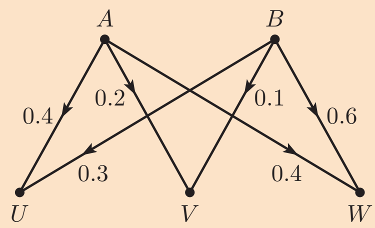


Figure 4 A network diagram representing the pipe system in Figure 3

The dots in a network diagram are called **nodes**, and the lines are called **arcs**. The arrows on the arcs in Figure 4 are included because this network diagram represents a network in which flow along an arc takes place in only one direction. When no confusion arises, we use the word 'network' to mean either 'network diagram' or 'physical network'. We'll use network diagrams to represent networks from now on in this section.

Example 2 *Calculating network outputs*

The network diagram in Figure 4 is repeated below.



How much water is output at each of nodes U, V and W if the only input is 1 litre of water at node B?

Solution

The label on the arc joining node B to node U indicates the proportion of the water input at node B that reaches node U.

The label on the arc from node B to node U is 0.3. So 0.3 litres of water is output at node U when 1 litre is input at node B.

Read the labels on the other arcs from node B.

Similarly, 0.1 litres of water is output at node V, and 0.6 litres of water is output at node W.

Now consider what happens if water is input at *both* node A and node B in the network in Example 2. The amount of water reaching an output node is the sum of the water reaching the node from node A and from node B. For example, if 1 litre of water is input at node A and another 1 litre is input at node B, then node U will collect 0.4 litres of water from node A and 0.3 litres from node B. So, in total, 0.7 litres of water will be output at node U. Similarly, $0.2 + 0.1 = 0.3$ litres of water will be output at node V, and $0.4 + 0.6 = 1$ litre of water will be output at node W.

In general, if x litres of water is input at node A and y litres of water is input at node B, then

$0.4x + 0.3y$ litres of water is output at node U,

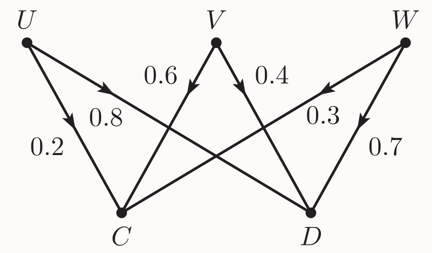
$0.2x + 0.1y$ litres of water is output at node V,

$0.4x + 0.6y$ litres of water is output at node W.

In the next activity you can practise calculating network outputs yourself.

Activity 14 *Calculating network outputs*

The following network diagram represents a network of water pipes.



(a) How much water is output at nodes C and D if 1 litre of water is input at node U, 2 litres of water is input at node V, and there is no input at node W?

(b) How much water is output at nodes C and D if x litres of water are input at node U, y litres of water is input at node V, and z litres of water is input at node W?

Network diagrams like those that you've seen in this subsection can be used to model many other kinds of flows, such as the movement of money in and out of companies. For example, consider three corporate donors, D, E and F, say, that every month make donations to two charities, J and K. Donor D donates 20% of its monthly designated sum to charity J and 80% to charity K, donor E donates 30% of its sum to charity J and 70% to charity K, and donor F donates 50% of its sum to charity J and 50% to charity K.

The flow of money from the three donors to the two charities can be represented by the network in Figure 5. For example, a total donation of £1000 from donor D in a given month is modelled by an input of £1000 at node D. The label of the arc connecting D to J tells you that 0.2, or 20%, of that sum, which is £200, reaches the output node J.

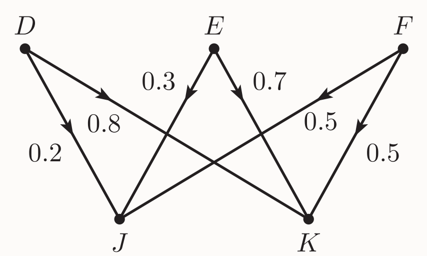
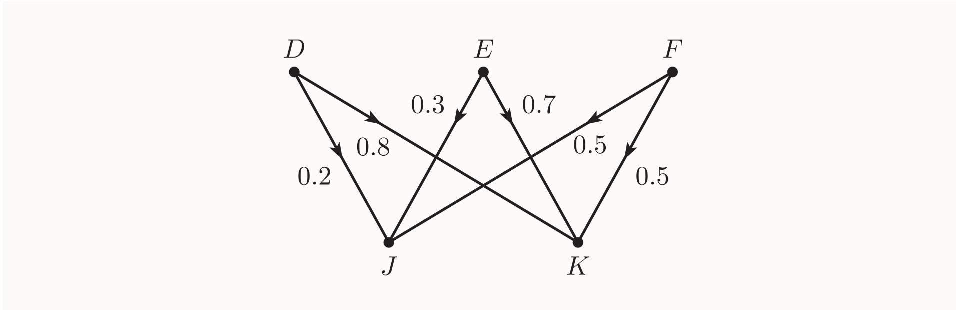


Figure 5 A network diagram representing the flow of money from three corporate donors to two charities

Activity 15 *Calculating more network outputs*

The network diagram in Figure 5 is repeated below.



It represents the flow of money from donors D, E and F to charities J and K, as described above. How much money is output at each of nodes J and K in each of the following situations?

(a) The only input is £2000 at node D.

(b) The input at node D is £2400, the input at node E is £1700, and the input at node F is £1100.

(c) The input (in £) at node D is x, the input (in £) at node E is y, and the input (in £) at node F is z.

2.2 From networks to matrices

In this subsection you'll see how network output calculations of the kinds that you saw in Subsection 2.1 can be expressed succinctly using matrices.

Consider again the first network representing water pipes that you saw in Subsection 2.1. It is repeated in Figure 6.

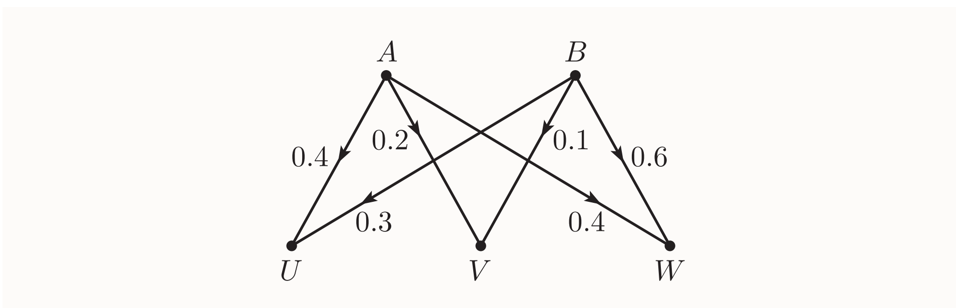


Figure 6 A network representing water pipes

The amounts of water (in litres) input at nodes A and B can be represented as a two-dimensional vector (a 2×1 matrix), whose first and second components are the inputs at A and B, respectively. For example, the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

represents 0 litres of water input at node A and 1 litre of water input at node B.

The outputs from the network can also be represented by a vector. For example, the vector

$$\begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}$$

represents an output of 0.4 litres of water at node U, 0.2 litres at node V, and 0.4 litres at node W.

The network itself can be represented by the matrix

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}.$$

Here the elements in each column are the labels of the arcs *from* a node (A or B), and the elements in each row are the labels of the arcs *to* a node (U, V or W), as shown below:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} U \\ V \\ W \end{array} & \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \end{array}.$$

For example, the element in the column labelled B and the row labelled U is the label of the arc from node B to node U. You'll see the reason for arranging the arc labels in this way in a moment.

You saw in Subsection 2.1 that if x litres of water is input at node A and y litres of water is input at node B, then

$0.4x + 0.3y$ litres of water is output at node U,

$0.2x + 0.1y$ litres of water is output at node V,

$0.4x + 0.6y$ litres of water is output at node W.

In other words,

$$\text{if the input vector is } \begin{pmatrix} x \\ y \end{pmatrix}, \text{ then the output vector is } \begin{pmatrix} 0.4x + 0.3y \\ 0.2x + 0.1y \\ 0.4x + 0.6y \end{pmatrix}.$$

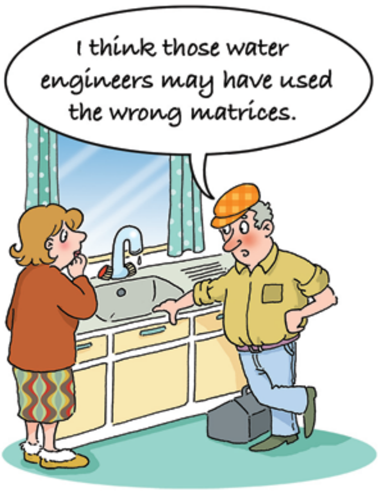
This output vector can be expressed as the product of the matrix that represents the network and the input vector:

$$\begin{pmatrix} 0.4x + 0.3y \\ 0.2x + 0.1y \\ 0.4x + 0.6y \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So we can say that

$$\text{if the input vector is } \begin{pmatrix} x \\ y \end{pmatrix}, \text{ then the output vector is } \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So, for any input vector, you can work out the corresponding output vector by multiplying the input vector by the matrix that represents the network. This is illustrated in the next example.



Example 3 *Using matrices to calculate network outputs*

Use the matrix that represents the network diagram in Figure 6 to calculate the outputs at nodes U, V and W if 2 litres of water is input at node A and there is no input at node B.

Solution

The matrix representing the network was found before this example.

The matrix representing the network is

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}.$$

The input vector (in litres) is

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

The output vector is the product of the matrix representing the network and the input vector.

Hence the output vector is

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \times 2 + 0.3 \times 0 \\ 0.2 \times 2 + 0.1 \times 0 \\ 0.4 \times 2 + 0.6 \times 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.4 \\ 0.8 \end{pmatrix}.$$

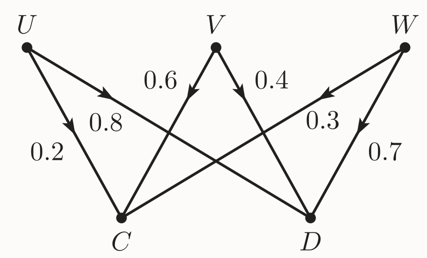
So the outputs at nodes U, V and W are 0.8 litres, 0.4 litres and 0.8 litres, respectively.

In general, to represent a network by a matrix in the way described above, you need a column for each input node and a row for each output node. So if the network has n input nodes and m output nodes, then the matrix has size $m \times n$.

When you find a matrix representing a network of the type considered in this section, it's useful to check that the elements in each column have sum 1. The matrix must have this property because the elements in each column are the labels on an arc *from* a particular node. You can see that the matrix representing the network in Example 3 has this property.

Activity 16 *Using matrices to calculate network outputs*

The network in Activity 14, which represents a network of water pipes, is shown again below.



(a) Write down the matrix that represents this network.

(b) Write down a vector that represents an input of 1 litre of water at each of nodes U, V and W.

(c) Use your answer to part (a) to calculate the output vector for the input vector in part (b), and interpret your answer in terms of the amounts of water output at nodes C and D.

2.3 Combining networks

Networks can be combined by making the outputs from one network become the inputs to another network. In this section you'll see how to use matrices to calculate the outputs from a combined network using the information about the original networks.

The network diagram in Figure 4 and the one in Activity 14 are repeated in Figure 7. Both diagrams represent networks of water pipes.

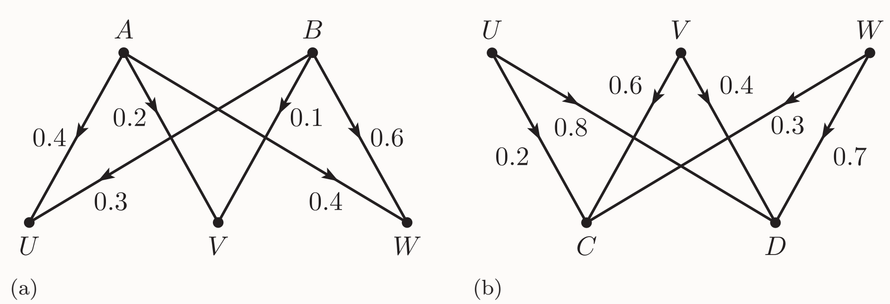


Figure 7 Two network diagrams

The first network has three output nodes, and the second network has three input nodes. We can join these nodes so that, for instance, the water output at U in the first network flows down the pipes that leave U in the second network. The combined network is shown in Figure 8.

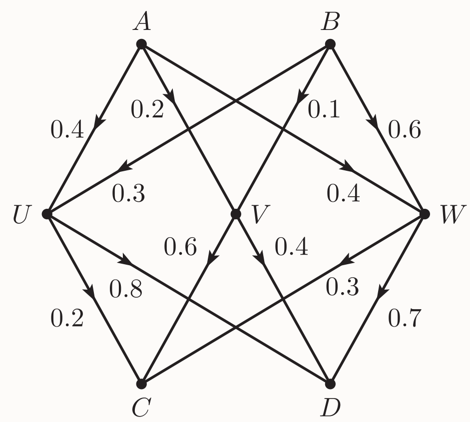


Figure 8 The combined network

The network obtained in this way can be considered as equivalent to a simpler network with arcs going directly from the input nodes A and B to the output nodes C and D, as illustrated in Figure 9. In general, we regard two networks of the type considered in this section as **equivalent** if they have the same input and output nodes, and all choices of inputs produce the same outputs in both networks.

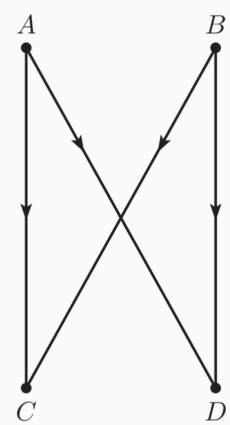


Figure 9 A network equivalent to the one in Figure 8

How can we find the labels for the arcs in the new, simpler network diagram? Here's how you can find the label for the arc from A to C, for example.

Suppose that 1 litre of water is input at node A in the combined network diagram, and there are no other inputs. Then, from the arc labels in the top half of the diagram, you know that the amounts of water passing through nodes U, V and W are 0.4, 0.2 and 0.4 litres, respectively. So, from the arc labels in the bottom half of the diagram, you can work out the following.

- Of the 0.4 litres of water that passes through node U, the amount that reaches node C is $0.2 \times 0.4 = 0.08$ litres.
- Of the 0.2 litres of water that passes through node V, the amount that reaches node C is $0.6 \times 0.2 = 0.12$ litres.
- Of the 0.4 litres of water that passes through node W, the amount that reaches node C is $0.3 \times 0.4 = 0.12$ litres.

To find the total amount of water output at node C, we add the contributions from the three different routes, which gives $0.08 + 0.12 + 0.12 = 0.32$ litres. So the proportion of the water input at node A that's output at node C is 0.32, and hence the arc from A to C should be labelled with the number 0.32.

You can find the labels for the remaining arcs in a similar way. However, these calculations are quite tedious, and it turns out that they can be simplified by using the matrices that represent the two original networks, which are

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{pmatrix},$$

respectively. These matrices were found in Subsection 2.2.

Suppose that the input and output vectors (in litres) for the combined network in Figure 8 are

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix},$$

respectively. Suppose also that the output vector (in litres) of the top part of the network, which is also the input vector for the bottom part, is

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Then

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Using the first of these equations to substitute in the second gives

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{pmatrix} \left(\begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right).$$

You know from Section 1 that matrix multiplication is associative, so

$$\begin{pmatrix} c \\ d \end{pmatrix} = \left(\begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \right) \begin{pmatrix} a \\ b \end{pmatrix};$$

that is,

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0.32 & 0.3 \\ 0.68 & 0.7 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2}$$

The conclusion of this discussion is that the matrix that represents the combined network in Figure 9 is the product of the matrices that represent the original networks. So matrix multiplication provides a neat way to calculate outputs of the combined network.

Notice that the matrix representing the *top* network is the *second* matrix in the product that represents the combined network.

The labels of the simpler network that is equivalent to the combined network can be read off the product matrix. This is easier if we label the rows and the columns with the nodes as below:

$$\begin{array}{cc} & \begin{array}{cc} A & \;\;B \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 0.32 & 0.3 \\ 0.68 & 0.7 \end{pmatrix} \end{array}.$$

Notice that each column of this matrix has sum 1, as expected in view of the comment after Example 3.

The original combined network and the equivalent simpler network, with the arc labels, are shown in Figure 10.

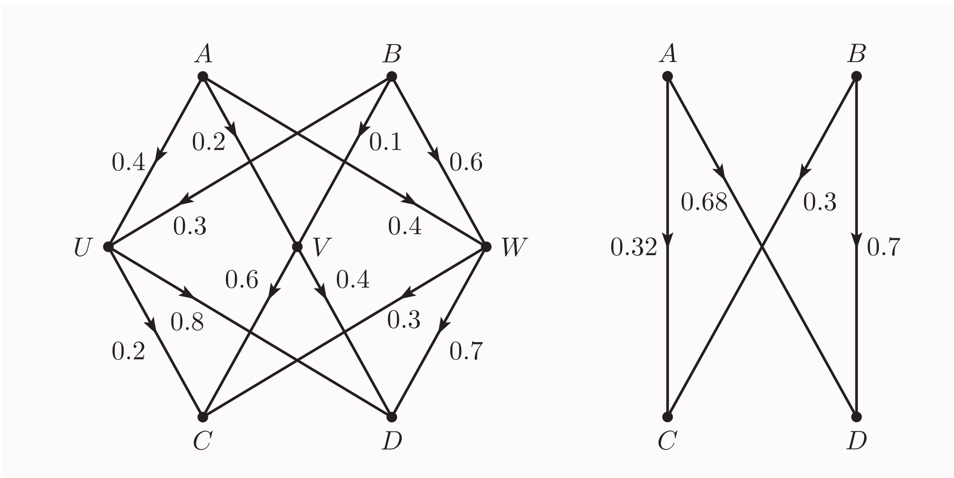
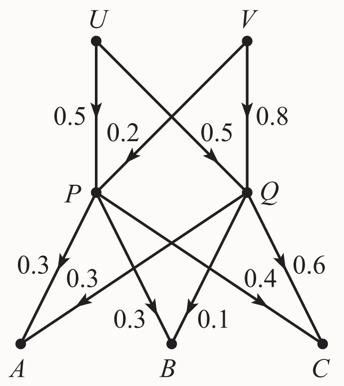


Figure 10 The two equivalent networks

Activity 17 *Finding the matrix representing a combined network*

The network below is the combination of two smaller networks, one with input nodes U and V and output nodes P and Q, and the other with input nodes P and Q and output nodes A, B, and C.



(a) Write down the matrix equation that gives the output vector $\begin{pmatrix} p \\ q \end{pmatrix}$ at nodes P and Q when the input vector is $\begin{pmatrix} u \\ v \end{pmatrix}$ at nodes U and V.

(b) Write down the matrix equation that gives the output vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ at nodes A, B and C when the input vector is $\begin{pmatrix} p \\ q \end{pmatrix}$ at nodes P and Q.

(c) Use your answers to parts (a) and (b) to find the matrix equation for the combined network when the input vector is $\begin{pmatrix} u \\ v \end{pmatrix}$ at nodes U and V.

(d) Draw a network with input nodes U and V and output nodes A, B and C that is equivalent to the given network.

(e) Suppose that the given network represents a network of water pipes. Determine the outputs at nodes A, B and C when 1 litre of water is input at U and 2 litres of water is input at V.

Joining networks is also useful in other contexts. For example, Figure 11 repeats the network diagram from Figure 5, which represents the flow of money from three corporate donors D, E and F to two charities J and K.

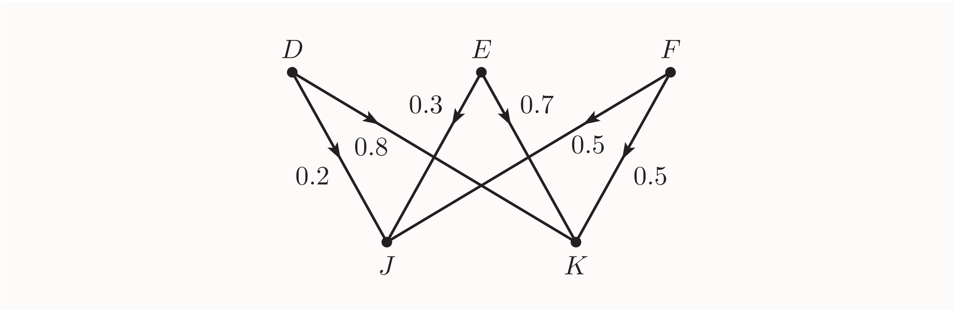


Figure 11 A network diagram representing the flow of money from three corporate donors to two charities

Suppose that donors D, E and F are the only donors to the two charities, and that the two charities spend their money in two different geographical regions, R and S, as indicated in Figure 12. So, for example, charity J spends 60% of its money in region R and 40% in region S.

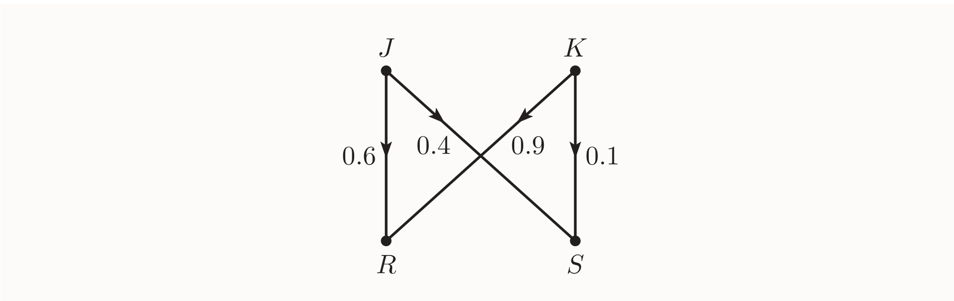


Figure 12 A network diagram representing the flow of money from two charities into two regions

In the next activity you're asked to combine the networks in Figures 11 and 12 to obtain a network that represents the flow of money from the three corporate donors to the two regions.

Activity 18 *Finding the matrix representing another combined network*

(a) Write down the matrices that represent the networks in Figures 11 and 12.

(b) Hence find the matrix that represents the network obtained by combining these two networks.

(c) Draw a simplified network that represents the combined network described in part (b).

(d) If, in a particular month, donor D donates £1700, donor E donates £1400 and donor F donates nothing, how much money is spent in region S?

In a real-life situation, networks like the one in Activity 18 would have many more input and output nodes, and the matrices representing them would be manipulated on a computer.

3 The inverse of a matrix

In Section 1 you met four matrix operations: addition, multiplication by a scalar, subtraction and matrix multiplication. No division was mentioned: it is not possible to divide a matrix by another matrix. However there is a way to manipulate matrices that in certain situations plays the role of division. To understand this method, you need to know about special matrices called *identity matrices*, which are introduced next.

3.1 Identity matrices

The number 1 has a special property among real numbers: multiplication by 1 leaves any real number unchanged. An identity matrix is a matrix that behaves like the number 1, in the sense that if a matrix $\mathbf{A}$ is multiplied by an identity matrix of an appropriate size then the result is again $\mathbf{A}$. You will meet an identity matrix in the next activity.

Activity 19 *Multiplying by an identity matrix*

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 7 & -3 \\ 8 & 1 & 2 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Evaluate each of the following products, where possible:

$$\mathbf{AI}, \quad \mathbf{IA}, \quad \mathbf{BI}, \quad \mathbf{IB}.$$

The matrix $\mathbf{I}$ in Activity 19 has the following two properties.

- If $\mathbf{A}$ is any matrix such that the product $\mathbf{AI}$ exists, then $\mathbf{AI} = \mathbf{A}$.
- If $\mathbf{A}$ is any matrix such that the product $\mathbf{IA}$ exists, then $\mathbf{IA} = \mathbf{A}$.

Any matrix $\mathbf{I}$ with these two properties is called an **identity matrix**.

For a matrix to be an identity matrix, it has to be square. To see this, suppose that $\mathbf{A}$ is an $m \times n$ matrix. Then in order for the product $\mathbf{AI}$ to be defined, $\mathbf{I}$ must have n rows. Moreover, in order for $\mathbf{AI}$ to have size $m \times n$ (which it must if it is to be equal to $\mathbf{A}$), $\mathbf{I}$ must have n columns:

$$m \times \boxed{n} \quad \text{times} \quad \boxed{n} \times n \quad \text{gives} \quad m \times n.$$

So $\mathbf{I}$ must have size $n \times n$.

It's straightforward to verify that the following 2×2, 3×3 and 4×4 matrices are identity matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In fact, for every natural number n, the $n \times n$ matrix that has ones down the **leading diagonal** (the diagonal that starts at the top left element and ends at the bottom right element), and zeros everywhere else, is an identity matrix.

Furthermore, for every natural number n, this matrix is the *only* $n \times n$ identity matrix. To see this, let $\mathbf{I}$ be the $n \times n$ identity matrix described above, and let $\mathbf{E}$ be an $n \times n$ identity matrix. Then $\mathbf{IE} = \mathbf{I}$ and $\mathbf{IE} = \mathbf{E}$, so $\mathbf{I} = \mathbf{E}$.

When we're working with matrices, we usually reserve the letter $\mathbf{I}$ to denote the $n \times n$ identity matrix.

Activity 20 *Multiplying by an identity matrix*

Let $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be general matrices, and let $\mathbf{I}$ be the 3×3 identity matrix. Verify that $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IB} = \mathbf{B}$.

The facts that you've seen about identity matrices are summarised in the following box.

Identity matrices

An **identity matrix** is a square matrix $\mathbf{I}$ such that

- for any matrix $\mathbf{A}$ for which the product $\mathbf{AI}$ is defined, $\mathbf{AI} = \mathbf{A}$
- for any matrix $\mathbf{A}$ for which the product $\mathbf{IA}$ is defined, $\mathbf{IA} = \mathbf{A}$.

Each identity matrix has the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

That is, it has ones down the leading diagonal and zeros elsewhere.

3.2 Matrix inverses

As mentioned earlier, it isn't possible to divide matrices, but it is sometimes possible to perform a manipulation with a similar effect.

Remember that, in the arithmetic of numbers, dividing by a number is the same as multiplying by its reciprocal. For example, dividing by 2 is the same as multiplying by $\frac{1}{2}$. The reciprocal of a non-zero number a is the number b such that $ab = 1$.

There is a notion in matrix arithmetic that is analogous to the notion of a reciprocal in ordinary arithmetic. Let $\mathbf{A}$ be a *square* matrix. If there is another matrix $\mathbf{B}$ of the same size with the property that

$$\mathbf{AB} = \mathbf{I} \quad \text{and} \quad \mathbf{BA} = \mathbf{I},$$

where $\mathbf{I}$ is an identity matrix, then we say that $\mathbf{A}$ is **invertible** and that $\mathbf{B}$ is an **inverse** of $\mathbf{A}$. Multiplying by an inverse of a matrix is useful in many situations where you might want to use a matrix operation analogous to division.

Activity 21 *Checking matrix inverses*

For each of the following pairs of matrices $\mathbf{A}$ and $\mathbf{B}$, check that $\mathbf{B}$ is an inverse of $\mathbf{A}$, by finding the products $\mathbf{AB}$ and $\mathbf{BA}$.

(a) $\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 2 & -1 \\ 1 & -8 & 5 \end{pmatrix}$

If a matrix $\mathbf{A}$ is invertible, then it has *exactly one* inverse. To see this, suppose that the matrices $\mathbf{B}$ and $\mathbf{C}$ are both inverses of the matrix $\mathbf{A}$. Then

$$\mathbf{AB} = \mathbf{I} \quad \text{and} \quad \mathbf{BA} = \mathbf{I}, \quad \text{and also} \quad \mathbf{AC} = \mathbf{I} \quad \text{and} \quad \mathbf{CA} = \mathbf{I}.$$

Now consider the matrix $\mathbf{BAC}$. Since matrix multiplication is associative,

$$\mathbf{BAC} = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C},$$

and also

$$\mathbf{BAC} = \mathbf{B}(\mathbf{AC}) = \mathbf{BI} = \mathbf{B}.$$

Hence $\mathbf{B}$ and $\mathbf{C}$ are the same matrix. That is, $\mathbf{A}$ has only one inverse.

We usually write the inverse of an invertible matrix $\mathbf{A}$ as $\mathbf{A}^{-1}$. So the following facts hold.

Inverse of a matrix

If $\mathbf{A}$ is an invertible matrix, then

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{A}$.

It follows from the definition of the inverse of a matrix that if $\mathbf{A}$ is an invertible matrix, then not only is $\mathbf{A}^{-1}$ the inverse of $\mathbf{A}$, but also $\mathbf{A}$ is the inverse of $\mathbf{A}^{-1}$. In other words, $\mathbf{A}$ and $\mathbf{A}^{-1}$ are inverses of each other.

In the rest of this subsection, you'll learn how to establish whether any particular 2×2 matrix has an inverse, and how to work out the inverse when it exists. You'll also use a computer to calculate inverses of larger matrices.

Inverses of 2×2 matrices

For 2×2 matrices, there is a useful formula for finding inverses.

Inverse of a 2×2 matrix

The inverse of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{provided that } ad - bc \neq 0.$$

In other words, to obtain the inverse when $ad - bc \neq 0$, swap the two elements on the leading diagonal and multiply the other two elements by -1, then multiply the resulting matrix by the scalar $1/(ad - bc)$.

You'll be asked to verify this general formula later in this subsection, but for now let's check that it works for the matrix

$$\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}$$

in Activity 21(a). Here $ad - bc = 8 \times 2 - 5 \times 3 = 1 \neq 0$, so the scalar $1/(ad - bc)$ is 1. You can see that the matrix

$$\mathbf{B} = \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix}$$

in Activity 21(a) is indeed obtained from $\mathbf{A}$ by swapping 8 and 2, multiplying the other two elements by -1, and multiplying the resulting matrix by the scalar 1.

Notice that the box above includes the condition $ad - bc \neq 0$. This is because the formula involves dividing by $ad - bc$, and division by 0 is not defined. If the elements of a 2×2 matrix do not satisfy this condition, then the matrix does *not* have an inverse. You'll see a proof of this fact in the next section.

In fact the quantity $ad - bc$ that appears in the formula is so important that it deserves a name, as follows.

Determinant of a 2×2 matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the number $ad - bc$ is called the **determinant** of $\mathbf{A}$, and written as $\det \mathbf{A}$.

If the determinant of $\mathbf{A}$ is not zero, then $\mathbf{A}$ is invertible.

If the determinant of $\mathbf{A}$ is zero, then $\mathbf{A}$ is not invertible.

Some texts denote $\det \mathbf{A}$ by $|\mathbf{A}|$, which looks like the notation for the modulus of a real number, but has a different meaning in this context. This notation is also used when a matrix is written out in full; for example,

$$\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = 2 \times 0 - 1 \times (-3).$$

When seeking the inverse of a 2×2 matrix, you should first check that the matrix is invertible by calculating the determinant. If the determinant is not zero, then you can use the rule for finding the inverse.

Example 4 *Finding inverses of 2×2 matrices*

For each of the matrices below, check whether its inverse exists, and find it if it does.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 6 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$

Solution

(a) Let $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 6 \end{pmatrix}$.

Check whether $\det \mathbf{A} = 0$.

Here $\det \mathbf{A} = 24 - 21 = 3$. Since $\det \mathbf{A} \neq 0$, the matrix has an inverse.

Use the formula to write down the inverse of $\mathbf{A}$.

The inverse is

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -7 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{pmatrix}.$$

If you wish, check the answer by verifying that $\mathbf{AA}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Check: we have

$$\begin{pmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 4 & 7 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) For $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$, we have

$$\det \mathbf{B} = 3 \times 8 - 4 \times 6 = 24 - 24 = 0.$$

The determinant of $\mathbf{B}$ is zero.

Thus $\mathbf{B}$ is not invertible; that is, $\mathbf{B}^{-1}$ does not exist.

You can practise finding the inverses of 2×2 matrices in the next activity.

Activity 22 *Finding inverses of 2×2 matrices*

Determine which of the following matrices are invertible. For each matrix that is invertible, write down its inverse.

(a) $\mathbf{A} = \begin{pmatrix} 13 & 5 \\ 5 & 2 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 3 & 0 \\ -5 & 2 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} -3 & \frac{1}{5} \\ 5 & -\frac{1}{3} \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} 1.5 & 2.5 \\ 0.5 & 1.5 \end{pmatrix}$

In the next activity you're asked to verify that the formula for the inverse of a 2×2 matrix works in general.

Activity 23 *Verifying the formula for the inverse of a 2×2 matrix*

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc \neq 0$, and $\mathbf{B} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Work out the products $\mathbf{AB}$ and $\mathbf{BA}$, and deduce that $\mathbf{B} = \mathbf{A}^{-1}$.

Hint: use the fact that $\mathbf{P}(k\mathbf{Q}) = k(\mathbf{PQ})$ for any matrices $\mathbf{P}$ and $\mathbf{Q}$ for which the product $\mathbf{PQ}$ is defined and for any scalar k.

A matrix that does not have an inverse is called a **non-invertible** matrix. In some texts, non-invertible matrices are called **singular** matrices, and invertible matrices are called **non-singular** matrices.

Determinants and inverses of larger matrices

You have seen that a 2×2 matrix has an inverse only if its determinant is not zero.

The determinant is defined for square matrices of any size, and it plays the same role as in the 2×2 case: any square matrix $\mathbf{A}$ such that $\det \mathbf{A} \neq 0$ has an inverse $\mathbf{A}^{-1}$. If $\det \mathbf{A} = 0$, then no inverse can be found.

It is possible to use hand calculations to work out the determinant and the inverse (when it exists) of a square matrix whose size is larger than 2×2, but the process is rather complicated, and beyond the scope of this module. However, you can use the computer algebra system to calculate determinants and inverses of larger matrices.

Activity 24 *Working with matrix inverses on a computer*

Work through Subsection 10.2 of the *Computer algebra guide*.

4 Simultaneous linear equations and matrices

In Section 3 of Unit 2, you met simultaneous linear equations and you saw two methods (substitution and elimination) for solving them. Here you'll meet a third method, which uses matrices.

4.1 Solving simultaneous linear equations in two unknowns

Consider the following simultaneous linear equations in the two unknowns x and y:

$$\begin{aligned} 0.4x + 0.2y &= 4 \\ 0.6x + 0.8y &= 11. \end{aligned} \tag{3}$$

There's a useful way to write equations (3) as a single equation involving two-dimensional vectors (which are 2×1 matrices). To do this, you write the sides of each equation as the components of a vector, as follows:

$$\begin{pmatrix} 0.4x + 0.2y \\ 0.6x + 0.8y \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

You can now write the left-hand side of this equation as the product of a 2×2 matrix and a two-dimensional vector:

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

The 2×2 matrix that appears in this equation is called the *coefficient matrix* of the simultaneous linear equations (3). Its elements are the numbers that appear on the left-hand sides of equations (3). Any pair of simultaneous linear equations in two unknowns can be written in this form.

Matrix form of simultaneous linear equations

The simultaneous linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

can be written as the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the **coefficient matrix**.

Note that the coefficient matrix of simultaneous linear equations depends on how the equations are written. For example, if you write equations (3) as

$$\begin{aligned} 0.6x + 0.8y &= 11 \\ 0.4x + 0.2y &= 4, \end{aligned}$$

then you obtain the coefficient matrix

$$\begin{pmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{pmatrix},$$

which is different from the coefficient matrix obtained above for the same equations.

Example 5 *Writing simultaneous linear equations in matrix form*

Write the following pair of equations in matrix form.

$$y = 2x - 1$$
$$3y = 2$$

Solution

Rearrange each equation so that the unknowns are on the left-hand side and in the same order in each equation, and the constant terms are on the right-hand sides.

The equations can be rewritten as follows:

$$\begin{aligned} 2x + (-1)y &= 1 \\ 0x + \quad 3y &= 2. \end{aligned}$$

Now read off the matrix form from the equations.

Therefore the equations can be written in matrix form as

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

There are other possible solutions to Example 5. For example, the equations can be rewritten in the form

$$\begin{aligned} -2x + \ y &= -1 \\ 0x + 3y &= 2, \end{aligned}$$

which gives the matrix form

$$\begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

In the next activity you can practise finding matrix forms of pairs of simultaneous linear equations.

Activity 25 *Writing simultaneous linear equations in matrix form*

Write each of the following pairs of equations in matrix form.

(a) $2x + 3y = 3$
$x + 4y = -1$

(b) $2x - 6 \quad = 0$
$3x + 6y = 15$

(c) $\frac{3}{5}x - \frac{4}{5}y = 18$
$\frac{4}{5}x + \frac{3}{5}y = -1$

The matrix form of a pair of simultaneous linear equations is not just convenient shorthand: it is the initial step in a method for solving the equations. Let's see how this method works for equations (3) from the beginning of this subsection, which have the matrix form

$$\begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

Let $\mathbf{A}$ be the coefficient matrix; that is,

$$\mathbf{A} = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix}.$$

Then the matrix form of the equations can be written as

$$\mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

Since $\det \mathbf{A} = 0.4 \times 0.8 - 0.2 \times 0.6 = 0.2$, the matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$. If we multiply both sides of the equation above by $\mathbf{A}^{-1}$, then the result is

$$\mathbf{A}^{-1} \left(\mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} \right) = \mathbf{A}^{-1} \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

By the associativity of matrix multiplication, this equation can be rearranged as:

$$(\mathbf{A}^{-1}\mathbf{A}) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix, the equation above can be rewritten as

$$\mathbf{I} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 4 \\ 11 \end{pmatrix}.$$

Since $\mathbf{I}$ is an identity matrix, our equation becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 4 \\ 11 \end{pmatrix}. \tag{4}$$

Therefore, if we work out $\mathbf{A}^{-1}$ and calculate the product on the right-hand side of this matrix equation, then we'll find a solution to the original equations. Now

$$\mathbf{A}^{-1} = \frac{1}{0.2} \begin{pmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}.$$

So substituting for $\mathbf{A}^{-1}$ in equation (4) gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

In other words,

$$x = 5 \quad \text{and} \quad y = 10.$$

The example above illustrates a general method for solving simultaneous linear equations, which can be summarised as follows.

Strategy:
To solve a pair of simultaneous linear equations in two unknowns using matrices

Write the simultaneous linear equations in matrix form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the corresponding vector of unknowns, and $\mathbf{b}$ is the vector whose components are the corresponding right-hand sides of the equations.

If the matrix $\mathbf{A}$ is invertible, then the solution is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Example 6 *Using matrices to solve simultaneous linear equations*

Use matrices to solve the following pair of simultaneous linear equations.

$$x - 5y = -3$$
$$x + 3y = 13$$

Solution

Find the matrix form of the equations.

The matrix form of the equations is

$$\begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 13 \end{pmatrix}.$$

Find the inverse of the coefficient matrix.

The determinant of the coefficient matrix is

$$1 \times 3 - (-5) \times 1 = 8,$$

so the inverse of the coefficient matrix is

$$\frac{1}{8}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} \\ -\frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

Use the inverse matrix to solve the simultaneous equations.

Hence

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{8}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 13 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 2 \end{pmatrix}.\end{aligned}$$

The solution is $x = 7$, $y = 2$.

In the calculation above it is convenient to use the form of the inverse matrix with the fraction $1/8$ outside the matrix, in order to simplify the working.

As usual, you can check the answer to Example 6 by substituting back into the original equations.

You can practise the method in Example 6 in the next activity.

Activity 26 *Using matrices to solve simultaneous linear equations*

Use matrices to solve the following pairs of simultaneous linear equations. (You were asked to write these pairs of equations in matrix form in Activity 25.)

(a) $\begin{aligned} 2x + 3y &= 3 \\ x + 4y &= -1 \end{aligned}$ (b) $\begin{aligned} 2x - 6 \quad &= 0 \\ 3x + 6y &= 15 \end{aligned}$ (c) $\begin{aligned} \tfrac{3}{5}x - \tfrac{4}{5}y &= 18 \\ \tfrac{4}{5}x + \tfrac{3}{5}y &= -1 \end{aligned}$

4.2 Non–invertible coefficient matrices

You saw in Unit 2 that a pair of simultaneous linear equations in two unknowns can have no solutions, exactly one solution or infinitely many solutions. If it has exactly one solution, then we say that it has a *unique* solution.

In this subsection you'll see that a simple way to tell whether a pair of simultaneous linear equations in two unknowns has a unique solution is to work out the determinant of its coefficient matrix.

If the determinant is non-zero, then the coefficient matrix has an inverse given by the formula that you saw in Subsection 3.2, and hence the equations have a unique solution, by what you saw in Subsection 4.1.

Let's now look at what happens when the determinant is zero. Let's start by looking at two examples.

First, consider the following pair of simultaneous equations:

$$\begin{aligned} 2x - 3y &= 5 \\ -4x + 6y &= 7. \end{aligned} \tag{5}$$

The coefficient matrix is

$$\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix},$$

which has determinant $2 \times 6 - (-3) \times (-4) = 0$. Let's look for a solution to the equations using the elimination method. If we multiply both sides of the first one of equations (5) by -2, then we get

$$\begin{aligned} -4x + 6y &= -10 \\ -4x + 6y &= 7. \end{aligned}$$

These equations have identical left-hand sides, but different right-hand sides. So they have no solution: no values of x and y can make $-4x + 6y$ equal to both -10 and 7.

Now consider the following pair of simultaneous equations:

$$\begin{aligned} x + 2y &= -6 \\ -3x - 6y &= 18. \end{aligned} \tag{6}$$

The coefficient matrix is

$$\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix},$$

which has determinant $1 \times (-6) - 2 \times (-3) = 0$. Again, let's look for a solution to the equations using the elimination method. If we multiply both sides of the first of equations (6) by -3, then we get

$$\begin{aligned} -3x - 6y &= 18 \\ -3x - 6y &= 18. \end{aligned}$$

These equations are identical. So they have infinitely many solutions: all values of x and y that satisfy the first equation also satisfy the second equation.

You've now seen two cases where the coefficient matrix has determinant zero. In the first case, the simultaneous equations had no solution. In the second case, they had infinitely many solutions. In fact, whenever the coefficient matrix of a pair of simultaneous linear equations has determinant zero, one of these two possibilities must occur.

To see why, consider any two simultaneous linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f, \end{aligned}$$

and suppose that the determinant of the coefficient matrix is zero; that is, $ad - bc = 0$. We can assume that in each equation at least one of the coefficients is non-zero; otherwise, the equation is of the form $0x + 0y =$ constant, so the pair of equations has either no solutions or infinitely many solutions.

Suppose that in the first equation $a \neq 0$; a similar argument holds if $b \neq 0$. Since $ad - bc = 0$, we have $d = bc/a$. Hence $c \neq 0$, since if $c = 0$ then $d = 0$ also. Multiplying the first equation by c/a gives

$$\begin{aligned} cx + (bc/a)y &= ce/a \\ cx + dy &= f. \end{aligned}$$

Since $d = bc/a$, these two equations can be rewritten as

$$\begin{aligned} cx + dy &= ce/a \\ cx + dy &= f. \end{aligned}$$

These equations have identical left-hand sides. So they (and hence the original equations) have either no solution (if $ce/a \neq f$) or infinitely many solutions (if $ce/a = f$).

So, in conclusion, if the coefficient matrix has determinant zero, then the equations have either no solution or infinitely many solutions.

We can deduce from this fact that if a 2×2 matrix has determinant zero, then it is *not* invertible. This is a fact that was mentioned in Subsection 3.2. To see how, consider any 2×2 matrix with determinant zero. If the matrix *were* invertible, then any pair of simultaneous equations with this matrix as its coefficient matrix would have a unique solution, which cannot be the case, since you saw above that if the determinant of the coefficient matrix is zero, then the equations have no solution or infinitely many solutions.

So you have learned the following important payoffs from the matrix method for simultaneous linear equations.

For a pair of simultaneous linear equations in two unknowns:

- if the determinant of the coefficient matrix is non-zero, then this matrix is invertible and the equations have a unique solution
- if the determinant of the coefficient matrix is zero, then this matrix is not invertible and the equations have no solution or infinitely many solutions.

Thus, calculating the determinant of the coefficient matrix gives a quick test for the existence of a unique solution.

Activity 27 *Determining whether simultaneous equations have unique solutions*

Which of the following pairs of simultaneous linear equations has a unique solution? (You are not asked to find the solution.)

(a) $\begin{aligned} 5x - 3y &= 5 \\ -x + 4y &= 7 \end{aligned}$ (b) $\begin{aligned} 3x - y &= -6 \\ -9x + 3y &= 18 \end{aligned}$

4.3 Simultaneous linear equations in more than two unknowns

The matrix method that you met in Subsection 4.2 can be used to find a solution for more than two simultaneous equations in more than two unknowns.

In Subsection 5.3 of Unit 1, you learned that a *linear* equation is one in which, after you've expanded any brackets and cleared any fractions, each term is either a constant term or a number times a variable. For example, the equation

$$x + 2y - z = 3$$

is linear, but none of the equations

$$x + y^2 = 2, \quad x + yz = 1 \quad \text{and} \quad x + 2^y = 1$$

are linear. In this subsection we'll usually write a linear equation in a form similar to the linear equation above, that is, with its left-hand side as a sum of terms, each of which is a number times a variable (and with each variable appearing just once), and with the right-hand side as a number.

We'll refer to a collection of linear equations in a given set of unknowns as a **system of linear equations**. For example, here's a system of three linear equations in three unknowns, x, y and z:

$$\begin{aligned} 2x + 2y + 3z &= 2 \\ 2x + 3y + 4z &= -1 \\ 4x + 7y + 10z &= 3. \end{aligned}$$

A **solution** to a system of linear equations is an assignment of values to the unknowns that makes all the equations hold simultaneously. As in Subsection 3.1 of Unit 2, the process of finding these values is known as *solving the equations simultaneously*. We'll look only at systems where the number of unknowns is equal to the number of equations.

Numerical problems that can be modelled by systems of linear equations often arise in applications. Consider, for example, a bank that offers three types of investment to its customers: a financial product A that pays guaranteed interest of 2.5% over two years and is quite safe, and two financial products B and C that in the same time period are likely to yield interests of 5% and 6% respectively, but are more risky.

Suppose that a customer is seeking to invest £9000 over two years, and is advised to spread his investment across the three types of financial products. He is risk-averse, so he wants to invest a total of only £4000 across the two riskier products B and C. Of this £4000, he wants to invest just enough in the higher-yielding product C to ensure that his total £9000 investment is likely to yield an overall return of 4%, with the remainder of the £4000 invested in the lower-yielding product B, to spread his risks. How should he spread his investment across the three products in order to achieve his desired return?

First of all, the amounts invested in each product must add up to £9000. So, if x, y and z represent the amounts invested (in £) in products A, B and C, respectively, then

$$x + y + z = 9000.$$

Moreover, the investor wants to invest a total of £4000 in products B and C, so

$$y + z = 4000.$$

Since the interest on his investment in product A is 2.5%, the investor's expected interest payment from his investment in product A is $0.025x$. Likewise, his expected interest payments from his investments in products B and C are $0.05y$ and $0.06z$, respectively. The investor is seeking an overall return of 4%, that is, an overall interest payment of $0.04 \times £9000 = £360$. Thus the amounts invested in the three products must also satisfy the equation

$$0.025x + 0.05y + 0.06z = 360.$$

Therefore, the amounts x, y and z must simultaneously satisfy the three linear equations

$$\begin{aligned} x + \quad y + \quad z &= 9000 \\ y + \quad z &= 4000 \\ 0.025x + 0.05y + 0.06z &= 360. \end{aligned} \tag{7}$$

Such systems of linear equations can be solved efficiently by methods similar to elimination and substitution – you can probably see that equations (7) can be solved fairly easily for x, y and z by

- first subtracting the second equation from the first to give $x = 5000$
- then substituting this value for x into the third equation and solving the resulting two simultaneous equations for y and z.

Methods of this sort are covered in detail in other modules. Systems of linear equations like those above can also be solved by using matrices, and the rest of this subsection covers this approach.

Any system of n linear equations in n unknowns can be written as a single matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ coefficient matrix, $\mathbf{x}$ is an n-dimensional vector whose components are the unknowns, and $\mathbf{b}$ is an n-dimensional vector, as you saw for the case $n = 2$ in Subsection 4.2. If the coefficient matrix $\mathbf{A}$ is invertible, then the matrix equation can be solved by multiplying both sides by the inverse of $\mathbf{A}$, giving $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, just as in the 2×2 case.

In Section 3 you learned how to use a computer to find the inverses of matrices of size 3×3 and larger. This enables you to use the matrix method to solve systems of three or more linear equations. In the next example, this method is used to solve equations (7).

Example 7 *Solving a system of three linear equations*

Use the matrix method to solve the following system of three linear equations.

$$\begin{aligned} x + \quad y + \quad z &= 9000 \\ y + \quad z &= 4000 \\ 0.025x + 0.05y + 0.06z &= 360 \end{aligned}$$

Solution

The matrix form of the system of three equations is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0.025 & 0.05 & 0.06 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9000 \\ 4000 \\ 360 \end{pmatrix}.$$

Use the computer algebra system to find the inverse of the coefficient matrix.

The inverse of the coefficient matrix is

$$\begin{pmatrix} 1 & -1 & 0 \\ 2.5 & 3.5 & -100 \\ -2.5 & -2.5 & 100 \end{pmatrix}.$$

Multiply each side of the matrix equation by the inverse of the coefficient matrix.

Therefore the solution is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 2.5 & 3.5 & -100 \\ -2.5 & -2.5 & 100 \end{pmatrix} \begin{pmatrix} 9000 \\ 4000 \\ 360 \end{pmatrix} = \begin{pmatrix} 5000 \\ 500 \\ 3500 \end{pmatrix}.$$

Note that you could use the computer algebra system to solve the simultaneous equations (7) directly, that is, without first finding the inverse of the coefficient matrix, and so check the answer to Example 7.

The solution to equations (7) found in this example tells our investor that the best way to spread his investment in order to achieve a 4% return is as follows:

£5000 on investment A
£500 on investment B
£3500 on investment C.

You can practise finding inverses of matrices and using them to solve systems of more than two linear equations in the next activity.

Activity 28 *Solving systems of three linear equations*

Solve each of the following systems of simultaneous linear equations by using the computer algebra system to find the inverse of the coefficient matrix.

(a)
$$\begin{aligned} 2x + 2y + 3z &= 2 \\ 2x + 3y + 4z &= -1 \\ 4x + 7y + 10z &= 3 \end{aligned}$$

(b)
$$\begin{aligned} 3x + 2y + z &= 1 \\ 4x + 3y + z &= 2 \\ 7x + 5y + z &= 1 \end{aligned}$$

In fact using matrix inverses is not usually the most efficient way to solve systems of linear equations, but the method has theoretical importance.

For example, it can be shown that, for all $n \geq 2$,

- if the determinant of the coefficient matrix of a system of n equations is non-zero, then this matrix is invertible and the equations have a unique solution
- if the determinant of the coefficient matrix of a system of n equations is zero, then this matrix is not invertible and the equations have no solution or infinitely many solutions.

You saw the case $n = 2$ of this result in Subsection 4.2.

Example 8 *Determining whether a system of linear equations has a unique solution*

Does the system

$$\begin{aligned} x + 3y - 2z &= 12 \\ \tfrac{1}{2}x - 8y + z &= 0 \\ 7x \phantom{{}- 8y} - z &= 2 \end{aligned}$$

have a unique solution?

Solution

The coefficient matrix of this system is

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ \frac{1}{2} & -8 & 1 \\ 7 & 0 & -1 \end{pmatrix}.$$

Using the computer algebra system we find that $\det \mathbf{A} = -81.5$. Since $\det \mathbf{A} \neq 0$, the system has a unique solution.

The following activity provides further practice on the significance of the determinant of the coefficient matrix for systems of three or more linear equations.

Activity 29 *Determining whether systems of linear equations have unique solutions*

Which of the following systems of linear equations have a unique solution?

(a)
$$\begin{aligned} x + 2y + 3z &= 5\\ 2x + 3y + z &= 6\\ 4x + 7y + 7z &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 2x + 2y + 3z &= 5\\ 2x + 3y + z &= 6\\ 4x + 7y + 7z &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x + y - w + z &= 0\\ x - y + w + z &= 2\\ 2x \phantom{{}+2y} + 3w - z &= 1\\ 6x + 2y + 4w \phantom{{}+z} &= 0 \end{aligned}$$

Learning outcomes

After studying this unit, you should be able to:

- use matrix notation
- evaluate sums, differences and scalar multiples of matrices
- establish whether two matrices can be multiplied together
- evaluate a matrix product both by hand and by using the computer algebra system
- determine whether a given $n \times n$ matrix has an inverse
- work out the determinant and inverse (if it exists) of a 2×2 matrix by hand
- use the computer algebra system to work out the determinant and inverse (if it exists) of a square matrix
- write a system of linear equations in matrix form and solve it using matrix methods where possible.

Solutions to activities

Solution to Activity 1

We have $a_{33} = 3$, $a_{34} = -4$, $a_{43} = -7$, $a_{14} = -1$, $a_{44} = -9$, $a_{42} = -5$.

The matrix $\mathbf{A}$ has four rows and four columns, so its size is 4×4.

Solution to Activity 2

(a) $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 6 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & -1 \end{pmatrix}$

(b) $$\begin{aligned}\mathbf{A}+\mathbf{B} &= \begin{pmatrix} 2 & 0 & -4 \\ -1 & 6 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & -3 \\ -1 & 7 & 1 \end{pmatrix}\end{aligned}$$

(c) $$\begin{aligned}\mathbf{A}+\mathbf{B} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} a+3 & b-1 \\ c-1 & d+2 \end{pmatrix}\end{aligned}$$

Solution to Activity 3

(a) $\mathbf{A}-\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -6 \\ 2 & -2 \end{pmatrix}$

(b) $\mathbf{B}-\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 4 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 6 \\ -2 & 2 \end{pmatrix}$

(c) $$\begin{aligned}\mathbf{B}-\mathbf{C} &= \begin{pmatrix} \frac{1}{2} & 4 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 4+x \\ -1-x & 2 \end{pmatrix}\end{aligned}$$

Solution to Activity 4

(a) (i) $3\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 6 & 3 \end{pmatrix}$

(ii) $\frac{1}{2}\begin{pmatrix} 4 & 6 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -\frac{1}{2} \end{pmatrix}$

(iii) $\frac{2}{3}\begin{pmatrix} 6 & x \\ 3 & y \end{pmatrix} = \begin{pmatrix} 4 & \frac{2}{3}x \\ 2 & \frac{2}{3}y \end{pmatrix}$

(b) (i) $\begin{pmatrix} 4.5 & 3 \\ 2 & 1.5 \end{pmatrix} = 0.5\begin{pmatrix} 9 & 6 \\ 4 & 3 \end{pmatrix}$

(ii) $\begin{pmatrix} \frac{3}{2} \\ \frac{2}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 9 \\ 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 2x & 3x \\ 0 & 5x \end{pmatrix} = x\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$

In each of the answers in this part, there is more than one way to choose the scalar multiple. For example, another correct solution to part (i) is

$$\begin{pmatrix} 4.5 & 3 \\ 2 & 1.5 \end{pmatrix} = 0.25\begin{pmatrix} 18 & 12 \\ 8 & 6 \end{pmatrix}.$$

Solution to Activity 5

We have

$$\begin{aligned}\mathbf{A}-\mathbf{C} &= \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 4 & -2 & 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} & 0 & -1 \\ -1 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 3 \\ 5 & -5 & -5 \end{pmatrix}.\end{aligned}$$

Next,

$$2\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 8 & -4 & 0 \end{pmatrix}.$$

The matrix $2\mathbf{A}$ has size 2×3 and $\mathbf{B}$ has size 2×2; hence the sum $\mathbf{B}+2\mathbf{A}$ is not defined.

Finally,

$$\begin{aligned}\mathbf{C}-2\mathbf{A} &= \begin{pmatrix} \frac{3}{2} & 0 & -1 \\ -1 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 4 \\ 8 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -2 & -5 \\ -9 & 7 & 5 \end{pmatrix}.\end{aligned}$$

Solution to Activity 6

For each pair of matrices in this question, the sizes of the matrices are written underneath the matrices, and the middle two numbers are boxed in order to use the method described in the text.

(a) $$\begin{pmatrix} 2 & 9 \\ 7 & 1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} -2 & 3 & 1 \\ 4 & 0 & 5 \end{pmatrix}$$

$$3 \times \boxed{2} \quad \boxed{2} \times 3$$

The middle numbers are the same, so these matrices can be multiplied together. Their product is

$$\begin{pmatrix} 2 & 9 \\ 7 & 1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} -2 & 3 & 1 \\ 4 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 32 & 6 & 47 \\ -10 & 21 & 12 \\ 44 & -18 & 34 \end{pmatrix}.$$

(b) $$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

$$1 \times \boxed{3} \quad \boxed{3} \times 1$$

The middle numbers are the same, so these matrices can be multiplied together. Their product is

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} = (46).$$

(c) $$\begin{pmatrix} 2 & 9 \\ 7 & 1 \\ -6 & 8 \end{pmatrix} \begin{pmatrix} 0 & 10 \\ 4 & 1 \\ -2 & 7 \end{pmatrix}$$

$$3 \times \boxed{2} \quad \boxed{3} \times 2$$

The middle numbers are different, so these matrices cannot be multiplied together.

(d) $$\begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$3 \times \boxed{1} \quad \boxed{1} \times 3$$

The middle numbers are the same, so these matrices can be multiplied together. Their product is

$$\begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 18 & 27 \\ 8 & 16 & 24 \\ 7 & 14 & 21 \end{pmatrix}.$$

(e) $$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$2 \times \boxed{3} \quad \boxed{3} \times 3$$

The middle numbers are the same, so these matrices can be multiplied together. Their product is

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 8 \\ 15 & 1 & 4 \end{pmatrix}.$$

Solution to Activity 7

(a) (i) Here $\mathbf{A}$ has size 2×2 and $\mathbf{B}$ has size 2×1. For $\mathbf{AB}$, the size check is

$$2 \times \boxed{2} \quad \boxed{2} \times 1.$$

The middle numbers are the same, so the product $\mathbf{AB}$ is defined.

For $\mathbf{BA}$, the size check is

$$2 \times \boxed{1} \quad \boxed{2} \times 2.$$

The middle numbers are different, so the product $\mathbf{BA}$ is undefined.

(ii) Here both $\mathbf{A}$ and $\mathbf{B}$ have size 2×2, so the products $\mathbf{AB}$ and $\mathbf{BA}$ are both defined.

(b) For the matrices in part (a)(i), the product $\mathbf{BA}$ is not defined.

For the matrices in part (a)(ii),

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}$$

and

$$\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix}.$$

Therefore $\mathbf{AB} \neq \mathbf{BA}$.

Solution to Activity 8

(a) The size of $\mathbf{A}$ is 1×2 and the size of $\mathbf{B}$ is 2×2. The size check for the product $\mathbf{AB}$ is

$$1 \times \boxed{2} \quad \boxed{2} \times 2.$$

Thus $\mathbf{AB}$ has size 1×2.

The size of $\mathbf{C}$ is 2×3. The size check for the product $\mathbf{BC}$ is

$$2 \times \boxed{2} \quad \boxed{2} \times 3.$$

Thus $\mathbf{BC}$ has size 2×3.

We have

$$\mathbf{AB} = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \end{pmatrix}, \text{ and}$$

$$\begin{aligned} \mathbf{BC} &= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ -3 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -1 & -1 \\ 5 & 2 & -1 \end{pmatrix}. \end{aligned}$$

(b) From part (a), the size of $\mathbf{AB}$ is 1×2. Since $\mathbf{C}$ has size 2×3, the size check for $(\mathbf{AB})\mathbf{C}$ is

$$1 \times \boxed{2} \quad \boxed{2} \times 3.$$

So the product $(\mathbf{AB})\mathbf{C}$ exists and has size 1×3.

From part (a), the size of $\mathbf{BC}$ is 2×3. Since $\mathbf{A}$ has size 1×2, the size check for $\mathbf{A}(\mathbf{BC})$ is

$$1 \times \boxed{2} \quad \boxed{2} \times 3.$$

So the product $\mathbf{A}(\mathbf{BC})$ exists and has size 1×3.

Using the answers for the matrices $\mathbf{AB}$ and $\mathbf{BC}$ found in part (a) we obtain

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ -3 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 1 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}(\mathbf{BC}) &= \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 & -1 \\ 5 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 1 & 1 \end{pmatrix}. \end{aligned}$$

We can see that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

Solution to Activity 9

We have

$$2\mathbf{B} = 2\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix},$$

so

$$\mathbf{A}(2\mathbf{B}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 22 & 8 \end{pmatrix}.$$

Also,

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix},$$

so

$$2(\mathbf{AB}) = 2\begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 22 & 8 \end{pmatrix}.$$

Therefore $\mathbf{A}(2\mathbf{B}) = 2(\mathbf{AB})$, as required.

Solution to Activity 10

We have

$$\mathbf{B} + \mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

so

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}.$$

Also,

$$\mathbf{AB} = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix},$$

$$\mathbf{AC} = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

so

$$\mathbf{AB} + \mathbf{AC} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}.$$

Thus $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, as required.

Solution to Activity 11

(a) The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ can be formed only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. Since $\mathbf{M}$ has two rows and three columns, the number of its rows is not equal to the number of its columns. Thus the product $\mathbf{MM}$ cannot be formed.

(b) The number of rows of $\mathbf{N}$ is equal to the number of columns of $\mathbf{N}$, so the product $\mathbf{NN}$ is defined. We have

$$\mathbf{NN} = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix}.$$

(c) A matrix $\mathbf{A}$ must have the same number of rows as columns in order for the product $\mathbf{AA}$ to be defined. So only matrices that have the same number of rows as columns can be squared. (You might remember that such matrices are called *square* matrices.)

Solution to Activity 12

(a) $$\begin{pmatrix} -1 & 4 \\ 2 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 4 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -4 \\ -2 & 8 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \end{aligned}$$

(b) From the text preceding this activity, we know that $\mathbf{A}^3 = \begin{pmatrix} 20 & 7 \\ 21 & 6 \end{pmatrix}$. Therefore

$$\begin{aligned}\mathbf{A}^6 &= \mathbf{A}^3\mathbf{A}^3 = (\mathbf{A}^3)^2 \\ &= \begin{pmatrix} 20 & 7 \\ 21 & 6 \end{pmatrix}^2 = \begin{pmatrix} 20 & 7 \\ 21 & 6 \end{pmatrix}\begin{pmatrix} 20 & 7 \\ 21 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 547 & 182 \\ 546 & 183 \end{pmatrix}.\end{aligned}$$

Solution to Activity 14

(a) Inputting 1 litre of water at node U gives 0.2 litres of water output at node C and 0.8 litres of water output at node D.

Inputting 2 litres of water at node V gives 2×0.6 litres of water output at node C and 2×0.4 litres of water output at node D.

So, in total, we have $0.2 + 2 \times 0.6 = 1.4$ litres of water output at node C, and $0.8 + 2 \times 0.4 = 1.6$ litres of water output at node D.

(b) Using a method similar to that of part (a) gives $0.2x + 0.6y + 0.3z$ litres of water output at node C and $0.8x + 0.4y + 0.7z$ litres of water output at node D.

Solution to Activity 15

(a) If £2000 is input at node D, then the output at node J is $0.2 \times £2000 = £400$.

Similarly, the output at node K is $0.8 \times £2000 = £1600$.

(b) If the input at node D is £2400, at node E is £1700, and at node F is £1100, then the output (in £) at node J is

$$0.2 \times 2400 + 0.3 \times 1700 + 0.5 \times 1100 = 1540,$$

and at node K is

$$0.8 \times 2400 + 0.7 \times 1700 + 0.5 \times 1100 = 3660.$$

(c) If the input (in £) at node D is x, at node E is y, and at node F is z, then the output (in £) at node J is

$$0.2x + 0.3y + 0.5z,$$

and at node K is

$$0.8x + 0.7y + 0.5z.$$

Solution to Activity 16

(a) Since there are three input nodes and two output nodes, the network can be represented by a 2×3 matrix.

The number that labels the arc from the first input node (U) to the first output node (C) is 0.2, and this goes in the first row and first column. The number that labels the arc from the second input node (V) to the first output node (C) is 0.6, and this goes in the first row and second column.

Proceeding similarly for all the inputs and outputs of the network gives the following matrix:

$$\begin{array}{cc} & \begin{array}{ccc} U & V & W \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{pmatrix} \end{array}.$$

(b) The vector that represents an input of 1 litre of water at nodes U, V and W is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(c) The output vector for the input vector found in part (b) is

$$\begin{aligned}&\begin{pmatrix} 0.2 & 0.6 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.2 \times 1 + 0.6 \times 1 + 0.3 \times 1 \\ 0.8 \times 1 + 0.4 \times 1 + 0.7 \times 1 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 1.9 \end{pmatrix}.\end{aligned}$$

This vector represents an output of 1.1 litres of water at node C and 1.9 litres of water at node D.

Solution to Activity 17

(a) The matrix that represents the network with nodes U, V, P and Q is

$$\begin{array}{c} \\ P \\ Q \end{array}\begin{array}{c} \begin{array}{cc} U & V \end{array} \\ \begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{pmatrix} \end{array}.$$

The matrix equation giving the output $\begin{pmatrix} p \\ q \end{pmatrix}$ for the input $\begin{pmatrix} u \\ v \end{pmatrix}$ is

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

(b) The matrix that represents the network with nodes P, Q, A, B and C is

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cc} P & Q \end{array} \\ \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \end{array}.$$

The matrix equation giving the output $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for the input $\begin{pmatrix} p \\ q \end{pmatrix}$ is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}.$$

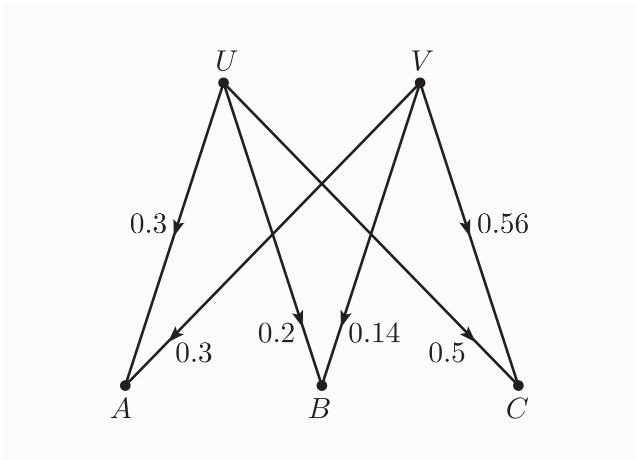
(c) The matrix equation giving the output $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for an input $\begin{pmatrix} u \\ v \end{pmatrix}$ in the combined network is obtained by using parts (a) and (b) as follows.

$$\begin{aligned}\begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \\ &= \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}\left(\begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}\right)\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.14 \\ 0.5 & 0.56 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\end{aligned}$$

(d) Labelling the matrix that represents the combined network gives

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cc} U & V \end{array} \\ \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.14 \\ 0.5 & 0.56 \end{pmatrix} \end{array}.$$

A simplified network equivalent to the original one is given below.



(e) An input of 1 litre at node U and 2 litres at node V is represented by the input vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using the equation for the combined network found in part (c), the corresponding output is given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.14 \\ 0.5 & 0.56 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.9 \\ 0.48 \\ 1.62 \end{pmatrix}.$$

Therefore the output is 0.9 litres at node A, 0.48 litres at node B and 1.62 litres at node C.

Solution to Activity 18

(a) The matrices (with row and column headings) are

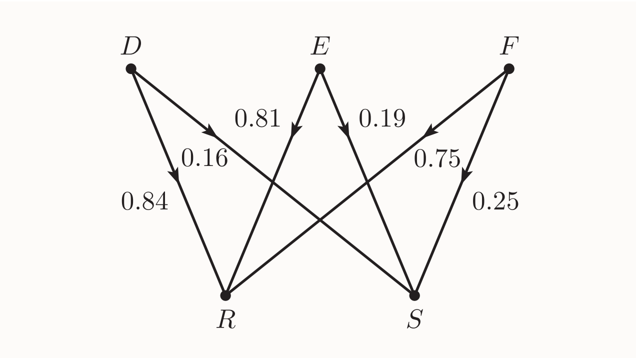
$$\begin{array}{c} \\ J \\ K \end{array}\begin{array}{c} \begin{array}{ccc} D & E & F \end{array} \\ \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.8 & 0.7 & 0.5 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \\ R \\ S \end{array}\begin{array}{c} \begin{array}{cc} J & K \end{array} \\ \begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \end{array},$$

respectively.

(b) The matrix that represents the combined network is

$$\begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix}\begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.8 & 0.7 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.84 & 0.81 & 0.75 \\ 0.16 & 0.19 & 0.25 \end{pmatrix}.$$

(c) The simplified network is shown below.



(d) If the input vector (in £) to the combined network is

$$\begin{pmatrix} 1700 \\ 1400 \\ 0 \end{pmatrix},$$

then the output vector (in £) is

$$\begin{pmatrix} 0.84 & 0.81 & 0.75 \\ 0.16 & 0.19 & 0.25 \end{pmatrix} \begin{pmatrix} 1700 \\ 1400 \\ 0 \end{pmatrix} = \begin{pmatrix} 2562 \\ 538 \end{pmatrix}.$$

So £538 is spent in region S.

Solution to Activity 19

The products $\mathbf{AI}$, $\mathbf{IA}$ and $\mathbf{IB}$ are defined, and they give the following results:

$$\mathbf{AI} = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix} = \mathbf{A},$$

$$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix} = \mathbf{A},$$

$$\mathbf{IB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 7 & -3 \\ 8 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 7 & -3 \\ 8 & 1 & 2 \end{pmatrix} = \mathbf{B}.$$

The product $\mathbf{BI}$ is not defined because $\mathbf{B}$ has 3 columns and $\mathbf{I}$ has 2 rows.

Solution to Activity 20

We have

$$\mathbf{AI} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

Also,

$$\mathbf{IB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Therefore $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IB} = \mathbf{I}$, as required.

Solution to Activity 21

(a) $$\mathbf{AB} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{BA} = \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) $$\mathbf{CD} = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 2 & -1 \\ 1 & -8 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{DC} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 2 & -1 \\ 1 & -8 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Activity 22

(a) We have $\det \mathbf{A} = 13 \times 2 - 5 \times 5 = 1 \neq 0$. Thus $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -5 \\ -5 & 13 \end{pmatrix}.$$

(b) We have $\det \mathbf{B} = 3 \times 2 - 0 \times (-5) = 6 \neq 0$. Thus $\mathbf{B}$ is invertible and

$$\mathbf{B}^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{5}{6} & \frac{1}{2} \end{pmatrix}.$$

(Either of these two forms of the inverse is acceptable as your final answer.)

(c) We have $\det \mathbf{C} = -3 \times (-\frac{1}{3}) - \frac{1}{5} \times 5 = 0$. Thus $\mathbf{C}$ is not invertible.

(d) We have $\det \mathbf{D} = 1.5 \times 1.5 - 2.5 \times 0.5 = 1 \neq 0$. Thus $\mathbf{D}$ is invertible and

$$\mathbf{D}^{-1} = \begin{pmatrix} 1.5 & -2.5 \\ -0.5 & 1.5 \end{pmatrix}.$$

Solution to Activity 23

Using the property of matrix products given in the hint, we have

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right) \\ &= \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= \frac{1}{ad-bc}\times(ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbf{BA} &= \left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \mathbf{I}.\end{aligned}$$

Hence $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, and it follows that $\mathbf{B} = \mathbf{A}^{-1}$.

Solution to Activity 25

(a) The coefficient matrix of the equations

$$\begin{aligned} 2x + 3y &= 3 \\ x + 4y &= -1 \end{aligned}$$

is $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Thus, the matrix form of the equations is

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

(b) First rearrange the first equation as

$$2x = 6.$$

The matrix form of the equations

$$\begin{aligned} 2x &= 6 \\ 3x + 6y &= 15 \end{aligned}$$

is

$$\begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}.$$

(c) The matrix form of the equations

$$\begin{aligned} \tfrac{3}{5}x - \tfrac{4}{5}y &= 18 \\ \tfrac{4}{5}x + \tfrac{3}{5}y &= -1 \end{aligned}$$

is

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -1 \end{pmatrix}.$$

Solution to Activity 26

(a) Activity 25(a) gives the matrix form of the equations as

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$2 \times 4 - 3 \times 1 = 5,$$

so the inverse of the coefficient matrix is

$$\frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

Hence

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \end{pmatrix}.\end{aligned}$$

The solution is $x = 3$, $y = -1$.

(b) Activity 25(b) gives the matrix form of the equations as

$$\begin{pmatrix} 2 & 0 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$2 \times 6 - 0 \times 3 = 12,$$

so the inverse of the coefficient matrix is

$$\frac{1}{12}\begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{6} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 15 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The solution is $x = 3$, $y = 1$.

(c) Activity 25(c) gives the matrix form of the equations as

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -1 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$\tfrac{3}{5} \times \tfrac{3}{5} - \left(-\tfrac{4}{5}\right) \times \tfrac{4}{5} = 1,$$

so the inverse of the coefficient matrix is

$$\frac{1}{1}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 18 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{50}{5} \\ -\frac{75}{5} \end{pmatrix} = \begin{pmatrix} 10 \\ -15 \end{pmatrix}.$$

The solution is $x = 10$, $y = -15$.

Solution to Activity 27

(a) The matrix form of the equations is

$$\begin{pmatrix} 5 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$5 \times 4 - (-3) \times (-1) = 17.$$

This is non-zero, so the coefficient matrix is invertible and the equations have a unique solution.

(b) The matrix form of the equations is

$$\begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 18 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$3 \times 3 - (-1) \times (-9) = 0.$$

This is zero, so the coefficient matrix is non-invertible and the equations do not have a unique solution.

Solution to Activity 28

(a) The matrix form of the equations is

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 7 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 7 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & 4 & -1 \\ 1 & -3 & 1 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & 4 & -1 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -11 \\ 8 \end{pmatrix}.$$

The solution is $x = 0$, $y = -11$, $z = 8$.

(b) The matrix form of the equations is

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 1 \\ 7 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 1 \\ 7 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 4 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 4 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}.$$

The solution is $x = -3$, $y = 4$, $z = 2$.

(Details of how to use the CAS to find the inverses of the matrices are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

Solution to Activity 29

(a) The matrix form of the equations is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 7 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}.$$

The determinant of the coefficient matrix is zero, so the coefficient matrix is non-invertible and the equations do not have a unique solution.

(b) The matrix form of the equations is

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 7 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}.$$

The determinant of the coefficient matrix is 14. This is non-zero, so the coefficient matrix is invertible and the equations have a unique solution.

(c) The matrix form of the equations is

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 2 & 0 & 3 & -1 \\ 6 & 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is zero, so the coefficient matrix is non-invertible and the equations do not have a unique solution.

(Details of how to use the CAS to find the determinants are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

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