Unit 14

Partial differential equations

Introduction

A **partial differential equation** is an equation relating a dependent variable and two or more independent variables through the *partial* derivatives of the dependent variable. Differential equations have played a very important role in the module so far. But until now, all the differential equations that you have met have involved just one independent variable, and have been equations containing one or more dependent variables and their *ordinary* derivatives with respect to that independent variable. Such equations are often called *ordinary* differential equations when it is necessary to distinguish them from partial differential equations. For many systems that we want to be able to model, ordinary differential equations are inadequate because the states of the system can be specified only in terms of two – or even more – independent variables. When we are trying to model the way in which such a system changes, we are inevitably led to consider partial differential equations.

Partial derivatives were introduced in Unit 7.

Ordinary differential equations are the subject of Units 1, 6 and 12.

This unit is an introduction to partial differential equations and their solution. The method of solution introduced here is called *separation of variables*. This idea is similar to, but distinct from, the method for solving first-order ordinary differential equations that is also called separation of variables, described in Unit 1. Section 1 introduces partial differential equations and concepts associated with them, then outlines the separation of variables method that is the core of this unit.

Partial differential equations have many applications, but here we describe just two of them. In Section 2 we look at a model of the transverse vibrations of a taut string, such as a guitar string. First the model is derived in terms of a partial differential equation called the *wave equation*, then the method of separation of variables is used to find particular solutions of the wave equation, such as the vibrations of a plucked string.

The vibrations of a guitar string are also considered in Unit 11.

Section 3 looks at a different application, namely the modelling of the flow of heat in a rod. This section introduces a physical law governing heat flow called *Newton's law of cooling*, then uses this law to derive a partial differential equation modelling the flow of heat that is called the *heat equation*. The section concludes by using the separation of variables method to find particular solutions of the heat equation. A physical phenomenon known as diffusion also satisfies the same partial differential equation, so the heat equation is sometimes known as the *diffusion equation*.

1 Solving partial differential equations

This section introduces partial differential equations and describes the method of separation of variables that is used to solve them.

1.1 Introducing partial differential equations

Both the wave equation and the heat equation are examples of second-order partial differential equations, where the notion of order for partial differential equations is defined in the same way as for ordinary differential equations.

The **order** of a partial differential equation is the order of the highest derivative that occurs.

The only issue worth highlighting here is that the order of mixed derivatives such as $\partial^2 u/\partial x\,\partial t$ is 2, because we count the total number of times that the function u is differentiated.

In this unit we consider only second-order partial differential equations, although the methods described are applicable to partial differential equations of any order. Furthermore, all the equations that we deal with are *linear*, where again linear is defined as it is for ordinary differential equations.

A **linear** partial differential equation is one that contains no products or non-linear functions of terms involving the dependent variable and its partial derivatives.

The following exercise asks you to apply these definitions to classify some partial differential equations.

Exercise 1

For each of the following differential equations, state whether it is linear or non-linear, and write down its order.

(a) $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ (b) $\dfrac{\partial^2 u}{\partial t^2} = x\dfrac{\partial u}{\partial x} + u$

(c) $\dfrac{\partial^2 u}{\partial t\,\partial x} = x^2 t$ (d) $u_{xxx} + u_x = 0$

We use the convention that a subscript is used to denote a partial derivative with respect to that variable. So the equation in part (d) could be written as $\dfrac{\partial^3 u}{\partial x^3} + \dfrac{\partial u}{\partial x} = 0$.

As we will see later, the wave equation is a good model of a plucked guitar string. We will use this context to introduce other aspects of partial differential equations. A partial differential equation model is required because the state of the string – by which is meant its shape at any given time after it has been plucked – requires a function $u(x,t)$ of two independent variables, x and t, where x is the distance along the straight line joining the two points at which the string is anchored (which we can consider as an axis with origin at one end of the string), and t is the time since the string was plucked.

The straight line joining the two points at which the string is anchored is the equilibrium position of the string. The dependent variable $u = u(x,t)$ is the transverse displacement of the string from the point on the axis determined by x, at time t. For fixed $t = t_1$ and varying x, $u(x,t_1)$ specifies the shape of the string at time t_1, as shown in Figure 1(a). On the other hand, we can think about a fixed $x = x_1$ and varying t, and then $u(x_1,t)$ tells us how the transverse displacement of the string from the fixed point $x = x_1$ on the axis varies with time (see Figure 1(b)).

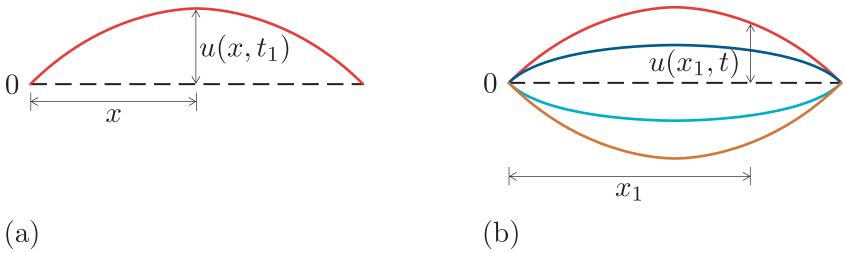

Figure 1 Motion of a plucked string: (a) string position with t fixed as t_1, and x varying; (b) string position with x fixed as x_1, and t varying – four snapshots in time are shown in different colours

The model of the motion of the string that we will develop will be a differential equation for the variable u. Since u depends on both x and t, an equation that models the motion of the string will involve partial derivatives of u with respect to both x and t. One such equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where c is a constant whose value depends on various physical characteristics of the string. This partial differential equation is called the **wave equation**. It is a very important equation of mathematical physics, in part because it occurs in many situations that involve vibrations of extended flexible objects like strings and springs, or indeed other wave motions such as sound waves and light waves.

The name *wave* equation is used because it models wave-like motions such as that of a plucked guitar string.

Checking whether or not a given function is a solution of a given partial differential equation is simply a matter of substituting the function into the equation, and seeing whether it is satisfied. The only difference from the case of an ordinary differential equation is that you have to calculate all the relevant partial derivatives.

Example 1

Check that

$$u(x,t) = \sin(kx)\cos(kct)$$

is a solution of the wave equation (1) for any constant k.

Solution

We have

$$\frac{\partial u}{\partial x} = k\cos(kx)\cos(kct), \quad \frac{\partial^2 u}{\partial x^2} = -k^2\sin(kx)\cos(kct),$$
$$\frac{\partial u}{\partial t} = -kc\sin(kx)\sin(kct), \quad \frac{\partial^2 u}{\partial t^2} = -k^2c^2\sin(kx)\cos(kct).$$

So when $u(x,t) = \sin(kx)\cos(kct)$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

and this function is indeed a solution of the wave equation.

Now try this yourself by attempting the following exercise.

Exercise 2

Check that $u(x,t) = \sin(x)\,e^{-\alpha t}$ is a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

This partial differential equation is known as the *heat equation*.

1.2 Initial conditions and boundary conditions

Initial conditions and boundary conditions for partial differential equations play the same role as they do for ordinary differential equations: they can be applied to a general solution to get a particular solution.

The initial conditions and/or boundary conditions appropriate to obtaining a particular solution of the wave equation or the heat equation depend on the context. For example, for the wave equation model of the vibrations of a guitar string, we need to use *two* boundary conditions and *two* initial conditions in order to have a unique solution. This is because the wave equation involves the second partial derivative with respect to x (hence the need for two boundary conditions) and also involves the second partial derivative with respect to t (hence the need for two initial conditions). In contrast, the heat equation (2) involves only the first partial derivative with respect to t, hence it requires only one initial condition in order to have a unique solution.

For the wave equation, if L is the equilibrium length of the string, then the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \quad t \geq 0; \tag{3}$$

these conditions correspond to the string being fixed at its ends. The initial conditions model the action of plucking the string, which sets it in motion. Plucking consists of holding the string in a certain shape, at rest, and then releasing it. If the initial shape of the string is given by a function $f(x)$, then the initial conditions may be specified in the form

$$u(x,0) = f(x), \quad 0 < x < L,$$
$$u_t(x,0) = 0, \quad 0 \le x \le L.$$

We use the range $0 < x < L$ for the initial condition $u(x,0) = f(x)$ because the values of $u(x,0)$ when $x = 0$ and $x = L$ are specified by boundary conditions (3).

The first initial condition models the initial shape of the string, while the second corresponds to it being at rest initially. The initial condition for a particular initial shape is given in the following example.

Example 2

A taut string of equilibrium length L is plucked at its midpoint, which is given an initial displacement $\frac{1}{2}$, as shown in Figure 2. It is then released from rest.

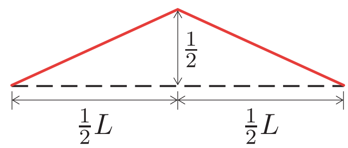

Figure 2 Initial position of a taut string plucked at its midpoint

Write down the initial conditions for the wave equation for the transverse vibrations of this string.

Solution

The displacement shown in Figure 2 has two linear sections, with slopes $\pm 1/L$. Hence the initial displacement is given by

$$u(x,0) = \begin{cases} x/L & \text{for } 0 < x \le \frac{1}{2}L, \\ (L-x)/L & \text{for } \frac{1}{2}L < x < L. \end{cases}$$

As the string is released from rest, the transverse component of the initial velocity is given by

$$u_t(x,0) = 0, \quad 0 \le x \le L.$$

Exercise 3

Suppose that the initial conditions for the transverse vibrations of a taut string are

$$u(x,0) = \begin{cases} -\dfrac{4d}{L}x & \text{for } 0 < x \le \frac{1}{4}L, \\ -\dfrac{4d}{3L}(L-x) & \text{for } \frac{1}{4}L < x < L, \end{cases}$$
$$u_t(x,0) = 0, \quad 0 \le x \le L.$$

Describe how the string has been set in motion.

Exercise 4

A taut string of equilibrium length L is plucked one-third of the way along its length, which is given an initial displacement d, as shown in Figure 3.

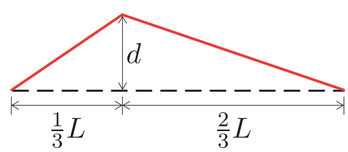

Figure 3 Initial position of a taut string plucked at a point a third of the way along its length

What is the corresponding initial condition, for this displacement, for the wave equation for the transverse vibrations of this string?

Exercise 5

A taut string is initially in its equilibrium position. At time $t = 0$, it is struck in such a way as to impart, instantaneously, a transverse velocity v (in the positive direction) to the middle third of the string. This is a simple model of a string that is hammered, such as a piano wire.

Modify the initial conditions for the wave equation for transverse vibrations of the string to model this situation.

Example 1 showed that any function of the form $u(x,t) = \sin(kx)\cos(kct)$ is a solution of the wave equation, for any constant k. In particular, therefore,

$$u(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) \qquad (4)$$

is a solution of the wave equation for transverse vibrations of a taut string, where L is the equilibrium length of the string. The following exercise asks you to show that this solution also satisfies fixed endpoint boundary conditions and the initial condition that the string starts from rest.

Exercise 6

Show that the solution given by equation (4) satisfies the boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t \geq 0,$$

and the initial condition

$$u_t(x,0) = 0, \quad 0 \leq x \leq L.$$

Equation (4) is not the only solution of the wave equation that satisfies the boundary conditions. You may like to verify that each member of the following family of functions $u_n(x,t)$ $(n = 1, 2, \ldots)$ also satisfies the wave equation, the fixed endpoint boundary conditions and the starting from rest initial condition:

$$u_n(x,t) = \sin\left(\frac{\pi nx}{L}\right)\cos\left(\frac{\pi nct}{L}\right). \qquad (5)$$

The first three members of this family of solutions are shown in Figure 4 at time $t = 0$.

(Note that $u_n(x,t)$ denotes a family of functions indexed by the discrete variable n; it is not the derivative with respect to n, as the notation would imply if n were a continuous variable.)

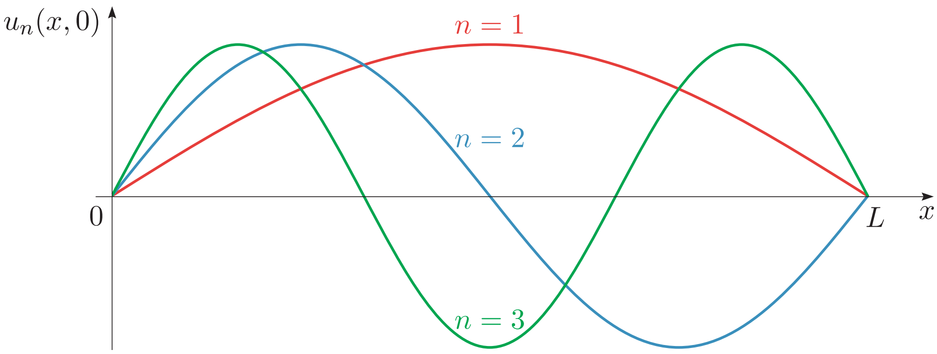

Figure 4 The first three transverse vibrations of a taut string

You may recognise the waves shown in Figure 4 from earlier physics or music theory as having the shape of the fundamental ($n = 1$) and harmonics ($n = 2, 3, \ldots$) of a vibrating string. You can observe that the larger the value of n, the higher the frequency (and the higher the pitch of the note).

In music we rarely hear pure tones such as the ones shown in Figure 4 as musical instruments each produce a characteristic mixture of pure tones that gives the characteristic sound (known as the timbre) of an instrument. In terms of a mathematical model of a taut string, the mixing of tones corresponds to taking linear combinations of the family of solutions (5). This is a key feature of the type of partial differential equations that we study, namely linear differential equations that are **homogeneous**, in that each additive term involves the dependent variable or its derivatives – there are no constant terms or terms involving solely the independent variables. The key result is as follows.

Principle of superposition

If u and v are solutions of a linear homogeneous partial differential equation, then $Au + Bv$ is also a solution of the same equation for any constants A and B. Furthermore, if u and v also satisfy homogeneous boundary conditions such as $u(0,t) = 0$ or $u_x(L,t) = 0$, then the linear combination $Au + Bv$ will also satisfy them.

The principle of superposition enables the construction of a general solution of a partial differential equation from particular solutions such as the family of solutions (5).

It may seem that this principle of superposition is of limited applicability since it applies only to homogeneous boundary conditions. The following exercises show that this is not the case, as inhomogeneous boundary conditions can be reduced to homogeneous boundary conditions.

Exercise 7

Consider the heat equation

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2},$$

subject to boundary conditions

$$\Theta(0,t) = \Theta(L,t) = \Theta_0, \quad t \geq 0,$$

where $\Theta_0 \neq 0$.

(a) Show that the function $\Theta(x,t) = \Theta_0$ satisfies the differential equation and boundary conditions.

(b) Define $u(x,t) = \Theta(x,t) - \Theta_0$. Show that $u(x,t)$ satisfies the heat equation with homogeneous boundary conditions.

Exercise 8

Consider the heat equation

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2},$$

subject to boundary conditions

$$\Theta(0,t) = \Theta_0, \quad \Theta(L,t) = \Theta_L, \quad t \geq 0,$$

where Θ_0 and Θ_L are non-zero constants.

(a) Show that the function

$$\Theta(x,t) = \frac{L-x}{L}\Theta_0 + \frac{x}{L}\Theta_L$$

satisfies the differential equation and the boundary conditions.

(b) Define the function

$$u(x,t) = \Theta(x,t) - \frac{L-x}{L}\Theta_0 - \frac{x}{L}\Theta_L.$$

Show that $u(x,t)$ satisfies the heat equation with homogeneous boundary conditions.

Results similar to those in Exercises 7 and 8 allow us to concentrate on the case of homogeneous boundary conditions, which we will mainly consider from now on.

The case of homogeneous boundary conditions is simpler because we can then apply the principle of superposition. This will be the strategy for solving partial differential equations: first find a family of particular solutions, then use the principle of superposition to find the general solution. The next subsection describes this strategy.

1.3 Separation of variables

This subsection is the core of this unit. It describes a method known as separation of variables, which is one of the few techniques available for solving linear partial differential equations; it is the only technique that we will study in this unit. As mentioned in the previous subsection, the strategy is to look for particular solutions that can be combined to give the general solution. We look for solutions of the form

$$u(x,t) = X(x)\,T(t).$$

We now have to find the relevant partial derivatives of the function $u(x,t)$ in terms of the functions $X(x)$ and $T(t)$.

Exercise 9

If the function u is defined as a product

$$u(x,t) = X(x)\,T(t),$$

find formulas for the partial derivatives $\partial u/\partial t$, $\partial^2 u/\partial t^2$, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ in terms of the functions X and T and their ordinary derivatives.

Using the derivatives obtained in Exercise 9, we can separate variables in a partial differential equation in a similar (but distinct) way to the separation of variables method for solving first-order differential equations that you studied in Unit 1. As an example, consider the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{6}$$

and look for solutions of the form $u(x,t) = X(x)\,T(t)$. From Exercise 9 we have

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t) \quad \text{and} \quad \frac{\partial u}{\partial t} = X(x)\,T'(t).$$

Substituting into the partial differential equation gives

$$X''(x)\,T(t) = X(x)\,T'(t).$$

Dividing by $X(x)\,T(t)$ gives

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

This has achieved our aim of separating the variables, as the left-hand side involves only x and the right-hand side involves only t. This equation must hold for all x and t, and the only way that this can happen is if both sides are equal to the same constant. Let μ be this constant, which is called the **separation constant**. So we have

$$\frac{X''(x)}{X(x)} = \mu \quad \text{and} \quad \frac{T'(t)}{T(t)} = \mu.$$

Rearranging these equations gives two ordinary differential equations:

$$X''(x) - \mu\, X(x) = 0 \quad \text{and} \quad T'(t) - \mu\, T(t) = 0. \tag{7}$$

Each of these equations can be solved separately by using the methods of Unit 1, but before we progress to doing this, try the following exercise to separate the variables for another partial differential equation.

Exercise 10

Consider the differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2u = \frac{\partial u}{\partial t}.$$

Use the substitution $u(x,t) = X(x)\,T(t)$ to separate the variables x and t, and hence find the two differential equations satisfied by $X(x)$ and $T(t)$.

Ordinary differential equations similar to equations (7) will occur frequently for the partial differential equations that we consider in this unit. These equations can be solved by using the methods of Unit 1, and for reference we recall the solutions here.

> The general solution of the equation $T'(t) - \mu\, T(t) = 0$ is
>
> $$T(t) = C \exp(\mu t), \tag{8}$$
>
> where C is a constant.

The form of the solution of the equation involving x depends on the value of the separation constant μ and splits into three cases.

> The general solution of the equation $X''(x) - \mu\, X(x) = 0$ is
>
> $$X(x) = \begin{cases} Ae^{cx} + Be^{-cx} & \text{for } \mu > 0, \\ Ax + B & \text{for } \mu = 0, \\ A\cos kx + B\sin kx & \text{for } \mu < 0, \end{cases} \tag{9}$$
>
> where A and B are constants, $c = \sqrt{\mu}$ and $k = \sqrt{-\mu}$.

Note that the constants c and k that appear in the box above are real numbers (in the cases where they apply) and are positive (since the square root function gives the positive root).

Now we turn our attention to the boundary conditions for the partial differential equation, which can be used to derive boundary conditions for the separated ordinary differential equations. As an example to show the general method, consider the boundary condition $u(0,t) = 0$ for $t \geq 0$. Substituting $u(x,t) = X(x)\,T(t)$ into the boundary condition gives

$$X(0)\,T(t) = 0.$$

This equation implies that either $X(0) = 0$ or $T(t) = 0$ for all t. The latter option gives $u(x,t) = X(x) \times 0 = 0$, which is known as the *trivial* solution (it is always a solution of a linear homogeneous differential equation). So for non-trivial solutions of the partial differential equation we must have $X(0) = 0$. Hence the boundary condition for $u(x,t)$ imposes a boundary condition on $X(x)$.

Similar results hold for other boundary conditions, as the next exercise shows.

Exercise 11

Consider the boundary condition $u_x(1,t) = 0$ for $t \geq 0$. If $u(x,t) = X(x)\,T(t)$, then what boundary condition must $X(x)$ satisfy in order to find non-trivial solutions of a partial differential equation?

Returning now to the differential equation (6), namely $u_{xx} = u_t$, we consider the solutions subject to the boundary conditions $u(0,t) = u(1,t) = 0$ for $t \geq 0$. In terms of X and T, the boundary conditions become $X(0)\,T(t) = 0$ and $X(1)\,T(t) = 0$, hence for non-trivial solutions we must have $X(0) = 0$ and $X(1) = 0$.

The aim is now to find non-trivial solutions of the differential equation $X''(x) - \mu\,X(x) = 0$ that satisfy the boundary conditions. In equation (9) there are three different forms of the solution depending on the sign of μ, and we consider each in turn.

- $\mu > 0$. Let $c = \sqrt{\mu}$. As stated in equation (9), the general solution in this case is

  $$X(x) = Ae^{cx} + Be^{-cx},$$

  where A and B are constants.

  The boundary condition $X(0) = 0$ gives the equation $Ae^0 + Be^0 = 0$, that is, $A + B = 0$ or $B = -A$.

  The boundary condition $X(1) = 0$ gives the equation

  $$Ae^c + Be^{-c} = 0.$$

  As $B = -A$, this simplifies to

  $$A(e^c - e^{-c}) = 0.$$

  Multiplying both sides by e^c gives

  $$A(e^{2c} - 1) = 0.$$

  Now we argue that the term in brackets is never zero. Since $c > 0$, we have $2c > 0$, and taking exponentials gives $\exp(2c) > \exp(0) = 1$ (this last step is a consequence of the fact that e^x is an increasing function). Thus the term $(e^{2c} - 1)$ is never zero, and we conclude that $A = 0$ and $B = -A = 0$.

  So the only solution is the trivial solution $X(x) = 0$.

- $\mu = 0$. As stated in equation (9), the general solution in this case is

  $$X(x) = Ax + B,$$

  where A and B are constants.

  The boundary condition $X(0) = 0$ gives $B = 0$. The boundary condition $X(1) = 0$ then gives $A = 0$.

  So the only solution is the trivial solution $X(x) = 0$.

Note that as μ is negative, $\sqrt{-\mu}$ is a positive real number.

- $\mu < 0$. Let $k = \sqrt{-\mu}$. As stated in equation (9), the general solution can be written as

  $$X(x) = A\cos kx + B\sin kx,$$

  where A and B are constants.

  The boundary condition $X(0) = 0$ yields $A\cos 0 + B\sin 0 = 0$, so $A = 0$. The boundary condition $X(1) = 0$ then gives $B\sin k = 0$. So either $B = 0$ or $\sin k = 0$. The option $B = 0$ leads to $X(x) = 0$ again, so for non-trivial solutions we must have

  $$\sin k = 0.$$

  The zeros of the sine function occur at integer multiples of π, so $k = n\pi$ for some integer n.

  So the function

  $$X(x) = B\sin(n\pi x) \quad \text{for } n = 1, 2, \ldots$$

  is a solution of the differential equation that satisfies the boundary condition for any positive integer n. The separation constant for this solution is $\mu = -k^2 = -n^2\pi^2$.

Note that n must be positive since k is positive.

Now we have found non-trivial solutions for $X(x)$, we proceed to find the corresponding solutions for $T(t)$.

Recall that the differential equation for $T(t)$ is $T'(t) - \mu\, T(t) = 0$, which becomes

$$T'(t) + n^2\pi^2\, T(t) = 0$$

as $\mu = -n^2\pi^2$. From equation (8), the general solution of this equation is

$$T(t) = C\exp(-n^2\pi^2 t),$$

where C is a constant.

Multiplying the solutions for $X(x)$ and $T(t)$ gives a function $u(x,t)$ that satisfies the partial differential equation and boundary conditions:

$$u(x,t) = a\sin(n\pi x)\exp(-n^2\pi^2 t),$$

where we have combined the two constants into one: that is, $a = BC$.

There is one solution for each positive integer n, so we have a family of solutions

$$u_n(x,t) = a_n \sin(n\pi x) \exp(-n^2\pi^2 t) \quad \text{for } n = 1, 2, \ldots,$$

where we have added a subscript n to the constant a to emphasise that these constants can have a different value for each value of n. These solutions of the partial differential equation are known as **normal mode solutions**.

As the partial differential equation and boundary conditions are homogeneous and linear, any superposition of these solutions is also a solution. So we can write the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \exp(-n^2\pi^2 t).$$

To determine the constants, we need an initial condition for the partial differential equation, such as

$$u(x,0) = \sin(2\pi x).$$

Substituting the general solution into this equation gives

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = \sin(2\pi x). \tag{10}$$

You should recognise this as a Fourier sine series that would result from computing the Fourier series of the odd periodic extension of $\sin(2\pi x)$ defined on the interval $[0,1]$. We can proceed, as in Unit 13, to determine the coefficients by integration:

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \sin(2\pi x) \sin(n\pi x)\, dx.$$

This integral has already been evaluated in Unit 13 – the integral over a complete period of a product of sines is zero unless the arguments are the same. So we have $a_2 = 1$, and $a_n = 0$ if $n \neq 2$. In fact, this result could have been written down 'by inspection' of equation (10), as the function on the right-hand side is one of the terms of the series on the left-hand side; equation (10) could be written as

$$a_1 \sin(\pi x) + a_2 \sin(2\pi x) + a_3 \sin(3\pi x) + \cdots = \sin(2\pi x).$$

So the particular solution that satisfies the given initial condition is

$$u(x,t) = \sin(2\pi x) \exp(-4\pi^2 t).$$

In finding this solution we have gone through several steps, and it is worthwhile summarising them now in the form of a procedure. The terms in the margin will be used in later examples to show how this procedure is being applied.

Procedure 1 Separation of variables

To find the solution of a homogeneous linear partial differential equation with dependent variable u and independent variables x and t, subject to boundary and initial conditions, carry out the following steps.

◀ Separate variables ▶

1. Separate the variables by substituting the trial solution

   $$u(x,t) = X(x)\,T(t)$$

   into the partial differential equation and rearranging so that each side of the equation involves only one of the variables. Both sides must then be equal to a separation constant μ.

   Rearranging then gives two separate ordinary differential equations for X and T. The boundary conditions for u will give boundary conditions for X.

◀ Solve ODEs ▶

2. Find the general solution of the ordinary differential equations for X and T found in Step 1. Use the boundary conditions for X to find the normal mode solutions $u_n(x,t)$.

◀ Initial conditions ▶

3. Write down the general solution as a linear combination of the normal mode solutions:

   $$u(x,t) = \sum_{n=0}^{\infty} a_n\,u_n(x,t).$$

   The initial conditions (and results about Fourier series) can be used to determine the constants a_n appearing in this solution.

The next exercise asks you to follow the first two steps of this procedure to find a family of normal mode solutions for a different partial differential equation.

Exercise 12

Use Procedure 1 to find an infinite family of normal mode solutions for the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0,$$

This equation is known as *Laplace's equation*.

subject to boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \geq 0.$$

Procedure 1 is the key technique introduced in this unit. The remaining sections apply this technique to analyse models in two different contexts: modelling the vibrations of a string and modelling the flow of heat. The next section concentrates on the first of these.

2 The wave equation

This section is devoted to deriving and solving the wave equation. In Subsection 2.1 we will see how the wave equation arises as a model of the transverse vibrations of a taut string, and also how damping can be incorporated into the model. Subsection 2.2 looks at modelling the vibrations of a plucked string. The model for this is the wave equation subject to fixed endpoint boundary conditions and initial conditions specifying a given initial string profile and zero initial velocity. Subsection 2.3 looks at what happens when damping is incorporated into the model.

2.1 Deriving the wave equation

In this subsection you will see how a continuous model of a guitar string – or indeed of any taut string – leads to a model of the transverse vibrations of the string as a second-order partial differential equation. This derivation is given for completeness – you will not be asked to reproduce or modify this derivation in the assessment of this module.

You saw a *discrete* model of a guitar string in Section 3 of Unit 11.

The assumptions needed to develop this model are as follows.

(a) The string is uniform, with total mass M and equilibrium length L.

(b) Each point of the string is subject only to a small, smoothly varying transverse displacement u, as shown in Figure 5.

(c) All external forces, such as weight, friction and air resistance, are negligible in comparison with the tension in the string.

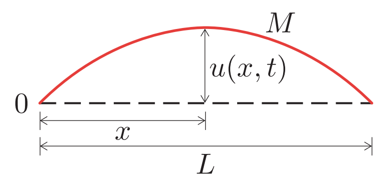

Figure 5 A taut string

Using assumption (c) that the weight of the string is negligible in comparison to the tension in the string implies that when it is in equilibrium, the string lies in a straight line. We will take this equilibrium line as the x-axis, with origin at the left-hand end; see Figure 5.

Consider a small segment of length δx a distance x along the string (so that in equilibrium it would occupy the interval $[x - \delta x/2, x + \delta x/2]$), as shown in Figure 6. Also shown in the figure are the two neighbouring segments.

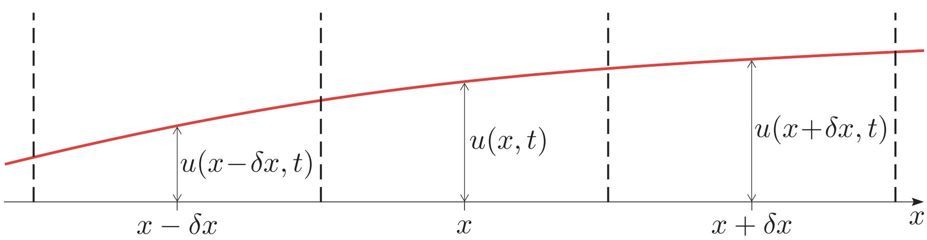

Figure 6 Splitting the string into small segments: imagine the string being split at the dashed lines and replaced by particles located at the midpoints

By assumption (a) the string is uniform, so the mass of each segment is $M\,\delta x/L$. So each small segment can be modelled as a particle of mass

$M\,\delta x/L$ located at the point $(x, u(x,t))$, as shown in Figure 7. Also shown in Figure 7 is the angle $\theta(x,t)$, which is the angle that the tension force due to the string makes to the left of the particle at position x at time t.

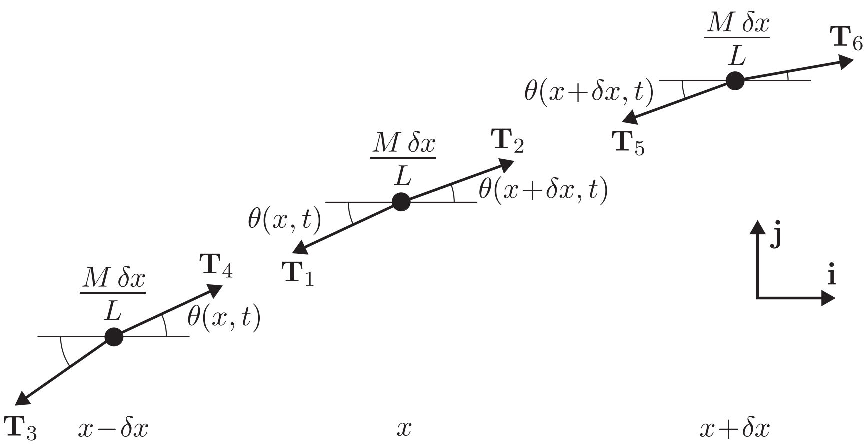

Figure 7 Force diagram for three neighbouring elements of string (vertical scale exaggerated for clarity)

Also shown in Figure 7 are the forces acting on the three segments. By assumption (c), all forces are negligible compared to the tension in the string, so we have modelled only the tension forces. As the tension forces between neighbouring particles (e.g. $\mathbf{T}_4$ and $\mathbf{T}_1$) are due to a portion of string at a fixed angle, they must be of equal magnitude and opposite direction, as shown. The total force on the central segment at position x is $\mathbf{T}_1 + \mathbf{T}_2$.

By assumption (b), the central segment moves vertically, so its acceleration is $u_{tt}(x,t)\,\mathbf{j}$. Applying Newton's second law gives

$$\frac{M\,\delta x}{L}\,u_{tt}(x,t)\,\mathbf{j} = \mathbf{T}_1 + \mathbf{T}_2. \tag{11}$$

We now need expressions for $\mathbf{T}_1$ and $\mathbf{T}_2$. Resolving in the $\mathbf{i}$-direction gives

$$0 = \mathbf{T}_1 \cdot \mathbf{i} + \mathbf{T}_2 \cdot \mathbf{i}.$$

From this equation we get $|\mathbf{T}_1 \cdot \mathbf{i}| = |\mathbf{T}_2 \cdot \mathbf{i}|$, that is, the magnitude of the horizontal component of the tension force is constant. As the string is horizontal in equilibrium, this is equal to the equilibrium tension, T_{eq}. This gives the diagram shown in Figure 8.

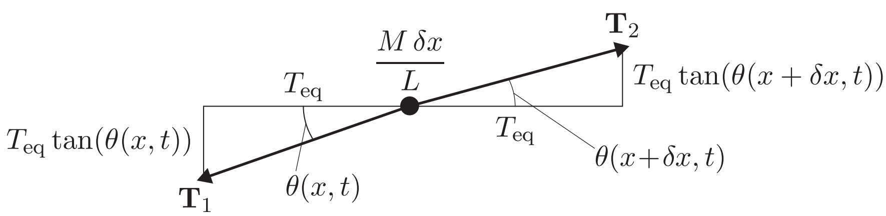

Figure 8 Resolving the forces in the central line segment (vertical scale exaggerated for clarity)

From Figure 8 we have

$$\mathbf{T}_1 = -T_{\text{eq}}\mathbf{i} - T_{\text{eq}}\tan(\theta(x,t))\,\mathbf{j}$$

and

$$\mathbf{T}_2 = T_{\text{eq}}\mathbf{i} + T_{\text{eq}}\tan(\theta(x+\delta x,t))\,\mathbf{j}.$$

What remains to be done is to relate the angle $\theta(x,t)$ to the string displacement $u(x,t)$. From the triangle shown in Figure 9, we obtain

$$\tan(\theta(x,t)) = \frac{u(x,t) - u(x-\delta x,t)}{\delta x}.$$

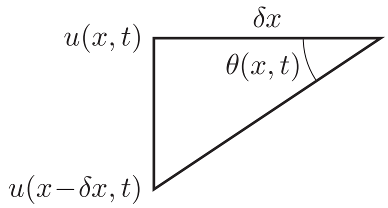

Figure 9 Determining the relationship between $\theta(x,t)$ and $u(x,t)$

As δx becomes smaller, the right-hand side of this equation tends to the first derivative of the displacement with respect to x, that is, $u_x(x,t)$. So $\tan(\theta(x,t)) = u_x(x,t)$, and similarly $\tan(\theta(x+\delta x,t)) = u_x(x+\delta x,t)$, hence we have

Recall from Unit 7 that
$$\lim_{h\to 0}\frac{f(x+h,t)-f(x,t)}{h} = f_x(x,t).$$
Here $f = u$ and $h = -\delta x$.

$$\mathbf{T}_1 = -T_{\text{eq}}\mathbf{i} - T_{\text{eq}}\,u_x(x,t)\,\mathbf{j}$$

and

$$\mathbf{T}_2 = T_{\text{eq}}\mathbf{i} + T_{\text{eq}}\,u_x(x+\delta x,t)\,\mathbf{j}.$$

Substituting into equation (11) and resolving in the **j**-direction gives

$$\frac{M\,\delta x}{L}\,u_{tt}(x,t) = T_{\text{eq}}\,u_x(x+\delta x,t) - T_{\text{eq}}\,u_x(x,t). \tag{12}$$

Rearranging yields

$$\frac{M}{T_{\text{eq}}L}\,u_{tt}(x,t) = \frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x}.$$

Taking the limit as $\delta x \to 0$ gives

$$\frac{M}{T_{\text{eq}}L}\,u_{tt}(x,t) = u_{xx}(x,t).$$

Now in
$$\lim_{h\to 0}\frac{f(x+h,t)-f(x,t)}{h} = f_x(x,t)$$
we have $f = u_x$ and $h = \delta x$.

This is the differential equation that we were aiming to derive, but it is more usual to write

$$c^2 = T_{\text{eq}}L/M \tag{13}$$

and present the equation as follows.

Wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2}. \tag{14}$$

Example 3

Use equation (14) to determine the dimensions of the constant c.

Solution

The dimensions of a derivative were dealt with in Exercise 21 of Unit 8.

$$[\partial^2 u/\partial x^2] = \mathrm{L\,L^{-2}} = \mathrm{L^{-1}},$$
$$[\partial^2 u/\partial t^2] = \mathrm{L\,T^{-2}}.$$

Therefore

$$[c^2] = \frac{[\partial^2 u/\partial t^2]}{[\partial^2 u/\partial x^2]} = \frac{\mathrm{L\,T^{-2}}}{\mathrm{L^{-1}}} = \mathrm{L^2T^{-2}},$$

so $[c] = \mathrm{L\,T^{-1}}$.

So c has the dimensions of speed, and equation (14) will be dimensionally consistent provided that $\sqrt{T_{\mathrm{eq}}L/M}$ has the same dimensions.

Exercise 13

Show that $\sqrt{T_{\mathrm{eq}}L/M}$ has the dimensions of speed.

This subsection concludes by deriving a more complicated model for the transverse vibrations of a string that includes damping. We start from equation (12), which is the equation of transverse motion of the string without damping, and now consider adding a linear damping force to the motion. A first model of the damping force assumes that its magnitude is proportional to the length of the segment δx and to the transverse component of the segment's velocity $u_t(x,t)$, and that it acts in the direction opposite to the velocity. Equation (12) then becomes

$$\frac{M\,\delta x}{L}\,u_{tt}(x,t) = T_{\mathrm{eq}}\,u_x(x+\delta x,t) - T_{\mathrm{eq}}\,u_x(x,t) - \alpha\,\delta x\,u_t(x,t),$$

where α is a constant to be determined experimentally. Dividing through by δx and taking the limit as $\delta x \to 0$, we obtain

$$\frac{M}{L}\,\frac{\partial^2 u}{\partial t^2}(x,t) = T_{\mathrm{eq}}\,\frac{\partial^2 u}{\partial x^2}(x,t) - \alpha\,\frac{\partial u}{\partial t}(x,t).$$

This is the partial differential equation for damped transverse oscillations, but it is more usually written with different parameters by writing $c^2 = T_{\mathrm{eq}}L/M$ and $\varepsilon = \alpha L/(2M)$ to obtain the following equation.

Damped wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\,\frac{\partial u}{\partial t}\right). \tag{15}$$

2.2 Solving the wave equation

In this subsection we will solve the wave equation for vibrations of a taut string.

Example 4

In this example we consider the vibrations of a guitar string of length L that is held fixed at its endpoints. The string is plucked, that is, released from rest with an initial profile given by a function $f(x)$, $0 < x < L$.

We model the guitar string as a taut string, so the transverse displacement of the string at position x and time t will satisfy the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{16}$$

The fixed endpoints of the string give the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \quad t \geq 0. \tag{17}$$

The initial profile of the string gives the initial condition

$$u(x,0) = f(x), \quad 0 < x < L, \tag{18}$$

and the fact that it is released from rest gives the initial condition

$$u_t(x,0) = 0, \quad 0 \leq x \leq L. \tag{19}$$

Find $u(x,t)$ that satisfies this model.

Solution

Applying Procedure 1, we look for solutions of the form

◀ Separate variables ▶

$$u(x,t) = X(x)\,T(t).$$

Here T is a function of time. It should not be confused with the magnitude of a force $\mathbf{T}$ or the dimension T.

Differentiating this expression gives

$$\frac{\partial u}{\partial t} = X(x)\,T'(t), \quad \frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t),$$
$$\frac{\partial u}{\partial x} = X'(x)\,T(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t).$$

Substituting into equation (16) gives

$$X''(x)\,T(t) = \frac{1}{c^2}\,X(x)\,T''(t),$$

which on rearranging gives

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2\,T(t)}.$$

Since the left-hand side is a function of x only, and the right-hand side is a function of t only, both sides must be constant and equal to a separation constant μ. This gives the two equations

$$\frac{X''(x)}{X(x)} = \mu \quad \text{and} \quad \frac{T''(t)}{c^2\,T(t)} = \mu.$$

Rearranging gives two ordinary differential equations:

$$X''(x) = \mu\,X(x), \quad T''(t) = \mu c^2\,T(t).$$

The given boundary conditions (17) become

$$X(0)\,T(t) = 0 \quad \text{and} \quad X(L)\,T(t) = 0, \quad t \geq 0.$$

One solution of these equations is $T(t) = 0$ for $t \geq 0$, which leads to the trivial solution $u(x,t) = 0$ that is always a possibility. For non-trivial solutions $T(t)$ is not always zero, so we must have

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

◀ Solve ODEs ▶

Now we consider solving the differential equation for X. The general solution is given by equation (9) and splits into three cases depending on the sign of μ.

- $\mu > 0$. The general solution can be written as

  $$X(x) = Ae^{cx} + Be^{-cx},$$

  where $c = \sqrt{\mu}$, and A and B are constants. The boundary condition $X(0) = 0$ gives $A + B = 0$, that is, $B = -A$. The boundary condition $X(L) = 0$ then gives

  $$A(e^{cL} - e^{-cL}) = 0.$$

  Multiplying by e^{cL} yields

  $$A(e^{2cL} - 1) = 0.$$

  Now we can argue as before that the term in brackets is never zero. As $2cL > 0$, we must have $\exp(2cL) > \exp(0) = 1$ as the exponential function is everywhere increasing. So $A = 0$ and the only solution in this case is the trivial solution $X(x) = 0$.

- $\mu = 0$. Now equation (9) gives the general solution as

  $$X(x) = Ax + B,$$

  where A and B are constants. The boundary condition $X(0) = 0$ gives $B = 0$. The boundary condition $X(L) = 0$ then gives $AL = 0$, which implies that $A = 0$ as L is non-zero (because it represents the length of the string). So the only solution in this case is the trivial solution $X(x) = 0$.

- $\mu < 0$. Let $k = \sqrt{-\mu}$. Then equation (9) gives the general solution as

  $$X(x) = A\cos kx + B\sin kx,$$

  where A and B are constants. The boundary condition $X(0) = 0$ gives $A = 0$. The boundary condition $X(L) = 0$ then gives

  $$B\sin kL = 0.$$

  So $B = 0$ or $\sin kL = 0$. The case where $B = 0$ gives the trivial solution $X(x) = 0$ for all x. So for non-trivial solutions we must have $\sin kL = 0$, which gives $kL = n\pi$ for some positive integer n, that is,

  $$k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots.$$

  So the non-trivial solution is

  $$X(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots. \tag{20}$$

In this case the separation constant μ is given by

$$\mu = -k^2 = -\frac{n^2\pi^2}{L^2}.$$

We now turn to the function $T(t)$. Substituting for the value of μ, the differential equation for T becomes

$$T''(t) + \frac{c^2n^2\pi^2}{L^2}\,T(t) = 0.$$

The general solution of this equation is stated in equation (9) (using the third option as μ is negative) as

$$T(t) = C\cos\left(\frac{cn\pi t}{L}\right) + D\sin\left(\frac{cn\pi t}{L}\right),$$

where C and D are constants.

Multiplying together the solutions for $X(x)$ and $T(t)$, and combining constants, gives the family of normal mode solutions

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{cn\pi t}{L}\right) + b_n\sin\left(\frac{cn\pi t}{L}\right)\right],$$

where $a_n = BC$ and $b_n = BD$ are constants; we have added the subscript n to emphasise that there are different constants for each value of n.

The general solution of the partial differential equation with boundary conditions is a linear combination of these normal mode solutions:

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{cn\pi t}{L}\right) + b_n\sin\left(\frac{cn\pi t}{L}\right)\right].$$

◀ Initial conditions ▶

Now we apply the initial conditions to find a particular solution for the plucked string model. Starting with the homogeneous initial condition, that the string starts from rest, we need the solution to satisfy $u_t(x,0) = 0$. We begin by computing the partial derivative of the general solution:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[-\frac{cn\pi}{L}\,a_n\sin\left(\frac{cn\pi t}{L}\right) + \frac{cn\pi}{L}\,b_n\cos\left(\frac{cn\pi t}{L}\right)\right].$$

So we have

$$u_t(x,0) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\frac{cn\pi}{L}\,b_n\right].$$

From the results of Unit 13, if $u_t(x,0) = 0$, then by inspection $b_n = 0$ for all n. (This is a consequence of a Fourier sine series giving a unique representation of a function.)

This gives the following solution that satisfies the partial differential equation, the boundary conditions and the homogeneous initial condition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right). \qquad (21)$$

Now we turn to the inhomogeneous initial condition, namely $u(x,0) = f(x)$ for $0 < x < L$. Substituting for $u(x,t)$ in this equation gives

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

This is a Fourier sine series for the function $f(x)$, which is given by the odd periodic extension of the function $f(x)$ on the interval $[-L, L]$. So the coefficients a_n will be given by the formula

$$a_n = \frac{2}{2L}\int_{-L}^{L} f_{\text{odd}}(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

As the odd extension f_{odd} is odd and the sine function is odd, the product is even, so the above integral can be written as twice the sum over the positive x-values:

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,$$

where we have written $f(x)$ instead of $f_{\text{odd}}(x)$ as these two functions are equal for $0 < x < L$.

Evaluating these integrals for a given initial profile $f(x)$ will give the coefficients a_n of the solution (21) for the plucked string model.

Example 2 looked at initial conditions for a string plucked at its centre, and in this case the initial displacement $f(x)$ is given by

$$f(x) = \begin{cases} x/L & \text{for } 0 \le x < \frac{1}{2}L, \\ (L-x)/L & \text{for } \frac{1}{2}L \le x \le L. \end{cases}$$

The Fourier sine series for this function was considered in Example 12 of Unit 13, where it was found that the coefficients are

$$a_n = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right),$$

so the values for the coefficients a_n, for $n = 1, 2, 3, 4, 5, 6, \ldots$, are given by

$$\frac{4}{\pi^2},\ 0,\ -\frac{4}{9\pi^2},\ 0,\ \frac{4}{25\pi^2},\ 0,\ \ldots .$$

Substituting into equation (21) gives the solution for our plucked string problem as

$$u(x,t) = \frac{4}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi ct}{L}\right) - \cdots\right].$$

This may seem a very complicated solution, and you may not be able to see immediately how this combination of terms behaves. The behaviour is best seen graphically, as in Figure 10.

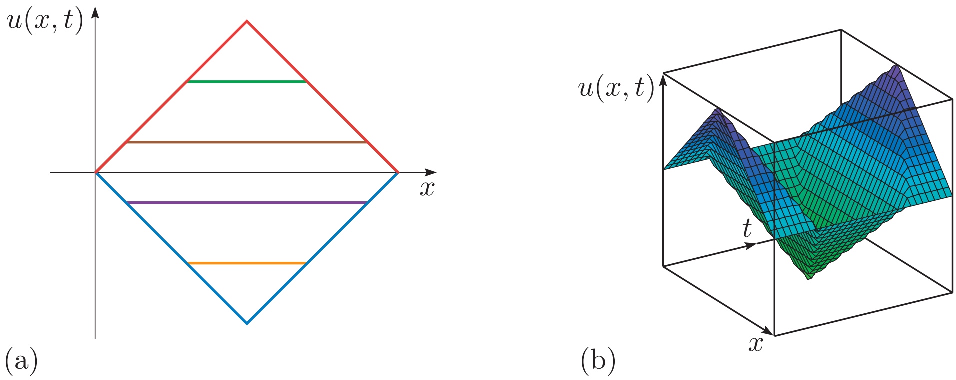

Figure 10 (a) Snapshots of the profile of the string initially (red), then after successive tenths of a cycle, green, brown, purple, orange, blue (and then back again). (b) The solution $u(x,t)$ of the damped wave equation plotted as a surface.

Figure 10 shows two different representations of the solution to the plucked string problem. Figure 10(a) shows snapshots of the position of the string at several points in time. This shows that initially only the centre portion of the string starts to move. As time progresses, more of the string moves, until after one-quarter of a cycle, the whole string is horizontal and moving downwards. Over the next quarter cycle this sequence is reversed, with the centre portion moving most, until after half a cycle the string regains its initial triangular shape. The sequence of shapes over the next half cycle is the reverse of the above, so that after one complete cycle the string has regained its initial position, then the cycle repeats. Figure 10(b) shows the same information. On the left-hand edge the string has its initial triangular shape. Moving left to right on the time axis, the string goes through one complete cycle, and on the right-hand edge we see that the string has regained its initial triangular shape. If you imagine slicing across the surface at regular time intervals, then you will see the curves shown in Figure 10(a).

This is a quite surprising prediction from the model, but modern high-speed photography confirms that this is indeed the motion of a plucked string initially. This model is only an idealised approximation of a real string, and over long time periods the approximation becomes worse. For example, real strings are not perfectly flexible and there is a force that resists the string forming a kink such as the kink at the centre of the triangular initial shape. This extra force is something that has not been modelled, and it can cause the predictions of the model to deviate from reality.

Using this model we can calculate the fundamental frequency of the guitar string that we considered in Unit 11, Section 3. There we constructed discrete models (consisting of separate springs and particles) of a guitar E string, with fundamental frequency 323 Hz, length 0.65 m and mass 0.25 g. We calculated the equilibrium tension T_{eq} of the string to be 68 N.

This model predicts that the fundamental angular frequency of the guitar string will be the smallest coefficient of t in the cosine terms of the series, that is, $\pi c/L$. The fundamental frequency f is obtained by dividing the fundamental angular frequency by 2π:

$$f = \frac{\omega}{2\pi} = \frac{\pi c/L}{2\pi} = \frac{c}{2L}.$$

Now c is related to the properties of the string by the relationship (13), namely $c^2 = T_{\text{eq}}L/M$. This can be substituted into the above equation to yield

$$f = \frac{1}{2L}\sqrt{\frac{T_{\text{eq}}L}{M}} = \frac{1}{2}\sqrt{\frac{T_{\text{eq}}}{ML}} = \frac{1}{2}\sqrt{\frac{68}{0.25 \times 10^{-3} \times 0.65}} = 323.4.$$

So this continuous model of the guitar string gives an accurate prediction of the fundamental frequency.

The following exercise looks at a different initial condition.

Exercise 14

Write down the solution for the plucked string model when the initial profile $f(x)$ is given by

$$f(x) = \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right).$$

2.3 Solving the damped wave equation

In Subsection 2.2 we solved the wave equation model for vibrations of a taut string initially plucked at its midpoint. In this subsection we generalise the method in order to solve the damped wave equation model for vibrations of a damped taut string plucked at its midpoint.

The following exercise asks you to begin by following the first step of the separation of variables procedure.

Exercise 15

Consider the damped wave equation involving the constants ε and c,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\,\frac{\partial u}{\partial t}\right),$$

together with the boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t \geq 0.$$

Apply the first step of Procedure 1 to separate the variables and obtain two ordinary differential equations together with corresponding boundary conditions.

The results of this exercise are used in the following example, which uses arguments about the relative sizes of parameters to find an approximate solution to the problem. This often happens with complicated mathematical models – either the exact equations cannot be solved or the solution is too complicated to give insight into the problem – and consequently an approximate solution to the problem is sought. This example is consequently harder than the other examples in this unit.

Example 5

Consider the model developed in Subsection 2.1 for damped vibrations of a string,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\,\frac{\partial u}{\partial t}\right), \tag{22}$$

where ε and c are constants. In this example we consider weak damping, which is the case when ε is small.

As in Example 4, the string is subject to fixed endpoint boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{23}$$

and initial conditions that the string starts from rest in a given shape:

$$u(x,0) = \begin{cases} x/L & \text{for } 0 < x \leq \frac{1}{2}L, \\ (L-x)/L & \text{for } \frac{1}{2}L < x < L, \end{cases} \tag{24}$$

$$u_t(x,0) = 0, \quad 0 \leq x \leq L. \tag{25}$$

Solution

Using the results of Exercise 15, the differential equations corresponding to the given partial differential equation are

◀ Separate variables ▶

$$X''(x) - \mu\,X(x) = 0 \tag{26}$$

and

$$T''(t) + 2\varepsilon\,T'(t) - c^2\mu\,T(t) = 0, \tag{27}$$

where μ is the separation constant. In addition, we have

$$X(0) = 0 \quad \text{and} \quad X(L) = 0, \tag{28}$$

which are boundary conditions for the ordinary differential equation (26).

The differential equation for X, and its boundary conditions, are the same as in Example 4. You have seen that a non-trivial solution occurs only if the separation constant μ is negative, and is given by

◀ Solve ODEs ▶

$$X(x) = A\cos kx + B\sin kx,$$

where A and B are constants, and $k = \sqrt{-\mu}$.

As in Example 4, the boundary conditions (28) tell us that $A = 0$ and $B \sin kL = 0$, so $B = 0$ or $\sin kL = 0$. The case $B = 0$ gives the trivial solution again, whereas the case $\sin kL = 0$ restricts k to take one of the values $n\pi/L$, where n is an integer. Hence, as in Example 4, we find again a family of solutions to the boundary-value problem for X of the form

$$X(x) = B \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots. \tag{29}$$

Now we must deal with the equation for T. We know from our discussion of the function X that to have a non-trivial solution u, we must have $\mu = -k^2$ and $k = n\pi/L$ (for positive integers n), so the differential equation (27) for T becomes

$$T''(t) + 2\varepsilon T'(t) + \omega^2 T(t) = 0, \quad \text{where } \omega = ck.$$

To solve this equation for T, we first solve the auxiliary equation

$$\lambda^2 + 2\varepsilon\lambda + \omega^2 = 0,$$

which gives

$$\lambda = -\varepsilon \pm \sqrt{\varepsilon^2 - \omega^2}.$$

Using the assumption that $\varepsilon \ll \omega$, that is, ε is much smaller than ω, which corresponds to very weak damping, this reduces to

$$\lambda \simeq -\varepsilon \pm i\omega.$$

The corresponding general solution is

$$T(t) \simeq e^{-\varepsilon t}(C \cos \omega t + D \sin \omega t),$$

where C and D are constants.

Using the allowed values for k (i.e. $n\pi/L$ for positive integers n), where $\omega = ck$, this gives the approximate solutions

$$T(t) \simeq e^{-\varepsilon t}\left(C \cos\left(\frac{cn\pi t}{L}\right) + D \sin\left(\frac{cn\pi t}{L}\right)\right), \quad n = 1, 2, 3, \ldots. \tag{30}$$

Now we can combine the two families (29) and (30) to obtain the family of approximate solutions

$$u_n(x,t) \simeq e^{-\varepsilon t} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right), \tag{31}$$
$$n = 1, 2, 3, \ldots,$$

where $a_n = BC$ and $b_n = BD$ are constants. By the principle of superposition, any linear combination of members of this family of solutions is (approximately) a solution of the original partial differential equation (22) and satisfies the boundary conditions (23).

◀ Initial conditions ▶

Now we try to satisfy the initial conditions. We begin with the homogeneous initial condition (25). This was automatically satisfied for the undamped problem, but it is not automatically satisfied here.

We know, from the principle of superposition, that if each member of family (31) satisfies the homogeneous initial condition (25), then so will any linear combination of members of the family. So we just need to consider $u_n(x,t)$ for an arbitrary n (in the range $n = 1, 2, 3, \ldots$). Differentiating (31) with respect to t, we obtain

$$\frac{\partial u_n}{\partial t} \simeq e^{-\varepsilon t} \sin\left(\frac{n\pi x}{L}\right)\left(-a_n \frac{cn\pi}{L} \sin\left(\frac{cn\pi t}{L}\right) + b_n \frac{cn\pi}{L} \cos\left(\frac{cn\pi t}{L}\right)\right)$$
$$- \varepsilon e^{-\varepsilon t} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right).$$

When $t = 0$, and writing $\omega = ck = cn\pi/L$ as earlier, we obtain

$$\left.\frac{\partial u_n}{\partial t}\right|_{t=0} \simeq \left(b_n \frac{cn\pi}{L} - \varepsilon a_n\right) \sin\left(\frac{n\pi x}{L}\right) = (\omega b_n - \varepsilon a_n) \sin\left(\frac{n\pi x}{L}\right).$$

For initial condition (25) to be satisfied, for $0 \leq x \leq L$, we must have $\omega b_n - \varepsilon a_n \simeq 0$. For very weak damping, as we noted earlier, we have $\varepsilon \ll \omega$. This means that to satisfy $\omega b_n - \varepsilon a_n \simeq 0$, we must have $b_n \simeq 0$. Thus to satisfy the initial condition (25), approximately, we need to restrict family (31) to

$$u_n(x,t) \simeq a_n e^{-\varepsilon t} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right).$$

We now need to satisfy the inhomogeneous initial condition (24). As in Procedure 1, we look for an approximate solution of the form

$$u(x,t) \simeq e^{-\varepsilon t} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right). \tag{32}$$

Setting $t = 0$ in approximation (32) gives

$$u(x,0) \simeq \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \tag{33}$$

while the given initial condition (24) is

$$u(x,0) = \begin{cases} x/L & \text{for } 0 < x \leq \frac{1}{2}L, \\ (L-x)/L & \text{for } \frac{1}{2}L < x < L. \end{cases}$$

The situation is exactly the same as in Subsection 2.2, so we obtain the same Fourier coefficients a_n as there. Substituting these into approximation (32) gives the final form of the approximate solution as

$$u(x,t) \simeq \frac{4}{\pi^2} e^{-\varepsilon t} \left[\sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi ct}{L}\right) - \frac{1}{9} \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{3\pi ct}{L}\right)\right.$$
$$\left. + \frac{1}{25} \sin\left(\frac{5\pi x}{L}\right) \cos\left(\frac{5\pi ct}{L}\right) - \cdots\right].$$

Remember that this is an approximate solution, under the assumption of very weak damping, where ε is small compared with ω.

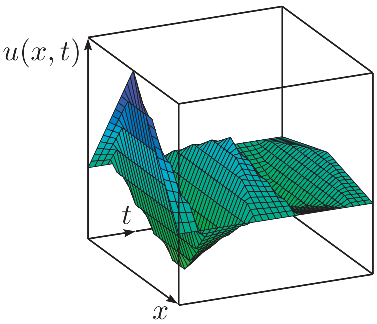

Figure 11 The solution $u(x,t)$ of the damped wave equation shown over two complete cycles

This is the same as the solution to the undamped problem except for the factor $e^{-\varepsilon t}$, which describes exponential decay. Hence the shape of the string is roughly the same as before, apart from the amplitude becoming progressively smaller by the factor $e^{-\varepsilon t}$. The shape determines the sound, so the sound stays the same, only now it gets progressively quieter, since the volume is determined by the amplitude. The solution is shown in Figure 11.

The figure shows that the exponential factor has the effect of reducing the amplitude of the vibrations of the string, which is to be expected for a solution of the damped wave equation.

3 The heat equation

In this section we look at a situation that can be modelled using a different second-order partial differential equation, known as the heat equation. In Subsection 3.1 we will see how the heat equation arises as a model of temperature variation in an insulated metal rod, that is, a rod that does not exchange heat with its surroundings. In addition, we will see how, for an uninsulated metal rod, heat exchange with its surroundings can be incorporated into the model.

Then in Subsection 3.2 we will see how the method of separating the variables, Procedure 1, can be used to solve the heat equation subject to appropriate boundary and initial conditions.

3.1 Deriving the heat equation

This derivation is given for completeness – you will not be asked to reproduce or modify this derivation in the assessment of this module.

In order to model the temperature changes in a rod, we need a formula to describe how temperature changes. Basic intuition tells us that hot things cool down and cold things heat up. Refining this a little, we might consider how we distinguish between hot and cold, and a moment's thought will reveal that hot and cold are relative to the temperature of the surrounding environment. Moreover, the hotter an object is, the quicker its temperature will decrease. Isaac Newton formulated this as an empirical law of temperature change.

Newton's law of cooling

For a given object, the rate of decrease of temperature is proportional to the excess temperature over the environment.

This law applies to objects that are not changing state in the process (such as a solid melting into a liquid). Although Newton called this a law of nature, it is really just a first model of how heat flows. It should be considered in a similar way to Hooke's law for springs, where we used the law to derive useful models of oscillating systems.

We will use Θ to denote the temperature, so this law can be formulated as

$$\frac{d\Theta}{dt} = -k(\Theta - \Theta_0),$$

where Θ_0 is the temperature of the surrounding environment, and k is a positive constant of proportionality. The negative sign in this equation is because hotter objects (i.e. with $\Theta > \Theta_0$) cool down (so $d\Theta/dt < 0$). The constant of proportionality k depends on many properties of the object (such as size and mass) and the contact between the object and its surroundings (such as whether there is direct contact or an air gap), but will be constant for a given object in a given situation.

The study of the flow of heat is called thermodynamics, and this is a substantial topic in modern physics. Here we are not concerned with the details of thermodynamics and the different methods of heat transfer. We will use Newton's law of cooling as a mathematical model of how everyday physical objects behave in an analogous way to how we use Hooke's law as a mathematical model to describe the behaviour of physical springs.

Armed with this mathematical model, we can derive a partial differential equation that models the change in temperature distribution in a uniform rod as it conducts heat. We will assume that no heat is lost from the sides of the rod and that heat is transferred only along the length of the rod (so that we have a one-dimensional problem). We are modelling a straight rod, and we choose an x-axis to be aligned with the rod with the origin at the left-hand end. Let $\Theta(x,t)$ be the temperature at position x along the rod at time t. Consider a small segment of rod of length δx with midpoint a distance x along the rod, as shown in Figure 12.

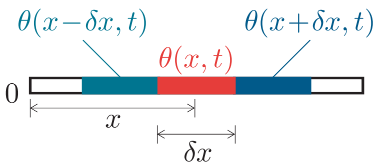

Figure 12 A conducting rod

We will now apply our model of cooling with the small central segment of the rod considered as the object. The temperature on the left of the central segment is $\Theta(x-\delta x, t)$ and this takes the role of the temperature of the environment for the interface between the central and left-hand segments, so the temperature change due to this left-hand interface is $-k\big(\Theta(x,t) - \Theta(x-\delta x,t)\big)$. Similarly, on the right-hand end of the central segment, the 'environment' temperature is $\Theta(x+\delta x, t)$, and the temperature change due to heat exchange at this interface is $-k\big(\Theta(x,t) - \Theta(x+\delta x,t)\big)$ (note that the constant of proportionality k is the same for both ends as the rod is uniform thus the ends are identical). The change in temperature of the central segment $\Theta_t(x,t)$ is given by the sum of these two contributions:

$$\begin{aligned}\Theta_t(x,t) &= -k\big(\Theta(x,t) - \Theta(x-\delta x,t)\big) - k\big(\Theta(x,t) - \Theta(x+\delta x,t)\big) \\ &= k\big(\Theta(x-\delta x,t) - 2\Theta(x,t) + \Theta(x+\delta x,t)\big). \qquad (34)\end{aligned}$$

This can be simplified by using the following Taylor series:

$$\Theta(x+\delta x,t) = \Theta(x,t) + \delta x\,\Theta_x(x,t) + \tfrac{1}{2}(\delta x)^2\,\Theta_{xx}(x,t) + \cdots\,,$$
$$\Theta(x-\delta x,t) = \Theta(x,t) - \delta x\,\Theta_x(x,t) + \tfrac{1}{2}(\delta x)^2\,\Theta_{xx}(x,t) - \cdots\,.$$

Adding these expressions gives

$$\Theta(x-\delta x,t) - 2\Theta(x,t) + \Theta(x+\delta x,t) = (\delta x)^2\,\Theta_{xx}(x,t) + \cdots\,,$$

so

$$\Theta_t(x,t) \simeq k\,(\delta x)^2\,\Theta_{xx}(x,t).$$

As δx becomes smaller, the Taylor approximation will become exact and we can replace the above approximation with an equality. Also, the constant $k\,(\delta x)^2$ must tend to a finite limit α so that the model of smoothly varying temperature applies. (If $k\,(\delta x)^2$ tends to infinity, then $\partial\Theta/\partial t$ tends to infinity, that is, the temperature jumps abruptly.) This gives the partial differential equation model that we are aiming to derive.

Note that k varies with δx, as it is the constant of proportionality for a segment of the rod of length δx.

Heat equation

$$\frac{\partial\Theta}{\partial t} = \alpha\,\frac{\partial^2\Theta}{\partial x^2}. \tag{35}$$

If the ends of the rod are kept at a steady temperature Θ_0, the boundary conditions are

$$\Theta(0,t) = \Theta(L,t) = \Theta_0, \quad t \ge 0. \tag{36}$$

Further, if the initial distribution of temperature along the rod is given by a function $f(x)$, then we have the initial condition

$$\Theta(x,0) = f(x), \quad 0 < x < L. \tag{37}$$

Note that we need only one initial condition because the equation is first-order in time.

These conditions are sufficient for us to be able to obtain a unique solution of the heat equation for the variation of temperature in an insulated rod, as you will see in Subsection 3.2.

Exercise 16

Show that the function

$$\Theta(x,t) = \sin\left(\frac{\pi x}{L}\right)\exp\left(-\frac{\alpha\pi^2 t}{L^2}\right)$$

satisfies the heat equation (35) and the boundary conditions (36) if $\Theta_0 = 0$.

Exercise 17

Suppose that initially, the temperature of a rod rises linearly with x towards a peak in the centre that is half a degree above the end temperature Θ_0, as shown in Figure 13.

Write down a formula describing the initial temperature function $f(x)$.

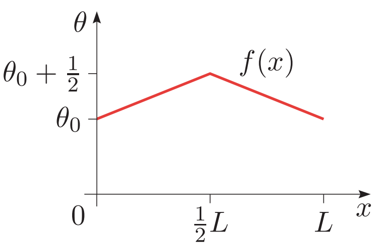

Figure 13 An initial temperature distribution

Exercise 18

Write down the initial condition describing the temperature distribution if the central third of a rod is initially heated to a temperature Θ_1 while the remainder of the rod stays at the background temperature Θ_0.

We conclude this subsection by considering a rod that is not insulated from the environment, so that the rod loses heat to an environment of constant temperature Θ_0 in addition to conducting heat along its length. In this situation, equation (34) must be modified to include a term that represents the heat transferred to its surroundings along the length of the segment in addition to the heat transferred at the ends. Applying our model of cooling, this term will be $-\gamma(\Theta - \Theta_0)$, where Θ_0 is the temperature of the surroundings, and we have written the constant of proportionality as γ to distinguish it from the different constant of proportionality k already in the equation. So we have

$$\Theta_t(x,t) = k\big(\Theta(x-\delta x,t) - 2\Theta(x,t) + \Theta(x+\delta x,t)\big) - \gamma(\Theta - \Theta_0),$$

and proceeding by using Taylor series as before, we can derive the following partial differential equation.

Uninsulated rod equation

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2} - \gamma(\Theta - \Theta_0). \tag{38}$$

3.2 Solving the heat equation

In this subsection we will use Procedure 1 to solve a particular case of the heat equation that was derived in Subsection 3.1. In Example 6 we consider a rod insulated from its surroundings with a given initial temperature.

Example 6

Apply Procedure 1 to the heat equation

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2},$$

subject to the boundary conditions

$$\Theta_x(0,t) = \Theta_x(L,t) = 0, \quad t \geq 0,$$

These boundary conditions model the situation where the ends of a hot rod are insulated.

and the initial condition

$$\Theta(x,0) = \tfrac{1}{2}\cos\left(\frac{2\pi x}{L}\right), \quad 0 < x < L.$$

Solution

◀ Separate variables ▶

We write $\Theta(x,t) = X(x)\,T(t)$, so

$$\frac{\partial\Theta}{\partial x} = X'T, \quad \frac{\partial^2\Theta}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial\Theta}{\partial t} = XT'.$$

Substituting into the partial differential equation gives $XT' = \alpha X''T$. Separate the variables to yield

$$\frac{X''}{X} = \frac{T'}{\alpha T}.$$

Arguing as before, both sides of this equation must be equal to a separation constant μ, so we have the two differential equations

$$X'' - \mu X = 0 \quad \text{and} \quad T' - \alpha\mu T = 0.$$

We have $\partial\Theta/\partial x = X'T$, so the boundary conditions become

$$X'(0)\,T(t) = X'(L)\,T(t) = 0, \quad t \geq 0,$$

hence

$$X'(0) = X'(L) = 0.$$

◀ Solve ODEs ▶

Consider the three cases $\mu = k^2 > 0$, $\mu = 0$ and $\mu = -k^2 < 0$.

- $\mu > 0$. Let $c = \sqrt{\mu}$. Then equation (9) gives the solution of the differential equation for $X(x)$ as

  $$X(x) = Ae^{cx} + Be^{-cx},$$

  where A and B are constants.

  Now we apply the boundary conditions, and to do this we first differentiate this solution:

  $$X'(x) = Ace^{cx} - Bce^{-cx}.$$

  The boundary condition $X'(0) = 0$ gives $Ac - Bc = 0$, so $A = B$ as $c > 0$. The boundary condition $X'(L) = 0$ gives

  $$Ace^{cL} - Bce^{-cL} = 0.$$

  As $A = B$ and $c > 0$, this simplifies to

  $$A(e^{cL} - e^{-cL}) = 0.$$

  Proceeding as before we can multiply by e^{cL} to obtain

  $$A(e^{2cL} - 1) = 0.$$

  Now we can see that the term in brackets is not zero as $2cL > 0$ so $\exp(2cL) > \exp(0) = 1$. So we have $A = B = 0$, and the only solution is the trivial solution.

- $\mu = 0$. In this case the differential equation for $X(x)$ becomes $X'' = 0$, with solution $X(x) = Ax + B$, where A and B are constants. So $X'(x) = A$, and both boundary conditions give the equation $A = 0$.

  So the solution $X(x) = B$ is a non-trivial solution of the equation.

- $\mu < 0$. Let $k = \sqrt{-\mu}$. Then by equation (9) the general solution of the equation for X is

  $$X(x) = A\cos kx + B\sin kx,$$

  where A and B are constants, so

  $$X'(x) = -Ak\sin kx + Bk\cos kx.$$

  Using the boundary conditions, we find that $B = 0$, and $A = 0$ or $k = n\pi/L$ for any non-zero integer n. This leads to the solution

  $$X(x) = A\cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots.$$

So we have *two* cases that give non-trivial solutions, namely the constant solutions that correspond to $\mu = 0$ and the sinusoidal solutions that correspond to $\mu < 0$. Here these two cases can be conveniently combined by adding $n = 0$ to the second set of solutions (using the fact that $\cos(0 \times x) = 1$ for all x). So we have the solutions

$$X(x) = A\cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, 3, \ldots.$$

With $\mu = n^2\pi^2/L^2$, the differential equation for $T(t)$ becomes

$$T'(t) + \alpha\,\frac{n^2\pi^2}{L^2}\,T(t) = 0.$$

Equation (8) gives the solution of this equation as

$$T(t) = C\exp\left(-\frac{\alpha n^2\pi^2 t}{L^2}\right), \quad n = 0, 1, 2, 3, \ldots,$$

where C is a constant.

This leads to the family of solutions

$$\Theta_n(x,t) = a_n\exp\left(-\frac{\alpha n^2\pi^2 t}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, 3, \ldots,$$

where the $a_n = AC$ are constants.

To find the solution that satisfies the initial condition, write

◀ Initial conditions ▶

$$\Theta(x,t) = \sum_{n=0}^{\infty} a_n\exp\left(-\frac{\alpha n^2\pi^2 t}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right), \tag{39}$$

and set $t = 0$ to give

$$\Theta(x,0) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right).$$

Now we use the given initial condition

$$\Theta(x,0) = \tfrac{1}{2}\cos\left(\frac{2\pi x}{L}\right).$$

By inspection we have $a_2 = \frac{1}{2}$, and $a_n = 0$ for $n \neq 2$, so the solution is

$$\Theta(x,t) = \tfrac{1}{2}\exp\left(-\frac{4\alpha\pi^2 t}{L^2}\right)\cos\left(\frac{2\pi x}{L}\right).$$

In Example 6, if the initial condition was not a cosine function, then at this point we would need to use the techniques of Unit 13 to find the Fourier cosine series in order to find a_n. Recall that the Fourier coefficient a_0 is equal to the average value of the given function over the interval. In this case this would mean that a_0 would be the average of the initial temperature distribution over the length of the rod. Looking at equation (39), we can see that the term involving a_0 is the only term that does not involve a negative exponential function of time. So the model predicts that the temperature of the rod tends to a constant temperature that is equal to the average of the initial temperatures. This is in accord with intuition for a model of a rod that is completely insulated from its surroundings.

The following exercises ask you to apply the separation of variables method.

Exercise 19

Solve the heat equation

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2}, \tag{40}$$

subject to the boundary conditions

These boundary conditions model the situation where the ends of the rod are held at a fixed temperature.

$$\Theta(0,t) = \Theta(L,t) = 0, \quad t \geq 0, \tag{41}$$

and the initial condition

$$\Theta(x,0) = \sin\left(\frac{\pi x}{L}\right). \tag{42}$$

Exercise 20

Find the solution of the uninsulated rod equation

This is equation (38) derived in Subsection 3.1, with $\Theta_0 = 0$.

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2} - \gamma\Theta, \tag{43}$$

subject to the boundary conditions

$$\Theta(0,t) = \Theta(L,t) = 0, \quad t \geq 0, \tag{44}$$

and the initial condition

$$\Theta(x,0) = \sin\left(\frac{\pi x}{L}\right). \tag{45}$$

Compare the solution with the solution to the insulated rod problem in Example 6.

Learning outcomes

After studying this unit, you should be able to:

- use the terms linear, homogeneous, order, initial condition and boundary condition as applied to partial differential equations
- show that a given function satisfies a given partial differential equation and/or boundary conditions and/or initial conditions
- use the method of separation of variables to find solutions of linear homogeneous second-order partial differential equations
- understand how the wave equation and heat equation can be used to model certain physical systems
- interpret solutions of partial differential equations in terms of a model.

Solutions to exercises

Solution to Exercise 1

(a) This equation is non-linear (because of the term that contains a product of u and u_x). It is a first-order equation.

(b) This equation is a linear second-order equation.

(c) This equation is a linear second-order equation.

(d) This equation is a linear third-order equation.

Solution to Exercise 2

Start by differentiating $u(x,t)$:

$$\frac{\partial u}{\partial x} = \cos(x)\, e^{-\alpha t}, \quad \frac{\partial^2 u}{\partial x^2} = -\sin(x)\, e^{-\alpha t}, \quad \frac{\partial u}{\partial t} = -\alpha \sin(x)\, e^{-\alpha t}.$$

So

$$\alpha \frac{\partial^2 u}{\partial x^2} = -\alpha \sin(x)\, e^{-\alpha t} = \frac{\partial u}{\partial t},$$

and the partial differential equation is satisfied.

Solution to Exercise 3

The initial velocity is zero, so the string is released from rest. When $x = \frac{1}{4}L$, $u(x,0) = -d$, so the point one-quarter of the way along the string has been displaced downwards by a distance d.

Solution to Exercise 4

The initial displacement has two linear sections, with slopes

$$\frac{d}{\frac{1}{3}L} = \frac{3d}{L} \quad \text{and} \quad -\frac{d}{\frac{2}{3}L} = -\frac{3d}{2L},$$

respectively. Hence the required initial condition is

$$u(x,0) = \begin{cases} \dfrac{3d}{L} x & \text{for } 0 < x \leq \frac{1}{3}L, \\ \dfrac{3d}{2L}(L - x) & \text{for } \frac{1}{3}L < x < L. \end{cases}$$

Solution to Exercise 5

Initially, there is no displacement, so

$$u(x,0) = 0, \quad 0 < x < L.$$

Since only the middle third is set in motion, the initial velocity is given by

$$u_t(x,0) = \begin{cases} v & \text{for } \frac{1}{3}L \leq x \leq \frac{2}{3}L, \\ 0 & \text{otherwise.} \end{cases}$$

Solution to Exercise 6

The boundary conditions are satisfied since

$$u(0,t) = \sin 0 \cos\left(\frac{\pi ct}{L}\right) = 0,$$

$$u(L,t) = \sin \pi \cos\left(\frac{\pi ct}{L}\right) = 0.$$

The initial condition is satisfied since

$$u_t(x,0) = -\frac{\pi c}{L}\sin\left(\frac{\pi x}{L}\right)\sin 0 = 0.$$

Solution to Exercise 7

(a) Since the function $\Theta(x,t) = \Theta_0$ is constant, all its derivatives are zero, and the differential equation reduces to $0 = 0$ thus is satisfied.

As $\Theta(x,t)$ is always equal to Θ_0, in particular it is equal to Θ_0 at the boundaries, that is, the boundary conditions are satisfied.

(b) As Θ_0 is constant, its derivatives are zero, so $u_t = \Theta_t$ and $u_{xx} = \Theta_{xx}$. So if Θ satisfies the heat equation, then so does u.

On the boundary we have $\Theta(0,t) = \Theta_0$, so $u(0,t) = \Theta_0 - \Theta_0 = 0$. Similarly, $\Theta(L,t) = \Theta_0$, so $u(L,t) = \Theta_0 - \Theta_0 = 0$.

So $u(x,t)$ satisfies the heat equation with homogeneous boundary conditions.

Solution to Exercise 8

(a) For

$$\Theta(x,t) = \frac{L-x}{L}\Theta_0 + \frac{x}{L}\Theta_L,$$

we have

$$\frac{\partial^2\Theta}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial\Theta}{\partial t} = 0,$$

so the differential equation reduces to $0 = 0$ thus is satisfied.

When $x = 0$ we have

$$\Theta(0,t) = \frac{L-0}{L}\Theta_0 + 0 = \Theta_0,$$

and when $x = L$ we have

$$\Theta(L,t) = \frac{L-L}{L}\Theta_0 + \frac{L}{L}\Theta_L = \Theta_L,$$

so the boundary conditions are satisfied.

(b) Start by differentiating:

$$\frac{\partial u}{\partial t} = \frac{\partial \Theta}{\partial t},$$
$$\frac{\partial u}{\partial x} = \frac{\partial \Theta}{\partial x} + \frac{1}{L}\Theta_0 - \frac{1}{L}\Theta_L,$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \Theta}{\partial x^2}.$$

So $u(x,t)$ satisfies the heat equation if $\Theta(x,t)$ satisfies the heat equation.

For the first boundary condition we have

$$u(0,t) = \Theta(0,t) - \frac{L-0}{L}\Theta_0 - 0 = \Theta_0 - \Theta_0 = 0,$$

and for the second boundary condition we have

$$u(L,t) = \Theta(L,t) - \frac{L-L}{L}\Theta_0 - \frac{L}{L}\Theta_L = \Theta_L - \Theta_L = 0.$$

So $u(x,t)$ satisfies the heat equation with homogeneous boundary conditions.

Solution to Exercise 9

$$\frac{\partial u}{\partial t} = X(x)\,T'(t), \quad \frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t),$$
$$\frac{\partial u}{\partial x} = X'(x)\,T(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t).$$

Solution to Exercise 10

Proceeding as in the main text, we use the derivatives found in Exercise 9, namely

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t) \quad \text{and} \quad \frac{\partial u}{\partial t} = X(x)\,T'(t).$$

Substituting these into the partial differential equation gives

$$X''(x)\,T(t) + 2\,X(x)\,T(t) = X(x)\,T'(t).$$

Dividing by $X(x)\,T(t)$ gives

$$\frac{X''(x)}{X(x)} + 2 = \frac{T'(t)}{T(t)}.$$

We chose to leave the 2 on the left-hand side. It would be equally correct to write the separated equations as

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} - 2.$$

This has separated the variables, so both sides must be equal to a constant, say μ. So we have the two equations

$$\frac{X''(x)}{X(x)} + 2 = \mu \quad \text{and} \quad \frac{T'(t)}{T(t)} = \mu.$$

Multiplying out the fractions and rearranging gives the required differential equations:

$$X''(x) + (2-\mu)\,X(x) = 0 \quad \text{and} \quad T'(t) - \mu\,T(t) = 0.$$

Solution to Exercise 11

As the boundary condition is defined by the partial derivative with respect to x, we first find

$$\frac{\partial u}{\partial x} = X'(x)\,T(t).$$

So the boundary condition becomes $X'(1)\,T(t) = 0$ for $t \geq 0$. The solution $T(t) = 0$ for $t \geq 0$ leads to the trivial solution. So for a non-trivial solution we must have $X'(1) = 0$.

Solution to Exercise 12

Setting $u(x,t) = X(x)\,T(t)$, the required partial derivatives are

◀ Separate variables ▶

$$\frac{\partial^2 u}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = XT''.$$

Substituting into the partial differential equation and dividing by XT gives

$$\frac{X''}{X} + \frac{T''}{T} = 0,$$

from which it follows that

$$\frac{X''}{X} = -\frac{T''}{T}.$$

Both sides of this equation must be a constant, say μ, giving

$$\frac{X''}{X} = \mu \quad \text{and} \quad -\frac{T''}{T} = \mu$$

or equivalently,

$$X'' - \mu X = 0 \quad \text{and} \quad T'' + \mu T = 0.$$

The boundary conditions become $X(0) = X(1) = 0$.

The boundary conditions are the same as those used in the main text. So arguing as in the text, only negative μ gives a non-trivial solution for X. In this case, the equation for X has general solution (see equation (9))

◀ Solve ODEs ▶

$$X(x) = A\cos kx + B\sin kx,$$

where $k = \sqrt{-\mu}$, and A and B are constants. The boundary condition $X(0) = 0$ implies $A = 0$. The boundary condition $X(1) = 0$ implies $B\sin k = 0$, so for non-trivial solutions we must have $\sin k = 0$, that is, $k = n\pi$ for some positive integer n. For this value of k we have

$$X(x) = B\sin(n\pi x), \quad n = 1, 2, 3, \ldots.$$

Since $\mu = -n^2\pi^2$, the equation for T can be written as

$$T''(t) - n^2\pi^2\,T(t) = 0.$$

Using equation (9), the solutions in this case are (as $n^2\pi^2$ is positive)

$$T(t) = Ce^{n\pi t} + De^{-n\pi t}, \quad n = 1, 2, 3, \ldots,$$

where C and D are constants.

So the required family of normal mode solutions is

$$u_n(x,t) = \sin(n\pi x)\,(a_n e^{n\pi t} + b_n e^{-n\pi t}),$$

where n is a positive integer and $a_n = BC$, $b_n = BD$ are constants.

Solution to Exercise 13

T_{eq} is a component of a force, so $[T_{\text{eq}}] = \mathrm{M\,L\,T^{-2}}$. Also, $[M] = \mathrm{M}$ and $[L] = \mathrm{L}$. Hence

$$[T_{\text{eq}}L/M] = \mathrm{M\,L\,T^{-2}} \times \mathrm{L/M} = \mathrm{L^2\,T^{-2}},$$

so $[\sqrt{T_{\text{eq}}L/M}] = \mathrm{L\,T^{-1}}$, as required.

Solution to Exercise 14

To find the solution we need to find the coefficient a_n appearing in the equation

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right).$$

This can be done by inspection as the term on the right-hand side appears as one of the terms on the left-hand side. So we get $a_3 = \frac{1}{2}$, and $a_n = 0$ for $n \neq 3$.

So the solution for the model is given by equation (21) with the above coefficients, that is,

$$u(x,t) = \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3c\pi t}{L}\right).$$

Solution to Exercise 15

Write the unknown function as

$$u(x,t) = X(x)\,T(t).$$

Differentiate to get the partial derivatives:

$$\frac{\partial u}{\partial t} = X(x)\,T'(t), \quad \frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t).$$

Substituting these into the damped wave equation gives

$$X''(x)\,T(t) = \frac{1}{c^2}\left(X(x)\,T''(t) + 2\varepsilon\,X(x)\,T'(t)\right).$$

As before, this equation can hold for all x and all t only if both sides are equal to a separation constant μ, that is,

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\left(\frac{T''(t)}{T(t)} + 2\varepsilon\,\frac{T'(t)}{T(t)}\right) = \mu,$$

giving a pair of equations.

The first of these is

$$\frac{X''(x)}{X(x)} = \mu,$$

which on rearranging becomes

$$X''(x) - \mu\, X(x) = 0.$$

Similarly, the second equation reduces to

$$T''(t) + 2\varepsilon\, T'(t) - c^2 \mu\, T(t) = 0.$$

Now we have completed the process of separating the single partial differential equation into two ordinary differential equations, but we still have to find boundary conditions for the function X. To do this, we substitute $u(x,t) = X(x)\,T(t)$ into the original boundary conditions.

The boundary condition $u(0,t) = 0$ becomes $X(0)\,T(t) = 0$ for all t. So for non-trivial solutions we must have $X(0) = 0$. Similarly, the boundary condition $u(L,t) = 0$ gives $X(L)\,T(t) = 0$ for all t, so non-trivial solutions must satisfy $X(L) = 0$. So the boundary conditions become

$$X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

which are boundary conditions for the ordinary differential equation for X.

Solution to Exercise 16

$$\frac{\partial \Theta}{\partial x} = \frac{\pi}{L} \cos\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right),$$

so

$$\frac{\partial^2 \Theta}{\partial x^2} = -\frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right) = -\frac{\pi^2}{L^2}\, \Theta(x,t).$$

Also,

$$\frac{\partial \Theta}{\partial t} = -\frac{\alpha \pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right) = -\frac{\alpha \pi^2}{L^2}\, \Theta(x,t).$$

Thus

$$\frac{\partial \Theta}{\partial t} = -\alpha\, \frac{\pi^2}{L^2}\, \Theta(x,t) = \alpha\, \frac{\partial^2 \Theta}{\partial x^2}.$$

Hence equation (35) is satisfied.

The boundary conditions are

$$\Theta(0,t) = \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right) \sin 0 = 0,$$

$$\Theta(L,t) = \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right) \sin \pi = 0,$$

so these are satisfied for a temperature at the rod ends of $\Theta_0 = 0$.

Solution to Exercise 17

The graph in Figure 13 looks just like the picture of the plucked string in Figure 2, so the required formula is

$$f(x) = \begin{cases} \Theta_0 + x/L & \text{for } 0 < x \le \frac{1}{2}L, \\ \Theta_0 + (L-x)/L & \text{for } \frac{1}{2}L < x < L. \end{cases}$$

As required, this function takes the value Θ_0 when x is 0 or L, and takes the value $\Theta_0 + \frac{1}{2}$ when x is $\frac{1}{2}L$.

Solution to Exercise 18

$$\Theta(x,0) = \begin{cases} \Theta_0 & \text{for } 0 < x < \frac{1}{3}L, \\ \Theta_1 & \text{for } \frac{1}{3}L \le x \le \frac{2}{3}L, \\ \Theta_0 & \text{for } \frac{2}{3}L < x < L. \end{cases}$$

Solution to Exercise 19

◀ Separate variables ▶

We begin as in Example 6, and obtain the equations

$$X'' - \mu X = 0 \quad \text{and} \quad T' - \alpha\mu T = 0.$$

To find boundary conditions for X, we put $x = 0$ and $x = L$ in $\Theta(x,t) = X(x)\,T(t)$ and substitute into the boundary conditions (41), which gives

$$X(0)\,T(t) = X(L)\,T(t) = 0, \quad t \ge 0,$$

hence

$$X(0) = X(L) = 0.$$

◀ Solve ODEs ▶

Next we solve the differential equations for X and T, and combine the families of solutions.

The differential equation for X, and its boundary conditions, are the same as in Example 4. You have seen that a non-trivial solution occurs only if the separation constant μ is negative, and is given by

$$X(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots,$$

where B is a constant. In this case the separation constant μ is $-n^2\pi^2/L^2$.

Our next task is to solve the equation for T when $\mu = -n^2\pi^2/L^2$, namely

$$T'(t) + \frac{\alpha n^2\pi^2}{L^2}\,T(t) = 0.$$

The general solution is given by equation (8) as

$$T(t) = C\exp\left(-\frac{\alpha n^2\pi^2 t}{L^2}\right),$$

where C is a constant.

Combining the solutions for X and T, we obtain the family of normal mode solutions

$$\Theta_n(x,t) = a_n \exp\left(-\frac{\alpha n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots,$$

where the $a_n = BC$ are constants.

As the partial differential equation and boundary conditions are linear and homogeneous, we can use the principle of superposition to form the more general solution

$$\Theta(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\alpha n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Now we use the initial condition (42) and results on Fourier series to determine the coefficients a_n. Setting $t = 0$ in this equation gives

◀ Initial conditions ▶

$$\sin\left(\frac{\pi x}{L}\right) = \Theta(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).$$

By inspection, $a_1 = 1$, and $a_n = 0$ for $n = 2, 3, \ldots$, as the term on the left-hand side is one member of the sum on the right-hand side.

So the solution is

$$\Theta(x,t) = \exp\left(-\frac{\alpha \pi^2 t}{L^2}\right) \sin\left(\frac{\pi x}{L}\right).$$

Solution to Exercise 20

Set $\Theta(x,t) = X(x)\,T(t)$. Then

◀ Separate variables ▶

$$\frac{\partial^2\Theta}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial\Theta}{\partial t} = XT'.$$

Equation (43) becomes

$$XT' = \alpha X''T - \gamma XT,$$

and dividing by XT gives

$$\frac{T'}{T} = \alpha\,\frac{X''}{X} - \gamma.$$

This can be rearranged as

$$\frac{1}{\alpha}\left(\frac{T'}{T} + \gamma\right) = \frac{X''}{X}.$$

Again, a function of x is equal to a function of t, so both must be constant. Choosing the constant to be $\mu = -k^2$, as before, the equations become

$$X'' + k^2X = 0 \quad \text{and} \quad T' + (\alpha k^2 + \gamma)T = 0.$$

The boundary conditions reduce to

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

◀ Solve ODEs ▶

Solving the differential equation for $X(x)$ subject to fixed endpoint boundary conditions leads again to $k = n\pi/L$, for any positive integer n, and then to the family of solutions for X given by

$$X(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots,$$

where B is a constant.

The solution for T is given by equation (8) with $k = n\pi/L$:

$$T(t) = A\exp\left(-\left(\frac{\alpha n^2\pi^2}{L^2} + \gamma\right)t\right), \quad n = 1, 2, 3, \ldots,$$

where A is a constant.

So the family of solutions is

$$\Theta_n(x,t) = a_n\exp\left(-\left(\frac{\alpha n^2\pi^2}{L^2} + \gamma\right)t\right)\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots,$$

where the $a_n = BA$ are constants.

◀ Initial conditions ▶

Now use the principle of superposition to write down a more general solution:

$$\Theta(x,t) = \sum_{n=1}^{\infty} a_n\exp\left(-\left(\frac{\alpha n^2\pi^2}{L^2} + \gamma\right)t\right)\sin\left(\frac{n\pi x}{L}\right).$$

Setting $t = 0$ gives

$$\sin\left(\frac{\pi x}{L}\right) = \Theta(x,0) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right).$$

By inspection, $a_1 = 1$, and $a_n = 0$ for $n = 2, 3, \ldots$, as the term on the left-hand side is one member of the sum on the right-hand side. Therefore the solution is

$$\begin{aligned}\Theta(x,t) &= \exp\left(-\left(\frac{\alpha\pi^2}{L^2} + \gamma\right)t\right)\sin\left(\frac{\pi x}{L}\right)\\ &= e^{-\gamma t}\exp\left(-\frac{\alpha\pi^2 t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right).\end{aligned}$$

This is the same as the solution for the insulated rod except for the $e^{-\gamma t}$ factor, so the uninsulated rod cools more quickly than the insulated rod, by a factor of $e^{-\gamma t}$.