Q 1.

(a) The gradient of a function T in Cartesian coordinates is

$$\mathbf{grad}T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

as per Unit 15, equation 7. The partial derivatives of T w.r.t. x, y, and z are

$$\frac{\partial T}{\partial x} = -4x - 3y + 5, \qquad \frac{\partial T}{\partial y} = -3x + 6y + 5, \qquad \frac{\partial T}{\partial z} = 0$$

Hence

$$\mathbf{grad}T = -(4x + 3y - 5)\mathbf{i} - (3x - 6y - 5)\mathbf{j}.$$

and at the point $(1, -1)$:

$$\mathbf{grad}T(1, -1) = 4\mathbf{i} - 4\mathbf{j}$$

(b) The derivative of T at $(1, -1)$ in the direction of a vector $\mathbf{d}$ is $\mathbf{grad}T(1, -1) \cdot \hat{\mathbf{d}}$ where $\hat{\mathbf{d}}$ is a unit vector in the direction of $\mathbf{d}$ (MST210 Book E, p.23). Hence, the derivative of $T(1, -1)$ in the direction of $\mathbf{d} = \mathbf{i} + 2\mathbf{j}$ is

$$\begin{aligned}
\mathbf{grad}T(1, -1) \cdot \hat{\mathbf{d}} &= (4\mathbf{i} - 4\mathbf{j}) \cdot \left(\frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{1^2 + 2^2}}\right) \\
&= (4\mathbf{i} - 4\mathbf{j}) \cdot \left(\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}\right) \\
&= -\frac{4}{\sqrt{5}}
\end{aligned}$$

(c) The maximum derivative of a scalar field T at a point (x, y) is $|\mathbf{grad}T(x, y)|$ (MST210 Book E, p.19), and so the maximum derivative at $T(1, -1)$ is

$$\begin{aligned}|\mathbf{grad}T(1, -1)| &= \sqrt{4^2 + (-4)^2} \\ &= 4\sqrt{2}\end{aligned}$$

Hence, the maximum rate of change of temperature at $T(1, -1)$ is in the direction

$$\begin{aligned}\frac{\mathbf{grad}T(1, -1)}{|\mathbf{grad}T(1, -1)|} &= \frac{4\mathbf{i} - 4\mathbf{j}}{4\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\end{aligned}$$

(d) Substituting $x = 1$ and $y = -1$ into $T(x, y)$ gives

$$\begin{aligned}T(1, -1) &= -2 + 3 + 5 + 3 - 5 \\ &= 4\end{aligned}$$

As $\mathbf{grad}T$ is normal to the contour curves of T, we look for a vector $\mathbf{R}$ that is perpendicular to $\mathbf{grad}T(1, -1)$, that is

$$\mathbf{grad}T(1, -1) \cdot \mathbf{R} = 0$$

As $\mathbf{grad}T(1, -1) = 4\mathbf{i} - 4\mathbf{j}$, $\mathbf{R} = \mathbf{i} + \mathbf{j}$ is a solution to the scalar product equation above.

If $\mathbf{r} = \mathbf{i} - \mathbf{j}$ is the position vector of the point (1, -1), $x = 0$ at $\mathbf{r} - \mathbf{R} = -2\mathbf{j}$, so the line corresponding to the tangent to the $T = 4$ contour at $T(1, -1)$ has a slope of 1 and a y intercept of -2. The equation of this tangent is therefore $y = x - 2$.

Q 2.

(a) Any point P can be represented in cylindrical space by the triple (ρ, ϕ, z), where ρ, ϕ, and z are related to Cartesian space by

$$\rho = (x^2 + y^2)^{1/2}, \qquad \phi = \cos^{-1}\left(\frac{x}{\rho}\right) = \sin^{-1}\left(\frac{y}{\rho}\right), \qquad z = z$$

(Unit 15 equations 13 and 14).

Hence, $h(x, y, z)$ in cylindrical coordinates is

$$\begin{aligned} h(\rho, \phi, z) &= \frac{z}{\sqrt{\rho^2 \cos^2\phi + \rho^2 \sin^2\phi + z^2}} + z \\ &= \frac{z}{\sqrt{\rho^2 + z^2}} + z \end{aligned}$$

(b) The gradient function in cylindrical coordinates of a scalar field h is

$$\mathbf{grad} h = \mathbf{e}_\rho \frac{\partial h}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial h}{\partial \phi} + \mathbf{e}_z \frac{\partial h}{\partial z}$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$ are the unit vectors in the ρ, ϕ, and z directions, respectively (Unit 15 equation 24). The partial derivatives of h w.r.t. ρ, ϕ, and z are

$$\frac{\partial h}{\partial \rho} = -\frac{\rho z}{(\rho^2 + z^2)^{3/2}}, \qquad \frac{\partial h}{\partial \phi} = 0, \qquad \frac{\partial h}{\partial z} = \frac{1}{\sqrt{\rho^2 + z^2}} - \frac{z^2}{(\rho^2 + z^2)^{3/2}} + 1$$

Hence

$$\mathbf{grad} h(\rho, \phi, z) = -\mathbf{e}_\rho \left(\frac{\rho z}{(\rho^2 + z^2)^{3/2}}\right) + \mathbf{e}_z \left(\frac{1}{\sqrt{\rho^2 + z^2}} - \frac{z^2}{(\rho^2 + z^2)^{3/2}} + 1\right)$$

(c) Any point P can be represented in spherical space by the triple (r, θ, ϕ), where r, θ, and ϕ are related to Cartesian space by

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \cos^{-1}\left(\frac{z}{r}\right), \qquad \phi = \cos^{-1}\left(\frac{x}{r\sin\theta}\right)$$

Hence, $h(x, y, z)$ in spherical coordinates is

$$\begin{aligned} h(r,\theta,\phi) &= \frac{r\cos\theta}{r} + r\cos\theta \\ &= \cos\theta + r\cos\theta \end{aligned}$$

(d) The gradient function in spherical coordinates of a scalar field h is

$$\mathbf{grad}h = \mathbf{e}_r\frac{\partial h}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial h}{\partial \phi}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ are unit vectors in the r, θ, and ϕ directions, respectively (Unit 15 equation 36). The partial derivatives of h w.r.t. r, θ, and ϕ are

$$\frac{\partial h}{\partial r} = \cos\theta, \qquad \frac{\partial h}{\partial \theta} = -\sin\theta - r\sin\theta, \qquad \frac{\partial h}{\partial \phi} = 0$$

Hence

$$\mathbf{grad}h(r,\theta,\phi) = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\left(\frac{\sin\theta}{r} + \sin\theta\right)$$

and

$$|\mathbf{grad}h(r,\theta,\phi)| = \sqrt{\cos^2\theta + (-\sin(\theta/r) - \sin\theta)^2}$$

Q 3. The scalar line interval of a vector field $\mathbf{F}(\mathbf{r})$ along a closed path C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t1)$ is

$$\oint_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} dt$$

(Unit 16 equation 27). Expressing $\mathbf{F}$ and $\mathbf{r}$ as functions of t gives

$$\mathbf{F}(t) = (2\cos t + 1)\mathbf{i} - 2\mathbf{j} + \cos t\mathbf{k}$$
$$\mathbf{r}(t) = \mathbf{i} + \cos t\mathbf{j} + (\sin t + 1)\mathbf{k}$$

Differentiating $\mathbf{r}$ w.r.t. t gives

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{j} + \cos t\mathbf{k}$$

Hence

$$\begin{aligned}
\oint_0^{2\pi} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} dt &= \oint_0^{2\pi} \Big((2\cos t + 1)\mathbf{i} - 2\mathbf{j} + \cos t\mathbf{k} \Big) \cdot \Big(-\sin t\mathbf{j} + \cos t\mathbf{k} \Big) dt \\
&= \oint_0^{2\pi} 2\sin t + \cos^2(t) dt \\
&= \left[\frac{\sin(2t) - 8\cos t + 2t}{4} \right]_0^{2\pi} \\
&= \pi
\end{aligned}$$

Therefore the line integral is π. $\mathbf{F}$ is not a conservative field as the line integral of any conservative field around any closed curve is zero (Unit 16, Property b).

Q 4.

(a) The **curl** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ in Cartesian space is

$$\begin{aligned}\mathbf{curl}\mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}\end{aligned} \tag{4.1}$$

(Unit 16, equation 14). Substituting $F_1 = 4xy^2 - yz + 1$, $F_2 = 4x^2y - xz$, and $F_3 = -xy$ into (4.1) gives

$$\begin{aligned}\mathbf{curl}\mathbf{F} &= (-x + x)\mathbf{i} + (-y + y)\mathbf{j} + (8xy - 8xy)\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

As $\mathbf{curl}\mathbf{F} = \mathbf{0}$ everywhere, $\mathbf{F}$ is a conservative field , provided that its domain is simply connected.

(b) Following Procedure 1, we start by taking C to be the direct path from $(0,0,0)$ to the general point (a,b,c) parametrised by

$$\mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \qquad (0 \leq t \leq 1)$$

With this choice of parametrisation we calculate the scalar line integral

$$U(a,b,c) = -\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$$

Substituting the expressions for $\mathbf{F}$ and $d\mathbf{r}/dt = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ gives

$$\begin{aligned}
U(a,b,c) &= -\int_0^1 \left(\left((4at(bt)^2 - btct + 1)\mathbf{i} + (4(at)^2bt - atct)\mathbf{j} - (atbt)\mathbf{k} \right) \cdot \left(a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \right) \right) dt \\
&= -\int_0^1 4a^2b^2t^3 - abct^2 + a + 4a^2b^2t^3 - abct^2 - abct^2 \ dt \\
&= -\int_0^1 8a^2b^2t^3 - 3abct^2 + a \ dt \\
&= -8a^2b^2[\tfrac{1}{4}t^4]_0^1 + 3abc[\tfrac{1}{3}t^3]_0^1 + a[t]_0^1 \\
&= -2a^2b^2 - abc + a \\
&= U(0,0,0) - U(a,b,c)
\end{aligned}$$

Setting the datum for the potential energy function at the origin such that $U(0,0,0) = 0$, we can deduce that a potential energy function of $\mathbf{F}$ is

$$U(x,y,z) = -2x^2y^2 + xyz - x$$

(c) Calculating $-\mathbf{grad}U$ gives

$$\begin{aligned}
-\mathbf{grad}U &= -\left(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k} \right) \\
&= -\left((-4xy^2 + yz - 1)\mathbf{i} + (-4x^2y + xz)\mathbf{j} + xy\mathbf{k} \right) \\
&= (4xy^2 - yz + 1)\mathbf{i} + (4x^2y - xz)\mathbf{j} - xy\mathbf{k}
\end{aligned}$$

as required.

Q 5. The divergence of a vector field $\mathbf{F}(\rho, \phi, z) = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z$ is given in cylindrical coordinates by

$$\mathrm{div}\mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho} F_\rho + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

(Unit 16, equation 2). The partial derivatives for the given vector field are

$$\frac{\partial F_\rho}{\partial \rho} = -\frac{z^2 + \cos\phi}{\rho^2}, \qquad \frac{\partial F_\phi}{\partial \phi} = -\rho \sin\phi, \qquad \frac{\partial F_z}{\partial z} = -2e^{-2z}$$

Hence

$$\begin{aligned} \mathrm{div}\mathbf{F} &= -\frac{z^2 + \cos\phi}{\rho^2} + \frac{z^2 + \cos\phi}{\rho^2} - \sin\phi - 2e^{-2z} \\ &= -\sin\phi - 2e^{-2z} \end{aligned}$$

Q 6. The **curl** of a vector field $\mathbf{v}(r, \theta, \phi) = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi$ in spherical space is

$$\begin{aligned}\nabla \times \mathbf{v} = \mathbf{u} &= u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_\phi\mathbf{e}_\phi \\ &= \left(\frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\cot\theta}{r}v_\phi\right)\mathbf{e}_r \\ &\quad + \left(-\frac{\partial v_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}v_\phi\right)\mathbf{e}_\theta \\ &\quad + \left(\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{1}{r}v_\theta\right)\mathbf{e}_\phi\end{aligned}$$

(Unit 16, equation 17). Calculating the components of $\mathbf{u}$ separately we get

$$\begin{aligned}u_r &= -\frac{2r\cos\phi\sin\theta}{3} - \frac{2r\cos\phi\cos(2\theta)}{3\sin\theta} + \frac{2r\cos\phi\cos^2\theta}{3\sin\theta} \\ &= \frac{2r\cos\phi\cos(2\theta)}{3\sin\theta} - \frac{2r\cos\phi\cos(2\theta)}{3\sin\theta} \\ &= 0\end{aligned}$$

$$\begin{aligned}u_\theta &= -2r\cos\phi\cos\theta + \frac{r\cos\phi\sin(2\theta)}{\sin\theta} \\ &= -2r\cos\phi\cos\theta + 2r\cos\phi\cos\theta \\ &= 0\end{aligned}$$

$$\begin{aligned}u_\phi &= \frac{4r\sin\phi\cos(2\theta)}{3} - 2r\sin\phi\cos(2\theta) + \frac{2r\sin\phi\cos(2\theta)}{3} \\ &= 2r\sin\phi\cos(2\theta) - 2r\sin\phi\cos(2\theta) \\ &= 0\end{aligned}$$

Therefore, $\nabla \times \mathbf{v} = \mathbf{0}$ and so the vector field $\mathbf{v}$ is conservative.

Q 7.

(a)

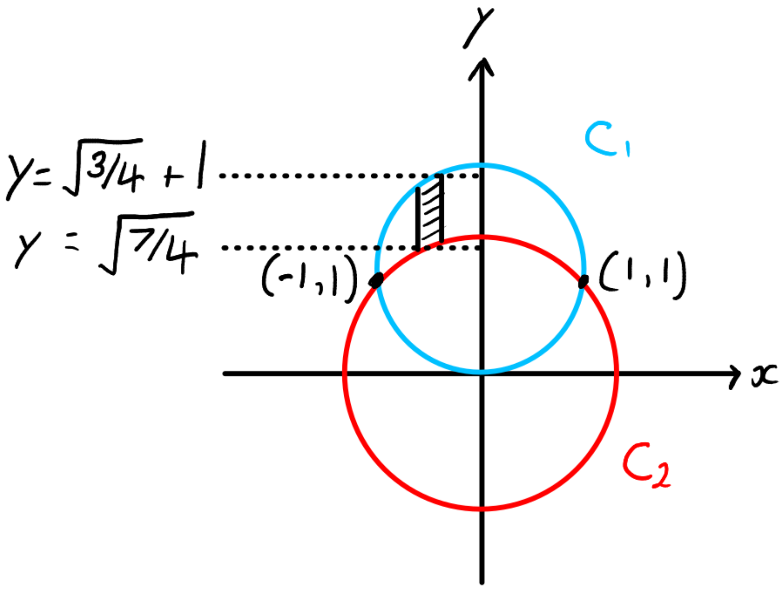

Figure 1: Diagram of the two overlapping circles. C_1 (blue) is centered at $(0, 1)$ with radius 1, and C_2 (red) is centered at $(0, 0)$ with radius $\sqrt{2}$. The two points of intersection between the circles are shown at $(-1, 1)$ and $(1, 1)$. A thin vertical strip centered over $x = -\frac{1}{2}$ inside the lune is shown shaded, with the values of y marked at its endpoints.

C_1 is defined by the equation $x^2 + (y - 1)^2 = 1$, and C_2 is defined by the equation $x^2+y^2 = 2$. The points of intersection were found by solving this pair of simultaneous equations, giving $(\pm 1, 1)$ as solutions (as shown in Figure 1).

When considering a vertical strip centered over $x = -\frac{1}{2}$ inside the lune, we substitute this value of x into each equation and solve for the upper value. At $x = -\frac{1}{2}$ the upper value of C_1 is $y = \sqrt{\frac{3}{4}} + 1$, and the upper value of C_2 is $y = \sqrt{\frac{7}{4}}$, as indicated in the figure.

(b) Following Procedure 1 of Unit 17, the upper limit of the lune S in the y direction is given by

$$x^2 + (y-1)^2 = 1$$
$$y = \sqrt{1-x^2} + 1 \qquad (-1 \leq x \leq 1)$$

and its lower limit in the y direction is given by

$$x^2 + y^2 = 2$$
$$y = \sqrt{2-x^2} \qquad (-1 \leq x \leq 1)$$

The upper and lower limits of S in the x direction are $x = \pm 1$.

With the upper and lower limits of S defined, its area can be expressed as the area integral

$$\int_S 1dA = \int_{x=-1}^{1} \left(\int_{y=\sqrt{2-x^2}}^{\sqrt{1-x^2}+1} 1dy \right) dx$$

(c) Evaluating the area integral from part (b) gives

$$\begin{aligned}
\int_S 1dA &= \int_{x=-1}^{1} \left(\sqrt{1-x^2} + 1 - \sqrt{2-x^2} \right) dx \\
&= \int_{x=-1}^{1} \sqrt{1-x^2}dx + \int_{x=-1}^{1} 1dx - \int_{x=-1}^{1} \sqrt{2-x^2}dx \\
&= \left[\frac{1}{2}\arcsin x + \frac{x}{2}\sqrt{1-x^2} \right]_{-1}^{1} + \Big[x \Big]_{-1}^{1} - \left[\arcsin\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{2}\sqrt{2-x^2} \right]_{-1}^{1} \\
&= \frac{\pi}{2} + 2 - \frac{\pi+2}{2} \\
&= 1
\end{aligned}$$

Hence, the area of the lune is 1 and does not involve π.

Q 8. We begin by converting the expression for the curved top into cylindrical space to make the subsequent integration simpler:

$$\begin{aligned} z &= 2a - \frac{x^2}{a} - \frac{y^2}{a} \\ &= 2a - \frac{(\rho\cos\phi)^2}{a} - \frac{(\rho\sin\phi)^2}{a} \end{aligned}$$

In cylindrical space, the volume integral of an object B centered on the z axis with its base on the xy plane is given by

$$\int_B D\ dV = \int_{\rho=0}^{a}\left(\int_{\phi=-\pi}^{\pi}\left(\int_{z=0}^{\beta(x,y)} D\rho\ dz\right)d\phi\right)d\rho$$

where D is a density function and $\beta(x, y)$ is a function that defines the upper limit of the object (Unit 17, equation 20).

Substituting $D = \frac{\rho}{a} + 1$ and $\beta(x, y) = 2a - \frac{(\rho\cos\phi)^2}{a} - \frac{(\rho\sin\phi)^2}{a}$ gives

$$\begin{aligned} \int_B DdV &= \int_{\rho=0}^{a}\left(\int_{\phi=-\pi}^{\pi}\left(\int_{z=0}^{2a-\frac{(\rho\cos\phi)^2}{a}-\frac{(\rho\sin\phi)^2}{a}} \frac{\rho^2}{a} + \rho\ dz\right)d\phi\right)d\rho \\ &= \int_{\rho=0}^{a}\left(\int_{\phi=-\pi}^{\pi}\left(\rho\left(\frac{\rho}{a}+1\right)\left(-\frac{\rho^2\sin^2\phi}{a} - \frac{\rho^2\cos^2\phi}{a} + 2a\right)\right)d\phi\right)d\rho \\ &= \int_{\rho=0}^{a}\left(-\frac{\rho\left(\frac{\rho}{a}+1\right)(2\pi\rho^2 - 4\pi a^2)}{a}\right)d\rho \\ &= \frac{73\pi a^3}{30} \end{aligned}$$

Therefore, the mass of the object is $\frac{73\pi a^3}{30}$Kg.

Q 9.

(a) In spherical coordinates, $z = r\cos\theta$ and $r = \sqrt{x^2 + y^2 + z^2}$. Substituting these into the density function ρ gives

$$\begin{aligned}\rho &= \frac{Ar^2\cos^2\theta}{r}\\ &= Ar\cos^2\theta\end{aligned}$$

as required.

(b) In spherical space, the volume integral of an object B centered on the origin is given by

$$\int_B \rho\ dV = \int_{r=R_1}^{R_2}\left(\int_{\theta=0}^{\pi}\left(\int_{\phi=-\pi}^{\pi} \rho r^2 \sin\theta\ d\phi\right) d\theta\right) dr$$

where ρ is a density function and R_1 and R_2 are the lower and upper distance from the origin for the region to be integrated, respectively (Unit 17, equation 20). Substituting $\rho = Ar\cos^2\theta$, $R_1 = a$, and $R_2 = 2a$ gives

$$\begin{aligned}\int_B \rho\ dV &= \int_{r=a}^{2a}\left(\int_{\theta=0}^{\pi}\left(\int_{\phi=-\pi}^{\pi} Ar^3\cos^2\theta\sin\theta\ d\phi\right) d\theta\right) dr\\ &= \int_{r=a}^{2a}\left(\int_{\theta=0}^{\pi}\left(2\pi Ar^3\cos^2\theta\sin\theta\right) d\theta\right) dr \qquad (9.1)\end{aligned}$$

The integral w.r.t. θ can be rearranged as

$$2\pi Ar^3 \int_{\theta=0}^{\pi} \sin\theta\cos^2\theta\ d\theta \qquad (9.2)$$

Let $u = \cos\theta$ then $du = -\sin\theta\ d\theta$. Substituting into (9.2) gives

$$\begin{aligned}-2\pi Ar^3 \int_{\theta=0}^{\pi} u^2\ du &= -2\pi Ar^3\left[\frac{1}{3}\cos^3\theta\right]_0^{\pi}\\ &= \frac{4\pi Ar^3}{3}\end{aligned}$$

We substitute this into (9.1) and complete the volume integral:

$$\begin{aligned}\int_{r=a}^{2a}\left(\frac{4\pi Ar^3}{3}\right) dr &= \frac{4\pi A}{3}\int_{r=a}^{2a} r^3\ dr\\ &= \frac{4\pi A}{3}\left[\frac{1}{4}r^4\right]_a^{2a}\\ &= 5\pi Aa^4\end{aligned}$$

Therefore, the mass of the shell is $5\pi Aa^4$ Kg.