MST210 iCME 81 Hefin Rhys

20)

a) The rabbit population changes over time according to

$$\frac{dN}{dt} = (a - bN)N - cN + d$$

b) At steady state, $\frac{dN}{dt} = 0$. Substituting into the differential equation gives

$$\begin{aligned} 0 &= (a - bN)N - cN + d \\ &= aN - bN^2 - cN + d \\ &= -bN^2 + (a-c)N + d \\ &= bN^2 + (c-a)N - d \end{aligned}$$

Using the quadratic formula gives

$$N = \frac{(a-c) \pm \sqrt{(c-a)^2 - 4b(-d)}}{2b}$$

21)

a) In matrix form this system can be represented as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 8 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

b) The eigenvalues are the solutions of

$$\begin{vmatrix} 2-\lambda & -4 \\ 8 & -10-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-10-\lambda) - (-32) = 0$$
$$\lambda^2 + 8\lambda + 12 = 0$$
$$(\lambda + 6)(\lambda + 2) = 0$$

So the matrix has eigenvalues $\lambda_1 = -6$ and $\lambda_2 = -2$

For $\lambda_1 = -6$ we have

$$\begin{pmatrix} 2-(-6) & -4 \\ 8 & -10-(-6) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which gives

$$8x - 4y = 0$$
$$2x = y$$

So $(1 \quad 2)^T$ is an eigenvector for $\lambda_1 = -6$

For $\lambda_2 = -2$ we have

$$\begin{pmatrix} 2-(-2) & -4 \\ 8 & -10-(-2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives

$$4x - 4y = 0$$
$$8x - 8y = 0$$

So $(1 \ 1)^T$ is an eigenvector for $\lambda_2 = -2$

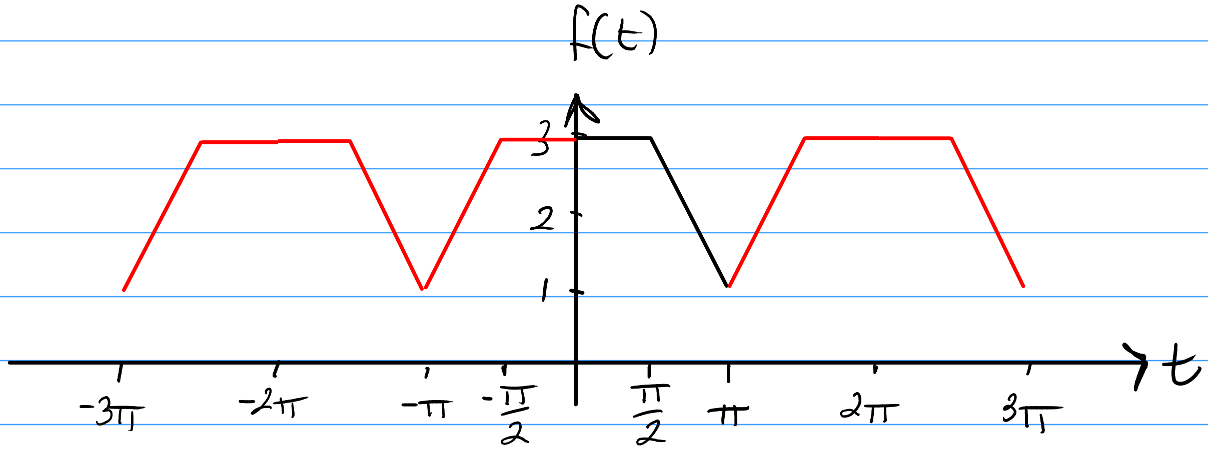
c) The general solution is therefore

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{-2t}$$

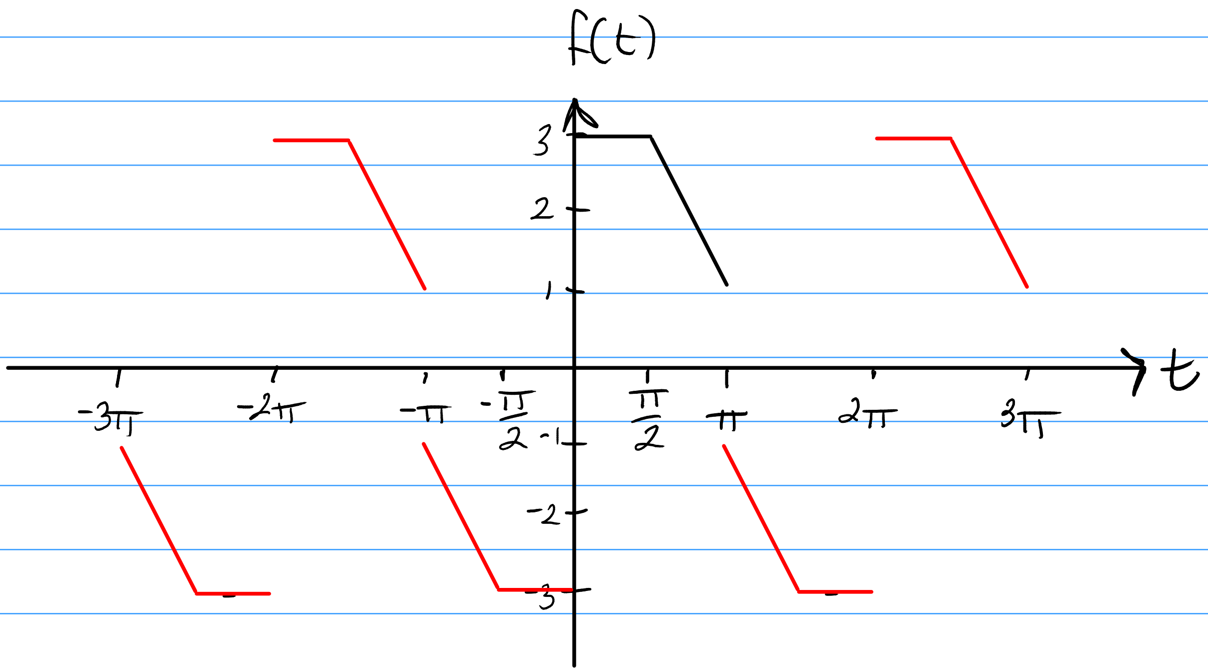
where α and β are arbitrary constants.

22)

a) Even extension:



Odd extension:



b) The constant A_0 is given by

$$A_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

where T is the period 2π. Evaluating the integral over the segments of $f(t)$ separately gives

$$A_0 = \frac{1}{2\pi}\left(\int_0^{\frac{\pi}{2}} 3\,dt + \int_{\frac{\pi}{2}}^{\pi} -4t+5\,dt\right)$$

$$= \frac{1}{2\pi}\left(\left[3t\right]_0^{\frac{\pi}{2}} + \left[-2t^2+5t\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$= \frac{1}{2\pi}\left(\frac{3\pi}{2} + \left((-2\pi^2+10\pi)-\left(-2\left(\frac{\pi}{2}\right)^2+\frac{5}{2}\pi\right)\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{3\pi}{2} - 2\pi^2 + 10\pi + \frac{\pi^2}{2} - \frac{5}{2}\pi\right)$$

$$= \frac{1}{2\pi}\left(9\pi - \frac{3\pi^2}{2}\right)$$

$$= \frac{9}{2} - \frac{3\pi}{4}$$

23)

a) Substituting the trial solution into the first boundary condition gives

$$X(0)T(t) = 0$$

So either $X(0)$ or $T(t)$ must be 0. $T(t) = 0$ gives a trivial solution so we must have $X(0) = 0$ as a boundary condition for $X(x)$.

For the second boundary condition we have

$$X''(1) = kX(1) = 0$$

$k = 0$ gives another trivial solution so $X(1) = 0$ is another boundary condition for $X(x)$.

b) If $k < 0$ then $X(x)$ has a general solution

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

where $\omega = \sqrt{-k}$

Substituting $X(0) = 0$ gives

$$0 = A$$

Substituting $X(1) = 0$ and $A = 0$ gives

$$0 = B\sin(\omega)$$

So either $B = 0$ or $\sin\omega = 0$. As $B = 0$ gives a trivial solution we must have

$\sin \omega = 0$, which implies
$$\omega = \pi n$$

This gives a family of solutions of the $X(x)$ differential equation of

$$X_n(x) = B \sin(\omega_n x) \qquad n = 1, 2, 3, \ldots$$

24) The scaler line integral of $\underline{F}(\underline{r})$ along a path C given by $\underline{r} = \underline{r}(t)$ from 0 to 2π is

$$\int_C \underline{F}(\underline{r}) \cdot \frac{d\underline{r}}{dt}\, dt$$

Let

$$\underline{r}(t) = \cos t\, \underline{i} + \sin t\, \underline{j}$$

$$\frac{d\underline{r}}{dt} = -\sin t\, \underline{i} + \cos t\, \underline{j}$$

and

$$\underline{F}(\underline{r}) = (\cos t - \sin t)\underline{i} + (3\cos t + \sin t)\underline{j}$$

then

$$\underline{F}(\underline{r}) \cdot \frac{d\underline{r}}{dt} = -\sin t(\cos t - \sin t) + \cos t(3\cos t + \sin t)$$

$$= -\sin t \cos t + \sin^2 t + 3\cos^2 t + \sin t \cos t$$

$$= \sin^2 t + 3\cos^2 t$$

Integrating wrt t gives

$$\int_C \sin^2 t + 3\cos^2 t \quad dt$$

$$= \int_C \frac{1}{2} - \frac{1}{2}\cos 2t + \frac{3}{2} + \frac{3}{2}\cos 2t \, dt$$

$$= \int_C \frac{1}{2} + 2\cos 2t \, dt$$

$$= \left[\frac{\sin(2t) + 4t}{2}\right]_0^{2\pi}$$

$$= 4\pi$$

So the line integral is 4π.

25) We calculate the volume integral in cylindrical coordinates, noting that

$$z = h + x = h + \rho\cos\phi$$

The limits of integration are

$z = 0$ and $z = h + \rho\cos\phi$

$\phi = -\pi$ and $\phi = \pi$

$\rho = 0$ and $\rho = a$

The volume integral in cylindrical space is therefore

$$\int_{\rho=0}^{a}\left(\int_{\phi=-\pi}^{\pi}\left(\int_{z=0}^{h+\rho\cos\phi} \frac{8z\rho}{h}\,dz\right)d\phi\right)d\rho$$

$$= \int_{\rho=0}^{a}\left(\int_{\phi=-\pi}^{\pi}\left[\frac{4z^2\rho}{h}\right]_0^{h+\rho\cos\phi} d\phi\right)d\rho$$

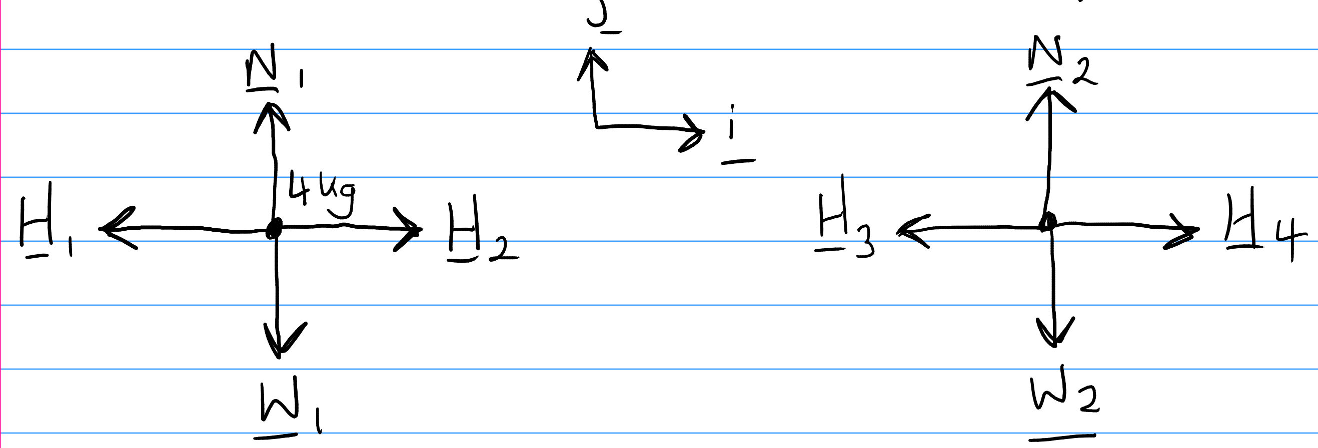
$$= \int_{\rho=0}^{a}\left(\int_{\phi=-\pi}^{\pi} 4\rho h + \frac{4\rho^3\cos^2\phi}{h}\,d\phi\right)d\rho$$

$$= \int_{\rho=0}^{a} \frac{4\rho(\pi\rho^2 + 2\pi h^2)}{h}\,d\rho$$

$$= \frac{4\pi a^2 h^2 + \pi a^4}{h}$$

26)

a) <u>Left-hand particle</u> : <u>Right-hand particle</u>



where $\underline{W}_1$ and $\underline{W}_2$ are the weights, $\underline{N}_1$ and $\underline{N}_2$ are normal reactions, and $\underline{H}_n$ is the nth spring force.

b) The spring forces are found in the table below

Spring	L	L_0	$L-L_0$	k	$\hat{\underline{s}}$	$\underline{H}$
$\underline{H}_1$	L_0+x	L_0	x	20	$-\underline{i}$	$-20x\underline{i}$
$\underline{H}_2$	L_0-x+y	L_0	$-x+y$	4	$\underline{i}$	$4(-x+y)\underline{i}$
$\underline{H}_3$	L_0-x+y	L_0	$-x+y$	4	$-\underline{i}$	$-4(-x+y)\underline{i}$
$\underline{H}_4$	L_0-y	L_0	$-y$	2	$\underline{i}$	$2(-y)\underline{i}$

Using $\underline{F} = m\ddot{\underline{r}} = \underline{W} + \underline{N} + \sum_{n=1}^{n} \underline{H}_n$, the equations of motion for each particle are

$$4\ddot{\underline{x}} = (|\underline{N}_1| - 4g)\underline{j} + (4(-x+y) - 20x)\underline{i}$$

$$\ddot{\underline{y}} = (|\underline{N}_2| - g)\underline{j} + (2(-y) - 4(-x+y))\underline{i}$$

Resolving in the $\underline{i}$ direction and rearranging gives

$$4\ddot{x} = -4x + 4y - 20x$$
$$\ddot{x} = -6x + y$$

$$\ddot{y} = 4x - 6y$$

which can be represented as

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -6 & 1 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

as required.

d) The eigenvalues of $\underline{A}$ are given by

$$\begin{vmatrix} -6-\lambda & 1 \\ 4 & -6-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)(-6-\lambda) - 4 = 0$$
$$\lambda^2 + 12\lambda + 32 = 0$$
$$(\lambda + 4)(\lambda + 8) = 0$$

So eigenvalues are -4 and -8 with respective normal mode angular frequencies $\sqrt{-(-4)} = 2$ and $\sqrt{-(-8)} = 2\sqrt{2}$.

e) For eigenvalue -4 we have

$$\begin{pmatrix} -6+4 & 1 \\ 4 & -6+4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which gives

$$-2x + y = 0$$
$$4x - 2y = 0$$

So $(1\ \ 2)^T$ is an eigenvector. As its components have the same sign, the motion is in phase.

For eigenvalue -8 we have

$$\begin{pmatrix} -6+8 & 1 \\ 4 & -6+8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

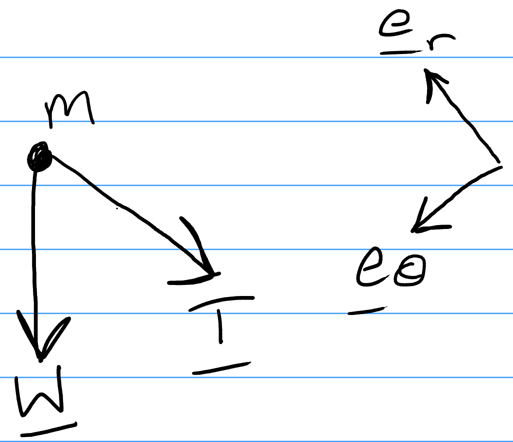
Which gives

$$2x + y = 0$$
$$4x + 2y = 0$$

So $(1\ \ -2)^T$ is an eigenvector. As its components have opposite sign, the motion is phase opposed.

27)

a)



Where $\underline{W}$ is the weight and $\underline{T}$ is the tension in the rod.

b) $\underline{T} = -|\underline{T}|\,\underline{e}_r$

$$\underline{W} = -mg\cos\theta\,\underline{e}_r + mg\sin\theta\,\underline{e}_\theta$$

c) The moment of inertia of the pendulum about O is

$$I = ml^2$$

d) The equation of rotational motion is

$$I\ddot{\theta} = \Gamma_{axis}$$

where Γ_{axis} is the component of total torque in the $\underline{e}_\theta$ direction

The torque can be calculated as

$$\underline{\Gamma} = \underline{r}_w \times \underline{W}$$

$$= L\underline{e}_r \times (-mg\cos\theta\, \underline{e}_r + mg\sin\theta\, \underline{e}_\theta)$$

$$= Lmg\sin\theta\, \underline{e}_\theta$$

And so Γ_{axis} is $Lmg\sin\theta$ and

$$I\ddot{\theta} = Lmg\sin\theta$$

$$mL^2\ddot{\theta} = Lmg\sin\theta$$

$$\ddot{\theta} = \frac{g\sin\theta}{L}$$

I seem to be missing a factor of $\frac{9}{8}$ somewhere sorry.

e)

Thanks for marking! :)