Unit 1

First- and second-order differential equations

Introduction

An important class of equations that arise in mathematics consists of those that feature the *rates of change* of one or more variables with respect to one or more others. These rates of change are expressed mathematically by *derivatives*, and the corresponding equations are called *differential equations*. Equations of this type crop up in a wide variety of situations. They are found, for example, in models of physical, electronic, economic, demographic and biological phenomena.

First-order differential equations, which are the particular topic of Section 1, feature derivatives of order one only; that is, if the rate of change of variable y with respect to variable x is involved, then the equations feature dy/dx but not d^2y/dx^2, d^3y/dx^3, etc.

When a differential equation arises, it is usually an important aim to *solve* the equation. For an equation that features the derivative dy/dx, this entails expressing the *dependent variable* y directly in terms of the *independent variable* x. You will see four possible approaches to finding a solution in Section 1.

This unit also considers *second-order* differential equations, that is, differential equations that involve a second (but no higher) derivative. Examples of second-order differential equations are

A second-order differential equation may or may not include a first derivative.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4e^x \quad \text{and} \quad 3\frac{d^2y}{dx^2} + y^2 \sin x = x^2.$$

1 First-order differential equations

The content of this section should be familiar to you if you have studied other modules in which first-order differential equations are discussed. For the most part, this section looks at methods for finding *analytic* solutions of first-order differential equations – that is, solutions expressed in terms of exact formulas in the three possible cases

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dx} = g(x)\,h(y), \quad \frac{dy}{dx} + g(x)\,y = h(x),$$

where f, g and h are specified functions in each case. The first equation can be solved by *direct integration*, the second by *separation of variables*, and the third by finding an *integrating factor*. For other types of equation, and indeed for some of the equations of this form where the integration cannot be performed analytically, numerical methods can be used to find an approximate solution. However, where a simple formula can be found, this is likely to be more informative than a solution found by the use of a numerical method.

In this module you will meet many examples of differential equations. Frequently these arise from studying the motion of physical objects, but we start with an example drawn from biology and show how this leads naturally to a particular differential equation.

Suppose that we are interested in the size of a particular population, and in how it varies over time. The first point to make is that any population size is measured in integers (whole numbers), so it is not clear how differentiation will be relevant. (Differentiable functions must be continuous, therefore defined on an interval of real numbers in $\mathbb{R}$.) Nevertheless, if the population is large, say in hundreds of thousands, a change of one unit will be relatively very small, and in these circumstances we may choose to model the population size as a *continuous* function of time. We write this function as $P(t)$, and our task is to show how $P(t)$ may be described by a differential equation.

Let us assume a fixed starting time (which we label $t = 0$). If the population is not constant, then there will be 'leavers' and 'joiners'. For example, in a population of humans in a particular country, the former will be those who die or emigrate, while the latter represent births and immigrants.

It is usual to express birth rates as a proportion of the current population size. Death rates are specified in a similar way. To emphasise that these rates are expressed as a *proportion* of the current population, we use the terms **proportionate birth rate** and **proportionate death rate**.

Note that in our model the proportionate birth rate is expressed as a proportion of the whole population, not just the number of women.

For our simple model we ignore immigration and emigration, and concentrate solely on births and deaths. Denote the proportionate birth rate by b and the proportionate death rate by c. Then in a short interval of time δt, we would expect

$$\text{number of births} \simeq b\,P(t)\,\delta t, \tag{1}$$

$$\text{number of deaths} \simeq c\,P(t)\,\delta t, \tag{2}$$

where $P(t)$ is the population size (in units of 100 000) at time t.

At this stage, we seek some relationship between the chosen variables. In order to find this, we make use of the **input–output principle**, which can be expressed as

$$\boxed{\text{accumulation}} = \boxed{\text{input}} - \boxed{\text{output}}.$$

This principle applies to any quantity whose change, over a given time interval, is due solely to the specified input and output.

The *accumulation* δP of population over the time interval δt is the population at the end of the interval minus the population at the start of the interval, that is,

$$\delta P = P(t + \delta t) - P(t).$$

The *input* is the number of births (equation (1)), and the *output* is the number of deaths (equation (2)). The input–output principle now enables us to express the accumulation δP of the population over the time interval δt as

$$\delta P \simeq b\,P(t)\,\delta t - c\,P(t)\,\delta t = (b-c)\,P(t)\,\delta t.$$

Dividing through by δt, we obtain

$$\frac{\delta P}{\delta t} \simeq (b-c)\,P(t).$$

The approximations involved in deriving this equation become progressively more accurate for shorter time intervals. So, finally, by letting δt tend to zero, we obtain

This is the step that requires P to be a continuous (rather than discrete) function of t.

$$\frac{dP}{dt} = (b-c)\,P(t).$$

(This follows because

$$\frac{dP}{dt} = \lim_{\delta t \to 0} \frac{P(t+\delta t) - P(t)}{\delta t}$$

is the *definition* of the derivative of P.)

This is a *differential* equation because it describes dP/dt rather than the eventual object of our interest (which is P itself). The purpose of this unit is to enable you to solve a wide variety of such equations.

We can simplify the above equation slightly by using the **proportionate growth rate** r, which is the difference between the proportionate birth and death rates: $r = b - c$. Then our model becomes

$$\frac{dP}{dt} = rP. \tag{3}$$

For very simple population models, r is taken to be a constant. As we will see, this leads to a prediction of exponential growth (or, if $r < 0$, decay) in population size with time, as illustrated in Figure 1. This may be a very good approximation for certain populations, but it cannot be sustained indefinitely if $r > 0$.

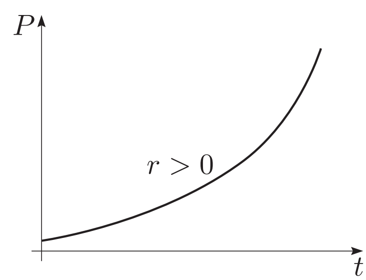


Figure 1 Population growth

In practice, both the proportionate birth rate and the proportionate death rate will vary, and so therefore will the proportionate growth rate. It turns out to be convenient to model these changes as being dependent on the population size, so the proportionate growth rate r becomes a function of P. The justification for this is as follows. When the population is low, one may assume that there is potential for it to grow (assuming a reasonable environment). The proportionate growth rate should therefore be high. However, as the population grows, there will be competition for resources. Thus the proportionate growth rate will decline, and in this way unlimited (exponential) growth does not occur.

A particularly useful model arises from taking $r(P)$ to be a decreasing linear function of P. We write this as

$$r(P) = k\left(1 - \frac{P}{M}\right), \tag{4}$$

You will see later why this particular form is chosen.

where k and M are positive constants. Looking at this formula, you can see that the proportionate growth rate r decreases linearly with P, from the value k (when $P = 0$) to the value 0 (when $P = M$).

Using this expression for r, the differential equation (3) satisfied by P becomes

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right). \tag{5}$$

This is well known to biologists as the **logistic equation** – we will consider it further in Subsection 1.1, and see how to solve it in Subsection 1.3. For now, we have achieved our objective of showing that differential equations arise naturally in modelling the real world.

Exercise 1

Suppose that a population obeys the logistic model (with the proportionate growth rate given by equation (4)), and that you are given the following information. When $P = 10$ the proportionate growth rate is 1, and when $P = 10\,000$ the proportionate growth rate is 0.

Find the corresponding values of k and M.

1.1 Solutions of differential equations

This notation is named after Gottfried Leibniz (1646–1716).

The *derivative* of a variable y with respect to another variable x is denoted in *Leibniz notation* by dy/dx. In this derivative expression we refer to y as the *dependent variable* and to x as the *independent variable*.

There are other notations in use for derivatives. If the relation between variables x and y is expressed in terms of a function f, so that $y = f(x)$, then the derivative may be written in function notation as $f'(x)$.

A further notation, attributed to Sir Isaac Newton, is restricted to cases in which the independent variable is time, denoted by t. The derivative of $y = f(t)$ could be written in this case as $\dot{y}$, in which the dot over the y stands for the d/dt of Leibniz notation. Thus we may express this derivative in any of the equivalent forms

$$\frac{dy}{dt} = \dot{y} = f'(t).$$

Further derivatives are obtained on differentiating this first derivative. The second derivative of $y = f(t)$ could be represented by any of the forms

$$\frac{d^2y}{dt^2} = \ddot{y} = f''(t).$$

These possible notations have different strengths and weaknesses, and which is most appropriate in any situation depends on the purpose at hand. You will see all of these notations employed at various times during the module.

It is common practice in applied mathematics to reduce the proliferation of symbols as far as possible. One aspect of this practice is that we often avoid allocating separate symbols to variables and to associated functions. Thus in place of the equation $y = f(t)$ (where y and t denote variables, and f denotes the function that relates them), we could write $y = y(t)$, which is read as 'y is a function of t'.

Strictly speaking, this is an abuse of notation, since there is ambiguity as to exactly what the symbol y represents: it is a variable on the left-hand side of $y = y(t)$, but a function on the right-hand side. However, it is a very convenient abuse.

The following definitions explain just what are meant by a *differential equation*, by the *order* of such an equation, and by a *solution* of it.

A **differential equation** for $y = y(x)$ is an equation that relates the independent variable x, the dependent variable y, and one or more derivatives of y.

The **order** of such a differential equation is the order of the highest derivative that appears in the equation. Thus a **first-order** differential equation for $y = y(x)$ features only the first derivative, dy/dx.

A **solution** of such a differential equation is a function $y = y(x)$ that satisfies the differential equation.

A function must *satisfy* a differential equation in order to be regarded as a solution of it. The differential equation is satisfied by the function provided that when the function is substituted into the equation, the left- and right-hand sides of the equation give an identical expression. This substitution includes the requirement that the function should be *differentiable* (i.e. that it should have a derivative) at all points where it is claimed to be a solution.

You are asked to verify in the next exercise that three functions are solutions of corresponding first-order differential equations.

Exercise 2

Verify that each of the functions given below is a solution of the corresponding differential equation.

(a) $y = 2e^x - (x^2 + 2x + 2)$; $\quad \dfrac{dy}{dx} = y + x^2$.

(b) $y = \tan x + \sec x$; $\quad \dfrac{dy}{dx} = y \tan x + 1$.

(c) $y = t + Ce^{-t}$; $\quad \dot{y} = -y + t + 1$. (Here C is an arbitrary constant.)

In the last part of Exercise 2 you were asked to verify that

$$y = t + Ce^{-t}$$

is a solution of the differential equation $\dot{y} = -y + t + 1$, where C is an arbitrary constant. In saying that C is *arbitrary*, we mean that it can assume any real value. Whatever number is chosen for C, the corresponding expression for $y(t)$ is always a solution of the differential equation. Choosing $C = 1$, for example, gives the particular function $y = t + e^{-t}$.

This demonstrates that solutions of a differential equation can exist in profusion; as a result, we need terms to distinguish between the totality of all these solutions for a given equation and the individual solutions that are completely specified.

The **general solution** of a differential equation is the collection of all possible solutions of that equation.

A **particular solution** of a differential equation is a single solution of the equation, and consists of a solution function whose rule contains no arbitrary constant.

In many cases it is possible to describe the general solution of a first-order differential equation by a single formula involving an arbitrary constant. For example, you will see from Exercise 2(c) that $y = t + Ce^{-t}$ is the general solution of the equation $\dot{y} = -y + t + 1$; this means that not only is $y = t + Ce^{-t}$ a solution whatever the value of C, but also *every* particular solution of the equation may be obtained by giving C a suitable value.

Exercise 3

(a) Verify that for any value of the constant C, the function $y = C - \frac{1}{3}e^{-3x}$ is a solution of the differential equation

$$\frac{dy}{dx} = e^{-3x}.$$

(b) Verify that for any value of the constant C, the function

$$P = \frac{CMe^{kt}}{1 + Ce^{kt}}$$

is a solution of the logistic equation

See equation (5).

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

As you have seen, there are many solutions of a differential equation. However, a particular solution of the equation, representing a definite relationship between the variables involved, is often what is needed. This is achieved by using a further piece of information in addition to the differential equation. Often the extra information takes the form of a pair of values for the independent and dependent variables.

For example, in the case of a population model, it would be natural to specify the starting population, P_0 say, and to start measuring time from $t = 0$. We could then write

$$P = P_0 \text{ when } t = 0, \quad \text{or equivalently,} \quad P(0) = P_0.$$

A requirement of this type is called an *initial condition*.

An **initial condition** associated with the differential equation

$$\frac{dy}{dx} = f(x, y)$$

specifies that the dependent variable y takes some value y_0 when the independent variable x takes some value x_0. This is written either as

$$y = y_0 \text{ when } x = x_0$$

or as

$$y(x_0) = y_0.$$

The numbers x_0 and y_0 are referred to as **initial values**.

The combination of a first-order differential equation and an initial condition is called an **initial-value problem**.

The word 'initial' in these definitions arises from those (frequent) cases in which the independent variable represents time. In such cases, the differential equation describes how the system being modelled behaves once started, while the initial condition specifies the configuration with which the system is started off. In fact, if the initial condition is $y(x_0) = y_0$, then we are often interested in solving the corresponding initial-value problem for $x > x_0$.

If x represents time, then $x > x_0$ is 'the future' after the system has been started off.

We usually require that an initial-value problem should have a *unique* solution, since then the outcome is completely determined by how the system behaves and its configuration at the start. Almost all the differential equations in this module do have unique solutions, and we will assume that all the initial-value problems in this unit have unique solutions.

Example 1

Using the result given in Exercise 3(a), solve the initial-value problem

$$\frac{dy}{dx} = e^{-3x}, \quad y(0) = 1.$$

Solution

From Exercise 3(a), a solution of the differential equation is

$$y = C - \tfrac{1}{3}e^{-3x},$$

In fact, as will be shown in Example 2, this is the general solution.

where C is a constant.

The initial condition says that $y = 1$ when $x = 0$, and on feeding these values into the above solution we find that

$$1 = C - \tfrac{1}{3}.$$

Hence $C = \frac{4}{3}$, and the particular solution of the differential equation that solves the initial-value problem is

$$y = \tfrac{4}{3} - \tfrac{1}{3}e^{-3x}.$$

Exercise 4

The size of a population (measured in units of hundreds of thousands) is modelled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad P(0) = 1,$$

where $k = 0.15$ and $M = 10$.

(a) Use your answer to Exercise 3(b) to solve this initial-value problem.

(b) Can you predict the long-term behaviour of the population size from your answer?

Finally in this subsection, note that one needs to keep an eye on the *domain* of the function defining the differential equation. 'Gaps' in the domain usually show up as some form of restriction on the nature of a solution curve. For example, consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x}. \tag{6}$$

It turns out that there are two distinct families of solutions of this equation, given by $y = \ln x + C$ (if $x > 0$) and $y = \ln(-x) + C$ (if $x < 0$). These two families of solutions are illustrated in Figure 2. Notice that the right-hand side of equation (6) is not defined at $x = 0$, and that there is no solution that crosses the y-axis.

Since $|x| = -x$ if $x < 0$, we can write
$$\int \frac{1}{x}\,dx = \ln|x|.$$

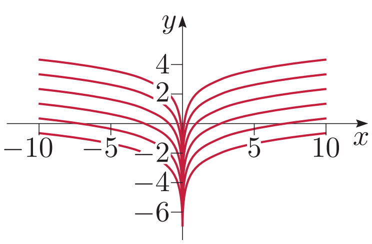


Figure 2 Solutions of equation (6)

1.2 Direct integration

Equations of the form

$$\frac{dy}{dx} = f(x)$$

can be solved by taking the indefinite integral of both sides of the equation.

The function $f(x)$ is assumed to be continuous (i.e. its graph has no breaks).

Example 2

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{-3x}.$$

(b) Find the particular solution of this differential equation that satisfies the initial condition $y(0) = \frac{5}{3}$.

Solution

(a) On applying direct integration, we obtain the general solution

$$y = \int e^{-3x}\,dx = -\tfrac{1}{3}e^{-3x} + C,$$

where C is an arbitrary constant.

(b) In order to satisfy the initial condition $y(0) = \frac{5}{3}$ (i.e. $y = \frac{5}{3}$ when $x = 0$), we must have

$$\tfrac{5}{3} = -\tfrac{1}{3}e^{0} + C,$$

so $C = 2$. The required particular solution is therefore

$$y = -\tfrac{1}{3}e^{-3x} + 2.$$

Procedure 1 Direct integration

The general solution of the differential equation

$$\frac{dy}{dx} = f(x)$$

is

$$y = \int f(x)\,dx = F(x) + C, \qquad (7)$$

where $F(x)$ is an integral of $f(x)$, and C is an arbitrary constant.

Once the general solution has been found, it is possible to single out a particular solution by specifying a value for the constant C. This value may be found by applying an initial condition.

Exercise 5

Solve each of the following initial-value problems.

(a) $\dfrac{dy}{dx} = 6x, \quad y(1) = 5.$

(b) $\dfrac{dv}{du} = e^{4u}, \quad v(0) = 2.$

(c) $\dot{y} = \dfrac{t}{1+t^2}, \quad y(0) = 2.$

(*Hint*: For the integral, try the substitution $u = 1 + t^2$.)

The answer to Exercise 5(c) can be generalised.

Any differential equation of the form

$$\frac{dy}{dx} = k\,\frac{f'(x)}{f(x)} \quad (f(x) \neq 0),$$

where k is a constant, can be integrated to give the general solution

$$y = k \ln|f(x)| + C, \tag{8}$$

where C is an arbitrary constant.

The proof of this result involves differentiating the solution and showing that y satisfies the differential equation.

In Exercise 5(c) we had $f(t) = 1 + t^2$ and $f'(t) = 2t$, with $k = \frac{1}{2}$. The initial right-hand side $t/(1+t^2)$ had to be manipulated slightly to get it into the right form. Spotting integrands of this form (or of this form apart from a constant multiple) can allow you to solve some quite tricky-looking problems.

Exercise 6

Find the general solution of each of the following differential equations, where a is a non-zero constant.

(a) $\dfrac{dy}{du} = \dfrac{1}{u-a} \quad (u \neq a)$

(b) $\dfrac{dy}{dx} = \dfrac{1}{x(1-ax)} \quad (x \neq 0,\ x \neq 1/a)$

$\left(\textit{Hint}\text{: First verify that } \dfrac{1}{x(1-ax)} = \dfrac{1}{x} + \dfrac{a}{1-ax}.\right)$

1.3 Separation of variables

Equations of the form

$$\frac{dy}{dx} = g(x)\,h(y),$$

The functions $g(x)$ and $h(y)$ are assumed to be continuous, and $h(y) \neq 0$.

where g is a function of x only, and h is a function of y only, can be solved by dividing both sides by $h(y)$ to give

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x),$$

and then taking the indefinite integral of both sides of the equation, as before. The final step is to rearrange the results to obtain the explicit solution in the form

$$y = \text{a function of } x,$$

if this is possible. It is necessary to be careful about the domain or image set of the solution obtained, as the following example illustrates.

Example 3

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2 y}{4 + x^3} \quad (y > 0).$$

(b) Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{3y} \quad (y > 0),$$

and the particular solution that satisfies the initial condition $y(0) = 3$.

Solution

(a) The equation is of the form

$$\frac{dy}{dx} = g(x)\,h(y),$$

where the obvious choices for g and h are

$$g(x) = \frac{3x^2}{4 + x^3} \quad \text{and} \quad h(y) = y.$$

Dividing both sides of the given differential equation by y gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3x^2}{4 + x^3}.$$

Integrating both sides with respect to x gives

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{y}\,dy = \int \frac{3x^2}{4 + x^3}\,dx,$$

where the second integral comes from applying the rule for integration by substitution.

Spotting that the right-hand integrand is of the form $f'(x)/f(x)$ with $f(x) = 4 + x^3$, we have

$$\ln|y| = \ln|4 + x^3| + C,$$

where the arbitrary constants associated with the two indefinite integrals have been lumped into a single arbitrary constant C.

The solution is not yet in the explicit form $y = F(x)$ for some function F. If we write $C = \ln A$ $(A > 0)$, then the right-hand side becomes $\ln|4 + x^3| + \ln A = \ln(A|4 + x^3|)$, and we can take exponentials of both sides to give the explicit form of the solution as

$$y = A(4 + x^3).$$

If necessary, A can be negative to take into account changes in sign for y or $(4 + x^3)$.

(b) The equation is of the form

$$\frac{dy}{dx} = g(x)\,h(y),$$

where

$$g(x) = -x \quad \text{and} \quad h(y) = 1/(3y).$$

Notice that since $y > 0$, $h(y)$ is never zero.

On dividing through by $h(y) = 1/(3y)$ (i.e. multiplying through by $3y$) and integrating with respect to x, the differential equation becomes

$$\int 3y\,dy = \int -x\,dx.$$

Evaluating the integrals gives

$$\tfrac{3}{2}y^2 = -\tfrac{1}{2}x^2 + C,$$

where C is an arbitrary constant. This is an implicit form of the general solution.

On solving for y (and noting the condition $y > 0$ given above, which determines the sign of the square root), we obtain the explicit general solution

$$y = \sqrt{\tfrac{1}{3}(2C - x^2)}.$$

This can be simplified slightly by writing A in place of $2C$, where A is another arbitrary constant. However, we need to recognise that the formula for y represents a real quantity greater than zero only when the argument of the square root is positive, so we must have $A - x^2 > 0$. This in turn means that A cannot be completely arbitrary, since it must at least be positive. The general solution in this case is therefore

Since $x^2 \geq 0$ for all x, $A - x^2 > 0$ implies $A > x^2 \geq 0$, so A must be positive.

$$y = \sqrt{\tfrac{1}{3}(A - x^2)} \quad (-\sqrt{A} < x < \sqrt{A}),$$

where A is a positive but otherwise arbitrary constant.

The initial condition is $y(0) = 3$, so we substitute $x = 0$ and $y = 3$ into the general solution above. This gives $3 = \sqrt{A/3}$, so $A = 27$, and the required particular solution is

$$y = \sqrt{\tfrac{1}{3}(27 - x^2)} \quad (-3\sqrt{3} < x < 3\sqrt{3}).$$

The method is summarised below.

Procedure 2 Separation of variables

This method applies to separable differential equations, which are of the form

$$\frac{dy}{dx} = g(x)\,h(y).$$

1. Divide both sides by $h(y)$ (where $h(y) \neq 0$), and integrate both sides with respect to x, to obtain
$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$
2. If possible, perform the integrations to obtain an implicit form of the general solution.
3. If possible, rearrange the formula found in Step 2 to give y in terms of x. This is the explicit (general) solution.

It is a good idea to check, by substitution into the original differential equation, that the function obtained is indeed a solution.

The separation of variables method is useful, but there are some difficulties with it. First, it may not be possible to perform the necessary integrations. Second, the general solution obtained is restricted to those values of y such that $h(y) \neq 0$. Third, it may not be possible to perform the necessary manipulations to obtain an explicit solution.

Of these difficulties, the first can be overcome by use of a numerical method, such as Euler's method (see Subsection 1.5). The second will be discussed shortly. The third will usually also need numerical techniques.

Exercise 7

A mass $m(t)$ of a uranium isotope, which is present in an object at time t, declines over time due to radioactive decay. Its behaviour is modelled by the differential equation

$$\frac{dm}{dt} = -\lambda m \quad (m > 0),$$

where the *decay constant* λ is a positive constant characteristic of the uranium isotope.

This model can be applied to other radioactive substances by selecting the appropriate value of the parameter λ.

Note the condition $m > 0$. You can see that $m = 0$ also satisfies the differential equation.

(a) Find the general solution of this differential equation.

(b) Find the particular solution for which the initial amount of uranium present (at time $t = 0$) is m_0.

The condition $m > 0$ in Exercise 7 arose from the modelling context. This condition enabled us to find the general solution without needing to worry about dividing by zero at Step 1 of the separation of variables method (and hence without needing to restrict the image set further).

We should also note that:

- the separation of variables method requires that $h(y) \neq 0$ and gives a family of solutions containing an arbitrary constant
- the case when $h(y) = 0$ is exceptional and can give extra solutions that may or may not have the same form as the family of general solutions.

The following exercises provide you with some practice at applying the separation of variables method and at completing the general solution for values of y such that $h(y) = 0$.

Exercise 8

Find the general solution of each of the following differential equations.

(a) $\dfrac{dy}{dx} = \dfrac{y-1}{x}$ $(x > 0)$ (b) $\dfrac{dy}{dx} = \dfrac{2y}{x^2+1}$

Exercise 9

The differential equation here is the logistic equation (5), which, as was pointed out earlier, may be used as a model for the size $P(t)$ of a population at time t.

(a) Solve the initial-value problem

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad P(0) = P_0 \text{ (where } P_0 > 0\text{)},$$

where k and M are positive constants.

(*Hint*: For the integral involving P, the solution to Exercise 6(b) should be useful.)

(b) Describe what happens to the solution $P(t)$ as t becomes large.

1.4 Integrating factor method

This subsection presents one final method of analytic solution for *linear* first-order differential equations.

A first-order differential equation for $y = y(x)$ is **linear** if it can be expressed in the form

$$\frac{dy}{dx} + g(x)\,y = h(x), \tag{9}$$

where $g(x)$ and $h(x)$ are given functions.

A linear first-order differential equation is said to be **homogeneous** if $h(x) = 0$ for all x, and **inhomogeneous** or **non-homogeneous** otherwise.

For example, the differential equation

$$\frac{dy}{dx} - x^2 y = x^3$$

is linear, with $g(x) = -x^2$ and $h(x) = x^3$, whereas the equation

$$\frac{dy}{dx} = xy^2$$

is not, due to the presence of the non-linear term y^2.

Exercise 10

Decide whether or not each of the following first-order differential equations is linear.

(a) $\dfrac{dy}{dx} = x \sin x$ (b) $\dot{y} + y^2 = t$

(c) $x\dfrac{dy}{dx} + y = y^2$ (d) $(1+x^2)\dfrac{dy}{dx} + 2xy = 3x^2$

An important existence and uniqueness theorem for *linear* first-order differential equation guarantees that an initial-value problem has a unique solution.

Theorem 1

If the functions $g(x)$ and $h(x)$ are continuous throughout an interval (a, b) and x_0 belongs to this interval, then the initial-value problem

$$\frac{dy}{dx} + g(x)\,y = h(x), \quad y(x_0) = y_0,$$

has a unique solution throughout the interval.

This includes the possibility that either $a = -\infty$ or $b = \infty$, so the interval might be all of the real line.

This is a very powerful result, since it means that once you have found a solution in a particular interval, that solution will be the *only* one.

The *integrating factor method* is a technique for solving linear differential equations. It derives from the rule for integration by parts or, equivalently, from the product rule for derivatives. To introduce the topic, consider the differential equation

$$(1+x^2)\frac{dy}{dx} + 2xy = 3x^2. \tag{10}$$

As you saw in Exercise 10(d), this differential equation is linear; but it is not soluble by direct integration or by separation of variables.

Note first that $2x$ (the coefficient of y) is the derivative of $1 + x^2$ (the coefficient of dy/dx). It follows from the product rule that

$$\frac{d}{dx}\big((1+x^2)y\big) = (1+x^2)\frac{dy}{dx} + 2xy.$$

The right-hand side of this equation is the same as the left-hand side of equation (10), so we can rewrite the latter as

$$\frac{d}{dx}\big((1+x^2)y\big) = 3x^2. \tag{11}$$

Now the left-hand side here is just the derivative of $(1+x^2)y$, so we can apply direct integration to equation (11) to obtain

$$(1+x^2)y = \int 3x^2\,dx = x^3 + C,$$

where C is an arbitrary constant. Division by $1+x^2$ then gives the general solution of equation (10) explicitly as

$$y = \frac{x^3+C}{1+x^2}.$$

This solution was arrived at by noting that the left-hand side of equation (10) is of the form

$$p\,\frac{dy}{dx} + \frac{dp}{dx}\,y, \tag{12}$$

where $p = 1+x^2$, and that this form can be re-expressed, using the product rule, as

$$\frac{d}{dx}(py).$$

Linear differential equations need not come in this convenient form. The left-hand side of the equation

$$\frac{dy}{dx} + g(x)\,y = h(x) \tag{13}$$

may not be of the form (12). An *integrating factor* $p = p(x)$ that enables us to transform the left-hand side of equation (13) into the form (12) can be found by writing down the two properties that such a function must satisfy, as follows.

- Multiplying equation (13) by p gives, on the left-hand side,

  $$p\,\frac{dy}{dx} + p\,g(x)\,y.$$

- The left-hand side must be of the form

  $$p\,\frac{dy}{dx} + \frac{dp}{dx}\,y.$$

Comparison of these two expressions shows that p must itself be a particular solution of the differential equation

$$\frac{dp}{dx} = g(x)\,p.$$

This is a homogeneous linear first-order differential equation, and we can solve it by separation of variables. Indeed, following Procedure 2, the equation becomes (for $p \neq 0$)

$$\int \frac{dp}{p} = \int g(x)\,dx.$$

Performing the left-hand integral gives

Because any constant multiple of an integrating factor is still an integrating factor, we may assume $p > 0$.

$$\ln p = \int g(x)\,dx,$$

so

$$p = \exp\left(\int g(x)\,dx\right),$$

which defines the **integrating factor** for equation (13).

When equation (13) is multiplied through by the integrating factor, the resulting differential equation is

$$p(x)\,\frac{dy}{dx} + p(x)\,g(x)\,y = p(x)\,h(x), \tag{14}$$

the left-hand side of which, by the definition of p, is of the form (12). So equation (14) can be re-expressed, using the product rule, as

$$\frac{d}{dx}(p(x)\,y) = p(x)\,h(x). \tag{15}$$

Direct integration can then be used on equation (15) to try to find the general solution.

The definition of p ensures that the left-hand side of equation (14) is of the form (12) since

$$\begin{aligned}\frac{dp}{dx} &= \frac{d}{dx}\left(\exp\left(\int g(x)\,dx\right)\right)\\ &= \exp\left(\int g(x)\,dx\right)g(x)\\ &= p(x)\,g(x).\end{aligned}$$

This integrating factor method is summarised below.

Procedure 3 Integrating factor method

This method applies to differential equations of the form

$$\frac{dy}{dx} + g(x)\,y = h(x). \tag{16}$$

1. Determine the integrating factor

$$p(x) = \exp\left(\int g(x)\,dx\right). \tag{17}$$

2. Multiply equation (16) by $p(x)$ to recast the differential equation as

$$p(x)\,\frac{dy}{dx} + p(x)\,g(x)\,y = p(x)\,h(x).$$

3. Rewrite the differential equation as

$$\frac{d}{dx}(p(x)\,y) = p(x)\,h(x).$$

4. Integrate this last equation to obtain

$$p(x)\,y = \int p(x)\,h(x)\,dx.$$

5. Divide through by $p(x)$ to obtain the general solution in explicit form.

The constant of integration is not needed here.

You can, if you wish, check that you have found p correctly by checking that

$$p(x)\,\frac{dy}{dx} + p(x)\,g(x)\,y = \frac{d}{dx}(p(x)\,y),$$

i.e. by checking that $dp/dx = p(x)\,g(x)$.

The integral in Step 4 will involve an arbitrary constant C.

It is a good idea to check, by substitution into the original equation, that the function obtained is indeed a solution.

As with the separation of variables method, it may not be possible to perform the necessary final integration.

Exercise 11

Find the general solution of each of the following differential equations.

(a) $\dfrac{dy}{dx} - y = e^x \sin x$ (b) $\dfrac{dy}{dx} = \dfrac{y-1}{x}$ $(x > 0)$

Exercise 12

Which method would you use to try to solve each of the following linear first-order differential equations?

(a) $\dfrac{dy}{dx} + x^3 y = x^5$ (b) $\dfrac{dy}{dx} = x \sin x$

(c) $\dfrac{dv}{du} + 5v = 0$ (d) $(1+x^2)\dfrac{dy}{dx} + 2xy = 1 + x^2$

1.5 Direction fields and Euler's method

The final method of this section is a numerical method that can be used when the analytic methods fail. First, you will see that qualitative information about the solutions of a first-order differential equation may be gleaned directly from the equation itself, without undertaking any form of integration process. The main concept here is the *direction field*, sketches of which usually give a good idea of how the graphs of solutions behave. Direction fields can also be regarded as the starting point for a numerical (i.e. calculational rather than algebraic) method of solution called *Euler's method*.

Consider what can be deduced about solutions of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

from direct observation of this equation.

Here we have

$$f(t, P) = kP\left(1 - \frac{P}{M}\right).$$

In Section 1 we encountered the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \tag{18}$$

where k and M are positive constants. In certain circumstances this is a useful mathematical model of population changes, in which $P(t)$ denotes the size of the population at time t. The right-hand side of this equation is equal to zero if either $P = 0$ or $P = M$. Hence, since $dP/dt = 0$ in both cases, each of the constant functions $P = 0$ and $P = M$ is a particular solution of equation (18). Within the model, these solutions correspond to a complete absence of the population ($P = 0$), and an equilibrium population level ($P = M$) for which the proportionate birth and death rates are equal.

Such spotting of constant functions that are particular solutions is useful on occasion but of limited applicability. In general, more useful information can be deduced from the observation that, for any given point (x, y) in the plane, the equation

$$\frac{dy}{dx} = f(x, y) \tag{19}$$

describes the *direction* in which the graph of the particular solution through that point is heading (see Figure 3). This is because if $y = y(x)$ is any solution of the differential equation, then dy/dx is the *gradient* or *slope* of the graph of that function. Equation (19) therefore tells us that $f(x, y)$ represents the slope at (x, y) of the graph of the particular solution that passes through (x, y). If the slope $f(x, y)$ at this point is positive, then the corresponding solution graph is increasing (rising) from left to right through the point (x, y); if the slope is negative, then the graph is decreasing (falling); and if the slope is zero, then the graph is horizontal at the point.

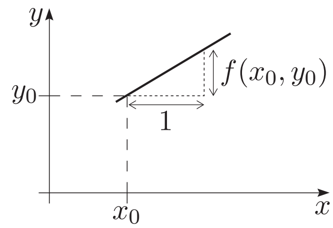


Figure 3 A graphical representation of the slope at the point (x_0, y_0)

For example, if $f(x, y) = x + y$, then the slope at the point $(1, 2)$ is $f(1, 2) = 1 + 2 = 3$, the slope at the point $(2, -7)$ is $f(2, -7) = 2 - 7 = -5$, and the slope at the point $(3, -3)$ is $f(3, -3) = 3 - 3 = 0$.

When looking at $f(x, y)$ in this light, it is referred to as a *direction field*, since it describes a *direction* (slope) for each point (x, y) where $f(x, y)$ is defined.

A **direction field** associates a unique direction to each point within a specified region of the (x, y)-plane. The direction corresponding to the point (x, y) may be thought of as the slope of a short line segment through the point.

In particular, the direction field for the differential equation

$$\frac{dy}{dx} = f(x, y)$$

associates the direction $f(x, y)$ with the point (x, y).

Direction fields can be visualised by constructing the short line segments referred to above for a finite set of points in an appropriate region of the plane, where typically the points are chosen to form a rectangular grid.

Example 4

(a) Part of the direction field for the logistic equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{1000}\right)$$

is sketched in Figure 4. Using this diagram, sketch the solution curves that pass through the points

$$(0, 1500),\ (0, 1000),\ (0, 100),\ (0, 0),\ (0, -100).$$

From your results, describe the graphs of particular solutions of the differential equation.

This is equation (18) with $k = 1$ and $M = 1000$.

We do not normally need to consider $P < 0$ since populations must be non-negative.

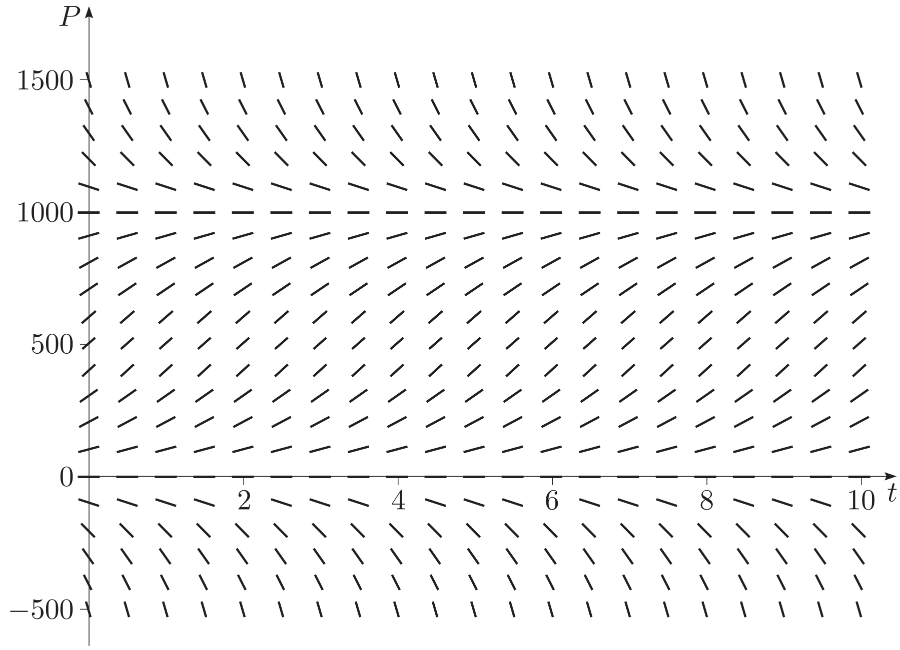


Figure 4 A direction field for the logistic equation

(b) What does your answer to part (a) tell you about the predicted behaviour of a population whose size $P(t)$ at time t is modelled by this logistic equation?

Solution

(a) The slope is shown to be zero at all points on the horizontal lines $P = 0$ and $P = 1000$, so these correspond to constant solutions of the differential equation. (As pointed out earlier in the text, these two solutions can also be spotted directly from the form of the differential equation.)

The graphs of solutions through starting points above the line $P = 1000$ appear to decrease, but at a slower and slower rate, tending from above towards the limit $P = 1000$ as t increases.

The graphs of solutions through starting points in the region $0 < P < 1000$ are increasing, with slope growing before the level $P = 500$ is reached and declining thereafter. For large values of t, these graphs tend from below towards the limit $P = 1000$.

For starting points in the region $P < 0$, the graphs decrease without limit and with steeper and steeper slope.

These various cases are illustrated by typical graphs in Figure 5.

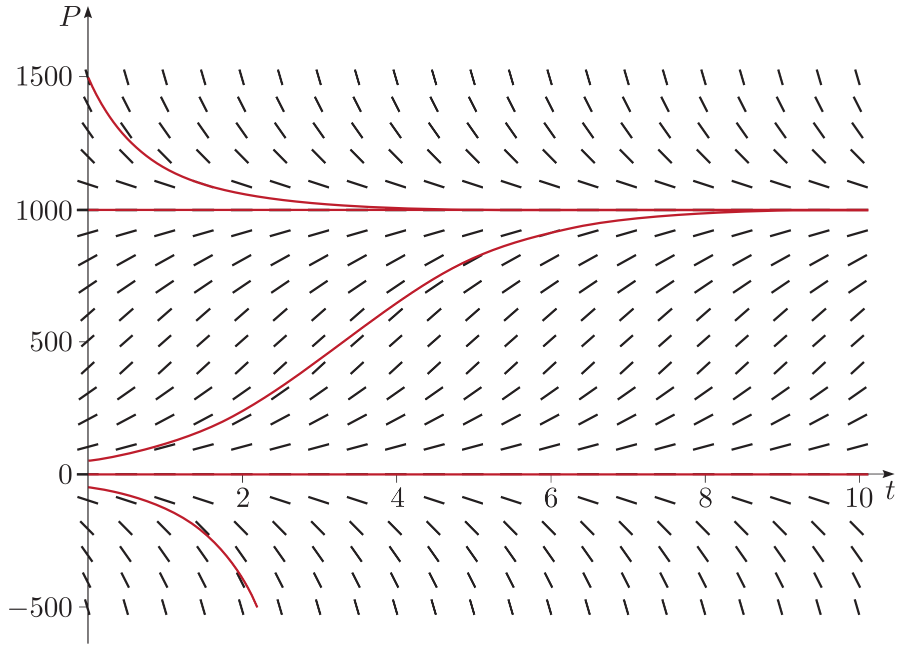


Figure 5 Some solution curves for the logistic equation

(b) If the differential equation is considered as a model of population behaviour, then the region $P < 0$ must be excluded. The analysis above leads to the following predictions for the population.

- If the population is zero at the start, then it remains zero.
- If the population size starts at 1000, then it remains fixed at this level.
- If the population starts at a level higher than 1000, then it declines (more and more gradually) towards 1000.
- If the population starts at a level below 1000 (but above 0), then it increases and eventually tends gradually towards 1000.

Drawing *by hand* precise grids of line segments to represent direction fields is not a good use of your time. However, it is a task that your computer can be programmed to perform. Furthermore, the concept of direction fields helps in constructing approximate numerical solutions for first-order differential equations.

The graphs of particular solutions of a differential equation

$$\frac{dy}{dx} = f(x, y)$$

can be 'mentally sketched' on a diagram of the direction field given by $f(x, y)$. The tangent to the solution curve is always 'parallel to the local slope' of the direction field. While this gives a good visual image of the connection between the direction field and the graph of a solution, it is somewhat short on precision. We could not expect, by this approach, to predict with any accuracy the actual solution to an initial-value problem.

Building on this idea, a solution to an initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \tag{20}$$

may be estimated by calculational means. The direction field diagram helps to explain how this numerical method arises.

Suppose that instead of trying to sketch a solution curve to fit the direction field, we move in a sequence of straight-line steps whose directions are governed by the direction field. The aim is to produce a sequence of points that provide approximate values of the solution function $y(x)$ for the initial-value problem at a sequence of x-values. The steps are constructed as follows.

Corresponding to the given initial condition $y(x_0) = y_0$, there is a point P_0 in the (x, y)-plane with coordinates (x_0, y_0), and this is our starting point. At P_0, the direction field $f(x, y)$ defines a particular slope, namely $f(x_0, y_0)$. We move off from P_0 along a straight line that has this slope, and continue until we have travelled a horizontal distance h to the right of P_0. The point that has now been reached is labelled P_1, as in Figure 6.

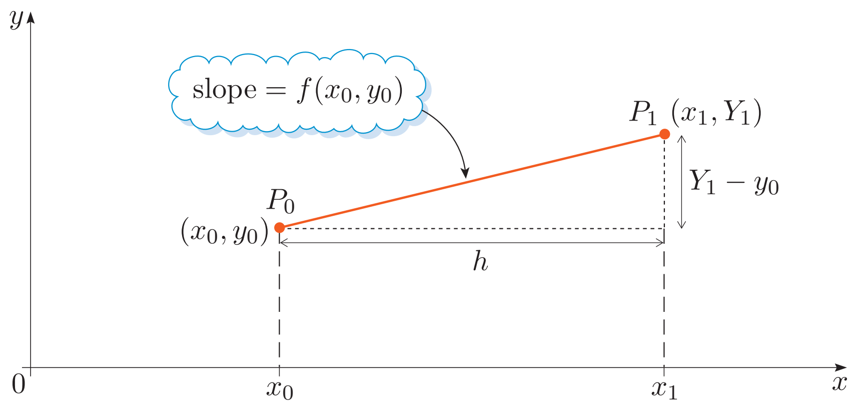


Figure 6 Using the slope at x_0 to estimate the solution at x_1

The reason for using Y_1 here, rather than y_1, will be explained shortly.

The idea is that the point P_1, whose coordinates have been denoted by (x_1, Y_1), provides an approximate value Y_1 of the solution function $y(x)$ at $x = x_1$. Now, unless the solution function follows a straight line as x moves from x_0 to x_1, Y_1 is unlikely to give the exact value of $y(x_1)$. However, the hope is that because we headed off from x_0 along the correct slope, as given by the direction field, Y_1 will be reasonably close to the exact value. Before worrying about accuracy, let us continue with the construction of the points in our sequence.

The next thing that we need to do, before proceeding to the second step in the construction process, is determine formulas for x_1 and Y_1 in terms of x_0, y_0, h and $f(x_0, y_0)$. By the construction described, as the point P_1 is reached from P_0 by taking a step to the right of horizontal length h, we have

$$x_1 = x_0 + h. \tag{21}$$

We can express Y_1 in terms of other quantities by equating two expressions for the slope of the line segment P_0P_1,

$$\frac{Y_1 - y_0}{h} = f(x_0, y_0),$$

and then rearranging to give

$$Y_1 = y_0 + h\, f(x_0, y_0). \tag{22}$$

This completes the first step, and we now take a second step to the right.

The direction of the second step is along the line with slope defined by the direction field at the point P_1, namely $f(x_1, Y_1)$. The second step moves us from P_1 through a further horizontal distance h to the right, to the point labelled P_2, as illustrated in Figure 7. This point provides an approximate value Y_2 of the solution function $y(x)$ at $x = x_2$.

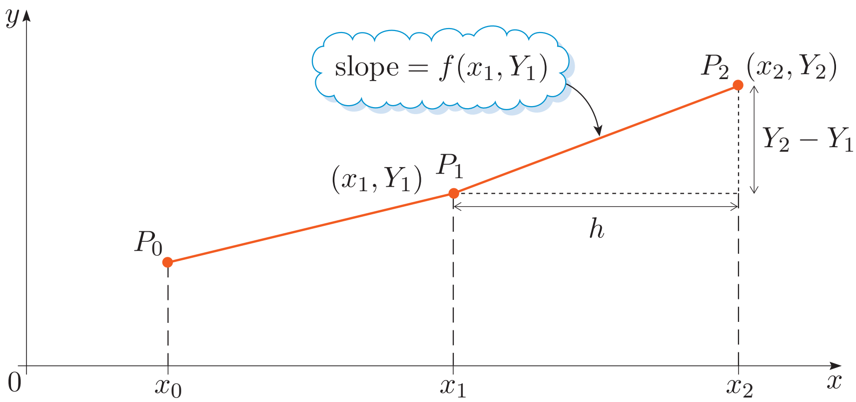


Figure 7 Estimating the value of the solution at x_2

As in the first step, we need now to express the coordinates (x_2, Y_2) of P_2 in terms of x_1, Y_1, h and $f(x_1, Y_1)$. We have

$$x_2 = x_1 + h \tag{23}$$

and (equating two expressions for the slope of the line segment P_1P_2)

$$\frac{Y_2 - Y_1}{h} = f(x_1, Y_1),$$

which can be rearranged to give

$$Y_2 = Y_1 + h\, f(x_1, Y_1). \tag{24}$$

Having carried out two steps of the process, it is possible to see that the same procedure can be applied to construct any number of further steps, and we next generalise to a description of what happens at the $(i+1)$th step, where i represents any non-negative integer.

Suppose that after i steps we have reached the point P_i, with coordinates (x_i, Y_i). For the $(i+1)$th step, we move away from P_i along the line with slope $f(x_i, Y_i)$, as defined by the direction field at P_i. After moving through a horizontal distance h to the right, we reach the point P_{i+1}, whose coordinates are denoted by (x_{i+1}, Y_{i+1}), as illustrated in Figure 8.

The point P_{i+1} provides an approximate value Y_{i+1} of the solution function $y(x)$ at $x = x_{i+1}$.

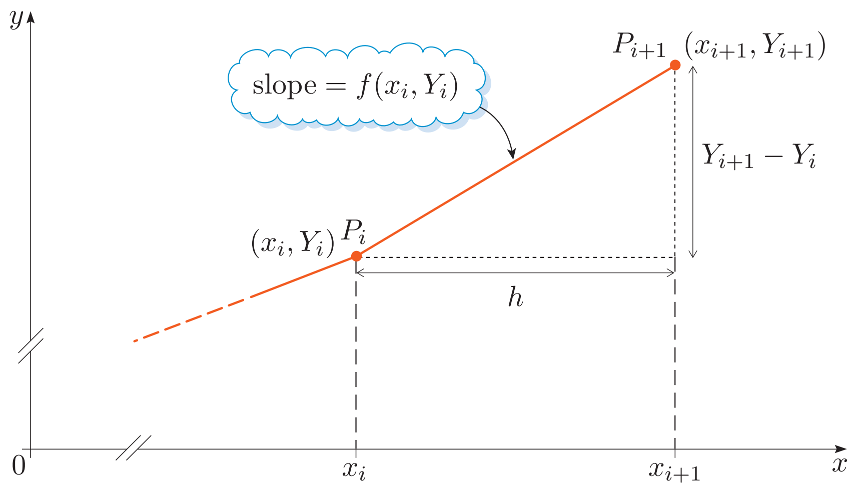


Figure 8 The $(i+1)$th step of the sequence

Arguing as before, we have

$$x_{i+1} = x_i + h \tag{25}$$

and (equating two expressions for the slope of the line segment P_iP_{i+1})

$$\frac{Y_{i+1} - Y_i}{h} = f(x_i, Y_i),$$

which can be rearranged to give

$$Y_{i+1} = Y_i + h\,f(x_i, Y_i). \tag{26}$$

Note that equations (21) and (23) are the special cases of equation (25) for $i = 0$ and $i = 1$, respectively, and that equation (24) is the special case of equation (26) for $i = 1$. If we also define Y_0 to be equal to the initial value y_0, then equation (22) is the special case of equation (26) for $i = 0$.

To sum up, for the initial-value problem (20), we have a procedure for constructing a sequence of points

$$P_i \text{ with coordinates } (x_i, Y_i) \quad (i = 1, 2, \ldots),$$

where the values of x_i and Y_i for each value of i are determined by the respective formulas (25) and (26). The starting point for the sequence is the point P_0 with coordinates (x_0, Y_0), where $Y_0 = y_0$. Because the procedure is based on the direction field, each Y_i provides an approximation at $x = x_i$ to the value of the solution function $y(x)$ for the initial-value problem. The horizontal distance h by which we move to the right at each stage of the procedure is called the **step size** (or **step length**).

Figure 9 shows the constructed sequence of points, and for comparison includes a curve representing the graph of the exact solution of the initial-value problem (20). This makes clear that the successive points $P_1, P_2, P_3, \ldots$ are only *approximations* to points on the solution curve.

In fact, the situation shown in Figure 9 is typical of the behaviour of the constructed approximations, in that they gradually move further and further from the exact solution curve. This is because at each step, the direction of movement is along the slope of the direction field at $P_i = (x_i, Y_i)$ and not along the slope of the direction field at (x_i, y_i), where $y_i = y(x_i)$ denotes the value of the exact solution function at $x = x_i$; that is, for each x_i, the slope for the next step is defined by a point close to the solution curve rather than by the point exactly on that curve.

The common use of $y_i = y(x_i)$ to represent the exact solution at $x = x_i$ explains why we use a different notation, namely Y_i, for the numerical approximation to $y(x_i)$.

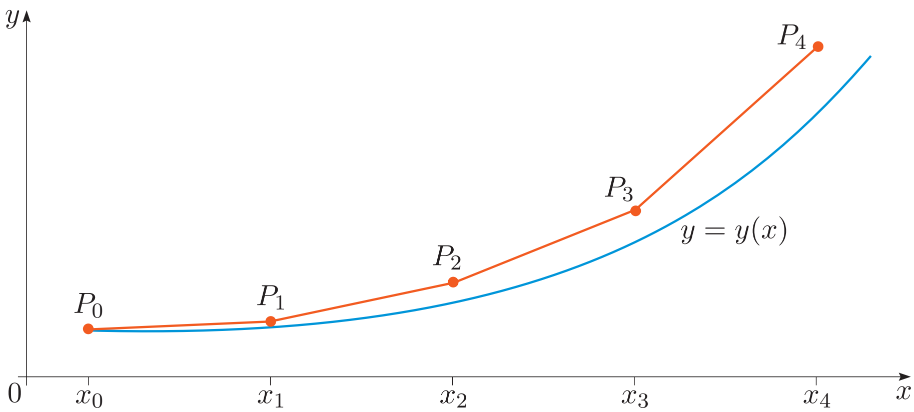


Figure 9 The exact solution and the approximate solution

Nevertheless, the formulas (25) and (26) provide a method for finding approximate solutions to the initial-value problem (20), in terms of numerical estimates $Y_1, Y_2, Y_3, \ldots$ at the respective domain values $x_1, x_2, x_3, \ldots$. This is called *Euler's method*, after Leonhard Euler (Figure 10). It is summarised below.

The accuracy of such approximate solutions, and ways of improving accuracy, will be considered shortly.

Procedure 4 Euler's method

To apply Euler's method to the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

proceed as follows.

1. Take x_0 and $Y_0 = y_0$ as starting values, choose a step size h, and set $i = 0$.
2. Calculate the x-coordinate x_{i+1}, using the formula
$$x_{i+1} = x_i + h.$$
3. Calculate a corresponding approximation Y_{i+1} to $y(x_{i+1})$, using the formula
$$Y_{i+1} = Y_i + h\, f(x_i, Y_i).$$
4. If further approximate values are required, increase i by 1 and return to Step 2.

Figure 10 Leonhard Euler (1707–1783) was one of the most prolific mathematicians of all time. (His surname is pronounced 'oiler'.) He first devised this method in order to compute the orbit of the Moon.

Example 5

Consider the initial-value problem

$$\frac{dy}{dx} = x + y, \quad y(0) = 1.$$

Use Euler's method, with step size $h = 0.2$, to obtain an approximation to $y(1)$.

Solution

We have $x_0 = 0$, $Y_0 = y_0 = 1$, and $f(x_i, Y_i) = x_i + Y_i$. The step size is given as $h = 0.2$. Equation (25) with $i = 0$ gives

$$x_1 = x_0 + h = 0 + 0.2 = 0.2,$$

and equation (26) with $i = 0$ gives

$$Y_1 = Y_0 + h\,f(x_0, Y_0) = 1 + 0.2 \times (0 + 1) = 1.2.$$

For the second step, we have (from equation (25) with $i = 1$)

$$x_2 = x_1 + h = 0.2 + 0.2 = 0.4,$$

and (from equation (26) with $i = 1$)

$$Y_2 = Y_1 + h\,f(x_1, Y_1) = 1.2 + 0.2 \times (0.2 + 1.2) = 1.48.$$

If more than a couple of steps of such a calculation have to be computed by hand, then it is a good idea to lay out the calculation as a table. In this case, by continuing as above and putting i in turn equal to 2, 3 and 4, we obtain Table 1.

Table 1

i	x_i	Y_i	$f(x_i, Y_i) = x_i + Y_i$	$Y_{i+1} = Y_i + h\,f(x_i, Y_i)$
0	0	1	1	1.2
1	0.2	1.2	1.4	1.48
2	0.4	1.48	1.88	1.856
3	0.6	1.856	2.456	2.347 2
4	0.8	2.347 2	3.147 2	2.976 64
5	1.0	2.976 64		

(After each value of Y_{i+1} has been calculated from the formula and entered in the last column, it is transferred to the Y_i column in the next row. Once the value at $x = 1.0$ has been found, no further calculations are necessary.)

So at $x = 1$, Euler's method with step size $h = 0.2$ gives the approximation $y(1) \simeq 2.976\,64$.

Exercise 13

Use Euler's method, with step size $h = 0.2$, to obtain an approximation to $y(1)$ for the initial-value problem

$$\frac{dy}{dx} = y, \quad y(0) = 1.$$

The solution to the initial-value problem in Example 5 is in fact known exactly, and is $y = 2e^x - x - 1$. Putting $x = 1$, this gives

$$y(1) = 2e - 1 - 1 = 3.436\,56,$$

correct to five decimal places. This value may be compared with the approximation 2.976 64 for $y(1)$ obtained by Euler's method in Example 5, and the comparison indicates that the approximation is not at all accurate. Indeed, the *absolute error* in this case is

$$|2.976\,64 - 3.436\,56| = 0.459\,92,$$

which is about 13% of the exact value, and indeed not even one decimal place accuracy is achieved.

The absolute error is defined to be the magnitude of the difference between the approximate value and the exact value.

Similarly, the other values Y_i $(i = 1, 2, 3, 4)$ found in Example 5 contain significant absolute errors when considered as approximations to the corresponding exact values $y_i = y(x_i)$. This is illustrated in general terms in Figure 9, where the absolute error in approximation Y_i is the vertical distance from the point P_i to the point directly below it on the exact solution curve. As shown here, and for reasons given earlier, the absolute error tends to increase as more and more steps are taken.

The realisation that Euler's method can produce values that are poor approximations to the exact solution to an initial-value problem invites us to ask whether the accuracy of the approximations can be improved using this method. In fact, it is not hard to see that improvements in accuracy ought to be achieved by *reducing the step size h*. Our earlier prescription for constructing the sequence of points $P_1, P_2, P_3, \ldots$ from the starting point P_0 and the given direction field amounts loosely to saying 'match the direction of the solution curve at the current point, take a step, then adjust direction so as to try not to move further away from the curve'. It seems natural, therefore, that the approximations will improve if we reduce the size of the steps taken and correspondingly 'adjust direction' more frequently. This is illustrated for a hypothetical case in Figure 11.

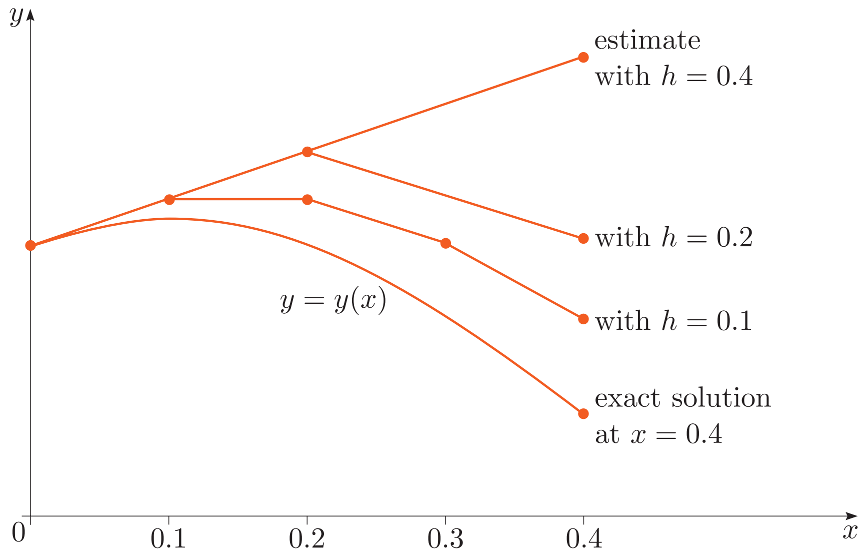


Figure 11 Comparing the approximate solution with the exact solution as h decreases

In fact, it can be shown that the accuracy of Euler's method does indeed usually improve when we take a smaller step size.

To demonstrate this, consider the initial-value problem from Exercise 13. This has the exact solution $y = e^x$ (as you can verify), and its value at $x = 1$ is $y(1) = e = 2.718\,282$, to six decimal places. In Exercise 13 you showed that with a step size $h = 0.2$, Euler's method gives the approximation 2.488 32 for $y(1)$. Table 2 shows the corresponding results (to six decimal places) obtained when we apply Euler's method to this same initial-value problem but with successively smaller step sizes h.

In Exercise 13, where $h = 0.2$, the value of $y(1)$ was approximated by Y_5. From the column for 'Number of steps' in Table 2, you can see that $y(1)$ is approximated by:

Y_{10} when $h = 0.1$;
Y_{100} when $h = 0.01$;
Y_{1000} when $h = 0.001$;
Y_{10000} when $h = 0.0001$.

Table 2

h	Approximation to $y(1)$	Absolute error	Number of steps
0.1	2.593 742	0.124 539	10
0.01	2.704 814	0.013 468	100
0.001	2.716 924	0.001 358	1000
0.0001	2.718 146	0.000 136	10000

As expected, the absolute errors in the third column of the table become progressively smaller as h is reduced.

Looking more carefully at these absolute errors, we notice that they seem to tend towards a sequence in which each number is a tenth of the previous one. Since each value of h in the table is a tenth of the previous one, this suggests that

absolute error is approximately proportional to step size h.

This turns out to be a general property of Euler's method, for sufficiently small values of the step size. So not only do we know that accuracy can be improved by decreasing the step size h, but this general property also tells us that by making h small enough, the absolute error in an approximation can be made as small as desired. In other words, the absolute error approaches the limit zero as h approaches zero (as you might have expected from the intuitive argument preceding Figure 11).

Exercise 14

Suppose that when Euler's method is applied to the problem in Exercise 13, the absolute error in approximating $y(1)$ is proportional to the step size h, for sufficiently small h.

Use the last row of Table 2 to estimate the constant of proportionality, k say, and hence estimate the step size required to compute $y(1)$ correct to five decimal places (i.e. so that the absolute error is less than 5×10^{-6}).

A few words of caution are necessary at this point. Although the absolute error can be made as small as we please by making the step size h sufficiently small, this is valid *only* if the arithmetic is performed using sufficient decimal places. Where a calculator or computer is involved, the number of decimal places that can be used is limited, and as a result *rounding errors* may be introduced into the calculations. After a certain point, any increase in accuracy brought about by reducing the size of h may be swamped by these rounding errors.

Moreover, rounding errors are not the only problem. Before concluding that h should always be chosen to be very small, we must also consider the cost of this additional accuracy. Now, by *cost* is meant the effort involved, which can be measured in a variety of ways; commonly for iterative methods (such as Euler's method) it is measured by the number of steps taken. In general for numerical methods, the greater the accuracy required, the greater the cost. To illustrate this, look back at Table 2. The last column of the table shows how the number of steps required for the calculation goes up in inverse proportion to the step size: to move from $x = 0$ to $x = 1$, it takes 10 steps with step size $h = 0.1$, 100 steps with step size $h = 0.01$, and so on. Since, for sufficiently small h, the error in Euler's method is approximately proportional to the step size, it follows that for this method a tenfold improvement in accuracy is paid for by a tenfold increase in the number of steps required.

In general, to move from a to b (where $b > a$) with step size h takes $(b - a)/h$ steps.

So for Euler's method and similar methods, the choice of step size has to be based on a compromise between the two opposing requirements of accuracy and cost. There are better numerical methods for solving initial-value problems that are considerably more *efficient* than Euler's method. In fact, Euler's method is not suitable for high-accuracy work. Its virtue lies rather in its simplicity and its clear illustration of the basic principles of how differential equations may be solved numerically.

Greater *efficiency* means that the same or better numerical accuracy is achieved with fewer numerical computations.

In any practical case, calculations of the type described in this subsection are ideally suited to being performed on a computer.

Exercise 15

(a) Without plotting the direction field, say what you can about the slopes defined by the differential equation

$$\frac{dy}{dx} = f(x,y) = y + x^2.$$

(b) Verify that your conclusions are consistent with the direction field diagram in Figure 12.

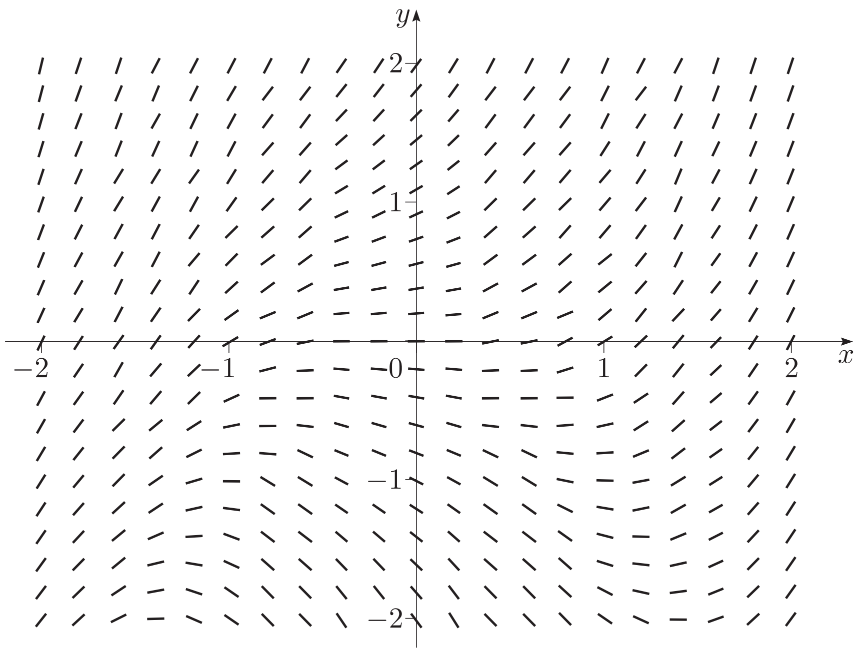


Figure 12 A direction field

(c) On the basis of the direction field, what can be said about the graphs of solutions of the differential equation?

(d) Write down the formulas required in order to apply Euler's method to the initial-value problem

$$\frac{dy}{dx} = y + x^2, \quad y(-1) = -0.2,$$

using a step size $h = 0.1$.

2 Homogeneous second-order differential equations

You will recall that a *particular solution* of a first-order differential equation is obtained by applying a *single* condition (known as an *initial condition*) to the general solution in order to find a particular value of the single arbitrary constant. In the case of a second-order differential equation, a particular solution is obtained by applying *two* conditions to the general solution in order to find particular values of the *two* arbitrary constants. The following example illustrates this.

Example 6

Suppose that a car is travelling with constant acceleration a along a straight road. If, at time t, its distance from a fixed point is s, then its velocity is given by ds/dt, its acceleration is given by d^2s/dt^2, and its motion is modelled by

$$\frac{d^2s}{dt^2} = a. \tag{27}$$

If the car is initially stationary at position $s = 0$ and thereafter has a constant acceleration of $2\,\mathrm{m\,s^{-2}}$, how long does it take for the car to attain a velocity of $30\,\mathrm{m\,s^{-1}}$, and what distance has it travelled in that time?

Solution

Integrating equation (27) leads to

$$\frac{ds}{dt} = at + C \quad \text{and} \quad s = \tfrac{1}{2}at^2 + Ct + D,$$

where C and D are arbitrary constants. To find these constants (and hence answer the questions asked), we need to make use of the conditions given. These are that the car is initially stationary (i.e. $ds/dt = 0$ when $t = 0$) at position $s = 0$ (i.e. $s = 0$ when $t = 0$). The first of these conditions together with the equation $ds/dt = at + C$ tells us that $C = 0$. With $C = 0$, the second equation becomes $s = \frac{1}{2}at^2 + D$, and this together with the second condition tells us that $D = 0$.

Therefore when $a = 2$, we have

$$\frac{d^2s}{dt^2} = 2, \quad \frac{ds}{dt} = 2t, \quad s = t^2.$$

In mathematical modelling, the parameters of the model, in this case a, are retained for as long as possible, in order to generate results that could be used in other situations, with different values.

So the velocity is $ds/dt = 30$ when $2t = 30$, that is, after 15 seconds, and in this time the car has travelled a distance $s = 15^2$, that is, 225 metres.

The solution of second-order differential equations is rarely as easy as the solution of equation (27) above. In fact, the approach of repeated direct integration works for only some equations of the form

$$\frac{d^2y}{dx^2} = f(x).$$

Most second-order differential equations cannot be solved by analytic methods at all, and numerical methods have to be employed instead. However, there is one important class of second-order differential equations that can be solved by analytic means; this is the topic of this section, and we introduce it next.

2.1 Linear constant-coefficient differential equations

The rest of this unit considers *linear constant-coefficient* second-order differential equations. But what exactly do the terms 'linear' and 'constant-coefficient' mean in this context? The answers lie in the following definitions.

Compare the definitions for first-order equations in Subsection 1.4. The important feature is the *linear* combination of y and its derivatives on the left-hand side.

A second-order differential equation for $y = y(x)$ is **linear** if it can be expressed in the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)\,y = f(x), \tag{28}$$

where $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are given continuous functions.

A linear second-order differential equation is **constant-coefficient** if the functions $a(x)$, $b(x)$ and $c(x)$ are all constant, so that the equation is of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{29}$$

where $a \neq 0$.

If $a = 0$, then the equation is first-order.

A linear constant-coefficient second-order differential equation is said to be **homogeneous** if $f(x) = 0$ for all x, and **inhomogeneous** (or **non-homogeneous**) otherwise.

Linear constant-coefficient second-order differential equations can be written in other ways. For example, we can divide equation (29) through by a to obtain an equation of the form

$$\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + \gamma y = \phi(x),$$

and this more closely resembles the definition of linear first-order differential equations from Section 1.

Exercise 16

Consider the following second-order differential equations.

(i) $\dfrac{d^2y}{dx^2} = x^2$ (ii) $3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = x^2$ (iii) $3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$

(iv) $xy'' + x^2y = 0$ (v) $2y\dfrac{d^2y}{dx^2} + 4y = 3\dfrac{dy}{dx}$

(vi) $2\dfrac{d^2t}{d\theta^2} + 3\dfrac{dt}{d\theta} + 4t = \sin\theta$ (vii) $\ddot{x} = -4t$ (viii) $\ddot{x} = -4x$

(a) Which of the equations are both linear and constant-coefficient?

(b) Which of the linear constant-coefficient equations are homogeneous?

(c) For each equation, identify the dependent and independent variables.

One of the main reasons for concentrating on linear constant-coefficient differential equations is that there is a large body of theory on which we can call in order to solve them. Subsection 2.2 illustrates this.

Principle of superposition

A key theoretical result will turn out to be extremely useful throughout this module. This is known as the *principle of superposition*, and it is a fundamental property of linear differential equations.

Suppose that we have a solution $y_1(x)$ of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_1(x),$$

Here a, b and c can be functions of x.

and a solution $y_2(x)$ of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_2(x).$$

Then we claim that the linear combination $k_1y_1 + k_2y_2$, where k_1 and k_2 are constants, is a solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = k_1\, f_1(x) + k_2\, f_2(x). \tag{30}$$

In fact, this is easy to see, for if we substitute $k_1y_1 + k_2y_2$ directly into equation (30), we obtain

$$\begin{aligned} &a\frac{d^2}{dx^2}\left(k_1y_1 + k_2y_2\right) + b\frac{d}{dx}\left(k_1y_1 + k_2y_2\right) + c\left(k_1y_1 + k_2y_2\right) \\ &\quad = a\left(k_1\frac{d^2y_1}{dx^2} + k_2\frac{d^2y_2}{dx^2}\right) + b\left(k_1\frac{dy_1}{dx} + k_2\frac{dy_2}{dx}\right) + c\left(k_1y_1 + k_2y_2\right) \\ &\quad = k_1\left(a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right) + k_2\left(a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2\right) \\ &\quad = k_1\, f_1(x) + k_2\, f_2(x), \end{aligned}$$

as required.

We summarise this important result as a theorem.

Theorem 2 Principle of superposition

If $y_1(x)$ is a solution of the linear second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_1(x),$$

and $y_2(x)$ is a solution of the linear second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f_2(x)$$

(with the same left-hand side), then the function

$$y(x) = k_1\, y_1(x) + k_2\, y_2(x),$$

where k_1 and k_2 are constants, is a solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = k_1\, f_1(x) + k_2\, f_2(x).$$

2.2 Method of solution

This subsection develops a method for solving homogeneous linear constant-coefficient second-order differential equations, that is, equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{31}$$

where a, b, c are constants and $a \neq 0$.

To see how this method arises, consider the *first*-order differential equation

$$b\frac{dy}{dx} + cy = 0, \tag{32}$$

where b and c are constants and $b \neq 0$. This homogeneous linear equation, which can be solved using the integrating factor method, has a general solution of the form $y = Ae^{\lambda x}$, where A is an arbitrary constant and λ is some fixed constant. To find λ, we can substitute $y = Ae^{\lambda x}$ and $dy/dx = \lambda Ae^{\lambda x}$ into equation (32) to give

$$b\frac{dy}{dx} + cy = b\lambda Ae^{\lambda x} + cAe^{\lambda x} = (b\lambda + c)Ae^{\lambda x}.$$

Therefore, for $y = Ae^{\lambda x}$ to be a solution, $(b\lambda + c)Ae^{\lambda x}$ must be zero for all x. Since A is arbitrary and $e^{\lambda x} > 0$ for all x, we must have $b\lambda + c = 0$, that is, $\lambda = -c/b$.

This useful idea of substituting $y = Ae^{\lambda x}$ as a possible solution can be applied to equation (31) as well. Let us suppose that equation (31) has a solution of the form $y = Ae^{\lambda x}$, for some value of λ. If so, then

$dy/dx = \lambda A e^{\lambda x}$ and $d^2y/dx^2 = \lambda^2 A e^{\lambda x}$, and substituting into the left-hand side of equation (31) gives

$$\begin{aligned} a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= a\lambda^2 A e^{\lambda x} + b\lambda A e^{\lambda x} + cAe^{\lambda x} \\ &= (a\lambda^2 + b\lambda + c)Ae^{\lambda x}. \end{aligned}$$

Hence $y = Ae^{\lambda x}$ is indeed a solution of equation (31), for any value of A, provided that λ satisfies

$$a\lambda^2 + b\lambda + c = 0.$$

Note that the discussion here applies irrespective of whether λ is real or complex. The consequences of λ being complex are explained later.

This equation plays such an important role in solving linear constant-coefficient second-order differential equations that it is given a special name.

The **auxiliary equation** of the homogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

is the quadratic equation

$$a\lambda^2 + b\lambda + c = 0. \tag{33}$$

The auxiliary equation is sometimes called the *characteristic equation*.

The auxiliary equation is obtained from the differential equation by replacing y by 1, $\dfrac{dy}{dx}$ by λ, and $\dfrac{d^2y}{dx^2}$ by λ^2.

Example 7

Write down the auxiliary equation of the differential equation

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0.$$

Solution

The auxiliary equation is

$$3\lambda^2 - 2\lambda + 4 = 0.$$

Exercise 17

Write down the auxiliary equation of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ (b) $y'' - 9y = 0$ (c) $\ddot{x} + 2\dot{x} = 0$

Now, so far, we know that $y = Ae^{\lambda x}$ is a solution of equation (31) provided that λ satisfies its auxiliary equation. But the auxiliary equation is a quadratic equation with real coefficients, so it has two roots (which in general are distinct). These two roots, λ_1 and λ_2 say, give two solutions $y_1 = Ce^{\lambda_1 x}$ and $y_2 = De^{\lambda_2 x}$ of equation (31), where C and D are arbitrary constants.

If $\lambda_1 = \lambda_2$, then we obtain only one solution. This case is dealt with separately below.

Example 8

(a) Write down the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$

and find its roots λ_1 and λ_2.

(b) Deduce that $y_1 = Ce^x$ and $y_2 = De^{2x}$ are both solutions of the differential equation, for any values of the two constants C and D.

(c) Show that $y = Ce^x + De^{2x}$ is also a solution of the differential equation, for any values of the two constants C and D.

Solution

(a) The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0.$$

This equation may be solved, for example, by factorising in the form $(\lambda - 1)(\lambda - 2) = 0$, to give the two roots $\lambda_1 = 1$ and $\lambda_2 = 2$.

Using the formula

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

produces the same answer.
It does not matter which of the roots is called λ_1 and which is called λ_2.

(b) Since $\lambda_1 = 1$ and $\lambda_2 = 2$ are the roots of the auxiliary equation, $y_1 = Ce^x$ and $y_2 = De^{2x}$ are solutions of the differential equation, for any values of C and D.

(c) To show that $y = Ce^x + De^{2x}$ is a solution of the differential equation, we differentiate and substitute into the differential equation. Differentiating to obtain the first and second derivatives of y gives

$$\frac{dy}{dx} = Ce^x + 2De^{2x} \quad \text{and} \quad \frac{d^2y}{dx^2} = Ce^x + 4De^{2x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\begin{aligned}
&\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \\
&\quad = \left(Ce^x + 4De^{2x}\right) - 3\left(Ce^x + 2De^{2x}\right) + 2\left(Ce^x + De^{2x}\right) \\
&\quad = C(1 - 3 + 2)e^x + D(4 - 6 + 2)e^{2x} \\
&\quad = 0.
\end{aligned}$$

Hence $y = Ce^x + De^{2x}$ is a solution of the differential equation, for any values of C and D.

It can be seen that if λ_1 and λ_2 are *distinct* roots of the auxiliary equation of a homogeneous linear constant-coefficient second-order differential equation, then any solution of the form

$$y = Ce^{\lambda_1 x} + De^{\lambda_2 x}, \tag{34}$$

for some choice of constants C and D, is also a solution. Furthermore, it can be shown that a solution of this form, for distinct roots of the auxiliary equation, and involving two arbitrary constants, is the general solution of the homogeneous linear second-order differential equation.

Exercise 18

Use the auxiliary equation to find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$ (b) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = 0$ (c) $\dfrac{d^2z}{du^2} - 4z = 0$

We now consider an example where the two roots of the auxiliary equation are *equal*, in which case the above method does not work! Indeed, in light of the earlier discussion, you might expect the solution always to be of the form $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$, where A and B are arbitrary constants. But if $\lambda_1 = \lambda_2$, this reduces to $y = (A + B)e^{\lambda_1 x} = Ce^{\lambda_1 x}$, where $C = A + B$ is a *single* arbitrary constant, so this cannot be the general solution of a *second*-order differential equation, which requires two arbitrary constants.

Example 9

(a) Write down the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0,$$

and find its roots λ_1 and λ_2.

(b) Deduce that $y_1 = Ce^{-3x}$ is a solution of the differential equation, for any value of the constant C.

(c) Show that $y_2 = Dxe^{-3x}$ is also a solution, for any value of the constant D.

(d) Deduce that $y = (C + Dx)e^{-3x}$ is also a solution of the differential equation, for any values of the two constants C and D.

Solution

(a) The auxiliary equation is

$$\lambda^2 + 6\lambda + 9 = 0.$$

The left-hand side is the perfect square $(\lambda + 3)^2$, so the auxiliary equation has equal roots $\lambda_1 = \lambda_2 = -3$.

Note that the 'other' root $\lambda_2 = -3$ gives the same solution.

(b) Since $\lambda_1 = -3$ is a root of the auxiliary equation, $y_1 = Ce^{-3x}$ is a solution of the differential equation, for any value of C.

(c) To show that $y_2 = Dxe^{-3x}$ is a solution of the differential equation, we differentiate and substitute into the differential equation. Differentiating to obtain the first and second derivatives of y_2 gives

Here we are using the product rule for differentiation.

$$\frac{dy_2}{dx} = De^{-3x} + Dx\left(-3e^{-3x}\right) = D(1-3x)e^{-3x},$$
$$\frac{d^2y_2}{dx^2} = -3De^{-3x} + D(1-3x)(-3e^{-3x}) = D(-6+9x)e^{-3x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_2}{dx^2} + 6\frac{dy_2}{dx} + 9y_2 &= D(-6+9x)e^{-3x} + 6D(1-3x)e^{-3x} + 9Dxe^{-3x} \\ &= D(-6+6)e^{-3x} + D(9-18+9)xe^{-3x} \\ &= 0.\end{aligned}$$

Hence $y_2 = Dxe^{-3x}$ is a solution of the differential equation, for any value of D.

(d) Since $y_1 = Ce^{-3x}$ and $y_2 = Dxe^{-3x}$ are both solutions of the differential equation, the principle of superposition tells us that so is $y = Ce^{-3x} + Dxe^{-3x} = (C + Dx)e^{-3x}$, for any values of C and D.

The solution in Example 9 is of the form $y = Ce^{\lambda_1 x} + Dxe^{\lambda_1 x}$. The extra x in the second term, $Dxe^{\lambda_1 x}$, is needed, in this special case, to incorporate the second arbitrary constant required by the general solution of a second-order differential equation.

In general, when $\lambda_1 = \lambda_2$, $y = xe^{\lambda_1 x}$ is a solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{35}$$

where a, b, c are constants and $a \neq 0$.

To see this, differentiate twice to obtain

$$\frac{dy}{dx} = e^{\lambda_1 x} + \lambda_1 xe^{\lambda_1 x} = (1 + \lambda_1 x)e^{\lambda_1 x},$$
$$\frac{d^2y}{dx^2} = \lambda_1 e^{\lambda_1 x} + \lambda_1(1 + \lambda_1 x)e^{\lambda_1 x} = (2\lambda_1 + \lambda_1^2 x)e^{\lambda_1 x},$$

and substitute into the left-hand side of equation (35) to obtain

$$\begin{aligned}&a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy \\ &\quad= a\big((2\lambda_1 + \lambda_1^2 x)e^{\lambda_1 x}\big) + b\big((1 + \lambda_1 x)e^{\lambda_1 x}\big) + c\big(xe^{\lambda_1 x}\big) \\ &\quad= e^{\lambda_1 x}\left(a(2\lambda_1 + \lambda_1^2 x) + b(1 + \lambda_1 x) + cx\right) \\ &\quad= e^{\lambda_1 x}\left((2a\lambda_1 + b) + (a\lambda_1^2 + b\lambda_1 + c)x\right).\end{aligned} \tag{36}$$

Since λ_1 is the solution of the auxiliary equation, we have $a\lambda_1^2 + b\lambda_1 + c = 0$. Also, the formula method for solving the auxiliary equation $a\lambda^2 + b\lambda + c = 0$ gives

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since in this case we have equal roots, we must have $b^2 - 4ac = 0$, so $\lambda_1 = -b/2a$, that is, $2a\lambda_1 + b = 0$. Thus the right-hand side of equation (36) is zero, and $y = xe^{\lambda_1 x}$ is indeed a solution of equation (35). Therefore when $\lambda_1 = \lambda_2$, by the principle of superposition,

$$y = Ce^{\lambda_1 x} + Dxe^{\lambda_1 x} = (C + Dx)e^{\lambda_1 x}, \tag{37}$$

where C and D are arbitrary constants, is always a solution of equation (35).

Exercise 19

Use the auxiliary equation to find the general solution of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ (b) $\ddot{s} - 4\dot{s} + 4s = 0$

Equations (34) and (37) give us the general solution of equation (35) for the cases where the roots λ_1 and λ_2 of the auxiliary equation are distinct and equal, respectively. However, the distinct roots of a quadratic equation may not be real – they could consist of a pair of complex conjugate roots $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. If the auxiliary equation has such a pair of roots, we can still write the general solution in the form

Recall that the complex conjugate of the complex number $\alpha + \beta i$ is $\alpha - \beta i$.

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x},$$

but we now have a complex-valued solution.

You will soon see why we use A and B for the arbitrary constants (rather than our usual choice of C and D).

Since equation (35) has real coefficients, we would like a real-valued solution. In order to achieve this, we need to allow A and B to be complex. Then we can use Euler's formula, which tells us that

Euler's formula is given in the Handbook.

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

Now

$$\begin{aligned}
y &= Ae^{\lambda_1 x} + Be^{\lambda_2 x} \\
&= Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \\
&= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\
&= Ae^{\alpha x}(\cos\beta x + i\sin\beta x) + Be^{\alpha x}(\cos\beta x - i\sin\beta x) \\
&= e^{\alpha x}\big((A+B)\cos\beta x + (Ai - Bi)\sin\beta x\big) \\
&= e^{\alpha x}(C\cos\beta x + D\sin\beta x),
\end{aligned}$$

The constants in the final expression are now C and D, in keeping with our previous solutions.

where $C = A + B$ and $D = (A - B)i$. Provided that any initial conditions are real-valued, C and D are real, and this is the required real-valued solution containing two arbitrary constants.

Example 10

(a) Write down the auxiliary equation of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0,$$

and show that its roots are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

(b) Confirm that $y_1 = e^{3x}\cos 2x$ and $y_2 = e^{3x}\sin 2x$ are both solutions of the differential equation.

(c) Deduce that $y = e^{3x}(C\cos 2x + D\sin 2x)$ is also a solution of the differential equation, for any values of the two constants C and D.

Solution

(a) The characteristic equation is

$$\lambda^2 - 6\lambda + 13 = 0.$$

The formula method gives

$$\lambda = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i,$$

so the two complex conjugate roots are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

With the previous notation we have $\alpha = 3$ and $\beta = 2$.

(b) To confirm that $y_1 = e^{3x}\cos 2x$ is a solution of the differential equation, we differentiate and substitute into the differential equation. Differentiating to obtain the first and second derivatives of y_1 gives

$$\frac{dy_1}{dx} = 3e^{3x}\cos 2x + e^{3x}(-2\sin 2x) = e^{3x}(3\cos 2x - 2\sin 2x),$$

$$\begin{aligned}\frac{d^2y_1}{dx^2} &= 3e^{3x}(3\cos 2x - 2\sin 2x) + e^{3x}(-6\sin 2x - 4\cos 2x)\\ &= e^{3x}(5\cos 2x - 12\sin 2x).\end{aligned}$$

Substituting these into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_1}{dx^2} - 6\frac{dy_1}{dx} + 13y_1 &= e^{3x}(5\cos 2x - 12\sin 2x)\\ &\quad - 6e^{3x}(3\cos 2x - 2\sin 2x) + 13e^{3x}\cos 2x\\ &= e^{3x}[(5 - 18 + 13)\cos 2x + (-12 + 12)\sin 2x]\\ &= 0.\end{aligned}$$

Hence $y_1 = e^{3x}\cos 2x$ is a solution.

Similarly, for $y_2 = e^{3x}\sin 2x$ we have

$$\frac{dy_2}{dx} = e^{3x}(2\cos 2x + 3\sin 2x), \qquad \frac{d^2y_2}{dx^2} = e^{3x}(12\cos 2x + 5\sin 2x),$$

and substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_2}{dx^2} - 6\frac{dy_2}{dx} + 13y_2 &= e^{3x}[(12 - 12)\cos 2x + (5 - 18 + 13)\sin 2x]\\ &= 0.\end{aligned}$$

Hence $y_2 = e^{3x} \sin 2x$ is also a solution.

(c) Since $y_1 = e^{3x} \cos 2x$ and $y_2 = e^{3x} \sin 2x$ are both solutions of the differential equation, the principle of superposition tells us that so is

$$y = Ce^{3x} \cos 2x + De^{3x} \sin 2x = e^{3x}(C \cos 2x + D \sin 2x),$$

for any values of C and D.

Exercise 20

Use the auxiliary equation to find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 8y = 0$ (b) $\dfrac{d^2\theta}{dt^2} + 9\theta = 0$

We now summarise the method for solving these differential equations as a procedure.

Procedure 5 Solving homogeneous linear constant-coefficient second-order differential equations

The general solution of the homogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where a, b, c are (real) constants and $a \neq 0$, may be found as follows.

1. Write down the auxiliary equation

   $$a\lambda^2 + b\lambda + c = 0,$$

   and find its roots λ_1 and λ_2.

2. - If the auxiliary equation has two distinct real roots λ_1 and λ_2, the general solution of the differential equation is

     $$y = Ce^{\lambda_1 x} + De^{\lambda_2 x}.$$

   - If the auxiliary equation has two equal real roots $\lambda_1 = \lambda_2$, the general solution of the differential equation is

     $$y = (C + Dx)e^{\lambda_1 x}.$$

   - If the auxiliary equation has a pair of complex conjugate roots $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, the general solution of the differential equation is

     $$y = e^{\alpha x}(C \cos \beta x + D \sin \beta x).$$

   In each case, C and D are arbitrary constants.

It is worth noting that the three cases in Step 2 of Procedure 5 correspond to three different possibilities that arise when solving the characteristic equation $a\lambda^2 + b\lambda + c = 0$. These three different possibilities relate to the value of the **discriminant** $b^2 - 4ac$, where $b^2 - 4ac > 0$ corresponds to the first case, $b^2 - 4ac = 0$ to the second, and $b^2 - 4ac < 0$ to the third.

Exercise 21

Find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} + 4y = 0$ (b) $u''(x) - 6u'(x) + 8u(x) = 0$

(c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$ (d) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$

(e) $\dfrac{d^2y}{dx^2} - \omega^2 y = 0$, where ω is a real constant

(f) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 29y = 0$

Exercise 22

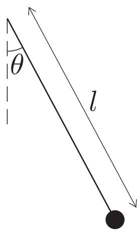


Figure 13 A pendulum

Small oscillations of the pendulum of a clock can be modelled by the differential equation

$$\ddot{\theta} = -\frac{g}{l}\theta,$$

where g is the magnitude of the acceleration due to gravity, l is the length of the pendulum, and θ is the angle that the pendulum makes with the vertical (see Figure 13).

Solve the differential equation to obtain an expression for θ in terms of g and l.

3 Inhomogeneous second-order differential equations

Section 2 was concerned with finding the general solution of homogeneous linear constant-coefficient second-order differential equations. This section concerns *inhomogeneous* linear constant-coefficient second-order differential equations, that is, equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{38}$$

where a, b, c are real constants, $a \neq 0$, and $f(x)$ is a given continuous real-valued function of x.

Subsection 3.1 gives the general method for constructing solutions of equation (38). Subsection 3.2 shows how to find an appropriate particular solution of the differential equation, for use in constructing the general solution, in cases where the function $f(x)$ takes one of a few particular forms. Subsection 3.3 deals with certain cases where complications can arise. Subsection 3.4 shows how to deal with cases where $f(x)$ is a combination of the functions discussed in Subsection 3.2.

3.1 General method of solution

The basic method used for finding the general solution of equation (38) depends on the principle of superposition (Theorem 2) and is illustrated in the following example.

Example 11

Show that $y = Ce^{-2x} + De^{-3x} + 2$ is a solution of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12, \tag{39}$$

for any values of the constants C and D.

Solution

We know from Exercise 18(a) that the homogeneous differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \tag{40}$$

has a general solution $y_\mathrm{c} = Ce^{-2x} + De^{-3x}$, where C and D are arbitrary constants.

The notation y_c and y_p will be explained shortly.

Now consider the constant function $y_\mathrm{p} = 2$. This is a particular solution of equation (39) since $d^2y_\mathrm{p}/dx^2 = dy_\mathrm{p}/dx = 0$ and $6y_\mathrm{p} = 12$.

Therefore, by the principle of superposition (Theorem 2),

$$y = y_\mathrm{c} + y_\mathrm{p} = Ce^{-2x} + De^{-3x} + 2$$

is a solution of equation (39), for any values of C and D.

Equation (40) is an example of an *associated homogeneous equation* – that is, the homogeneous equation associated with the inhomogeneous equation (39) by making its right-hand side zero. The solutions y_c and y_p also have special names in this context: y_c, the general solution of the associated homogeneous equation (40), is called the *complementary function*, and y_p, a particular solution of the inhomogeneous equation (39), is called a *particular integral*.

(The term *particular integral* is used here, rather than the term particular solution used in some other texts, to distinguish it from the particular solution of equation (41) that satisfies given initial or boundary conditions (see Section 4).)

Let

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{41}$$

be an inhomogeneous linear constant-coefficient second-order differential equation.

Its **associated homogeneous equation** is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

The general solution y_{c} of the associated homogeneous equation is known as the **complementary function** for the original inhomogeneous equation (41).

Any particular solution y_{p} of the original inhomogeneous equation (41) is referred to as a **particular integral** for that equation.

Later in this section we will show how to find particular integrals for a wide variety of equations. Before we do that, it is important that you realise the full significance of finding just one particular integral.

Exercise 23

Suppose that we have found two different particular integrals y_{p_1}, y_{p_2} for equation (41). Use the principle of superposition to show that the function $y_{\mathrm{p}_1} - y_{\mathrm{p}_2}$ is then a solution of the associated homogeneous equation.

The result of Exercise 23 shows the true significance of finding a particular integral. For if we do so, then since from Section 2 we know how to solve the associated homogeneous equation, we can find *all* particular integrals simply by adding the complementary function (which involves two arbitrary constants). We have the following important result.

That is, Section 2 enables us to find the complementary function.

Theorem 3

If y_{c} is the complementary function for an inhomogeneous linear constant-coefficient second-order differential equation, and y_{p} is a particular integral for that equation, then $y_{\mathrm{c}} + y_{\mathrm{p}}$ is the general solution of that equation.

Note that y_{c}, being the general solution of the associated homogeneous equation, will contain *two* arbitrary constants, whereas y_{p}, being a particular solution, will contain none.

Let us now see how the method based on the above theorem can be applied.

Example 12

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 9x + 9. \tag{42}$$

Solution

The associated homogeneous equation is

$$\frac{d^2y}{dx^2} + 9y = 0,$$

which has the general solution $y_c = C\cos 3x + D\sin 3x$, where C and D are arbitrary constants. This is the complementary function for equation (42).

See Exercise 20(b), although there different symbols were used for the variables.

A particular integral for equation (42) is $y_p = x + 1$.

You will see in the next subsection how to find such a particular integral.

This may be verified by differentiation and substitution: $y_p' = 1$ and $y_p'' = 0$, and substituting into the left-hand side of equation (42) gives

$$y_p'' + 9y_p = 0 + 9(x+1) = 9x + 9,$$

which is the same as the right-hand side of equation (42), as required.

The general solution of equation (42) is therefore, by Theorem 3,

$$y = y_c + y_p = C\cos 3x + D\sin 3x + x + 1,$$

where C and D are arbitrary constants.

The method of Example 12 may be summarised as follows.

Procedure 6 Solving inhomogeneous linear constant-coefficient second-order differential equations

To find the general solution of the inhomogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

carry out the following steps.

1. Find the complementary function y_c, that is, the general solution of the associated homogeneous equation

   $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

   using Procedure 5.
2. Find a particular integral y_p.

The general solution is

$$y = y_c + y_p.$$

The reason why y_c is found first will become clear in Subsection 3.3.

It is worth noting that *any* choice of particular integral in Procedure 6 gives the *same* general solution. Formulas obtained for the general solution may look different for different choices of particular integral, but they are in fact always equivalent. For example, in Example 12 the particular integral $y_p = x + 1$ was chosen, and the form of the general solution was obtained as $y = C\cos 3x + D\sin 3x + x + 1$. It would have been equally valid to have chosen as a particular integral $y_p = x + 1 + \sin 3x$. In that case, the form of the general solution would have been obtained as $y = C\cos 3x + D\sin 3x + x + 1 + \sin 3x$. This looks a little different, but it may be written in the form $y = C\cos 3x + (D+1)\sin 3x + x + 1$; and since C and D are arbitrary constants, this form of the general solution represents exactly the same family of solutions.

Exercise 24

For each of the following equations:

- write down its associated homogeneous equation and its complementary function y_c
- find a particular integral of the form $y_p = p$, where p is a constant
- write down the general solution.

See Exercise 21(a) and Example 8.

(a) $\dfrac{d^2y}{dx^2} + 4y = 8$ (b) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 6$

When using Procedure 6, the complementary function is found by using Procedure 5. However, the procedures for finding a particular integral are another matter. In Exercise 24, where the right-hand sides of the equations are constants, it was possible to find a particular integral almost 'by inspection'; but this method is generally inadequate. Fortunately, there exist procedures for finding a particular integral for equations involving wide classes of right-hand-side functions $f(x)$. The remainder of this section considers some of the simpler cases, where it is possible to determine the *form* of a particular integral by inspection, although some manipulation is required in order to determine the values of certain coefficients.

3.2 Finding a particular integral by the method of undetermined coefficients

In the previous subsection you saw that the inhomogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

can be solved by first solving the associated homogeneous equation, using the methods of Section 2, and then finding a particular integral of the original equation, which depends on the function $f(x)$. This subsection and the next two subsections show you how to find a particular integral when $f(x)$ is a polynomial, exponential or sinusoidal function, or a sum of such functions.

There exist procedures for finding a particular integral for fairly general types of continuous function $f(x)$, but these are not considered in this module.

You saw an example of the approach in Exercise 24. There the functions $f(x)$ were *constants* and you tried a *constant* function $y = p$ as a particular integral, substituting into the differential equation to find a suitable value for p. In general, we try a function of the same form as $f(x)$ as a particular integral, and substitute into the differential equation to find suitable values for its unknown coefficients. The function that we try is known as a **trial solution**, and the method is known as the method of **undetermined coefficients**.

The following examples illustrate the method. Bear in mind, though, that the method (and hence each example) finds only a particular integral for the differential equation; to find the general solution you would need to find the complementary function and combine it with the particular integral, according to Procedure 6.

A polynomial function

We consider $f(x) = m_n x^n + m_{n-1}x^{n-1} + \cdots + m_1 x + m_0$.

Let us first consider a case where $f(x)$ is a linear function (i.e. a polynomial of degree 1).

Example 13

Find a particular integral for

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4x + 2.$$

Solution

We try a solution of the form

$$y = p_1 x + p_0,$$

where p_1 and p_0 are coefficients to be determined so that the differential equation is satisfied. To try this solution, we need the first and second derivatives of y:

$$\frac{dy}{dx} = p_1, \quad \frac{d^2y}{dx^2} = 0.$$

Substituting these into the left-hand side of the differential equation gives

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3 \times 0 - 2p_1 + (p_1 x + p_0) = p_1 x + (p_0 - 2p_1).$$

Therefore for $y = p_1 x + p_0$ to be a solution of the differential equation, we require that

$$p_1 x + (p_0 - 2p_1) = 4x + 2 \quad \text{for all } x. \tag{43}$$

To find the two unknown coefficients p_1 and p_0, we compare the coefficients on both sides of equation (43). Comparing the terms in x gives $p_1 = 4$. Comparing the constant terms gives $p_0 - 2p_1 = 2$, so $p_0 = 2 + 2p_1 = 2 + 2 \times 4 = 10$.

Comparing coefficients works because two polynomials are equal if and only if all their corresponding coefficients are the same.

Therefore we have the particular integral

$$y_{\mathrm{p}} = 4x + 10.$$

Check: If $y = 4x + 10$, then $dy/dx = 4$, $d^2y/dx^2 = 0$, and substituting into the left-hand side of the differential equation gives

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3 \times 0 - 2 \times 4 + (4x + 10) = 4x + 2,$$

as required.

You will have noticed in Example 13 that substituting a *linear* trial solution $y = p_1x + p_0$ into the left-hand side of the differential equation resulted in a *linear* function, namely $p_1x + (p_0 - 2p_1)$, whose coefficients could be compared with those of the *linear* target function $4x + 2$ to obtain values for p_1 and p_0. This is really the key to the method. If the target function is linear, then choosing a linear trial solution ensures that substituting into the left-hand side of the differential equation results in a linear function whose coefficients can be compared with those of the target function. Similarly, as you will see below, if the target function belongs to one of certain other classes of functions, then choosing as a trial solution a *general* function from that class ensures that substitution into the left-hand side of the differential equation produces another function from the same class, whose coefficients can be compared with those of the target function, thus enabling values to be given to the coefficients of the trial solution. The method will work provided that *all* the derivatives of functions in the class are also in the class.

Exercise 25

Find particular integrals of the form $y = p_1x + p_0$ for each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 2x + 3$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2x$

Try it and see what goes wrong.

Note that in Exercise 25(b), although $f(x)$ is just a multiple of x, it is not possible to find a solution of the form $y(x) = p_1x$. It is necessary for the trial solution to contain terms like those in $f(x)$ *and all its derivatives*, so in this case the trial solution must be of the form $y = p_1x + p_0$. So in general, even if $m_0 = 0$ in $f(x) = m_1x + m_0$, so that $f(x) = m_1x$, you should use a trial solution of the form $y = p_1x + p_0$.

You saw examples of this in Exercise 24.

However, if $f(x) = m_0$ is a constant function, then the trial solution need only be a constant function $y = p_0$.

In general, if $f(x) = m_n x^n + m_{n-1}x^{n-1} + \cdots + m_1 x + m_0$, where $m_n \neq 0$, then a trial solution of the form $y = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ should be used.

Exercise 26

Find a particular integral for

$$\ddot{y} - y = t^2.$$

An exponential function

We consider $f(x) = me^{kx}$.

Example 14

Find a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x}.$$

Solution

We try a solution of the form

$$y = pe^{3x},$$

where p is a coefficient to be determined so that the differential equation is satisfied. Differentiating $y = pe^{3x}$ gives

Since the derivative of e^{3x} is $3e^{3x}$, the exponent $(3x)$ appearing in $y(x)$ should be the same as that appearing in $f(x)$, and only the coefficient p is to be determined.

$$\frac{dy}{dx} = 3pe^{3x}, \quad \frac{d^2y}{dx^2} = 9pe^{3x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} + 9y = 9pe^{3x} + 9pe^{3x} = 18pe^{3x}.$$

Therefore for $y = pe^{3x}$ to be a solution of the differential equation, we require that $18pe^{3x} = 2e^{3x}$ for all x. Hence $p = \frac{1}{9}$, and

$$y_{\mathrm{p}} = \tfrac{1}{9}e^{3x}$$

is a particular integral for the given differential equation.

Exercise 27

Find a particular integral for

$$2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^{-x}.$$

A sinusoidal function

This type of function is particularly important in many practical applications.

We consider $f(x) = m\cos\Omega x + n\sin\Omega x$.

Following on from earlier ideas, the trial solution must contain terms like those in $f(x)$ *and all its derivatives*; so even if $f(x)$ contains only a sine or only a cosine, the trial solution $y(x)$ must contain both a sine and a cosine. However, the value of the parameter Ω appearing in $y(x)$ should be the same as that appearing in $f(x)$.

Example 15

Find a particular integral for

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 10\sin 2x.$$

Solution

We try a solution of the form

$$y = p\cos 2x + q\sin 2x,$$

where p and q are coefficients to be determined so that the differential equation is satisfied. Differentiating y gives

$$\frac{dy}{dx} = -2p\sin 2x + 2q\cos 2x, \quad \frac{d^2y}{dx^2} = -4p\cos 2x - 4q\sin 2x.$$

Substituting these into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = {} & (-4p\cos 2x - 4q\sin 2x)\\ & + 2\,(-2p\sin 2x + 2q\cos 2x)\\ & + 2(p\cos 2x + q\sin 2x)\\ = {} & (-2p + 4q)\cos 2x + (-4p - 2q)\sin 2x.\end{aligned}$$

Therefore for $y = p\cos 2x + q\sin 2x$ to be a solution of the differential equation, we require that

$$(-2p + 4q)\cos 2x + (-4p - 2q)\sin 2x = 10\sin 2x \quad \text{for all } x. \tag{44}$$

Comparing coefficients works because the cosine and sine functions are *linearly independent*: if $a\sin rx + b\cos rx = 0$ for all x, then $a = b = 0$.

To find p and q, we compare the coefficients of $\cos 2x$ and $\sin 2x$ on both sides of equation (44). Comparing $\cos 2x$ terms gives $-2p + 4q = 0$, and comparing $\sin 2x$ terms gives $-4p - 2q = 10$. Solving these simultaneous equations gives $p = -2$, $q = -1$. Hence

$$y_{\text{p}} = -2\cos 2x - \sin 2x$$

is a particular integral for the given differential equation.

Exercise 28

Find a particular integral for

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = \cos 3t + \sin 3t.$$

Method of undetermined coefficients

The following procedure summarises the results of this subsection.

Procedure 7 Method of undetermined coefficients

To find a particular integral for the inhomogeneous linear constant-coefficient second-order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

use a trial solution $y(x)$ that has a form similar to that of $f(x)$. For simple forms of $f(x)$, the following table gives the appropriate form of trial solution. (In the table, m_i, p_i ($i = 1, \ldots, n$), m, p, q and Ω are all constants.)

$f(x)$	Trial solution $y(x)$
$m_n x^n + m_{n-1}x^{n-1} + \cdots + m_1 x + m_0$	$p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$
me^{kx}	pe^{kx}
$m\cos\Omega x + n\sin\Omega x$	$p\cos\Omega x + q\sin\Omega x$

To determine the coefficient(s) in $y(x)$, differentiate $y(x)$ twice, substitute into the left-hand side of the differential equation, and equate coefficients of corresponding terms.

Note that there are exceptional cases where these trial solutions do not work; see Subsection 3.3.

Exercise 29

What form of trial solution should you use in order to find a particular integral for each of the following differential equations?

(a) $\dfrac{d^2y}{dx^2} - y = e^{2x}$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4y = \sin 4x$

Exercise 30

Use Procedures 6 and 7 to find the *general* solution of each of the following differential equations.

The roots of the auxiliary equation are $-1 \pm i$ for part (a), and 0 and -3 for part (b).

(a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 4$

(b) $\dfrac{d^2\theta}{dt^2} + 3\dfrac{d\theta}{dt} = 13\cos 2t$

Exercise 31

A long horizontal rectangular beam of length l rests on rigid supports at each end. It is important in civil engineering to determine how much such a beam 'sags'. A simple model of this 'sag', or vertical displacement y, is the differential equation

$$Ry'' - Sy + \tfrac{1}{2}Q(l - x)x = 0,$$

where R, S and Q are constants related to the structure of the beam, and x is the distance from one end of the beam (as illustrated in Figure 14).

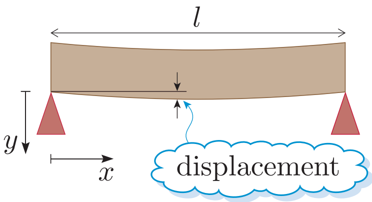


Figure 14 A horizontal beam resting on supports

Find the *general* solution of the differential equation in the case where R, S and Q are all equal to 1.

In Subsection 3.4 you will see how the principle of superposition can be used in combination with Procedure 7 to solve differential equations whose right-hand-side functions $f(x)$ are sums of polynomial, exponential and sinusoidal functions. But first let us look at some exceptional cases for which Procedure 7 does not work and needs to be adapted.

3.3 Exceptional cases

There are some exceptional cases for which Procedure 7 fails. The aim of this subsection is to indicate when such difficulties arise, and how a particular integral may be found in those circumstances. Let us begin with an example.

Example 16

Find a particular integral for

$$\frac{d^2y}{dx^2} - 4y = 2e^{2x}.$$

Solution

Using Procedure 7, let us try $y = pe^{2x}$. Differentiating this gives

$$\frac{dy}{dx} = 2pe^{2x}, \quad \frac{d^2y}{dx^2} = 4pe^{2x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} - 4y = 4pe^{2x} - 4pe^{2x} = 0.$$

So there is no value of p that gives a particular integral of the form $y = pe^{2x}$.

The trouble is that the complementary function, that is, the general solution of the associated homogeneous equation

$$\frac{d^2y}{dx^2} - 4y = 0,$$

See Exercise 18(c).

is $y = Ce^{-2x} + De^{2x}$, where C and D are arbitrary constants, thus the trial solution is a solution of the associated homogeneous equation (with

$C = 0$, $D = p$). Hence on substituting the trial solution $y = pe^{2x}$ into the inhomogeneous equation, the left-hand side is zero for any value of p, so it cannot be equal to the non-zero right-hand side.

In such circumstances, the difficulty can generally be overcome by multiplying the trial solution suggested in Procedure 7 by x. Thus in this case, the trial solution should be modified to take the form $y = pxe^{2x}$. Differentiating this gives

There is an analogy here with the case of the homogeneous differential equation when the characteristic equation has equal roots; in that case, when $e^{\lambda x}$ is one solution of the equation, another solution is given by $xe^{\lambda x}$.

$$\frac{dy}{dx} = pe^{2x} + 2pxe^{2x} = p(1+2x)e^{2x},$$
$$\frac{d^2y}{dx^2} = 2pe^{2x} + 2p(1+2x)e^{2x} = 4p(1+x)e^{2x}.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} - 4y = 4p(1+x)e^{2x} - 4pxe^{2x} = 4pe^{2x}.$$

Therefore $y = pxe^{2x}$ is a solution of the differential equation provided that $4pe^{2x} = 2e^{2x}$ for all x. Hence $p = \frac{1}{2}$, and

$$y_{\text{p}} = \tfrac{1}{2}xe^{2x}$$

is a particular integral for the given differential equation.

The problem with the trial solution being a solution of the associated homogeneous equation can occur irrespective of the form of the trial solution (i.e. polynomial, exponential or sinusoidal), but in most cases it can be overcome by multiplying the trial solution suggested in Procedure 7 by x. When using Procedure 7, you should check whether the proposed trial solution is a solution of the associated homogeneous equation, and if so try multiplying it by x. (This is why it is important to find y_{c} before y_{p} in Procedure 6.)

Exercise 32

Find a particular integral for each of the following differential equations.

The complementary functions are given in the solutions to Example 8 and Exercise 18(b).

(a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4e^x$ (b) $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = 1$

Exercise 33

The motion of a marble dropped from the Clifton Suspension Bridge into the River Avon can be modelled by the differential equation

$$m\ddot{x} + r\dot{x} - mg = 0,$$

where m is the mass of the marble, r is a constant related to air resistance, g is the magnitude of the acceleration due to gravity, and x is the vertical distance from the point of dropping (as shown in Figure 15).

Find an expression for x in terms of the time t.

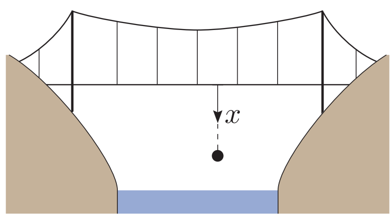


Figure 15 A marble dropped from the Clifton Suspension Bridge

We have seen that Procedure 7 fails if the trial solution is a solution of the associated homogeneous differential equation; in such cases we multiply the suggested trial solution by the independent variable and use this as the trial solution. Another situation in which it is necessary to multiply the suggested trial solution by the independent variable is illustrated in the following example.

Example 17

Find a particular integral for

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2x + 2.$$

Solution

Using Procedure 7, let us try $y = p_1x + p_0$. Differentiating this gives

$$\frac{dy}{dx} = p_1, \quad \frac{d^2y}{dx^2} = 0.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2p_1.$$

But there is no value of p_1 that satisfies $2p_1 = 2x + 2$ for *all* x.

This example again illustrates why it is better to find the complementary function before looking for a particular integral.

The problem this time is that (from Exercise 21(c)) the complementary function is $y = C + De^{-2x}$, where C and D are arbitrary constants, so the p_0 part of the trial solution is a solution of the associated homogeneous equation (with $C = p_0$, $D = 0$). Hence on substituting the trial solution $y = p_1x + p_0$ into the inhomogeneous equation, the p_0 part disappears, and the trial solution effectively reduces to $y = p_1x$. The result in this case is that after substituting the trial solution and its derivatives into the left-hand side of the equation, there are not enough terms on the left-hand side to compare with the terms in the right-hand-side function.

Again, the difficulty can be overcome by multiplying the trial solution suggested by Procedure 7 by x, to give $y = p_1x^2 + p_0x$. Differentiating this gives

$$\frac{dy}{dx} = 2p_1x + p_0, \quad \frac{d^2y}{dx^2} = 2p_1.$$

Substituting these into the left-hand side of the differential equation gives

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2p_1 + 2(2p_1x + p_0) = 4p_1x + (2p_1 + 2p_0).$$

Therefore $y = p_1x^2 + p_0x$ is a solution of the differential equation provided that $4p_1x + (2p_1 + 2p_0) = 2x + 2$ for all x. This gives $p_1 = \frac{1}{2}$, $p_0 = \frac{1}{2}$, so

$$y_{\text{p}} = \tfrac{1}{2}(x^2 + x)$$

is a particular integral for the given differential equation.

To summarise, Procedure 7 will fail if *all* or *part* of the suggested trial solution is a solution of the associated homogeneous equation. In such cases, a particular integral can usually be found by multiplying the trial solution by the independent variable.

However, it may sometimes be necessary to multiply the trial function by the independent variable more than once, as explained below in Procedure 8 and Exercise 34.

Procedure 8 Exceptional cases in the method of undetermined coefficients

Suppose that you try using the method of undetermined coefficients (described in Procedure 7) for finding a particular integral for an inhomogeneous linear constant-coefficient second-order differential equation.

If this fails because all or part of the trial solution is a solution of the associated homogeneous equation, then try multiplying the trial solution by the independent variable.

If all or part of the resulting trial solution is still a solution of the associated homogeneous equation, then try multiplying by the independent variable again.

Exercise 34

Find a particular integral for

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

You found the complementary function in Exercise 21(d).

3.4 Combining cases

You have seen how to find a particular integral when the right-hand-side function $f(x)$ is polynomial, exponential or sinusoidal. In this subsection you will see how to find a particular integral when $f(x)$ is a combination of such functions, by using the principle of superposition.

Example 18

Find a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x} + 18x + 18. \qquad (45)$$

Solution

In Example 14 you saw that $y_{\mathrm{p}} = \frac{1}{9}e^{3x}$ is a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 2e^{3x},$$

and in Example 12 you saw that $y_{\mathrm{p}} = x + 1$ is a particular integral for

$$\frac{d^2y}{dx^2} + 9y = 9x + 9.$$

Therefore, by the principle of superposition, a particular integral for equation (45) is

$$y_{\mathrm{p}} = \tfrac{1}{9}e^{3x} + 2 \times (x + 1) = \tfrac{1}{9}e^{3x} + 2x + 2.$$

The approach of Example 18 is to find particular integrals for differential equations involving each part of $f(x)$ separately, and then to use the principle of superposition to combine the two.

Exercise 35

Find particular integrals for each of the following differential equations.

See Exercise 34.

(a) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 4e^x - 3e^{2x}$

(b) $2\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = 12\cos 2t + 10$

Exercise 36

You will find some help for parts (a), (d), (e) and (f) in Exercises 18 and 21, and Example 8.

Find the *general* solution of each of the following differential equations.

(a) $\dfrac{d^2\theta}{dt^2} + 4\theta = 2t$

(b) $u''(t) + 4u'(t) + 5u(t) = 5$

(c) $3\dfrac{d^2Y}{dx^2} - 2\dfrac{dY}{dx} - Y = e^{2x} + 3$

(d) $\dfrac{d^2y}{dx^2} - 4y = e^{-2x}$

(e) $\dfrac{d^2y}{dx^2} + 4y = \sin 2x + 3x$

(f) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2e^x - 5e^{2x}$

---

4 Initial conditions and boundary conditions

In Section 3 you saw how to find the general solution of an inhomogeneous linear constant-coefficient second-order differential equation as a combination of a complementary function and a particular integral. In practice, however, we usually want a *particular* solution that satisfies certain additional conditions. Recall that a particular solution is one that does not involve arbitrary constants. In Section 1 you saw how one additional condition (called an initial condition) was needed to find a value for the single arbitrary constant in the general solution of a first-order differential equation in order to obtain a particular solution. In the case of second-order differential equations, in order to obtain a particular solution, *two* additional conditions are needed to obtain values for the *two* arbitrary constants in the general solution.

There are two types of additional conditions for second-order differential equations: *initial conditions* and *boundary conditions*. Problems involving such conditions are called *initial-value problems* and *boundary-value problems*, respectively, and are discussed in Subsections 4.1 and 4.2.

4.1 Initial-value problems

For a *first-order* differential equation, an initial condition consists of specifying the value of the dependent variable ($y = y_0$, say) at a given value of the independent variable ($x = x_0$), and is often written in the form $y(x_0) = y_0$.

See Section 1.

One fairly obvious way of specifying *two* additional conditions for a *second-order* differential equation is to give the values of both the dependent variable ($y = y_0$) and its derivative ($dy/dx = z_0$) for the *same* given value of the independent variable ($x = x_0$).

There are many examples of such a pair of initial conditions occurring naturally as part of a problem. One example is provided by the marble being dropped from the Clifton Suspension Bridge in Exercise 33. In that example, x is the vertical distance from the point of dropping. The obvious choice of origin for the time t is the time at which the marble is dropped. Therefore a naturally occurring pair of initial conditions is that at time $t = 0$, we know both the position $x = 0$ and the speed $\dot{x} = 0$ (since the marble is dropped, i.e. is released with zero initial velocity). Another example is provided by the clock pendulum in Exercise 22. In this example, when the pendulum changes direction, its rate of change of angle θ is momentarily zero; also, when it changes direction, it makes its greatest angle θ_0 with the vertical (see Figure 16). Therefore if we measure time t from the moment when the pendulum changes direction, we have the initial conditions $\theta = \theta_0$ and $\dot{\theta} = 0$ when $t = 0$.

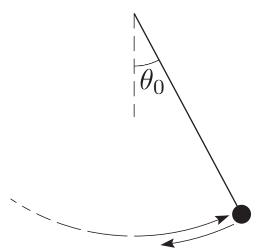


Figure 16 A pendulum changing direction

Initial conditions associated with a second-order differential equation with dependent variable y and independent variable x specify that y and dy/dx take values y_0 and z_0, respectively, when x takes the value x_0. These conditions can be written as

$$y = y_0 \text{ and } \frac{dy}{dx} = z_0 \text{ when } x = x_0$$

or as

$$y(x_0) = y_0, \quad y'(x_0) = z_0.$$

The numbers x_0, y_0 and z_0 are often referred to as **initial values**.

The combination of a second-order differential equation and initial conditions is called an **initial-value problem**.

Let us now see how initial conditions can be used to determine values for the two arbitrary constants and hence find a particular solution.

Example 19

Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

that satisfies the initial conditions $y = 0$ and $dy/dx = 1$ when $x = 0$.

Solution

From Example 8 we know that the general solution is

$$y = Ce^x + De^{2x}, \tag{46}$$

where C and D are arbitrary constants. One of the initial conditions involves the derivative of the solution, so we need to obtain that derivative:

$$\frac{dy}{dx} = Ce^x + 2De^{2x}. \tag{47}$$

The initial conditions state that $y(0) = 0$, $y'(0) = 1$. Substituting $x = 0$, $y = 0$ into equation (46) gives

$$0 = Ce^0 + De^0 = C + D,$$

while substituting $x = 0$, $dy/dx = 1$ into equation (47) gives

$$1 = Ce^0 + 2De^0 = C + 2D.$$

Solving these equations gives $C = -1$, $D = 1$, so the required particular solution is

$$y = -e^x + e^{2x}.$$

Note that when you check a particular solution, you should check that it satisfies the initial conditions as well as the differential equation.

Exercise 37

Find solutions to the following initial-value problems.

(a) $u''(t) + 9u(t) = 0, \quad u\left(\frac{\pi}{2}\right) = 0,\ u'\left(\frac{\pi}{2}\right) = 1.$

See Exercise 20(b).

(b) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4e^x, \quad y(0) = 4,\ y'(0) = 2.$

See Exercise 32(a).

(c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 4e^x - 3e^{2x}, \quad y(0) = 4,\ y'(0) = -1.$

See Exercises 21(d) and 35(a).

You saw in Subsection 1.4 that an initial-value problem involving a linear first-order differential equation has a unique solution under certain circumstances. (Such circumstances hold for nearly every such initial-value problem that you are likely to come across in practice.) The same is true of initial-value problems involving a linear constant-coefficient second-order differential equation, as the following theorem makes clear.

Theorem 4

The initial-value problem

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \quad y(x_0) = y_0,\ y'(x_0) = z_0,$$

where a, b, c are real constants with $a \neq 0$, and $f(x)$ is a given continuous real-valued function on an interval (r, s), with $x_0 \in (r, s)$, has a unique solution on that interval.

Note that one consequence of this theorem is that if the differential equation is *homogeneous* and the initial conditions are of the form $y(x_0) = 0$ and $y'(x_0) = 0$, then the unique solution must be the zero function $y = 0$, since it satisfies the differential equation and the initial conditions.

4.2 Boundary-value problems

The two conditions in an initial-value problem (the value of the dependent variable y and its derivative dy/dx) both relate to the same value of x. However, the two conditions that are required to determine values for the arbitrary constants need not relate to the same value of x. We could have one condition for $x = x_0$ and another for $x = x_1$, say. For example, consider again the 'sagging' beam from Exercise 31. Two known conditions on this beam are its zero displacements at the ends of the beam, where it rests on the rigid supports: that is, its *boundary* conditions are $y(0) = 0$ and $y(l) = 0$ (where l is the length of the beam). This pair of boundary conditions gives the value of y at two different points, but in general each boundary condition could specify the value of either y or dy/dx (or even a relationship between them).

Boundary conditions associated with a second-order differential equation with dependent variable y and independent variable x specify that y or dy/dx (or some combination of the two) takes values y_0 and y_1 at two different values x_0 and x_1, respectively, of x. The numbers x_0, x_1, y_0 and y_1 are often referred to as **boundary values**.

The combination of a second-order differential equation and boundary conditions is called a **boundary-value problem**.

The conditions are referred to as 'boundary' conditions because, as in the beam example, they often relate to conditions at the endpoints x_0 and x_1 of an interval $[x_0, x_1]$ on which we are interested in exploring the differential equation.

Let us now see how boundary conditions can be used to determine values for the two arbitrary constants and hence find a particular solution.

Example 20

Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 0$$

that satisfies the boundary conditions $y = 0$ when $x = 0$ and $dy/dx = 1$ when $x = \frac{\pi}{3}$.

Solution

From Exercise 20(b), the general solution is

$$y = C\cos 3x + D\sin 3x, \tag{48}$$

where C and D are arbitrary constants.

One of the boundary conditions involves the derivative of the solution, so we need to obtain that derivative:

$$\frac{dy}{dx} = -3C\sin 3x + 3D\cos 3x. \tag{49}$$

The boundary conditions state that $y(0) = 0$, $y'\left(\frac{\pi}{3}\right) = 1$. Substituting $x = 0$, $y = 0$ into equation (48) gives

$$0 = C\cos 0 + D\sin 0 = C,$$

so $C = 0$. Substituting $x = \frac{\pi}{3}$, $y' = 1$ and $C = 0$ into equation (49) gives

$$1 = 3D\cos\pi = -3D.$$

Therefore $C = 0$, $D = -\frac{1}{3}$, so the required particular solution is

$$y = -\tfrac{1}{3}\sin(3x).$$

Exercise 38

Solve the boundary-value problem

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4e^x, \quad y'(0) = 2,\ y(1) = 0.$$

See Exercise 32(a).

Exercise 39

Use the differential equation of Exercise 31, with $R = S = Q = 1$, namely

$$y'' - y + \tfrac{1}{2}(l - x)x = 0,$$

to determine the vertical displacement at the centre of a beam of length 2 metres resting on rigid supports at its ends.

Unlike the case of initial-value problems, boundary-value problems may not have solutions even when the differential equation is linear and constant-coefficient with a continuous real-valued right-hand-side function, as the following example illustrates.

Example 21

Try to solve the boundary-value problem

$$\frac{d^2y}{dx^2} + 4y = 0, \quad y(0) = 0,\ y\left(\tfrac{\pi}{2}\right) = 1.$$

Solution

From Exercise 21(a), the general solution is

$$y = C\cos 2x + D\sin 2x,$$

where C and D are arbitrary constants.

The boundary conditions state that $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$. Substituting each of these into the general solution in turn gives

$$0 = C\cos 0 + D\sin 0 = C,$$
$$1 = C\cos\pi + D\sin\pi = -C.$$

There is no solution for which $C = 0$ and $C = -1$, so there is no solution of the differential equation that satisfies the given boundary conditions.

Fortunately it is rare for a boundary-value problem that models a real-life situation to have no solution (and in such cases it is usually possible to reformulate the model to overcome the difficulty).

Not only is it possible for boundary-value problems to have no solution, but it is also possible for them to have solutions that are not unique, as the following example illustrates.

Example 22

Solve the boundary-value problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5, \quad y(0) = 1,\ y(\pi) = 1.$$

Solution

From Exercise 36(b), the general solution is

$$y = e^{-2x}(C\cos x + D\sin x) + 1,$$

where C and D are arbitrary constants.

The boundary conditions state that $y(0) = 1$, $y(\pi) = 1$. Substituting each of these into the general solution in turn gives

$$1 = e^0(C\cos 0 + D\sin 0) + 1 = C + 1,$$
$$1 = e^{-2\pi}(C\cos \pi + D\sin \pi) + 1 = -Ce^{-2\pi} + 1.$$

Both of these equations reduce to $C = 0$, but D can take any value, so any solution of the form

$$y = De^{-2x}\sin x + 1$$

satisfies the differential equation and the boundary conditions.

In Example 22, there is no unique solution of the differential equation that satisfies the given boundary conditions, but instead there is an infinite family of possible solutions.

Finally, a word of reassurance: most of the boundary-value problems that you will come across in this module will have a unique solution.

The final exercises of the unit test your understanding of the whole of this section.

Exercise 40

You found the general solution of the differential equation in Exercise 21(a).

For each of the following problems, identify the conditions as either initial conditions or boundary conditions, and solve the problem.

(a) $u''(x) + 4u(x) = 0, \quad u(0) = 1,\ u'(0) = 0.$

(b) $u''(x) + 4u(x) = 0, \quad u(0) = 0,\ u\left(\frac{\pi}{2}\right) = 0.$

(c) $u''(x) + 4u(x) = 0, \quad u\left(\frac{\pi}{2}\right) = 0,\ u'\left(\frac{\pi}{2}\right) = 0.$

(d) $u''(x) + 4u(x) = 0, \quad u(-\pi) = 1,\ u\left(\frac{\pi}{4}\right) = 2.$

(e) $u''(x) + 4u(x) = 0, \quad u'(0) = 0,\ u'\left(\frac{\pi}{4}\right) = 1.$

Exercise 41

Find the solution (if any) to each of the following problems.

(a) $u''(t) + 4u'(t) + 5u(t) = 0, \quad u(0) = 0,\ u'(0) = 2.$

The roots of the auxiliary equation are $-2 \pm i$.

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0, \quad$ where $y = 0$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.

Roots are $-1 \pm i$.

(c) $\ddot{x} + 9x = 3(1 - \pi t), \quad x(0) = \frac{1}{3},\ \dot{x}\left(\frac{\pi}{3}\right) = 0.$

Roots are $\pm 3i$.

Learning outcomes

After studying this unit, you should be able to:

- understand and use the basic terminology relating to differential equations and their solutions
- check by substitution whether a given function is a solution of a given first-order differential equation or initial-value problem
- find from the general solution of a first-order differential equation the particular solution that satisfies a given initial condition
- appreciate the difficulties with domains and image sets for the solution of some differential equations
- deduce the qualitative behaviour of solutions from consideration of a first-order differential equation itself, as visualised from its direction field
- set up the formulas required by Euler's method for solving an initial-value problem, and carry out a few steps of the method
- recognise when a first-order differential equation is soluble by direct integration, and carry out that integration when appropriate, in simple cases
- recognise when a first-order differential equation is separable, and apply the method of separation of variables in simple cases
- recognise when a first-order differential equation is linear, and solve such an equation by the integrating factor method in simple cases
- understand and use the terminology relating to linear constant-coefficient second-order differential equations
- understand the key role of the principle of superposition in the solution of linear constant-coefficient second-order differential equations
- obtain the general solution of a homogeneous linear constant-coefficient second-order differential equation using the solutions of its auxiliary equation

- use the method of undetermined coefficients to find a particular integral for an inhomogeneous linear constant-coefficient second-order differential equation with certain simple forms of right-hand-side function
- obtain the general solution of an inhomogeneous linear constant-coefficient second-order differential equation by combining its complementary function and a particular integral
- use the general solution together with a pair of initial or boundary conditions to obtain, when possible, a particular solution of a linear constant-coefficient second-order differential equation.

Solutions to exercises

Solution to Exercise 1

We have

$$r(P) = k\left(1 - \frac{P}{M}\right),$$

so we need to solve the pair of simultaneous equations

$$k\left(1 - \frac{10}{M}\right) = 1, \quad k\left(1 - \frac{10\,000}{M}\right) = 0.$$

From the second equation, since $k > 0$, we see immediately that $M = 10\,000$. Substituting in the first equation leads to

$$k\frac{999}{1000} = 1, \quad \text{so} \quad k = \frac{1000}{999}.$$

Solution to Exercise 2

(a) In each case we need to show that the given function satisfies the differential equation, that is, it gives the same expression for either side of the equation.

If $y = 2e^x - (x^2 + 2x + 2)$, then differentiating y gives

$$\frac{dy}{dx} = 2e^x - 2x - 2,$$

and substituting the expression for y into the right-hand side of the given differential equation gives

$$y + x^2 = 2e^x - (x^2 + 2x + 2) + x^2 = 2e^x - 2x - 2,$$

as required.

(b) If $y = \tan x + \sec x$, then

$$\frac{dy}{dx} = \sec^2 x + \tan x \sec x,$$

and substituting the expression for y into the right-hand side of the given differential equation gives

$$\begin{aligned} y\tan x + 1 &= (\tan x + \sec x)\tan x + 1 \\ &= (\tan^2 x + 1) + \sec x \tan x \\ &= \sec^2 x + \tan x \sec x, \end{aligned}$$

as required.

(c) If $y = t + Ce^{-t}$, then

$$\dot{y} = \frac{dy}{dt} = 1 - Ce^{-t},$$

and substituting the expression for y into the right-hand side of the given differential equation gives

$$-y + t + 1 = -(t + Ce^{-t}) + t + 1 = 1 - Ce^{-t},$$

as required.

Solution to Exercise 3

(a) In each case we need to show that the given function satisfies the differential equation, that is, it gives the same expression for either side of the equation.

If $y = C - \frac{1}{3}e^{-3x}$, then

$$\frac{dy}{dx} = e^{-3x},$$

as required.

(b) If

$$P = \frac{CMe^{kt}}{1 + Ce^{kt}},$$

Rules such as the quotient rule can be found in the Handbook.

then, using the quotient rule for differentiation,

$$\begin{aligned}
\frac{dP}{dt} &= \frac{(1 + Ce^{kt})(CMke^{kt}) - (CMe^{kt})(Cke^{kt})}{(1 + Ce^{kt})^2} \\
&= k\left(\frac{CMe^{kt}}{1 + Ce^{kt}}\right)\left(\frac{1 + Ce^{kt} - Ce^{kt}}{1 + Ce^{kt}}\right) \\
&= k\left(\frac{CMe^{kt}}{1 + Ce^{kt}}\right)\left(1 - \frac{Ce^{kt}}{1 + Ce^{kt}}\right) \\
&= kP\left(1 - \frac{P}{M}\right),
\end{aligned}$$

as required.

Solution to Exercise 4

(a) From Exercise 3(b) and using $k = 0.15$ and $M = 10$, we know that

$$P = \frac{CMe^{kt}}{1 + Ce^{kt}} = \frac{10Ce^{0.15t}}{1 + Ce^{0.15t}}$$

is a solution of the differential equation, where C is a constant. The initial condition $P(0) = 1$ then implies

$$1 = \frac{10C}{1 + C}, \quad \text{so} \quad C = \tfrac{1}{9}.$$

A particular solution is therefore

$$P = \frac{\frac{10}{9}e^{0.15t}}{1 + \frac{1}{9}e^{0.15t}} = \frac{10e^{0.15t}}{9 + e^{0.15t}}.$$

(b) Dividing top and bottom by $e^{0.15t}$, we see that

$$P = \frac{10}{9e^{-0.15t} + 1}.$$

For large values of t, the exponential term on the bottom will be very small. The result is that P will approach the value 10 in the long term.

Solution to Exercise 5

(a) In each case, we apply direct integration to find the general solution, and C is an arbitrary constant.

The differential equation $dy/dx = 6x$ has general solution

$$y = \int 6x\,dx = 3x^2 + C.$$

From the initial condition $y(1) = 5$, we have $5 = 3 + C$, so $C = 2$. The solution to the initial-value problem is therefore

$$y = 3x^2 + 2.$$

(b) The differential equation $dv/du = e^{4u}$ has general solution

$$v = \int e^{4u}\,du = \tfrac{1}{4}e^{4u} + C.$$

From the initial condition $v(0) = 2$, we have $2 = \frac{1}{4} + C$, so $C = \frac{7}{4}$. The solution to the initial-value problem is therefore

$$v = \tfrac{1}{4}e^{4u} + \tfrac{7}{4}.$$

(c) The differential equation $\dot{y} = t/(1 + t^2)$ has general solution

$$y = \int \frac{t}{1 + t^2}\,dt.$$

Using the hint, we make the substitution $u = 1 + t^2$, for which $du/dt = 2t$. This gives

$$\begin{aligned}\int \frac{t}{1+t^2}\,dt &= \tfrac{1}{2}\int \frac{1}{1+t^2}(2t)\,dt\\ &= \tfrac{1}{2}\int \frac{1}{u}\,du\\ &= \tfrac{1}{2}\ln u + C \quad (\text{since } u = 1 + t^2 > 0)\\ &= \tfrac{1}{2}\ln(1 + t^2) + C.\end{aligned}$$

The $\frac{1}{2}$ is included outside the integral to match the 2 inside the integral that enables the substitution to be made.

The general solution of the differential equation is therefore

$$y = \tfrac{1}{2}\ln(1 + t^2) + C.$$

The initial condition $y(0) = 2$ gives $C = 2$, so the particular solution is

$$y = \tfrac{1}{2}\ln(1 + t^2) + 2.$$

Solution to Exercise 6

(a) Each of the differential equations is soluble by direct integration.

The general solution of $dy/du = 1/(u - a)$, where $u \neq a$, is given by

$$y = \int \frac{1}{u - a}\,du.$$

Using equation (8), integration produces the general solution

$$y = \ln|u - a| + C,$$

where C is an arbitrary constant.

(b) To verify the equation in the hint, taking a common denominator, we have

$$\frac{1}{x}+\frac{a}{1-ax}=\frac{(1-ax)+ax}{x(1-ax)}=\frac{1}{x(1-ax)}.$$

Then the general solution of $dy/dx = 1/(x(1-ax))$, where $x \neq 0$, $x \neq 1/a$, is given by

$$\begin{aligned} y=\int \frac{1}{x(1-ax)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{a}{1-ax}\,dx \\ &= \ln|x| - \ln|1-ax| + C \\ &= \ln\left|\frac{x}{1-ax}\right| + C, \end{aligned}$$

where C is an arbitrary constant. This can also be written as

$$y = C - \ln\left|\frac{1-ax}{x}\right| = C - \ln\left|\frac{1}{x}-a\right|.$$

Solution to Exercise 7

(a) The differential equation is $dm/dt = -\lambda m$, where $m > 0$. Following Procedure 2, we obtain

$$\int \frac{1}{m}\,dm = \int (-\lambda)\,dt,$$

and since $m > 0$, integration produces

$$\ln m = -\lambda t + C,$$

where C is an arbitrary constant. On solving this equation for m, by taking the exponential of both sides, we obtain

$$m = e^{-\lambda t + C} = e^{C}e^{-\lambda t} = Be^{-\lambda t},$$

where $B = e^{C}$ is a positive (since $e^{C} > 0$ for all C) but otherwise arbitrary constant. The general solution is therefore

$$m = Be^{-\lambda t},$$

where B is a positive but otherwise arbitrary constant.

(b) The initial condition is $m(0) = m_0$, from which we have $m_0 = Be^{0}$, so $B = m_0$. The required particular solution is therefore

$$m = m_0 e^{-\lambda t}.$$

Solution to Exercise 8

(a) The differential equation is

$$\frac{dy}{dx} = \frac{y-1}{x}, \quad \text{where } x > 0.$$

In order to apply the separation of variables method, we need to exclude the case where $y = 1$.

So for $y \neq 1$, on applying Procedure 2 we have

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x}\,dx.$$

Since $x > 0$, for $y \neq 1$ (so that $y - 1 \neq 0$), integration produces

$$\ln|y-1| = \ln x + C,$$

where C is an arbitrary constant. Writing $C = \ln B$ ($B > 0$) and using properties of the logarithm function, the right-hand side becomes $\ln x + \ln B = \ln(Bx)$, so we can deduce that

$$|y-1| = Bx.$$

So $y - 1 = Bx$ if $y > 1$, and $y - 1 = -Bx$ if $y < 1$. We thus have

$$y = 1 + Ax,$$

where the sign can be absorbed into A, which is a non-zero but otherwise arbitrary constant.

Examination of the differential equation shows that $A = 0$ also gives a solution (the constant function $y = 1$).

The general solution is therefore

$$y = 1 + Ax,$$

where A is an arbitrary constant.

(b) The differential equation is $dy/dx = 2y/(x^2+1)$. In order to apply the separation of variables method, we need to exclude the case where $y = 0$. So for $y \neq 0$, on applying Procedure 2 we have

$$\int \frac{1}{y}\,dy = \int \frac{2}{x^2+1}\,dx.$$

Since $y \neq 0$, integration produces

$$\ln|y| = 2\arctan x + C,$$

where C is an arbitrary constant. On solving this equation for y, we obtain

$$y = \pm e^{2\arctan x + C} = \pm e^{C} e^{2\arctan x} = Be^{2\arctan x},$$

where $B = \pm e^{C}$ is a non-zero but otherwise arbitrary constant.

Examination of the differential equation shows that $B = 0$ also gives a solution (the zero function $y = 0$).

The general solution is therefore

$$y = Be^{2\arctan x},$$

where B is an arbitrary constant.

Solution to Exercise 9

(a) The given equation is $dP/dt = kP(1 - P/M)$. First, note that the constant functions $P = 0$ and $P = M$ are both solutions. Assuming that we are considering neither of these possibilities (we are certainly not interested in $P = 0$ since we know that $P_0 > 0$), we can use the separation of variables method to obtain

$$\frac{1}{k}\int \frac{1}{P(1 - P/M)}\,dP = \int 1\,dt.$$

The integral on the left-hand side is of the form evaluated in Exercise 6(b), with $1/M$ in place of a. Hence we have

$$-\frac{1}{k}\ln\left|\frac{1}{P} - \frac{1}{M}\right| = t + C,$$

where C is an arbitrary constant. On solving for P, we find first that

$$\frac{1}{P} - \frac{1}{M} = \pm e^{-k(t+C)} = \pm e^{-kC}e^{-kt} = Be^{-kt},$$

where $B = \pm e^{-kC}$ is a non-zero but otherwise arbitrary constant. However, note that $B = 0$ corresponds to the constant solution $P = M$ already noted, so the restriction $B \neq 0$ may be dropped. Hence we obtain

$$P = \left(\frac{1}{M} + Be^{-kt}\right)^{-1} \qquad (e^{kt} \neq -MB),$$

where B is an arbitrary constant.

From the initial condition $P(0) = P_0$, we have

$$P_0 = \left(\frac{1}{M} + Be^{0}\right)^{-1}, \quad \text{so} \quad B = \frac{1}{P_0} - \frac{1}{M}.$$

The solution to the initial-value problem is therefore

$$P = \left(\frac{1}{M} + \left(\frac{1}{P_0} - \frac{1}{M}\right)e^{-kt}\right)^{-1},$$

which yields

$$P = \frac{M}{1 + (M/P_0 - 1)e^{-kt}}.$$

(b) As $t \to \infty$, we have $e^{-kt} \to 0$, and consequently the value of $P(t)$ approaches M. This is true whether the starting value P_0 is greater than or less than M.

Solution to Exercise 10

(a) The equation $dy/dx = x \sin x$ is linear, with $g(x) = 0$ (for all x) and $h(x) = x \sin x$.

(b) The equation $\dot{y} + y^2 = t$ is not linear (because of the y^2 term).

(c) The equation $x(dy/dx) + y = y^2$ is not linear (because of the y^2 term).

(d) The equation $(1+x^2)(dy/dx)+2xy=3x^2$ is linear, since we can divide through by $1+x^2$ to obtain

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{3x^2}{1+x^2},$$

which is of the defined form with $g(x)=2x/(1+x^2)$ and $h(x)=3x^2/(1+x^2)$.

Solution to Exercise 11

(a) The given equation is $dy/dx-y=e^x\sin x$. Comparison with equations (16) and (17) shows that the integrating factor is

$$p(x)=\exp\left(\int(-1)\,dx\right)=\exp(-x)=e^{-x}.$$

Multiplying through by $p(x)$ gives

$$e^{-x}\frac{dy}{dx}-e^{-x}y=\sin x.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}(e^{-x}y)=\sin x.$$

On integrating, we find the general solution

$$e^{-x}y=-\cos x+C,$$

or equivalently,

$$y=e^x(C-\cos x),$$

where C is an arbitrary constant.

(b) The given equation, when rearranged into form (16), is $dy/dx-y/x=-1/x$. Thus $g(x)=-1/x$ and $h(x)=-1/x$, and the integrating factor is

$$\begin{aligned}p(x)=\exp\left(\int g(x)\,dx\right)&=\exp\left(\int-\frac{1}{x}\,dx\right)\\&=\exp\left(-\ln x\right)=\exp\left(\ln\left(\frac{1}{x}\right)\right)=\frac{1}{x}.\end{aligned}$$

Multiplying through by $p(x)$ gives

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=-\frac{1}{x^2}.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}\left(\frac{y}{x}\right)=-\frac{1}{x^2}.$$

Hence

$$\frac{y}{x}=\int-\frac{1}{x^2}\,dx=\frac{1}{x}+C,$$

where C is an arbitrary constant.

After multiplying through by x, the general solution in explicit form is

$$y = 1 + Cx.$$

Solution to Exercise 12

(a) This requires the integrating factor method.

(b) This is best solved by direct integration.

(c) This can be solved by separation of variables or the integrating factor method.

(d) This requires the integrating factor method.

Solution to Exercise 13

For the initial-value problem

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

we have $x_0 = 0$, $Y_0 = y_0 = 1$ and $f(x_i, Y_i) = Y_i$. The step size is given as $h = 0.2$. Equation (25) with $i = 0$ gives

$$x_1 = x_0 + h = 0 + 0.2 = 0.2,$$

and equation (26) with $i = 0$ gives

$$Y_1 = Y_0 + h\, f(x_0, Y_0) = 1 + 0.2 \times 1 = 1.2.$$

Applying equations (25) and (26) in turn for $i = 1, 2, 3, 4$, we obtain the following table.

i	x_i	Y_i	$f(x_i, Y_i) = Y_i$	$Y_{i+1} = Y_i + h\, f(x_i, Y_i)$
0	0	1	1	1.2
1	0.2	1.2	1.2	1.44
2	0.4	1.44	1.44	1.728
3	0.6	1.728	1.728	2.073 6
4	0.8	2.073 6	2.073 6	2.488 32
5	1.0	2.488 32		

The approximation to $y(1)$ is 2.488 32.

Solution to Exercise 14

Since we are told that for a sufficiently small step size h, the absolute error is proportional to h, we can deduce from the last row of Table 2 that there exists a constant k such that

$$0.000\,136 = 0.0001k,$$

so $k = 1.36$. In order to determine $y(1)$ correct to five decimal places, h must be such that

$$1.36h < 5 \times 10^{-6}$$

or

$$h < \frac{5 \times 10^{-6}}{1.36} \simeq 3.7 \times 10^{-6}.$$

So a suitable choice of h would be $10^{-6} = 0.000\,001$.

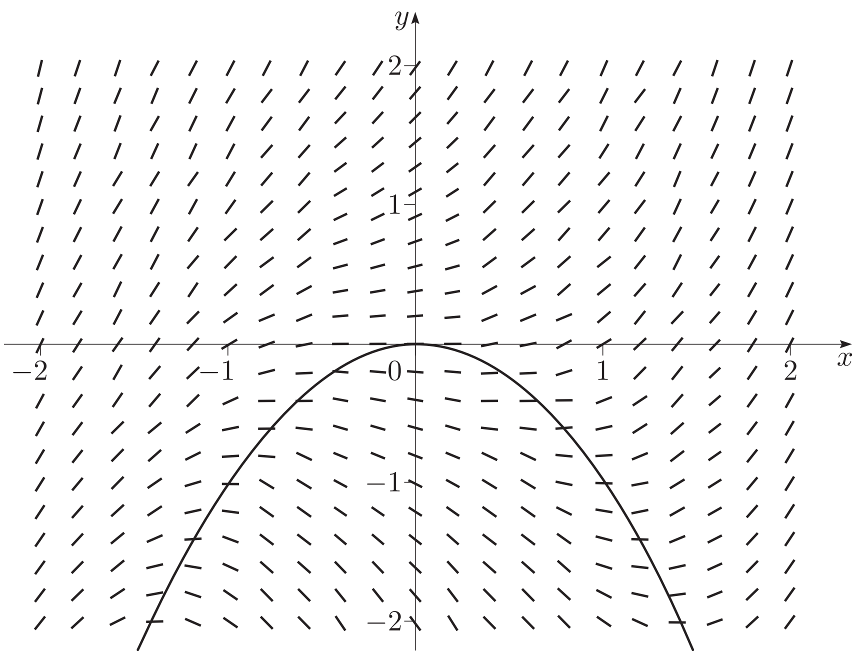
(In fact, using this value of h gives an approximation to $y(1)$ of $2.718\,280$, which *is* correct to five decimal places.)

Solution to Exercise 15

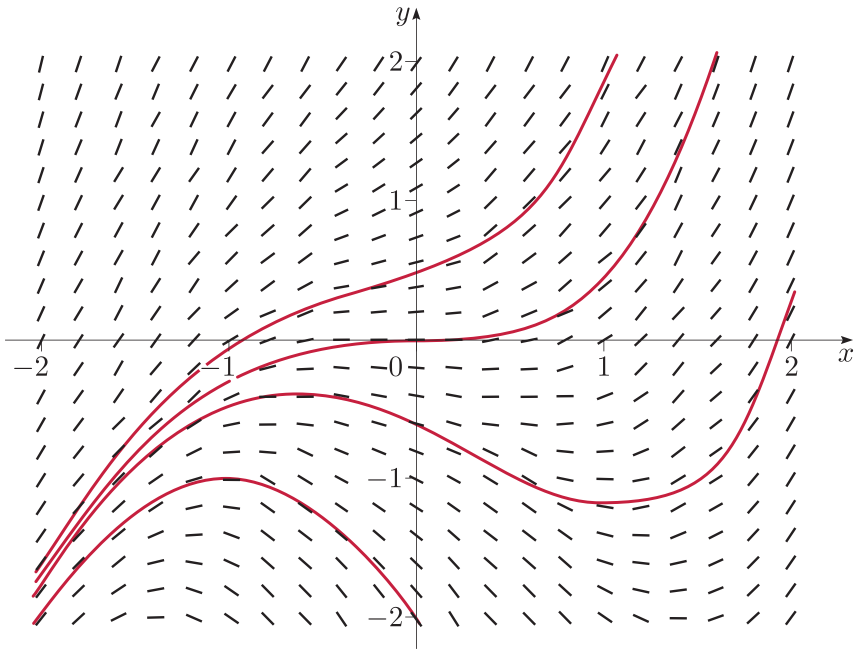
(a) The slope defined by the direction field $f(x, y) = y + x^2$ is zero when $y = -x^2$, which is a parabola in the lower half-plane with vertex at the origin. Below this parabola we have $y < -x^2$ and $f(x, y) < 0$, while above the parabola we have $y > -x^2$ and $f(x, y) > 0$. Thus all slopes for points of the plane below the parabola $y = -x^2$ are negative, and all slopes for points above it are positive.

Also, if x is fixed, then $f(x, y) = y + x^2$ is an increasing function as y increases. If instead y is fixed, then for $x > 0$, $f(x, y)$ increases as x increases, and for $x < 0$, $f(x, y)$ increases as x becomes more negative. These observations indicate that the slope given by the direction field increases as we move from bottom to top along any vertical line, whereas on moving along any horizontal line, the slope increases with distance from the y-axis.

(b) The features described in the solution to part (a) are all apparent on the direction field diagram. This direction field diagram is repeated below, with the parabola $y = -x^2$ superimposed on it. (Note that this parabola does not represent a solution of the differential equation.)



(c) It appears from the direction field that there are several types of solution. Any solution whose graph cuts the y-axis above the origin has positive slope at all points. The solution graph that passes through the origin has zero slope there, but positive slope everywhere else. Any solution graph that cuts the y-axis below the origin has a maximum (where it meets $y = -x^2$ for $x < 0$). Some of these graphs also have a minimum (where they meet $y = -x^2$ for $x > 0$). Others have no minimum (though this is not clear from the diagram given). A solution graph of each type is sketched below.



(d) The initial-value problem is

$$\frac{dy}{dx} = y + x^2, \quad y(-1) = -0.2.$$

From equations (25) and (26), the necessary formulas are

$$x_{i+1} = x_i + h,$$
$$Y_{i+1} = Y_i + h\, f(x_i, Y_i).$$

For the current problem, $x_0 = -1$, $Y_0 = y_0 = -0.2$, $f(x_i, Y_i) = Y_i + x_i^2$ and $h = 0.1$. The particular formulas needed here are therefore

$$x_{i+1} = x_i + 0.1, \quad \text{where } x_0 = -1,$$
$$Y_{i+1} = Y_i + 0.1(Y_i + x_i^2), \quad \text{where } Y_0 = -0.2.$$

The second of these formulas can also be written as

$$Y_{i+1} = 1.1Y_i + 0.1x_i^2, \quad \text{where } Y_0 = -0.2.$$

Solution to Exercise 16

(a) Equations (i), (ii), (iii), (vi), (vii) and (viii) are linear and constant-coefficient. (Equation (v) is non-linear; (iv) is linear but not constant-coefficient.)

(b) Of the linear constant-coefficient equations, only (iii) and (viii) are homogeneous.

(c) In equations (i)–(v) the (dependent, independent) variable pairs are all (y, x). In equations (vi), (vii) and (viii) they are (t, θ), (x, t) and (x, t), respectively.

Solution to Exercise 17

(a) $\lambda^2 - 5\lambda + 6 = 0$

(b) $\lambda^2 - 9 = 0$

(c) $\lambda^2 + 2\lambda = 0$

Solution to Exercise 18

(a) The auxiliary equation is $\lambda^2 + 5\lambda + 6 = 0$. Solving this by factorisation as $(\lambda + 2)(\lambda + 3) = 0$ gives the roots $\lambda_1 = -2$ and $\lambda_2 = -3$. The general solution is therefore

$$y = Ce^{-2x} + De^{-3x},$$

where C and D are arbitrary constants.

(b) The auxiliary equation is $2\lambda^2 + 3\lambda = 0$. This can be factorised as $\lambda(2\lambda + 3) = 0$, so its roots are $\lambda_1 = 0$ and $\lambda_2 = -\frac{3}{2}$. The general solution is therefore

$$y = Ce^0 + De^{-3x/2} = C + De^{-3x/2},$$

where C and D are arbitrary constants.

(c) The auxiliary equation is $\lambda^2 - 4 = 0$, that is, $\lambda^2 = 4$, so its roots are $\lambda_1 = -2$ and $\lambda_2 = 2$. The general solution is therefore

$$z = Ce^{-2u} + De^{2u},$$

where C and D are arbitrary constants.

Solution to Exercise 19

(a) The auxiliary equation is $\lambda^2 + 2\lambda + 1 = 0$, which can be factorised as $(\lambda + 1)^2 = 0$, giving equal roots $\lambda_1 = \lambda_2 = -1$. The general solution is therefore

$$y = (C + Dx)e^{-x},$$

where C and D are arbitrary constants.

(b) The auxiliary equation $\lambda^2 - 4\lambda + 4 = 0$ factorises as $(\lambda - 2)^2 = 0$, which has equal roots $\lambda_1 = \lambda_2 = 2$. The general solution is therefore

$$s = (C + Dt)e^{2t},$$

where C and D are arbitrary constants.

Solution to Exercise 20

(a) The auxiliary equation is $\lambda^2 + 4\lambda + 8 = 0$, which has solutions

$$\lambda = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i.$$

The general solution is therefore

$$y = e^{-2x}(C \cos 2x + D \sin 2x),$$

where C and D are arbitrary constants.

(b) The auxiliary equation is $\lambda^2 + 9 = 0$, which has solutions

$$\lambda = \pm 3i.$$

The general solution is therefore

$$\theta = e^0(C \cos 3t + D \sin 3t) = C \cos 3t + D \sin 3t,$$

where C and D are arbitrary constants.

Solution to Exercise 21

(a) The auxiliary equation is $\lambda^2 + 4 = 0$, which has solutions $\lambda = \pm 2i$. The general solution is therefore

$$y = C \cos 2x + D \sin 2x,$$

where C and D are arbitrary constants.

(b) The auxiliary equation is $\lambda^2 - 6\lambda + 8 = 0$, which has solutions $\lambda_1 = 4$ and $\lambda_2 = 2$. The general solution is therefore

$$u = Ce^{4x} + De^{2x},$$

where C and D are arbitrary constants.

(c) The auxiliary equation is $\lambda^2 + 2\lambda = 0$, which has solutions $\lambda_1 = 0$ and $\lambda_2 = -2$. The general solution is therefore

$$y = C + De^{-2x},$$

where C and D are arbitrary constants.

(d) The auxiliary equation is $\lambda^2 - 2\lambda + 1 = 0$, which has solutions $\lambda_1 = \lambda_2 = 1$. The general solution is therefore

$$y = (C + Dx)e^x,$$

where C and D are arbitrary constants.

(e) The auxiliary equation is $\lambda^2 - \omega^2 = 0$, which has solutions $\lambda = \pm\omega$. The general solution is therefore

$$y = Ce^{\omega x} + De^{-\omega x},$$

where C and D are arbitrary constants.

(f) The auxiliary equation is $\lambda^2 + 4\lambda + 29 = 0$, which has solutions $\lambda = -2 \pm 5i$. The general solution is therefore

$$e^{-2x}(C\cos 5x + D\sin 5x),$$

where C and D are arbitrary constants.

Solution to Exercise 22

The auxiliary equation is $\lambda^2 + g/l = 0$, which has solutions $\lambda = \pm i\sqrt{g/l}$. The general solution is therefore

$$\theta = C\cos\left(\sqrt{\frac{g}{l}}\,t\right) + D\sin\left(\sqrt{\frac{g}{l}}\,t\right).$$

This problem will be discussed again in Unit 10.

Solution to Exercise 23

We could check this directly, by substituting $y = y_{\mathrm{p}_1} - y_{\mathrm{p}_2}$ into the associated homogeneous equation. However, it is easier to appeal to the principle of superposition. Since y_{p_1} and y_{p_2} both satisfy

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

Theorem 2 shows that the combination $y = y_{\mathrm{p}_1} - y_{\mathrm{p}_2}$ indeed satisfies

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) - f(x) = 0,$$

as required.

Solution to Exercise 24

(a) The associated homogeneous equation is $d^2y/dx^2 + 4y = 0$. The complementary function (see Exercise 21(a)) is $y_{\mathrm{c}} = C\cos 2x + D\sin 2x$, where C and D are arbitrary constants.

Trying a solution of the form $y_{\mathrm{p}} = p$, where p is a constant, in the original equation $d^2y/dx^2 + 4y = 8$ gives $0 + 4p = 8$, so $p = 2$. Thus a particular integral is $y_{\mathrm{p}} = 2$.

By Procedure 6, the general solution is

$$y = C\cos 2x + D\sin 2x + 2.$$

(b) The associated homogeneous equation is $d^2y/dx^2 - 3\,dy/dx + 2y = 0$. The complementary function (see Example 8) is $y_{\mathrm{c}} = Ce^{x} + De^{2x}$, where C and D are arbitrary constants.

Trying a solution of the form $y_{\mathrm{p}} = p$ in the original equation $d^2y/dx^2 - 3\,dy/dx + 2y = 6$ gives $0 - 0 + 2p = 6$, so $p = 3$. Thus a particular integral is $y_{\mathrm{p}} = 3$.

By Procedure 6, the general solution is

$$y = Ce^{x} + De^{2x} + 3.$$

Solution to Exercise 25

(a) Substituting $y = p_1x + p_0$ and its derivatives into the differential equation gives

$$0 - 2p_1 + 2(p_1x + p_0) = 2p_1x + (2p_0 - 2p_1) = 2x + 3.$$

Equating coefficients gives $p_1 = 1$, $p_0 = \frac{5}{2}$. Therefore a particular integral is

$$y_\text{p} = x + \tfrac{5}{2}.$$

(b) Substituting $y = p_1x + p_0$ and its derivatives into the differential equation gives

$$0 + 2p_1 + (p_1x + p_0) = p_1x + (2p_1 + p_0) = 2x.$$

Hence $p_1 = 2$, $p_0 = -4$, and a particular integral is

$$y_\text{p} = 2x - 4.$$

Solution to Exercise 26

We try $y = p_2t^2 + p_1t + p_0$, which has derivatives $\dot{y} = 2p_2t + p_1$, $\ddot{y} = 2p_2$. Substituting these into the differential equation gives

$$2p_2 - (p_2t^2 + p_1t + p_0) = -p_2t^2 - p_1t + (2p_2 - p_0) = t^2.$$

Hence $p_2 = -1$, $p_1 = 0$, $p_0 = -2$, and a particular integral is

$$y_\text{p} = -t^2 - 2.$$

Solution to Exercise 27

We try a solution of the form $y = pe^{-x}$, which has derivatives $dy/dx = -pe^{-x}$, $d^2y/dx^2 = pe^{-x}$. Substituting these into the differential equation gives

$$2pe^{-x} + 2pe^{-x} + pe^{-x} = 5pe^{-x} = 2e^{-x}.$$

Hence $p = \frac{2}{5}$, and a particular integral is

$$y_\text{p} = \tfrac{2}{5}e^{-x}.$$

Solution to Exercise 28

We try $y = p\cos 3t + q\sin 3t$, which has derivatives

$$\frac{dy}{dt} = -3p\sin 3t + 3q\cos 3t, \quad \frac{d^2y}{dt^2} = -9p\cos 3t - 9q\sin 3t.$$

Substituting into the differential equation gives

$$\begin{aligned}(-9p\cos 3t &- 9q\sin 3t) - (-3p\sin 3t + 3q\cos 3t)\\ &= -(9p + 3q)\cos 3t + (3p - 9q)\sin 3t\\ &= \cos 3t + \sin 3t.\end{aligned}$$

Hence we have a pair of simultaneous equations

$$\begin{aligned}-9p - 3q &= 1,\\ 3p - 9q &= 1.\end{aligned}$$

Adding three times the second equation to the first, to eliminate p, gives $-30q = 4$, so $q = -\frac{2}{15}$, whence $p = -\frac{1}{15}$. A particular integral is thus

$$y_{\mathrm{p}} = -\tfrac{1}{15}\cos 3t - \tfrac{2}{15}\sin 3t.$$

Solution to Exercise 29

(a) Try $y = pe^{2x}$.

(b) Try $y = p\cos 4x + q\sin 4x$.

Solution to Exercise 30

(a) The complementary function is $y_{\mathrm{c}} = e^{-x}(C\cos x + D\sin x)$, where C and D are arbitrary constants.

To find a particular integral, try $y = p_0$. Substituting into the differential equation gives

$$0 + 0 + 2p_0 = 2p_0 = 4.$$

Hence $p_0 = 2$, and a particular integral is $y_{\mathrm{p}} = 2$.

Therefore the general solution is

$$y = e^{-x}(C\cos x + D\sin x) + 2.$$

(b) The complementary function is $\theta_{\mathrm{c}} = C + De^{-3t}$, where C and D are arbitrary constants.

To find a particular integral, try $\theta = p\cos 2t + q\sin 2t$. Differentiating gives

$$\frac{d\theta}{dt} = -2p\sin 2t + 2q\cos 2t, \quad \frac{d^2\theta}{dt^2} = -4p\cos 2t - 4q\sin 2t.$$

Substituting into the differential equation gives

$$\begin{aligned}&(-4p\cos 2t - 4q\sin 2t) + 3(-2p\sin 2t + 2q\cos 2t)\\ &\quad = (6q - 4p)\cos 2t - (4q + 6p)\sin 2t\\ &\quad = 13\cos 2t.\end{aligned}$$

Comparing the coefficients of $\cos 2t$ and $\sin 2t$ gives a pair of simultaneous equations to solve:

$$\begin{aligned}-4p + 6q &= 13,\\ -6p - 4q &= 0.\end{aligned}$$

Hence $p = -1$, $q = \frac{3}{2}$, and a particular integral is $\theta_{\mathrm{p}} = -\cos 2t + \frac{3}{2}\sin 2t$.

Therefore the general solution is

$$\theta = C + De^{-3t} - \cos 2t + \tfrac{3}{2}\sin 2t.$$

Solution to Exercise 31

Putting the equation into the usual form and using $R = S = Q = 1$ gives

$$y'' - y = -\tfrac{1}{2}(l - x)x = -\tfrac{1}{2}lx + \tfrac{1}{2}x^2.$$

The associated homogeneous equation is $y'' - y = 0$, which has auxiliary equation $\lambda^2 - 1 = 0$. This has roots $\lambda = \pm 1$, so the complementary function is

$$y_c = Ce^x + De^{-x},$$

where C and D are arbitrary constants.

To obtain a particular integral, we try a function of the form $y = p_2x^2 + p_1x + p_0$. Its derivatives are $y' = 2p_2x + p_1$, $y'' = 2p_2$. Substituting into the differential equation gives

$$\begin{aligned} 2p_2 - (p_2x^2 + p_1x + p_0) &= -p_2x^2 - p_1x + (2p_2 - p_0) \\ &= \tfrac{1}{2}x^2 - \tfrac{1}{2}lx. \end{aligned}$$

Hence $p_2 = -\frac{1}{2}$, $p_1 = \frac{1}{2}l$, $p_0 = -1$, and a particular integral is

$$y_p = -\tfrac{1}{2}x^2 + \tfrac{1}{2}lx - 1.$$

Therefore the general solution is

$$y = Ce^x + De^{-x} - \tfrac{1}{2}x^2 + \tfrac{1}{2}lx - 1.$$

Solution to Exercise 32

(a) From Example 8, the associated homogeneous equation has general solution $y = Ce^x + De^{2x}$, where C and D are arbitrary constants, and the trial solution $y = pe^x$ suggested by Procedure 7 is a solution of this equation (with $C = p$, $D = 0$). So we try $y = pxe^x$ instead. Differentiating twice gives

$$\begin{aligned} \frac{dy}{dx} &= pe^x + pxe^x = p(1 + x)e^x, \\ \frac{d^2y}{dx^2} &= pe^x + p(1 + x)e^x = p(2 + x)e^x. \end{aligned}$$

Substituting into the differential equation gives

$$p(2 + x)e^x - 3p(1 + x)e^x + 2pxe^x = -pe^x = 4e^x.$$

Hence $p = -4$, and a particular integral is

$$y_p = -4xe^x.$$

(b) From Exercise 18(b), the associated homogeneous equation has general solution $y = C + De^{-3x/2}$, where C and D are arbitrary constants, and the trial solution $y = p_0$ suggested by Procedure 7 is a solution of this equation (with $C = p_0$, $D = 0$). So we try $y = p_0x$. Differentiating twice gives

$$\frac{dy}{dx} = p_0, \quad \frac{d^2y}{dx^2} = 0.$$

Substituting into the differential equation gives $3p_0 = 1$, so $p_0 = \frac{1}{3}$, and a particular integral is

$$y_{\text{p}} = \tfrac{1}{3}x.$$

Solution to Exercise 33

The auxiliary equation of the differential equation $m\ddot{x} + r\dot{x} = mg$ is

$$m\lambda^2 + r\lambda = 0,$$

with solutions $\lambda = 0$ and $\lambda = -r/m$. The complementary function is therefore

$$x_{\text{c}} = C + De^{-rt/m},$$

where C and D are arbitrary constants.

The inhomogeneous term is mg, so Procedure 7 suggests a trial solution $x = p_0$. However, this is a solution of the associated homogeneous equation (with $C = p_0$, $D = 0$). Hence we try $x = p_0 t$ instead, where t is the independent variable in this problem. Differentiating and substituting gives

$$rp_0 = mg,$$

so

$$p_0 = \frac{mg}{r}.$$

Hence a particular integral is

$$x_{\text{p}} = \frac{mgt}{r},$$

and the general solution is

$$x = C + De^{-rt/m} + \frac{mgt}{r}.$$

Solution to Exercise 34

From Exercise 21(d), the associated homogeneous equation has general solution $y = (C + Dx)e^x$, where C and D are arbitrary constants. So not only is the trial solution $y = pe^x$ suggested by Procedure 7 a solution of the associated homogeneous differential equation (with $C = p$, $D = 0$), but so is $y = pxe^x$ (with $C = 0$, $D = p$). So we try $y = px^2e^x$. Differentiating twice gives

$$\frac{dy}{dx} = 2pxe^x + px^2e^x = p(2x + x^2)e^x,$$
$$\frac{d^2y}{dx^2} = p(2 + 2x)e^x + p(2x + x^2)e^x = p(2 + 4x + x^2)e^x.$$

Substituting into the differential equation gives

$$p(2 + 4x + x^2)e^x - 2p(2x + x^2)e^x + px^2e^x = 2pe^x = e^x.$$

Hence $p = \frac{1}{2}$, and a particular integral is

$$y_{\text{p}} = \tfrac{1}{2}x^2e^x.$$

Solution to Exercise 35

(a) From Exercise 34, $y_{\mathrm{p}} = \frac{1}{2}x^2e^x$ is a particular integral for

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

So, using the principle of superposition, we can find a particular integral for the given differential equation if we can find one for

$$\frac{d^2y}{dx} - 2\frac{dy}{dx} + y = -3e^{2x}.$$

We try $y = pe^{2x}$, which has derivatives

$$\frac{dy}{dx} = 2pe^{2x}, \quad \frac{d^2y}{dx^2} = 4pe^{2x}.$$

Substituting into the differential equation gives

$$4pe^{2x} - 4pe^{2x} + pe^{2x} = pe^{2x} = -3e^{2x}.$$

Hence $p = -3$, and $y_{\mathrm{p}} = -3e^{2x}$ is a particular integral for the differential equation with right-hand side $-3e^{2x}$.

Thus, using the principle of superposition, a particular integral for the given differential equation is

$$y_{\mathrm{p}} = 4\left(\tfrac{1}{2}x^2e^x\right) - 3e^{2x} = 2x^2e^x - 3e^{2x}.$$

(b) This time we do not have a particular integral for any part of the right-hand-side function, so we need to start from scratch.

First consider the $12\cos 2t$ term on the right-hand side, and try $x = p\cos 2t + q\sin 2t$ as a trial solution. This has derivatives

$$\frac{dx}{dt} = -2p\sin 2t + 2q\cos 2t, \quad \frac{d^2x}{dt^2} = -4p\cos 2t - 4q\sin 2t.$$

Substituting into the differential equation gives

$$\begin{aligned}
&2(-4p\cos 2t - 4q\sin 2t) + 3(-2p\sin 2t + 2q\cos 2t)\\
&\qquad + 2(p\cos 2t + q\sin 2t)\\
&\quad = 6(q-p)\cos 2t - 6(p+q)\sin 2t\\
&\quad = 12\cos 2t.
\end{aligned}$$

So $p + q = 0$, $q - p = 2$, hence $p = -1$, $q = 1$, and a particular integral is

$$x_{\mathrm{p}} = -\cos 2t + \sin 2t.$$

Now consider the 10 term, and try $x = p_0$. Substituting into the differential equation gives $2p_0 = 10$, so $p_0 = 5$, and a particular integral is

$$x_{\mathrm{p}} = 5.$$

Therefore, using the principle of superposition, a particular integral for the differential equation with $f(t) = 12\cos 2t + 10$ is

$$x_{\mathrm{p}} = -\cos 2t + \sin 2t + 5.$$

Solution to Exercise 36

(a) From Exercise 21(a), the complementary function is

$$\theta_c = C\cos 2t + D\sin 2t,$$

where C and D are arbitrary constants.

To find a particular integral, try $\theta = p_1 t + p_0$. Substituting this and its derivatives into the differential equation gives

$$4(p_1 t + p_0) = 2t.$$

Hence $p_1 = \frac{1}{2}$, $p_0 = 0$, and a particular integral is

$$\theta_p = \tfrac{1}{2}t.$$

Therefore the general solution is

$$\theta = C\cos 2t + D\sin 2t + \tfrac{1}{2}t.$$

(b) The auxiliary equation is $\lambda^2 + 4\lambda + 5 = 0$, which has solutions $\lambda = -2 \pm i$. So (using Procedure 5) the complementary function is

$$u_c = e^{-2t}(C\cos t + D\sin t),$$

where C and D are arbitrary constants.

To find a particular integral, try $u = p_0$. Substituting gives $5p_0 = 5$. Hence $p_0 = 1$, and a particular integral is

$$u_p = 1.$$

Therefore the general solution is

$$u = e^{-2t}(C\cos t + D\sin t) + 1.$$

(c) The auxiliary equation is $3\lambda^2 - 2\lambda - 1 = 0$, which has solutions $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{3}$. So the complementary function is

$$Y_c = Ce^x + De^{-x/3},$$

where C and D are arbitrary constants.

Consider first the e^{2x} term on the right-hand side of the equation. To find a particular integral, try $Y = pe^{2x}$. The derivatives are $dY/dx = 2pe^{2x}$ and $d^2Y/dx^2 = 4pe^{2x}$. Substituting gives

$$3(4pe^{2x}) - 2(2pe^{2x}) - pe^{2x} = 7pe^{2x} = e^{2x}.$$

Hence $p = \frac{1}{7}$, and a particular integral is

$$Y_p = \tfrac{1}{7}e^{2x}.$$

Now consider the 3 term on the right-hand side of the equation, and try $Y = p_0$. Substituting gives $-p_0 = 3$, so $p_0 = -3$, and a particular integral is

$$Y_p = -3.$$

Therefore, using the principle of superposition, a particular integral for the differential equation with $f(x) = e^{2x} + 3$ is

$$Y_p = \tfrac{1}{7}e^{2x} - 3.$$

Therefore the general solution is

$$Y = Ce^{x} + De^{-x/3} + \tfrac{1}{7}e^{2x} - 3.$$

(d) From Exercise 18(c), the complementary function is

$$y_{\mathrm{c}} = Ce^{-2x} + De^{2x},$$

where C and D are arbitrary constants.

To find a particular integral, since e^{-2x} is a solution of the associated homogeneous equation, try $y = pxe^{-2x}$. The derivatives are $dy/dx = p(1-2x)e^{-2x}$ and $d^2y/dx^2 = 4p(x-1)e^{-2x}$. Substituting gives

$$4p(x-1)e^{-2x} - 4pxe^{-2x} = -4pe^{-2x} = e^{-2x}.$$

Hence $p = -\frac{1}{4}$, and a particular integral is

$$y_{\mathrm{p}} = -\tfrac{1}{4}xe^{-2x}.$$

Therefore the general solution is

$$y = Ce^{-2x} + De^{2x} - \tfrac{1}{4}xe^{-2x}.$$

(e) From Exercise 21(a), the complementary function is

$$y_{\mathrm{c}} = C\cos 2x + D\sin 2x,$$

where C and D are arbitrary constants.

To find a particular integral, we note that, from part (a) of this exercise, a particular integral for $d^2y/dx^2 + 4y = 2x$ is $y_{\mathrm{p}} = \frac{1}{2}x$, so for a right-hand side of $3x$ we will have $y_{\mathrm{p}} = \frac{3}{2}(\frac{1}{2}x) = \frac{3}{4}x$. So we need to consider only the $\sin 2x$ term, and then use the principle of superposition.

For this term, noting the form of the complementary function, try $y = x(p\cos 2x + q\sin 2x)$. The derivatives are

$$\frac{dy}{dx} = (p + 2qx)\cos 2x + (q - 2px)\sin 2x,$$
$$\frac{d^2y}{dx^2} = (4q - 4px)\cos 2x - (4p + 4qx)\sin 2x.$$

Substituting gives

$$\begin{aligned}(4q - 4px)&\cos 2x - (4p + 4qx)\sin 2x + 4x(p\cos 2x + q\sin 2x)\\ &= 4q\cos 2x - 4p\sin 2x\\ &= \sin 2x.\end{aligned}$$

Hence $p = -\frac{1}{4}$, $q = 0$, and a particular integral is

$$y_{\mathrm{p}} = -\tfrac{1}{4}x\cos 2x.$$

Therefore, using the principle of superposition, a particular integral for the given differential equation is

$$y_{\mathrm{p}} = \tfrac{3}{4}x - \tfrac{1}{4}x\cos 2x.$$

Thus the general solution is

$$y = C\cos 2x + D\sin 2x + \tfrac{3}{4}x - \tfrac{1}{4}x\cos 2x.$$

(f) From Example 8, the complementary function is

$$y_{\rm c} = Ce^{x} + De^{2x},$$

where C and D are arbitrary constants.

Consider first the $2e^x$ term on the right-hand side of the equation. To find a particular integral, since e^x appears in the complementary function, try $y = pxe^x$, which has derivatives

$$\frac{dy}{dx} = p(1+x)e^{x}, \quad \frac{d^2y}{dx^2} = p(2+x)e^{x}.$$

Substituting into the differential equation gives

$$p(2+x)e^{x} - 3p(1+x)e^{x} + 2pxe^{x} = -pe^{x} = 2e^{x}.$$

Hence $p = -2$, and a particular integral is

$$y_{\rm p} = -2xe^{x}.$$

Now consider the $-5e^{2x}$ term on the right-hand side of the equation. To find a particular integral, since e^{2x} appears in the complementary function, try $y = pxe^{2x}$, which has derivatives

$$\frac{dy}{dx} = p(1+2x)e^{2x}, \quad \frac{d^2y}{dx^2} = p(4+4x)e^{2x}.$$

Substituting into the differential equation gives

$$p(4+4x)e^{2x} - 3p(1+2x)e^{2x} + 2pxe^{2x} = pe^{2x} = -5e^{2x}.$$

Hence $p = -5$, and a particular integral is

$$y_{\rm p} = -5xe^{2x}.$$

Therefore, using the principle of superposition, a particular integral for the differential equation with $f(x) = 2e^x - 5e^{2x}$ is

$$y_{\rm p} = -2xe^{x} - 5xe^{2x}.$$

Therefore the general solution is

$$y = Ce^{x} + De^{2x} - 2xe^{x} - 5xe^{2x}.$$

Solution to Exercise 37

(a) From Exercise 20(b), the general solution is

$$u = C\cos 3t + D\sin 3t,$$

where C and D are arbitrary constants. Its derivative is

$$u' = -3C\sin 3t + 3D\cos 3t.$$

Substituting the initial condition $t = \frac{\pi}{2}$, $u = 0$ into the general solution gives $D = 0$. Substituting the initial condition $t = \frac{\pi}{2}$, $u' = 1$ into the derivative gives $C = \frac{1}{3}$. Hence the required particular solution is

$$u = \tfrac{1}{3}\cos 3t.$$

(b) From Exercise 32(a), the general solution is

$$y = Ce^x + De^{2x} - 4xe^x,$$

where C and D are arbitrary constants. Its derivative is

$$y' = Ce^x + 2De^{2x} - 4(1 + x)e^x.$$

Substituting the initial condition $x = 0$, $y = 4$ into the general solution gives $C + D = 4$. Substituting the initial condition $x = 0$, $y' = 2$ into the derivative gives $C + 2D - 4 = 2$. Hence $C = 2$, $D = 2$, and the required particular solution is

$$y = 2e^x + 2e^{2x} - 4xe^x = (2 + 4x)e^x + 2e^{2x}.$$

(c) From Exercises 21(d) and 35(a), the general solution is

$$y = (C + Dx)e^x + 2x^2e^x - 3e^{2x},$$

where C and D are arbitrary constants. Its derivative is

$$y' = (C + D + Dx)e^x + (4x + 2x^2)e^x - 6e^{2x}.$$

Substituting the initial condition $x = 0$, $y = 4$ into the general solution gives $C - 3 = 4$. Substituting the initial condition $x = 0$, $y' = -1$ into the derivative gives $C + D - 6 = -1$. Hence $C = 7$, $D = -2$, and the required particular solution is

$$y = (7 - 2x)e^x + 2x^2e^x - 3e^{2x} = (7 - 2x + 2x^2)e^x - 3e^{2x}.$$

Solution to Exercise 38

From Exercise 32(a), the general solution is

$$y = Ce^x + De^{2x} - 4xe^x,$$

where C and D are arbitrary constants, and its derivative is

$$y' = Ce^x + 2De^{2x} - 4(1 + x)e^x.$$

Substituting the boundary condition $x = 0$, $y' = 2$ into the derivative gives $C + 2D - 4 = 2$. Substituting $x = 1$, $y = 0$ into the general solution gives $Ce + De^2 - 4e = 0$, which can be rearranged to give $C + eD = 4$. Hence $C = (8 - 6e)/(2 - e)$, $D = 2/(2 - e)$, and the required particular solution is

$$y = \frac{8 - 6e}{2 - e}e^x + \frac{2}{2 - e}e^{2x} - 4xe^x.$$

Solution to Exercise 39

From Exercise 31, the general solution of the differential equation is $y = Ce^x + De^{-x} - \frac{1}{2}x^2 + \frac{1}{2}lx - 1$, which for $l = 2$ becomes

$$y = Ce^x + De^{-x} - \tfrac{1}{2}x^2 + x - 1,$$

where C and D are arbitrary constants.

The boundary conditions, resulting from the beam resting on supports at its two ends, are $y(0) = 0$, $y(2) = 0$.

Substitution of these into the general solution gives $C + D - 1 = 0$ and $Ce^2 + De^{-2} - 1 = 0$. Multiplying the second equation by e^2 gives $C + D = 1$ and $Ce^4 + D = e^2$ as the equations to solve. Subtracting the equations gives $C(e^4 - 1) = e^2 - 1$, thus

$$C = \frac{e^2 - 1}{e^4 - 1} = \frac{e^2 - 1}{(e^2 + 1)(e^2 - 1)} = \frac{1}{e^2 + 1},$$

$$D = 1 - C = 1 - \frac{1}{e^2 + 1} = \frac{e^2 + 1 - 1}{e^2 + 1} = \frac{e^2}{e^2 + 1}.$$

Hence the required particular solution is

$$y = \frac{1}{e^2 + 1}(e^x + e^{2-x}) - \tfrac{1}{2}x^2 + x - 1.$$

You can check that this is symmetric about the centre of the beam by replacing x with $2 - x$.

At the centre of the beam, $x = 1$, so $y \simeq 0.148$. The displacement or 'sag' at the centre of the beam is approximately $0.148\,\mathrm{m}$ or about $14.8\,\mathrm{cm}$.

Solution to Exercise 40

(a) The differential equation is the same in each case, and from Exercise 21(a) its general solution is

$$u = C\cos 2x + D\sin 2x,$$

where C and D are arbitrary constants, and the derivative is

$$u' = -2C\sin 2x + 2D\cos 2x.$$

In this case, we have an initial-value problem.

The condition $u(0) = 1$ gives $C = 1$. The condition $u'(0) = 0$ gives $D = 0$. The required solution is therefore

$$u = \cos 2x.$$

(b) This is a boundary-value problem.

The condition $u(0) = 0$ gives $C = 0$. The condition $u\left(\frac{\pi}{2}\right) = 0$ gives $C = 0$ also. D therefore remains arbitrary, so there is an infinite number of solutions, of the form

$$u = D\sin 2x.$$

(c) This is an initial-value problem.

The condition $u\left(\frac{\pi}{2}\right) = 0$ gives $C = 0$. The condition $u'\left(\frac{\pi}{2}\right) = 0$ gives $D = 0$. The required solution is therefore the zero function

$$u = 0.$$

(Alternatively, since the differential equation is homogeneous and the initial values are both equal to zero, by the remarks after Theorem 4, the solution is the zero function $u = 0$.)

(d) This is a boundary-value problem.

The condition $u(-\pi) = 1$ gives $C = 1$. The condition $u\left(\frac{\pi}{4}\right) = 2$ gives $D = 2$. The required solution is therefore

$$u = \cos 2x + 2\sin 2x.$$

(e) This is a boundary-value problem.

The condition $u'(0) = 0$ gives $D = 0$. The condition $u'\left(\frac{\pi}{4}\right) = 1$ gives $C = -\frac{1}{2}$. The required solution is therefore

$$u = -\tfrac{1}{2}\cos 2x.$$

Solution to Exercise 41

(a) This is an initial-value problem, therefore by Theorem 4 it has a unique solution.

The general solution is

$$u = e^{-2t}(C\cos t + D\sin t),$$

where C and D are arbitrary constants. Its derivative is

$$u' = e^{-2t}\big((-2C + D)\cos t - (C + 2D)\sin t\big).$$

The condition $u(0) = 0$ gives $C = 0$. The condition $u'(0) = 2$ gives $D = 2$. The solution is therefore

$$u = 2e^{-2t}\sin t.$$

(b) This is an initial-value problem, therefore by Theorem 4 it has a unique solution.

The differential equation is homogeneous and the initial values are both equal to zero. Hence the solution is the zero function $y = 0$.

(c) This is a boundary-value problem, which may have no solution, a unique solution, or an infinite number of solutions.

The complementary function is

$$x_{\mathrm{c}} = C\cos 3t + D\sin 3t,$$

where C and D are arbitrary constants.

To find a particular integral, try $x = p_1 t + p_0$. Substituting into the differential equation gives

$$9(p_1 t + p_0) = 3(1 - \pi t).$$

Hence $p_1 = -\frac{\pi}{3}$, $p_0 = \frac{1}{3}$, and a particular integral is

$$x_{\mathrm{p}} = -\tfrac{\pi}{3}t + \tfrac{1}{3}.$$

Therefore the general solution is

$$x = C\cos 3t + D\sin 3t - \tfrac{\pi}{3}t + \tfrac{1}{3},$$

and its derivative is

$$\dot{x} = -3C\sin 3t + 3D\cos 3t - \tfrac{\pi}{3}.$$

The condition $x(0) = \frac{1}{3}$ gives $C = 0$. The condition $\dot{x}\left(\frac{\pi}{3}\right) = 0$ gives $D = -\frac{\pi}{9}$. The solution is therefore

$$x = -\tfrac{\pi}{9}\sin 3t - \tfrac{\pi}{3}t + \tfrac{1}{3}.$$

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