Unit 17

Multiple integrals

Introduction

You know that in one dimension the definite integral of a function $f(x)$ on an interval of the x-axis can be defined as the limit of a sum of function values on the interval. In Unit 16 you saw how this idea could be generalised to line integrals, and used to calculate properties of vector fields. In this unit we will generalise the idea of an integral still further to deal with two and three dimensions by introducing two new kinds of integrals, called *area integrals* and *volume integrals*.

Area integrals arise in calculations of quantities such as populations over regions. For example, suppose that you know the number of bacteria per unit area at each point on a glass plate. If this area density is uniform, then the total number of bacteria on any region of the plate is the product of the surface density and the area of that region. More generally, if the population density is non-uniform, then the total number of bacteria on a region of the plate can be expressed as an *area integral of the population density* over that region. Because a surface is a two-dimensional region, an area integral involves integrating with respect to two independent variables. Section 1 shows how area integrals can be evaluated as combinations of two ordinary integrals, while Section 2 describes applications of area integrals.

Volume integrals arise in the calculation of the mass of an object. For example, suppose that you know the mass per unit volume of a solid body. If the density is uniform, then the total mass of the body is the product of its density and its volume. More generally, the mass of the body is the *volume integral of the density* over the three-dimensional region occupied by the body. Section 3 shows how volume integrals can be expressed as combinations of three ordinary integrals.

Area integrals and volume integrals are examples of *multiple integrals*, the title of this unit. One of the major applications of multiple integrals is the calculation of *moments of inertia*. If you have not come across this term before, do not worry. Unit 21 will give a full treatment of moments of inertia and will describe their role in the rotational dynamics of solid bodies. In this unit we will give a definition of the moment of inertia but thereafter we will be concerned only with the techniques of calculating moments of inertia as multiple integrals.

Section 4 shows how area integrals can be used to compute the area of a curved surface.

1 Area integrals

In Unit 16 you saw that a scalar line integral involves integrating with respect to just one variable along a path. We now turn to *area integrals* in which the function being integrated is defined over an area, rather than along a path, so that it is a function of two variables.

This variable is usually the path parameter.

We can use an area integral to calculate, for example, the mass of a thin plate when the composition of the plate is non-uniform. Also, suppose that the height of a three-dimensional region Ω is represented by a height function $z = h(x, y)$ over an area S of the (x, y)-plane. Then the area integral of $h(x, y)$ over S will give the volume occupied by Ω.

We can also use area integrals to calculate populations. The distribution of, say, bacteria on a microscope slide can be described by a *population density* function giving the number of individuals per square metre at each point. Then the total population on the slide is the area integral of the population density over the slide. We will consider this and other applications of area integrals in Section 2. In this section we define area integrals (in Subsection 1.1) and see how to compute them using Cartesian coordinates (in Subsections 1.2 and 1.3).

1.1 Defining area integrals

Materials such as plate glass are usually sold by the square metre. By ignoring the fact that the material is a three-dimensional solid and treating it instead as an infinitely thin sheet, or **lamina**, we can describe the 'weight' of such a material by a *surface mass density* or mass per unit area of the sheet. The surface mass density, or **surface density** for short, is a constant for a uniform sheet, but will vary with position when the material is of non-uniform composition. If we imagine the lamina placed in the (x, y)-plane, then the surface density is a **surface density function** $f(x, y)$ defined on the region S of the (x, y)-plane occupied by the lamina.

We assume that we are carrying out a theoretical investigation rather than simply weighing the lamina to find its mass.

Suppose that we wish to find the mass of this lamina. We know that when $f(x, y)$ is a constant, the total mass of the lamina is simply the constant density (in $\mathrm{kg\,m^{-2}}$) times the total area (in $\mathrm{m^2}$) of the lamina. When the surface density varies, we can divide the region S into N small **area elements** such that the density can be considered constant on each element (see Figure 1). Let the area of the ith element be δA_i, and in this element select a point with coordinates (x_i, y_i). Then the mass of this element of the plate is approximately $f(x_i, y_i)\,\delta A_i$, and the total mass of the lamina is approximately the sum of the masses of all the elements:

$$\sum_{i=1}^{N} f(x_i, y_i)\,\delta A_i.$$

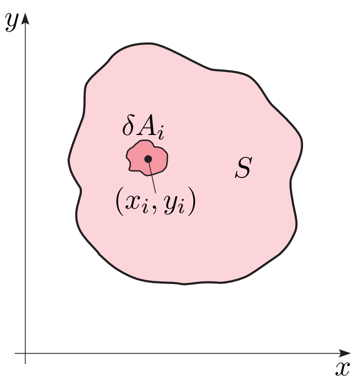


Figure 1 Lamina occupying region S of the (x, y)-plane

Now if we take the limit of this sum, as the number N of area elements increases indefinitely and the size of the elements tends to zero, we obtain an integral called an area integral, which we denote by

$$\int_S f(x, y)\, dA.$$

This area integral gives the mass of the lamina. Note that if we take $f(x, y) = 1$, then the area integral will give the area of the lamina.

We can define an area integral for any function of two variables over a region S in its domain.

Area integral

The **area integral** of $f(x, y)$ over a **region of integration** S in the (x, y)-plane, subdivided into N area elements where element i contains the point (x_i, y_i) and is of area δA_i, is

$$\int_S f(x,y)\,dA = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\,\delta A_i, \tag{1}$$

where $\delta A_i \to 0$ for each i as $N \to \infty$.

We will look at other applications of area integrals in Section 2. Next we consider how to evaluate area integrals, first over rectangular regions and then over non-rectangular regions. Since $f(x, y)$ is a function of two variables, we will find that the evaluation of an area integral involves evaluating *two* definite integrals, one after the other.

Area integrals are sometimes referred to as *double integrals* or *surface integrals*.

1.2 Area integrals over rectangular regions

In this subsection we consider how to apply equation (1) to evaluate an area integral, and in particular how to subdivide the region of integration S in a convenient way when working with Cartesian coordinates.

We begin by considering the case when the region S is the rectangle formed by the lines $x = a$, $x = b$ and $y = c$, $y = d$, as in Figure 2.

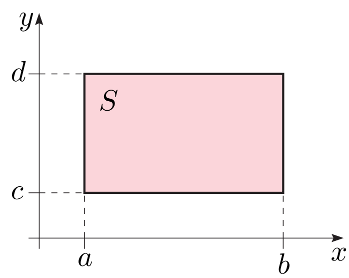


Figure 2 Rectangular region S

The most natural way to evaluate the sum in equation (1) is to divide the rectangle into small rectangular elements. We proceed as follows (see Figure 3).

- Divide the interval $[a, b]$ of the x-axis into n subintervals of widths $\delta x_1, \delta x_2, \ldots, \delta x_j, \ldots, \delta x_n$, and choose numbers $x_1, x_2, \ldots, x_n$ such that x_j is in the jth subinterval.
- Divide the interval $[c, d]$ of the y-axis into m subintervals of widths $\delta y_1, \delta y_2, \ldots, \delta y_k, \ldots, \delta y_m$, and choose numbers $y_1, y_2, \ldots, y_m$ such that y_k is in the kth subinterval.
- The area elements are the small rectangles so formed; the general element has side lengths $\delta x_j, \delta y_k$ and area $\delta x_j\,\delta y_k$, and contains the point (x_j, y_k).

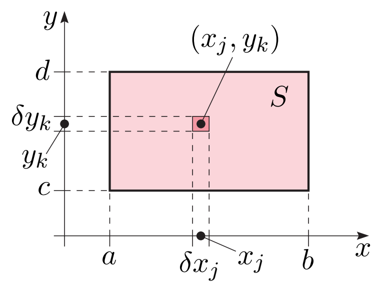


Figure 3 Small rectangular element within the region S

The rectangle S has been divided into $n \times m$ rectangular area elements, each one identified by a point (x_j, y_k), and we need to find a systematic way of including all $N = n \times m$ of them in the summation. Let us organise the rectangular area elements into thin strips. There are two straightforward ways. We can consider vertical strips of widths $\delta x_1, \delta x_2, \ldots, \delta x_j, \ldots, \delta x_n$ (as in Figure 4(a)), counting all the segments of length δy_k within each strip, or we can consider horizontal strips of widths $\delta y_1, \delta y_2, \ldots, \delta y_k, \ldots, \delta y_m$ (as in Figure 4(b)), counting all the segments of length δx_j in each.

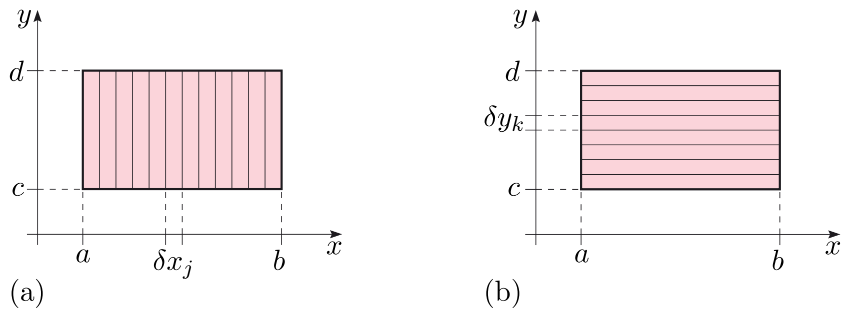


Figure 4 Rectangular region S divided into (a) vertical strips, (b) horizontal strips

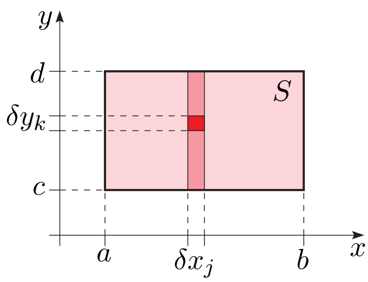


Figure 5 Rectangular region S showing the jth vertical strip

We choose vertical strips. The jth vertical strip (of width δx_j) is shown shaded in Figure 5. So a typical element with subscript i in equation (1) is included by virtue of its position y_k in the strip and the label x_j for the strip. Thus a typical term in the sum in equation (1) is $f(x_j, y_k)\,\delta x_j\,\delta y_k$. So the contribution to the sum from the m rectangular area elements in the jth vertical strip, of width δx_j, is

$$\left(\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right)\delta x_j.$$

In this summation, x_j and δx_j remain unchanged, and we are summing over the y_k terms, that is, over the subscript k. Now we need to add the contributions of the n vertical strips that lie between $x = a$ and $x = b$, so we are summing over the x_j terms, that is, over the subscript j. We then have an expression involving two summations:

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right)\delta x_j.$$

The summation over k is called the *inner summation*, and the summation over j is called the *outer summation*. The area integral of the function f over the region S is the limit of this double summation as the number of rectangles increases indefinitely, that is, as n and m increase indefinitely. So

$$\int_S f(x,y)\,dA = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right)\delta x_j\right). \qquad (2)$$

First let us look at the inner summation over k, remembering that x_j is constant here. In the limit as the lengths δy_k go to zero, this summation equals the definite integral of $f(x_j, y)$ between $y = c$ and $y = d$, that is, between the bottom and top of the jth vertical strip. We write

$$\lim_{m\to\infty}\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k = \int_{y=c}^{y=d} f(x_j, y)\,dy. \qquad (3)$$

You can think of the integration over y as *partial integration* by analogy with partial differentiation. You are integrating with respect to one variable while keeping the other variable fixed.

Remember that x_j is a constant in the summation and is held constant when we evaluate the integral. So the value of the integral will depend on x_j but not on y: that is, the value of the integral is a function of x_j. We illustrate this with an example.

Example 1

Find the value of the definite integral

$$\int_{y=1}^{y=2} x_j y \, dy,$$

where x_j is a constant.

Solution

The function to be integrated is $f(x_j, y) = x_j y$. In the integration we are varying y (between the limits 1 and 2) but keeping x_j fixed. So

$$\int_{y=1}^{y=2} x_j y \, dy = x_j \int_{y=1}^{y=2} y \, dy = x_j \left[\tfrac{1}{2}y^2\right]_{y=1}^{y=2} = \tfrac{3}{2}x_j.$$

Suppose that we denote the value of the integral $\int_{y=c}^{y=d} f(x_j, y)\, dy$ in equation (3) by $g(x_j)$. Equation (2) then becomes

$$\int_S f(x,y)\, dA = \lim_{n\to\infty} \sum_{j=1}^{n} g(x_j)\, \delta x_j.$$

In this summation we are adding vertical strips between $x = a$ and $x = b$ (see Figure 6), and in the limit the summation approaches the definite integral of $g(x)$ between the limits $x = a$ and $x = b$:

$$\int_{x=a}^{x=b} g(x)\, dx = \lim_{n\to\infty} \sum_{j=1}^{n} g(x_j)\, \delta x_j.$$

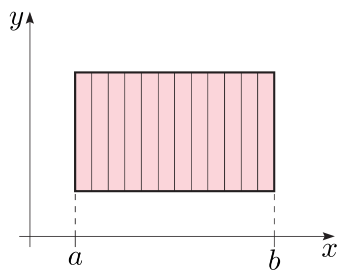


Figure 6 Rectangular region S divided into vertical strips

In the limit of the double summation, as δx_j and δy_k go to zero in equation (2), the value of the area integral, in terms of two single integrals, the first over y and the second over x, is

$$\int_S f(x,y)\, dA = \int_{x=a}^{x=b} g(x)\, dx, \quad \text{where } g(x) = \int_{y=c}^{y=d} f(x,y)\, dy. \tag{4}$$

The two integrals in equation (4) can be evaluated using standard techniques.

Area integral over a rectangular region

The area integral of a function $f(x,y)$ over a rectangular region S contained between the lines $x = a$, $x = b$ and $y = c$, $y = d$ is obtained as two successive integrals as follows:

$$\int_S f(x,y)\, dA = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\, dy \right) dx.$$

Remember that in the integral over y, which is computed first, we treat x as a constant.

Note that the limits are stated with respect to the variable over which the integration is performed. This is done for clarity.

Example 2

Find the value of the area integral of the function $f(x,y) = xy$ over the rectangle bounded by the lines $x = 0$, $x = 3$ and $y = 1$, $y = 2$.

Solution

The region of integration S is shown in Figure 7. The area integral is

$$\int_S xy\,dA = \int_{x=0}^{x=3} \left(\int_{y=1}^{y=2} xy\,dy \right) dx.$$

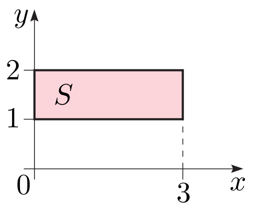


Figure 7 Region of integration

The integral over y was evaluated in Example 1, that is,

$$\int_{y=1}^{y=2} xy\,dy = \tfrac{3}{2}x,$$

so

$$\int_S xy\,dA = \int_{x=0}^{x=3} \left(\int_{y=1}^{y=2} xy\,dy \right) dx = \int_{x=0}^{x=3} \tfrac{3}{2}x\,dx = \left[\tfrac{3}{4}x^2\right]_0^3 = \tfrac{27}{4}.$$

The value of the area integral of xy over the rectangle is $\frac{27}{4}$.

Exercise 1

Find the value of the area integral of the function $f(x,y) = x^2y^3$ over the square bounded by the lines $x = 0$, $x = 2$ and $y = 1$, $y = 3$.

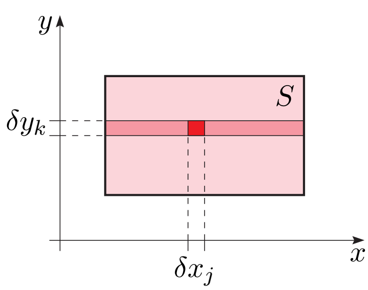


Figure 8 Rectangular region S showing the kth horizontal strip

So far we have evaluated area integrals by first integrating over y and then integrating over x. In the summations this is equivalent to summing first over k and then over j. However, we could have used another order for the summations, by first drawing a strip parallel to the x-axis of width δy_k containing y_k, and summing over the rectangles of width δx_j for all j running from 1 to n (see Figure 8). The result would be

$$\int_S f(x,y)\,dA = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy.$$

This time in the inner integral we integrate over x, treating y as a constant, to give a function of y, and then we integrate over y, to complete the evaluation of the integral.

Exercise 2

Find the value of the area integral in Example 2, but integrate over x first and then over y.

The value of the area integral in Exercise 2 is the same as in Example 2, which you may have expected since the function and the region of integration are the same. Either order of evaluating the single integrals will give the same answer.

1.3 Area integrals over non-rectangular regions

The area integral of a function over a rectangular region involves two single integrals for which the limits of integration are constants. For non-rectangular regions, we must be more careful in setting up the integrals because the strips are no longer of the same length and the limits on the inner integral depend on the variable in the outer integral. To illustrate this, consider the following example.

Example 3

Find the value of the area integral of the function $f(x,y) = xy$ over the region S bounded by the curves $y = 2x$, $y = x^2$ and the line $x = 1$.

Solution

We begin by drawing a diagram to show the region of integration – it is the shaded region in Figure 9. We choose to integrate first over y and then over x since the geometry of the region of integration suggests that this is likely to lead to the easier calculation. If we choose to integrate over x first, then the nature of the upper limit of the x-integral would depend on whether $y < 1$ or $y > 1$. We would therefore need to break up the region of integration into two subregions, one with $0 \leq y \leq 1$ and another with $1 \leq y \leq 2$. So, to avoid this, we integrate over y first.

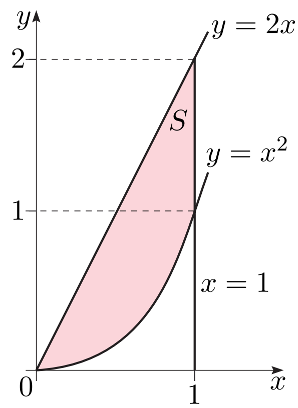


Figure 9 Region of integration

To decide on the limits of the integration over y, consider a vertical strip drawn at an arbitrary value of x, as shown in Figure 10(a). The ends of the strip lie on the curves $y = x^2$ and $y = 2x$. Hence the lower and upper limits for the y-integration are (the functions) $y = x^2$ and $y = 2x$, respectively. The y-integration sums all area elements in the strip of Figure 10(b).

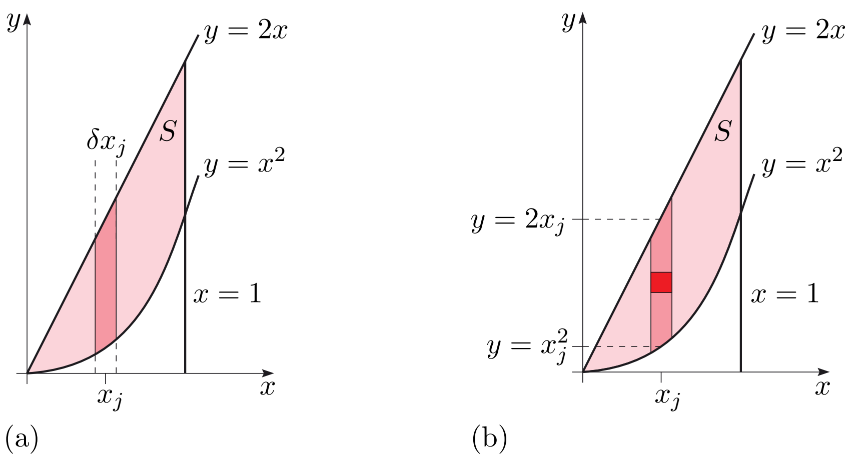


Figure 10 (a) Vertical strip at $x = x_j$ in the region of integration. (b) An area element in the strip.

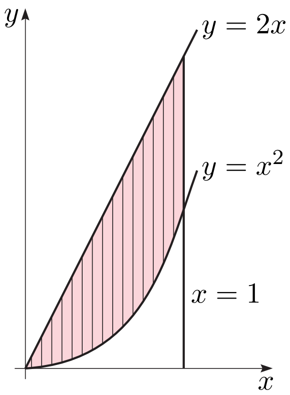


Figure 11 Vertical strips dividing the region of integration

The x-integration sums over all the strips (see Figure 11). The first strip is at $x = 0$, and the last strip is at $x = 1$. Hence the lower and upper limits for the x-integration are $x = 0$ and $x = 1$, respectively, and in the limit equation (2) becomes

$$\int_S xy\,dA = \int_{x=0}^{x=1}\left(\int_{y=x^2}^{y=2x} xy\,dy\right)dx. \qquad (5)$$

The inner integral, in the brackets, is an integral with respect to y and can be evaluated by treating x as a constant, to give

$$\int_{y=x^2}^{y=2x} xy\,dy = x\left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=2x} = x\left(\tfrac{1}{2}(2x)^2 - \tfrac{1}{2}(x^2)^2\right) = 2x^3 - \tfrac{1}{2}x^5.$$

Notice that the result of evaluating this integral is a function of x, so the area integral in equation (5) is reduced to a single integral over x, and

$$\int_S xy\,dA = \int_{x=0}^{x=1}\left(2x^3 - \tfrac{1}{2}x^5\right)dx = \left[\tfrac{2}{4}x^4 - \tfrac{1}{12}x^6\right]_0^1 = \tfrac{5}{12}.$$

In Example 3, the limits on the integral over y depend on the variable x. Drawing a diagram of the region of integration is very helpful for getting the limits correct, and the method of solution used in the example works for all area integrals. We will now generalise the procedure for evaluating area integrals.

Consider a non-rectangular region S like one of the two shown in Figure 12.

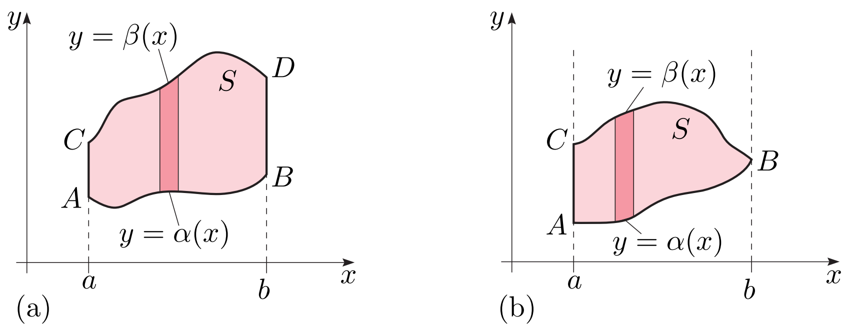


Figure 12 Non-rectangular regions of integration S. In (a) the maximum value of x on the boundary of S lies along a line, whereas in (b) it is at a point.

The minimum and maximum values can be straight lines, as shown in Figure 12(a), or points. One such point is shown in Figure 12(b).

Suppose that a and b are the minimum and maximum values of x for the points on the boundary of S. Let $y = \alpha(x)$ and $y = \beta(x)$ be the equations of the boundary curves AB and CD, respectively, as shown in Figure 12(a). (In Figure 12(b), points B and D coincide, so in this case, $y = \beta(x)$ is the equation of the boundary curve CB.) Then

$$\int_S f(x,y)\,dA = \int_{x=a}^{x=b}\left(\int_{y=\alpha(x)}^{y=\beta(x)} f(x,y)\,dy\right)dx.$$

The area integral can be written as two single definite integrals, but we must be careful with the limits. The limits on the inner integral are functions of x rather than constants.

We can summarise the steps for evaluating an area integral using vertical strips, as follows.

Procedure 1 Evaluating an area integral (integration with respect to y first)

To evaluate an area integral

$$\int_S f(x,y)\,dA,$$

carry out the following steps.

1. Draw a diagram showing the region of integration S.
2. Draw on the diagram a strip parallel to the y-axis, and show the lower limit $y=\alpha(x)$ and the upper limit $y=\beta(x)$ of this strip. These are the limits of the y-integration, that is, for the 'inner' integral.
3. Determine the lower limit a and upper limit b of x for points on the boundary of S. These are the limits of the x-integration, that is, for the 'outer' integral.
4. Write the area integral as two single integrals, making sure that the outer limits are constants:
$$\int_S f(x,y)\,dA = \int_{x=a}^{x=b}\left(\int_{y=\alpha(x)}^{y=\beta(x)} f(x,y)\,dy\right)dx.$$
5. Evaluate the inner integral, holding x constant, to give
$$g(x) = \int_{y=\alpha(x)}^{y=\beta(x)} f(x,y)\,dy.$$
6. Evaluate the outer integral to give
$$\int_S f(x,y)\,dA = \int_{x=a}^{x=b} g(x)\,dx.$$

Note that there should be no holes in region S, and that $\alpha(x)$ and $\beta(x)$ should be single-valued functions.

Exercise 3

Use Procedure 1 to find the value of the area integral of the function $f(x,y)=y$ over the region S bounded by the curves $y=x^2$ and $y=x+2$.

Exercise 4

Find the value of the area integral of the function $f(x,y)=x-y$ over the triangle bounded by the lines $y=x-1$, $x=3$ and $y=0$.

Exercise 5

Find the value of the area integral of the function $f(x, y) = x + y$ over the triangle S bounded by the lines $y = 1 - x$, $x = 0$ and $y = 0$.

In the course of this unit, you will often come across single integrals that contain both a function $f(x)$ and its derivative $f'(x)$, and can be cast in the form

$$\int k\, f'(x)\, \big(f(x)\big)^n \, dx.$$

There is a trick to integrate this, based on recognising that, from the chain rule,

$$\frac{d}{dx}\big(f(x)\big)^{n+1} = (n+1)\, \big(f(x)\big)^n\, f'(x),$$

so

$$\int k\, f'(x)\, \big(f(x)\big)^n \, dx = \frac{k}{n+1}\, \big(f(x)\big)^{n+1}. \tag{6}$$

Exercise 6

Find the value of the area integral of the function $f(x, y) = x$ over the quarter-disc S given by $x^2 + y^2 \leq 1$ $(x \geq 0,\ y \geq 0)$.

Exercise 7

Determine the region of integration for, and evaluate,

$$\int_{x=1}^{x=2} \left(\int_{y=x^2}^{y=x+2} 1\, dy \right) dx.$$

Procedure 1 is easily adapted for this reversed order of integration by interchanging x and y.

So far we have chosen to organise the area elements into vertical strips and integrate over y first and then x. We can reverse the order of integration, carrying out the x-integration first and then the y-integration by imagining the area divided into horizontal strips. The following example illustrates how this is done.

Example 4

Find the value of the area integral of the function xy^2 over the region S bounded by the curve $y = x^2$ and the line $y = x$ by integrating first over x and then over y.

Solution

The region of integration is shown in Figure 13(a), and one horizontal strip is shown in Figure 13(b).

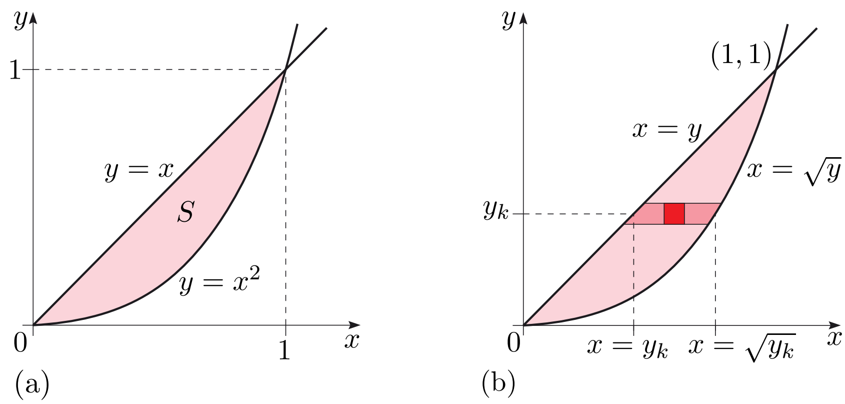


Figure 13 (a) Region of integration S. (b) The kth horizontal strip.

The x-integral effectively sums area elements along each horizontal strip. To decide on the limits for the x-integral, look at the left-hand and right-hand endpoints of the strip. These endpoints lie on the graphs of $x = y$ and $x = \sqrt{y}$, so the lower and upper limits of the x-integral are the functions $x = y$ and $x = \sqrt{y}$, respectively. The limits for the y-integral are the y-values where the graphs intersect, where $y = x = x^2$, that is, where $x^2 - x = x(x-1) = 0$. These points of intersection are therefore $(0,0)$ and $(1,1)$, as can be seen in Figure 13(a). Hence the area integral is

Note that we take the *positive* root of $y = x^2$ here as we have $x > 0$.

$$\int_S f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=y}^{x=\sqrt{y}} xy^2\,dx \right) dy.$$

The x-integral can be evaluated, treating y as a constant, to give

$$\int_{x=y}^{x=\sqrt{y}} xy^2\,dx = y^2 \left[\tfrac{1}{2}x^2\right]_{x=y}^{x=\sqrt{y}} = \tfrac{1}{2}(y^3 - y^4),$$

so evaluation of the area integral is reduced to a single integral in y, giving

$$\int_S f(x,y)\,dA = \int_{y=0}^{y=1} \tfrac{1}{2}(y^3 - y^4)\,dy = \left[\tfrac{1}{8}y^4 - \tfrac{1}{10}y^5\right]_{y=0}^{y=1} = \tfrac{1}{40}.$$

Exercise 8

Evaluate

$$\int_S (x^2 + y^2)\,dA,$$

where S is the triangle formed by the lines $y = 0$, $y = x - 1$ and $x = 2$, in the following ways.

(a) Integrate over y first, then over x.

(b) Integrate over x first, then over y.

Exercise 8 illustrates that either order of integration is acceptable, although the calculations are often easier for one particular order. It is important to be careful with the limits on the two single integrals. The limits on the outer integral are constants, whereas the limits on the inner integral will, in general, be non-constant functions. Only for rectangular regions of integration will all four limits be constants.

Exercise 9

By performing the integration with respect to x first, integrate $f(x,y) = x$ over the region bounded by the positive x-axis and the parabolas $y = x^2$ and $y = 4 - x^2$.

2 Applications of area integrals

Surface density functions were mentioned at the beginning of Section 1.

In Section 1 we were concerned with the mathematical problem of evaluating area integrals. In this section we consider applications of area integrals to a variety of problems involving surface density functions. Some of the applications involve integrating over a circular region of the (x,y)-plane, and in such cases it is usually easier to evaluate these area integrals using polar coordinates, as we demonstrate in Subsection 2.2. In Subsection 2.3, we introduce the idea of the *moment of inertia* of a body, a subject of great importance for Unit 21, and show how the moments of inertia of laminas can be calculated as area integrals.

2.1 Integrating surface density functions

Surface density functions can describe surface distributions of mass, populations and many other scalar quantities. The area integral of a surface density function then gives the total mass, population or whatever of the surface.

Example 5

A population of bacteria is grown on a rectangular glass plate. The plate is placed in the (x,y)-plane with its edges on the lines $y = 0$, $y = 0.04$ and $x = 0$, $x = 0.03$ (see Figure 14).

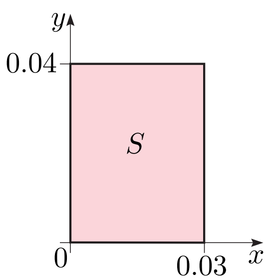


Figure 14 Rectangular glass plate

Determine the total number N_{b} of bacteria on the plate, given that the distribution of the population (in bacteria per square metre) is described by the surface density function

$$f(x,y) = (1 - 1000x^2) \times 10^{11}.$$

Solution

We integrate the surface density function over the rectangular region S occupied by the plate, choosing arbitrarily to integrate over x first. Thus

$$\begin{aligned}
N_{\rm b} &= \int_S f(x,y)\,dA \\
&= 10^{11}\int_{y=0}^{y=0.04}\left(\int_{x=0}^{x=0.03}(1-1000x^2)\,dx\right)dy \\
&= 10^{11}\int_{y=0}^{y=0.04}\left[x-\tfrac{1000}{3}x^3\right]_{x=0}^{x=0.03}\,dy \\
&= 10^{11}\int_0^{0.04} 0.021\,dy = 8.4\times 10^7.
\end{aligned}$$

Exercise 10

The density of bacteria on the surface of a glass plate in the (x,y)-plane is given by

$$f(x,y) = A\left(2-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right) \quad (x^2+y^2<2a^2),$$

where A and a are positive constants.

How many bacteria are there on a triangular region of the plate bounded by the x- and y-axes and the line $y=a-x$, when $A=10^{12}$ and $a=0.01$?

(In this exercise the constants are represented by symbols that can be used throughout the calculation. You are advised to do this, putting in the numbers only at the end in order to answer the question. The symbolic answer gives a more general result, which could be used with other values of the constants.)

Suppose for the bacteria in Exercise 10 that we want to know the number within a circular region S of radius a centred on the origin, as shown in Figure 15. Let us choose to integrate with respect to y first. The lower and upper ends of a vertical strip lie on the lower and upper semicircles

$$y=-\sqrt{a^2-x^2} \quad \text{and} \quad y=\sqrt{a^2-x^2}.$$

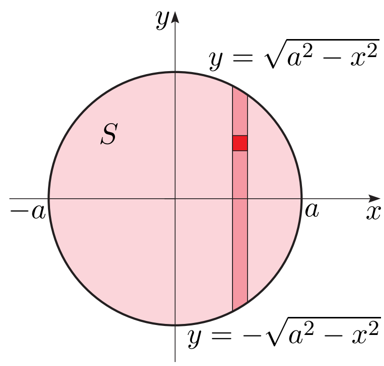


Figure 15 Circular region of integration, showing a vertical strip

These functions are the limits of the y-integration. The x-integral represents the addition of vertical strips starting at $x=-a$ and ending at $x=a$. Hence the total number of bacteria within the circle (over the disc) is

$$\int_S f\,dA = A\int_{x=-a}^{x=a}\left(\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}}\left(2-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right)dy\right)dx. \qquad (7)$$

Evaluating the integral in equation (7) is not particularly difficult, but it is more difficult than it needs to be. You will see in the next subsection that this double integral, and many other area integrals over circular regions, become much easier when we use polar coordinates. As you will see in Example 6, the total number of bacteria in this case is given by $N_{\rm b}=\frac{3}{2}\pi Aa^2$.

2.2 Changing to polar coordinates

In order to see how to set up an area integral in polar coordinates, recall first how a rectangular region was divided up in Cartesian coordinates by a rectangular grid of lines, $x =$ constant and $y =$ constant. A small rectangular element, the **Cartesian area element**, has area $\delta A = \delta y\,\delta x$. This element is represented symbolically as $dy\,dx$ in the area integral

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

Now consider the case of polar coordinates r and θ. Figure 16(a) shows the subdivision of a circular region (disc) by a grid of circles $r =$ constant and 'spokes' $\theta =$ constant.

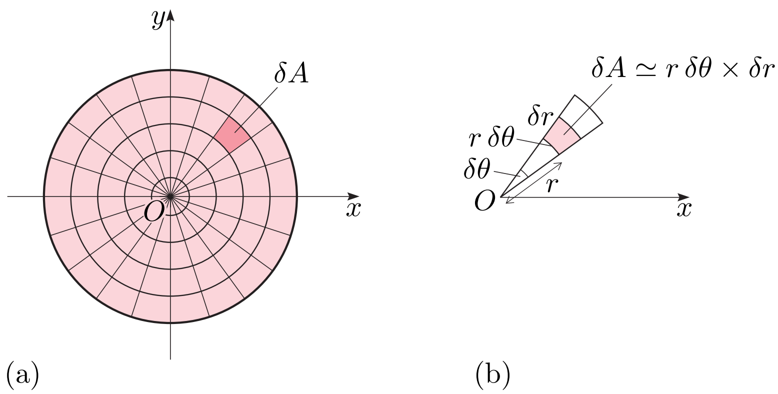


Figure 16 (a) Circular region divided into area elements using a grid based on polar coordinates. (b) A single area element shown in more detail.

As illustrated in Figure 16(b), each *area element* is nearly a rectangle of area $\delta A \simeq r\,\delta\theta \times \delta r$, and this approximation improves as δr and $\delta\theta$ tend to zero. To appreciate this, recall that an arc length is given by the radius multiplied by the subtended angle, so here the arc length is $r\,\delta\theta$, hence the area is $\delta A \simeq r\,\delta\theta\,\delta r$. The value of a function f at a point of this element is $f(r,\theta)$, so the contribution of this area element to the area integral is approximately $f(r,\theta)\,r\,\delta r\,\delta\theta$. Summing over all area elements and taking the limit in equation (1), we have the following definition.

Area integral in polar coordinates

The area integral of a function $f(r,\theta)$ over a disc D of radius a is

$$\int_D f\,dA = \int_{\theta=-\pi}^{\theta=\pi} \left(\int_{r=0}^{r=a} f(r,\theta)\,r\,dr \right) d\theta. \tag{8}$$

The factor r comes from the approximation $r\,\delta r\,\delta\theta$ of the area of the area element. It is very easy to forget this factor!

We can choose to integrate with respect to r first and then θ as in equation (8), or we can reverse the order of integration, as is convenient.

Because the limits are constants, they will not change when the order of integration is reversed.

Example 6

Evaluate the number of bacteria on the circular region of the (x, y)-plane of radius a centred at the origin when the surface density of bacteria is given by

$$f(x,y) = A\left(2 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right).$$

What is the number N_b of bacteria in the circle when $A = 10^{12}$ and $a = 0.01$?

Solution

We first express the surface density function in polar coordinates. Recognising that $x^2 + y^2 = r^2$, we have $f(r,\theta) = A(2 - r^2/a^2)$. The region of integration is a disc of radius a centred on the origin. Using equation (8), we can express the number of bacteria as

Note the deliberate abuse of notation where $f(x, y)$, in Cartesian coordinates, is written $f(r, \theta)$ in polar coordinates.

$$N_\mathrm{b} = A \int_{\theta=-\pi}^{\theta=\pi} \left(\int_{r=0}^{r=a} \left(2 - \frac{r^2}{a^2}\right) r\, dr \right) d\theta. \tag{9}$$

The inner integral, the integral with respect to r with θ constant, represents summing elements of length δr along the narrow sector shown in Figure 16(b). The θ-integral then represents the summing of all sectors of the circle.

The r-integral gives

$$\int_{r=0}^{r=a} \left(2 - \frac{r^2}{a^2}\right) r\, dr = \left[r^2 - \frac{r^4}{4a^2} \right]_0^a = \tfrac{3}{4}a^2.$$

The area integral in equation (9) is now reduced to a single integral, so

$$N_\mathrm{b} = \tfrac{3}{4}Aa^2 \int_{-\pi}^{\pi} d\theta = \tfrac{3}{4}Aa^2 \left[\theta\right]_{-\pi}^{\pi} = \tfrac{3}{2}\pi A a^2.$$

For the given values of A and a, the total number N_b of bacteria in the circular region is 4.7×10^8 to two significant figures.

There are two points to notice about Example 6. First, the area integral has constant limits and is very easy to evaluate compared with the Cartesian equivalent, equation (7). Second, the function $f(r, \theta)$ has circular symmetry. It is independent of the angular coordinate θ and depends on r only, and consequently the integration over θ in the area integral yields the factor

$$\int_{-\pi}^{\pi} d\theta = \left[\theta\right]_{-\pi}^{\pi} = 2\pi.$$

Thus for any function $f(r,\theta)$ that varies with r only, the area integral over a disc D of radius a centred on the origin is

$$\int_D f\,dA = 2\pi \int_0^a f(r)\,r\,dr, \tag{10}$$

in which we have written $f(r)$ for $f(r,\theta)$ since f depends on r only.

Exercise 11

A thin circular magnifying glass of radius a can be modelled as a thin disc with a surface mass density given by $f(r,\theta) = k(1 - r^2/a^2)$, where the constant k is the surface density at the centre.

Determine the total mass of the lens when $a = 2.5 \times 10^{-2}$ and $k = 3$.

Exercise 12

Consider the integral

$$J = \int_{-\infty}^{\infty} \exp(-x^2)\,dx,$$

called the *Gaussian integral*, which plays an important role in probability theory and statistics.

(a) Show that J^2 is the area integral of $\exp\big(-(x^2+y^2)\big)$ over the whole of the (x,y)-plane.

(b) Use polar coordinates to evaluate J^2, and hence show that $J = \sqrt{\pi}$.

Exercise 13

A drop of coloured chemical falls onto a plane sheet of blotting paper and spreads to form a stain. Taking the origin to be the point where the drop hits the blotting paper, the density of the chemical in the stain is described by $f(r,\theta) = A\exp(-r^2/a^2)$, in polar coordinates, where $A = 10^{-5}$ (in $\mathrm{kg\,m^{-2}}$) and $a = 1.5 \times 10^{-2}$ (in m).

Find the mass of chemical within a radius of $b = 0.01$ from the origin.

2.3 Moments of inertia

We now turn to an important application of area integrals, the calculation of *moments of inertia* for laminas. The central role played by moments of inertia in the dynamics of rotating bodies is fully described in Unit 21. Here, and in the next section, we are concerned only with calculations of moments of inertia using multiple integrals, so we give only a minimal introduction to the topic.

Figure 17 shows a particle of mass m moving in a plane with constant speed v along a circular path of radius d centred on a point O. If O is the origin of the (x, y)-plane, then such a motion can be thought of as rotation about the z-axis. Circular motion will be discussed more fully in Unit 20, but for the purpose here, you should note the following. If θ is the angle (in radians) subtended by an arc of a circle of radius d, then θd is the length of the arc. Therefore $v = |\dot{\theta}|d$, since d is constant for circular motion, so $v = \omega d$, where $\omega = |\dot{\theta}|$, the *angular speed* of the particle.

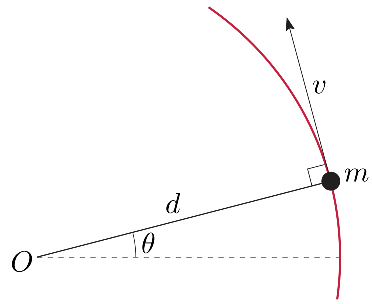


Figure 17 Particle moving in a circular path

You know that the kinetic energy of the particle is $\frac{1}{2}mv^2$. Now we can write this as $\frac{1}{2}m\omega^2 d^2$, where $\omega = v/d$ is the angular speed.

We make the following definition.

Moment of inertia of a particle

The quantity $I = md^2$, the product of a particle's mass m and the square of its perpendicular distance d from a fixed axis, is called the **moment of inertia** of the particle about the axis.

So the kinetic energy of the particle can also be written as $\frac{1}{2}mv^2 = \frac{1}{2}I\omega^2$. Comparing this with the more familiar $\frac{1}{2}mv^2$, you can see that if the motion of the particle is described by an angular speed ω rather than a speed v, then I replaces m as the parameter describing the inertia of the particle.

Consider now an N-particle system rotating about an axis. Each particle rotates in a circle of radius given by its perpendicular distance from the axis (see Figure 18). The ith particle of the system has moment of inertia $m_i d_i^2$, where m_i is the mass of the particle and d_i is its perpendicular distance from the axis. So the moment of inertia of the N-particle system about the axis is

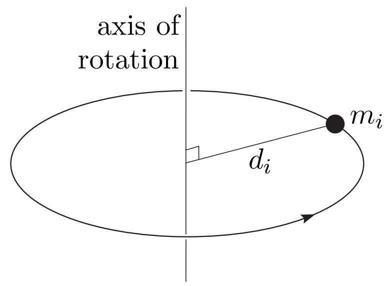


Figure 18 Circular motion of the ith particle

$$I = \sum_{i=1}^{N} m_i d_i^2. \qquad (11)$$

In Unit 21 we will be interested in the motion of rigid bodies that can rotate about fixed axes, and we will need to consider, for each rigid body, its moment of inertia. We can define the *moment of inertia of a rigid body* about an axis to be the limit of the sum of the moments of inertia of its N constituent particles when $N \to \infty$. Thus, using equation (11), the moment of inertia of the rigid body about the fixed axis is

$$I = \lim_{N\to\infty} \sum_{i=1}^{N} m_i d_i^2. \qquad (12)$$

If the rigid body is a lamina in the (x, y)-plane that can rotate about the z-axis, then any variation of composition can be described by a surface density function $f(x, y)$.

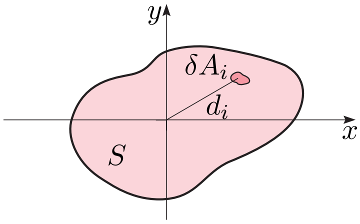


Figure 19 Lamina in the (x, y)-plane

We subdivide the lamina into a large number N of small area elements δA_i, modelled as particles (see Figure 19). Each element has mass $\delta m_i = f_i\,\delta A_i$, where f_i is the surface density at a point in the area element δA_i. The moment of inertia of this element about the z-axis is $\delta m_i\, d_i^2 = f_i\,\delta A_i\, d_i^2$, where d_i is the distance of the element from the origin. From equation (12), the moment of inertia of the whole lamina about the z-axis is

$$I = \lim_{N\to\infty} \sum_{i=1}^{N} f_i\,\delta A_i\, d_i^2,$$

the area integral of fd^2, where $d(x, y)$ is a function giving the distance of the point (x, y) from the axis of rotation. Thus if a lamina occupies a region S of the (x, y)-plane and has surface density function f, then its moment of inertia about the z-axis is given as follows.

Moment of inertia of a lamina

The moment of inertia about the z-axis of a lamina with surface density function f occupying a region S of the (x, y)-plane is

$$I = \int_S f d^2\, dA, \tag{13}$$

where d is the perpendicular distance from the z-axis, so

$$fd^2 = \begin{cases} f(x,y)\,(x^2+y^2) & \text{in Cartesian coordinates,} \\ f(r,\theta)\, r^2 & \text{in polar coordinates.} \end{cases} \tag{14}$$

Example 7

Determine the moment of inertia of a square plate of side length a, constant surface density f, and negligible thickness, about an axis that is perpendicular to the plane of the plate, passing through its centre.

Solution

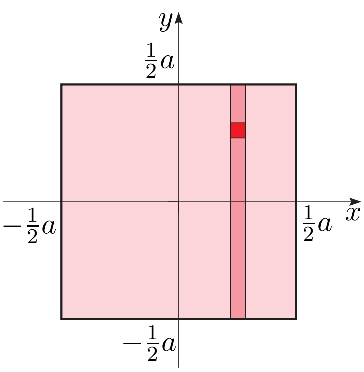


Figure 20 Square plate in the (x, y)-plane

Figure 20 shows the plate or lamina in the (x, y)-plane with its centre at the origin. Since the region of integration is a square, we use Cartesian coordinates. The surface density function f is constant, so we can take it outside the area integral of equation (13), and we have

$$\begin{aligned} I &= f\int_{x=-a/2}^{x=a/2}\left(\int_{y=-a/2}^{y=a/2} (x^2+y^2)\,dy\right)dx \\ &= f\int_{x=-a/2}^{x=a/2} \left[x^2y + \tfrac{1}{3}y^3\right]_{y=-a/2}^{y=a/2} dx \\ &= 2f\int_{-a/2}^{a/2} \left(\tfrac{1}{2}x^2a + \tfrac{1}{24}a^3\right) dx \\ &= 2f\left[\tfrac{1}{6}ax^3 + \tfrac{1}{24}a^3x\right]_{-a/2}^{a/2} = \tfrac{1}{6}fa^4. \end{aligned}$$

The mass of the plate is $M = fa^2$, so we can express the moment of inertia of the plate as $I = \frac{1}{6}Ma^2$.

Exercise 14

Determine the moment of inertia of a flat ruler of length $a = 0.30$ and width $b = 0.04$ about an axis that is perpendicular to the plane of the ruler, passing through its centre. Assume that the mass of the ruler can be modelled by a constant surface density function $f = 10^{-2}$.

(*Hint*: Evaluate the area integral using the symbols a, b and f, then substitute the numbers into the answer.)

The ruler is considered as a lamina in that its thickness is assumed to be negligible.

It is usual to quote the moment of inertia of a body in terms of its total mass M and its dimensions. For a square of side length a, the moment of inertia about an axis through its centre that is perpendicular to the plane of the square is

We did this in Example 7 and in the solution to Exercise 14. Note that the SI units of moment of inertia are kg m^2.

$$I_{\text{square}} = \tfrac{1}{6}Ma^2, \tag{15}$$

and for a rectangle of length a and width b, the moment of inertia about an axis through its centre that is perpendicular to the plane of the rectangle is

$$I_{\text{rectangle}} = \tfrac{1}{12}M(a^2 + b^2). \tag{16}$$

This last result can be used to find the moment of inertia of a thin rod of mass M and length a about an axis through its centre at right angles to its length. We take the limit as $b \to 0$ in equation (16) to obtain

$$I_{\text{rod}} = \tfrac{1}{12}Ma^2. \tag{17}$$

We used Cartesian coordinates in Example 7 and Exercise 14 because the regions of integration were rectangular. To find the moment of inertia of a disc, it is simpler to use polar coordinates.

Example 8

Express the moment of inertia of a disc of radius a with surface density function f, about an axis perpendicular to its plane and passing through its centre, as an area integral in polar coordinates.

Solution

Using polar coordinates in equation (13), which entails replacing f in equation (8) by fr^2, the moment of inertia of the disc about its centre is given by the area integral

$$I_{\text{disc}} = \int fr^2\,dA = \int_{\theta=-\pi}^{\theta=\pi} \left(\int_{r=0}^{r=a} fr^3\,dr \right) d\theta.$$

Here we have left the surface density function f inside the integral in case it is not a constant.

Exercise 15

Evaluate the area integral in Example 8 for the case of a uniform disc of radius a, and hence show that the moment of inertia of the disc about an axis perpendicular to its plane and passing through its centre is $\frac{1}{2}Ma^2$, where M is the mass of the disc.

Exercise 16

Find the moment of inertia of a flat circular washer of uniform material, negligible thickness and total mass M, in the shape of an annulus of internal radius a and external radius b, about an axis perpendicular to the plane of the washer through its centre.

Exercise 17

Determine the moment of inertia about the z-axis of the magnifying glass in Exercise 11, which has surface density

$$f(r,\theta) = k(1 - r^2/a^2),$$

where k is a constant, and a is the radius of the magnifying glass.

Laminas, by definition, have negligible thickness.

In the above examples and exercises we have modelled plates and discs as laminas with variations of composition described by surface density functions. In an improved model of a plate, we recognise that the thickness cannot be ignored and model the plate as a solid body, forming a *volume integral* by summing over small *volume elements* of the body. We do this in Section 3.

3 Volume integrals

In Section 2 you saw how to calculate the masses and moments of inertia of laminas where the distribution of mass was modelled by a surface density function defined on a region of the (x, y)-plane. A more fundamental approach is to recognise at the outset that a real body is a three-dimensional object. Masses and moments of inertia are then expressed as *volume integrals*. In this section we define the volume integral and show how volume integrals are evaluated and used to find masses and moments of inertia.

3.1 Defining and evaluating volume integrals

You know that the mass of an object of uniform density is simply the product of its volume times its density, where the *density* here is the *mass per unit volume*, measured in $\mathrm{kg\,m^{-3}}$. To find the mass of an object of non-uniform density, we subdivide the object into very small **volume elements**, estimate the mass of an element as the product of its volume times the local value of the density, form the sum over all elements, and then go to the limit of an infinitely large number of small elements. The summation then becomes an integral called the *volume integral*, which we can evaluate.

Consider a scalar field $f(x,y,z)$ defined in a three-dimensional region B, and subdivide B into N elements, where element i has volume δV_i. We make the following definition.

Volume integral

The **volume integral** of $f(x,y,z)$ over a **region of integration** B, subdivided into N elements where element i contains the point (x_i, y_i, z_i) and is of volume δV_i, is

$$\int_B f(x,y,z)\,dV = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,y_i,z_i)\,\delta V_i, \tag{18}$$

where $\delta V_i \to 0$ as $N \to \infty$.

If $f(x,y,z) = 1$, then we obtain the volume of region B.

Before we can evaluate a volume integral, we must choose a shape for the volume elements. The shape that we choose depends on whether it is best to work in the Cartesian, cylindrical or spherical coordinate system. We begin with the Cartesian system.

Volume integral over a rectangular cuboid

The volume integral of a function $f(x,y,z)$ over a rectangular cuboid B whose faces lie in coordinate planes $x = a$, $x = b$, $y = c$, $y = d$, $z = p$, $z = q$ is obtained as three successive integrals as follows:

$$\int_B f(x,y,z)\,dV = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \left(\int_{z=p}^{z=q} f(x,y,z)\,dz \right) dy \right) dx. \tag{19}$$

A *rectangular cuboid* is just a rectangular block.

The method for evaluating the volume integral is similar to that for evaluating surface integrals over rectangular regions in the (x,y)-plane, except that now we must integrate over three variables. The method is best illustrated by an example.

Volume integrals are sometimes referred to as *triple integrals*.

Example 9

The density inside a cube whose faces are the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ is given by the density function

$$f(x, y, z) = c(x^2 + y^2 + z^2),$$

where c is a constant.

Determine the mass of the cube.

Solution

The mass of the cube is given by the volume integral

$$\int_B f\, dV = \int_B c(x^2 + y^2 + z^2)\, dV.$$

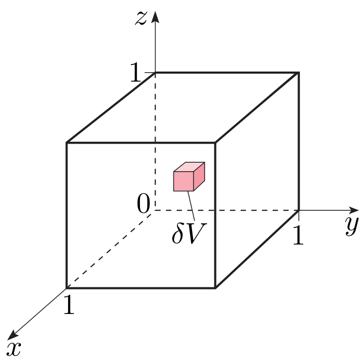


Figure 21 Cubic region of integration

The region of integration B is shown in Figure 21. To evaluate this volume integral, we form three single integrals. The method for finding the limits for these single integrals is similar to the approach that we used for area integrals. The volume element in Cartesian coordinates is a small block with volume $\delta V = \delta x\, \delta y\, \delta z$. If we draw within the region a vertical column of rectangular cross-section, then the volume of the column is found by summing the volume elements along this column. This column intersects the bottom and top faces of the region at the lower and upper limits of the z-integration. In this case, the limits are $z = 0$ and $z = 1$ (see Figure 22(a)).

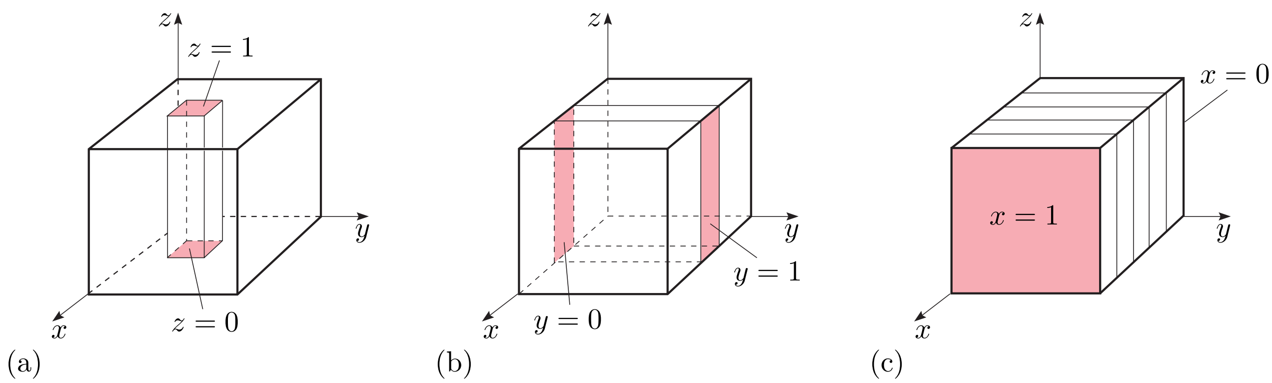


Figure 22 Cubic region of integration showing (a) a vertical column, (b) a slice parallel to the (y, z)-plane, and (c) the region divided into slices parallel to the (y, z)-plane

Now we form the y-integral by summing all the columns in a 'slice' parallel to the (y, z)-plane. The limits of the y-integration are therefore $y = 0$ and $y = 1$ (see Figure 22(b)). Finally, we complete the volume by adding together all possible slices so that the limits of the x-integration are $x = 0$ and $x = 1$ (see Figure 22(c)).

Hence the mass M of the cube can be found from three successive integrals:

$$M = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} \left(\int_{z=0}^{z=1} c(x^2 + y^2 + z^2)\, dz \right) dy \right) dx.$$

The inner integral is evaluated first. This is an integral over z with x and y treated as constants. Hence

$$\begin{aligned} M &= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} c\left[(x^2+y^2)z + \tfrac{1}{3}z^3\right]_{z=0}^{z=1} dy \right) dx \\ &= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} c\left(x^2+y^2+\tfrac{1}{3}\right) dy \right) dx. \end{aligned}$$

Now we are left with an area integral that we know how to evaluate. Thus

$$\begin{aligned} M &= \int_{x=0}^{x=1} c\left[x^2 y + \tfrac{1}{3}y^3 + \tfrac{1}{3}y\right]_{y=0}^{y=1} dx \\ &= \int_{x=0}^{x=1} c\left(\tfrac{2}{3} + x^2\right) dx \\ &= c\left[\tfrac{2}{3}x + \tfrac{1}{3}x^3\right]_{x=0}^{x=1} = c. \end{aligned}$$

The main point to notice is that after each integration, the number of variables is reduced by one.

Exercise 18

The density of the material in a block B with faces $x=0$, $x=2$, $y=1$, $y=2$, $z=2$, $z=5$ is given by $f(x,y,z) = x+y+z$.

Find the mass of the block.

Exercise 19

The density of a rectangular block B bounded by the planes $x=1$, $x=2$, $y=0$, $y=3$, $z=-1$, $z=0$ is given by $f(x,y,z) = x(y+1) - z$.

Find the mass of the block.

Sometimes it is more appropriate to evaluate a volume integral using cylindrical coordinates (introduced in Subsection 4.1 of Unit 15). We must then divide the region of integration into volume elements with surfaces $\rho = \text{constant}$, $\phi = \text{constant}$ and $z = \text{constant}$. A typical volume element is shown in Figure 23. It is approximately a rectangular cuboid of sides δz, $\rho\,\delta\phi$ and $\delta\rho$, so its volume is $\delta V \simeq \rho\,\delta z\,\delta\phi\,\delta\rho$, and this approximation improves as δz, $\delta\phi$ and $\delta\rho$ tend to zero. Note the factor ρ again, coming from the fact that $\rho\,\delta\phi$ is an arc length.

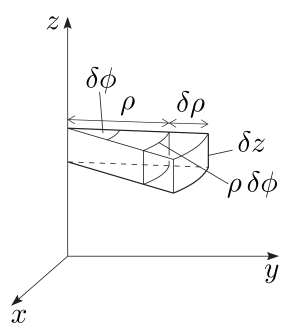


Figure 23 Volume element in cylindrical coordinates

If the region of integration is a cylinder of radius a and height h, with the base of the cylinder on the (x,y)-plane, then in cylindrical coordinates the limits of integration are $z=0$, $z=h$, $\phi=-\pi$, $\phi=\pi$, $\rho=0$, $\rho=a$. So the volume integral of a scalar field f over the cylinder is as follows.

Volume integral in cylindrical coordinates

$$\begin{aligned}\int_B f\,dV &= \int_B f(\rho,\phi,z)\,\rho\,dz\,d\phi\,d\rho \\ &= \int_{\rho=0}^{\rho=a}\left(\int_{\phi=-\pi}^{\phi=\pi}\left(\int_{z=0}^{z=h} f\,\rho\,dz\right)d\phi\right)d\rho. \qquad (20)\end{aligned}$$

In most applications that you will meet in this module, the scalar field f is independent of the azimuthal angle ϕ, so the ϕ-integration simply gives a factor $\int_{-\pi}^{\pi} d\phi = 2\pi$, and equation (20) becomes

$$\int_B f\,dV = 2\pi\int_{\rho=0}^{\rho=a}\left(\int_{z=0}^{z=h} f\,\rho\,dz\right)d\rho, \qquad (21)$$

which is an area integral.

Example 10

The density of a cylinder of height h and radius a is given by $f = k\rho^2$, where k is a positive constant and ρ is the perpendicular distance from the axis of the cylinder.

Find the total mass of the cylinder when $h = 2$, $a = 0.5$ and $k = 4$.

Solution

Use cylindrical coordinates with the axis of the cylinder aligned along the z-axis and its base on the (x, y)-plane. The total mass of the cylinder is given by

$$M = \int_B f\,dV = \int_B f\,\rho\,dz\,d\phi\,d\rho = \int_B k\rho^3\,dz\,d\phi\,d\rho.$$

Using equation (21),

$$M = 2\pi\int_{\rho=0}^{\rho=a}\left(\int_{z=0}^{z=h} k\rho^3\,dz\right)d\rho = 2\pi k\int_0^a \rho^3 h\,d\rho = \tfrac{1}{2}\pi k a^4 h.$$

When $h = 2$, $a = 0.5$ and $k = 4$, the mass of the cylinder is $\frac{1}{4}\pi$ or about 0.79 kg.

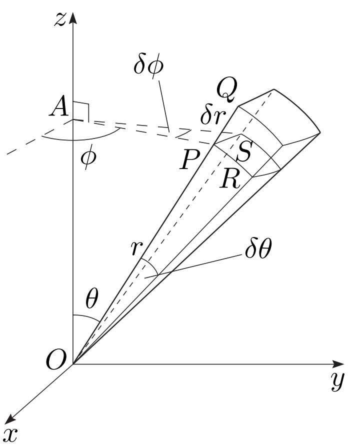


Figure 24 Volume element in spherical coordinates

Now we turn our attention to spherical coordinates (introduced in Subsection 4.2 of Unit 15). In order to evaluate volume integrals in this system, we need to find the volume δV of the volume element that is shown in Figure 24.

This volume is approximately a rectangular cuboid with sides PQ, PR and PS. Hence the volume of the element is

$$\delta V \simeq PQ \times PR \times PS.$$

From Figure 24 we have line $PQ = \delta r$, arc $PR = r\,\delta\theta$, and arc $PS = AP\,\delta\phi = (OP\sin\theta)\,\delta\phi = r\sin\theta\,\delta\phi$. So the volume element is

$$\delta V \simeq r^2 \sin\theta\,\delta\phi\,\delta\theta\,\delta r,$$

and this approximation improves as $\delta\phi$, $\delta\theta$ and δr tend to zero. Hence the volume integral of a scalar field f, over a region B, is given by the following.

Volume integral in spherical coordinates

$$\int_B f\,dV = \int_B f(r,\theta,\phi)\,r^2 \sin\theta\,d\phi\,d\theta\,dr. \tag{22}$$

Once the function $f(r,\theta,\phi)$ is given, equation (22) can be evaluated as three successive integrals over ϕ, θ and r.

In many cases, the scalar field f is spherically symmetric so that f depends on r only, not on θ or ϕ. In such cases, and for a region contained between spherical shells of radii R_1 and R_2 ($R_1 < R_2$), equation (22) takes the form

$$\int_B f\,dV = \int_{r=R_1}^{r=R_2} \left(\int_{\theta=0}^{\theta=\pi} \left(\int_{\phi=-\pi}^{\phi=\pi} f(r)\,r^2 \sin\theta\,d\phi \right) d\theta \right) dr.$$

The ϕ-integral gives a factor 2π, and the θ-integral gives $\left[-\cos\theta\right]_0^\pi = 2$. Thus in such cases,

$$\int_B f\,dV = 4\pi \int_{R_1}^{R_2} f(r)\,r^2\,dr. \tag{23}$$

Exercise 20

By evaluating a volume integral, find the mass of a sphere of radius R that is centred at the origin and whose density is given by $f = c + \alpha r$, where c and α are positive constants, and r is the distance from the origin.

In the examples and exercises that we have considered in this subsection, the regions of integration have been rectangular cuboids, cylinders and spheres. The boundaries of these regions coincide with coordinate surfaces in the appropriate coordinate system, so the limits of integration are all constants. In Subsection 3.3 we will look at examples where the regions of integration cut across coordinate surfaces, and as a result some of the limits of integration are functions. First, in the next subsection, we consider an application of volume integrals to the calculation of moments of inertia of rigid bodies.

3.2 Moments of inertia of rigid bodies

In Subsection 2.3 we calculated the moments of inertia of laminas in the (x, y)-plane, with mass distributions described by surface density functions $f(x, y)$. We now consider a rigid body that can rotate about the z-axis.

The mass distribution in a rigid body is modelled by a density function $f(x, y, z)$ with units of $\mathrm{kg\,m^{-3}}$. As in Subsection 2.3, our starting point is equation (12), giving the moment of inertia of a rigid body as the limit of a summation.

Suppose that a rigid body is divided into a very large number N of volume elements, modelled as particles. The mass of an element is approximately $f_i\,\delta V_i$, where δV_i is the volume of the element, and f_i is the density at a point inside the element. So the moment of inertia of the element about the z-axis is approximately $(f_i\,\delta V_i)d_i^2 = f_i d_i^2\,\delta V_i$, where d_i is the perpendicular distance of the ith element from the z-axis (see Figure 25). The moment of inertia of the whole body about the z-axis is found by summing over all elements and taking the limit as $N \to \infty$. The moment of inertia about the z-axis is then the volume integral of the function fd^2 over the region of space B occupied by the body.

The function $d(x, y, z)$ gives the distance of the point (x, y, z) from the point z on the z-axis, which is the perpendicular distance of the element from the z-axis.

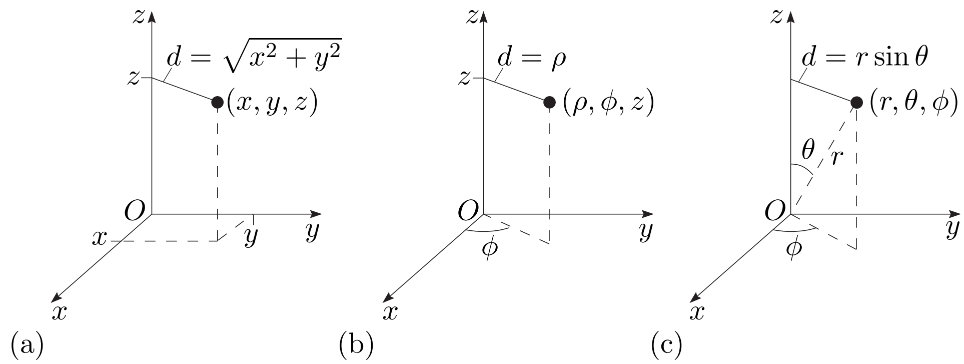


Figure 25 Perpendicular distance d from the z-axis in (a) Cartesian, (b) cylindrical and (c) spherical coordinate systems

Moment of inertia of a rigid body

The moment of inertia about the z-axis of a rigid body with density function f occupying a region B of space is

$$I = \int_B f d^2\,dV, \tag{24}$$

where d is the perpendicular distance from the z-axis, so

$$fd^2 = \begin{cases} f(x, y, z)\,(x^2 + y^2) & \text{in Cartesian coordinates,} \\ f(\rho, \phi, z)\,\rho^2 & \text{in cylindrical coordinates,} \\ f(r, \theta, \phi)\,(r\sin\theta)^2 & \text{in spherical coordinates.} \end{cases} \tag{25}$$

Example 11

Starting from equation (24), evaluate the moment of inertia I of a uniform sphere of radius R and constant density D about an axis through its centre.

Solution

Let the centre of the sphere be at the origin, and we will calculate the moment of inertia about the z-axis. The density function $f = D$ is constant, and the distance d from the z-axis in spherical coordinates is $d = r\sin\theta$ (see Figure 25(c)). The moment of inertia of the sphere is

Since the region of integration B is a sphere, we use spherical coordinates.

$$\begin{aligned}
I &= \int_{\text{sphere}} D(r\sin\theta)^2\, dV \\
&= D\int_{\text{sphere}} (r\sin\theta)^2 r^2 \sin\theta\, dr\, d\theta\, d\phi \\
&= D\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{\phi=-\pi}^{\phi=\pi} r^4\sin^3\theta\, d\phi\right) d\theta\right) dr.
\end{aligned}$$

The ϕ-integral yields the factor

$$\int_{-\pi}^{\pi} d\phi = 2\pi,$$

the θ-integral yields the factor

$$\int_0^{\pi} \sin^3\theta\, d\theta = \int_0^{\pi} \sin\theta(1-\cos^2\theta)d\theta = \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi} = \tfrac{4}{3},$$

and the r-integral gives

$$\int_0^R r^4\, dr = \tfrac{1}{5}R^5.$$

Hence the volume integral yields the moment of inertia

$$I = \tfrac{8}{15}\pi D R^5.$$

Note that in Example 11 the limits are all constant so the three integrals can be carried out in any order without changing the limits.

Exercise 21

Determine the mass M of a uniform sphere of radius R and density D by evaluating a volume integral, and hence show that the moment of inertia of the sphere about an axis through its centre is $I = \frac{2}{5}MR^2$.

Exercise 22

Show that the moment of inertia of a uniform solid cylinder of mass M, height h and radius a about its axis is $I = \frac{1}{2}Ma^2$.

Exercise 23

Find the moment of inertia of a uniform spherical shell of mass M, outer radius b and inner radius a about an axis through its centre.

The answer to Exercise 23 can be used to find the moment of inertia of a very thin spherical shell, such as a ping-pong ball, about an axis through its centre. We put $b = a + h$, where h, the thickness of the shell, is very small compared with a. Then we have

$$I_{\text{shell}} = \frac{2M(b^5 - a^5)}{5(b^3 - a^3)} = \frac{2M((a+h)^5 - a^5)}{5((a+h)^3 - a^3)} = \frac{2Ma^5\left(\left(1+\frac{h}{a}\right)^5 - 1\right)}{5a^3\left(\left(1+\frac{h}{a}\right)^3 - 1\right)}.$$

We can expand

$$\left(1+\frac{h}{a}\right)^5 \simeq 1 + \frac{5h}{a} \quad \text{and} \quad \left(1+\frac{h}{a}\right)^3 \simeq 1 + \frac{3h}{a}$$

See Unit 7 for Taylor polynomials.

by using a Taylor polynomial, keeping only the first-degree (i.e. linear) terms in h, since h/a is very much less than 1. Thus

$$I_{\text{thin shell}} \simeq \frac{2Ma^5(5h/a)}{5a^3(3h/a)} = \tfrac{2}{3}Ma^2.$$

The disc was assumed to be a lamina in the (x, y)-plane.

You may have noticed that the moment of inertia of the uniform cylinder in Exercise 22 is the same as the moment of inertia of the uniform disc in Exercise 15. This is not surprising since the moment of inertia is the sum over all elements of the quantity (mass) $\times$ (distance from z-axis)^2, which is independent of the distribution of mass in the z-direction. It follows that the moment of inertia of a uniform body is unchanged by projecting the mass distribution of the body onto the (x, y)-plane, provided that distances from the z-axis do not change.

The moment of inertia of any solid body about the z-axis can always, in principle, be evaluated as an area integral over a region of the (x, y)-plane, *provided* that the equivalent projected surface density in the (x, y)-plane is known. So, for example, the z-integration of the volume integral in the solution to Exercise 22 yields a factor Dh, the mass per unit volume times the height. This is equivalent to a mass per unit area – that is, a surface density – in the (x, y)-plane. The integrals over ρ and ϕ that are left after the z-integral is evaluated are the same as those that you evaluated in Exercise 15 to get the moment of inertia of a disc in the (x, y)-plane.

Exercise 24

Find the moment of inertia of a uniform cylindrical shell of mass M, outer radius b and inner radius a about its axis.

Exercise 25

Show that the moment of inertia of a uniform solid cylinder of mass M, length L and radius R, about an axis through the centre of the cylinder that is *normal* to the axis of the cylinder, is given by

$$I = \frac{1}{12} M(3R^2 + L^2).$$

(*Hint:* Use cylindrical coordinates with the z-axis aligned along the cylinder axis so that the perpendicular distance to the axis of rotation is *no longer* the perpendicular distance to the z-axis.)

Exercise 26

The density of a sphere of radius R varies as $f(\mathbf{r}) = c|\mathbf{r}|^2$, where c is a constant and $\mathbf{r}$ is measured from the sphere's centre.

(a) Determine the mass of the sphere.

(b) Find the moment of inertia of the sphere about an axis through its centre.

3.3 Volume integrals with irregular boundaries

Here we evaluate volume integrals where the boundaries of the region of integration do not coincide with coordinate surfaces. The limits on some of the integrals are then functions that must be determined. Thus the first stage in the evaluation of a volume integral is to decide on a coordinate system and the limits of integration. The second stage is to evaluate the resulting single integrals.

In this subsection we will illustrate how limits are decided on in cases where we use Cartesian coordinate systems for non-rectangular regions of integration. The procedure is to carry out the z-integration first. When this is done, we are left with an area integral in the (x, y)-plane, which we can evaluate as in Subsection 1.3.

So instead of the volume integral with constant limits as in equation (19), we have variable limits in the z-direction (and possibly the y-direction).

Volume integral over a general volume

$$\int_B f(x,y,z)\,dV$$
$$= \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} \left(\int_{z=\gamma(x,y)}^{z=\psi(x,y)} f(x,y,z)\,dz \right) dy \right) dx. \qquad (26)$$

To illustrate the method, we consider a volume integral of a function f over the cylindrical region of radius a shown in Figure 26(a). Note that this cylinder has its axis on the y-axis, not the z-axis, and the curved boundary does not coincide with any Cartesian coordinate surface. Of course, depending on the form of the integrand f, it is often more convenient to use a cylindrical coordinate system here, given the shape of the region. (This would require redefining the y-axis as the z-axis, and introducing the ρ and ϕ coordinates accordingly.) However, for the purpose of this illustration, we assume that the form of f is such that there is no disadvantage in using Cartesian coordinates.

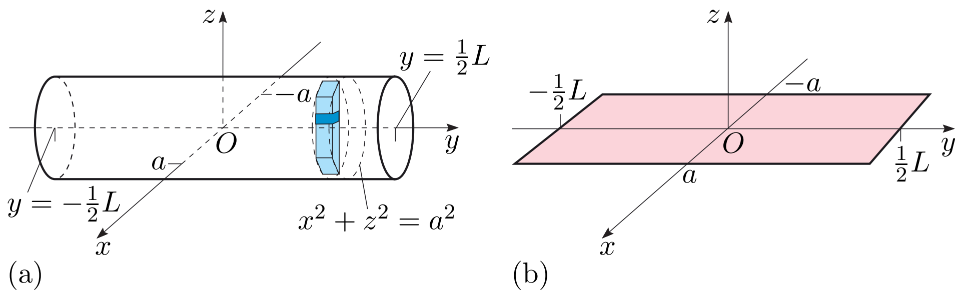


Figure 26 Cylindrical region of integration: (a) a vertical column; (b) projection of the cylinder onto the (x, y)-plane

Recall that the volume elements in Cartesian coordinates are small blocks of volume $\delta V = \delta z\,\delta y\,\delta x$. We choose to integrate first over z. The z-integration then represents summing up the volume elements in the vertical column in Figure 26(a). Where this column intersects the bottom and top curved boundaries of the cylinder are the lower and upper limits of the z-integration. In this case the limits are $z = -\sqrt{a^2 - x^2}$ and $z = \sqrt{a^2 - x^2}$. Hence the z-integral is

The curved boundary has the equation $x^2 + z^2 = a^2$ so is made up of $z = \sqrt{a^2 - x^2}$ $(z \geq 0)$ and $z = -\sqrt{a^2 - x^2}$ $(z < 0)$.

$$\int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} f(x, y, z)\,dz = h(x, y).$$

The result of the integration over z is $h(x, y)$, some function of x and y only. To complete the volume integral, we must evaluate the area integral

$$\int_S h(x, y)\,dy\,dx,$$

where the region S is the projection of the cylinder onto the (x, y)-plane. This projection is just the rectangle shown in Figure 26(b). The following example illustrates how this method works.

Example 12

Use Cartesian coordinates to evaluate the volume integral of the function $x^2 y^2 (a^2 - x^2)^{3/2}$ over the cylinder of radius a, length L, centred at the origin and with its axis along the y-axis (i.e. as shown in Figure 26(a)).

Solution

Figure 26(a) shows the cylinder lying with its centre at the origin and its axis on the y-axis. With $f(x,y,z) = x^2y^2(a^2-x^2)^{3/2}$, we require

$$\int_{\text{cylinder}} f(x,y,z)\,dV = \int_{\text{cylinder}} x^2y^2(a^2-x^2)^{3/2}\,dz\,dy\,dx.$$

The limits of the z-integral are chosen as described above, so the z-integral gives

$$\begin{aligned}
h(x,y) &= \int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} x^2y^2(a^2-x^2)^{3/2}\,dz \\
&= x^2y^2(a^2-x^2)^{3/2}\int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} 1\,dz \\
&= 2x^2y^2(a^2-x^2)^2.
\end{aligned}$$

We are now left with the x- and y-integrations in equation (26). We must evaluate the area integral of $h(x,y)$ over the projection of the cylinder on the (x,y)-plane, which is the rectangle shown in Figure 26(b). Following Procedure 1 for evaluating area integrals, we first note that the limits of the x-integral are $-a$ and a, and the limits on the y-integral are $-\frac{1}{2}L$ and $\frac{1}{2}L$. Thus we have

$$\begin{aligned}
\int_{\text{cylinder}} f(x,y,z)\,dV &= \int_{\text{rectangle}} h(x,y)\,dy\,dx \\
&= \int_{x=-a}^{x=a}\left(\int_{y=-L/2}^{y=L/2} 2x^2y^2(a^2-x^2)^2\,dy\right)dx \\
&= 2\int_{x=-a}^{x=a} x^2(a^2-x^2)^2\left[\tfrac{1}{3}y^3\right]_{y=-L/2}^{y=L/2}\,dx \\
&= \tfrac{1}{6}L^3\left[\tfrac{1}{3}a^4x^3 - \tfrac{2}{5}a^2x^5 + \tfrac{1}{7}x^7\right]_{-a}^{a} \\
&= \tfrac{8}{315}L^3a^7.
\end{aligned}$$

Finally, it should be noted that had we chosen a cylindrical coordinate system to perform this volume integral, we would have been left with much harder single integrals to evaluate than those given above. So in this case, as a result of the form of the integrand $f(x,y,z)$, we really are justified in using Cartesian coordinates.

The steps that we have used to evaluate a volume integral are summarised in the following procedure.

Procedure 2 Evaluating volume integrals

To evaluate a volume integral

$$\int_B f(x,y,z)\,dV,$$

carry out the following steps.

1. Draw two diagrams, showing the region of integration B with the equations of the upper and lower boundaries marked, and the projection S of this region onto the (x,y)-plane.
2. Within the region B, draw a column perpendicular to the (x,y)-plane and determine the limits of the z-integration, $z = \gamma(x,y)$ and $z = \psi(x,y)$, say.
3. Evaluate the single integral of $f(x,y,z)$ over z between $z = \gamma(x,y)$ and $z = \psi(x,y)$, keeping x and y constant, to find the function $h(x,y)$ defined by
$$h(x,y) = \int_{z=\gamma(x,y)}^{z=\psi(x,y)} f(x,y,z)\,dz.$$
4. Evaluate the area integral of $h(x,y)$ over the region S using Procedure 1.

In Example 12 we had $\gamma(x,y) = -\sqrt{a^2 - x^2}$ and $\psi(x,y) = \sqrt{a^2 - x^2}$.

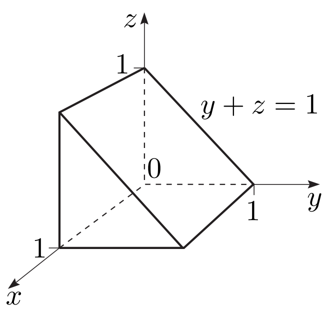


Figure 27

Exercise 27

Find the value of the volume integral of the function

$$f(x,y,z) = x^2yz$$

over the wedge-shaped region bounded by the planes $z = 0$, $y = 0$, $x = 0$, $x = 1$ and $y + z = 1$ (as shown in Figure 27).

Exercise 28

Find the value of the volume integral of the function

$$f(x,y,z) = z + 3x - 2$$

over the region bounded by the parabolic surfaces $y = 1 - x^2$ and $y = x^2 - 1$, and the planes $z = 0$ and $z = 1$.

One often encounters volume integrals over regions with boundaries that do not coincide with coordinate surfaces but which are best evaluated using cylindrical or spherical coordinates as opposed to Cartesian coordinates. To illustrate how this is done, we now determine the moment of inertia of a solid cone about its axis.

Example 13

Find the moment of inertia of a solid uniform cone of mass M, height h and base radius a about its axis (see Figure 28).

(*Hint*: Use cylindrical coordinates and integrate first over z; express the height z of the vertical column in terms of a and h.)

Solution

Let the base of the cone lie in the (x, y)-plane with the axis of the cone on the z-axis, as shown in Figure 28. Use cylindrical coordinates and integrate over z first. This represents first summing elements in the column starting at the base, $z = 0$, and ending at $z = h(1 - \rho/a)$, the top of the column. Thus taking the uniform density to be D, the moment of inertia of the cone is

$$\begin{aligned} I_{\text{cone}} &= \int_{\text{cone}} D\rho^2 \, dV \\ &= D \int_{\rho=0}^{\rho=a} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=0}^{z=h(1-\rho/a)} \rho^3 \, dz \right) d\phi \right) d\rho. \end{aligned}$$

Integrating over z leaves the area integral over the circular base in the (x, y)-plane, which using equation (10) gives

$$I_{\text{cone}} = 2\pi D \int_0^a \rho^3 h(1 - \rho/a) \, d\rho = 2\pi Dh \left[\tfrac{1}{4}\rho^4 - \tfrac{1}{5}\rho^5/a\right]_0^a = \tfrac{1}{10}\pi Dha^4.$$

The mass of the cone is simply the volume integral of D over the region occupied by the cone, which is

$$\begin{aligned} M &= D \int_{\rho=0}^{\rho=a} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=0}^{z=h(1-\rho/a)} \rho \, dz \right) d\phi \right) d\rho \\ &= 2\pi D \int_0^a \rho h(1 - \rho/a) \, d\rho = 2\pi Dh \left[\tfrac{1}{2}\rho^2 - \tfrac{1}{3}\rho^3/a\right]_0^a = \tfrac{1}{3}D\pi a^2 h. \end{aligned}$$

Hence

$$I_{\text{cone}} = \tfrac{3}{10}Ma^2.$$

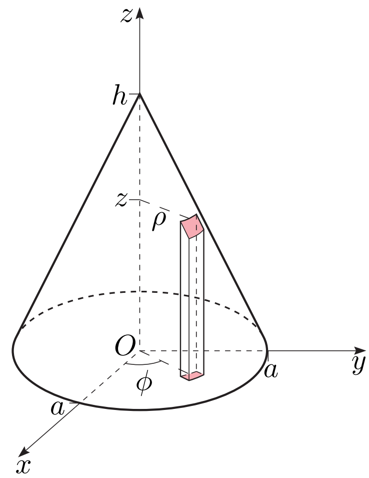


Figure 28 A cone

Exercise 29

Consider a region B of space bounded by the circular paraboloid $z/h = (x^2 + y^2)/a^2$ $(h > 0)$ and the plane $z = h$, as shown in Figure 29.

(a) Determine the volume of region B.

(b) If region B consists of a uniform solid of mass M, show that its moment of inertia about the z-axis is given by $I = \frac{1}{3}Ma^2$.

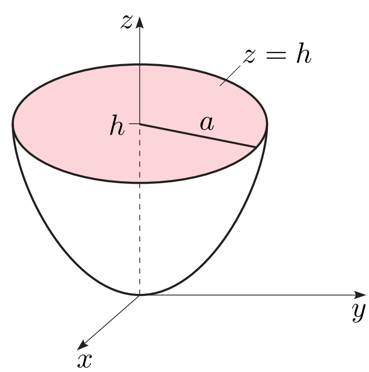


Figure 29 A circular paraboloid

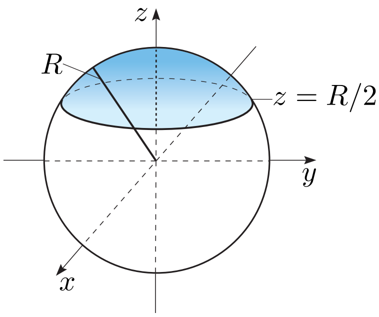


Figure 30 A spherical cap

Exercise 30

Consider a uniform solid of mass M whose shape is given by the interior of a *spherical cap* defined by the region of space described by the inequalities $x^2 + y^2 + z^2 \leq R^2$ and $z \geq R/2 > 0$ (see Figure 30). By using cylindrical coordinates, do the following.

(a) Show that the volume of the solid is $\frac{5}{24}\pi R^3$.

(b) Show that the moment of inertia of the solid about its axis of symmetry is $\frac{53}{200}MR^2$.

(*Hint*: You may use the indefinite integral

$$\int x^3\sqrt{a^2 - x^2}\,dx = \tfrac{1}{5}(a^2 - x^2)^{5/2} - \tfrac{1}{3}a^2(a^2 - x^2)^{3/2}.)$$

4 Area of a surface

In Unit 16 we studied the scalar line integral, and showed that as a particular application it can be used to find the length of a curve. In this section we find a corresponding application of the area integrals studied in Section 1; this enables us to calculate the area of a curved surface.

In Section 1 we defined and calculated many area integrals. However, the areas that we considered were always in a plane. Sometimes it is useful to be able to calculate areas of surfaces in three-dimensional space – for example, we may need to know the area of a curved surface like part of a sphere. Fortunately, there is a way to find such an area in terms of area integrals.

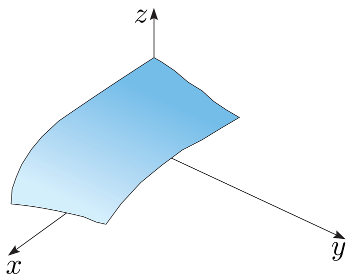


Figure 31 Portion of the surface $z = f(x, y)$

Consider a surface in space defined by the equation $z = f(x, y)$, a portion of which is shown in Figure 31.

We use Σ to denote the *surface area* of the *curved surface*. This upper-case sigma should not be confused with the summation symbol $\sum$; context should make the distinction clear.

Suppose that we wish to calculate the area Σ of that part of the surface that lies above the rectangle defined by $a \leq x \leq b$ and $c \leq y \leq d$. Take a small element of the area above the point (x_0, y_0), as shown in Figure 32.

Note that this approach and the notation used differs somewhat from the derivation of area integrals in Subsection 1.2.

If small enough, the element will be approximately rectangular with vertices

$$\begin{aligned}
P &= (x_0, y_0, f(x_0, y_0)),\\
Q &= (x_0 + \delta x, y_0, f(x_0 + \delta x, y_0)),\\
R &= (x_0, y_0 + \delta y, f(x_0, y_0 + \delta y)),\\
S &= (x_0 + \delta x, y_0 + \delta y, f(x_0 + \delta x, y_0 + \delta y)).
\end{aligned}$$

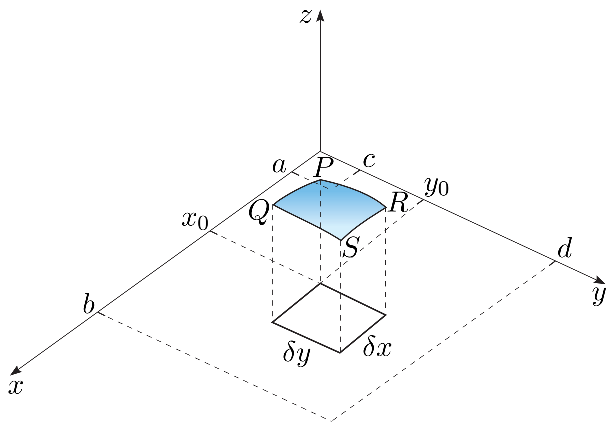


Figure 32 Small element of the surface $z = f(x, y)$

Now recall that a point (x, y, z) in space can be represented as a position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. We can therefore think of the edges PQ and PR as forming two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$. We have

$$\begin{aligned}\mathbf{v}_1 &= \overrightarrow{PQ}\\ &= \big((x_0 + \delta x)\,\mathbf{i} + y_0\,\mathbf{j} + f(x_0 + \delta x, y_0)\,\mathbf{k}\big) - \big(x_0\,\mathbf{i} + y_0\,\mathbf{j} + f(x_0, y_0)\,\mathbf{k}\big)\\ &= \delta x\,\mathbf{i} + \big(f(x_0 + \delta x, y_0) - f(x_0, y_0)\big)\,\mathbf{k}\\ &\simeq \delta x\,\mathbf{i} + \delta x\,\frac{\partial f}{\partial x}\,\mathbf{k},\end{aligned}$$

where we have used the approximation

$$\frac{\partial f}{\partial x} \simeq \frac{f(x_0 + \delta x, y_0) - f(x_0, y_0)}{\delta x}.$$

Similarly,

$$\begin{aligned}\mathbf{v}_2 &= \overrightarrow{PR}\\ &= \big(x_0\,\mathbf{i} + (y_0 + \delta y)\,\mathbf{j} + f(x_0, y_0 + \delta y)\,\mathbf{k}\big) - \big(x_0\,\mathbf{i} + y_0\,\mathbf{j} + f(x_0, y_0)\,\mathbf{k}\big)\\ &= \delta y\,\mathbf{j} + \big(f(x_0, y_0 + \delta y) - f(x_0, y_0)\big)\,\mathbf{k}\\ &\simeq \delta y\,\mathbf{j} + \delta y\,\frac{\partial f}{\partial y}\,\mathbf{k}.\end{aligned}$$

Now the area of the parallelogram formed by two vectors is given by the magnitude of their cross product. In our context this becomes

This was covered in Unit 2, Subsection 2.3.

$$\begin{aligned}\delta\Sigma &= |\mathbf{v}_1 \times \mathbf{v}_2|\\ &\simeq \left|\left(\delta x\,\mathbf{i} + \delta x\,\frac{\partial f}{\partial x}\,\mathbf{k}\right) \times \left(\delta y\,\mathbf{j} + \delta y\,\frac{\partial f}{\partial y}\,\mathbf{k}\right)\right|\\ &= \left|-\delta x\,\delta y\,\frac{\partial f}{\partial x}\,\mathbf{i} - \delta x\,\delta y\,\frac{\partial f}{\partial y}\,\mathbf{j} + \delta x\,\delta y\,\mathbf{k}\right|\\ &= \delta x\,\delta y\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2},\end{aligned}$$

where $\delta\Sigma$ is the area of the small element.

The area Σ of the whole surface is given approximately by adding up the areas of all the small elements, so

$$\Sigma \simeq \sum_{\text{small elements}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, \delta y\, \delta x.$$

As the size of the elements tends to zero, we can pass to the limit, and the sum becomes an area integral in the sense of Section 1:

$$\Sigma = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy \right) dx.$$

As in Section 1 the integrations can be performed in either order.

We have dealt with a portion of surface lying over a rectangular region of the plane, but as in Section 1 this can be extended to more general regions (see Example 15). The limits of the two integrals are then determined by the context, but often, as we found in Section 1, the limits of the inner integral will be functions rather than constants.

Procedure 3 Finding the area of a surface

Suppose that a surface is described in the form $z = f(x, y)$. To find the area Σ of a portion of the surface lying over the region S in the (x, y)-plane, carry out the following steps.

1. Form the function

$$g(x, y) = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

2. Calculate the area integral

$$\Sigma = \int_S g(x, y)\, dA,$$

as in Procedure 1.

The idea is simple enough, although the integrals can be extremely messy – even for fairly simple surfaces.

Example 14

Express the area of the surface $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$, lying over the square defined by $-1 \le x \le 1$, $-1 \le y \le 1$, as an area integral. (Do *not* attempt to evaluate the integral.)

Solution

With $f(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2$, we have

$$\frac{\partial f}{\partial x}=x \quad\text{and}\quad \frac{\partial f}{\partial y}=y.$$

Thus, following Procedure 3, we write

$$g(x,y)=\sqrt{1+x^2+y^2},$$

and obtain the integral

$$\Sigma=\int_S g(x,y)\,dA.$$

With the appropriate limits this becomes

$$\Sigma=\int_{x=-1}^{x=1}\left(\int_{y=-1}^{y=1}\sqrt{1+x^2+y^2}\,dy\right)dx.$$

Note that this integral cannot easily be evaluated by analytical means, even with the assistance of a computer algebra package. However, a numerical computation gives it a value of about 5.123.

As in Section 2, the evaluation of a surface integral may sometimes be dramatically simplified by using polar coordinates.

Example 15

Find the area of the portion of a sphere of radius a, centred at the origin, lying above the (x,y)-plane and defined by the angle ψ as shown in Figure 33 (where $0\le\psi\le\pi/2$). (Recall that the volume and moment of inertia of a spherical cap were calculated in Exercise 30.)

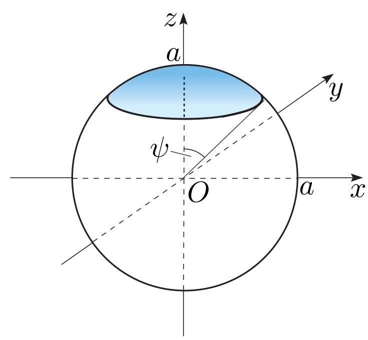


Figure 33 Spherical cap defined in spherical coordinates by $r=a$ and $0\le\theta\le\psi$

Solution

This surface is given by the equation

$$f(x,y)=\sqrt{a^2-x^2-y^2},$$

since $x^2+y^2+z^2=a^2$, and the corresponding region in the (x,y)-plane is a circular disc of radius $a\sin\psi$. Now

$$\frac{\partial f}{\partial x}=\frac{-x}{\sqrt{a^2-x^2-y^2}} \quad\text{and}\quad \frac{\partial f}{\partial y}=\frac{-y}{\sqrt{a^2-x^2-y^2}},$$

and we must compute

$$\Sigma=\int_{x=a}^{x=b}\left(\int_{y=\alpha(x)}^{y=\beta(x)}\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dy\right)dx$$

over the disc in the (x,y)-plane given by

$$x^2+y^2\le a^2\sin^2\psi.$$

Compare with Figure 15.

Following the discussion after Exercise 10, we see that the limits are

$$-a\sin\psi \le x \le a\sin\psi, \quad -\sqrt{a^2\sin^2\psi - x^2} \le y \le \sqrt{a^2\sin^2\psi - x^2},$$

and we must evaluate

$$\begin{aligned}\Sigma &= \int_{x=-a\sin\psi}^{x=a\sin\psi}\left(\int_{y=-\sqrt{a^2\sin^2\psi-x^2}}^{y=\sqrt{a^2\sin^2\psi-x^2}}\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dy\right)dx\\ &= \int_{x=-a\sin\psi}^{x=a\sin\psi}\left(\int_{y=-\sqrt{a^2\sin^2\psi-x^2}}^{y=\sqrt{a^2\sin^2\psi-x^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dy\right)dx.\end{aligned}$$

It is clear that this will be considerably easier to deal with if we use polar coordinates, so we proceed as in Section 2.

The function to be integrated is simply

$$g(r,\theta) = \frac{a}{\sqrt{a^2-r^2}}.$$

Note that for the r-integral we use the trick given in equation (6), with $k = -\frac{1}{2}$, $n = -\frac{1}{2}$ and $f(r) = a^2 - r^2$.

The limits for the r-integration are 0 and $a\sin\psi$, while for θ the limits are simply $-\pi$ and π. Therefore the required integral is

$$\begin{aligned}\Sigma &= \int_{r=0}^{r=a\sin\psi}\left(\int_{\theta=-\pi}^{\theta=\pi}\frac{a}{\sqrt{a^2-r^2}}\,r\,d\theta\right)dr\\ &= 2\pi a\int_{r=0}^{r=a\sin\psi}\frac{r}{\sqrt{a^2-r^2}}\,dr\\ &= 2\pi a\left[-\sqrt{a^2-r^2}\right]_0^{a\sin\psi}\\ &= 2\pi a\left(-a\cos\psi-(-a)\right)\\ &= 2\pi a^2(1-\cos\psi).\end{aligned}$$

We can confirm that this is a plausible answer by considering some specific values of ψ. When $\psi = 0$ we obtain 0, and when $\psi = \pi/2$ we obtain $2\pi a^2$ (which is the correct formula for the area of a hemisphere of radius a).

Exercise 31

Show that the area of the surface $z = \frac{1}{2}(x^2+y^2)$, lying over a disc centred at the origin with radius a, is given by

$$\Sigma = \tfrac{2}{3}\pi\big((1+a^2)^{3/2}-1\big).$$

Exercise 32

Find the surface area of the curved surface of a cone of height h and base radius a.

Learning outcomes

After studying this unit, you should be able to:

- evaluate area integrals over regions of the (x, y)-plane using Cartesian or polar coordinate systems
- use area integrals for evaluating population models in the (x, y)-plane, and for obtaining the masses and moments of inertia of laminas
- evaluate volume integrals over regions of space using Cartesian, cylindrical or spherical coordinate systems
- use volume integrals to evaluate the masses and moments of inertia of solids
- use area integrals to find surface areas.

Solutions to exercises

Solution to Exercise 1

$$\begin{aligned}\int_S x^2y^3\,dA &= \int_{x=0}^{x=2}\left(\int_{y=1}^{y=3} x^2y^3\,dy\right)dx\\ &= \int_{x=0}^{x=2}\left(\left[x^2(\tfrac{1}{4}y^4)\right]_{y=1}^{y=3}\right)dx\\ &= \int_{x=0}^{x=2}(20x^2)\,dx\\ &= \left[\tfrac{20}{3}x^3\right]_{x=0}^{x=2}\\ &= \tfrac{160}{3}.\end{aligned}$$

Solution to Exercise 2

We must evaluate the expression

$$\int_{y=1}^{y=2}\left(\int_{x=0}^{x=3} xy\,dx\right)dy.$$

Taking the integral over x first, treating y as a constant, we have

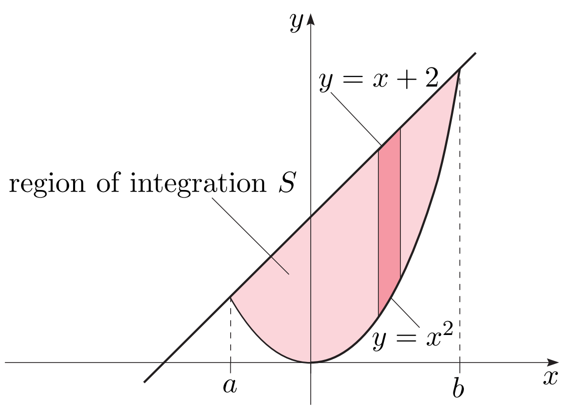
$$\int_{x=0}^{x=3} xy\,dx = y\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=3} = \tfrac{9}{2}y.$$

Now the y-integration gives

$$\int_{y=1}^{y=2} \tfrac{9}{2}y\,dy = \tfrac{9}{2}\left[\tfrac{1}{2}y^2\right]_{y=1}^{y=2} = \tfrac{27}{4}.$$

Solution to Exercise 3

Step 1: The diagram is shown below.



Step 2: To find the y-limits, we draw in a vertical strip at some general position in the region, as shown in the figure above. The lower point of this strip is then $y = x^2$, and the upper limit is $y = x + 2$. Note that for each vertical strip, the lower and upper vertical limits are on the parabola and straight line, respectively.

Step 3: The limits of the x-integration are $x = a$ and $x = b$, where these values of x are solutions of the simultaneous equations $y = x^2$ and $y = x + 2$, that is, of $x^2 - x - 2 = (x-2)(x+1) = 0$, giving $x = 2$ and $x = -1$. So $a = -1$ and $b = 2$.

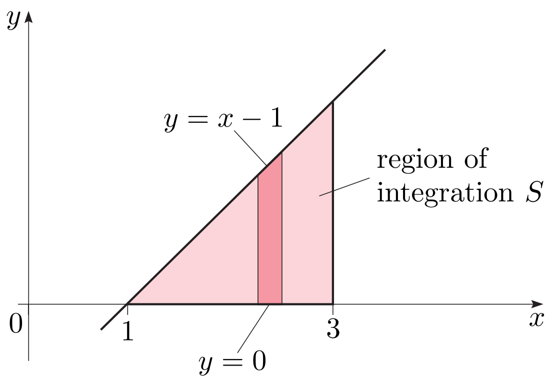
Step 4: $\displaystyle\int_S y\,dA = \int_{x=-1}^{x=2} \left(\int_{y=x^2}^{y=x+2} y\,dy \right) dx.$

Step 5: $\displaystyle\int_{y=x^2}^{y=x+2} y\,dy = \left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=x+2} = \tfrac{1}{2}(x+2)^2 - \tfrac{1}{2}x^4.$

Step 6: $\displaystyle\int_{x=-1}^{x=2} \left(\tfrac{1}{2}(x+2)^2 - \tfrac{1}{2}x^4\right) dx = \left[\tfrac{1}{6}(x+2)^3 - \tfrac{1}{10}x^5\right]_{x=-1}^{x=2} = \tfrac{36}{5}.$

Solution to Exercise 4

Step 1: The diagram is shown below.



Step 2: The y-limits are $y = 0$ and $y = x - 1$.

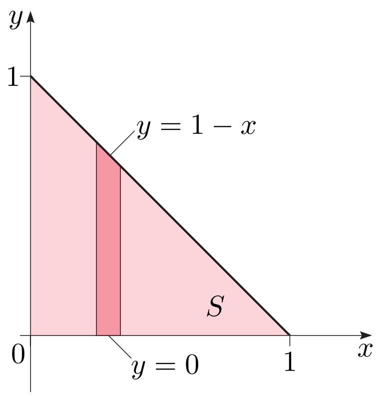
Step 3: The x-limits are $x = 1$ and $x = 3$.

Step 4: $\displaystyle\int_S (x-y)\,dA = \int_{x=1}^{x=3} \left(\int_{y=0}^{y=x-1} (x-y)\,dy \right) dx.$

Step 5: $\displaystyle\int_{y=0}^{y=x-1} (x-y)\,dy = \left[-\tfrac{1}{2}(x-y)^2\right]_{y=0}^{y=x-1} = \tfrac{1}{2}x^2 - \tfrac{1}{2}.$

Step 6: $\displaystyle\int_{x=1}^{x=3} \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}\right) dx = \left[\tfrac{1}{6}x^3 - \tfrac{1}{2}x\right]_{x=1}^{x=3} = \tfrac{10}{3}.$

Solution to Exercise 5



A diagram showing the region of integration S and a strip for the y-integration is shown in the margin.

Combining the steps of Procedure 1 gives

$$\begin{aligned}\int_S f\,dA &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1-x}(x+y)\,dy\right)dx\\ &= \int_{x=0}^{x=1}\left[\tfrac{1}{2}(x+y)^2\right]_{y=0}^{y=1-x}dx\\ &= \int_0^1\left(\tfrac{1}{2}-\tfrac{1}{2}x^2\right)dx\\ &= \left[\tfrac{1}{2}x-\tfrac{1}{6}x^3\right]_0^1=\tfrac{1}{3}.\end{aligned}$$

Note that we can write integration limits as simple constants once we reach the final variable.

Solution to Exercise 6

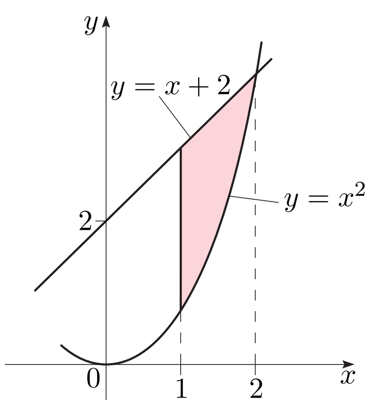
We have

$$\begin{aligned}\int_S f\,dA &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=\sqrt{1-x^2}}x\,dy\right)dx\\ &= \int_{x=0}^{x=1}\left[xy\right]_{y=0}^{y=\sqrt{1-x^2}}dx\\ &= \int_0^1 x\sqrt{1-x^2}\,dx\\ &= \left[-\tfrac{1}{3}(1-x^2)^{3/2}\right]_0^1=\tfrac{1}{3}.\end{aligned}$$

For the x-integral, we have used the trick given in equation (6), with $k=-\frac{1}{2}$, $n=\frac{1}{2}$ and $f(x)=1-x^2$.

Solution to Exercise 7

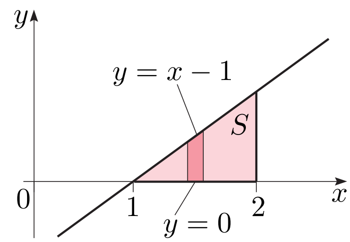
The region is enclosed by the straight lines $x=1$, $x=2$, $y=x+2$ and the parabola $y=x^2$, as shown in the diagram below.



So we have

$$\int_{x=1}^{x=2}\left(\int_{y=x^2}^{y=x+2} 1\,dy\right)dx = \int_{x=1}^{x=2}\big[y\big]_{y=x^2}^{y=x+2}\,dx$$
$$= \int_1^2 (x+2-x^2)\,dx$$
$$= \left[\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\right]_1^2$$
$$= \tfrac{7}{6}.$$

Solution to Exercise 8



(a) Step 1: If we integrate over y first, then we *fix* x and draw in a *vertical* strip as shown in the figure in the margin.

Step 2: The y-limits are $y = 0$ and $y = x - 1$.

Step 3: The x-limits are $x = 1$ and $x = 2$.

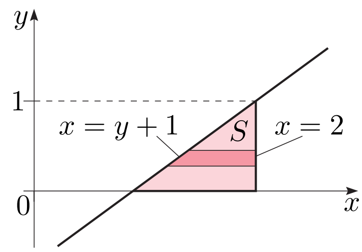
Step 4: $\displaystyle\int_S (x^2+y^2)\,dA = \int_{x=1}^{x=2}\left(\int_{y=0}^{y=x-1}(x^2+y^2)\,dy\right)dx.$

Step 5: $\displaystyle\int_{y=0}^{y=x-1}(x^2+y^2)\,dy = \left[x^2y + \tfrac{1}{3}y^3\right]_{y=0}^{y=x-1}$

$= x^2(x-1) + \tfrac{1}{3}(x-1)^3.$

Step 6: $\displaystyle\int_{x=1}^{x=2}\left(x^3 - x^2 + \tfrac{1}{3}(x-1)^3\right)dx$

$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{12}(x-1)^4\right]_{x=1}^{x=2} = \tfrac{3}{2}.$



(b) Step 1: If we integrate over x first, then we *fix* y and draw in a *horizontal* strip as shown in the figure in the margin.

Step 2: The x-limits are $x = y + 1$ and $x = 2$.

Step 3: The y-limits are $y = 0$ and $y = 1$.

Step 4: $\displaystyle\int_S (x^2+y^2)\,dA = \int_{y=0}^{y=1}\left(\int_{x=y+1}^{x=2}(x^2+y^2)\,dx\right)dy.$

Step 5: $\displaystyle\int_{x=y+1}^{x=2}(x^2+y^2)\,dx = \left[\tfrac{1}{3}x^3 + xy^2\right]_{x=y+1}^{x=2}$

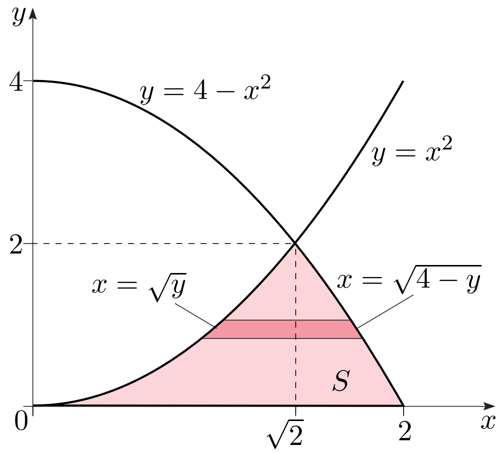
$= \tfrac{8}{3} + 2y^2 - \tfrac{1}{3}(y+1)^3 - (y+1)y^2.$

Step 6: $\displaystyle\int_{y=0}^{y=1}\left(\tfrac{8}{3} - \tfrac{1}{3}(y+1)^3 + y^2 - y^3\right)dy$

$= \left[\tfrac{8}{3}y - \tfrac{1}{12}(y+1)^4 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right]_{y=0}^{y=1} = \tfrac{3}{2}.$

Solution to Exercise 9

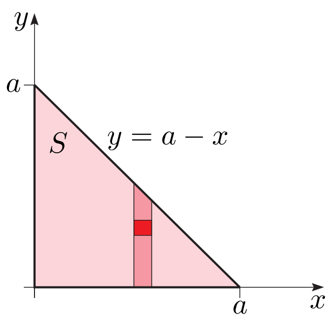
The region of integration S is shown shaded in the following figure.



The upper limit of the y-integral (outer integral) is given by the intersection of the two parabolas, where $y = x^2 = 4 - x^2$, that is, $x^2 = 2$. So the point of intersection is at $(\sqrt{2}, 2)$, as shown in the figure. Thus the area integral is

$$\begin{aligned}\int_S f\,dA &= \int_{y=0}^{y=2}\left(\int_{x=\sqrt{y}}^{x=\sqrt{4-y}} x\,dx\right)dy\\ &= \int_{y=0}^{y=2}\left[\tfrac{1}{2}x^2\right]_{\sqrt{y}}^{\sqrt{4-y}}\,dy\\ &= \int_0^2 (2-y)\,dy = \left[2y - \tfrac{1}{2}y^2\right]_0^2 = 2.\end{aligned}$$

Solution to Exercise 10



The total population is the area integral of the surface density function over the triangular region of the plate shown in the figure in the margin.

Integrating first over y, that is, using vertical strips, the lower and upper limits of the y-integral are $y = 0$ and $y = a - x$. The strips begin at $x = 0$ and end at $x = a$. Hence the total number of bacteria is given by the area integral

$$\begin{aligned}&A\int_{x=0}^{x=a}\left(\int_{y=0}^{y=a-x}\left(2 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)dy\right)dx\\ &= A\int_{x=0}^{x=a}\left[2y - \frac{x^2 y}{a^2} - \frac{y^3}{3a^2}\right]_{y=0}^{y=a-x}\,dx\\ &= A\int_0^a\left(2(a-x) - \frac{x^2(a-x)}{a^2} - \frac{(a-x)^3}{3a^2}\right)dx\\ &= A\left[2ax - x^2 - \frac{x^3}{3a} + \frac{x^4}{4a^2} + \frac{(a-x)^4}{12a^2}\right]_0^a\\ &= \tfrac{5}{6}Aa^2.\end{aligned}$$

When $A = 10^{12}$ and $a = 0.01$, the number of bacteria is 8.3×10^7.

Solution to Exercise 11

The mass of the lens is

$$\begin{aligned}\int_{\text{disc}} f\,dA &= 2\pi k \int_{r=0}^{r=a} \left(1 - \frac{r^2}{a^2}\right) r\,dr \\ &= 2\pi k \left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a = \tfrac{1}{2}\pi k a^2 = 2.9 \times 10^{-3}\end{aligned}$$

for $a = 2.5 \times 10^{-2}$ and $k = 3$.

Solution to Exercise 12

(a) We have

$$\begin{aligned}J^2 &= \int_{-\infty}^{\infty} \exp(-x^2)\,dx \times \int_{-\infty}^{\infty} \exp(-y^2)\,dy \\ &= \int_{x=-\infty}^{x=\infty} \left(\int_{y=-\infty}^{y=\infty} \exp\left(-(x^2+y^2)\right)dy\right) dx,\end{aligned}$$

showing that J^2 is an area integral of $\exp\left(-(x^2+y^2)\right)$ over the whole of the (x, y)-plane.

(b) In polar coordinates, $\exp\left(-(x^2+y^2)\right) = \exp(-r^2)$ and the whole of the (x, y)-plane can be regarded as a disc of infinite radius, denoted D. Thus

$$\begin{aligned}J^2 &= \int_D \exp(-r^2)\,dA \\ &= 2\pi \int_0^{\infty} \exp(-r^2)\,r\,dr \\ &= \pi \int_0^{\infty} e^{-u}\,du = \pi\left[-e^{-u}\right]_0^{\infty} = \pi,\end{aligned}$$

where the substitution $u = r^2$ was used in the evaluation of the r-integral. Hence $J = \sqrt{\pi}$.

Solution to Exercise 13

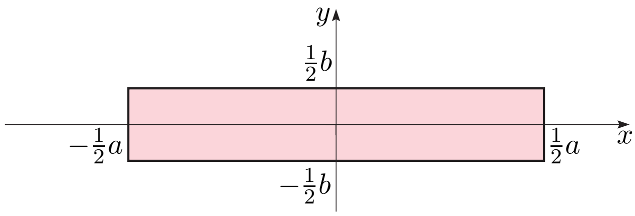
The required mass is the area integral of f on a disc of radius b. Noting that f depends on r only, we have

$$\begin{aligned}\int_S f\,dA &= 2\pi A \int_0^b \exp(-r^2/a^2) r\,dr \\ &= -\pi A a^2 \left[\exp(-r^2/a^2)\right]_0^b \\ &= \pi a^2 A(1 - \exp(-b^2/a^2)) \\ &= \pi \times (1.5 \times 10^{-2})^2 \times 10^{-5}(1 - \exp(-1/1.5^2)) \\ &= 2.5 \times 10^{-9}\end{aligned}$$

for $A = 10^{-5}$, $a = 1.5 \times 10^{-2}$ and $b = 0.01$.

Solution to Exercise 14

The ruler is modelled as a lamina in the (x, y)-plane with its centre at the origin. The region of integration is shown in the figure below.



Using Cartesian coordinates, we have

$$\begin{aligned} I &= f \int_{x=-a/2}^{x=a/2} \left(\int_{y=-b/2}^{y=b/2} (x^2 + y^2)\, dy \right) dx \\ &= f \int_{x=-a/2}^{x=a/2} \left[x^2 y + \tfrac{1}{3} y^3 \right]_{y=-b/2}^{y=b/2} dx \\ &= 2f \int_{-a/2}^{a/2} \left(\tfrac{1}{2} x^2 b + \tfrac{1}{24} b^3 \right) dx \\ &= 2f \left[\tfrac{1}{6} b x^3 + \tfrac{1}{24} b^3 x \right]_{-a/2}^{a/2} \\ &= \tfrac{1}{12} f a b (a^2 + b^2). \end{aligned}$$

Substituting in the particular values given, the moment of inertia of the ruler is

$$\begin{aligned} I &= \frac{10^{-2} \times 0.30 \times 0.04}{12} (0.30^2 + 0.04^2) \\ &= 9.16 \times 10^{-7}. \end{aligned}$$

Note that fab is the total mass M of the ruler, so we can also write

$$I = \tfrac{1}{12} M (a^2 + b^2).$$

Solution to Exercise 15

Taking the constant surface density f outside the area integral, and using equation (10), we have

$$I_{\text{disc}} = f \int_{\theta=-\pi}^{\theta=\pi} \left(\int_{r=0}^{r=a} r^3\, dr \right) d\theta = 2\pi f \int_0^a r^3 dr = \tfrac{1}{2} \pi f a^4.$$

The mass of the disc is $M = f\pi a^2$, so we have $I_{\text{disc}} = \frac{1}{2} M a^2$, as required.

Solution to Exercise 16

Let f be the density of the washer. Then using equation (10) again, we have

$$I_{\text{washer}} = 2\pi f \int_a^b r^3\, dr = 2\pi f \left[\tfrac{1}{4} r^4 \right]_a^b = \tfrac{1}{2} \pi f (b^4 - a^4).$$

The mass of the washer is $M = f\pi(b^2 - a^2)$, so $I_{\text{washer}} = \frac{1}{2} M (b^2 + a^2)$.

Solution to Exercise 17

The moment of inertia of the magnifying glass is

$$\begin{aligned} I &= 2\pi k \int_0^a \left(1 - \frac{r^2}{a^2}\right) r^3\,dr \\ &= 2\pi k \left[\frac{r^4}{4} - \frac{r^6}{6a^2}\right]_0^a \\ &= \tfrac{1}{6}\pi k a^4 = 6.1 \times 10^{-7}, \end{aligned}$$

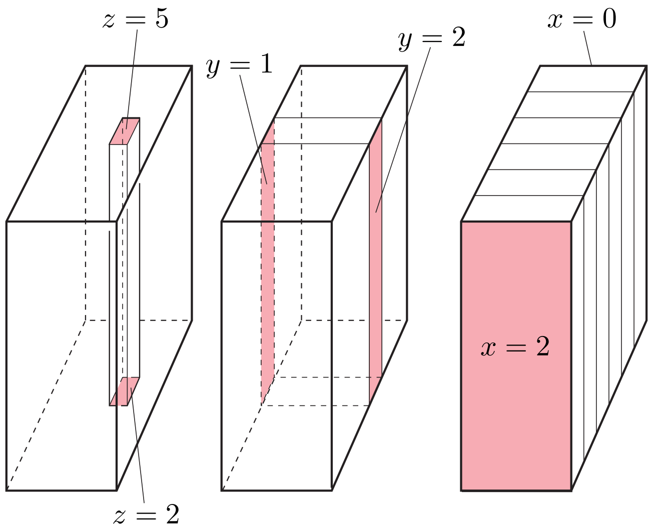
for the given values of k and a.

Solution to Exercise 18

The mass of the block is the volume integral

$$M = \int_B f\,dV = \int_B (x+y+z)\,dV.$$

The region of integration is shown in the following figure.



So the mass is

$$\begin{aligned} M &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left(\int_{z=2}^{z=5} (x+y+z)\,dz \right) dy \right) dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left[(x+y)z + \tfrac{1}{2}z^2\right]_{z=2}^{z=5} dy \right) dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left(3x + 3y + \tfrac{21}{2}\right) dy \right) dx \\ &= \int_{x=0}^{x=2} \left[\left(3x + \tfrac{21}{2}\right) y + \tfrac{3}{2}y^2\right]_{y=1}^{y=2} dx \\ &= \int_{x=0}^{x=2} (3x+15)\,dx = \left[\tfrac{3}{2}x^2 + 15x\right]_{x=0}^{x=2} = 36. \end{aligned}$$

Solution to Exercise 19

The mass of the block is

$$\begin{aligned}\int_B (x(y+1)-z)\,dV &= \int_{x=1}^{x=2}\left(\int_{y=0}^{y=3}\left(\int_{z=-1}^{z=0}(x(y+1)-z)\,dz\right)dy\right)dx\\ &= \int_{x=1}^{x=2}\left(\int_{y=0}^{y=3}\left(x(y+1)+\tfrac{1}{2}\right)dy\right)dx\\ &= \int_1^2\left(\tfrac{9}{2}x+3x+\tfrac{3}{2}\right)dx\\ &= \left[\tfrac{15}{4}x^2+\tfrac{3}{2}x\right]_1^2 = \tfrac{51}{4}.\end{aligned}$$

Solution to Exercise 20

The scalar function $f = c + \alpha r$ is spherically symmetric so, using equation (23),

$$\begin{aligned}M &= 4\pi\int_{r=0}^{r=R}(c+\alpha r)r^2\,dr\\ &= 4\pi\left[\tfrac{1}{3}cr^3+\tfrac{1}{4}\alpha r^4\right]_0^R\\ &= \tfrac{1}{3}\pi R^3(4c+3\alpha R).\end{aligned}$$

(The case $c = 1$ and $\alpha = 0$ gives the volume of a sphere.)

Solution to Exercise 21

The mass of the sphere of density D is

$$M = D\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{\phi=-\pi}^{\phi=\pi} r^2\sin\theta\,d\phi\right)d\theta\right)dr.$$

Using equation (23), $M = \frac{4}{3}\pi DR^3$, and using this result in the solution to Example 11, we find

$$I = \tfrac{2}{5}MR^2.$$

Solution to Exercise 22

Let the constant density of the cylinder be D. The volume integral in cylindrical coordinates can be carried out in any order, so equation (21) becomes

$$I = 2\pi\int_{\rho=0}^{\rho=a}\left(\int_{z=0}^{z=h} D\,dz\right)\rho^3\,d\rho.$$

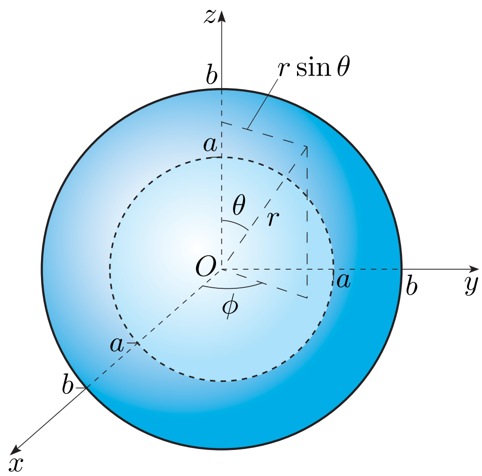
The z-integral yields a factor Dh, and the ρ-integral yields $\frac{1}{4}a^4$. Hence $I = \frac{1}{2}\pi Dha^4$. The mass M of the cylinder is given by a similar volume integral but with ρ replacing ρ^3 in the integrand. Hence $M = \pi Dha^2$, thus

$$I = \tfrac{1}{2}Ma^2.$$

Note that the volume of the cylinder is given by $M/D = \pi ha^2$.

Solution to Exercise 23

We use spherical coordinates, and the shell is as shown in the figure below.



Using equation (24) we have, with $f = D$,

$$I = D \int_{\text{shell}} (r \sin\theta)^2 r^2 \sin\theta \, d\phi \, d\theta \, dr.$$

The function to be integrated does not depend on the azimuthal angle ϕ, so the ϕ-integral yields a factor 2π. This leaves the area integral

$$2\pi D \int_{r=a}^{r=b} \left(\int_{\theta=0}^{\theta=\pi} r^4 \sin^3\theta \, d\theta \right) dr.$$

We can write $\sin^3\theta = \sin\theta(1 - \cos^2\theta)$, so the θ-integral yields

$$\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi} = \tfrac{4}{3}.$$

Hence

$$\begin{aligned} I &= \tfrac{8}{3}\pi D \int_a^b r^4 \, dr \\ &= \tfrac{8}{15}\pi D(b^5 - a^5). \end{aligned}$$

The mass of the shell is $M = \frac{4}{3}\pi D(b^3 - a^3)$, since this is just the mass of a sphere of radius b minus that of radius a (where the mass of a sphere of radius R is given in the solution to Exercise 21), so we have

$$I = \frac{2M(b^5 - a^5)}{5(b^3 - a^3)}.$$

(Note that this answer gives $\frac{2}{5}Mb^2$ in the limit $a \to 0$, which is the moment of inertia of a uniform solid sphere of radius b as found in Exercise 21.)

Solution to Exercise 24

With the axis of the cylinder on the z-axis, and since the cylinder is uniform, the moment of inertia must be the same as the moment of inertia of the (uniform) washer in Exercise 16, since the projection of the cylindrical shell onto the (x, y)-plane occupies the same region as the washer. Thus we have

$$I_{\text{shell}} = I_{\text{washer}} = \tfrac{1}{2}M(b^2 + a^2).$$

Solution to Exercise 25

We use cylindrical coordinates with the z-axis pointing along the axis of the cylinder, and let the rotation axis be the x-axis. So the perpendicular distance d of a volume element from the rotation axis is now given by $d^2 = y^2 + z^2 = \rho^2 \sin^2 \phi + z^2$. Hence with D denoting the constant density of the cylinder, and using the double-angle formula $\sin^2 \phi = (1 - \cos 2\phi)/2$, we have

$$\begin{aligned}
I &= \int_{\text{cylinder}} d^2 D\, dV \\
&= D \int_{\text{cylinder}} \left(\rho^2 \sin^2 \phi + z^2\right) dV \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=-L/2}^{z=L/2} \left(\rho^2 \sin^2 \phi + z^2\right) \rho\, dz \right) d\phi \right) d\rho \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left[\rho^2 z \sin^2 \phi + \tfrac{1}{3} z^3\right]_{z=-L/2}^{z=L/2} \rho\, d\phi \right) d\rho \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(L\rho^2 \sin^2 \phi + \tfrac{1}{12} L^3\right) \rho\, d\phi \right) d\rho \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(L\rho^2 \tfrac{1}{2}(1 - \cos 2\phi) + \tfrac{1}{12} L^3\right) \rho\, d\phi \right) d\rho \\
&= \pi D \int_0^R \left(L\rho^3 + \tfrac{1}{6} L^3 \rho\right) d\rho \\
&= \pi D \left(\tfrac{1}{4} L R^4 + \tfrac{1}{12} L^3 R^2\right) \\
&= \tfrac{1}{12} \pi D L R^2 (3R^2 + L^2).
\end{aligned}$$

Now recall that the volume of a cylinder of length L and radius R is given by $\pi R^2 L$ (see Exercise 22), so the density is $D = M/(\pi R^2 L)$. Thus substituting for D in the above expression for I gives the required result:

$$I = \tfrac{1}{12} M (3R^2 + L^2).$$

Note that in the limit as $R \to 0$, this becomes $\frac{1}{12} M L^2$, the moment of inertia of a thin rod of length L about an axis through its centre perpendicular to its length, as found in Subsection 2.3.

Solution to Exercise 26

(a) We use spherical coordinates. The density function is $f(\mathbf{r}) = cr^2$, so using equation (23), the mass of the sphere is

$$M = 4\pi \int_0^R (cr^2)r^2\,dr = \tfrac{4}{5}\pi c R^5.$$

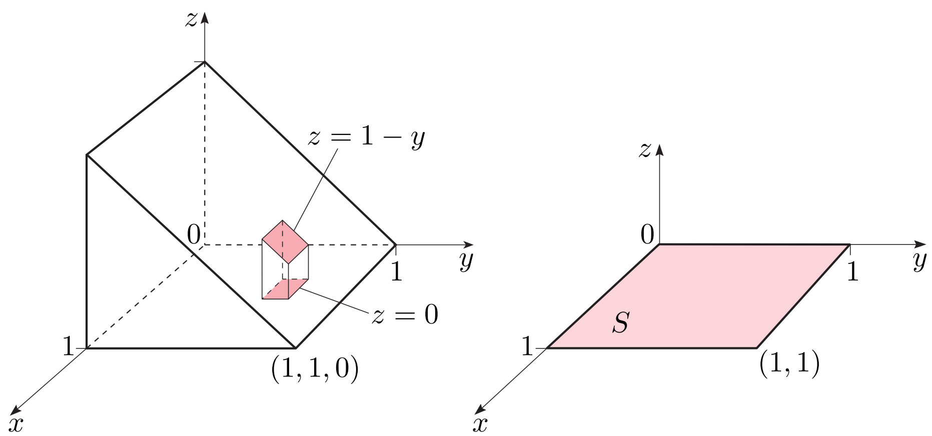
(b) The moment of inertia I is the volume integral of the function $g(r,\theta,\phi) = f(r)\,d^2 = (cr^2)(r\sin\theta)^2$ (which is *not* spherically symmetric). Hence

$$\begin{aligned}
I &= \int_{\text{sphere}} cr^2(r\sin\theta)^2\,dV \\
&= c\int_{\text{sphere}} r^2(r\sin\theta)^2 r^2\sin\theta\,d\phi\,d\theta\,dr \\
&= c\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=\pi}\int_{\phi=-\pi}^{\phi=\pi} r^6\sin^3\theta\,d\phi\,d\theta\,dr \\
&= c\times 2\pi\times\tfrac{4}{3}\times\tfrac{1}{7}R^7 = \tfrac{8}{21}c\pi R^7 = \tfrac{10}{21}MR^2,
\end{aligned}$$

using $\int_0^\pi \sin^3\theta\,d\theta = \frac{4}{3}$ from the solution to Exercise 23 or Example 11.

Solution to Exercise 27

The region and the projection onto the (x,y)-plane are shown below.



The integral over z gives

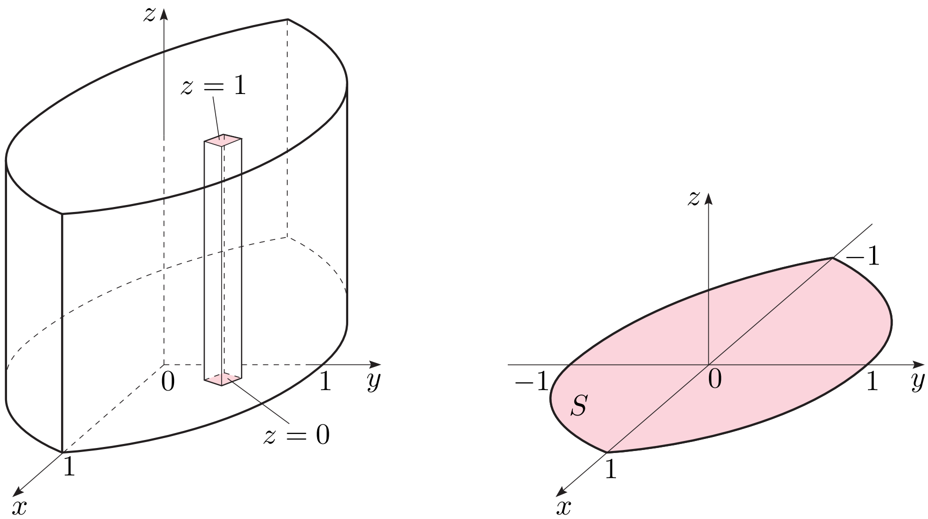
$$\begin{aligned}
h(x,y) &= \int_{z=0}^{z=1-y} x^2yz\,dz \\
&= \left[\tfrac{1}{2}x^2yz^2\right]_{z=0}^{z=1-y} = \tfrac{1}{2}x^2y(1-y)^2.
\end{aligned}$$

Now following the steps in Procedure 1 for evaluating the area integral of $h(x,y)$ over S, we have

$$\int_S h(x,y)\,dA = \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1}\tfrac{1}{2}x^2y(1-y)^2\,dy\right)dx$$
$$= \int_{x=0}^{x=1}\left[\tfrac{1}{2}x^2\left(\tfrac{1}{2}y^2-\tfrac{2}{3}y^3+\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1}dx$$
$$= \int_0^1 \tfrac{1}{24}x^2\,dx = \tfrac{1}{72}.$$

Solution to Exercise 28

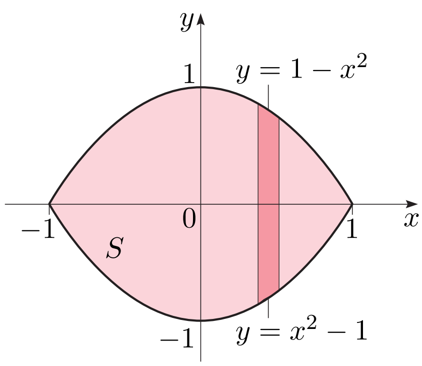
The region and its projection onto the (x,y)-plane (bounded by the parabolas $y=1-x^2$ and $y=x^2-1$) are illustrated below.



The column is shown in the left-hand figure. The z-limits are $z=0$ and $z=1$. The integral over z is

$$h(x,y) = \int_{z=0}^{z=1}(z+3x-2)\,dz$$
$$= \left[\tfrac{1}{2}z^2+(3x-2)z\right]_{z=0}^{z=1} = \tfrac{1}{2}+3x-2 = 3x-\tfrac{3}{2}.$$

We now evaluate the area integral over the shape shown below.



So we have

$$\begin{aligned}\int_S h(x,y)\,dA &= \int_{x=-1}^{x=1}\left(\int_{y=x^2-1}^{y=1-x^2}\left(3x-\tfrac{3}{2}\right)dy\right)dx\\ &= 3\int_{-1}^{1}(2x-1)(1-x^2)\,dx\\ &= 3\left[x^2-\tfrac{1}{2}x^4-x+\tfrac{1}{3}x^3\right]_{-1}^{1}\\ &= -4.\end{aligned}$$

Solution to Exercise 29

(a) Use cylindrical coordinates and note that the curved boundary of the paraboloid is given by the equation $z = h\rho^2/a^2$, which can also be expressed as $\rho = a\sqrt{z/h}$. The volume is given by the volume integral $\int_B 1\,dV$, which is evaluated in cylindrical coordinates by performing the ρ-integral before the z-integral, that is, the opposite order to Example 13, as this leads to slightly easier single integrals. Also, since the integrand and the boundary of B are independent of ϕ, the ϕ-integral can be evaluated independently of the z- and ρ-integrals, leading to an overall factor of 2π. Thus we have

$$\begin{aligned}\int_B 1\,dV &= 2\pi\int_{z=0}^{z=h}\left(\int_{\rho=0}^{\rho=a\sqrt{z/h}}\rho\,d\rho\right)dz\\ &= \frac{\pi a^2}{h}\int_0^h z\,dz\\ &= \tfrac{1}{2}\pi a^2 h.\end{aligned}$$

(b) Let the density of the uniform solid be D. Then

$$D = M\Big/\int_B 1\,dV = 2M/(\pi a^2 h).$$

The moment of inertia about the z-axis is given by

$$I = \int_B \rho^2 D\,dV,$$

and again, since the integrand and boundary of B are independent of ϕ, we have

$$\begin{aligned}I &= 2\pi\int_{z=0}^{z=h}\left(\int_{\rho=0}^{\rho=a\sqrt{z/h}}\rho^3 D\,d\rho\right)dz\\ &= 2\pi D\int_{z=0}^{z=h}\left(\left[\tfrac{1}{4}\rho^4\right]_{\rho=0}^{\rho=a\sqrt{z/h}}\right)dz\\ &= \tfrac{1}{2}\pi D\frac{a^4}{h^2}\int_0^h z^2\,dz\\ &= \tfrac{1}{6}\pi D a^4 h.\end{aligned}$$

Substituting for D then leads directly to $I = \tfrac{1}{3}Ma^2$, as required.

Solution to Exercise 30

(a) We use cylindrical coordinates and integrate over z first as was done in Example 13. The curved surface of the spherical cap is given by the equation $\rho^2 + z^2 = R^2$. Thus for a given ρ, the upper and lower limits of the z-integral are $\sqrt{R^2-\rho^2}$ and $R/2$, respectively. Moreover, the largest value of ρ, that is, the radius of the circular base of the cap, is given by $\sqrt{R^2-(R/2)^2} = \sqrt{3}R/2$. Hence the volume of the spherical cap is given by

$$\begin{aligned}
\int_{\text{spherical cap}} 1\,dV &= \int_{\rho=0}^{\rho=\sqrt{3}R/2} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=R/2}^{z=\sqrt{R^2-\rho^2}} \rho\,dz \right) d\phi \right) d\rho \\
&= 2\pi \int_0^{\sqrt{3}R/2} \left(\sqrt{R^2-\rho^2} - \tfrac{1}{2}R \right) \rho\,d\rho \\
&= 2\pi \left[-\tfrac{1}{3}(R^2-\rho^2)^{3/2} - \tfrac{1}{4}R\rho^2 \right]_0^{\sqrt{3}R/2} \\
&= 2\pi \left(\tfrac{1}{3} - \tfrac{1}{3}\left(\tfrac{1}{4}\right)^{3/2} - \tfrac{3}{16} \right) R^3 \\
&= \tfrac{5}{24}\pi R^3.
\end{aligned}$$

(b) Let D denote the density of the solid. Then the moment of inertia I of the spherical cap about the z-axis (axis of symmetry) is given by

$$\begin{aligned}
I &= \int_{\text{spherical cap}} \rho^2 D\,dV \\
&= \int_{\rho=0}^{\rho=\sqrt{3}R/2} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=R/2}^{z=\sqrt{R^2-\rho^2}} D\rho^3\,dz \right) d\phi \right) d\rho \\
&= 2\pi D \int_0^{\sqrt{3}R/2} \left(\sqrt{R^2-\rho^2} - \tfrac{1}{2}R \right) \rho^3\,d\rho.
\end{aligned}$$

Using the hint, we can write

$$\begin{aligned}
I &= 2\pi D \left[\tfrac{1}{5}(R^2-\rho^2)^{5/2} - \tfrac{1}{3}R^2(R^2-\rho^2)^{3/2} - \tfrac{1}{8}R\rho^4 \right]_0^{\sqrt{3}R/2} \\
&= 2\pi D R^5 \left(\tfrac{1}{5}\left(\tfrac{1}{4}\right)^{5/2} - \tfrac{1}{3}\left(\tfrac{1}{4}\right)^{3/2} - \tfrac{9}{128} - \tfrac{1}{5} + \tfrac{1}{3} \right) \\
&= \tfrac{53}{960}\pi D R^5.
\end{aligned}$$

The density D is the mass per unit volume. Hence

$$D = M \Big/ \int_{\text{spherical cap}} 1\,dV = 24M/(5\pi R^3),$$

and substituting this into the above expression for I gives

$$I = \frac{53\pi}{960} \times \frac{24M}{5\pi R^3} \times R^5 = \frac{53}{200} M R^2,$$

as required.

Solution to Exercise 31

This is the same surface as in Example 14, but over a different region of the (x, y)-plane. Thus $g(x, y) = \sqrt{1 + x^2 + y^2}$ and

$$\Sigma = \int_D \sqrt{1 + x^2 + y^2}\, dA,$$

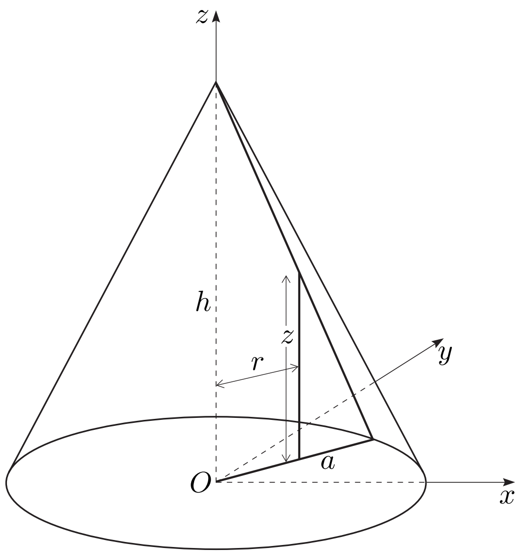
where D denotes the disc centred at the origin with radius a. Given the shape of the region D, the area integral is best evaluated using polar coordinates, noting that $\sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}$, which depends *only* on the radial coordinate r. Thus, from equation (10) (Subsection 2.2), we have

$$\Sigma = 2\pi \int_0^a \sqrt{1 + r^2}\, r\, dr = \tfrac{2}{3}\pi\big[(1 + r^2)^{3/2}\big]_0^a = \tfrac{2}{3}\pi\big((1 + a^2)^{3/2} - 1\big).$$

Again, we have used the trick given in equation (6), with $k = \frac{1}{2}$, $n = \frac{1}{2}$ and $f(r) = 1 + r^2$.

Solution to Exercise 32

A diagram of the cone is shown below.



The equation of the cone can easily be found by using similar triangles. Since $r = \sqrt{x^2 + y^2}$, we have

$$\frac{z}{a - r} = \frac{h}{a}.$$

Thus $z = h - hr/a$, or

$$z = h - \frac{h}{a}\sqrt{x^2 + y^2}.$$

The partial derivatives are therefore

$$\frac{\partial z}{\partial x} = \frac{-hx}{a\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-hy}{a\sqrt{x^2+y^2}},$$

and the area of the cone is given by

$$\begin{aligned}\Sigma &= \int_S \sqrt{1 + \frac{h^2x^2}{a^2(x^2+y^2)} + \frac{h^2y^2}{a^2(x^2+y^2)}}\, dA \\ &= \int_S \sqrt{\frac{a^2+h^2}{a^2}}\, dA,\end{aligned}$$

where S is a disc of radius a lying in the (x, y)-plane.

Note that the integrand $\sqrt{(a^2+h^2)/a^2}$ is constant throughout the region S, therefore we can write

$$\Sigma = \frac{\sqrt{a^2+h^2}}{a} \int_S 1\, dA.$$

But $\int_S 1\, dA$ is just the area πa^2 of region S. Hence

$$\Sigma = \frac{\sqrt{a^2+h^2}}{a} \times \pi a^2 = \pi a\sqrt{a^2+h^2}.$$