Unit 3

Dynamics

Introduction

Think of an object that is moving in some way. It might be a car accelerating on a motorway, a tennis ball flying through the air, a comet hurtling through space, or a pendulum swinging to and fro. Why does the object move as it does? How will it move in the future? To what extent can you influence its motion? Questions like these are very important from a practical point of view. The control that the human race exerts over the environment depends, to a large extent, on our ability to find the right answers. For example, in Figure 1, can we predict whether the comet will collide with the Earth in time for something to be done about it?

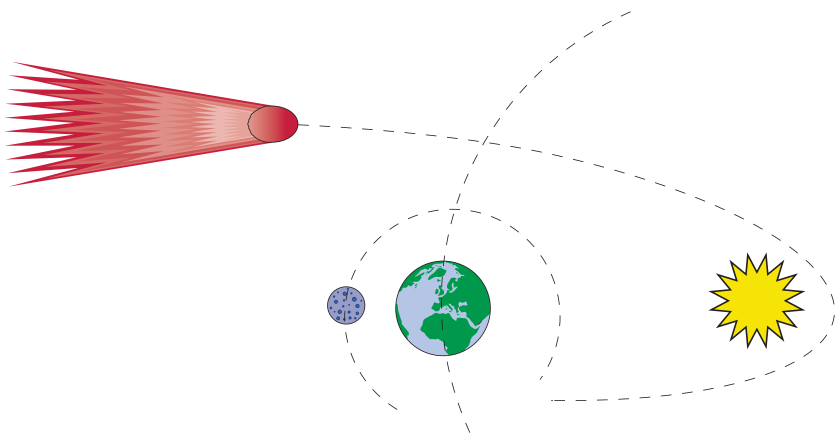


Figure 1 Collision course?

Fortunately, the systematic, organised body of knowledge called *mechanics*, which you began to study in Unit 2, provides answers to many questions about motion and the forces that cause it.

Unit 2 concentrated on whether objects will stay put; this unit looks at moving objects, that is, **dynamics**. The starting point for all of Newtonian mechanics is Newton's laws of motion, which were stated in words in the Introduction of Unit 2. To apply these laws, we need to be able to translate them into useful mathematical equations – which is one of the aims of this unit.

You should bear in mind that this unit and Unit 2 provide only an introduction to the ideas of mechanics. However, the framework of mechanics presented in these units is of great significance as it provides the foundations from which, in later units, more complex ideas are developed.

Section 1 of this unit is concerned with concepts like position, velocity and acceleration, which describe the way an object moves. Section 2 discusses Newton's laws of motion, which predict the motion of an object when the forces acting on the object are known. Section 3 shows how Newton's second law of motion can be used to predict the motion of objects. Section 4 is concerned with modelling some of the forces that occur in nature, which enables more realistic situations to be analysed.

In Section 5, we turn to motion in more than one dimension. In Unit 2 you met vectors – quantities with magnitude and direction. In modelling motion in more than one dimension, the role of vectors is more crucial, as it was in Unit 2 when modelling forces in equilibrium in two and three dimensions.

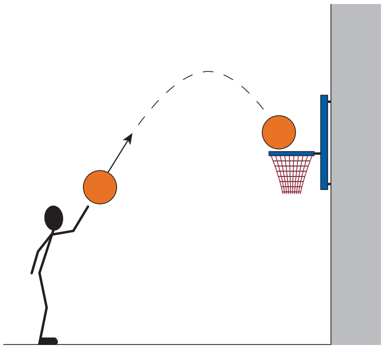


Figure 2 A basketball in flight

A thrown object, such as a basketball (see Figure 2) or a shot, from the point of release until it hits the ground, is subject only to the force of gravity and to any force exerted on it by the air (broadly referred to as *air resistance*). Such objects are examples of **projectiles**, where we know the forces acting on the object and want to deduce the path of the object through space. Of course, such thrown objects may happen to execute motion that is purely vertical (straight up and straight down), but our interest here is in cases where the motion is horizontal as well as vertical. Athletic and sporting activities provide a wide variety of examples of projectile motion. As well as the throwing or striking of objects such as basketballs or golf balls, sports may involve humans themselves acting as projectiles, as in diving, long-jumping and ski-jumping.

Here we will deal with models of projectiles without air resistance. Section 5 examines a variety of aspects of such motion.

In many of the examples considered in this unit, the size of the moving object is of significance. For example, in a football free kick, the ball is often given a spinning motion to make it swerve through the air. We will not consider such aspects of motion here, however. In this unit we consider only objects modelled as particles.

Throughout this unit we make use of Newton's second law of motion in vector form, given by

Remember to underline vectors in handwritten work.

$$\mathbf{F} = m\mathbf{a},$$

where the total force $\mathbf{F}$ and the acceleration $\mathbf{a}$ are vectors. In Sections 1 to 4, this law is used almost exclusively in one dimension. In Section 5, we will be concerned with its use in more than one dimension. If the position vector of a particle is known as a function of time, we can obtain its velocity and acceleration vectors by differentiation. In more than one dimension, we use the position vector $\mathbf{r}(t)$ to represent the position of a particle at time t. To differentiate $\mathbf{r}(t)$ with respect to t, we differentiate separately each of the functions giving its $\mathbf{i}$-, $\mathbf{j}$- and $\mathbf{k}$-components. We usually employ the notation $\dot{\mathbf{r}}(t)$ (or just $\dot{\mathbf{r}}$) for the velocity $\mathbf{v} = d(\mathbf{r}(t))/dt$. Similarly, we usually write $\ddot{\mathbf{r}}(t)$ (or just $\ddot{\mathbf{r}}$) for the second derivative of $\mathbf{r}(t)$ that gives the acceleration $\mathbf{a}$.

1 Describing motion

This section is devoted to describing the motion of objects modelled as particles. In Subsection 1.1 the motion is described by giving a position vector at each instant in time. The ideas in Subsection 1.1 apply whether the particle moves along a straight line or along a curve of some sort; in Subsection 1.2, however, and for Sections 2 to 4, only motion along a straight line is considered.

The subject matter of this section, the description of motion, is often referred to by the technical term *kinematics*.

1.1 Motion of a particle

The motion of a real object, say a leaf that is falling to the ground, is very difficult to describe exactly. The leaf may rotate, bend or vibrate while moving along a complicated path in three-dimensional space. And its motion may be affected by the presence of other moving objects, such as other falling leaves. It would be foolhardy to try to meet all of these difficulties head-on, so we start by making a number of simplifications, some of which will be relaxed later.

The motion of rigid bodies is discussed in Unit 21, and many-particle systems are discussed in Unit 19.

Simplification 1 Objects will be modelled as particles.

Simplification 2 Only the motion of *single* particles is considered.

Modelling objects as particles means that we neglect an object's size and internal structure. Neglecting an object's size means that its location at any given time may be described by a single point in space, and that its motion may be described by a single curve. Neglecting the internal structure of an object, and hence also any internal motion, amounts to saying that the curve described in time by the particle gives the only information of interest about the way in which the object moves. In mathematical terms, this means that a particle's motion is completely described by its *position vector* $\mathbf{r}$, relative to some fixed origin O, at those times t in the time interval of interest.

The representation of the motion of a particle by a position vector that changes with time leads naturally to the representation of such motion by a *vector function*.

> A **vector function** $\mathbf{r}(t)$ of some variable t is a vector whose components are functions of t.

Unlike the components of a vector, which are constant, the components of a vector function vary as the independent variable changes. In the case of the motion of a particle, a vector function $\mathbf{r}(t)$, whose components at each time t represent the position vector $\mathbf{r}$ of the particle at that time, completely describes the particle's motion. The idea is illustrated in Figure 3 and in the following example of two-dimensional motion.

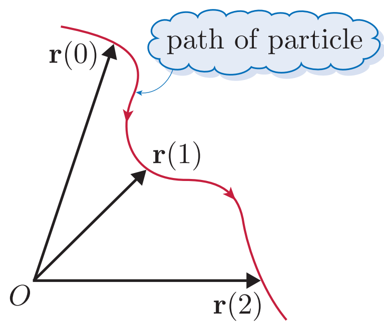


Figure 3 The path of a particle described by the vector function $\mathbf{r}(t)$

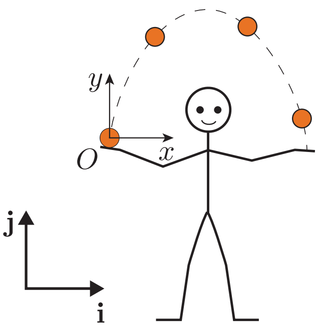


Figure 4 A juggler with a ball

Example 1

A juggler throws a ball from one hand to the other, in a vertical plane, as shown in Figure 4. The ball is modelled as a particle, and its motion, with respect to the horizontal and vertical axes shown in Figure 4, is described by the two-dimensional vector function

$$\mathbf{r}(t) = 1.5t\mathbf{i} + t(4-5t)\mathbf{j} \quad (0 \le t \le 1),$$

where distances are measured in metres, and time t is measured in seconds after the ball was thrown. The origin O is the juggler's right hand just as he throws the ball, which occurs at time $t = 0$.

Let $x(t)$ be the component of $\mathbf{r}(t)$ in the $\mathbf{i}$-direction, that is, $x(t) = \mathbf{r}(t) \cdot \mathbf{i} = 1.5t$. Similarly, let $y(t)$ be the component of $\mathbf{r}(t)$ in the $\mathbf{j}$-direction, that is, $y(t) = \mathbf{r}(t) \cdot \mathbf{j} = t(4-5t)$. These give the horizontal distance travelled and the height of the ball, respectively, at time t.

(a) Calculate $x(t)$ and $y(t)$ at times $t = 0, 0.2, 0.4, 0.6, 0.8, 1$, and sketch the graphs of $y(t)$ against t and against $x(t)$. Comment on what the graphs represent.

(b) Using your graphs, or otherwise, answer the following questions.

(i) How high does the ball go, and what is its position at its highest point?

(ii) Does the juggler catch the ball?

Solution

(a) The values are tabulated in Table 1, and the graphs are shown in Figure 5.

Table 1

t	0	0.2	0.4	0.6	0.8	1
$x(t)$	0	0.3	0.6	0.9	1.2	1.5
$y(t)$	0	0.6	0.8	0.6	0	-1

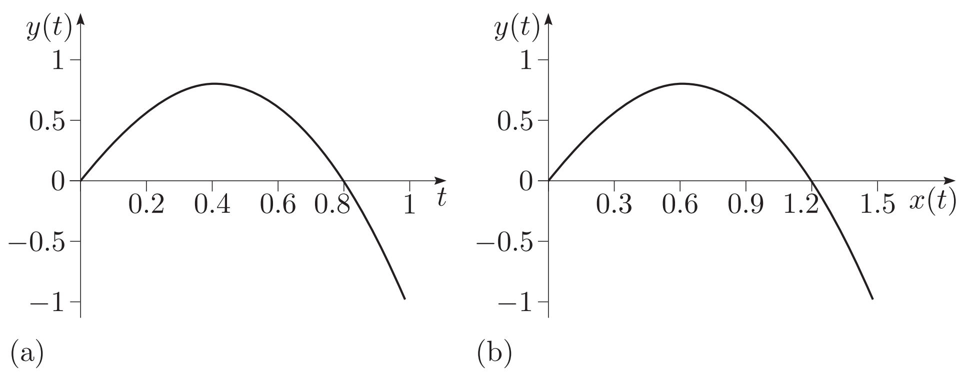


Figure 5 Graphs of (a) height against time, (b) height against horizontal distance

Figure 5(a) is a distance–time graph, representing the *height* $y(t)$ of the ball as the time t varies over the interval $[0,1]$. Figure 5(b) represents the *position* of the ball in the (x,y)-plane as t varies; the coordinates $(x(t),y(t))$ of points on this curve are the components of the vector function $\mathbf{r}(t)$ describing the motion of the ball. In this case, since $x(t)=1.5t$ is just a multiple of t, the two curves appear identical – the only difference is in the scales on the horizontal axes. However, the curves *represent* different things.

You may recognise $(x(t),y(t))$ as a *parametrisation* of the path of the ball, in which the Cartesian coordinates of the ball are expressed in terms of another variable (in this case time t).

(b) (i) The quadratic function $y(t)=t(4-5t)$ represents a parabola with maximum value in the interval $[0,0.8]$ (see Figure 5(a)). From the symmetry of the parabola, this maximum must occur at $t=0.4$, at which time $y(t)=0.8$ (see Table 1). So the ball reaches a maximum height of 0.8 m above the juggler's hand after 0.4 s. At this maximum height, $x(t)=0.6$ (see Table 1).

So the ball's position at its maximum height is given by the coordinates $(0.6,0.8)$ (see Figure 5(b)), or equivalently, by the position vector $\mathbf{r}(0.4)=0.6\mathbf{i}+0.8\mathbf{j}$.

(ii) The juggler does not catch the ball because the data indicate that the ball continues to travel downwards until it is one metre below the juggler's left hand (see Table 1 and Figure 5).

In general, motion in three-dimensional space is represented by a three-dimensional vector function. However, because the motion in Example 1 was in a plane, it was possible by careful choice of axes to represent that motion in three-dimensional space by a two-dimensional vector function. A similarly careful choice of axes can enable certain types of motion in three-dimensional space to be represented by a *one*-dimensional vector function, as the following exercise illustrates.

Exercise 1

An ice hockey player aims to hit the puck towards the goal, as shown in Figure 6, but misses. The puck hits the back wall and bounces straight back, and is then hit by a second player who (incredibly) hits it along the same path as the first player. Let the origin O be the point of impact of the puck by the first player, which occurs at time $t=0$. The motion of the puck, with respect to the axis shown in Figure 6, is described by the one-dimensional vector function $\mathbf{r}(t)=x(t)\,\mathbf{i}$, where the function $x(t)$ is defined by

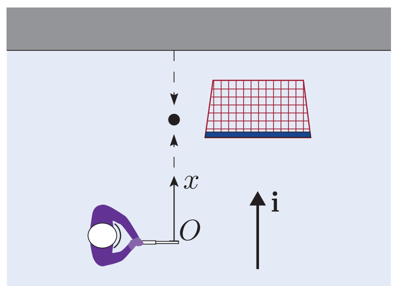


Figure 6 The path of the puck

$$x(t)=\begin{cases} 5t, & 0\le t<0.4,\\ 4-5t, & 0.4\le t<1,\\ 6t-7, & 1\le t\le 1.5,\end{cases}$$

where x is measured in metres, and t is measured in seconds.

(a) Sketch the graph of the function $x(t)$.

(b) Using your graph, or otherwise, answer the following questions.

(i) How far is the first player from the back wall when the puck is struck?

(ii) Does the second player hit the puck from a position closer to or further from the back wall than the first player?

(iii) Does the second player give the puck more speed than the first player?

You saw in Unit 2 that some large static objects can be modelled as particles.

In Example 1 and Exercise 1, objects were modelled as particles. The question arises as to when this is appropriate. It seems obvious that small objects can be modelled as particles, but what about large objects such as the Earth? The answer is that it depends on the context. For example, to calculate the Earth's orbit around the Sun, it is permissible to model the Earth as a particle. One of Newton's great achievements was to realise that it is appropriate to model the Earth as a particle in this context.

The Earth's diameter (about 13 000 km) is small compared with the Earth–Sun distance (about 1.5×10^8 km).

Whether a particle model will be satisfactory is not just a question of size. For example, if a ball is placed on a rough sloping table, then it will roll down the slope. A particle model could be used to describe the path of the ball's centre, but it would not be adequate to keep track of the rolling motion that takes place about the centre. Note that this inadequacy of the particle model occurs regardless of the ball's size, since the same consideration would apply to a football, a tennis ball, a marble or a ball-bearing. However, if the ball is not rotating, or if it is assumed that the rotations can be ignored, then the particle model is appropriate.

You might think from this last example that the particle model is of very limited use for moving objects, but in fact the example hints at how this model can be extended. You will see in Unit 19 that the motion of an object can be described well by specifying:

You met the idea of centre of mass in Unit 2. For the symmetric objects considered in that unit, the centre of mass is the geometric centre.

- the motion of the *centre of mass* of an object
- the motion of the whole object relative to its centre of mass.

As you will see in Unit 19, the motion of the centre of mass may be predicted by considering a particle of the same mass as the object, placed at the point defined by the centre of mass, and subjected to all of the forces that act on the object. So even in this more complicated situation, the concept of a particle is important. Alternatively, it may be appropriate to think of an object as being composed of a number of elements, each of which can be modelled individually as a particle.

Position, velocity and acceleration

As in Unit 1, we may sometimes wish to consider position as a variable rather than as a function, and write $\mathbf{r}$ rather than $\mathbf{r}(t)$.

We have seen how a particle's motion can be completely described by a vector function $\mathbf{r}(t)$, the position vector of the particle at any given time t. For this reason, we often refer to particles as having **position** $\mathbf{r}(t)$.

However, other quantities, such as the velocity and acceleration of a particle, are often of more interest than its position. For example, an aggressive motorist might be proud of his acceleration away from traffic lights, whereas a police officer would probably be more interested in the motorist's velocity. These quantities can be calculated directly from a particle's position $\mathbf{r}(t)$ by differentiating. But before we can do that, we need to define the derivative of a vector function.

The **derivative** of a vector function $\mathbf{r}(t)$ whose components are smooth functions of t, is the vector function

$$\frac{d\mathbf{r}(t)}{dt} = \lim_{h\to 0}\left(\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}\right). \qquad (1)$$

The derivative $d\mathbf{r}(t)/dt$ is often written as $\mathbf{r}'(t)$ or, where t represents time, as $\dot{\mathbf{r}}(t)$. Sometimes it is written more succinctly as $d\mathbf{r}/dt$, $\mathbf{r}'$ or $\dot{\mathbf{r}}$.

Compare this with the definition of the derivative of a function as

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{y(x+h)-y(x)}{h}\right).$$

Newton's notation $\dot{\mathbf{r}}(t)$ is commonly used in mechanics.

This definition makes use of the concept of the **limit** of a vector function. As you might expect, the limit of a vector function $\mathbf{f}(h)$ as $h \to 0$ is the vector function whose components are the limits, as $h \to 0$, of the components of $\mathbf{f}(h)$.

Now, as you will recall, velocity is defined to be rate of change of position, so the definition of the derivative of a vector function can be used to define the **velocity** of a particle with position $\mathbf{r}(t)$ as

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}.$$

The **speed** of the particle, which as you will recall is defined to be the magnitude of the velocity, is therefore given by $|\mathbf{v}(t)|$.

We sometimes consider velocity as a variable rather than as a function, and write $\mathbf{v}$ rather than $\mathbf{v}(t)$.

To find the derivative of a vector function, we make use of the following theorem.

Theorem 1

If a vector function $\mathbf{r}(t)$ has the component form

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the (fixed-direction) Cartesian unit vectors, then its derivative is given by

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \qquad (2)$$

Exercise 2

For each of the particles whose positions are given, calculate the velocity and speed of the particle at $t = 1$, correct to two decimal places.

(a) $\mathbf{r}(t) = t^2\mathbf{i} + 10t\mathbf{j}$ (b) $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$

In many situations, velocity is less important than *changes* in velocity. For example, if you are on board a train travelling at a steady speed, you may not even notice that you are moving. You will have no difficulty in, say, drinking a cup of tea. However, this operation becomes more hazardous if the driver changes the velocity of the train by putting on the brakes! Similarly, if the train goes round a bend at constant speed, you will notice the change in velocity (hot tea in your lap again!). In both cases, the rate of change of velocity is an important factor.

Now, you will recall that just as velocity is defined to be rate of change of position, so acceleration is defined to be rate of change of velocity. Therefore, along similar lines to the definition of the velocity of a particle, the **acceleration** of a particle with velocity $\mathbf{v}(t)$ is defined as

As with position and velocity, we sometimes write $\mathbf{a}$ rather than $\mathbf{a}(t)$.

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}.$$

Notice that the definition of acceleration as the rate of change of *velocity* differs from its common everyday meaning as the rate of increase of *speed*. For example, consider the train moving at constant speed around a bend mentioned above. Its speed is constant, so it is not accelerating in the everyday sense. However, it is accelerating in a mathematical sense, because its velocity is changing direction. Similarly, in the everyday sense, the braking train is not accelerating but decelerating, in that its speed is decreasing. However, in a mathematical sense it is accelerating since its velocity is changing.

It is important to understand the difference between the mathematical and everyday meanings of acceleration, and also to be able to interpret the meaning of the components of the vectors defining the position, velocity and acceleration of a particle. The following exercise should help you to do this.

Exercise 3

Three particles, A, B and C, are moving along three different straight lines. In each case, the straight line is chosen as the x-axis, so the vectors defining the position, velocity and acceleration of the three particles have only x-components. The graphs of the x-components $x(t)$ of the position of particle A, $\dot{x}(t)$ of the velocity of particle B, and $\ddot{x}(t)$ of the acceleration of particle C are shown in Figure 7. Use the graphs to answer the questions that follow.

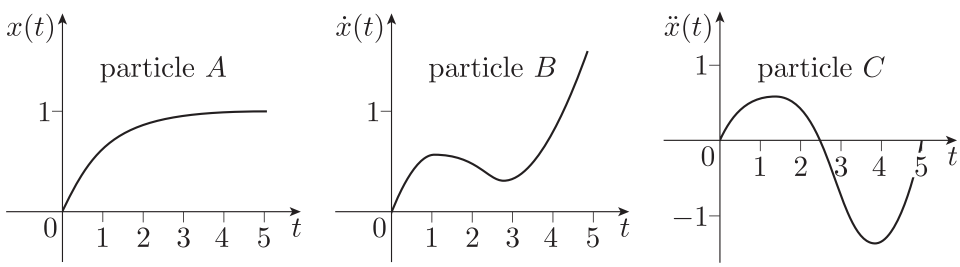


Figure 7 Graphs against t of x for particle A, $\dot{x}$ for particle B, and $\ddot{x}$ for particle C

(a) When is particle A travelling fastest?

(b) Does particle B change its direction of motion in the time interval shown?

(c) Particle C starts from rest at $x = 0$ at time $t = 0$. Is particle C momentarily stationary at any later time in the time interval shown?

(*Hint*: Think about the meaning of the area beneath a curve.)

1.2 One-dimensional motion

The ideas in Subsection 1.1 apply to motion in three-dimensional space generally. For the rest of this section and in the next three sections, we make the simplification that the motion will be in one dimension, so that the fundamental theory is not obscured.

Restriction to one-dimensional motion still allows a wide range of situations to be covered. One such situation, which is analysed fully in Section 3, is the following.

If a marble is dropped from the Clifton Suspension Bridge, how long does it take to fall into the River Avon below? What is its velocity just before it hits the water?

You met this situation in Unit 1.

To answer these questions, we need expressions for the position and velocity of the falling marble. It is reasonable to assume that the path of the marble is a straight line; so, under this assumption, although the marble is moving in three-dimensional space, this is a one-dimensional problem.

With badly chosen axes, all three components of the position vector of the marble could be changing. (Imagine fixing your axes on a car as it crossed the bridge at constant speed, or simply axes inclined to the vertical.) What makes this a one-dimensional problem is that if we choose the axes well, then only one component of the position vector is changing. Normally this axis is labelled as the x-axis, with the result that the vector has only an $\mathbf{i}$-component. Furthermore, if the situation being considered has an obvious direction of positive motion, then it is usual to choose this direction as the direction of the positive x-axis. So in the case of the Clifton Suspension Bridge problem (see Figure 8), it is sensible to choose the x-axis Ox pointing vertically downwards, so that the marble moves in the positive x-direction.

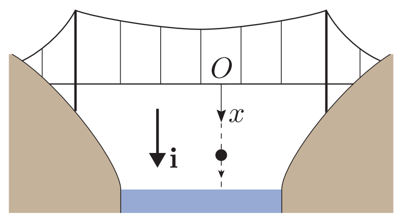


Figure 8 A marble dropped from the Clifton Suspension Bridge

For one-dimensional problems in which the x-axis corresponds to the direction of motion, the position, velocity and acceleration of the particle can be expressed as

As with $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$, we will sometimes find it convenient to write x, v and a instead of $x(t)$, $v(t)$ and $a(t)$.

$$\mathbf{r}(t) = x(t)\,\mathbf{i}, \quad \mathbf{v}(t) = \frac{dx(t)}{dt}\,\mathbf{i}, \quad \mathbf{a}(t) = \frac{d^2x(t)}{dt^2}\,\mathbf{i},$$

respectively. So for such problems, the position of the particle can be described by the function $x(t)$, the velocity by the function $v(t) = dx(t)/dt$, and the acceleration by the function $a(t) = dv(t)/dt = d^2x(t)/dt^2$.

In fact, for one-dimensional motion *along a given x-axis*, the vector quantities position, velocity and acceleration are completely described by the corresponding (scalar) functions $x(t)$, $v(t)$ and $a(t)$, respectively. For example, the magnitude of the velocity is given by $|v(t)|$, its orientation by the given x-axis, and its sense by the sign of $v(t)$. (Remember from Unit 2 that the direction of a vector comprises its orientation and its sense.) Therefore for the one-dimensional motion of an object along a given x-axis, we can – and frequently will – refer to $x(t)$ (or x), $v(t)$ (or v) and $a(t)$ (or a) as the position, velocity and acceleration of the object.

In most of the examples and exercises in this unit so far, the position of the object was known. It is more usual for the forces acting on the object to be known. As you will see in Section 2, this gives information about the acceleration of the object. So in the following examples and exercises we practise finding the position and velocity of a particle, given its acceleration.

Example 2

A particle is moving in a straight line along the x-axis. The acceleration $\mathbf{a}$ of the particle at time t is given by

$$\mathbf{a}(t) = (12t^2 + 2)\mathbf{i}.$$

Note that the 'initial' condition for a problem does not have to be at $t = 0$.

After 1 second, the particle is 3 metres from the origin and has velocity $2\,\mathrm{m\,s^{-1}}$.

(a) Find the velocity $\mathbf{v}(t)$ and the position $\mathbf{r}(t)$ of the particle.

(b) Find the velocity and position of the particle at time $t = 2$.

Solution

(a) The acceleration is known and is a one-dimensional vector function. Using the notation above, we have

$$a(t) = \mathbf{a}(t) \cdot \mathbf{i} = 12t^2 + 2.$$

In general, when trying to find the position of a particle given its acceleration, it is often easier to first find its velocity and then find its position, and thus solve two first-order differential equations, rather than to find its position directly by solving a second-order differential equation.

Since $a(t) = dv(t)/dt$, we have the first-order differential equation

$$\frac{dv}{dt} = 12t^2 + 2.$$

This equation can be solved by direct integration, giving

$$v = \int (12t^2 + 2)\,dt = 4t^3 + 2t + A,$$

where A is a constant.

To find the value of the constant A, we use the fact that $v = 2$ when $t = 1$, so $2 = 4 + 2 + A$, giving $A = -4$. This gives

$$v(t) = 4t^3 + 2t - 4.$$

Since $v(t) = dx(t)/dt$, we have the first-order differential equation

$$\frac{dx}{dt} = 4t^3 + 2t - 4.$$

Solving this equation by direct integration gives

$$x = \int (4t^3 + 2t - 4)\, dt = t^4 + t^2 - 4t + B,$$

where B is a constant.

The initial condition $x = 3$ when $t = 1$ can be used to find the constant B, so $3 = 1 + 1 - 4 + B$, giving $B = 5$. This gives

$$x(t) = t^4 + t^2 - 4t + 5.$$

So the velocity and position of the particle are

$$\mathbf{v}(t) = (4t^3 + 2t - 4)\mathbf{i}, \quad \mathbf{r}(t) = (t^4 + t^2 - 4t + 5)\mathbf{i}.$$

(b) When $t = 2$, the velocity and position have values

$$\mathbf{v}(2) = (4 \times 2^3 + 2 \times 2 - 4)\mathbf{i} = 32\mathbf{i},$$
$$\mathbf{r}(2) = (2^4 + 2^2 - 4 \times 2 + 5)\mathbf{i} = 17\mathbf{i}.$$

Hence at time 2 seconds, the particle has position 17 metres along the positive x-axis and velocity 32 metres per second in the direction of the positive x-axis.

Exercise 4

A particle is moving in a straight line along the x-axis. At time t the particle has an acceleration given by

$$\mathbf{a}(t) = (18t - 20)\mathbf{i} \quad (t \geq 0).$$

Initially, at $t = 0$, the particle has position $\mathbf{r}(0) = 7\mathbf{i}$ and velocity $\mathbf{v}(0) = 3\mathbf{i}$.

Find the position, velocity and speed of the particle at time $t = 10$.

Exercise 5

A particle is moving in a straight line along the x-axis. At time t the particle has an acceleration given by

$$\mathbf{a}(t) = ge^{-kt}\mathbf{i} \quad (t \geq 0),$$

where g and k are positive constants. Initially, at $t = 0$, the particle is at the origin ($\mathbf{r}(0) = \mathbf{0}$) and is at rest ($\mathbf{v}(0) = \mathbf{0}$).

Find the velocity and position of the particle as vector functions.

In general, an **equation of motion** is any equation relating two or more of acceleration, velocity, position and time. The rest of this subsection is devoted to problems involving the solution of equations of motion. We begin with the case of constant acceleration, which occurs frequently.

Example 3

The subscript 0 is used to distinguish the constants a_0 and v_0 from the variables a and v.

A particle moves in a straight line along the x-axis with constant acceleration $\mathbf{a}(t) = a_0\mathbf{i}$ ($a_0 \neq 0$). The particle starts from $x = x_0$ at time $t = 0$ with initial velocity $\mathbf{v}(0) = v_0\mathbf{i}$.

(a) Show that the velocity vector and position vector of the particle are given by $\mathbf{v}(t) = v(t)\,\mathbf{i}$ and $\mathbf{r}(t) = x(t)\,\mathbf{i}$, where

$$v(t) = v = v_0 + a_0 t,$$
$$x(t) = x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2.$$

(b) By eliminating t between these two equations, show that

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

Solution

(a) We have

$$a(t) = \frac{dv}{dt} = a_0,$$

which on integration yields

$$v = \int a_0\,dt = a_0 t + A,$$

where A is a constant.

The initial condition $v(0) = v_0$ gives $A = v_0$, so the velocity is given by

$$v(t) = v = v_0 + a_0 t. \qquad (3)$$

Hence

$$\frac{dx}{dt} = v_0 + a_0 t,$$

from which

$$x = \int (v_0 + a_0 t)\,dt = v_0 t + \tfrac{1}{2}a_0 t^2 + B,$$

where B is a constant.

Since the particle starts at $x = x_0$, we have $B = x_0$, so the position is given by

$$x(t) = x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2. \qquad (4)$$

(b) Rearranging equation (3) gives, since $a_0 \neq 0$,

$$t = \frac{v - v_0}{a_0}.$$

Substituting this into equation (4) yields

$$x = x_0 + v_0\left(\frac{v - v_0}{a_0}\right) + \tfrac{1}{2}a_0\left(\frac{v - v_0}{a_0}\right)^2.$$

Multiplying through by $2a_0$ and expanding the brackets gives

$$2a_0(x - x_0) = 2(v_0 v - v_0^2) + (v^2 - 2v_0 v + v_0^2) = v^2 - v_0^2,$$

which can be rearranged to give

$$v^2 = v_0^2 + 2a_0(x - x_0),$$

as required.

The results of Example 3 can be summarised as follows.

Constant acceleration

If a particle is moving in a straight line along the x-axis with constant acceleration $\mathbf{a}(t) = a_0\mathbf{i}$, and at time $t = 0$ it has initial position $\mathbf{r}(0) = x_0\mathbf{i}$ and initial velocity $\mathbf{v}(0) = v_0\mathbf{i}$, then the components of acceleration, velocity and position along the x-axis are given, respectively, by

$$a = a_0, \tag{5}$$

$$v = v_0 + a_0 t, \tag{6}$$

$$x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2. \tag{7}$$

Furthermore,

$$v^2 = v_0^2 + 2a_0(x - x_0). \tag{8}$$

We know that, by definition, $a = dv/dt = d^2x/dt^2$. There is a useful alternative expression for a that can be derived using the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\,\frac{dx}{dt} = \frac{dv}{dx}\,v.$$

Alternative expressions for $a(t)$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\,\frac{dv}{dx}. \tag{9}$$

The formula $a = v\,dv/dx$ can be used to obtain equation (8) directly, without having to find v and x first, as the following exercise asks you to demonstrate.

Exercise 6

A particle moves in a straight line along the x-axis with constant acceleration $\mathbf{a}(t) = a_0\mathbf{i}$. Initially, at time $t = 0$, the particle is at $x = x_0$ and has velocity $\mathbf{v}(0) = v_0\mathbf{i}$.

Use the relationship $a = v\,dv/dx$ to show that

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

In general, given an equation of motion relating acceleration to one or more of velocity, position and time, we want to obtain an equation relating velocity to position and/or time, or an equation relating position to time. In the case of one-dimensional motion, we can do this by using one of

$$a = \frac{dv}{dt}, \quad a = \frac{d^2x}{dt^2}, \quad a = v\,\frac{dv}{dx}$$

to substitute for a and then solving the resulting differential equation. The following exercise asks you to decide which formula for a provides the most appropriate substitution in a variety of typical cases.

Exercise 7

How would you use the above formulas to substitute for a in the following equations of motion in order to obtain the specified information? (You are not expected to solve the resulting equations.)

(a) The x-component of the equation of motion is $a = \cos t$; it is required to find velocity and position in terms of time.

(b) The x-component of the equation of motion is $a = -x$; it is required to find a relationship between velocity and position.

(c) The x-component of the equation of motion is $a = -2x - 3v + \cos t$; it is required to find position in terms of time.

The models developed in this section are used in practice as shown in the next exercise.

Exercise 8

The data in Table 2, taken from the United Kingdom *Highway Code*, show the shortest stopping distances of cars travelling along a straight road.

The thinking distance is defined to be the distance travelled by a car in the maximum time that it takes for an alert driver to react to a hazardous situation.

Table 2

Speed (mph)	Thinking distance (metres)	Braking distance (metres)	Overall stopping distance (metres)
20	6	6	12
30	9	14	23
40	12	24	36
50	15	38	53
60	18	55	73
70	21	75	96

The British Imperial unit for speed is miles per hour (mph), and $1\,\text{mph} \simeq 0.447\,\text{m}\,\text{s}^{-1}$.

(a) The data in Table 2 are not from an experiment; they are the predictions of models. Your task is to discover what models were used.

(i) What model (using SI units) was used to obtain the thinking distance data?

(*Hint*: Think about the speed of the car before and after the thinking phase, and use the constant acceleration formula $x = x_0 + v_0 t + \frac{1}{2}a_0 t^2$.)

(ii) What model (using SI units) was used to obtain the braking distance data?

(*Hint*: Think about the speed of the car before and after the braking phase, and use the constant acceleration formula $v^2 = v_0^2 + 2a_0 x$.)

(b) Use your models from part (a) to predict the overall stopping distance (in metres) for a speed of 45 mph.

2 A theory of motion

Section 1 introduced the basic concepts of position, velocity and acceleration that are needed to *describe* motion. In this section, two concepts introduced in Unit 2, *force* and *mass*, enable us to go beyond the mere description of motion and formulate laws *predicting* what motions take place.

At first sight it might seem that a different set of rules of motion would be required for each type of object – one set for tennis balls, another set for planets, and so on. Fortunately, there is a simple underlying pattern. Newton was able to see beyond individual cases, and his three laws of motion form a framework, or theory, for predicting the motion of all objects.

Newton's three laws of motion were stated in the Introduction of Unit 2.

You met the idea of friction in Unit 2. It is discussed further in Subsection 4.1 of this unit.

Our instinctive ideas about motion are shaped by the presence of friction in almost all things in our everyday lives. Aristotle (384–322 BC) constructed a theory out of this experience that turned out to be completely wrong. This theory and the subsequent development of ideas of motion led eventually to Newton's laws. Consider the following thought experiment to imagine motion without friction.

Once the toboggan has been released, the force of the push ceases to act on it.

Consider a toboggan on a horizontal icy surface such as a frozen lake. Left undisturbed, the toboggan remains static; it must be pushed or pulled in some way if it is to be set in motion, that is, a force must act on the toboggan. However, if you give the toboggan a push and then release it, the toboggan will move across the ice at almost constant speed in the direction in which it has been pushed.

This suggests that under ideal (i.e. frictionless) conditions, the following applies:

> in the absence of a force, the toboggan remains at rest or moves with constant speed in a straight line.

Air resistance is discussed in Subsection 4.2.

In real life, the toboggan does eventually slow down, partly due to air resistance and partly due to friction between the toboggan runners and the ice. In competitive tobogganing, the tobogganers go to great lengths to reduce these resistive forces (i.e. forces resisting motion) by streamlining the toboggan to reduce air resistance, and waxing the runners to reduce friction.

Exercise 9

A car on a flat, straight road requires a motive force (supplied by its engine) in order to maintain a constant speed of 70 miles per hour; if the engine is switched off, then the car slows down. It might be thought from the above example that if an object is moving with constant velocity, then there is no force acting on it. Try to explain this apparent contradiction.

Returning to the example above, suppose that you apply a force by pushing the toboggan continuously. You cannot quantify this force, but the sensations in your muscles and nerves will reveal whether you are pushing gently or firmly. From experience, you know that:

> the harder you continue to push, the further and faster the toboggan moves in a given time.

This suggests that there is a link between the force that is applied and the way in which the toboggan moves.

Exercise 10

A toboggan on an icy slope may accelerate even when it is not being pushed. Try to identify the force that causes this acceleration.

Next, imagine pushing two identical toboggans, one of which is empty while the other carries a heavy person. If you apply the same force to the two toboggans, then the laden toboggan will move more sluggishly. To achieve the same motion in each case, it is necessary to apply a greater force to the laden toboggan. In other words:

> if you apply the same force to the two toboggans, the laden toboggan does not travel as far or as fast in a given time as the empty toboggan;
>
> in order for the two toboggans to move in the same way, a greater force must be applied to the laden toboggan than to the empty toboggan.

In general, it seems that three concepts are linked together:

- the *force* that is applied to an object
- the *mass* of the object
- the *motion* of the object.

Newton proposed in his book *Principia* that this link takes the form

$$\text{force} = \text{mass} \times \text{acceleration}. \tag{10}$$

The validity of this equation is shown by its success at predicting motion – nearly the whole of nineteenth-century science and engineering rested on it! It also provides a more formal definition of **force** than was given in Unit 2.

Equation (10) can be written in the vector form

$$\mathbf{F} = m\mathbf{a},$$

where $\mathbf{F}$ is the vector quantity force, m is the scalar quantity mass, and $\mathbf{a}$ is the vector quantity acceleration. This equation is the bedrock of Newtonian mechanics, and it is usually referred to as **Newton's second law**.

Strictly speaking, the argument concerning the toboggan justifies only the statement that force is *proportional* to mass times acceleration, that is, $\mathbf{F} = km\mathbf{a}$ with k some constant of proportionality. However, in the SI system of units, k is *chosen* to be 1 by an appropriate definition of the unit of force, the *newton*, which makes use of Newton's second law.

You met the newton, though not a formal definition of it, in Unit 2.

> A force of magnitude one **newton** ($1\,\mathrm{N}$) is the force required to accelerate a mass of one kilogram at one metre per second per second (i.e. $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$). The direction of the force is the direction in which the mass accelerates.

This is a good point at which to summarise the discussion so far in this section into precise laws comprising the foundations of Newtonian mechanics.

You met one of these laws, the law of addition of vectors (forces), in Unit 2.

Fundamental laws of Newtonian mechanics

These laws concern a **particle**, which is a mathematical model for any material object whose size and internal structure may be neglected.

The **mass** of the particle is expressed by a single positive number m. This number is an inherent property of the particle and does not depend on time, position, force or any other variable.

Law of addition of mass

If an object modelled as a particle is composed of a number of parts, then the mass m of the particle is the sum of the masses of the parts.

Law of addition of forces

It is very important to take into account the directions of the individual forces as well as their magnitudes.

If several forces act simultaneously on a particle, then the resultant force is the vector sum of the individual forces.

Newton's second law

This statement of Newton's second law applies to particles of constant mass, which are the only particles studied in this module.

If a particle has a constant mass m and experiences a total force $\mathbf{F}$, then its acceleration $\mathbf{a}$ is given by

$$\mathbf{F} = m\mathbf{a}. \tag{11}$$

Newton's first law

Newton's first law is a special case of Newton's second law. It forms the basis of the equilibrium conditions in Unit 2.

When $\mathbf{F}$ is zero, $\mathbf{a}$ is zero: in the absence of a force, a particle either stays permanently at rest or moves at constant velocity, that is, at a constant speed in a straight line.

Strictly speaking, Newton's laws hold in an *inertial frame of reference* – a frame of reference (coordinate system) that is not accelerating. It is assumed throughout this module that reference frames are inertial.

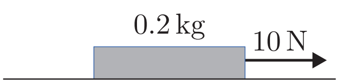


Figure 9 An object on a smooth horizontal surface

Exercise 11

If a mass of 200 grams on a smooth horizontal surface (see Figure 9) is subjected to a horizontal force of magnitude 10 newtons, what is the magnitude of the acceleration produced?

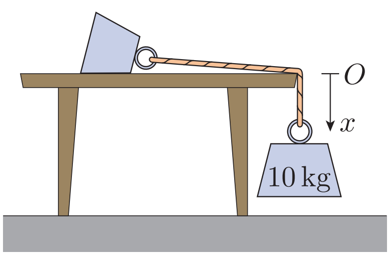


Figure 10 Two objects connected by a string

Exercise 12

An object of mass 10 kilograms is attached to a string hanging over the edge of a table, as shown in Figure 10. The other end of the string is attached to another object, on top of the table. The object hanging over the edge of the table is observed to be accelerating at $1\,\mathrm{m\,s^{-2}}$ downwards.

(a) What is the resultant force on the hanging object?

(b) Apply Newton's second law to the hanging object, and hence calculate the tension force due to the string acting on the hanging object.

Exercise 13

A fighter pilot can experience an acceleration of magnitude approximately six times the magnitude of the acceleration due to gravity before being rendered unconscious. If a fighter of mass 4000 kilograms is subjected to a force $50\,000\,\mathbf{i} + 60\,000\,\mathbf{j} + 100\,000\,\mathbf{k}$ (with magnitude in newtons) during an aerobatic manoeuvre, will the pilot remain conscious?

3 Predicting motion

Newton's second law of motion can be used to help to solve a huge variety of mechanics problems. The example below considers the motion of an object falling under gravity alone.

The steps involved in the solution are similar to those in Procedure 2 of Unit 2. As in Unit 2, the steps are highlighted by labels in the margin.

Example 4

A marble, initially at rest, is dropped from the Clifton Suspension Bridge and falls into the River Avon, 77 metres below. Assume that the bridge is fixed and the only force acting on the marble is its weight due to gravity.

In Section 4 this example will be remodelled to include air resistance.

(a) Find the time taken before the marble hits the water.

(b) Find the speed of the marble just before it hits the water.

Solution

(a) The first step is always to draw a diagram that includes all the relevant information given in the problem, as in Figure 11.

◀ Draw picture ▶

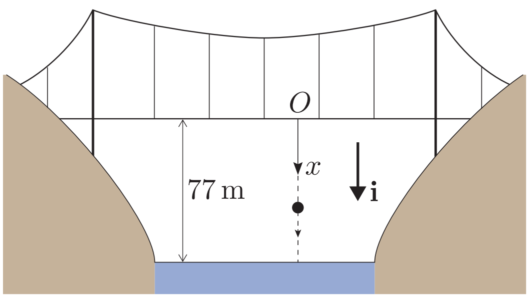


Figure 11 Sketch of the Clifton Suspension Bridge with the data marked on it

Choose an x-axis pointing vertically downwards, with its origin O at the fixed point where the object is released, as shown in Figure 11. (This makes the algebra simpler as all quantities are positive.)

◀ Choose axes ▶

The marble is modelled as a particle with its weight the only force acting on it. This is represented by the force diagram in Figure 12, in which m denotes the mass of the marble and $\mathbf{W}$ denotes its weight.

◀ State assumptions ▶

◀ Draw force diagram ▶

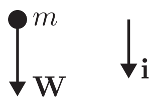


Figure 12 Force diagram for the marble

In Unit 2 the downwards vertical was usually in the $-\mathbf{j}$ direction.

The resultant force acting on the marble is $\mathbf{W} = mg\mathbf{i}$ (where, as usual, $\mathbf{i}$ is a unit vector pointing in the direction of the positive x-axis, in this case vertically downwards).

◀ Apply Newton's 2nd law ▶

The acceleration $\mathbf{a}$ is obtained by applying Newton's second law:

$$m\mathbf{a} = mg\mathbf{i}.$$

From this equation we see that only the $\mathbf{i}$-component of $\mathbf{a}$ could be varying, so we have $\mathbf{a} = a(t)\,\mathbf{i}$. So the equation becomes $ma = mg$, and dividing through by m gives

$$a = g.$$

(Notice that since m cancels out, the results apply to a marble of any mass.)

◀ Solve differential equation ▶

The constant acceleration formulas from Subsection 1.2 could be used since $a = g$ is constant, but we choose a more general method that applies even when a is not constant.

There are several approaches to obtaining the equations needed to solve the problem. One approach requires that we first replace a by dv/dt to obtain

$$\frac{dv}{dt} = g.$$

This is integrated to give

$$v = gt + A,$$

where A is a constant.

The initial condition that the marble is initially at rest (i.e. $v = 0$ when $t = 0$) gives $A = 0$, so

$$v = gt. \tag{12}$$

Now replacing v by dx/dt gives

$$\frac{dx}{dt} = gt.$$

This is integrated to obtain

$$x = \tfrac{1}{2}gt^2 + B,$$

where B is a constant.

The origin was chosen so that $x = 0$ when $t = 0$, so $B = 0$, which gives

$$x = \tfrac{1}{2}gt^2. \tag{13}$$

◀ Interpret solution ▶

When the marble hits the water, $x = 77$ and equation (13) gives $77 = \frac{1}{2}gt^2$, which on putting $g = 9.81\,\mathrm{m\,s^{-2}}$ gives

$$t = \sqrt{\frac{2 \times 77}{9.81}} \simeq 3.962.$$

So the model predicts that the marble hits the water approximately 3.96 seconds after being released.

(b) Putting $t = 3.962$ into equation (12) gives

$$v = 9.81 \times 3.962 \simeq 38.87.$$

So the model predicts that the marble has a speed of approximately 38.9 metres per second just before it hits the water.

Try the following exercise, following the same steps as given in the margin in the above example.

Exercise 14

A stone, dropped from rest, takes 3 seconds to reach the bottom of a well. Assume that the only force acting on the stone is gravity.

Use the more general approach of Example 4 rather than the constant acceleration formulas of Subsection 1.2.

(a) Estimate the depth of the well.

(b) Estimate the speed of the stone when it reaches the bottom.

The steps highlighted in Example 4 and in the solution to Exercise 14 are re-stated in the following procedure for solving mechanics problems involving one-dimensional motion using Newton's second law. Remember that it is intended to be a guide rather than a rigid set of rules.

Procedure 1 Applying Newton's second law in one dimension

Given a mechanics problem involving one-dimensional motion in which a question regarding the motion is to be answered, proceed as follows.

1. Draw a sketch of the physical situation, and annotate it with any relevant information. ◀ Draw picture ▶
2. Choose the x-axis to lie along the direction of motion, and select an origin. Mark the x-axis, its direction and the origin on your sketch. ◀ Choose axes ▶
3. State any assumptions that you make about the object and the forces acting on it. ◀ State assumptions ▶
4. Draw a force diagram. ◀ Draw force diagram ▶
5. Apply Newton's second law to obtain a vector equation. Resolve each force along the chosen axis in order to resolve the vector equation into a scalar equation. ◀ Apply Newton's 2nd law ▶
6. Substitute $v\,dv/dx$, dv/dt or d^2x/dt^2 for the acceleration a in the equation of motion, as appropriate, and solve the resulting differential equation(s) to obtain velocity v in terms of position x or time t, or position x in terms of time t, as required. ◀ Solve differential equation ▶
7. Interpret the solution in terms of the original problem. ◀ Interpret solution ▶

We make the following notes about this procedure.

- The procedure assumes that the question to be answered is given. When modelling real-world situations, deriving a suitable question is half the work. This part of the modelling process is looked at in detail in Unit 8.
- State any assumptions that you make (e.g. that the object is modelled as a particle and the only force acting on it is its weight).
- The importance of drawing a picture cannot be stressed too strongly. Include in the picture all relevant information from the problem, e.g. distances and masses.
- The choices of the origin and the direction of the x-axis are arbitrary, and will have no effect on the final outcome of your calculations. However, try to make these choices so that the position x and/or the velocity v are positive for the particle's motion, as this will simplify the algebra. If there is a clear starting point for the motion, then this is often a suitable choice for the origin, provided that it is a fixed point.

See Subsection 4.1.

- For some problems it may be necessary to choose a y-axis and even a z-axis as well as an x-axis.

You experienced the interchangeability of these steps in Unit 2.

- Steps 2 and 4 are interchangeable; the force diagram does not change with a different choice of axes. If you have difficulty choosing an x-axis, then draw the force diagram first.
- Your choice of substitution for a will depend on what question you want to answer (e.g. you may want an equation linking velocity to time, or velocity to position). (See Exercise 7 for examples of this.)
- When the acceleration is constant, you can use the general constant acceleration formulas from Subsection 1.2 (with a reference to any formula that you use) to solve the differential equation.
- Perform any readily available checks on your working, and consider whether your answers are physically reasonable. For example, you could check that the units of your answer are correct, or use common sense to tell you whether your answer is in the correct range.
- When you have finished interpreting the solution, write out your conclusion in words, and remember to include the physical units for any quantities given (as in Example 4). Also look back at the problem and check that you have fully answered the question asked.

Try using Procedure 1 in the following exercise.

Exercise 15

A ball is thrown vertically upwards from ground level with an initial speed of 10 metres per second. Assume that gravity is the only force acting on the ball.

(a) Find the time taken for the ball to reach its maximum height.

(b) Find the maximum height attained.

(c) Find the time taken for the ball to return to the ground.

(d) Find the speed of the ball as it reaches the ground on its return.

Exercise 16

A man leaning out from a window throws a ball vertically upwards from a point 4.4 metres above the ground. The initial speed of the ball is 7.6 metres per second. It travels up and then down in a straight vertical line, and eventually reaches the ground. Assume that its weight is the only force acting on the ball.

(a) Estimate the time that elapses before the ball reaches the ground.

(b) Estimate the speed of the ball when it strikes the ground.

In Example 4, which considered the motion of a marble falling from the Clifton Suspension Bridge, first the speed and then the position were found as functions of time (by using the substitutions dv/dt for a and dx/dt for v). However, as was indicated in the solution to that example, there are other approaches to solving the problem. You are asked to adopt one of these other approaches in the following exercise.

Exercise 17

A marble, initially at rest, is dropped from the Clifton Suspension Bridge and falls into the River Avon, 77 metres below. Assume that the only force acting on the marble is its weight due to gravity.

(a) By putting $a = v\,dv/dx$ in Newton's second law and solving the resulting differential equation, find the marble's speed v as a function of the distance x through which it has fallen.

(b) By putting $v = dx/dt$ in your answer to part (a), find the time t that the object takes to fall a distance x.

(c) Hence find the time taken before the marble hits the water, and the speed of the marble just before it hits the water.

In the previous examples and exercises, all of the forces acting on an object were in the same direction; but this does not have to be so for the motion to be in one dimension. This is illustrated in the following example.

Example 5

A crate of empty bottles of total mass 30 kilograms is being hauled by rope up a smooth ramp from the cellar of a pub. The ramp makes an angle of $\pi/6$ radians with the horizontal. When the crate has been hauled 2 metres up the ramp (i.e. 2 metres along the slope of the ramp), the rope suddenly breaks. It is estimated that if the crate hits the bottom of the ramp at a speed of 5 metres per second or greater, then the bottles in the crate will break.

A similar problem was considered in Unit 2. The problem is reconsidered in the presence of friction between the crate and the ramp in Section 4.

Assuming that the crate can be modelled as a particle, that there is no friction between the crate and the ramp, and that air resistance can be neglected, will the bottles break?

Solution

◀ Draw picture ▶

The situation is sketched in Figure 13.

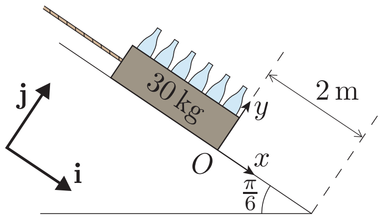


Figure 13 The crate of bottles on the ramp

◀ Choose axes ▶

The crate moves down the slope, so we choose the x-axis to point down the slope, with the origin O at the crate's position when the rope breaks. (We model the crate as a particle, so its position is at a point.) Choose a y-axis perpendicular to this, as shown in Figure 13. Also shown in the picture are the unit vectors $\mathbf{i}$ and $\mathbf{j}$ implied by this choice of axes.

◀ State assumptions ▶

The only forces on the crate are the weight $\mathbf{W}$ and the normal reaction $\mathbf{N}$ of the ramp (since we are neglecting friction and air resistance in this example). So modelling the crate as a particle of mass m, the force

◀ Draw force diagram ▶

diagram (showing also how the relevant angles are calculated) is as shown in Figure 14.

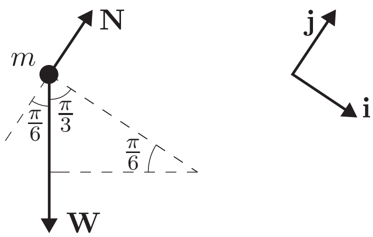


Figure 14 The force diagram

◀ Apply Newton's 2nd law ▶

As we have done elsewhere, we use the letter m rather than the value 30 kg for the mass since, as will be discussed in Unit 8, inserting actual data values too early in the modelling process can obscure features of the resulting model.

Since there is no motion in the $\mathbf{j}$-direction, we do not need to resolve in this direction. If we did, we would have

$$|\mathbf{N}| - mg\tfrac{\sqrt{3}}{2} = 0.$$

Now, Newton's second law for this system gives

$$m\mathbf{a} = \mathbf{W} + \mathbf{N}. \qquad (14)$$

From the force diagram, $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$. The weight can be resolved into components using the formula from Unit 2, Example 9:

$$\mathbf{W} = |\mathbf{W}|\cos\tfrac{\pi}{3}\,\mathbf{i} - |\mathbf{W}|\sin\tfrac{\pi}{3}\,\mathbf{j} = \tfrac{1}{2}|\mathbf{W}|\,\mathbf{i} - \tfrac{\sqrt{3}}{2}|\mathbf{W}|\,\mathbf{j}.$$

The motion is along the slope, so $\mathbf{a} = a\mathbf{i}$. (All we are saying here is that there is no resultant force in the $\mathbf{j}$-direction.) Now that we have all the vectors in component form, we can immediately resolve equation (14) in the $\mathbf{i}$-direction:

$$ma = \tfrac{1}{2}|\mathbf{W}| + 0.$$

Substituting $|\mathbf{W}| = mg$ gives $ma = \frac{1}{2}mg$, so

$$a = \tfrac{1}{2}g.$$

Again m cancels out, so the results apply to a crate of any mass.

We want an equation for the velocity v in terms of the distance travelled x. Since the acceleration is constant, we can use equation (8), that is,

◀ Solve differential equation ▶

$$v^2 = v_0^2 + 2ax.$$

This gives the final velocity in terms of known quantities: the initial velocity v_0 is $0\,\mathrm{m\,s^{-1}}$, the distance travelled x is $2\,\mathrm{m}$, and the constant acceleration is $a = \frac{1}{2}g$.

Putting the data into the equation gives

◀ Interpret solution ▶

$$v^2 = 0^2 + 2 \times \tfrac{1}{2}g \times 2 \simeq 19.62 \quad \text{to 2 d.p.}$$

So $v \simeq 4.4$ to one decimal place, which means that the crate is travelling at about 4.4 metres per second when it hits the bottom of the ramp. This speed is just below the estimated speed at which the bottles will break, so – provided that the estimate of the speed at which the bottles will break is a good one – this model predicts that the bottles will not break.

Exercise 18

A skier of mass 65 kilograms starts from rest at the top of a 120-metre ski slope (i.e. 120 metres is the vertical distance from top to bottom), as shown in Figure 15.

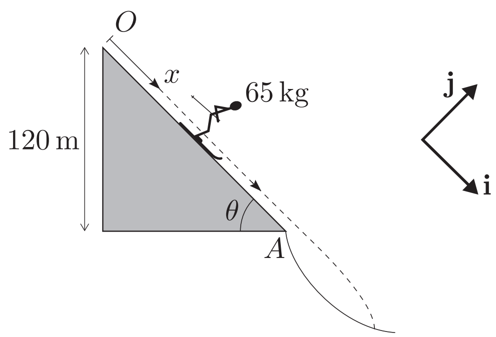


Figure 15 The skier on the ski slope

The skier travels down the slope before leaving the ground and then landing some distance further down the hill. Use the x-axis shown, which points down the slope with the origin O at the top of the slope and point A at the bottom of the slope. Let θ be the angle that the slope makes with the horizontal. Assume that friction and air resistance can be neglected.

This problem is reconsidered in the presence of friction in Section 4.

(a) If $\theta = \frac{\pi}{4}$, what is the speed of the skier at point A at the bottom of the slope?

(b) If $\theta = \frac{\pi}{3}$, what is the speed of the skier at point A at the bottom of the slope?

4 Some more force models

In Section 3 we described a procedure for predicting the motion of a particle using Newton's second law. To use the procedure, we must first model every force acting on the particle. In this section, models of forces due to friction and air resistance are described. This greatly extends the range of problems that can be solved using the procedure.

4.1 Friction

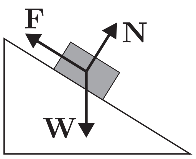


Figure 16 The forces acting on a block at rest on a slope

In Unit 2 you encountered a model for the friction force that acts on an object at rest in contact with a surface. This force $\mathbf{F}$ has magnitude less than or equal to $\mu|\mathbf{N}|$, that is, $|\mathbf{F}| \leq \mu|\mathbf{N}|$, where $\mathbf{N}$ is the normal reaction force and μ is the coefficient of static friction between the object and the surface. The direction of the friction force opposes any possible motion (e.g. is up the slope in Figure 16).

The names *coefficient of dynamical friction* or *coefficient of kinetic friction* are sometimes used instead of the name *coefficient of sliding friction*.

The coefficient μ is called the coefficient of *static* friction to distinguish it from the **coefficient of sliding friction**, denoted by μ', that is used when an object is moving along a surface. In this situation, experiments show that the magnitude of the friction force $\mathbf{F}$ is equal to $\mu'|\mathbf{N}|$, that is, $|\mathbf{F}| = \mu'|\mathbf{N}|$, where $\mathbf{N}$ is the normal reaction force and μ' is the coefficient of sliding friction; the direction of the friction force is opposite to the direction of motion.

Modelling sliding friction

- The friction force $\mathbf{F}$ acts in a direction perpendicular to the normal reaction $\mathbf{N}$ and opposite to the motion.
- $|\mathbf{F}| = \mu'|\mathbf{N}|$, where μ' is the coefficient of sliding friction for the two surfaces involved.

The numerical value of the coefficient of sliding friction μ' is always smaller than the numerical value of the coefficient of static friction μ. (It is harder to get objects moving than to keep them moving.) A generalisation that is often useful is

$$\mu' \simeq \tfrac{3}{4}\mu.$$

The use of the sliding friction model is best explained by an example.

Example 6

Consider again the problem of Example 5 concerning the crate of empty bottles sliding down a cellar ramp, but this time assume that there is friction between the crate and the ramp, with coefficient of sliding friction $\mu' = 0.15$.

(a) Estimate the speed of the crate when it reaches the bottom of the ramp. Will the bottles break?

(b) Compare the answer to part (a) with the answer to Example 5, and comment.

In Example 5 it was assumed the bottles would break if the speed at the bottom of the ramp exceeded $5\,\mathrm{m\,s^{-1}}$.

Solution

(a) The situation is sketched in Figure 17.

◀ Draw picture ▶

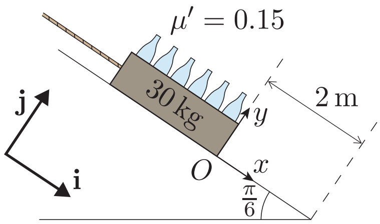


Figure 17 The crate of bottles on the ramp

Choose the x-axis parallel to the ramp and the y-axis perpendicular to it, as shown in Figure 17. Choose the origin O to be the crate's position when the rope breaks.

◀ Choose axes ▶

The crate is modelled as a particle. The forces acting on the crate are its weight $\mathbf{W}$, the normal reaction of the ramp on the crate $\mathbf{N}$, and the friction force between the ramp and the crate $\mathbf{F}$, giving the force diagram in Figure 18.

◀ State assumptions ▶

◀ Draw force diagram ▶

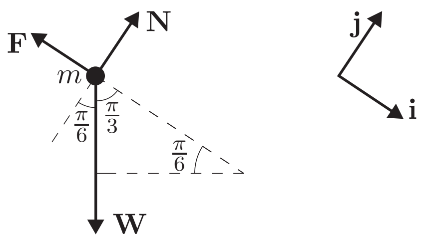


Figure 18 The force diagram

Using m for the mass of the crate (as in Example 5), applying Newton's second law to this system gives

◀ Apply Newton's 2nd law ▶

$$m\mathbf{a} = \mathbf{W} + \mathbf{N} + \mathbf{F}. \tag{15}$$

From the force diagram, $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$ and $\mathbf{F} = -|\mathbf{F}|\,\mathbf{i}$. It is also apparent that the motion is one-dimensional parallel to the x-axis, that is, $\mathbf{a} = a\mathbf{i}$. The weight is resolved into components as

$$\begin{aligned}\mathbf{W} &= |\mathbf{W}|\cos\tfrac{\pi}{3}\,\mathbf{i} - |\mathbf{W}|\sin\tfrac{\pi}{3}\,\mathbf{j}\\ &= \tfrac{1}{2}mg\,\mathbf{i} - \tfrac{\sqrt{3}}{2}mg\,\mathbf{j}.\end{aligned}$$

Now we can immediately resolve equation (15) in the $\mathbf{i}$-direction to obtain

$$ma = \tfrac{1}{2}mg + 0 - |\mathbf{F}|.$$

Using the friction model $|\mathbf{F}| = \mu'|\mathbf{N}|$ gives

$$ma = \tfrac{1}{2}mg - \mu'|\mathbf{N}|. \tag{16}$$

So to find a, we need to find $|\mathbf{N}|$. To do this, we resolve equation (15) in the $\mathbf{j}$-direction to obtain

$$0 = -\tfrac{\sqrt{3}}{2}mg + |\mathbf{N}| + 0,$$

which gives $|\mathbf{N}| = \frac{\sqrt{3}}{2}mg$. Substituting this into equation (16) gives

$$ma = \tfrac{1}{2}mg - \tfrac{\sqrt{3}}{2}\mu' mg,$$

so

$$a = \tfrac{1}{2}g - \tfrac{\sqrt{3}}{2}\mu' g.$$

◀ Solve differential equation ▶

As in Example 5, the acceleration is constant, so we can use

$$v^2 = v_0^2 + 2ax.$$

This gives the final speed in terms of known quantities: the initial velocity $v_0 = 0$, the distance travelled $x = 2$, and the constant acceleration $a = \frac{1}{2}g - \frac{\sqrt{3}}{2}\mu' g$, where $\mu' = 0.15$.

◀ Interpret solution ▶

Putting the data into the equation gives

$$v^2 = 0^2 + 2 \times (\tfrac{1}{2}g - \tfrac{\sqrt{3}}{2} \times 0.15 \times g) \times 2 \simeq 14.52 \quad \text{to 2 d.p.}$$

So $v \simeq 3.8$ to one decimal place, which means that the crate is travelling at about 3.8 metres per second when it hits the bottom of the ramp. This speed is well below the estimated speed at which the bottles will break, so this model predicts that the bottles will not break.

(b) The model that neglected friction (Example 5) predicted a speed of $4.4\,\mathrm{m\,s^{-1}}$. The new prediction of $3.8\,\mathrm{m\,s^{-1}}$ is slower, as should be expected after the incorporation of friction into the model (this is a good quick check of the solution), and is significantly slower than the breaking threshold of $5\,\mathrm{m\,s^{-1}}$. So in this case, even if the estimated breaking threshold is not very accurate, we can be reasonably confident that our prediction that the bottles will not break is correct.

Table 3 shows some values of the coefficient of sliding friction that may be useful in problems involving sliding objects.

Table 3 Coefficients of sliding friction

Object	Surface	μ'
Waxed ski	Dry snow	0.03
Brass	Ice	0.02
Vulcanised rubber	Dry tarmac	1.07
Vulcanised rubber	Wet tarmac	0.95

Exercise 19

A tip-up truck is delivering a concrete block to a building site. The driver increases the angle of tip of the carrier until the concrete block begins to slide, then keeps the carrier at this constant angle. The coefficient of static friction between a concrete block and metal is approximately $\mu = 0.4$, and the coefficient of sliding friction is approximately $\mu' = 0.3$.

(a) Calculate the angle at which the concrete block begins to slide.

(b) If the concrete block has 3 metres to travel before leaving the carrier, how long will it take to unload it?

Exercise 20

Repeat Exercise 18 under the new modelling assumption that friction cannot be neglected; the coefficient of sliding friction between waxed skis and dry snow is 0.03. Compare your answers here with those for Exercise 18, and comment.

4.2 Air resistance

In many situations it is adequate to treat a falling object taking into account only the force of gravity. For example, we predicted a time of fall of a marble from the Clifton Suspension Bridge of 3.96 seconds, which is close to an experimental value of 4.1 seconds. Most of the time difference can be attributed to air resistance, which is modelled in this subsection.

The idea of air resistance is quite familiar in everyday life. For example, any experienced cyclist knows the following.

- Air resistance tends to slow one down and resists one's attempts to increase speed.
- At low speeds air resistance has little effect, but at higher speeds it becomes more noticeable, making it difficult to cycle faster than about 40 kilometres per hour.
- Air resistance can be reduced by crouching over the handlebars to present a smaller profile to the wind.

From these observations we conclude that air resistance is a force $\mathbf{R}$ whose direction is opposite to that of the motion of an object and whose magnitude $|\mathbf{R}|$ depends on the object's speed, shape and size.

To simplify the discussion, we will restrict our attention to smooth spherical objects. For such an object, we would expect $|\mathbf{R}|$ to increase as the object's speed $|\mathbf{v}|$ and diameter D increase. In fact, experiments show that the force on a sphere depends only on the product $D|\mathbf{v}|$. The results of such experiments are shown as the solid black line in Figure 19 (where, because of the wide range of values, a log–log graph has been used).

Notice the 'kink' in Figure 19 when $\log_{10}(D|\mathbf{v}|)$ is between 0 and 1: in this region there is less air resistance as the sphere travels faster! The physical reason for this is that a highly chaotic layer of air develops around the sphere.

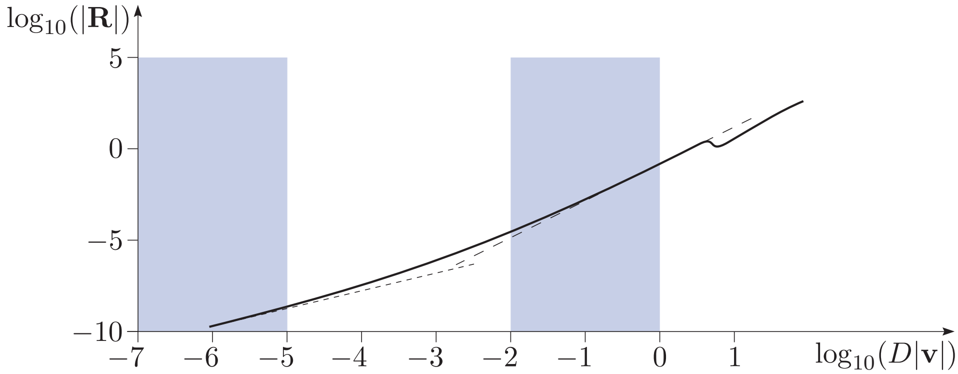


Figure 19 The experimental data for air resistance together with broken lines showing the two simplest models and their ranges of validity (shaded)

The graph has a complicated shape, so the air resistance force is modelled by a complicated function of $D|\mathbf{v}|$. However, there are two simple models that fit the experimental data for wide ranges of situations, namely the linear and quadratic models

$$|\mathbf{R}| = c_1 D|\mathbf{v}| \quad \text{and} \quad |\mathbf{R}| = c_2 D^2|\mathbf{v}|^2,$$

where c_1 and c_2 are positive constants. The best fit of these models to the experimental data (when using SI units) is given by $c_1 = 1.7 \times 10^{-4}$ and $c_2 = 0.20$ over certain ranges:

The symbol $\lesssim$ means 'less than about'.

$$\begin{aligned} &|\mathbf{R}| \simeq 1.7 \times 10^{-4} D|\mathbf{v}| && \text{for } D|\mathbf{v}| \lesssim 10^{-5}; \\ &|\mathbf{R}| \simeq 0.2 D^2|\mathbf{v}|^2 && \text{for } 10^{-2} \lesssim D|\mathbf{v}| \lesssim 1. \end{aligned}$$

These approximations for air resistance are shown as broken lines in Figure 19, together with their ranges of validity (shaded).

Consider a sphere of given diameter D. From Figure 19 it can be seen that the linear model applies for low velocities, and the quadratic model applies for an intermediate range of velocities. Notice also that there are large ranges of velocities where neither of these simple models applies.

Note that the vector and scalar statements of the quadratic air resistance model agree since

$$\begin{aligned} \left|-c_2 D^2|\mathbf{v}|\mathbf{v}\right| &= c_2 D^2 |\mathbf{v}|\,|\mathbf{v}| \\ &= c_2 D^2 |\mathbf{v}|^2. \end{aligned}$$

The air resistance force $\mathbf{R}$ always opposes the motion, that is, it is in the direction of $-\mathbf{v}$. So the vector equations for the air resistance models given above are

$$\mathbf{R} = -c_1 D\mathbf{v} \quad \text{and} \quad \mathbf{R} = -c_2 D^2|\mathbf{v}|\mathbf{v}.$$

Air resistance

The air resistance force $\mathbf{R}$ on a smooth spherical object of diameter D travelling with velocity $\mathbf{v}$ can be modelled as follows:

$$\begin{aligned} &\text{linear model} && \mathbf{R} = -c_1 D\mathbf{v} && \text{for } D|\mathbf{v}| \lesssim 10^{-5}, && (17) \\ &\text{quadratic model} && \mathbf{R} = -c_2 D^2|\mathbf{v}|\mathbf{v} && \text{for } 10^{-2} \lesssim D|\mathbf{v}| \lesssim 1, && (18) \end{aligned}$$

where $c_1 \simeq 1.7 \times 10^{-4}$ and $c_2 \simeq 0.20$.

These models can be also used for other fluids, with different ranges of applicability. For example, in water we have $c_1 = 9.4 \times 10^{-3}$ for $D|\mathbf{v}| \lesssim 10^{-6}$ and $c_2 = 156$ for $10^{-3} \lesssim D|\mathbf{v}| \lesssim 10^{-1}$.

The use of these models is illustrated by examples, the first of which adds air resistance to Example 4.

Example 7

In Example 4, the falling time for a marble dropped from the Clifton Suspension Bridge into the River Avon was calculated to be 3.96 seconds. The experimental value is 4.1 seconds. Can the discrepancy be accounted for by a linear air resistance model?

The investigation is subdivided into the following steps.

(a) Find how, under a linear air resistance model, the distance from the point of release varies with time for an arbitrary spherical object of mass m and diameter D.

(b) Calculate the time, under a linear air resistance model, that a marble of diameter 2 cm and mass 13 g takes to fall the 77 m from the Clifton Suspension Bridge into the River Avon below.

(c) Comment on the validity of the linear air resistance model for this problem.

Solution

(a) The picture is the same as in Example 4 – see Figure 11. ◀ Draw picture ▶

The x-axis points vertically downwards, with its origin O at the point where the marble is released. The unit vector $\mathbf{i}$ acts downwards. ◀ Choose axes ▶

There are two forces acting on the marble, namely its weight due to gravity downwards and air resistance upwards, as shown in the force diagram in Figure 20. ◀ State assumptions ▶ ◀ Draw force diagram ▶

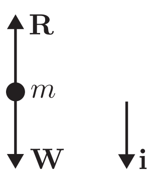


Figure 20 The force diagram

Applying Newton's second law to the marble gives ◀ Apply Newton's 2nd law ▶

$$m\mathbf{a} = \mathbf{W} + \mathbf{R}. \tag{19}$$

From the force diagram, $\mathbf{W} = |\mathbf{W}|\,\mathbf{i} = mg\mathbf{i}$. Using the linear air resistance model, we have $\mathbf{R} = -c_1 D\mathbf{v} = -c_1 Dv\mathbf{i}$, since $\mathbf{v} = v\mathbf{i}$ with $v \geq 0$. The acceleration is also downwards, so $\mathbf{a} = a\mathbf{i}$. Now we can resolve equation (19) in the $\mathbf{i}$-direction to obtain

$$ma = mg - c_1 Dv.$$

◀ Solve differential equation ▶

Notice that the acceleration is not constant here, so we cannot use the constant acceleration formulas.

We want the distance x in terms of the time t. One way of obtaining this is to first use the substitution $a = dv/dt$, and then substitute dx/dt for v later. So we have

$$m\,\frac{dv}{dt} = mg - c_1Dv.$$

This assumption is checked later. It is certainly true initially as the marble is dropped from rest.

The integrating factor method could also be used.

We now make the assumption that $mg - c_1Dv > 0$ (i.e. the air resistance never becomes so strong as to overcome the marble's weight). Under this assumption, the differential equation can be solved by the separation of variables method, to obtain

$$\int \frac{m}{mg - c_1Dv}\,dv = \int 1\,dt,$$

which, on dividing the numerator and denominator by m, and for convenience writing $k = C_1D/m$, gives

$$\int \frac{1}{g - kv}\,dv = \int 1\,dt.$$

Note that $g - kv > 0$ if $mg - c_1Dv > 0$.

Integrating, using the assumption that $g - kv > 0$, leads to

$$-\frac{1}{k}\ln(g - kv) = t + A, \qquad (20)$$

where A is an arbitrary constant. Rearranging gives

$$g - kv = e^{(-kt-kA)} = Be^{-kt},$$

where $B = e^{-kA}$ is another constant. The initial condition that the marble is initially at rest (i.e. $v = 0$ when $t = 0$) gives $B = g$, so

$$v = \frac{g}{k}(1 - e^{-kt}). \qquad (21)$$

Since c_1, D, t and m are all greater than or equal to zero, so that $kt > 0$, this equation predicts that $v < g/k = mg/c_1D$ for all t, which is consistent with the assumption made in obtaining equation (20).

Writing dx/dt for v gives

$$\frac{dx}{dt} = \frac{g}{k}(1 - e^{-kt}).$$

Integrating directly gives

$$x = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} + C,$$

where C is a constant.

The initial condition $x = 0$ when $t = 0$ gives $C = -g/k^2$, so

$$x = \frac{g}{k}t - \frac{g}{k^2}(1 - e^{-kt}), \qquad (22)$$

where $k = c_1D/m$.

Since $e^{-kt} \to 0$ as $t \to \infty$, the speed v tends to g/k, and the position x tends to $gt/k - g/k^2 \to gt/k$.

◀ Interpret solution ▶

(b) We want to find t given $x = 77$. To do this, we could try to rearrange equation (22) to give t in terms of x; however, no such rearrangement is possible. But we can use a numerical method. First we use the given data to interpret equation (22) in the context of the current problem.

We have

$$k = \frac{c_1 D}{m} = \frac{1.7 \times 10^{-4} \times 0.02}{0.013} \simeq 2.6 \times 10^{-4},$$

so

$$x \simeq \frac{9.81}{2.6 \times 10^{-4}}\, t - \frac{9.81}{(2.6 \times 10^{-4})^2}\left(1 - e^{-2.6 \times 10^{-4} \times t}\right). \qquad (23)$$

Now we already have two values for the time taken for the marble to fall 77 metres: 3.96 seconds estimated in Example 4, and 4.1 seconds given by experiment. Substituting these values for t into equation (23), we obtain

$$x(3.96) \simeq 76.89 \quad \text{and} \quad x(4.1) \simeq 82.42.$$

So it looks like $t = 3.96$ is close to the solution of equation (23) for $x = 77$.

So $t = 3.96$ (to two decimal places) when $x = 77$, that is, the marble takes about 3.96 seconds to reach the River Avon under the linear air resistance model. This is exactly the same value, to two decimal places, as for the model without air resistance (Example 4).

More detailed calculation shows that the linear air resistance model predicts that the marble will hit the water approximately one thousandth of a second later than it will in the case of the model without air resistance.

(c) A condition for the linear air resistance model to be valid is that the product of the diameter of the marble and its speed is less than about 10^{-5}. Using equation (21) gives $|\mathbf{v}(3.96)| \simeq 38.8$, so $D|\mathbf{v}| \simeq 0.78$, which is much greater than $10^{-5}\,\mathrm{m\,s^{-1}}$. So the linear air resistance model is not appropriate.

In fact, $D|\mathbf{v}(t)| \lesssim 10^{-5}$ for $t \lesssim 5.1 \times 10^{-5}$, so the linear model applies for only about the first 51 microseconds of the motion of the marble.

From Example 7 it seems that the linear air resistance model may apply only to objects moving very slowly – not to the speeds experienced in everyday life. So our attention turns to the quadratic air resistance model. In general, the differential equations that arise from this model are harder to solve than the ones that arise from the linear model. However, some of the differential equations are easily soluble, as the following example shows.

Example 8

Revisit the Clifton Suspension Bridge problem in Example 7 using the quadratic air resistance model.

(a) Derive an expression for the marble's velocity in terms of its position.

(b) Use the expression derived in part (a) to estimate the speed of the marble just before it hits the water.

(c) Is the quadratic air resistance model valid for this problem?

Solution

(a) Everything is the same as in Example 7 up to the point where we apply Newton's second law to the marble, to obtain

◀ Apply Newton's 2nd law ▶

$$m\mathbf{a} = \mathbf{W} + \mathbf{R}. \qquad (24)$$

As before, $\mathbf{W} = mg\mathbf{i}$, but now we use the quadratic air resistance model $\mathbf{R} = -c_2D^2|\mathbf{v}|\mathbf{v}$. Since the motion is downwards, $\mathbf{v} = v\mathbf{i}$ with $v > 0$ and hence $|\mathbf{v}| = v$, so $\mathbf{R} = -c_2D^2v^2\mathbf{i}$. Resolving equation (24) in the **i**-direction gives

$$ma = mg - c_2D^2v^2.$$

◀ Solve differential equation ▶

The question asks for v in terms of x, so we use the substitution $a = v\,dv/dx$ to obtain

This equation is non-linear in v thus cannot be solved by the integrating factor method.

$$mv\frac{dv}{dx} = mg - c_2D^2v^2.$$

This assumption is checked later.

We now make the assumption that $mg - c_2D^2v^2 > 0$ (i.e. the air resistance never becomes so strong as to overcome the marble's weight). Under this assumption, the differential equation can be solved by separation of variables,

$$\int \frac{mv}{mg - c_2D^2v^2}\,dv = \int 1\,dx,$$

which is equivalent to

$$\int \frac{v}{g - kv^2}\,dv = \int 1\,dx,$$

Note that $g - kv^2 > 0$ if $mg - c_2D^2v^2 > 0$, as assumed. The case where $g - kv^2 < 0$ is considered later.

where $k = c_2D^2/m$. Integrating, using the assumption that $g - kv^2 > 0$, and noting that

$$\frac{d}{dv}(g - kv^2) = -2kv,$$

so the integral is of the form

$$-\frac{1}{2k}\int \frac{f'(v)}{f(v)}\,dv,$$

we have

$$-\frac{1}{2k}\ln(g - kv^2) = x + A,$$

where A is an arbitrary constant. So

$$g - kv^2 = e^{(-2kx-2kA)} = Be^{-2kx},$$

where $B = e^{-2kA}$ is another constant. The initial condition that the marble is initially at rest at the origin (i.e. $v = 0$ when $x = 0$) gives $B = g$, so

$$g - kv^2 = ge^{-2kx}, \quad \text{or equivalently,} \quad v^2 = \frac{g}{k}(1 - e^{-2kx}).$$

Note that c_2, D, x and m are all greater than or equal to zero, so $0 < \exp(-2kx) \leq 1$, and equation (25) predicts that $v < \sqrt{g/k} = \sqrt{mg/c_2D^2}$ for all t, which is consistent with the assumption made earlier.

Therefore

$$v = \sqrt{\frac{g}{k}}\sqrt{1 - e^{-2kx}}, \tag{25}$$

where $k = c_2D^2/m$ and we take the positive square root since we must have $v \geq 0$ from the description of the problem.

Since $e^{-2kx} \to 0$ as $x \to \infty$, the speed v tends to $\sqrt{g/k}$.

(b) Substituting the values for the marble (see Example 7) gives

◀ Interpret solution ▶

$$k = \frac{c_2 D^2}{m} = \frac{0.2 \times (0.02)^2}{0.013} \simeq 6.2 \times 10^{-3},$$

so from equation (25),

$$v = \sqrt{\frac{9.81}{6.2 \times 10^{-3}}}\sqrt{1 - e^{-2\times 6.2\times 10^{-3}\times 77}} \simeq 31.2.$$

So the model predicts that the speed of the marble is about 31 metres per second just before it hits the water.

(c) To test whether the quadratic air resistance model applies to the motion, we must calculate $D|\mathbf{v}|$ and check that it is in the range 0.01 to 1. At the end of the motion, $D|\mathbf{v}(77)| \simeq 0.62$, which is within the limits of applicability of the model. The motion starts from rest, and $D|\mathbf{v}(0)| = 0$ is obviously outside the limits of the model. However, $D|\mathbf{v}(0.01)| \simeq 0.01$, so after the marble has dropped approximately one centimetre, the quadratic model applies. So the quadratic model applies for almost all of the motion.

Admittedly, this calculation has been done using the quadratic model, but it can't be far out!

The given air resistance models apply only to smooth spherical objects. To apply these models to other objects, we have to model the objects as smooth spheres. The diameter of the sphere used to model an object is referred to as the **effective diameter** of the object. The following exercise makes use of this idea.

The determination of an effective diameter for an object is beyond the scope of this module.

Exercise 21

In the discussion leading up to the statement of Newton's second law in Section 2, the motion of an empty toboggan sliding on ice was considered. In this exercise, the effect of air resistance on the motion is examined.

Assume that the quadratic air resistance model applies, and that the effective diameter of the toboggan is 5 centimetres. If the toboggan has mass 2 kilograms and an initial speed of 2 metres per second, by what percentage has its speed diminished after it has travelled 100 metres?

Terminal speed

The general equation for the velocity $\mathbf{v} = v\mathbf{i}$ of an object falling from rest under gravity, ignoring the effects of air resistance, was derived in Example 4 as

$$v = gt.$$

This predicts that v increases indefinitely as t increases.

The general equation for the velocity $\mathbf{v} = v\mathbf{i}$ of an object falling from rest under gravity with linear air resistance was derived in Example 7 to be

$$v = \frac{g}{k}(1 - e^{-kt}),$$

where $k = c_1 D/m$. The exponential term decreases to zero as time increases, so

$$v \to \frac{g}{k} = \frac{mg}{c_1 D} \quad \text{as } t \to \infty.$$

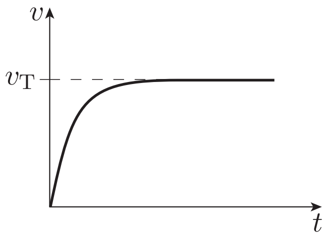


Figure 21 Velocity graph

This behaviour, which is quite different from the case of the model that neglects air resistance, is illustrated in Figure 21. The limiting value is called the **terminal speed** of the object and is denoted by v_{T}. So in the case of the linear air resistance model, the terminal speed (in SI units) is

$$v_{\mathrm{T}} = \frac{g}{k} = \frac{mg}{c_1 D}.$$

The quadratic air resistance model predicts results that are qualitatively similar to the linear case. The general equation for the speed of an object falling from rest under gravity with quadratic air resistance was derived in Example 8 to be

$$v = \sqrt{\frac{g}{k}}\sqrt{1 - e^{-2kx}},$$

where $k = c_2 D^2/m$. The exponential term decreases to zero as x increases, so the quadratic model predicts a terminal speed (in SI units) given by

$$v_{\mathrm{T}} = \sqrt{\frac{g}{k}} = \sqrt{\frac{mg}{c_2 D^2}}.$$

There is another way of looking at the concept of terminal speed. The equation of motion of an object falling from rest under gravity with linear air resistance is $ma = mg - c_1 D v$; thus $v_{\mathrm{T}} = mg/c_1 D$ is exactly that value of v for which $a = 0$. The equation of motion of an object falling from rest under gravity with quadratic air resistance is $ma = mg - c_2 D^2 v^2$; again $v_{\mathrm{T}} = \sqrt{mg/c_2 D^2}$ is such that $a = 0$. Thus in each case the terminal speed is the speed at which the object can fall without accelerating, or in other words at which the air resistance just balances the object's weight. This way of looking at terminal speed enables us to derive equations for the terminal speed of any falling object, irrespective of whether or not the object is falling from rest, as the following example illustrates.

See Example 7.

See Example 8.

Example 9

A small spider is blown by the wind into the high atmosphere and, once the wind has died away, falls back to Earth. The spider may be modelled as a sphere of effective diameter 2 millimetres.

If the spider has mass 0.004 grams, calculate the speed at which it will land, assuming that the fall is long enough for the spider effectively to reach terminal speed.

Solution

The first step is to draw a picture and mark the x-axis on it, then draw the force diagram (see Figure 22).

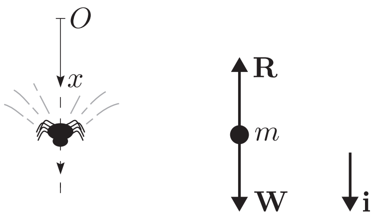


Figure 22 The spider descending to Earth, and the force diagram

When the spider is travelling at its terminal speed, it is not accelerating. Therefore by Newton's second law the resultant force on the spider must be zero, that is,

$$\mathbf{W} + \mathbf{R} = \mathbf{0}. \tag{26}$$

Suppose that the linear air resistance model is valid.

The force models are $\mathbf{W} = mg\mathbf{i}$ and $\mathbf{R} = -c_1 D\mathbf{v}_\mathrm{T}$. The motion is downwards, so $\mathbf{v}_\mathrm{T} = v_\mathrm{T}\mathbf{i}$. Resolving equation (26) in the $\mathbf{i}$-direction gives

$$mg - c_1 D v_\mathrm{T} = 0,$$

so

$$v_\mathrm{T} = \frac{mg}{c_1 D} = \frac{4 \times 10^{-6} \times 9.81}{1.7 \times 10^{-4} \times 0.002} \simeq 115.$$

So the terminal speed of the spider under the linear air resistance model is about 115 metres per second. This gives $D|\mathbf{v}_\mathrm{T}| \simeq 0.2$, which is much greater than 10^{-5}, so the linear air resistance model is not valid.

Now suppose that the quadratic air resistance model holds.

The weight is $\mathbf{W} = mg\mathbf{i}$, as before. The air resistance force is now given by the equation $\mathbf{R} = -c_2 D^2 |\mathbf{v}_\mathrm{T}|\mathbf{v}_\mathrm{T}$. As before, the motion is downwards, so $\mathbf{v}_\mathrm{T} = v_\mathrm{T}\mathbf{i}$ and $\mathbf{R} = -c_2 D^2 v_T^2 \mathbf{i}$. Resolving equation (26) in the $\mathbf{i}$-direction gives

$$mg - c_2 D^2 v_T^2 = 0,$$

which can be rearranged to obtain

$$v_\mathrm{T} = \sqrt{\frac{mg}{c_2 D^2}} = \sqrt{\frac{4 \times 10^{-6} \times 9.81}{0.2 \times (0.002)^2}} \simeq 7.$$

So the terminal speed of the spider under the quadratic air resistance model is about 7 metres per second. This gives $D|\mathbf{v}_T| \simeq 0.014$, which is in the range of validity for the quadratic model. So the quadratic air resistance model is valid and gives a terminal speed of about 7 metres per second.

The results about the terminal speed of objects falling under the influence of gravity and air resistance alone are summarised in the following box.

Terminal speed under air resistance

The terminal speed of an object falling under gravity and air resistance is the constant velocity that the object will acquire as time tends to infinity, and is the velocity at which air resistance just balances the object's weight. For an object of mass m and effective diameter D it is given (using SI units) as follows:

under the linear air resistance model $$v_{\mathrm{T}} = \frac{mg}{c_1 D}, \tag{27}$$

under the quadratic air resistance model $$v_{\mathrm{T}} = \sqrt{\frac{mg}{c_2 D^2}}, \tag{28}$$

where g is the magnitude of the acceleration due to gravity, $c_1 \simeq 1.7 \times 10^{-4}$ and $c_2 \simeq 0.20$.

The terminal speed of an object is an important concept that is often the only quantity of interest in a problem including air resistance. For example, it is often assumed that the landing speed of a parachute is essentially its terminal speed, so considering terminal speed is a crucial aspect of the design of parachutes.

Exercise 22

A parachutist of mass 65 kilograms has a parachute of effective diameter 10 metres when fully opened. Estimate the landing speed of the parachutist, assuming that the parachute jump is long enough for the terminal speed effectively to be reached.

Exercise 23

The maximum speed at which a parachutist can land safely is about 13 metres per second. Assuming that the parachute jump is long enough for the terminal speed effectively to be reached, calculate the effective diameter of a parachute that will enable a parachutist of mass 70 kilograms to land safely.

The parachutist will start in freefall before pulling the rip cord, thus will be approaching the terminal speed from above. It will be important that when the rip cord is pulled, there is sufficient time to slow down.

Exercise 24

Revisit the Clifton Suspension Bridge problem described in Example 8, with different initial conditions. Assume now that instead of it being dropped from rest, the marble is catapulted downwards with an initial velocity of $50\,\mathrm{m\,s^{-1}}$.

The initial stages of the solution will be exactly the same as in the example, since the initial conditions are not used until the step solving the differential equation. So the equation of motion of the marble is still

$$v \frac{dv}{dx} = g - kv^2,$$

where $k = c_2 D^2/m$ is a positive constant.

(a) Derive an expression for the marble's velocity in terms of its position.

(b) Describe the motion of the marble.

5 Projectiles

In Section 1 we discussed motion in one dimension. In this section we discuss two-dimensional motion – more specifically, the motion of projectiles modelled as particles. We will use the following terminology in our discussion. During the period that the projectile is off the ground and subject only to the force of gravity (and possibly air resistance), it is said to be *in flight*. The start of the flight is the *launch*, and the initial velocity is the *launch velocity*. The flight ends with an *impact* (often hitting the ground again, but possibly hitting some other target). The *time of flight* is the time between the moment of launch and the moment of impact. The path of the projectile while in flight is its **trajectory**.

Whenever we mention a projectile in this section, it will be modelled as a particle.

In this section, we consider only models in which the effect of air resistance is assumed to be negligible.

In ignoring air resistance, we also ignore effects such as the swerve that may occur when a football is kicked with spin, or when a golf ball is sliced.

In Subsection 5.1, we model the forces acting on a projectile and solve its equation of motion. In Subsection 5.2, we look at the trajectory of a projectile and consider a variety of examples.

5.1 Motion of a projectile

We consider the motion of a particle, called a projectile, launched into the air. The only force acting on the projectile while it is in flight is that due to gravity – we are assuming that air resistance is negligible and can be ignored. So the subsequent motion is in two dimensions, that is, in the vertical plane determined by the direction of the launch velocity. We normally choose the origin to be at the point of launch, the x-axis to be in the direction of the launch velocity, and the y-axis to be vertically upwards. However, it is sometimes convenient to choose an origin that is not the point of projection. Figure 23 shows two examples of projectiles.

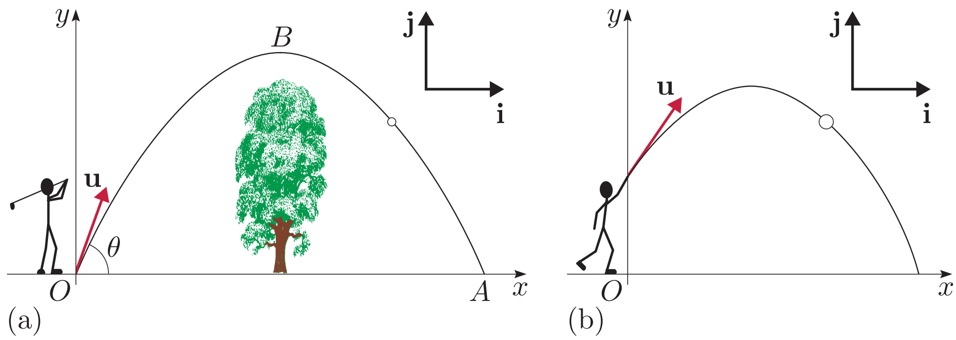


Figure 23 (a) Golf shot (b) Shot put

In Figure 23(a) the projectile is launched from the origin, while Figure 23(b) illustrates the case where the origin is not the launch position. Furthermore, it may be convenient to take $t = 0$ at some time other than the moment of projection. All these complications will change the initial conditions of the projectile motion. None changes the fundamental equation of motion, however. So long as we choose the y-axis (and the unit vector $\mathbf{j}$) to be vertically upwards, and ignore air resistance, the only force on a projectile of mass m is that due to gravity, $-mg\mathbf{j}$, and Newton's second law gives

$$m\,\ddot{\mathbf{r}}(t) = \mathbf{F} = -mg\mathbf{j},$$

that is,

$$\ddot{\mathbf{r}}(t) = -g\mathbf{j}. \tag{29}$$

We will integrate this equation twice, using the initial conditions, to obtain $\mathbf{r}(t)$.

In this subsection we look first at projectile motion of the type illustrated in Figure 23(a), where the launch point and the impact point are in the same horizontal plane. Assume that the projectile is launched with velocity $\mathbf{u}$, at an angle θ above the horizontal. We will derive expressions in terms of $\mathbf{u}$ and θ for the 'range' of the projectile's flight (OA in Figure 23(a)), and for the coordinates of the highest point of its trajectory (B in Figure 23(a)).

We refer to θ as the *launch angle*.

To solve the differential equation (29), we can integrate it twice. We will work in the vector form. Integrating once gives

$$\dot{\mathbf{r}}(t) = -gt\mathbf{j} + \mathbf{c}, \tag{30}$$

where $\mathbf{c}$ is a constant vector. Integrating equation (30) gives

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\mathbf{j} + \mathbf{c}t + \mathbf{d}, \tag{31}$$

where $\mathbf{d}$ is a constant vector. We will take $t = 0$ to be the moment of launch and the origin to be the point of launch. Then we have the initial condition

$$\mathbf{r}(0) = \mathbf{0}.$$

Substituting this into equation (31) gives $\mathbf{d} = \mathbf{0}$, so we have

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\mathbf{j} + \mathbf{c}t.$$

The constant vector $\mathbf{c}$ can be determined from a knowledge of the initial launch velocity. Suppose that the launch velocity is $\mathbf{u}$. Substituting $t = 0$ in equation (30) gives $\dot{\mathbf{r}}(0) = \mathbf{u} = \mathbf{c}$, so

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\mathbf{j} + \mathbf{u}t.$$

From this equation, we can see that the motion of the projectile is in the vertical plane that contains the launch velocity $\mathbf{u}$, as you might expect. We choose the horizontal x-axis to lie in this plane, as shown in Figures 23 and 24.

To express the launch velocity $\mathbf{u}$ in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, recall from Unit 2 that we can resolve the vector $\mathbf{u}$ into its $\mathbf{i}$-component vector $(|\mathbf{u}|\cos\theta)\,\mathbf{i}$ and its $\mathbf{j}$-component vector $(|\mathbf{u}|\sin\theta)\,\mathbf{j}$ (see Figure 24). Since the launch speed is $|\mathbf{u}| = u$, we have $\mathbf{u} = (u\cos\theta)\,\mathbf{i} + (u\sin\theta)\,\mathbf{j}$. So the solution of equation (29) satisfying these initial conditions is

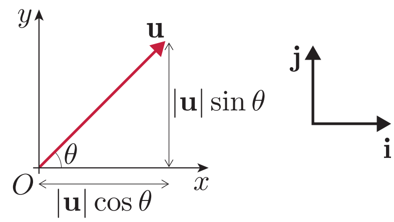


Figure 24 The components of the launch velocity

$$\begin{aligned}\mathbf{r}(t) &= -\tfrac{1}{2}gt^2\mathbf{j} + (u\cos\theta\,\mathbf{i} + u\sin\theta\,\mathbf{j})t\\ &= (ut\cos\theta)\mathbf{i} + (ut\sin\theta - \tfrac{1}{2}gt^2)\mathbf{j}.\end{aligned} \tag{32}$$

The vector solution given by equation (32) can be expressed as separate equations for the x- and y-coordinates as

$$x = ut\cos\theta, \tag{33}$$

$$y = ut\sin\theta - \tfrac{1}{2}gt^2. \tag{34}$$

So long as the ground is horizontal, we can define the **range** of the projectile to be the horizontal distance between the point of launch and the point of impact. To determine the projectile's range when the launch point and the impact point are in the same horizontal plane, note that the vertical coordinate of its position y will be zero at the launch and again when it hits the ground.

This definition of range is unlikely to be suitable where launch and impact are on an inclined plane, as in the ski-jump example in Exercise 18.

Putting $y = 0$ into equation (34) gives

$$\begin{aligned}0 &= ut\sin\theta - \tfrac{1}{2}gt^2\\ &= t(u\sin\theta - \tfrac{1}{2}gt).\end{aligned}$$

This equation has two solutions, $t = 0$ and $t = (2u\sin\theta)/g$. At this latter time, the horizontal coordinate of the position x gives the range R of the projectile. From equation (33), this is

$t = 0$ is the instant of launch.

$$R = u\,\frac{2u\sin\theta}{g}\cos\theta = \frac{2u^2\sin\theta\cos\theta}{g}.$$

Now $\sin 2\theta = 2\sin\theta\cos\theta$, so we have

$$R = \frac{u^2\sin 2\theta}{g}. \tag{35}$$

For a launch on horizontal ground, we must have $\theta > 0$. (For a launch from above ground level, such as from a cliff or bridge, we could have a launch angle below the horizontal, when θ would be negative.)

The sine function never exceeds 1 in value, and we have $\sin 2\theta = 1$ when $2\theta = \frac{\pi}{2}$, that is, when $\theta = \frac{\pi}{4}$. Since the launch angle must be between 0 and $\frac{\pi}{2}$, other solutions can be ignored. So for a given launch speed u, the *maximum range* R_{max} for a projectile launched from a horizontal surface (ignoring air resistance) is obtained using a launch angle of $\frac{\pi}{4}$ to the horizontal, and this maximum range is

$$R_{\text{max}} = \frac{u^2}{g}.$$

The condition $\dot{y}(t) = 0$ is equivalent to asserting that the vertical component of the velocity is zero when the projectile is at its maximum height.

From equation (34), we see that y is a quadratic function of t, with a negative coefficient of t^2. So the graph of y against t is part of a parabola opening downwards. Such a parabola has a single stationary point where $\dot{y}(t) = 0$, and this will give the maximum value of y. Differentiating equation (34) with respect to t gives

$$\dot{y}(t) = u\sin\theta - gt,$$

and this is 0 when $u\sin\theta - gt = 0$, that is, when $t = (u\sin\theta)/g$. Substituting $t = (u\sin\theta)/g$ into the right-hand side of equation (34), the corresponding *maximum height* is given by

$$\begin{aligned} H &= u\left(\frac{u\sin\theta}{g}\right)\sin\theta - \frac{g}{2}\left(\frac{u\sin\theta}{g}\right)^2 \\ &= \frac{u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{2g} \\ &= \frac{u^2\sin^2\theta}{2g}. \end{aligned} \tag{36}$$

Substituting $t = (u\sin\theta)/g$ into the right-hand side of equation (33), the x-component of the projectile's position at the point of maximum height is, using $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\begin{aligned} x &= u\left(\frac{u\sin\theta}{g}\right)\cos\theta \\ &= \frac{u^2\sin\theta\cos\theta}{g} \\ &= \frac{u^2\sin 2\theta}{2g}. \end{aligned}$$

Thus the maximum height occurs when x is half the range R, as you might expect.

The results derived above apply to any projectile where the points of launch and impact are in the same horizontal plane (and air resistance is ignored). Some problems involving projectiles can conveniently be solved by direct use of these results.

Example 10

During a particular downhill run, a short but sharp rise causes a skier to leave the ground at $25\,\text{m}\,\text{s}^{-1}$ at an angle of $\frac{\pi}{6}$ above the horizontal. The ground immediately beyond the rise is horizontal for 60 metres. After this, the slope is again downhill. Will the skier land on the level ground or on the downhill slope beyond it?

Solution

Model the skier as a particle launched from the end of the rise (see Figure 25). With $u = 25$ and $\theta = \frac{\pi}{6}$, the expression in equation (35) for the range on a horizontal surface gives

$$R = \frac{25^2 \sin\frac{\pi}{3}}{9.81} \simeq 55.17.$$

Since this is less than 60 metres, it would seem that the skier will land on the flat part of the run.

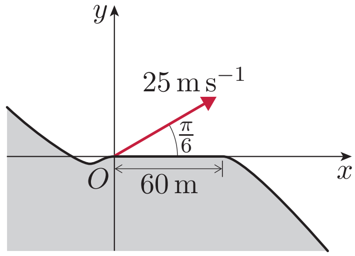


Figure 25 The skier at take-off

(This conclusion is expressed cautiously because of the underlying modelling assumptions. Drag forces may reduce the range of a projectile, but a skier may also experience aerodynamic lift forces that would increase the range. Also, the skier may change the position of her skis relative to the position of her centre of mass – for example, by bending or straightening her legs – which would affect the validity of the model of the skier as a particle.)

This example illustrates how efficiently some questions can be answered by use of general results, such as those giving the range and the maximum height of a projectile when the launch point and the impact point are in the same horizontal plane. When using the results in this subsection in this way, it is important to ensure that they are applicable. We could not, for example, use equation (35) to determine how far the shot putter illustrated in Figure 23(b) would send the shot, since there the point of impact is *not* in the same horizontal plane as that of the launch.

It is important to pay attention to the methodology used in deriving these results, since it can be used in other cases of projectile motion. In the absence of air resistance, we always arrive at the same vector differential equation (29) from Newton's second law. In general, we then need to identify the initial conditions appropriate to the particular situation, find the solution of equation (29) satisfying those initial conditions, and use the solution to address the specific question(s) of interest in the particular problem.

Exercise 25

A ball kicked from a flat piece of ground at an angle of $\frac{\pi}{6}$ above the horizontal lands 40 metres away from where it was kicked. What is the greatest height above the ground that the ball will have reached?

Exercise 26

Consider a projectile launched at an angle θ above the horizontal, from a point at a height h above the origin O, with speed u. Take $t = 0$ to be the moment of launch, and use the coordinate system (and associated unit vectors) shown in Figure 26. Find the solution of $\ddot{\mathbf{r}}(t) = -g\mathbf{j}$ satisfying these initial conditions.

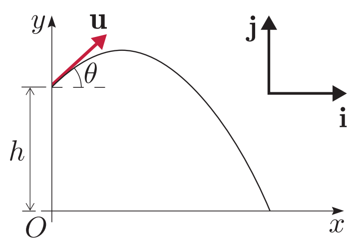


Figure 26 A projectile at launch

Exercise 27

After a road accident, a crashed car is found on a sandy beach at the base of a cliff. The cliff is vertical and is 18 metres high. The investigating police officer finds that the marks in the sand resulting from the car's impact on the beach start 8 metres from the base of the cliff, and that the point of impact is at roughly the same horizontal level as the cliff base. The car appears to have been travelling at right angles to the cliff when it went over. Assuming that the car was travelling in a horizontal direction when it left the cliff, estimate the speed with which it went over.

The vector solution obtained in Exercise 26 can be expressed as separate equations for the components x and y of the position of such a projectile at time t:

Equation (37) is identical to equation (33) as the differential equation for x and its initial conditions are unchanged.

$$x = ut\cos\theta, \tag{37}$$

$$y = h + ut\sin\theta - \tfrac{1}{2}gt^2. \tag{38}$$

As one might expect, launch at a height h simply adds a term h to the y-coordinate.

We now summarise the main results of this subsection.

Projectiles

The equation of motion of a projectile subject only to the force of gravity is

$$\ddot{\mathbf{r}}(t) = -g\mathbf{j}, \tag{39}$$

where $\mathbf{j}$ is a unit vector pointing vertically upwards.

If the projectile is launched at time $t = 0$, from the point $x = 0$, $y = 0$, with launch speed u in the (x, y)-plane and launch angle θ above the horizontal, then the solution of the equation of motion satisfying these initial conditions is

$$x = ut\cos\theta, \tag{40}$$

$$y = ut\sin\theta - \tfrac{1}{2}gt^2. \tag{41}$$

The maximum height H reached by such a projectile is

$$H = \frac{u^2\sin^2\theta}{2g}. \tag{42}$$

If you encounter different initial conditions, you should go back to equation (39) and find the appropriate solution by integration. However, if launch is from $x(0) = 0$, $y(0) = h$, then we need to modify the solutions of the equation of motion (39) given in equations (40) and (41) by adding a term h to the right-hand side of equation (41) as given in equation (38). (In this case, the results for the range and the maximum height are *not* applicable.)

There are also two results for the range of a projectile launched from a level surface.

Range of a projectile launched from a level surface

The range R of a projectile launched from a level surface, subject only to the force of gravity, is

$$R = \frac{u^2 \sin 2\theta}{g}. \tag{43}$$

The maximum range R_{max} for a launch speed u is achieved with a launch angle $\theta = \frac{\pi}{4}$ and is

$$R_{\text{max}} = \frac{u^2}{g}. \tag{44}$$

Exercise 28

A shot putter launches a shot at a speed of $13\,\mathrm{m\,s^{-1}}$ at an angle of $\frac{\pi}{6}$ above the horizontal from a height of 1.8 metres above ground level. How far will the shot travel in the horizontal direction before it hits the ground, assuming that the ground is horizontal?

Exercise 29

A stone is thrown from a height of 1.5 metres above horizontal ground at an angle of $\frac{\pi}{4}$ above the horizontal and lands at a distance of 30 metres from the point where it was thrown. Estimate the speed with which it was thrown.

5.2 Trajectory of a projectile

A variety of problems can be set about the motion of projectiles. Some can be 'pigeonholed', perhaps requiring you to find the range, or to ensure that a target is hit. Others may require you to bring information about the flight of a projectile to bear on the problem in less predictable ways. We start this subsection with problems that involve hitting a target.

The trajectory of a projectile is the path that it traces. To hit some target, say at P, we require that the point P lies on the trajectory. Suppose that a projectile has launch speed u and launch angle θ above the horizontal, and that it is launched from $(0,0)$ at time $t = 0$. Then, from work in the previous subsection, we have

A launch angle *below* the horizontal would correspond to a negative value of θ.

$$x = ut\cos\theta, \tag{45}$$

$$y = ut\sin\theta - \tfrac{1}{2}gt^2. \tag{46}$$

If we have no interest in *when* the projectile hits a target, it is efficient to eliminate t to obtain an equation for the trajectory relating y and x directly. Equation (45) gives $t = x/(u\cos\theta)$, and substituting this into equation (46) gives

$$\begin{aligned} y &= u\,\frac{x}{u\cos\theta}\sin\theta - \frac{g}{2}\left(\frac{x}{u\cos\theta}\right)^2 \\ &= x\tan\theta - x^2\frac{g}{2u^2}\sec^2\theta. \end{aligned}$$

We see that y is a quadratic function of x, and hence that the trajectory is part of a parabola.

As illustrated in Example 11 below, it is often convenient to replace $\sec^2\theta$ by $1+\tan^2\theta$, giving

$$y = x\tan\theta - x^2\frac{g}{2u^2}(1+\tan^2\theta), \tag{47}$$

which is a quadratic equation in $\tan\theta$.

Example 11

A golfer wants to play a recovery shot through a copse of trees. There is a small gap in the foliage at a height of 12 metres and 40 metres in front of him. He knows that with his usual swing, he hits the ball at about $35\,\mathrm{m\,s^{-1}}$. What angle of launch will enable the ball to hit the gap in the foliage?

Solution

We make the usual choice of axes, with origin at the point of launch. The golfer wants the trajectory of the ball to pass through the point $x = 40$, $y = 12$ (working in SI units). So from equation (47) with $u = 35$, we have

$$\begin{aligned} 12 &= 40\tan\theta - 40^2\frac{9.81}{2\times 35^2}(1+\tan^2\theta) \\ &\simeq 40\tan\theta - 6.407(1+\tan^2\theta), \end{aligned}$$

so

$$6.407\tan^2\theta - 40\tan\theta + 18.407 = 0.$$

This is a quadratic equation for $\tan\theta$, with the two solutions

$$\tan\theta = 0.500, \quad \tan\theta = 5.743 \quad \text{to 3 d.p.}$$

Each of these gives a single value for θ in the range $0 \le \theta \le \frac{\pi}{2}$, namely

$$\theta = 0.4638\ (26.58^\circ), \quad \theta = 1.398\ (80.12^\circ).$$

We see that there are two possible launch angles that strike the target (see Figure 27). In this example, the choice of a launch angle of about 27° is perhaps more likely to be suitable, since the other choice would have the ball descending through the foliage at a steep angle, when it would be more likely to hit part of a tree.

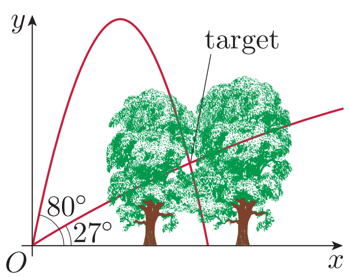


Figure 27 Two launch angles to hit the target

In any problem where we need to find a launch angle (given the launch speed) to hit a specified target, we arrive at a quadratic equation for $\tan\theta$, namely equation (47). So long as this equation has two distinct real roots, there will be two launch angles that enable the target to be hit. Of course, the roots may coincide. In either case, we say that the target is **achievable**. On the other hand, we may arrive at a quadratic equation with no real roots. In this case, the target is not achievable – for the given launch speed, there is no launch angle that enables the target to be hit.

Exercise 30

Suppose that a projectile has launch speed u at an angle θ above the horizontal, and that it is launched from $(0, h)$ at time $t = 0$. (That is, the projectile is launched at a height h above the origin.) By eliminating t from equations (37) and (38), obtain an equation (relating y to x) for the trajectory of such a projectile.

We see from Exercise 30 that for a launch at height h above the origin, we need to add only a term h to the right-hand side of equation (47) for the trajectory. This result is frequently useful.

Trajectory of a projectile

For a projectile launched at time $t = 0$ from $(0, h)$ with launch speed u and launch angle θ above the horizontal, the trajectory has the equation

$$y = h + x\tan\theta - x^2\frac{g}{2u^2}\sec^2\theta. \tag{48}$$

It is often convenient to use the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$ to give

$$y = h + x\tan\theta - x^2\frac{g}{2u^2}(1 + \tan^2\theta), \tag{49}$$

which is a quadratic equation in $\tan\theta$.

Exercise 31

A basketball player is 2.6 metres (horizontally) from the basket. The basket is 3 metres above ground level. The player launches the ball at $7\,\mathrm{m\,s^{-1}}$, from a height of 1.8 metres above ground level. What angle of launch should the player choose?

We now look at an example that requires more work to answer the posed question.

Example 12

This example concerns baseball fielders throwing the ball back to the catcher. Assume throughout that the point of launch and the point of impact are at the same horizontal level.

(a) A fielder can just throw a ball a distance of 60 metres. How fast can the fielder throw the ball?

(b) A fielder needs to throw a ball to the catcher from a distance of 58 metres. Assuming that the fielder throws directly to the catcher at the speed calculated in part (a), what is the shortest time in which the fielder can return the ball to the catcher?

(c) Suppose that a second fielder is midway between the first fielder and the catcher (so that each gap is 29 metres), and that each fielder throws at the speed calculated in part (a). The first fielder throws to the second fielder, then the second fielder throws to the catcher. As well as the time in flight, the second fielder requires 0.3 seconds to catch and throw the ball. Would this 'relaying' result in a quicker return of the ball to the catcher?

Solution

(a) In the previous subsection we found that the maximum range of a projectile (for launch speed u) is achieved at a launch angle $\frac{\pi}{4}$, and this maximum range is u^2/g (see equation (44)). So for the fielder, we have $u^2/g = 60$, which gives

$$\begin{aligned} u = \sqrt{60g} &= \sqrt{60 \times 9.81} \\ &= 24.26 \quad \text{to 2 d.p.} \end{aligned}$$

So the fielder can throw the ball at a speed of approximately $24.3\,\mathrm{m\,s^{-1}}$.

(b) Taking the point of launch as origin, and working in metres, the trajectory of the ball needs to pass through the point $(58, 0)$. So using equation (47), we need a launch angle θ where

It is simpler, as well as more accurate, to use $u = \sqrt{60g}$ here.

$$0 = 58\tan\theta - 58^2 \frac{g}{2 \times 60g}(1 + \tan^2\theta).$$

This can be rearranged as

$$\tan^2\theta - \frac{120}{58}\tan\theta + 1 = 0.$$

This quadratic equation has solutions $\tan\theta = 0.7696$ and $\tan\theta = 1.299$. The corresponding values of θ (between 0 and $\frac{\pi}{2}$) are 0.656 (37.6°) and 0.915 (52.4°). Throwing the ball at either of these angles will return it to the catcher.

To find the time that it takes the ball to reach the catcher, we can use equation (45). When $x = 58$, the time t must satisfy

$$58 = ut\cos\theta = \sqrt{60g}\,t\cos\theta.$$

We obtain different times depending on the choice of the launch angle θ. With $\theta = 0.656$, the time is 3.02 seconds. With $\theta = 0.915$, the time is 3.92 seconds. As one might expect, the lower angle of launch gives the shorter flight time, so the fastest possible return time is 3.02 seconds.

(c) The total time to return the ball to the catcher is $2T + 0.3$ seconds, where T is the time (in seconds) to throw the ball 29 metres at the launch speed calculated in part (a). To ensure that the ball passes through the point $(29, 0)$, we need a launch angle θ satisfying equation (47) with these coordinates, that is,

$$0 = 29\tan\theta - 29^2 \frac{g}{2 \times 60g}(1 + \tan^2\theta),$$

so

$$\tan^2\theta - \frac{120}{29}\tan\theta + 1 = 0.$$

This has solutions $\tan\theta = 0.2577$ and $\tan\theta = 3.880$. The corresponding launch angles are 0.252 (14.5°) and 1.319 (75.6°). Flight times are 1.23 seconds and 4.79 seconds. The lower launch angle gives the shorter flight time, and the total return time to the catcher for the 'relay' is

$$2 \times 1.23 + 0.3 = 2.77.$$

This time of 2.77 seconds *is* shorter than the time found in part (b) for a direct throw.

In the previous subsection we saw that for a launch speed u on a horizontal surface, the maximum range for a projectile is obtained with a launch angle $\frac{\pi}{4}$ and is u^2/g (equation (44)). We next consider the angle of launch required to give the maximum range when launch is from *above* ground level.

We continue to define the range to be the horizontal displacement between the point of launch and the point of impact, even for a launch above ground level. The following calculation of the maximum range brings together ideas about projectiles and methods from calculus. The approaches that we use are chosen to minimise the complexity of the algebra; they are not always those that first come to mind.

This horizontal displacement measures the length of a shot put, for example.

Consider a projectile launched with speed u from a point at height h above the ground, which we assume is horizontal. We choose the origin and axes as shown in Figure 28, and $t = 0$ as the time of launch. If the launch angle is θ above the horizontal, the equation of the trajectory is given by equation (49) as

$$y = h + x\tan\theta - x^2 \frac{g}{2u^2}(1 + \tan^2\theta).$$

Let R be the range. Then since $(R, 0)$ lies on the trajectory, we have

$$0 = h + R\tan\theta - R^2 \frac{g}{2u^2}(1 + \tan^2\theta).$$

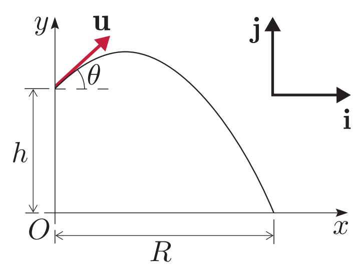


Figure 28 The range of the projectile

Defining $z = \tan\theta$ and $L = u^2/g$ gives

$$0 = h + Rz - \frac{R^2}{2L}(1 + z^2). \qquad (50)$$

We assume that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Remember that a negative value for θ corresponds to a launch angle below the horizontal.

For a given launch speed and height, h and L are constants. We want to maximise the range R by choice of the launch angle θ or, equivalently, by choice of z. The global maximum of a function can occur at a stationary point or at an endpoint of its domain. However, the endpoints of this domain, $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$, both lead to a range of $R = 0$ (the motion is straight up and down), so the maximum range must occur at a stationary point. In order to simplify the algebra, we consider R as a function of z, so we want to find values of z for which $dR/dz = 0$.

Now implicit differentiation of equation (50) with respect to z gives

$$0 = \frac{d}{dz}\big(z\,R(z)\big) - \frac{1}{2L}\frac{d}{dz}\big((1+z^2)(R(z))^2\big),$$

or

$$0 = \left(R + z\,\frac{dR}{dz}\right) - \frac{1}{2L}\left(2zR^2 + (1+z^2)2R\,\frac{dR}{dz}\right).$$

Setting $dR/dz = 0$ reduces this equation to

$$0 = R - \frac{zR^2}{L}.$$

So the maximum range occurs when $z = L/R$. Substituting $z = L/R$ into equation (50) gives

$$\begin{aligned}
0 &= h + R\,\frac{L}{R} - \frac{R^2}{2L}\left(1 + \frac{L^2}{R^2}\right)\\
&= h + L - \frac{R^2}{2L} - \frac{L}{2}\\
&= h + \frac{L}{2} - \frac{R^2}{2L}.
\end{aligned}$$

Hence

$$R = \sqrt{L^2 + 2Lh}.$$

This maximum range is achieved when

$$\tan\theta = z = \frac{L}{R} = \frac{L}{\sqrt{L^2 + 2Lh}} = \frac{1}{\sqrt{1 + 2h/L}},$$

that is, when

$$\theta = \arctan\left(\frac{1}{\sqrt{1 + 2h/L}}\right).$$

Strictly speaking, we have not shown that the range given above *is* a maximum. All we have shown is that it is a stationary value, which could also be a minimum value, for example. However, physically we know that the projectile does have a maximum range, and as $z = L/R$ is the only stationary point, this stationary point must be a maximum. Alternatively, we could show mathematically that the stationary point given by $z = L/R$ is a maximum by considering the sign of the second derivative d^2R/dz^2.

Maximum range for an elevated launch

A projectile launched with speed u from a height h above ground level has a maximum range on horizontal ground given by

$$R_{\max} = \sqrt{L^2 + 2Lh}, \tag{51}$$

where $L = u^2/g$. This maximum range is achieved using the launch angle

$$\theta = \arctan\left(\frac{1}{\sqrt{1 + 2h/L}}\right). \tag{52}$$

If $h = 0$, equation (51) gives $R_{\max} = L$, as it should, since L is the maximum range for a launch from ground level (when $h = 0$). Also, if $h > 0$, then $R_{\max} > L$. As one might expect, a launch from above ground level achieves a maximum range greater (for the same launch velocity) than a launch at ground level. More generally, the greater the height of the launch point, the greater the maximum range.

With $h = 0$, equation (52) gives $\theta = \arctan 1 = \frac{\pi}{4}$, again corresponding to our previous result for a launch from ground level.

Exercise 32

At a tutorial, one of the students, who happens to be an expert shot putter, asserts that aiming to launch at an angle $\frac{\pi}{4}$ has always been good enough for him. He says that improving launch speed is the key to good shot putting. Assume that the student launches the shot from a height of 2 metres above ground level.

(a) The student can put a shot 17 metres with a launch angle $\frac{\pi}{4}$ above the horizontal. Calculate the speed at which the shot is being launched to achieve this range.

(b) For launch at the speed calculated in part (a), use equations (51) and (52) to find the optimum launch angle and the corresponding range.

(c) If the student achieves a launch speed 1% higher than that calculated in part (a) and launches at an angle $\frac{\pi}{4}$ above the horizontal, what range will he achieve?

Exercise 33

As usual in this unit, model the ball as a particle. This means that any swerve that a footballer may achieve by kicking the ball with spin will be overlooked.

A footballer taking a free kick launches the ball from ground level so that it just clears a player who is 10 metres away and 2 metres high. The ball enters the goal 30 metres away at a height of 2.4 metres.

(a) Take as the origin the point from which the ball was kicked. Let the launch speed be u, and let the launch angle above the horizontal be θ. Use equation (48) (with $h = 0$) twice to obtain two equations that u and θ must satisfy.

Multiply one of these equations by a suitable constant, so that the term $(g\sec^2\theta)/2u^2$ has the same coefficient in each equation. Then subtract one equation from the other to eliminate u, and thus obtain an equation that is satisfied by $\tan\theta$. Hence find the launch angle θ.

(b) At what speed was the ball kicked? What period of time elapsed from the moment the ball was kicked until it entered the goal?

Learning outcomes

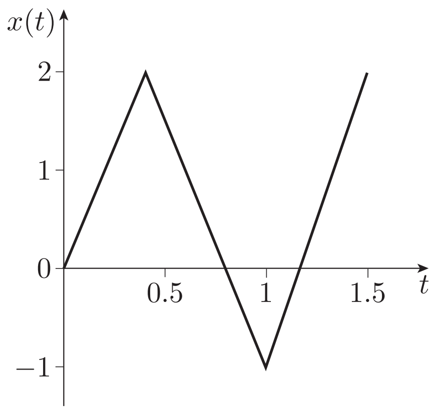
After studying this unit, you should be able to:

- understand and use the basic terms for the description of the motion of particles: position, velocity and acceleration
- understand and use vector functions
- understand the fundamental laws of Newtonian mechanics
- solve mechanics problems in one dimension by drawing a sketch, choosing a suitable x-axis and origin, making assumptions, drawing a force diagram, applying Newton's second law, taking the x-component, and making suitable substitutions
- solve mechanics problems in one dimension that involve one or more of the forces of gravity, friction and air resistance
- understand the concept of terminal speed, and use it in solving mechanics problems in one dimension
- apply Newton's second law in vector form to problems in more than one dimension
- solve problems relating to the motion of a projectile in the absence of air resistance.

Solutions to exercises

Solution to Exercise 1

(a) The graph of x against t for the puck is shown below.



(b) (i) The distance between the first player and the back wall is the distance between where the puck starts and where it first changes direction. This can be read off the graph as 2 m.

(ii) The second player must be further away than the first player since the x-coordinate is -1 when the puck changes direction for a second time, indicating that the second player is 3 m from the wall.

(iii) The speed at which the puck is travelling is given by the slope of the distance–time graph. The slope after the second player hits the puck is greater than the slope after the first player hits the puck, so the second player gives the puck more speed.

Solution to Exercise 2

(a) $\dfrac{d\mathbf{r}(t)}{dt} = \left(\dfrac{d}{dt}t^2\right)\mathbf{i} + \left(\dfrac{d}{dt}10t\right)\mathbf{j} = 2t\mathbf{i} + 10\mathbf{j}.$

So the velocity of the particle at $t = 1$ is

$$\mathbf{v} = 2\mathbf{i} + 10\mathbf{j},$$

and the speed is

$$|\mathbf{v}| = \sqrt{4 + 100} = \sqrt{104} \simeq 10.20 \quad \text{to 2 d.p.}$$

(b) $\dfrac{d\mathbf{r}(t)}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k}.$

So the velocity of the particle at $t = 1$ is

$$\mathbf{v} = (\cos 1)\mathbf{i} - (\sin 1)\mathbf{j} + \mathbf{k},$$

and the speed is

$$|\mathbf{v}| = \sqrt{\cos^2 1 + \sin^2 1 + 1} = \sqrt{2} \simeq 1.41 \quad \text{to 2 d.p.}$$

Solution to Exercise 3

(a) The graph for particle A is its position–time graph. The component of velocity of the particle along the x-axis, dx/dt, is given by the slope of this graph: it can be seen that the slope starts at a high value and decreases as time increases. So particle A is travelling fastest when $t = 0$.

(b) The graph for particle B shows the variation with time of the particle's component of velocity along the x-axis. For the particle to change direction, this velocity must change from positive to negative or from negative to positive (i.e. the graph must cross the t-axis). The graph shown does not do this, so particle B does not change its direction.

(c) The graph for particle C shows the variation with time of the particle's component of acceleration along the x-axis. On the interval $[0, 2.5]$ the acceleration is positive, so the velocity increases in this interval. On the interval $[2.5, 5]$ the acceleration is negative, so the velocity decreases in this interval. To discover whether the particle is ever stationary for $t > 0$, we need to know whether its velocity along the x-axis is ever zero for $t > 0$. Now, since acceleration is obtained by differentiating velocity, we can obtain velocity by integrating acceleration. We also know that the definite integral of a function over a given interval gives the 'area' under the graph of the function, where 'areas' below the t-axis are negative. So the 'area' under the graph gives us the velocity.

Now, on the interval $[0, 2.5]$, the 'area' under the curve is positive, and since the particle starts from rest, the velocity after 2.5 seconds is positive. However, on the interval $[2.5, 5]$, the 'area' under the curve is negative, and furthermore the magnitude of this 'area' is greater than the magnitude of the 'area' for the interval $[0, 2.5]$; therefore the velocity after 5 seconds is negative. Hence, since the particle has both positive and negative velocity in the given time interval, it must be momentarily stationary at some point towards the end of the time interval.

Solution to Exercise 4

Since

$$a(t) = \frac{dv}{dt} = 18t - 20,$$

we have

$$v = \int (18t - 20)\, dt = 9t^2 - 20t + A,$$

where A is a constant.

Using the initial condition $v(0) = 3$, we obtain $A = 3$. Hence the component of the velocity of the particle along the x-axis is given by

$$v(t) = 9t^2 - 20t + 3.$$

Now $v(t) = dx/dt$, so

$$x = \int (9t^2 - 20t + 3)\, dt = 3t^3 - 10t^2 + 3t + B,$$

where B is a constant.

The initial condition $x(0) = 7$ gives $B = 7$. Hence the component of the position of the particle along the x-axis is given by

$$x(t) = 3t^3 - 10t^2 + 3t + 7.$$

Substituting $t = 10$ into the expressions for $x(t)$ and $v(t)$ gives

$$x(10) = 3000 - 1000 + 30 + 7 = 2037,$$
$$v(10) = 9 \times 10^2 - 20 \times 10 + 3 = 703,$$

so at time $t = 10$ the particle is 2037 metres from the origin, with a speed of $703\,\mathrm{m\,s^{-1}}$ along the positive x-axis. Thus the position is $\mathbf{r}(10) = 2037\,\mathbf{i}$ and the velocity is $\mathbf{v}(10) = 703\,\mathbf{i}$.

Solution to Exercise 5

We have

$$a(t) = \frac{dv}{dt} = ge^{-kt}.$$

Integrating this gives

$$v = \int ge^{-kt}\, dt = A - \frac{g}{k}e^{-kt},$$

where A is a constant.

The initial condition $v(0) = 0$ gives $A = g/k$, so the velocity is given by

$$v(t) = \frac{g}{k} - \frac{g}{k}e^{-kt} = \frac{g}{k}(1 - e^{-kt}).$$

Then from $v(t) = dx/dt$ we have

$$x = \int \left(\frac{g}{k} - \frac{g}{k}e^{-kt}\right) dt = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} + B,$$

where B is a constant.

The initial condition $x(0) = 0$ gives $B = -g/k^2$, so the position is given by

$$x(t) = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2} = \frac{g}{k}t - \frac{g}{k^2}(1 - e^{-kt}).$$

Therefore the velocity and position of the particle are given by the vector functions

$$\mathbf{v}(t) = \frac{g}{k}\left(1 - e^{-kt}\right)\mathbf{i},$$
$$\mathbf{r}(t) = \left(\frac{g}{k}t - \frac{g}{k^2}(1 - e^{-kt})\right)\mathbf{i}.$$

Solution to Exercise 6

Given that $a(t) = a_0$, the relationship $a = v\,dv/dx$ gives

$$v\,\frac{dv}{dx} = a_0.$$

Applying the separation of variables method to this differential equation gives

$$\int v\,dv = \int a_0\,dx,$$

so

$$\tfrac{1}{2}v^2 = a_0 x + C,$$

where C is a constant.

Using the initial condition that the velocity is v_0 along the x-axis at $x = x_0$ gives $C = \frac{1}{2}v_0^2 - a_0 x_0$, so

$$\tfrac{1}{2}v^2 = a_0 x + \tfrac{1}{2}v_0^2 - a_0 x_0.$$

Multiplying through by 2 and rearranging gives

$$v^2 = v_0^2 + 2a_0(x - x_0),$$

as required.

Solution to Exercise 7

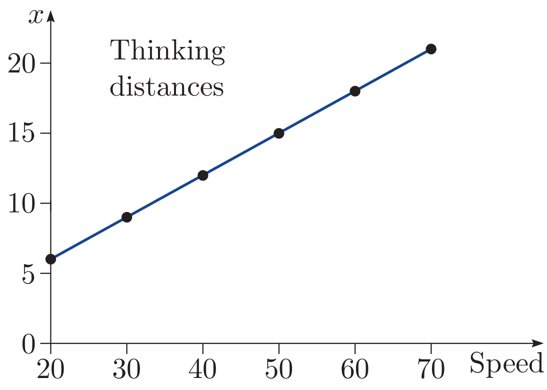
(a) To find v in terms of t, substitute dv/dt for a; then v may be found by direct integration. To find x in terms of t, substitute dx/dt for v and integrate again.

(b) To find a relationship between v and x, substitute $v\,dv/dx$ for a. The result is an equation that can be solved by separation of variables, whose solution will give the required relationship.

(c) To find x in terms of t, substitute d^2x/dt^2 for a, and dx/dt for v. The result is a linear constant-coefficient second-order differential equation for the variable x, namely $\ddot{x} + 3\dot{x} + 2x = \cos t$. This can be solved by the methods of Unit 1. (The general solution is $x = Ae^{-2t} + Be^{-t} + \frac{1}{10}\cos t + \frac{3}{10}\sin t$.)

Solution to Exercise 8

(a) (i) The figure below shows the thinking distance in metres against the speed in miles per hour. The speed of a car before the thinking phase is the value given in the table. The speed after the thinking phase is exactly the same, because the driver has not yet reacted to the hazard. So the acceleration is zero during this phase, and with $x_0 = 0$, the formula $x = x_0 + v_0 t + \frac{1}{2}a_0 t^2$ reduces to $x = v_0 t$.



The values of v_0 and the thinking distance x, in SI units, can be calculated from the values given in the table (using the conversion factor given in the margin next to the question); the only unknown is the thinking time t. The value of t for each pair of speeds and distances can be calculated from $t = x/v_0$.

The calculation, given that the thinking distance at 20 mph is 6 m, and using the given conversion factor, is (in seconds)

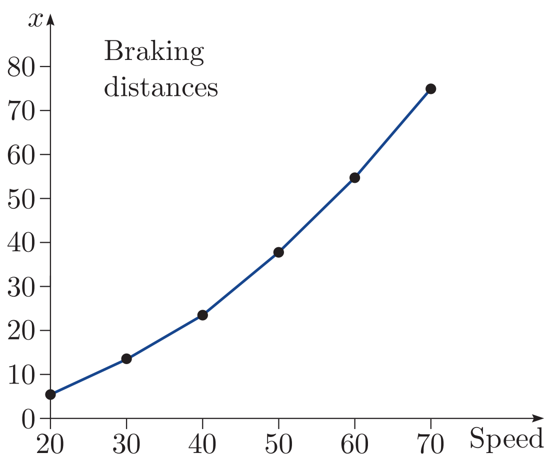
$$t = \frac{x}{v_0} = \frac{6}{20 \times 0.447} \simeq 0.67.$$

(Other pairs give the same value for the thinking time.)

So the model used for calculating the thinking distance data is

$$x = v_0 t = 0.67 v_0.$$

(ii) The figure below shows the braking distance in metres against the speed in miles per hour. The relationship is clearly not linear. The speed of a car at the start of the braking phase is the speed at the end of the thinking phase (i.e. the value given in the table, converted to SI units). The speed at the end of the braking phase is zero ($v = 0$). Assuming that the braking is uniform, so that the acceleration is constant, we can use the formula $v^2 = v_0^2 + 2a_0x$, which for $v = 0$ reduces to $0 = v_0^2 + 2a_0x$.



The values of v_0 and the braking distance x, in SI units, can be calculated from the values given in the table; the only unknown is the acceleration a_0. The value of a_0 for each pair of speeds and distances can be calculated from $a_0 = -v_0^2/(2x)$. (Note that from this equation the acceleration is negative, which is a good check because the car is stopping!)

The calculation for the last pair of values given in the table is

$$a_0 = -\frac{v_0^2}{2x} = -\frac{(70 \times 0.447)^2}{2 \times 75} \simeq -6.53.$$

(Other pairs give values between $-6.42\,\mathrm{m\,s^{-2}}$ and $-6.66\,\mathrm{m\,s^{-2}}$ for the acceleration, which is about two-thirds of the magnitude of the acceleration due to gravity.)

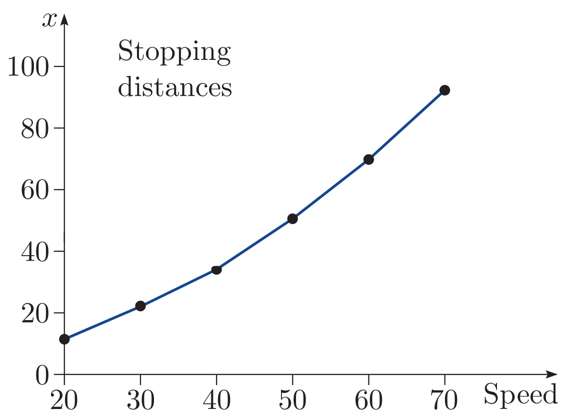
So the model used for calculating the braking distance data is

$$x \simeq -\frac{v_0^2}{2a_0} = \frac{v_0^2}{13.1}.$$

(b) The overall stopping distance is equal to the thinking distance plus the braking distance, and these distances are calculated separately using the models in part (a) to give

$$x = 0.67v_0 + \frac{v_0^2}{13.1}.$$

The figure below shows the stopping distance in metres against the speed in miles per hour.



At 45 mph, the total stopping distance is calculated as

$$\begin{aligned} x &= 0.67 \times (45 \times 0.447) + \frac{(45 \times 0.447)^2}{13.1} \\ &\simeq 13.5 + 30.9 \\ &= 44.4 \quad \text{to 1 d.p.} \end{aligned}$$

Thus the stopping distance at 45 mph is approximately 44.4 m. (Note that the calculated distance is nearly halfway between the tabulated values for 40 mph and 50 mph – it is not *exactly* halfway, because the braking distance is a quadratic function of v_0.)

Solution to Exercise 9

Any moving car is subject to resistive forces, namely air resistance, the internal frictional forces in the car's engine, transmission and wheel bearings, and the external frictional forces between the car's tyres and the road. In order to maintain a constant velocity, it is necessary to apply a motive force that balances these resistive forces.

Solution to Exercise 10

The component of the force of gravity in the direction of a unit vector pointing down the slope is non-zero. If the slope is steep enough, then this force down the slope will be greater than the resistive force of friction, causing the toboggan to accelerate.

Solution to Exercise 11

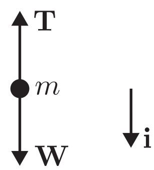
Newton's second law, $\mathbf{F} = m\mathbf{a}$, is a vector equation. Taking the magnitude of both sides of the equation (and using the fact that mass is always positive) gives the scalar equation $|\mathbf{F}| = m|\mathbf{a}|$, into which the values given in the question can be substituted (after converting the mass from grams into the SI unit kilograms) to obtain

$$10 = 0.2|\mathbf{a}|.$$

This gives $|\mathbf{a}| = 50$, so the force produces an acceleration of magnitude $50\,\mathrm{m\,s^{-2}}$.

Solution to Exercise 12

(a) The only forces on the hanging object are the weight $\mathbf{W}$ of the object (a downward force, in the direction of the positive x-axis shown) and the tension force $\mathbf{T}$ due to the string (an upward force, in the direction of the negative x-axis). The resultant force on the object is the sum $\mathbf{W} + \mathbf{T}$ of these forces. This information is shown in the force diagram in the margin.



(b) Applying Newton's second law to this system gives

$$m\mathbf{a} = \mathbf{W} + \mathbf{T}.$$

From the force diagram, we have $\mathbf{W} = |\mathbf{W}|\,\mathbf{i} = mg\mathbf{i}$, where g is the magnitude of the acceleration due to gravity. Similarly, the tension force due to the string is $\mathbf{T} = -|\mathbf{T}|\,\mathbf{i}$. The given acceleration of the hanging object is $1\,\mathrm{m\,s^{-2}}$ downwards, so $\mathbf{a} = \mathbf{i}$. With this information we can resolve the equation above in the $\mathbf{i}$-direction to obtain

$$m \times 1 = mg - |\mathbf{T}|.$$

Substituting $m = 10$ and rearranging gives

$$|\mathbf{T}| = 10g - 10 \simeq 88.1 \quad (\text{using } g = 9.81\,\mathrm{m\,s^{-2}}).$$

So $\mathbf{T} = -|\mathbf{T}|\,\mathbf{i} = -88.1\mathbf{i}$, that is, the tension force due to the string is 88.1 newtons in the upward direction.

Solution to Exercise 13

Substitute the given force and mass into the equation for Newton's second law to obtain

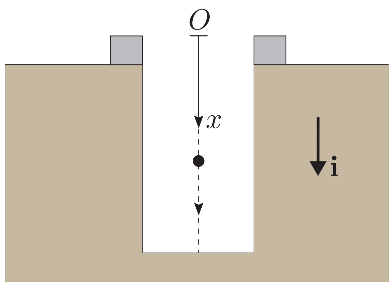
$$4000\,\mathbf{a} = 50\,000\,\mathbf{i} + 60\,000\,\mathbf{j} + 100\,000\,\mathbf{k}.$$

So $\mathbf{a} = 12.5\mathbf{i} + 15\mathbf{j} + 25\mathbf{k}$, which has magnitude (in $\mathrm{m\,s^{-2}}$) $\sqrt{12.5^2 + 15^2 + 25^2} \simeq 32$. This is well below the threshold of $6g \simeq 59\,\mathrm{m\,s^{-2}}$, so the pilot should remain conscious.

Solution to Exercise 14

◀ Draw picture ▶

(a) First, we draw a picture.



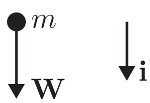
◀ Choose axes ▶

The x-axis is chosen to point vertically downwards, with the origin O at the top of the well, as shown above.

◀ State assumptions ▶

◀ Draw force diagram ▶

The stone is modelled as a particle, and the model assumes that the only force is the stone's weight due to gravity. The force diagram is as follows.



◀ Apply Newton's 2nd law ▶

Applying Newton's second law to the stone gives $\mathbf{W} = m\mathbf{a}$. Since $\mathbf{W} = mg\mathbf{i}$, we have $m\mathbf{a} = mg\mathbf{i}$, and resolving in the $\mathbf{i}$-direction gives

$$a = g.$$

◀ Solve differential equation ▶

Using $a = dv/dt$, we obtain

$$\frac{dv}{dt} = g.$$

Integrating this gives

$$v = gt + A,$$

where A is a constant.

The initial condition that the stone is dropped from rest ($v = 0$ when $t = 0$) gives $A = 0$. Hence

$$v = gt.$$

Now using $v = dx/dt$, we have

$$\frac{dx}{dt} = gt.$$

Integrating this gives

$$x = \tfrac{1}{2}gt^2 + B,$$

where B is a constant.

The initial condition $x = 0$ when $t = 0$ gives $B = 0$. So

$$x = \tfrac{1}{2}gt^2.$$

Using this equation with $t = 3$ gives ◀ Interpret solution ▶

$$x = \tfrac{1}{2} \times 9.81 \times 3^2 = 44.145.$$

So the well is estimated to be about 44 metres deep.

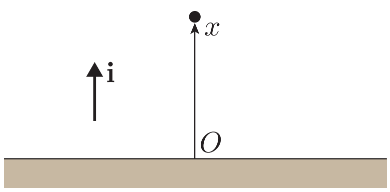
(b) Using the equation for v with $t = 3$ gives

$$v = gt = 9.81 \times 3 = 29.43.$$

So the predicted speed of the stone as it reaches the bottom is about $29\,\mathrm{m\,s^{-1}}$.

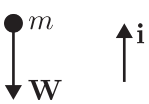
Solution to Exercise 15

(a) First, we draw a picture. ◀ Draw picture ▶



Choose the x-axis to point vertically upwards, with its origin O at the point from which the ball is thrown, as shown above. ◀ Choose axes ▶

The model assumes that the weight due to gravity is the only force acting on the ball, so the force diagram is as follows. ◀ State assumptions ▶ ◀ Draw force diagram ▶



Applying Newton's second law to the ball gives ◀ Apply Newton's 2nd law ▶

$$m\mathbf{a} = \mathbf{W}.$$

Since the x-axis points upwards, the weight of the ball is given by $\mathbf{W} = -mg\mathbf{i}$. The acceleration is downwards, so $\mathbf{a} = a\mathbf{i}$, where a is negative. Resolving in the $\mathbf{i}$-direction gives $ma = -mg$. Dividing by the mass gives

$$a = -g.$$

◀ Solve differential equation ▶

To answer the question, we need an equation relating x to t, and another relating v to t or to x. Since the acceleration $a = -g$ is constant, one approach is to use the constant acceleration formulas of Subsection 1.2. The initial velocity is $10\,\mathrm{m\,s^{-1}}$ upwards from the origin. Using the notation of Subsection 1.2, the initial velocity is $\mathbf{v}_0 = 10\mathbf{i}$, so $v_0 = 10$ and $a_0 = -g$, and hence

$$v = v_0 + a_0 t = 10 - gt,$$
$$x = v_0 t + \tfrac{1}{2}a_0 t^2 = 10t - \tfrac{1}{2}gt^2.$$

(The approach of Example 4, using the substitutions $a = dv/dt$ and $v = dx/dt$ and integrating, leads to the same pair of equations.)

◀ Interpret solution ▶

The ball reaches its maximum height when $v = 0$, and from the equation for v this occurs at time

$$t = \frac{10}{g} \simeq 1.02 \quad \text{to 2 d.p.}$$

Since the motion started at $t = 0$, the duration of the upward flight of the ball is about 1 second.

(b) Substituting $t = 10/g$ into the equation for x, we have

$$x = \frac{100}{g} - \frac{100}{2g} \simeq 5.10 \quad \text{to 2 d.p.}$$

So the maximum height attained by the ball is about 5.1 metres.

The motion up and down has the same acceleration $-g\mathbf{i}$, so it can be treated as one motion. There is no need to consider the upward and downward motions separately.

(c) The ball reaches the ground when $x = 0$, and from the equation for x this occurs when

$$0 = 10t - \tfrac{1}{2}gt^2 = t(10 - \tfrac{1}{2}gt).$$

Hence $t = 0$ or $t = 20/g \simeq 2.04$ to two decimal places. Now $t = 0$ corresponds to the time when the ball is thrown, so the ball *returns* to the ground after approximately 2 seconds.

(d) Substituting $t = 20/g$ into the equation for v gives

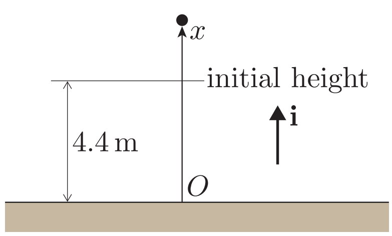
$$v = 10 - gt = 10 - 20 = -10, \quad \text{so } \mathbf{v} = 10(-\mathbf{i}).$$

The ball reaches the ground on its return with a speed of $10\,\mathrm{m\,s^{-1}}$. (The $(-\mathbf{i})$ indicates that the ball is now travelling in the direction of decreasing x, i.e. downwards.)

Solution to Exercise 16

◀ Draw picture ▶

(a) First, we draw a picture.



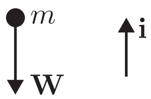
The x-axis is chosen to point vertically upwards, with the origin O at ground level, as shown in the diagram. (The other obvious choice for the origin, which you may well have chosen, is at the point where the ball is thrown. Both choices of course lead to the same answers below.)

◀ Choose axes ▶

The model assumes that the only force acting on the ball is its weight due to gravity, so the force diagram is as follows.

◀ State assumptions ▶

◀ Draw force diagram ▶



Applying Newton's second law to the ball gives $m\mathbf{a} = \mathbf{W}$, where $\mathbf{a} = a\mathbf{i}$ and $\mathbf{W} = -mg\mathbf{i}$ (as the x-axis points upwards). Resolving in the $\mathbf{i}$-direction gives

◀ Apply Newton's 2nd law ▶

$$ma = -mg, \quad \text{so } a = -g.$$

We want equations relating x to t, and v to x or t. One approach is to use the substitution $a = dv/dt$, to obtain

◀ Solve differential equation ▶

$$\frac{dv}{dt} = -g.$$

Integrating this, we obtain

$$v = -gt + A,$$

where A is a constant.

The initial velocity of the ball is $7.6\mathbf{i}$, so $v = 7.6$ when $t = 0$, which gives $A = 7.6$. So

$$v = -gt + 7.6.$$

Therefore using the substitution $v = dx/dt$ gives

$$\frac{dx}{dt} = -gt + 7.6.$$

Integrating this gives

$$x = -\tfrac{1}{2}gt^2 + 7.6t + B,$$

where B is a constant.

The initial condition that $x = 4.4$ when $t = 0$ leads to $B = 4.4$. So

$$x = -\tfrac{1}{2}gt^2 + 7.6t + 4.4.$$

(Since the acceleration is constant, another approach is to use the constant acceleration formulas to obtain the equations for v and x. Note that with the choice of origin used here, $x_0 = 4.4$, $v_0 = 7.6$ and $a_0 = -9.81$. You could use the equation $x = x_0 + v_0t + \frac{1}{2}a_0t^2$ to obtain the equation for x.)

The ball reaches the ground when $x = 0$. Substituting this into the equation for x gives a quadratic equation for the time t,

◀ Interpret solution ▶

$$4.905t^2 - 7.6t - 4.4 = 0,$$

whose solution is

$$t = \frac{7.6 \pm \sqrt{(-7.6)^2 - 4 \times 4.905 \times (-4.4)}}{2 \times 4.905},$$

so $t \simeq 1.998$ or $t \simeq -0.4489$. The negative time is before the ball is thrown and may therefore be ignored.

So the ball lands about 2 seconds after being thrown.

It may be easier to find the speed v at ground level by solving

$$v\,dv/dx = -g$$

to obtain

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = -g(x - x_0),$$

so when $x = 0$,

$$v^2 = v_0^2 + 2gx_0.$$

(b) Substituting this time into the equation for v gives

$$v = -gt + 7.6 = -9.81 \times 1.998 + 7.6 \simeq -12.00 \quad \text{to 2 d.p.}$$

So the ball lands with a speed of about $12.0\,\mathrm{m\,s^{-1}}$. (The negative sign for v confirms that the ball is moving downwards, i.e. the speed is $12.0\,\mathrm{m\,s^{-1}}$ in the $-\mathbf{i}$ direction.)

Solution to Exercise 17

(a) Choose the same x-axis and the same origin as in Example 4, and proceed in exactly the same way as before until you reach the equation $a = g$. Now write a as $v\,dv/dx$ to obtain

$$v\frac{dv}{dx} = g.$$

Solving this differential equation by the method of separation of variables, we have

$$\int v\,dv = \int g\,dx, \quad \text{so} \quad \tfrac{1}{2}v^2 = gx + A,$$

where A is a constant.

Now the marble starts from rest, so $v = 0$ when $x = 0$, which leads to $A = 0$. Hence $\frac{1}{2}v^2 = gx$, or equivalently,

$$v = \sqrt{2gx},$$

where we have taken the positive square root because the velocity is positive throughout the motion.

(This equation could also have been obtained from the constant acceleration formula (8).)

(b) Putting $v = dx/dt$ in the equation for v gives

$$\frac{dx}{dt} = \sqrt{2gx}.$$

Again we use the method of separation of variables to solve this differential equation. So we have

$$\int 1\,dt = \int \frac{1}{\sqrt{2gx}}\,dx = \frac{1}{\sqrt{2g}}\int x^{-1/2}dx = \frac{1}{\sqrt{2g}}\,2x^{1/2} + B,$$

where B is a constant. So

$$t = \sqrt{\frac{2x}{g}} + B.$$

The marble starts at the origin, so $x = 0$ when $t = 0$, which gives $B = 0$. Hence

$$t = \sqrt{\frac{2x}{g}}.$$

(This equation could also have been obtained from the constant acceleration formula (7).)

(c) At $x = 77$, the equation for t yields

$$t = \sqrt{\frac{2x}{g}} = \sqrt{\frac{2 \times 77}{9.81}} \simeq 3.962,$$

and the equation for v gives

$$v = \sqrt{2gx} = \sqrt{2 \times 9.81 \times 77} \simeq 38.87.$$

We take the positive square root as the motion is in the **i**-direction.

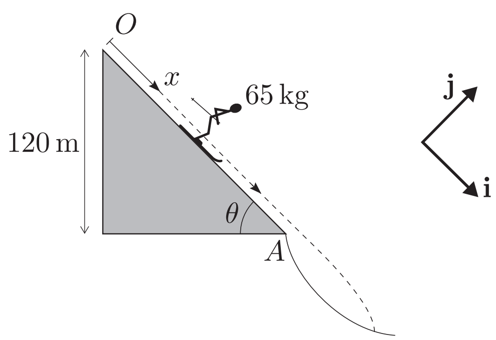
So the object hits the water after about 3.96 seconds, with a speed of about $38.9\,\mathrm{m\,s^{-1}}$. (The answers are, of course, the same as those in Example 4.)

Solution to Exercise 18

(a) As the question asks for the speed for two different angles of the slope of the jump, it is sensible (as suggested in Figure 15) to use θ to be an arbitrary angle of slope and substitute for θ at the interpretation stage.

A diagram of the situation is shown with the question as Figure 15, and is repeated here.

◀ Draw picture ▶

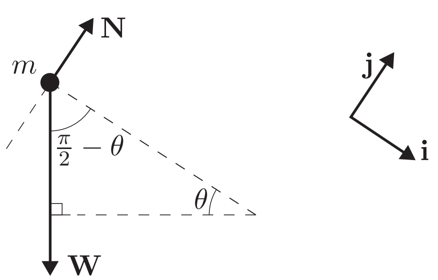


The x-axis is given as pointing down the slope with origin O at the top of the slope, as shown above. The model assumes that the skier can be treated as a particle, and that the only forces acting on the skier are her weight due to gravity and the normal reaction from the slope.

◀ Choose axes ▶

◀ State assumptions ▶

◀ Draw force diagram ▶



◀ Apply Newton's 2nd law ▶

Applying Newton's second law to the skier gives

$$m\mathbf{a} = \mathbf{W} + \mathbf{N}.$$

From the force diagram, $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$. We can resolve $\mathbf{W}$ into components:

$$\begin{aligned}\mathbf{W} &= |\mathbf{W}|\cos(\tfrac{\pi}{2}-\theta)\,\mathbf{i} - |\mathbf{W}|\sin(\tfrac{\pi}{2}-\theta)\,\mathbf{j}\\ &= mg\sin\theta\,\mathbf{i} - mg\cos\theta\,\mathbf{j}.\end{aligned}$$

Now we can resolve in the **i**-direction to obtain

$$ma = mg\sin\theta, \quad \text{so} \quad a = g\sin\theta.$$

◀ Solve differential equation ▶

As we want the velocity when the skier has travelled a vertical distance of 120 metres, it is best to find v as a function of x. Since the acceleration is constant, we can use equation (8) to obtain

$$v^2 = v_0^2 + 2ax = 0 + 2(g\sin\theta)x,$$

so

$$v = \sqrt{2gx\sin\theta}.$$

◀ Interpret solution ▶

The task is to find the velocity when the skier has travelled a vertical distance of 120 metres. Now x is the distance travelled down the slope; so, using trigonometry, $x = 120/\sin\theta$. Substituting for x in the equation for v gives

$$v = \sqrt{2g \times \frac{120}{\sin\theta} \times \sin\theta} = \sqrt{240g} \simeq 48.52 \quad \text{to 2 d.p.}$$

So the speed of the skier at the bottom of the slope angled at $\frac{\pi}{4}$ is about $48.5\,\mathrm{m\,s^{-1}}$.

The substitution of θ at an early stage would have led to a lot of extra work here as v is independent of θ.

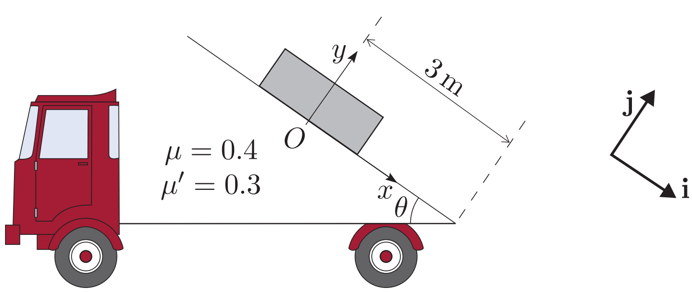
(b) In part (a), all mention of θ cancelled from the final expression. So the answer remains the same: the speed at the bottom of the $\frac{\pi}{3}$ slope is about $48.5\,\mathrm{m\,s^{-1}}$.

(In the equation for v, note that $x\sin\theta = h$ is the vertical height thus the speed at the bottom depends on the height of the slope and not the angle of the slope.)

Solution to Exercise 19

◀ Draw picture ▶

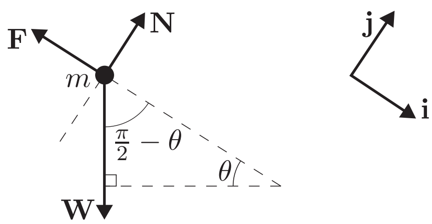
(a) Let θ be the angle that the carrier makes with the horizontal, as shown below, and let m be the mass of the block.



Choose the axes to be parallel and perpendicular to the carrier, with origin O at the rest position of the concrete block, as shown above. ◀ Choose axes ▶

The block is modelled as a particle, and we assume that the only forces acting on the block are the block's weight $\mathbf{W}$, the normal reaction with the carrier $\mathbf{N}$, and the friction force up the slope $\mathbf{F}$. ◀ State assumptions ▶

This gives the following force diagram. ◀ Draw force diagram ▶



The equilibrium condition for the block is ◀ Apply law(s) ▶

$$\mathbf{F} + \mathbf{N} + \mathbf{W} = \mathbf{0}.$$

When the block is on the point of moving, the magnitude of the friction force is given by

$$|\mathbf{F}| = \mu|\mathbf{N}|.$$

From the force diagram, $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$ and $\mathbf{F} = -|\mathbf{F}|\,\mathbf{i}$. The weight can be resolved into components as ◀ Solve equation(s) ▶

$$\begin{aligned}\mathbf{W} &= |\mathbf{W}|\cos\left(\tfrac{\pi}{2} - \theta\right)\,\mathbf{i} + |\mathbf{W}|\sin\left(\tfrac{\pi}{2} - \theta\right)\,(-\mathbf{j})\\ &= mg\sin\theta\,\mathbf{i} - mg\cos\theta\,\mathbf{j}.\end{aligned}$$

Now we can resolve the equilibrium equation in the $\mathbf{i}$-direction, giving

$$-|\mathbf{F}| + 0 + mg\sin\theta = 0,$$

so

$$|\mathbf{F}| = mg\sin\theta.$$

Resolving in the $\mathbf{j}$-direction gives

$$0 + |\mathbf{N}| - mg\cos\theta = 0,$$

so

$$|\mathbf{N}| = mg\cos\theta.$$

But when the block is on the point of slipping, the friction equation applies, and substituting the values of $|\mathbf{F}|$ and $|\mathbf{N}|$ gives

$$mg\sin\theta = \mu mg\cos\theta.$$

Rearranging this gives

$$\tan\theta = \mu.$$

Substituting $\mu = 0.4$ into this equation gives $\theta = \arctan 0.4 \simeq 0.381$ to three decimal places. ◀ Interpret solution ▶

So the angle at which the concrete block begins to slide is about 0.38 radians or about 22°.

(b) To solve the dynamics problem when the block is in motion down the carrier, we start in exactly the same way as for the statics problem. So we start the analysis of the motion by applying Newton's second law, using the same axes, with the origin O at the point at which slipping first occurs.

◀ Apply Newton's 2nd law ▶

Applying Newton's second law to the block gives

$$m\mathbf{a} = \mathbf{F} + \mathbf{N} + \mathbf{W}.$$

The acceleration is down the carrier, so $\mathbf{a} = a\mathbf{i}$ and all the forces are resolved in exactly the same way as in the statics problem. So we can resolve in the **i**-direction to obtain

$$ma = mg\sin\theta - |\mathbf{F}|.$$

Now from the moment at which the block begins to slide, $|\mathbf{F}| = \mu'|\mathbf{N}|$, so this equation becomes

$$ma = mg\sin\theta - \mu'|\mathbf{N}|.$$

To find $|\mathbf{N}|$, we resolve the original equation in the **j**-direction to obtain

$$0 = -mg\cos\theta + |\mathbf{N}|.$$

Therefore $|\mathbf{N}| = mg\cos\theta$ (as before). Thus

$$ma = mg\sin\theta - \mu' mg\cos\theta,$$

so

$$a = g\sin\theta - \mu' g\cos\theta.$$

◀ Solve differential equation ▶

For a fixed angle θ, the acceleration is constant, so we can use the constant acceleration formulas from Section 1. Since we want to relate time to distance travelled, the appropriate formula is equation (7):

$$x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2.$$

Initially, the block is at rest at $x_0 = 0$, so $v_0 = 0$ and this becomes

$$x = \tfrac{1}{2}g(\sin\theta - \mu'\cos\theta)t^2.$$

◀ Interpret solution ▶

The time before the block slides off the back of the truck is calculated from this equation using the value of θ calculated in part (a). Substituting the distance travelled ($x = 3$) and $\mu' = 0.3$, we obtain

Using the exact angle $\theta = \arctan 0.4$ gives $t = 2.57$ to two decimal places.

$$3 \simeq \tfrac{1}{2}g[\sin(0.381) - 0.3\cos(0.381)]t^2 \simeq 0.458t^2,$$

so $t \simeq 2.56$ to two decimal places.

So once the concrete block begins to slide, it takes about 2.6 seconds to slide off the back of the truck.

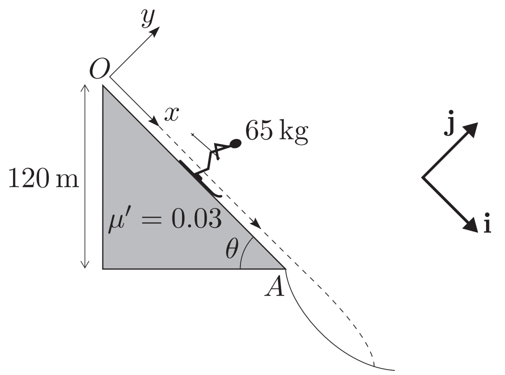
Solution to Exercise 20

◀ Draw picture ▶

First, we draw a picture.

◀ Choose axes ▶

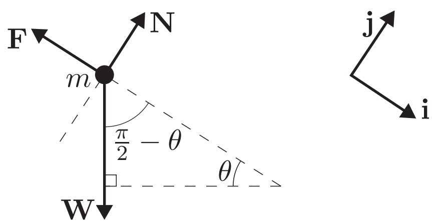
The x-axis is given to be parallel to the slope, with origin O at the top of the slope. Choose the y-axis to be perpendicular to the slope, as shown below.



The skier is modelled as a particle, and we assume that the only forces acting on the skier are her weight, the normal reaction with the slope, and the friction force up the slope. This gives the following force diagram.

◀ State assumptions ▶

◀ Draw force diagram ▶



Applying Newton's second law to the skier gives

◀ Apply Newton's 2nd law ▶

$$m\mathbf{a} = \mathbf{W} + \mathbf{N} + \mathbf{F}.$$

From the force diagram, $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$ and $\mathbf{F} = -|\mathbf{F}|\,\mathbf{i}$. The weight can be resolved into components as

$$\begin{aligned}\mathbf{W} &= |\mathbf{W}|\cos(\tfrac{\pi}{2}-\theta)\,\mathbf{i} + |\mathbf{W}|\sin(\tfrac{\pi}{2}-\theta)\,(-\mathbf{j})\\ &= mg\sin\theta\,\mathbf{i} - mg\cos\theta\,\mathbf{j}.\end{aligned}$$

The acceleration is down the slope, so $\mathbf{a} = a\mathbf{i}$, and we can resolve in the $\mathbf{i}$-direction to obtain

$$ma = mg\sin\theta + 0 - |\mathbf{F}|.$$

Using $|\mathbf{F}| = \mu'|\mathbf{N}| = 0.03|\mathbf{N}|$, this becomes

$$ma = mg\sin\theta - 0.03|\mathbf{N}|.$$

Resolving in the $\mathbf{j}$-direction leads to

$$0 = -mg\cos\theta + |\mathbf{N}| + 0,$$

so $|\mathbf{N}| = mg\cos\theta$. Substituting gives

$$ma = mg\sin\theta - 0.03 \times mg\cos\theta,$$

so

$$a = g\sin\theta - 0.03g\cos\theta.$$

Using equation (8), we obtain

◀ Solve differential equation ▶

$$v^2 = v_0^2 + 2ax = 0 + 2(g\sin\theta - 0.03g\cos\theta)x,$$

so

$$v = \sqrt{2g(\sin\theta - 0.03\cos\theta)x}.$$

◀ Interpret solution ▶

As in Exercise 18, $x = 120/\sin\theta$, so

$$\begin{aligned} v &= \sqrt{2g(\sin\theta - 0.03\cos\theta)\frac{120}{\sin\theta}} \\ &= \sqrt{240g - 7.2g\cot\theta}. \end{aligned}$$

Substituting $\theta = \frac{\pi}{4}$ into this equation gives $v \simeq 47.79$ to two decimal places.

So the speed of the skier at the bottom of the slope angled at $\frac{\pi}{4}$ is about $47.8\,\mathrm{m\,s^{-1}}$.

Substituting $\theta = \frac{\pi}{3}$ into the equation gives $v \simeq 48.10$ to two decimal places.

Note that the calculations are easier here since θ was used in developing and solving the model. The solution can be used to answer the question for the two different values of θ.

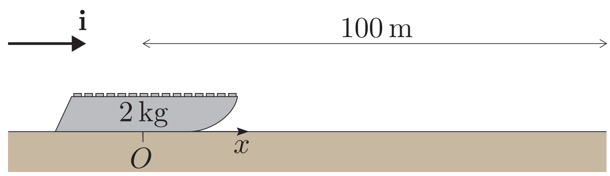
So the speed of the skier at the bottom of the slope angled at $\frac{\pi}{3}$ is about $48.1\,\mathrm{m\,s^{-1}}$.

From these answers, it can be seen that the steeper the slope, the faster the final speed of the skier. The final speed is always less than the $48.5\,\mathrm{m\,s^{-1}}$ calculated in Exercise 18, which omitted friction from the model.

Solution to Exercise 21

◀ Draw picture ▶

First, we draw a picture.



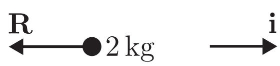
◀ Choose axes ▶

Choose the x-axis along the direction of motion, with origin O at the point where the toboggan has speed $2\,\mathrm{m\,s^{-1}}$, as shown in the diagram.

◀ State assumptions ▶

We model the toboggan as a particle. There are three forces acting on the toboggan: its weight, the normal reaction from the horizontal surface, and the air resistance force. The vertical forces $\mathbf{W}$ and $\mathbf{N}$ are in equilibrium and do not affect the motion. This leaves only the air resistance force $\mathbf{R}$, which opposes the motion, as shown in the force diagram below.

◀ Draw force diagram ▶



◀ Apply Newton's 2nd law ▶

Applying Newton's second law gives

$$m\mathbf{a} = \mathbf{R} + \mathbf{N} + \mathbf{W} = \mathbf{R}.$$

The question states that the quadratic air resistance model should be used, so $\mathbf{R} = -c_2 D^2 |\mathbf{v}|\mathbf{v} = -c_2 D^2 v^2 \mathbf{i}$, since $\mathbf{v} = v\mathbf{i}$, $v > 0$, thus $|\mathbf{v}| = v$. Resolving in the $\mathbf{i}$-direction gives

$$ma = -c_2 D^2 v^2.$$

◀ Solve differential equation ▶

The question requires the velocity of the toboggan after 100 m, so we use the substitution $a = v\,dv/dx$ to obtain

$$mv\,\frac{dv}{dx} = -c_2D^2v^2.$$

Solving by separation of variables,

$$\int \frac{1}{v}\,dv = -\int k\,dx,$$

where $k = c_2D^2/m$, so

$$\ln v = -kx + A,$$

where A is a constant. Hence

$$v = Be^{-kx},$$

where $B = e^A$ is another constant. The initial condition that the toboggan is initially moving at $2\,\mathrm{m\,s^{-1}}$ (i.e. $v = 2$ when $x = 0$) gives $B = 2$, so

$$v = 2e^{-kx},$$

where $k = c_2D^2/m$.

Substitute the data given in the question to find the velocity after 100 m: ◀ Interpret solution ▶

$$k = \frac{c_2D^2}{m} = \frac{0.2 \times (0.05)^2}{2} = 2.5 \times 10^{-4},$$

$$v(100) = 2\exp(-2.5 \times 10^{-4} \times 100) \simeq 1.95.$$

So the percentage decrease in velocity is about

$$100\,\frac{v(0) - v(100)}{v(0)} = 100\,\frac{2 - 1.95}{2} = 2.5,$$

that is, after 100 metres the toboggan has lost only about 2.5% of its speed.

Solution to Exercise 22

First suppose that the linear air resistance model applies, so that the terminal speed is

$$v_{\mathrm{T}} = \frac{mg}{c_1D} = \frac{65 \times 9.81}{1.7 \times 10^{-4} \times 10} \simeq 4 \times 10^5.$$

So $Dv_{\mathrm{T}} \simeq 4 \times 10^6$, which is greater than 10^{-5}, so the linear model does not apply.

Now suppose that the quadratic model applies, so that the terminal speed is

$$v_{\mathrm{T}} = \sqrt{\frac{mg}{c_2D^2}} = \sqrt{\frac{65 \times 9.81}{0.2 \times 10^2}} \simeq 5.6.$$

So $Dv_{\mathrm{T}} \simeq 56$, which is greater than 1, so the quadratic model does not apply either.

The condition for the quadratic model is closer to being satisfied than the condition for the linear model, so the quadratic model is likely to produce the better estimate. So the conclusion is that the landing speed of the parachutist is approximately $6\,\mathrm{m\,s^{-1}}$.

(Looking at Figure 19, it can be seen that $6\,\mathrm{m\,s^{-1}}$ (with $D = 10$, so that $\log_{10}(D|\mathbf{v}|) \simeq 1.8$) falls to the right of the range of validity of the quadratic model. In this region, you can see that the quadratic model lies just above the experimental curve, so that it gives a slight overestimate of the air resistance. So it should be expected that the actual landing speed is slightly greater than $6\,\mathrm{m\,s^{-1}}$. Also note that this landing speed would be achieved by a particle falling under gravity, without air resistance, from a height of $1.8\,\mathrm{m}$.)

Solution to Exercise 23

You saw in Exercise 22 that the quadratic air resistance model is better than the linear model for problems of this type. Rearrangement of the quadratic model $v_{\mathrm{T}} = \sqrt{mg/c_2D^2}$ gives

$$D = \sqrt{\frac{mg}{c_2 v_{\mathrm{T}}^2}} = \sqrt{\frac{70 \times 9.81}{0.2 \times 13^2}} \simeq 4.5.$$

So the effective diameter needs to be at least 4.5 metres.

As in the solution to Exercise 22, we have $\log_{10}(D|\mathbf{v}|) = \log_{10}(4.5 \times 13) \simeq 1.8$, so the quadratic model overestimates the air resistance in this case. So to be safe, a parachute with effective diameter of 5 or even 6 metres would probably be needed.

Solution to Exercise 24

(a) The equation of motion of the marble is given in the question as

$$v\,\frac{dv}{dx} = g - kv^2,$$

where $k = c_2D^2/m$ is a positive constant.

Proceeding as in Example 8, we solve the differential equation by separation of variables:

$$\int \frac{v}{g - kv^2}\,dv = \int 1\,dx.$$

Now the first difference due to the changing initial conditions occurs, since if $v = 50$, then

$$g - kv^2 = 9.81 - \frac{0.2 \times (0.02)^2 \times (50)^2}{0.013} \simeq -5.6.$$

So the denominator of the first integrand is negative (whereas it was positive before). Rewriting this to make the denominator positive by taking out the factor -1 gives

$$-\int \frac{v}{kv^2 - g}\,dv = \int 1\,dx,$$

so

$$-\frac{1}{2k}\ln(kv^2 - g) = x + A,$$

where A is a constant.

To determine A, use the initial condition that $v = v_0$ when $x = 0$:

$$-\frac{1}{2k}\ln(kv_0^2 - g) = 0 + A.$$

Substituting for A and rearranging gives

$$-2kx = \ln(kv^2 - g) - \ln(kv_0^2 - g),$$

so

$$e^{-2kx} = \frac{kv^2 - g}{kv_0^2 - g}$$

or

$$e^{-2kx}(kv_0^2 - g) = kv^2 - g,$$

thus

$$kv^2 = g + e^{-2kx}(kv_0^2 - g).$$

So

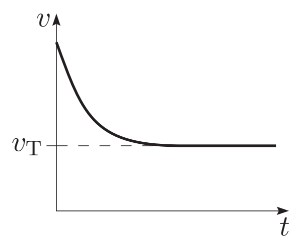
$$v = \sqrt{\frac{g}{k} + e^{-2kx}\frac{kv_0^2 - g}{k}}.$$

(b) The eventual behaviour of the velocity $\mathbf{v} = v\mathbf{i}$ can be seen from the equation for kv^2 above, since as x becomes large, the exponential term will become vanishingly small. So as x becomes large, v satisfies the equation

$$g = kv^2.$$

This means that the speed will tend to $\sqrt{g/k}$ as x becomes large.

This is the same eventual behaviour as in Example 8, that is, the marble's speed tends to a terminal speed. Moreover, the terminal speed $\sqrt{g/k}$ has the same value as for the marble falling from rest. Now, however, the speed *decreases* exponentially towards the terminal speed. The behaviour is shown in the sketch in the margin.



Solution to Exercise 25

In the text, we obtained equation (35) for the range, namely $R = u^2 \sin 2\theta/g$. With $\theta = \frac{\pi}{6}$, the kick has range 40 metres, so the launch speed u must satisfy

$$u^2 = \frac{40g}{\sin\frac{\pi}{3}} = \frac{80g}{\sqrt{3}}.$$

We also obtained the expression (36) for the greatest height $H = (u^2 \sin^2\theta)/2g$ reached by a projectile. With $\theta = \frac{\pi}{6}$ and $u^2 = 80g/\sqrt{3}$, this gives

$$H = \frac{80g \times \sin^2\frac{\pi}{6}}{2g\sqrt{3}} = \frac{10}{\sqrt{3}} \simeq 5.77 \quad \text{to 2 d.p.}$$

So the greatest height reached by the ball is about 5.8 metres.

Solution to Exercise 26

We need to find the solution of $\ddot{\mathbf{r}}(t) = -g\mathbf{j}$ satisfying $\dot{\mathbf{r}}(0) = \mathbf{u}$, where $\mathbf{u} = (u\cos\theta)\mathbf{i} + (u\sin\theta)\mathbf{j}$ and $\mathbf{r}(0) = h\mathbf{j}$ (since the point of launch is $(0, h)$). The integral of $\ddot{\mathbf{r}}(t) = -g\mathbf{j}$ is

$$\dot{\mathbf{r}}(t) = -gt\mathbf{j} + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector.

Substituting $t = 0$ and using the initial condition $\dot{\mathbf{r}}(0) = \mathbf{u}$, we must have $\mathbf{c} = \mathbf{u}$.

Integrating again gives

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\mathbf{j} + \mathbf{u}t + \mathbf{d},$$

where $\mathbf{d}$ is a constant vector.

Substituting $t = 0$ and using the initial condition $\mathbf{r}(0) = h\mathbf{j}$, we must have $\mathbf{d} = h\mathbf{j}$.

Hence the required solution is

$$\begin{aligned}\mathbf{r}(t) &= -\tfrac{1}{2}gt^2\mathbf{j} + (ut\cos\theta)\mathbf{i} + (ut\sin\theta)\mathbf{j} + h\mathbf{j}\\ &= (ut\cos\theta)\mathbf{i} + (h + ut\sin\theta - \tfrac{1}{2}gt^2)\mathbf{j}.\end{aligned}$$

Solution to Exercise 27

We choose the y-axis vertically upwards and the x-axis horizontal. We choose the origin at the bottom of the cliff at beach level. Suppose that the car left the cliff with a speed u, travelling in a horizontal direction (at right angles to the cliff). Then the initial conditions are $\mathbf{r}(0) = 18\mathbf{j}$ and $\dot{\mathbf{r}}(0) = u\mathbf{i}$. We can use the solution of $\ddot{\mathbf{r}}(t) = -g\mathbf{j}$ derived in Exercise 26, with $\theta = 0$, which is

$$\mathbf{r}(t) = ut\mathbf{i} + (h - \tfrac{1}{2}gt^2)\mathbf{j},$$

or, separated into components,

$$\begin{aligned}x &= ut,\\ y &= h - \tfrac{1}{2}gt^2.\end{aligned}$$

We know that $x = 8$ when $y = 0$ (assuming that the car hit the ground exactly 8 metres from the cliff and modelling the car as a particle). So from the first equation, the car hit the ground at $t = 8/u$. Substituting this into the second equation gives

$$0 = 18 - \tfrac{1}{2}g(64/u^2).$$

Thus, as u is positive,

$$u = \sqrt{\frac{64 \times 9.81}{2 \times 18}} \simeq 4.18 \quad \text{to 2 d.p.}$$

So the car left the cliff at a speed of just over $4\,\mathrm{m\,s^{-1}}$.

Solution to Exercise 28

Taking the origin to be at ground level, and using equations (37) and (38), the position of the shot at a time t after the launch is given by

$$x = 13t\cos\tfrac{\pi}{6},$$
$$y = 1.8 + 13t\sin\tfrac{\pi}{6} - \tfrac{1}{2}\times 9.81t^2.$$

To find the time when the shot hits the ground, we substitute $y = 0$ in the second equation and solve the resulting quadratic equation for t. The solutions are $t = 1.560$ and $t = -0.2352$. The negative solution represents a time before the shot is launched and so can be rejected. At $t = 1.560$, we have $x = 17.57$.

So the shot lands at a horizontal distance of 17.57 metres from the point of launch.

Solution to Exercise 29

Taking the origin to be at ground level, we can use equations (37) and (38) with $h = 1.5$ and $\theta = \frac{\pi}{4}$. Suppose that the launch speed is u. Then equation (37) gives $x = ut\cos\frac{\pi}{4} = ut/\sqrt{2}$. If the stone hits the ground when $t = T$, we have $30 = uT/\sqrt{2}$, so $T = 30\sqrt{2}/u$. We know that $y = 0$ when $t = T$, so substituting into equation (38) gives

$$\begin{aligned} 0 &= 1.5 + u\,\frac{30\sqrt{2}}{u}\frac{1}{\sqrt{2}} - \frac{g}{2}\left(\frac{30\sqrt{2}}{u}\right)^2 \\ &= 31.5 - \frac{900g}{u^2}. \end{aligned}$$

This gives $u = 30\sqrt{9.81/31.5} \simeq 16.74$.

So the launch speed was approximately $16.74\,\mathrm{m\,s^{-1}}$.

Solution to Exercise 30

Equations (37) and (38) are

$$x = ut\cos\theta,$$
$$y = h + ut\sin\theta - \tfrac{1}{2}gt^2.$$

From the first equation, $t = x/(u\cos\theta)$. Substituting this into the second equation gives

$$\begin{aligned} y &= h + u\,\frac{x}{u\cos\theta}\sin\theta - \frac{g}{2}\left(\frac{x}{u\cos\theta}\right)^2 \\ &= h + x\tan\theta - x^2\frac{g}{2u^2}\sec^2\theta. \end{aligned}$$

Alternatively, using $\sec^2\theta = 1 + \tan^2\theta$,

$$y = h + x\tan\theta - x^2\frac{g}{2u^2}(1 + \tan^2\theta).$$

Solution to Exercise 31

We choose the origin to be at ground level, vertically below the point of launch. So the equation of the trajectory of the basketball is equation (49) with $h = 1.8$ and $u = 7$ (using SI units). In order for the ball to pass through the hoop, we want the point $x = 2.6$, $y = 3$ to be on this trajectory. Hence

$$3 = 1.8 + 2.6\tan\theta - (2.6)^2 \frac{9.81}{2 \times 7^2}(1 + \tan^2\theta).$$

This simplifies to the quadratic equation

$$0.6767\tan^2\theta - 2.6\tan\theta + 1.877 = 0.$$

(Alternatively, and more efficiently, you may have chosen the origin to be the point from which the ball was launched. However, this leads to the same equation for $\tan\theta$.)

This equation for $\tan\theta$ has the two solutions

$$\tan\theta = 2.879 \quad \text{and} \quad \tan\theta = 0.963.$$

Each of these gives a single value for θ in the range $0 \le \theta \le \frac{\pi}{2}$:

$$\theta = 1.236\ (70.9^\circ) \quad \text{and} \quad \theta = 0.767\ (43.9^\circ).$$

We see that there are two possible launch angles that enable the target to be hit. In this example, the choice of a launch angle of approximately 71° is more likely to be suitable, since this has the ball descending towards the net at the steeper angle, so the ball is less likely to catch on the rim of the basket.

Solution to Exercise 32

(a) We choose the origin to be at ground level, vertically below the point of launch. So the equation of the trajectory of the shot is equation (48) with $h = 2$ and $\theta = \frac{\pi}{4}$ (using SI units). The trajectory must pass through the point of impact, namely $x = 17$, $y = 0$. So the launch speed u must satisfy the equation

$$0 = 2 + 17 - \frac{17^2 \times 9.81 \times 2}{2u^2}.$$

(Alternatively, you may have chosen the origin to be the point from which the shot is launched. Then the equation of the trajectory is equation (48) with $h = 0$, and the point of impact is $x = 17$, $y = -2$. However, you should arrive at the same equation for u as above.)

We have $u^2 = (17^2 \times 9.81)/19$, so $u \simeq 12.22$.

So the launch speed is about $12.22\,\mathrm{m\,s^{-1}}$.

(b) With u as calculated in part (a), the parameter L in equations (51) and (52) has the value $u^2/g \simeq 15.21$. We also have $h = 2$, so the value of θ giving the maximum range is (from equation (52))

$$\begin{aligned}\theta &= \arctan\left(\frac{1}{\sqrt{1+2h/L}}\right)\\ &= \arctan\left(\frac{1}{\sqrt{1+4/15.21}}\right)\\ &= 0.727\ (41.7^\circ).\end{aligned}$$

The range achieved with this optimum launch angle is (from equation (51))

$$\begin{aligned}R &= \sqrt{L^2+2Lh}\\ &= \sqrt{15.21^2+4\times 15.21}\\ &\simeq 17.09.\end{aligned}$$

So the optimum launch angle is about 41.7°, with a range of approximately 17.1 metres.

(c) An improvement of 1% on the launch speed $12.22\,\mathrm{m\,s^{-1}}$ calculated in part (a) would give launch speed $12.22 \times 1.01 = 12.34\,\mathrm{m\,s^{-1}}$. With this launch speed and launch angle $\frac{\pi}{4}$, choosing the origin to be at ground level, vertically below the point of launch, the equation of the trajectory of the shot (equation (48)) is

$$y = 2 + x - x^2\frac{9.81}{2\times 12.34^2}\times 2.$$

At the point of impact $y = 0$, which leads to the quadratic equation

$$0 = 2 + x - 0.0644x^2.$$

This has solutions $x = 17.32$ and $x = -1.79$.

We can reject the negative solution, which represents the point behind the putter where the trajectory intersects ground level. So the range of the put will be approximately 17.3 metres.

(We can see from the answers to parts (b) and (c) that the student is right in saying that a small increase in launch speed is more effective in increasing the range than is getting the optimum launch angle. However, although slight, the improvement in range (of 9 cm) resulting from putting at the optimum angle could be the difference between winning and coming nowhere! So one might as well try to achieve the optimum launch angle.)

Solution to Exercise 33

(a) The trajectory must pass through the points $(10, 2)$ and $(30, 2.4)$. So using equation (48) twice, we have

$$2 = 10\tan\theta - 100\,\frac{g}{2u^2}\sec^2\theta,$$
$$2.4 = 30\tan\theta - 900\,\frac{g}{2u^2}\sec^2\theta.$$

To eliminate the $\sec^2\theta$ term, we multiply the first equation by 9 to obtain

$$18 = 90\tan\theta - 900\,\frac{g}{2u^2}\sec^2\theta.$$

Subtracting gives

$$15.6 = 60\tan\theta.$$

So $\tan\theta = 0.26$ and $\theta = 0.2544$ (14.6°).

(b) Substituting $\theta = 0.2544$ into the first equation in part (a) gives

$$2 = 10\tan(0.2544) - \frac{50g}{u^2}\sec^2(0.2544),$$

so

$$u^2 = \frac{50\times 9.81\sec^2(0.2544)}{10\tan(0.2544)-2} = 872.28$$

and $u \simeq 29.53$.

So the ball was kicked at a speed of approximately $29.5\,\mathrm{m\,s^{-1}}$ to one decimal place.

Using equation (45), we have $x = ut\cos\theta$. We know that the ball entered the goal when $x = 30$, so $t = 30/(29.53\cos(0.2544)) \simeq 1.05$ to two decimal places.

So just over 1 second after having been kicked, the ball entered the goal.