Unit 5

Statics

Introduction

How strong does a tow rope need to be to pull a car up a steep hill? Is it safe to put a grand piano in the middle of an upstairs room and, if so, what kind of lifting gear is needed to place it there? How do engineers design bridges and cranes that are strong enough not to collapse in use?

Questions like these belong to an area of applied mathematics called **mechanics**, which is concerned with why some objects stay still, and why and how others move. Mechanics can be split into the two areas of **statics**, which deals with why objects stay still, and **dynamics**, which deals with why and how objects move. You will learn about statics in this unit, and you will study dynamics in Unit 10.

These two units are about *Newtonian mechanics*, which is just one of the three main approaches to analysing whether and how objects move. These three approaches are as follows.

Quantum mechanics, which deals with very *small* objects (such as atoms, which have diameters of about 10^{-10} metres).

Relativistic mechanics, which deals with very *fast* objects (such as electrons travelling at speeds of about 10^8 metres per second, close to the speed of light).

Newtonian mechanics, which deals with more familiar everyday objects, which are larger than atoms, and which move at speeds of less than a few million metres per second.

Each of these approaches is a mathematical representation of aspects of the real world, that is, it is a **mathematical model**, often called just a **model**, of reality. The process of developing a mathematical model involves identifying the important features of a real-world situation, and then simplifying these features so that mathematical methods can be applied to them.

Sir Isaac Newton (1642–1727)

Quantum mechanics and relativistic mechanics were developed only recently, in the twentieth century. Newtonian mechanics is based on three *laws of motion* proposed by Sir Isaac Newton in the seventeenth century, which are traditionally stated as below. They involve the idea of a *force*, which, roughly speaking, means a push or a pull.

Law I Every body continues in a state of rest, or moves with constant velocity in a straight line, unless a resultant force is applied to it.

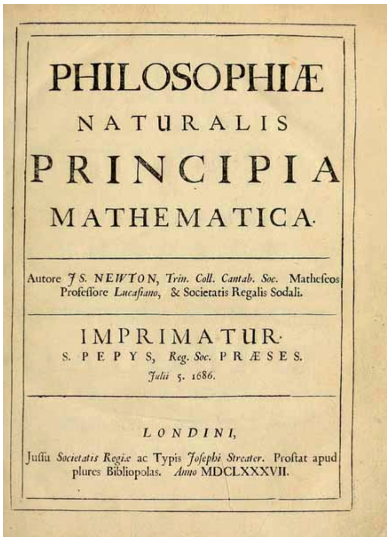
Law II The rate of change of the velocity of a body is proportional to the resultant force applied to the body, and is made in the direction of the resultant force.

Law III To every action (that is, force) by one body on another there is always an opposed equal reaction (force).

In this unit you will see how statics relies on Newton's first and third laws, and in Unit 10 you will see how dynamics relies on Newton's second law.

Force is a vector quantity: it has both magnitude and direction. So you can use the methods of vectors, as recapped in Section 3 of Unit 1, to solve problems in Newtonian mechanics.

In Section 1 you will see how a force is defined mathematically, and meet three types of real-world force: weight, tension and normal reaction. You will learn how to deal with forces that act on objects at rest whose size can be ignored. In Section 2 you will see how to apply the methods that you learned in Section 1 to some simple real-world statics problems. Then in Section 3 you will meet another real-world force, friction, and in Section 4 you will learn how to solve problems involving more than one object.



The title page of the original edition of *Philosophiae Naturalis Principia Mathematica*

History of Newton's laws of motion

Isaac Newton proposed his three laws of motion in his 1687 book *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*), which is known as *Principia* for short. In this book he showed for the first time how motion on earth and motion in space obey the same laws.

Newton's work drew on that of earlier investigators, notably the Italian mathematician and scientist Galileo Galilei (1564–1642) and the French polymath René Descartes (1596–1650). But it was Newton who realised that these three laws were a sufficient basis for mechanics.

1 Balancing forces

In this section you will learn some ideas and methods that are fundamental to solving all problems in statics.

1.1 Forces

You encounter many instances of forces being exerted every day. Pushing an object involves the exertion of a force, as does pulling an object. Your own experience of pushing and pulling should give you an idea of the characteristics of forces. The effect of a force on an object depends on the following.

1. *The magnitude of the force.* For example, pushing a supermarket trolley hard will make it accelerate more rapidly than pushing it gently.
2. *The direction of the force.* For example, pushing a trolley makes it move forwards, whereas pulling it makes it move backwards.
3. *How heavy the object is.* For example, the same effort makes a heavily laden trolley accelerate more slowly than an empty one.
4. *Where on the object the force acts.* For example, pushing in the middle of the handle of a trolley makes the trolley accelerate forward, whereas pushing at the side of the handle makes the trolley rotate.

A supermarket trolley

Points 1 and 2 above show that, like *displacement* and *velocity*, force has both magnitude and direction. In fact, like displacement and velocity, force is a *vector quantity*. This is confirmed by the fact that forces can be combined using the rules of vector arithmetic, as can be checked experimentally. We define force as follows.

A **force** is an influence that can cause an object to accelerate. It is a vector quantity.

The SI unit for force is the *newton*, whose symbol is N. It is named after Isaac Newton. (Remember that SI units are units from a standard, internationally-agreed set of units. The initials SI stand for *Système International d'Unités*.) To understand the definition of the newton, you need to understand what *mass* and *acceleration* mean in mathematics, and know what their SI units are.

The **mass** of an object is a measure of the amount of matter that constitutes the object. The SI unit of mass is the **kilogram** (kg).

The **acceleration** of an object is its rate of change of velocity. For example, suppose that an object is pushed along a straight path, and the magnitude of its velocity increases by 0.2 metres per second in each second that passes. Then the magnitude of the acceleration of the object is 0.2 metres per second per second. The units here, **metres per second per second**, which are usually called **metres per second squared** and written as $\mathrm{m\,s^{-2}}$, are the SI units for acceleration. (We read $\mathrm{m\,s^{-2}}$ as 'metres per second squared'.) Since velocity is a vector quantity, acceleration is also a vector quantity: an object accelerates in a particular direction.

The newton is defined in terms of the units for mass and acceleration as follows.

> One **newton** is the magnitude of force that, when acting on an object of mass 1 kg, causes the object to accelerate at 1 metre per second per second ($1\,\mathrm{m\,s^{-2}}$) in the direction of the force.

In this unit, most forces are denoted by bold upper-case letters, such as $\mathbf{F}$, $\mathbf{W}$, $\mathbf{T}$, $\mathbf{N}$ and so on. They are in bold to indicate that they are vectors. When you hand write a letter that represents a force, you should indicate that it is a vector by underlining it. For example, you should write $\underline{F}$, $\underline{W}$, $\underline{T}$, $\underline{N}$ and so on.

The magnitude of a force is denoted using the usual notation for the magnitude of a vector. For example, the magnitude of the force $\mathbf{F}$ is $|\mathbf{F}|$.

As mentioned in point 4 at the beginning of this subsection, the effect of a force on an object depends on where on the object the force acts. However, in this unit we will simplify all situations in which forces act on an object by modelling the object as a *particle* (sometimes called a *point mass*), which is defined as follows.

> A **particle** is an object that has mass but no size, and so occupies a single point in space.

When an object is modelled as a particle, there is only one point on the object at which a force can act. So such a model does not take account of any rotational effects of the force on the object.

Whether it is reasonable to model an object as a particle depends on the situation that you are modelling, and on the questions that you want to answer. It does not depend on the size of the object – for example, it can be reasonable to model the Earth as a particle if you are modelling the orbit of the Earth around the Sun! If you go on to study more applied mathematics, physics or engineering after this module, then you may learn how to model objects as *extended bodies* rather than particles, and hence take account of the rotational effects of forces.

As mentioned in the introduction to this unit, the two fundamental principles in statics are *Newton's first* and *third laws of motion*. In this section you will see how to use Newton's first law of motion in statics problems. You'll see how to use the third law in Section 4. The first law is stated again in the following box, in slightly different wording. Note that an object is said to be **at rest** if its position does not change.

Newton's first law of motion

A particle remains at rest or continues to move with a constant velocity, unless acted on by a resultant force.

Note the importance of the word 'resultant' in Newton's first law. The **resultant** force, sometimes called the **net force**, on an object is the vector sum of the forces that act on it. Newton's first law does not mean that if an object is at rest, then it is not being acted on by any forces. Instead it means that the forces acting on the object *balance*, giving a resultant force of zero magnitude. For example, if you push a supermarket trolley then it will start to move, but if someone else also pushes the trolley, just as hard but in the opposite direction and in line with your push, then the trolley will remain at rest, because the forces balance.

In the rest of this subsection you will meet three types of force that commonly act on objects, and start to see some ways in which forces can balance to keep an object at rest.

Weight

If an object is dropped from above, but not too far from, the surface of the Earth, then it will start to fall. All objects fall at the same rate (except for differences caused by forces such as air resistance, which are often small and so can often be ignored). The acceleration with which objects fall is known as the **acceleration due to gravity**. The magnitude of this acceleration is usually denoted by g, and has value $9.8\,\mathrm{m\,s^{-2}}$, to two significant figures. We will use this approximation for g throughout this unit. The precise value of g actually varies slightly according to the geographical location of the object on the Earth. The direction of the acceleration due to gravity is vertically downwards.

The force that causes an object to accelerate downwards when it is dropped is called the **weight** of the object. It is exerted due to gravity acting on the object.

In everyday language the word 'weight' is often used to mean 'mass', so it is important that you clearly understand the difference between these two concepts. Mass is the amount of matter that makes up an object, and is a scalar quantity that can be measured in kilograms. Weight is the force exerted on an object due to gravity, and is a vector quantity whose magnitude can be measured in newtons.

Mass and weight are related, however. An object of mass m (in kg) has a weight of magnitude mg (in N), where g is the magnitude (in $\mathrm{m\,s^{-2}}$) of the acceleration due to gravity.

Here is a summary of the definition of weight.

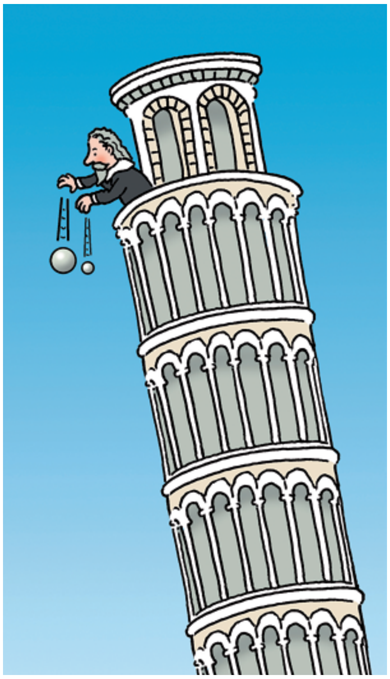
The **weight** of an object is a force that acts on the object and is directed vertically downwards. If the object has mass m (in kg), then its weight has magnitude mg (in N), where $g = 9.8\,\mathrm{m\,s^{-2}}$.

Take care not to confuse g, the magnitude of the acceleration due to gravity, with the symbol g for the gram (a unit of mass).

Activity 1 *Calculating weights and masses*

(a) What is the magnitude of the weight of an object of mass 5 kg?

(b) If an object has a weight of magnitude 8 N, what is its mass?



You can obtain a rough idea of the size of a newton by considering the masses, and hence the weights, of everyday objects. For example, a typical tomato has a mass of about 0.1 kg, so the magnitude of its weight is about 1 N. So if you place a tomato on your palm, then it will press down with a force of magnitude approximately one newton.

The fact that, in the absence of other forces such as air resistance, all objects fall at the same rate under the influence of gravity was first realised by the Italian scientist Galileo Galilei (1564–1642).

He is reputed to have confirmed it by dropping two balls of different masses from the top of the Leaning Tower of Pisa, and noting that they reached the ground at the same instant. In fact, Galileo made his discovery after making careful measurements of the speeds of balls rolling down slopes.

The astronaut David Scott demonstrated during the 1971 Apollo 15 moon landing that on the Moon, with no air resistance, a hammer and feather do indeed fall at the same rate (a different rate from on the Earth). There is a video of this demonstration on the module website.

Dropping a hammer and feather on the Moon

The reasons for the variation in the value of the magnitude g of the acceleration due to gravity on the Earth include the rotation of the Earth, its shape, its geology, and the altitude of each location on its surface. To two decimal places, g is $9.78\,\mathrm{m\,s^{-2}}$ at the Equator and $9.83\,\mathrm{m\,s^{-2}}$ at the North Pole.

Since every object on the Earth is acted on by its weight, an object that does not move is also being acted on by at least one force that balances its weight. You will now meet two types of force that can do this.

Tension

Consider the situation shown in Figure 1, in which an object of mass 2 kg is suspended from a string whose top end is tied to a hook on the ceiling. The object remains at rest, even though it is acted on by its weight, directed vertically downwards. The string is therefore providing a balancing force, which must be in the opposite direction, vertically upwards. This force is called the **tension** in the string.

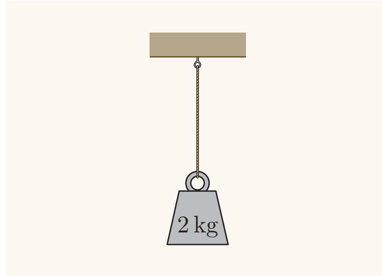


Figure 1 An object hanging from a string

If the mass of the suspended object is increased, then the magnitude of the tension in the string also increases, to balance the increased weight. However, if the mass of the object is increased beyond a certain limit, then the string will break.

In this unit we make various assumptions about strings, to simplify the models that we are working with. We assume that a string is **inextensible**; that is, it does not stretch. We also assume that it has no mass; this assumption is reasonable for a real string whose mass is insignificant compared with the masses of the objects involved. A string with no mass is referred to as a **light** string, and an inextensible light string is called a **model string**.

Any taut string (or rope, wire or chain) exerts a tension at each of its two points of attachment. The tension at each point of attachment acts along the line of the string, and is directed away from the point of attachment. So, for example, the string in Figure 1 exerts an upward force on the 2 kg object, and a downward force on the hook. For a model string running directly between two points of attachment, the tensions at the two points of attachment have the same magnitude but opposite directions.

Here is a summary of the definition of tension.

Tension

A **tension** in a taut string (or rope, wire or chain) is a force that pulls at a point of attachment of the string, along the line of the string. Its magnitude adjusts to balance other forces, up to some limit.

The word 'tension' is sometimes used to refer to the *magnitude* of a tension force, which is a scalar quantity, rather than to the force itself. Its meaning should be clear from the context.

When you want to analyse a problem involving forces acting on an object, it is usually useful to draw a diagram that represents these forces and how they act. Such a diagram is known as a **force diagram**. Figure 2 shows a force diagram for the situation in Figure 1. The 2 kg object has been modelled as a particle, and the dot represents this particle.

The arrows represent the two forces that act on the object. The weight and the tension in the string are denoted by $\mathbf{W}$ and $\mathbf{T}$, respectively, and the arrows are labelled accordingly.

Figure 2 A force diagram for the situation in Figure 1

In a force diagram, the arrows that represent the forces are usually drawn with the arrowheads at their tips, rather than in the middle, as shown in Figure 2. As you would expect, the lengths of the arrows represent the magnitudes of the forces, and the directions of the arrows represent the directions of the forces. However, when you draw a force diagram you often do not know the magnitudes and directions of some of the forces, since usually you are trying to determine some of these. So any force diagram is just a rough approximation of the situation that it represents, and the lengths and directions of the arrows may be far from true representations.

Since the particle in Figure 2 remains at rest, it follows from Newton's first law of motion that the sum of the forces acting on it has zero magnitude. In other words, the vector equation $\mathbf{T} + \mathbf{W} = \mathbf{0}$, or, equivalently, $\mathbf{T} = -\mathbf{W}$, holds. So the weight $\mathbf{W}$ and the tension $\mathbf{T}$ have the same magnitude but act in opposite directions. You can use this fact to find the tension in the string, as shown in the example below. This simple example illustrates the sorts of techniques that you will be using to solve more complicated statics problems in this unit.

Example 1 *Calculating a tension*

Find the tension in the string that supports the 2 kg object in Figure 1. Give your answer in newtons to two significant figures.

Solution

Define the symbols to be used.

Let $\mathbf{T}$ be the tension in the string and let $\mathbf{W}$ be the weight of the object.

Use the force diagram to find a vector equation relating the forces.

Then $\mathbf{T} = -\mathbf{W}$.

Use this equation to deduce the unknown magnitude.

So $|\mathbf{T}| = |\mathbf{W}|$.

But

$$|\mathbf{W}| = 2g = 2 \times 9.8 = 19.6,$$

so $|\mathbf{T}| = 19.6$.

State a conclusion.

The tension in the string has magnitude 20 N (to 2 s.f.) and is directed vertically upwards.

Activity 2 *Calculating another tension*

A chandelier of mass 40 kg hangs from a light inextensible cable attached to the ceiling of a room.

(a) Draw a diagram of the situation, and a separate force diagram that shows all the forces acting on the chandelier.

(b) Find the magnitude of the tension in the cable, in newtons to two significant figures.

(c) A replacement cable is needed. An available cable can sustain tensions of magnitude up to 350 N, but will break at greater tensions. Is this cable suitable?

You will now meet another type of force that can balance the weight of an object.

Normal reaction

Suppose that an object of mass 2 kg is resting on a horizontal table top, as shown in Figure 3.

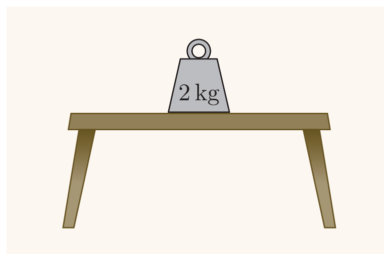


Figure 3 An object at rest on a table top

The object remains at rest, even though it is acted on by its weight, directed vertically downwards. The table top is therefore providing a balancing force, which must be in the opposite direction, vertically upwards. This force is called the **normal reaction** of the table top on the object.

If the mass of the object is increased, then the magnitude of the normal reaction also increases, to balance the increased weight. However, if the mass of the object is increased beyond a certain limit, then the table top will break.

Whenever an object rests on a surface, there is a normal reaction of the surface on the object. This force acts at right angles to the surface. The word *normal* in 'normal reaction' means 'at right angles'.

George's reaction to his brother's weight was perfectly normal

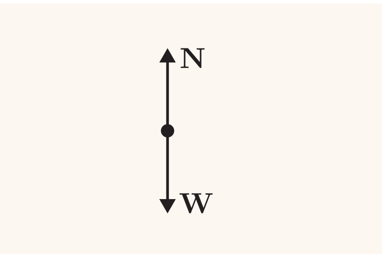


Figure 4 A force diagram for the situation in Figure 3

Here is a summary of the definition of normal reaction.

Normal reaction

The **normal reaction** of a surface on an object resting on the surface is a force that pushes on the object, at right angles to the surface. Its magnitude adjusts to balance other forces, up to some limit.

Figure 4 shows a force diagram for the situation in Figure 3. The weight of the object is denoted by $\mathbf{W}$, and the normal reaction is denoted by $\mathbf{N}$.

Since the particle that models the object remains at rest, the forces are balanced and hence $\mathbf{W} + \mathbf{N} = \mathbf{0}$, or, equivalently, $\mathbf{N} = -\mathbf{W}$. You can use this equation to calculate the magnitude of the normal reaction.

Activity 3 *Calculating a normal reaction*

Find the normal reaction of the table top on the 2 kg object in Figure 3, in newtons to two significant figures.

1.2 Equilibrium of objects modelled as particles

An object is said to be in **equilibrium** if the sum of the forces that act on it is the zero vector, $\mathbf{0}$. By Newton's first law of motion, if an object is in equilibrium, then the object either remains at rest or moves with a constant velocity. If an object is in equilibrium and remains at rest, then it is said to be in **static equilibrium**. In this unit, we use the phrase 'in equilibrium' as shorthand for 'in static equilibrium'.

You have seen some examples where two forces acting on an object have sum $\mathbf{0}$, so the object is in equilibrium. The condition for an object to be in equilibrium, for any number of forces acting on it, can be stated as follows.

Equilibrium condition for a particle

If an object modelled by a particle is acted on by the n forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$, and is in equilibrium, then

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}.$$

This condition is the key to solving problems in statics. In the rest of this subsection you will see various ways in which you can use it. The examples and activities in this subsection involve abstract forces, but in the next section you will see how to use the equilibrium condition with real forces such as weights, tensions and normal reactions. The equilibrium condition applies to forces in both two and three dimensions, though in this unit we will work with forces only in two dimensions.

Since force is a vector quantity, you can specify a force either by giving its magnitude and direction, or by giving its component form in terms of Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$. The example below illustrates how to apply the equilibrium condition to forces in component form.

Example 2 *Finding a force by using the equilibrium condition*

A particle, which remains at rest, is acted on by three forces $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$, where $\mathbf{F} = 7\mathbf{i} + 2\mathbf{j}$ and $\mathbf{G} = -9\mathbf{i} + 3\mathbf{j}$. Find the force $\mathbf{H}$ in component form.

Solution

Apply the equilibrium condition.

The particle is in equilibrium, so $\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}$. Hence

$$\begin{aligned}\mathbf{H} &= -\mathbf{F} - \mathbf{G}\\ &= -(7\mathbf{i} + 2\mathbf{j}) - (-9\mathbf{i} + 3\mathbf{j})\\ &= -7\mathbf{i} - 2\mathbf{j} + 9\mathbf{i} - 3\mathbf{j}\\ &= 2\mathbf{i} - 5\mathbf{j}.\end{aligned}$$

In the next activity, and throughout the unit, remember that when you are handwriting you should underline all symbols that represent vectors. In this activity this applies to the symbols $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{i}$ and $\mathbf{j}$.

Activity 4 *Finding a force by using the equilibrium condition*

A particle, which remains at rest, is acted on by four forces $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, where $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{B} = -\mathbf{j}$ and $\mathbf{C} = \mathbf{i} + 2\mathbf{j}$. Find the force $\mathbf{D}$ in component form.

In practical statics problems, you usually know the magnitudes and directions of the forces involved, rather than their component forms. However it is often easier to apply the equilibrium condition to forces in component form. So you need to be able to convert between these two ways of specifying vectors. You saw how to do this in MST124 Unit 5, and there is some revision of it in MST125 Unit 1, but you will now have a chance to revise and practise the techniques in the particular context of force vectors.

The process of converting from the magnitude and direction of a force to its component form in terms of the Cartesian unit vectors **i** and **j** is known as **resolving the force in the i- and j-directions**. If you know the angle θ that a force makes with the direction of the vector **i**, then you can resolve the force by using the fact below. It is illustrated in Figure 5 in the case of an acute angle θ, but remember that it applies for any angle θ (including when θ is negative).

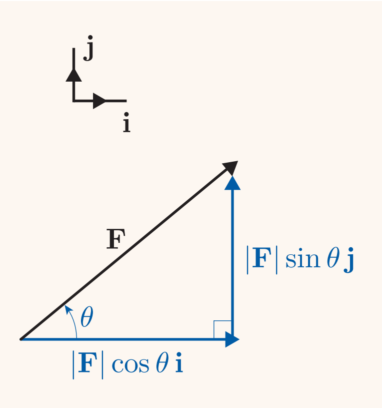


Figure 5 A force **F** and its components in the **i**- and **j**-directions

If the force **F** makes the angle θ with the direction of the Cartesian unit vector **i** (where θ is measured anticlockwise from the direction of **i**), then

$$\mathbf{F} = |\mathbf{F}|\cos\theta\,\mathbf{i} + |\mathbf{F}|\sin\theta\,\mathbf{j}.$$

If you do not know the direction of a force relative to the **i**-direction, but you do know it relative to some other direction, then one way to resolve the force in the **i**- and **j**-directions is to first use geometric methods to find its direction relative to the **i**-direction, and then use the formula in the box above. However, with practice, it's usually simpler to use the alternative method demonstrated in the next example.

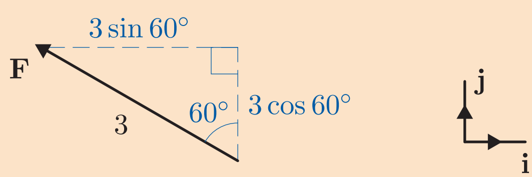
Example 3 *Resolving a force in the* **i**- *and* **j**-*directions*

Find the component form of the force **F** of magnitude 3 N that acts to the left and up at an angle of 60° to the vertical. Assume that **i** points to the right, and **j** points up.

Solution

Draw **F** as the hypotenuse of a right-angled triangle whose other two sides are parallel to **i** and **j**. Mark the size of an acute angle in this triangle, and use simple trigonometry to find the lengths of the other sides.

The force **F** is shown below.



The lengths found are the magnitudes of the components. Use the triangle and the directions of $\mathbf{i}$ and $\mathbf{j}$ to find the signs of the components.

From the diagram,

$$\begin{aligned}\mathbf{F} &= -3\sin 60^\circ\mathbf{i} + 3\cos 60^\circ\mathbf{j}\\ &= -3\times\frac{\sqrt{3}}{2}\mathbf{i} + 3\times\frac{1}{2}\mathbf{j}\\ &= -\frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}.\end{aligned}$$

When you use the method of Example 3 to resolve a force $\mathbf{F}$, and θ is an acute angle in the right-angled triangle, the side of the triangle *opposite* θ always has length $|\mathbf{F}|\sin\theta$, and the side *adjacent* to θ always has length $|\mathbf{F}|\cos\theta$. It's useful to remember this, to help you speed up your working.

Activity 5 *Resolving forces in the* $\mathbf{i}$*- and* $\mathbf{j}$*-directions*

Find the component forms of the following forces, assuming that $\mathbf{i}$ points to the right and $\mathbf{j}$ points up.

(a) The force $\mathbf{F}$ of magnitude 10 N that acts to the right and down at 30° to the vertical.

(b) The force $\mathbf{G}$ of magnitude 6 N that acts to the left and down at 30° to the horizontal.

The next example demonstrates how to carry out the reverse process to the process of resolving a force in the $\mathbf{i}$- and $\mathbf{j}$-directions; that is, how to use the component form of a force to find its magnitude and direction.

Finding the magnitude is straightforward; you use the standard formula below.

The force $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$ has magnitude

$$|\mathbf{F}| = \sqrt{a^2+b^2}.$$

Example 4 *Finding the magnitude and direction of a force from its component form*

Find the magnitude and direction of the force $\mathbf{F} = 7\mathbf{i} - 3\mathbf{j}$ (given in newtons). Give the magnitude in newtons to one decimal place, and the direction to the nearest degree. Assume that $\mathbf{i}$ points to the right, and $\mathbf{j}$ points up.

Solution

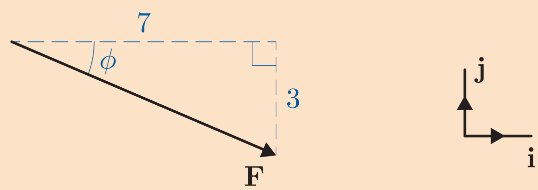
To find the magnitude, use the standard formula.

The magnitude is given by

$$|\mathbf{F}| = \sqrt{7^2 + (-3)^2} = \sqrt{49+9} = \sqrt{58} = 7.6157\ldots = 7.6 \text{ (to 1 d.p.)}.$$

To find the direction, draw $\mathbf{F}$ as the hypotenuse of a right-angled triangle whose other two sides are parallel to $\mathbf{i}$ and $\mathbf{j}$. Find an acute angle in this triangle, and hence find the direction of $\mathbf{F}$.

The force $\mathbf{F}$ is shown below.



For the angle ϕ shown in the diagram,

$$\tan\phi = \frac{3}{7},$$

so

$$\phi = \tan^{-1}\left(\tfrac{3}{7}\right) = 23.198\ldots^\circ = 23^\circ \text{ (to the nearest degree)}.$$

The force $\mathbf{F}$ has magnitude $7.6\,\mathrm{N}$ (to 1 d.p.) and is directed to the right and down, at an angle of 23° (to the nearest degree) to the horizontal.

In the solution to Example 4 the direction of the force was described relative to the horizontal. It would be just as acceptable to describe it relative to the vertical. For a practical situation involving forces, it might be appropriate to describe the direction of a force relative to some other direction that is intrinsic to the situation.

Activity 6 *Finding the magnitudes and directions of forces from their component forms*

Find the magnitude and direction of each of the following forces (given in newtons). Give the magnitude in newtons to one decimal place, and the direction to the nearest degree. Assume that $\mathbf{i}$ points to the right, and $\mathbf{j}$ points up.

(a) $\mathbf{F} = 4\mathbf{i} + 9\mathbf{j}$ (b) $\mathbf{G} = -5\mathbf{i} + 2\mathbf{j}$

Now that you have practised converting between the magnitude and direction of a force and its component form, you are ready to apply the equilibrium condition to forces that are given in terms of their magnitudes and directions. The next example demonstrates how to do this.

Example 5 *Using the equilibrium condition for forces expressed in terms of magnitude and direction*

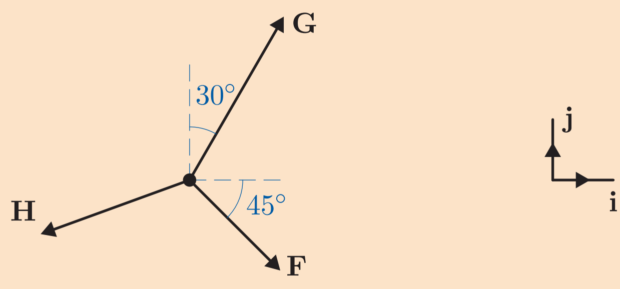
A particle, which remains at rest, is acted on by three forces $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$. The force $\mathbf{F}$ has magnitude 40 N and is directed to the right and down, at 45° to the horizontal. The force $\mathbf{G}$ has magnitude 60 N and is directed to the right and up, at 30° to the vertical.

Find the magnitude and direction of the force $\mathbf{H}$, giving the magnitude to the nearest newton and the direction to the nearest degree.

Solution

Draw a force diagram, indicating the known angles. You don't know the direction of $\mathbf{H}$, so draw it anywhere that seems reasonable. State the known magnitudes.

A force diagram is shown below.



We know that $|\mathbf{F}| = 40$ and $|\mathbf{G}| = 60$.

Choose suitable directions for **i** and **j**, since these are not specified.

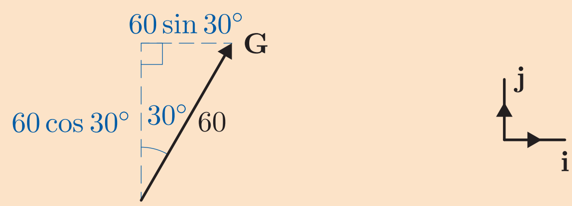
Let **i** and **j** point right and up, respectively, as shown.

Express the known forces in component form.

The force **F** makes an angle of $-45°$ with the **i**-direction, so

$$\begin{aligned}\mathbf{F} &= |\mathbf{F}|\cos(-45°)\,\mathbf{i} + |\mathbf{F}|\sin(-45°)\,\mathbf{j}\\ &= 40 \times \frac{1}{\sqrt{2}}\,\mathbf{i} + 40 \times \left(-\frac{1}{\sqrt{2}}\right)\mathbf{j}\\ &= 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j}.\end{aligned}$$

To find **G** in component form, we can sketch it as below.



So

$$\begin{aligned}\mathbf{G} &= 60\sin 30°\,\mathbf{i} + 60\cos 30°\,\mathbf{j}\\ &= 60 \times \frac{1}{2}\,\mathbf{i} + 60 \times \frac{\sqrt{3}}{2}\,\mathbf{j}\\ &= 30\,\mathbf{i} + 30\sqrt{3}\,\mathbf{j}.\end{aligned}$$

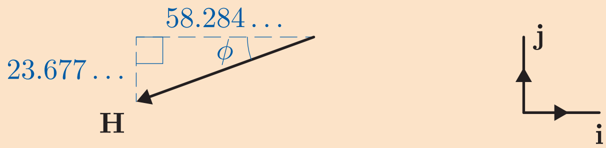
Use the equilibrium condition to find **H**.

Since the particle is at rest, $\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}$. This gives

$$\begin{aligned}\mathbf{H} &= -\mathbf{F} - \mathbf{G}\\ &= -(20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j}) - (30\,\mathbf{i} + 30\sqrt{3}\,\mathbf{j})\\ &= -20\sqrt{2}\,\mathbf{i} + 20\sqrt{2}\,\mathbf{j} - 30\,\mathbf{i} - 30\sqrt{3}\,\mathbf{j}\\ &= (-20\sqrt{2} - 30)\,\mathbf{i} + (20\sqrt{2} - 30\sqrt{3})\,\mathbf{j}\\ &= -58.284\ldots\,\mathbf{i} - 23.677\ldots\,\mathbf{j}.\end{aligned}$$

Finally, find the magnitude and direction of **H**.

The force **H** is shown below.



The magnitude of **H** is given by

$$|\mathbf{H}| = \sqrt{(-58.284\ldots)^2 + (-23.677\ldots)^2} = 62.910\ldots.$$

For the acute angle ϕ in the diagram above,

$$\tan\phi = \frac{23.677\ldots}{58.284\ldots},$$

so

$$\phi = \tan^{-1}\left(\frac{23.677\ldots}{58.284\ldots}\right) = 22.108\ldots^\circ.$$

The force **H** has magnitude 63 N (to the nearest newton) and is directed to the left and down, at 22° (to the nearest degree) to the horizontal.

Here is a similar example for you to try.

Activity 7 *Using the equilibrium condition for forces expressed in terms of magnitude and direction*

Three forces act on a particle at rest. The force **F** has magnitude 3 N and is directed vertically downwards, and the force **G** has magnitude 2 N and is directed to the left and up at 30° to the horizontal.

Find the magnitude (in newtons, to one decimal place) and direction (to the nearest degree) of the third force **H**.

When you are working with a force diagram, and you want to use a right-angled triangle to convert between the magnitude and direction of a force and its component form, you might find it convenient to sketch the triangle directly on top of the force diagram. This is done in many of the solutions to examples and activities later in this unit. In fact, as you become more familiar with working with forces in this way, you may find that you can sometimes manage without actually drawing the triangle at all. Instead you can just picture such a triangle in your head, on top of the force diagram. A few of the solutions to examples and activities later in this unit do not include such triangles, for this reason. However, you should keep drawing the triangles, in whatever way is convenient for you, until and unless you are confident that you can manage with only the force diagram. The important thing is to get the right answer!

In each of Example 5 and Activity 7, three forces acted on a particle in equilibrium. You knew the magnitudes and directions of two of the forces, and the problem was to find the magnitude and direction of the third force.

The next example is a little different, though it still relies on the techniques that you have been practising. Again it involves three forces that act on a particle in equilibrium. However in this case you know the directions of all three forces, and the magnitude of one of them, and the problem is to find the magnitudes of the other two forces.

Notice that in the solution to this example the magnitude $|\mathbf{P}|$ of a force $\mathbf{P}$ is denoted by P, and similarly the magnitude $|\mathbf{Q}|$ of a force $\mathbf{Q}$ is denoted by Q. In general, it is often convenient to denote the magnitude of a force by the same upper-case letter as denotes the force, but not bold or underlined. This is done throughout the rest of this unit.

Example 6 *Finding the magnitudes of two forces*

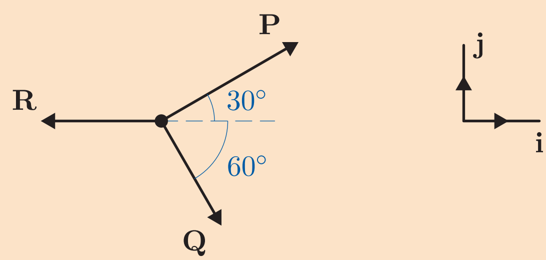
A particle, which remains at rest, is acted on by three forces, $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. The force $\mathbf{P}$ has magnitude 50 N and is directed to the right and up, at 30° above the horizontal. The force $\mathbf{Q}$ is directed to the right and down, at 60° below the horizontal, and the force $\mathbf{R}$ is directed horizontally to the left.

Find the magnitudes of the forces $\mathbf{Q}$ and $\mathbf{R}$, in newtons to one decimal place.

Solution

Draw a force diagram, indicating the known angles. State the known magnitude. Choose directions for $\mathbf{i}$ and $\mathbf{j}$.

A force diagram is shown below. Let $\mathbf{i}$ point right and $\mathbf{j}$ point up, as shown.



We know that $|\mathbf{P}| = 50$.

Express each force in component form, in terms of an unknown magnitude where necessary.

Let $Q = |\mathbf{Q}|$ and $R = |\mathbf{R}|$. Expressing each force in component form gives

$$\begin{aligned}\mathbf{P} &= |\mathbf{P}|\cos 30^\circ\,\mathbf{i} + |\mathbf{P}|\sin 30^\circ\,\mathbf{j}\\ &= 50 \times \frac{\sqrt{3}}{2}\,\mathbf{i} + 50 \times \frac{1}{2}\,\mathbf{j}\\ &= 25\sqrt{3}\,\mathbf{i} + 25\,\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{Q} &= |\mathbf{Q}|\cos(-60^\circ)\,\mathbf{i} + |\mathbf{Q}|\sin(-60^\circ)\,\mathbf{j}\\ &= \frac{1}{2}\,Q\,\mathbf{i} - \frac{\sqrt{3}}{2}\,Q\,\mathbf{j}\end{aligned}$$

$$\mathbf{R} = -R\,\mathbf{i}.$$

Use the equilibrium condition.

Since the particle is at rest, $\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{0}$. This gives

$$\left(25\sqrt{3}\,\mathbf{i} + 25\,\mathbf{j}\right) + \left(\frac{1}{2}\,Q\,\mathbf{i} - \frac{\sqrt{3}}{2}\,Q\,\mathbf{j}\right) + (-R\,\mathbf{i}) = \mathbf{0}.$$

Collecting the terms in $\mathbf{i}$ and $\mathbf{j}$ gives

$$\left(25\sqrt{3} + \frac{1}{2}\,Q - R\right)\mathbf{i} + \left(25 - \frac{\sqrt{3}}{2}\,Q\right)\mathbf{j} = \mathbf{0}.$$

Since the vector on the left-hand side is equal to the zero vector, its $\mathbf{i}$-component and $\mathbf{j}$-component are both zero.

So

$$25\sqrt{3} + \frac{1}{2}Q - R = 0,$$

$$25 - \frac{\sqrt{3}}{2}\,Q = 0.$$

Solve this pair of simultaneous equations to find Q and R.

The second of these equations gives

$$\frac{\sqrt{3}}{2}Q = 25$$

so

$$Q = 25 \times \frac{2}{\sqrt{3}} = \frac{50}{\sqrt{3}} = 28.9 \text{ (to 1 d.p.)}.$$

Substituting $Q = \dfrac{50}{\sqrt{3}}$ into the first of the two equations gives

$$25\sqrt{3} + \frac{1}{2} \times \frac{50}{\sqrt{3}} - R = 0,$$

so

$$\begin{aligned} R &= 25\sqrt{3} + \frac{25}{\sqrt{3}} \\ &= 25\sqrt{3}\left(1 + \frac{1}{3}\right) \\ &= 25\sqrt{3} \times \frac{4}{3} \\ &= \frac{100\sqrt{3}}{3} \\ &= 57.7 \ \text{(to 1 d.p.)}. \end{aligned}$$

So, to one decimal place, $\mathbf{Q}$ has magnitude 28.9 N and $\mathbf{R}$ has magnitude 57.7 N.

In Example 6, the equilibrium condition gave rise to a vector equation containing two unknown scalar quantities, Q and R. To find the values of these quantities, the **i**- and **j**-components of the vector equation were considered separately. This gave a pair of simultaneous *scalar* equations, which were solved to find the unknown quantities. This important technique is useful in most of the examples and activities in this unit.

The process of using a vector equation to obtain simultaneous scalar equations in this way is sometimes referred to as **resolving an equation into components**, or simply **taking components**.

Here is a similar activity for you to try. You can solve the simultaneous equations that you obtain in it by using standard methods that should be familiar to you.

Activity 8 *Finding the magnitudes of two more forces*

A particle, which remains at rest, is acted on by three forces, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

The force $\mathbf{F}$ is directed to the left and down, at $120°$ to the vertical, and the force $\mathbf{G}$ is directed to the right and down, at $150°$ to the vertical. The force $\mathbf{H}$ has magnitude $70\,\mathrm{N}$ and is directed vertically upwards.

Find the magnitudes of the forces $\mathbf{F}$ and $\mathbf{G}$, giving your answers as exact values. In your working take $\mathbf{i}$ to point right and $\mathbf{j}$ to point up.

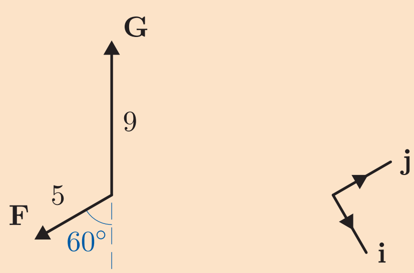
In all the example and activity solutions up till now, the directions of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ have been chosen to point in the familiar directions of right and up, respectively. However, when you are solving a statics problem, you can often make your working simpler by choosing the directions of $\mathbf{i}$ and $\mathbf{j}$ more strategically. For example, if the problem involves two perpendicular forces, then it is sometimes useful to choose the directions of $\mathbf{i}$ and $\mathbf{j}$ to be parallel to these forces. This makes it easy to write down the component forms of these forces, and may simplify some of the working.

Finding the component form of a force when $\mathbf{i}$ and $\mathbf{j}$ do not point right and up is really no different from finding it when they do, as illustrated in the next example. You might find it helpful to rotate the page (or your head) to help you see this!

Example 7 *Resolving forces when* $\mathbf{i}$ *and* $\mathbf{j}$ *do not point right and up*

In the diagram below, the force $\mathbf{F}$ has magnitude $5\,\mathrm{N}$ and is directed to the left and down, at $60°$ to the vertical. The force $\mathbf{G}$ has magnitude $9\,\mathrm{N}$ and points vertically up. The vectors $\mathbf{i}$ and $\mathbf{j}$ have been chosen to be as shown, with $\mathbf{j}$ parallel to $\mathbf{F}$ (but in the opposite direction).

Find the component forms of the forces $\mathbf{F}$ and $\mathbf{G}$.



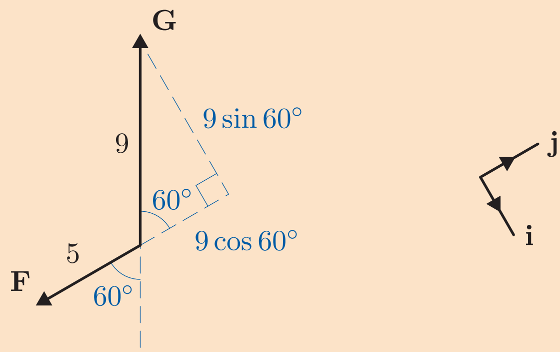
Solution

The component form of $\mathbf{F}$ is

$$\mathbf{F} = -5\mathbf{j}.$$

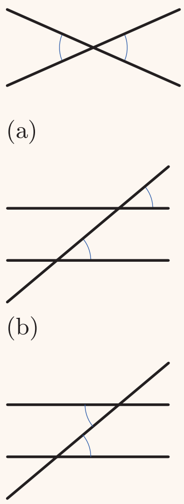
To find the component form of $\mathbf{G}$, draw a right-angled triangle with hypotenuse $\mathbf{G}$ and shorter sides parallel to $\mathbf{i}$ and $\mathbf{j}$. Use geometric properties of angles to work out an acute angle in this triangle, and mark it on your diagram. Then proceed as in Example 3 on page 122.

The forces are shown below.



From the diagram,

$$\begin{aligned}\mathbf{G} &= -9\sin 60^\circ\,\mathbf{i} + 9\cos 60^\circ\,\mathbf{j}\\ &= -9 \times \frac{\sqrt{3}}{2}\,\mathbf{i} + 9 \times \frac{1}{2}\,\mathbf{j}\\ &= -\frac{9\sqrt{3}}{2}\,\mathbf{i} + \frac{9}{2}\,\mathbf{j}.\end{aligned}$$



(a)

(b)

(c)

Figure 6 (a) opposite angles (b) corresponding angles (c) alternate angles (in (b) and (c) two lines are parallel)

An alternative way to work out the component form of the force $\mathbf{G}$ in Example 7 is to use the formula in the box on page 122. The angle that $\mathbf{G}$ makes with the $\mathbf{i}$-direction is $60^\circ + 90^\circ = 150^\circ$.

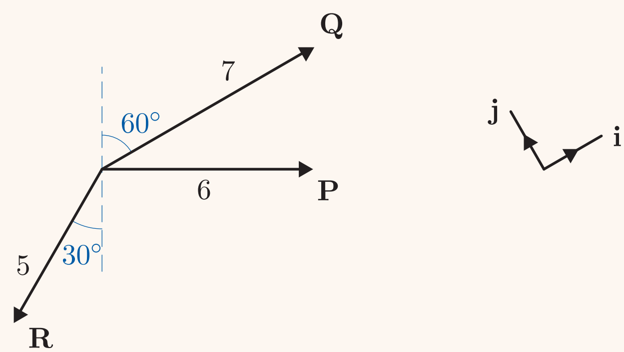
In Example 7, the size of an acute angle in the right-angled triangle was found by using the fact that *opposite* angles are equal, as illustrated in Figure 6(a). In general when you are trying to find the size of an acute angle in such a right-angled triangle, you may need to use this and other properties of the angles in your particular diagram. For example, there may be angles that add up to 90° or 180°. Occasionally you may be able to use the fact that *corresponding* angles are equal, as illustrated in Figure 6(b), or the fact that *alternate* angles are equal, as illustrated in Figure 6(c). You will see examples of this sort of angle calculation throughout the rest of this unit.

Note that the two forces **F** and **G** in Example 7 do not have sum **0**, so if they were the only forces acting on a particle, then the particle would not be in equilibrium. The same is true of the three forces **P**, **Q** and **R** in Activity 9 below.

Activity 9 *Resolving forces when* **i** *and* **j** *do not point right and up*

In the diagram below, the force **P** has magnitude 6 N, and points horizontally to the right. The force **Q** has magnitude 7 N and is directed to the right and up, at 60° to the vertical. The force **R** has magnitude 5 N and is directed to the left and down, at 30° to the vertical. The vectors **i** and **j** have been chosen to be as shown, with **i** parallel to **Q**.

Find the component forms of the forces **P**, **Q** and **R**.



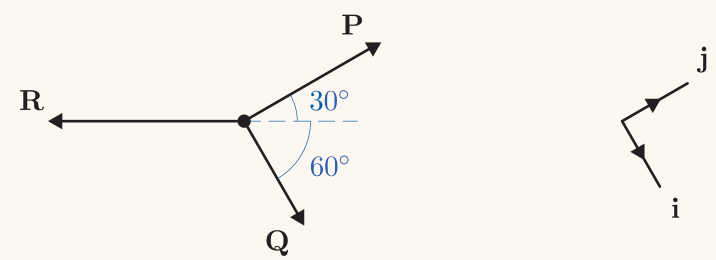
In the next activity, you are asked to solve the problem in Example 6 on page 128 again, using the same method as in Example 6, but with **i** and **j** chosen to be parallel to the forces **P** and **Q**, which are at right angles to each other. You'll see that this makes some of the working simpler. Unfortunately, it also makes one part of the working, namely finding the component form of **R**, a little harder, but the benefits outweigh the disadvantages!

Activity 10 *Finding the magnitudes of two forces, again*

A particle, which remains at rest, is acted on by three forces, $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. The force $\mathbf{P}$ has magnitude $50\,\mathrm{N}$ and is directed to the right and up, at $30°$ to the horizontal. The force $\mathbf{Q}$ is directed to the right and down, at $60°$ to the horizontal, and the force $\mathbf{R}$ is directed horizontally to the left.

Find the magnitudes of the forces $\mathbf{Q}$ and $\mathbf{R}$, in newtons to one decimal place.

In your working, take $\mathbf{i}$ to have the same direction as $\mathbf{Q}$, and $\mathbf{j}$ to have the same direction as $\mathbf{P}$, as shown below.



You should have found that your working in Activity 10 was simpler than the working in Example 6. The final answers should be the same, of course, since the questions are the same.

When you are choosing directions for $\mathbf{i}$ and $\mathbf{j}$, it's usually a good idea to choose directions that are obtained by *rotating them together* from their familiar directions of right and up. For example, if your problem involves two forces as shown in Figure 7(a), then you might choose $\mathbf{i}$ and $\mathbf{j}$ to be as shown in Figure 7(b) or (c), in each of which $\mathbf{i}$ and $\mathbf{j}$ have been rotated together, but not as shown in Figure 7(d), in which $\mathbf{i}$ and $\mathbf{j}$ have been 'flipped' as well as rotated. Note that if you choose directions for $\mathbf{i}$ and $\mathbf{j}$ that are *not* obtained by rotating them together, then the formula for the component form of a force in the box on page 122 *does not hold*, though you can still find component forms of forces by using the method of drawing a right-angled triangle, as illustrated in Examples 3 and 7.

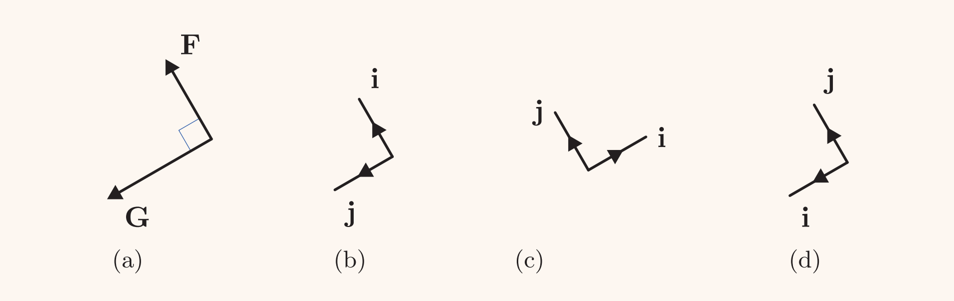


Figure 7 Two forces **F** and **G** and three possible choices for the directions of **i** and **j**. With choice (d) the formula in the box on page 122 does not hold

When you are trying to solve a statics problem, one of the main benefits of choosing the directions of **i** and **j** strategically is that you can often make the simultaneous scalar equations that you obtain easier to solve. If the problem involves finding the magnitude of a force whose direction you know, then you might like to consider choosing the directions of **i** and **j** so that either **i** or **j** is parallel to this force. Then, usually, the symbol that represents the unknown magnitude of the force will appear just once in the simultaneous equations, which makes them easier to solve. If the problem involves finding the magnitudes of *two* forces, and these two forces are perpendicular, then you should certainly consider choosing **i** and **j** to be parallel to these two forces.

In the next activity, you are asked to solve the problem in Activity 8 again. This time, try to choose the directions of **i** and **j** to make the calculations as easy as possible.

Activity 11 *Finding the magnitudes of two more forces*

A particle, which remains at rest, is acted on by three forces, **F**, **G** and **H**.

The force **F** is directed to the left and down, at 120° to the vertical, and the force **G** is directed to the right and down, at 150° to the vertical. The force **H** has magnitude 70 N and is directed vertically upwards.

Find the magnitudes of **F** and **G**, giving your answers as exact values.

You should have found that your working in Activity 11 was significantly simpler than your working in Activity 8.

2 Using tensions and normal reactions

In this section, you will see how to apply the techniques that you learned in Subsection 1.2 to practical statics problems involving weights, tensions and normal reactions. To be able to solve a statics problem in this way, you need to make various simplifications to the situation involved, as described in Subsection 1.1. For example, you have to model objects as particles, and assume that strings are model strings. When we simplify a real-world situation to obtain a model that we can deal with mathematically, we say that we are making **modelling assumptions**.

A general strategy for solving statics problems is given below. It is used in the solutions to most of the examples and activities in the rest of this unit.

Strategy:
To solve a statics problem

1. Make any necessary modelling assumptions.
2. Draw a diagram of the physical situation, annotating it with any relevant information.
3. Identify all the forces acting and draw a separate force diagram, stating any known magnitudes of forces.
4. Choose directions for $\mathbf{i}$ and $\mathbf{j}$ and draw them on the force diagram.
5. Express the forces in component form, in terms of unknown quantities where necessary.
6. Use the equilibrium condition and any other appropriate laws to obtain one or more equations.
7. Solve the equations.
8. State a conclusion.

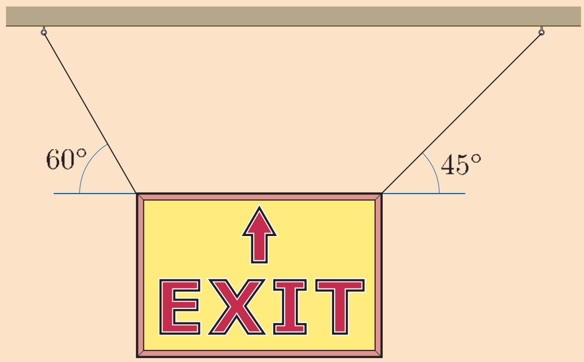
This strategy is a guide rather than a rigid set of rules, and you may need to adapt it for a particular problem that you want to solve. For example, your problem may already include a diagram of the physical situation, or may specify the modelling assumptions to be used or the directions of the Cartesian unit vectors. Also, you have not yet seen how to apply any laws other than the equilibrium condition to statics problems, but you will do so later in this unit.

2.1 Problems involving tensions

The following example demonstrates how to apply the strategy above to a problem involving weight and tension. It is followed by several activities for you to try.

Example 8 *Finding magnitudes of tensions*

A sign of mass 5 kg is supported by two wires making angles of 60° and 45° with the horizontal respectively, as shown below.



Find the tension in each wire, in newtons to two significant figures.

Solution

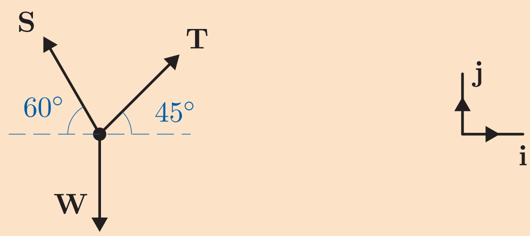
State any modelling assumptions.

Model the sign as a particle, and the wires as model strings.

A diagram of the situation is already provided.

Identify the forces acting. Draw a force diagram. Make sure that you have defined any symbols that you use.

The forces acting are the weight **W** of the sign, the tension **S** in the left-hand wire and the tension **T** in the right-hand wire.



State any known magnitudes of forces. Remember that the magnitude of the weight is mg, where m is the mass and g is the magnitude of the acceleration due to gravity. Since the value of g is not a round number, work in terms of the symbol g rather than immediately using its value.

We know that $|\mathbf{W}| = 5g$.

Choose directions for $\mathbf{i}$ and $\mathbf{j}$. You could consider choosing $\mathbf{i}$ or $\mathbf{j}$ to be parallel to $\mathbf{S}$ or $\mathbf{T}$, but we'll take them to point right and up, so that it's straightforward to work out the component forms of the forces.

Let $\mathbf{i}$ point right and $\mathbf{j}$ point up, as shown.

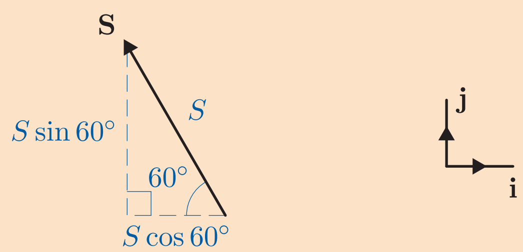
Express each force in component form, using symbols to denote the unknown quantities, which are the magnitudes of $\mathbf{S}$ and $\mathbf{T}$.

Let $S = |\mathbf{S}|$ and $T = |\mathbf{T}|$.

Then

$$\begin{aligned}\mathbf{T} &= |\mathbf{T}|\cos 45^\circ\,\mathbf{i} + |\mathbf{T}|\sin 45^\circ\,\mathbf{j}\\ &= \frac{1}{\sqrt{2}}T\,\mathbf{i} + \frac{1}{\sqrt{2}}T\,\mathbf{j}.\end{aligned}$$

The force $\mathbf{S}$ is shown below.



From the diagram,

$$\begin{aligned}\mathbf{S} &= -S\cos 60^\circ\,\mathbf{i} + S\sin 60^\circ\,\mathbf{j}\\ &= -\frac{1}{2}S\,\mathbf{i} + \frac{\sqrt{3}}{2}S\,\mathbf{j}.\end{aligned}$$

Also,

$$\mathbf{W} = -5g\,\mathbf{j}.$$

Apply the equilibrium condition

Since the particle is in equilibrium, $\mathbf{T}+\mathbf{S}+\mathbf{W}=\mathbf{0}$. So

$$\frac{1}{\sqrt{2}}T\,\mathbf{i}+\frac{1}{\sqrt{2}}T\,\mathbf{j}-\frac{1}{2}S\,\mathbf{i}+\frac{\sqrt{3}}{2}S\,\mathbf{j}-5g\,\mathbf{j}=\mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives the simultaneous equations

$$\frac{1}{\sqrt{2}}T-\frac{1}{2}S=0$$

$$\frac{1}{\sqrt{2}}T+\frac{\sqrt{3}}{2}S-5g=0.$$

Solve these simultaneous equations to find the unknown quantities S and T. Since both equations contain the term $\frac{1}{\sqrt{2}}T$ on the left-hand side, subtracting one equation from the other will eliminate T.

Subtracting the first equation from the second gives

$$\frac{\sqrt{3}}{2}S+\frac{1}{2}S-5g=0;$$

that is,

$$\left(\frac{\sqrt{3}+1}{2}\right)S=5g,$$

which gives

$$S=5g\times\frac{2}{\sqrt{3}+1}=5\times 9.8\times\frac{2}{\sqrt{3}+1}=35.87\ldots.$$

The first simultaneous equation can be written as

$$\frac{1}{\sqrt{2}}T=\frac{1}{2}S$$

and substituting the value of S into this equation gives

$$T=\frac{1}{\sqrt{2}}S=\frac{35.87\ldots}{\sqrt{2}}=25.36\ldots.$$

State a conclusion.

To two significant figures, the magnitudes of the tensions in the left- and right-hand wires are 36 N and 25 N, respectively.

Notice that, in Example 8, instead of substituting in $g=9.8\,\mathrm{m\,s^{-2}}$ and $\sqrt{3}=1.732\ldots$ near the start of the calculation, we worked with the symbols g and $\sqrt{3}$ until the end. This sort of approach is often helpful, because it is quicker to write down a symbol than a detailed numerical value, and because working with exact values until the end of a calculation can help you avoid rounding errors and can sometimes simplify working.

Here is a similar problem for you to try.

Activity 12 *Finding magnitudes of tensions*

A child of mass 30 kg holds on to a playground rope that is suspended at each end. On one side of the child, the rope is at an angle of 45° to the horizontal, and on the other side it is at an angle of 30° to the horizontal.

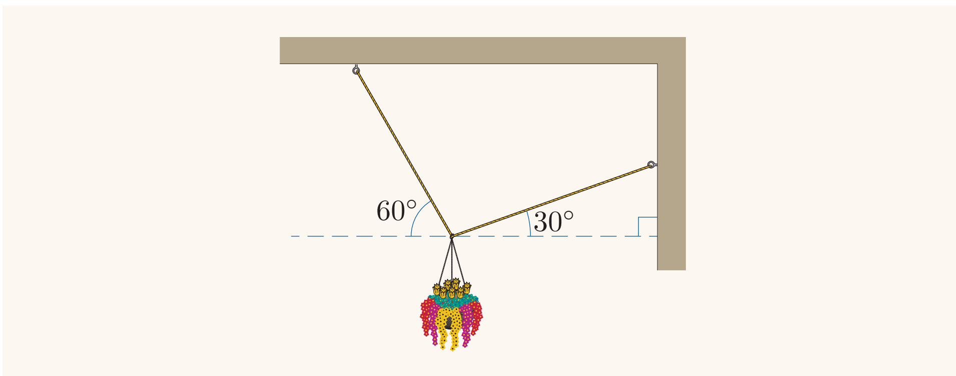
Calculate the tension in each side of the rope, in newtons to two significant figures.

In the next activity, the two unknown quantities are the magnitude of a tension and the mass of a hanging object, so denote these quantities by symbols when you apply the strategy for solving a statics problem.

Activity 13 *Finding a mass*

A flower basket hangs at rest from two straight cords, as shown. One cord, fixed to the ceiling, makes an angle of 60° with the horizontal. The other cord, fixed to a wall, makes an angle of 30° with the horizontal. The second cord applies a tension of magnitude 21 N.

Find the mass of the flower basket, in kilograms to two significant figures.



A wrecking ball

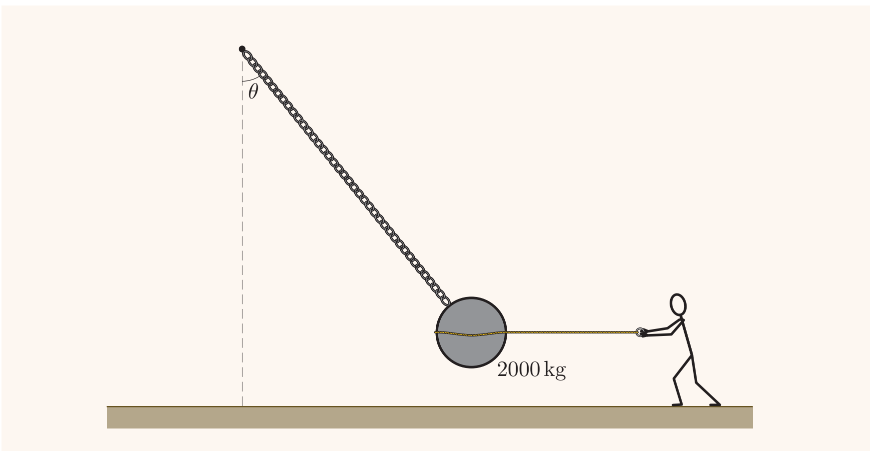
Notice that in the solution to Activity 13, with **i** and **j** chosen to be parallel to the cords, you need to resolve the vector equation only in the **i**-direction to obtain the required answer.

The final activity in this subsection is about a *wrecking ball*, which is a heavy, solid metal ball used for knocking down walls when buildings are demolished. The ball is suspended on a chain, which hangs from a support such as the top of a crane. The ball is pulled to one side and then released to swing and hit its target.

The two unknown quantities in the problem in part (a) of the activity are the magnitude and direction of the tension in the chain that supports the wrecking ball. You can solve the problem by applying the strategy for solving a statics problem in the usual way, denoting these two unknown quantities by symbols. However, you may find that the two simultaneous equations that you obtain seem a little tricky to solve at first sight. See the solution for ideas if you get stuck at this point. An alternative way to solve the problem is to adapt the strategy slightly. In the activity you know the magnitudes and directions of two of the three forces that act on an object in equilibrium, and the problem is to find the magnitude and direction of the third force. A straightforward way to do this is to use the approach of Example 5 in Subsection 1.2. That is, you use the fact that, by the equilibrium condition, the unknown force is the sum of the negatives of the other two forces. Two different solutions are included for this activity, which use the two slightly different methods just described. Either approach is acceptable.

Activity 14 *Finding the magnitude and direction of a tension*

A wrecking ball of mass 2000 kg is suspended from a chain whose other end is attached to a crane. A workman pulls a rope attached to the ball horizontally as hard as he can. This produces a tension of magnitude 350 N in the horizontal rope, and holds the ball at rest, as shown below.



(a) Calculate the magnitude of the tension in the chain (in newtons, to two significant figures) and the angle θ between the chain and the vertical (in degrees, to two significant figures).

(b) Given that the chain has length 6 m, find the horizontal distance of the ball from the vertical line through the top end of the chain.

You may be surprised that in Activity 14 the workman can pull the ball sideways by only 11 cm. This is partly because the ball is so heavy, of course. But the length of the chain also comes into it. You can see from part (a) that the angle to which the workman can pull the chain does not depend on the length of the chain, but is always approximately 1.0°. So, the longer the chain is, the greater is the distance by which the workman can pull the ball sideways. More precisely, the sideways distance is proportional to the length of the chain, as you can see by considering similar triangles, or from the working in part (b) of the solution to Activity 14. So, for example, if the length of the chain is doubled, then the distance by which the workman can pull the ball sideways is also doubled.

2.2 Problems involving normal reactions

In this subsection you will use the strategy that you met at the beginning of this section to solve statics problems that involve normal reactions, as well as other forces such as weights and tensions.

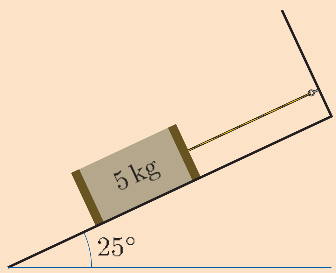
You saw in Subsection 1.1 that whenever an object rests on a surface, there is a pushing force from the surface on the object, called the **normal reaction** of the surface on the object. This force acts at right angles to the surface, and its magnitude adjusts to balance other forces, up to some limit. For example, if an object rests on a horizontal surface, then the normal reaction of the surface on the object acts vertically upwards to balance the weight of the object.

The next example shows you how to deal with a normal reaction from a flat sloping surface, sometimes called an **inclined plane**. Throughout this subsection, whenever we deal with an object resting on a sloping surface, we will make the modelling assumption that the surface imposes no resistance to motion parallel to itself; that is, the surface is *frictionless*. Such a surface is referred to as a **smooth** surface.

In reality, every surface imposes some resistance to motion parallel to itself, due to the force of *friction*. You will see in Section 3 how to include friction in your models for statics problems.

Example 9 *Working with a normal reaction*

A box of mass 5 kg rests on a flat but sloping smooth surface that makes an angle of 25° with the horizontal. The box is prevented from sliding down the slope by a rope attached to it, which runs parallel to the slope, as shown in the diagram below.



Calculate the magnitudes of the tension in the rope and the normal reaction of the surface on the box. Give your answers in newtons, to two significant figures.

Solution

Use the strategy for solving a statics problem. State the modelling assumptions.

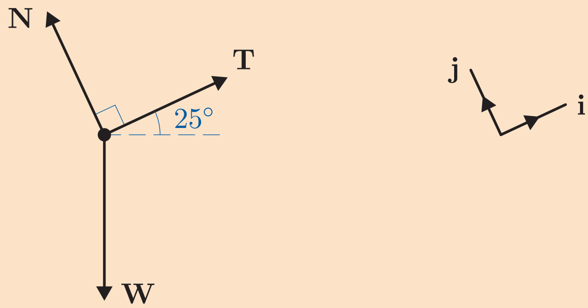
Model the box as a particle and the rope as a model string. From the question, the surface is smooth.

A diagram has already been provided.

Identify all the forces acting and draw a force diagram. State any known magnitudes of forces.

The forces acting on the box are its weight **W**, the tension **T** in the rope and the normal reaction **N** of the surface on the box.

A force diagram is shown below.



We know that $|\mathbf{W}| = 5g$.

Choose directions for **i** and **j**. Notice that we want to find the magnitudes of **T** and **N**, and these forces are at right angles.

Take **i** and **j** to be parallel and perpendicular, respectively, to the slope, as shown above.

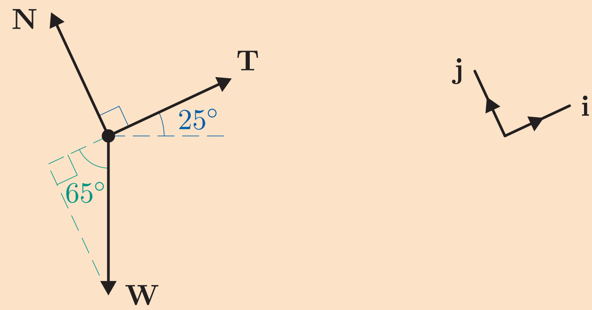
Express each force in component form.

Let $T = |\mathbf{T}|$ and $N = |\mathbf{N}|$. Then

$$\mathbf{N} = N\,\mathbf{j}$$

$$\mathbf{T} = T\,\mathbf{i}.$$

To express **W** in component form, first draw a suitable right-angled triangle on the diagram, and work out one of its acute angles.



Since **W** is vertical, the angle between **W** and the horizontal is 90°, and hence the angle at the tail of **W** in the right-angled triangle is

$$180^\circ - 90^\circ - 25^\circ = 65^\circ,$$

as marked.

So

$$\begin{aligned}\mathbf{W} &= -|\mathbf{W}|\cos 65^\circ\mathbf{i} - |\mathbf{W}|\sin 65^\circ\mathbf{j}\\ &= -5g\cos 65^\circ\mathbf{i} - 5g\sin 65^\circ\mathbf{j}.\end{aligned}$$

Since $\cos 65^\circ$ and $\sin 65^\circ$ cannot easily be expressed in any other form, except as approximations, keep them in this form until the end of the working.

Use the equilibrium condition.

Since the particle is in equilibrium, $\mathbf{N} + \mathbf{T} + \mathbf{W} = \mathbf{0}$. So

$$N\,\mathbf{j} + T\,\mathbf{i} - 5g\cos 65^\circ\mathbf{i} - 5g\sin 65^\circ\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives

$$T - 5g\cos 65^\circ = 0$$
$$N - 5g\sin 65^\circ = 0.$$

Hence

$$T = 5g\cos 65^\circ = 5 \times 9.8 \times \cos 65^\circ = 20.708\ldots.$$

and

$$N = 5g\sin 65^\circ = 5 \times 9.8 \times \sin 65^\circ = 44.409\ldots.$$

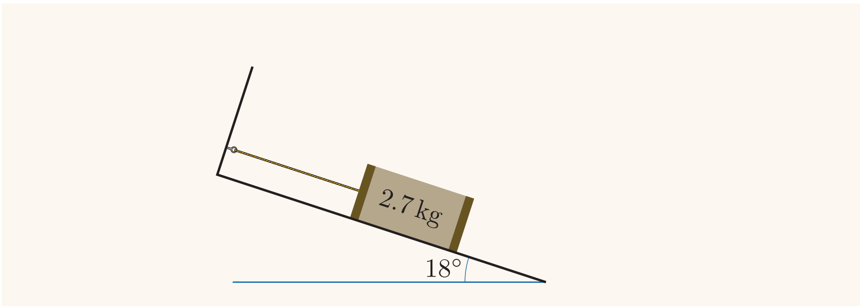
The magnitude of the tension in the rope is 21 N and the magnitude of the normal reaction is 44 N (both to 2 s.f.).

Notice that in Example 9 the directions of **i** and **j** were chosen to be parallel and perpendicular to the slope. This is usually helpful when you are dealing with an inclined plane. It does not matter which of **i** or **j** is parallel to the slope, and which is perpendicular.

In Example 9 the symbol N was used to denote the magnitude of a normal reaction $\mathbf{N}$. When you do this, be careful not to confuse N with the symbol N for the newton.

Activity 15 *Working with a normal reaction*

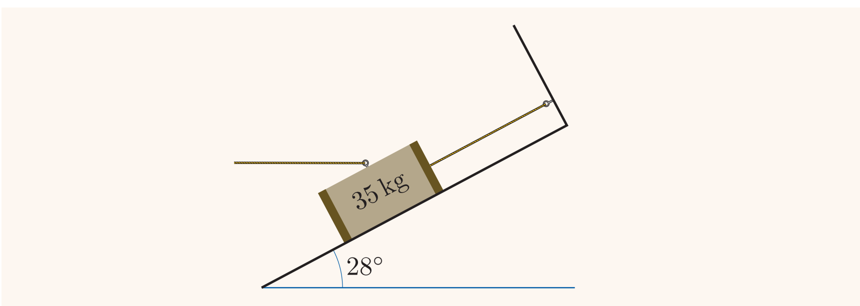
A block of mass 2.7 kg rests on a flat but sloping smooth surface that makes an angle of 18° with the horizontal. The block is prevented from sliding down the slope by a rope attached to it, which runs parallel to the slope, as shown in the diagram below.



Calculate the magnitudes of the tension in the rope and the normal reaction of the surface on the block. Give your answers in newtons, to two significant figures.

Activity 16 *Working with another normal reaction*

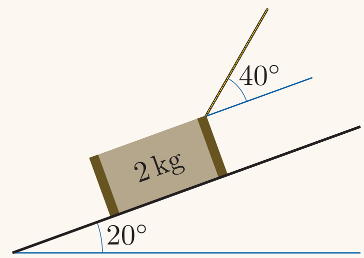
A block of mass 35 kg is held at rest on a smooth inclined plane by an attached rope parallel to the plane. The block is also pulled by a horizontal rope, as shown in the diagram below, which applies a force of 110 N. The plane is inclined at 28° to the horizontal.



Calculate the magnitudes of the tension in the sloping rope and the normal reaction of the plane on the block. Give your answers in newtons, to two significant figures.

Activity 17 *Working with a normal reaction, again*

A block of mass 2 kg is held at rest on a smooth inclined plane by an attached string. The plane is inclined at 20° to the horizontal, and the string is angled at 40° to the plane, as shown in the diagram below.



Calculate the magnitudes of the tension in the string and the normal reaction of the plane on the block. Give your answers in newtons, to two significant figures.

In the last activity of this section you can apply your statics problem-solving skills to an archaeological problem concerning the construction of the Great Pyramid of Cheops at Giza in Egypt.

The construction of the Great Pyramid of Cheops

The Great Pyramid of Cheops

The Egyptian pyramids are the only surviving 'wonder' of the Seven Wonders of the Ancient World. The construction of the Great Pyramid of Cheops, about 4500 years ago, was a stupendous feat of engineering. The pyramid was originally about 150 metres high, with a square base of side length 230 metres. It is made up principally of about 2.5 million blocks of limestone, with an average mass of 2.6 tonnes (2600 kg) each.

The Egyptians of the time were unaware of the wheel and had no sophisticated machines, only human strength. It has been debated for many years just how each of these massive blocks of stone was raised up from the ground to its final position on the pyramid. There are no surviving records to give a definite answer to this question. One theory, backed by some archaeological evidence, is that the blocks were placed on sledges, attached to ropes and then pulled up ramps by a number of workers, as illustrated in Figure 8. The slope would have been made slippery by some form of lubrication.

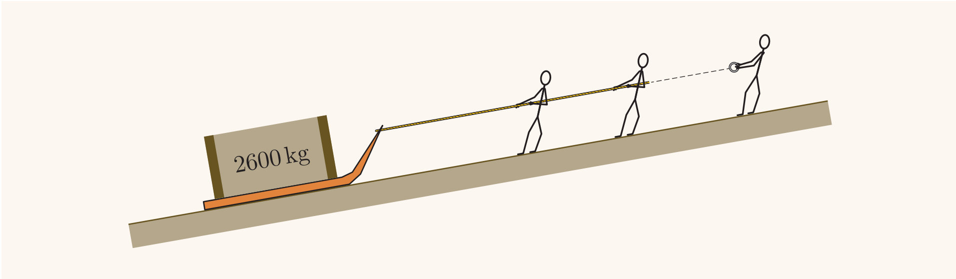


Figure 8 Workers pulling a block of stone up a ramp

Let us try to work out how many workers are needed to raise each block, to test whether this theory seems realistic. We will assume that the workers do not slip on the ramp – they may have stood slightly to the side of the slippery part, though to keep our model simple we will ignore this complication and assume that they stand directly up the ramp from the block.

If the ramp is slippery enough, then the total force required to pull a block *up* the slope is only a little more than the force needed to hold the block *at rest* on the ramp. This is because any net force up the slope will cause the block to move.

A typical angle between the ramp and the horizontal might be 10°. We will assume that the mass of the sledge is small compared with the mass of the block of stone, for which the average mass was around 2600 kg. So the sledge and block together can be modelled as a particle of mass 2600 kg.

In the activity below you are asked to use what you have learned about statics to find an estimate for the number of workers needed to pull the block up the ramp. To do this you need to know how hard each worker can pull. A typical maximum pulling force of a man has magnitude 350 N. (This figure was used earlier, in Activity 14, which was about a workman pulling a wrecking ball.) Since all the workers pull in the same direction, the magnitude of the total pulling force exerted by the workers is the sum of the magnitudes of the pulling forces exerted by the individual workers.

Activity 18 *Finding the number of workers needed to pull a block up a ramp*

A block of stone, of mass 2600 kg, on a light sledge, is held at rest on a smooth ramp by workers pulling on a rope attached to the block. The plane makes an angle of 10° with the horizontal, and the rope is parallel to the plane. Each worker can pull with a force of magnitude at most 350 N.

(a) Calculate the tension in the rope. Give your answer in newtons to two significant figures.

(b) Hence find the minimum number of workers needed to hold the block at rest.

It has to be emphasised that the model used in Activity 18 assumed a complete absence of friction between the sledge and the ramp. In reality, even with a well-lubricated ramp, friction would remain significant. In the next section you will see how you can incorporate friction in your models for statics problems.

3 Static friction

In the previous section we modelled all surfaces as *smooth* – that is, frictionless. In this section you will see how to include the effects of friction in your models. This will allow you to model physical situations more realistically, and extend the range of statics problems that you can solve.

3.1 Modelling static friction

A surface that is not smooth is often referred to as **rough**, though the roughness might be at a microscopic scale, and the surface might *look* smooth.

To understand the force of friction, consider an object, perhaps a book, resting on a rough horizontal surface, such as a table top. Suppose that you push the object gently, parallel to the surface. From your experience you know that if you push only very gently, then the object will not move. This situation is illustrated in Figure 9, which shows the object at rest but pushed by a force that is directed to the right.

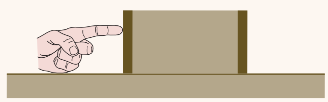


Figure 9 An object at rest on a rough surface but being pushed gently

Since the object remains at rest, there must be a force that balances the pushing force. This force is the force due to friction. There is always a friction force acting whenever two rough surfaces are in contact and 'trying' to move across each other. For the object in Figure 9, the rough surfaces are the bottom of the object and the surface on which the object rests. A friction force that acts on an object at rest on a surface is directed parallel to the surface, in the opposite direction to any possible motion. Since the object in Figure 9 would move to the right if no friction force were acting, the friction force is directed to the left.

Note that a friction force acts only when there is motion or the possibility of motion. If the pushing force in Figure 9 were absent, then there would be no possible motion along the surface and hence no friction force.

Figure 10 shows a force diagram for the situation in Figure 9. Here **P** is the pushing force, **F** is the friction force, **W** is the weight of the object and **N** is the normal reaction of the surface on the object.

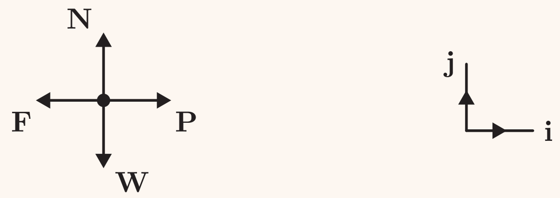


Figure 10 A force diagram for the situation in Figure 9

Resolving the equilibrium condition for Figure 10 in the **i**- and **j**-directions shown gives the equations

$$|\mathbf{P}| - |\mathbf{F}| = 0,$$
$$|\mathbf{N}| - |\mathbf{W}| = 0.$$

So, as you would expect, the pushing force is balanced by the friction force, and the weight is balanced by the normal reaction.

If the object in Figure 9 is pushed harder, then the magnitude of the friction force will increase to balance the increased magnitude of the pushing force. But if the object is pushed hard enough, then it will start to move. So the friction force can have any magnitude up to some maximum value. If the magnitude of the pushing force exceeds the maximum magnitude of the friction force, then the friction force can no longer balance the pushing force, and the object will move to the right.

Let's consider what the maximum magnitude of the friction force depends on. First, think about what happens if the object is made of metal and the surface consists of oil-covered metal. In this case you need to apply only a small force to move the object. This means that the maximum magnitude of the friction force is small. Now think about what happens if the object and the surface are made of rubber (but the mass of the object is the same as before). In this case you need to apply a large force to move the object. This means that the maximum magnitude of the friction force is large. So the maximum magnitude of the friction force seems to depend, at least partly, on the materials in contact.

However, the maximum magnitude of the friction force also depends on something else. Think about what happens if the materials in contact remain the same, but the object is heavier. For example, you might place a heavy object on top of the first object. You now need to apply a larger force to move the object. This means that the maximum magnitude of the friction force is larger than before. It suggests that the maximum magnitude of the friction force also depends on the weight of the object. However, things are not quite so simple. In fact, the maximum magnitude of the friction force also depends not on the weight of the object, but on the *normal reaction* of the surface on the object.

To see this, consider the situation in Figure 11. Here the object is lifted slightly by the tension in an attached taut vertical string, but is not lifted enough to break contact with the surface. On the right is a force diagram for this situation, in which the tension force is denoted by $\mathbf{T}$, and the other forces are denoted by the same symbols as earlier. Note that when two forces act on the same object in the same direction, we usually draw them in a force diagram as overlapping arrows, as is done for $\mathbf{N}$ and $\mathbf{T}$ in Figure 11.

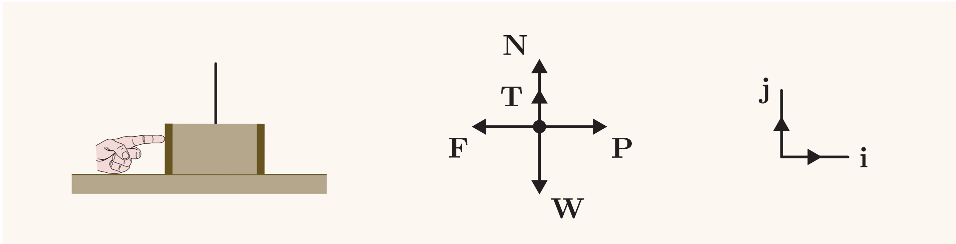


Figure 11 An object on a rough surface being slightly lifted

With the taut string present, you need to apply a smaller pushing force to move the object than if the string were not present. So the maximum magnitude of the friction force is smaller. But the weight of the object has not changed!

What *has* changed is the normal reaction. Resolving the equilibrium condition for Figure 11 in the $\mathbf{j}$-direction shown gives the equation

$$|\mathbf{N}| + |\mathbf{T}| - |\mathbf{W}| = 0;$$

that is,

$$|\mathbf{N}| = |\mathbf{W}| - |\mathbf{T}|.$$

So the tension in the string has reduced the magnitude of the normal reaction. This suggests that the maximum magnitude of the friction force depends on the normal reaction of the surface on the object, rather than on the weight of the object. Experiments confirm that this is indeed the case.

In fact, experiments show that the maximum magnitude of the friction force exerted by a surface on an object at rest can be modelled by the expression $\mu|\mathbf{N}|$, where $\mathbf{N}$ is the normal reaction of the surface on the object, and μ is a constant whose value depends on the properties of the materials in contact. The constant μ is known as the **coefficient of static friction**. The symbol μ is the lower-case Greek letter mu, which is pronounced 'mew'. The phrase 'static friction' is used here because we are considering a friction force that acts on an object *at rest*. Similar, but slightly different, friction forces act on a moving object. Some typical values of μ are given in Table 1.

Table 1 Typical values of the coefficient μ of static friction

Materials in contact	μ
Steel on steel (dry)	0.74
Steel on steel (oiled)	0.14
Plastic on plastic	0.35
Rubber on tarmac	0.85
Steel on wood	0.55
Wood on wood	0.42

Notice that the maximum magnitude of the friction force that a surface exerts on an object does *not* depend on the area of contact between the object and the surface.

If the maximum value $\mu|\mathbf{N}|$ of the magnitude of the friction force on an object is insufficient for the friction force to balance the other forces acting on the object and hence prevent motion, then the object will start to move. If the magnitude of the friction force on an object is exactly at its maximum value $\mu|\mathbf{N}|$, then we say that the object is **on the point of slipping**, or **held by limiting friction**.

The properties of friction acting on an object at rest that you have met are summarised below.

Static friction

If an object at rest is in contact with a flat rough surface, then any friction force $\mathbf{F}$ on the object acts in a direction parallel to the surface and opposite to any possible motion along the surface. Its magnitude adjusts to balance other forces, up to its maximum magnitude.

The maximum magnitude of the friction force is $\mu|\mathbf{N}|$, where μ is the coefficient of static friction for the object and the surface involved, and $\mathbf{N}$ is the normal reaction of the surface on the object.

If $|\mathbf{F}| = \mu|\mathbf{N}|$, then the object is on the point of slipping; that is, it is held by limiting friction.

You can use the strategy for solving statics problems from the beginning of Section 2 to solve statics problems where friction forces are present, as demonstrated in the following example.

Example 10 *Working with limiting static friction*

A full cardboard box of mass 60 kg rests on a horizontal wooden pallet. The coefficient of static friction between the box and the pallet is 0.22.

With what magnitude of force does a worker need to push the box to make it be on the point of slipping? Give your answer to two significant figures.

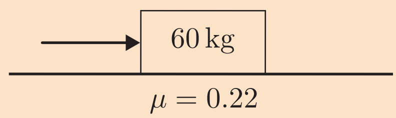
Solution

Start by assuming that the situation asked about is actually happening.

Assume that the worker pushes the box so that it is on the point of slipping.

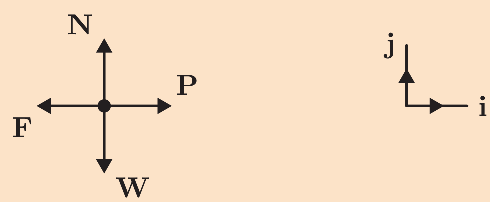
Make modelling assumptions, and draw a diagram of the situation.

Model the box as a particle.



Identify the forces acting, and draw a force diagram, stating any known magnitudes of forces. Remember that friction acts in a direction to oppose any motion. Choose directions for $\mathbf{i}$ and $\mathbf{j}$.

The forces acting are the weight $\mathbf{W}$ of the box, the normal reaction $\mathbf{N}$, the pushing force $\mathbf{P}$ and the friction force $\mathbf{F}$.



We know that $|\mathbf{W}| = 60g$.

Take $\mathbf{i}$ to point right and $\mathbf{j}$ to point up, as shown.

Express each force in component form, in terms of unknown quantities where necessary.

Let $P = |\mathbf{P}|$, $N = |\mathbf{N}|$ and $F = |\mathbf{F}|$.

Expressing the forces in component form gives

$$\mathbf{P} = P\,\mathbf{i}$$
$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{F} = -F\,\mathbf{i}$$
$$\mathbf{W} = -60g\,\mathbf{j}.$$

Apply the equilibrium condition.

Since the box is at rest, $\mathbf{P} + \mathbf{W} + \mathbf{N} + \mathbf{F} = \mathbf{0}$. This gives

$$P\,\mathbf{i} + N\,\mathbf{j} - F\,\mathbf{i} - 60g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives the pair of equations

$$P - F = 0,$$
$$N - 60g = 0.$$

Hence find N. Then use the fact that the box is on the point of slipping to find F and hence P.

The second equation gives $N = 60g$.

Since the box is on the point of slipping, we have

$$F = \mu N = 0.22 \times 60g = 13.2g.$$

So the first of the pair of equations above gives

$$P = F = 13.2g = 13.2 \times 9.8 = 129.36.$$

The required magnitude of the pushing force is 130 N (to 2 s.f.).

Here are two similar activities for you to try. As in Example 10, you should apply the usual strategy for solving statics problems.

Activity 19 *Working with limiting static friction*

A plastic box of mass 10 kg rests on a wooden floor. The coefficient of static friction between the box and the floor is 0.5.

What is the maximum horizontal force that can be applied to the box without the box moving? Give your answer in newtons to two significant figures.

Activity 20 *Working with limiting static friction, again*

A wooden block of mass 5.5 kg rests on a horizontal plank of wood. It is pushed by a horizontal force of magnitude 23 N, and is on the point of slipping.

Determine the coefficient of static friction between the block and the plank, to two significant figures.

3.2 Static friction on objects on inclined planes

If you place an object on an inclined plane, and the object remains at rest, then this is because the friction force is acting to prevent the object from sliding down the plane. In this situation, it is clear that the friction force acts *up* the slope (parallel to the slope). If you know the mass of the object and the angle of the slope, then you can calculate the magnitude of the friction force by using the strategy for solving statics problems in the usual way.

Sometimes, however, it is not clear in which direction a friction force acts. Consider an object that is held at rest on a rough inclined plane by a string, as illustrated in Figure 12.

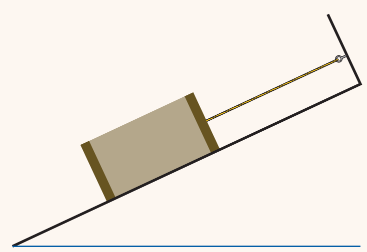


Figure 12 An object held on a rough inclined plane by a string

One possibility is that the friction force and the tension in the string are both contributing to preventing the object from sliding down the plane. In this case, the friction force acts *up* the slope, as illustrated in the force diagram in Figure 13(a). An alternative possibility, if the magnitude of the tension force is larger, is that the tension force may be 'trying' to pull the block up the plane, and the force of friction is preventing this from happening. In this case, the friction force acts *down* the slope, as illustrated in the force diagram in Figure 13(b). In these diagrams, as usual $\mathbf{W}$ is the weight of the object, $\mathbf{N}$ is the normal reaction, $\mathbf{T}$ is the tension and $\mathbf{F}$ is the friction force.

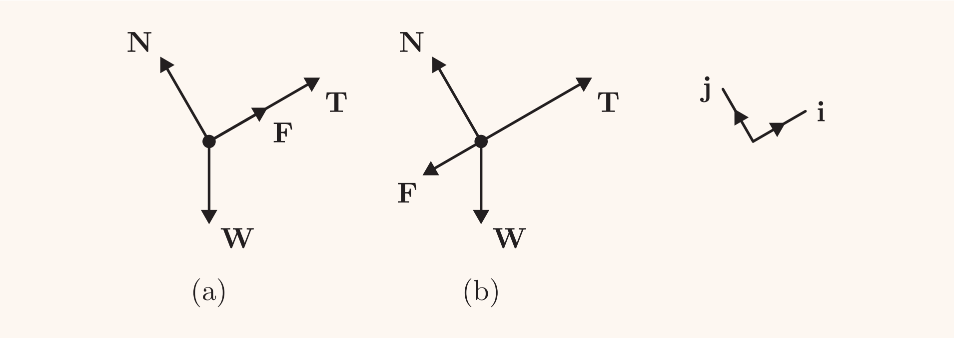


Figure 13 Possible force diagrams for the object in Figure 12, with friction acting (a) up the slope (b) down the slope

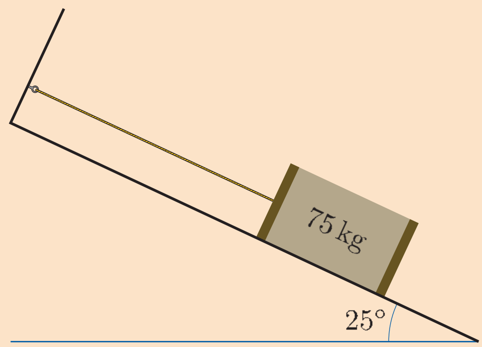
So, when you want to solve a problem involving friction exerted by an inclined plane, you may not initially know whether the friction force acts up the slope or down the slope. In such a case, how can you draw a force diagram, and express the friction force in component form?

Here is how you can get round this problem. You start by choosing $\mathbf{i}$ to point parallel to the slope, and $\mathbf{j}$ perpendicular to it, as shown in Figure 13. You draw a force diagram with the friction force $\mathbf{F}$ acting in the $\mathbf{i}$-direction. Then, instead of taking an unknown F to represent the *magnitude* of the friction force $\mathbf{F}$, you take F to represent the $\mathbf{i}$-*component* of $\mathbf{F}$; that is, you write $\mathbf{F} = F\,\mathbf{i}$. You can then proceed in the usual way. If F turns out to be positive, then $\mathbf{F}$ acts in the $\mathbf{i}$-direction; if F turns out to be negative, then $\mathbf{F}$ acts in the opposite direction. (You can choose $\mathbf{i}$ to be perpendicular to the slope and $\mathbf{j}$ to be parallel to it, of course, rather than the other way round – this might be convenient in some situations. If you do this, then you write $\mathbf{F} = F\,\mathbf{j}$.)

This method is demonstrated in the next example. Note that unlike the example and activities that you saw in the previous subsection, this example is not about *limiting* static friction. In this example, the friction force is just large enough to balance the other forces.

Example 11 *Finding the magnitude and direction of a friction force*

A box of mass 75 kg is held on a plane inclined at 25° to the horizontal by a rope parallel to the plane, as shown below.



If the tension in the rope is 135 N, what is the magnitude and direction of the friction force?

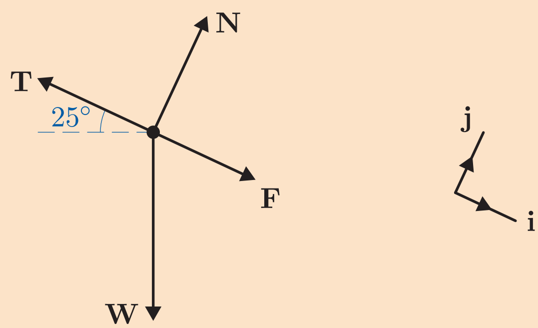
Solution

Make modelling assumptions.

Model the box as a particle and the string as a model string.

Identify the forces acting, and draw a force diagram. Choose **i** to be parallel to the slope, and draw the friction force acting in the **i**-direction. State the known magnitudes of forces.

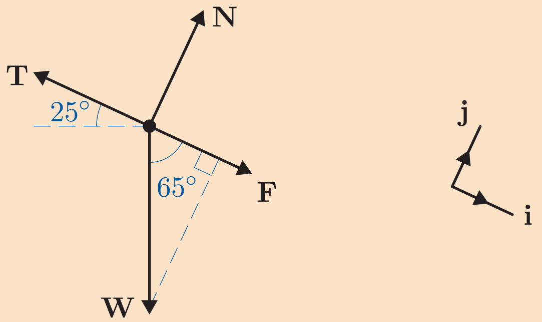
The forces acting are the weight **W** of the box, the normal reaction **N**, the tension **T** in the string and the friction force **F**. A force diagram is shown below.



We know that $|\mathbf{T}| = 135$ and $|\mathbf{W}| = 75g$.

Take **i** to point down the slope and **j** to point perpendicular to the slope, as shown.

Express each force in component form, in terms of unknown quantities where necessary. To help find **W** in component form, draw a suitable right-angled triangle on the force diagram, and work out an acute angle in it.



The angle between **F** and **W** is $180° - 25° - 90° = 65°$, as marked in the diagram above.

Let $N = |\mathbf{N}|$ and let F be the **i**-component of **F**. Then

$$\begin{aligned} \mathbf{T} &= -135\,\mathbf{i} \\ \mathbf{F} &= F\,\mathbf{i} \\ \mathbf{N} &= N\,\mathbf{j} \\ \mathbf{W} &= 75g\cos 65°\mathbf{i} - 75g\sin 65°\mathbf{j}. \end{aligned}$$

Use the equilibrium condition.

Since the block is at rest, $\mathbf{T} + \mathbf{F} + \mathbf{N} + \mathbf{W} = \mathbf{0}$. This gives

$$-135\,\mathbf{i} + F\,\mathbf{i} + N\,\mathbf{j} + 75g\cos 65°\mathbf{i} - 75g\sin 65°\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**-direction gives

$$-135 + F + 75g\cos 65° = 0.$$

So

$$F = 135 - 75 \times 9.8\cos 65° = -175.624\ldots = -180 \text{ (to 2 s.f.)}.$$

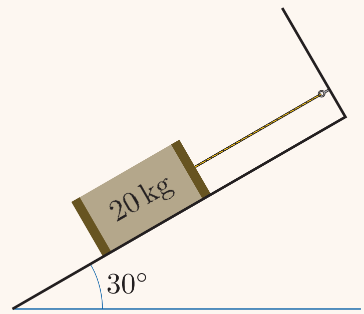
Since F is negative, **F** points in the *opposite* direction to **i**.

State a conclusion.

The friction force has magnitude 180 N (to 2 s.f.) and acts up the slope.

Activity 21 *Finding the magnitude and direction of a friction force*

A box of mass 20 kg is held on a plane inclined at 30° to the horizontal by a string parallel to the plane, as shown below.



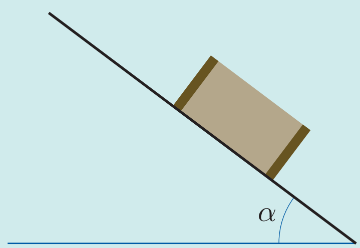
If the tension in the string is 120 N, what is the magnitude and direction of the friction force? Give the magnitude in newtons to two significant figures.

There is a neat experimental method for finding an estimate of the coefficient of static friction between two materials. You place an object whose surface is made of one material on a plane made of the other material, and increase the angle of inclination of the plane until the object starts to slip. Then you use the fact below.

Coefficient of static friction in terms of angle of inclination

If an object rests on a plane inclined at the angle α, and is on the point of slipping, then the coefficient μ of static friction between the object and the plane is given by

$$\mu = \tan\alpha.$$



You can confirm this fact by using the usual strategy for solving a statics problem. Figure 14 shows a force diagram for the object described in the box above. Here $\mathbf{W}$ is the weight of the object, $\mathbf{N}$ is the normal reaction and $\mathbf{F}$ is the friction force.

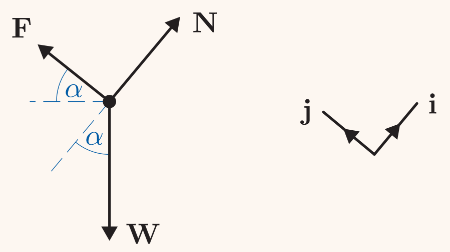


Figure 14 A force diagram for the object in the box above

Since $\mathbf{F}$ is parallel to the inclined plane, the angle between $\mathbf{F}$ and the horizontal is α, as marked. It follows that the acute angle between $\mathbf{W}$ and a line parallel to $\mathbf{N}$ is also α, as also marked. (To see this, imagine rotating the first angle through 90° anticlockwise.)

Let $N = |\mathbf{N}|$ and $W = |\mathbf{W}|$. Since the object is on the point of slipping, $|\mathbf{F}| = \mu|\mathbf{N}| = \mu N$.

Let $\mathbf{i}$ and $\mathbf{j}$ have the same directions as $\mathbf{N}$ and $\mathbf{F}$, respectively, as shown. Then

$$\mathbf{N} = N\,\mathbf{i}$$
$$\mathbf{F} = \mu N\,\mathbf{j}$$
$$\mathbf{W} = -W\cos\alpha\,\mathbf{i} - W\sin\alpha\,\mathbf{j}.$$

Applying the equilibrium condition $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$ gives

$$N\,\mathbf{i} + \mu N\,\mathbf{j} - W\cos\alpha\,\mathbf{i} - W\sin\alpha\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives

$$N = W\cos\alpha$$
$$\mu N = W\sin\alpha.$$

Dividing the second of these equations by the first gives the equation $\mu = \tan\alpha$ stated in the box above.

Activity 22 *Using the relationship between the coefficient of static friction and the angle of a slope*

A plate on a tray is found to start slipping when the tray is tilted at 12° to the horizontal. What is the coefficient of static friction between the plate and the tray, to two decimal places?

Notice in particular that the fact in the box above tells you that if you place an object on a plane and gradually increase the angle of inclination of the plane, then the angle α at which the object starts to slip does not depend on the mass of the object. The angle α is given simply by $\alpha = \tan^{-1}\mu$, where μ is the coefficient of static friction between the object and the plane.

Activity 23 *Using the relationship between the coefficient of static friction and the angle of a slope, again*

A wooden block is placed on a rough plane. The coefficient of static friction between the block and the plane is 0.34. If the angle of inclination of the plane is gradually increased, at what angle will the block start slipping? Give your answer to the nearest degree.

The construction of the Great Pyramid of Cheops revisited

In Subsection 2.2 you were asked to calculate the number of Egyptian workers needed to pull a massive stone up a ramp during the construction of the Great Pyramid of Cheops. Let's now revisit this problem and incorporate the effect of friction in our model, which should lead to a more accurate answer.

To do this, we need an estimate for the coefficient μ of static friction that applies for contact between the sledge containing the stone block and the lubricated ramp. To obtain such an estimate, let's refer to the Egyptian relief picture shown in Figure 15. It depicts the transport of a huge alabaster statue on a sledge across horizontal ground. Liquid is being poured from pots in front of the statue, to reduce friction, and 172 workers are pulling the statue along with ropes. The value of μ that applies here should be about the same as the value of μ that applies for the sledge on the lubricated ramp. The mass of the statue has been estimated to be 60 tonnes, that is, 60 000 kg. In the next activity you are asked to use this information to estimate the value of μ. To do this, use the usual strategy for solving a statics problem – the problem is similar to the one in Activity 20. Assume that all the workers are pulling in the same direction, so that the total magnitude of the pulling force is the sum of the magnitudes of the 172 individual pulling forces.

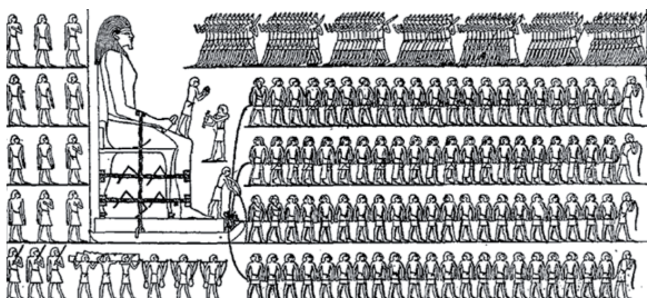


Figure 15 A picture (restored) at the tomb of the XIIth dynasty noble Djehutihotep

Activity 24 *Finding an estimate for the coefficient of static friction that applies in the pyramid construction problem*

A stone statue of mass 60 000 kg on a light sledge is pulled along a horizontal surface by 172 workers pulling on ropes. Each worker pulls with a force of magnitude 350 N, and the combined effort of the workers is just sufficient to overcome friction and move the statue. (In other words, the total pulling force is the pulling force needed for the statue to be on the point of slipping.) Calculate, to two decimal places, the coefficient μ of static friction between the statue and the lubricated surface along which it is pulled.

You can now use the coefficient of static friction found in Activity 24 to revise your earlier calculation of the number of workers needed to pull the stone up the ramp.

Activity 25 *Finding a more accurate estimate for the number of workers needed in the pyramid problem*

A stone block of mass 2600 kg, on a light sledge, is held on a ramp by workers pulling on a rope attached to the block. The plane makes an angle of $10°$ with the horizontal, and the rope is parallel to the plane. The coefficient of static friction between the sledge and the ramp is $\mu = 0.10$. Each worker can pull with a force of magnitude at most 350 N.

(a) If the block is at the point of slipping *up* the ramp, in what direction does the friction force act?

(b) Calculate the tension in the rope for the situation where the block is at the point of slipping up the ramp. Give your answer in newtons to two significant figures.

(c) Hence find the minimum number of workers needed to exert a pulling force sufficient for the block to be on the point of slipping up the ramp.

In Activity 18 in Subsection 2.2 you saw that if you do not take friction into account, then the minimum number of workers needed to overcome the effect of gravity and pull the block up the ramp is 13. The result of Activity 25 shows that in fact another 7 workers are needed to overcome the friction force.

You can use these facts to estimate the minimum number of workers needed to just *hold* the block at rest on the ramp, rather than pull it up, when friction is taken into account. In this situation the block is at the point of slipping *down* the slope, so the friction force acts *up* the slope. The normal reaction remains the same, so the maximum magnitude of the friction force remains the same. So in this case friction provides the same force up the slope as would be provided by about 7 workers pulling on the rope, and hence only about 13 workers minus 7 workers, that is, 6 workers, are needed to prevent the block sliding down the slope. You might like to confirm this by repeating the working in Activity 25, but with the friction force acting in the opposite direction.

4 More than one object

So far in this unit you have looked how forces act on single objects that can be modelled as particles. In this section you will consider situations that involve more than one object. You will also see how to deal with physical situations involving *pulleys*.

4.1 Actions and reactions

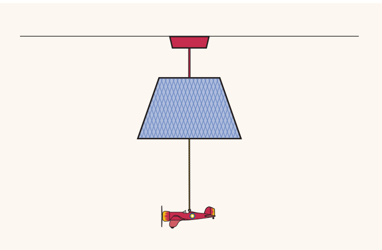


Figure 16 A child's bedroom ceiling lamp

As a simple example of the type of situation involving more than one object that we will be looking at here, consider a child's bedroom ceiling lamp that consists of a light suspended from the ceiling, with a model aeroplane suspended below the light, as illustrated in Figure 16. This is a system consisting of one suspended object with a further suspended object hanging from it.

You can model the light and the model aeroplane as particles, and the cable supporting the light and the string supporting the model aeroplane as model strings, as illustrated in Figure 17.

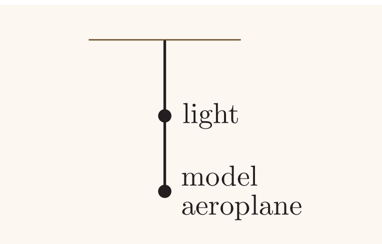


Figure 17 A particle model of the child's bedroom ceiling lamp

The forces acting on the lower particle are its weight, say $\mathbf{W}_1$, and the tension in the lower string, say $\mathbf{T}_1$. A force diagram for this particle is shown on the right of Figure 18. The forces acting on the upper particle are its weight, say $\mathbf{W}_2$, the tension in the upper string, say $\mathbf{T}_2$, and the tension in the lower string, say $\mathbf{T}_3$, which pulls down on the upper particle. A force diagram for this particle is shown on the left of Figure 18.

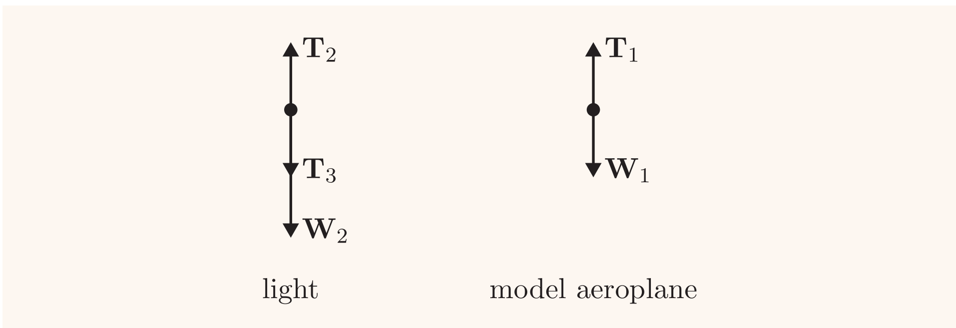


Figure 18 Force diagrams for the particles modelling the light and the model aeroplane

Each of the two particles is in equilibrium, so we can apply the equilibrium condition separately to each particle. For the lower particle, this gives

$$\mathbf{W}_1 + \mathbf{T}_1 = \mathbf{0}.$$

For the upper particle, it gives

$$\mathbf{W}_2 + \mathbf{T}_2 + \mathbf{T}_3 = \mathbf{0}.$$

There is a link between these two equations, because the forces $\mathbf{T}_1$ and $\mathbf{T}_3$ are different manifestations of the tension in the same string. The force $\mathbf{T}_1$ is the upwards force on the lower particle due to the upper particle, and the force $\mathbf{T}_3$ is the downwards force on the upper particle due to the lower particle. Since these forces are tensions in the same model string, they have the same magnitude but act in opposite directions. That is,

$$\mathbf{T}_1 = -\mathbf{T}_3.$$

Together, these two forces constitute an example of *Newton's third law of motion*, which you met in the introduction to this unit. It is stated again below, in different wording.

Newton's third law of motion

For each force exerted by one object on a second object, there is a force of equal magnitude in the opposite direction, exerted by the second object on the first.

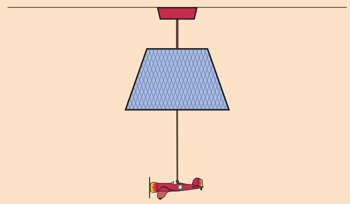
Two forces equal in magnitude and opposite in direction that act according to Newton's third law are sometimes referred to as an **action** and an **equal and opposite reaction**.

It is important to appreciate that the two forces in Newton's third law *act on different objects*. So, for example, the weight of an object that rests on a flat surface and the normal reaction on the object from the surface are *not* an action and an equal and opposite reaction. However, Newton's third law tells you that, since the surface exerts an upward force on the object, the object also exerts a downward force of equal magnitude on the surface. These two forces are an action and an equal and opposite reaction.

The next example demonstrates how you can solve a statics problem that involves a system of two objects, each of which can be modelled as a particle. The system in the example is a child's ceiling lamp of the kind described near the beginning of this subsection.

Example 12 *Solving a statics problem involving two objects*

A light of mass 0.5 kg hangs from the ceiling on a cable. A model aeroplane of mass 0.3 kg is suspended from the light by a string, as shown below.



By modelling the light and the model aeroplane as particles, and the cable and the string as model strings, calculate the magnitudes of the tensions in the cable and string, in newtons to two significant figures.

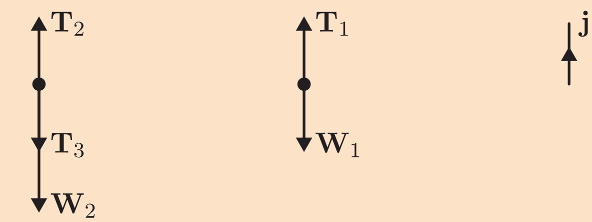
Solution

Identify the forces acting, and draw a force diagram for each particle.

The forces acting on the model aeroplane are its weight $\mathbf{W}_1$ and the tension $\mathbf{T}_1$ in the lower string pulling upwards.

The forces acting on the light are its weight $\mathbf{W}_2$, the tension $\mathbf{T}_2$ in the cable, and the tension $\mathbf{T}_3$ in the lower string pulling downwards.

Force diagrams are shown below.



light   model aeroplane

We know that $|\mathbf{W}_1| = 0.3g$ and $|\mathbf{W}_2| = 0.5g$.

Choose the direction for $\mathbf{j}$. Since all the forces act vertically, there is no need for $\mathbf{i}$.

Take $\mathbf{j}$ to point vertically upwards.

Express each force in component form, in terms of unknown quantities where necessary.

We have $|\mathbf{T}_1| = |\mathbf{T}_3|$ (since $\mathbf{T}_1$ and $\mathbf{T}_3$ are tensions in the same string), so let $T = |\mathbf{T}_1| = |\mathbf{T}_3|$. Also, let $T_2 = |\mathbf{T}_2|$.

From the force diagram for the lower particle we have

$$\mathbf{W}_1 = -0.3g\,\mathbf{j}$$
$$\mathbf{T}_1 = T\,\mathbf{j}.$$

From the force diagram for the upper particle we have

$$\mathbf{W}_2 = -0.5g\,\mathbf{j}$$
$$\mathbf{T}_2 = T_2\,\mathbf{j}$$
$$\mathbf{T}_3 = -T\,\mathbf{j}.$$

First apply the equilibrium condition to the lower particle to find the value of T, the magnitude of each of the two tension forces that form the action and reaction pair.

Since the lower particle is at rest, $\mathbf{W}_1 + \mathbf{T}_1 = \mathbf{0}$. This gives

$$-0.3g\,\mathbf{j} + T\,\mathbf{j} = \mathbf{0},$$

which gives

$$-0.3g + T = 0.$$

Hence

$$T = 0.3g = 0.3 \times 9.8 = 2.94.$$

Now apply the equilibrium condition to the upper particle, and use the value of T, to find the magnitude of the third tension force.

Since the upper particle is at rest, $\mathbf{W}_2 + \mathbf{T}_2 + \mathbf{T}_3 = \mathbf{0}$. This gives

$$-0.5g\,\mathbf{j} + T_2\,\mathbf{j} - T\,\mathbf{j} = \mathbf{0},$$

which gives

$$-0.5g + T_2 - T = 0.$$

Hence

$$\begin{aligned} T_2 &= 0.5g + T \\ &= 0.5g + 0.3g \\ &= 0.8g \\ &= 0.8 \times 9.8 \\ &= 7.84. \end{aligned}$$

So the tension in the cable has magnitude 7.8 N (to 2 s.f.) and the tension in the string has magnitude 2.9 N (to 2 s.f.).

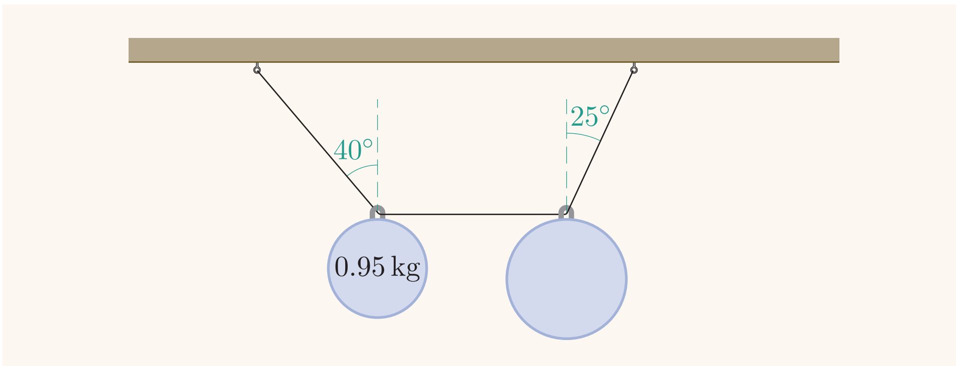
An alternative approach to the problem in Example 12 is to first find the tension in the cable (the upper model string) by modelling all of the light, the model aeroplane and the string as a single particle of weight $\mathbf{W}_1 + \mathbf{W}_2$, and then find the tension in the string (the lower model string) by separately considering the model aeroplane and the string supporting it.

When you are trying to solve a statics problem that involves two objects, it is often helpful to proceed by applying the equilibrium condition to one of the two force diagrams to find (in terms of unknown quantities if necessary) the magnitude of the force in this force diagram that is one of the action and reaction pair that links the two force diagrams. Then you can apply the equilibrium condition to the other force diagram, and use this magnitude, to solve the problem. Usually, the first force diagram to which you should apply the equilibrium condition should be the one for which you have the most information. This approach was illustrated in Example 12, and you can use it in the next activity.

Activity 26 *Solving a statics problem involving two objects*

Two decorations hang from a string that is suspended at each end, as shown below. The left- and right-hand ends of the string make angles of 40° and 25° with the vertical, respectively, and the middle section is horizontal. The left-hand decoration has mass 0.95 kg.

Find the mass of the right-hand decoration, in kilograms to two significant figures.



Notice that the answer to Activity 26 does not depend on the value of g, since g is cancelled out at the end of the working in the given solution.

4.2 Pulleys

A **pulley** is a commonly-used device consisting of a wheel over which a string can run, as shown in Figure 19. The string touches part of the rim of the wheel, but can leave the wheel at any point, redirecting the tension force in the string. For example, the operating string of a roller blind runs over a pulley, and pulleys are used to help raise and lower sails on boats.

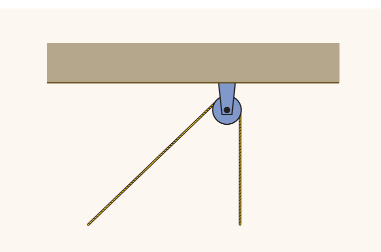


Figure 19 A string running over a pulley

A pulley is normally designed so that the string does not slip relative to the wheel, and so that the wheel spins freely on its axle. When we model a physical situation involving a pulley, we usually model the pulley as a *model pulley*. This is defined as follows.

A **model pulley** is an object with no mass or size, over which a model string may pass without any resistance to motion. The magnitude of the tension in a model string passing over a model pulley is the same on each side of the model pulley.

A model pulley is a reasonable model of a real pulley, provided that the dimensions of the real pulley are small compared with the length of the rope or cable passing over it, and the weight of the real pulley is small compared with the other forces involved.

However, a model pulley has uses other than for modelling actual pulleys. It is useful for modelling any situation where the direction of a tension force changes, and friction where this change occurs is negligible. For example, if an object hanging from a string attached to the ceiling is pulled to one side by the string dangling over the edge of a shelf, as illustrated in Figure 20, then it may be reasonable to model the edge of the shelf as a model pulley. You will see an example of this sort of use of a model pulley in the second of the three activities that follow the example below.

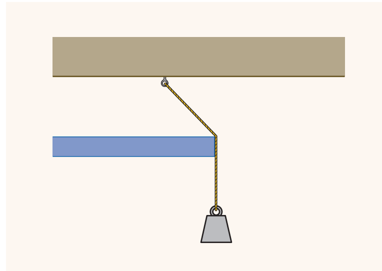


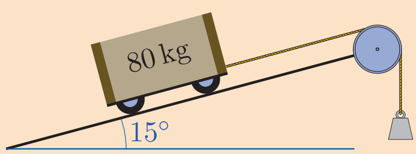
Figure 20 An object hanging on a string dangling over a shelf

The key property of a model pulley, whether it models a real pulley or something else, is that it changes the direction of a tension force without changing its magnitude.

Example 13 *Using a model pulley*

A loaded trolley of mass 80 kg is held at rest on a ramp inclined at 15° to the horizontal by a rope parallel to the ramp, as shown below. The rope passes over a pulley at the top of the ramp, and a metal weight hangs from the other end of the rope.

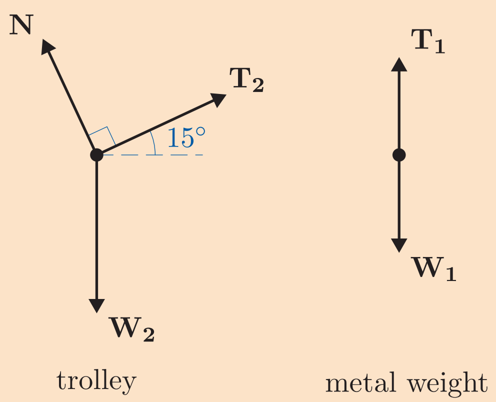
Model the trolley and the metal weight as particles, the rope as a model string, the ramp as a smooth inclined plane, and the pulley as a model pulley. Find the mass of the metal weight, in kilograms to two significant figures.



Solution

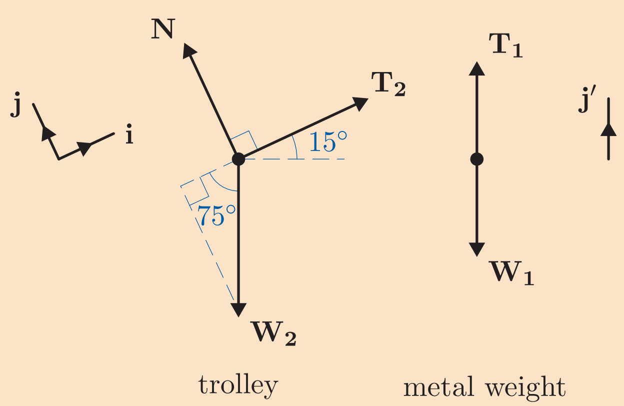
Identify the forces acting on each particle, and draw force diagrams.

The forces acting on the metal weight are its weight $\mathbf{W}_1$ and the tension $\mathbf{T}_1$ in the rope. The forces acting on the trolley are its weight $\mathbf{W}_2$, the tension $\mathbf{T}_2$ in the rope, and the normal reaction $\mathbf{N}$ from the plane. Force diagrams are shown below.



We know that $|\mathbf{W}_2| = 80g$.

Choose directions for $\mathbf{i}$ and $\mathbf{j}$. In this particular example, the directions that will simplify the working for one force diagram are different from the directions that will simplify the working for the other force diagram. So choose different directions in each case. Use the labels $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{i}'$, $\mathbf{j}'$ to avoid confusion (though here only $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{j}'$ are needed).



Take $\mathbf{i}$ and $\mathbf{j}$ to be parallel and perpendicular to the slope, respectively, as shown. Take $\mathbf{j}'$ to point vertically up.

Express each force in component form, in terms of unknown quantities where necessary.

Since $\mathbf{T}_1$ and $\mathbf{T}_2$ are tensions in the same string passing over a model pulley, their magnitudes are equal. So let $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Also, let $N = |\mathbf{N}|$ and let m be the mass of the metal weight.

For the metal weight we have

$$\mathbf{T}_1 = T\,\mathbf{j}'$$
$$\mathbf{W}_1 = -mg\,\mathbf{j}'.$$

The acute angle between $\mathbf{W}_2$ and a line parallel to $\mathbf{T}_2$ is

$$180° - 90° - 15° = 75°,$$

as marked on the force diagram.

For the trolley we have

$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{T}_2 = T\,\mathbf{i}$$
$$\mathbf{W}_2 = -80g\cos 75°\,\mathbf{i} - 80g\sin 75°\,\mathbf{j}.$$

First apply the equilibrium condition to the metal weight. Aim to find (in terms of the unknown mass m) an expression for T, the magnitude of each of the two tension forces that form the action and reaction pair.

The equilibrium condition for the metal weight is $\mathbf{T}_1 + \mathbf{W}_1 = \mathbf{0}$, which gives

$$T\,\mathbf{j}' - mg\,\mathbf{j}' = \mathbf{0}.$$

Hence $T - mg = 0$, which gives

$$T = mg.$$

Now apply the equilibrium condition to the trolley, and use the expression for T, to find the required mass.

The equilibrium condition for the trolley is $\mathbf{N} + \mathbf{T}_2 + \mathbf{W}_2 = \mathbf{0}$, which gives

$$N\,\mathbf{j} + T\,\mathbf{i} - 80g\cos 75^\circ\mathbf{i} - 80g\sin 75^\circ\mathbf{j} = \mathbf{0}.$$

Substituting for T using the equation $T = mg$ gives

$$N\,\mathbf{j} + mg\,\mathbf{i} - 80g\cos 75^\circ\mathbf{i} - 80g\sin 75^\circ\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**-direction gives

$$mg - 80g\cos 75^\circ = 0.$$

Hence

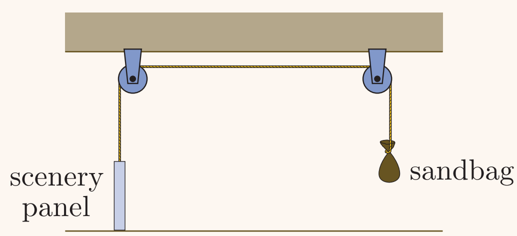
$$\begin{aligned} m &= 80\cos 75^\circ \\ &= 20.705\ldots . \end{aligned}$$

The mass of the metal weight is 21 kg (to 2 s.f.).

Here are three problems for you to solve by using model pulleys.

Activity 27 *Using model pulleys*

A theatre scenery panel of mass 100 kg resting on the floor is held by a light rope, which passes over two pulleys fixed to the ceiling, and is attached at its other end to a sandbag of mass 80 kg that hangs backstage without touching the floor.



Model the scenery panel and the sandbag as particles, the pulleys as model pulleys and the rope as a model string. Calculate the normal reaction of the floor on the scenery panel, in newtons to two significant figures.

If the sandbag and the pulleys were not present in the situation described in Activity 27, then the magnitude of the normal reaction of the floor on the scenery panel would be equal to the magnitude of the weight of the scenery panel, which is about 980 N, rather than the answer of about 200 N found in the solution to Activity 27. The effect of the sandbag and the pulleys is as if the magnitude of the weight were reduced by the magnitude of the weight of the sandbag, which is about 780 N.

Activity 27 illustrates a real-world use of pulleys. The sandbag and the pulleys make it possible for a stage hand to raise the scenery panel by applying only a small pulling force, namely about 200 N, in addition to the force provided by the weight of the sandbag. In general, pulleys can be used in various ways to reduce the magnitudes of forces that need to be applied.

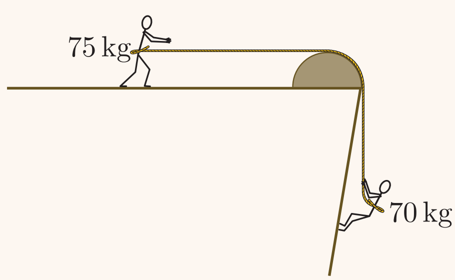
The situation in the next activity does not involve a real pulley, but you can model it by using a model pulley.

Activity 28 *Using a model pulley and finding a coefficient of static friction*

After slipping from a mountain, a climber of mass 70 kg is suspended from a rope attached to another climber, of mass 75 kg, who stands on a ledge. The rope passes over a boulder on the edge of the ledge, and is horizontal between the standing climber and the boulder.

If the boulder is modelled as a model pulley, what is the minimum coefficient of static friction between the soles of the standing climber's boots and the top surface of the ledge that will stop the two climbers falling from the ledge?

Give your answer to two decimal places.



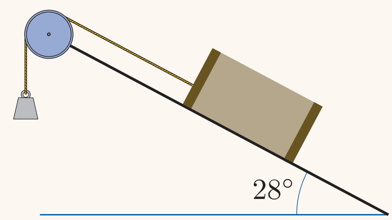
Activity 29 *Solving another problem involving a model pulley and friction*

A wooden block rests on a rough ramp inclined at 28° to the horizontal. A rope is attached to the block and runs parallel to the ramp and over a pulley at the top of the ramp, as shown below. A metal weight of mass 3.3 kg hangs from the other end of the rope, and is just heavy enough to prevent the wooden block from slipping down the ramp. The coefficient of static friction between the wooden block and the ramp is 0.24.

Model the wooden block and the metal weight as particles, the rope as a model string and the pulley as a model pulley.

(a) Find the mass of the wooden block, in kilograms to two significant figures.

(b) By how much would you need to increase the mass of the metal weight to make the wooden block be on the point of slipping *up* the plane? Give your answer in kilograms to two significant figures.



Learning outcomes

After studying this unit, you should be able to:

- understand the concept of a force
- understand and model forces of weight, normal reaction, tension and friction
- recognise and model the forces that act on an object in equilibrium
- model objects as particles
- use model strings and model pulleys to model systems involving forces
- draw force diagrams and choose appropriate directions for the Cartesian unit vectors **i** and **j**
- use the equilibrium conditions for particles
- model and solve a variety of problems involving objects that can be modelled as particles in equilibrium.

Solutions to activities

Solution to Activity 1

(a) The magnitude of the weight of the object is given by

$$mg = 5 \times 9.8 = 49,$$

so it is 49 N.

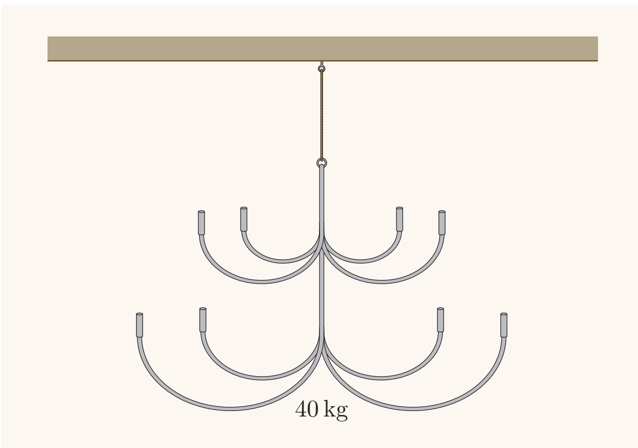
(b) If the mass of the object is m, then $8 = mg$, so

$$m = \frac{8}{g} = \frac{8}{9.8} = 0.8163\ldots .$$

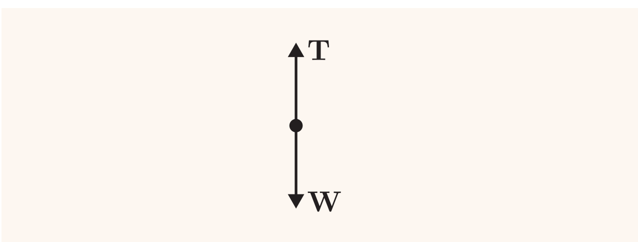
So the mass of the object is 0.82 kg (to 2 s.f.).

Solution to Activity 2

(a) The situation is illustrated below.



The forces acting on the chandelier are its weight, $\mathbf{W}$, and the tension in the cable, $\mathbf{T}$. We model the chandelier as a particle. A force diagram is shown below.



(b) Since the chandelier remains at rest, we have

$$\mathbf{T} = -\mathbf{W}.$$

So $|\mathbf{T}| = |\mathbf{W}|$.

But

$$|\mathbf{W}| = 40g = 40 \times 9.8 = 392,$$

so $|\mathbf{T}| = 392$.

The tension in the cable has magnitude 390 N (to 2 s.f.) and acts vertically upwards.

(c) Since the magnitude of the tension is greater than the maximum magnitude of a tension that the available cable can sustain, this cable is not suitable.

Solution to Activity 3

The forces acting on the object are its weight, $\mathbf{W}$, and the normal reaction from the table top, $\mathbf{N}$. We model the object as a particle. Since the object remains at rest, we have

$$\mathbf{N} = -\mathbf{W}.$$

So $|\mathbf{N}| = |\mathbf{W}|$.

But

$$|\mathbf{W}| = 2g = 2 \times 9.8 = 19.6,$$

so $|\mathbf{N}| = 19.6$.

So the normal reaction has magnitude 20 N (to 2 s.f.) and is directed vertically upwards.

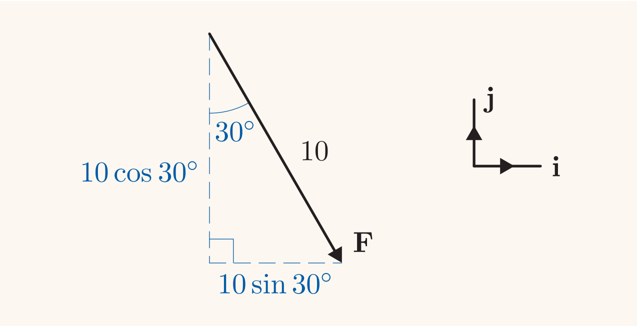
Solution to Activity 4

The particle is in equilibrium, so $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = \mathbf{0}$. Hence

$$\begin{aligned}\mathbf{D} &= -\mathbf{A} - \mathbf{B} - \mathbf{C} \\ &= -(2\mathbf{i} + \mathbf{j}) - (-\mathbf{j}) - (\mathbf{i} + 2\mathbf{j}) \\ &= -2\mathbf{i} - \mathbf{j} + \mathbf{j} - \mathbf{i} - 2\mathbf{j} \\ &= -3\mathbf{i} - 2\mathbf{j}.\end{aligned}$$

Solution to Activity 5

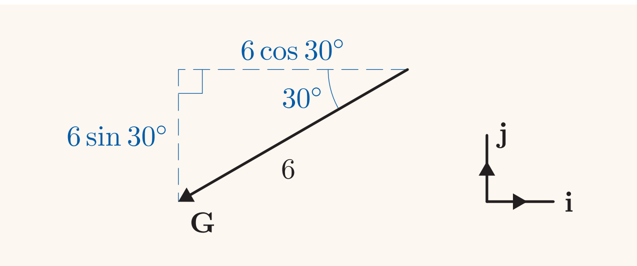
(a) The force $\mathbf{F}$ is shown below.



From the diagram,

$$\begin{aligned}\mathbf{F} &= 10\sin 30^\circ\,\mathbf{i} - 10\cos 30^\circ\,\mathbf{j}\\ &= 10\times\frac{1}{2}\,\mathbf{i} - 10\times\frac{\sqrt{3}}{2}\,\mathbf{j}\\ &= 5\,\mathbf{i} - 5\sqrt{3}\,\mathbf{j}.\end{aligned}$$

(b) The force $\mathbf{G}$ is shown below.



From the diagram,

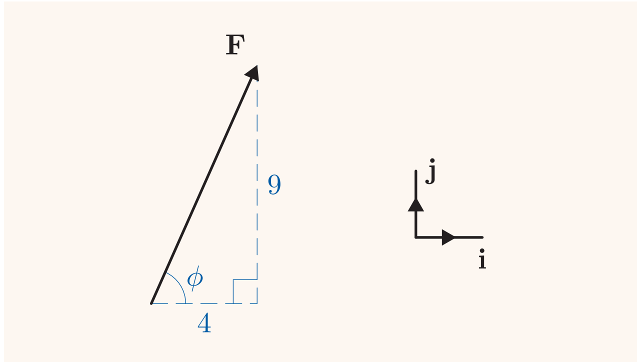
$$\begin{aligned}\mathbf{G} &= -6\cos 30^\circ\,\mathbf{i} - 6\sin 30^\circ\,\mathbf{j}\\ &= -6\times\frac{\sqrt{3}}{2}\,\mathbf{i} - 6\times\tfrac{1}{2}\,\mathbf{j}\\ &= -3\sqrt{3}\,\mathbf{i} - 3\,\mathbf{j}.\end{aligned}$$

Solution to Activity 6

(a) The magnitude is given by

$$\begin{aligned}|\mathbf{F}| &= \sqrt{4^2+9^2}\\ &= \sqrt{16+81}\\ &= \sqrt{97}\\ &= 9.8 \text{ (to 1 d.p.)}.\end{aligned}$$

The force $\mathbf{F}$ is shown below.



For the angle ϕ shown,

$$\tan\phi = \frac{9}{4},$$

so

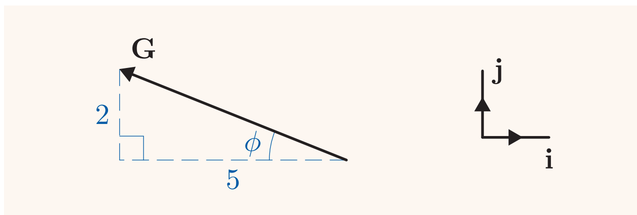
$$\phi = \tan^{-1}\left(\tfrac{9}{4}\right) = 66^\circ \text{ (to the nearest degree)}.$$

The force $\mathbf{F}$ has magnitude 9.8 N (to 1 d.p.) and is directed to the right and up, at an angle of 66° (to the nearest degree) to the horizontal.

(b) The magnitude is given by

$$\begin{aligned}|\mathbf{G}| &= \sqrt{(-5)^2+2^2}\\ &= \sqrt{25+4}\\ &= \sqrt{29}\\ &= 5.4 \text{ (to 1 d.p.)}.\end{aligned}$$

The force $\mathbf{G}$ is shown below.



For the angle ϕ shown,

$$\tan\phi = \frac{2}{5},$$

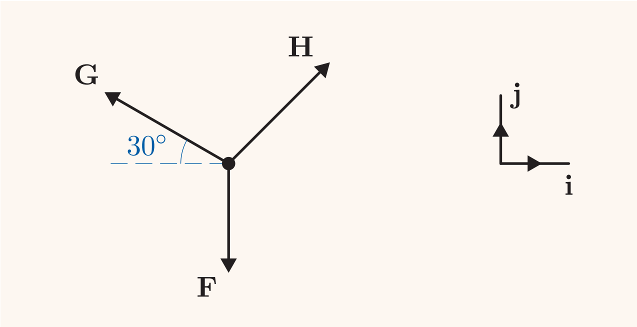
so

$$\phi = \tan^{-1}\left(\tfrac{2}{5}\right) = 22^\circ \text{ (to the nearest degree)}.$$

The force $\mathbf{G}$ has magnitude 5.4 N (to 1 d.p.) and is directed to the left and up, at an angle of 22° (to the nearest degree) to the horizontal.

Solution to Activity 7

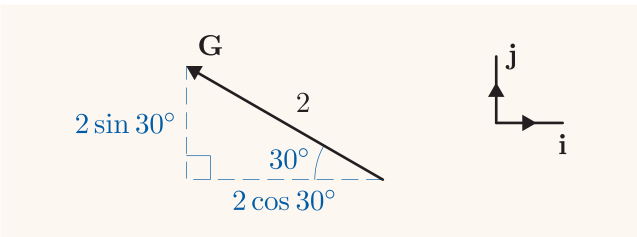
A force diagram is shown below. The Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ have been chosen to point right and up, respectively.



We know that $|\mathbf{F}| = 3$ and $|\mathbf{G}| = 2$.

The component form of $\mathbf{F}$ is $\mathbf{F} = -3\,\mathbf{j}$.

To find $\mathbf{G}$ in component form, we can use the diagram below.



So

$$\begin{aligned}\mathbf{G} &= -2\cos 30°\mathbf{i} + 2\sin 30°\mathbf{j}\\ &= -2 \times \frac{\sqrt{3}}{2}\,\mathbf{i} + 2 \times \frac{1}{2}\,\mathbf{j}\\ &= -\sqrt{3}\,\mathbf{i} + \mathbf{j}.\end{aligned}$$

(Alternatively, notice that the anticlockwise angle that $\mathbf{G}$ makes with the $\mathbf{i}$-direction is $180° - 30° = 150°$, so

$$\begin{aligned}\mathbf{G} &= 2\cos 150°\,\mathbf{i} + 2\sin 150°\,\mathbf{j}\\ &= 2 \times \left(-\frac{\sqrt{3}}{2}\right)\,\mathbf{i} + 2 \times \frac{1}{2}\,\mathbf{j}\\ &= -\sqrt{3}\,\mathbf{i} + \mathbf{j},\end{aligned}$$

as before.)

Since the particle is at rest, we have

$$\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0},$$

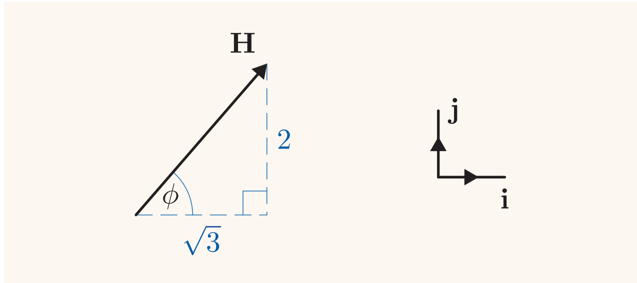
so

$$\begin{aligned}\mathbf{H} &= -\mathbf{F} - \mathbf{G}\\ &= -(-3\mathbf{j}) - (-\sqrt{3}\,\mathbf{i} + \mathbf{j})\\ &= \sqrt{3}\,\mathbf{i} + 2\mathbf{j}.\end{aligned}$$

The magnitude of $\mathbf{H}$ is given by

$$|\mathbf{H}| = \sqrt{\left(\sqrt{3}\right)^2 + 2^2} = \sqrt{7} = 2.6 \text{ (to 1 d.p.)}.$$

The force $\mathbf{H}$ is shown below.



For the angle ϕ in the diagram,

$$\tan\phi = \frac{2}{\sqrt{3}},$$

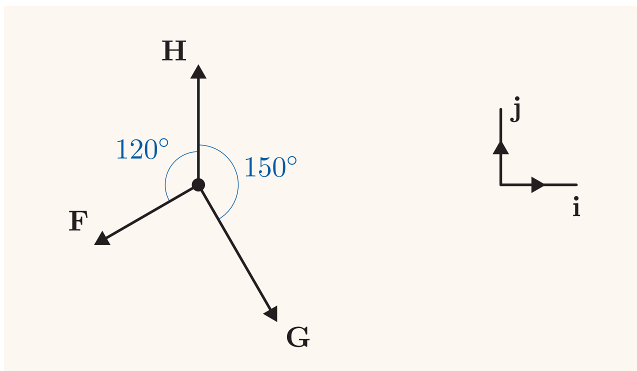
so

$$\phi = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = 49° \text{ (to the nearest degree)}.$$

So $\mathbf{H}$ has magnitude 2.6 N (to 1 d.p.) and is directed to the right and up, at 49° (to the nearest degree) to the horizontal.

Solution to Activity 8

A force diagram is shown below. The vectors $\mathbf{i}$ and $\mathbf{j}$ have been chosen to point right and up, respectively, as specified in the question.



We know that $|\mathbf{H}| = 70$.

Let $F = |\mathbf{F}|$ and $G = |\mathbf{G}|$.

Since $\mathbf{F}$ makes an angle of $90° + 120° = 210°$ with the $\mathbf{i}$-direction,

$$\begin{aligned}\mathbf{F} &= |\mathbf{F}|\cos 210°\mathbf{i} + |\mathbf{F}|\sin 210°\mathbf{j}\\ &= -\frac{\sqrt{3}}{2}F\,\mathbf{i} - \frac{1}{2}F\,\mathbf{j}.\end{aligned}$$

Similarly, since $\mathbf{G}$ makes an angle of $-(150° - 90°) = -60°$ with the $\mathbf{i}$-direction,

$$\begin{aligned}\mathbf{G} &= |\mathbf{G}|\cos(-60°)\,\mathbf{i} + |\mathbf{G}|\sin(-60°)\,\mathbf{j}\\ &= \frac{1}{2}G\,\mathbf{i} - \frac{\sqrt{3}}{2}G\,\mathbf{j}.\end{aligned}$$

Also,

$$\mathbf{H} = |\mathbf{H}|\,\mathbf{j} = 70\,\mathbf{j}.$$

Since the particle is at rest, $\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}$. So

$$-\frac{\sqrt{3}}{2}F\,\mathbf{i} - \frac{1}{2}F\,\mathbf{j} + \frac{1}{2}G\,\mathbf{i} - \frac{\sqrt{3}}{2}G\,\mathbf{j} + 70\,\mathbf{j} = \mathbf{0};$$

that is,

$$\left(-\frac{\sqrt{3}}{2}F + \frac{1}{2}G\right)\,\mathbf{i} + \left(-\frac{1}{2}F - \frac{\sqrt{3}}{2}G + 70\right)\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the simultaneous equations

$$-\frac{\sqrt{3}}{2}F + \frac{1}{2}G = 0 \qquad (1)$$

$$-\frac{1}{2}F - \frac{\sqrt{3}}{2}G + 70 = 0. \qquad (2)$$

To solve these simultaneous equations, we can start by eliminating G. Multiplying equation (1) by $\sqrt{3}$ gives

$$-\frac{3}{2}F + \frac{\sqrt{3}}{2}G = 0. \qquad (3)$$

Adding equations (2) and (3) gives

$$-2F + 70 = 0.$$

Hence

$$F = 35.$$

Substituting the value of F into equation (1) gives

$$-\frac{\sqrt{3}}{2} \times 35 + \frac{1}{2}G = 0$$

which can be simplified to

$$-35\sqrt{3} + G = 0.$$

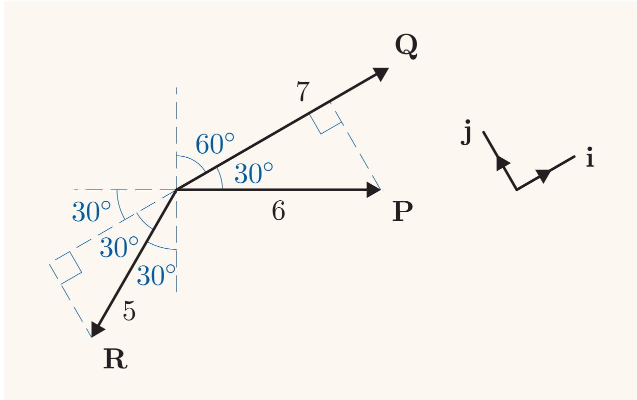
Hence

$$G = 35\sqrt{3}.$$

The force $\mathbf{F}$ has magnitude 35 N, and the force $\mathbf{G}$ has magnitude $35\sqrt{3}$ N.

Solution to Activity 9

The forces are shown below, with some further angles worked out and marked.



The component form of $\mathbf{P}$ is

$$\begin{aligned}\mathbf{P} &= 6\cos 30°\,\mathbf{i} - 6\sin 30°\,\mathbf{j}\\ &= 6 \times \frac{\sqrt{3}}{2}\,\mathbf{i} - 6 \times \frac{1}{2}\,\mathbf{j}\\ &= 3\sqrt{3}\,\mathbf{i} - 3\,\mathbf{j}.\end{aligned}$$

The component form of $\mathbf{Q}$ is

$$\mathbf{Q} = 7\,\mathbf{i}.$$

The component form of $\mathbf{R}$ is

$$\begin{aligned}\mathbf{R} &= -5\cos 30°\,\mathbf{i} - 5\sin 30°\,\mathbf{j}\\ &= -5 \times \frac{\sqrt{3}}{2}\,\mathbf{i} - 5 \times \frac{1}{2}\,\mathbf{j}\\ &= -\frac{5\sqrt{3}}{2}\,\mathbf{i} - \frac{5}{2}\,\mathbf{j}.\end{aligned}$$

(An alternative way to work out the component forms of $\mathbf{P}$ and $\mathbf{R}$ is to use the formula in the box on page 122. For $\mathbf{P}$ this gives

$$\begin{aligned}\mathbf{P} &= 6\cos(-30°)\,\mathbf{i} + 6\sin(-30°)\,\mathbf{j}\\ &= 6 \times \frac{\sqrt{3}}{2}\,\mathbf{i} + 6 \times \left(-\frac{1}{2}\right)\,\mathbf{j}\\ &= 3\sqrt{3}\,\mathbf{i} - 3\,\mathbf{j}.\end{aligned}$$

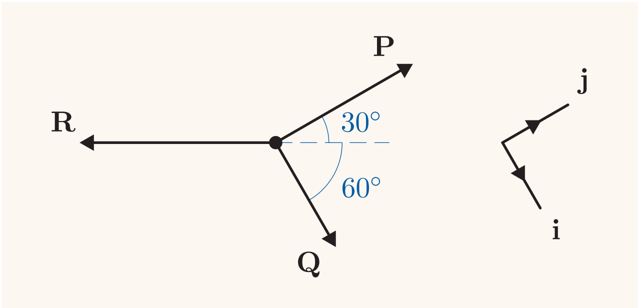
The angle from the $\mathbf{i}$-direction to $\mathbf{R}$ is $-(30° + 90° + 30°) = -150°$. So for $\mathbf{R}$ the formula gives

$$\begin{aligned}\mathbf{R} &= 5\cos(-150°)\,\mathbf{i} + 5\sin(-150°)\,\mathbf{j}\\ &= 5 \times \left(-\frac{\sqrt{3}}{2}\right)\,\mathbf{i} - 5 \times \left(-\frac{1}{2}\right)\,\mathbf{j}\\ &= -\frac{5\sqrt{3}}{2}\,\mathbf{i} - \frac{5}{2}\,\mathbf{j}.\end{aligned}$$

As expected, these answers are the same as above.)

Solution to Activity 10

The force diagram, as given in the question, is shown below. Here $\mathbf{i}$ is parallel to $\mathbf{Q}$ and $\mathbf{j}$ is parallel to $\mathbf{P}$.



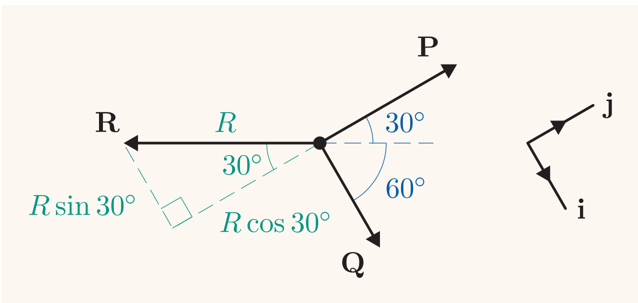
We know that $|\mathbf{P}| = 50$.

Let $Q = |\mathbf{Q}|$ and $R = |\mathbf{R}|$. We now express the three forces in component form. First,

$$\mathbf{P} = |\mathbf{P}|\,\mathbf{j} = 50\,\mathbf{j}$$

$$\mathbf{Q} = |\mathbf{Q}|\,\mathbf{i} = Q\,\mathbf{i}.$$

To find the component form of $\mathbf{R}$, we can draw a right-angled triangle as shown below.



This gives

$$\begin{aligned}\mathbf{R} &= -R\sin 30^\circ\mathbf{i} - R\cos 30^\circ\mathbf{j}\\ &= -\frac{1}{2}R\,\mathbf{i} - \frac{\sqrt{3}}{2}R\,\mathbf{j}.\end{aligned}$$

Since the particle is at rest, $\mathbf{P}+\mathbf{Q}+\mathbf{R}=\mathbf{0}$, so

$$50\,\mathbf{j} + Q\,\mathbf{i} + \left(-\frac{1}{2}R\,\mathbf{i} - \frac{\sqrt{3}}{2}R\,\mathbf{j}\right) = \mathbf{0}.$$

Collecting the terms in $\mathbf{i}$ and $\mathbf{j}$ gives

$$\left(Q - \frac{1}{2}R\right)\mathbf{i} + \left(50 - \frac{\sqrt{3}}{2}R\right)\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the simultaneous equations

$$Q - \frac{1}{2}R = 0$$

$$50 - \frac{\sqrt{3}}{2}R = 0.$$

The second of these equations gives

$$\frac{\sqrt{3}}{2}R = 50,$$

so

$$R = \frac{100}{\sqrt{3}} = 57.7 \text{ (to 1 d.p.)}.$$

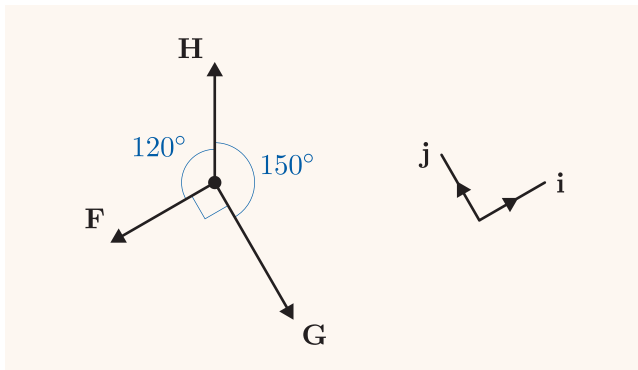
Substituting $R = 100/\sqrt{3}$ into the first of the pair of simultaneous equations gives

$$Q = \frac{1}{2}R = \frac{50}{\sqrt{3}} = 28.9 \text{ (to 1 d.p.)}.$$

So, to one decimal place, $\mathbf{Q}$ has magnitude 28.9 N and $\mathbf{R}$ has magnitude 57.7 N.

Solution to Activity 11

A force diagram is shown below. The vectors $\mathbf{i}$ and $\mathbf{j}$ have been chosen to be parallel to $\mathbf{F}$ and $\mathbf{G}$, respectively, as shown.



We know that $|\mathbf{H}| = 70$.

Let $F = |\mathbf{F}|$ and $G = |\mathbf{G}|$.

We have

$$\mathbf{F} = -F\,\mathbf{i},$$

$$\mathbf{G} = -G\,\mathbf{j}.$$

Also, since $\mathbf{H}$ makes an angle of $150^\circ - 90^\circ = 60^\circ$ with the $\mathbf{i}$-direction,

$$\begin{aligned}\mathbf{H} &= |\mathbf{H}|\cos 60^\circ\mathbf{i} + |\mathbf{H}|\sin 60^\circ\mathbf{j}\\ &= 70 \times \frac{1}{2}\,\mathbf{i} + 70 \times \frac{\sqrt{3}}{2}\,\mathbf{j}\\ &= 35\,\mathbf{i} + 35\sqrt{3}\,\mathbf{j}.\end{aligned}$$

Since the particle is at rest, $\mathbf{F}+\mathbf{G}+\mathbf{H}=\mathbf{0}$. So

$$-F\,\mathbf{i}-G\,\mathbf{j}+35\,\mathbf{i}+35\sqrt{3}\,\mathbf{j}=\mathbf{0};$$

that is,

$$(-F+35)\,\mathbf{i}+\left(-G+35\sqrt{3}\right)\,\mathbf{j}=\mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the simultaneous equations

$$-F+35=0$$
$$-G+35\sqrt{3}=0.$$

These equations give

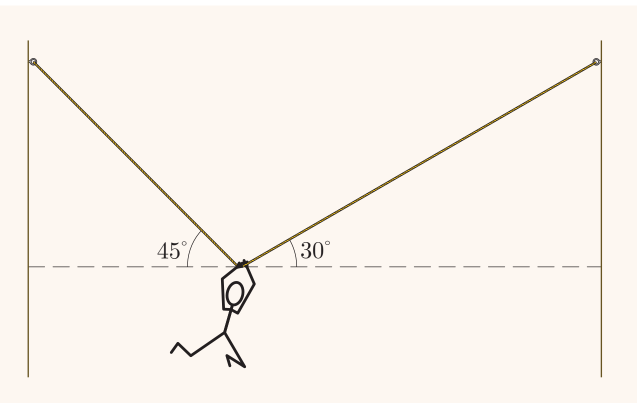
$$F=35 \quad\text{and}\quad G=35\sqrt{3}.$$

The force $\mathbf{F}$ has magnitude 35 N, and the force $\mathbf{G}$ has magnitude $35\sqrt{3}$ N.

Solution to Activity 12

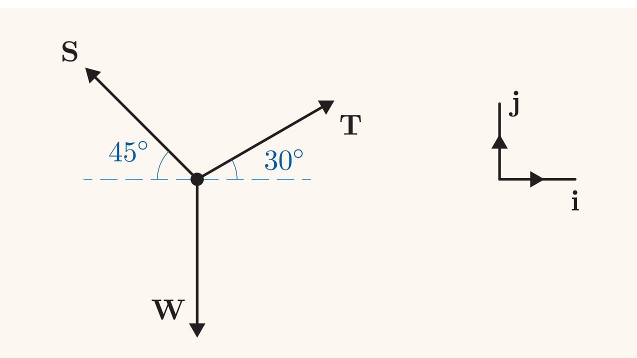
Model the child as a particle of mass 30 kg, and assume that each side of the rope is a model string.

A diagram of the situation is shown below.



The forces acting on the particle are the weight $\mathbf{W}$ of the child, the tension $\mathbf{S}$ in the left-hand rope, and the tension $\mathbf{T}$ in the right-hand rope.

A force diagram is shown below.



We know that $|\mathbf{W}|=30g$.

Take $\mathbf{i}$ to point right and $\mathbf{j}$ to point up, as shown.

Let $S=|\mathbf{S}|$ and $T=|\mathbf{T}|$. The forces can be written in component form as

$$\begin{aligned}\mathbf{S}&=-S\cos 45^\circ\mathbf{i}+S\sin 45^\circ\mathbf{j}\\&=-\frac{1}{\sqrt{2}}S\,\mathbf{i}+\frac{1}{\sqrt{2}}S\,\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{T}&=T\cos 30^\circ\mathbf{i}+T\sin 30^\circ\mathbf{j}\\&=\frac{\sqrt{3}}{2}T\,\mathbf{i}+\frac{1}{2}T\,\mathbf{j}\end{aligned}$$

$$\mathbf{W}=-30g\,\mathbf{j}.$$

Since the particle is in equilibrium,

$$\mathbf{S}+\mathbf{T}+\mathbf{W}=\mathbf{0}.$$

So

$$-\frac{1}{\sqrt{2}}S\,\mathbf{i}+\frac{1}{\sqrt{2}}S\,\mathbf{j}+\frac{\sqrt{3}}{2}T\,\mathbf{i}+\frac{1}{2}T\,\mathbf{j}-30g\,\mathbf{j}=\mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the simultaneous equations

$$-\frac{1}{\sqrt{2}}S+\frac{\sqrt{3}}{2}T=0$$
$$\frac{1}{\sqrt{2}}S+\frac{1}{2}T-30g=0.$$

We now solve these equations to find S and T.

Adding the equations gives

$$\left(\frac{\sqrt{3}+1}{2}\right)T-30g=0,$$

so

$$\begin{aligned}T&=30g\times\frac{2}{\sqrt{3}+1}\\&=30\times 9.8\times\frac{2}{\sqrt{3}+1}\\&=215.22\ldots.\end{aligned}$$

From the first simultaneous equation,

$$\frac{1}{\sqrt{2}}S=\frac{\sqrt{3}}{2}T,$$

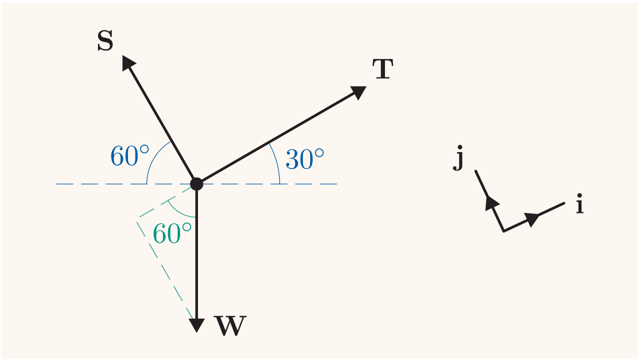
so

$$\begin{aligned}S&=\frac{\sqrt{2}\sqrt{3}}{2}T\\&=\frac{\sqrt{2}\sqrt{3}}{2}\times 215.22\ldots\\&=263.59\ldots.\end{aligned}$$

So the tension in the rope at 45° to the horizontal has magnitude 260 N (to 2 s.f.) and the tension in the rope at 30° to the horizontal has magnitude 220 N (to 2 s.f.).

Solution to Activity 13

Model the flower basket as a particle and the cords as model strings. The forces acting on the particle are the weight $\mathbf{W}$ of the flower basket, the tension $\mathbf{S}$ in the first cord and the tension $\mathbf{T}$ in the second cord. A force diagram is shown below.



We know that $|\mathbf{T}| = 21$.

Take $\mathbf{i}$ and $\mathbf{j}$ to be parallel to $\mathbf{T}$ and $\mathbf{S}$ (which are perpendicular), as shown.

Let $S = |\mathbf{S}|$ and let m be the mass of the flower basket, in kilograms. Then $|\mathbf{W}| = mg$.

The acute angle between $\mathbf{W}$ and a line parallel to $\mathbf{i}$ is $180° - 30° - 90° = 60°$, as marked on the force diagram.

Expressing the forces in component form gives

$$\mathbf{S} = S\,\mathbf{j}$$
$$\mathbf{T} = 21\,\mathbf{i}$$

and

$$\begin{aligned}\mathbf{W} &= -|\mathbf{W}|\cos 60°\,\mathbf{i} - |\mathbf{W}|\sin 60°\,\mathbf{j}\\ &= -mg \times \frac{1}{2}\,\mathbf{i} - mg \times \frac{\sqrt{3}}{2}\,\mathbf{j}\\ &= -\frac{1}{2}mg\,\mathbf{i} - \frac{\sqrt{3}}{2}mg\,\mathbf{j}.\end{aligned}$$

Since the particle is at rest, $\mathbf{S} + \mathbf{T} + \mathbf{W} = \mathbf{0}$. This gives

$$S\,\mathbf{j} + 21\,\mathbf{i} - \frac{1}{2}mg\,\mathbf{i} - \frac{\sqrt{3}}{2}mg\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$-direction gives

$$21 - \frac{1}{2}mg = 0.$$

Hence

$$m = \frac{2 \times 21}{g} = \frac{42}{9.8} = 4.285\ldots.$$

The mass of the flower basket is 4.3 kg (to 2 s.f.).

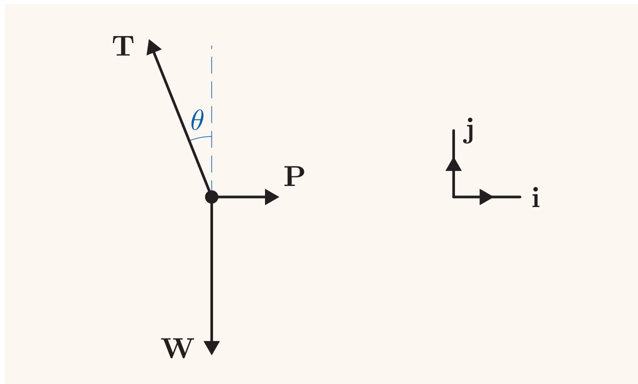
Solution to Activity 14

(a) Two alternative solutions are given here. In the first solution, the unknown magnitude and direction of the tension in the chain are denoted by symbols, and found by solving two simultaneous equations. In the second solution, the unknown magnitude and direction are found by using the approach of Example 5 in Subsection 1.2.

First possible method of solution

Model the ball as a particle and the chain as a model string. The forces acting on the particle are the weight $\mathbf{W}$ of the wrecking ball, the tension $\mathbf{T}$ in the chain, and the tension $\mathbf{P}$ in the rope.

A force diagram is shown below.



We know that $|\mathbf{P}| = 350$ and $|\mathbf{W}| = 2000g$.

Let $\mathbf{i}$ point right and $\mathbf{j}$ point up, as shown.

Let θ be the angle between the force $\mathbf{T}$ and the vertical, as marked on the force diagram. (The angle θ marked on the force diagram and the angle θ marked on the diagram in the question are alternate angles and hence equal.)
Let $T = |\mathbf{T}|$.

The forces can be written in component form as

$$\mathbf{T} = -T\sin\theta\,\mathbf{i} + T\cos\theta\,\mathbf{j}$$
$$\mathbf{P} = 350\,\mathbf{i}$$
$$\mathbf{W} = -2000g\,\mathbf{j}.$$

The equilibrium condition, $\mathbf{T} + \mathbf{P} + \mathbf{W} = \mathbf{0}$, gives

$$-T\sin\theta\,\mathbf{i} + T\cos\theta\,\mathbf{j} + 350\,\mathbf{i} - 2000g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives the simultaneous equations

$$-T\sin\theta + 350 = 0$$
$$T\cos\theta - 2000g = 0,$$

which can be written as

$$T\sin\theta = 350 \qquad (4)$$
$$T\cos\theta = 2000g. \qquad (5)$$

Squaring and adding these equations to eliminate θ gives

$$T^2\cos^2\theta + T^2\sin^2\theta = 350^2 + (2000g)^2$$
$$T^2(\cos^2\theta + \sin^2\theta) = 350^2 + (2000\times 9.8)^2$$
$$T^2 \times 1 = 384\,282\,500$$
$$T^2 = 384\,282\,500$$
$$T = 20\,000 \quad \text{(to 2 s.f.)}.$$

(We take the positive square root since T, the magnitude of a force, is positive.)

Dividing equation (4) by equation (5) to eliminate T gives

$$\tan\theta = \frac{350}{2000g},$$

so

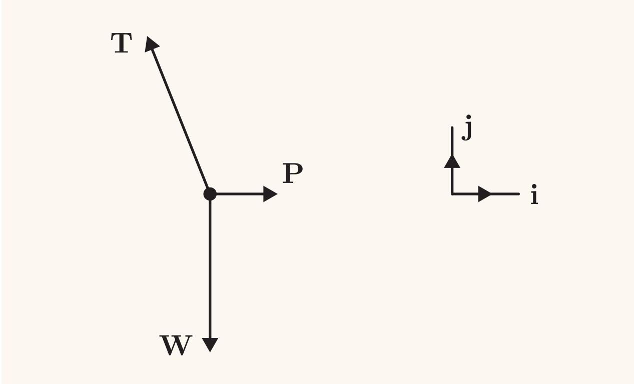
$$\theta = \tan^{-1}\left(\frac{350}{2000\times 9.8}\right)$$
$$= 1.023\,030\ldots$$
$$= 1.0^\circ \text{ (to 2 s.f.)}.$$

So the tension in the chain has magnitude 20 000 N (to 2 s.f.), and the chain is pulled to the side at an angle of 1.0° (to 2 s.f.) to the vertical.

Second possible method of solution

Model the ball as a particle and the chain as a model string. The forces acting on the particle are the weight **W** of the wrecking ball, the tension **T** in the chain, and the tension **P** in the rope.

A force diagram is shown below.



We know that $|\mathbf{P}| = 350$ and $|\mathbf{W}| = 2000g$.

Let **i** point right and **j** point up, as shown.

The forces **P** and **W** can be written in component form as

$$\mathbf{P} = 350\,\mathbf{i}$$
$$\mathbf{W} = -2000g\,\mathbf{j}.$$

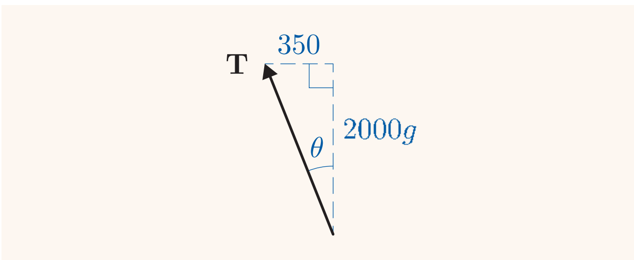
The equilibrium condition, $\mathbf{T} + \mathbf{P} + \mathbf{W} = \mathbf{0}$, gives

$$\mathbf{T} = -\mathbf{P} - \mathbf{W}$$
$$= -350\,\mathbf{i} + 2000g\,\mathbf{j}.$$

The magnitude of **T** is

$$|\mathbf{T}| = \sqrt{(-350)^2 + (2000g)^2}$$
$$= \sqrt{350^2 + (2000\times 9.8)^2}$$
$$= 20\,000 \text{ (to 2 s.f.)}.$$

The force **T** is shown below.



For the acute angle θ shown,

$$\tan\theta = \frac{350}{2000g},$$

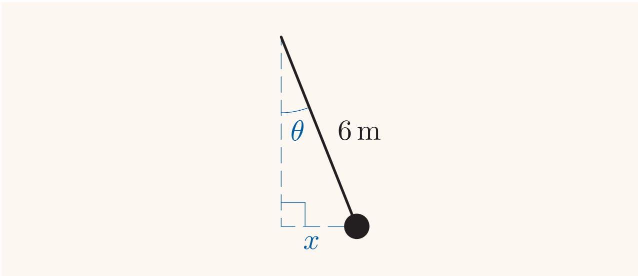
so

$$\theta = \tan^{-1}\left(\frac{350}{2000\times 9.8}\right)$$
$$= 1.023\,030\ldots^\circ$$
$$= 1.0^\circ \text{ (to 2 s.f.)}.$$

The angle θ here is equal to the angle θ between the chain and the vertical marked on the diagram in the question, as they are alternate angles.

So the tension in the chain has magnitude 20 000 N (to 2 s.f.), and the chain is pulled to the side at an angle of 1.0° (to 2 s.f.) to the vertical.

(b) Let x be the horizontal distance of the ball from the vertical line through the top end of the chain. The situation is as shown below.



We have

$$x = 6\sin\theta$$
$$= 6\sin 1.023\,030\ldots°$$
$$= 0.11\,\text{m (to 2 s.f).}$$

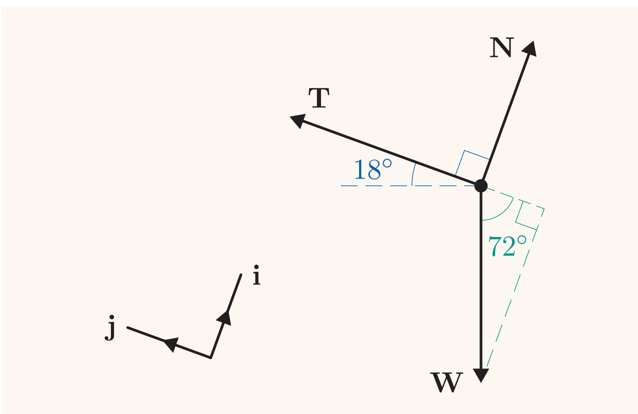
So the horizontal distance of the ball from the vertical line through the top end of the chain is about 11 cm.

Solution to Activity 15

Model the block as a particle and the rope as a model string. From the question, the surface is smooth.

The forces acting on the block are its weight **W**, the tension **T** in the rope and the normal reaction **N** of the surface on the block.

A force diagram is shown below.



We know that $|\mathbf{W}| = 2.7g$.

Take **i** and **j** to be perpendicular and parallel, respectively, to the slope, as shown above.

Let $T = |\mathbf{T}|$ and $N = |\mathbf{N}|$. Then

$$\mathbf{N} = N\,\mathbf{i}$$
$$\mathbf{T} = T\,\mathbf{j}.$$

Since **W** is vertical, the angle between **W** and the horizontal is 90°, and hence the angle at the tail of **W** in the right-angled triangle shown is

$$180° - 90° - 18° = 72°,$$

as marked.

So

$$\mathbf{W} = -|\mathbf{W}|\sin 72°\,\mathbf{i} - |\mathbf{W}|\cos 72°\,\mathbf{j}$$
$$= -2.7g\sin 72°\,\mathbf{i} - 2.7g\cos 72°\,\mathbf{j}.$$

Since the particle is in equilibrium,

$$\mathbf{N} + \mathbf{T} + \mathbf{W} = \mathbf{0}.$$

So

$$N\,\mathbf{i} + T\,\mathbf{j} - 2.7g\sin 72°\,\mathbf{i} - 2.7g\cos 72°\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives

$$N - 2.7g\sin 72° = 0$$
$$T - 2.7g\cos 72° = 0.$$

Hence

$$N = 2.7g\sin 72° = 2.7 \times 9.8 \times \sin 72° = 25.164\ldots.$$

and

$$T = 2.7g\cos 72° = 2.7 \times 9.8 \times \cos 72° = 8.176\ldots.$$

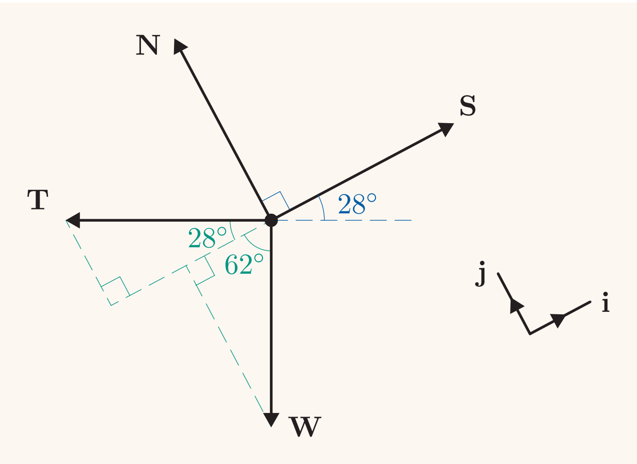
The magnitude of the tension in the rope is 8.2 N and the magnitude of the normal reaction is 25 N (both to 2 s.f.).

Solution to Activity 16

Model the block as a particle and the ropes as model strings. From the question, the inclined plane is smooth.

The forces acting on the particle are the weight $\mathbf{W}$ of the block, the normal reaction $\mathbf{N}$ from the plane, the tension $\mathbf{S}$ in the sloping string and the tension $\mathbf{T}$ in the horizontal string.

A force diagram is shown below.



We know that $|\mathbf{T}| = 110$ and $|\mathbf{W}| = 35g$.

Take $\mathbf{i}$ and $\mathbf{j}$ to be parallel and perpendicular, respectively, to the slope, as shown above.

Let $S = |\mathbf{S}|$ and $N = |\mathbf{N}|$.

Expressing the four forces in component form gives

$$\mathbf{S} = S\,\mathbf{i}$$

$$\mathbf{N} = N\,\mathbf{j}$$

$$\mathbf{T} = -110\cos 28°\,\mathbf{i} + 110\sin 28°\,\mathbf{j}$$

$$\mathbf{W} = -35g\cos 62°\,\mathbf{i} - 35g\sin 62°\,\mathbf{j}.$$

The particle is in equilibrium, so $\mathbf{S} + \mathbf{N} + \mathbf{T} + \mathbf{W} = \mathbf{0}$. This gives

$$S\,\mathbf{i} + N\,\mathbf{j} - 110\cos 28°\,\mathbf{i} + 110\sin 28°\,\mathbf{j} - 35g\cos 62°\,\mathbf{i} - 35g\sin 62°\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives

$$S - 110\cos 28° - 35g\cos 62° = 0$$

$$N + 110\sin 28° - 35g\sin 62° = 0.$$

The first of these equations gives

$$\begin{aligned} S &= 110\cos 28° + 35g\cos 62° \\ &= 110\cos 28° + 35 \times 9.8 \times \cos 62° \\ &= 258.152\ldots. \end{aligned}$$

The second of the equations gives

$$\begin{aligned} N &= -110\sin 28° + 35g\sin 62° \\ &= -110\sin 28° + 35 \times 9.8 \times \sin 62° \\ &= 251.209\ldots. \end{aligned}$$

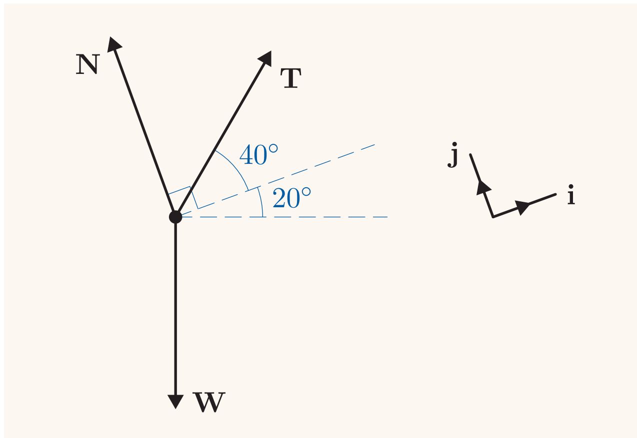
So, to two significant figures, the magnitude of the tension in the sloping rope is 260 N and the magnitude of the normal reaction is 250 N.

Solution to Activity 17

Model the block as a particle and the string as a model string. From the question, the inclined plane is smooth.

The forces acting on the particle are the weight $\mathbf{W}$ of the block, the normal reaction $\mathbf{N}$ from the plane, and the tension $\mathbf{T}$ in the string.

A force diagram is shown below.



We know that $|\mathbf{W}| = 2g$.

Take $\mathbf{i}$ and $\mathbf{j}$ to be parallel and perpendicular, respectively, to the slope, as shown above.

Let $T = |\mathbf{T}|$ and $N = |\mathbf{N}|$.

We now express the three forces in component form.

We have

$$\mathbf{N} = N\,\mathbf{j}$$

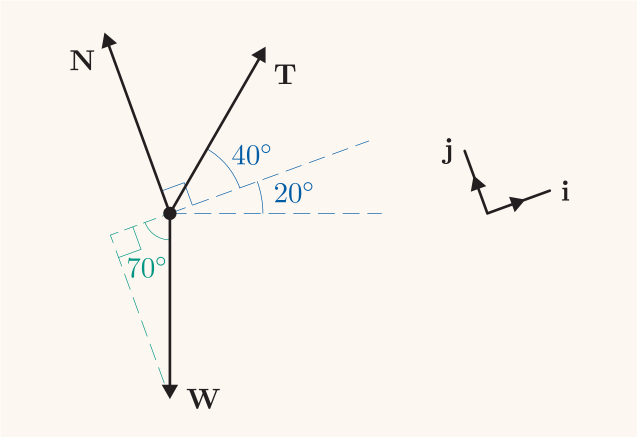
and

$$\mathbf{T} = T\cos 40°\,\mathbf{i} + T\sin 40°\,\mathbf{j}.$$

The acute angle between $\mathbf{W}$ and a line parallel to $\mathbf{i}$ is

$$180° - 90° - 20° = 70°,$$

as marked in the next diagram.



This gives

$$\mathbf{W} = -2g\cos 70°\,\mathbf{i} - 2g\sin 70°\,\mathbf{j}.$$

The particle is in equilibrium, so $\mathbf{N}+\mathbf{T}+\mathbf{W}=\mathbf{0}$. This gives

$$N\,\mathbf{j} + T\cos 40°\,\mathbf{i} + T\sin 40°\,\mathbf{j} \\ - 2g\cos 70°\,\mathbf{i} - 2g\sin 70°\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives the pair of equations

$$T\cos 40° - 2g\cos 70° = 0$$

$$N + T\sin 40° - 2g\sin 70° = 0.$$

The first of these equations gives

$$T = \frac{2g\cos 70°}{\cos 40°} = \frac{2\times 9.8\times\cos 70°}{\cos 40°} = 8.750\ldots.$$

The second of the equations gives

$$N = -T\sin 40° + 2g\sin 70°$$

and substituting for T in this equation gives

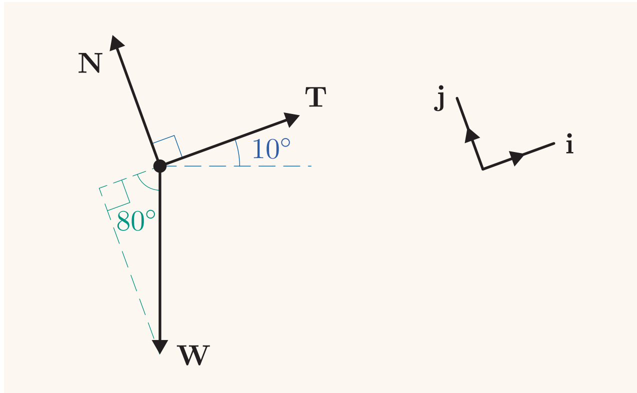
$$N = -\frac{2g\cos 70°}{\cos 40°}\sin 40° + 2g\sin 70° \\ = 2g(-\cos 70°\tan 40° + \sin 70°) \\ = 2\times 9.8(-\cos 70°\tan 40° + \sin 70°) \\ = 12.7929\ldots.$$

So, to two significant figures, the magnitude of the tension in the string is 8.8 N and the magnitude of the normal reaction is 13 N.

Solution to Activity 18

(a) Model the block and sledge together as a particle.

The forces acting on the particle are its weight **W**, the normal reaction **N**, and the tension **T** in the rope. A force diagram is shown below.



We know that $|\mathbf{W}| = 2600g$.

Take **i** and **j** to be parallel and perpendicular, respectively, to the slope, as shown above.

The acute angle between **W** and a line parallel to **i** is

$$180° - 10° - 90° = 80°,$$

as marked on the force diagram.

Let $T = |\mathbf{T}|$ and $N = |\mathbf{N}|$. The forces can be expressed in component form as

$$\mathbf{T} = T\,\mathbf{i}$$

$$\mathbf{N} = N\,\mathbf{j}$$

$$\mathbf{W} = -2600g\cos 80°\,\mathbf{i} - 2600g\sin 80°\,\mathbf{j}.$$

The particle is at rest, so, by the equilibrium condition,

$$\mathbf{T}+\mathbf{N}+\mathbf{W}=\mathbf{0}.$$

This gives

$$T\,\mathbf{i} + N\,\mathbf{j} - 2600g\cos 80°\,\mathbf{i} - 2600g\sin 80°\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**-direction gives

$$T - 2600g\cos 80° = 0$$

which gives

$$T = 2600g\cos 80° = 4424.555\ldots.$$

Hence the tension in the rope is 4400 N (to 2 s.f.).

(b) Since the pulling force of each worker is 350 N, the number of workers needed is

$$\frac{T}{350} = \frac{4424.555\ldots}{350} = 12.64\ldots.$$

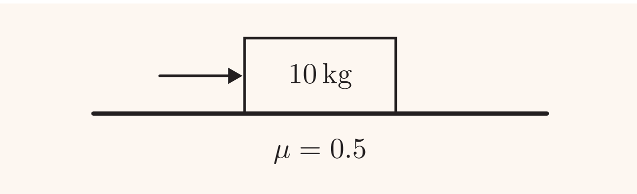
Rounding up to a whole number of workers shows that 13 workers are needed.

Solution to Activity 19

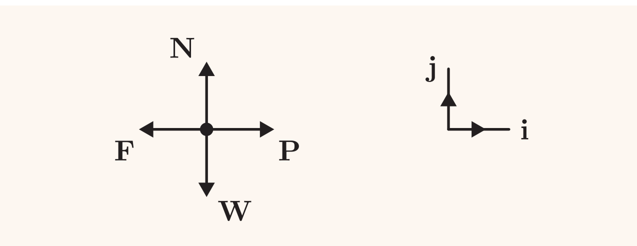
Model the block as a particle.

Assume that the maximum horizontal force is applied to the box, so that it is on the point of slipping.

A diagram of the situation is shown below.



The forces acting are the friction force **F**, the applied horizontal force **P**, the weight **W** and the normal reaction **N**. A force diagram is shown below.



We know that $|\mathbf{W}| = 10g$.

Take $\mathbf{i}$ to point right and $\mathbf{j}$ to point up, as shown.

Let $P = |\mathbf{P}|$, $N = |\mathbf{N}|$ and $F = |\mathbf{F}|$.

Expressing the forces in component form gives

$$\mathbf{P} = P\,\mathbf{i}$$
$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{F} = -F\,\mathbf{i}$$
$$\mathbf{W} = -10g\,\mathbf{j}.$$

Since the box is at rest, $\mathbf{P} + \mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This gives

$$P\,\mathbf{i} + N\,\mathbf{j} - F\,\mathbf{i} - 10g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the pair of equations

$$P - F = 0,$$
$$N - 10g = 0.$$

The second equation gives $N = 10g$.

Since the box is on the point of slipping, we have

$$F = \mu N = 0.5 \times 10g = 5g.$$

So the first of the pair of equations above gives

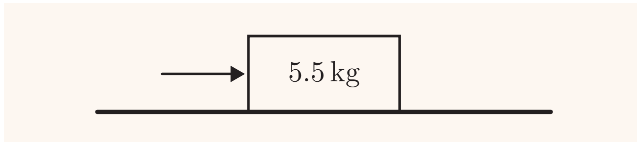
$$P = F = 5g = 5 \times 9.8 = 49.$$

The maximum pushing force that can be applied without the box moving is 49 N.

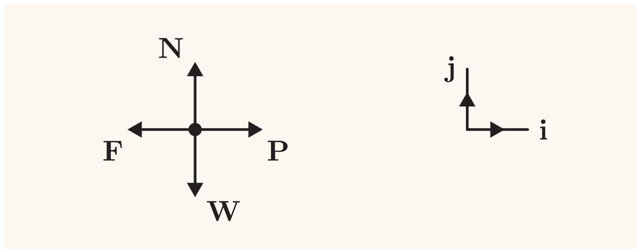
Solution to Activity 20

Model the block as a particle.

A diagram of the situation is shown below.



The forces acting are the weight **W** of the block, the normal reaction **N**, the pushing force **P** and the friction force **F**.



We know that $|\mathbf{P}| = 23$ and $|\mathbf{W}| = 5.5g$.

Take $\mathbf{i}$ to point right and $\mathbf{j}$ to point up, as shown.

Let $N = |\mathbf{N}|$ and $F = |\mathbf{F}|$. We have

$$\mathbf{P} = 23\,\mathbf{i}$$
$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{F} = -F\,\mathbf{i}$$
$$\mathbf{W} = -5.5g\,\mathbf{j}.$$

Since the block is at rest, $\mathbf{P} + \mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This gives

$$23\,\mathbf{i} + N\,\mathbf{j} - F\,\mathbf{i} - 5.5g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives

$$23 - F = 0,$$
$$N - 5.5g = 0.$$

These equations give

$$F = 23 \quad \text{and} \quad N = 5.5g.$$

Since the box is on the point of slipping, we have

$$F = \mu N.$$

This gives

$$\mu = \frac{F}{N} = \frac{23}{5.5g} = \frac{23}{5.5 \times 9.8} = 0.4267\ldots.$$

The coefficient of static friction is 0.43 (to 2 s.f.).

Solution to Activity 21

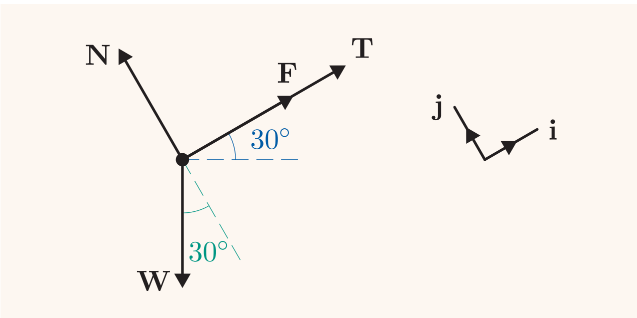
Model the box as a particle and the string as a model string.

The forces acting are the weight $\mathbf{W}$ of the box, the normal reaction $\mathbf{N}$, the tension $\mathbf{T}$ in the string and the friction force $\mathbf{F}$.

Take $\mathbf{i}$ to point up the slope and $\mathbf{j}$ to be perpendicular to the slope, as shown below.

A force diagram is shown below. Here $\mathbf{F}$ is shown acting in the $\mathbf{i}$-direction, though we do not know at this point whether it acts in the $\mathbf{i}$-direction or the opposite direction.

(One way to see why the acute angle between $\mathbf{W}$ and a line parallel to $\mathbf{N}$ is 30° is to imagine rotating the angle between $\mathbf{T}$ and the horizontal clockwise by 90°.)



We know that $|\mathbf{T}| = 120$ and $|\mathbf{W}| = 20g$.

Let $N = |\mathbf{N}|$ and let F be the $\mathbf{i}$-component of $\mathbf{F}$. Then

$$\mathbf{T} = 120\,\mathbf{i}$$
$$\mathbf{F} = F\,\mathbf{i}$$
$$\mathbf{N} = N\,\mathbf{j}$$

and

$$\begin{aligned}\mathbf{W} &= -20g\sin 30°\,\mathbf{i} - 20g\cos 30°\,\mathbf{j}\\ &= -20g \times \frac{1}{2}\,\mathbf{i} - 20g \times \frac{\sqrt{3}}{2}\,\mathbf{j}\\ &= -10g\,\mathbf{i} - 10\sqrt{3}g\,\mathbf{j}.\end{aligned}$$

Since the block is at rest, $\mathbf{T} + \mathbf{F} + \mathbf{N} + \mathbf{W} = \mathbf{0}$. This gives

$$120\,\mathbf{i} + F\,\mathbf{i} + N\,\mathbf{j} - 10g\,\mathbf{i} - 10\sqrt{3}g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$-direction gives

$$120 + F - 10g = 0.$$

So

$$F = 10g - 120 = 10 \times 9.8 - 120 = -22.$$

Since F is negative, $\mathbf{F}$ points in the *opposite* direction to $\mathbf{i}$.

The friction force has magnitude 22 N and acts down the slope.

Solution to Activity 22

The coefficient of static friction is

$$\mu = \tan 12° = 0.21 \text{ (to 2 d.p.)}.$$

Solution to Activity 23

The block will start slipping when the angle is

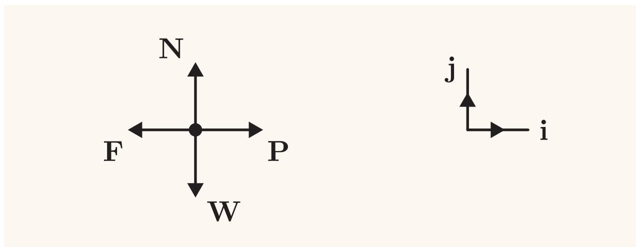
$$\tan^{-1} 0.34 = 19° \text{ (to the nearest degree)}.$$

Solution to Activity 24

Model the statue and the sledge together as a particle.

Assume that the statue is on the point of slipping.

The forces acting are the pulling force $\mathbf{P}$ exerted by the 172 workers, the weight $\mathbf{W}$ of the statue, the normal reaction $\mathbf{N}$ and the friction force $\mathbf{F}$. A force diagram is shown below.



We know that

$$|\mathbf{P}| = 172 \times 350 = 60\,200,$$

and $|\mathbf{W}| = 60\,000g$.

Take $\mathbf{i}$ to point right and $\mathbf{j}$ to point up, as shown.

Let $N = |\mathbf{N}|$ and $F = |\mathbf{F}|$.

Expressing the forces in component form gives

$$\mathbf{P} = 60\,200\,\mathbf{i}$$
$$\mathbf{W} = -60\,000g\,\mathbf{j}$$
$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{F} = -F\,\mathbf{i}.$$

Since the particle is at rest, the equilibrium condition holds, so

$$\mathbf{P} + \mathbf{W} + \mathbf{N} + \mathbf{F} = \mathbf{0}.$$

This gives

$$60\,200\,\mathbf{i} - 60\,000g\,\mathbf{j} + N\,\mathbf{j} - F\,\mathbf{i} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives

$$60\,200 - F = 0$$
$$-60\,000g + N = 0.$$

These equations give

$$F = 60\,200 \quad \text{and} \quad N = 60\,000g.$$

Since the statue is on the point of slipping, we have

$$F = \mu N,$$

which gives

$$\mu = \frac{F}{N} = \frac{60\,200}{60\,000g} = \frac{60\,200}{60\,000 \times 9.8} = 0.1023\ldots.$$

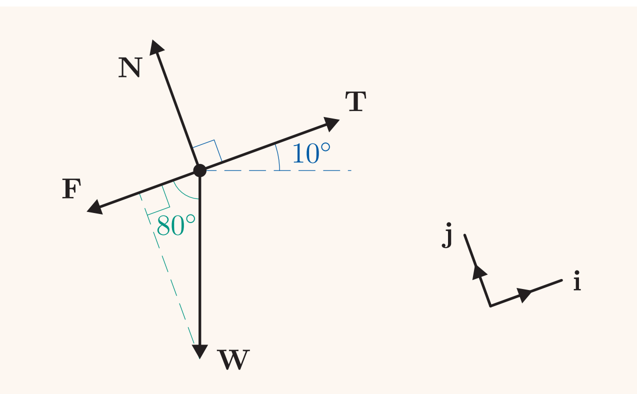
The coefficient of static friction is 0.10 (to 2 d.p.).

Solution to Activity 25

(a) If the block is at the point of slipping up the ramp, then the friction force acts down the ramp.

(b) Model the sledge and the block together as a particle, and the rope as a model string. Assume that the block is on the point of slipping up the ramp.

The forces acting are the weight **W**, the normal reaction **N**, the friction force **F** and the tension **T** in the rope. A force diagram is shown below.



(The acute angle between **W** and a line parallel to **F** is $180° - 10° - 90° = 80°$, as shown.)

Take **i** to point up the slope and **j** to point perpendicular to the slope, as shown.

We know that $|\mathbf{W}| = 2600g$. Let $N = |\mathbf{N}|$, $F = |\mathbf{F}|$ and $T = |\mathbf{T}|$.

Expressing the forces in component form gives

$$\mathbf{T} = T\,\mathbf{i}$$
$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{F} = -F\,\mathbf{i}$$
$$\mathbf{W} = -2600g\cos 80°\,\mathbf{i} - 2600g\sin 80°\,\mathbf{j}.$$

Since the block is at rest, the equilibrium condition holds. This is

$$\mathbf{T} + \mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0},$$

which gives

$$T\,\mathbf{i} + N\,\mathbf{j} - F\,\mathbf{i} - 2600g\cos 80°\,\mathbf{i} - 2600g\sin 80°\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives the pair of equations

$$T - F - 2600g\cos 80° = 0$$
$$N - 2600g\sin 80° = 0.$$

The second equation gives

$$N = 2600g\sin 80°.$$

Since the block is on the point of slipping, we have

$$\begin{aligned} F &= \mu N \\ &= 0.10 \times 2600g\sin 80° \\ &= 260g\sin 80°. \end{aligned}$$

So the first of the pair of equations above gives

$$\begin{aligned} T &= F + 2600g\cos 80° \\ &= 260g\sin 80° + 2600g\cos 80° \\ &= 260g(\sin 80° + 10\cos 80°) \\ &= 260 \times 9.8 \times (\sin 80° + 10\cos 80°) \\ &= 6933.845\ldots \end{aligned}$$

So when the block is on the point of slipping up the ramp, the tension in the rope is 6900 N (to 2 s.f.).

(c) Since the maximum pulling force of each worker is 350 N, the number of workers needed is

$$\frac{T}{350} = \frac{6933.845\ldots}{350} = 19.81\ldots.$$

Rounding up to a whole number of workers shows that 20 workers are needed.

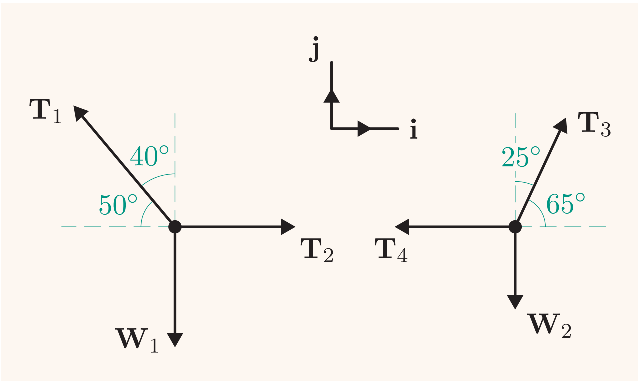
Solution to Activity 26

Model the decorations as particles and the three sections of the string as model strings.

The forces acting on the left-hand decoration are its weight $\mathbf{W}_1$, the tension $\mathbf{T}_1$ in the left-hand string and the tension $\mathbf{T}_2$ in the middle string.

Similarly, the forces acting on the right-hand decoration are its weight $\mathbf{W}_2$, the tension $\mathbf{T}_3$ in the right-hand string and the tension $\mathbf{T}_4$ in the middle string.

Force diagrams for the two decorations are shown below.



We know that $|\mathbf{W}_1| = 0.95g$.

Take $\mathbf{i}$ and $\mathbf{j}$ to point right and up, as shown.

We have $|\mathbf{T}_2| = |\mathbf{T}_4|$ (since $\mathbf{T}_2$ and $\mathbf{T}_4$ are tensions in the same string), so let $T = |\mathbf{T}_2| = |\mathbf{T}_4|$. Also, let $T_1 = |\mathbf{T}_1|$ and $T_3 = |\mathbf{T}_3|$, and let m be the mass of the right-hand decoration, which gives $|\mathbf{W}| = mg$.

From the force diagram for the left-hand decoration we have

$$\mathbf{T}_1 = -T_1 \cos 50^\circ \mathbf{i} + T_1 \sin 50^\circ \mathbf{j}$$
$$\mathbf{T}_2 = T\,\mathbf{i}$$
$$\mathbf{W}_1 = -0.95g\,\mathbf{j}.$$

From the force diagram for the right-hand decoration we have

$$\mathbf{T}_3 = T_3 \cos 65^\circ \mathbf{i} + T_3 \sin 65^\circ \mathbf{j}$$
$$\mathbf{T}_4 = -T\,\mathbf{i}$$
$$\mathbf{W}_2 = -mg\,\mathbf{j}.$$

First we aim to find the value of T, the magnitude of the tension in the middle section of the string.

Since the left-hand decoration is at rest,

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W}_1 = \mathbf{0}.$$

This gives

$$-T_1 \cos 50^\circ \mathbf{i} + T_1 \sin 50^\circ \mathbf{j} + T\,\mathbf{i} - 0.95g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives

$$-T_1 \cos 50^\circ + T = 0$$
$$T_1 \sin 50^\circ - 0.95g = 0.$$

The second of these equations gives

$$T_1 = \frac{0.95g}{\sin 50^\circ}.$$

Hence the first of the equations gives

$$\begin{aligned} T &= T_1 \cos 50^\circ \\ &= \frac{0.95g \cos 50^\circ}{\sin 50^\circ} \\ &= 0.95g \cot 50^\circ. \end{aligned}$$

Now we use the value of T to find the required mass.

Since the right-hand decoration is at rest,

$$\mathbf{T}_3 + \mathbf{T}_4 + \mathbf{W}_2 = \mathbf{0}.$$

This gives

$$T_3 \cos 65^\circ \mathbf{i} + T_3 \sin 65^\circ \mathbf{j} - T\,\mathbf{i} - mg\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives

$$T_3 \cos 65^\circ - T = 0$$
$$T_3 \sin 65^\circ - mg = 0.$$

The first of these equations gives

$$T_3 = \frac{T}{\cos 65^\circ}.$$

Hence, by the second of the equations,

$$\begin{aligned} m &= \frac{T_3 \sin 65^\circ}{g} \\ &= \frac{T \sin 65^\circ}{g \cos 65^\circ} \\ &= \frac{T \tan 65^\circ}{g}. \end{aligned}$$

Substituting the expression for T found above into this equation gives

$$m = \frac{0.95g \cot 50^\circ \tan 65^\circ}{g}.$$

Cancelling the factor g in the top and bottom of the fraction gives

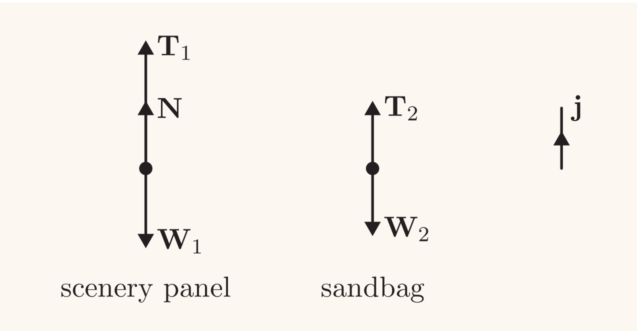
$$\begin{aligned} m &= 0.95 \cot 50^\circ \tan 65^\circ \\ &= 1.709\,482\ldots. \end{aligned}$$

So the mass of the right-hand decoration is 1.7 kg (to 2 s.f.).

Solution to Activity 27

The forces acting on the scenery panel are its weight $\mathbf{W}_1$, the tension $\mathbf{T}_1$ in the rope and the normal reaction $\mathbf{N}$ from the floor. The forces acting on the sandbag are its weight $\mathbf{W}_2$ and the tension $\mathbf{T}_2$ in the rope.

Force diagrams are shown below.



We know that $|\mathbf{W}_1| = 100g$ and $|\mathbf{W}_2| = 80g$.

Take $\mathbf{j}$ to point up. (Since all the forces are vertical, there is no need for $\mathbf{i}$.)

Since $\mathbf{T}_1$ and $\mathbf{T}_2$ are tensions in the same string passing over model pulleys, their magnitudes are equal. So let $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Also, let $N = |\mathbf{N}|$.

From the force diagram for the scenery panel we have

$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{T}_1 = T\,\mathbf{j}$$
$$\mathbf{W}_1 = -100g\,\mathbf{j}.$$

From the force diagram for the sandbag we have

$$\mathbf{T}_2 = T\,\mathbf{j}$$
$$\mathbf{W}_2 = -80g\,\mathbf{j}.$$

Since the sandbag is at rest, $\mathbf{T}_2 + \mathbf{W}_2 = \mathbf{0}$. This gives

$$T\,\mathbf{j} - 80g\,\mathbf{j} = \mathbf{0}.$$

So $T - 80g = 0$ and hence

$$T = 80g.$$

Since the scenery panel is at rest, $\mathbf{N} + \mathbf{T}_1 + \mathbf{W}_1 = \mathbf{0}$. This gives

$$N\,\mathbf{j} + T\,\mathbf{j} - 100g\,\mathbf{j} = \mathbf{0},$$

which gives

$$N + T - 100g = 0.$$

This equation gives

$$\begin{aligned} N &= 100g - T \\ &= 100g - 80g \\ &= 20g \\ &= 20 \times 9.8 \\ &= 196 = 200 \text{ (to 2 s.f.)}. \end{aligned}$$

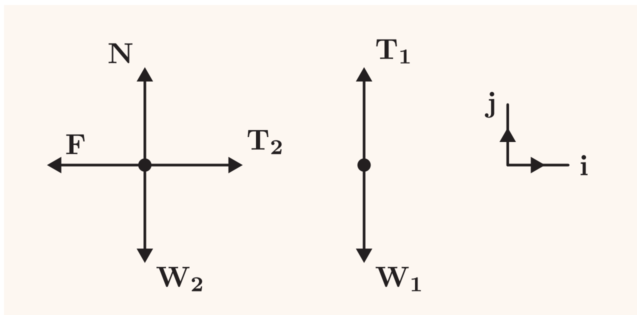
So the normal reaction of the floor on the scenery panel has magnitude 200 N (to 2 s.f.) and is directed upwards.

Solution to Activity 28

Model the climbers as particles, the rope as a model string and the boulder as a model pulley. To find the minimum coefficient of static friction, assume that each climber is at rest, and the standing climber is at the point of slipping.

The forces acting on the suspended climber are his weight $\mathbf{W}_1$, and the tension $\mathbf{T}_1$ in the rope. The forces acting on the standing climber are his weight $\mathbf{W}_2$, the tension $\mathbf{T}_2$ in the rope, the normal reaction $\mathbf{N}$ from the ledge and the friction force $\mathbf{F}$.

Force diagrams are shown below.



We know that $|\mathbf{W}_1| = 70g$ and $|\mathbf{W}_2| = 75g$.

Take $\mathbf{i}$ and $\mathbf{j}$ to point right and up, as shown.

To find the coefficient μ of static friction between the soles of the standing climber's boots and the top surface of the ledge, we need to use the equation $|\mathbf{F}| = \mu|\mathbf{N}|$, which applies since the standing climber is on the point of slipping. So first we need to use the force diagrams to find the values of $|\mathbf{F}|$ and $|\mathbf{N}|$.

We have $|\mathbf{T}_1| = |\mathbf{T}_2|$ (since $\mathbf{T}_1$ and $\mathbf{T}_2$ are tensions in the same string running over a model pulley), so let $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Also, let $F = |\mathbf{F}|$ and $N = |\mathbf{N}|$.

From the force diagram for the suspended climber we have

$$\mathbf{T}_1 = T\,\mathbf{j}$$
$$\mathbf{W}_1 = -70g\,\mathbf{j}.$$

From the force diagram for the standing climber we have

$$\mathbf{F} = -F\,\mathbf{i}$$
$$\mathbf{N} = N\,\mathbf{j}$$
$$\mathbf{T}_2 = T\,\mathbf{i},$$
$$\mathbf{W}_2 = -75g\,\mathbf{j}.$$

Since the suspended climber is at rest,

$$\mathbf{T}_1 + \mathbf{W}_1 = \mathbf{0}.$$

This gives

$$T\,\mathbf{j} - 70g\,\mathbf{j} = \mathbf{0},$$

which gives

$$T - 70g = 0,$$

and hence $T = 70g$.

Since the standing climber is at rest,

$$\mathbf{F} + \mathbf{N} + \mathbf{T}_2 + \mathbf{W}_2 = \mathbf{0}.$$

This gives

$$-F\,\mathbf{i} + N\,\mathbf{j} + T\,\mathbf{i} - 75g\,\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the **i**- and **j**-directions gives

$$-F + T = 0$$
$$N - 75g = 0.$$

We now solve these equations to find F and N. The first equation gives

$$F = T$$

and hence

$$F = 70g.$$

The second equation gives

$$N = 75g.$$

The equation $|\mathbf{F}| = \mu|\mathbf{N}|$, that is, $F = \mu N$, gives

$$70g = \mu \times 75g,$$

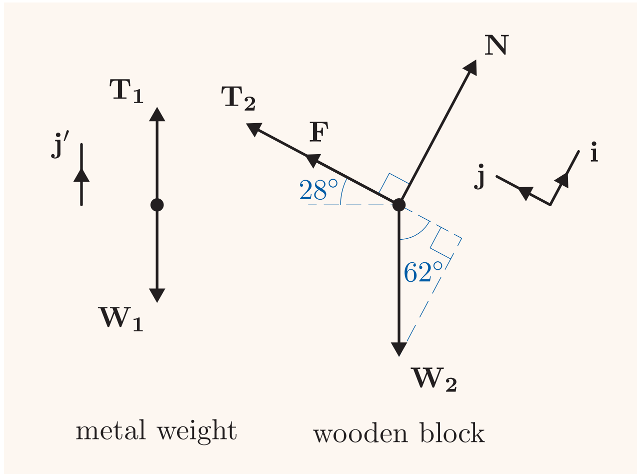
which gives

$$\mu = \frac{70g}{75g} = \frac{70}{75} = 0.933\ldots.$$

So the minimum coefficient of static friction is 0.93 (to 2 d.p.).

Solution to Activity 29

(a) The forces acting on the metal weight are its weight $\mathbf{W}_1$ and the tension $\mathbf{T}_1$ in the rope. The forces acting on the wooden block are its weight $\mathbf{W}_2$, the tension $\mathbf{T}_2$ in the rope, the friction force $\mathbf{F}$ and the normal reaction $\mathbf{N}$ from the plane. Since the wooden block is on the point of slipping down the slope, the friction force acts up the slope. Force diagrams are shown below.



We know that $|\mathbf{W}_1| = 3.3g$.

Take **i** and **j** to be perpendicular and parallel to the slope, respectively, as shown. Take $\mathbf{j}'$ to point vertically up. (We choose different directions for the Cartesian unit vectors for each force diagram to simplify the working, as in Example 13.)

Since $\mathbf{T}_1$ and $\mathbf{T}_2$ are tensions in the same string passing over a model pulley, their magnitudes are equal. So let $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Also, let $N = |\mathbf{N}|$ and $F = |\mathbf{F}|$, and let m be the mass of the wooden block, so $|\mathbf{W}_2| = mg$.

For the metal weight we have

$$\mathbf{T}_1 = T\,\mathbf{j}'$$
$$\mathbf{W}_1 = -3.3g\,\mathbf{j}'.$$

The acute angle between $\mathbf{W}_2$ and a line parallel to $\mathbf{T}_2$ is

$$180^\circ - 90^\circ - 28^\circ = 62^\circ,$$

as marked on the force diagram.

For the wooden block we have

$$\mathbf{N} = N\,\mathbf{i}$$
$$\mathbf{T}_2 = T\,\mathbf{j}$$
$$\mathbf{F} = F\,\mathbf{j}$$
$$\mathbf{W}_2 = -mg\sin 62^\circ\mathbf{i} - mg\cos 62^\circ\mathbf{j}.$$

First we find the value of T.

The equilibrium condition for the metal weight is $\mathbf{T}_1 + \mathbf{W}_1 = \mathbf{0}$, which gives

$$T\,\mathbf{j}' - 3.3g\,\mathbf{j}' = \mathbf{0}.$$

Hence $T - 3.3g = 0$, which gives

$$T = 3.3g.$$

Now we use the value of T to find the value of m.

The equilibrium condition for the wooden block is $\mathbf{N} + \mathbf{T}_2 + \mathbf{F} + \mathbf{W}_2 = \mathbf{0}$, which gives

$$N\,\mathbf{i} + T\,\mathbf{j} + F\,\mathbf{j} - mg\sin 62^\circ\mathbf{i} - mg\cos 62^\circ\mathbf{j} = \mathbf{0}.$$

Since the block is on the point of slipping, and the coefficient of static friction is 0.24, we have $F = 0.24N$. Using this equation and the equation $T = 3.3g$ to substitute for F and T in the vector equation above gives

$$N\,\mathbf{i} + 3.3g\,\mathbf{j} + 0.24N\,\mathbf{j} - mg\sin 62^\circ\mathbf{i} - mg\cos 62^\circ\mathbf{j} = \mathbf{0}. \tag{6}$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the pair of equations

$$N - mg\sin 62^\circ = 0$$
$$3.3g + 0.24N - mg\cos 62^\circ = 0.$$

The first of these equations gives

$$N = mg\sin 62^\circ.$$

Using this equation to substitute for N in the second of the pair of equations gives

$$3.3g + 0.24mg\sin 62^\circ - mg\cos 62^\circ = 0.$$

Dividing through by g gives

$$3.3 + 0.24m\sin 62^\circ - m\cos 62^\circ = 0.$$

Solving for m gives

$$m(0.24\sin 62^\circ - \cos 62^\circ) = -3.3;$$

that is,

$$\begin{aligned} m &= \frac{-3.3}{0.24\sin 62^\circ - \cos 62^\circ} \\ &= 12.8123\ldots. \end{aligned}$$

The mass of the wooden block is 13 kg (to 2 s.f.).

(b) To work out the mass of a metal weight that would cause the block to be on the point of slipping up the plane, the analysis of the problem is the same as in part (a), except for three things, as follows.

- The metal weight has an unknown mass, say M, rather than mass 3.3 kg.
- The mass m of the wooden block is known, as found in part (a).
- The friction force acts down the slope rather than up the slope. This gives $\mathbf{F} = -F\mathbf{j}$ rather than $\mathbf{F} = F\mathbf{j}$. Hence, since $F = 0.24N$, we have $\mathbf{F} = -0.24N\mathbf{j}$ rather than $\mathbf{F} = 0.24N\mathbf{j}$.

So equation (6) becomes

$$N\,\mathbf{i} + Mg\,\mathbf{j} - 0.24N\,\mathbf{j} - mg\sin 62^\circ\mathbf{i} - mg\cos 62^\circ\mathbf{j} = \mathbf{0}.$$

Resolving this equation in the $\mathbf{i}$- and $\mathbf{j}$-directions gives the pair of equations

$$N - mg\sin 62^\circ = 0$$
$$Mg - 0.24N - mg\cos 62^\circ = 0.$$

The first of these equations gives

$$N = mg\sin 62^\circ.$$

Using this equation to substitute for N in the second of the pair of equations gives

$$Mg - 0.24mg\sin 62^\circ - mg\cos 62^\circ = 0.$$

Dividing through by g gives

$$M - 0.24m\sin 62^\circ - m\cos 62^\circ = 0.$$

Solving for M gives

$$\begin{aligned} M &= 0.24m\sin 62^\circ + m\cos 62^\circ \\ &= m(0.24\sin 62^\circ + \cos 62^\circ) \\ &= 12.8123\ldots \times (0.24\sin 62^\circ + \cos 62^\circ) \\ &= 8.7300\ldots. \end{aligned}$$

Now

$$M - 3.3 = 8.7300\ldots - 3.3 = 5.4300\ldots.$$

So the mass of the metal weight has to be increased by 5.4 kg (to 2 s.f.) to make the wooden block be on the point of slipping up the slope.

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