MST210 TMA07 - modelling report

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1 Specify the purpose

The purpose of this task is to mitigate the likelihood of an accident occurring on a playground swing, be it a child falling or an object falling from the child. This will be done by deriving the minimum area around the swing that should be established as a soft-landing zone, by considering the maximum distance a child or object could reasonably travel while falling.

2 Create the model

The model will consider a 2-dimensional scenario where a child on a swing is pulled back and released from rest at a particular angle from the vertical, before the child (or an object held by the child) falls off the swing. Modelling the motion of the projectile after release will give an expression for its initial velocity, which in turn will give an expression for its horizontal range. Maximizing this expression for the projectile's range with respect to the angle of release will allow estimation of a maximum horizontal range in a worst-case scenario.

2.1 Assumptions

To formulate the model we make the following assumptions:

1. The swing is released at rest
2. Air resistance and friction are negligible and can be ignored
3. The only force acting on the child and the projectile is the force of gravity
4. The seat and the child can be modeled together as a particle
5. The chains are modeled as a rigid body that undergoes rotation about a fixed point
6. The swing, child, and projectiles move in the vertical plane only
7. The projectile leaves the swing on a tangent to the swing's path
8. The projectile stops moving the instant it touches the ground

2.2 Variables and parameters

Table 1 lists the quantities used in the modelling process and Figure 1 shows a diagram of the swing in the vertical plane.

Symbol	Quantity	Unit
v	Magnitude of the velocity of the swing/child	ms^{-1}
g	Acceleration due to gravity (taken to be 9.81)	ms^{-2}
α	Initial angle between the swing and the vertical	rad
θ	Angle between the swing and the vertical at time t	rad
h	Vertical distance between the swing seat and the ground at time t	m
h_0	Vertical distance between the swing seat and the ground at rest	m
l	Length of the swing chain	m
R	Horizontal range of the projectile from the center of the swing	m
m	Mass of the swing seat/child	kg
t	Time since the projectile's release from the swing	s
t_{flight}	The time between the projectile's release and it touching the ground	s
E	Total mechanical energy of the swing/child	J
T	Kinetic energy of the swing/child	J
U	Potential energy of the swing/child	J

Table 1: Quantities used in the modelling process, with their symbols and units.

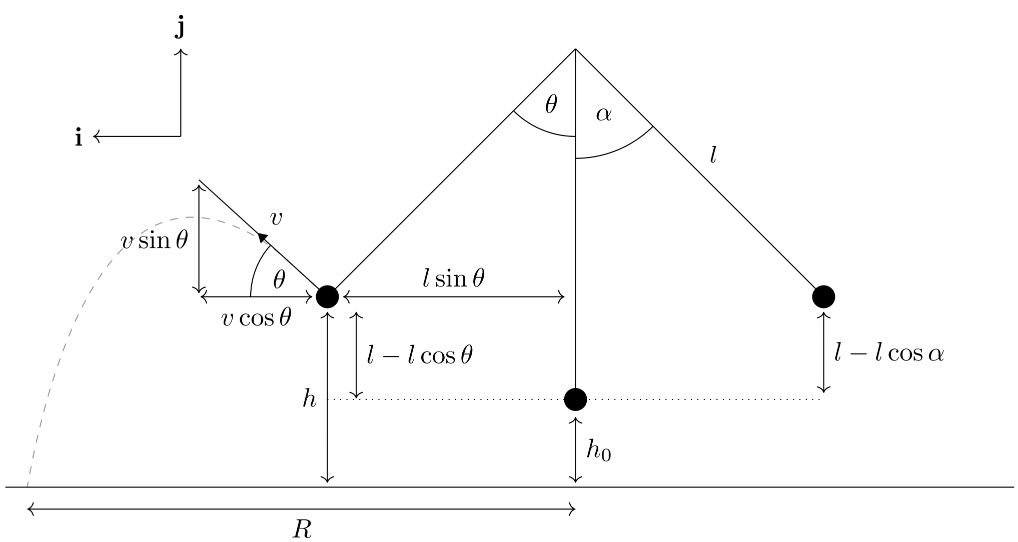

Figure 1: Diagram of the swing in the vertical plane. The filled circle represents the swing/child at the end of the chains. The swing is pulled back until it forms the angle α and is released from rest. The dashed line represents the path of a projectile released at a tangent to the motion of the swing. The direction of the **i** and **j** unit vectors are shown.

3 Formulate the mathematical relationships

As shown in Figure 1, the child and swing (henceforth referred to as just "the swing") are pulled back until they form an angle α with the vertical and are released from rest. The origin is taken to be the point on the ground vertically below the swing at rest, with the x and y axes pointing to the left and up, respectively. The $\mathbf{i}$ and $\mathbf{j}$ unit vectors point in the x and y axis directions, respectively.

We model the swing and the child together as a particle, as per assumption 4. As the force acting on the swing depends only on its position, the law of conservation of mechanical energy[1] states that its total mechanical energy, which is the sum of its kinetic and potential energy, is constant, that is

$$\begin{aligned} E &= T + U \\ &= \tfrac{1}{2}mv^2 + mgh \end{aligned} \tag{1}$$

where, by assumption 5:

$$h = l - l\cos\alpha + h_0$$

By assumption 1, at the point immediately before the swing is released from rest, it has no kinetic energy and so

$$\begin{aligned} E &= U \\ &= mgh \\ &= mg(l - l\cos\alpha + h_0) \end{aligned} \tag{2}$$

Since, by assumption 2, the total mechanical energy is constant, we can equate (1) and (2) and write

$$mg(l - l\cos\alpha + h_0) = \tfrac{1}{2}mv^2 + mg(l - l\cos\theta + h_0)$$

which can be rearranged as

$$v = \sqrt{2gl(\cos\theta - \cos\alpha)}$$

where v is the magnitude of the velocity (speed) of the swing, and the initial speed of the projectile, released at angle θ with the horizontal.

4 Do the mathematics

4.1 Derive a first model

From the moment the projectile is released from the swing, by assumption 3, the only force acting on it is gravity. If we take $\mathbf{r}(t)$ to be the position vector of the projectile at time t since it left the swing then the projectile's acceleration can be expressed as

$$\ddot{\mathbf{r}} = -g\mathbf{j}$$

Integrating wrt t gives

$$\dot{\mathbf{r}} = -gt\mathbf{j} + \mathbf{C}$$

where $\mathbf{C}$ is an arbitrary vector constant. Using the initial condition (by assumptions 6 and 7, and illustrated in Figure 1) that $\dot{\mathbf{r}}(0) = (v\cos\theta)\mathbf{i} + (v\sin\theta)\mathbf{j}$ gives

$$\dot{\mathbf{r}} = (v\cos\theta)\mathbf{i} + (v\sin\theta - gt)\mathbf{j}$$

Further integrating wrt t gives

$$\mathbf{r} = (vt\cos\theta)\mathbf{i} + \big(-\tfrac{1}{2}gt^2 + vt\sin\theta\big)\mathbf{j} + \mathbf{D}$$

where $\mathbf{D}$ is an arbitrary vector constant. Using the initial condition (also shown in Figure 1) that

$$\mathbf{r}(0) = (l\sin\theta)\mathbf{i} + (h_0 + l - l\cos\theta)\mathbf{j}$$

gives

$$\mathbf{r} = (l\sin\theta + vt\cos\theta)\mathbf{i} + \big(-\tfrac{1}{2}gt^2 + v\sin\theta t + h_0 + l - l\cos\theta\big)\mathbf{j}$$

The projectile hits the ground when the $\mathbf{j}$-component of its position vector is 0 and so we have the quadratic equation in terms of t:

$$\tfrac{1}{2}gt^2 - v\sin\theta t - (h_0 + l - l\cos\theta) = 0$$

Applying the quadratic formula and only considering the positive solution for time, we get an expression for the time of flight of the projectile

$$t_{\text{flight}} = \frac{1}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2g(h_0 + l - l\cos\theta)}\right)$$

By assumption 8, the range of the projectile is the $\mathbf{i}$-component of its position vector at t_{flight}:

$$R = l\sin\theta + vt_{\text{flight}}\cos\theta$$

which we can maximise wrt θ to find the maximum horizontal range of a projectile, in a worst-case scenario.

In summary, the model has three steps:

1. Calculate the speed of the swing given its release angle α, current angle θ, and chain length l:

$$v = \sqrt{2gl(\cos\theta - \cos\alpha)}$$

2. Calculate the time of flight of a projectile given its initial speed v, release angle θ, chain length l, and swing height above the ground at rest h_0:

$$t_{\text{flight}} = \frac{1}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2g(h_0 + l - l\cos\theta)}\right)$$

3. Maximise the horizontal range of a projectile wrt to its release angle θ, given the time of flight t_{flight} and chain length l:

$$R = l\sin\theta + vt_{\text{flight}}\cos\theta$$

4.2 Graphs of typical relationships

Figure 2 shows the speed of the swing as a function of θ for three different possible release angles (using a fixed chain length $l = 1.5$m). The speed is zero at the angle of release, increases until the chain is vertical, and then decreases to zero at the negative of the angle of release. Greater angles of release result in greater speed, as expected.

Figure 3 shows the speed of the swing as a function of θ for three different chain lengths (using an angle of release of $\frac{\pi}{2}$). Longer chain lengths result in greater speed, as expected. The behaviour of the graphs of these functions seems congruent with our experience of how we would expect the swing to behave with these parameter changes.

The speed of the swing doesn't depend on the height of the swing above the ground at rest h_0, but the time of flight of the projectile does. Figure 4 shows the time of flight of a projectile as a function of its release angle θ for three different seat heights. As we might expect, higher seat heights result in longer times of flight. The time of flight is lowest when the swing is vertical and has a global maximum at a particular release angle.

4.3 Dimensional analysis

Applying the method of dimensional analysis to the quantity to be determined (the range of the projectile) gives

$$[R] = [l]\sin\theta + [v][t_{\text{flight}}]\cos\theta$$

The range and chain length have dimensions of L, the release angle is dimensionless, the speed has dimension LT^{-1} and the time of flight has dimension T. Thus

$$L = L + LT^{-1}T$$
$$L = L$$

As the terms on the LHS and RHS have dimensions of length, the model is dimensionally consistent.

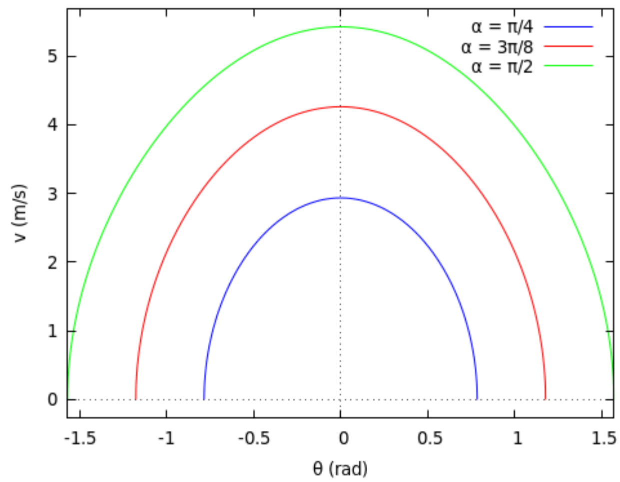

Figure 2: The model's predicted relationship between the speed of the swing v and its angle with the vertical θ, for three values of release angle α, using a constant chain length of $l = 1.5$m.

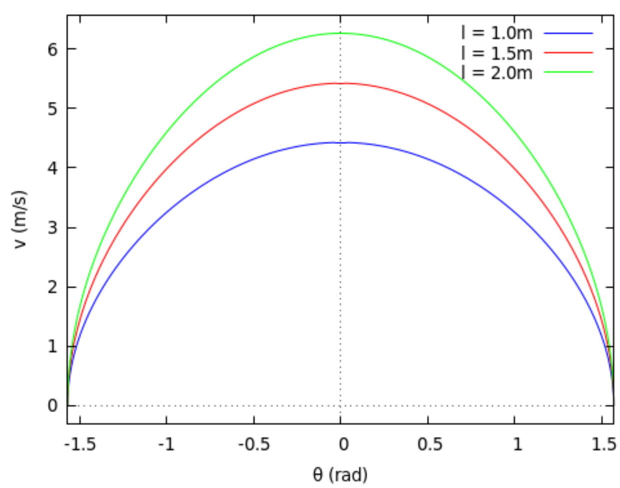

Figure 3: The model's predicted relationship between the speed of the swing v and its angle with the vertical θ for three values of chain length l, using a constant release angle of $\pi/2$.

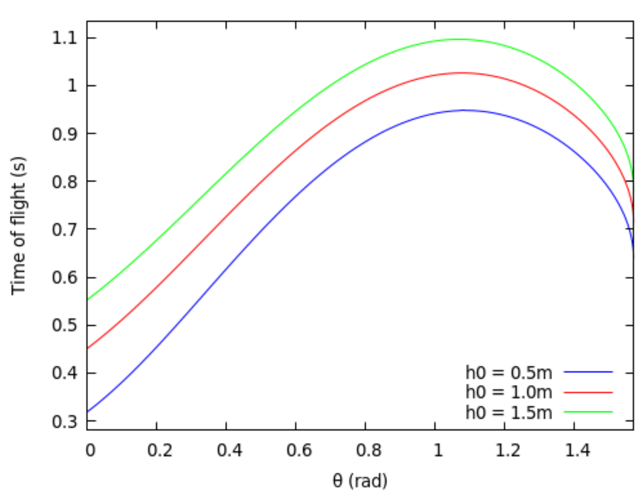

Figure 4: The model's predicted relationship between the time of flight of a projectile and its angle with the horizontal θ for three heights of the swing above the ground at rest h_0, using a constant chain length of $l = 1.5$m and angle of release $\alpha = \pi/2$. Note the change in x axis scale to the figures above.

5 Interpret the results

5.1 Collect relevant data for parameter values

The model has three input parameters that must be chosen to estimate the maximum range:

- The release angle α
- The chain length l
- The height of the swing above the ground at rest h_0

As it is best to plan for a worst-case scenario, we consider the situation where the swing is released from rest at an angle $\frac{\pi}{2}$ as this is more extreme than is likely to be used. To obtain reasonable values for l and h_0, a small sample of swing set products and, where available, their relevant dimensions have been compiled in Table 2.

Retailer (Product)	Overall height (m)	Seat height (m)	Chain length (m)
Jungle Gym (211158)[2]	2.00	Min. 0.35	Max. 1.65
SutcliffePlay (SBN083)[3]	2.40	Min. 0.40	Max. 2.00
Wicksteed (6040-099)[4]	1.46	Min. 0.35	Max. 1.11
Argos (389/2040)[5]	2.03	Not given	Not given
OutdoorToys (SWNG-Venus)[6]	2.12	Not given	Not given
The Outdoor Toy Center (TP 511)[7]	2.00	Not given	Not given

Table 2: A sample of swing set products with their overall height (vertical distance from the ground to the pivot point of the chains), seat height, and chain length. Min. = minimum, Max. = maximum.

Most of the products sampled have a height around 2m, recommend a minimum height above the ground of 0.35m or 0.40m, and recommend a maximum chain length of 1.1m to 2.0m. Given these data, we choose to use input values of $h_0 = 0.4$m and $l = 1.6$m as reasonable inputs to an initial model.

5.2 Describe the mathematical solution

We substitute $\alpha = \frac{\pi}{2}$, $h_0 = 0.4$m, and $l = 1.6$m into the model and maximise the range R wrt θ using the Maxima computer algebra system. This was done by computing the first derivative of the range wrt θ, and using command `find_root` to find its maximum. This gives the angle 0.669... rads (38.376...°) as the angle of release of the projectile that results in its maximum horizontal range (see Figure 5). The time of flight of a projectile released at this angle is $t_{\text{flight}} = 0.814...$s, giving a range of

$$\begin{aligned} R &= 1.6\sin(0.669...) + \sqrt{3.2g\cos(0.669)} \times 0.814... \times \cos(0.669...) \\ &= 4.161... \end{aligned}$$

So with these input parameters, the model predicts a minimum safe distance of 4.2m (to 1 d.p.) from the origin. Based on this result (and as we expect minimal velocity perpendicular to the direction of the swing as per assumption 6), we would recommend the Council implements a minimum safety area around the swing that is 8.4m long (centered at the origin and in the natural direction of the swing) and twice the width of the swing frame. For example, a swing set 2m wide would give a final area of $(4.161...\text{m} \times 2) \times (2\text{m} \times 2) = 33.3\text{m}^2$ (to 1 d.p.).

Given that these input parameters will require adaptation based on the swing set chosen, Table 3 gives predictions (to 1 d.p.) for swings with different combinations of chain length and seat height at rest (for a release angle of $\alpha = \frac{\pi}{2}$ and swing set width of 4m).

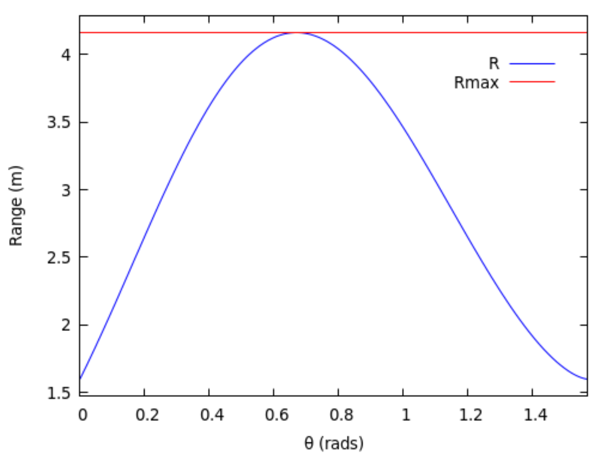

Figure 5: Graph illustrating the solution to the maximisation problem. The blue line shows the range R of the projectile given the input parameters, for values of θ in $[0, \frac{\pi}{2}]$. The red line indicates the maximum range of ≈ 4.2m at $\theta \approx 0.67$rads.

Chain length (m)	Seat height (m)	Maximum range (m)	Recommended safe area (m^2)
1.2	0.4	3.2	25.6
1.2	0.6	3.3	26.8
1.6	0.4	4.2	33.3
1.6	0.6	4.3	34.5
2.0	0.4	5.1	41.0
2.0	0.6	5.3	42.2

Table 3: Predicted maximum range and recommended safe area (both to 1 d.p.) for a 2m-wide swing set with a release angle of $\alpha = \frac{\pi}{2}$.

6 Evaluate the model

6.1 Collect data to compare with the model

Table 4 shows the recommended length and width of the safe landing area around swings, from a UK standard, US standard, and three products. The UK standard[8] states the length of the landing zone should cover 1.5m from the point of falling. The maximum horizontal distance the swing can be from the origin is when $\alpha = \frac{\pi}{2}$, giving a safe landing zone of $1.5 + l\sin(\pi/2) = l + 1.5$. Using our value of $l = 1.6$, we get a safe landing length of 3.1m either side of the origin for a total length of 6.2m.

The US standard[9] states the length of the landing zone should cover a distance of twice the distance from the ground to the pivot point of the chains, in either direction. Assuming a swing set of height 2m, we get a safe landing length of 4m either side of the origin for a total length of 8m.

Source	Recommended length	Recommended Width
BS EN 1176-1:2008 Section 4.2.8.2.4[8]	1.5m from point of fall	1.5m either side of seat
Public Playground Safety Handbook[9]	$4 \times$ height of pivot	6ft from perimeter
SBN083 datasheet[3]	7.0m	5.3m (for 2 seats)
Jungle Gym (211158) datasheet[2]	7.0m	6.1m (for 2 seats)
Argos (389-2040) web page[5]	5.6m	5.5m (for 1 seat)

Table 4: Data sources for the length and width of safe landing areas around swing sets.

The predicted length of the safety area from our initial model is 8.4m. This estimate is greater than all the sources of data collected in Table 4. While it is better to overestimate the dimensions of the safety zone than underestimate it, this discrepancy may suggest the predicted zone is larger than it needs to be. As the model predicts the *length* of the safety zone only, the suggested width was arbitrarily chosen to be a multiple of the width of the swing frame. The sources in Table 4 instead suggest the width of the swing frame plus a constant.

To test the model against real-world data, an experiment could be set up where a mechanism is attached to a swing that can be remotely triggered to release a solid ball. The swing would be released at $\alpha = \frac{\pi}{2}$ from stationary, the ball release mechanism triggered, and the angle of release could be measured using a video camera centered at the origin and facing perpendicular to the motion of the swing. If the test is repeated many times for a variety of release angles and the range of the ball recorded for each test, these values can be compared against the predictions of the model. The test could also be repeated for different release angles, chain lengths, and seat heights above the ground at rest.

6.2 Criticise the model

The model assumes that the swing is released from rest, but this is not how swings are principally used. Usually, the child will propel themselves by kicking off the ground or be pushed by someone, such that their initial speed is > 0. Releasing the swing with non-zero speed would increase the maximum range of any projectile leaving the swing.

The assumption that air resistance and friction are negligible are demonstrably not true. However accounting for a worst-case scenario where the projectile leaves the swing within half a period of motion from being released/pushed, it seems reasonable that neither air resistance nor friction will have a large impact on the speed of the swing and subsequent range. Nevertheless, a single damping term to account for both of these forces could be included in a revised model to test this hypothesis.

Finally, the chosen release angle of $\alpha = \frac{\pi}{2}$, while accommodating a worst-case scenario, is quite improbable. The model could adopt a more likely release angle, such as $\alpha = \frac{\pi}{4}$. This choice of excessive release angle could explain why the model's predicted range is greater than the published requirements.

7 Revise the first model

7.1 Assumptions, variables and parameters

Based on the arguments of the previous section, a revised form of the model is justified. In particular, assumptions 1 and 2 are altered to:

1. The swing is initially pushed and released with non-zero velocity

2a. The swing experiences air resistance and friction opposing its direction of motion

2b. The effect of air resistance on the *projectile* is negligible and can be ignored

and we introduce the additional quantities shown in Table 5.

Symbol	Quantity	Unit
v_0	Initial magnitude of the velocity of the swing/child	ms^{-1}
β	Angle the swing changes direction at	rad
d	Damping constant	dimensionless

Table 5: Additional quantities inroduced in the revised model, with their symbols and units.

7.2 Derive the revised model

We start by accounting for the possibility of the swing having positive velocity when it is released. Suppose that the swing is pulled back, pushed and released when it forms angle α with the vertical, with such initial velocity (tangential to the path of the swing at α) that it reaches angle β with the vertical where it is momentarily stationary (see Figure 6).

When the swing makes the angle β with the vertical, it has no kinetic energy and so

$$E = mg(l - l\cos\beta + h_0)$$

When the swing is released, it's total energy is given by

$$E = \tfrac{1}{2}mv_0^2 + mg(l - l\cos\alpha + h_0)$$

where v_0 is its initial speed. By the law of conservation of mechanical energy, the energy of the swing at these two points is equal and so

$$\begin{aligned}
\tfrac{1}{2}mv_0^2 + mg(l - l\cos\alpha + h_0) &= mg(l - l\cos\beta + h_0) \\
\tfrac{1}{2}v_0^2 + g(l - l\cos\alpha + h_0) &= g(l - l\cos\beta + h_0) \\
\tfrac{1}{2}v_0^2 &= gl\cos\alpha - gl\cos\beta \\
v_0 &= \sqrt{2gl(\cos\alpha - \cos\beta)} \qquad (\beta > \alpha)
\end{aligned}$$

Thus, we have an expression for the initial velocity of the swing required for it to be released at α and change direction at β. At any point during the motion of the swing we have

$$E = \tfrac{1}{2}mv^2 + mg(l - l\cos\theta + h_0)$$

Again, applying the law of conservation of energy tells us that this is equal to the total energy when the swing is released and so

$$\begin{aligned}
\tfrac{1}{2}mv_0^2 + mg(l - l\cos\alpha + h_0) &= \tfrac{1}{2}mv^2 + mg(l - l\cos\theta + h_0) \\
\tfrac{1}{2}v_0^2 + g(l - l\cos\alpha + h_0) &= \tfrac{1}{2}v^2 + g(l - l\cos\theta + h_0) \\
\tfrac{1}{2}v^2 &= \tfrac{1}{2}v_0^2 + gl\cos\theta - gl\cos\alpha \\
v &= \sqrt{v_0^2 + 2gl(\cos\theta - \cos\alpha)}
\end{aligned}$$

which gives us an expression for the speed of the swing at any angle θ, given the swing's release angle and initial speed. Figure 7 shows the speed of the swing as a function of θ for four different possible release speeds (using a release angle of $\alpha = \frac{\pi}{4}$ and fixed chain length of $l = 1.6$). When $v_0 = 0$, the expression is equivalent to the original model that assumed the swing was released at rest. Higher initial speeds correspond to a higher speed through the period of the motion.

Thus far we have relied on the law of conservation of mechanical energy to model the speed of the swing through its motion. This law only applies to situations where the force on a particle depends only on its position and breaks down in the presence of non-conservative or dissipative forces like friction and air resistance. To allow for the combined effect of air resistance and friction on the motion of the swing, the expression for the speed of the swing can be modified to

$$v = e^{-d(\alpha-\theta)}\sqrt{v_0^2 + 2gl(\cos\theta - \cos\alpha)}$$

where $\alpha - \theta$ is the phase angle between the instantaneous angle of the swing with the vertical and its starting angle (increasing from 0 at the point of release to a maximum of β), and d is a decay factor[10]. As shown in Figure 8, when $d = 0$ there is no damping effect and the predicted speed remains the same as the original model. Larger values of d model a damping of the speed of the swing, an effect that increases during the period of the swing. While the addition of this damping effect allows the combined effects of air resistance and friction to be accounted for, d is another parameter that needs to be chosen (possibly through experimentation) and makes the model more complex.

All the criticisms of the original model were of the equation for the speed of the swing, criticisms that we have tried to address with the revised equation above. The rest of the model remains unchanged, and so we cam summarise the revised model as

1. Calculate the initial speed of the swing given its release angle α and the angle at which it changes direction β

$$v_0 = \sqrt{2gl(\cos\alpha - \cos\beta)} \qquad (\beta > \alpha)$$

2. Calculate the speed of the swing given its release angle α, current angle θ, initial speed v_0, damping constant d, and chain length l:

$$v = e^{-d(\alpha-\theta)}\sqrt{v_0^2 + 2gl(\cos\theta - \cos\alpha)}$$

3. Calculate the time of flight of a projectile given its initial speed v, release angle θ, chain length l, and swing height above the ground at rest h_0:

$$t_{\text{flight}} = \frac{1}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2g(h_0 + l - l\cos\theta)}\right)$$

4. Maximise the horizontal range of a projectile wrt to its release angle θ, given the time of flight t_{flight} and chain length l:

$$R = l\sin\theta + vt_{\text{flight}}\cos\theta$$

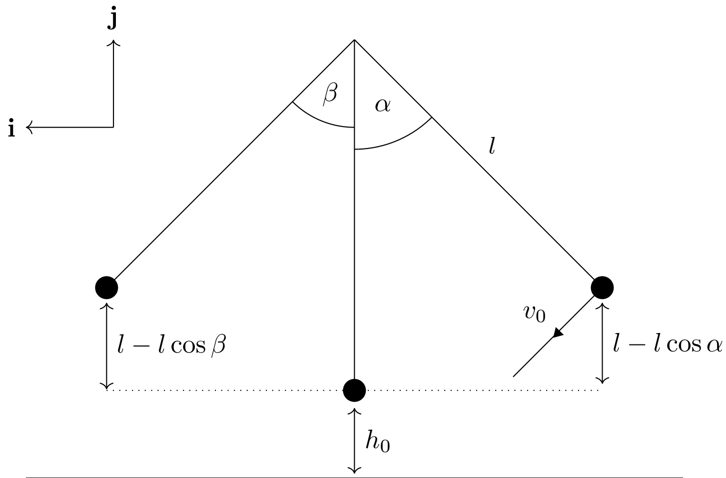

Figure 6: Diagram of the swing in the vertical plane. The filled circle represents the swing/child at the end of the chains. The swing is pulled back, pushed forward until it forms the angle α and is released with speed v_0. The swing reaches the angle β, at which point it momentarily comes to rest and then changes direction.

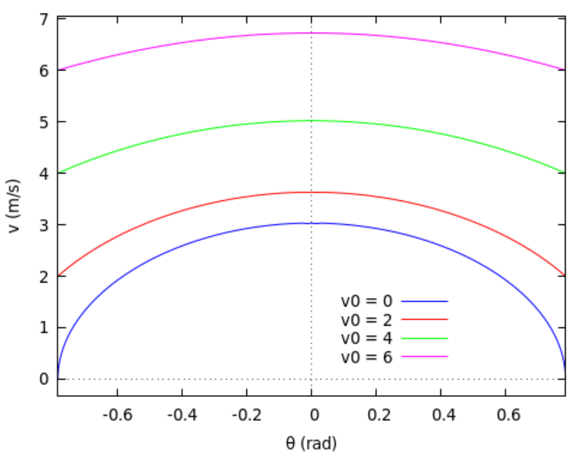

Figure 7: The updated model's predicted relationship between the speed of the swing v and its angle with the vertical θ, for four values of initial speed v_0, using a constant chain length of $l = 1.6$m and angle of release $\alpha = \pi/4$.

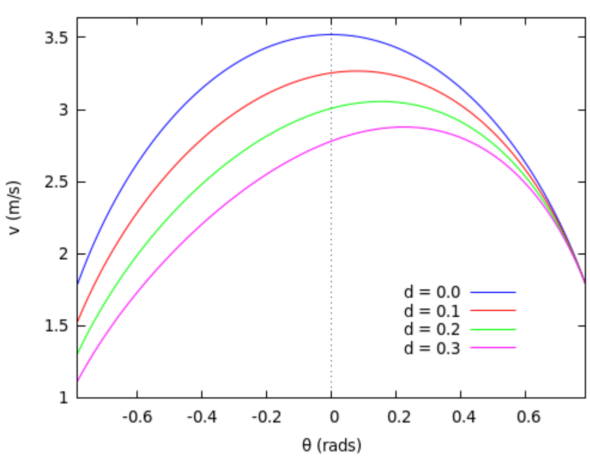

Figure 8: The model's predicted relationship between the speed of the swing v and its angle with the vertical θ after incorporating the effect of damping, using a constant chain length of $l = 1.6$m, release angle of $\alpha = \pi/4$ and initial speed $v_0 = 3.2\text{ms}^{-1}$. Each line represents a different value of the damping constant d.

7.3 Describe the mathematical solution of the revised model

The revised model has the same input parameters as the first, plus the angle the swing reaches after its first push β, and a damping constant d, both of which must be chosen. This time, we choose the more realistic release angle of $\alpha = \frac{\pi}{4}$. The British standard states the "maximum swing angle ... considered for swing seats suspended from ropes or chains is 80° from the vertical position". Therefore, we model the worst-case scenario and choose $\beta = \frac{4\pi}{9}$ (80°). The damping constant can only be estimated by experiment, but for the purpose of illustrating the revised model, we choose a value of $d = 0.1$.

Substituting $\alpha = \frac{\pi}{4}$, $\beta = \frac{4\pi}{9}$ and $l = 1.6$ (as before) into the equation for the initial speed gives

$$\begin{aligned} v_0 &= \sqrt{3.2g\big(\cos(\tfrac{\pi}{4}) - \cos(\tfrac{4\pi}{9}\big)} \\ &= 4.092... \end{aligned}$$

We substitute this initial speed into the model and maximise the range R wrt θ using Maxima, as before. This gives the angle 0.682... rads (39.116...°) as the angle of release of the projectile that results in the maximum horizontal range (see Figure 9). The time of flight of a projectile released at this angle is $t_{\text{flight}} = 0.757...$s, giving a range of 3.5m (to 1 d.p.) from the origin (lower than the original model's prediction).

Based on these input parameters to the revised model, we would recommend the Council implements a minimum safety area around the swing that is 7m long (centered at the origin and in the natural direction of the swing) and twice the width of the swing frame. Taking the example from earlier, a swing set 2m wide would give a final area of 7m $\times$ (2m $\times 2$) = 28m^2.

Table 6 gives predictions (to 1 d.p.) for swings with different combinations of chain length, seat height at rest, and damping constant (for $\alpha = \frac{\pi}{4}$ and $\beta = \frac{4\pi}{9}$).

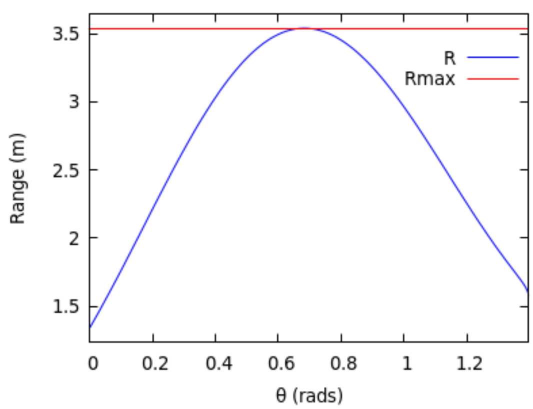

Figure 9: Graph illustrating the solution to the revised maximisation problem. The blue line shows the range R of the projectile given the input parameters, for values of θ in $[0, \frac{4\pi}{9}]$. The red line indicates the maximum range of ≈ 3.5m at ≈ 0.68rads.

Chain length (m)	Seat height (m)	Damping constant	Maximum range (m)	Recommended safe area (m^2)
1.2	0.4	0.1	2.7	21.6
1.2	0.4	0.2	2.7	21.6
1.2	0.6	0.1	2.9	23.2
1.2	0.6	0.2	2.8	22.4
1.6	0.4	0.1	3.5	28.0
1.6	0.4	0.2	3.5	28.0
1.6	0.6	0.1	3.7	29.6
1.6	0.6	0.2	3.6	28.8
2.0	0.4	0.1	4.4	35.2
2.0	0.4	0.2	4.3	34.4
2.0	0.6	0.1	4.5	36.0
2.0	0.6	0.2	4.4	35.2

Table 6: Predicted maximum range and recommended safe area (both to 1 d.p.) for a 2m-wide swing set with $\alpha = \frac{\pi}{4}$ and $\beta = \frac{4\pi}{9}$.

8 Conclusions

The goal of this modelling process was to find the maximum range a child or object might travel when falling off a swing in motion, with reasonable input parameters. The initial model used the law of conservation of mechanical energy to express the speed of the swing as a function of its instantaneous angle with the vertical. By integrating the acceleration of the projectile wrt time, we could determine the **j**-component of its position vector, and solve for its time of flight. Finding the **i**-component of the projectile's position vector gave its range, and we maximised this expression wrt θ.

With reasonable input parameters, the original model appeared overly cautious when compared to other data sources for safety zoning. The original model also assumed the swing was released from rest, and that no dissipative force acted against the motion of the swing. The revised model included an initial speed (calculated based on the amount of energy needed to start at angle α and change direction at angle β), and a damping term to model the combined effect of air resistance and friction. These revisions only affected the calculation for the velocity of the swing, and so the rest of the optimization problem remained the same. With reasonable input parameters, the revised model predicts slightly shorter safety zones compared with the original model, more in line with published guidelines. However, the choice of the damping constant is difficult to make without performing an experiment.

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