Q 1.

(a) On the first line, type the heading 'MST125 TMA 01 Question 1', making it centred and bold. On the second line, type and centre your name and personal identifier. On the third line, type and centre the name of the typesetting software that you are using.

MST125 TMA 01 Question 1

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I am typesetting this document using LaTeX

(b) Typeset this draft solution, using the same content and a similar layout. Take care to centre or indent each equation that is displayed on a line by itself, and to align equals signs where appropriate.

1. The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Hence the distance between $A(-3, 1)$ and $B(2, -2)$ is

$$\begin{aligned} AB &= \sqrt{(2-(-3))^2 + (-2-1)^2} \\ &= \sqrt{5^2 + (-3)^2} \\ &= \sqrt{34}\ . \end{aligned}$$

2. The gradient m of the line through (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}\ .$$

Hence the gradient of the line through $(-3, 1)$ and $(2, -2)$ is

$$m = \frac{-2-1}{2-(-3)} = -\frac{3}{5}\ .$$

3. The gradient of the line is $-\frac{3}{5}$. Hence $\tan\alpha = -\frac{3}{5}$.

Let ϕ be the acute angle that the line makes with the negative direction of the x-axis. Then

$$\tan\phi = \frac{3}{5}\ ,$$

so

$$\phi = \tan^{-1}\left(\frac{3}{5}\right) = 0.540\ldots$$

Hence

$$\alpha = \pi - 0.540\ldots = 2.601\ldots\ .$$

Therefore the angle α is 2.60 radians (to 2 d.p.).

(c) Do you intend to typeset your TMA solutions? Justify your decision briefly.

Yes I intend to typeset my TMA solutions because I find it clearer to see my solutuions laid out neatly in a typed document. I will do my rough workings on paper and then type them up as part of double-checking my answers.

Q 2.

(a) Find the least residue of 64^7 modulo 15.

The least residue of a modulo n is the remainder that you obtain when you divide a by n. Using the power rule for congruences, since

$$64 \equiv 4 \pmod{15}$$

then

$$64^7 \equiv 4^7 \pmod{15} \ .$$

Using the multiplication rule for congruences, since

$$4^2 \equiv 1 \pmod{15}$$

then

$$(4^2)^3 \equiv 1^3 \equiv 1 \pmod{15}$$

and

$$4^7 \equiv 1 \times 4 \equiv 4 \pmod{15} \ .$$

Therefore the least residue of 64^7 modulo 15 is 4.

(b) Use Fermat's little theorem to find the least residue of 30^{51} modulo 13.

Fermat's little theorem states that if p is a prime number, and a is an integer that is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$. As 13 is a prime number, Fermat's little theorem tells us that

$$30^{12} \equiv 1 \pmod{13} \ .$$

Using the power rule for congruences we can say that

$$(30^{12})^4 \equiv 1^4 \pmod{13}$$

and using the multiplication rule

$$30^{48} \times 30^3 \equiv 1 \times 30^3 \equiv 30^3 \pmod{13} \ .$$

Recognising that $30^3 = 27 \times 1000$ and that $27 \equiv 1 \pmod{13}$ we can say

$$10 \times 100 \equiv 10 \times 9 \equiv 90 \equiv 12 \pmod{13} \ .$$

Therefore the least residue of 30^{51} modulo 13 is 12.

Q 3.

(a)

(i) Use Euclid's algorithm to find a multiplicative inverse of 17 modulo 90, and hence solve the linear congruence

$$17x \equiv 9 \pmod{90} .$$

A multiplicative inverse of a modulo n is an integer v such that $av \equiv 1 \pmod{n}$. Using Euclid's algorithm

$$\begin{aligned} 90 &= 5 \times 17 + 5 \\ 17 &= 3 \times 5 + 2 \\ 5 &= 2 \times 2 + 1 . \end{aligned}$$

Since their highest common factor is 1, 90 and 17 are coprime and a multiplicative inverse exists. Rearranging to make the remainders the subjects gives

$$\begin{aligned} 5 &= 90 - 5 \times 17 \\ 2 &= 17 - 3 \times 5 \\ 1 &= 5 - 2 \times 2 \end{aligned}$$

Applying backwards substitution gives

$$\begin{aligned} 1 &= 5 - 2 \times 2 \\ &= 5 - 2(17 - 3 \times 5) \\ &= 7 \times 5 - 2 \times 17 \\ &= 7(90 - 5 \times 17) - 2 \times 17 \\ &= 7 \times 90 - 37 \times 17 \end{aligned}$$

Since

$$-35 \times 17 = 1 + 7 \times 90$$

we obtain

$$-35 \times 17 \equiv 1 \pmod{90} .$$

As -37 is a multiplicative inverse of 17 modulo 90, the solutions to the linear congruence $17x \equiv 9 \pmod{90}$ are given by

$$x \equiv -37 \times 9 \equiv -333 \equiv 27 \pmod{90}$$

(ii) Explain why the following linear congruence has no solutions:

$$9x \equiv 12 \pmod{90}\ .$$

9 and 90 are not coprime but have a highest common factor of 9. As 12 is not divisible by 9, the linear congruence has no solutions.

(iii) Solve the linear congruence

$$27x \equiv 72 \pmod{90}\ .$$

27 and 90 are not coprime but have a highest common factor of 9. As 72 is divisible by 9 the linear congruence has a solution. The linear congruence can be simplified by dividing all terms by the highest common factor of 27 and 90

$$\frac{27}{9} \equiv \frac{72}{9} \left(\text{mod}\frac{90}{9}\right)$$

$$3 \equiv 8 \pmod{10}$$

As $-3 \times 3 = 1 - 1 \times 10$, -3 is a multiplicative inverse of $3 \equiv 8 \pmod{10}$. The solution to the linear congruence $3 \equiv 8 \pmod{10}$ is given by

$$x \equiv 8 \times (-3) \equiv 6 \pmod{10}\ .$$

Therefore, the solution to the equivalent linear congruence $27x \equiv 72 \pmod{90}$ is $x \equiv 6 \pmod{90}$.

(b)

(i) Show that 15 is a multiplicative inverse of 7 modulo 26.

A multiplicative inverse of 7 modulo 26 is an integer v such that $7v \equiv 1 \pmod{26}$. Since

$$7 \times 15 = 105$$

and

$$105 = 4 \times 26 + 1\ ,$$

$$7 \times 15 \equiv 1 \pmod{26}\ .$$

Therefore, 15 is a multiplicative inverse of 7 modulo 26.

(ii) Decipher the message and deduce the word that was sent.

The deciphering rule for an affine cipher with enciphering rule

$$D(x) \equiv ax + b \pmod{n}$$

is given by

$$D(y) \equiv v(y - b) \pmod{n}$$

where v is the multiplicative inverse of a modulo n. As the multiplicative inverse of 7 modulo 26 is 15, the deciphering rule for this affine cipher is

$$D(y) \equiv 15(y - 12) \pmod{26} .$$

Substituting the integers of the enciphered message gives

$$\begin{aligned} D(8) &\equiv 15(8 - 12) \equiv 18 \pmod{26} \\ D(16) &\equiv 15(16 - 12) \equiv 8 \pmod{26} \\ D(25) &\equiv 15(25 - 12) \equiv 13 \pmod{26} \\ D(2) &\equiv 15(2 - 12) \equiv 6 \pmod{26} . \end{aligned}$$

Looking up these values in the conversion table gives the message SING.

Q 4.

(a)

(i) Find the vertices and asymptotes of the hyperbola.

The equation for the hyperbola can be rearranged to give

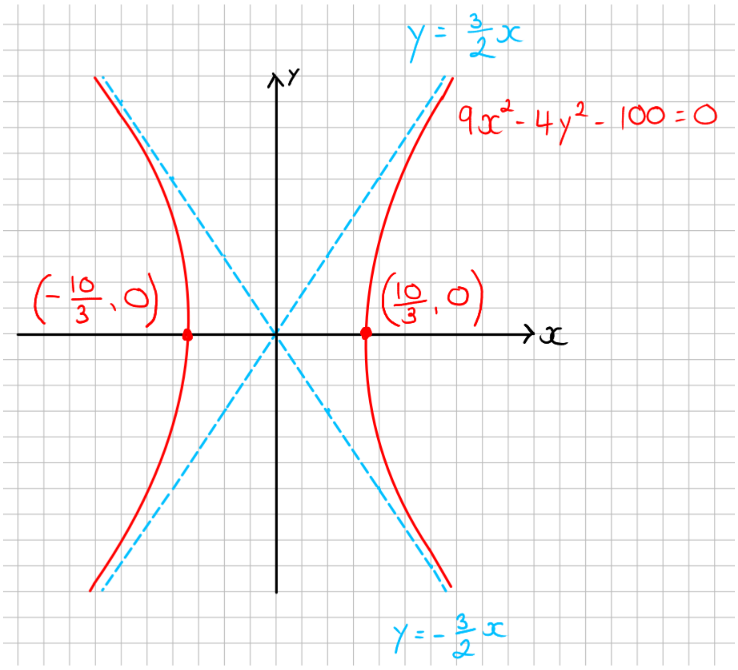
$$\begin{aligned} 9x^2 - 4y^2 &= 100 \\ \frac{9x^2}{100} - \frac{y^2}{25} &= 1 . \end{aligned}$$

Therefore $a^2 = \frac{100}{9}$ and $b^2 = 25$. The vertices of the hyperbola are at $\big(\pm a, 0\big)$ and so are positioned at $\big(\pm \frac{10}{3}, 0\big)$. The asymptotes of a hyperbola are given by

$$\begin{aligned} y &= \pm\frac{b}{a}x \\ &= \pm 5 \times \frac{3}{10}x \\ &= \pm\frac{3}{2}x . \end{aligned}$$

Therefore, the vertices are found at $\big(\pm \frac{10}{3}, 0\big)$ and the asymptotes are given by $y = \pm\frac{3}{2}x$.

(ii) Sketch the hyperbola, labelling its vertices with their coordinates and its asymptotes with their equations.



(iii) Find the eccentricity, foci and directrices of the hyperbola.

The eccentricity of the hyperbola is given by

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + 25 \times \frac{9}{100}} \\ &= \frac{\sqrt{13}}{2} \,. \end{aligned}$$

The foci of the hyperbola are at $(\pm ae, 0)$ giving $\left(\pm \frac{5\sqrt{13}}{3}, 0\right)$.

The directrices have equations $x = \pm\frac{a}{e}$ giving

$$\begin{aligned} x &= \pm\frac{10}{3} \times \frac{2}{\sqrt{13}} \\ &= \pm\frac{20}{3\sqrt{13}} \,. \end{aligned}$$

In summary, the hyperbola has eccentricity $e = \frac{\sqrt{13}}{2}$, foci at $\left(\pm \frac{5\sqrt{13}}{3}, 0\right)$, and directrices with equations $x = \pm\frac{20}{3\sqrt{13}}$.

(iv) Find a parametrisation of the part of this hyperbola that lies in the second quadrant (not including any points that lie on the x-axis or y-axis). Make sure to find and include the restrictions that are needed on the values of the parameters.

A hyperbola in standard position has the parametrisation

$$x = a\sec t, \quad y = b\tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}, \frac{\pi}{2} < t < \frac{3\pi}{2}\right)$$

but the domain of t corresponding to the part of the hyperbola that lies in the second quadrant is $\left(\pi < t < \frac{3\pi}{2}\right)$. The parametrisation is therefore

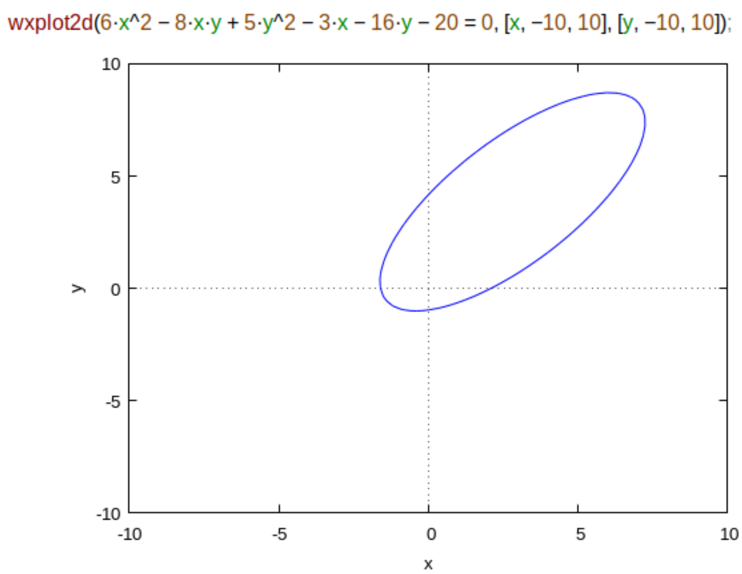
$$x = \frac{10}{3}\sec t, \quad y = 5\tan t \quad \left(\pi < t < \frac{3\pi}{2}\right).$$

(b)

(i) Determine the type of conic.

This equation represents an ellipse.

(ii) Plot the conic using Maxima, and provide a screenshot of your plot, including the command that you used to produce the plot.



Q 5.

(a) Find the equation of the line that A moves along.

The x-coordinate of object A at time t is given as $4t+2$. Expressing t in terms of x gives

$$\begin{aligned} x &= 4t+2 \\ t &= \frac{x-2}{4} \ . \end{aligned}$$

Substituting this expression for t into the expression for the y-coordinate gives the equation of the line that A moves along, which is $y = \frac{x-2}{4} - 1$.

(b) Let d be the distance between A and B at time t. Show that an expression for d^2 in terms of t is given by

$$d^2 = 8t^2 - 12t + 9 \ .$$

The distance formula states that the distance between the two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1) + (y_2 - y_1)} \ .$$

Substituting the expressions for the x and y-coordinates of the objects at time t gives

$$\begin{aligned} d &= \sqrt{(2t + 5 - (4t+2))^2 + (3t - 1 - (t-1))^2} \\ d^2 &= (2t + 5 - (4t+2))^2 + (3t - 1 - (t-1))^2 \\ &= 4t^2 - 12t + 9 + 4t^2 \\ &= 8t^2 - 12t + 9 \ . \end{aligned}$$

Therefore $d^2 = 8t^2 - 12t + 9$ is an expression for d^2 in terms of t.

(c) Find and describe the two mistakes. Write out a correct solution, stating when the minimum value of d^2 occurs and the shortest distance between A and B.

The first mistake is between the second and third lines where the term $-\frac{9}{16}$ is not multiplied by 8. Instead the completed square expression for d^2 is

$$\begin{aligned} d^2 &= 8t^2 - 12t + 9 \\ &= 8\left(t^2 - \frac{3}{2}t\right) + 9 \\ &= 8\left(\left(t - \frac{3}{4}\right)^2 - \frac{9}{16}\right) + 9 \\ &= 8\left(t - \frac{3}{4}\right)^2 - \frac{9}{2} + 9 \\ &= 8\left(t - \frac{3}{4}\right)^2 + \frac{9}{2} \ . \end{aligned}$$

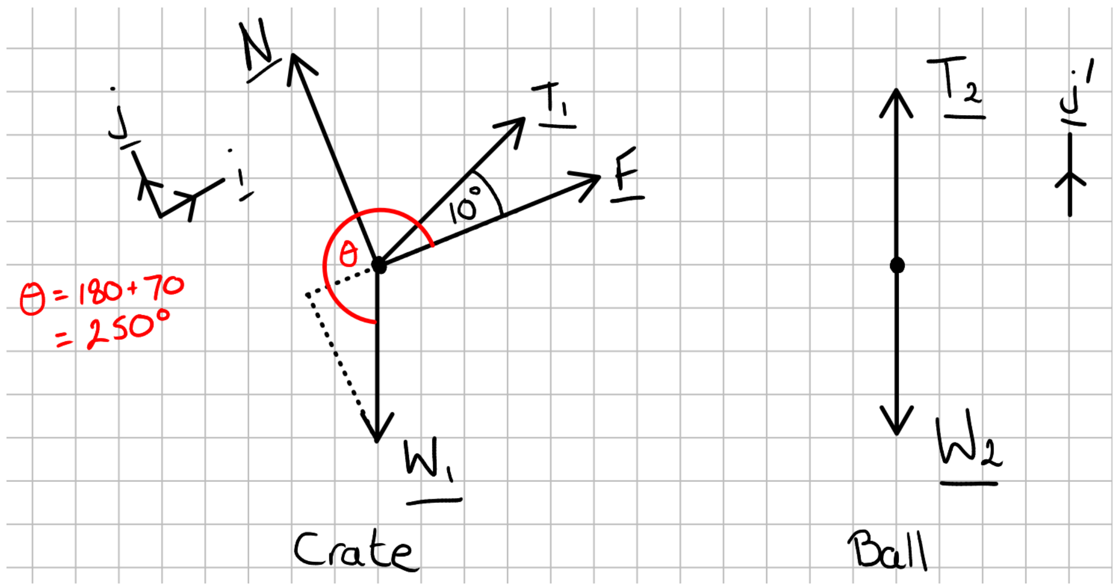
The second mistake is that the value given for the minumum distance is in units of m^2, not m. The value of d^2 is minimised when $t = \frac{3}{4}$, at which $d^2 = \frac{9}{2}$ and $d = \frac{3\sqrt{2}}{2}$. Therefore the minimum distance between objects A and B is 2.1m (to 2 s.f.), which occurs at $t = \frac{3}{4}$ minutes.

Q 6.

(a) State the four forces that act on the crate and draw a force diagram showing them, including angles that show their directions.

The four forces that act on the crate are its weight $\mathbf{W_1}$, the normal reaction from the ramp $\mathbf{N}$, the tension in the rope $\mathbf{T_1}$, and the friction force $\mathbf{F}$. These forces and the weight of the ball $\mathbf{W_2}$ and the tension of the rope suspending the ball $\mathbf{T_2}$ are shown in the figure below.

$\mathbf{F}$ and $\mathbf{N}$ point in the positive $\mathbf{i}$ and $\mathbf{j}$ directions, respectively. $\mathbf{T_1}$ makes a 10° angle with the positive $\mathbf{i}$ direction and $\mathbf{W_1}$ makes an acute angle of $90 - 20 = 70°$ with the $\mathbf{i}$ direction.



(b) Express the forces in component form, in terms of unknown magnitudes where appropriate. Hence find the magnitude of the tension in the rope in newtons to two significant figures.

Let $N = |\mathbf{N}|$, $T = |\mathbf{T_1}|$, and g be the acceleration due to gravity, then

$$\begin{aligned}
\mathbf{N} &= N\mathbf{j} \\
\mathbf{F} &= 0.21N\mathbf{i} \\
\mathbf{W_1} &= -17g\cos 250\mathbf{i} - 17g\sin 250\mathbf{j} \\
\mathbf{T} &= T\cos 10\mathbf{i} + T\sin 10\mathbf{j}
\end{aligned}$$

As the crate is at rest, $\mathbf{N} + \mathbf{F} + \mathbf{W_1} + \mathbf{T_1} = 0$. Resolving this equation in the $\mathbf{i}$ and $\mathbf{j}$ directions gives

$$0.21N - 17g\cos 250 + T\cos 10 = 0 \tag{6.1}$$
$$N - 17g\sin 250 + T\sin 10 = 0 \tag{6.2}$$

To solve these simultaneous equations I start by multiplying equation 6.2 by 0.21 and subtracting equation 6.1, taking g to be equal to 9.8:

$$\begin{aligned}
0.21 \times 17g\sin 250 - 17g\cos 250 + 0.21\sin 10 - T\cos 10 &= 0 \\
24.10\ldots - 0.94\ldots T &= 0 \\
-0.94\ldots T &= -24.10\ldots \\
T &= \frac{-24.10\ldots}{-0.94\ldots} \\
T &= 25.41\ldots
\end{aligned}$$

Substituting the exact value for T into equation 6.2 gives

$$\begin{aligned}
N &= -17g\sin 250 - 25.41\sin 10\ldots \\
&= 152.13\ldots
\end{aligned}$$

Solving these simultaneous equations gives $N = 152.13\ldots$ and $T = 25.41\ldots$. Therefore the tension in the rope is 25N (to 2 s.f.).

(c) By considering the forces acting on the ball, find the mass of the ball in kilograms to two significant figures.

Assuming the ball is suspended by a model pulley, then the tension of the rope it's attached to $\mathbf{T_2}$ has the same magnitude as $\mathbf{T_1}$. Let $\mathbf{j}'$ point vertically upward, then we have

$$\mathbf{T_2} = T\mathbf{j}'$$
$$\mathbf{W_2} = -mg\mathbf{j}'$$

and

$$\mathbf{T_2} + \mathbf{W_2} = \mathbf{0}$$
$$T\mathbf{j}' - mg\mathbf{j}' = \mathbf{0}$$

where m is the mass of the ball in Kg and g is the force of acceleration due to gravity. Resolving in the $\mathbf{j}'$ direction and substituting $T = 25.41\ldots$ and $g = 9.8$ gives

$$\begin{aligned} T - mg &= 0 \\ m &= \frac{T}{g} \\ &= \frac{25.4175}{9.8} \\ &= 2.59\ldots \end{aligned}$$

Therefore the mass of the ball is 2.6Kg (to 2 s.f.).

Q 7.

(a) Provide a printout or take a screenshot from your computer that shows the 'Summary of attempt' table.

Unit 5 Practice quiz

Summary of attempt

Question	Status	Marks
1	Correct	1.00
2	Incorrect	0.00
3	Correct	1.00
4	Correct	1.00
5	Correct	1.00
6	Correct	0.67
7	Incorrect	0.00
8	Incorrect	0.00

(b) Based on your results from the practice quiz and your other work on Unit 5, state the ideas in this unit that you feel you now understand well and also any topics or questions that you still find difficult.

I feel I understand how to construct a force diagram, the equilibrium condition, and how to resolve a vector equation into its i and j components. I still find it difficult to identify which angles should be used when writing a vector in component form, and I struggle to solve the simultaneous equations that result from resolving vector equations into scalar ones.

(c) What further work (if any) do you plan to do to help your understanding of the ideas in Unit 5?

I plan to contact my tutor to discuss the two specific areas of confusion I have (choosing angles and solving the simultaneous equations). Once I feel I have a better understanding of these I will retake the unit 5 practice quiz and work through the exercise book until the process becomes second nature.

Q 8. Five marks on this assignment are allocated for good mathematical communication in Questions 2 to 6.

NOTE TO TUTOR: I particularly struggled with Question 6, but I understand the process much better after your email, thank you.