Unit 19

Systems of particles

Introduction

Figure 1 shows three positions of a diver performing a forward somersault, and the curves trace out the paths of the diver's head and feet. The motion of the diver is very complicated, which is reflected in the shapes of these curves; but in spite of this complexity, it is possible to obtain some simple information that adds considerably to our understanding of the dive. Associated with each position of the diver, there is a point known as the diver's *centre of mass*, and this point moves along a simpler path. (You met the idea of centre of mass in Unit 2.)

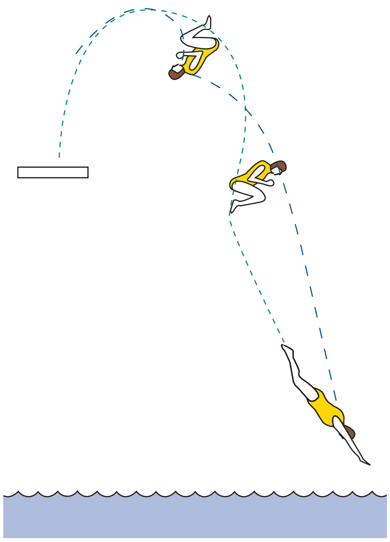


Figure 1 The paths of the head and feet of a diver

Our first objective in this unit is to obtain some useful information on the complicated motion of an object or system, and the concept of centre of mass is crucial to this process. It transpires that in a number of circumstances, we can justify modelling a complicated system, possibly composed of very many objects, by a single particle (of the same mass as the system) placed at the centre of mass of the system. This process has the dual benefit of making the behaviour of quite complicated systems much easier to understand and of considerably reducing the work involved when analysing them. It will form a central theme of the unit.

We start, in Section 1, by examining some specific examples of two-particle systems, from which we will be able to extract some general principles that apply to all two-particle systems and that can be extended to systems involving any number of particles. We move on, in Section 2, to discuss general systems of particles and the important concept of centre of mass. Then, in Section 3, we discuss the behaviour of objects during a collision, and here we introduce the principle of conservation of linear momentum. Section 4 concerns Newton's law of restitution, which models the effects of inelasticity in collisions.

1 Two-particle systems

The three mechanical systems illustrated in Figure 2 appear to have little in common. However, they all illustrate an important principle of mechanics – a principle that we intend to develop in this unit.

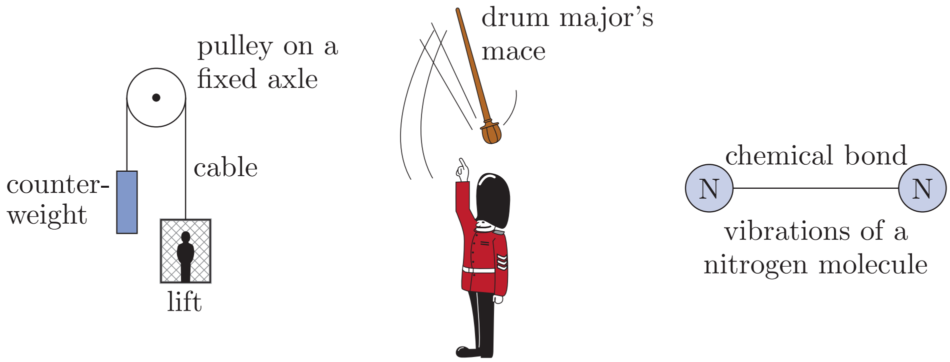


Figure 2 Three mechanical systems

In Subsection 1.1 we will use various techniques from previous units to examine each system in turn. Then, in Subsection 1.2, we will begin to draw together the ideas common to all three examples. This drawing together will lead to the statement of a general principle for two-particle systems, based on Newton's second law. First, we give a definition.

You have met two-particle systems before. For example, in Unit 2, Example 11, you saw how a scarf draped over the edge of a table can be modelled as a two-particle system. In Unit 11, many of the spring–mass systems are two-particle systems.

A **two-particle system** is a system modelled so that the total mass of the system is divided between two particles.

1.1 Examples of two-particle systems

A simple design for a lift

In this example we investigate a (rather impractical) design for a lift, as shown in Figure 3(a). The mechanism consists of a lift compartment and a counterweight, supported by a cable passing over a pulley. We wish to investigate this design with the simplifying assumptions that all frictional forces and air resistance can be ignored.

In order to model this system we will need to make some more simplifying assumptions. Figure 3(b) shows our mathematical model, which consists of two particles of masses m_1 and m_2 connected by a model string passing over a model pulley with its axle fixed at P. We assume that m_2 is greater than m_1, so when the system is released from rest, the lift (i.e. the particle of mass m_2) will begin to move down. We denote the vertical displacements of the particles of masses m_1 and m_2 from P by x_1 and x_2, respectively. Since the model string is inextensible, we have $x_1 + x_2 = \text{constant}$.

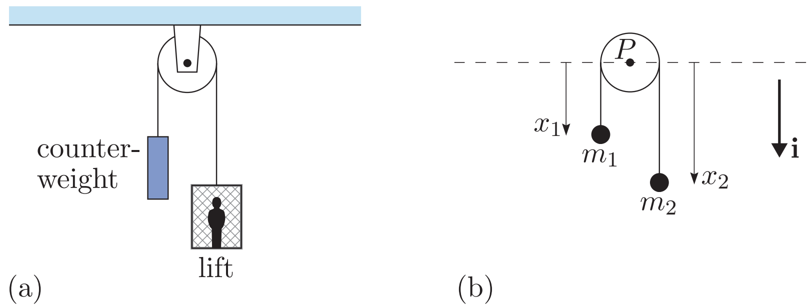


Figure 3 (a) A simple lift, and (b) a mathematical model for it

Before investigating the motion of this system, we will look at the system as a whole, in order to derive a result that will be generalised later. The force diagram in Figure 4 shows the forces acting on the two particles and on the model pulley.

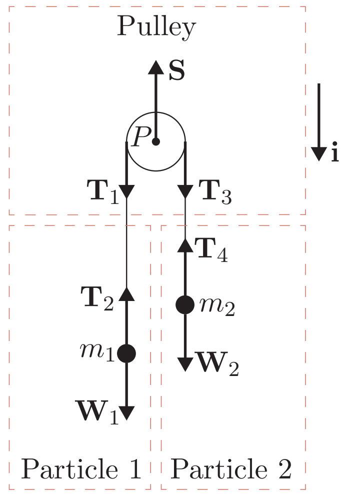


Figure 4 Force diagram for the simple lift

Now we apply Newton's second law to each particle. The accelerations of the particles of masses m_1 and m_2 are $\ddot{x}_1\mathbf{i}$ and $\ddot{x}_2\mathbf{i}$. The forces acting on the particles are $\mathbf{W}_1 + \mathbf{T}_2$ and $\mathbf{W}_2 + \mathbf{T}_4$, where $\mathbf{W}_1 = m_1g\mathbf{i}$ and $\mathbf{W}_2 = m_2g\mathbf{i}$. So Newton's second law gives

$$\mathbf{W}_1 + \mathbf{T}_2 = m_1g\mathbf{i} + \mathbf{T}_2 = m_1\ddot{x}_1\mathbf{i}, \tag{1}$$

$$\mathbf{W}_2 + \mathbf{T}_4 = m_2g\mathbf{i} + \mathbf{T}_4 = m_2\ddot{x}_2\mathbf{i}. \tag{2}$$

We also observe that the axle of the pulley is stationary (i.e. in equilibrium). Introducing the reaction force $\mathbf{S}$ to balance the other forces on the pulley, we have

$$\mathbf{S} + \mathbf{T}_1 + \mathbf{T}_3 = \mathbf{0}. \tag{3}$$

On adding equations (1), (2) and (3), we obtain

$$\mathbf{S} + \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4 + m_1g\mathbf{i} + m_2g\mathbf{i} = m_1\ddot{x}_1\mathbf{i} + m_2\ddot{x}_2\mathbf{i}.$$

The left-hand side of this equation simplifies because we have a model string, which implies that $\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}$ and $\mathbf{T}_3 + \mathbf{T}_4 = \mathbf{0}$. So we obtain

$$\mathbf{S} + m_1g\mathbf{i} + m_2g\mathbf{i} = m_1\ddot{x}_1\mathbf{i} + m_2\ddot{x}_2\mathbf{i}.$$

The right-hand side of this equation is almost of the mass times acceleration form that we expect from Newton's second law. A little rearrangement puts the equation in this form:

$$\mathbf{S} + m_1g\mathbf{i} + m_2g\mathbf{i} = \underbrace{(m_1 + m_2)}_{\text{the total mass}} \frac{m_1\ddot{x}_1 + m_2\ddot{x}_2}{m_1 + m_2}\,\mathbf{i}. \tag{4}$$

The significance of equation (4) will become clear shortly, but for the moment just notice that the right-hand side is the total mass of the system multiplied by an expression that has the dimensions of acceleration.

Exercise 1

This exercise completes the analysis of the simple design for a lift described above. Use the notation of Figure 4.

(a) Use the facts that the model string is inextensible (giving $x_1 + x_2 = \text{constant}$, $|\mathbf{T}_1| = |\mathbf{T}_2|$ and $|\mathbf{T}_3| = |\mathbf{T}_4|$) and that the pulley is a model pulley (so $|\mathbf{T}_1| = |\mathbf{T}_3|$) together with equations (1) and (2) to determine the acceleration of the lift, $\ddot{x}_2$.

(b) Use the result of part (a) and equation (2) to determine the magnitude of the tension in the cable.

(c) Use the result of part (b) and equation (3) to determine the magnitude of the reaction force on the pulley, $\mathbf{S}$.

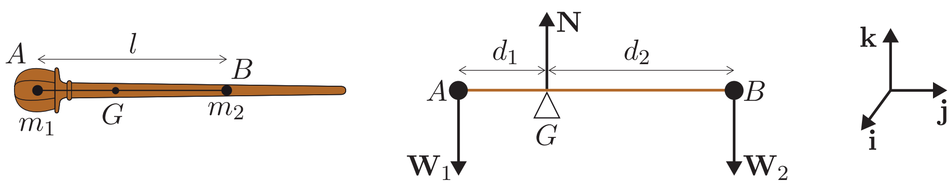
(d) Suppose that the lift and the counterweight are each of mass 1000 kg, and that a person of mass 65 kg steps into the lift. Calculate the numerical value of the acceleration of the lift and also the velocity of the lift after it starts from rest and travels 100 m. Comment on the suitability of the lift design.

A drum major's mace

In this example we attempt to model the motion of a drum major's mace. You may have seen a military band with the drum major striding out in front, twirling a long rod with a heavy weight at one end: his mace. Every few strides, he twirls the mace and then hurls it high into the air; it then appears to follow a complicated path until he finally catches it and continues with the twirling.

Recall from Unit 2 that a model rod is a rigid body with length but no breadth or depth. In this case we have a *light* model rod, so it has no mass either.

We model the mace as two particles A and B of unequal masses m_1 and m_2, connected by a light model rod, where l is the distance between the two masses, as in Figure 5(a). Our purpose is to try to discover some pattern in the behaviour of the mace (and perhaps provide some useful advice to would-be drum majors).



(a) (b)

Figure 5 (a) A drum major's mace. (b) A force diagram for the mace at rest on its balance point.

As you will see later, G is the centre of mass of the system.

We assume that the drum major holds the mace at its 'balance point' G, by which we mean that if we were to balance the model rod, plus the two particles, on a model pivot at G, then the rod would be in equilibrium.

The equilibrium condition for rigid bodies tells us that the sum of the torques on the rod when balanced in this way is zero. This situation and the associated forces are shown in Figure 5(b).

You met torques and the equilibrium condition for rigid bodies in Unit 2.

Take the origin for the torques at G, so that the torque due to $\mathbf{N}$ is zero. The position vectors of A and B are $\mathbf{r}_1 = -d_1\mathbf{j}$ and $\mathbf{r}_2 = d_2\mathbf{j}$. The weights are $\mathbf{W}_1 = -m_1g\mathbf{k}$ and $\mathbf{W}_2 = -m_2g\mathbf{k}$. The torque equilibrium equation is

$$\mathbf{r}_1 \times \mathbf{W}_1 + \mathbf{r}_2 \times \mathbf{W}_2 = (-d_1\mathbf{j}) \times (-m_1g\mathbf{k}) + (d_2\mathbf{j}) \times (-m_2g\mathbf{k}) = \mathbf{0}.$$

This simplifies to

$$d_1m_1\mathbf{i} - d_2m_2\mathbf{i} = \mathbf{0},$$

and resolving in the $\mathbf{i}$-direction gives $d_1m_1 - d_2m_2 = 0$, or

$$d_1/d_2 = m_2/m_1. \tag{5}$$

From this we see that the ratio d_1/d_2 will be small if m_1 is much larger than m_2. In other words, if the head of the mace is very heavy, then the balance point will be near this end.

Drum majors hold their maces near the end, whereas drum majorettes hold their batons near the centre. Presumably this is because a drum majorette's baton is more centrally balanced.

We now suppose that the drum major throws the mace into the air and at the same time gives it a flick in such a way that the mace turns end-over-end in a vertical plane. If we ignore the effects of air resistance, Figure 6 shows the forces acting on the mace and its position at time t (in terms of position vectors relative to an origin O, the point at which the mace was released). The rod gives rise to a force $\mathbf{R}_1$ on particle A and a force $\mathbf{R}_2$ on particle B.

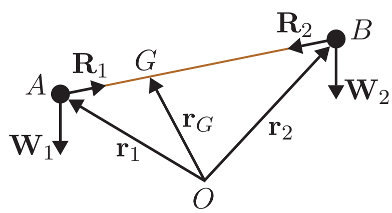


Figure 6 Force diagram for the mace in the air

It turns out that the position of G is of crucial importance. It will be useful later if we express the position vector of G in terms of the position vectors of A and B. Consider the displacement vectors $\overrightarrow{AG} = \mathbf{r}_G - \mathbf{r}_1$ and $\overrightarrow{GB} = \mathbf{r}_2 - \mathbf{r}_G$ (see Figure 6). We know that $|\overrightarrow{AG}| = d_1$ and $|\overrightarrow{GB}| = d_2$, so $|\overrightarrow{AG}|/|\overrightarrow{GB}| = d_1/d_2 = m_2/m_1$, by equation (5). Hence $m_1\,\overrightarrow{AG} = m_2\,\overrightarrow{GB}$, or equivalently,

$$m_1(\mathbf{r}_G - \mathbf{r}_1) = m_2(\mathbf{r}_2 - \mathbf{r}_G),$$

which can be rearranged to give the position vector of G as

$$\mathbf{r}_G = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}. \tag{6}$$

Differentiating equation (6) twice with respect to time, we obtain the following expression for the acceleration of the point G in terms of the accelerations $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$ of the points A and B:

$$\ddot{\mathbf{r}}_G = \frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2}. \tag{7}$$

The total force on particle A is $\mathbf{R}_1 + \mathbf{W}_1$, and the total force on particle B is $\mathbf{R}_2 + \mathbf{W}_2$. So, applying Newton's second law to particles A and B in turn, we obtain

$$\mathbf{R}_1 + \mathbf{W}_1 = \mathbf{R}_1 - m_1g\mathbf{k} = m_1\ddot{\mathbf{r}}_1,$$
$$\mathbf{R}_2 + \mathbf{W}_2 = \mathbf{R}_2 - m_2g\mathbf{k} = m_2\ddot{\mathbf{r}}_2.$$

Adding these equations gives

$$\mathbf{R}_1 - m_1 g\mathbf{k} + \mathbf{R}_2 - m_2 k\mathbf{k} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2.$$

But as the mace is modelled by a light model rod, $\mathbf{R}_1 + \mathbf{R}_2 = \mathbf{0}$, so this equation simplifies to

$$-m_1 g\mathbf{k} - m_2 g\mathbf{k} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2.$$

This can be rearranged as

$$-(m_1 + m_2)g\mathbf{k} = (m_1 + m_2)\,\frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2}.$$

Substituting from equation (7), we obtain

$$\ddot{\mathbf{r}}_G = -g\mathbf{k},$$

which is the equation of motion of the point G.

This result may appear to be simple, but it is really quite surprising. It tells us that the balance point G – the centre of mass of the mace – behaves as if it were a particle acted on solely by the force of gravity. Thus if the initial velocity of G is vertically upwards, then G will simply move up and down in a vertical line. However, if the initial velocity has a horizontal component, then G will follow the parabolic trajectory of a projectile. In both cases, the other points on the mace may move along much more complicated trajectories. Figure 7 illustrates the simple parabolic trajectory of G and the more complicated trajectory of another point on the mace when there is a horizontal component to the initial velocity.

See Unit 3 for details about projectiles. We assume here that air resistance can be neglected.

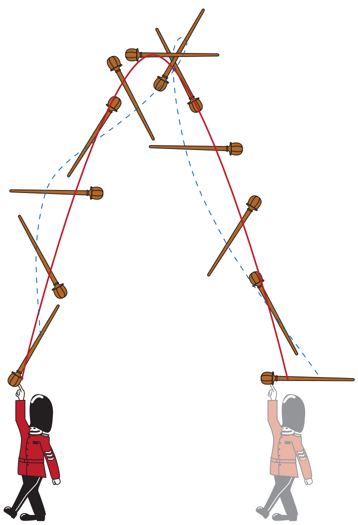


Figure 7 Trajectory of the balance point

What does all this mean for the drum major? First, if he wants to hold the mace horizontally, then he should hold it at its balance point G, where he will not need to apply any torques to keep it level. Second, if he wants to be able to catch it after throwing it into the air, then catching it at G would be best, as G is the only point whose trajectory in the air he can easily predict.

So the drum major knows where to hold and catch his mace. But how should he throw it? When he is stationary, it is clear that the initial upwards velocity of G should be vertical; but what if he is marching? Should he throw the mace upwards and slightly forwards? The answer is no, he should simply throw it upwards. This is because when the drum major is marching (at a constant velocity) holding the mace, the mace has a constant horizontal component of velocity that is the same as that of the drum major. Then the horizontal distance travelled by the mace, after being thrown upwards, will be the same as that travelled by the marching drum major. So the mace should return to hand level just as the drum major's hand gets there. If the mace were thrown slightly forwards, then the drum major would have to increase his marching pace to match, or else the mace would, embarrassingly, fall to the ground. However, when throwing the mace (vertically upwards), the drum major needs to make sure that he keeps his marching velocity constant in order to catch the mace safely. It would be a mistake, for example, to throw the mace while marching and then stop to wait for it to fall, or to change direction while the mace is in the air.

Vibrations of a nitrogen molecule

A nitrogen molecule consists of two nitrogen atoms, each of mass m, joined by a chemical bond, as illustrated in Figure 8(a). The atoms vibrate relative to each other along the direction of this bond. The molecule can be modelled as a pair of particles joined by a model spring, which exerts a force $\mathbf{H}_1$ on the left-hand particle, labelled A, and a force $\mathbf{H}_2$ on the right-hand particle, labelled B, as illustrated in Figure 8(b). Here we are concerned with the vibrations of the molecule in isolation and hence do not consider the effects of gravity (for example, you could consider that the molecule is vibrating in deep space, far away from any gravitational influences). So the total force acting on the *system* is $\mathbf{F} = \mathbf{H}_1 + \mathbf{H}_2$.

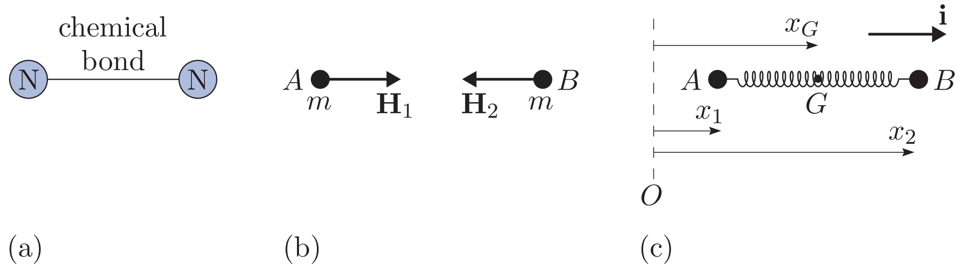


Figure 8 (a) A nitrogen molecule, (b) its force diagram, and (c) a spring–mass system model for vibrations of the molecule

You saw how to model the vibrations of such a system in terms of normal modes in Unit 11. Here we want to look at what information we can glean from the motion of the centre G of the spring, which is the geometric centre of this symmetric system. We begin as usual by defining axes: we choose a fixed origin O and a direction $\mathbf{i}$ aligned with the spring. We measure x_1, x_2 and x_G along this direction, as shown in Figure 8(c).

As you will see later, the centre of the spring is the centre of mass of the system.

To derive the equation of motion of G, we start by expressing the position vector of G relative to O. From equation (6) with $m_1 = m_2$, we have $\mathbf{r}_G = (\mathbf{r}_1 + \mathbf{r}_2)/2$.

Applying Newton's second law to particles A and B in turn gives

$$\mathbf{H}_1 = m\ddot{\mathbf{r}}_1, \quad \mathbf{H}_2 = m\ddot{\mathbf{r}}_2.$$

Proceeding as before, we add together the above equations to obtain

$$\mathbf{H}_1 + \mathbf{H}_2 = m\ddot{\mathbf{r}}_1 + m\ddot{\mathbf{r}}_2.$$

But $\mathbf{H}_1$ and $\mathbf{H}_2$ are forces exerted by the same model spring, so $\mathbf{H}_1 = -\mathbf{H}_2$. Hence the left-hand side of the above equation is zero, so

You saw this before, in Unit 11.

$$\mathbf{0} = m(\ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2).$$

We now write the right-hand side of this equation in the form of Newton's second law, that is, a mass multiplied by the acceleration of a single point. To do this, we differentiate $\mathbf{r}_G = (\mathbf{r}_1 + \mathbf{r}_2)/2$ twice to obtain $\ddot{\mathbf{r}}_G = (\ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2)/2$.

Substituting for $\ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2$ in the above equation then gives

$$\underbrace{\mathbf{0}}_{\text{total force}} = \underbrace{2m}_{\text{total mass}} \times \underbrace{\ddot{\mathbf{r}}_G}_{\text{acceleration}}.$$

Now $\mathbf{r}_G = x_G\mathbf{i}$, since the motion is one-dimensional. So $\ddot{x}_G = 0$, and this can be integrated to give $\dot{x}_G =$ constant, that is, G moves at a constant speed along the direction of the axis of the spring. The net result is that the geometric centre of the molecule moves just like a particle with no forces acting on it, that is, it either remains at rest or travels with constant speed along a straight line (Newton's first law). It is worth emphasising that the motion of G remains the same irrespective of how much the atoms at the ends of the bond vibrate. We have not discussed the motion of the individual particles, which will depend on the forces acting on them.

Newton's first law was stated in Units 2 and 3.

1.2 Some general conclusions

Now let us try to extract some general principles from the three examples of two-particle systems discussed in the previous subsection.

The first point to note is that when we obtained an expression for the total force on the system, certain forces on the components of the system balance due to properties of model springs and rods, and Newton's third law. The forces that balance in this way are known as **internal forces** of the system, while the remaining forces are referred to as **external forces**.

The internal forces for the three examples from the previous subsection are shown as yellow arrows in Figure 9, and the external forces are shown as blue arrows. Notice that internal forces occur in equal and opposite pairs (i.e. each force in the pair has the same magnitude but opposite direction to the other). We have altered the notation of the previous subsection slightly in Figure 9 to reflect this.

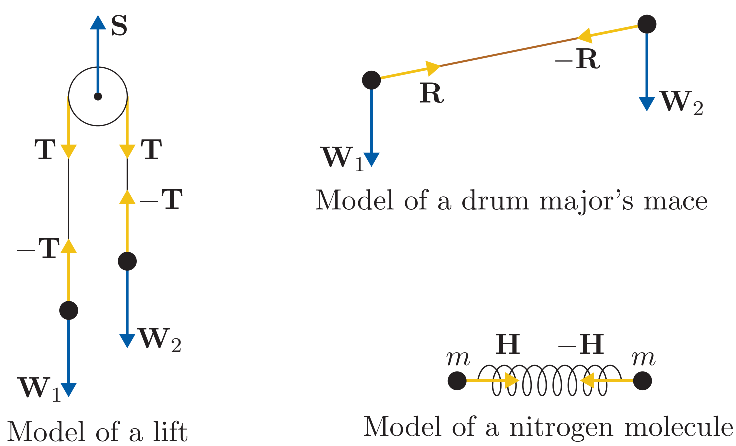


Figure 9 Internal and external forces

Exercise 2

Consider the system of weights, pulleys and strings shown in Figure 10(a). Assume that the pulleys are model pulleys (i.e. light and frictionless) and that the strings are model strings (i.e. light and inextensible). Modelling the weights as particles, we can therefore model the system and the forces acting on it as shown in Figure 10(b).

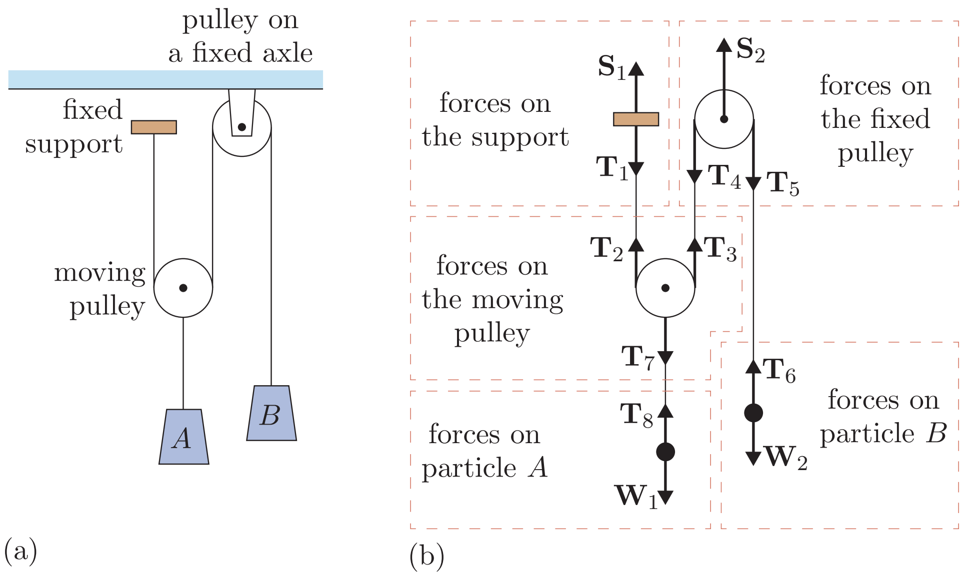


Figure 10 (a) A system of weights and pulleys, and (b) its force diagram

Decide which of the forces are internal and which are external. Hence write down the total force acting on the whole system.

In all three examples of two-particle systems in the previous subsection, we not only obtained an expression for the total force $\mathbf{F}$ on the system, but also used Newton's second law to obtain an equation involving that force. In the lift example we obtained equation (4):

$$\mathbf{F} = (m_1 + m_2)\,\frac{m_1\ddot{x}_1 + m_2\ddot{x}_2}{m_1 + m_2}\,\mathbf{i}.$$

If we write $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$, so that $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the lift and counterweight particles relative to the point P in Figure 3, then we obtain

$$\mathbf{F} = (m_1 + m_2)\,\frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2}.$$

This is exactly the equation that we obtained for the drum major's mace, where again $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the two particles.

For the nitrogen molecule we obtained

$$\mathbf{0} = 2m\,\ddot{\mathbf{r}}_G,$$

where $\mathbf{r}_G$ is the position vector of the geometric centre of the position of the two particles, $\mathbf{r}_G = (\mathbf{r}_1 + \mathbf{r}_2)/2$. At first sight this looks different, but it is of the same form because there are no external forces acting (so $\mathbf{F} = \mathbf{0}$), both particles have the same mass (so $m_1 + m_2 = 2m$), and also

$$\frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2} = \frac{m\ddot{\mathbf{r}}_1 + m\ddot{\mathbf{r}}_2}{m + m} = \frac{\ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2}{2} = \ddot{\mathbf{r}}_G.$$

In summary, for each two-particle system, the right-hand side of the Newton's second law equation is a product of the total mass of the system and a term of the form

$$\frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2},$$

which is the acceleration of the point G, with position vector

$$\mathbf{r}_G = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2},$$

within that system. We call this point the **centre of mass**.

In the case of symmetric systems (where the two particles have equal mass), such as the nitrogen molecule, the centre of mass is at the geometric centre. For systems where the particles are joined by a light model rod, such as the drum major's mace, the centre of mass is at the balance point.

Exercise 3

The mass of the Earth is approximately 5.97×10^{24} kg, and its diameter is about 1.27×10^4 km, while the mass of the Moon is approximately 7.34×10^{22} kg, and its diameter is about 3.5×10^3 km. The mean distance from the centre of the Earth to the centre of the Moon is about 3.84×10^5 km.

How far above the Earth's surface would you estimate the centre of mass of the Earth and the Moon to lie?

The equations obtained by applying Newton's second law to our examples, which led to the definition of centre of mass, are all of the form $\mathbf{F} = M\ddot{\mathbf{r}}_G$, where $\mathbf{F}$ is the total force on the system and M is its total mass. We should now verify that this equation holds for the centre of mass of *any* two-particle system.

Consider a general system consisting of two particles, labelled 1 and 2, of masses m_1 and m_2, and with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to an origin O. In general, each particle is subject to internal and external forces. Let us denote the resultant of the external forces on each particle by $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. The internal force on each particle is exerted by the other particle by means of a model string, a model rod or a model spring, or in some other way. Let us denote the internal force exerted on particle 1 by particle 2 by $\mathbf{I}_{12}$, and similarly denote the internal force exerted on particle 2 by particle 1 by $\mathbf{I}_{21}$. The situation is illustrated in Figure 11.

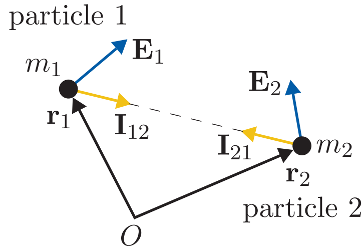


Figure 11 Internal and external forces

Applying Newton's second law to each particle, we obtain

$$\mathbf{E}_1 + \mathbf{I}_{12} = m_1\ddot{\mathbf{r}}_1, \quad \mathbf{E}_2 + \mathbf{I}_{21} = m_2\ddot{\mathbf{r}}_2.$$

We can add these equations to obtain

$$\mathbf{E}_1 + \mathbf{I}_{12} + \mathbf{E}_2 + \mathbf{I}_{21} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2.$$

Since $\mathbf{I}_{12}$ and $\mathbf{I}_{21}$ are internal forces, we can apply Newton's third law to obtain $\mathbf{I}_{12} + \mathbf{I}_{21} = \mathbf{0}$. So the above equation simplifies to

$$\mathbf{F}^{\text{ext}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2, \tag{8}$$

where $\mathbf{F}^{\text{ext}} = \mathbf{E}_1 + \mathbf{E}_2$ is the sum of the external forces.

The centre of mass of this system is defined by the position vector

$$\mathbf{r}_G = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M},$$

where $M = m_1 + m_2$ is the total mass of the system.

Differentiating twice gives

$$\ddot{\mathbf{r}}_G = \frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{M}. \tag{9}$$

From equations (8) and (9), we have

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{r}}_G,$$

which is an important result that is worth re-stating.

Motion of the centre of mass of a two-particle system

The motion of the centre of mass G with position vector $\mathbf{r}_G$ of a two-particle system of total mass M subject to external forces with sum $\mathbf{F}^{\text{ext}}$ is given by

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{r}}_G. \tag{10}$$

Example 1

Two particles of masses m_1 and m_2 are attached to a model spring of natural length l_0 and stiffness k (as shown in Figure 12). The particle–spring system moves vertically.

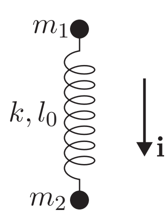


Figure 12 Two particles joined by a model spring

Determine the equation of motion of the centre of mass of the system. What would be the effect on this equation of motion if the stiffness of the spring were doubled?

Solution

The spring forces are internal forces. The only external forces are the weights of the two particles, given by $m_1 g\mathbf{i}$ and $m_2 g\mathbf{i}$. Thus equation (10) gives

$$(m_1 + m_2)g\mathbf{i} = (m_1 + m_2)\ddot{\mathbf{r}}_G, \tag{11}$$

where $\mathbf{r}_G$ is the position vector of the centre of mass of the system. This simplifies to $\ddot{\mathbf{r}}_G = g\mathbf{i}$, so the centre of mass moves as if it is a particle falling under gravity.

Since the spring forces are internal forces, they have no effect on the equation of motion (11), so changing the stiffness of the spring will have no effect on the equation of motion of the centre of mass of the system.

Exercise 4

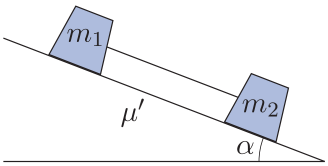


Figure 13 Two objects connected by a light model rod, on an inclined plane

Two objects of masses m_1 and m_2, connected by a light model rod, are sliding down an inclined plane in the direction of the axis of the rod, as shown in Figure 13. The coefficient of sliding friction between each object and the plane is μ'. The objects may be modelled as particles and the connecting rod as a model rod.

(a) Draw a force diagram for the system, and identify any internal forces.

(b) Determine the equation of motion for the centre of mass of this system.

(c) How would your solution to part (b) change in the following situations?

- The rod is replaced by a light spring.
- The rod is removed, leaving just two disconnected objects.

(d) Compare your equation of motion in part (b) with that for a single particle of mass $m_1 + m_2$ sliding down the same inclined plane.

The two-particle system in Exercise 4 is actually a rigid body and does not rotate, so its motion is completely described by the equation of motion of its centre of mass. The drum major's mace is also a rigid body, but it is subject to torques to make it rotate as the drum major twirls it. Nevertheless, the motion of the centre of mass still provides useful information for the drum major. In other cases, knowing about the motion of the centre of mass may not be directly useful. However, as the following example illustrates, it can form the basis for an analysis of the complete motion of such a system.

Example 2

Figure 14(a) shows two particles A and B, each of mass m, lying on a smooth horizontal surface and connected by a model spring of natural length $2l_0$ and stiffness k.

Initially, the particles are at rest a distance $2l_0$ apart, so the spring is unstretched, and with centre of mass at the origin. Then a constant horizontal force $\mathbf{P}$ is applied to B in the direction AB, as shown in Figure 14(b). At time t, the particles are a distance $2d$ apart.

It is important to measure coordinates from a *fixed* point O. It is tempting to measure distances from the position of G or A or B, but as the whole system is moving, none of these is appropriate.

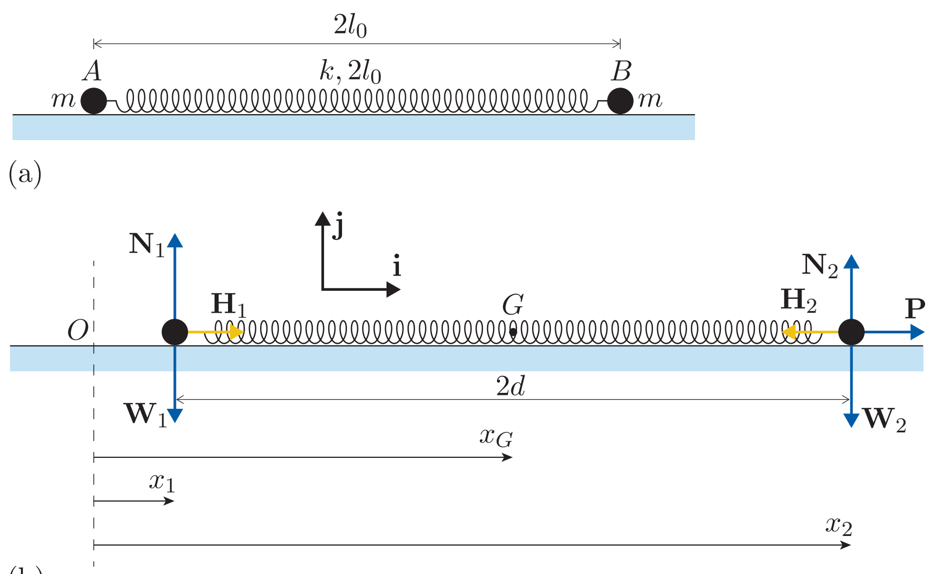


Figure 14 A system of two masses connected by a spring, subjected to a constant force $\mathbf{P}$ at one end: (a) unstretched spring, (b) stretched spring

(a) Find the position of the centre of mass at time t.

(b) Find a differential equation for d.

(c) Find the x-coordinates of A and B at time t.

(d) Interpret your solution.

Solution

(a) The force diagrams are shown in Figure 14(b). The two spring forces are internal forces, so equation (10) gives

$$\mathbf{P} + \mathbf{N}_1 + \mathbf{W}_1 + \mathbf{N}_2 + \mathbf{W}_2 = 2m\ddot{x}_G\mathbf{i}.$$

But both particles are in vertical equilibrium, so $\mathbf{N}_1 + \mathbf{W}_1 = \mathbf{0}$ and $\mathbf{N}_2 + \mathbf{W}_2 = \mathbf{0}$. Thus resolving in the **i**-direction, we find that the centre of mass moves to the right with constant acceleration

$$\ddot{x}_G = \frac{|\mathbf{P}|}{2m}. \qquad (12)$$

Note that $\mathbf{N}_1$ and $\mathbf{W}_1$ are not internal forces, so we cannot invoke Newton's third law to obtain $\mathbf{N}_1 + \mathbf{W}_1 = \mathbf{0}$; this equation is true because the particles are not moving vertically.

Since the system starts from rest with the centre of mass at the origin, integrating twice gives

$$x_G = \frac{|\mathbf{P}|}{4m}t^2.$$

Alternatively, we can use the constant acceleration equation $x = x_0 + v_0t + \frac{1}{2}a_0t^2$ from Unit 3, with $x_0 = 0$, $v_0 = 0$ and $a_0 = \ddot{x}_G$.

(b) Applying Newton's second law to particle B gives

$$\mathbf{H}_2 + \mathbf{P} = m\ddot{x}_2\mathbf{i}.$$

If we started by applying Newton's second law to particle A, we would obtain the same differential equation for d.

Now we need to model the forces. We are given that the force $\mathbf{P}$ acts in the direction AB, so $\mathbf{P} = |\mathbf{P}|\mathbf{i}$. Also, by Hooke's law, for a spring of length l, natural length $2l_0$ and stiffness k, and a unit vector $\widehat{\mathbf{s}}$ in the direction from the particle to the centre of the spring, we have

$$\mathbf{H}_2 = k(l - 2l_0)\,\widehat{\mathbf{s}} = k(2d - 2l_0)(-\mathbf{i}).$$

Therefore we obtain

$$-2k(d - l_0)\mathbf{i} + |\mathbf{P}|\mathbf{i} = m\ddot{x}_2\mathbf{i}.$$

Resolving in the **i**-direction and rearranging gives

$$m\ddot{x}_2 + 2kd = 2kl_0 + |\mathbf{P}|. \tag{13}$$

To proceed further, we need to express the acceleration $\ddot{x}_2$ of particle B in terms of d and known parameters. We start by noting that throughout the motion, the centre of mass is at the point

$$x_G\mathbf{i} = \frac{mx_1\mathbf{i} + mx_2\mathbf{i}}{2m} = \tfrac{1}{2}(x_1 + x_2)\mathbf{i},$$

that is, the centre of mass is always halfway between A and B.

This gives $x_2 = x_G + d$, which we can differentiate twice to obtain $\ddot{x}_2 = \ddot{x}_G + \ddot{d}$. Now we can use equation (12) to obtain $\ddot{x}_2 = |\mathbf{P}|/(2m) + \ddot{d}$, which is the desired expression for $\ddot{x}_2$ in terms of d and known parameters. Substituting this into equation (13) and rearranging gives

$$m\ddot{d} + 2kd = 2kl_0 + |\mathbf{P}|/2, \tag{14}$$

which is the required differential equation for d.

(c) You should recognise equation (14) as the differential equation for simple harmonic motion, which has solution

See Unit 9.

$$d = C_1\cos(\omega t) + C_2\sin(\omega t) + l_0 + |\mathbf{P}|/(4k),$$

where $\omega = \sqrt{2k/m}$ and the values of the constants C_1 and C_2 are determined by initial conditions. When $t = 0$, before the force $\mathbf{P}$ is applied, we have $d = l_0$ and $\dot{d} = 0$. This gives $C_1 = -|\mathbf{P}|/(4k)$ and $C_2 = 0$, so

$$d = -\frac{|\mathbf{P}|}{4k}\cos(\omega t) + l_0 + \frac{|\mathbf{P}|}{4k}.$$

We use $x_2 = x_G + d$ together with $x_G = |\mathbf{P}|t^2/(4m)$ and the above expression for d, to obtain

$$x_2 = \frac{|\mathbf{P}|}{4m}t^2 + \left(l_0 + \frac{|\mathbf{P}|}{4k} - \frac{|\mathbf{P}|}{4k}\cos(\omega t)\right).$$

Similarly, $x_1 = x_G - d$, so we have

$$x_1 = \frac{|\mathbf{P}|}{4m}t^2 - \left(l_0 + \frac{|\mathbf{P}|}{4k} - \frac{|\mathbf{P}|}{4k}\cos(\omega t)\right).$$

(d) The equations for x_1 and x_2 show that particle A oscillates according to the function $(|\mathbf{P}|/(4k))\cos(\omega t)$ about a point a fixed distance $l_0 + |\mathbf{P}|/(4k)$ to the left of the moving position $x_G = |\mathbf{P}|t^2/(4m)$ of the centre of mass, and that particle B oscillates according to the function $-(|\mathbf{P}|/(4k))\cos(\omega t) = (|\mathbf{P}|/(4k))\cos(\omega t + \pi)$ about a point the same fixed distance to the right of the moving position of the centre of mass.

The oscillations are simple harmonic with the same angular frequency $\omega = \sqrt{2k/m}$ and amplitude $|\mathbf{P}|/(4k)$, but are phase-opposed (in the terminology of Unit 11) because the terms (in large brackets above) describing the oscillations have different signs; this means that, relative to the centre of mass, the particles are always moving in opposite directions.

The situation is illustrated in Figure 15.

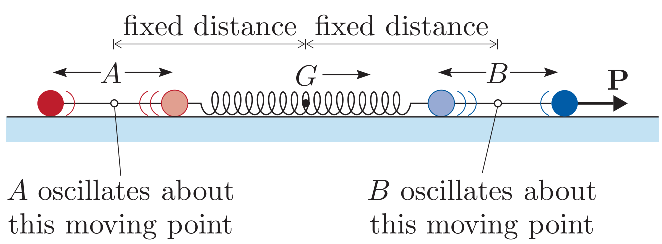


Figure 15 The effect of applying a force $\mathbf{P}$ to one end of a spring–mass system

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Exercise 5

Suppose that the model spring in Example 2 is replaced by a light model rod of length $2l_0$.

What can you say about the motion of the centre of mass of this new system, and about the motion of A and B? What forces are exerted by the rod on particles A and B?

Exercise 6

Two particles A and B, each of mass m, move along the x-axis, and although each particle exerts a force on the other, there are no external forces. Initially, A is at rest at the origin, while B is at $x = 3$ and moving away from the origin with speed $6\,\mathrm{m\,s^{-1}}$.

If particle B returns to its initial position at time $t = 0.5$, where is A at this time?

Exercise 7

Long ago and in a galaxy far, far away, an alien was alone in his spaceship, floating aimlessly through the void. He cursed his luck for running out of fuel, but then came up with a brilliant idea. Fortunately, he weighed half as much as the entire ship and he was quick on his (many) feet, so, starting from the back of the ship, he would run as fast as he could and hurl himself at the forward bulkhead. If he did this a few thousand times, he thought, the ship would gradually pick up speed and he would eventually arrive home. Comment on his chances of success.

Exercise 8

Two particles A and B, each of mass 50 kg, are at rest on a smooth horizontal surface and connected by a model string, which is stretched tightly between them. The breaking strain of the string is 10 N (i.e. it cannot sustain a tension greater than this value). A horizontal force $\mathbf{P}$ is applied to B in the direction AB, as shown in Figure 16.

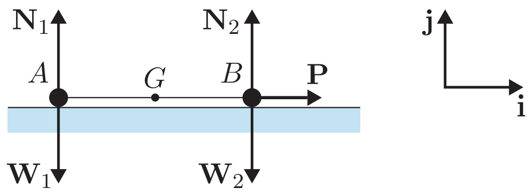

Figure 16 Two particles connected by a model string

What is the greatest magnitude of the force $\mathbf{P}$ that can be applied to B before the string breaks?

Exercise 9

Two particles A and B slide along a straight rough horizontal wire, and the coefficient of sliding friction between each of the particles and the wire is $\mu' = 0.4$. Particle A has mass 3 kg and particle B has mass 0.5 kg, and each particle exerts a force on the other.

Find the acceleration of the centre of mass when both particles move along the wire in the same direction.

Exercise 10

Two particles move along a smooth straight horizontal track, and each exerts a force on the other. The first particle has mass 4 kg and experiences no external forces. The second particle has mass 1 kg and is pulled along the track by an external force of magnitude 20 N. It is found that the two particles accelerate along the track a fixed distance apart.

Find the common acceleration of the particles and the magnitude of the internal forces.

2 Many-particle systems

In this section we generalise the ideas of Section 1 to systems of more than two particles. We also look at how to find the centre of mass of a variety of different systems and solid objects.

2.1 Motion of the centre of mass

We generalise here the results of Subsection 1.2 to systems with more than two particles. In fact, we consider a system of n particles, where n may be very large.

An n-**particle system** is a system modelled so that the total mass of the system is divided among n particles.

We assume that particle i has mass m_i and that its position vector is $\mathbf{r}_i$ (relative to some fixed origin O). We begin our discussion by extending the notion of the centre of mass to a system of n particles.

The **centre of mass** of an n-particle system, whose particles have masses $m_1, m_2, \ldots, m_n$, and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ relative to an origin O, has position vector

$$\mathbf{r}_G = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^n m_i\mathbf{r}_i}{M} \tag{15}$$

relative to O, where $M = \sum_{i=1}^n m_i$ is the total mass of the system.

Exercise 11

Determine the centre of mass of each of the systems of particles in Figure 17, where each particle has the same mass m.

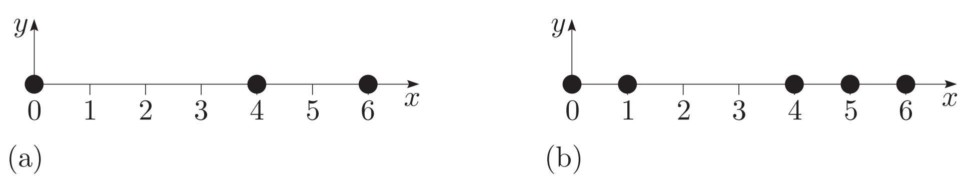

Figure 17 Particles on the x-axis

Each particle may be subject to a number of external forces, but these can always be added together to give a single resultant force, so we may assume that particle i is acted on by a single external force $\mathbf{E}_i$. The total external force on the whole system will then be

$$\mathbf{F}^{\text{ext}} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n = \sum_{i=1}^{n} \mathbf{E}_i. \tag{16}$$

In addition to the external forces, we assume that each particle exerts a force on every other (and that this force acts along the line joining them). Each particle is therefore acted on by $n - 1$ internal forces. We denote the force acting on particle i due to particle j by $\mathbf{I}_{ij}$. Then (from Newton's third law) we have

$$\mathbf{I}_{ij} = -\mathbf{I}_{ji}. \tag{17}$$

It will make the summations that follow easier to write if we also introduce a term $\mathbf{I}_{ii}$ (the force exerted by the ith particle on itself), which we assume to be zero.

Now we apply Newton's second law to the ith particle and obtain

$$\underbrace{\mathbf{E}_i}_{\substack{\text{external force}\\ \text{on the } i\text{th particle}}} + \underbrace{\sum_{j=1}^{n} \mathbf{I}_{ij}}_{\substack{\text{sum of the internal}\\ \text{forces on the}\\ i\text{th particle}}} = \underbrace{m_i}_{\substack{\text{mass of the}\\ i\text{th particle}}} \times \underbrace{\ddot{\mathbf{r}}_i}_{\substack{\text{acceleration of}\\ \text{the } i\text{th particle}}}.$$

Adding together all of the equations like this, for all choices of i, we obtain

$$\underbrace{\sum_{i=1}^{n} \mathbf{E}_i}_{\substack{\text{sum of all the}\\ \text{external forces}\\ \text{on the system}}} + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{I}_{ij}}_{\substack{\text{sum of all the}\\ \text{internal forces}\\ \text{in the system}}} = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i. \tag{18}$$

The first term on the left-hand side is simply $\mathbf{F}^{\text{ext}}$, by equation (16). The second term, involving a double summation, looks rather complicated. But it simply represents the sum of all the internal forces in the system. From equation (17) we know that these forces balance in pairs, or are zero when $i = j$. The net result is that this term is zero. We now use equation (15) to simplify the right-hand side of equation (18), and we are left with the following important result.

Motion of the centre of mass of an n-particle system

The motion of the centre of mass $\mathbf{r}_G$ of an n-particle system of total mass M subject to external forces with sum $\mathbf{F}^{\text{ext}}$ is given by

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{r}}_G. \tag{19}$$

Notice that this formula is identical to the one for the two-particle case given by equation (10).

Thus the centre of mass of an n-particle system moves as if it is a particle with mass the same as the total mass of the system and with all the external forces on the system applied to this particle.

Equation (19) is critical to the modelling of real objects as single particles. It tells us that an object composed of a very large number of components (modelled as particles) can be modelled by a single particle. However, if we do this, then the single-particle model will predict only the motion of the centre of mass. For example, consider the diver mentioned in the Introduction, moving under gravity alone. As you know from Unit 3, the single-particle model predicts that the particle (i.e. the diver's centre of mass) moves along a parabolic path. This model does not give any information about the motion of any other part of the diver, which may move along a complicated path (see Figure 1).

2.2 Locating the centre of mass

In this subsection you will see how to find the centre of mass of a variety of systems and objects. One method for finding the centre of mass is to use the definition directly (equation (15)). If the system is symmetric, then we can locate the centre of mass by finding the geometric centre (this was the method used in Unit 2).

Try the following exercise to find the centre of mass of two systems.

Exercise 12

Determine the centre of mass of each of the systems of particles in Figure 18, where each particle has the same mass m.

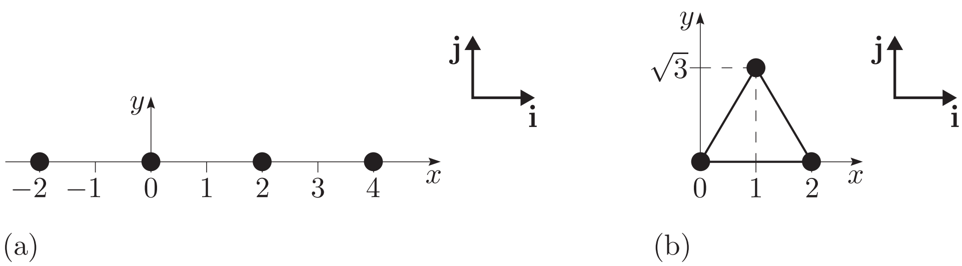

Figure 18 Two systems of particles

So far, we have looked at finding the centres of mass only of systems of particles. But you saw in Unit 2 that sometimes it is more appropriate to model an object as a rigid body. To find the centre of mass of a rigid body, we simply suppose that the object is made up of a very large number of very small chunks of material, each of which can be modelled as a particle – that is, we model the object as a system of particles. Then we can apply equation (15) to find its centre of mass. However, this poses questions about how many particles we should use, what mass to give them, and where to locate them.

Recall that *density* is mass per unit volume.

In general, this is a difficult problem. But there are many objects whose physical properties can be used to help in this process. In particular, many objects are of *uniform density*, in that the mass of any small chunk of the object is proportional to its volume. Such an object is said to be *homogeneous*.

The use of 'homogeneous' here is different from its use in the context of differential equations.

A **homogeneous rigid body** is a rigid body of uniform density.

For a symmetric homogeneous rigid body, we can see at once that the centre of mass is at the geometric centre. We exploited this fact in Unit 2 in locating the centres of mass of objects with some simple geometric shapes. Figure 19 shows six examples of symmetric homogeneous rigid bodies with their centres of mass G marked. Notice that the centre of mass in each case is located on an axis of symmetry – this is true in general of any symmetric homogeneous rigid body.

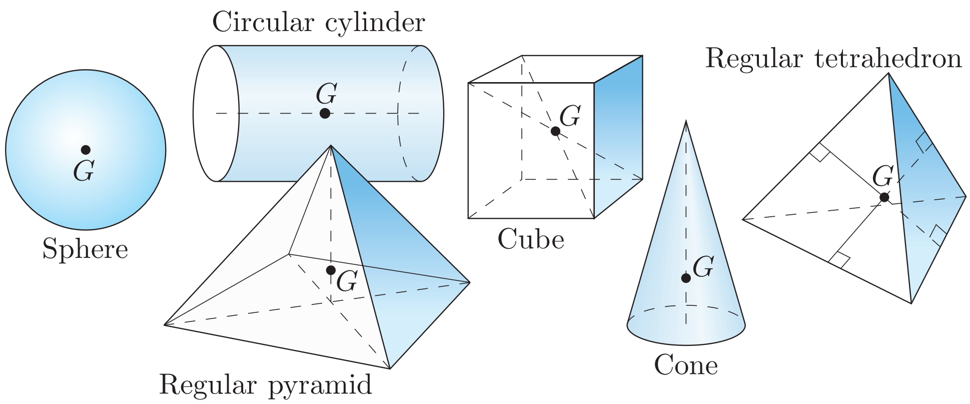

Figure 19 Six common solids, where G is the centre of mass in each case

A homogeneous rigid body that is not symmetric can often be broken down into symmetric parts. The centre of mass of each symmetric part is at its geometric centre. If we model each part as a particle of appropriate mass located at its geometric centre, then we can use equation (15) to find the centre of mass of the whole rigid body.

Example 3

Determine the centre of mass of the L-shaped piece of uniform cardboard shown in Figure 20.

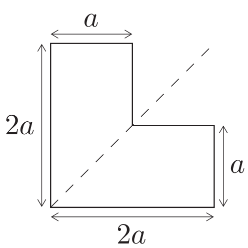

Figure 20 An L-shaped region

Solution

Since the cardboard is uniform, its centre of mass must lie halfway through its thickness, so we can ignore the thickness and treat this as a two-dimensional problem.

Thus we model the cardboard as a homogeneous two-dimensional rigid body. We know that its centre of mass must lie on the axis of symmetry shown in Figure 20. To find where on this axis it lies, we divide the L-shape into two parts, a square and a rectangle, as in Figure 21. Since we have homogeneity, their individual centres of mass must be at their geometric centres G_1 and G_2, which have position vectors

$$\mathbf{r}_{G_1} = \tfrac{1}{2}a\mathbf{i} + \tfrac{3}{2}a\mathbf{j},$$
$$\mathbf{r}_{G_2} = a\mathbf{i} + \tfrac{1}{2}a\mathbf{j},$$

with respect to O, where $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors in the positive x- and y-directions. We now model each part as a particle: the square as a particle of mass m (say) at $\mathbf{r}_{G_1}$, and the rectangle as a particle of mass $2m$ (since the rectangle is twice the size of the square) at $\mathbf{r}_{G_2}$. Equation (15) then gives the centre of mass as

$$\begin{aligned}\mathbf{r}_G &= \frac{m\left(\frac{1}{2}a\mathbf{i} + \frac{3}{2}a\mathbf{j}\right) + 2m\left(a\mathbf{i} + \frac{1}{2}a\mathbf{j}\right)}{3m} \\ &= \tfrac{5}{6}a\mathbf{i} + \tfrac{5}{6}a\mathbf{j},\end{aligned}$$

which lies on the axis of symmetry (the line $y = x$), as expected.

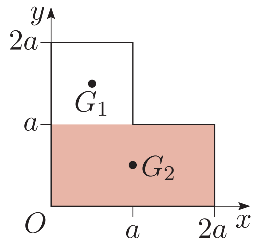

Figure 21 The L-shaped region as two rectangles

Exercise 13

The shape in Figure 20 can be divided into three squares as shown in Figure 22. Use this division of the shape to find its centre of mass.

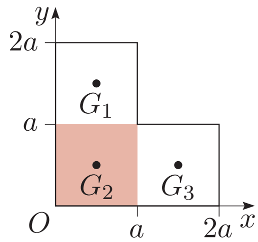

Figure 22 The L-shaped region as three squares

It is often the case that a three-dimensional problem can be reduced to one in two dimensions by taking account of certain uniformities or symmetries in the problem, as in Example 3. But even then there is no guarantee that we can divide the resulting plane figure into parts whose geometric centres we can find easily. Squares and rectangles are no problem, as Example 3 illustrates. But what about triangles?

You saw other examples of this in Unit 2.

Figure 23(a) shows an isosceles triangle, and clearly its geometric centre lies on its axis of symmetry. Figure 23(b) shows the shape divided into a large number of very thin horizontal strips, and Figure 23(c) shows these strips pushed to the right to form another triangle. The centre of each strip lies on the broken line in Figure 23(c), so the geometric centre of this triangular shape lies on this line. Such a line passes through a vertex of the triangle and bisects the opposite side, and is known as a *median* of the triangle. This argument works for thin strips parallel to any of the three sides of the triangle. It follows that the geometric centre must lie on all three medians, that is, the geometric centre must be the point of intersection of the medians, as shown in Figure 23(d). Any triangle may be constructed in this way, so this result holds for all triangles.

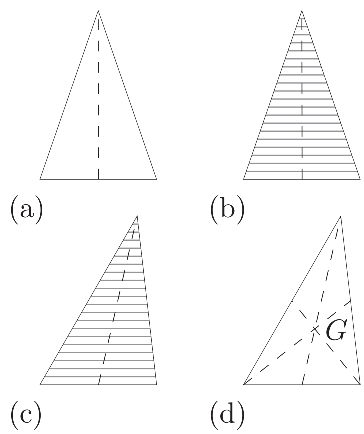

Figure 23 For a triangle, the centre of mass lies on the median

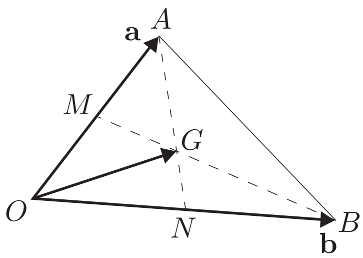

Figure 24 Finding the centre of mass of a triangle

We can find the precise location of this point of intersection by using vectors. Figure 24 shows an arbitrary triangle OAB, and we know that the geometric centre G lies at the intersection of the medians AN and BM. It follows from the triangle rule for adding vectors (defined in Unit 2) that for some numbers λ and μ, we have

$$\overrightarrow{OG} = \mathbf{a} + \lambda\,\overrightarrow{AN} = \mathbf{b} + \mu\,\overrightarrow{BM}. \tag{20}$$

But $\overrightarrow{AN} = -\mathbf{a} + \frac{1}{2}\mathbf{b}$ and $\overrightarrow{BM} = -\mathbf{b} + \frac{1}{2}\mathbf{a}$, so equation (20) gives

$$\mathbf{a} + \lambda\left(-\mathbf{a} + \tfrac{1}{2}\mathbf{b}\right) = \mathbf{b} + \mu\left(-\mathbf{b} + \tfrac{1}{2}\mathbf{a}\right).$$

Equating the coefficients of $\mathbf{a}$ and $\mathbf{b}$ gives

$$1 - \lambda = \tfrac{1}{2}\mu,$$
$$1 - \mu = \tfrac{1}{2}\lambda,$$

so $\lambda = \mu = \frac{2}{3}$. Thus the geometric centre lies on a median at the point two-thirds of the distance from the vertex to the centre of the opposite side, or equivalently, one-third of the way up a median from its base.

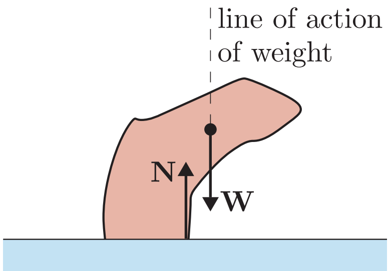

Figure 25 Toppling condition

Now we mention one simple application of the centre of mass. There is a simple test to determine whether an object with a square base resting on a flat horizontal surface will topple over. Consider the situation shown in Figure 25. The forces acting on the object here are its weight $\mathbf{W}$ and the normal reaction force $\mathbf{N}$ due to the horizontal surface, as shown.

Recall from Unit 2 that the weight acts through the centre of mass of the object. The point of action of the normal reaction force can be any point of contact between the object and the surface. If the object does not topple, then the equilibrium conditions of Unit 2 apply, that is, both the resultant force and the resultant torque must be zero. The condition that the resultant force must be zero fixes the magnitude of the normal reaction and can always be satisfied. For the resultant torque to be zero requires that the two forces have the same line of action. So the line of action of the weight of the object must pass through the base of the object, that is, the centre of mass must be vertically above the base. This gives the following useful result.

Toppling condition

Consider an object with a square base resting on a flat horizontal surface. The object will topple over if and only if the centre of mass of the object is not vertically above the base.

Try to apply this result in the following exercise.

Exercise 14

Figure 26(a) shows the design for a sculpture. It is to be made of concrete of uniform density, to be 1 m thick, 3 m high, and to have constant cross-sectional shape as shown in Figure 26(b), where $a = \sqrt{3}/2 \simeq 0.866$.

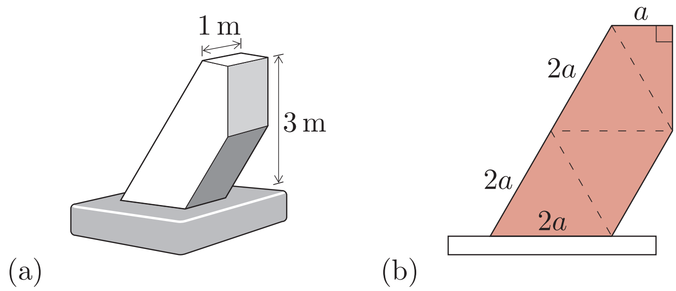

Figure 26 Design for a sculpture

The sculpture is intended to rest on a horizontal surface, as shown in the figure. Will it topple over?

You have seen how to exploit the geometry of certain homogeneous rigid bodies to determine their centres of mass. In some cases, however, we may not be able to subdivide the object into parts whose geometric centres are known or easily found. In such cases we need to resort to integration to help us to find the centre of mass, using the methods of Unit 17. We explore this in the next subsection.

2.3 Centres of mass for laminas

We now turn to the calculation of centres of mass using area integrals. How can we find the centre of mass of a lamina such as a semicircular plate that is bounded by curved lines? We start with the definition of centre of mass. The position vector $\mathbf{r}_G = (x_G, y_G)$ of the centre of mass of a system of n particles each with mass m_i and position vector $\mathbf{r}_i$ is

Area integrals are defined in Unit 17.

$$\mathbf{r}_G = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M} = x_G\mathbf{i} + y_G\mathbf{j}, \tag{21}$$

where M is the total mass. If the particles lie in the (x, y)-plane, then we can project equation (21) onto the unit vectors $\mathbf{i}$ and $\mathbf{j}$ to obtain the coordinates of the centre of mass:

$$\mathbf{r}_G \cdot \mathbf{i} = x_G = \frac{\sum_{i=1}^{n} m_i x_i}{M} \quad \text{and} \quad \mathbf{r}_G \cdot \mathbf{j} = y_G = \frac{\sum_{i=1}^{n} m_i y_i}{M}. \tag{22}$$

Now, expressions (22) are for the centre of mass of a system of n discrete particles lying in the (x, y)-plane. Suppose that the number of particles n increases indefinitely. Then in the limit, the particles will coalesce to form a lamina, and the summations in (22) will become integrals and give the centre of mass of the lamina.

Let us look at the expression for x_G in detail. First, consider the denominator. This is the total mass of the lamina and can be considered as either the limit of the sum of n particles each of mass m_i or as the limit of the sum of area elements $\delta x_j\,\delta y_k$ each of surface density $f(x_j, y_k)$. So

$$\begin{aligned} M = \lim_{n\to\infty}\sum_{i=1}^{n} m_i &= \lim_{p\to\infty}\left(\sum_{j=1}^{p}\left(\lim_{q\to\infty}\sum_{k=1}^{q} f(x_j,y_k)\,\delta y_k\right)\delta x_j\right)\\ &= \int_S f(x,y)\,dA, \end{aligned}$$

where S is the region of the (x, y)-plane occupied by the lamina. In a similar way, the numerator can be considered as the limit as $n \to \infty$ for the particle model and as both $p \to \infty$ and $q \to \infty$ for the area element model. So we have

$$\begin{aligned} \lim_{n\to\infty}\sum_{i=1}^{n} m_i x_i &= \lim_{p\to\infty}\left(\sum_{j=1}^{p}\left(\lim_{q\to\infty}\sum_{k=1}^{q} f(x_j,y_k)\,\delta y_k\right)x_j\,\delta x_j\right)\\ &= \int_S x\,f(x,y)\,dA. \end{aligned}$$

The coordinates (x_G, y_G) of the **centre of mass of a lamina** S are

$$x_G = \frac{\int_S xf\,dA}{M} = \frac{\int_S xf\,dA}{\int_S f\,dA} \quad\text{and}\quad y_G = \frac{\int_S yf\,dA}{M} = \frac{\int_S yf\,dA}{\int_S f\,dA}.$$

For a uniform lamina, the surface density f is a constant and can be taken outside the integral.

Example 4

Determine the position of the centre of mass of the uniform semicircular plate in Figure 27.

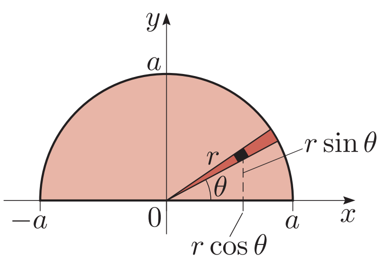

Figure 27

Solution

The y-axis is a symmetry axis of the semicircle, so we must have $x_G = 0$. To find the y-coordinate, we must evaluate $\int_S yf\,dA$. We can use polar coordinates with $y = r\sin\theta$ and surface density $f =$ constant. Then the area integral is

Recall that $\delta A = r\,\delta r\,\delta\theta$.

$$\begin{aligned} \int_S yf\,dA &= f\int_{\theta=0}^{\theta=\pi}\left(\int_{r=0}^{r=a} r\sin\theta\, r\,dr\right)d\theta\\ &= f\int_{\theta=0}^{\theta=\pi}\left[\tfrac{1}{3}r^3\right]_{r=0}^{r=a}\sin\theta\,d\theta\\ &= \tfrac{1}{3}fa^3\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} = \tfrac{2}{3}fa^3. \end{aligned}$$

The mass is the area $\frac{1}{2}\pi a^2$ times the (constant) surface density f.

The mass M of the semicircle is $\frac{1}{2}\pi a^2 f$, so we have $y_G = 4a/3\pi$. Hence the centre of mass of the semicircular plate is $\mathbf{r} = 4a\mathbf{j}/3\pi$.

Exercise 15

Confirm the answer for the coordinate y_G in Example 4 by working in Cartesian coordinates (see Figure 28).

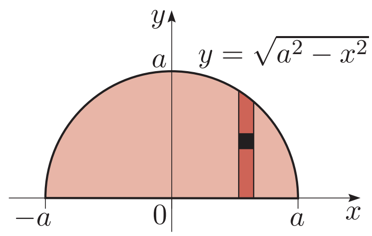

Figure 28

Exercise 16

Confirm the result of the previous subsection, that the centre of mass of an isosceles triangle is one-third of the way up a median from its base, by evaluating an area integral.

2.4 Potential energy

In this short subsection we look at another important application of the centre of mass – calculating potential energy.

Consider the homogeneous solid block shown in two positions in Figure 29. Clearly the block has a lower potential energy in the position shown on the right than in that shown on the left, but how are we to calculate it?

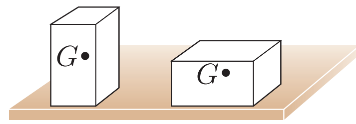

Figure 29 Two positions for a block

Let us return to the general system of particles introduced in Subsection 2.1, where the ith particle has mass m_i and position vector $\mathbf{r}_i$ (relative to some fixed origin O), but this time imagine the particles to lie in the Earth's gravitational field. Choose a Cartesian coordinate system using x, y and z, with origin at O and z-axis pointing vertically upwards, and take corresponding Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The height of the ith particle above the (x, y)-plane is the $\mathbf{k}$-component of $\mathbf{r}_i$, given by $\mathbf{r}_i \cdot \mathbf{k}$, so the potential energy of the ith particle (relative to the datum O) is $m_i g(\mathbf{r}_i \cdot \mathbf{k})$. From equation (15) we have

$$\sum_{i=1}^{n} m_i \mathbf{r}_i = M\mathbf{r}_G = M(x_G\mathbf{i} + y_G\mathbf{j} + h\mathbf{k}).$$

Taking the dot product of each side of this equation with $\mathbf{k}$, and then multiplying by g, we obtain

$$\sum_{i=1}^{n} m_i g(\mathbf{r}_i \cdot \mathbf{k}) = Mg(\mathbf{r}_G \cdot \mathbf{k}) = Mgh, \tag{23}$$

where h is the height of the centre of mass of the system above O. The left-hand side of this equation is the total potential energy of the system of particles, and the right-hand side is the potential energy of a single particle of mass M (equal to the total mass of the system) placed at the centre of mass. Equation (23) thus tells us that to find the potential energy of a system of particles or of a homogeneous rigid body of total mass M, all we need to do is to find the height h of the centre of mass above the datum and then use the formula Mgh.

Exercise 17

If the homogeneous solid block in Figure 29 has mass M, height $4h$, width h and depth h, calculate the change in its potential energy in the two positions shown in Figure 29.

Exercise 18

A uniform ball of mass M is dropped from rest and falls under gravity. What is the speed of its centre of mass when the ball has fallen a distance h? What is the change in the potential energy when the ball has fallen a distance h?

Exercise 19

Find the centre of mass of the homogeneous rigid body shown in Figure 30. Will the object topple over when it is placed on a horizontal surface as shown on the right in Figure 30?

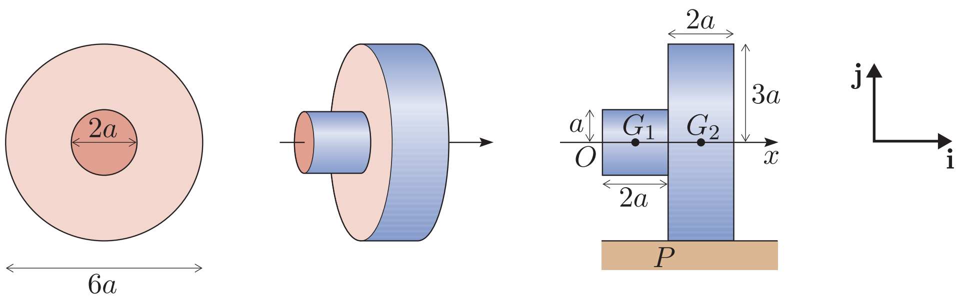

Figure 30 Three images of a rigid body

Exercise 20

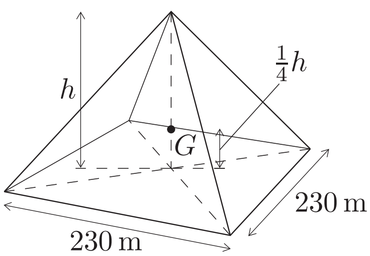

Figure 31 Great Pyramid of Cheops

Figure 31 shows the Great Pyramid of Cheops, with its centre of mass G a quarter of its height h above the square base. Originally the pyramid was 147 m high and built on a square base with sides of length 230 m. The stones of the pyramid are made of limestone.

(a) Assuming that the density of limestone is approximately $2500\,\mathrm{kg\,m^{-3}}$, what is the approximate mass of the pyramid? (*Hint*: Recall that the volume of a pyramid is $\frac{1}{3}$ × base area × vertical height.)

(b) Estimate the total energy required to lift all the stones of the pyramid into place.

(c) Given that a man can lift approximately 1000 kg of stones through a height of 1 m in a day, estimate how long a gang of 1000 men would have taken to lift all the stones of the pyramid into place.

3 Collisions

In this section we are concerned with the behaviour of objects when they collide. We begin in Subsection 3.1 by defining the concept of linear momentum, and we obtain a result that tells us when linear momentum is conserved. In Subsection 3.2 we go on to define and explore elastic and inelastic collisions.

3.1 Conservation of linear momentum

Suppose that two balls A and B, with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to a fixed origin, are moving far out in space, away from any outside influence, so there are no external forces. Let m_1 be the mass of ball A, and let m_2 be the mass of ball B. The balls collide, and just after the collision A has velocity $\dot{\mathbf{R}}_1$ and B has velocity $\dot{\mathbf{R}}_2$, as shown in Figure 32. (We will denote correspondingly the position vectors of A and B, relative to the origin, after the collision by $\mathbf{R}_1$ and $\mathbf{R}_2$.)

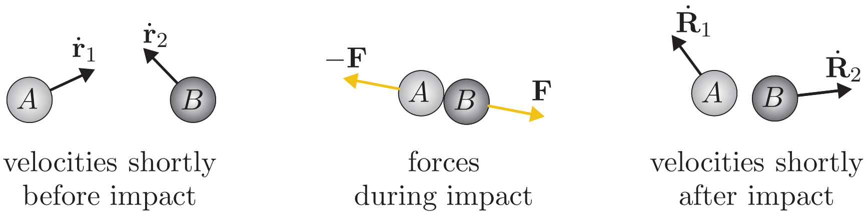

Figure 32 Two balls collide

The system under consideration consists of the two balls A and B, which we model as two particles. The impact happens over a very small interval of time. During that time interval, B experiences a force $\mathbf{F}$ (due to A) and, by Newton's third law, A experiences an equal and opposite force $-\mathbf{F}$ (due to B). These forces may vary over the very small time interval when the balls are in contact, but the important point to appreciate is that they are internal forces. Thus, because there are no external forces, the discussion in Section 2 tells us that the centre of mass of the system moves with constant velocity before, during and after the collision.

If we denote the velocity of the centre of mass before and after the collision by $\dot{\mathbf{r}}_G$ and $\dot{\mathbf{R}}_G$, respectively, then we have $\dot{\mathbf{r}}_G = \dot{\mathbf{R}}_G$, where

$$\dot{\mathbf{r}}_G = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1 + m_2} \quad \text{and} \quad \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2}.$$

It follows that

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2. \tag{24}$$

The quantity

$$(\text{mass of } A) \times (\text{velocity of } A) + (\text{mass of } B) \times (\text{velocity of } B)$$

has been unchanged by the collision, and this leads us to make the following general definitions.

The SI units for the magnitude of linear momentum are $\mathrm{kg\,m\,s^{-1}}$.

Here *linear* is being used in a translational sense as an object moves through space with velocity $\dot{\mathbf{r}}$. 'Linear momentum' is often abbreviated to 'momentum' when the meaning is clear.

The **linear momentum p** of a particle with position vector $\mathbf{r}$ is a vector quantity given as the product of its mass m and its velocity $\dot{\mathbf{r}}$, so

$$\mathbf{p} = m\dot{\mathbf{r}}.$$

The **linear momentum P** of an n-particle system is the vector sum of the linear momenta of the individual particles, so

$$\mathbf{P} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = M\dot{\mathbf{r}}_G, \tag{25}$$

where m_i and $\mathbf{r}_i$ represent the mass and position of particle i, M is the total mass of the system, and $\mathbf{r}_G$ is its centre of mass.

Thus equation (24) tells us that the linear momentum of our two-ball system with no external forces is the same before and after the collision, that is, it is *conserved*.

We can link linear momentum to the motion of the centre of mass of a system, given by equation (19) as $\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{r}}_G$. Differentiating equation (25) with respect to time gives $\dot{\mathbf{P}} = M\ddot{\mathbf{r}}_G$, so equation (19) becomes

$$\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}. \tag{26}$$

This tells us that the sum of the external forces on an n-particle system is equal to the rate of change of linear momentum of the system. The case when there are no external forces is particularly simple, for then we have $\dot{\mathbf{P}} = \mathbf{0}$, so the linear momentum is constant throughout the motion.

As we observed above, the forces acting during a collision are *internal* forces, thus do not affect equations (19) and (26). Therefore equation (19) tells us that the motion of the centre of mass of an n-particle system is unaffected by any collisions between the particles. Equation (26) tells us that the rate of change of linear momentum of the system is also unaffected by any collisions.

In the case where there is no net external force, equation (26) also tells us that the linear momentum remains constant despite any collisions, as you saw in the two-ball example. Let us now consider what equation (26) tells us about the linear momentum (rather than its rate of change) when our two-ball system is subject to external forces. With this aim, suppose now that the two balls are in the Earth's gravitational field, as shown in Figure 33, where there are no other external forces.

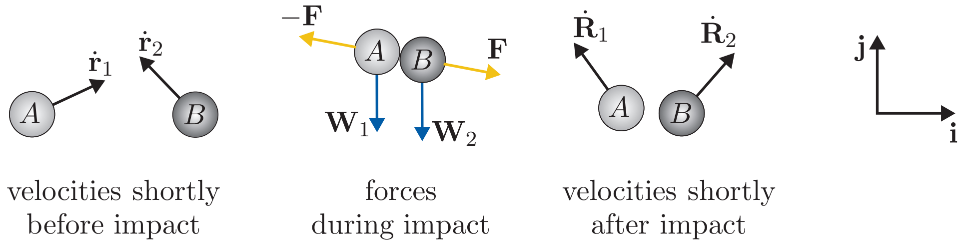

Figure 33 Two balls collide, subject to gravity

In this case there is a net external force, the combined weight of the balls, so $\mathbf{F}^{\text{ext}} = \mathbf{W}_1 + \mathbf{W}_2 = (m_1 + m_2)g(-\mathbf{j})$, which is constant, and equation (26) becomes $-(m_1 + m_2)g\mathbf{j} = \dot{\mathbf{P}}$. The collision occurs over a very small time interval of length T. If we integrate this equation with respect to time over this time interval, we obtain

$$\int_0^T -(m_1 + m_2)g\mathbf{j}\,dt = \int_0^T \dot{\mathbf{P}}\,dt = [\mathbf{P}]_0^T,$$

and it follows that $-(m_1 + m_2)Tg\mathbf{j} = \mathbf{P}_{\text{after}} - \mathbf{P}_{\text{before}}$. If we suppose that the collision is instantaneous, so $T = 0$ (i.e. if we take the limit as $T \to 0$), then $\mathbf{P}_{\text{after}} = \mathbf{P}_{\text{before}}$; in other words, the linear momentum is conserved.

This result generalises to any n-particle system. Suppose that we have such a system on which the resultant of the external forces is $\mathbf{F}^{\text{ext}}$. Over a very small time interval of length T, we can take $\mathbf{F}^{\text{ext}}$ to be constant, so integrating equation (26) with respect to time gives

$$\int_0^T \mathbf{F}^{\text{ext}}\,dt = \int_0^T \dot{\mathbf{P}}\,dt = [\mathbf{P}]_0^T,$$

thus $\mathbf{F}^{\text{ext}}T = \mathbf{P}_{\text{after}} - \mathbf{P}_{\text{before}}$. Hence, for an instantaneous collision, with $T = 0$ (i.e. if we take the limit as $T \to 0$), we have $\mathbf{P}_{\text{after}} = \mathbf{P}_{\text{before}}$.

This means that in the presence of external forces, the total linear momentum of an n-particle system may change over time, but the instantaneous collisions of the particles within the system have no effect on the total linear momentum.

Principle of conservation of linear momentum

The total linear momentum of an n-particle system is not affected by (instantaneous) collisions among the particles. Furthermore, in the absence of external forces, the total linear momentum of the system remains constant.

By modelling a rigid body as a system of particles (as we did in Section 2), this principle can be extended to systems involving rigid bodies and/or particles. In particular, it means that if two rigid bodies collide, then the collision causes no change in the total linear momentum of the bodies.

Example 5

A railway truck A of mass 50 000 kg rolls down a slight incline and collides with a stationary truck B of mass 30 000 kg. After the collision, the trucks are coupled together (so A and B have the same velocity).

If the velocity of A immediately before the collision is $2\mathbf{i}$ (in $\mathrm{m\,s^{-1}}$), what is the combined speed of the trucks immediately after the collision (assuming that the collision is instantaneous)?

Solution

We ignore the motion of the wheels and model the trucks as rigid bodies. Let $v\mathbf{i}$ be the velocity of the trucks after impact. Then, using the notation $\mathbf{P}_{\text{before}}$ and $\mathbf{P}_{\text{after}}$ for the total linear momentum before and after impact, we have

$$\mathbf{P}_{\text{before}} = 50\,000 \times 2\mathbf{i},$$
$$\mathbf{P}_{\text{after}} = (50\,000 + 30\,000) \times v\mathbf{i}.$$

It follows from the principle of conservation of linear momentum that $80\,000v = 100\,000$, so $v = 1.25$. Hence the combined speed after impact is $1.25\,\mathrm{m\,s^{-1}}$.

Exercise 21

A railway engine of mass M, moving on a straight horizontal track, collides with a stationary truck of mass m. The truck becomes attached to the engine, and both move off together.

Express the speed v of the engine and truck immediately after the collision in terms of M, m and the speed u of the engine immediately prior to the collision.

3.2 Elastic and inelastic collisions

Sometimes in a collision between objects, the total kinetic energy of the system before and after the collision remains the same, that is, energy is conserved. However, in other cases, some of the kinetic energy may be transformed into other forms of energy by the collision. We use different terms for the two types of collision.

If the kinetic energy of a system is the same before and after a collision within the system, then the collision is said to be **elastic**; otherwise, it is said to be **inelastic**.

Notice that here we use the terms 'elastic' and 'inelastic' in a technical sense that is somewhat different from their everyday usage.

Example 6

Suppose that a snooker ball A, of mass m, moving with speed $5\,\mathrm{m\,s^{-1}}$ in a straight line across a smooth snooker table, collides elastically with a similar ball B, of the same mass, at rest. Suppose that A hits B head on, that is, they both move along the same straight line after impact as A was moving along before impact.

Modelling each ball as a particle, predict the speeds of the balls after the collision.

Solution

Define an x-axis along the line of motion of the balls, with positive direction in the direction of motion of A before the collision. Then the velocities of A and B before the collision can be written as $\dot{\mathbf{r}}_1 = \dot{x}_1\mathbf{i}$ and $\dot{\mathbf{r}}_2 = \dot{x}_2\mathbf{i}$, and their velocities after the collision as $\dot{\mathbf{R}}_1 = \dot{X}_1\mathbf{i}$ and $\dot{\mathbf{R}}_2 = \dot{X}_2\mathbf{i}$, where $\mathbf{i}$ is a unit vector in the positive x-direction.

By the principle of conservation of linear momentum, we have

$$m\dot{x}_1\mathbf{i} + m\dot{x}_2\mathbf{i} = m\dot{X}_1\mathbf{i} + m\dot{X}_2\mathbf{i}.$$

Dividing by m and resolving in the $\mathbf{i}$-direction gives

$$\dot{x}_1 + \dot{x}_2 = \dot{X}_1 + \dot{X}_2,$$

which, since $\dot{x}_1 = 5$ and $\dot{x}_2 = 0$, gives

$$\dot{X}_1 + \dot{X}_2 = 5. \tag{27}$$

The kinetic energies of the balls A and B before impact are $\frac{1}{2}m\dot{x}_1^2 = \frac{25}{2}m$ and $\frac{1}{2}m\dot{x}_2^2 = 0$, and the kinetic energies after impact are $\frac{1}{2}m\dot{X}_1^2$ and $\frac{1}{2}m\dot{X}_2^2$. Since the collision is elastic, kinetic energy is conserved, so we have

$$\tfrac{25}{2}m = \tfrac{1}{2}m\dot{X}_1^2 + \tfrac{1}{2}m\dot{X}_2^2,$$

hence

$$\dot{X}_1^2 + \dot{X}_2^2 = 25. \tag{28}$$

From equation (27), $\dot{X}_2 = 5 - \dot{X}_1$. Substituting for $\dot{X}_2$ in equation (28), we obtain

$$\dot{X}_1^2 + (5 - \dot{X}_1)^2 = 25,$$

which simplifies to

$$\dot{X}_1(\dot{X}_1 - 5) = 0.$$

So there are two possibilities: either $\dot{X}_1 = 5$ or $\dot{X}_1 = 0$. Substituting $\dot{X}_1 = 5$ into equation (27) gives $\dot{X}_2 = 0$, which is physically impossible (since B is in front of A). So $\dot{X}_1 = 0$ and $\dot{X}_2 = 5$ (again from equation (27)). In other words, after the collision, ball A comes to rest, while ball B moves across the table with a speed equal to the original speed of ball A.

Exercise 22

A railway truck of mass 15 000 kg is moving at $4\,\mathrm{m\,s^{-1}}$ along a straight track and collides with a stationary truck of mass 10 000 kg. After impact the trucks move together along the track.

(a) What is the combined speed of the trucks after impact?

(b) Is the collision elastic?

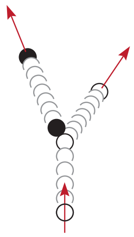

Figure 34 Oblique collision

In Example 6 and Exercise 22, the motion before and after impact was along the same straight line. Also, only one object was moving before impact. In the next example there is still only one object moving before impact, but the directions of motion of the two objects after impact are along different straight lines. This sort of collision, referred to as an *oblique* collision, occurs, for example, when one ball strikes a glancing blow on another, as illustrated in Figure 34.

Example 7

A white snooker ball A, moving parallel to the sides of the table with speed u, collides obliquely with the stationary black ball B (which is equidistant from two of the end pockets), as shown in Figure 35. As a result, the black ball moves off at an angle of $\frac{\pi}{4}$ to the original direction of motion of the white ball, and ends up in the pocket at the top of Figure 35.

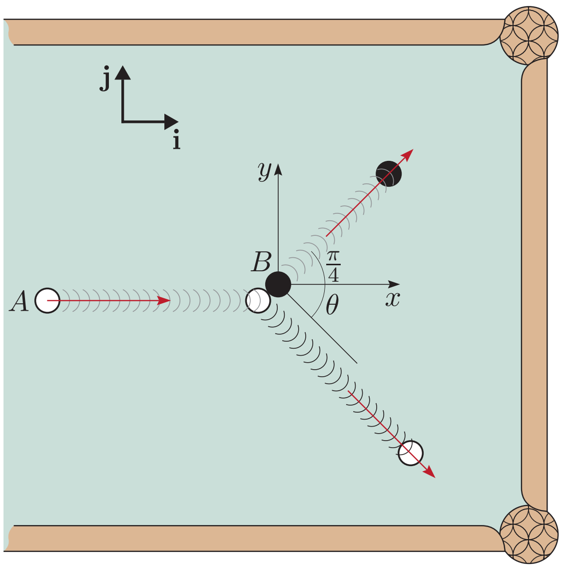

Figure 35 Balls on a snooker table

If the collision is elastic and each ball has mass m, find the speeds of the balls after impact, and decide if the white ball is likely to enter a pocket.

Solution

We choose axes as shown in Figure 35, with origin at the original position of the black ball. Modelling the balls as particles and using $\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2$ and

$\dot{\mathbf{R}}_1, \dot{\mathbf{R}}_2$ for the velocities of A, B just before and just after impact, respectively, we have

$$\dot{\mathbf{r}}_1 = u\mathbf{i} \quad \text{and} \quad \dot{\mathbf{r}}_2 = \mathbf{0},$$

where $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors in the positive x- and y-directions, respectively.

We are told that the direction of motion of the black ball after impact is at an angle of $\frac{\pi}{4}$ to the x-axis, so

$$\dot{\mathbf{R}}_2 = \left(|\dot{\mathbf{R}}_2| \cos \tfrac{\pi}{4}\right)\mathbf{i} + \left(|\dot{\mathbf{R}}_2| \sin \tfrac{\pi}{4}\right)\mathbf{j} = \tfrac{1}{\sqrt{2}}|\dot{\mathbf{R}}_2|\mathbf{i} + \tfrac{1}{\sqrt{2}}|\dot{\mathbf{R}}_2|\mathbf{j} = V\mathbf{i} + V\mathbf{j},$$

where $V = |\dot{\mathbf{R}}_2|/\sqrt{2}$. Also, if θ is the angle made with the x-axis by the white ball's direction of motion after impact, then

$$\dot{\mathbf{R}}_1 = \left(|\dot{\mathbf{R}}_1| \cos \theta\right)\mathbf{i} - \left(|\dot{\mathbf{R}}_1| \sin \theta\right)\mathbf{j} = V_x\mathbf{i} - V_y\mathbf{j},$$

We write $-V_y\mathbf{j}$ because we expect the $\mathbf{j}$-component of the white ball to be negative.

where $V_x = |\dot{\mathbf{R}}_1| \cos \theta$ and $V_y = |\dot{\mathbf{R}}_1| \sin \theta$ denote the components of the white ball's velocity after impact.

The principle of conservation of linear momentum gives

$$mu\mathbf{i} = m(V_x\mathbf{i} - V_y\mathbf{j}) + m(V\mathbf{i} + V\mathbf{j}).$$

Resolving in the $\mathbf{i}$- and $\mathbf{j}$-directions in turn, and dividing by m, leads to

$$V_x = u - V \quad \text{and} \quad V_y = V. \tag{29}$$

As the collision is elastic, kinetic energy is conserved, so we have

$$\begin{aligned}
\tfrac{1}{2}mu^2 &= \tfrac{1}{2}m|\dot{\mathbf{R}}_1|^2 + \tfrac{1}{2}m(\sqrt{2}V)^2 \\
&= \tfrac{1}{2}m\big((V_x\mathbf{i} - V_y\mathbf{j}) \cdot (V_x\mathbf{i} - V_y\mathbf{j})\big) + mV^2 \\
&= \tfrac{1}{2}m(V_x^2 + V_y^2) + mV^2.
\end{aligned}$$

If we substitute the values for V_x and V_y obtained in equations (29) and divide by $\frac{1}{2}m$, then this equation becomes

$$u^2 = (u - V)^2 + V^2 + 2V^2,$$

which simplifies to $uV = 2V^2$.

We know that $V \neq 0$, therefore $V = \frac{1}{2}u$, and equations (29) give $V_x = V_y = \frac{1}{2}u$. Hence the speeds of the white ball and the black ball after impact are, respectively,

$$|\dot{\mathbf{R}}_1| = \sqrt{V_x^2 + V_y^2} = \tfrac{1}{\sqrt{2}}u \quad \text{and} \quad |\dot{\mathbf{R}}_2| = \sqrt{2}V = \tfrac{1}{\sqrt{2}}u.$$

Also,

$$\tan \theta = \left(|\dot{\mathbf{R}}_1| \sin \theta\right)/\left(|\dot{\mathbf{R}}_1| \cos \theta\right) = V_y/V_x = 1,$$

hence $\theta = \frac{\pi}{4}$, so the white ball moves off at an angle of $\frac{\pi}{4}$ to its original direction of motion, towards the bottom pocket shown in Figure 35.

However, although the white ball is travelling in the general direction of the bottom pocket, notice from Figure 35 that its diagonal motion does not start at exactly the same point as that of the black ball, but rather starts slightly below and to the left of it. So although the white ball goes close to the bottom pocket, we need to know more about the size of snooker balls, the dimensions of snooker tables and the geometry of snooker table pockets before we can make any firm conclusion about whether it enters the bottom pocket. (Even with all this information, we still could not be sure because our two-particle model takes no account of aspects of the physical situation such as the rolling motion of the balls.)

Exercise 23

You could try an experiment at home with two identical coins and a piece of paper on a flat surface. Draw a circle on the paper round one coin. Place the other coin on the paper a little way from the first coin, and flick it towards the stationary coin in the circle. Measure the angle between the paths of the coins after the collision.

Show that if two identical balls of mass m collide in an elastic collision, where one is initially stationary and the other is travelling with velocity $\mathbf{u}$, then immediately after the collision, either one ball becomes stationary or the balls go off at right angles to each other.

(*Hint*: Write the initial kinetic energy of the ball as $\frac{1}{2}m|\mathbf{u}|^2 = \frac{1}{2}m\mathbf{u} \cdot \mathbf{u}$.)

Exercise 24

A white snooker ball, travelling with speed u, collides with a stationary green ball. As a result, the green ball moves off at an angle of $\frac{\pi}{3}$ to the original direction of motion of the white ball.

Model the balls as particles of equal mass m, and assume that the collision is elastic. Determine the direction of motion of the white ball after the collision. (*Hint*: Use the result of the previous exercise.) Also, find the velocity of each ball after the collision.

Exercise 25

A particle of mass m_1 travels in the positive x-direction with speed u. Another particle of mass m_2 travels in the positive y-direction with the same speed u. The particles collide and then move off together with the same velocity.

Find the velocity of the particles after the collision. Is the collision elastic?

Exercise 26

A particle of mass m_1 collides with a particle of mass m_2. The velocities of the particles are $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, before impact, and $\mathbf{V}_1$ and $\mathbf{V}_2$, respectively, after impact.

(a) If $m_1 = m_2 = 3$, $\mathbf{v}_1 = 2\mathbf{i}$, $\mathbf{v}_2 = \mathbf{0}$, $\mathbf{V}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{V}_2 = \mathbf{i} - \mathbf{j}$, show that the collision is elastic.

(b) If $m_1 = 1$, $m_2 = 3$, $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v}_2 = \mathbf{j}$ and $\mathbf{V}_1 = \mathbf{V}_2 = \frac{1}{2}\mathbf{i} + \mathbf{j}$, show that the collision is inelastic, and find the decrease in kinetic energy due to the collision, assuming that all quantities are measured in SI units.

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4 Newton's law of restitution

Here we look at the relationship between the relative velocities of objects before and after collisions.

Although some collisions are elastic, usually we expect there to be some transfer of energy when objects collide, to heat energy or sound energy, for example. So most collisions are inelastic. To obtain information about velocities after an inelastic collision, where kinetic energy is not conserved, we can use *Newton's law of restitution*, a law that is well supported by empirical evidence.

The starting point for this law is the experimental observation that if you drop a ball onto a flat fixed horizontal solid surface, then the height to which the ball rebounds appears to be a fixed fraction of the original height. If the ball is dropped from a height H and rebounds to a height h, then it appears that

$$h = cH,$$

where c is a constant (with $0 \leq c \leq 1$) depending on the material of both the ball and the surface. If $c = 1$, then the ball rebounds to its original height; if $c = 0$, then the ball does not bounce. A steel ball dropped onto a steel plate would provide an example of a collision where c is close to 1; on the other hand, a ball of a material like putty would hardly bounce, so c would be almost 0.

If we model the ball as a particle of mass m, then (relative to a datum at the surface) the potential energy of the ball as it is dropped is mgH, and its potential energy at the top of the first bounce is $mgh = cmgH$. So with each successive bounce, a fixed fraction of the ball's energy is lost (we will see later where it has gone). Let $\mathbf{i}$ be a unit vector in the upwards direction. Suppose that the velocities are $\dot{x}\mathbf{i}$ (with $\dot{x} < 0$) and $\dot{X}\mathbf{i}$ (with $\dot{X} > 0$) just before and after the first impact, as shown in Figure 36. We can apply the law of conservation of mechanical energy (see Unit 9) to the one-dimensional motion of the ball before impact and after impact (but not during impact, as then there is an extra force due to the impact). The law applied to the motion before impact gives

$$\tfrac{1}{2}m\dot{x}^2 = mgH,$$

and after impact it gives

$$\tfrac{1}{2}m\dot{X}^2 = mgh = cmgH.$$

Therefore $\dot{X}^2 = c\dot{x}^2$, so $\dot{X}\mathbf{i} = -\sqrt{c}\,\dot{x}\mathbf{i}$ (since $\dot{X}$ and $\dot{x}$ have opposite signs). We normally replace the constant $\sqrt{c}$ by e, which is known as the *coefficient of restitution* between the ball and the surface, so

$$\dot{X}\mathbf{i} = -e\dot{x}\mathbf{i}.$$

This means that the velocity of the ball after impact is $-e$ times its velocity before impact.

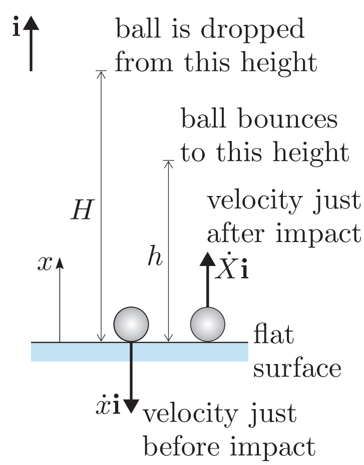


Figure 36 A ball bouncing off a flat surface

This result holds not only for balls bouncing on surfaces, but also for any instantaneous collision between any two objects moving along the same line, in which case it takes the form

$$\begin{aligned}&\text{relative velocity after impact}\\&= -e \times \text{relative velocity before impact.}\end{aligned} \tag{30}$$

Notice that if $e = 1$, then the speed is the same before and after impact, hence so is the kinetic energy. Therefore $e = 1$ corresponds to an elastic collision, while $e < 1$ corresponds to an inelastic one.

Exercise 27

Suppose that there is a collision between a ball A of mass m_1, which is moving with velocity $\dot{\mathbf{r}}_1$, and a ball B of mass m_2, which is moving with velocity $\dot{\mathbf{r}}_2$. Suppose also that both balls are moving along the same straight line, and that the coefficient of restitution between the balls is e.

(a) Find the velocities of the balls after impact in terms of the velocities before impact.

(b) Use your results to provide an alternative solution to Example 6.

(c) Show that if the balls coalesce, then $e = 0$.

Exercise 28

A ball of mass m_1 moves with speed u across a smooth table and collides head on with a stationary ball of mass m_2 (so the motion before and after impact takes place along the same straight line).

Given that the coefficient of restitution between the balls is e, and assuming that the balls can be modelled as particles, show that the kinetic energy lost from the system due to the collision is

$$\frac{m_1 m_2 (1 - e^2) u^2}{2(m_1 + m_2)}.$$

Consider a ball bouncing on a smooth fixed horizontal solid surface. When the ball is dropped onto the surface, the only forces acting on the ball at the moment of impact are the weight of the ball and the normal reaction of the surface on the ball. These forces are vertical and act along the line of motion, so their effect changes the velocity of the ball but not its line of motion, which remains vertical.

Suppose now that the ball is not dropped but is thrown so that it strikes the surface at an angle, as in Figure 37. At the moment of impact, the forces acting on the ball are still just the weight and the normal reaction, and these are both vertical. So the impact affects just the vertical component of the ball's velocity – there is no force with a horizontal component to affect the horizontal component of the ball's velocity.

There is no friction as the surface is smooth.

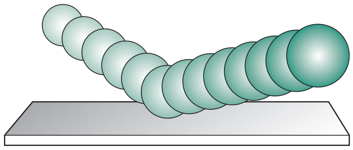


Figure 37 A ball striking a surface at an angle

If the velocities before and after impact are $\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and $\dot{\mathbf{R}} = \dot{X}\mathbf{i} + \dot{Y}\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical Cartesian unit vectors, then we have $\dot{X} = \dot{x}$. Also, applying equation (30) to the vertical components of the velocity, we have $\dot{Y} = -e\dot{y}$, where e is the coefficient of restitution between the ball and the surface. The kinetic energy just before impact is $\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$, and the kinetic energy just after impact is $\frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right) = \frac{1}{2}m\left(\dot{x}^2 + e^2\dot{y}^2\right)$.

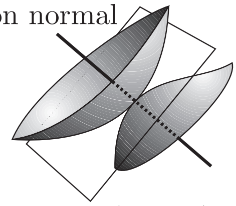


Figure 38 Collision between two smooth objects

As before, these ideas can be extended to collisions between any two objects, one or both of which may be moving, provided that the areas of contact between the objects during impact are both smooth (i.e. frictionless) and lie on the same tangent plane, as illustrated in Figure 38. In our mathematical models of colliding objects, the objects are often spheres or planes, in which case the proviso about the areas of contact lying on the same tangent plane is automatically satisfied, but the ideas can be applied to collisions between objects of other shapes too. We will also need to assume that neither object is rotating, or at least that any rotation can be ignored – this assumption is reasonable in many cases. (The assumption that an object can be modelled as a particle implies that any rotation of the object can be ignored – see Unit 3.)

If there is significant rotation, then other forces come into play and affect the results.

As for the ball bouncing obliquely on a surface, only the component of the velocity perpendicular to the tangent plane of contact is affected by the collision. At a point of contact, the direction perpendicular to the common tangent plane is referred to as the **common normal**; this is illustrated in Figure 38. Thus we have the following result, which as we noted earlier is well supported by empirical evidence.

Newton's law of restitution

In an (instantaneous) collision between two smooth non-rotating objects, where the area of contact at the moment of impact lies on a common tangent plane, the velocities parallel to the tangent plane remain unchanged before and after impact. Also, we have

$$\begin{pmatrix}\text{relative velocity component}\\ \text{parallel to the common}\\ \text{normal after impact}\end{pmatrix} = -e \times \begin{pmatrix}\text{relative velocity component}\\ \text{parallel to the common}\\ \text{normal before impact}\end{pmatrix},$$

where e is the **coefficient of restitution** for a collision between the two objects.

To see how we can make use of this law, suppose that two smooth balls A and B of the same radius, of masses m_1 and m_2, are moving along a smooth horizontal surface before colliding. Assume that any rotation can be ignored. The situation is illustrated in Figure 39, which also shows the common tangent plane at the point of impact and Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the direction of the tangent plane and perpendicular to it (in the direction of the common normal). Figure 39 also shows the velocities of the balls before and after impact.

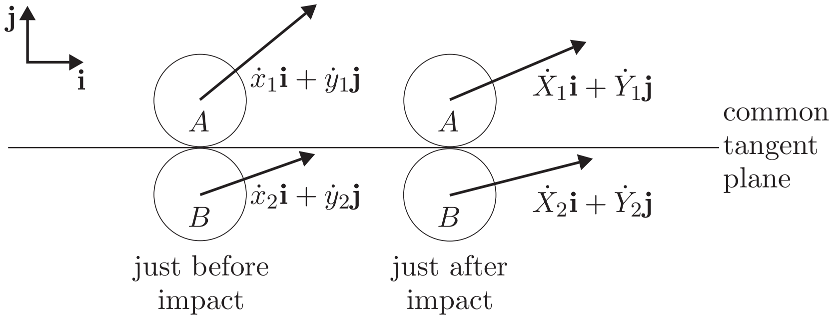


Figure 39 Oblique impact between two balls

Since there are four components of velocity after the collision, we need four equations to determine them. Two equations come from the tangential conditions, one from conservation of linear momentum in the $\mathbf{j}$-direction, and one from Newton's law of restitution.

From the tangential conditions and Newton's law of restitution, we have

$$\dot{x}_1\mathbf{i} = \dot{X}_1\mathbf{i}, \quad \dot{x}_2\mathbf{i} = \dot{X}_2\mathbf{i}, \quad (\dot{Y}_1 - \dot{Y}_2)\mathbf{j} = -e(\dot{y}_1 - \dot{y}_2)\mathbf{j}, \tag{31}$$

where e is the coefficient of restitution for the balls. Also, the principle of conservation of linear momentum gives

$$m_1(\dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j}) + m_2(\dot{x}_2\mathbf{i} + \dot{y}_2\mathbf{j}) = m_1(\dot{X}_1\mathbf{i} + \dot{Y}_1\mathbf{j}) + m_2(\dot{X}_2\mathbf{i} + \dot{Y}_2\mathbf{j}). \tag{32}$$

Resolving in the $\mathbf{j}$-direction in equations (31) and (32) gives

$$\dot{Y}_1 - \dot{Y}_2 = -e(\dot{y}_1 - \dot{y}_2),$$
$$m_1\dot{y}_1 + m_2\dot{y}_2 = m_1\dot{Y}_1 + m_2\dot{Y}_2.$$

You solved the equivalent equations using xs rather than ys in Exercise 27.

These equations can be solved for $\dot{Y}_1$ and $\dot{Y}_2$ to give

$$\dot{Y}_1 = \frac{(m_1 - em_2)\dot{y}_1 + m_2(1+e)\dot{y}_2}{m_1 + m_2},$$
$$\dot{Y}_2 = \frac{m_1(1+e)\dot{y}_1 + (m_2 - em_1)\dot{y}_2}{m_1 + m_2}.$$

So, given e and the velocities before impact, we can find the velocities after impact.

Example 8

Figure 40 shows the head of a golf club A just before and just after it strikes a golf ball B. Just before impact, the ball is stationary and the head of the club is moving horizontally with speed u.

The mass of A is $9m$ and the mass of B is m, the coefficient of restitution between the club and the ball is 0.8, and the face of the club is inclined at an angle α, as shown in the figure.

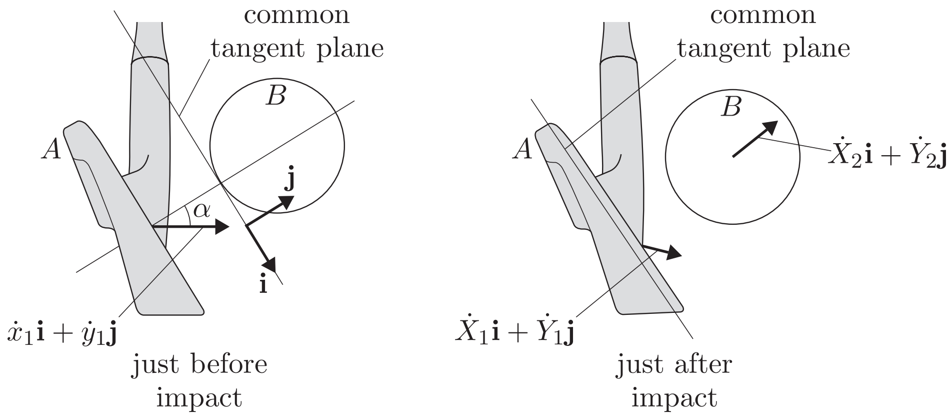


Figure 40 Golf club striking a ball

Model the club head and the ball as smooth objects, and ignore any rotational effects. Estimate the loss of kinetic energy caused by the collision.

The 'spin' imparted to a golf ball by a golf club can be very important, as any golfer will tell you, but we ignore it here.

Solution

Choosing the Cartesian unit vectors as shown in Figure 40, we can write the velocity of A just before impact as

$$\dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j} = (u\sin\alpha)\mathbf{i} + (u\cos\alpha)\mathbf{j},$$

while the velocity of B just before impact is $\mathbf{0}$.

The common tangent plane is in the **i**-direction. From the tangential conditions and Newton's law of restitution, we have

$$\dot{X}_1\mathbf{i} = \dot{x}_1\mathbf{i}, \quad \dot{X}_2\mathbf{i} = \dot{x}_2\mathbf{i}, \quad (\dot{Y}_1 - \dot{Y}_2)\mathbf{j} = -e(\dot{y}_1 - \dot{y}_2)\mathbf{j}.$$

Resolving in the **i**- and **j**-directions, and substituting in known values, we obtain

$$\begin{aligned}\dot{X}_1 &= \dot{x}_1 = u\sin\alpha,\\ \dot{X}_2 &= \dot{x}_2 = 0,\\ \dot{Y}_1 - \dot{Y}_2 &= -0.8(u\cos\alpha - 0) = -0.8u\cos\alpha. \end{aligned} \tag{33}$$

Also, the principle of conservation of linear momentum gives

$$9m(\dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j}) + \mathbf{0} = 9m(\dot{X}_1\mathbf{i} + \dot{Y}_1\mathbf{j}) + m(\dot{X}_2\mathbf{i} + \dot{Y}_2\mathbf{j}),$$

which, resolving in the **j**-direction, using known values and dividing through by m, gives

$$9u\cos\alpha = 9\dot{Y}_1 + \dot{Y}_2. \tag{34}$$

Solving equations (33) and (34) for $\dot{Y}_1$ and $\dot{Y}_2$, we obtain

$$\dot{Y}_1 = 0.82u\cos\alpha, \quad \dot{Y}_2 = 1.62u\cos\alpha.$$

We now have enough information to be able to compare the kinetic energy just after impact with the kinetic energy just before impact. The kinetic energy of the system just before impact is

$$\tfrac{1}{2}(9m)u^2 = 4.5mu^2.$$

The velocity of the club head just after impact is $\dot{X}_1\mathbf{i} + \dot{Y}_1\mathbf{j} = (u\sin\alpha)\mathbf{i} + (0.82u\cos\alpha)\mathbf{j}$, and the velocity of the ball just after impact is $\dot{X}_2\mathbf{i} + \dot{Y}_2\mathbf{j} = 0\mathbf{i} + (1.62u\cos\alpha)\mathbf{j}$, so their combined kinetic energy just after impact is

$$\begin{aligned}&\tfrac{1}{2}(9m)(u^2\sin^2\alpha + (0.82)^2u^2\cos^2\alpha) + \tfrac{1}{2}m(1.62)^2u^2\cos^2\alpha\\ &= (4.5\sin^2\alpha + 4.338\cos^2\alpha)mu^2\\ &= (4.5 - 0.162\cos^2\alpha)mu^2.\end{aligned}$$

So the model estimates that the kinetic energy of the system has decreased by $0.162mu^2\cos^2\alpha$.

In Unit 9 we showed that the total mechanical energy of certain one-particle systems is conserved, but in this section we have shown that the mechanical energy decreases in an inelastic collision. What has happened to this energy?

It is a basic assumption of physics that the total energy of a system is conserved. So either our model is inadequate, or some of the mechanical energy has been converted into another form of energy. Actually, it is a bit of both. During a collision, a little of the mechanical energy is converted into sound energy, and some is converted into heat energy; but much of the 'missing' energy still exists in the form of mechanical energy – it is just that our particle model is too crude to detect it.

For the golf ball, some of the 'missing' energy is in the spinning motion of the ball.

Exercise 29

Two smooth non-rotating balls A and B of equal mass m slide on a frictionless horizontal table and undergo an inelastic collision with coefficient of restitution $e = 0$. Before the collision, A has speed u while B is stationary. After the collision, B moves off at an angle $\frac{\pi}{4}$ measured clockwise from the direction of approach of A. Let $\mathbf{i}$ be a unit vector in the direction of motion of B. Let $\mathbf{j}$ be the unit vector obtained by rotating $\mathbf{i}$ through $\frac{\pi}{2}$ anticlockwise in the plane of the horizontal table.

Such a collision is sometimes called a *plastic* collision.

Find the velocities of both balls after the collision. (*Hint*: The vector $\mathbf{j}$ lies in the common tangent plane for the collision.)

Exercise 30

A rubber ball is dropped from rest onto a horizontal floor, and after bouncing twice it rebounds to half its original height.

Calculate the coefficient of restitution between the ball and the floor.

Learning outcomes

After studying this unit, you should be able to:

- find the centre of mass of a system of particles, a lamina, and some homogeneous rigid bodies
- determine the motion of the centre of mass of a system of particles
- use the centre of mass to calculate the potential energy of a homogeneous rigid body
- apply the principle of conservation of linear momentum to collisions between objects
- use the change in kinetic energy during a collision to determine whether or not the collision is elastic
- apply Newton's law of restitution to collisions between objects.

Solutions to exercises

Solution to Exercise 1

(a) Resolving the forces into components gives $\mathbf{T}_2 = -|\mathbf{T}_2|\mathbf{i}$ and $\mathbf{T}_4 = -|\mathbf{T}_4|\mathbf{i}$. Resolving equations (1) and (2) in the $\mathbf{i}$-direction then gives

$$m_1 g - |\mathbf{T}_2| = m_1\ddot{x}_1,$$
$$m_2 g - |\mathbf{T}_4| = m_2\ddot{x}_2.$$

To progress further, we need to relate $\ddot{x}_1$ and $\ddot{x}_2$, and to relate $|\mathbf{T}_2|$ and $|\mathbf{T}_4|$.

Using the assumption that the cable is inextensible gives $x_1 + x_2 = \text{constant}$, as quoted in the question. Differentiating this twice with respect to t gives the desired relation between the accelerations of the lift and the counterweight: $\ddot{x}_1 = -\ddot{x}_2$.

To find relationships between the magnitudes of the tension forces, we use the assumptions that the cable is a model string and the pulley is a model pulley to obtain $|\mathbf{T}_2| = |\mathbf{T}_1|$ (model string), $|\mathbf{T}_1| = |\mathbf{T}_3|$ (model pulley) and $|\mathbf{T}_3| = |\mathbf{T}_4|$ (model string again). Putting these together gives $|\mathbf{T}_2| = |\mathbf{T}_4|$, which we can substitute into the above equations to obtain

$$m_1 g - |\mathbf{T}_2| = -m_1\ddot{x}_2,$$
$$m_2 g - |\mathbf{T}_2| = m_2\ddot{x}_2.$$

Eliminating $|\mathbf{T}_2|$ by subtracting the first equation from the second gives

$$m_2 g - m_1 g = m_2\ddot{x}_2 + m_1\ddot{x}_2.$$

Rearrangement gives the acceleration of the lift:

$$\ddot{x}_2 = \frac{(m_2 - m_1)g}{m_1 + m_2}.$$

(b) Substituting for $\ddot{x}_2$ in equation (2) gives

$$m_2 g\,\mathbf{i} + \mathbf{T}_4 = m_2\left(\frac{(m_2 - m_1)g}{m_1 + m_2}\right)\mathbf{i}.$$

Rearranging gives

$$\begin{aligned}\mathbf{T}_4 &= m_2 g\left(\frac{m_2 - m_1}{m_1 + m_2} - 1\right)\mathbf{i}\\ &= m_2 g\left(\frac{m_2 - m_1 - m_1 - m_2}{m_1 + m_2}\right)\mathbf{i}\\ &= -\frac{2m_1 m_2 g}{m_1 + m_2}\,\mathbf{i}.\end{aligned}$$

From part (a) we have $|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{T}_3| = |\mathbf{T}_4|$, so the magnitude of the tension in the cable is $2m_1 m_2 g/(m_1 + m_2)$.

(c) Resolving equation (3) in the $\mathbf{i}$-direction, where $\mathbf{S} = |\mathbf{S}|(-\mathbf{i})$, gives

$$-|\mathbf{S}| + |\mathbf{T}_1| + |\mathbf{T}_3| = 0.$$

From part (a) we have $|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{T}_3| = |\mathbf{T}_4|$, so

$$|\mathbf{S}| = 2|\mathbf{T}_2| = \frac{4m_1m_2g}{m_1+m_2}.$$

(d) Substituting the given values $m_1 = 1000$ and $m_2 = 1065$ into the equation for the acceleration of the lift in part (a) gives

$$\ddot{x}_2 = 65g/2065 \simeq 0.31.$$

The speed of the lift after it has travelled 100 m can be calculated using the constant acceleration formula $v^2 = v_0^2 + 2a_0(x - x_0)$, where v_0 is the initial speed of the lift (zero in this case). Substituting in the values gives

This constant acceleration formula was derived in Unit 3, Subsection 1.2.

$$v^2 \simeq 0^2 + 2 \times 0.31 \times 100 = 62.$$

So the speed of the lift after travelling 100 m is $\sqrt{62} \simeq 7.9\,\mathrm{m\,s^{-1}}$.

This speed is more than fast enough to give the occupant a nasty bump as the lift hits the floor, despite the acceleration being quite small! This explains why such a design is impractical: a lift that travels with constant acceleration (either up or down) could be quite dangerous.

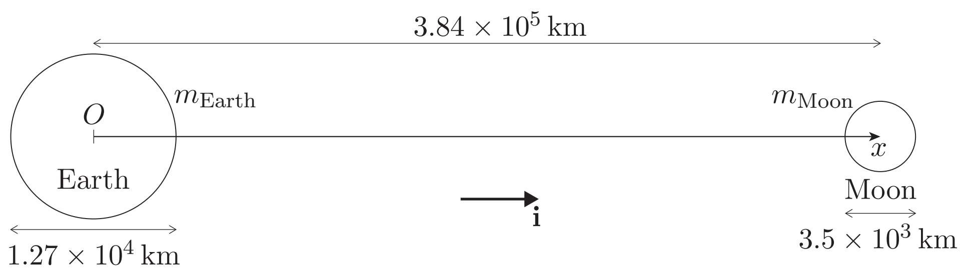
Solution to Exercise 2

The internal forces are $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$, $\mathbf{T}_5$, $\mathbf{T}_6$, $\mathbf{T}_7$ and $\mathbf{T}_8$. The external forces are $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{W}_1$ and $\mathbf{W}_2$.

The total force is $\mathbf{F} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{W}_1 + \mathbf{W}_2$.

Solution to Exercise 3

Since the radii of the Earth and the Moon are small compared with their distance apart, we can model each as a particle so that we have a two-particle system. The system is pictured in the diagram below (not to scale!).



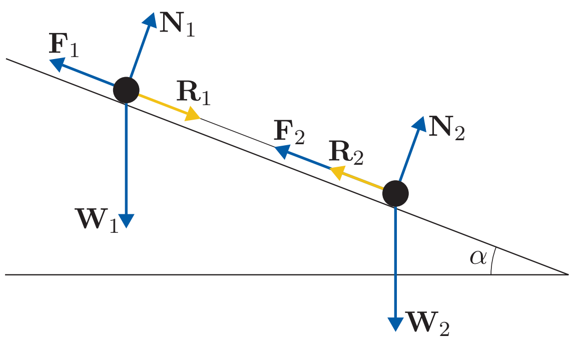
If we choose an x-axis joining the centre of the Earth to the centre of the Moon, as shown in the figure, with the centre of the Earth at the origin, then using the definition of centre of mass we have, working in kilometres,

$$\begin{aligned}\mathbf{r}_G &= \frac{m_{\text{Earth}}\mathbf{r}_{\text{Earth}} + m_{\text{Moon}}\mathbf{r}_{\text{Moon}}}{m_{\text{Earth}} + m_{\text{Moon}}}\\ &= \frac{(5.97\times 10^{24})\times \mathbf{0} + (7.34\times 10^{22})\times(3.84\times 10^5)\mathbf{i}}{5.97\times 10^{24} + 7.34\times 10^{22}}\\ &\simeq 4660\mathbf{i}.\end{aligned}$$

So the distance from the centre of the Earth to the centre of mass of the Earth/Moon system is about 4660 km. But the radius of the Earth is about 6350 km, so the centre of mass of this system is about 1690 km *below* the Earth's surface.

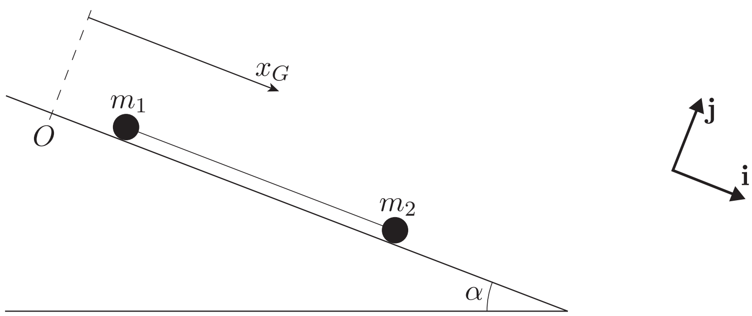
Solution to Exercise 4

(a) The force diagram is shown below.



The only internal forces are those due to the rod, $\mathbf{R}_1$ and $\mathbf{R}_2$. The remaining forces – the weights $\mathbf{W}_1$ and $\mathbf{W}_2$, the normal reaction forces $\mathbf{N}_1$ and $\mathbf{N}_2$, and the friction forces $\mathbf{F}_1$ and $\mathbf{F}_2$ – are all external forces.

(b) We define axes as shown below, with origin at the initial position of the upper particle.



Equation (10), $\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{r}}_G$, gives the equation of motion

$$\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{F}_1 + \mathbf{F}_2 = (m_1 + m_2)\ddot{\mathbf{r}}_G,$$

where $\mathbf{r}_G = x_G\mathbf{i}$ is the position vector of the centre of mass relative to O. We have

$$\begin{aligned}
\mathbf{W}_1 &= m_1 g(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j}),\\
\mathbf{W}_2 &= m_2 g(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j}),\\
\mathbf{N}_1 &= |\mathbf{N}_1|\mathbf{j}, \quad \mathbf{N}_2 = |\mathbf{N}_2|\mathbf{j},\\
\mathbf{F}_1 &= -|\mathbf{F}_1|\mathbf{i}, \quad \mathbf{F}_2 = -|\mathbf{F}_2|\mathbf{i}.
\end{aligned}$$

Also, $|\mathbf{F}_1| = \mu'|\mathbf{N}_1|$ and $|\mathbf{F}_2| = \mu'|\mathbf{N}_2|$.

Thus the equation of motion becomes

$$\begin{aligned}
&m_1 g(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j}) + m_2 g(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j})\\
&\quad + |\mathbf{N}_1|\mathbf{j} + |\mathbf{N}_2|\mathbf{j} - \mu'|\mathbf{N}_1|\mathbf{i} - \mu'|\mathbf{N}_2|\mathbf{i} = (m_1 + m_2)\ddot{x}_G\mathbf{i}.
\end{aligned}$$

Resolving in the **i**- and **j**-directions in turn gives the two equations

$$\begin{aligned}
&(m_1 + m_2)g\sin\alpha - \mu'(|\mathbf{N}_1| + |\mathbf{N}_2|) = (m_1 + m_2)\ddot{x}_G,\\
&-(m_1 + m_2)g\cos\alpha + |\mathbf{N}_1| + |\mathbf{N}_2| = 0.
\end{aligned}$$

Hence, from the second equation, we have

$$|\mathbf{N}_1| + |\mathbf{N}_2| = (m_1 + m_2)g\cos\alpha.$$

Substituting this into the first equation gives

$$(m_1 + m_2)g(\sin\alpha - \mu'\cos\alpha) = (m_1 + m_2)\ddot{x}_G,$$

which simplifies to

$$\ddot{x}_G = g(\sin\alpha - \mu'\cos\alpha).$$

This is the required equation of motion for the centre of mass.

(c) Changing the rod to a light spring would change only the internal forces; the external forces would remain the same. Similarly, removing the rod would not change the external forces. So in both cases the equation of motion of the centre of mass will be as in part (b).

(d) You saw in Unit 3 (e.g. Example 6 and Exercise 19) that the equation of motion of a particle sliding down an inclined plane is

$$a = g(\sin\alpha - \mu'\cos\alpha),$$

where a is the acceleration of the particle and α is the angle of incline. This is identical to the equation of motion for the centre of mass found in part (b) with $\ddot{x}_G$ replaced by a. Thus the motion of the centre of mass of a two-particle system sliding down an inclined plane is the same as the motion of a single particle sliding down the plane.

Solution to Exercise 5

The equation $\mathbf{P} = 2m\ddot{x}_G\mathbf{i}$ still holds, so $\ddot{x}_G = |\mathbf{P}|/(2m)$. But now d is constant, so $x_1 = x_G - d$ and $x_2 = x_G + d$ give $\ddot{x}_1 = \ddot{x}_G = \ddot{x}_2$. Hence A, B and G all move with the same fixed acceleration $|\mathbf{P}|/(2m)$. The system moves as a rigid body and there is no oscillatory motion.

We denote the forces on particles A and B due to the model rod by $\mathbf{R}_1$ and $\mathbf{R}_2$. Applying Newton's second law to particle A gives

$$\mathbf{R}_1 = m\ddot{x}_1\mathbf{i} = m\ddot{x}_G\mathbf{i} = \tfrac{1}{2}|\mathbf{P}|\mathbf{i}.$$

Also, since $\mathbf{R}_2 = -\mathbf{R}_1$, we have $\mathbf{R}_2 = -\frac{1}{2}|\mathbf{P}|\mathbf{i}$. So the forces exerted by the rod on A and B have half the magnitude of $\mathbf{P}$ and are directed from each particle towards the centre of the rod.

Solution to Exercise 6

There are no external forces, so from equation (10) we have $\ddot{\mathbf{r}}_G = \mathbf{0}$, and it follows that $\dot{x}_G = c$ (a constant). Now

$$x_G = \frac{mx_1 + mx_2}{m + m} = \frac{x_1 + x_2}{2},$$

where $\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = x_2\mathbf{i}$ are the position vectors of A and B. So

$$\dot{x}_G = \frac{\dot{x}_1 + \dot{x}_2}{2} = c.$$

Initially, we have $\dot{x}_1 = 0$ and $\dot{x}_2 = 6$, so $c = 3$. Integrating the equation $\dot{x}_G = 3$ with respect to time gives $x_G = 3t + 1.5$ (because the centre of mass is initially at $x = 1.5$). Substituting $t = 0.5$ into this gives $x_G = (3 \times 0.5) + 1.5 = 3$ and $x_2 = 3$, therefore we must also have $x_1 = 3$, that is, the particles collide at $t = 0.5$.

Solution to Exercise 7

Not a chance! If we consider the ship and the alien as a two-particle system, then from Newton's third law, any force that the alien exerts on the ship is met by an equal and opposite force from the ship on the alien. In other words, nothing that the alien does can affect the external forces acting on the system, so he cannot change the motion of the centre of mass of the two-particle system.

Solution to Exercise 8

We can use notation similar to that of Example 2, using $x_G\mathbf{i}$ as the position vector of the centre of mass G (which is midway between A and B), and $x_1\mathbf{i}, x_2\mathbf{i}$ as the position vectors of A and B, all with respect to a horizontal x-axis with origin at the position of A before the force is applied.

As in Example 2, the motion is in a horizontal plane, so the vertical forces must sum to zero, that is, $\mathbf{N}_1 = -\mathbf{W}_1$ and $\mathbf{N}_2 = -\mathbf{W}_2$. So from equation (10) we have

$$\mathbf{P} = \mathbf{F}^{\text{ext}} = (m_1 + m_2)\ddot{\mathbf{r}}_G = 100\ddot{x}_G\mathbf{i}.$$

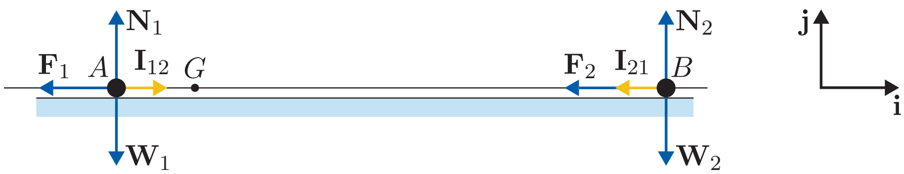
The string is taut and inextensible, so A, B and G have the same acceleration, and in particular $\ddot{x}_2\mathbf{i} = \ddot{x}_G\mathbf{i}$. Let $\mathbf{T}$ be the force exerted by the string on B. Then applying Newton's second law to B gives

$$\mathbf{P} + \mathbf{T} = m_2\ddot{x}_2\mathbf{i} = 50\ddot{x}_G\mathbf{i} = \tfrac{1}{2}\mathbf{P}.$$

It follows that $\mathbf{T} = -\frac{1}{2}\mathbf{P}$, so if the string is not to break, the magnitude of $\mathbf{P}$ must not exceed 20 N.

Solution to Exercise 9

Choose the wire to lie along the x-axis, and suppose that the particles are moving in the positive direction (left to right). As shown in the figure below, the forces acting on the particles are the weights $\mathbf{W}_1$ and $\mathbf{W}_2$, the normal reactions of the wire $\mathbf{N}_1$ and $\mathbf{N}_2$, and friction forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (which act at right angles to the other forces).



We have

$$\mathbf{N}_1 = -\mathbf{W}_1, \quad \mathbf{N}_2 = -\mathbf{W}_2,$$
$$|\mathbf{F}_1| = \mu'|\mathbf{N}_1|, \quad |\mathbf{F}_2| = \mu'|\mathbf{N}_2|.$$

If particle A has mass m_1 and particle B has mass m_2, then the friction force on A is $\mathbf{F}_1 = -\mu' m_1 g\mathbf{i}$ and the friction force on B is $\mathbf{F}_2 = -\mu' m_2 g\mathbf{i}$. Since the normal reactions and weights balance each other, the total external force is

$$\mathbf{F}^{\text{ext}} = \mathbf{F}_1 + \mathbf{F}_2 = -\mu'(m_1 + m_2)g\mathbf{i}.$$

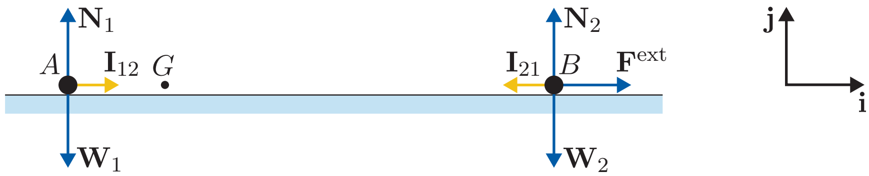
Hence, by equation (10), we have

$$-\mu'(m_1 + m_2)g\mathbf{i} = (m_1 + m_2)\ddot{\mathbf{r}}_G,$$

giving $\ddot{\mathbf{r}}_G = -\mu' g\mathbf{i} = -0.4g\mathbf{i}$. So the centre of mass accelerates at $0.4g\,\text{m s}^{-2}$ in the direction opposite to that of the motion of the particles. (Notice that the masses of the particles do not affect this result.)

Solution to Exercise 10

We choose the x-axis to lie along the track in the direction of the external force – see the figure below.



Using equation (10) with $\mathbf{F}^{\text{ext}} = 20\mathbf{i}$ and total mass $M = 5$, we have $20\mathbf{i} = 5\ddot{\mathbf{r}}_G$, so $\ddot{\mathbf{r}}_G = 4\mathbf{i}$. Therefore the common acceleration of the particles is $4\,\mathrm{m\,s^{-2}}$.

Applying Newton's second law to the first particle, we obtain

$$\text{internal force} = 4 \times 4\mathbf{i} = 16\mathbf{i},$$

so the internal forces have magnitude $16\,\mathrm{N}$.

Solution to Exercise 11

(a) If we let $\mathbf{i}$ denote a unit vector in the positive x-direction, then we have particles of mass m at the points $\mathbf{r}_1 = \mathbf{0}$, $\mathbf{r}_2 = 4\mathbf{i}$ and $\mathbf{r}_3 = 6\mathbf{i}$, relative to the origin. From equation (15) we have

$$\mathbf{r}_G = \frac{m\mathbf{r}_1 + m\mathbf{r}_2 + m\mathbf{r}_3}{m+m+m} = \frac{\mathbf{0} + 4\mathbf{i} + 6\mathbf{i}}{3} = \tfrac{10}{3}\mathbf{i}.$$

(b) Similarly,

$$\mathbf{r}_G = \frac{\mathbf{0} + \mathbf{i} + 4\mathbf{i} + 5\mathbf{i} + 6\mathbf{i}}{5} = \tfrac{16}{5}\mathbf{i}.$$

Solution to Exercise 12

(a) Let $\mathbf{i}$ be a unit vector in the positive x-direction. The most obvious way to calculate the centre of mass is to use equation (15), giving

$$\mathbf{r}_G = \frac{m \times (-2\mathbf{i}) + m \times \mathbf{0} + m \times 2\mathbf{i} + m \times 4\mathbf{i}}{m+m+m+m} = \frac{4m\mathbf{i}}{4m},$$

so $\mathbf{r}_G = \mathbf{i}$.

Alternatively, in this case the centre of mass can also be found by inspection, since the system is symmetric about the point $(1, 0)$. So this is the centre of mass, that is, $\mathbf{r}_G = \mathbf{i}$.

(b) To find the position vector of the centre of mass, relative to the origin, consider Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the positive x- and y-directions. Now use equation (15) to obtain

$$\mathbf{r}_G = \frac{m\mathbf{r}_1 + m\mathbf{r}_2 + m\mathbf{r}_3}{m+m+m} = \frac{\mathbf{0} + 2\mathbf{i} + (\mathbf{i} + \sqrt{3}\mathbf{j})}{3} = \mathbf{i} + \tfrac{1}{\sqrt{3}}\mathbf{j}.$$

Alternatively, you could have used symmetry. In this case notice that the particles are at the corners of an equilateral triangle (Pythagoras' theorem tells us that the sloping sides are each of length 2). So the centre of mass is at the geometric centre of the triangle.

If you know that the geometric centre of an equilateral triangle is one-third of the way up the perpendicular from a base to an apex, then you can write down the equation immediately as $\mathbf{r}_G = \mathbf{i} + \frac{1}{3}\left(\sqrt{3}\mathbf{j}\right) = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j}$.

Solution to Exercise 13

The squares are of equal size and hence of equal mass, m say, with centres of mass at

$$\mathbf{r}_{G_1} = \tfrac{1}{2}a\mathbf{i} + \tfrac{3}{2}a\mathbf{j}, \quad \mathbf{r}_{G_2} = \tfrac{1}{2}a\mathbf{i} + \tfrac{1}{2}a\mathbf{j}, \quad \mathbf{r}_{G_3} = \tfrac{3}{2}a\mathbf{i} + \tfrac{1}{2}a\mathbf{j}.$$

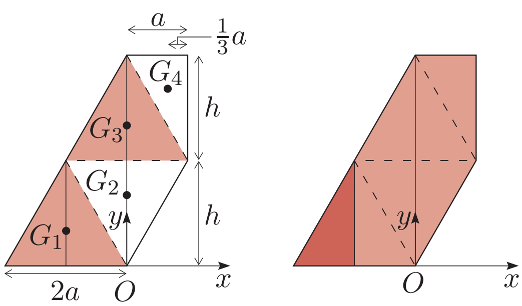
Hence, using equation (15), the centre of mass of the whole shape is at position vector

$$\mathbf{r}_G = \frac{m\left(\frac{1}{2}a\mathbf{i} + \frac{3}{2}a\mathbf{j}\right) + m\left(\frac{1}{2}a\mathbf{i} + \frac{1}{2}a\mathbf{j}\right) + m\left(\frac{3}{2}a\mathbf{i} + \frac{1}{2}a\mathbf{j}\right)}{3m} = \tfrac{5}{6}a\mathbf{i} + \tfrac{5}{6}a\mathbf{j},$$

which, as expected, is the same as the result obtained in Example 3.

Solution to Exercise 14

The uniform density and constant cross-sectional area mean that the problem reduces to a two-dimensional one. One way to find the centre of mass of the two-dimensional cross-section is to divide it into triangles as in Figure 26(b), and to find the centre of mass of each triangle (see the figure below).



Relative to the origin at O shown above, the position vectors of the centres of mass G_1, G_2, G_3 and G_4 are

$$\mathbf{r}_{G_1} = -a\mathbf{i} + \tfrac{1}{3}h\mathbf{j}, \quad \mathbf{r}_{G_2} = \tfrac{2}{3}h\mathbf{j}, \quad \mathbf{r}_{G_3} = \tfrac{4}{3}h\mathbf{j}, \quad \mathbf{r}_{G_4} = \tfrac{2}{3}a\mathbf{i} + \tfrac{5}{3}h\mathbf{j},$$

where h is half the height of the sculpture (as shown), and $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors in the positive x- and y-directions.

If we model the equilateral triangles as particles of mass m (say) at G_1, G_2 and G_3, then the fourth triangle can be modelled as a particle of mass $\frac{1}{2}m$ at G_4. Then equation (15) gives the centre of mass of the cross-section as

$$\begin{aligned}\mathbf{r}_G &= \frac{m(-a\mathbf{i}+\frac{1}{3}h\mathbf{j})+m(\frac{2}{3}h\mathbf{j})+m(\frac{4}{3}h\mathbf{j})+\frac{1}{2}m(\frac{2}{3}a\mathbf{i}+\frac{5}{3}h\mathbf{j})}{m+m+m+\frac{1}{2}m}\\ &= \frac{-\frac{2}{3}a\mathbf{i}+\frac{19}{6}h\mathbf{j}}{\frac{7}{2}}\\ &= -\tfrac{4}{21}a\mathbf{i}+\tfrac{19}{21}h\mathbf{j}.\end{aligned}$$

Since $\mathbf{r}_G$ lies to the left of O, over the base, we find that the sculpture will not topple over.

Alternatively, you could notice that the lightly shaded part of the cross-section on the right above is symmetric about the y-axis, so its centre of mass will lie on the y-axis. Hence the whole cross-section, with the darker shaded triangle added, must have its centre of mass to the left of O, so the sculpture will not topple over.

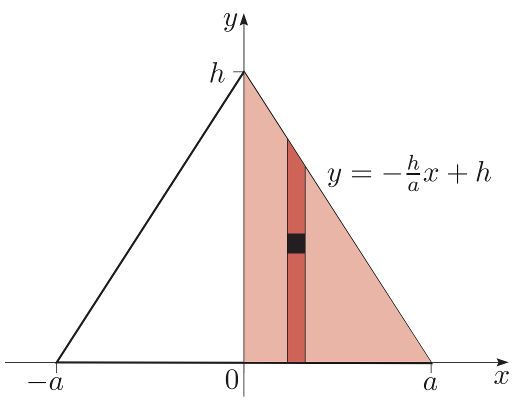
Solution to Exercise 15

The area integral is

$$\begin{aligned}\int_S yf\,dA &= f\int_{x=-a}^{x=a}\left(\int_{y=0}^{y=\sqrt{a^2-x^2}} y\,dy\right)dx\\ &= f\int_{x=-a}^{x=a}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{a^2-x^2}}\,dx\\ &= \tfrac{1}{2}f\int_{-a}^{a}(a^2-x^2)\,dx\\ &= \tfrac{1}{2}f\left[a^2x-\tfrac{1}{3}x^3\right]_{-a}^{a}\\ &= \tfrac{2}{3}fa^3,\end{aligned}$$

and since $M=\frac{1}{2}\pi a^2 f$, $y_G = 4a/3\pi$ as in Example 4.

Solution to Exercise 16



We have $x_G = 0$ by symmetry. Also by symmetry, the y-coordinates of the centres of mass of the two triangles in the positive and negative x-regions are the same (and equal to the y-coordinate of the centre of mass of the whole triangle), so we only need to find y_G for the triangle in the positive x-region. This is given by

$$y_G = \frac{1}{M} f \int_{x=0}^{x=a} \left(\int_{y=0}^{y=-\frac{h}{a}x+h} y\, dy \right) dx,$$

where the mass M of the half-triangle is $\frac{1}{2}ahf$. Thus

$$y_G = \frac{2}{ah} \int_{x=0}^{x=a} \tfrac{1}{2} \left(-\frac{h}{a}x + h \right)^2 dx = \tfrac{1}{3}h.$$

Note that
$$\int (\alpha x + \beta)^2\, dx = \frac{(\alpha x + \beta)^3}{3\alpha} + C.$$

Solution to Exercise 17

Taking a datum at ground level, the height of the centre of mass above the datum is $2h$ when upright and $\frac{1}{2}h$ when lying down. So the change in potential energy is $Mg \times 2h - Mg \times \frac{1}{2}h = \frac{3}{2}Mgh$.

Solution to Exercise 18

Let the x-axis point downwards, and let $\mathbf{i}$ be a unit vector in this direction. The total external force on the ball is its weight $Mg\mathbf{i}$, so from equation (19), we have

$$Mg\mathbf{i} = M\ddot{\mathbf{r}}_G,$$

which gives $\ddot{\mathbf{r}}_G = g\mathbf{i}$, so the centre of mass of the ball accelerates downwards with the acceleration due to gravity. We have $\mathbf{r}_G = x_G\mathbf{i}$ and

$$\ddot{x}_G = g.$$

So the acceleration is constant and we may use the constant acceleration formula $v^2 = v_0^2 + 2a_0(x - x_0)$ from Unit 3. In this case we have

$$\dot{x}_G^2 = \dot{x}_G(0)^2 + 2gh = 2gh,$$

since the ball starts from rest. So the speed of the ball after falling a distance h is $\sqrt{2gh}$.

(Notice that these equations are exactly what we would have obtained if we had modelled the ball as a particle, using the energy methods of Unit 9.)

The change in the potential energy is $-Mgh$.

Solution to Exercise 19

The solid shape is composed of two cylinders with radii a and $3a$, each with thickness $2a$. The centre of mass lies on the axis of symmetry (the x-axis in Figure 30). The positions of the centres of mass of the two cylinders, G_1 and G_2, relative to O are given by

$$\mathbf{r}_{G_1} = a\mathbf{i} \quad \text{and} \quad \mathbf{r}_{G_2} = 3a\mathbf{i}.$$

The volume of the smaller cylinder is given by

$$V_1 = \pi a^2 \times 2a = 2\pi a^3,$$

while the volume of the larger cylinder is given by

$$V_2 = \pi(3a)^2 \times 2a = 18\pi a^3.$$

If the density of the material is f, then the corresponding masses are

$$m_1 = 2\pi f a^3 \quad \text{and} \quad m_2 = 18\pi f a^3.$$

From equation (15) we have

$$\mathbf{r}_G = \frac{2\pi f a^3(a\mathbf{i}) + 18\pi f a^3(3a\mathbf{i})}{20\pi f a^3} = \tfrac{14}{5}a\mathbf{i}.$$

The centre of mass is a little to the left of G_2, as we would expect. The object will not topple over because $14a/5 > 2a$, so the centre of mass is to the right of the point P in Figure 30 (the leftmost point of contact with the surface).

Solution to Exercise 20

(a) The volume V of a pyramid is given by

$$V = \tfrac{1}{3} \times \text{base area} \times \text{vertical height},$$

so in this case we have (in m^3)

$$V = \tfrac{1}{3} \times (230)^2 \times 147 = 2\,592\,100.$$

Hence the total mass (in kg) is approximately

$$2\,592\,100 \times 2500 \simeq 6.48 \times 10^9.$$

(b) The centre of mass is a quarter of the height above the base, which in this case is $h/4$. The total energy required to lift all the stones from ground level is (in J)

$$\begin{aligned} Mg(h/4) &= (6.48 \times 10^9) \times 9.81 \times 147/4 \\ &\simeq 2.34 \times 10^{12}. \end{aligned}$$

(c) According to the question, the energy that a man can expend in a day is $1000g$ J. So the total number of days required by 1000 men is approximately

$$\frac{2.34 \times 10^{12}}{1000 \times 9.81 \times 1000} \simeq 2.4 \times 10^5.$$

A gang of 1000 men would therefore take about 652 years. (This figure is certainly a low estimate, since it takes no account of friction in the system used to lift the stones, or the time taken to construct the lifting mechanism. However, we can be reasonably sure that it would have taken 1000 men more than 600 years to lift the stones to the appropriate heights, which gives some idea of the scale of the enterprise.)

Solution to Exercise 21

If we take $\mathbf{i}$ to be a unit vector in the direction of motion, then the total linear momentum of the system just before the collision is $\mathbf{P}_{\text{before}} = Mu\mathbf{i} + m\mathbf{0} = Mu\mathbf{i}$, and the total linear momentum just after the collision is $\mathbf{P}_{\text{after}} = (M+m)v\mathbf{i}$. If the collision is instantaneous, then the principle of conservation of linear momentum tells us that

$$Mu\mathbf{i} = (M+m)v\mathbf{i}.$$

Resolving in the $\mathbf{i}$-direction and rearranging, we obtain

$$v = Mu/(M+m).$$

Solution to Exercise 22

(a) We define an x-axis along the track in the direction of motion of the first truck before impact. Let the trucks have masses m_1 and m_2, and velocities before impact $\dot{\mathbf{r}}_1 = \dot{x}_1\mathbf{i}$ and $\dot{\mathbf{r}}_2 = \dot{x}_2\mathbf{i}$. Then the total linear momentum before impact is

$$m_1\dot{x}_1\mathbf{i} + m_2\dot{x}_2\mathbf{i} = 15\,000 \times 4\mathbf{i} + 10\,000 \times \mathbf{0} = 60\,000\mathbf{i}.$$

After impact, the combined mass of 25 000 kg moves with speed V, say, so the linear momentum is $25\,000V\mathbf{i}$. By the principle of conservation of linear momentum,

$$60\,000\mathbf{i} = 25\,000V\mathbf{i},$$

so $V = 2.4$. Thus the combined speed after impact is $2.4\,\mathrm{m\,s^{-1}}$.

(b) The kinetic energy before impact is

$$\tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 = \tfrac{1}{2} \times 15\,000 \times 4^2 = 120\,000,$$

while after impact the kinetic energy is

$$\tfrac{1}{2} \times 25\,000 \times 2.4^2 = 72\,000,$$

so the collision is not elastic.

Solution to Exercise 23

Let the velocities of the balls after collision be $\mathbf{U}$ and $\mathbf{V}$. By conservation of linear momentum,

$$m\mathbf{u} = m\mathbf{U} + m\mathbf{V},$$

so

$$\mathbf{u} = \mathbf{U} + \mathbf{V}.$$

Since the collision is elastic, there is no loss of kinetic energy during the collision, and we have

$$\tfrac{1}{2}m\mathbf{u}\cdot\mathbf{u} = \tfrac{1}{2}m\mathbf{U}\cdot\mathbf{U} + \tfrac{1}{2}m\mathbf{V}\cdot\mathbf{V},$$

so

$$\mathbf{u}\cdot\mathbf{u} = \mathbf{U}\cdot\mathbf{U} + \mathbf{V}\cdot\mathbf{V}.$$

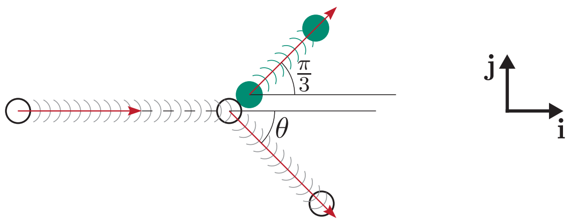
Since $\mathbf{u} = \mathbf{U} + \mathbf{V}$, we also have

$$\mathbf{u} \cdot \mathbf{u} = (\mathbf{U} + \mathbf{V}) \cdot (\mathbf{U} + \mathbf{V}) = \mathbf{U} \cdot \mathbf{U} + 2\mathbf{U} \cdot \mathbf{V} + \mathbf{V} \cdot \mathbf{V}.$$

Hence we must have $\mathbf{U} \cdot \mathbf{V} = 0$. This can be achieved only if $\mathbf{U} = \mathbf{0}$, $\mathbf{V} = \mathbf{0}$, or the angle between $\mathbf{U}$ and $\mathbf{V}$ is $\pi/2$, as required.

Solution to Exercise 24

Choose the positive x-direction to be along the direction of motion of the white ball before the collision, and the positive y-direction to correspond to a positive $\mathbf{j}$-component for the velocity of the green ball after the collision, where $\mathbf{i}$ and $\mathbf{j}$ are Cartesian unit vectors in the positive x- and y-directions. The situation is illustrated in the figure below.



From Exercise 23, since the collision is elastic and the balls are identical apart from their colour, the white ball moves at right angles to the green ball, so we can deduce that $\theta = \pi/6$.

If $\dot{\mathbf{R}}_1$ and $\dot{\mathbf{R}}_2$ are the velocities of the white and green balls, respectively, after the collision, then we have

$$\begin{aligned}\dot{\mathbf{R}}_1 &= (|\dot{\mathbf{R}}_1| \cos\theta)\mathbf{i} + (|\dot{\mathbf{R}}_1| \sin\theta)(-\mathbf{j}) \\ &= U \cos\tfrac{\pi}{6}\,\mathbf{i} - U \sin\tfrac{\pi}{6}\,\mathbf{j} \\ &= \tfrac{\sqrt{3}}{2}U\mathbf{i} - \tfrac{1}{2}U\mathbf{j},\end{aligned}$$

where $U = |\dot{\mathbf{R}}_1|$ is the speed of the white ball, and

$$\begin{aligned}\dot{\mathbf{R}}_2 &= (|\dot{\mathbf{R}}_2| \cos\tfrac{\pi}{3})\mathbf{i} + (|\dot{\mathbf{R}}_2| \sin\tfrac{\pi}{3})\mathbf{j} \\ &= \tfrac{1}{2}V\mathbf{i} + \tfrac{\sqrt{3}}{2}V\mathbf{j},\end{aligned}$$

where $V = |\dot{\mathbf{R}}_2|$ is the speed of the green ball. Also, if $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ are the velocities of the white and green balls before the collision, then we have

$$\dot{\mathbf{r}}_1 = u\mathbf{i} \quad \text{and} \quad \dot{\mathbf{r}}_2 = \mathbf{0}.$$

Therefore, by the principle of conservation of linear momentum,

$$mu\mathbf{i} = m\left(\tfrac{\sqrt{3}}{2}U\mathbf{i} - \tfrac{1}{2}U\mathbf{j}\right) + m\left(\tfrac{1}{2}V\mathbf{i} + \tfrac{\sqrt{3}}{2}V\mathbf{j}\right).$$

Resolving in the $\mathbf{i}$- and $\mathbf{j}$-directions, and dividing by m, gives

$$u = \tfrac{\sqrt{3}}{2}U + \tfrac{1}{2}V, \quad 0 = -\tfrac{1}{2}U + \tfrac{\sqrt{3}}{2}V.$$

Thus $U = \sqrt{3}V$, hence we have $V = \frac{1}{2}u$ and $U = \frac{\sqrt{3}}{2}u$, so

$$\dot{\mathbf{R}}_1 = u\left(\tfrac{3}{4}\mathbf{i} - \tfrac{\sqrt{3}}{4}\mathbf{j}\right) \quad \text{and} \quad \dot{\mathbf{R}}_2 = u\left(\tfrac{1}{4}\mathbf{i} + \tfrac{\sqrt{3}}{4}\mathbf{j}\right).$$

Solution to Exercise 25

Suppose that the velocity of the particles immediately after the collision is $\mathbf{V}$. Then, from the principle of conservation of linear momentum, we have

$$m_1u\mathbf{i} + m_2u\mathbf{j} = (m_1 + m_2)\mathbf{V}.$$

It follows that

$$\begin{aligned}\mathbf{V} &= \frac{u(m_1\mathbf{i} + m_2\mathbf{j})}{m_1 + m_2} \\ &= \frac{m_1u}{m_1 + m_2}\,\mathbf{i} + \frac{m_2u}{m_1 + m_2}\,\mathbf{j}.\end{aligned}$$

The kinetic energy before impact is $\frac{1}{2}(m_1 + m_2)u^2$, while the kinetic energy after impact is

$$\tfrac{1}{2}(m_1 + m_2)\left[\left(\frac{m_1u}{m_1 + m_2}\right)^2 + \left(\frac{m_2u}{m_1 + m_2}\right)^2\right] = \tfrac{1}{2}\,\frac{m_1^2 + m_2^2}{m_1 + m_2}\,u^2.$$

The change in kinetic energy is

$$\begin{aligned}\tfrac{1}{2}(m_1 + m_2)u^2 - \tfrac{1}{2}\left(\frac{m_1^2 + m_2^2}{m_1 + m_2}\right)u^2 &= \tfrac{1}{2}\left(\frac{2m_1m_2}{m_1 + m_2}\right)u^2 \\ &= \frac{m_1m_2u^2}{m_1 + m_2}.\end{aligned}$$

The collision is not elastic since the loss of kinetic energy is greater than zero.

Solution to Exercise 26

(a) If $m_1 = m_2 = 3$, $\mathbf{v}_1 = 2\mathbf{i}$, $\mathbf{v}_2 = \mathbf{0}$ and $\mathbf{V}_1 = \mathbf{i} + \mathbf{j}$, $\mathbf{V}_2 = \mathbf{i} - \mathbf{j}$, then the kinetic energy just before impact is

$$\tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2 = \tfrac{1}{2}(3 \times 4 + 3 \times 0) = 6,$$

and the kinetic energy just after impact is

$$\tfrac{1}{2}m_1|\mathbf{V}_1|^2 + \tfrac{1}{2}m_2|\mathbf{V}_2|^2 = \tfrac{1}{2}(3 \times 2 + 3 \times 2) = 6.$$

Therefore the collision is elastic since no kinetic energy is lost during impact.

(b) If $m_1 = 1$, $m_2 = 3$, $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v}_2 = \mathbf{j}$ and $\mathbf{V}_1 = \mathbf{V}_2 = \frac{1}{2}\mathbf{i} + \mathbf{j}$, then the kinetic energy just before impact is

$$\tfrac{1}{2}m_1|\mathbf{v}_1|^2 + \tfrac{1}{2}m_2|\mathbf{v}_2|^2 = \tfrac{1}{2}(1 \times 5 + 3 \times 1) = 4,$$

and the kinetic energy just after impact is

$$\tfrac{1}{2}m_1|\mathbf{V}_1|^2 + \tfrac{1}{2}m_2|\mathbf{V}_2|^2 = \tfrac{1}{2}(1 \times 1.25 + 3 \times 1.25) = 2.5.$$

As there is a loss of kinetic energy, the collision is inelastic.

The decrease in kinetic energy is 1.5 J.

Solution to Exercise 27

(a) Take the positive x-axis to be in the direction of motion of ball A before impact, and let $\mathbf{i}$ be a unit vector in this direction. Then we can write the velocities of the balls before impact as $\dot{\mathbf{r}}_1 = \dot{x}_1\mathbf{i}$ and $\dot{\mathbf{r}}_2 = \dot{x}_2\mathbf{i}$, and after impact as $\dot{\mathbf{R}}_1 = \dot{X}_1\mathbf{i}$ and $\dot{\mathbf{R}}_2 = \dot{X}_2\mathbf{i}$. Modelling the balls as particles, we can use the principle of conservation of linear momentum to obtain

$$m_1\dot{x}_1\mathbf{i} + m_2\dot{x}_2\mathbf{i} = m_1\dot{X}_1\mathbf{i} + m_2\dot{X}_2\mathbf{i}.$$

Also, equation (30) gives the relative velocity after impact as

$$\dot{X}_1\mathbf{i} - \dot{X}_2\mathbf{i} = -e(\dot{x}_1\mathbf{i} - \dot{x}_2\mathbf{i}).$$

Resolving both equations in the $\mathbf{i}$-direction gives

$$\begin{aligned} m_1\dot{x}_1 + m_2\dot{x}_2 &= m_1\dot{X}_1 + m_2\dot{X}_2, \\ \dot{X}_1 - \dot{X}_2 &= -e(\dot{x}_1 - \dot{x}_2). \end{aligned}$$

Eliminating $\dot{X}_1$ between these equations, we obtain

$$\begin{aligned} m_1\dot{x}_1 + m_2\dot{x}_2 &= m_1(\dot{X}_2 - e(\dot{x}_1 - \dot{x}_2)) + m_2\dot{X}_2 \\ &= (m_1 + m_2)\dot{X}_2 - em_1(\dot{x}_1 - \dot{x}_2), \end{aligned}$$

so

$$\dot{X}_2 = \frac{m_1(1+e)\dot{x}_1 + (m_2 - em_1)\dot{x}_2}{m_1 + m_2}.$$

Therefore

$$\begin{aligned} \dot{X}_1 &= \dot{X}_2 - e(\dot{x}_1 - \dot{x}_2) \\ &= \frac{(m_1 - em_2)\dot{x}_1 + m_2(1+e)\dot{x}_2}{m_1 + m_2}. \end{aligned}$$

So the velocities after impact are

$$\begin{aligned} \dot{\mathbf{R}}_1 &= \frac{(m_1 - em_2)\dot{x}_1 + m_2(1+e)\dot{x}_2}{m_1 + m_2}\,\mathbf{i}, \\ \dot{\mathbf{R}}_2 &= \frac{m_1(1+e)\dot{x}_1 + (m_2 - em_1)\dot{x}_2}{m_1 + m_2}\,\mathbf{i}. \end{aligned}$$

(b) The situation here is the same as in Example 6 with $m_1 = m_2 = m$, $\dot{x}_1 = 5$, $\dot{x}_2 = 0$ and $e = 1$ (since the collision is elastic), so we have

$$\begin{aligned} \dot{X}_1 &= \frac{(m-m)5 + m(1+1)0}{m+m} = 0, \\ \dot{X}_2 &= \frac{m(1+1)5 + (m-m)0}{m+m} = 5, \end{aligned}$$

as in Example 6.

(c) If the balls coalesce, then $\dot{X}_1 = \dot{X}_2$, so $e(\dot{x}_1 - \dot{x}_2) = 0$. So either $e = 0$ or $\dot{x}_1 = \dot{x}_2$. However, the balls will not collide if $\dot{x}_1 = \dot{x}_2$, so $e = 0$.

Solution to Exercise 28

Let U and V be the speeds after the collision of the balls of masses m_1 and m_2, respectively. If we let $\mathbf{i}$ be a unit vector in the direction of motion, then the principle of conservation of linear momentum gives

$$m_1u\mathbf{i} = m_1U\mathbf{i} + m_2V\mathbf{i},$$

so resolving in the $\mathbf{i}$-direction, we obtain

$$m_1u = m_1U + m_2V.$$

Equation (30) gives the relative velocity after impact as

$$V\mathbf{i} - U\mathbf{i} = -e(\mathbf{0} - u\mathbf{i}),$$

so resolving in the $\mathbf{i}$-direction, we obtain

$$V - U = -e(0 - u) = eu.$$

Solving these equations for U and V, we find that

$$U = \frac{m_1 - em_2}{m_1 + m_2}\,u$$

and

$$V = \frac{m_1(1+e)}{m_1 + m_2}\,u.$$

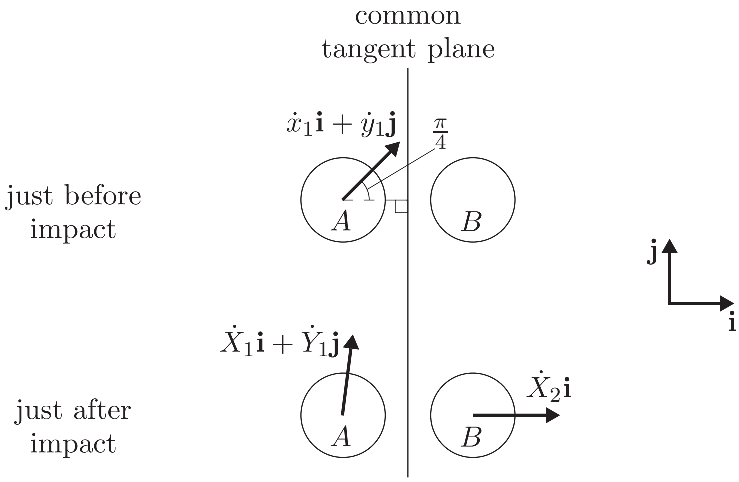
The change in kinetic energy is

$$\begin{aligned}
&\tfrac{1}{2}m_1u^2 - \left(\tfrac{1}{2}m_1U^2 + \tfrac{1}{2}m_2V^2\right)\\
&= \tfrac{1}{2}m_1u^2 - \tfrac{1}{2}m_1\left(\frac{m_1 - em_2}{m_1 + m_2}\right)^2 u^2 - \tfrac{1}{2}m_2\left(\frac{m_1(1+e)}{m_1 + m_2}\right)^2 u^2\\
&= \tfrac{1}{2}m_1u^2 - \tfrac{1}{2}\,\frac{m_1^3 - 2em_1^2m_2 + e^2m_1m_2^2 + m_1^2m_2(1 + 2e + e^2)}{(m_1 + m_2)^2}\,u^2\\
&= \tfrac{1}{2}m_1u^2 - \tfrac{1}{2}\,\frac{m_1^2(m_1 + m_2) + e^2m_1m_2(m_1 + m_2)}{(m_1 + m_2)^2}\,u^2\\
&= \tfrac{1}{2}\,\frac{m_1(m_1 + m_2) - m_1^2 - e^2m_1m_2}{m_1 + m_2}\,u^2\\
&= \frac{m_1m_2(1 - e^2)u^2}{2(m_1 + m_2)}.
\end{aligned}$$

(Notice that if $e = 1$, then no kinetic energy is lost from the system and the collision is elastic. If $e = 0$, then the kinetic energy lost is a maximum.)

Solution to Exercise 29

We choose Cartesian unit vectors as shown in the figure below, which also shows the common tangent plane at the moment of impact, and the velocities of A and B just before and just after impact.



Using the fact that A has speed u just before impact while B is stationary, from the figure we see that

$$\dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j} = \left(u\cos\tfrac{\pi}{4}\right)\mathbf{i} + \left(u\sin\tfrac{\pi}{4}\right)\mathbf{j} = \tfrac{1}{\sqrt{2}}u\mathbf{i} + \tfrac{1}{\sqrt{2}}u\mathbf{j},$$
$$\dot{x}_2\mathbf{i} + \dot{y}_2\mathbf{j} = \mathbf{0}.$$

So, by Newton's law of restitution and noticing that the common normal is parallel to $\mathbf{i}$, we have

$$\dot{X}_1 - \dot{X}_2 = -e(\dot{x}_1 - \dot{x}_2) = -e\,\tfrac{1}{\sqrt{2}}u,$$
$$\dot{Y}_1 = \dot{y}_1 = \tfrac{1}{\sqrt{2}}u, \quad \dot{Y}_2 = \dot{y}_2 = 0.$$

Since $e = 0$, we have $\dot{X}_1 = \dot{X}_2$. Now, using the principle of conservation of linear momentum, we obtain

$$m(\dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j}) + m(\dot{x}_2\mathbf{i} + \dot{y}_2\mathbf{j}) = m(\dot{X}_1\mathbf{i} + \dot{Y}_1\mathbf{j}) + m(\dot{X}_2\mathbf{i} + \dot{Y}_2\mathbf{j}),$$

so, resolving in the $\mathbf{i}$-direction and dividing by m, we have

$$\dot{x}_1 + \dot{x}_2 = \dot{X}_1 + \dot{X}_2,$$

hence

$$\dot{X}_1 + \dot{X}_2 = \tfrac{1}{\sqrt{2}}u.$$

Therefore, since $\dot{X}_1 = \dot{X}_2$, we have

$$\dot{X}_1 = \dot{X}_2 = \tfrac{1}{2\sqrt{2}}u,$$

and the velocities just after impact are

$$\dot{X}_1\mathbf{i} + \dot{Y}_1\mathbf{j} = \tfrac{1}{2\sqrt{2}}u\mathbf{i} + \tfrac{1}{\sqrt{2}}u\mathbf{j}, \quad \dot{X}_2\mathbf{i} + \dot{Y}_2\mathbf{j} = \tfrac{1}{2\sqrt{2}}u\mathbf{i}.$$

(In the question you were told that the common tangent plane at the moment of impact is perpendicular to the subsequent direction of motion of B. This piece of information is, in fact, redundant. Ball B was originally *stationary*, so after impact it must move in the direction perpendicular to the common tangent plane, i.e. in the direction of the common normal.)

Solution to Exercise 30

Let the ball have mass m and start from height h_1. By using conservation of energy we have $mgh_1 = \frac{1}{2}mv_1^2$, where v_1 is the speed of the ball just before the first bounce. The speed of the ball just after the first bounce is given by $v_2 = ev_1$, using Newton's law of restitution, where e is the coefficient of restitution. By conservation of energy, we have $mgh_2 = \frac{1}{2}mv_2^2$, where h_2 is the maximum height after the first bounce. Putting these pieces of information together, we have

$$mgh_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}m(ev_1)^2 = e^2 \times mgh_1,$$

that is, $h_2 = e^2 h_1$.

Similarly, we find that the height h_3 after the second bounce is related to the height before the second bounce by $h_3 = e^2 h_2$, so $h_3 = e^4 h_1$. But we are told that $h_3 = \frac{1}{2}h_1$, so $e^4 = \frac{1}{2}$, hence

$$e = \frac{1}{\sqrt[4]{2}} \simeq 0.84 \quad \text{(correct to 2 d.p.)}.$$