Unit 6

Geometric transformations

Introduction

So far in this module most of the functions you have met have been real functions; however, the mathematical concept of a *function* is much broader than this. In this unit, you'll learn about some of the functions that arise from geometry.

Broadly speaking, geometry is about the properties of objects in space or in a plane. For example, one of the geometric properties of the minaret in Figure 1 is that it has 8-fold symmetry – that is, its shape stays the same when it is rotated through an eighth of a complete turn.

Figure 1 A minaret with 8-fold symmetry

This idea of applying a rotation or some other kind of transformation to an object in order to check its properties occurs frequently in geometry. For example, given two triangles in a plane (see Figure 2), a typical geometric question is to ask whether they're *congruent* – that is, have the same size and shape. A simple thought experiment suggests a way to answer this question. Just pick up one of the triangles and attempt to superimpose it on the other. If the two triangles coincide exactly, then you can conclude that they do indeed have the same size and shape.

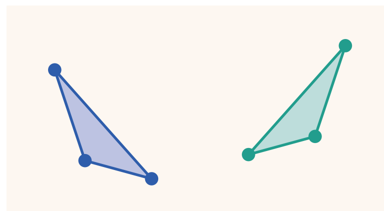


Figure 2 Two triangles

Some objects, such as the hexagons in Figure 3, have the property of being *similar* rather than congruent: they have the same shape but different sizes. You can check that any two of the hexagons are similar by enlarging the smaller one until it just covers the larger one.

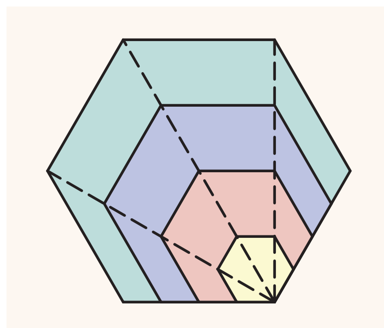


Figure 3 Similar hexagons

These remarks raise several questions. What is the best way to describe an object such as a triangle or a hexagon (or even a minaret) so that you can manipulate it mathematically? How can you represent a transformation such as a rotation or enlargement mathematically? These and many other questions can be answered using the language of functions.

By the end of this unit you'll have met a whole class of geometric functions known as *affine transformations*. Roughly speaking, these are functions that preserve 'parallelism'; that is, they map parallel lines to parallel lines. This kind of transformation is found in many computer graphics packages for manipulating polygonal figures on screen. With them, you can translate, rotate, reflect, scale and shear objects, as illustrated in Figure 4.

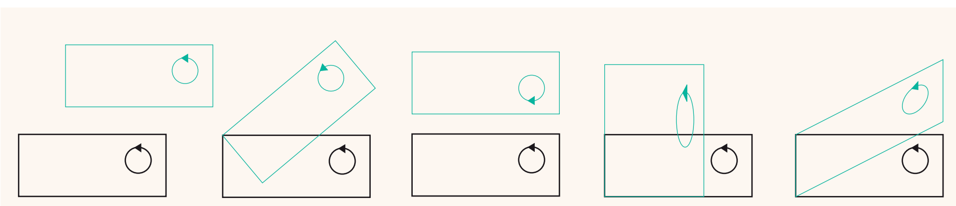


Figure 4 Some affine transformations

The ideas explored in this unit arise wherever the geometric configuration of objects is important, including areas as diverse as choreography, architectural and engineering drawing, the design of wallpaper patterns and the development of video games. The tools used here are also foundational for branches of mathematics known as *group theory* and *linear algebra*, which have many applications and which you will meet if you pursue your mathematical studies further.

This unit focuses on transformations of the plane, but you should bear in mind that many of the ideas involved can also be applied in three-dimensional space.

1 Transformations of the plane

In this section you'll revise the idea of a function, and you'll see how functions can be used to manipulate objects in the plane.

1.1 What is a transformation of the plane?

Roughly speaking, a *transformation of the plane* is a function that rearranges the points of the plane, a bit like the way the points on a flat sheet of rubber would be rearranged if the sheet were moved, flipped and/or stretched in some way.

The plane is often denoted by $\mathbb{R}^2$ to emphasise the fact that each of its points is specified by a pair of real Cartesian coordinates (x, y). Just as $\mathbb{R}$ is the set of real numbers, each of which can be thought of as a point on a line, so $\mathbb{R}^2$ is the set of pairs of real numbers, each of which can be thought of as the Cartesian coordinates of a point in the plane. Strictly, a point is not the same as its Cartesian coordinates. However, it is convenient to use both terms interchangeably and to refer to 'the point $(2, 4)$', for example.

The use of Cartesian coordinates creates a bridge between geometry and algebra that enables us to specify transformations of the plane in algebraic terms. To understand how this is done, recall the general definition of a *function*. This was covered in MST124 and revised in MST125 Unit 1.

A **function** consists of:

- a set of allowed input values, called the **domain** of the function
- a set of values in which every output value lies, called the **codomain** of the function
- a process, called the **rule** of the function, for converting each input value into *exactly one* output value.

If x is a point in the domain of a function f, then the point $f(x)$ in the codomain of f is called the **image** of x under f. We also say that the function f **maps** the point x in its domain to the point $f(x)$ in its codomain.

Since a transformation of the plane is a function that rearranges points of the plane, its input and output values are elements of $\mathbb{R}^2$, and its rule relates the coordinates of the output point to those of the input point. As an example, suppose f is a transformation with the rule

$$f(x, y) = (x + 2, y).$$

Here, $f(x, y)$ is a commonly used abbreviation for $f((x, y))$.

You can think of f as a processor that accepts a point (x, y) as input and returns the point $(x + 2, y)$ as output, as illustrated in Figure 5.

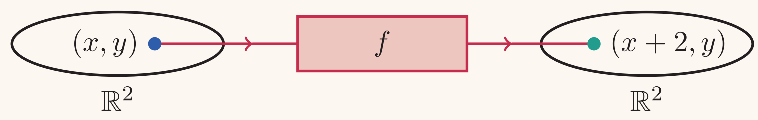


Figure 5 The processor for a transformation of the plane

For example, if you input the point $(2, 3)$ into this processor, then you'll obtain the point $(4, 3)$, and if you input $(0, 0)$, then you'll obtain $(2, 0)$. Since the output point $(x + 2, y)$ always lies 2 units to the right of the input point (x, y), the overall effect of f is to translate the whole plane 2 units to the right.

But what about the domain and codomain of f? Many choices are possible, but since the rule of f can be applied to every point (x, y) in $\mathbb{R}^2$, we'll let the domain of f be as large as possible and take it to be $\mathbb{R}^2$. The codomain has to accommodate all possible output points, and the simplest way to achieve this is to let the codomain be $\mathbb{R}^2$.

This illustrates the following simplifying assumptions that we'll adopt in this unit with regards to the domains and codomains of transformations. They're analogous to the simplifying assumptions adopted for real functions, and in most cases they'll enable you to specify a transformation simply by writing down its rule.

In this unit:

- we use the word 'transformation' to mean 'transformation of the plane'
- we take the codomain of every transformation to be the set $\mathbb{R}^2$, since this set contains every possible output point
- when a transformation is specified by *just a rule*, it is understood that the domain of the transformation is the largest possible set of points for which the rule is applicable.

For example, you can apply the rule

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

to all points of the plane apart from $(0,0)$ (applying it to $(0,0)$ would involve division by 0). The domain of f is therefore the set $\mathbb{R}^2$ excluding the origin.

Activity 1 *Understanding the domain, codomain and rule of a transformation*

For each of the following transformations, describe its domain and codomain and find the image of the point $(1,2)$.

(a) $f(x,y) = (x+3, y+6)$ (b) $g(x,y) = (3x+y, x)$

(c) $h(x,y) = (1/y, 1/x)$

In fact, all of the transformations from now on in this unit have rules that are applicable to every point (x,y) in $\mathbb{R}^2$. Their domains and codomains will therefore be $\mathbb{R}^2$.

You saw earlier in this section that you can interpret the transformation $f(x,y) = (x+2, y)$ geometrically as a translation. The next example explores the geometric effect of two more transformations.

Example 1 *Describing transformations geometrically*

Describe the geometric effect of each of the following transformations on the plane.

(a) $f(x,y) = (2x, y)$ (b) $g(x,y) = (x, -y)$

Solution

(a) The output point has the same y-coordinate as the input point, but it lies twice as far from the y-axis. Points on the y-axis do not move.

Here f 'stretches' the plane in the x-direction by a factor of 2.

(b) The output point has the same x-coordinate as the input point, but lies as far below the x-axis as the input point lies above it (or as far above the x-axis as the input point lies below it). Points on the x-axis do not move. It's as though the whole plane has been flipped over through 180° about the x-axis. Equivalently, it's as though the plane has been reflected in a two-way mirror standing vertically upright on the x-axis.

Here g flips the plane about the x-axis. Equivalently, it reflects the plane in the x-axis.

In cases such as Example 1(b), some people find the idea of 'flipping' easier to visualise than 'reflecting'. However, reflecting is probably the more usual terminology and has the advantage that it can also be applied in three-dimensional space.

Here are some transformations for you to interpret geometrically.

Activity 2 *Describing transformations geometrically*

Describe the geometric effect that each of the following transformations has on the plane.

(a) $f(x,y) = (x+2, y+3)$ (b) $g(x,y) = (-x, y)$

(c) $h(x,y) = (x, 2y)$

1.2 Images of plane figures under transformations

The rule of a function f tells you how to find the image of any element in its domain. Often, however, you need to consider the images of the elements of a whole subset A of the domain. Collectively these images form a subset of the codomain, which we call the **image** of the set A and denote by $f(A)$. We also say that f **maps** A to $f(A)$.

This is illustrated in Figure 6. Each element x in A has an image $f(x)$ in the codomain of f. As x ranges through A, its image $f(x)$ ranges through the image $f(A)$ of A.

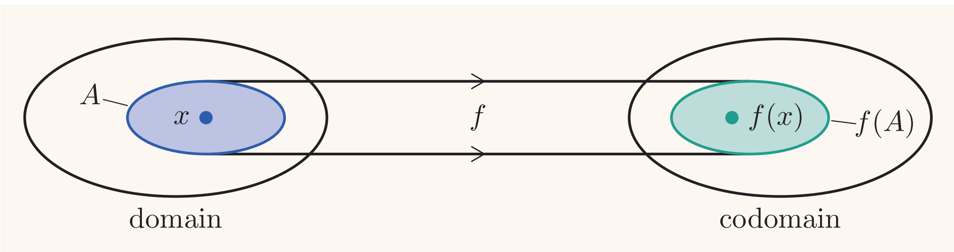


Figure 6 The image $f(A)$ of a set A

It's important to realise that $f(x)$ and $f(A)$ are two quite different uses of notation:

- $f(x)$ is the image of an *element* x of the domain. It's an *element* of the codomain.
- $f(A)$ is the image of a *subset* A of the domain. It's a *subset* of the codomain.

The *image set of a function* (which was discussed in MST124 and revised in MST125 Unit 1) is simply the image of the whole domain of the function. It contains every element of the codomain that is the image of an element of the domain. The image set of a function is also called the **range** of the function.

The concept of an image set is particularly useful for illustrating the effect that a transformation has on subsets of $\mathbb{R}^2$ like lines, squares or ellipses. Such geometric subsets are called *figures*.

In geometric contexts, subsets of $\mathbb{R}^2$ are called **(plane) figures**.

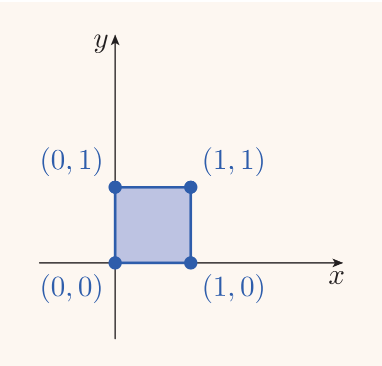


Figure 7 The unit square

One figure that you'll meet repeatedly is the square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, shown in Figure 7. Because this figure occurs so frequently, we'll give it a special name – the **unit square**.

Example 2 *Sketching images of the unit square*

Sketch the effect that each of the following transformations has on the unit square. In each case draw two copies of the plane $\mathbb{R}^2$, one showing the unit square before the transformation has been applied, the other showing its image set under the transformation.

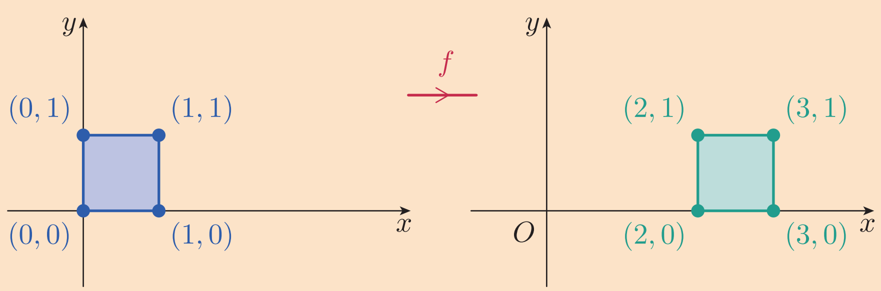
(a) $f(x, y) = (x + 2, y)$ (b) $g(x, y) = (2x, y)$

(c) $h(x, y) = (x, -y)$

Solution

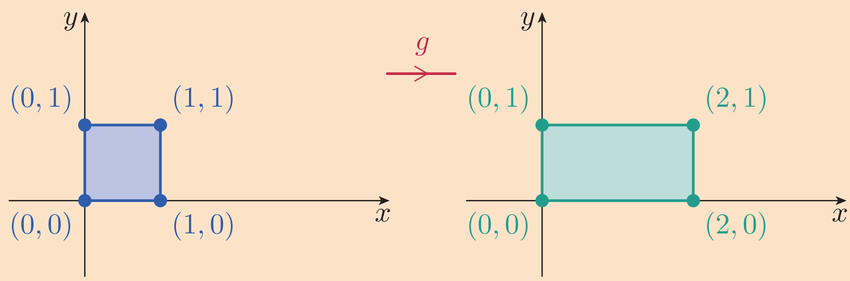
(a) Under f, the points that make up the unit square are all mapped 2 units to the right. It follows that their positions relative to each other do not change; they are mapped collectively, like a rigid object, 2 units to the right. Calculate the coordinates of the translated vertices and use them to sketch the image.

f maps the unit square 2 units to the right to produce a square image with vertices at $f(0, 0) = (2, 0)$, $f(1, 0) = (3, 0)$, $f(1, 1) = (3, 1)$ and $f(0, 1) = (2, 1)$.



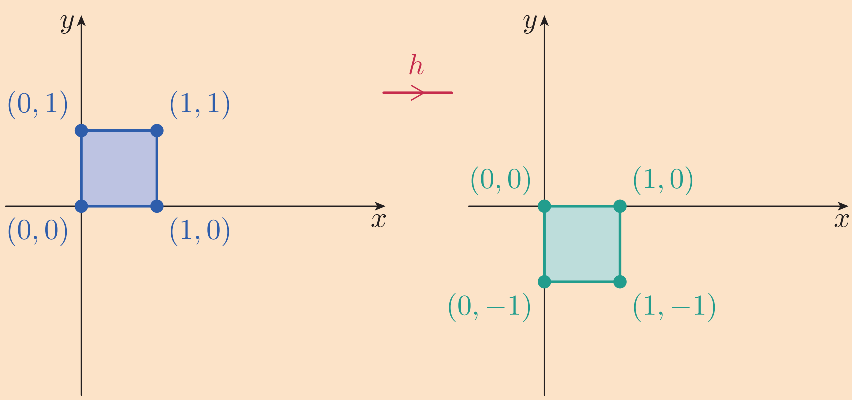
(b) Under g, distances parallel to the x-axis are doubled, whereas distances parallel to the y-axis are unchanged. So the unit square is stretched in the x-direction to produce a 2 by 1 rectangle. Calculate the coordinates of the vertices of this stretched rectangle and use them to sketch the image.

g stretches the unit square in the x-direction to produce a rectangular image with vertices at $g(0,0) = (0,0)$, $g(1,0) = (2,0)$, $g(1,1) = (2,1)$ and $g(0,1) = (0,1)$.



(c) Under h, the points that make up the unit square are collectively reflected in the x-axis. Calculate the coordinates of the reflected vertices and use them to sketch the image.

h reflects the unit square in the x-axis to produce a square image with vertices at $h(0,0) = (0,0)$, $h(1,0) = (1,0)$, $h(1,1) = (1,-1)$ and $h(0,1) = (0,-1)$.



By sketching the effect that a transformation has on the unit square, you gain some impression of the way the transformation behaves. Admittedly, the sketches only show the effect of the transformation on the unit square but, as you'll see later, the effect that a transformation has on this square often reveals far more about the transformation than you might expect.

Activity 3 *Sketching images of the unit square*

Sketch the effect that each of the following transformations has on the unit square.

(a) $f(x, y) = (x + 2, y + 3)$ (b) $g(x, y) = (-x, y)$

(c) $h(x, y) = (x, 2y)$ (d) $k(x, y) = (x, y)$

In Activity 3(d), you met the seemingly trivial transformation that leaves all points of the plane fixed. This transformation is called the **identity transformation** and it turns out to be particularly important – you'll see why in Subsection 3.2.

2 Isometries

In this section you'll learn about transformations known as *isometries* that act rigidly on the plane. You'll see that there are four types of isometries, namely reflections, rotations, translations and glide-reflections.

2.1 What is an isometry?

In geometry there are many occasions when you'll need to change the position of figures. For example, you may decide to investigate whether two triangles have the same size and shape by checking whether you can superimpose one of the triangles on the other. You may want to investigate the symmetry of a figure, by checking which changes in position leave the figure looking the same. Or you may decide to investigate the ways in which motifs (small designs or symbols) can be moved around the plane to form attractive patterns, like the tiling from the Saadian Tombs in Marrakech shown in Figure 8. (Here, and later, minor differences in shape and colour between the motifs are ignored.)

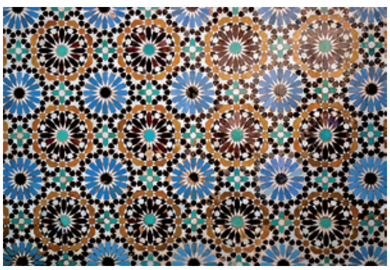


Figure 8 Tiling from the Saadian Tombs

In each case the idea is to move figures *rigidly* around the plane – that is, without changing their size or shape. But how can this be expressed mathematically? A clue is provided by comparing the transformations

$$f(x, y) = (x + 2, y) \quad \text{and} \quad g(x, y) = (2x, y)$$

that you met in Example 2. The transformation f translates all points by the same amount, leaving distances between points unchanged. So, for example, the unit square is mapped rigidly to a square of the same size and shape, as indicated at the top of Figure 9. By contrast, the transformation g fails to map the unit square rigidly, because it doubles lengths in the x-direction. This results in the unit square being stretched to give a rectangle of length 2, as indicated at the bottom of Figure 9. Notice that Figure 9 also includes line segments to illustrate the effect of the transformation on the distance between two points P and Q with the same y-coordinate, and adopts a commonly used convention in which the images of the points P and Q are denoted by P' and Q', respectively.

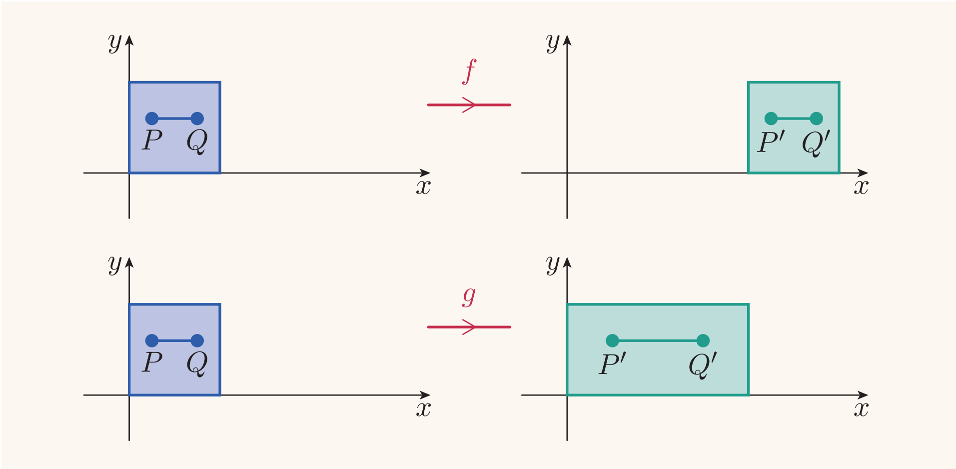


Figure 9 The effect of f and g on the unit square

In general, transformations that map the plane rigidly, and therefore preserve the size and shape of figures, are those that preserve the distances between any two points. Such transformations are called *isometries*, from the Greek words 'iso', meaning 'same', and 'metron', meaning 'measure'. For example, the transformation f shown in Figure 9 is an isometry, but g is not.

A transformation f is an **isometry** if the distance between any two points P and Q is equal to the distance between their images $P' = f(P)$ and $Q' = f(Q)$.

Since isometries map figures rigidly around the plane, they map any polygon – a flat shape whose boundary consists of line segments – to a polygon of the same size and shape. In particular, the images of the vertices of the polygon must be the vertices of its image (see Figure 10).

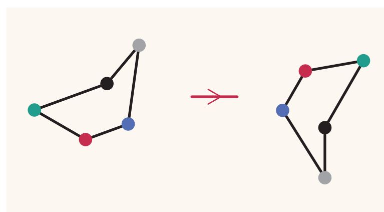


Figure 10 Vertices are mapped to vertices

Images of polygons

An isometry maps any polygon to a polygon of the same size and shape. In particular, the vertices of the polygon are mapped to the vertices of the image polygon.

The next example applies the above fact to a triangle.

Example 3 *Finding the image of a triangle under an isometry*

Draw a sketch showing the effect of the isometry

$$f(x, y) = (4 - y, 3 - x)$$

on the triangle with vertices at $(1, 1)$, $(2, 2)$, $(3, 1)$.

Solution

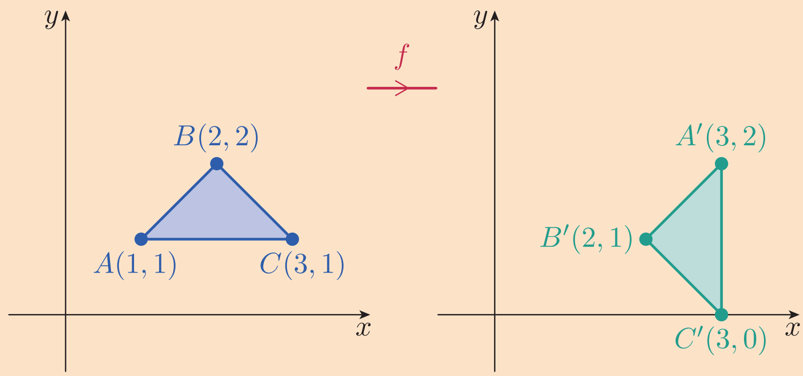
The vertices of the triangle are mapped to the vertices of the image triangle, so first calculate the coordinates of these image vertices.

The images of the vertices of the triangle under f are:

$$\begin{aligned} f(1, 1) &= (4 - 1, 3 - 1) = (3, 2), \\ f(2, 2) &= (4 - 2, 3 - 2) = (2, 1), \\ f(3, 1) &= (4 - 1, 3 - 3) = (3, 0). \end{aligned}$$

Draw a pair of Cartesian planes, using the same scale on all the axes. Choose the scale to ensure that the vertices and their images all fit on the sketch. Draw the original triangle on the left-hand plane and its image on the right-hand plane.

So the effect of f is as shown below.



As a check notice that the original triangle has the same size and shape as its image.

Activity 4 *Finding the image of a quadrilateral under an isometry*

Draw a sketch showing the effect that the isometry

$$f(x, y) = (4 - y, 1 + x)$$

has on the quadrilateral with vertices at $(1, 3)$, $(2, 1)$, $(3, 1)$, $(2, 3)$.

2.2 Types of isometries

It turns out that every isometry is one of four basic types: a *translation*, a *rotation*, a *reflection*, or a *glide-reflection*. A proof of this fact is beyond the scope of this module. In this subsection, we'll look in turn at the first three types of isometries. We'll consider the fourth type in the next section.

Translations

One of the easiest isometries to understand is a translation. A *translation* is a transformation that displaces each point of the plane through a fixed distance in a fixed direction, and as such it can be characterised by a displacement vector like the one shown in Figure 11(a) (vectors were covered in MST124 and revised in MST125 Unit 1). This particular displacement vector corresponds to the translation

$$f(x, y) = (x + 2, y + 3),$$

which displaces each point of the plane 2 units to the right and 3 units up.

Changing the sign of the x-component of the displacement vector changes the direction of the horizontal displacement, whereas changing the sign of the y-component changes the direction of the vertical displacement. So, for example, the translation

$$f(x, y) = (x - 2, y + 3)$$

displaces each point of the plane 2 units to the *left* and 3 units up (see Figure 11(b)) and the translation

$$f(x, y) = (x - 2, y - 3)$$

displaces each point of the plane 2 units to the left and 3 units *down* (see Figure 11(c)).

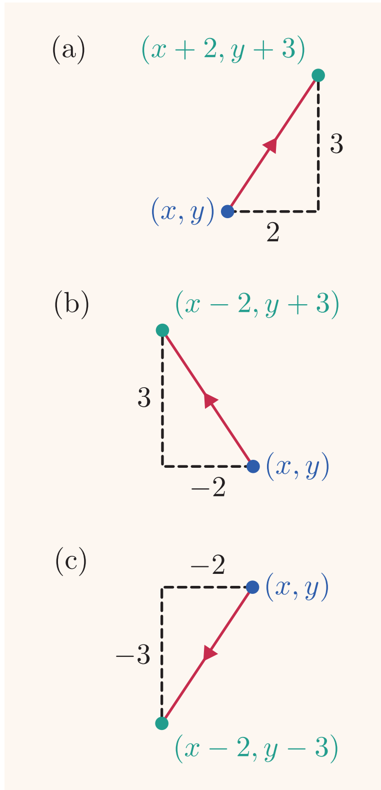


Figure 11 Translations

In general, we make the following definition.

> A **translation** p units horizontally and q units vertically is a transformation of the form
>
> $$f(x, y) = (x + p, y + q),$$
>
> where p and q can each be positive, negative or zero.
>
> The displacement vector $p\mathbf{i} + q\mathbf{j}$ is called the **associated vector** of the translation.

For example, the translation $f(x, y) = (x - 2, y + 3)$ has associated vector $-2\mathbf{i} + 3\mathbf{j}$. Here, $\mathbf{i}$ and $\mathbf{j}$ are the usual Cartesian unit vectors.

Activity 5 *Writing down a translation and its associated vector*

Write down the translation that maps points 1 unit down and 2 units to the right, together with its associated vector.

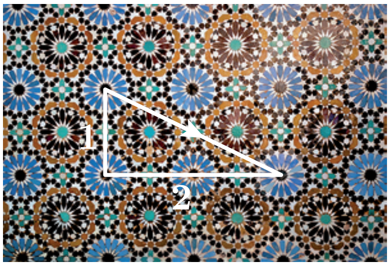


Figure 12 A translation that leaves the appearance of the tiling unchanged

Figure 12 illustrates how you can use the translation in Activity 5 to start to build up an understanding about the structure of the repeating pattern of the tiling from the Saadian Tombs that you met earlier.

Since it is the pattern that is of interest here, rather than the properties of any border or boundary, the first thing you need to do is to imagine that the pattern continues to repeat indefinitely in all directions so as to fill the entire plane. You'll then be able to see in your mind's eye that a translation 2 units to the right and 1 unit down leaves the appearance of the pattern unchanged. (Here, the unit of measurement is the distance between the centres of adjacent large blue circular motifs.)

In fact, there are many other translations that leave the appearance of the pattern unchanged. Any translation an integer number of units to the right or left and an integer number of units up or down leaves the appearance unchanged.

In general, a translation has the property that it displaces all points of the plane by the same (non-zero) amount, so there are no points that remain fixed (unless we allow a zero displacement to count as a translation, which can sometimes be convenient). This is quite different from a rotation, where one point always remains fixed.

Rotations

A rotation is another type of isometry (see Figure 13).

> A **rotation** is a transformation that moves each point of the plane through a fixed angle about a fixed point, called the **centre of rotation**.

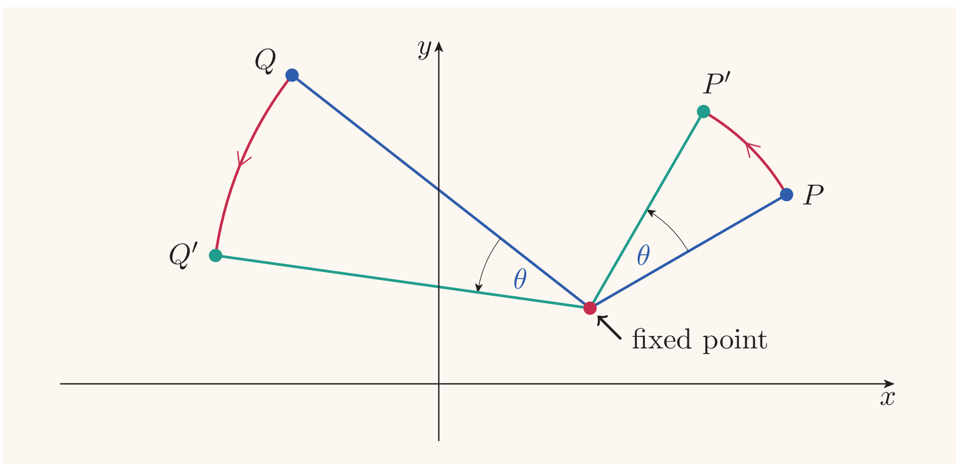


Figure 13 Images of points P and Q under rotation through the angle θ about a fixed point

By convention, a rotation through an angle θ about a fixed point is taken to be anticlockwise if θ is positive and clockwise if θ is negative. For example, the rotation through the angle $-3\pi/2$ about a point is the clockwise rotation through three quarters of a turn about the point (see Figure 14). (In this unit angles are measured in radians.)

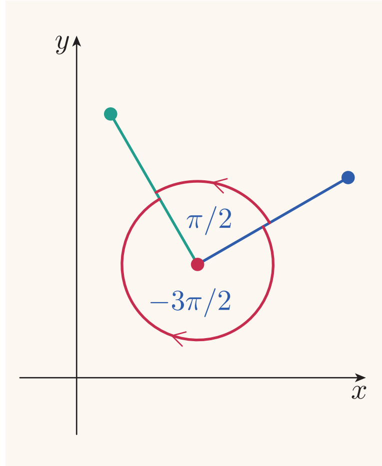


Figure 14 Rotations through $\pi/2$ and $-3\pi/2$ are the same

Notice, however, that the rotation through $-3\pi/2$ about a point is the same transformation as the rotation through $\pi/2$ about the point. The method of describing the transformation may be different, but the outcome in terms of the position of the image points is the same.

Activity 6 *Identifying different descriptions of the same rotation*

Think about rotations through the following angles about a fixed centre of rotation. Which describe the same transformations?

$\pi/2$, $-\pi/4$, 3π, $3\pi/2$, 2π, -4π, 0, $7\pi/4$, $5\pi/4$, $-3\pi/4$, $-\pi$

Rotations provide a further way of building up an understanding of the structure of the tiling pattern from the Saadian Tombs. Suppose, for example, that we choose coordinate axes that intersect at one of the blue circular motifs (which is therefore at the origin). Then you can see from Figure 15 that a rotation through $-\pi/2$ about the origin leaves the appearance of the pattern unchanged.

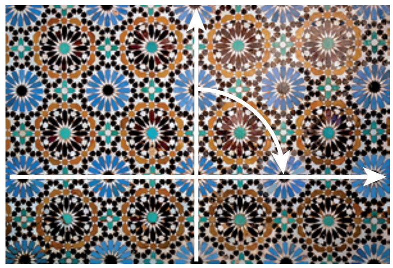


Figure 15 The appearance of the tiling is unchanged by rotation through $-\pi/2$ about the origin

Activity 7 *Identifying rotations that leave the tiling pattern unchanged*

Describe all of the rotations about the origin through angles between 0 and 2π that leave the appearance of the tiling pattern in Figure 15 unchanged.

Rotations about the origin through integer multiples of $\pi/2$ have fairly simple rules when expressed in terms of coordinates. To help you understand how to find such rules, let's start by looking at an example of the effect that a rotation of this type has on a particular point.

Example 4 *Finding the image of a point under a rotation about the origin*

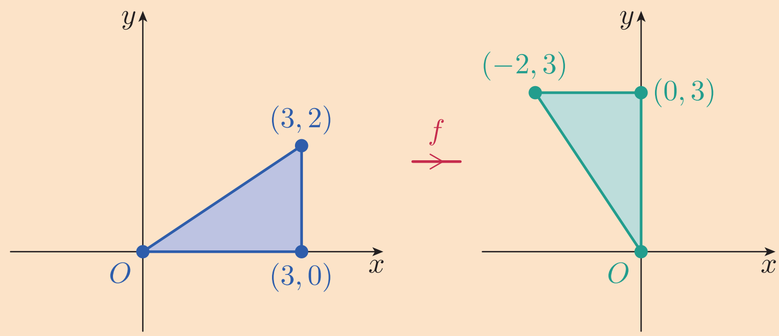
Let f be the rotation through the angle $\pi/2$ about the origin.

(a) Sketch the effect of f on the triangle with vertices $(0,0)$, $(3,0)$, $(3,2)$.

(b) Hence find the image of the point $(3,2)$ under f.

Solution

(a) Under f the triangle turns anticlockwise about the origin O, without changing shape, until its base (of length 3) aligns with the y-axis.



(b) The diagram in part (a) shows that $f(3,2) = (-2,3)$.

The next activity asks you to find the images of some other points under the same rotation f.

Activity 8 *Finding the images of points under a rotation about the origin*

Let f be the rotation through the angle $\pi/2$ about the origin O, as in Example 4. Find the images of each of the following points P under f. (In each case, start by sketching the effect of f on the right-angled triangle with hypotenuse OP and with its base along the x-axis.)

(a) $P = (-3,2)$ (b) $P = (-3,-2)$ (c) $P = (3,-2)$

Having practised how to find the images of various points under the rotation through $\pi/2$ about the origin, let's now consider its effect on a general point (x,y), and hence discover an algebraic expression for its rule.

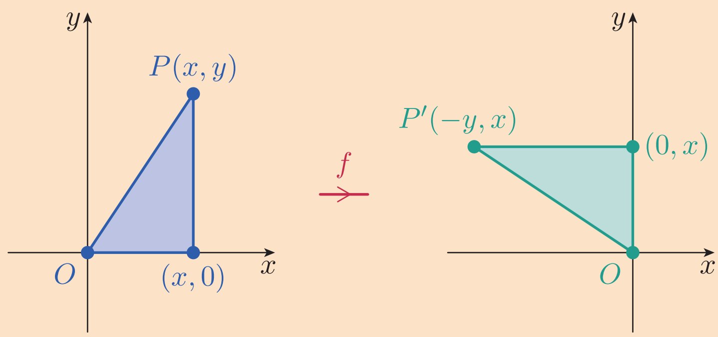
Example 5 *An algebraic specification for a rotation*

Find an algebraic specification for the rotation f through the angle $\pi/2$ about the origin O.

Solution

The effect that f has on a general point $P(x, y)$ is illustrated below.

Draw the right-angled triangle, with hypotenuse OP, and with base along the x-axis (left-hand diagram). Under f, this triangle turns anticlockwise about O, without changing shape, until its base aligns with the y-axis (right-hand diagram).



The diagram shows that $f(x, y) = (-y, x)$.

Although Example 5 only illustrated the case where the point P lay in the first quadrant, a comparison with the solution to Activity 8 should convince you that the same specification of f applies for P in any quadrant.

In the next activity, you're asked to find algebraic specifications for the other two rotations through integer multiples of $\pi/2$ about O.

Activity 9 *Finding algebraic specifications for more rotations*

Find an algebraic specification for each of the following rotations:

(a) g is the rotation through the angle π about the origin O.

(b) h is the rotation through the angle $3\pi/2$ about O.

You'll see later in this unit how you can find algebraic expressions for the rules of rotations through any angle and with any centre of rotation.

Whereas a rotation leaves just one point fixed, the next type of isometry, *reflection*, leaves a whole line of points fixed.

Reflections

Imagine that a mirror is held vertically along the central line of the photograph in Figure 16(a). Provided you ignore minor differences, you can regard the right-hand side of the photograph as the mirror image of its left-hand side and vice versa. This suggests the (mathematical) definition of a reflection illustrated in Figure 16(b) and given in the box below.

(a) (b)

Figure 16 (a) The Taj Mahal (b) a reflection

The **reflection** (of the plane) in a line ℓ is the transformation that maps each point P of the plane to the point P' on the other side of ℓ in such a way that ℓ is the perpendicular bisector of the line segment PP'. Points on ℓ remain fixed. We'll refer to the line ℓ as **the line of reflection**.

If you prefer, you can think of a reflection as a transformation that flips the plane over about the line of reflection ℓ.

You've already seen several examples of reflections earlier in this unit. In Example 1 you met the transformation

$$g(x, y) = (x, -y),$$

which reflects points in the x-axis (see Figure 17(a)), and in Activity 2 you met the transformation

$$g(x, y) = (-x, y),$$

which reflects points in the y-axis (see Figure 17(b)).

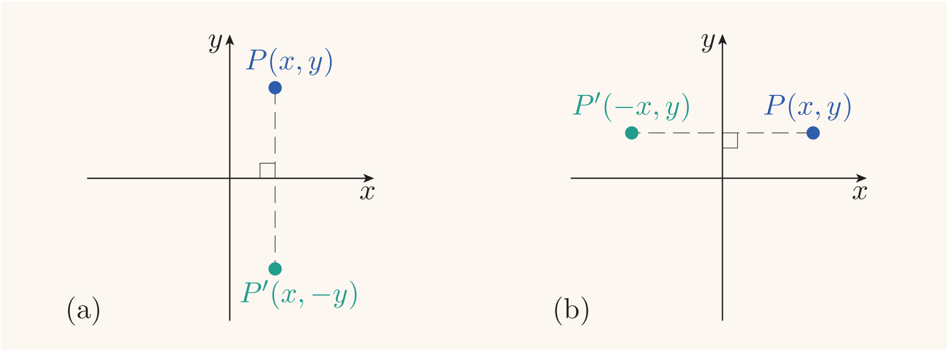


Figure 17 (a) Reflection in the x-axis (b) reflection in the y-axis

There are two other reflections with fairly simple rules, namely reflection in the line $y = x$ and reflection in the line $y = -x$. As you'll see in Example 6 and Activity 10, algebraic specifications for these reflections can be found by using an approach similar to the one used earlier for rotations. Later in the unit you'll see how to find the rule for a reflection in any line.

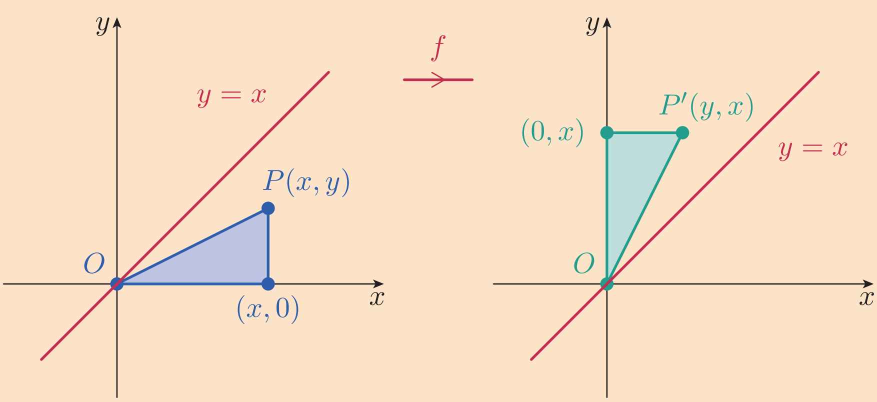
Example 6 *An algebraic specification for a reflection*

Find an algebraic specification for the reflection f in the line $y = x$.

Solution

The effect that f has on a general point $P(x, y)$ is illustrated below.

Draw a right-angled triangle, in the usual way, with hypotenuse OP and with its base along the x-axis (see left-hand diagram). Under f, this triangle flips about the line $y = x$ and ends up with its base along the y-axis (see right-hand diagram).



The diagram shows that $f(x, y) = (y, x)$.

Notice that the reflection in the line $y = x$ swaps the x- and y-coordinates of each point in the plane. (You may recall that this observation is used when finding the graph of the inverse of a real function by reflecting its graph in the line $y = x$.)

Activity 10 *Finding an algebraic specification for another reflection*

Find an algebraic specification for the reflection f in the line $y = -x$.

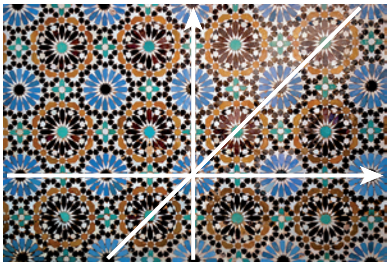


Figure 18 A reflection that leaves the appearance of the tiling unchanged

Reflections provide yet another tool for building up an understanding of repeated patterns. If the tiling from the Saadian Tombs is equipped with Cartesian coordinates as before, then the appearance of the tiling remains unchanged by reflection in the line $y = x$ (see Figure 18).

Activity 11 *Identifying other reflections that leave the tiling pattern unchanged*

What other reflections fix the origin and leave the appearance of the tiling pattern in Figure 18 unchanged?

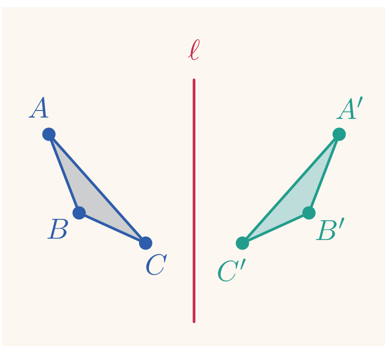


Figure 19 Reversing orientation

One characteristic of a reflection that distinguishes it from a rotation or a translation is that it reverses the *orientation* of figures. To see what this means, consider what happens to a triangle with vertices at A, B and C when it is reflected in a line ℓ, as shown in Figure 19. Under the reflection, the vertices A, B, C are mapped to the vertices A', B', C' of the image triangle on the right. The vertices A, B, C of the original triangle are arranged anticlockwise, but their images A', B', C' are arranged clockwise.

When a transformation changes the ordering of the vertices of a figure in this way, we say it reverses **orientation**.

Whereas reflections reverse the orientation of figures, rotations and translations preserve orientation. Actually, there is one other type of isometry that reverses orientation, namely the *glide-reflections*. You'll learn about these in the next section, after composite transformations have been introduced.

Symmetries

Finally in this section we introduce an important term that is used in connection with isometries.

You've already seen several examples of isometries that leave the appearance of the Saadian tomb tiling unchanged. You've also seen that the photograph of the Taj Mahal in Figure 16(a) looks the same after it has been reflected in a central line (well, almost).

Similarly, the regular pentagon in Figure 20 has several isometries that leave its appearance unchanged: in fact, any rotation through a multiple of $2\pi/5$ about its centre leaves it looking the same, as does any reflection in the perpendicular bisector of one of its sides. Such isometries are known as *symmetries*.

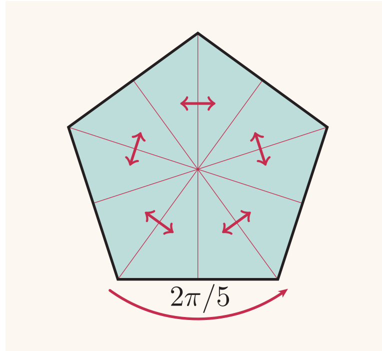


Figure 20 The symmetries of a regular pentagon

A **symmetry** of a pattern or figure is an isometry that leaves the appearance of the pattern or figure unchanged.

Depending on the pattern or figure, a symmetry can be any of the different types of isometries and, as we have seen in the case of the Saadian tomb tiling or the regular pentagon in Figure 20, a given figure can have many different symmetries.

3 Composite and inverse transformations

In this section you'll learn how you can combine two transformations by applying them one after another, and also how one transformation can undo the effect of another.

3.1 Composite transformations

The idea of a composite function was covered in MST124 and revised in MST125 Unit 1. It's a function that you obtain when you apply one function after another, and it's defined as follows.

Composite functions

Suppose that f and g are functions. The **composite function** $g \circ f$ is the function whose rule is

$$(g \circ f)(x) = g(f(x)),$$

and whose domain consists of all the values x in the domain of f such that $f(x)$ is in the domain of g.

So far in your studies, the focus has been on composites of real functions; however, the definition of a composite function is also useful in other contexts, such as in geometry when you want to consider the effect of one transformation followed by another. As an example, suppose that f and g are the transformations specified by

$$f(x, y) = (2x, y) \quad \text{and} \quad g(x, y) = (x, 2y).$$

Figure 21 illustrates how you can think of $g \circ f$ as a 'composite' processor that links f and g together by using the output from f as the input for g. For example, if the point $(2, 1)$ is fed into the composite processor, it is first processed by f to produce the intermediate point $f(2, 1) = (4, 1)$, and then this point is processed by g to give the final output point $g(4, 1) = (4, 2)$.

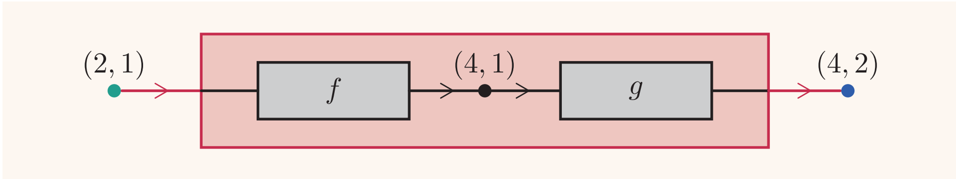


Figure 21 The composite processor $g \circ f$

You can feed any point of $\mathbb{R}^2$ into the processor, because its image under f lies in $\mathbb{R}^2$, the domain of g. So the domain of $g \circ f$ is also $\mathbb{R}^2$.

In fact, if you compose any two transformations that have domain and codomain $\mathbb{R}^2$, then by the same reasoning you'll obtain a composite transformation with domain $\mathbb{R}^2$. You can take this for granted from now on.

The two-stage diagram in Figure 22 illustrates the effect of the composite transformation $g \circ f$ above on the unit square. Under f, the square is first scaled (that is, stretched) in the x-direction by the factor 2 to give the rectangle shown in the middle. This rectangle is then scaled by the factor 2 in the y-direction to give the final image shown on the right.

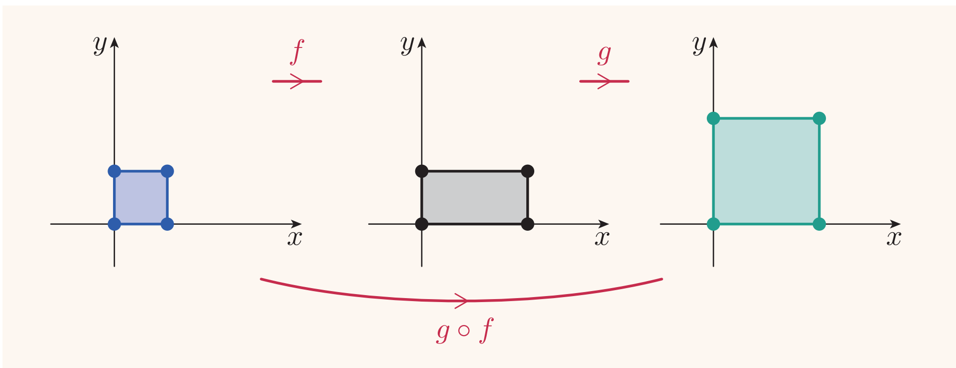


Figure 22 The effect of the composite transformation $g \circ f$

Overall, it appears that the unit square has simply been enlarged, giving a square with sides of length 2. To check this, let's find an algebraic expression for the rule of $g \circ f$.

Example 7 *Composing transformations algebraically*

Find an algebraic specification for the composite $g \circ f$ of the transformations f and g discussed above.

Solution

Calculate the effect that $g \circ f$ has on a general point (x, y) in its domain $\mathbb{R}^2$. Remember that $f(x, y) = (2x, y)$ and $g(x, y) = (x, 2y)$.

The composite transformation $g \circ f$ is given by

$$\begin{aligned}(g \circ f)(x, y) &= g(f(x, y)) \\ &= g(2x, y) \\ &= (2x, 2y).\end{aligned}$$

You can see from the algebraic specification for $g \circ f$ found in Example 7 that the composite transformation doubles the distance between each point of the plane and the origin. In particular, the sides of the unit square double in length, which agrees with what you observed from the two-stage diagram in Figure 22.

From now on in this unit, we will often refer to a composite transformation simply as a *composite*.

In Example 7, the transformations f and g are not isometries and nor is the composite $g \circ f$. However, if you compose two isometries, then you'll get another isometry. To see this, suppose f and g are isometries. If P and Q are points in the domain of f then, by the definition of an isometry, the distance between P and Q is the same as the distance between $f(P)$ and $f(Q)$. This in turn is the same as the distance between $g(f(P))$ and $g(f(Q))$. Overall, therefore, the distance between P and Q is the same as the distance between $(g \circ f)(P)$ and $(g \circ f)(Q)$. It follows that $g \circ f$ is an isometry.

A composite of two isometries is an isometry.

Many of the composites that you'll be working with later in this unit involve transformations that are not isometries, but for the rest of this section we'll confine our attention to composites of isometries. We'll look in turn at the effect of composing two rotations, two translations and two reflections, and then consider composites of different types of isometries, including a special composite of a reflection and a translation that leads to a kind of isometry you've not met before – a glide-reflection.

Composing rotations

The next activity asks you to explore the effect of composing two rotations about the same point.

Activity 12 *Composing rotations about the origin*

Let f and g be the rotations about the origin through $\pi/2$ and π, respectively, so that

$$f(x, y) = (-y, x) \quad \text{and} \quad g(x, y) = (-x, -y).$$

(a) Find the composite $g \circ f$.

(b) Draw a two-stage diagram to show the effect of $g \circ f$ on the triangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$.

(c) Describe $g \circ f$ geometrically.

The result in this activity is exactly what you'd expect. If you rotate through an angle θ about a point, and then rotate through an angle ϕ about the same point, the overall effect is to rotate through the angle $\theta + \phi$ about the point (see Figure 23). A moment's thought should also convince you that the order in which you carry out the two rotations does not matter.

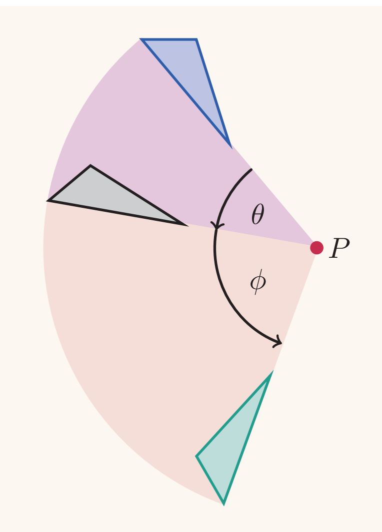


Figure 23 Composing rotations about a point

Composing rotations

When the rotation through an angle θ about a point P is composed with the rotation through an angle ϕ about the same point P (in either order), then the composite is the rotation through the angle $\theta + \phi$ about P.

It may interest you to know that, even if the rotations through θ and ϕ are about *different* points, P and Q say, then their composite is still a rotation through the angle $\theta + \phi$, but the centre of rotation is neither P nor Q. The only exception to this is if $\theta + \phi$ is a multiple of 2π, in which case the composite is a translation. We won't consider composites of rotations about different points any further in this section.

Composing translations

When composing two translations, it is helpful to think in terms of their associated vectors. For example, consider the translations

$$f(x, y) = (x + 2, y + 3) \quad \text{and} \quad g(x, y) = (x + 1, y - 2).$$

Figure 24 illustrates the composite $g \circ f$ in terms of the vectors associated with f and g.

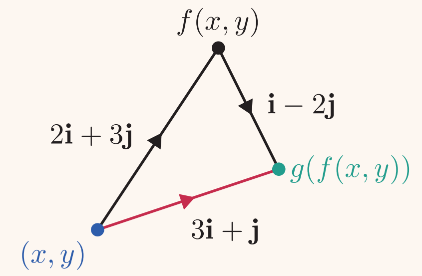


Figure 24 Using vectors to compose two translations

Since f has associated vector $2\mathbf{i} + 3\mathbf{j}$, its effect on an arbitrary point (x, y) is to displace it 2 units to the right and 3 units up, giving the intermediate point $f(x, y)$. Similarly, since g has associated vector $\mathbf{i} - 2\mathbf{j}$, its effect on $f(x, y)$ is to displace it 1 unit to the right and 2 units down, giving the final point $g(f(x, y))$. The overall displacement of (x, y) is specified by the vector sum

$$(2\mathbf{i} + 3\mathbf{j}) + (\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} + \mathbf{j}.$$

This must be the vector associated with the composite $g \circ f$, which is therefore specified by

$$(g \circ f)\,(x, y) = (x + 3, y + 1). \tag{1}$$

Of course, you can also work out an algebraic specification for the composite of two translations in the usual way, without reference to vectors.

Activity 13 *Composing translations without reference to vectors*

Check the result in equation (1) by finding an algebraic specification for $g \circ f$ without reference to vectors. Also give a specification for $f \circ g$. What do you notice?

The general result for composing translations using vectors is as follows.

Composing translations

The composite of two translations f and g (in either order) is the translation whose associated vector is the sum of the vectors associated with f and g.

Example 8 *Using vectors to compose translations*

Use vectors to find an algebraic specification for the composite $f \circ g$ of the following translations:

$$f(x, y) = (x - 2, y + 4) \quad \text{and} \quad g(x, y) = (x + 7, y - 5).$$

Solution

The vector associated with $f \circ g$ is the sum of the vectors associated with f and g.

The sum of the vectors associated with f and g is

$$(-2\mathbf{i} + 4\mathbf{j}) + (7\mathbf{i} - 5\mathbf{j}) = 5\mathbf{i} - \mathbf{j},$$

so

$$(f \circ g)(x, y) = (x + 5, y - 1).$$

Now try using vectors to find the composites in the following activity.

Activity 14 *Using vectors to compose translations*

Find $f \circ g$ for each of the following pairs of translations.

(a) $f(x, y) = (x + 3, y - 6)$ and $g(x, y) = (x - 3, y + 3)$

(b) $f(x, y) = (x - 4, y + 5)$ and $g(x, y) = (x + 2, y)$

(c) $f(x, y) = (x + 4, y - 5)$ and $g(x, y) = (x - 4, y + 5)$

Composing reflections

To see what happens when we compose two reflections, let's start by composing two reflections with fairly simple formulas that you saw earlier.

Example 9 *Composing reflections*

Let f be the reflection in the x-axis and g be the reflection in the line $y = x$, so that

$$f(x, y) = (x, -y) \quad \text{and} \quad g(x, y) = (y, x).$$

(a) Find the composite $g \circ f$.

(b) Draw a two-stage diagram to show the effect of $g \circ f$ on the triangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$.

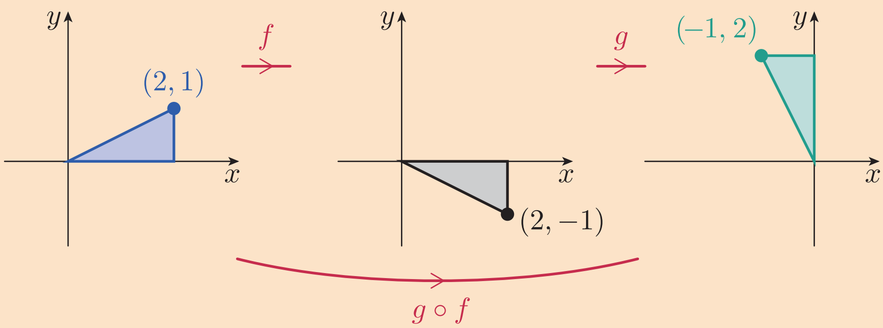
(c) Describe $g \circ f$ geometrically.

Solution

(a) The composite $g \circ f$ is given by

$$\begin{aligned}(g \circ f)(x, y) &= g(f(x, y)) \\ &= g(x, -y) \\ &= (-y, x).\end{aligned}$$

(b)



(c) The diagram in part (b) shows that the overall effect of $g \circ f$ is a rotation through the angle $\pi/2$ about the origin.

Now consider what happens if you compose the reflections in Example 9 in the reverse order.

Activity 15 *Composing reflections*

The reflections f and g in Example 9 were specified by

$$f(x, y) = (x, -y) \quad \text{and} \quad g(x, y) = (y, x).$$

(a) Find the composite $f \circ g$.

(b) Draw a two-stage diagram to show the effect of $f \circ g$ on the triangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$.

(c) Describe $f \circ g$ geometrically.

Example 9 and Activity 15 illustrate the fact that the order in which you compose two reflections is important. Indeed, although you have previously seen that the order in which you compose two translations or two rotations about the same point does not matter, in general, if f and g are two transformations, then $g \circ f$ need *not* be the same as $f \circ g$. We say that composition of transformations is *not commutative*.

Activity 16 *Investigating composites of reflections*

Open the *Composing reflections* applet. It displays a two-stage diagram that shows the effect of composing two reflections, f and g.

The first diagram shows a triangle. The middle diagram shows a faint copy of this triangle, together with its image under the first reflection, and the line in which it is reflected. Similarly, the third diagram shows a faint copy of the middle triangle, together with its image under the second reflection and the line in which it is reflected.

You can change the angle of inclination of each line of reflection, and you can opt to show either $g \circ f$ or $f \circ g$.

(a) Make sure that the 'Show angles' option is *not* selected and that the applet displays $g \circ f$. Also, ensure that f is the reflection in the x-axis (the line though the origin with angle of inclination 0) and g is the reflection in the line $y = x$ (the line through the origin with angle of inclination $\pi/4$). Check that the resulting two-stage diagram agrees with the one in Example 9.

(b) Now display $f \circ g$, and check that the two-stage diagram agrees with the one you obtained in Activity 15.

(c) Display $g \circ f$ again, and experiment with changing the angles of inclination of the two lines of reflection. Observe that in all cases the overall effect of $g \circ f$ is to *rotate* the original triangle about the origin.

(d) Now select the 'Show angles' option. Two angles are displayed:

- the angle from the line of reflection of f to the line of reflection of g
- the angle through which $g \circ f$ rotates the original triangle.

Each of these angles may be positive, negative or zero. Experiment with changing the angles of inclination of the two lines of reflection, and observe the relationship between the two angles displayed.

Activity 16 illustrates the following general fact.

Composing reflections

Let ℓ and m be lines that meet at a point P. Then the composite transformation formed from the reflection in ℓ followed by the reflection in m is the rotation about P through twice the angle from ℓ to m.

To see why this is true, consider Figure 25, in which we follow the images of the point Q through two reflections, first in the line ℓ and then in the line m.

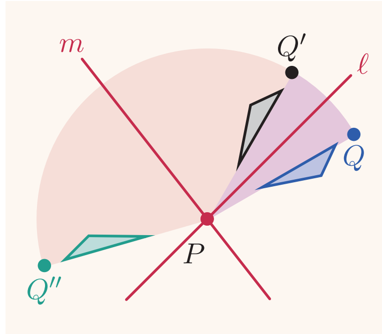


Figure 25 Composing reflections

The point Q is first reflected in the line ℓ to give the point Q'. This in turn is reflected again in the line m, to give the point Q'' (the coloured triangles in Figure 25 may help you to visualise the effects of these reflections).

Now the line ℓ bisects the angle QPQ', and the line m bisects the angle $Q'PQ''$. Taken together, these facts show that the angle QPQ'' must be twice the angle from the line ℓ to the line m. Moreover, the distances PQ, PQ' and PQ'' are all equal, so the composite of the two reflections has the effect of turning the line segment PQ about the point P until it lies on PQ''. Overall, therefore, the effect is that Q is mapped onto Q'' by rotating it about the point P through twice the angle from ℓ to m. This is the fact stated in the box above.

Note that it follows from the fact in the box above that if you swap the order in which two reflections are performed, then the angle of the resulting rotation changes to its negative.

The next activity asks you to consider a case where the lines of reflection ℓ and m do not meet at a point; that is, where they are parallel.

Activity 17 *Composing reflections in parallel lines*

Let f and g be the reflections specified by

$$f(x,y) = (-x,y) \quad \text{and} \quad g(x,y) = (2-x,y).$$

(a) Find the composite $g \circ f$.

(b) Draw a two-stage diagram to show the effect of $g \circ f$ on the triangle with vertices at $(0,0)$, $(1,0)$, $(1,2)$. Notice that this is not the same triangle as in Activity 15.

(c) Describe $g \circ f$ geometrically.

The lines of reflection in Activity 17 are parallel and are separated by a distance of 1 unit. The resulting composite is the translation through twice this distance in a direction perpendicular to the lines. This illustrates the following general result, which is justified geometrically in Figure 26.

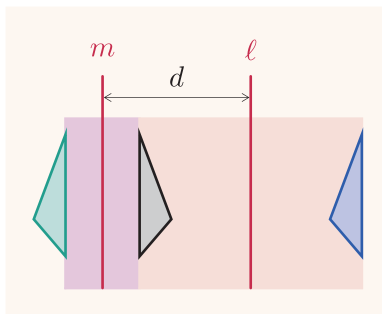


Figure 26 Composing reflections in parallel lines

Composing reflections in parallel lines

Let ℓ and m be parallel lines a distance d apart. Then the composite of the reflection in ℓ followed by the reflection in m is the translation through a distance $2d$ in the direction of the perpendicular from ℓ to m. Reversing the order in which the reflections are performed reverses the direction of the translation.

So the composite of two reflections is either a rotation or a translation. Intuitively this makes sense: the first reflection reverses orientation and the second reflection reverses it back again, so the composite is an isometry that preserves orientation. Hence it must be either a translation or a rotation. Which of these it is depends on the lines of reflection. If they intersect, then the composite fixes the point of intersection, and is therefore a rotation. If they don't intersect, then the composite is a translation.

Glide-reflections and other composite isometries

So far we have thought about composing two isometries of the same type. What happens if we compose two isometries of different types?

As an initial example, suppose f is the reflection in the line $y = x$ and g is the rotation through $\pi/2$ about the origin, so that

$$f(x, y) = (y, x) \quad \text{and} \quad g(x, y) = (-y, x). \tag{2}$$

The two-stage diagram in Figure 27 illustrates the effect that the composite transformation $g \circ f$ has on a right-angled triangle. Under f the triangle is reflected in the line $y = x$ to give the intermediate triangle shown in the middle. This triangle is then rotated by g through $\pi/2$ about the origin to give the final image shown on the right.

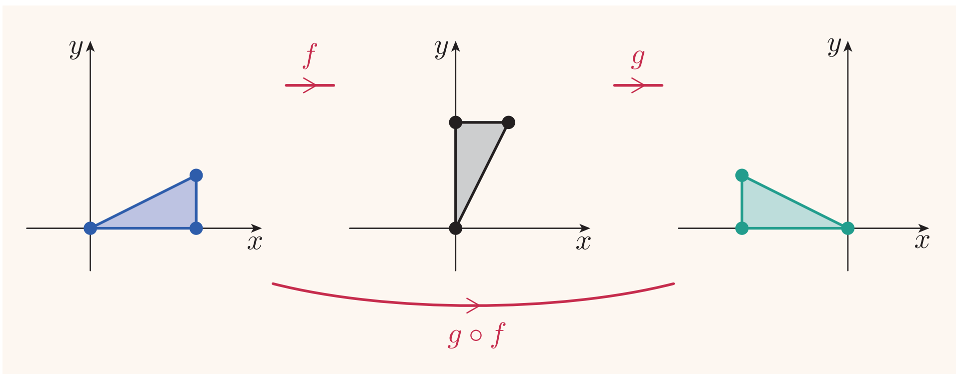


Figure 27 A reflection followed by a rotation

Overall, it appears that the original triangle has been reflected in the y-axis. (It's as though the rotation causes the original line of reflection $y = x$ to be realigned along the y-axis. We'll explore this idea further in Section 5.)

There are often clues to help you check your interpretation of a composite transformation. For example, you can consider orientation. Under the reflection f above, orientation is reversed. This reversed orientation is then preserved by the rotation g. It follows that the overall effect of the composite transformation is to reverse orientation, and this is consistent with $g \circ f$ being a reflection. As a further check, you can always calculate the composite transformation algebraically.

Activity 18 *Composing a reflection and a rotation*

Find the algebraic specification for the composite $g \circ f$ of the transformations f and g specified by equations (2).

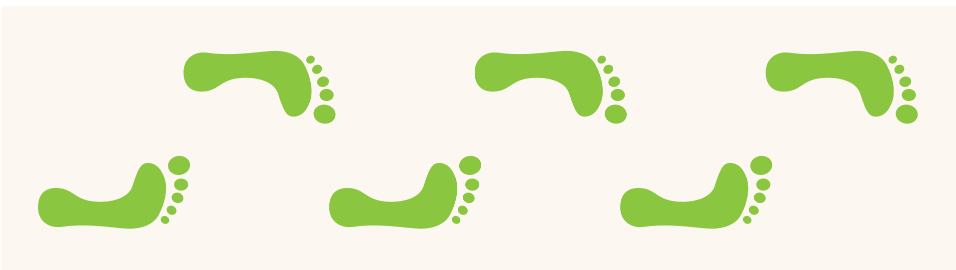
Whether you use the algebraic approach to find the composite of two isometries, or the geometric approach involving a two-stage diagram, will depend on the situation. Often it is a matter of preference.

You've seen above that, when you compose a reflection with a rotation, the resulting isometry reverses orientation. The same is true when you compose a reflection with a translation, because the reflection first reverses orientation, and this reversed orientation is then preserved by the translation, so overall the composite reverses orientation.

It's therefore tempting to conclude that the isometry that results when you compose a reflection with either a rotation or a translation is always a reflection, but is this necessarily the case? The next activity asks you to explore this question further.

Activity 19 *A new kind of isometry?*

In the following design, consider the process of superimposing the footprint of one step onto the footprint of the next.



(a) Try to describe this process as a composite of two isometries.

(b) Say whether or not your composite preserves orientation and explain why.

(c) Does your composite leave any points fixed?

So what kind of isometry is the composite in Activity 19? It can't be a rotation or a translation because it reverses orientation, and it can't be a reflection because there are no fixed points. It follows that it must be a type of isometry that we have not yet considered. It is known as a *glide-reflection*.

A **glide-reflection** in a line ℓ is a reflection in ℓ followed by a translation parallel to ℓ (or vice versa – the order of composition does not matter in a glide-reflection).

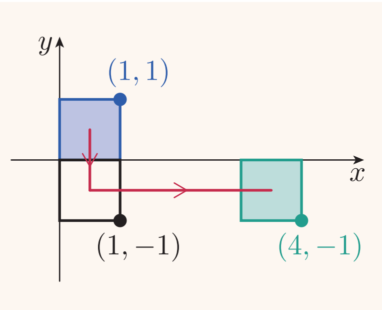


Figure 28 A glide-reflection in the x-axis

As an example, Figure 28 shows the effect of a particular glide-reflection on the unit square. It's the composite of a reflection in the x-axis followed by a translation 3 units to the right, and its specification is calculated in the next example.

Example 10 *Finding the specification of a glide-reflection*

Find the specification for the glide-reflection formed by a reflection in the x-axis followed by a translation 3 units to the right.

Solution

The glide-reflection is the composite $g \circ f$, where f and g are specified by

$$f(x, y) = (x, -y) \quad \text{and} \quad g(x, y) = (x + 3, y).$$

The rule of the composite $g \circ f$ is therefore given by

$$\begin{aligned}(g \circ f)(x, y) &= g(f(x, y)) \\ &= g(x, -y) \\ &= (x + 3, -y).\end{aligned}$$

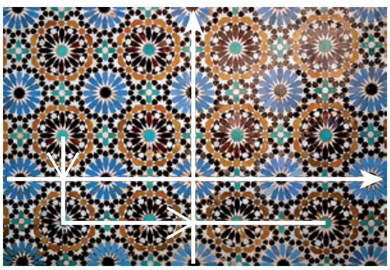


Figure 29 A glide-reflection that leaves the appearance of the tiling unchanged

Glide-reflections provide a further tool for building up an understanding of repeated patterns. If the tiling from the Saadian Tombs is equipped with Cartesian coordinates as before, then the appearance of the tiling is unchanged by the glide-reflection described in Example 10: the pattern is reflected in the x-axis and then translated 3 units to the right, as in Figure 29.

The next activity asks you to find the specification for a glide-reflection in a different direction that also leaves the appearance of the pattern unchanged.

Activity 20 *Finding the specification for another glide-reflection*

Find the specification for the glide-reflection formed by a reflection in the line $y = x$ followed by a translation 2 units to the right and 2 units up.

Now let's return to the question of what happens when a reflection is followed by a rotation, or when a rotation is followed by a reflection. You've already seen an example (Figure 27 and Activity 18) where a reflection followed by a rotation gives another reflection. However, this is not always the case. For example, Figure 30 shows the effect of a particular reflection f followed by a particular rotation g. (The reflection f is reflection in the vertical line through the centre of the base of the triangle shown at the left, and the rotation g is rotation through a half-turn about the origin.) The composite transformation $g \circ f$ is a glide-reflection.

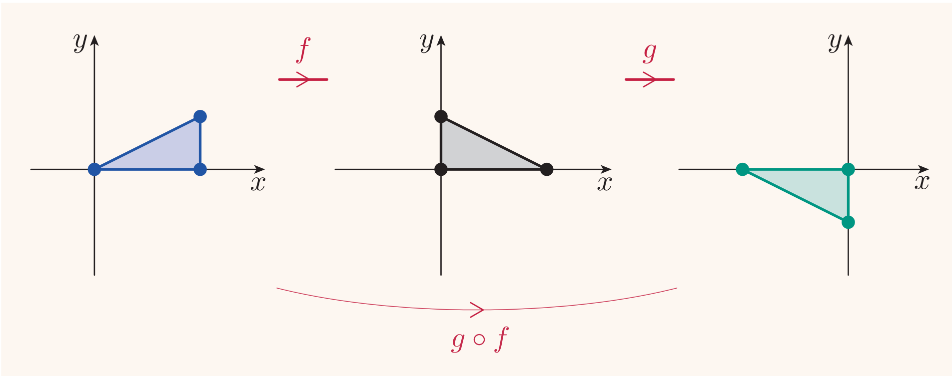


Figure 30 Another reflection followed by a rotation

In fact it can be shown that, if you choose a reflection at random and compose it with either a rotation or a translation, then the chances are that the composite will be a glide-reflection. Only in certain special cases, where the line of reflection passes through the centre of rotation (see Figure 27), or where the line of reflection is at a right angle to the direction of translation, will the composite turn out to be a reflection. Showing why this is true is beyond the scope of this unit.

Different types of isometries can be composed in many other ways. Here is one last example for you to consider.

Activity 21 *Composing a reflection with a glide-reflection*

What types of isometries would you expect to obtain when you compose a reflection with a glide-reflection?

Composing symmetries

In the previous section, you met the idea of a *symmetry* of a pattern or figure, which is an isometry that leaves the appearance of the pattern or figure unchanged.

The composite $g \circ f$ of any two symmetries f and g is also a symmetry. This is because the appearance of the pattern or figure is first left unchanged by f, and this unchanged appearance is then left unchanged by g, so overall $g \circ f$ leaves the appearance unchanged.

The composite of any two symmetries of a pattern or figure is itself a symmetry of the pattern or figure.

We often express this result by saying that the set of all symmetries of a pattern or figure is *closed* under composition. Studying this set of symmetries provides a powerful insight into the structure of the pattern or figure. It can be shown, for example, that there are only seventeen different ways of designing a repeated pattern such as the one found in the Saadian Tombs. Of course the motifs that make up the pattern can be changed indefinitely, but the different ways that the motifs can be fitted together to form a repeating pattern are limited to just seventeen. Proving this is beyond the scope of this unit.

On the module website there is a video exploring some of the repeated patterns that typically arise in cloth designs.

The study of symmetry forms part of a branch of mathematics known as *group theory*, which you will meet if you go on to study further pure mathematics (for example, by studying the Open University pure mathematics modules). It's a branch of mathematics that has many applications. For example, it is used in crystallography to analyse the structure of crystals such as the sodium chloride crystal illustrated in Figure 31. The symmetries involved in this application are symmetries of three-dimensional space rather than the plane, but the principles are much the same.

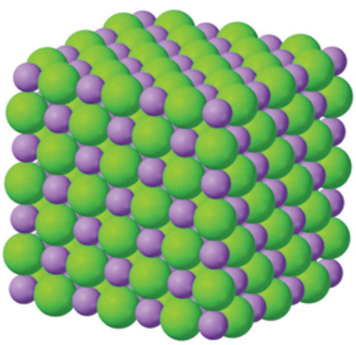


Figure 31 A model of a sodium chloride crystal

3.2 Inverse transformations

Inverse functions were covered in MST124 and revised in MST125 Unit 1. There the focus was on real functions, but the idea of an inverse function is also useful in geometrical contexts.

Remember that only *one-to-one* functions have inverses. A real function f is **one-to-one** if, for any two different real numbers x_1 and x_2 in its domain, $f(x_1)$ and $f(x_2)$ are also different. In other words, f maps exactly one element of its domain to each element of its codomain.

Figure 32 illustrates the idea of the inverse of a transformation f. Under f (see Figure 32(a)) each point (x, y) in the domain of f has an image $f(x, y)$. We'll denote the coordinates of this image point by (x', y'). As the point (x, y) ranges through the domain of f, the point (x', y') ranges through the image set (or range) of f.

If f is a one-to-one transformation, then there is by definition only one point (x, y) that is mapped to each point (x', y'). This makes it possible to reverse the direction of the mapping so as to obtain the so-called **inverse transformation** f^{-1} illustrated in Figure 32(b). The transformation f is then said to be **invertible**. The domain of f^{-1} is the image set of f, and each point (x', y') in this domain is mapped to a unique point (x, y) in the domain of f, which is also the image set of f^{-1}.

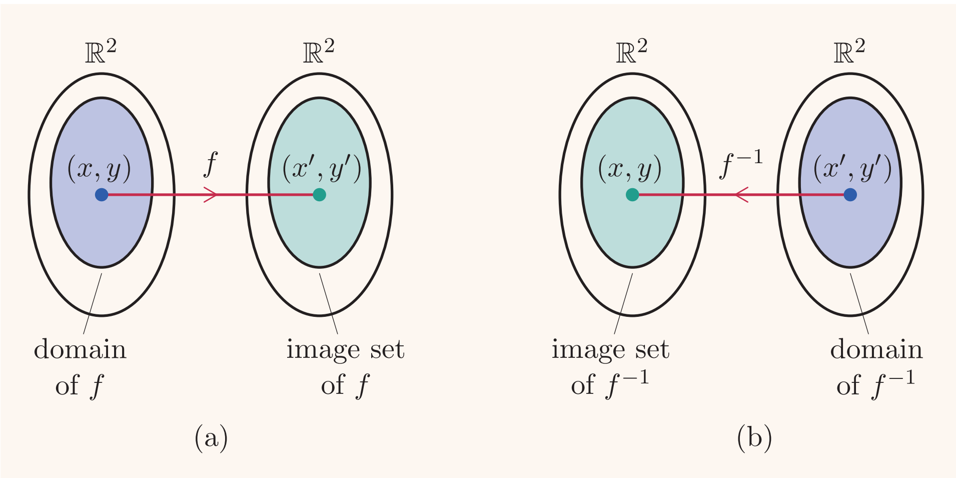


Figure 32 A transformation f and its inverse f^{-1}

But how do you find the rule of f^{-1}? You write down the equation $f(x, y) = (x', y')$ and solve it to obtain a formula for (x, y) in terms of (x', y'). This formula is the rule for f^{-1}.

Example 11 *Finding the rule of an inverse transformation*

Find the inverse of the transformation $f(x, y) = (3x, -y)$.

Solution

Consider a general point (x', y') and try to find a point (x, y) that is mapped by f to (x', y'); that is, solve the equation $f(x, y) = (x', y')$ to obtain (x, y) in terms of (x', y').

The equation $f(x, y) = (x', y')$ gives

$$(3x, -y) = (x', y').$$

This equation holds provided the corresponding coordinates are equal.

Equating coordinates gives

$$3x = x' \quad \text{and} \quad -y = y'.$$

By rearranging both equations, express each of x and y in terms of x' and y'.

Rearranging both equations then gives

$$x = \tfrac{1}{3}x' \quad \text{and} \quad y = -y',$$

so

$$f(\tfrac{1}{3}x', -y') = (x', y').$$

This equation shows that for each (x', y') in $\mathbb{R}^2$ there is a single point $(\frac{1}{3}x', -y')$ that is mapped by f to (x', y'). In other words, f is a one-to-one function with image set $\mathbb{R}^2$. It follows that f^{-1} exists and has domain $\mathbb{R}^2$.

Hence f^{-1} is specified by

$$f^{-1}(x', y') = (\tfrac{1}{3}x', -y'),$$

Now drop the dashes so that the rule is written in the usual form.

which, on replacing x' and y' by x and y, respectively, becomes

$$f^{-1}(x, y) = (\tfrac{1}{3}x, -y).$$

In general, when finding the rule for an inverse transformation you'll have to solve a system of two equations in two unknowns. As you'll see in Section 5, you can sometimes express these equations in matrix form and use the inverse of the matrix to solve the equations and so obtain the rule.

In the process of solving the equations, if you discover that there is *more than one solution* (x, y) for some values of (x', y'), then you'll know that f

isn't one-to-one, and consequently that it doesn't have an inverse. Also, if you discover that there are *no solutions* for some particular point (x', y'), then you'll know that this point isn't in the image set of f and therefore has to be excluded from the domain of f^{-1}. (All the transformations in the following activity have inverses with domain $\mathbb{R}^2$.)

Activity 22 *Finding the inverses of some transformations*

Find the inverse of each of the following transformations.

(a) $f(x, y) = (x - 4, y - 2)$ (b) $g(x, y) = (-y, x)$

(c) $h(x, y) = (3x, \frac{1}{2}y)$ (d) $k(x, y) = (x + 3y, -y)$

Inverting isometries

In the case of an isometry it is often easiest to think about its inverse geometrically. For example, the translation

$$f(x, y) = (x + 2, y - 3)$$

maps points 2 units to the right and 3 units down. So to undo the effect of f you'll need the translation that maps points 2 units to the left and 3 units up. That is,

$$f^{-1}(x, y) = (x - 2, y + 3).$$

In particular, for translations you can think of this process in terms of vectors. The vector associated with f^{-1} is the negative of the vector associated with f.

Inverse of a translation

The inverse of a translation $f(x, y) = (x + a, y + b)$ is the translation $f^{-1}(x, y) = (x - a, y - b)$. Its associated vector is the negative of the vector associated with f.

Similarly, to undo the effect of a rotation through an angle θ about a point, you'll need a rotation through the angle $-\theta$ about the point. For example, the transformation

$$f(x, y) = (-y, x)$$

is the rotation through $\pi/2$ about the origin, so its inverse is the rotation through $-\pi/2$ about the origin, that is,

$$f^{-1}(x, y) = (y, -x).$$

Inverse of a rotation

The inverse of the rotation through an angle θ about a point is the rotation through the angle $-\theta$ about the point.

Reflections are even easier. A moment's thought should convince you that any reflection is its own inverse; that is, it is **self-inverse**.

Inverse of a reflection

Every reflection is self-inverse.

By using the observations in these boxes, you can write down the rules for the inverses of many isometries directly.

Activity 23 *Finding the inverses of some isometries*

Write down the inverse of each of the following isometries.

(a) $f(x, y) = (-x, -y)$ (b) $g(x, y) = (x + 1, y - 7)$

(c) $h(x, y) = (y, x)$

Even if you can't see how to undo the effect of a given isometry, you'll still know that the inverse exists. This is because of the following result.

Every isometry is a one-to-one transformation, and therefore has an inverse.

You can see this by thinking about what would happen if an isometry mapped a pair of different points in its domain to a single point in its codomain. The distance between the pair of points would be a positive number, and yet the distance between their images would be zero, which cannot happen under an isometry. It follows that every point in the codomain of an isometry is the image of at most one point in its domain.

Moreover, if an isometry is a symmetry of a pattern or figure, then its inverse is also a symmetry. This is because a symmetry doesn't alter the appearance of the pattern (or figure), so undoing the symmetry won't alter the appearance either.

The inverse of a symmetry of a pattern or figure is a symmetry of the same pattern or figure.

4 Linear transformations

So far we've concentrated on one type of transformation, namely the isometries. Now you're going to meet another type, the linear transformations. You'll see that some isometries are linear transformations, while others are not. You'll also see many examples of linear transformations that are not isometries. The Venn diagram in Figure 33 illustrates the relationship between the two types of transformations, thought of as subsets of the set of all transformations of the plane.

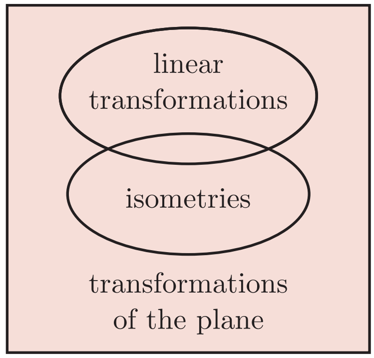


Figure 33 The sets of isometries and linear transformations overlap

4.1 What is a linear transformation?

To help answer this question, let's start by looking again at the way we write down the rule of a transformation. Up to now we've done this in terms of Cartesian coordinates, but in this section you'll see that it's often more convenient to express the rule of a transformation in terms of position vectors.

Remember that the position vector of a point $P(x, y)$ is the displacement vector $\overrightarrow{OP}$, where O is the origin, as shown in Figure 34. The components of $\overrightarrow{OP}$ have the same values as the coordinates of P:

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

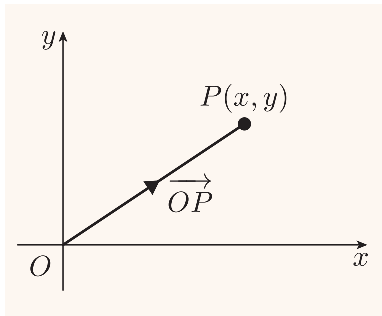


Figure 34 The position vector $\overrightarrow{OP}$

For this reason, pairs of real numbers in $\mathbb{R}^2$ are sometimes interpreted as the coordinates of a point, and sometimes as the components of the position vector of the point. Which interpretation is most useful depends on the context – both identify the same point in $\mathbb{R}^2$.

Many of the transformations that you've met so far are specified by a rule of the form

$$f(x, y) = (ax + by, cx + dy), \tag{3}$$

where a, b, c and d are real constants. For example, if $a = d = 0$ and $b = c = 1$, then $f(x, y) = (y, x)$, which is the rule for the reflection in the line $y = x$.

It's helpful to express equation (3) in the form of a matrix product. Suppose we let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the 2×2 matrix whose elements are the constants in the rule for f, and

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

be the position vector of the point (x, y), in column vector form. Then you'll remember that

$$\mathbf{Ax} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

and if you compare this with equation (3) you'll see that the matrix product enables us to express the rule for f in the column vector form

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

For example, the rule for reflection in the line $y = x$ is

$$f(x, y) = (y, x) = (0x + 1y, 1x + 0y),$$

so an equivalent column vector form of the rule is $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Transformations that can be written in the form of equation (3), and hence in the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, are called *linear transformations*.

A **linear transformation** of the plane $\mathbb{R}^2$ is a transformation whose rule is of the form

$$f(x, y) = (ax + by, cx + dy),$$

where a, b, c and d are real constants, or equivalently of the form

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We say that f is **represented** by $\mathbf{A}$, and that $\mathbf{A}$ is the **matrix of** f.

Example 12 *Finding the matrix of a linear transformation*

Find the matrix $\mathbf{A}$ of the linear transformation $f(x, y) = (-y, x)$. (Remember that f is the rotation through $\pi/2$ about the origin.)

Solution

Write f in the form $f(x, y) = (ax + by, cx + dy)$, and hence write down its matrix.

Since $f(x, y) = (0x - 1y, 1x + 0y)$, it follows that $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Activity 24 *Finding the matrices of some linear transformations*

Find the matrix of each of the following linear transformations.

(a) $f(x, y) = (y, -x)$ (rotation through $3\pi/2$ about the origin)

(b) $g(x, y) = (2x, y)$ (scaling in the x-direction by the factor 2)

(c) $h(x, y) = (x, y)$ (the identity transformation)

(d) $k(x, y) = (x + 2y, y)$ (you haven't met this type of transformation before – it's a *shear*, which you'll meet later in this section)

Having found the matrix of a linear transformation, you can use it to calculate the images of points under the transformation. Just write the position vector of the point as a column vector, and multiply it by the matrix of the transformation. For example, to find the image of the point $(2, 3)$ under the transformation in Activity 24(d), you calculate

$$k\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}8\\3\end{pmatrix}.$$

In terms of coordinates, this tells you that the image is $k(2, 3) = (8, 3)$.

Activity 25 *Finding the images of points under a linear transformation*

Let f be the linear transformation represented by the matrix $\begin{pmatrix}2 & 6\\3 & 5\end{pmatrix}$.

Find the image of each of the following points under f.

(a) $(2, -1)$ (b) $(0, 0)$ (c) $(1, 0)$ (d) $(0, 1)$

A couple of things may have struck you when you were doing Activity 25. The first is that the images (in vector form) of $(1, 0)$ and $(0, 1)$ under f are the columns of the matrix of f. This holds for all linear transformations, because

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}.$$

This result is worth remembering.

Images of $(1, 0)$ and $(0, 1)$

If f is the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}, \quad \text{then} \quad f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \quad \text{and} \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}.$$

Conversely, if f is the linear transformation such that $f(1, 0) = (a, c)$ and $f(0, 1) = (b, d)$, then the matrix of f is $\mathbf{A}$.

The second thing that you may have noticed in Activity 25 is that the origin remains fixed under the linear transformation f. Again, this is true for all linear transformations, because multiplying the zero vector $\mathbf{0}$ by any matrix $\mathbf{A}$ gives $\mathbf{0}$. That is,

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

A linear transformation leaves the origin fixed.

This fact provides a useful way to show that a transformation is *not* linear. For example, it tells us that translations and glide-reflections are not linear transformations. They move all points of the plane, and so certainly do not fix the origin.

Example 13 *Investigating whether transformations are linear*

For each of the following transformations, either find its matrix or explain why it is not linear.

(a) f is the reflection in the line $x = 1$

(b) $g(x, y) = (2x - y, x)$

(c) $h(x, y) = (2 - y, x)$

Solution

(a) Check whether f leaves the origin fixed.

The reflection f maps the origin to the point $(2, 0)$, so it does not fix the origin. It follows that f is not a linear transformation.

(b) This transformation seems to be of the form $g(x, y) = (ax + by, cx + dy)$.

$g(x, y) = (2x - 1y, 1x + 0y)$, so the matrix of g is $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

(c) $h(0, 0) = (2, 0) \neq (0, 0)$, so h is not a linear transformation.

You need to be careful when using the approach in Example 13. If you can show that a transformation doesn't fix the origin, then you'll know that it is *not* a linear transformation. However, showing that a transformation fixes the origin isn't enough to show that it *is* a linear transformation. For example, the transformation $f(x, y) = (x, y^2)$ fixes the origin, but it isn't linear because it can't be written in the form $f(x, y) = (ax + by, cx + dy)$.

Activity 26 *Investigating whether transformations are linear*

For each of the following transformations, either find its matrix or explain why it is not linear.

(a) f is the reflection in the x-axis

(b) g is the rotation through π about the point $(1, 0)$

(c) $h(x, y) = (2, -1)$

(d) $k(x, y) = (0, 0)$

The solution to Activity 26(d) shows that the transformation

$$k(x, y) = (0, 0)$$

is a linear transformation whose matrix is the **zero matrix** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This transformation is known as the **zero transformation**. It is quite different from any of the other transformations we've been discussing. For example, unlike the isometries, which are all one-to-one transformations, the zero transformation maps the entire plane to a single point. We'll return to such transformations later in this section, but for now we're going to concentrate on one-to-one linear transformations.

It's easy to check whether a linear transformation is one-to-one by looking at the determinant of its matrix. Remember that the *determinant* $\det \mathbf{A}$ of the 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the real number $ad - bc$.
(Determinants were covered in MST124 and revised in MST125 Unit 1.)

Identifying one-to-one linear transformations

A linear transformation f with matrix $\mathbf{A}$ is one-to-one if $\det \mathbf{A} \neq 0$ and is not one-to-one if $\det \mathbf{A} = 0$.

If you'd like to see why this result is true, have a look at the argument at the end of this subsection.

Activity 27 *Deciding whether linear transformations are one-to-one*

Decide which of the following linear transformations are one-to-one.

(a) $f(x, y) = (x + 2y, 2x + 4y)$ (b) $g(x, y) = (2x - 3y, 4x + y)$

(c) $h(x, y) = (2x - 4y, -x + 2y)$

An important characteristic of one-to-one linear transformations is that they map lines to lines. Transformations with this property are said to preserve **linearity**.

One-to-one linear transformations preserve linearity

One-to-one linear transformations map lines to lines.

In fact, you can use the matrix of the transformation to find the image of any given line under a one-to-one linear transformation. There's more about this below, but often we're just interested in the observation that a one-to-one linear transformation always maps lines to lines.

One-to-one linear transformations also have another related property. If f is a one-to-one linear transformation, and ℓ and m are parallel lines, then $f(\ell)$ and $f(m)$ are also parallel lines. Transformations with this property are said to preserve **parallelism**. This result is true because any intersection point of $f(\ell)$ and $f(m)$ would have to be the image under f of a point that lies on both ℓ and m. This can't happen if ℓ and m are parallel and f is one-to-one.

One-to-one linear transformations preserve parallelism

One-to-one linear transformations map parallel lines to parallel lines.

The rest of this subsection is optional. If you want to, you can now go straight to the beginning of Subsection 4.2, where you'll learn more about the behaviour of linear transformations. However, if you'd like to see some arguments in support of the results stated in the boxes above, then read on.

First, we'll look at why a linear transformation f with matrix $\mathbf{A}$ is one-to-one if $\det \mathbf{A} \neq 0$, and not one-to-one if $\det \mathbf{A} = 0$.

Remember that, if $\det \mathbf{A} \neq 0$, the matrix $\mathbf{A}$ has an *inverse* $\mathbf{A}^{-1}$ with the property that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix. Multiplying the position vector of a point in the plane by $\mathbf{I}$ leaves the position vector unchanged.

Now suppose that $\det \mathbf{A} \neq 0$, and that there are points with position vectors $\mathbf{u}$ and $\mathbf{v}$ such that $f(\mathbf{u}) = f(\mathbf{v})$. If f is one-to-one, then we would expect to be able to show that $\mathbf{u} = \mathbf{v}$, since two different points can never be mapped to the same point under a one-to-one transformation.

The argument goes like this. Since $f(\mathbf{u}) = f(\mathbf{v})$, we have

$$\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v},$$

and multiplying both sides by $\mathbf{A}^{-1}$ then gives

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}^{-1}\mathbf{A}\mathbf{v},$$

which is the same thing as

$$\mathbf{I}\mathbf{u} = \mathbf{I}\mathbf{v}.$$

But this means that $\mathbf{u} = \mathbf{v}$, so we have now proved that f is one-to-one if $\det \mathbf{A} \neq 0$.

We still have to show that, if $\det \mathbf{A} = 0$, then f is *not* one-to-one. Now f will certainly not be one-to-one if we can show that it maps more than one point to the origin. This will be the case if the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{4}$$

has a solution other than $\mathbf{x} = \mathbf{0}$ (remember that $\mathbf{x} = \mathbf{0}$ *must* be a solution, because every linear transformation fixes the origin).

Now

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix},$$

so if the matrix equation (4) holds, then we obtain the following two simultaneous linear equations:

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$

It was shown in MST124 Unit 9 that, if $\det \mathbf{A} = ad - bc = 0$, then these equations either have no solutions at all, or else they have infinitely many solutions. These are the only possibilities. Since we know that there is at least one solution (the zero vector $\mathbf{x} = \mathbf{0}$), it follows that there are infinitely many. In other words, f maps infinitely many points to the origin, so f is not one-to-one. This completes the argument.

We also stated above that linear transformations map lines to lines, and that you can use the matrix of the transformation to find the image of any given line. The following box explains how.

Image of a line

Let $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ be a one-to-one linear transformation. Then the image of the line $rx + sy = t$ under f is the line $r'x + s'y = t$, where r' and s' are given by the matrix equation

$$\begin{pmatrix} r' & s' \end{pmatrix} = \begin{pmatrix} r & s \end{pmatrix} \mathbf{A}^{-1}.$$

First, a word about the equation of a line. You may be more familiar with the equation of a line being expressed in the form $y = mx + c$, rather than in the form $rx + sy = t$ used in the box above. However, you should be able to see that, provided $s \neq 0$, the equation $rx + sy = t$ can be rearranged to give

$$y = -\left(\frac{r}{s}\right)x + \frac{t}{s},$$

which is now in the familiar form, with $m = -r/s$ and $c = t/s$. If $s = 0$, then the equation $rx + sy = t$ is equivalent to $x = t/r$ (provided $r \neq 0$), and is therefore the equation of a vertical line through the point $(t/r, 0)$. (If $r = s = 0$, then $rx + sy = t$ isn't the equation of a line.)

Now suppose that (x', y') is the image of a general point (x, y) under f, so that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then with $\begin{pmatrix} r' & s' \end{pmatrix}$ defined as in the box above, we have

$$\begin{aligned}
\begin{pmatrix} r' & s' \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} r' & s' \end{pmatrix}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} r & s \end{pmatrix}\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} r & s \end{pmatrix}\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} r & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

Multiplying out the matrices then gives

$$r'x' + s'y' = rx + sy,$$

so each point (x, y) on the line $rx + sy = t$ is mapped to a point (x', y') on the line $r'x + s'y = t$, and conversely each point (x', y') on the line $r'x + s'y = t$ is the image of a point (x, y) on the line $rx + sy = t$. This is the result we wanted.

The following example and activity illustrate the result in the box above. Remember that they are optional.

Example 14 *Finding the image of a line under a one-to-one linear transformation*

Find the image of the line $2x + 3y = 1$ under the linear transformation f represented by the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$.

Solution

First find the inverse of $\mathbf{A}$ and then multiply it by the matrix $\begin{pmatrix} 2 & 3 \end{pmatrix}$ formed from the coefficients of the equation of the line.

We have

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \end{pmatrix}.$$

The image of the line $2x + 3y = 1$ is therefore the line with the equation $-x + 2y = 1$.

Activity 28 *Finding the image of a line under a one-to-one linear transformation*

Find the image of the line $x + 2y = 1$ under the linear transformation f represented by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

4.2 Behaviour of linear transformations

Having seen that linear transformations always fix the origin and that one-to-one linear transformations preserve parallelism, you are now in a position to investigate their behaviour in more detail. A useful starting point is to consider the effect they have on the unit square.

Activity 29 *The image of the unit square under a linear transformation*

Let f be the linear transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Illustrate the effect that f has on the unit square.

As you saw in Activity 29, finding the image of the unit square under a linear transformation is fairly straightforward. The origin $(0, 0)$ is mapped to itself, and the points $(1, 0)$ and $(0, 1)$ are mapped to points whose position vectors are respectively the first and second columns of the matrix of the transformation. So you just need to work out the image of the remaining vertex $(1, 1)$.

The image of the unit square is a parallelogram. This is because the parallel sides of the unit square are mapped to parallel sides. So it's actually not even necessary to calculate the image of the vertex $(1, 1)$, since its coordinates are automatically determined by completing the image in the form of a parallelogram.

At first sight, this parallelogram appears to provide only limited information about the transformation. However, it actually conveys more than is immediately apparent. The key to unlocking this information is to equip the domain and codomain of the transformation with grids, as illustrated in Figure 35 for the transformation f from Activity 29.

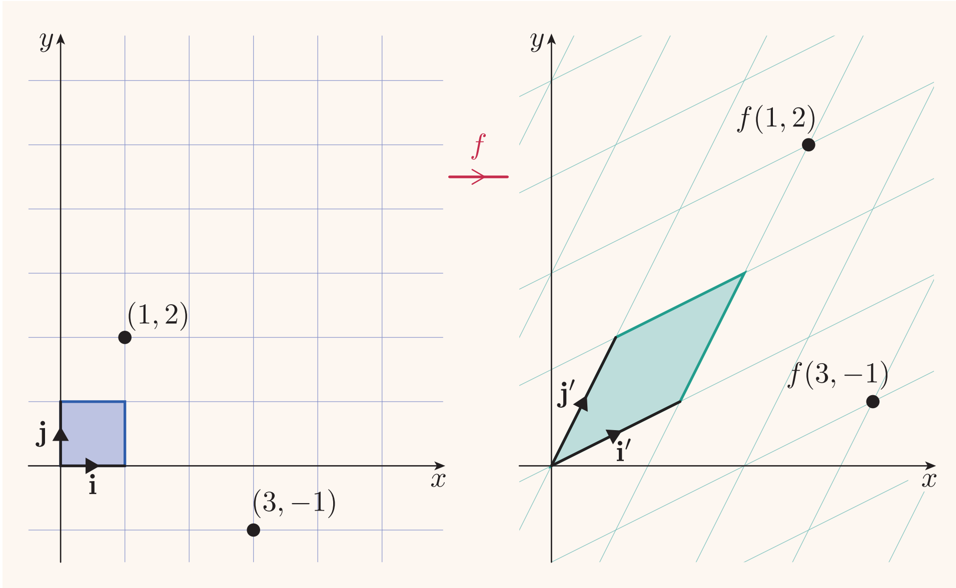


Figure 35 The unit grid and its image under f

The grid on the left in Figure 35 consists of two families of parallel lines, one in the direction of the Cartesian unit vector $\mathbf{i}$, the other in the direction of the Cartesian unit vector $\mathbf{j}$. These lines partition the plane into squares, each congruent to the unit square. We'll refer to this grid as the **unit grid**. The grid on the right consists of two families of parallel lines, one in the direction of the image $\mathbf{i}'$ of $\mathbf{i}$ under f, and the other in the direction of the image $\mathbf{j}'$ of $\mathbf{j}$ under f. These lines partition the plane into parallelograms, each congruent to the parallelogram that is the image of the unit square under f.

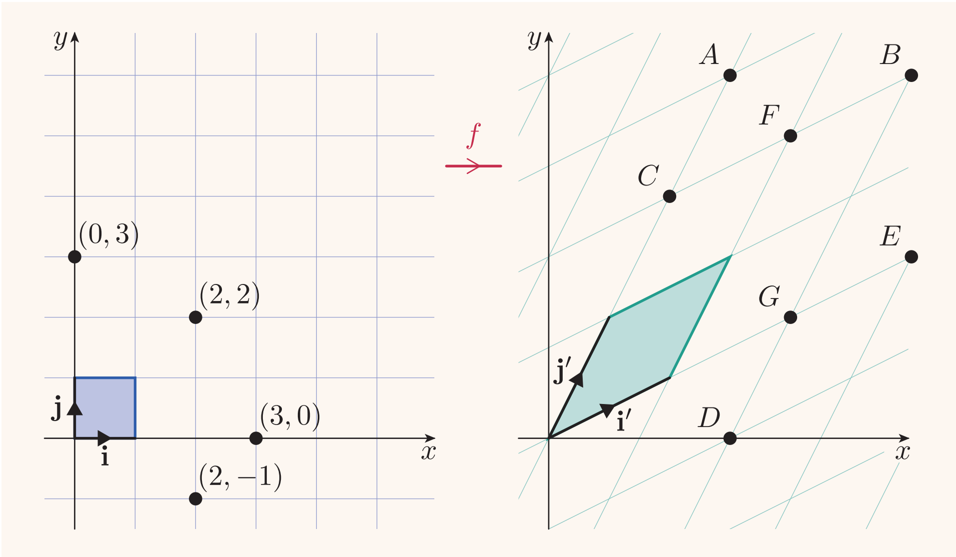
The individual squares or parallelograms into which a grid partitions the plane are known as **grid cells**. A grid is said to be **rectangular** if its grid cells are rectangles (or squares), and **skewed** otherwise.

As you may have guessed, the grid on the right in Figure 35 is the image under f of the unit grid on the left. In particular, each intersection point of the grid on the right is the image of the corresponding intersection point of the grid on the left. For example, the images $f(1,2)$ and $f(3,-1)$ of the points $(1,2)$ and $(3,-1)$ are located as indicated in Figure 35.

The next activity asks you to locate the images under f of some other intersection points on the unit grid.

Activity 30 *Finding images of intersection points on the unit grid*

The following figure illustrates the unit grid and its image under the linear transformation f from Activity 29.



Identify which of the points A, B, C, D, E, F or G is the image of each of the following points.

(a) $(0,3)$ (b) $(2,2)$ (c) $(2,-1)$ (d) $(3,0)$

So by using grids you can visualise where the image of each intersection point on the unit grid is located. In fact, you can locate the image of *any* point in the plane in the same way.

To see why this works, suppose that f is a linear transformation with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then f maps $\mathbf{i}$ to $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\mathbf{j}$ to $\begin{pmatrix} b \\ d \end{pmatrix}$, so using our previous notation $\mathbf{i}' = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\mathbf{j}' = \begin{pmatrix} b \\ d \end{pmatrix}$. A general point (x,y) in the plane, with position vector $x\mathbf{i} + y\mathbf{j}$, is therefore mapped to the point whose position vector is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = x\mathbf{i}' + y\mathbf{j}'.$$

So, whatever the values of x and y, f maps the point with position vector $x\mathbf{i} + y\mathbf{j}$ to the point with position vector $x\mathbf{i}' + y\mathbf{j}'$.

If you want to locate the positions of images with greater precision than can be achieved with the unit grid, then you can use a 'finer' grid that subdivides each grid cell into smaller cells. Figure 36, for example, shows the unit grid and its image subdivided, like a sheet of graph paper, so that each of the grid cells is partitioned into 25 smaller congruent cells.

By using fine enough grids you can get any degree of precision that you want. As an example, the image of the point $(1.4, 0.6)$ is shown in Figure 36.

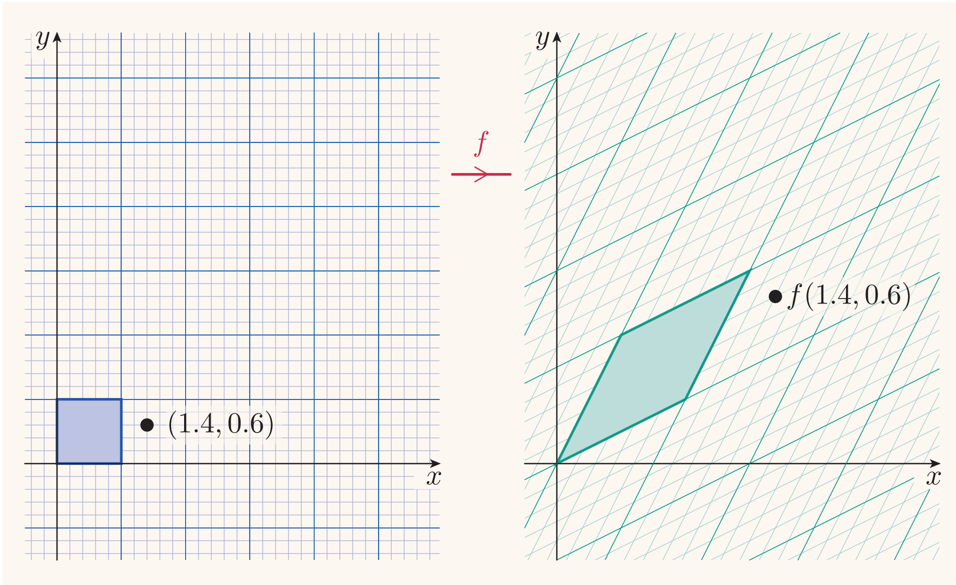


Figure 36 Refining the grids

An immediate consequence of these observations is that the one-to-one linear transformation f scales the areas of all figures by the same factor (see Figure 37). Indeed, if F is a figure that surrounds N grid squares, then the image of F will be a figure that surrounds the N corresponding parallelograms ($N = 6.1$ in Figure 37). The factor by which the area of F is scaled is therefore equal to the factor by which f scales the area of the unit square S. But the area of S is 1, so this factor is equal to the area of the parallelogram $f(S)$.

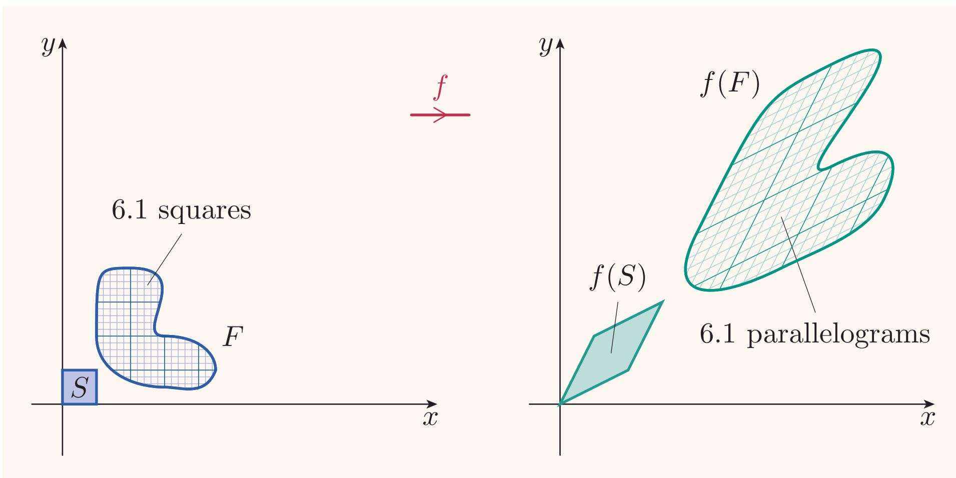


Figure 37 Scaling areas

Note also that a linear transformation may either preserve orientation, in which case the vertices of the parallelogram that is the image of the unit square appear in the same order as the vertices of the unit square, or else it may reverse orientation, in which case the vertices of the image parallelogram appear in the reverse order. These two possibilities are illustrated in the middle and right-hand diagrams in Figure 38.

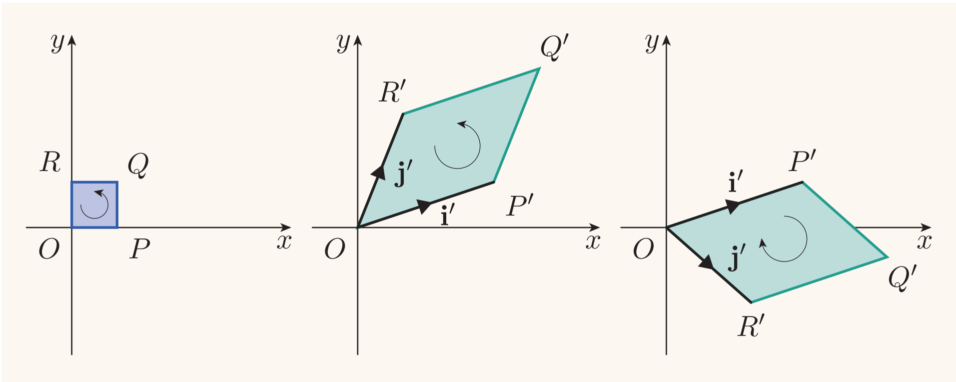


Figure 38 A linear transformation may either preserve or reverse orientation

In short, the behaviour of a one-to-one linear transformation is completely determined by the parallelogram that is the image of the unit square.

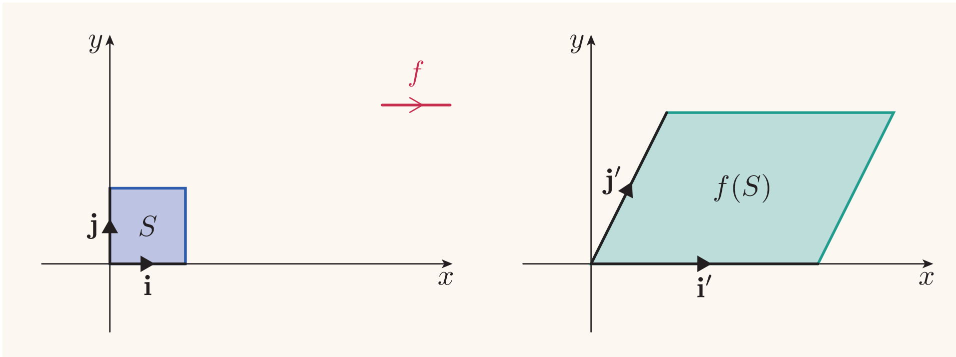
Behaviour of a one-to-one linear transformation f

- The unit grid is mapped to a grid consisting of two families of parallel lines that partition the plane into parallelograms, each congruent to the image $f(S)$ of the unit square S.
- One-to-one linear transformations either preserve or reverse orientation.
- Each point on the unit grid is mapped to the corresponding point on the image grid, as illustrated in Figure 36. Areas are scaled by a factor equal to the area of $f(S)$.

Before you can use the observation about areas above, you'll have to know a way of calculating the area of the parallelogram $f(S)$. The following activity asks you to do this for a special case where the base of the parallelogram lies along the x-axis.

Activity 31 *Calculating the area scaling factor of a one-to-one linear transformation*

Let f be the transformation represented by the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. The effect that f has on the unit square S is illustrated below. By calculating the area of the parallelogram $f(S)$, find the factor by which f scales areas. (Remember that the area of a parallelogram is base $\times$ height.)



In general, the parallelogram $f(S)$ may not lie along the x-axis. The method for calculating the area scaling factor of a general one-to-one linear transformation is explained in the box below.

Scaling of areas under a one-to-one linear transformation

Let f be the one-to-one linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then f scales areas by the factor

$$|\det \mathbf{A}| = |ad - bc|.$$

The orientation of figures is preserved if $\det \mathbf{A}$ is positive, and reversed if $\det \mathbf{A}$ is negative.

Let's see if we can explain why this result is true. You can skip this explanation if you like and go straight to Example 15, but you may find it interesting to read on.

For a linear transformation f represented by the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the area of the parallelogram $f(S)$ will still be equal to the length of its base multiplied by its perpendicular height, but we'll have to find a way to express this product in terms of the components of the vectors $\mathbf{i}'$ and $\mathbf{j}'$ that form the columns of $\mathbf{A}$ (see Figure 39).

One way to do this is to rotate the point (a, c) with position vector $\mathbf{i}'$ through $\pi/2$ about the origin to obtain the point N with coordinates $(-c, a)$, whose position vector we'll call $\mathbf{n}$. So $\mathbf{n}$ and $\mathbf{i}'$ lie at right angles to each other and have the same magnitude $|\mathbf{n}|$, which is equal to the length of the base of the parallelogram.

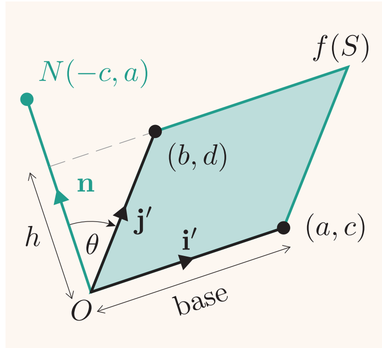


Figure 39 Finding the area of $f(S)$

The perpendicular height of the parallelogram, shown as h in Figure 39, is equal to the magnitude of $\mathbf{j}'$ multiplied by the absolute value of the cosine of the angle θ between $\mathbf{j}'$ and $\mathbf{n}$. (If θ is acute then $\cos\theta$ is positive, but if θ is obtuse then $\cos\theta$ is negative, so we need to take the absolute value of $\cos\theta$ to make sure we always end up with a positive value of h.)

It follows that

$$\text{area of } f(S) = \text{base} \times \text{height} = |\mathbf{n}| \times |\mathbf{j}'||\cos\theta| = |\mathbf{n} \cdot \mathbf{j}'|,$$

where $\mathbf{n} \cdot \mathbf{j}'$ is the scalar product of the vectors $\mathbf{n}$ and $\mathbf{j}'$.

Remarkably, the scalar product $\mathbf{n} \cdot \mathbf{j}'$ here turns out to be equal to $\det \mathbf{A}$:

$$\mathbf{n} \cdot \mathbf{j}' = \begin{pmatrix} -c \\ a \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = ad - bc = \det \mathbf{A}.$$

It follows that

$$\text{area of } f(S) = |\det \mathbf{A}|.$$

Furthermore, $\mathbf{n} \cdot \mathbf{j}'$ (and hence $\det \mathbf{A}$) is positive if the angle θ between $\mathbf{j}'$ and $\mathbf{n}$ is acute and negative if θ is obtuse. These two possibilities are illustrated in the middle and right-hand diagrams in Figure 40.

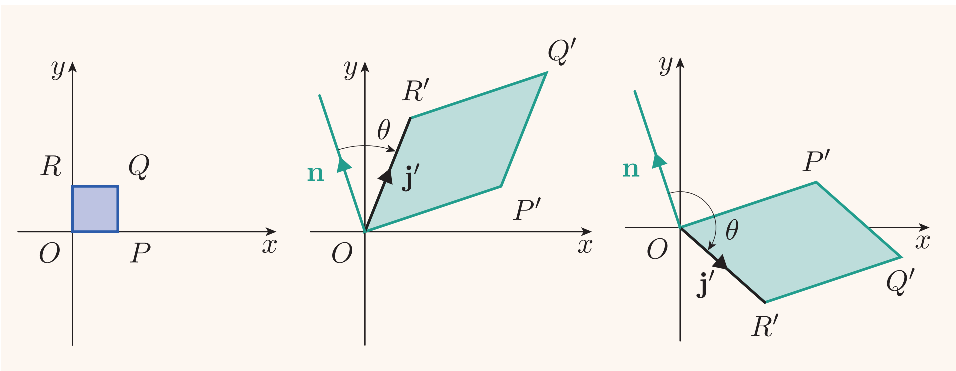


Figure 40 Preserving and reversing orientation

In the left-hand diagram in Figure 40, the vertices of the unit square are labelled O, P, Q and R in an anticlockwise direction. The corresponding images (in the middle and right-hand diagrams) are labelled O, P', Q' and R'. In the middle diagram, where the angle θ is acute, the labels appear in an anticlockwise direction, but in the right-hand diagram, where θ is obtuse, the labels appear in a clockwise direction. This tells you that f preserves orientation if $\det \mathbf{A}$ is positive and reverses orientation if $\det \mathbf{A}$ is negative.

Example 15 *Finding properties of a linear transformation*

Let f be the linear transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$. Find the factor by which f scales areas. Does f preserve orientation?

Solution

The matrix of f has determinant

$$\det \mathbf{A} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 3 \times 3 = -1.$$

So f scales areas by the factor $|-1| = 1$ (which means that areas are unchanged), and reverses orientation.

The linear transformation f in Example 15 leaves the areas of figures unchanged and reverses orientation, so you may be tempted to think that it is a reflection. However, that's not the case. This is because it changes the shapes of figures. You can easily see this because the scalar product of the column vectors $\mathbf{i}'$ and $\mathbf{j}'$ of the matrix is non-zero, and therefore the angle between them isn't a right angle. It follows that the image of the unit square cannot be a square.

Activity 32 *Finding properties of some linear transformations*

For the transformations represented by each of the following matrices, find the factor by which the transformation scales areas, and state whether it preserves orientation.

(a) $\begin{pmatrix} 4 & -2 \\ 3 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The next activity asks you to use the determinant of the matrix representing a linear transformation to calculate the area of a triangle that has one vertex at the origin. You'll see more general examples like this in Sections 5 and 6.

Activity 33 *Finding the area of a triangle*

Let f be the linear transformation that maps $(1, 0)$ to $(2, 5)$ and $(0, 1)$ to $(3, 1)$. Write down the matrix that represents f and calculate its determinant. Hence calculate the area of the triangle with vertices at $(0, 0)$, $(2, 5)$, $(3, 1)$.

4.3 Some standard types of linear transformations

Having explored how one-to-one linear transformations behave in general, let's now focus on some special types of linear transformations whose behaviour can easily be predicted from the elements of their matrices.

Horizontal and vertical scalings that fix the origin

As a first example, consider the linear transformation f represented by the diagonal matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Remember that a **diagonal matrix** is a square matrix such that each entry not on its leading diagonal is zero.

The transformation f maps the point $(1,0)$ to the point with position vector $\mathbf{i}' = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (the first column of $\mathbf{A}$) and it maps the point $(0,1)$ to the point with position vector $\mathbf{j}' = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ (the second column of $\mathbf{A}$), so the unit grid is mapped as shown in Figure 41.

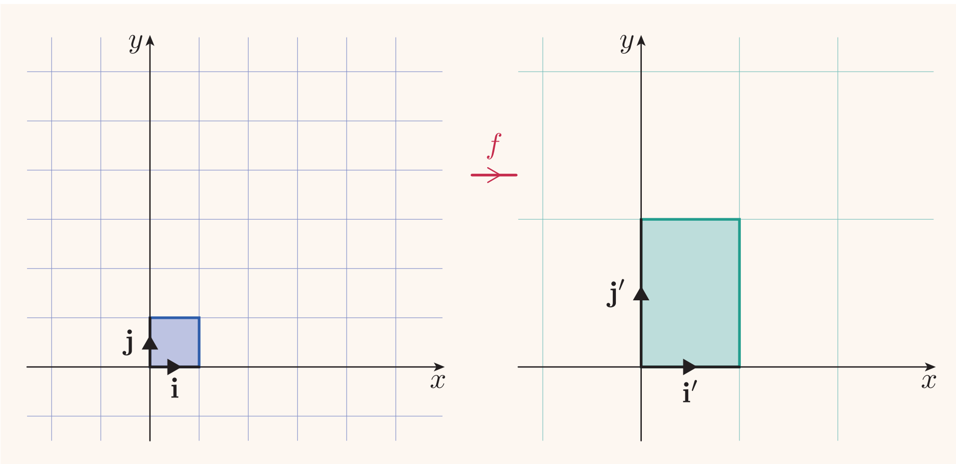


Figure 41 The effect of a scaling

Here the image grid is rectangular rather than skewed. Distances in the horizontal direction are scaled by the factor 2, and distances in the vertical direction are scaled by the factor 3. In particular, the unit square is mapped to the rectangle with vertices at $(0,0)$, $(2,0)$, $(2,3)$, $(0,3)$. The transformation therefore increases areas by the factor $2 \times 3 = 6$, the product of the diagonal elements of $\mathbf{A}$. Of course, this result can be obtained directly from the matrix by calculating

$$|\det \mathbf{A}| = |2 \times 3 - 0 \times 0| = 6.$$

(It is worth mentioning a general result that this calculation illustrates: the determinant of a diagonal matrix is the product of its diagonal entries.)

The linear transformation f is called a **scaling** because of the effect it has on distances in the horizontal and vertical directions. More specifically it is called a $(2,3)$-scaling, where the first number refers to the factor by which horizontal distances are scaled, and the second number refers to the factor by which vertical distances are scaled. These are the numbers that appear down the diagonal of the matrix, and they are known as the **horizontal scale factor** and **vertical scale factor**, respectively.

The next activity asks you to investigate the behaviour of scalings with various different horizontal and vertical scale factors.

Activity 34 *Investigating horizontal and vertical scalings*

Open the *Visualising linear transformations* applet.

(a) Select the 'Scaling' option, and set $k = 2$ and $l = 3$. Check that the diagram obtained matches Figure 41.

(b) Change the values of k and l to investigate the behaviour of scalings for various values of the diagonal matrix elements. In particular, investigate the effects of positive, negative and zero values of k and l.

Notice that the applet has an option that allows you to determine the effect on orientation.

The discussion above leads to the following definition.

A linear transformation represented by a diagonal matrix of the form

$$\begin{pmatrix} k & 0 \\ 0 & l \end{pmatrix}, \qquad \text{where } k \neq 0 \text{ and } l \neq 0,$$

is called a (k,l)-**scaling**.

The following properties of scalings were illustrated by Activity 34.

Properties of (k,l)-scalings

Under a (k,l)-scaling:

- the unit grid is not skewed
- horizontal distances are scaled by the factor $|k|$
- vertical distances are scaled by the factor $|l|$
- areas are scaled by the factor $|kl|$
- if kl is negative then orientation is reversed.

Horizontal and vertical scalings aren't the only kinds of scalings; you can scale in any two perpendicular directions. In fact, as you'll see in MST125 Unit 11, you can scale in directions that aren't perpendicular – but that's a story for another day.

Dilations that fix the origin

One case of scaling that deserves a special mention is where the diagonal elements of the matrix representing the scaling are equal; that is, where the matrix is

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},$$

for some number k. This matrix can be written as $k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The linear transformation represented by this matrix scales both horizontal and vertical distances by the same factor $k \neq 0$. Such a transformation is called a k-**dilation**. Figure 42 illustrates the 2-dilation specified by

$$f(\mathbf{x}) = 2\mathbf{I}\mathbf{x}.$$

Here, a general point with position vector $\mathbf{x}$ moves outwards to the point with position vector $2\mathbf{x}$, thereby doubling distances, irrespective of their direction. Areas are scaled by the factor $2^2 = 4$.

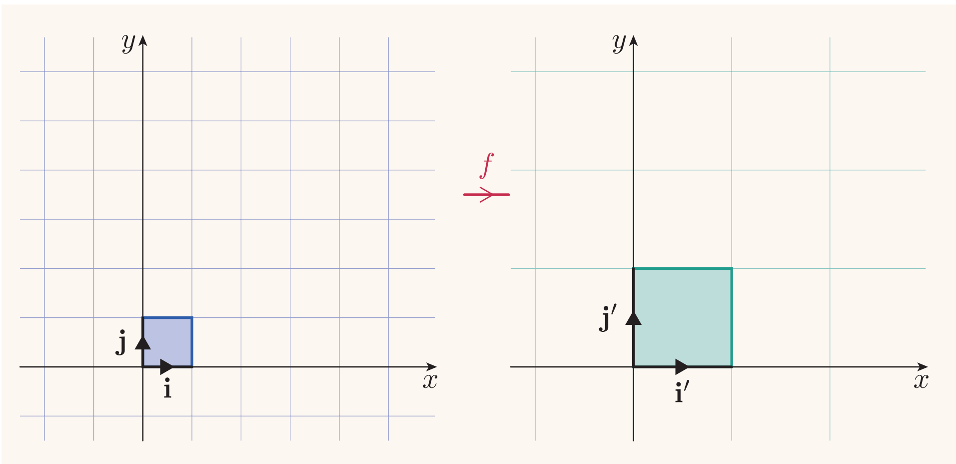


Figure 42 The effect of a 2-dilation

Activity 35 *Investigating dilations*

Use the *Visualising linear transformations* applet.

(a) Select the 'Dilation' option, and set $k = 2$. Check that the diagram obtained matches Figure 42.

(b) Now change the value of k and investigate the behaviour of dilations for various values of k. In particular, investigate the effects of positive, negative and zero values of k.

In general, the k-dilation represented by the matrix $k\mathbf{I}$ scales all distances by the factor $|k|$ and scales areas by the factor k^2. Orientation is preserved for all non-zero values of k.

Horizontal and vertical shears that fix the origin

Now consider the linear transformation f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Using the fact that the columns of $\mathbf{A}$ are the images under f of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, you can see that the effect of f on the unit grid is as shown in Figure 43.

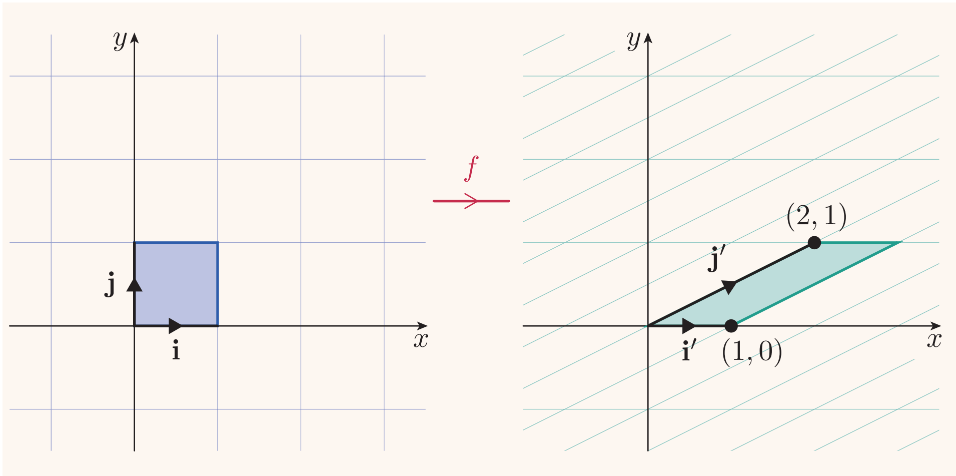


Figure 43 The effect of a shear

Under f each point (x, y) is mapped to the point with position vector

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ y \end{pmatrix};$$

that is, to the point $(x + 2y, y)$. So the effect of f is to displace each point horizontally by $2y$ units. All points on the x-axis remain fixed, while other points are displaced by amounts that are proportional to their y-coordinate, where the constant of proportionality is 2. Points that have a positive y-coordinate are mapped to the right, while those with a negative y-coordinate are mapped to the left. We say that f *shears* the plane about the x-axis with shear factor 2. Its effect on the unit square is a bit like pushing a stack of cards in the way illustrated in Figure 44.

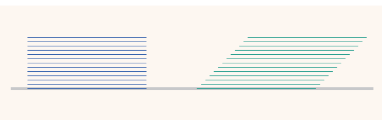


Figure 44 A sheared stack of cards

Shears about the x-axis aren't the only type that occur; shears can be defined about any line ℓ, as illustrated in Figure 45.

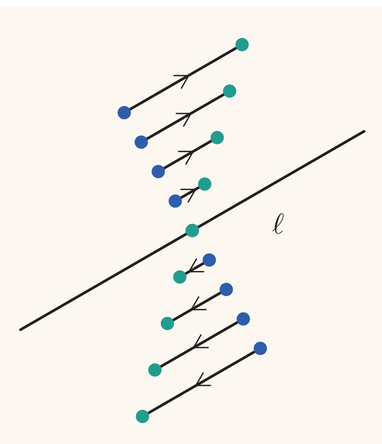


Figure 45 A shear about ℓ

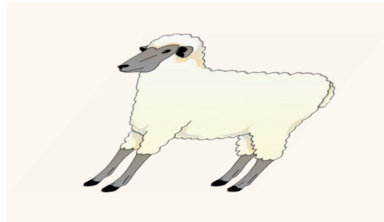
A **shear about a line** ℓ is a transformation of the plane that displaces each point P in a direction parallel to ℓ, by an amount proportional to its perpendicular distance from ℓ. The constant of proportionality is called the **shear factor**. Points are mapped in a 'clockwise' direction (as in Figure 45) if the shear factor is positive, and in an 'anticlockwise' direction if the shear factor is negative.

Note, however, that only shears about lines through the origin are linear transformations (since a linear transformation fixes the origin).

In particular, shears about the x-axis or the y-axis are linear transformations.

Shears about the x-axis, such as the one illustrated in Figure 43, are known as *horizontal shears*. In general a **horizontal shear** is a linear transformation represented by a matrix of the form

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$



A sheared sheep

Similarly, shears about the y-axis are known as *vertical shears*. A **vertical shear** is a linear transformation represented by a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

Notice that, for a horizontal or vertical shear, the determinant of the matrix of the transformation is always equal to 1.

The next activity asks you to compare these two types of shears.

Activity 36 *Investigating horizontal and vertical shears*

Use the *Visualising linear transformations* applet.

(a) Select the 'Horizontal shear' option, and set $k = 2$. Check that the diagram obtained matches Figure 43.

(b) Change the value of k to investigate the behaviour of horizontal shears for various values of k. In particular, investigate the effects of positive, negative and zero values of k.

(c) Now select 'Vertical shear' and investigate the effect of different values of k in this case.

The effects you saw in Activity 36 show that a horizontal shear leaves the horizontal grid lines looking the same, but it tilts all the vertical grid lines about the points where they intersect the x-axis. They are tilted clockwise if $k > 0$ and anticlockwise if $k < 0$. Points on the x-axis remain fixed. The shear factor is equal to k.

Similarly, a vertical shear leaves the vertical grid lines looking the same, but it tilts all the horizontal grid lines about the points where they intersect the y-axis. In this case, however, they are tilted anticlockwise if $k > 0$ and clockwise if $k < 0$. Points on the y-axis remain fixed. Since, by definition, the shear factor is positive when points are mapped in a clockwise direction, in this case the shear factor is equal to $-k$. (Note that in some texts the shear factor of both a horizontal and a vertical shear is defined to be its non-zero off-diagonal matrix entry k. However, such a definition would lead to inconsistencies later in the unit when we use a technique known as *conjugation* to calculate the matrix of a general shear.)

In both cases the unit square and its image parallelogram share the same base and perpendicular height, so their areas are both equal to 1. Also, their vertices have the same ordering, so orientation is preserved.

Properties of horizontal and vertical shears

Under a horizontal or vertical shear about the line ℓ (where ℓ is the x-axis in the case of a horizontal shear and the y-axis in the case of a vertical shear):

- points on ℓ remain fixed
- other points are moved parallel to ℓ through a distance proportional to their perpendicular distance from ℓ
- areas and orientation are preserved.

Rotations and reflections that fix the origin

In Section 2 you met isometries, which are transformations of the plane that keep the distances between pairs of points unchanged. The four types of isometries are rotations, reflections, translations and glide-reflections. Here we'll look at when an isometry is a linear transformation.

Now any isometry that moves the origin cannot be a linear transformation, but what about an isometry that fixes the origin? You'll see that such an isometry is indeed a linear transformation, and that it's either a rotation about the origin or a reflection in a line through the origin.

To see this, let f be any isometry that fixes the origin. Because f maps the plane rigidly, it leaves the size and shape of the unit grid unchanged, as illustrated in Figure 46. In particular, it maps the unit square to another square with sides of length 1 and a vertex at the origin. Also, it maps the unit vectors $\mathbf{i}$ and $\mathbf{j}$ along the sides of the unit square to a new pair of unit vectors $\mathbf{i}' = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\mathbf{j}' = \begin{pmatrix} b \\ d \end{pmatrix}$ along the sides of its image.

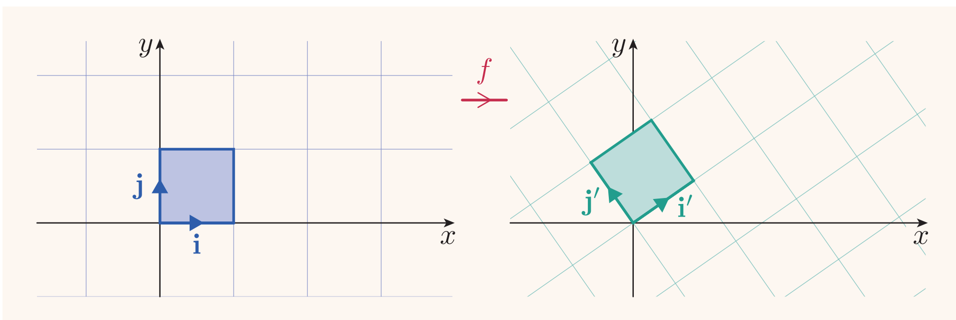


Figure 46 The effect of an isometry that fixes the origin on the unit grid

Now consider the effect that f has on a general point P with position vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ (see Figure 47). Because f maps the plane rigidly, it maps the point P to the corresponding point P' with position vector $\mathbf{x}' = x\mathbf{i}' + y\mathbf{j}'$. That is,

$$\mathbf{x}' = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \mathbf{A}\mathbf{x}, \quad \text{where } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It follows that f is a linear transformation, and that it is represented by the matrix $\mathbf{A}$ whose columns are $\mathbf{i}'$ and $\mathbf{j}'$.

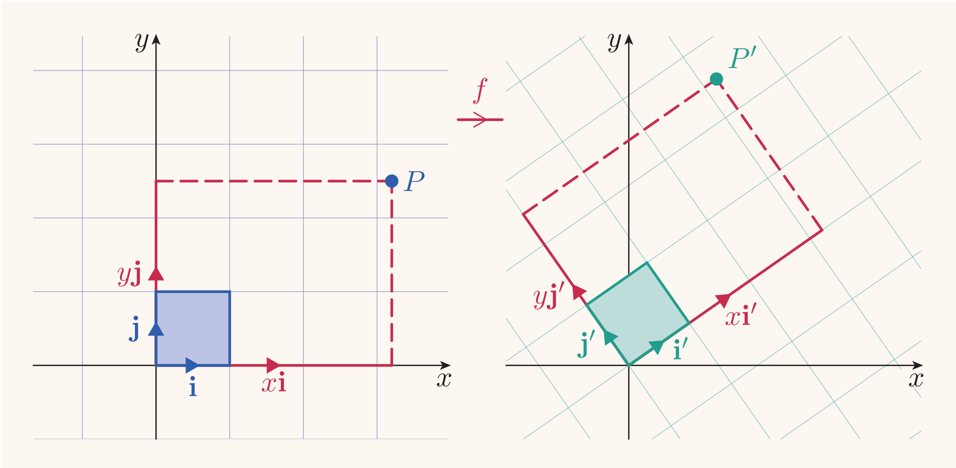


Figure 47 Isometries that fix the origin are linear transformations

To see how to write down the matrix $\mathbf{A}$ for a general isometry that fixes the origin, let θ be the angle that $\mathbf{i}'$ makes with the positive direction of the x-axis, as shown in Figure 48. Since $\mathbf{i}'$ is a unit vector, it follows that $\mathbf{i}' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, so this is the first column of $\mathbf{A}$.

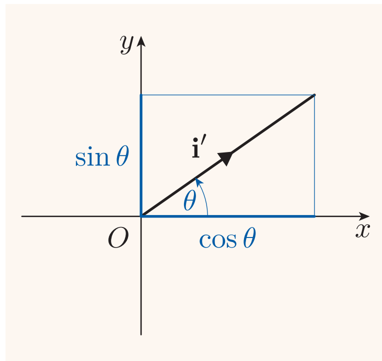


Figure 48 Components of $\mathbf{i}'$

To obtain the second column, let's focus on the unit square and its image under f. There are two possibilities for the position of this image, depending on the direction of the vector $\mathbf{j}'$. This is because $\mathbf{j}'$ can be perpendicular to $\mathbf{i}'$ in two ways, each the negative of the other (see Figure 49). One determines the image square in which orientation is preserved (shown in green), and the other the image square in which orientation is reversed (shown in purple).

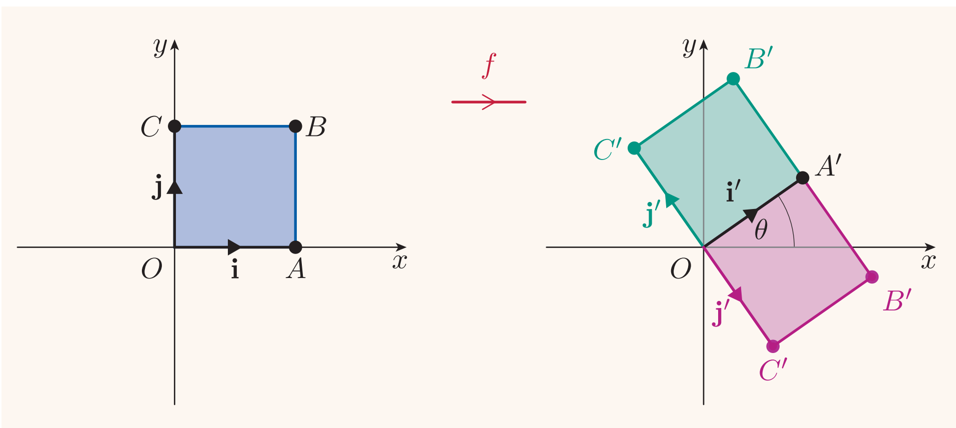


Figure 49 Isometries that fix the origin

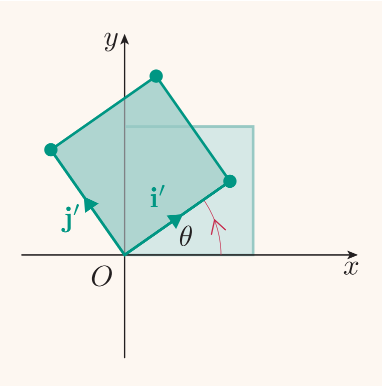


Figure 50 Rotation through θ about O

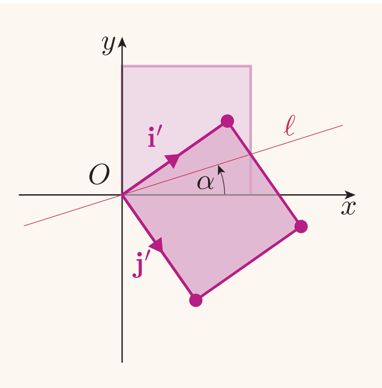


Figure 51 Reflection in the line ℓ through O

In the case of the image square in which orientation is preserved, the angle from $\mathbf{i}'$ to $\mathbf{j}'$ is $\pi/2$, so

$$\mathbf{j}' = \begin{pmatrix}\cos(\theta+\pi/2)\\ \sin(\theta+\pi/2)\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}, \quad \text{and hence} \quad \mathbf{A} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

This is the case shown in Figure 50, in which f is the rotation through the angle θ about the origin. Notice that

$$\det \mathbf{A} = \cos^2\theta - (-\sin^2\theta) = 1.$$

In the case of the image square in which orientation is reversed, the angle from $\mathbf{i}'$ to $\mathbf{j}'$ is $-\pi/2$, so

$$\mathbf{j}' = \begin{pmatrix}\cos(\theta-\pi/2)\\ \sin(\theta-\pi/2)\end{pmatrix} = \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}, \quad \text{and hence} \quad \mathbf{A} = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}.$$

This is the case shown in Figure 51, in which f is the reflection in the line ℓ through the origin that bisects the angle marked θ in Figure 49.

It's conventional to express the matrix $\mathbf{A}$ of this reflection in terms of the angle $\alpha = \frac{1}{2}\theta$ that ℓ makes with the positive direction of the x-axis (see Figure 51). This is achieved by replacing each θ in the matrix $\mathbf{A}$ by 2α to obtain the matrix

$$\mathbf{A} = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}.$$

Notice that

$$\det \mathbf{A} = -\cos^2 2\alpha - \sin^2 2\alpha = -1.$$

You may remember from MST124 Unit 4 that α is known as the **angle of inclination** of ℓ.

So any isometry that fixes the origin is a linear transformation. It is either a rotation about the origin, or a reflection in a line through the origin. All other isometries move the origin and so are not linear transformations. The matrices of isometries that fix the origin are summarised in the box below.

Rotations and reflections that fix the origin

Every isometry that fixes the origin is a linear transformation. It is either:

- a rotation through an angle θ about the origin, represented by the matrix
$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix};$$
or
- a reflection in a line ℓ through the origin, represented by the matrix
$$\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix},$$
where α is the angle of inclination of ℓ.

Using this result, you can calculate the matrix of any rotation or reflection that fixes the origin.

Example 16 *Finding matrices that represent isometries that fix the origin*

(a) Find the matrix that represents the rotation through $\pi/3$ about the origin.

(b) Find the matrix that represents the reflection in the line $y = x\sqrt{3}$.

Solution

(a) Substitute $\theta = \pi/3$ into the matrix of a general rotation about the origin, and simplify. Unless you are using the matrix in an application where numerical entries are needed, leave the entries in surd form.

The matrix is

$$\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Alternatively you can leave the $\frac{1}{2}$ inside the matrix and dispense with the final step.

(b) Use the gradient of the line to calculate its angle of inclination.

The line has gradient $\sqrt{3}$, so its angle of inclination is $\tan^{-1}(\sqrt{3}) = \pi/3$.

Substitute $\alpha = \pi/3$ into the matrix of a general reflection in a line through the origin, and simplify.

The matrix is

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ \sin(2\pi/3) & -\cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Activity 37 *Finding matrices that represent isometries that fix the origin*

(a) Find the matrix that represents the rotation through $\pi/4$ about the origin.

(b) Find the matrix that represents the reflection in the line $y = x/\sqrt{3}$.

Flattening the plane

So far, we've concentrated on one-to-one linear transformations, for which the determinant of the matrix representing the transformation is not zero. Now let's consider linear transformations $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for which $\det \mathbf{A} = 0$. A zero determinant suggests that all areas are reduced to zero under the transformation, and this in turn suggests that the transformation collapses the plane in some way.

A simple case is the zero transformation $f(x, y) = (0, 0)$. Under this transformation all points are mapped to $(0, 0)$, so the entire plane collapses onto a single point, as illustrated in Figure 52.

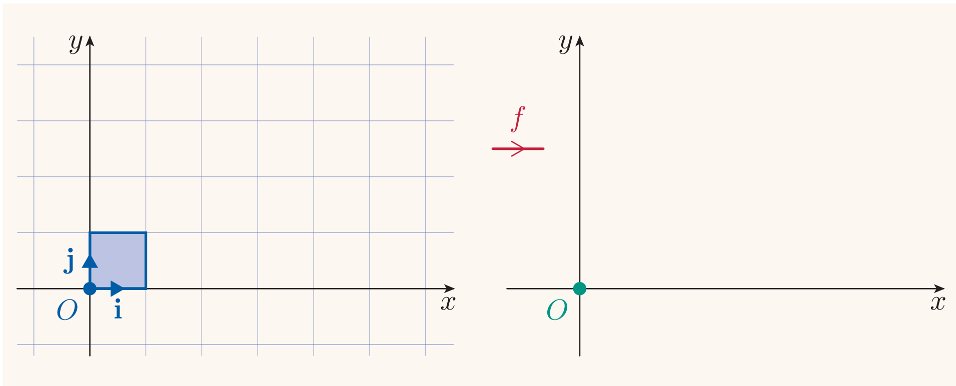


Figure 52 Under the zero transformation, the plane collapses onto O

There are other ways in which linear transformations can collapse the plane. For example, consider the linear transformation f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}.$$

The position vectors of the images of $(1, 0)$ and $(0, 1)$ are

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

which lie in the same direction. So the unit grid (and hence the entire domain $\mathbb{R}^2$) collapses onto the line through $(0, 0)$ and $(2, 1)$, as in the right-hand diagram of Figure 53.

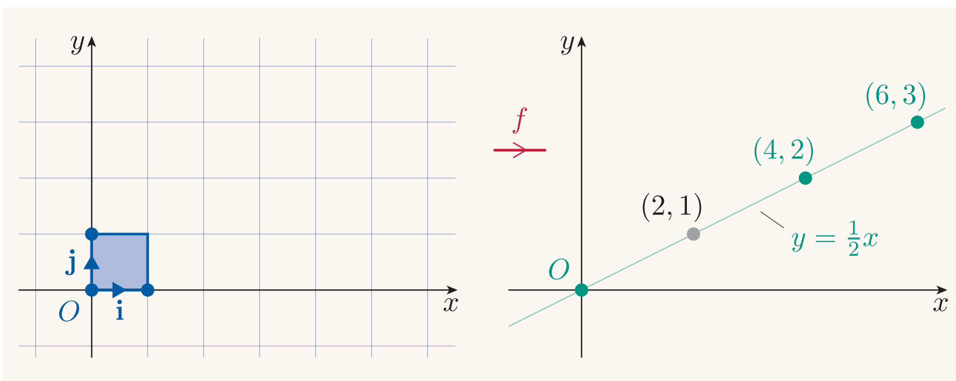


Figure 53 A transformation that collapses the plane onto the line $y = \frac{1}{2}x$

In fact, the image of any point (x, y) has the position vector given by

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 4 \\ 2 \end{pmatrix} + y\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 2x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3y\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (2x + 3y)\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So any point (x, y) is mapped onto a point whose position vector is a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Overall, therefore, the plane is *flattened* onto the line through the points $(0, 0)$ and $(2, 1)$. This line has equation $y = \frac{1}{2}x$.

The two examples illustrated in Figures 52 and 53 suggest the following definition.

A linear transformation is a **flattening** if its image set is either a line through the origin or the set containing the origin alone.

To check whether a linear transformation f represented by the matrix $\mathbf{A}$ is a flattening, think again about the image of the unit square (see Figure 54).

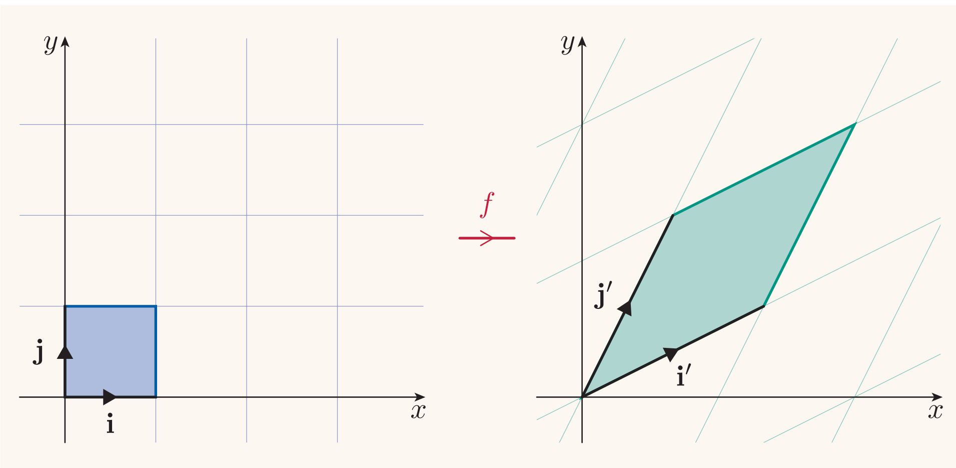


Figure 54 The image of the unit square

As usual $\mathbf{i}'$ and $\mathbf{j}'$ are the position vectors of $f(1, 0)$ and $f(0, 1)$ given by the columns of $\mathbf{A}$. The condition for f to be a flattening is that f collapses the plane onto a single line (or, in the case of the zero transformation, onto the origin). For this to happen, one of the two position vectors $\mathbf{i}'$ or $\mathbf{j}'$ along the edges of the parallelogram must be a scalar multiple of the other. But these two position vectors are the columns of $\mathbf{A}$, so you can check whether f is a flattening by checking whether one column of $\mathbf{A}$ is a scalar multiple of the other.

Alternatively, you can simply check whether $\det \mathbf{A} = 0$. If this is the case, then the area of the image parallelogram is 0, which implies that one of the vectors $\mathbf{i}'$ or $\mathbf{j}'$ is a multiple of the other.

Earlier you saw that, if f is not one-to-one, then $\det \mathbf{A} = 0$, so a third criterion for a flattening is that f is not one-to-one.

Criteria for a linear transformation to be a flattening

Let $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ be a linear transformation. Then the following are equivalent criteria for f to be a flattening:

- one column of $\mathbf{A}$ is a scalar multiple (possibly the zero multiple) of the other
- the determinant of $\mathbf{A}$ is zero
- f is not one-to-one.

You can use the facts below to find the image set of a flattening.

The image set of a flattening

Let f be a flattening, with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the image set of f contains the origin alone.
- Otherwise, the image set of f is the line that passes through the origin and the points (a, c) and (b, d).

The second fact holds because (a, c) and (b, d) are the images of $(1, 0)$ and $(0, 1)$ under f, so they lie in the image set.

Example 17 *Finding the image set of a flattening of the plane*

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}.$$

(a) Show that f is a flattening.

(b) Find the image set of f.

Solution

(a) Check whether the determinant of $\mathbf{A}$ is zero.

The determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = 4 \times 9 - (-6) \times (-6) = 36 - 36 = 0,$$

so f is a flattening.

(b) Use a column of $\mathbf{A}$.

The image set is the line that passes through $(0, 0)$ and $(4, -6)$. This line has gradient $(-6)/4 = -\frac{3}{2}$ and passes through the origin, so its equation is $y = -\frac{3}{2}x$.

When you use the method above to find the image set of a flattening $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A} \neq \mathbf{0}$, you can usually use either column of $\mathbf{A}$. However if one column is the zero vector, then you need to use the other column.

Activity 38 *Finding the image set of a flattening of the plane*

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}.$$

(a) Check that f is a flattening.

(b) Find the image set of f.

Recognising a linear transformation from its matrix

To end this section, suppose you are given a 2×2 matrix. Could you recognise the type of linear transformation it represents? Often this isn't possible without further work, but there are several things to look out for that can aid recognition.

- **The 'shape' of the matrix**
  If the matrix is diagonal, with entries k and l down the main diagonal, then the transformation is a (k, l)-scaling.

  In particular, if $k = l$, then it's a k-dilation, and if $k = l = 1$, then it's the identity transformation.

  If the matrix has a zero in the bottom left or top right corner and 1s down the main diagonal, then the transformation is a horizontal or a vertical shear, respectively.
- **The determinant of the matrix**
  If the determinant is 0, then the transformation is a flattening. In particular, if all the entries of the matrix are 0, then the transformation is the zero transformation.

  If the determinant is not equal to 1, then the transformation cannot be a shear or a rotation, and if it's not equal to -1 then the transformation cannot be a reflection.
- **The columns of the matrix**
  If one column is a multiple of the other then the transformation is a flattening.

  If, as in Figures 49, 50 and 51, the columns $\mathbf{i}'$ and $\mathbf{j}'$ of the matrix are unit vectors that are at right angles to each other, so that $|\mathbf{i}'| = |\mathbf{j}'| = 1$ and $\mathbf{i}' \cdot \mathbf{j}' = 0$, then the transformation is an isometry that fixes the origin. It's a rotation if the determinant of the matrix is 1, and a reflection if the determinant is -1.

Example 18 *Identifying linear transformations from their matrices*

Identify the type of linear transformation represented by each of the following matrices.

(a) $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ (b) $\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$

Solution

(a) The bottom left entry of the matrix is zero and there are 1s down the main diagonal, so the transformation is a horizontal shear.

The transformation is a horizontal shear with shear factor 5.

(b) Since both entries off the main diagonal are non-zero, this matrix cannot be a (k, l)-scaling, nor can it be a horizontal or vertical shear. Perhaps it's a rotation or a reflection.

The determinant of the matrix is

$$\left(\tfrac{3}{5} \times \tfrac{3}{5}\right) - \left(\tfrac{-4}{5} \times \tfrac{4}{5}\right) = 1.$$

The first and second columns of the matrix are vectors with lengths

$$\sqrt{\left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2} = 1 \quad \text{and} \quad \sqrt{\left(\tfrac{-4}{5}\right)^2 + \left(\tfrac{3}{5}\right)^2} = 1,$$

and moreover

$$\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \cdot \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} = \tfrac{3}{5} \times \left(-\tfrac{4}{5}\right) + \tfrac{4}{5} \times \tfrac{3}{5} = 0.$$

So the columns of the matrix are vectors of unit length and are perpendicular to each other. It follows that the transformation is a rotation about the origin.

(c) Here again, the matrix cannot be a (k, l)-scaling, or a horizontal or vertical shear, but the determinant is zero.

The matrix is non-zero but it has determinant $2 \times 6 - 3 \times 4 = 0$, so the transformation is a flattening onto a line through the origin.

(d) This is a diagonal matrix so the transformation is a $(5, 2)$-scaling.

Activity 39 *Recognising linear transformations*

Identify the type of linear transformation represented by each of the following matrices.

(a) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ (c) $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$

You might like to check your answers to Activity 39 by using the *Visualising linear transformations* applet.

5 Composite and inverse linear transformations

In Section 4 you saw how a linear transformation of the plane can be represented by a 2×2 matrix. In this section, you'll discover that there is more to this matrix representation of linear transformations than is immediately apparent.

You'll see that you can compose and invert linear transformations by multiplying and inverting their matrices. Then you'll see how this algebraic way of composing and inverting linear transformations can help you find the images of curves, and the matrices of shears that are neither horizontal nor vertical.

5.1 Composite linear transformations

In Section 3, you saw that if one transformation f is followed by another transformation g, then the overall effect is the *composite* transformation $g \circ f$. For example, if f is the rotation through $\pi/3$ about the origin and g is the rotation through $\pi/6$ about the origin, then $g \circ f$ is the rotation through $\pi/2$ about the origin, as illustrated in Figure 55.

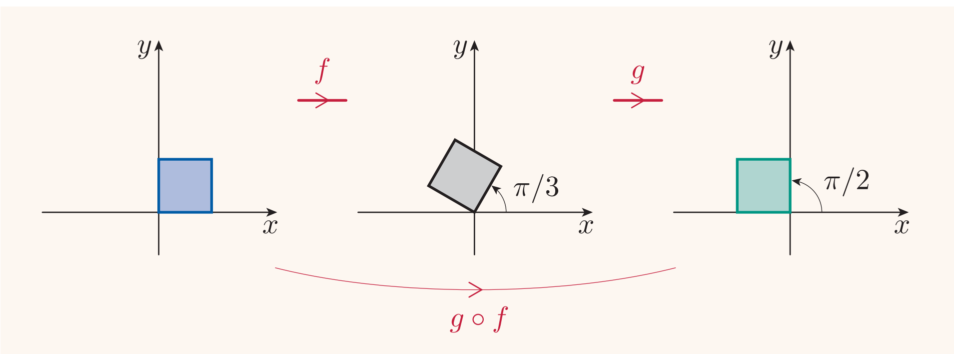


Figure 55 Composing rotations about the origin

The next activity asks you to investigate the relationship between the matrices of these rotations.

Activity 40 *Comparing the matrices of two rotations about the origin and their composite*

(a) Find the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ that represent rotations through $\pi/3$, $\pi/6$ and $\pi/2$ about the origin, respectively.

(b) Verify that $\mathbf{BA} = \mathbf{C}$.

The result of Activity 40 is a particular case of the following general fact.

Composition of linear transformations

If f and g are linear transformations represented by the matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, then $g \circ f$ is the linear transformation represented by the matrix $\mathbf{BA}$.

To see why this is true, suppose that f and g are linear transformations of the plane, with matrices $\mathbf{A}$ and $\mathbf{B}$ respectively. Then, if $\mathbf{x}$ is the position vector of any point in the plane,

$$f(\mathbf{x}) = \mathbf{Ax} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{Bx},$$

so

$$\begin{aligned}(g \circ f)(\mathbf{x}) &= g(f(\mathbf{x})) \\ &= g(\mathbf{Ax}) \\ &= \mathbf{B}(\mathbf{Ax}) \\ &= (\mathbf{BA})\mathbf{x}.\end{aligned}$$

So the composite transformation $g \circ f$ is a linear transformation and it is represented by the product matrix $\mathbf{BA}$.

The fact in the box above extends in the obvious way to composites of more than two linear transformations. For example, if f, g and h are linear transformations represented by the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, respectively, then $h \circ g \circ f$ is the linear transformation represented by the matrix $\mathbf{CBA}$. Note that you don't need brackets here because $(h \circ g) \circ f = h \circ (g \circ f)$ and $(\mathbf{CB})\mathbf{A} = \mathbf{C}(\mathbf{BA})$; that is, composition of transformations and multiplication of matrices are both associative.

These results further reinforce the bridge between geometry and algebra that was created when Descartes introduced coordinates. They tell you that the geometric effect of composing linear transformations can be expressed algebraically in terms of matrix multiplication. From a theoretical point of view this is important, because it enables you to use matrix algebra to study linear transformations. More practically, it provides a systematic way of manipulating linear transformations.

Example 19 *Finding the matrix of a composite linear transformation*

Let f and g be the linear transformations represented by the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix},$$

respectively. Find the matrix of the composite transformation $g \circ f$.

Solution

To find the matrix of $g \circ f$, multiply the matrix of g by the matrix of f.

$g \circ f$ is the linear transformation represented by the matrix

$$\mathbf{BA} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 19 \\ 18 & 23 \end{pmatrix}.$$

Activity 41 *Finding the matrix of a composite linear transformation*

Let f and g be the horizontal and vertical shears represented by the matrices

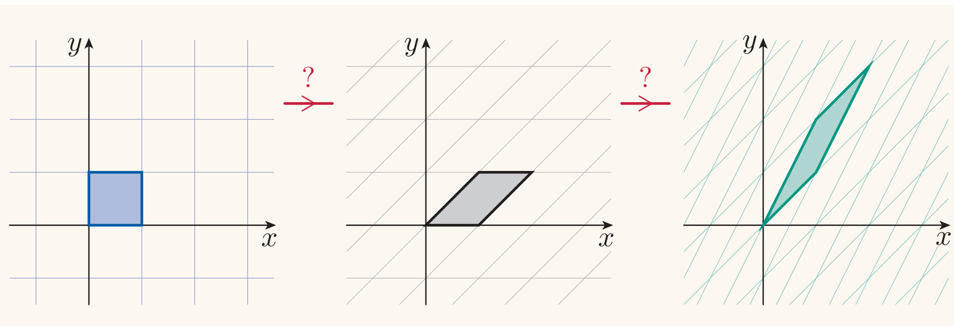
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively.

(a) Find the matrix representing each of the following composite transformations.

(i) $g \circ f$ (ii) $f \circ g$

(b) Which of the composite transformations in part (a) is illustrated by the two-stage diagram below?



As a further illustration of the interplay between the algebra of matrices and the geometry of transformations, let's revisit the following result about the composite of two reflections that you met on page 220.

Composition of two reflections

Let ℓ and m be lines that meet at a point P. Then the composite transformation formed from the reflection in ℓ followed by the reflection in m is the rotation about P through twice the angle from ℓ to m.

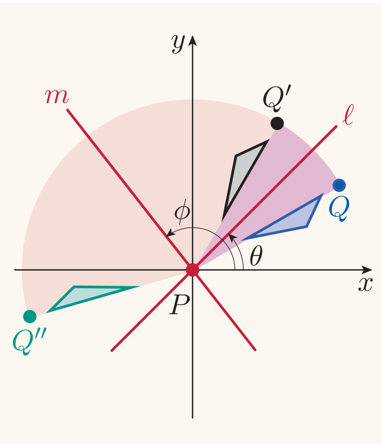


Figure 56 Composing reflections

You may remember that we justified this result geometrically. If, as shown in Figure 56, we now introduce a pair of Cartesian axes into the plane with the origin at P, and let θ and ϕ be the angles of inclination of ℓ and m, respectively, then we can also justify the result algebraically. This is done in the example below.

Example 20 *Composing two reflections*

Let f be the reflection in the line ℓ through the origin with angle of inclination θ, and let g be the reflection in the line m through the origin with angle of inclination ϕ. Use matrix multiplication to show that the composite $g \circ f$ is a rotation about the origin.

Interpret your result geometrically.

Solution

Remember that the angle of inclination of a line is the angle between 0 and π that it makes with the x-axis, measured anticlockwise from the positive direction of the x-axis. Use the expression for a general reflection in a line through the origin to write down the matrices of f and g.

The reflections f and g are represented by the matrices

$$\mathbf{A} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix},$$

respectively. So the composite $g \circ f$ is represented by the product matrix

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\phi \cos 2\theta + \sin 2\phi \sin 2\theta & \cos 2\phi \sin 2\theta - \sin 2\phi \cos 2\theta \\ \sin 2\phi \cos 2\theta - \cos 2\phi \sin 2\theta & \sin 2\phi \sin 2\theta + \cos 2\phi \cos 2\theta \end{pmatrix} \end{aligned}$$

Use the angle sum and angle difference trigonometric identities to simplify each element of the matrix.

$$= \begin{pmatrix} \cos(2\phi - 2\theta) & -\sin(2\phi - 2\theta) \\ \sin(2\phi - 2\theta) & \cos(2\phi - 2\theta) \end{pmatrix}.$$

Compare this with the matrix of a general rotation about the origin.

This is the matrix that represents a rotation about the origin through the angle $2\phi - 2\theta = 2(\phi - \theta)$, where $\phi - \theta$ is the angle from ℓ to m. Hence the composite $g \circ f$ is a rotation through twice the angle from ℓ to m.

On page 225 you saw that, if a rotation is composed with a reflection, and the centre of rotation lies on the line of reflection, then the resulting composite is a reflection. The following activity asks you to prove this result algebraically. It assumes that a pair of axes have been introduced into the plane with the origin on the line of reflection at the centre of rotation.

Activity 42 *Composing a reflection and a rotation*

Let f be the reflection in the line through the origin with angle of inclination θ, and let g be the rotation through the angle ϕ about the origin.

Use matrix multiplication to show that the composite $g \circ f$ is a reflection in a line ℓ through the origin. What is the angle of inclination of ℓ?

You can use a similar method to show that, if the linear transformations f and g in Activity 42 are composed in the reverse order to obtain $f \circ g$, then the result is a reflection in the line obtained by rotating ℓ about P through the angle $-\phi/2$. These results are summarised in the box below.

Composition of a reflection and a rotation

Let g be the rotation through the angle ϕ about a point P, and let f be the reflection in a line ℓ through P. Then $g \circ f$ is a reflection in a line obtained by rotating ℓ about P through the angle $\phi/2$, and $f \circ g$ is a reflection in a line obtained by rotating ℓ about P through the angle $-\phi/2$.

5.2 Inverse linear transformations

Remember that the inverse of a one-to-one transformation f is a transformation that undoes the effect of f. For example, the inverse of a rotation about the origin through the angle θ is a rotation about the origin through the angle $-\theta$. As another example, a reflection is its own inverse.

By contrast, a flattening of the plane has no inverse. Its effect cannot be undone because it is not one-to-one, and it is impossible for a second transformation to send a point (such as the origin) back to more than one point.

In Subsection 3.2, you saw how to find the inverse of a one-to-one transformation algebraically. This is particularly straightforward in the case of a linear transformation, because you can make use of the algebraic properties of matrices and their inverses, as described in the box below.

Inverse of a linear transformation

If f is an invertible linear transformation represented by the matrix $\mathbf{A}$, then its inverse f^{-1} is the linear transformation represented by the matrix $\mathbf{A}^{-1}$.

To see why this result is true, suppose that f is an invertible linear transformation represented by the matrix $\mathbf{A}$. Since f is invertible it is one-to-one, and so $\det \mathbf{A} \neq 0$. It follows that $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$. So, if $\mathbf{x}'$ is the position vector of a point in the codomain of f such that $f(\mathbf{x}) = \mathbf{x}'$, then

$$\mathbf{A}\mathbf{x} = \mathbf{x}'.$$

Multiplying both sides of this equation by $\mathbf{A}^{-1}$ therefore gives

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}',$$

and since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{I}\mathbf{x} = \mathbf{x}$, this means that

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}'.$$

Thus $f(\mathbf{A}^{-1}\mathbf{x}') = \mathbf{x}'$, so f^{-1} is specified by

$$f^{-1}(\mathbf{x}') = \mathbf{A}^{-1}\mathbf{x}'.$$

On dropping the dashes as usual, this becomes

$$f^{-1}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x},$$

so the matrix of the inverse transformation f^{-1} is $\mathbf{A}^{-1}$, as stated in the box above.

Example 21 *Finding the inverse of a linear transformation*

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}.$$

(a) Show that f is invertible.

(b) Find f^{-1}.

(c) Find the point (x, y) such that $f(x, y) = (2, 1)$.

Solution

(a) Check whether the determinant of $\mathbf{A}$ is non-zero. If it is, then f^{-1} exists.

The determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = 3 \times 4 - 2 \times 5 = 2.$$

Since $\det \mathbf{A}$ is non-zero, f is invertible.

(b) Remember that the inverse of $\mathbf{A}$ is obtained by swapping its diagonal elements, changing the sign of the off-diagonal elements and dividing by the determinant.

The inverse f^{-1} is the linear transformation represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix}.$$

(c) The required point is $f^{-1}(2, 1)$ with position vector

$$\mathbf{A}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{7}{2} \end{pmatrix},$$

that is, the point $\left(3, -\frac{7}{2}\right)$.

Activity 43 *Finding the inverse of a linear transformation*

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 9 & 5 \end{pmatrix}.$$

(a) Show that f is invertible.

(b) Find f^{-1}.

(c) Find the point (x, y) such that $f(x, y) = (6, 3)$.

You've already seen how you can find the image of a polygon under a transformation by finding the images of its vertices. However, many curves are specified by an equation rather than by a collection of vertices, and a different technique is required in such cases.

Rather surprisingly, when a curve is specified by an equation, you can find its image under an invertible transformation f by using the inverse transformation f^{-1}. Intuitively you might expect finding the equation of the image curve to involve using f rather than its inverse, but this turns out to be wrong. You'll see why in the following example.

Example 22 *Finding the image of the unit circle under an invertible linear transformation*

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & -3 \end{pmatrix}.$$

(a) Show that f is invertible, and find f^{-1}.

(b) Find the equation of the image $f(\mathcal{C})$ of the unit circle $\mathcal{C}$.

(c) Calculate the area enclosed by $f(\mathcal{C})$.

Solution

(a) f is invertible if $\det \mathbf{A} \neq 0$, and f^{-1} is the linear transformation represented by the matrix $\mathbf{A}^{-1}$.

The determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = 1 \times (-3) - (-2) \times 1 = -1,$$

so f has an inverse transformation f^{-1} represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} -3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}.$$

(b) Find the coordinates of the point in the domain of f that is mapped to a general point (x, y) in the codomain.

Each point (x, y) is the image under f of the point $f^{-1}(x, y)$, which has position vector

$$\mathbf{A}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 2y \\ x - y \end{pmatrix}.$$

Hence f maps the point $(3x - 2y, x - y)$ to the point (x, y).

Substitute these coordinates in the equation of the unit circle to find the equation of the image $f(\mathcal{C})$.

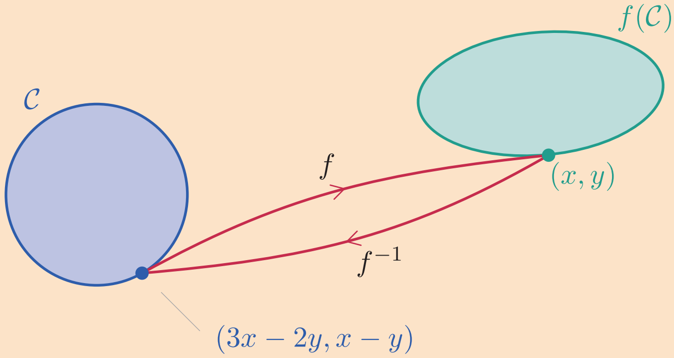
So if $(3x-2y, x-y)$ lies on $\mathcal{C}$, then (x,y) lies on $f(\mathcal{C})$, and conversely, if (x,y) lies on $f(\mathcal{C})$, then $(3x-2y, x-y)$ lies on $\mathcal{C}$, as illustrated in the figure below. It follows that

$$(3x-2y)^2+(x-y)^2=1.$$

Multiplying out the brackets and simplifying then gives

$$9x^2-12xy+4y^2+x^2-2xy+y^2=1$$
$$10x^2-14xy+5y^2=1.$$

Hence this is the equation of $f(\mathcal{C})$.



(c) Use the fact that the area scaling factor of a linear transformation is $|\det \mathbf{A}|$.

The area enclosed by $\mathcal{C}$ is π. Since f scales areas by the factor $|\det \mathbf{A}|=|-1|=1$, it follows that the area of $f(\mathcal{C})$ is also π.

Since the points $(1,0)$ and $(0,1)$ lie on the unit circle, a useful check on the answer to Example 22(b) is to check that the images of these points, which are $(1,1)$ and $(-2,-3)$ (the columns of $\mathbf{A}$), satisfy the equation found.

The next activity asks you to find the image of the unit circle under a different linear transformation.

Activity 44 *Finding the image of the unit circle under an invertible linear transformation*

Let f be the linear transformation represented by the matrix

$$\mathbf{A}=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}.$$

(a) Show that f is invertible, and find f^{-1}.

(b) Find the equation of the image $f(\mathcal{C})$ of the unit circle $\mathcal{C}$.

(c) Calculate the area enclosed by $f(\mathcal{C})$.

Now suppose that you are given the equation of an ellipse E and that you want to work out the area enclosed by E. Provided you can find a linear transformation that maps the unit circle onto E, then you will be able to calculate this area. The next activity asks you to do this.

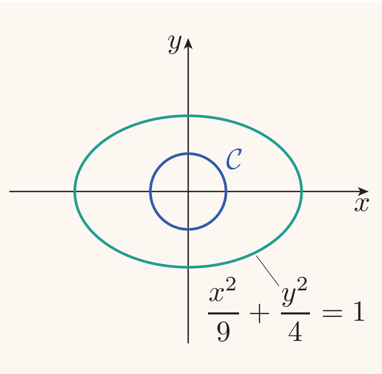


Figure 57 The ellipse E in Activity 45

Activity 45 *Finding the area of an ellipse*

(a) Let E be the ellipse

$$\frac{x^2}{9}+\frac{y^2}{4}=1,$$

shown in Figure 57.

(i) Write down the coordinates of the centre and vertices of E.

(ii) Give a geometric description of a linear transformation f that maps the unit circle $\mathcal{C}$ to E.

(iii) Hence write down the matrix that represents f, and use it to find the area enclosed by the ellipse.

(b) If $a>0$ and $b>0$, show that the area of an ellipse of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is πab.

As a final application of inverses, let's consider how to use them to find the matrix of a shear that is neither a horizontal nor a vertical shear. (Similar techniques can be used to find the matrix of a scaling that is neither horizontal nor vertical.)

Suppose you want to find the matrix of the linear transformation f that is a shear by the factor 3 about the line $y=x$. Rather than deal with this shear directly, you can compose three transformations that, overall, have the same effect as f:

1. Apply a rotation g about the origin so that the line $y=x$ ends up along the x-axis.
2. Apply a horizontal shear h with shear factor 3.
3. Apply the inverse rotation g^{-1}, so that the line along the x-axis is sent back to the line $y=x$.

Since g is an isometry, it preserves distances and therefore all the properties that are used to define a shear, so overall

$$f=g^{-1}\circ h\circ g$$

is the required shear by the factor 3 about the line $y=x$. The details are given in the next example.

Example 23 *Finding the matrix of a shear about the line $y = x$*

(a) Find the matrix of the rotation g that maps the line $y = x$ to the x-axis, and find its inverse.

(b) Hence find the matrix of the shear about the line $y = x$ with shear factor 3.

Solution

(a) Use the general form for a rotation about the origin.

The rotation g through $-\pi/4$ about the origin maps the line $y = x$ to the x-axis as required. It is represented by the matrix

$$\mathbf{R} = \begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Use the rule for inverting matrices to write down the inverse of $\mathbf{R}$. Alternatively use the general form for a rotation about the origin, this time through $\pi/4$ (or use the solution to Activity 37).

Since $\mathbf{R}$ represents a rotation, $\det \mathbf{R} = 1$, so the inverse rotation is represented by the inverse matrix

$$\mathbf{R}^{-1} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

(b) The horizontal shear by the factor 3 is represented by the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

The required shear about the line $y = x$ is therefore represented by the matrix

$$\begin{aligned} \mathbf{R}^{-1}\mathbf{H}\mathbf{R} &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ -1 & 1 \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix}. \end{aligned}$$

Here is a similar example for you to try.

Activity 46 *Finding the matrix of a shear about the line $y = -x$*

(a) Find the matrix of a rotation g that maps the line $y = -x$ to the x-axis, and find its inverse.

(b) Find the matrix of the shear about the line $y = -x$ with shear factor 2.

This idea of composing a transformation with an isometry and its inverse in order to perform the same action at a different location of the plane is known as **conjugation**. Here you've seen how the matrix of a shear about a line ℓ can be obtained by using a rotation to relocate ℓ. In the next section, you'll see how translations can be used in a similar way – for example, to relocate the line of a reflection or the centre of a rotation.

6 Affine transformations

Linear transformations suffer from a serious limitation, namely that they leave the origin fixed. So, although you can use linear transformations to describe rotations, dilations or scalings about the origin, you can't use them to describe rotations, dilations or scalings about other points. Nor can you use them to describe reflections in lines that don't pass through the origin, or shears about such lines.

6.1 What is an affine transformation?

To overcome this limitation of linear transformations, we use a new type of transformation known as an *affine transformation*. You can think of an affine transformation as the composite of a linear transformation followed by a translation. For example, suppose that

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

and let g be the linear transformation whose matrix is $\mathbf{A}$, and h be the translation with associated vector $\mathbf{a}$. Then $g(\mathbf{x}) = \mathbf{A}\mathbf{x}$ and $h(\mathbf{x}) = \mathbf{x} + \mathbf{a}$, so the composite $f = h \circ g$ is given by

$$f(\mathbf{x}) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{5}$$

which can be written more succinctly as $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$.

An **affine transformation** is a transformation of the form

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a},$$

where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a 2×1 column vector.

To find the image of a point under an affine transformation, you simply substitute the position vector of the point into the definition above in place of $\mathbf{x}$. For example, the image of the point $(2, 3)$ under the affine transformation f specified by equation (5) has position vector

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \end{pmatrix}.$$

So $f(2, 3) = (11, 8)$.

Activity 47 *Finding the images of points under an affine transformation*

Let f be the affine transformation specified by equation (5).

(a) Find the images of the points $(4, 3)$ and $(0, 0)$ under f.

(b) What do you notice about the position vector of the image of $(0, 0)$?

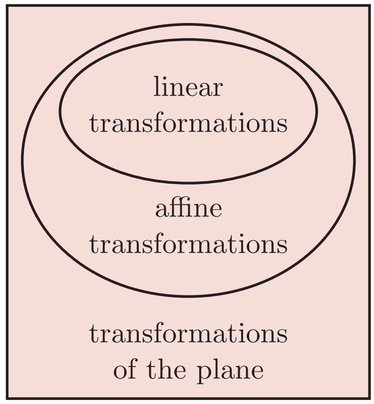


Figure 58 All linear transformations are affine transformations

As illustrated by the Venn diagram in Figure 58, the set of all affine transformations contains the set of all linear transformations. This is because each linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ can be written as the affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{0}$, where $\mathbf{0}$ is the zero vector, representing a zero translation of the plane. Many affine transformations aren't linear transformations because they do not fix the origin. Indeed, as illustrated by the solution to Activity 47(b), a general affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ maps the origin to the point with position vector $\mathbf{A}\mathbf{0} + \mathbf{a} = \mathbf{a}$.

You saw earlier that an isometry is only a linear transformation if it fixes the origin. However, the following result holds.

Every isometry is an affine transformation.

To see why this is true, suppose that f is an isometry, and suppose it maps the origin to a point with position vector $\mathbf{a}$. Then f will only be a linear transformation if $\mathbf{a}$ is the zero vector. However, the isometry defined by $g(\mathbf{x}) = f(\mathbf{x}) - \mathbf{a}$ is a linear transformation, because

$$g(\mathbf{0}) = f(\mathbf{0}) - \mathbf{a} = \mathbf{a} - \mathbf{a} = \mathbf{0}.$$

It follows that $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{a} = \mathbf{A}\mathbf{x} + \mathbf{a}$, where $\mathbf{A}$ is the matrix of g, which shows that f is an affine transformation.

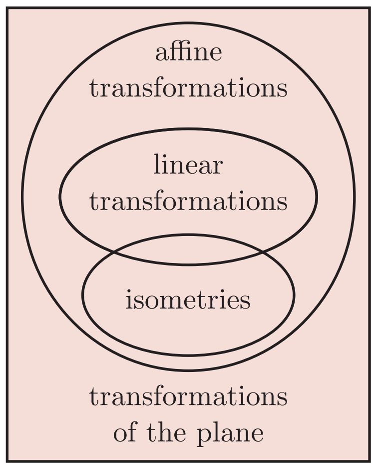


Figure 59 Sets of transformations

The Venn diagram in Figure 59 shows the relationships between all the sets of transformations we have been considering.

You saw in Section 4 that a linear transformation is completely determined by its effect on the two points $(1, 0)$ and $(0, 1)$, and that the position vectors of the images of these points are respectively the first and second columns of the matrix of the transformation. The box below gives the corresponding result for affine transformations.

Finding an affine transformation

The affine transformation f that maps the points with position vectors $\mathbf{0}$, $\mathbf{i}$ and $\mathbf{j}$ to the points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, is

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a},$$

where $\mathbf{A}$ is a 2×2 matrix whose first column is $\mathbf{b} - \mathbf{a}$ and whose second column is $\mathbf{c} - \mathbf{a}$.

Here (as usual), $\mathbf{0}$, $\mathbf{i}$ and $\mathbf{j}$ are the position vectors of the points $(0, 0)$, $(1, 0)$ and $(0, 1)$, respectively.

To see that this is true, consider Figure 60, which illustrates the effect of an affine transformation f on the unit square and the unit grid. The images of the points $(0, 0)$, $(1, 0)$ and $(0, 1)$ have been labelled A, B and C, respectively.

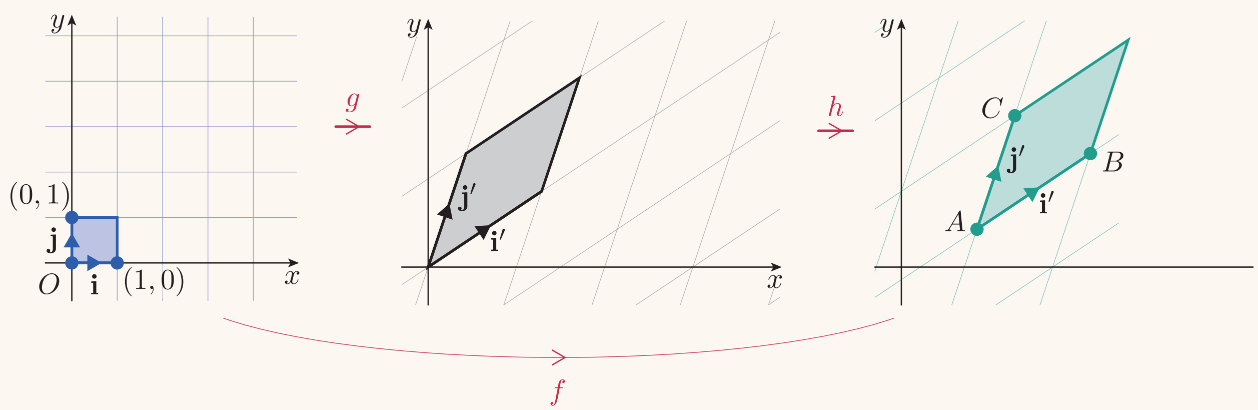


Figure 60 An affine transformation as the composite of a linear transformation and a translation

You can see that the unit grid is mapped in the way shown by thinking about f as the composite of the linear transformation $g(\mathbf{x}) = \mathbf{A}\mathbf{x}$ followed by the translation $h(\mathbf{x}) = \mathbf{x} + \mathbf{a}$. The unit square is first mapped by g to the parallelogram in the middle diagram in Figure 60, where $\mathbf{i}'$ is the first column of $\mathbf{A}$ and $\mathbf{j}'$ is the second column. Then, under h, this parallelogram is translated through the vector $\mathbf{a}$.

In particular, the vertex of the parallelogram at the origin is mapped to A, which therefore has position vector $\mathbf{a}$. Also, if $\mathbf{b}$ is the position vector of the point B, and $\mathbf{c}$ is the position vector of the point C, then $\mathbf{i}' = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\mathbf{j}' = \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$. This shows that the first column of $\mathbf{A}$ is given by $\mathbf{b} - \mathbf{a}$ and the second column of $\mathbf{A}$ is given by $\mathbf{c} - \mathbf{a}$, which is the result stated in the box above.

Example 24 *Finding an affine transformation*

Find the affine transformation f that maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(1,4)$, $(7,4)$ and $(3,5)$, respectively.

Solution

Write the position vectors $\mathbf{b}$ and $\mathbf{c}$ of $f(1,0)$ and $f(0,1)$ down the columns of a matrix that multiplies $\mathbf{x}$, leaving enough room so that you can then subtract the position vector $\mathbf{a}$ of $f(0,0)$ from each column. Finally add $\mathbf{a}$ at the end of the calculation.

The required affine transformation is given by

$$f(\mathbf{x}) = \begin{pmatrix} 7-1 & 3-1 \\ 4-4 & 5-4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

that is

$$f(\mathbf{x}) = \begin{pmatrix} 6 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

The following activity can be tackled in exactly the same way, but see whether you can spot anything unusual.

Activity 48 *Finding an affine transformation*

Find the affine transformation f that maps the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $(5,3)$, $(7,5)$, $(8,6)$, respectively.

As with a linear transformation, it is possible for an affine transformation to flatten the plane. Indeed, this is the case for the affine transformation

$$f(\mathbf{x}) = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

obtained in Activity 48. In cases like this, where the columns of the matrix are multiples of each other, the vectors $\mathbf{i}'$ and $\mathbf{j}'$ in Figure 60 point in the same direction, so that f collapses the plane onto a single line.

There are several ways that you can check whether or not an affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ is a flattening. Indeed, Figure 60 shows that f will be a flattening precisely when the linear transformation $g(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is a flattening. This is because the translation $h(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ won't affect whether the overall transformation f is a flattening. It follows that the criteria for an affine transformation to be a flattening are the same as for a linear transformation.

Criteria for an affine transformation to be a flattening

Let $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ be an affine transformation. Then the following are equivalent criteria for f to be a flattening:

- one column of $\mathbf{A}$ is a scalar multiple (possibly a zero multiple) of the other
- the determinant of $\mathbf{A}$ is zero
- f is not one-to-one.

So, if f is invertible, and therefore one-to-one, none of the equivalent criteria in the box above can hold. It follows that the matrix of f has a non-zero determinant, and therefore has an inverse. You can use this matrix inverse to find the inverse of f, as described below.

Inverse of an affine transformation

If $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ is an invertible affine transformation, then the inverse of f is specified by

$$f^{-1}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{a}.$$

To see why this result holds, let $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ be an invertible affine transformation and suppose that f maps the point with position vector $\mathbf{x}$ to the point with position vector $\mathbf{x}'$. Then $\mathbf{x}' = f(\mathbf{x})$, that is,

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{a},$$

or

$$\mathbf{A}\mathbf{x} = \mathbf{x}' - \mathbf{a}.$$

If we can solve this equation for $\mathbf{x}$ in terms of $\mathbf{x}'$, then we'll have obtained the rule for f^{-1}, as required. Since f is invertible, $\det \mathbf{A} \neq 0$, so $\mathbf{A}$ is invertible. Multiplying both sides of the above equation by $\mathbf{A}^{-1}$ then gives

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}(\mathbf{x}' - \mathbf{a}),$$

so

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}' - \mathbf{A}^{-1}\mathbf{a}.$$

Hence the inverse of f is specified by

$$f^{-1}(\mathbf{x}') = \mathbf{A}^{-1}\mathbf{x}' - \mathbf{A}^{-1}\mathbf{a},$$

which, on dropping the dashes, gives the result in the box above.

Example 25 *Finding the inverse of an affine transformation*

Show that the following affine transformation f is invertible, and find f^{-1}.

$$f(\mathbf{x}) = \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution

For ease of reference introduce a name for the matrix.

Let

$$\mathbf{A} = \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix}.$$

To show that f is invertible, find the determinant of $\mathbf{A}$.

Then

$$\det \mathbf{A} = 9 \times 3 - 5 \times 5 = 2 \neq 0,$$

so f is invertible.

To find f^{-1}, first find $\mathbf{A}^{-1}$ and $\mathbf{A}^{-1}\mathbf{a}$.

$$\mathbf{A}^{-1} = \tfrac{1}{2} \begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix}$$

$$\text{so} \quad \mathbf{A}^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} -9 \\ 17 \end{pmatrix}.$$

Substitute $\mathbf{A}^{-1}$ and $\mathbf{A}^{-1}\mathbf{a}$ into the formula for the inverse of an affine transformation. Be particularly careful to *subtract* $\mathbf{A}^{-1}\mathbf{a}$.

Hence

$$\begin{aligned} f^{-1}(\mathbf{x}) &= \tfrac{1}{2} \begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix} \mathbf{x} - \tfrac{1}{2} \begin{pmatrix} -9 \\ 17 \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix} \mathbf{x} + \tfrac{1}{2} \begin{pmatrix} 9 \\ -17 \end{pmatrix}. \end{aligned}$$

The final step above is not essential, and should be considered only if it reduces the total number of minus signs. It is also not essential for the fraction $\frac{1}{2}$ to be taken outside the matrix and the translation vector.

Activity 49 *Finding the inverse of an affine transformation*

Show that the following affine transformation f is invertible, and find f^{-1}.

$$f(\mathbf{x}) = \begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Invertible affine transformations share many of the properties possessed by invertible linear transformations.

Properties of an invertible affine transformation

Let $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ be an invertible affine transformation. Then:

- f maps lines to lines
- f maps parallel lines to parallel lines
- f scales areas by the factor $|\det \mathbf{A}|$
- f preserves orientation if $\det \mathbf{A} > 0$, and reverses orientation if $\det \mathbf{A} < 0$.

These properties hold because f can be thought of as a composite of the invertible linear transformation $g(\mathbf{x}) = \mathbf{A}\mathbf{x}$ followed by the translation $h(\mathbf{x}) = \mathbf{x} + \mathbf{a}$. Since both g and h map lines to lines and parallel lines to parallel lines, so does f. Also, g scales areas by the factor $|\det \mathbf{A}|$ and h leaves areas unchanged (it's an isometry), so f scales areas by the factor $|\det \mathbf{A}|$. Finally, h preserves orientation. So f preserves orientation if g does, and vice versa. But g preserves orientation if $\det \mathbf{A} > 0$, and reverses orientation if $\det \mathbf{A} < 0$, so the result follows.

You can sometimes use the result about areas in the box above to find the area of a figure. In Activity 33 you used a linear transformation to find the area of a triangle that had a vertex at the origin, but by using an affine transformation you can now find the area of any triangle.

Example 26 *Finding the area of a triangle*

(a) Find the affine transformation that maps the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $(2,5)$, $(3,-7)$, $(1,-3)$, respectively.

(b) Hence find the area of the triangle T with vertices at $(2,5)$, $(3,-7)$, $(1,-3)$.

Solution

(a) The required affine transformation is $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$, where

$$\mathbf{A} = \begin{pmatrix} 3-2 & 1-2 \\ -7-5 & -3-5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -12 & -8 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

(b) It follows that f scales areas by the factor

$$|\det \mathbf{A}| = |1 \times (-8) - (-1) \times (-12)| = 20.$$

The triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ has area $\frac{1}{2}$, so the area of the triangle T must be $20 \times \frac{1}{2} = 10$.

Activity 50 *Finding the area of a triangle*

(a) Write down the affine transformation that maps the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $(4,-3)$, $(-1,5)$, $(3,7)$, respectively.

(b) Hence find the area of the triangle T with vertices at $(4,-3)$, $(-1,5)$, $(3,7)$.

Having seen that the inverse of an affine transformation is itself an affine transformation, let's consider the composite of two affine transformations. You can compose two affine transformations by applying the rules of matrix algebra. The following example illustrates the approach.

Example 27 *Composing affine transformations*

Find the composite $g \circ f$, where

$$f(\mathbf{x}) = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad g(\mathbf{x}) = \begin{pmatrix} 7 & 5 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution

$$(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$$

Substitute the formula for f.

$$= g\left(\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right)$$

Now use the formula for g.

$$= \begin{pmatrix} 7 & 5 \\ 3 & -2 \end{pmatrix} \left(\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Multiply out the brackets.

$$= \begin{pmatrix} 7 & 5 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 5 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Evaluate the matrix products.

$$= \begin{pmatrix} 29 & 33 \\ 0 & 10 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 39 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Add the column vectors.

$$= \begin{pmatrix} 29 & 33 \\ 0 & 10 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 40 \\ -2 \end{pmatrix}.$$

You can use the approach in Example 27 to compose any two affine transformations. You can see that the resulting composite transformations will always be an affine transformation.

Composites of affine transformations

A composite of affine transformations is an affine transformation.

Activity 51 *Composing affine transformations*

(a) Find the composite $g \circ f$, where

$$f(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad g(\mathbf{x}) = \begin{pmatrix} 2 & 3 \\ 1 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

(b) Find the composites $g \circ f$ and $f \circ g$, where

$$f(\mathbf{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{x} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

6.2 Specifying isometries that do not fix the origin

So far, we have confined our attention mainly to rotations and reflections that fix the origin. But with the introduction of affine transformations, this constraint of a fixed origin is no longer necessary. Suppose you wish to investigate the transformation that rotates the plane about the point $(2, 3)$ through an (anticlockwise) angle $\pi/2$. How can you describe this transformation algebraically?

You have already seen (in Example 12) that an (anticlockwise) rotation through $\pi/2$ about the origin can be represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

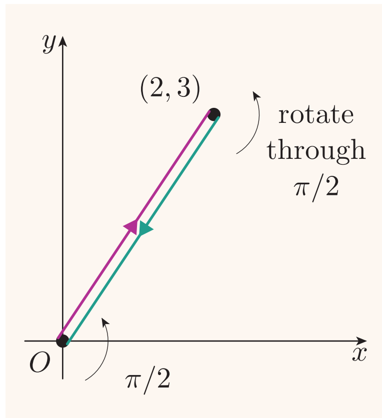


Figure 61 Conjugating a rotation

One way you can find the rotation through $\pi/2$ about $(2, 3)$ is to use the idea of conjugation (doing the same thing in a different location) that you met on page 272. This time, the three steps in the process are as follows (see Figure 61):

1. Translate the plane so that the point $(2, 3)$ ends up at the origin.
2. Rotate the translated plane through $\pi/2$ about the origin.
3. Translate the plane so as to map the origin back to the point $(2, 3)$.

This allows you to express the rotation f about $(2, 3)$ as the composite of three simpler transformations (two translations and a rotation about the origin), and you can use the formulas for these simpler transformations to find the formula for f.

Example 28 *Using conjugation by a translation to find a rotation*

Find the affine transformation f that describes the rotation through $\pi/2$ about the point $(2,3)$.

Solution

Express f in terms of the rotation through $\pi/2$ about the origin.

Let:

- h be the translation that maps the point $(2,3)$ to the origin
- g be the rotation through $\pi/2$ about the origin
- h^{-1} be the translation that maps the origin to the point $(2,3)$.

Then

$$h(\mathbf{x}) = \mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad g(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x}, \qquad h^{-1}(\mathbf{x}) = \mathbf{x} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The required affine transformation is

$$(h^{-1} \circ g \circ h)(\mathbf{x}) = h^{-1}(g(h(\mathbf{x})))$$

Substitute the formula for h.

$$= h^{-1}\left(g\left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)\right)$$

Now use the formula for g.

$$= h^{-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)\right)$$

Multiply out the brackets.

$$= h^{-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right)$$

Finally, use the formula for h^{-1}.

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Add the column vectors.

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Activity 52 *Using conjugation by a translation to find a rotation*

Find the affine transformation that describes an anticlockwise rotation through $\pi/4$ about the point $(3,1)$.

The idea of conjugation as a way of performing 'the same action somewhere else' can also be applied to reflections. The next activity asks you to find an algebraic description for a reflection in a line that does not pass through the origin. In this case, you can choose to translate any point on the line of reflection to the origin, and then make use of your knowledge of the matrix for a reflection in a line through the origin.

Activity 53 *Using conjugation by a translation to find a reflection*

(a) Determine the matrix that represents the reflection in the line $y = x$.

(b) By using the translation h that maps the point $(0,4)$ to the origin, and its inverse h^{-1}, find the affine transformation that reflects points in the line $y = x + 4$.

In the same way, you can use conjugation to find many other transformations of the plane, such as dilations about points other than the origin and shears about lines that do not pass through the origin.

6.3 Recognising affine transformations

Finally, in this short subsection, suppose you have an affine transformation and you know that it is an isometry. How can you decide what kind of isometry it is?

You can start by finding its fixed points. Remember that the four types of isometries differ in the points they leave fixed: a reflection has a whole line of fixed points (the line of reflection), a rotation has a single fixed point (the centre of rotation), and translations and glide-reflections have no fixed points.

Translations are easy to spot because they have the form $f(\mathbf{x}) = \mathbf{x} + \mathbf{a}$; that is, the matrix that multiplies $\mathbf{x}$ is the identity matrix (which is therefore usually omitted). If an affine transformation has no fixed points and the matrix is not the identity matrix, then it must be a glide-reflection.

Example 29 *Recognising an affine transformation that is an isometry*

The following affine transformation f is an isometry (it's the composite of the reflection in the line $y = x$ followed by the translation through $-2\mathbf{i} + 2\mathbf{j}$). Find its fixed points and hence describe it geometrically.

$$f(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

Solution

Write down the condition that a point (x, y) must satisfy if it is fixed by the transformation f.

Let (x, y) be a fixed point. Then

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Using the formula for f gives

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

so that

$$\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and hence

$$\begin{pmatrix} y - 2 \\ x + 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Express this as a pair of simultaneous equations and look for solutions.

This matrix equation is equivalent to the system of equations

$$y - 2 = x$$
$$x + 2 = y.$$

These two equations are rearrangements of each other, so there are infinitely many solutions.

Deduce properties of f.

It follows that f has infinitely many fixed points, so it is a reflection.

Each fixed point (x, y) satisfies the equation $y = x + 2$, and it follows that f is the reflection in the line $y = x + 2$.

Activity 54 *Recognising affine transformations that are isometries*

For each of the following affine transformations, find its fixed points (if any) and hence describe it geometrically. All are isometries.

(a) $f(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $g(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

(c) $h(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Learning outcomes

After studying this unit, you should be able to:

- give a geometric interpretation of a transformation specified algebraically
- give an algebraic specification for a transformation described geometrically
- write down the matrix that represents a given linear transformation
- find the image of a polygonal figure under a linear transformation or an isometry
- find the matrix of a rotation (about the origin), a reflection (in a line through the origin), a scaling (horizontal or vertical), a shear (horizontal or vertical) and a flattening
- compose transformations and find their inverses
- compose linear transformations by multiplying their matrices
- invert an invertible linear transformation by inverting its matrix
- compose affine transformations algebraically
- invert affine transformations algebraically
- use the inverse of a linear transformation f to find the image of a curve under f
- use determinants to calculate the area of the image of a figure under a linear or affine transformation
- determine the affine transformation that represents a rotation, reflection, dilation or shear that does not fix the origin
- use fixed points to determine the nature of an isometry.

Solutions to activities

Solution to Activity 1

(a) By the convention, the codomain is $\mathbb{R}^2$. The rule is applicable to all points (x, y), so the domain is $\mathbb{R}^2$.

$$f(1,2) = (1+3, 2+6) = (4,8).$$

(b) As part (a), the codomain is $\mathbb{R}^2$. The rule is applicable to all points (x, y), so the domain is $\mathbb{R}^2$.

$$g(1,2) = \big((3 \times 1) + 2, 1\big) = (5,1).$$

(c) As part (a), the codomain is $\mathbb{R}^2$. The rule is applicable to all points except those on the x- and y- axes, so the domain is $\mathbb{R}^2$ with all the points on the axes removed.

$$h(1,2) = \left(\tfrac{1}{2}, 1\right).$$

Solution to Activity 2

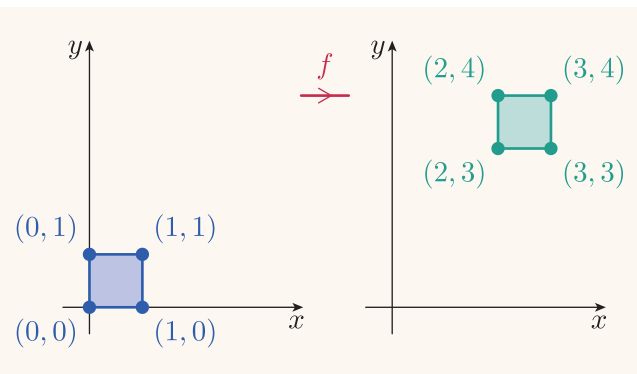
(a) f translates all points of the plane 2 units to the right and 3 units up.

(b) g reflects points in the y-axis.

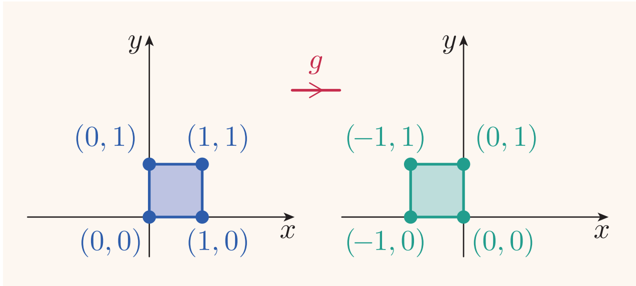
(c) h stretches the plane in the y-direction by a factor of 2, leaving distances in the x-direction unchanged.

Solution to Activity 3

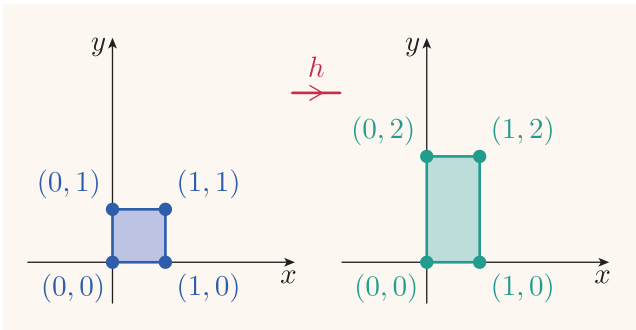
(a) f translates all points of the plane 2 units to the right and 3 units up, so the unit square is mapped to a square with vertices at $f(0,0) = (2,3)$, $f(1,0) = (3,3)$, $f(1,1) = (3,4)$ and $f(0,1) = (2,4)$.



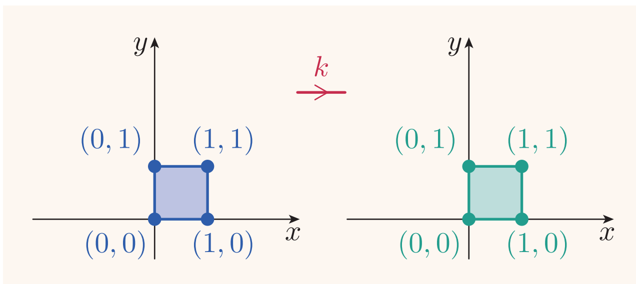
(b) g reflects the plane (and hence the unit square) in the y-axis.



(c) h 'stretches' the plane in the y-direction by a factor of 2. So the unit square is mapped to a rectangle of the same width, but double the height.



(d) k leaves all points of the plane unchanged. So, in particular, the unit square is mapped to itself.

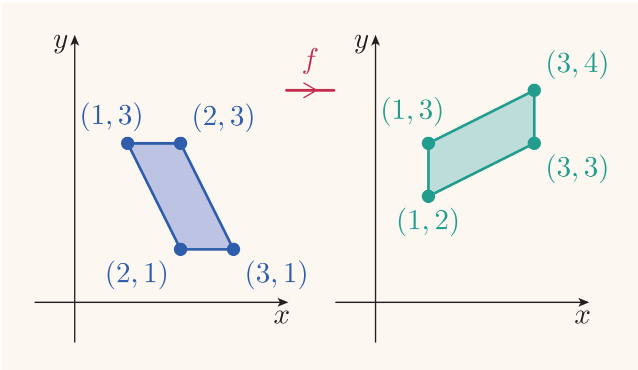


Solution to Activity 4

The images of the vertices under f are:

$$f(1,3) = (4-3, 1+1) = (1,2),$$
$$f(2,1) = (4-1, 1+2) = (3,3),$$
$$f(3,1) = (4-1, 1+3) = (3,4),$$
$$f(2,3) = (4-3, 1+2) = (1,3).$$

So the quadrilateral and its image are as shown below.



Solution to Activity 5

The required translation is

$$f(x, y) = (x+2, y-1).$$

Its associated vector is $2\mathbf{i} - \mathbf{j}$.

Solution to Activity 6

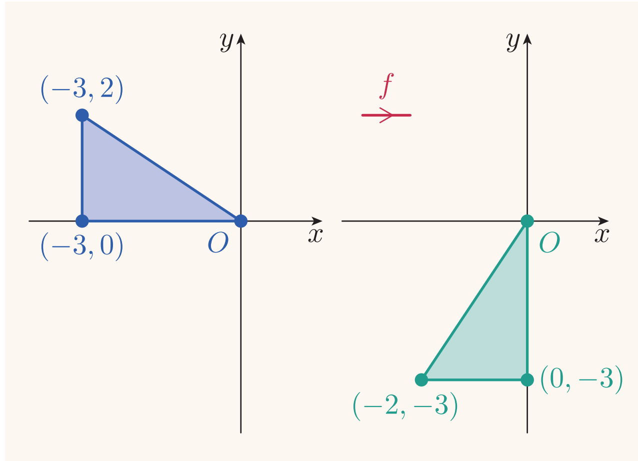
- $\pi/2$ (quarter of a turn anticlockwise)
- $7\pi/4$ and $-\pi/4$ (seven eighths of a turn anticlockwise)
- $3\pi/2$ (three quarters of a turn anticlockwise)
- $5\pi/4$ and $-3\pi/4$ (five eighths of a turn anticlockwise)
- 2π, -4π, 0 (no turn)
- 3π and $-\pi$ (half a turn)

Solution to Activity 7

Rotations through the angles 0, $\pi/2$, π, $3\pi/2$ and 2π about the origin leave the tiling pattern unchanged. (Actually, as you saw in Activity 6, rotation through 0 is the same transformation as rotation through 2π.)

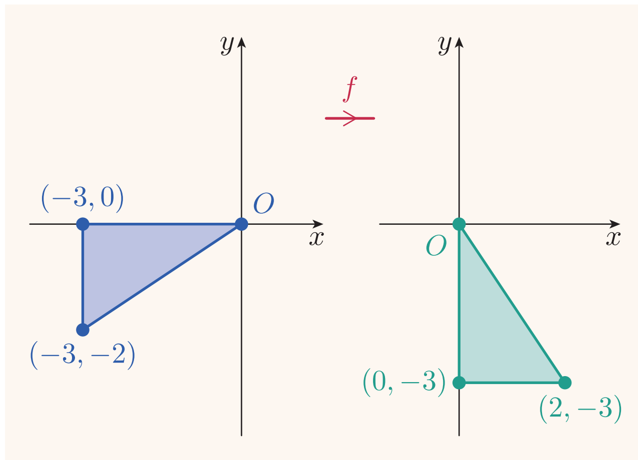
Solution to Activity 8

(a) The effect that f has on the point $(-3, 2)$ is shown below.



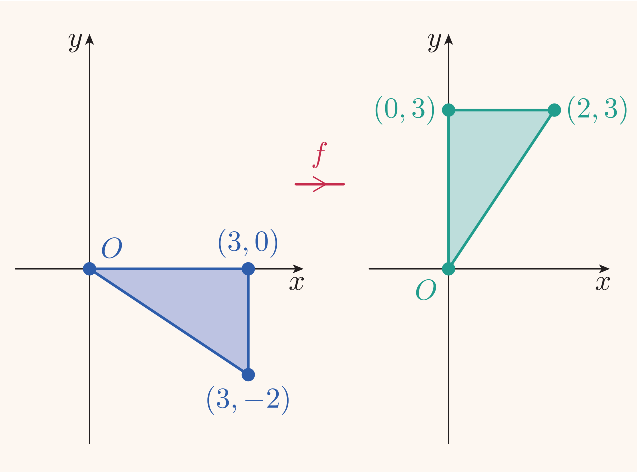
The diagram shows that $f(-3,2) = (-2,-3)$.

(b) The effect that f has on the point $(-3,-2)$ is shown below.



The diagram shows that $f(-3,-2) = (2,-3)$.

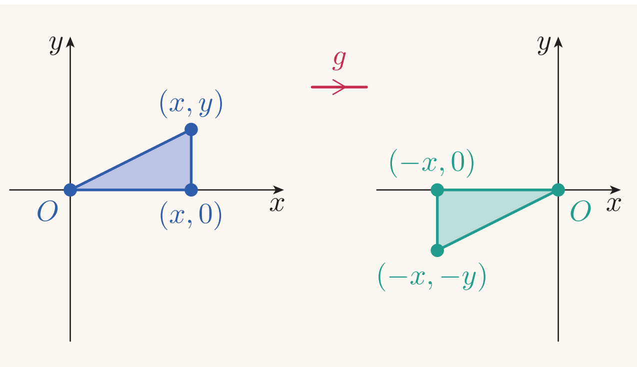
(c) The effect that f has on the point $(3, -2)$ is shown below.



The diagram shows that $f(3, -2) = (2, 3)$.

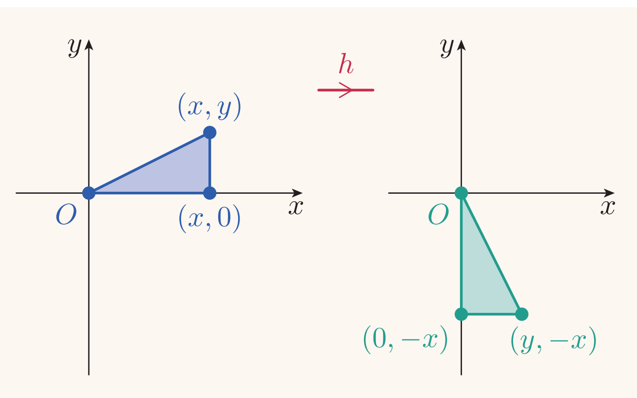
Solution to Activity 9

(a) The effect that g has on a general point (x, y) is illustrated below.



The diagram shows that $g(x, y) = (-x, -y)$.

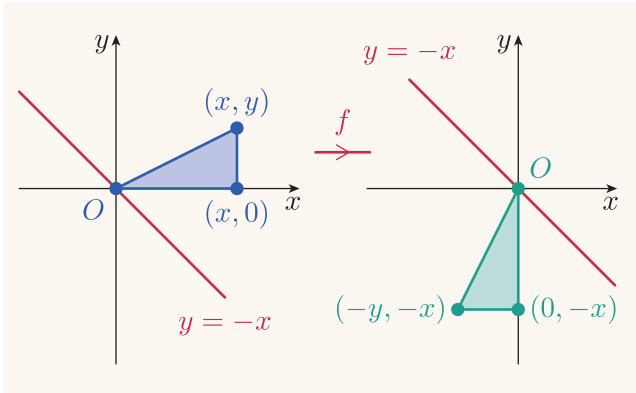
(b) The effect that h has on a general point (x, y) is illustrated below.



The diagram shows that $h(x, y) = (y, -x)$.

Solution to Activity 10

The effect that f has on a general point (x, y) is illustrated below.



The diagram shows that $f(x, y) = (-y, -x)$.

Solution to Activity 11

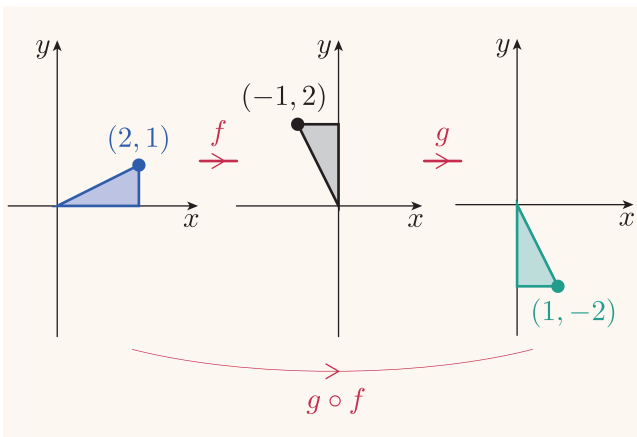
The appearance of the pattern is also unchanged by the reflections in the x- and y-axes, and in the line $y = -x$.

Solution to Activity 12

(a) The composite $g \circ f$ is given by

$$\begin{aligned}(g \circ f)(x, y) &= g(f(x, y))\\ &= g(-y, x)\\ &= \big(-(-y), -x\big)\\ &= (y, -x).\end{aligned}$$

(b)



(c) The diagram in part (b) shows that the overall effect of $g \circ f$ is the rotation through $3\pi/2$ about the origin.

Solution to Activity 13

The composite $g \circ f$ is given by

$$\begin{aligned}(g \circ f)\,(x,y) &= g(f(x,y))\\ &= g(x+2, y+3)\\ &= (x+2+1, y+3-2)\\ &= (x+3, y+1),\end{aligned}$$

as before. The composite $f \circ g$ is given by

$$\begin{aligned}(f \circ g)\,(x,y) &= f(g(x,y))\\ &= f(x+1, y-2)\\ &= (x+1+2, y-2+3)\\ &= (x+3, y+1).\end{aligned}$$

Thus $g \circ f = f \circ g$, so the order in which the two translations are composed does not matter.

Solution to Activity 14

(a) The sum of the vectors associated with f and g is

$$(3\mathbf{i} - 6\mathbf{j}) + (-3\mathbf{i} + 3\mathbf{j}) = 0\mathbf{i} - 3\mathbf{j},$$

so

$$(f \circ g)\,(x,y) = (x, y-3).$$

(b) The sum of the vectors associated with f and g is

$$(-4\mathbf{i} + 5\mathbf{j}) + 2\mathbf{i} = -2\mathbf{i} + 5\mathbf{j},$$

so

$$(f \circ g)\,(x,y) = (x-2, y+5).$$

(c) The sum of the vectors associated with f and g is

$$(4\mathbf{i} - 5\mathbf{j}) + (-4\mathbf{i} + 5\mathbf{j}) = \mathbf{0},$$

so

$$(f \circ g)\,(x,y) = (x,y),$$

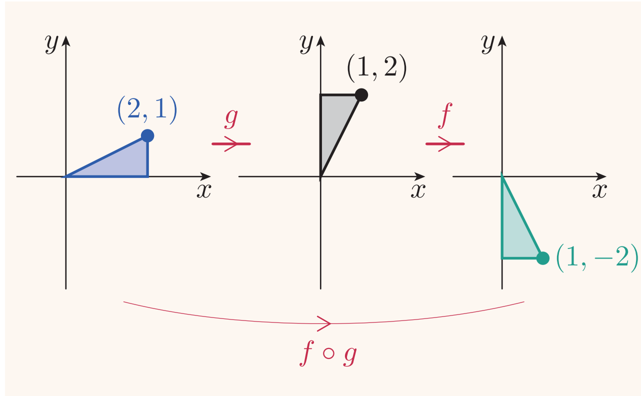
the identity transformation.

Solution to Activity 15

(a) The composite $f \circ g$ is given by

$$\begin{aligned}(f \circ g)(x,y) &= f(g(x,y))\\ &= f(y,x)\\ &= (y,-x).\end{aligned}$$

(b)



(c) The diagram in part (b) shows that the overall effect of $f \circ g$ is the rotation through $-\pi/2$ about the origin.

Solution to Activity 16

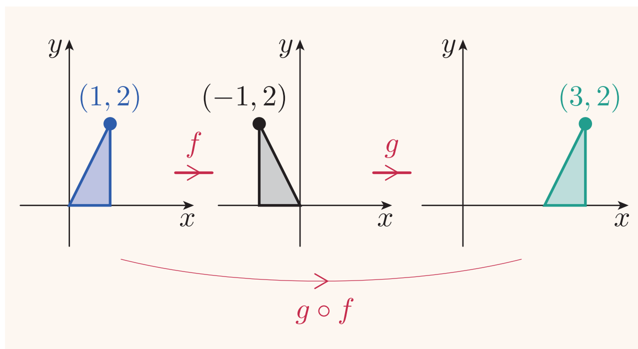
(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 17

(a) The composite $g \circ f$ is given by

$$\begin{aligned}(g \circ f)(x,y) &= g(f(x,y))\\ &= g(-x,y)\\ &= \big(2-(-x), y\big)\\ &= (x+2, y).\end{aligned}$$

(b)



(c) The diagram in part (b) shows that the overall effect of $g \circ f$ is a translation by 2 units to the right.

Solution to Activity 18

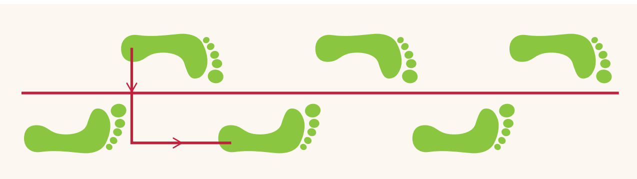
The composite $g \circ f$ is given by

$$\begin{aligned}(g \circ f)(x, y) &= g(f(x, y)) \\ &= g(y, x) \\ &= (-x, y).\end{aligned}$$

(The solution to this activity confirms that the composite $g \circ f$ above is indeed the reflection in the y-axis.)

Solution to Activity 19

(a) Each footprint can be sent onto the next by first reflecting it in a horizontal line and then translating the reflected footprint to the right, as shown below. (Alternatively, the translation could precede the reflection.)



(b) The reflection reverses the orientation of figures and the translation preserves this reversed orientation. So, overall, the composite reverses orientation.

(c) The reflection changes the position of points vertically, but leaves their horizontal position unchanged. The translation, on the other hand, ensures that the horizontal position of each point is then changed. There are therefore no points that remain fixed.

Solution to Activity 20

The glide-reflection is the composite $g \circ f$, where f and g are specified by

$$f(x, y) = (y, x) \quad \text{and} \quad g(x, y) = (x + 2, y + 2).$$

The composite is therefore given by

$$\begin{aligned}(g \circ f)(x, y) &= g(f(x, y)) \\ &= g(y, x) \\ &= (y + 2, x + 2).\end{aligned}$$

Solution to Activity 21

Reflections and glide-reflections both reverse orientation and so the composite of a reflection with a glide-reflection will preserve orientation. It follows that the composite will be either a translation or a rotation.

Solution to Activity 22

(a) The equation $f(x, y) = (x', y')$ gives

$$(x - 4, y - 2) = (x', y').$$

Equating coordinates gives

$$x - 4 = x' \quad \text{and} \quad y - 2 = y'.$$

Rearranging both equations then gives

$$x = x' + 4 \quad \text{and} \quad y = y' + 2.$$

Hence f^{-1} is specified by

$$f^{-1}(x', y') = (x' + 4, y' + 2),$$

which, on replacing x' and y' by x and y, respectively, becomes

$$f^{-1}(x, y) = (x + 4, y + 2).$$

(Here f is a translation 4 units to the left and 2 units down. Its inverse is a translation 4 units to the right and 2 units up.)

(b) The equation $g(x, y) = (x', y')$ gives

$$(-y, x) = (x', y').$$

Equating coordinates gives

$$-y = x' \quad \text{and} \quad x = y'.$$

Rearranging the first equation and leaving the second as it is then gives

$$x = y' \quad \text{and} \quad y = -x'.$$

Hence g^{-1} is specified by

$$g^{-1}(x', y') = (y', -x'),$$

which, on replacing x' and y' by x and y, respectively, becomes

$$g^{-1}(x, y) = (y, -x).$$

(Here g is a rotation through the angle $\pi/2$ about the origin. Its inverse is a rotation through $-\pi/2$ about the origin.)

(c) The equation $h(x, y) = (x', y')$ gives

$$\left(3x, \tfrac{1}{2}y\right) = (x', y').$$

Equating coordinates gives

$$3x = x' \quad \text{and} \quad \tfrac{1}{2}y = y'.$$

Rearranging both equations then gives

$$x = \tfrac{1}{3}x' \quad \text{and} \quad y = 2y'.$$

Hence h^{-1} is specified by

$$h^{-1}(x', y') = \left(\tfrac{1}{3}x', 2y'\right),$$

which, on replacing x' and y' by x and y, respectively, becomes

$$h^{-1}(x, y) = \left(\tfrac{1}{3}x, 2y\right).$$

(Here h stretches the plane in the x-direction by the factor 3, and compresses the plane in the y-direction by the factor $\frac{1}{2}$. Its inverse compresses the plane in the x-direction by the factor $\frac{1}{3}$, and stretches the plane in the y-direction by the factor 2.)

(d) The equation $k(x,y)=(x',y')$ gives

$$(x+3y,-y)=(x',y').$$

Equating coordinates gives

$$x+3y=x' \quad \text{and} \quad -y=y'.$$

Rearranging the second equation gives

$$y=-y'.$$

Substituting in the first equation then gives

$$x-3y'=x'.$$

Rearranging this gives

$$x=x'+3y'.$$

To summarise, $x=x'+3y'$ and $y=-y'$, and so k^{-1} is specified by

$$k^{-1}(x',y')=(x'+3y',-y'),$$

which, on replacing x' and y' by x and y, respectively, becomes

$$k^{-1}(x,y)=(x+3y,-y).$$

Solution to Activity 23

(a) $f^{-1}(x,y)=(-x,-y)$
(f is the rotation through the angle π about the origin, so its inverse is the rotation through the angle $-\pi$ about the origin. But this is the same as f, which is therefore self-inverse.)

(b) $g^{-1}(x,y)=(x-1,y+7)$
(g has associated vector $\mathbf{i}-7\mathbf{j}$, so g^{-1} has associated vector $-\mathbf{i}+7\mathbf{j}$.)

(c) $h^{-1}(x,y)=(y,x)$
(h is a reflection, so it is self-inverse.)

Solution to Activity 24

(a) $f(x,y)=(0x+1y,-1x+0y)$, so the matrix of f is $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

(b) $g(x,y)=(2x+0y,0x+1y)$, so the matrix of g is $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$.

(c) $h(x,y)=(1x+0y,0x+1y)$, so the matrix of h is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

(d) $k(x,y)=(1x+2y,0x+1y)$, so the matrix of k is $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$.

Solution to Activity 25

(a) $\begin{pmatrix}2 & 6\\ 3 & 5\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix}=\begin{pmatrix}-2\\ 1\end{pmatrix}$, so $f(2,-1)=(-2,1)$.

(b) $\begin{pmatrix}2 & 6\\ 3 & 5\end{pmatrix}\begin{pmatrix}0\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$, so $f(0,0)=(0,0)$.

(c) $\begin{pmatrix}2 & 6\\ 3 & 5\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}2\\ 3\end{pmatrix}$, so $f(1,0)=(2,3)$.

(d) $\begin{pmatrix}2 & 6\\ 3 & 5\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}6\\ 5\end{pmatrix}$, so $f(0,1)=(6,5)$.

Solution to Activity 26

(a) Example 1 showed that $f(x,y)=(x,-y)$, which can be written

$$f(x,y)=(1x+0y,0x-1y).$$

So f is the linear transformation represented by the matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

(b) g is not a linear transformation because it maps the origin to the point $(2,0)$.

(c) h is not a linear transformation because it maps the origin to the point $(2,-1)$.

(d) k is a linear transformation because it can be written in the form

$$k(x,y)=(0x+0y,0x+0y).$$

The matrix of k is

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Solution to Activity 27

(a) $\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}=0$, so f is not one-to-one.

(b) $\begin{vmatrix}2 & -3\\ 4 & 1\end{vmatrix}=14\neq 0$, so f is one-to-one.

(c) $\begin{vmatrix}2 & -4\\ -1 & 2\end{vmatrix}=0$, so f is not one-to-one.

Solution to Activity 28

We have

$$\mathbf{A}^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix}.$$

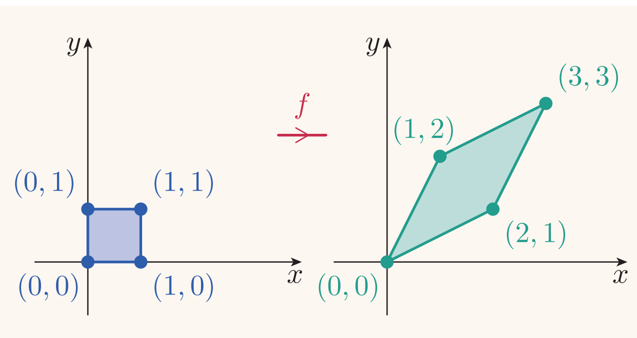
The image of the line $x + 2y = 1$ is therefore the line with the equation $0x + \frac{1}{2}y = 1$, or equivalently $y = 2$.

Solution to Activity 29

Since f is a linear transformation, the vertex of the unit square at the origin remains fixed. Also, the images of the vertices at $(1,0)$ and $(0,1)$ have coordinates given by the columns of the matrix, namely $(2,1)$ and $(1,2)$, respectively. Finally, the image of the vertex at $(1,1)$ is

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Since f maps lines to lines, we deduce that the image of the unit square is the parallelogram shown on the right-hand side of the diagram below (we know that the image is a parallelogram because the parallel sides of the unit square are mapped to parallel sides).



Solution to Activity 30

(a) A (b) B (c) D (d) E

Solution to Activity 31

The images under f of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ that form two sides of the unit square are the position vectors of $f(1,0)$ and $f(0,1)$ respectively. These position vectors are given by the columns of the matrix representing f, so if we call the image vectors $\mathbf{i}'$ and $\mathbf{j}'$ as usual, we have

$$\mathbf{i}' = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j}' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus the parallelogram has base $b = 3$ and perpendicular height $h = 2$. It follows that the area of the parallelogram is $bh = 3 \times 2 = 6$. So f scales areas by the factor 6.

Solution to Activity 32

(a) The matrix has determinant

$$\begin{vmatrix} 4 & -2 \\ 3 & 6 \end{vmatrix} = 4 \times 6 - (-2) \times 3 = 30.$$

So f scales areas by the factor $|30| = 30$, and preserves orientation.

(b) The matrix has determinant

$$\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = 3 \times 2 - 2 \times 4 = -2.$$

So f scales areas by the factor $|-2| = 2$, and reverses orientation.

(c) The matrix has determinant

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \times 0 - 1 \times 1 = -1.$$

So f leaves areas unchanged, and reverses orientation.

(This should not come as too much of a surprise, since the matrix represents reflection in the line $y = x$.)

Solution to Activity 33

Since f maps $(1,0)$ to $(2,5)$ and $(0,1)$ to $(3,1)$, it follows from the boxed result on page 233 that f is represented by the matrix

$$\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}.$$

This has determinant $2 \times 1 - 3 \times 5 = -13$, so f scales areas by a factor of 13. The triangle with vertices at $(0,0)$, $(2,5)$, $(3,1)$ is the image under f of the right-angled triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$. Since this triangle has area $\frac{1}{2}$, it follows that the required area is $13 \times \frac{1}{2} = 6\frac{1}{2}$.

Solution to Activity 34

As k and l change, the image grid remains rectangular.

Negative values of k cause the image of the unit square to flip across the y-axis, and negative values of l cause the image to flip across the x-axis. Each flip results in orientation being reversed. If *both* k and l are negative, then orientation is preserved. Orientation is also preserved when k and l are both positive.

When $k = 0$, the image grid collapses onto the y-axis, and when $l = 0$ the image grid collapses onto the x-axis. Only non-zero values of k and l can therefore be associated with a scaling. For all non-zero values of k and l, the transformation scales the width of the unit square by the factor $|k|$ and the height of the unit square by the factor $|l|$. The area of the unit square is therefore scaled by the factor $|kl|$.

Solution to Activity 35

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 36

(The effects that you should have seen are described in the text after the activity.)

Solution to Activity 37

(a) The matrix is

$$\begin{aligned}\begin{pmatrix}\cos(\pi/4) & -\sin(\pi/4)\\ \sin(\pi/4) & \cos(\pi/4)\end{pmatrix} &= \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.\end{aligned}$$

(Alternatively you can leave the $1/\sqrt{2}$ inside the matrix and dispense with the final step.)

(b) The line has gradient $1/\sqrt{3}$, so its angle of inclination is $\tan^{-1}(1/\sqrt{3}) = \pi/6$.

The matrix is

$$\begin{aligned}\begin{pmatrix}\cos(\pi/3) & \sin(\pi/3)\\ \sin(\pi/3) & -\cos(\pi/3)\end{pmatrix} &= \begin{pmatrix}1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{pmatrix}\\ &= \tfrac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}.\end{aligned}$$

(Alternatively you can leave the $\frac{1}{2}$ inside the matrix and dispense with the final step.)

Solution to Activity 38

(a) The determinant of $\mathbf{A}$ is

$$\begin{aligned}\det\mathbf{A} &= 6\times 3 - 2\times 9\\ &= 18 - 18 = 0,\end{aligned}$$

so f is a flattening.

(b) The second column of $\mathbf{A}$ is $\begin{pmatrix}2\\3\end{pmatrix}$, so the image set of f is the line that passes through $(0,0)$ and $(2,3)$. This line has gradient $\frac{3}{2}$ and passes through the origin, so its equation is $y = \frac{3}{2}x$.

(Alternatively, since the first column of $\mathbf{A}$ is $\begin{pmatrix}6\\9\end{pmatrix}$, you can work out the image set of f by using the fact that it is the line that passes through $(0,0)$ and $(6,9)$.)

Solution to Activity 39

(a) This is a diagonal matrix with 4s down the main diagonal, so the transformation is a 4-dilation.

(b) The matrix is non-zero but it has determinant $1\times 4 - (-2)\times(-2) = 4 - 4 = 0$, so the transformation is a flattening onto a line through the origin.

(c) The determinant of this matrix is

$$(1/2)\times(-1/2) - (\sqrt{3}/2)\times(\sqrt{3}/2) = -1.$$

The first and second columns are vectors with lengths

$$\tfrac{1}{2}\sqrt{1^2 + (\sqrt{3})^2} = 1$$

and

$$\tfrac{1}{2}\sqrt{(\sqrt{3})^2 + (-1)^2} = 1,$$

and moreover

$$\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}\cdot\begin{pmatrix}\sqrt{3}\\ -1\end{pmatrix} = 1\times\sqrt{3} + \sqrt{3}\times(-1) = 0.$$

So the columns of the matrix are unit vectors that are perpendicular to each other. It follows that the transformation is a reflection in a line through the origin.

(d) The transformation is a vertical shear, with shear factor 5.

Solution to Activity 40

(a) A rotation about the origin gives

$$\mathbf{A} = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix},$$

and

$$\mathbf{C} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) The results from part (a) give

$$\mathbf{BA} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\sqrt{3} - \frac{1}{4}\sqrt{3} & -\frac{3}{4} - \frac{1}{4} \\ \frac{1}{4} + \frac{3}{4} & -\frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{C}.$$

Solution to Activity 41

(a) (i) The composite $g \circ f$ is the linear transformation represented by the matrix

$$\mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(ii) The composite $f \circ g$ is the linear transformation represented by the matrix

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b) The two-stage diagram shows the effect of first doing f, the horizontal shear, then doing g. That is, it illustrates the transformation $g \circ f$.

Solution to Activity 42

The matrix of the reflection f is

$$\mathbf{A} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix},$$

and the matrix of the rotation g is

$$\mathbf{B} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

so $g \circ f$ is represented by the matrix

$$\begin{aligned}\mathbf{BA} &= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos 2\theta - \sin\phi\sin 2\theta & \cos\phi\sin 2\theta + \sin\phi\cos 2\theta \\ \sin\phi\cos 2\theta + \cos\phi\sin 2\theta & \sin\phi\sin 2\theta - \cos\phi\cos 2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi + 2\theta) & \sin(\phi + 2\theta) \\ \sin(\phi + 2\theta) & -\cos(\phi + 2\theta) \end{pmatrix}.\end{aligned}$$

This matrix represents the reflection in a line ℓ through the origin. If α is the angle from the x-axis to ℓ, then $\phi + 2\theta = 2\alpha$, so

$$\alpha = \tfrac{1}{2}\phi + \theta.$$

Geometrically, it is as if the rotation has turned the original mirror through the angle $\phi/2$ about the origin. If α lies in the interval $[0, \pi)$, then the angle of inclination of ℓ is α; otherwise, the angle of inclination of ℓ is obtained by subtracting from α the integer multiple of π that results in an angle in the interval $[0, \pi)$.

Solution to Activity 43

(a) The determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = 3 \times 5 - 2 \times 9 = -3.$$

Since $\det \mathbf{A}$ is non-zero, f is invertible.

(b) The inverse f^{-1} is the linear transformation represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} 5 & -2 \\ -9 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix}.$$

(c) The required point is $f^{-1}(6, 3)$ with position vector

$$\mathbf{A}^{-1} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 15 \end{pmatrix},$$

that is, the point $(-8, 15)$.
(As a check you may like to confirm that $f(-8, 15) = (6, 3)$.)

Solution to Activity 44

(a) The determinant of $\mathbf{A}$ is

$$\det \mathbf{A} = 2 \times 3 - 1 \times 1 = 5,$$

so f has an inverse transformation f^{-1} represented by the matrix

$$\mathbf{A}^{-1} = \tfrac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

(b) Each point (x, y) in the codomain of f is the image under f of the point $f^{-1}(x, y)$ with position vector

$$\begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{5}x - \frac{1}{5}y \\ -\frac{1}{5}x + \frac{2}{5}y \end{pmatrix}.$$

So if $(\frac{3}{5}x - \frac{1}{5}y, -\frac{1}{5}x + \frac{2}{5}y)$ lies on $\mathcal{C}$, then (x, y) lies on $f(\mathcal{C})$, and conversely if (x, y) lies on $f(\mathcal{C})$ then $(\frac{3}{5}x - \frac{1}{5}y, -\frac{1}{5}x + \frac{2}{5}y)$ lies on $\mathcal{C}$. It follows that

$$(\tfrac{3}{5}x - \tfrac{1}{5}y)^2 + (-\tfrac{1}{5}x + \tfrac{2}{5}y)^2 = 1.$$

Multiplying out the brackets and simplifying then gives

$$\begin{aligned} &\tfrac{1}{25}(9x^2 - 6xy + y^2 + x^2 - 4xy + 4y^2) = 1 \\ &\tfrac{1}{25}(10x^2 - 10xy + 5y^2) = 1 \\ &2x^2 - 2xy + y^2 = 5. \end{aligned}$$

So this is the equation of $f(\mathcal{C})$.

(Check: $f(1, 0) = (2, 1)$ (obtained from the first column of $\mathbf{A}$). Substituting $(x, y) = (2, 1)$ in the left-hand side of the equation above gives

$$2 \times 2^2 - 2 \times 2 \times 1 + 1^2 = 8 - 4 + 1 = 5.$$

So $(2, 1)$ satisfies the equation, as expected. You can do a similar check with $f(0, 1) = (1, 3)$, obtained from the second column of $\mathbf{A}$.)

(c) The area enclosed by $\mathcal{C}$ is π and f scales areas by the factor $|\det \mathbf{A}| = 5$, so the area enclosed by $f(\mathcal{C})$ is 5π.

Solution to Activity 45

(a) (i) E is centred at the origin and has vertices at the points $(3, 0)$, $(-3, 0)$, $(0, 2)$ and $(0, -2)$.

(ii) $\mathcal{C}$ can be mapped to E using a scaling by the factor 3 parallel to the x-axis and by the factor 2 parallel to the y-axis. That is, f is $(3, 2)$-scaling.

(iii) The matrix of f is

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

So f scales areas by the factor $|\det \mathbf{A}| = 6$.

Since $\mathcal{C}$ has area π, it follows that the ellipse has area 6π.

(b) In this case $\mathcal{C}$ is mapped to E by applying an (a, b)-scaling, that is, the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This transformation scales areas by the factor $|\det \mathbf{A}| = ab$, so the area of the ellipse is πab.

Solution to Activity 46

(a) The rotation through $\pi/4$ about the origin maps the line $y = -x$ to the x-axis as required. So in this activity the roles of the matrices $\mathbf{R}$ and $\mathbf{R}^{-1}$ in Example 23 are reversed. The matrix of the rotation through $\pi/4$ about the origin is

$$\mathbf{R} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and the matrix of the inverse rotation through $-\pi/4$ about the origin is

$$\mathbf{R}^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(b) The horizontal shear by the factor 2 is represented by the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

So the required shear by the factor 2 about the line $y = -x$ is represented by the matrix

$$\begin{aligned} \mathbf{R}^{-1}\mathbf{H}\mathbf{R} &= \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Solution to Activity 47

(a) The images of $(4,3)$ and $(0,0)$ are given by
$$\begin{pmatrix}3&1\\2&1\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}15\\11\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}17\\12\end{pmatrix}$$
and
$$\begin{pmatrix}3&1\\2&1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix},$$
respectively. So $f(4,3)=(17,12)$ and $f(0,0)=(2,1)$.

(b) If the affine transformation is thought of as a linear transformation followed by a translation, then the position vector of $f(0,0)$ is the vector associated with the translation.

Solution to Activity 48

The required affine transformation is given by
$$f(\mathbf{x})=\begin{pmatrix}7-5&8-5\\5-3&6-3\end{pmatrix}\mathbf{x}+\begin{pmatrix}5\\3\end{pmatrix},$$
that is
$$f(\mathbf{x})=\begin{pmatrix}2&3\\2&3\end{pmatrix}\mathbf{x}+\begin{pmatrix}5\\3\end{pmatrix}.$$
(Notice that the columns of the matrix are multiples of each other.)

Solution to Activity 49

Let
$$\mathbf{A}=\begin{pmatrix}4&3\\7&5\end{pmatrix}.$$
Then
$$\det\mathbf{A}=4\times5-7\times3=-1.$$
Since this is non-zero, f is invertible.
$$\mathbf{A}^{-1}=-\begin{pmatrix}5&-3\\-7&4\end{pmatrix}=\begin{pmatrix}-5&3\\7&-4\end{pmatrix},$$
so
$$\mathbf{A}^{-1}\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}-5&3\\7&-4\end{pmatrix}\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}-16\\23\end{pmatrix}.$$
Hence
$$\begin{aligned}f^{-1}(\mathbf{x})&=\begin{pmatrix}-5&3\\7&-4\end{pmatrix}\mathbf{x}-\begin{pmatrix}-16\\23\end{pmatrix}\\&=\begin{pmatrix}-5&3\\7&-4\end{pmatrix}\mathbf{x}+\begin{pmatrix}16\\-23\end{pmatrix}.\end{aligned}$$
(The final step is a little tidier, but not essential.)

Solution to Activity 50

(a) The required affine transformation is $f(\mathbf{x})=\mathbf{A}\mathbf{x}+\mathbf{a}$, where
$$\mathbf{A}=\begin{pmatrix}-1-4&3-4\\5+3&7+3\end{pmatrix}=\begin{pmatrix}-5&-1\\8&10\end{pmatrix}$$
and
$$\mathbf{a}=\begin{pmatrix}4\\-3\end{pmatrix}.$$

(b) It follows that the affine transformation scales areas by the factor
$$|\det\mathbf{A}|=|-5\times10-(-1)\times8|=42.$$
The triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ has area $\frac{1}{2}$, so the area of the triangle T must be $42\times\frac{1}{2}=21$.

Solution to Activity 51

(a) Here
$$\begin{aligned}(g\circ f)(\mathbf{x})&=g\left(\begin{pmatrix}1&1\\2&3\end{pmatrix}\mathbf{x}+\begin{pmatrix}2\\1\end{pmatrix}\right)\\&=\begin{pmatrix}2&3\\1&-3\end{pmatrix}\left(\begin{pmatrix}1&1\\2&3\end{pmatrix}\mathbf{x}+\begin{pmatrix}2\\1\end{pmatrix}\right)+\begin{pmatrix}2\\4\end{pmatrix}\\&=\begin{pmatrix}2&3\\1&-3\end{pmatrix}\begin{pmatrix}1&1\\2&3\end{pmatrix}\mathbf{x}+\begin{pmatrix}2&3\\1&-3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}\\&=\begin{pmatrix}8&11\\-5&-8\end{pmatrix}\mathbf{x}+\begin{pmatrix}7\\-1\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}.\end{aligned}$$
So
$$(g\circ f)(\mathbf{x})=\begin{pmatrix}8&11\\-5&-8\end{pmatrix}\mathbf{x}+\begin{pmatrix}9\\3\end{pmatrix}.$$

(b) For $g\circ f$ we have
$$\begin{aligned}(g\circ f)(\mathbf{x})&=g\left(\begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbf{x}+\begin{pmatrix}1\\1\end{pmatrix}\right)\\&=\begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbf{x}+\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}5\\1\end{pmatrix}.\end{aligned}$$
So
$$(g\circ f)(\mathbf{x})=\begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbf{x}+\begin{pmatrix}6\\2\end{pmatrix}.$$

For $f \circ g$ we have

$$\begin{aligned}(f \circ g)(\mathbf{x}) &= f\left(\mathbf{x} + \begin{pmatrix}5\\1\end{pmatrix}\right)\\ &= \begin{pmatrix}2&1\\1&2\end{pmatrix}\left(\mathbf{x} + \begin{pmatrix}5\\1\end{pmatrix}\right) + \begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbf{x} + \begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbf{x} + \begin{pmatrix}11\\7\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}.\end{aligned}$$

So

$$(f \circ g)(\mathbf{x}) = \begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbf{x} + \begin{pmatrix}12\\8\end{pmatrix}.$$

Solution to Activity 52

Let:

- h be the translation that maps $(3,1)$ to $(0,0)$
- g be the rotation through $\pi/4$ about $(0,0)$
- h^{-1} be the translation that maps $(0,0)$ to $(3,1)$.

Then

$$h(\mathbf{x}) = \mathbf{x} - \begin{pmatrix}3\\1\end{pmatrix},$$

$$g(\mathbf{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\mathbf{x},$$

$$h^{-1}(\mathbf{x}) = \mathbf{x} + \begin{pmatrix}3\\1\end{pmatrix}.$$

The required affine transformation is

$$\begin{aligned}(h^{-1} \circ g \circ h)(\mathbf{x}) &= h^{-1}(g(h(\mathbf{x})))\\ &= h^{-1}\left(g\left(\mathbf{x} - \begin{pmatrix}3\\1\end{pmatrix}\right)\right)\\ &= h^{-1}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}3\\1\end{pmatrix}\right)\right)\\ &= h^{-1}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\mathbf{x} - \frac{1}{\sqrt{2}}\begin{pmatrix}2\\4\end{pmatrix}\right)\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\mathbf{x} - \begin{pmatrix}\sqrt{2}\\2\sqrt{2}\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\mathbf{x} + \begin{pmatrix}3-\sqrt{2}\\1-2\sqrt{2}\end{pmatrix}.\end{aligned}$$

(You can easily check that the centre of rotation, at $(3,1)$, remains fixed under f.)

Solution to Activity 53

(a) Since the line $y = x$ makes an angle $\pi/4$ with the positive x-axis, it follows that reflection in this line is the linear transformation represented by the matrix

$$\begin{pmatrix}\cos(\pi/2) & \sin(\pi/2)\\ \sin(\pi/2) & -\cos(\pi/2)\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

(Alternatively, notice that the images of $(1,0)$ and $(0,1)$ under the reflection are $(0,1)$ and $(1,0)$, respectively. The position vectors of these images form the columns of the matrix.)

(b) Let:

- h be the translation that maps $(0,4)$ to $(0,0)$
- g be the reflection in the line $y = x$
- h^{-1} be the translation that maps $(0,0)$ to $(0,4)$.

Then

$$h(\mathbf{x}) = \mathbf{x} - \begin{pmatrix}0\\4\end{pmatrix},$$

$$g(\mathbf{x}) = \begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{x},$$

$$h^{-1}(\mathbf{x}) = \mathbf{x} + \begin{pmatrix}0\\4\end{pmatrix}.$$

The required affine transformation is

$$\begin{aligned}(h^{-1} \circ g \circ h)(\mathbf{x}) &= h^{-1}(g(h(\mathbf{x})))\\ &= h^{-1}\left(g\left(\mathbf{x} - \begin{pmatrix}0\\4\end{pmatrix}\right)\right)\\ &= h^{-1}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}0\\4\end{pmatrix}\right)\right)\\ &= h^{-1}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{x} - \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix}\right)\\ &= h^{-1}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{x} - \begin{pmatrix}4\\0\end{pmatrix}\right)\\ &= \begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{x} - \begin{pmatrix}4\\0\end{pmatrix} + \begin{pmatrix}0\\4\end{pmatrix}\\ &= \begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4\\4\end{pmatrix}.\end{aligned}$$

(You can easily check that a point on the line $y = x + 4$, such as $(0,4)$, remains fixed under f.)

Solution to Activity 54

(a) Let (x, y) be a fixed point. Then

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Using the formula for f gives

$$\begin{aligned}
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} -y+3 \\ x+5 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

This is equivalent to the system of equations

$$x = -y + 3$$
$$y = x + 5,$$

that is,

$$y + x = 3$$
$$y - x = 5.$$

Adding the two equations gives $2y = 8$, so $y = 4$. Substituting into the first equation gives $x = -1$. So $(-1, 4)$ is the only fixed point.

Hence f is a rotation about the point $(-1, 4)$.

(b) The matrix in the specification of g is the identity matrix, so g is simply the translation associated with the vector $7\mathbf{i} + 4\mathbf{j}$. (There are no fixed points.)

(c) Let (x, y) be a fixed point. Then

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Using the formula for h gives

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} y+3 \\ x+4 \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

This is equivalent to the system of equations

$$y + 3 = x$$
$$x + 4 = y,$$

that is,

$$x - y = 3$$
$$y - x = 4$$

Adding the first and second equations gives $0 = 7$, so there are no solutions.

Since the matrix of h is not the identity matrix, h is not a translation. It follows that h is a glide-reflection (and has no fixed points).

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