20)

a) Let $\frac{dx}{dt}$ be the rate of change of the amount of chemical per unit time. By the input-output principle the accumulation of chemical is $\alpha - \beta$, therefore a differential equation satisfied by $x$ would be

$$\frac{dx}{dt} = \alpha - \beta$$

b) At steady-state $\frac{dx}{dt} = 0$, so the model predicts $\alpha = \beta$, i.e. the rates of input and decay are equal.

21)

a) Rearranging into the form $\frac{dy}{dx} = g(x)h(y)$

$$\frac{dy}{dx} = \frac{1}{x^2} \times \frac{1}{y}$$

applying the separation of variables method

$$\int y \, dy = \int x^{-2} \, dx$$

$$\tfrac{1}{2} y^2 = -\frac{1}{x} + C$$

$$y^2 = 2C - \frac{2}{x}$$

$$y = \sqrt{D - \frac{2}{x}}$$

where $D$ is an arbitrary constant

b) Subsituting $y = 1$ and $x = 1$ into the solution gives

$$1 = \sqrt{D - 2}$$

$$1 = D - 2$$

$$D = 3$$

So the particular solution satisfying $y(1) = 1$ is

$$y = \sqrt{3 - \frac{2}{x}}$$

22

a) In matrix form this system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

b) The coefficient matrix is triangular, so the eigenvalues are the diagonal elements 1 and −1. An eigenvector for $\lambda = 1$ is a solution of

$$\begin{pmatrix} 1-1 & 0 \\ 1 & -1-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underline{0}$$

which gives

$$x - 2y = 0$$
$$x = 2y$$

So $(2\ 1)^T$ is an eigenvector for $\lambda = 1$

An eigenvector for $\lambda = -1$ is a solution of

$$\begin{pmatrix} 1-(-1) & 0 \\ 1 & -1-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underline{0}$$

which gives

$$2x = 0$$
$$x = 0$$

So $(0\ 1)^T$ is an eigenvector for $\lambda = -1$

c) The general solution is therefore

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

where $\alpha$ and $\beta$ are arbitrary constants.

23

a) $X(x)$ must satisfy the boundary conditions

$$X(0) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x} X(\pi) = 0$$

b) If $K$ is negative then the solution takes the form

$$X(x) = A\cos(\sqrt{-K}\,x) + B\sin(\sqrt{-K}\,x)$$

Using the boundary condition $X(0) = 0$:

$$0 = A$$

Therefore

$$X(x) = B\sin(\sqrt{-K}\,x)$$

And

$$\frac{\partial}{\partial x} X(x) = \sqrt{-K}\, B\cos(\sqrt{-K}\,x)$$

Using the second boundary condition

$$0 = \sqrt{-K}\, B\cos(\sqrt{-K}\,\pi)$$

As $K \neq 0$ and $B \neq 0$, $\cos(\sqrt{-K}\,\pi)$ must equal zero for a non-trivial solution.

$$\cos(\sqrt{-K}\,x) = 0$$

$$\Rightarrow \sqrt{-K}\,x = \frac{1}{2}\pi(2n-1)$$

$$\sqrt{-K} = \frac{1}{2x}\pi(2n-1)$$

24

The volume integral in cylindrical coordinates is

$$\int_D f \, dV = \int_D f(\rho, \phi, z) \; \rho \, dz \, d\phi \, d\rho$$

In this example, $f(\rho, \phi, z) = \cos\phi$. The limits of integration are

$z = 0$, $z = h + x$
$\phi = -\pi$, $\phi = \pi$
$\rho = 0$, $\rho = a$

Hence the volume integral is

$$\int_{\rho=0}^{a} \left( \int_{z=0}^{h+x} \left( \int_{\phi=-\pi}^{\pi} \rho\cos\phi \; d\phi \right) dx \right) d\rho$$

$$= \int_{\rho=0}^{a} \left( \int_{z=0}^{h+x} \right.$$

$$\left[ \rho \sin \right.$$

25

a) The linear moment of the two particles before collision is

$$\underline{p}_1 = m_1\dot{\underline{r}}_1 = mu\,\underline{i}$$
$$\underline{p}_2 = m_2\dot{\underline{r}}_2 = mu\,\underline{j}$$

After collision the particles have mass $(m_1+m_2)$ and velocity $\underline{w}$. The principle of conservation of linear momentum tells us that

$$mu\,\underline{i} + mu\,\underline{j} = 2m\underline{w}$$

$$\therefore \quad u\,\underline{i} + u\,\underline{j} = 2\underline{w}$$
$$\underline{w} = \frac{u}{2}\underline{i} + \frac{u}{2}\underline{j}$$

The speed after collision is

$$|\underline{w}| = \sqrt{\left(\frac{u}{2}\right)^2 + \left(\frac{u}{2}\right)^2}$$
$$= \sqrt{\frac{u^2}{2}}$$

b) The kinetic energy before collision is

$$E_1 = \frac{1}{2}mu + \frac{1}{2}mu$$

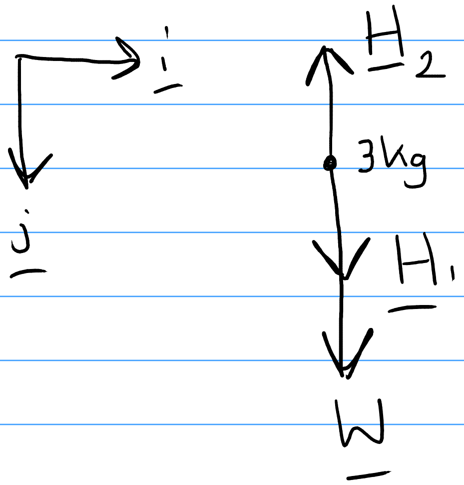
The kinetic energy after collision is

$$E_2 = \frac{1}{2}(2m)\sqrt{\frac{u^2}{2}}$$
$$= mu\sqrt{\frac{1}{2}}$$

The ratio of kinetic energy is $\dfrac{mu\sqrt{\frac{1}{2}}}{2mu} = \dfrac{\sqrt{2}}{4}$

26

a)




where $\underline{W}$ is the weight of P, $\underline{H}_1$ is the spring force of BP and $\underline{H}_2$ is the spring force of AP.

b) $\underline{W} = 3g\underline{j}$, where $g$ is acceleration due to gravity. $\underline{H}_1$ and $\underline{H}_2$ are found in the table below.

| Spring | $L$ | $l_0$ | $l - l_0$ | $K$ | $\underline{\hat{s}}$ | $\underline{H}$ |
|---|---|---|---|---|---|---|
| $\underline{H}_1$ | $\frac{3}{2} - x$ | 2 | $-\frac{1}{2} - x$ | 30 | $\underline{j}$ | $30(-\frac{1}{2} - x)\underline{j}$ |
| $\underline{H}_2$ | $x$ | 1 | $x - 1$ | 18 | $-\underline{j}$ | $-18(x-1)\underline{j}$ |

Therefore

$\underline{H}_1 = 30(-\frac{1}{2} - x)\underline{j}$

$\underline{H}_2 = -18(x-1)\underline{j}$

c) Applying Newton's second law gives

$$3\ddot{\underline{x}} = 3g\underline{j} + 30\left(-\tfrac{1}{2} - x\right)\underline{j} - 18(x-1)\underline{j}$$

If the system is in equilibrium the $\ddot{\underline{x}} = \underline{0}$ so

$$\underline{0} = 3g\underline{j} + 30\left(-\tfrac{1}{2} - x\right)\underline{j} - 18(x-1)\underline{j}$$

Resolving in the $\underline{j}$ direction and rearraging gives

$$0 = 3g - 15 - 30x - 18x + 18$$

$$48x = 3g + 3$$

$$x = \frac{3g+3}{48}$$

$$= \frac{g+1}{16}$$

d) When the particle is in motion we have

$$3\underline{\ddot{x}} = 3g\underline{j} + 30\left(-\tfrac{1}{2} - x\right)\underline{j} - 18(x-1)\underline{j}$$

Resolving in the $\underline{j}$ direction and rearranging

$$3\ddot{x} = 3g - 15 - 30x - 18x + 18$$
$$3\ddot{x} + 48x = 3g + 3$$
$$\ddot{x} + 16x = g + 1$$

e) Let $\omega^2 = 16$ then the equation of motion can be rewritten as

$$\ddot{x} + \omega^2 x = \omega^2 x_{eq}$$

with $x_{eq} = \frac{g+1}{16}$

The general solution is

$$x(t) = B\cos(\omega t) + C\sin(\omega t) + x_{eq}$$
$$= B\cos(4t) + C\sin(4t) + \frac{g+1}{16}$$

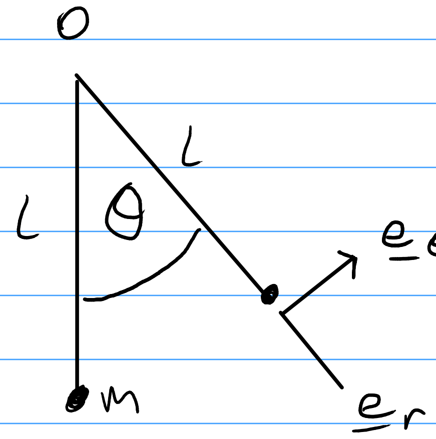
where $B$ and $C$ are constants. At $t=0$ we have $x(0) = \frac{g+1}{16} - \frac{1}{4}$

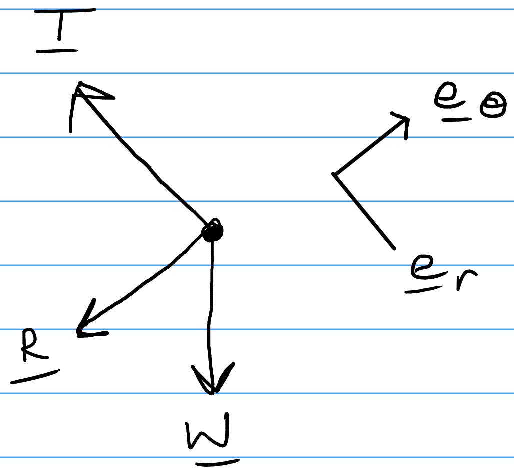
$$\frac{g+1}{16} - \frac{1}{4} = B\cos(0) + C\sin(0) + \frac{g+1}{16}$$

$$-\frac{1}{4} = B$$

27

a)




b) The tension in the rod is

$$\underline{T} = -|\underline{T}|\underline{e}_r$$

The resistive force is

$$\underline{R} = -Kv\,\underline{e}_\theta$$

The weight of the bob is

$$\underline{W} = mg\cos\theta\,\underline{e}_r - mg\sin\theta\,\underline{e}_\theta$$

The position of the bob in polar coordinates is

$$\underline{r} = l\underline{e}_r$$

c) The total torque about $O$ is the sum of the torques of all forces acting on the bob. As the lines of action of $\underline{T}$ and $\underline{R}$ pass through $O$, they contribute no torque. The torque due to $\underline{W}$ is

$$\tau = l\underline{e}_r \times (mg\cos\theta\, \underline{e}_r - mg\sin\theta\, \underline{e}_\theta)$$

$$= mgl\sin\theta\, \underline{k}$$