Unit 12

Systems of differential equations

Introduction

Systems of linear first-order differential equations were considered in Unit 6, and for these systems we were able to find an explicit solution. Here we consider qualitative, graphical methods that are applicable to all systems of first-order differential equations, but are of greatest value for those that we cannot solve explicitly, namely non-linear systems.

Recall from Unit 6 that the system of differential equations

$$\dot{x} = -y, \quad \dot{y} = 2x + y,$$

is called *linear* because the right-hand side of each equation does not contain the variables raised to a power (such as x^2) or as the argument of a non-linear function (such as $\sin x$), or products of the variables (such as xy). The systems of differential equations

$$\dot{x} = x^2 + y, \quad \dot{y} = y + t, \tag{1}$$

and

$$\dot{x} = xy, \quad \dot{y} = x^2 y, \tag{2}$$

are both *non-linear* systems of equations.

There is one difference between systems (1) and (2) that is important for this unit. In equations (1) the independent variable t appears explicitly on the right-hand side, whereas in equations (2) only the two dependent variables x and y occur. Systems such as (2), where t does not appear explicitly, are said to be *autonomous*. In this unit we will consider only autonomous equations of the general form

$$\dot{x} = u(x, y), \quad \dot{y} = v(x, y).$$

The methods developed in this unit are widely applicable to a wide variety of situations as differential equation models are common. For this reason we will generally consider the behaviour of systems of differential equations without any specific context, but we do develop one context to illustrate ideas. The situation that we develop is a model of the behaviour of two interacting populations of animals, usually with one variable $x = x(t)$ representing the number of individuals of a predator species, and the other variable $y = y(t)$ representing the number of individuals of its prey.

We use the notations x or $x(t)$, $\dot{x}$ or $\dot{x}(t)$, etc., interchangeably to suit the context.

The graphical methods that we develop use a diagram to give information about solutions of a system without first needing to calculate the solutions. From such diagrams we can answer questions such as deciding whether two populations of animals can coexist with stable populations or whether one population will die out. A solution to the predator and prey model $x(t) = X$, $y(t) = Y$, where X and Y are constants, describes a situation where the two populations coexist with stable populations. Such constant solutions are usually significant, so we give them a name, *equilibrium solutions*, and we call the point (X, Y) in the (x, y)-plane an *equilibrium point*.

Section 1 begins by looking at a way of visualising systems of differential equations and then derives a mathematical model for the interacting population model described above. Section 2 focuses on finding any equilibrium points, and Section 3 looks at classifying the nature of the solutions near an equilibrium point. Section 4 then considers the behaviour of solutions far from equilibrium points.

Note that it is also possible to use a computer to calculate numerical solutions to non-linear equations that cannot be solved explicitly, which is sometimes very useful. However, these numerical solutions are particular solutions with particular initial conditions and cannot answer questions about the set of all solutions in the same way that the graphical methods described in this unit can. These two approaches, graphical and numerical, are in many ways complementary.

1 Visualising systems of differential equations

It is not possible to find algebraic solutions of all systems of differential equations, so we introduce a graphical approach, based on the notion that a point $(x, y) = (x(t), y(t))$ in the plane may be used to represent two variables $x = x(t)$ and $y = y(t)$ at time t. As t increases, the point $(x, y) = (x(t), y(t))$ traces a path that represents the variation of both variables with time. This section also introduces the predator and prey model mentioned in the Introduction, and concludes by describing a method that can be used to derive plenty more examples of systems of differential equations.

1.1 Direction fields revisited

Before considering graphical methods for systems of first-order differential equations, we recall the graphical method for first-order differential equations described in Unit 1, namely direction fields. Consider the differential equation

$$\dot{x} = kx, \quad x > 0, \tag{3}$$

where k is a constant.

This simple differential equation arises in many contexts that involve growth or decay. Among these, it arises as a model for the population size x – which we usually simply refer to as the population x (omitting the word 'size') – as a function of time t. A population x can take only integer values, so we say that x is a discrete variable. It is often convenient to approximate a discrete variable by a variable that can take any real value, referred to as a continuous variable. This approximation will be good if the population size is large. Here we measure populations in hundreds or thousands, as appropriate, so we are able to use quite small numbers to represent large populations in our models. In the continuous model, the derivative $\dot{x}$ represents the rate of increase of the population, which we often refer to as the **growth rate** (even though if $\dot{x} < 0$, it actually represents a decay rate – compare the use in mechanics of ‘acceleration’ to cover both of the everyday terms ‘acceleration’ and ‘deceleration’).

Equation (3) can be solved using the methods of Unit 1, which we ask you to do now.

Exercise 1

Use the methods of Unit 1 to find the particular solution of equation (3) that satisfies the initial condition $x = x_0$ when $t = 0$.

Here we will be concerned not with finding explicit solutions of differential equations such as the one found in Exercise 1, but rather with using graphical methods to determine general features of the solution. As a first step along this path, consider the *direction field* of a differential equation that was introduced in Unit 1.

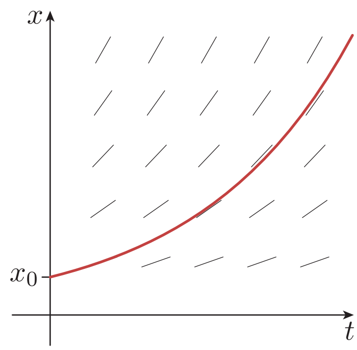


Figure 1 Direction field for $\dot{x} = kx$, with $k > 0$; the red curve shows the solution with initial condition $x(0) = x_0$

Figure 1 shows a direction field for $\dot{x} = kx$ together with a graph of the solution that you obtained in Exercise 1. Recall from Unit 1 that the solution is a curve whose tangent at any point has a slope that is equal to the value of the direction field at that point.

The direction field shown at a point indicates the slope of the solution curve passing through that point. Once the direction field is plotted, a sketch of a solution curve can be obtained by smoothly drawing a curve along the lines of the direction field, as shown in Figure 1. This time we calculated the exact solution first, using Exercise 1, but the key point is that we could sketch an approximate solution using the direction field as a guide. In the next section we generalise this from differential equations to systems of differential equations.

1.2 Pictures of solutions

Here we describe a graphical method that can be used to visualise the solutions of systems of first-order differential equations in a similar way to the way in which direction fields can be used to visualise differential equations.

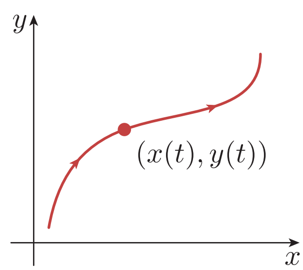


Figure 2 The evolution of two variables is represented by a path in the plane

Consider the two variables x and y, which could represent the number of individuals in a pair of populations. Our purpose is to determine how these variables evolve with time. At a particular time t, we suppose that these populations are $x(t)$ and $y(t)$, respectively. We represent this system by a point in the (x, y)-plane. The evolution of the two populations can be represented as a **path**, as shown in Figure 2, where the directions of the arrows on the path indicate the directions in which the point $(x(t), y(t))$ moves along the path with increasing time. Note that this representation does not show how quickly or slowly the point moves along the path but merely the direction of travel.

For the purposes of this discussion, suppose that the populations are evolving independently (perhaps on separate islands) with *no interactions*. The reason for this is that it makes the equations easy to solve, so the solutions can be compared with the graphical method that we are about to introduce. So consider the equations

$$\dot{x} = 0.2x, \quad \dot{y} = 0.3y. \tag{4}$$

(If the variables x and y represent populations, then we must have $x \geq 0$ and $y \geq 0$ in order to be physically reasonable. Here we are interested in developing general methods, so we do not impose this restriction.)

Note that each of equations (4) is in the form of the differential equation (3) considered previously, so they both represent exponential growth.

Equations (4) form a system of linear differential equations, which you met in Unit 6. Using vector notation, the pair of populations may be represented by the vector $\mathbf{x} = (x \quad y)^T$. The system of equations (4) then becomes the vector equation

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0.2x \\ 0.3y \end{pmatrix}. \tag{5}$$

Vector fields are discussed in more detail in Unit 15.

It is helpful now to introduce the notion of a **vector field**, which is similar to a direction field. In a plane, a direction field associates a direction $f(x,y)$ with each point (x,y), whereas a vector field associates a vector $\mathbf{u}(x,y)$ with each point (x,y). The vector field associated with equation (5) is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} 0.2x \\ 0.3y \end{pmatrix}, \tag{6}$$

so equation (5) becomes $\dot{\mathbf{x}} = \mathbf{u}(x,y)$, which will sometimes be written more simply as $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$. We could use this definition to calculate a direction at any point. For example, at the point $(1,2)$, the vector field has the direction $\mathbf{u}(1,2) = (0.2 \times 1 \quad 0.3 \times 2)^T = (0.2 \quad 0.6)^T$. Calculating vectors at several points enables us to construct Figure 3, which shows a plot of this vector field where an arrow in the direction of each vector $\mathbf{u}$ is placed with its midpoint at the point where the vector was calculated.

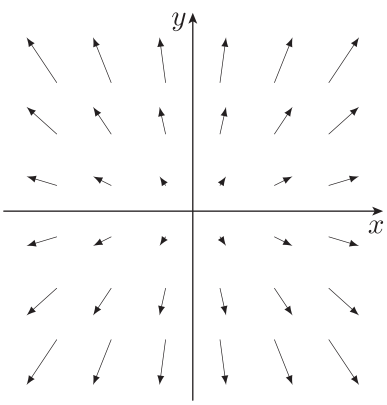


Figure 3 A representation of the vector field $\mathbf{u}(x,y)$

A direction field $f(x,y)$ represents the slope of a particular solution of the differential equation $dy/dx = f(x,y)$ at the point (x,y). Similarly, $\mathbf{u}(x,y)$ is the vector $\dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ that is tangential to a particular solution of $\dot{\mathbf{x}} = \mathbf{u}(x,y)$ at the point (x,y), because the slope of the tangent is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}.$$

This suggests a geometric way of finding a particular solution of equation (5): choose a particular starting point (x_0, y_0), then follow the directions of the tangent vectors. (An exception, which we discuss later, occurs when $\dot{x} = \dot{y} = 0$ at (X,Y), so $\mathbf{u}(X,Y) = \mathbf{0}$ and there is no tangent vector to follow.)

The essential difference between a direction field and a vector field is that the former consists of line segments, and the latter consists of *directed* line segments (which we indicate by arrows) whose lengths indicate the magnitude of $\mathbf{u}(x, y)$. However, since the magnitudes of $\mathbf{u}(x, y)$ may vary considerably and so make the diagram difficult to interpret, we often use arrows of a fixed length to show the direction of a vector field. This is acceptable because, in many cases, it is the direction of $\mathbf{u}(x, y)$ that is our primary concern, rather than its magnitude. Consider Figure 3, where the arrows become longer as distance from the origin increases, so if the arrows are scaled so that the longer arrows do not overlap, then it is hard to discern the direction of the arrows near the origin. To overcome this problem, we use the convention that all arrows will be scaled to the same length. The arrows shown in Figure 4 represent the same vector field but use this scaling convention.

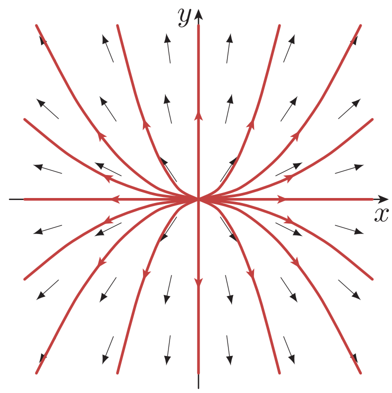


Figure 4 The vector field $\mathbf{u}(x, y)$ together with paths from the family of solutions

Using the methods of Unit 6, we can find the general solution of equations (4) as

$$x(t) = Ce^{0.2t}, \quad y(t) = De^{0.3t}, \tag{7}$$

where C and D are constants.

This general solution gives the position $(x(t), y(t))$ at time t. It is instructive to plot this solution for a range of time t for various values of the constants C and D. This will show the particular solutions for various initial conditions among the family of general solutions. These particular solutions are overlaid on the vector field plot in Figure 4. The arrows on the solution curves indicate the directions in which the curves are traversed with increasing time.

A solution curve along which the coordinates x and y vary as t increases is called a **phase path** (or **orbit**). The (x, y)-plane containing the solution curves is called the **phase plane**, and a diagram, such as Figure 4, showing the phase paths is called a **phase portrait**. In other words, a phase portrait is a collection of phase paths that illustrates the behaviour of the differential equations.

You may have noticed in Figure 4 that the paths radiate *outwards* from the origin in all directions. For this reason, we refer to the origin as a **source**.

A source can occur at a point other than the origin.

We now look at the phase paths for a similar system, for which

$$\mathbf{u}(x, y) = \begin{pmatrix} -0.2x \\ -0.3y \end{pmatrix}.$$

Again by using the methods of Unit 6, the general solution can be found as

$$x = Ce^{-0.2t}, \quad y = De^{-0.3t},$$

where C and D are constants.

This solution is in terms of exponential functions with negative exponents, so it represents exponential decay. As this general solution is the same as equations (7) except for the change in sign of the multiples of t, we can see that the paths in the phase plane are the same as before, but with the arrows reversed.

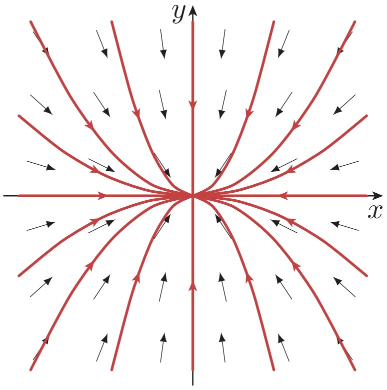


Figure 5 The vector field and solution paths of a *sink*

Another way of looking at this is to say that at each point in the phase plane, the vector field that represents these equations points in the opposite direction to the previous vector field. This gives the diagram shown in Figure 5. Now the paths radiate inwards *towards* the origin, and for this reason we refer to the origin as a **sink**.

Exercise 2

Write down and solve the system of differential equations $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ given by the vector field

$$\mathbf{u}(x, y) = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

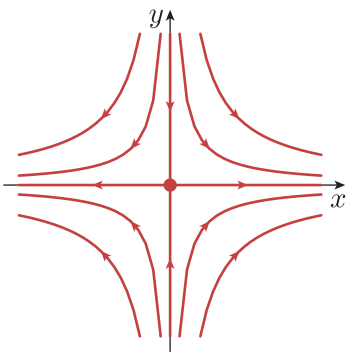


Figure 6 Phase paths for the vector field in Exercise 2, which shows a *saddle point* at the origin

You should find that if you multiply together your solutions for $x(t)$ and $y(t)$ found in Exercise 2, the result is a constant. So the phase paths have the equation $xy = A$, for some constant A. These paths are a family of rectangular hyperbolas (by choosing A to be non-zero) or lines along the axes (by choosing A to be zero). Hence we can sketch the phase portrait for the vector field examined in Exercise 2 to be as shown in Figure 6. You can see that the vast majority of the paths do not radiate into or out of the origin. On these paths, a point initially travels towards the origin, but eventually travels away from it again. The only paths that actually radiate inwards towards or outwards from the origin are those along the x- and y-axes. In this case we call the origin a **saddle point** (the paths look like the contours around a saddle point as defined in Unit 7).

This unit is concerned with sketching phase portraits. Before moving on to give general methods for doing this, we introduce some examples to which we can apply our methods.

1.3 Modelling populations of predators and prey

Here we develop a model for the populations of a predator and its prey. A predator population depends for its survival on there being sufficient prey to provide it with food. Intuition suggests that when the number of predators is low, the prey population may increase quickly, and that this in turn will result in an increase in the predator population. On the other hand, a large number of predators may diminish the prey population until it is very small, and this in turn will lead to a collapse in the predator population. Our mathematical model will need to reflect this behaviour.

To make the discussion more concrete, we will consider modelling populations of foxes and rabbits as the predators and prey. Let $x(t)$ be the number of rabbits, and let $y(t)$ be the number of foxes.

When this model was first proposed, the interaction was between two species of fish in the Adriatic Sea.

For a population x of rabbits in a fox-free environment, our first model for population change is given by the equation $\dot{x} = kx$, where k is a positive constant. This represents exponential growth. However, if there is a population y of predator foxes, then you would expect the growth rate $\dot{x}$ of rabbits to be reduced.

Similarly, for a population y of foxes in a rabbit-free environment, our first model for the population change is given by the equation $\dot{y} = -hy$, where h is a positive constant. This represents exponential decay. However, if there is a population x of rabbits for the foxes to eat, then you would expect the growth rate $\dot{y}$ of foxes to increase.

In our mathematical model we make the following assumptions.

- There is plenty of vegetation for the rabbits to eat.
- The rabbits are the only source of food for the foxes.
- An encounter between a fox and a rabbit contributes to the fox's larder, which leads directly to a decrease in the rabbit population and indirectly to an increase in the number of foxes.
- The number of encounters between foxes and rabbits is proportional to the number of rabbits multiplied by the number of foxes – that is, it is proportional to xy.
- The growth rate $\dot{x}$ of rabbits decreases by a factor that is proportional to the number of encounters between rabbits and foxes.
- The growth rate $\dot{y}$ of foxes increases by a factor that is proportional to the number of encounters between foxes and rabbits.

These assumptions lead to a differential equation that models the population x of rabbits as

$$\dot{x} = kx - Axy,$$

This equation can be derived using the input–output principle, which was introduced in Unit 1, Section 1. In a period of time δt, the change in the rabbit population is the number $kx\,\delta t$ of additional rabbits born, taking into account those dying from natural causes, less the number $Axy\,\delta t$ of rabbits eaten.

for some positive constant A. As we will see later, it is convenient to write $A = k/Y$ for some positive constant Y, giving

$$\dot{x} = kx\left(1 - \frac{y}{Y}\right). \tag{8}$$

This is a non-linear equation, since the right-hand side contains an xy term.

Similarly, our revised model for the foxes is given by

$$\dot{y} = -hy + Bxy$$

for some positive constant B. Again, it is convenient to write $B = h/X$ for some positive constant X, so that this equation becomes

$$\dot{y} = -hy\left(1 - \frac{x}{X}\right). \tag{9}$$

Again, this equation is non-linear because of the xy term on the right-hand side.

Together, the differential equations (8) and (9) model the pair of interacting populations.

Exercise 3

Sketch the graph of the proportionate growth rate $\dot{x}/x$ of rabbits as a function of the population y of foxes, and the graph of the proportionate growth rate $\dot{y}/y$ of foxes as a function of the population x of rabbits. Interpret these graphs.

Lotka–Volterra equations

The evolution of two interacting populations x and y can be modelled by the **Lotka–Volterra equations**

$$\dot{x} = kx\left(1 - \frac{y}{Y}\right), \quad \dot{y} = -hy\left(1 - \frac{x}{X}\right) \quad (x \geq 0,\ y \geq 0), \tag{10}$$

where x is the population of the prey and y is the population of the predators, and k, h, X and Y are positive constants.

Modelling success

This model was one of the first successful applications of mathematical models to biological systems. It was independently proposed in 1925 by the American biophysicist Alfred Lotka and in 1926 by the Italian mathematician Vito Volterra.

Now we begin to explore the Lotka–Volterra equations.

Exercise 4

(a) Write down the vector field $\mathbf{u}(x, y)$ that corresponds to the Lotka–Volterra equations.

(b) Now suppose that the variables x and y represent thousands of individuals (so that $x = 1$ represents one thousand rabbits, for example) and further suppose that the constants in equations (10) have the values $k = 1$, $h = \frac{1}{2}$, $X = 3$ and $Y = 2$. Complete the following table.

x	y	$\mathbf{u}(x, y)$
0	0	
0	2	
2	0	
2	2	
3	1	
3	2	
3	3	
4	2	

(c) Draw the vectors that you obtained in part (b) as a vector field in the phase plane.

Previously, we were able to solve the pairs of differential equations that arose from our mathematical model, but for equations (10) no explicit formulas for $x(t)$ and $y(t)$ are available. We will use graphical methods to describe the solutions. The vector field plot that you drew in Exercise 4 (partly reproduced as Figure 7(a)) is a starting point. From this plot it may seem possible that the phase path forms a closed loop about the point $(3, 2)$, but this is far from certain. Figure 7(b) shows a sketch of a possible phase path.

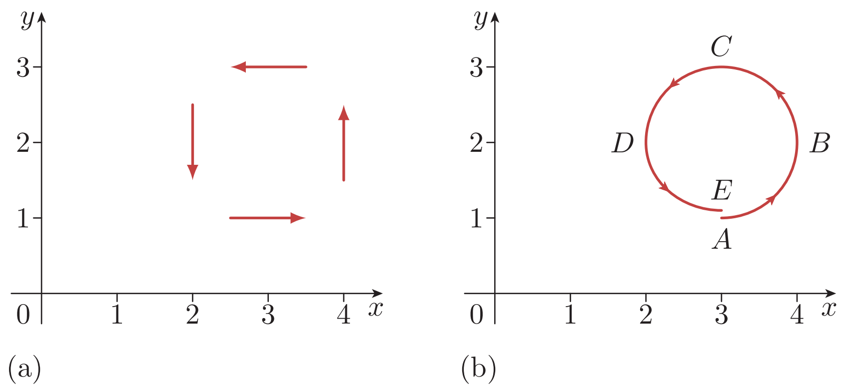


Figure 7 (a) Part of the vector field drawn in Exercise 4. (b) A possible path for the changes in rabbit and fox populations that follows the vector field.

Consider a phase path starting at point A, which is the point $(3, 1)$ representing 3000 rabbits and 1000 foxes. From the vector field plot in Exercise 4, we see that the rabbit population increases and so does the fox population, until at the point B we have reached a maximum rabbit population. As the fox population continues to rise, the rabbit population goes into decline. At C, the fox population has reached its maximum, while the rabbits decline further. After this point, there are not enough rabbits available to sustain the number of foxes, and the fox population also goes into decline. At D, the declining fox population gives some relief to the rabbit population, which begins to pick up. Finally, at E, the decline of the fox population is halted as the rabbit population continues to increase. We may even return exactly to the point A, in which case the cycle will repeat indefinitely.

In order to decide whether the cycle will repeat indefinitely, we need to do more than simply plotting a few arrows in the phase plane. This question is discussed further in Section 3 when we have introduced the ideas that are needed to resolve it. Here we conclude this section by looking at a way of generating many more examples of systems of differential equations.

1.4 More examples

In this subsection we describe a method for converting a higher-order differential equation into a system of first-order differential equations. Using this technique we can apply the graphical methods developed in this unit to a much wider range of problems.

As a first example, consider the simple harmonic motion equation

$$\ddot{x} + \omega^2 x = 0, \tag{11}$$

which is a linear second-order differential equation with one dependent variable x and one constant ω. Define $y = \dot{x}$ to be a second variable. Then by differentiation with respect to t we have $\dot{y} = \ddot{x}$. Substituting for $\ddot{x}$ in equation (11) gives $\dot{y} + \omega^2 x = 0$. The two differential equations relating x and y are then

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x,$$

which is a system of two first-order differential equations.

This idea of introducing new variables to represent derivatives of the independent variable can be used to transform any higher-order equation to a system of first-order equations. Try this yourself by attempting the following exercise.

Exercise 5

Convert each of the following differential equations into a system of first-order equations.

(a) $\ddot{x} + \sin x = 0$ (b) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x^2 = 0$ (c) $\dfrac{d^3u}{dx^3} = 6u\dfrac{du}{dx}$

Using this method you can see that any situation that leads to a differential equation model (such as any dynamics problem) can be converted into a form to which we can apply the graphical methods described in this unit. These methods are widely applicable.

Note that not every system of first-order differential equations can be converted into a single higher-order differential equation by *reversing* the above strategy. For example, the Lotka–Volterra equations cannot be written as a single second-order differential equation in this way.

This strategy of converting a higher-order differential equation to a system of first-order differential equations is also useful in the numerical solution of differential equations. Many numerical methods (e.g. Euler's method, which you met in Unit 1) apply only to first-order differential equations. These methods can be extended in a straightforward manner to apply to systems of first-order equations (by replacing scalar variables with vector variables), and this is the main method for solving higher-order differential equations on a computer. The numerical solutions of the weather prediction model employed to produce daily weather forecasts use exactly this strategy.

Now we have plenty of examples of systems of first-order differential equations and a graphical picture (the phase plane) to represent them. We go on to analyse the features of the graphical picture. First, we look at steady-state solutions of the equations (such as the point $(3, 2)$ in Exercise 4) where two populations can coexist in equilibrium.

2 Equilibrium points

The previous section introduced phase portraits as a method of visualising systems of non-linear differential equations. Now we begin to investigate features of phase plane portraits, and we start with the features called *equilibrium points*.

2.1 Finding equilibrium points

There is one and only one phase path through a point in the phase plane – this means that phase paths do not cross. This is because the vector field has only one direction at each point in the phase plane. But what is happening at the origin in Figure 8? Here there seem to be phase paths along the coordinate axes that appear to cross – but all is not as it seems. The solution along the positive x-axis is given by $x = Ce^t$, $y = 0$, and for all values of t, the x-coordinate is positive (since the exponential function is never zero). So the origin is not included on this phase path. Similarly, the phase paths along the other three axis directions do not include the origin. The origin is a constant solution of the system of differential equations that forms a separate path consisting of just a single point. So the phase paths do not cross at the origin as the origin is on its own path and all the paths along the coordinate axes approach but never reach the origin.

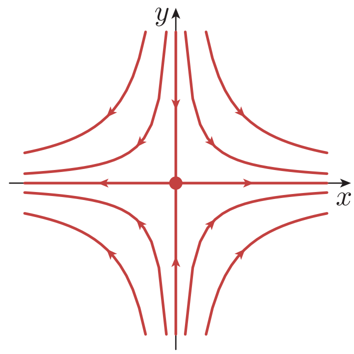


Figure 8 A saddle point at the origin

These single points, such as the origin in Figure 8, are important features of phase portraits, so we make the following definition.

An **equilibrium point** of a system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x, y)$$

is a point (X, Y) such that $x(t) = X$, $y(t) = Y$ is a constant solution of the system, that is, (X, Y) is a point at which $\dot{x}(t) = 0$ and $\dot{y}(t) = 0$.

Equilibrium points generally represent important physical properties of the system being modelled. For example, the equilibrium point $(3, 2)$ in Exercise 4 corresponds to a solution where the populations of rabbits and foxes are in stable coexistence, that is, they are not changing with time.

The definition of an equilibrium point leads directly to the following procedure for finding equilibrium points.

Procedure 1 Finding equilibrium points

To find the equilibrium points of the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x, y)$$

for some vector field $\mathbf{u}$, solve the equation

$$\mathbf{u}(x, y) = \mathbf{0}.$$

Solving $\mathbf{u}(x, y) = \mathbf{0}$ requires the solution of two simultaneous equations, which are generally non-linear, for the unknowns x and y, as the following example shows.

Example 1

Find the equilibrium points for the Lotka–Volterra equations (10) for the rabbit and fox populations. (Remember that h, k, X and Y are *positive* constants.)

Solution

Using Procedure 1, we need to solve the equation $\mathbf{u}(x, y) = \mathbf{0}$, which becomes

$$\begin{pmatrix} kx\left(1 - \dfrac{y}{Y}\right) \\ -hy\left(1 - \dfrac{x}{X}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the simultaneous equations

$$kx\left(1 - \frac{y}{Y}\right) = 0, \tag{12}$$

$$-hy\left(1 - \frac{x}{X}\right) = 0. \tag{13}$$

It is important to be methodical when solving simultaneous non-linear equations, in order to avoid missing solutions. Here we will first solve equation (12) and then substitute each solution into equation (13) to find all solutions. Equation (12) is already factorised and the solutions arise when either term is zero, so the solutions are $x = 0$ or $y = Y$.

Substituting $x = 0$ into equation (13) gives $-hy = 0$, so $y = 0$ and hence $(0, 0)$ is an equilibrium point.

Substituting $y = Y$ into equation (13) gives $-hY(1 - x/X) = 0$, so $x = X$ and hence (X, Y) is an equilibrium point. As both X and Y are positive, this equilibrium point is always in the first quadrant (thus is always in the quadrant $x > 0$, $y > 0$ that is physically relevant for population models).

Thus there are two possible equilibrium points for the pair of populations. The first has both the rabbit and fox populations zero, that is, the equilibrium point is at $(0, 0)$; there are no births or deaths – nothing happens. However, the other equilibrium point occurs when there are

X rabbits and Y foxes, that is, the equilibrium point is at (X, Y); the births and deaths exactly cancel out and both populations remain constant.

This explains our choice of constants X and Y in Subsection 1.3.

Now try this yourself by attempting the following exercises.

Exercise 6

Suppose that two variables x and y evolve according to the system of differential equations

$$\dot{x} = x(20 - y), \quad \dot{y} = y(10 - y)(10 - x).$$

Find the equilibrium points of the system.

Exercise 7

Find the equilibrium points of the system of differential equations

$$\dot{x} = 2x^2y + 7xy^2 + 2y + 1,$$
$$\dot{y} = xy - x.$$

2.2 Stability of equilibrium points

In a real ecosystem it is unlikely that predator and prey populations are in perfect harmony. What if equilibrium is disturbed by a small deviation caused perhaps by a severe winter or hunting? If the number of rabbits is reduced, there would be a decreased food supply for the foxes, and the population of foxes could decrease to zero as a consequence. On the other hand, if the number of foxes is reduced, the birth rate for rabbits would then exceed their death rate, and the number of rabbits could increase without limit.

If a small change or *perturbation* in the populations of rabbits and foxes from their equilibrium values, no matter what the cause, results in subsequent populations that remain close to their equilibrium values, then we say that the equilibrium point is **stable**. On the other hand, if a perturbation results in a catastrophic change, with, for example, the population of foxes or rabbits collapsing to zero or increasing without limit, then we say that the equilibrium point is **unstable**.

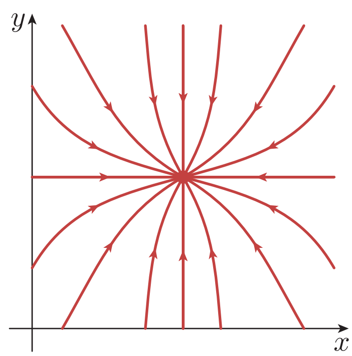


Figure 9 A stable equilibrium point

In the phase portrait in Figure 9, where the equilibrium point is a sink, you can see that any slight perturbation from the equilibrium point will result in a point that returns to the equilibrium point as time t increases. So this is a *stable* equilibrium point. Similarly, the point $(3, 2)$ in Exercise 4 is a *stable* equilibrium point. In this case a perturbation from the equilibrium point does not result in a point that returns to the equilibrium point as t increases, but it does result in a point that remains in the neighbourhood of the equilibrium point.

On the other hand, the origin in the phase portrait of a source shown in Figure 4 is an *unstable* equilibrium point. Any perturbation from the origin will result in the point travelling further and further away from the origin with time. Similarly, the origin in the phase portrait of a saddle shown in Figure 6 is an *unstable* equilibrium point. Apart from increases or decreases in y with x unchanged, any perturbation will result in a point that travels further and further away from the origin with time.

Stability of equilibrium points

An equilibrium point is said to be:

- **stable** when all points in the neighbourhood of the point remain in the neighbourhood of the point as time increases
- **unstable** otherwise.

Exercise 8

Consider the paths shown in Figure 10.

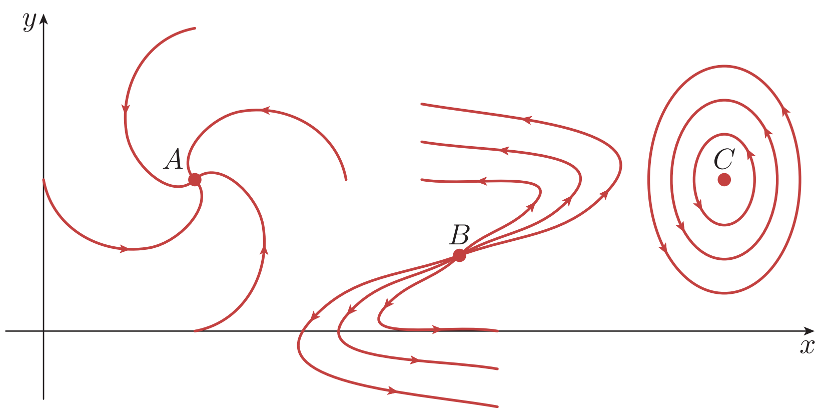


Figure 10 Three equilibrium points labelled A, B and C, together with phase paths in the neighbourhood of each point

Classify as stable or unstable each of the equilibrium points A, B and C.

2.3 Behaviour close to equilibrium

To investigate the behaviour of a non-linear system in the neighbourhood of equilibrium points, in this subsection we develop *linear* approximations to the system that are applicable close to the equilibrium points.

An example will make this clearer, so consider the system of non-linear equations

$$\dot{x} = y(x-1), \quad \dot{y} = x^2(y-2). \tag{14}$$

This system of equations has an equilibrium point $(1, 2)$. We wish to investigate the behaviour of solutions close to this equilibrium point, so we change coordinates by letting

$$x = 1 + p, \quad y = 2 + q. \tag{15}$$

If x and y represent populations, then they cannot be negative. However, the perturbations p and q can (usually) be negative as this would represent populations that are less than the equilibrium values.

For solutions close to equilibrium, (x, y) is close to the point $(1, 2)$, so (p, q) is close to $(0, 0)$, which is another way of saying that p and q should be small. By differentiating equations (15) we obtain $\dot{p} = \dot{x}$ and $\dot{q} = \dot{y}$, so we can write equations (14) as

$$\dot{p} = (2 + q)p = 2p + pq,$$
$$\dot{q} = (1 + p)^2 q = q + 2pq + p^2 q.$$

If p and q are small, then pq is much smaller (and $p^2 q$ is much smaller than this). So neglecting terms that are very small compared to p and q gives the linear system of equations

$$\dot{p} = 2p, \quad \dot{q} = q. \tag{16}$$

As the terms that we neglected were small compared to the terms that we retained, we would expect the solutions of this linear system to approximate the solutions of equations (14) near the equilibrium point. Equations (16) can be solved by separation of variables to give the solutions $p = Ae^{2t}$, $q = Be^t$, where A and B are constants. These equations are exponential growth equations, so the equilibrium point at $p = 0$, $q = 0$ is unstable. The behaviour of the solution for (x, y) close to $(1, 2)$ is just a translation of the behaviour of (p, q) close to $(0, 0)$ thus should be of the same type, so the point $(1, 2)$ is an unstable equilibrium point.

For equations (14) it was straightforward to see which terms to neglect since the right-hand sides of both differential equations were polynomials, but this would not be the case if the right-hand sides involve sinusoidal terms. We now show how Taylor polynomials can be used to find linear approximations in general.

Taylor polynomials were introduced in Unit 7.

If (X, Y) is an equilibrium point, consider small perturbations p and q giving new populations x and y defined by

$$x = X + p, \quad y = Y + q. \tag{17}$$

We can find the time development of the small perturbations p and q by linearising the differential equation $\dot{\mathbf{x}} = \mathbf{u}(x, y)$. We will make use of Taylor polynomials to achieve this. In order to do so, we must write each component of the vector $\mathbf{u}(x, y)$ as a function of the two variables x and y:

$$\mathbf{u}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

At the equilibrium point (X, Y), we have $\mathbf{u}(X, Y) = \mathbf{0}$, that is,

$$u(X, Y) = 0 \quad \text{and} \quad v(X, Y) = 0.$$

Now, for small perturbations p and q, we can use the linear Taylor polynomial for functions of two variables to approximate each of $u(x, y)$ and $v(x, y)$ near the equilibrium point (X, Y). Here we use the more compact notation u_x for $\partial u/\partial x$ etc.:

$$\begin{aligned} u(X+p, Y+q) &\simeq u(X, Y) + p\, u_x(X, Y) + q\, u_y(X, Y) \\ &= p\, u_x(X, Y) + q\, u_y(X, Y), \end{aligned}$$

since $u(X, Y) = 0$, and

$$\begin{aligned} v(X+p, Y+q) &\simeq v(X, Y) + p\, v_x(X, Y) + q\, v_y(X, Y) \\ &= p\, v_x(X, Y) + q\, v_y(X, Y), \end{aligned}$$

since $v(X, Y) = 0$.

The above two equations appear rather unwieldy, but are much more succinctly represented in matrix form:

$$\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} u_x(X, Y) & u_y(X, Y) \\ v_x(X, Y) & v_y(X, Y) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Since $x(t) = X + p(t)$ and $y(t) = Y + q(t)$, we also have

$$\dot{x} = \dot{p}, \quad \dot{y} = \dot{q}.$$

Putting the pieces together, substituting in $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ gives a system of *linear* differential equations for the perturbations p and q:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} u_x(X, Y) & u_y(X, Y) \\ v_x(X, Y) & v_y(X, Y) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{18}$$

Some examples will help to make this clear.

Example 2

Suppose that two variables x and y evolve according to the system of differential equations

The equilibrium points for these equations were found in Exercise 6.

$$\dot{x} = x(20 - y), \quad \dot{y} = y(10 - y)(10 - x).$$

Find the linear approximation to these equations near the equilibrium point $(0, 10)$.

Solution

Here we have

$$u(x, y) = x(20 - y), \quad v(x, y) = y(10 - y)(10 - x).$$

So the partial derivatives are

$$\begin{aligned} &u_x(x, y) = 20 - y, \quad u_y(x, y) = -x, \\ &v_x(x, y) = -y(10 - y), \quad v_y(x, y) = (10 - y)(10 - x) - y(10 - x). \end{aligned}$$

Evaluating these at the given point $(0, 10)$ yields

$$\begin{aligned} &u_x(x, y) = 10, \quad u_y(x, y) = 0, \\ &v_x(x, y) = 0, \quad v_y(x, y) = -100. \end{aligned}$$

So the linear system that approximates the given system near the point $(0, 10)$ is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & -100 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Example 3

Transform the Lotka–Volterra equations (10) into a system of linear differential equations for the perturbations p and q from the equilibrium point (X, Y).

Solution

Here we have

$$u(x,y) = kx\left(1 - \frac{y}{Y}\right), \quad v(x,y) = -hy\left(1 - \frac{x}{X}\right).$$

First, we compute the partial derivatives, obtaining

$$u_x(x,y) = k\left(1 - \frac{y}{Y}\right), \quad u_y(x,y) = -\frac{kx}{Y},$$
$$v_x(x,y) = \frac{hy}{X}, \quad v_y(x,y) = -h\left(1 - \frac{x}{X}\right).$$

Evaluating these at the point (X, Y) gives

$$u_x(X,Y) = 0, \quad u_y(X,Y) = -\frac{kX}{Y},$$
$$v_x(X,Y) = \frac{hY}{X}, \quad v_y(X,Y) = 0.$$

Thus the required system of linear differential equations is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -kX/Y \\ hY/X & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{19}$$

Note that equation (19) can be written as the pair of equations

$$\dot{p} = -\frac{kX}{Y}q, \quad \dot{q} = \frac{hY}{X}p,$$

so we have replaced a system of non-linear equations, for which we have no algebraic solution, with a pair of *linear* equations that we can solve using the methods of Unit 6. We should expect the solutions of equation (19) to provide a good approximation to the original system only when p and q are small (i.e. when the system is close to equilibrium).

Now seems to be an appropriate time to summarise what we have done and to give a name to the matrix that arises. The matrix

$$\mathbf{J}(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix}$$

is called the **Jacobian matrix** of the vector field

$$\mathbf{u}(x,y) = (u(x,y) \quad v(x,y))^T.$$

The 2×2 matrix on the right-hand side of equation (18) is this Jacobian matrix evaluated at the equilibrium point (X, Y), so equation (18) can be written succinctly as

$$\dot{\mathbf{p}} = \mathbf{J}\mathbf{p},$$

where $\mathbf{p} = (p \quad q)^T$ is the perturbation from the equilibrium point (X, Y), and $\mathbf{J}$ is the Jacobian matrix evaluated at the equilibrium point.

Procedure 2 Linearising a system of differential equations near an equilibrium point

Suppose that the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{u}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

has an equilibrium point at $x = X$, $y = Y$.

To linearise this system, carry out the following steps.

1. Find the Jacobian matrix
$$\mathbf{J}(x, y) = \begin{pmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{pmatrix}. \tag{20}$$
2. In the neighbourhood of the equilibrium point (X, Y), the differential equations can be approximated by the linearised form
$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \mathbf{J} \begin{pmatrix} p \\ q \end{pmatrix}, \tag{21}$$
where $x(t) = X + p(t)$ and $y(t) = Y + q(t)$, and $\mathbf{J} = \mathbf{J}(X, Y)$.

Exercise 9

Write down the linear approximations to the Lotka–Volterra equations (10) near the equilibrium point $(0, 0)$.

Exercise 10

This system was also considered in Exercise 6 and Example 2.

Consider the equations

$$\dot{x} = x(20 - y), \quad \dot{y} = y(10 - y)(10 - x).$$

Find the linear approximations to these equations near the equilibrium point $(10, 20)$.

Exercise 11

Find the equilibrium point of the system of differential equations

$$\dot{x} = 3x + 2y - 8, \quad \dot{y} = x + 4y - 6.$$

Find a system of linear differential equations satisfied by small perturbations p and q from the equilibrium point.

Exercise 12

Suppose that the pair of populations x and y can be modelled by the system of differential equations

$$\dot{x} = 0.5x - 0.000\,05x^2,$$
$$\dot{y} = -0.1y + 0.0004xy - 0.01y^2$$
$$(x \geq 0,\ y \geq 0).$$

(a) Find the three equilibrium points of the system.

(b) Find the Jacobian matrix of the system.

(c) For each of the three equilibrium points, find the linear differential equations that give the approximate behaviour of the system near the equilibrium point.

We have reduced the discussion of the behaviour of a system near an equilibrium point to an examination of the behaviour of a pair of linear differential equations. In the next section we use the techniques from Unit 6 to solve these differential equations.

3 Classifying equilibrium points

In the previous section you saw how a system of non-linear differential equations $\dot{\mathbf{x}} = \mathbf{u}(x, y)$ may be approximated near an equilibrium point by a linear system $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$, where $\mathbf{J}$ is the Jacobian matrix evaluated at the equilibrium point, and $\mathbf{p}$ is a vector of perturbations from the equilibrium point. In this section we develop an algebraic method of classification, based on the eigenvalues of the matrix of coefficients that arises from the linear approximations. Our overall strategy for classifying an equilibrium point of a non-linear system will then be:

- near an equilibrium point, approximate the non-linear system by a linear system
- find the eigenvalues of the matrix of coefficients for this linear approximation
- classify the equilibrium point of the linear system using these eigenvalues
- deduce the behaviour of the original system in the neighbourhood of the equilibrium point.

This section develops the steps of this overall strategy in turn, beginning by looking at the behaviour of systems with Jacobian matrices with two distinct real eigenvalues, complex eigenvalues, and a repeated eigenvalue. These results are then summarised to provide a procedure for classifying equilibrium points of linear systems and then finally non-linear systems.

3.1 Matrices with two distinct real eigenvalues

Let us first consider the system of differential equations $\dot{\mathbf{p}} = \mathbf{Jp}$ where

$$\mathbf{J} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \qquad (22)$$

See Unit 6, Procedure 1.

The matrix $\mathbf{J}$ is diagonal, so the eigenvalues are 2 and 3. The corresponding eigenvectors are $(1 \quad 0)^T$ and $(0 \quad 1)^T$, respectively. The general solution can be written as

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = C \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + D \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t},$$

where C and D are constants, or as

$$p(t) = Ce^{2t}, \quad q(t) = De^{3t}.$$

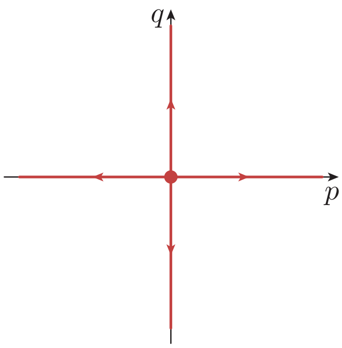


Figure 11 Paths in the (p, q) phase plane; the dot marks the equilibrium point

We are interested in the behaviour of phase paths near the equilibrium point at $p = 0$, $q = 0$. Consider, for example, the paths with $D = 0$ (and $C \neq 0$). On these paths we have $p(t) = Ce^{2t}$ and $q(t) = 0$, so the point $(p(t), q(t))$ moves away from the origin along the p-axis as t increases.

On the other hand, consider the paths with $C = 0$ (and $D \neq 0$). On these paths we have $p(t) = 0$ and $q(t) = De^{3t}$, so the point $(p(t), q(t))$ moves away from the origin along the q-axis as t increases.

Hence we have seen that there are phase paths along the axes, in the directions of the eigenvectors $(1 \quad 0)^T$ and $(0 \quad 1)^T$. A line in the direction of an eigenvector is called an **eigenline**. As t increases, a point on one of the axes moves away from the origin as shown in Figure 11.

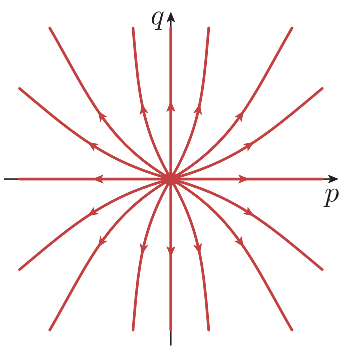


Figure 12 The phase plane near a source

For general values of C and D, where neither $C = 0$ nor $D = 0$, the point (Ce^{2t}, De^{3t}) still moves away from the origin as t increases, but not along a straight line. As t increases, the point moves along a path that radiates outwards from the origin. An equilibrium point with this type of qualitative behaviour in its neighbourhood is a **source**. This is illustrated in Figure 12, where we have incorporated the fact that the only straight-line paths are along the axes, in the directions of the eigenvectors of the matrix $\mathbf{J}$.

This behaviour occurs for any linear system $\dot{\mathbf{p}} = \mathbf{Jp}$ where the matrix of coefficients $\mathbf{J}$ has *positive distinct eigenvalues*. The only straight-line paths are in the directions of the eigenvectors of the matrix $\mathbf{J}$, as these are the eigenlines (although these will not, in general, be along the axes!).

Exercise 13

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues of the matrix of coefficients.

(b) Classify the equilibrium point $p = 0$, $q = 0$ of the system.

Consider now the system of differential equations $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$ where

$$\mathbf{J} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}. \tag{23}$$

The change in sign for matrix $\mathbf{J}$ from equation (22) to equation (23) changes the solution from one involving positive exponentials to one involving negative exponentials. You can think of this as replacing t by $-t$, so the solutions describe the same paths, but traversed in opposite directions. A change in the sign of both eigenvalues changes the directions of the arrows along the paths in Figure 12.

If the matrix of coefficients for a linear system has *negative distinct eigenvalues*, the equilibrium point is a **sink**. The only straight-line paths are along the directions of the eigenvectors of the matrix of coefficients.

Exercise 14

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues of the matrix of coefficients.

(b) Classify the equilibrium point $p = 0$, $q = 0$ of the system.

So far in this section we have considered the case where the matrix of coefficients has two distinct positive eigenvalues and the case where the matrix has two distinct negative eigenvalues. We now consider the case where the matrix has *one positive eigenvalue and one negative eigenvalue*. For example, consider the matrix

$$\mathbf{J} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix},$$

which has eigenvalues 2 and -3, and corresponding eigenvectors $(4 \quad 1)^T$ and $(1 \quad -1)^T$. The general solution of the linear system of differential equations $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$ is

$$\begin{pmatrix} p \\ q \end{pmatrix} = C \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + D \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}, \tag{24}$$

where C and D are constants. When $D = 0$ (and $C \neq 0$), we have $p(t) = 4Ce^{2t}$ and $q(t) = Ce^{2t}$, and the point $(p(t), q(t))$ moves away from the origin along the straight-line path $q = \frac{1}{4}p$ as t increases; this line is an eigenline. On the other hand, when $C = 0$ (and $D \neq 0$), the solution is $p(t) = De^{-3t}$, $q(t) = -De^{-3t}$, so the point $(p(t), q(t))$ approaches the origin along the straight-line path $q = -p$ as t increases.

Hence we have seen that there are two straight-line paths. On the line $q = \frac{1}{4}p$ (which corresponds to the eigenvector $(4 \quad 1)^T$), the point moves away from the origin as t increases. However, on the line $q = -p$ (which corresponds to the eigenvector $(1 \quad -1)^T$), the point moves towards the origin as t increases. These paths are shown in Figure 13.

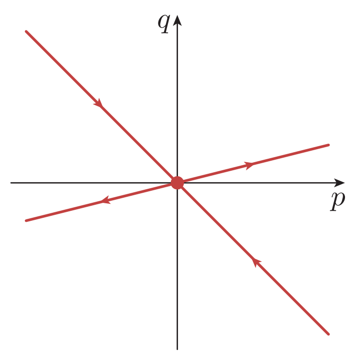


Figure 13 Paths to and from an equilibrium point

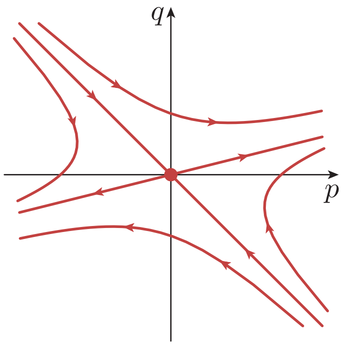


Figure 14 Paths near a *saddle point*; the dot marks the equilibrium point

Now we consider the behaviour of a general point $(p(t), q(t))$, where $p(t)$ and $q(t)$ are given by equation (24), and neither C nor D is zero. For large positive values of t, the terms involving e^{2t} dominate, so $p(t) \simeq 4Ce^{2t}$ and $q(t) \simeq Ce^{2t}$. So for large positive values of t, the general path approaches the line $q = \frac{1}{4}p$. On the other hand, for large negative values of t, the terms involving e^{-3t} dominate, so $p(t) \simeq De^{-3t}$ and $q(t) \simeq -De^{-3t}$. So for large negative values of t, the general path approaches the line $q = -p$. Using this information we can add to Figure 13 to obtain Figure 14.

We can see that the equilibrium point is a **saddle**. The type of behaviour shown in Figure 14 occurs when the matrix of coefficients has *one positive eigenvalue and one negative eigenvalue*. Again, the straight-line paths are in the directions of the eigenvectors of the matrix.

Exercise 15

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix of coefficients.

(b) Classify the equilibrium point $p = 0$, $q = 0$.

(c) Sketch the phase paths of the solutions of the differential equations in the neighbourhood of $(0, 0)$.

3.2 Matrices with complex eigenvalues

In Unit 5 you saw that some matrices have complex eigenvalues and eigenvectors. In Unit 6 you saw that these complex quantities can be used to construct the *real* solutions of the corresponding system of linear differential equations. Our next example involves such a system.

Example 4

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix of coefficients.

(b) Hence write down the general solution of the system of differential equations.

(c) Show that the phase paths for these differential equations are the ellipses

$$p^2 + \tfrac{1}{4}q^2 = K,$$

where K is a positive constant.

Solution

(a) The matrix of coefficients has characteristic equation

$$\begin{vmatrix} -\lambda & -1 \\ 4 & -\lambda \end{vmatrix} = 0,$$

that is, $\lambda^2 + 4 = 0$. So the eigenvalues are $\lambda = 2i$ and $\lambda = -2i$.

When $\lambda = 2i$, the eigenvector $(a \quad b)^T$ satisfies the equation

$$\begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so an eigenvector corresponding to the eigenvalue $\lambda = 2i$ is $(1 \quad -2i)^T$.

Similarly, an eigenvector corresponding to the eigenvalue $\lambda = -2i$ is $(1 \quad 2i)^T$.

(b) Using the eigenvalues and corresponding eigenvectors from part (a), and Procedure 3 of Unit 6, the general solution of the differential equations is

$$\begin{pmatrix} p \\ q \end{pmatrix} = C \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} + D \begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix},$$

where C and D are constants.

(c) We have

$$\begin{aligned} p(t) &= C\cos 2t + D\sin 2t, \\ q(t) &= 2C\sin 2t - 2D\cos 2t, \end{aligned}$$

so

$$\begin{aligned} p^2 + \tfrac{1}{4}q^2 &= (C\cos 2t + D\sin 2t)^2 + (C\sin 2t - D\cos 2t)^2 \\ &= (C^2\cos^2 2t + 2CD\cos 2t\sin 2t + D^2\sin^2 2t) \\ &\quad + (C^2\sin^2 2t - 2CD\cos 2t\sin 2t + D^2\cos^2 2t) \\ &= C^2(\cos^2 2t + \sin^2 2t) + D^2(\cos^2 2t + \sin^2 2t) \\ &= C^2 + D^2 = K, \end{aligned}$$

where $K = C^2 + D^2$ is a positive constant.

So the phase paths are ellipses, as shown in Figure 15. The direction of the arrows can be deduced from the original differential equations. For example, in the first quadrant $\dot{p} < 0$ and $\dot{q} > 0$.

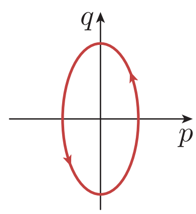


Figure 15 An elliptical path in the phase plane near a *centre*

In Example 4, we saw that all the phase paths are ellipses. This type of behaviour corresponds to any linear system of differential equations where the eigenvalues of the matrix of coefficients are *purely imaginary*. An equilibrium point that has this behaviour in its neighbourhood is called a **centre**.

Exercise 16

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues of the matrix of coefficients.

(b) Classify the equilibrium point $p = 0$, $q = 0$.

In general, when the eigenvalues of a matrix are complex, they are not purely imaginary but also contain a real part. This has a significant effect on the solution of the corresponding system, as you will see in the following example.

Example 5

Find the general solution of the system of equations $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$, where

$$\mathbf{J} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}.$$

Sketch some paths corresponding to the solutions of the system.

Solution

The characteristic equation of the matrix of coefficients is $(2+\lambda)^2 + 9 = 0$, so the eigenvalues are $-2+3i$ and $-2-3i$. Corresponding eigenvectors are $(1 \quad -i)^T$ and $(1 \quad i)^T$, respectively, so the general solution is given by

$$\begin{pmatrix} p \\ q \end{pmatrix} = Ce^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + De^{-2t} \begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix},$$

where C and D are constants.

If we neglect, for the time being, the e^{-2t} terms, the solution is

$$p = C\cos 3t + D\sin 3t,$$
$$q = C\sin 3t - D\cos 3t,$$

from which it follows that

$$p^2 + q^2 = C^2 + D^2.$$

So in the absence of the e^{-2t} terms, the paths would be circles with centre at the origin. The effect of the e^{-2t} terms on these paths is to reduce the radius of the circles gradually. In other words, the paths spiral in towards the origin as t increases, as shown in Figure 16.

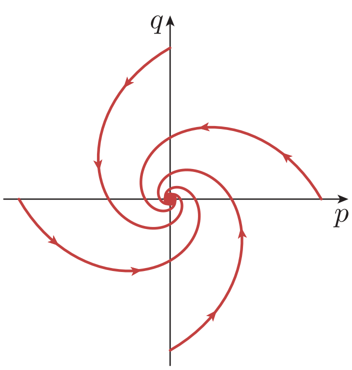


Figure 16 Paths near a *spiral sink*; the dot marks the equilibrium point

In Example 5 (and Figure 16) the paths spiral in towards the origin, so the origin is a sink (called a **spiral sink**) and therefore is a *stable* equilibrium point.

If the paths spiralled away from the origin, we should have a **spiral source** (see Figure 17) with the equilibrium point *unstable*.

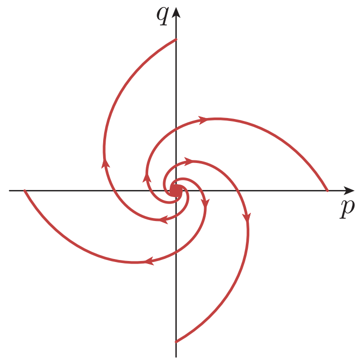


Figure 17 Paths near a *spiral source*; the dot marks the equilibrium point

The stability is determined by the sign of the real part of the complex eigenvalues. To summarise, if the real part is positive, then the general solution involves e^{kt} terms (where k is positive) and the equilibrium point is a spiral source; if the real part is negative, then the general solution involves e^{-kt} terms and the equilibrium point is a spiral sink.

Exercise 17

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues of the matrix of coefficients.

(b) Classify the equilibrium point $p = 0$, $q = 0$.

3.3 Matrices with repeated eigenvalues

So far we have considered the cases where the matrix of coefficients for a linear system of differential equations has two real distinct eigenvalues or complex eigenvalues. In this subsection we consider a third possibility: the case where the matrix has a real repeated eigenvalue. In fact, there are two separate cases, depending on how many independent eigenvectors there are.

First, we consider the case where there are two linearly independent eigenvectors. In this case, the matrix must be diagonal and of the form

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

(see Unit 5, Exercise 13(a)).

Exercise 18

Consider the linear system of differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of the coefficient matrix.

(b) Find the general solution of the system of differential equations.

(c) By eliminating t, find the equations of the paths, and describe them.

(d) Is the equilibrium point $p = 0$, $q = 0$ stable or unstable?

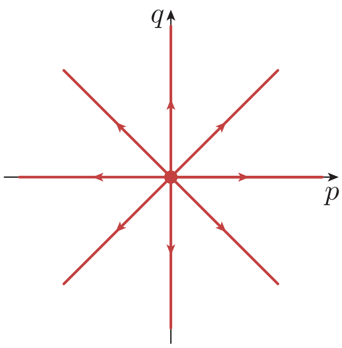


Figure 18 Paths near a *star source*; the dot marks the equilibrium point

In Exercise 18, we have seen that when the matrix of coefficients has two real *identical positive eigenvalues* and *two linearly independent eigenvectors*, all the paths are straight lines radiating away from the origin, as shown in Figure 18. The equilibrium point at $p = 0$, $q = 0$ is called a **star source**. If there are *two identical negative eigenvalues* (but still *two linearly independent eigenvectors*), then the arrows on the paths in Figure 18 are reversed, and the equilibrium point at $p = 0$, $q = 0$ is called a **star sink**.

We turn finally to the case where there are two identical eigenvalues but only one independent eigenvector.

Exercise 19

(a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{J} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

(b) Find the general solution of the system of differential equations $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$.

In Exercise 19 we have seen that the general solution of the system of linear differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

is

$$p(t) = Ce^{2t}, \quad q(t) = (Ct + D)e^{2t},$$

where C and D are constants.

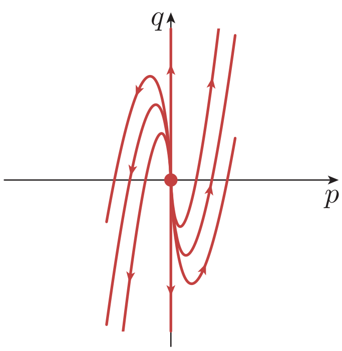


Figure 19 Paths near an *improper source*; the dot marks the equilibrium point

Figure 19 shows some typical paths of this system, and we conclude that the equilibrium point at $p = 0$, $q = 0$ is unstable. It is called an **improper source**. Observe that there is only one straight-line path leading away from the equilibrium point (the y-axis in this case), which is a consequence of the fact that the Jacobian matrix has only one eigenvector. If the (repeated) eigenvalue is *negative*, then the phase paths are obtained by reversing the arrows in Figure 19 to obtain an **improper sink**.

This completes the discussion of our range of examples, which were chosen to illustrate most types of behaviour that you might meet. These are summarised in the next subsection.

3.4 Classifying equilibrium points of linear systems

In Procedure 3 we summarise the results of the previous three subsections. Note that this procedure is not exhaustive because it does not include a number of degenerate cases where one of the eigenvalues is zero, which are not considered in this module.

Procedure 3 Equilibrium point classification for a linear system

Consider the linear system $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$ for a 2×2 matrix $\mathbf{J}$. The nature of the equilibrium point at $p = 0$, $q = 0$ is determined by the eigenvalues and eigenvectors of $\mathbf{J}$, and a decision tree for this is shown in Figure 20.

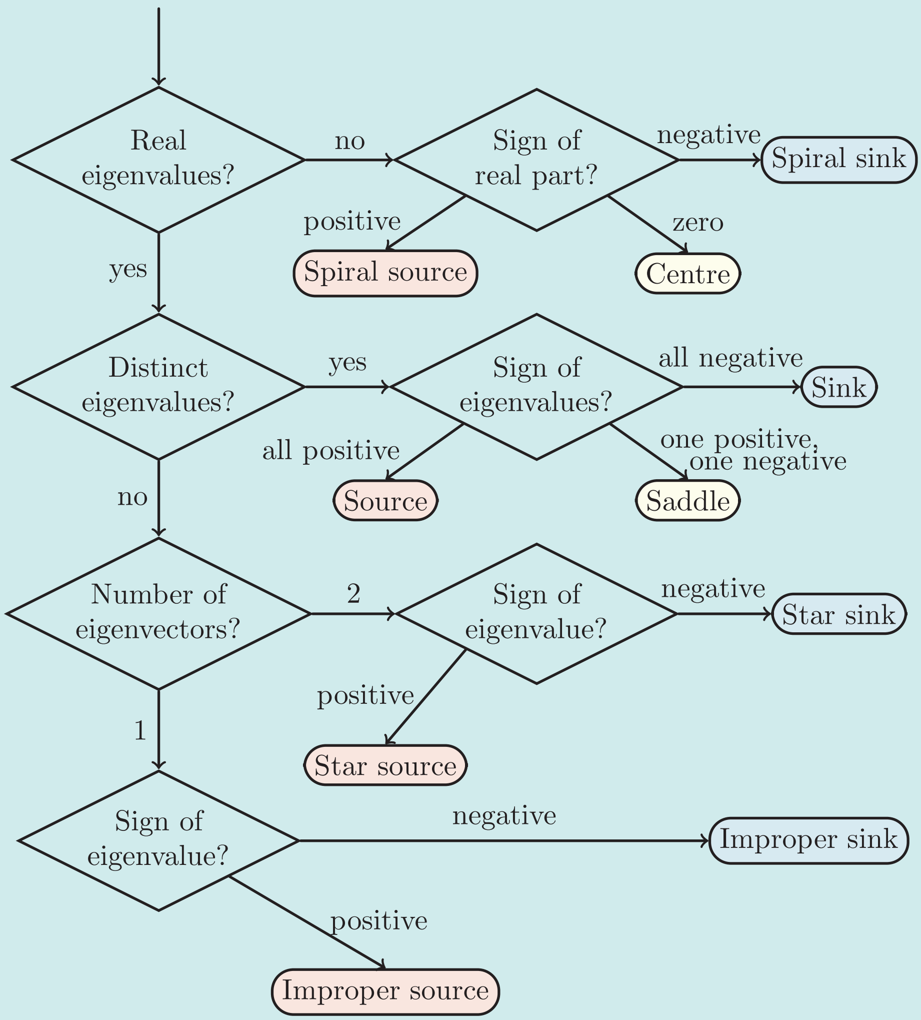


Figure 20 A decision tree for classifying equilibrium points

Exercise 20

In Example 3 we saw that the Lotka–Volterra equations can be approximated by the system of linear differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -kX/Y \\ hY/X & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

in the neighbourhood of the equilibrium point (X, Y). Find the eigenvalues of the matrix of coefficients, and hence classify the equilibrium point $p = 0$, $q = 0$.

Exercise 21

In Exercise 9 we saw that the Lotka–Volterra equations can be approximated by the system of linear differential equations

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

in the neighbourhood of the equilibrium point $(0, 0)$. Find the eigenvalues of the matrix of coefficients, and hence classify the equilibrium point $p = 0$, $q = 0$.

3.5 Classifying equilibrium points of non-linear systems

In Section 2 we saw how to find the equilibrium points of non-linear systems of differential equations $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})$, and how to find the linear system $\dot{\mathbf{p}} = \mathbf{J}\mathbf{p}$ that approximates the system in the neighbourhood of the equilibrium point. In this section we have seen how to classify the equilibrium point of the linear system by finding the eigenvalues and eigenvectors of the matrix $\mathbf{J}$. But is the behaviour of the non-linear system near the equilibrium point the same as the behaviour of the linear system that approximates it? It can be shown that, not surprisingly, the answer is yes, *except* when the equilibrium point of the approximating linear system is a centre.

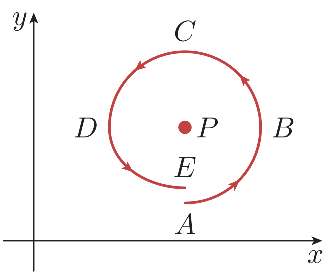


Figure 21 One circuit around an equilibrium point P of a non-linear system where the linearised system is a centre

If the approximating linear system is a centre, then the dominant behaviour of the original system is to circulate around the equilibrium point. Consider what happens when we follow one circuit of a path around the equilibrium point of a non-linear system, such as that shown in Figure 21. This figure shows the equilibrium point at point P and a path $ABCDE$ that circles around it. After one circuit, we compare the distance from the equilibrium point EP with the starting distance AP.

There are three possibilities: $AP < EP$, $AP = EP$ and $AP > EP$ (the latter is shown in the figure). These three possibilities correspond to a spiral source, a centre or a spiral sink, respectively. We now give examples to show that all three possibilities can occur.

A very simple example of a system with a centre at the origin is the linear system

$$\dot{x} = -y, \quad \dot{y} = x. \tag{25}$$

To see that the origin is a centre, differentiate the second equation with respect to t to get $\ddot{y} = \dot{x}$, and use the first equation to substitute for $\dot{x}$ to obtain $\ddot{y} = -y$. You should recognise this equation as the simple harmonic motion equation, which has solution $y = A\cos(t + \phi)$ where A and ϕ are constants. Hence we have $x = \dot{y} = -A\sin(t + \phi)$. So the general solution of the system of differential equations is $(x, y) = (-A\sin(t + \phi), A\cos(t + \phi))$, which is the family of concentric circles $x^2 + y^2 = A^2$.

There are other non-linear systems of equations that have equations (25) as a linear approximation, for example all equations using these linear terms with some higher powers of x or y added. Consider one such example:

$$\dot{x} = -y, \quad \dot{y} = x + y^3. \tag{26}$$

If we proceed as before and differentiate the second equation and substitute for $\dot{x}$, then we obtain $\ddot{y} - 3y^2\dot{y} + y = 0$. This non-linear equation is harder to solve, but we can get an idea of the solution by approximating the equation. Since y^2 is always positive, we can get a feel for the solutions by looking at the solutions of the equation $\ddot{y} - k\dot{y} + y = 0$, where k is a positive constant. The constant k is small for paths close to the equilibrium point, so the solutions of the characteristic equation are complex with positive real part, which implies that the solutions of the linear second-order equation increase exponentially. So we expect the solutions of equations (26) to also spiral outwards (as can be confirmed by plotting the solution on a computer). So (26) is an example of a non-linear system that is a spiral source where the linear approximation (equations (25)) has a centre.

Changing the sign in front of the y^3 term gives a non-linear system that has a spiral sink at the origin with the same linear approximation:

$$\dot{x} = -y, \quad \dot{y} = x - y^3.$$

Plotting the solutions of this system on a computer confirms that this is a non-linear system that has a spiral sink at the origin.

Thus if the linear approximation has a centre, we cannot immediately deduce the nature of the equilibrium point of the original non-linear system: it may be a centre, a spiral sink or a spiral source. This is summarised in the following procedure.

Procedure 4 Equilibrium point classification for a non-linear system

To classify the equilibrium points of a system of non-linear differential equations, carry out the following steps.

1. Find the equilibrium points by using Procedure 1.
2. Use Procedure 2 to find a linear system that approximates the original non-linear system in the neighbourhood of each equilibrium point.
3. For each equilibrium point, use Procedure 3 to classify the linear system.
4. For each equilibrium point, the behaviour of the original non-linear system is the same as that of the linear approximation, except when the linear system has a centre. If the linear system has a centre, then the equilibrium point of the original non-linear system may be a centre, a spiral sink or a spiral source.

The following example shows how this procedure is used.

Example 6

Consider the non-linear system of differential equations

$$\dot{x} = -4y + 2xy - 8, \quad \dot{y} = 4y^2 - x^2.$$

(a) Find the equilibrium points of the system.

(b) Compute the Jacobian matrix of the system.

(c) In the neighbourhood of each equilibrium point:

- linearise the system of differential equations
- classify the equilibrium point of the linearised system.

(d) What can you say about the classification of the equilibrium points of the original (non-linear) system of differential equations?

Solution

(a) The equilibrium points are given by

$$-4y + 2xy - 8 = 0,$$
$$4y^2 - x^2 = 0.$$

The second equation gives

$$x = \pm 2y.$$

When $x = 2y$, substitution into the first equation gives

$$-4y + 4y^2 - 8 = 0,$$

or $y^2 - y - 2 = 0$, which factorises to give

$$(y-2)(y+1) = 0.$$

Hence

$$y = 2 \quad \text{or} \quad y = -1.$$

When $y = 2$, $x = 2y = 4$. When $y = -1$, $x = 2y = -2$. So we have found two equilibrium points, namely $(4, 2)$ and $(-2, -1)$.

When $x = -2y$, substitution into the first equation gives

$$-4y - 4y^2 - 8 = 0,$$

or $y^2 + y + 2 = 0$. This quadratic equation has no real solutions, so there are no more equilibrium points.

(b) Differentiating the right-hand sides of the given differential equations gives the Jacobian matrix as

$$\mathbf{J} = \begin{pmatrix} 2y & 2x - 4 \\ -2x & 8y \end{pmatrix}.$$

(c) At the equilibrium point $(4, 2)$, the Jacobian matrix is

$$\begin{pmatrix} 4 & 4 \\ -8 & 16 \end{pmatrix},$$

so the linearised system is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -8 & 16 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

The characteristic equation of the matrix of coefficients is

$$(4 - \lambda)(16 - \lambda) + 32 = 0,$$

or $\lambda^2 - 20\lambda + 96 = 0$, which factorises to give

$$(\lambda - 8)(\lambda - 12) = 0,$$

so the eigenvalues are

$$\lambda = 8 \quad \text{and} \quad \lambda = 12.$$

The two eigenvalues are positive and distinct, so the equilibrium point $p = 0$, $q = 0$ is a source.

At the equilibrium point $(-2, -1)$, the Jacobian matrix is

$$\begin{pmatrix} -2 & -8 \\ 4 & -8 \end{pmatrix},$$

so the linearised system is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -2 & -8 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

The characteristic equation of the matrix of coefficients is

$$(-2 - \lambda)(-8 - \lambda) + 32 = 0,$$

which simplifies to

$$\lambda^2 + 10\lambda + 48 = 0.$$

The roots of this quadratic equation are

$$\lambda = -5 \pm i\sqrt{23},$$

so the eigenvalues are complex with negative real part. Hence the equilibrium point $p = 0$, $q = 0$ is a spiral sink.

(d) As neither of the equilibrium points in part (c) is a centre, the non-linear system has an equilibrium point $(4, 2)$ that is a source, and an equilibrium point $(-2, -1)$ that is a spiral sink.

The following exercises ask you to follow Procedure 4 to classify some equilibrium points.

Exercise 22

A certain system of differential equations has an equilibrium point where the Jacobian matrix evaluates to

$$\mathbf{J} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of this matrix, and hence classify the equilibrium point.

(b) If this Jacobian matrix is a linear approximation to a non-linear system $\dot{\mathbf{x}} = \mathbf{u}(x, y)$, what can you say about the equilibrium point of the non-linear system?

Exercise 23

Consider the non-linear system of differential equations

$$\dot{x} = (1 + x - 2y)x, \quad \dot{y} = (x - 1)y.$$

(a) Find the equilibrium points of the system.

(b) Find the Jacobian matrix of the system.

(c) In the neighbourhood of each equilibrium point:

- find the linear system of differential equations that gives the approximate behaviour of the system near the equilibrium point
- find the eigenvalues of the matrix of coefficients
- use the eigenvalues to classify the equilibrium point of the linearised system.

(d) What can you say about the classification of the equilibrium points of the original non-linear system of differential equations?

This section concludes with a brief discussion of the nature of the equilibrium point (X, Y) of the Lotka–Volterra equations (10). Near this equilibrium point, the linear system of differential equations that approximates the Lotka–Volterra equations has a centre (as you saw in Exercise 20). So using Procedure 4, the equilibrium point of the Lotka–Volterra equations is a centre, a spiral sink or a spiral source.

Methods for deciding between these three options are beyond the scope of this module, but for completeness we outline an argument for the Lotka–Volterra equations. The key here is to find a relationship between x and y that the phase paths satisfy, in the same way that we showed that the paths were ellipses in Example 4(c). For the Lotka–Volterra equations, we consider the function

$$f(x,y) = h\ln x - \frac{h}{X}x + k\ln y - \frac{k}{Y}y \quad (x > 0,\ y > 0).$$

How this function arises is explained below; all we are doing here is starting from this given function.

The key fact about this particular function f is that it is constant along the phase paths. To show this, differentiate with respect to t:

$$\begin{aligned}\frac{df}{dt} &= \frac{dx}{dt}\frac{d}{dx}\left(h\ln x - \frac{h}{X}x\right) + \frac{dy}{dt}\frac{d}{dy}\left(k\ln y - \frac{k}{Y}y\right)\\ &= \dot{x}\left(\frac{h}{x} - \frac{h}{X}\right) + \dot{y}\left(\frac{k}{y} - \frac{k}{Y}\right)\\ &= \dot{x}\frac{h}{x}\left(1 - \frac{x}{X}\right) + \dot{y}\frac{k}{y}\left(1 - \frac{y}{Y}\right).\end{aligned}$$

Now substitute for $\dot{x}$ and $\dot{y}$ using the Lotka–Volterra equations (10):

$$\begin{aligned}\frac{df}{dt} &= \left[kx\left(1 - \frac{y}{Y}\right)\right]\frac{h}{x}\left(1 - \frac{x}{X}\right) + \left[-hy\left(1 - \frac{x}{X}\right)\right]\frac{k}{y}\left(1 - \frac{y}{Y}\right)\\ &= hk\left(1 - \frac{y}{Y}\right)\left(1 - \frac{x}{X}\right) - hk\left(1 - \frac{x}{X}\right)\left(1 - \frac{y}{Y}\right)\\ &= 0.\end{aligned}$$

So the function f is unchanging with time, which means that each path in the phase plane must correspond to a different value of f. Thus the phase paths are the contours of f traversed in some direction, as shown in Figure 22. Hence the equilibrium point is a centre. The front cover of this book shows the vector field of the Lotka–Volterra equations together with coloured contours of f.

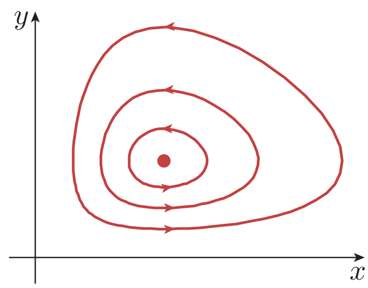


Figure 22 A phase portrait for the Lotka–Volterra equations; the dot marks an equilibrium point

The above argument could use *any* function that is constant along paths in order to show that the non-linear system has a centre. For a mechanical system, an argument based on conservation of energy could produce a function with the correct properties. For the Lotka–Volterra equations, the function $f(x,y)$ arises from the following argument. Start with the observation that

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}.$$

Now substitute for $\dot{x}$ and $\dot{y}$ using the Lotka–Volterra equations to get

$$\frac{dy}{dx} = \frac{-hy(1 - x/X)}{kx(1 - y/Y)}.$$

This first-order differential equation can be solved by separation of variables:

$$k\int\frac{1 - y/Y}{y}\,dy = -h\int\frac{1 - x/X}{x}\,dx.$$

Performing the integrations gives

$$k \ln y - \frac{k}{Y} y = -h \ln x + \frac{h}{X} x + C,$$

where C is a constant. Rearranging this equation gives $f(x, y)$.

4 Phase portraits

A system of non-linear differential equations may have several equilibrium points. Using the methods of the previous section, we can sketch paths in the neighbourhood of each equilibrium point. In this section we aim to knit together these sketches into a complete picture, or *portrait*, of the solutions of a system of differential equations.

We start by considering the equations of the predator–prey model again, and look at what general features about the phase plane can be deduced directly from the equations.

Exercise 24

Consider the system of differential equations defined in Exercise 4, namely

$$\dot{x} = x\left(1 - \frac{y}{2}\right), \quad \dot{y} = -\frac{1}{2} y \left(1 - \frac{x}{3}\right).$$

As we are interested in general features of these equations, consider both positive and negative values of x and y rather than restricting attention to those values that are physically reasonable for a population model (i.e. $x \geq 0$ and $y \geq 0$).

(a) For what values of x and y is $\dot{x}$ zero? In what regions of the plane is $\dot{x}$ positive? Sketch these regions on the phase plane.

(b) For what values of x and y is $\dot{y}$ zero? In which regions is $\dot{y}$ positive? Sketch these regions on the phase plane.

The regions of the plane that you sketched in Exercise 24 are significant. The regions where $\dot{x}$ is positive are the regions where the vector field arrows are pointing generally to the right. In these regions the phase paths will curve to the right. Similarly, the regions where $\dot{y}$ is positive are the regions where the phase paths curve upwards. This information is really useful when sketching phase portraits.

The boundaries of the regions are the key concept, as these boundaries are the only places where curves can change between curving leftwards/rightwards and upwards/downwards, so these are given a name.

The solutions of $\dot{x} = 0$ or $\dot{y} = 0$ are called **nullclines**. When we need to distinguish between the two sets of nullclines, the solutions of $\dot{x} = 0$ will be called the **nullclines for $\dot{x}$**, and the solutions of $\dot{y} = 0$ will be called the **nullclines for $\dot{y}$**.

Now we will use nullclines to plot the phase portrait of the predator–prey equations. The process that we go through will later be formalised in a procedure, and we will mark the steps of the procedure in the margin as we go along.

Example 7

Sketch the phase portrait of the system of differential equations

$$\dot{x} = x\left(1 - \frac{y}{2}\right), \quad \dot{y} = -\frac{1}{2}y\left(1 - \frac{x}{3}\right).$$

Solution

The first step in sketching a phase portrait is to find and classify the equilibrium points. In Example 1 we found that these equations have two equilibrium points, in this case (with $X = 3$ and $Y = 2$) $(0, 0)$ and $(3, 2)$. The given equations are the Lotka–Volterra equations for specific parameters; in Exercises 20 and 21 we classified the equilibrium points and found that $(0, 0)$ is a saddle point and $(3, 2)$ is a centre. These are marked in green in Figure 23 (see below). ◀ Equilibrium points ▶

The next step is to find the nullclines. In Exercise 24 we found that the nullclines for $\dot{x}$ are $x = 0$ and $y = 2$. These are marked by red lines in Figure 23. Also in Exercise 24 we found that the nullclines for $\dot{y}$ are $y = 0$ and $x = 3$. These are marked by blue lines in Figure 23. ◀ Nullclines ▶

Phase paths cross nullclines for $\dot{x}$ vertically (since $\dot{x} = 0$). We need to determine whether each crossing is upwards or downwards – this is determined by the sign of $\dot{y}$ on the nullcline. This sign can change only at the equilibrium points, as the right-hand sides of the system of differential equations are continuous. So all we need to do is determine the sign of $\dot{y}$ on either side of each equilibrium point. To do this, we compute the following values. ◀ Nullcline crossings ▶

(x, y)	$(0, 1)$	$(0, -1)$	$(2, 2)$	$(4, 2)$
$\dot{y}$	$-1/2$	$1/2$	$-1/3$	$1/3$

In Figure 23, the positive signs are represented by upwards arrows, and the negative signs are represented by downward arrows.

Similarly, the phase paths cross the nullclines for $\dot{y}$ horizontally. We can evaluate $\dot{x}$ at points on either side of equilibrium points to find the direction of crossing as follows.

(x, y)	$(1, 0)$	$(-1, 0)$	$(3, 1)$	$(3, 3)$
$\dot{x}$	1	-1	$3/2$	$-3/2$

Again, we represent the signs of these values by arrows in Figure 23, where positive signs are represented by rightwards arrows and negative signs are represented by leftwards arrows.

◀ Nullcline regions ▶

The nullclines divide the phase plane into regions. In each region the vector field arrows will point in the same general direction (up/down and left/right) as $\dot{x}$ and $\dot{y}$ can change sign only on the nullclines. In Exercise 24 the regions shaded red are the regions where $\dot{x} > 0$, and the regions shaded blue are the regions where $\dot{y} > 0$. So the overlap regions are the regions where $\dot{x} > 0$ and $\dot{y} > 0$, that is, the arrows will point rightwards and upwards. Similarly, the regions that are not shaded by either colour are regions where $\dot{x} < 0$ and $\dot{y} < 0$, that is, regions where the arrows point leftwards and downwards. These signs are represented by arrows pointing north-east, north-west, south-east and south-west in Figure 23. Note that the arrows in this figure are purely representative of the general direction of the arrows in each region; they are not the computed arrows at this point.

We often abbreviate these directions to NE, NW, SE and SW.

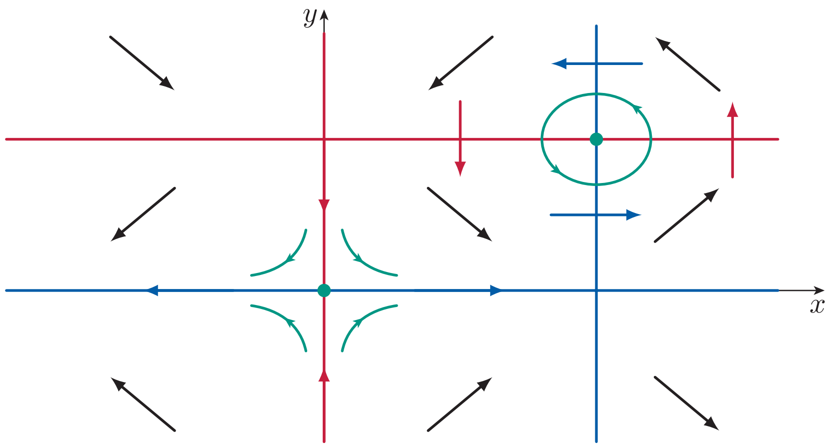


Figure 23 Accumulated information about the predator–prey equations. Nullclines for $\dot{x}$ are red, and nullclines for $\dot{y}$ are blue; arrows indicate crossings. Green dots and paths mark equilibrium points and their classification. Black arrows indicate the general direction in regions.

◀ Complete paths ▶

All that remains is to use this information to sketch typical paths in the phase plane. The aim here is not to fill the phase plane with paths, but to add sufficient paths to show the important features. This will usually involve showing the paths that start at equilibrium points (i.e. start infinitesimally close to but away from equilibrium points) and exploring each of the regions defined by the nullclines. The phase portrait for these equations is shown in Figure 24.

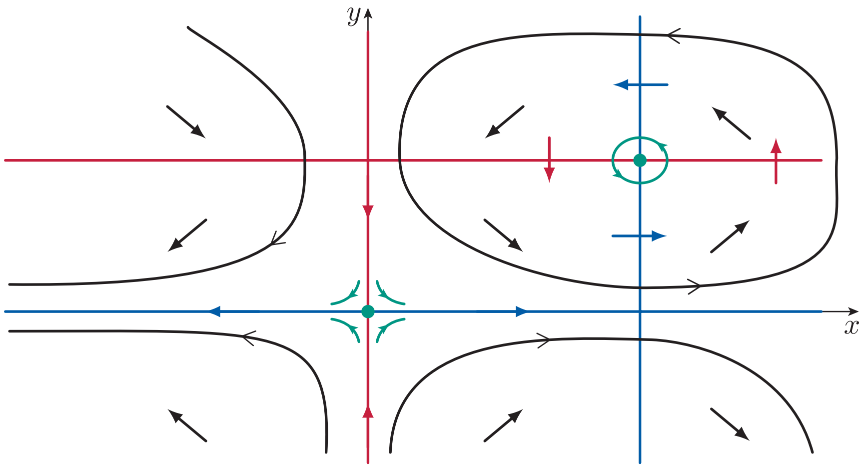


Figure 24 Nullclines for the predator–prey equations, together with the regions where $\dot{x} > 0$ and the regions where $\dot{y} > 0$

The steps used in Example 7 form the basis of a general procedure for sketching phase portraits.

Procedure 5 Sketching phase portraits

In order to sketch the phase portrait of a system of differential equations in x and y, carry out the following steps.

1. Use Procedure 4 to find and classify the equilibrium points. Mark these points on a sketch, and draw small sketches of the local behaviour of the paths near the equilibrium points. ◀ Equilibrium points ▶
2. Find the nullclines by solving $\dot{x} = 0$ and $\dot{y} = 0$. Draw these on your sketch, using two different colours for the two sets of nullclines. (Note that the equilibrium points occur where nullclines of different colours intersect – this is a useful check.) ◀ Nullclines ▶
3. For each equilibrium point on the nullclines for $\dot{x}$, evaluate the sign of $\dot{y}$ on either side. Mark this on your sketch by adding up or down arrows. ◀ Nullcline crossings ▶

   For each equilibrium point on the nullclines for $\dot{y}$, evaluate the sign of $\dot{x}$ on either side. Mark this on your sketch by adding left or right arrows.
4. The nullclines divide the phase plane into several regions. Label each region with a NE, NW, SE or SW arrow to show the general direction of the arrows in that region. ◀ Nullcline regions ▶
5. Extend the paths from the equilibrium points by curving the paths in the direction of the arrows in each region. Add any extra paths that do not start or end at equilibrium points, so that each region is crossed by at least one path. ◀ Complete paths ▶

The following harder example illustrates the use of this procedure.

Example 8

Consider the system of differential equations that we looked at in Example 6, namely

$$\dot{x} = -4y + 2xy - 8, \quad \dot{y} = 4y^2 - x^2,$$

◀ Equilibrium points ▶

where we found that these equations have a source at $(4, 2)$ and a spiral sink at $(-2, -1)$.

Sketch the phase portrait.

Solution

◀ Nullclines ▶

The nullclines for $\dot{x}$ are given by

$$-4y + 2xy - 8 = 0.$$

Rearranging to make y the subject of this equation gives

$$y = \frac{4}{x-2}.$$

So these nullclines are the two branches of a rectangular hyperbola obtained by taking the graph of $y = 1/x$, scaling by a factor 4, and translating 2 units to the right.

Now consider the nullclines for $\dot{y}$, which are given by

$$4y^2 - x^2 = 0.$$

This equation has solutions $x = 2y$ and $x = -2y$. So the nullclines in this case are a pair of straight lines.

A sketch of the nullclines is shown in Figure 25(a). One interesting feature to note from the diagram is that the equilibrium points (marked by dots) occur at the intersections of the red and blue lines. This will always happen, as the red lines correspond to $\dot{x} = 0$ and the blue lines correspond to $\dot{y} = 0$, so the intersection points are when both $\dot{x} = 0$ and $\dot{y} = 0$, that is, the equilibrium points.

◀ Nullcline crossings ▶

Now we determine the directions of the nullcline crossings. We start with the nullclines for $\dot{x}$, which are the two branches of the rectangular hyperbola. Both equilibrium points lie on these nullclines, so we need to evaluate $\dot{y}$ at four points (one on each side of each equilibrium point). We choose the points to simplify the arithmetic.

(x, y)	$(-6, -1/2)$	$(0, -2)$	$(3, 4)$	$(6, 1)$
$\dot{y}$	-35	16	55	-32

The signs of these values are represented by red upwards and downwards pointing arrows in Figure 25(b).

Now consider the nullclines for $\dot{y}$. The nullcline $x = 2y$ also has two equilibrium points lying on it, but this time we can reduce the effort by considering only three points as the curve is continuous. So we calculate the following.

(x,y)	$(-4,-2)$	$(0,0)$	$(6,3)$
$\dot{x}$	16	-8	16

The signs of these values are represented by blue leftwards and rightwards pointing arrows in Figure 25(b).

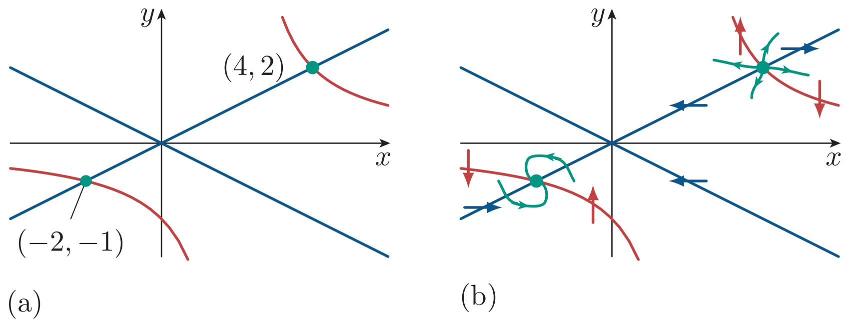


Figure 25 (a) Nullclines of the differential equations, with nullclines for $\dot{x}$ red and nullclines for $\dot{y}$ blue; the two dots mark the equilibrium points. (b) Nullclines plus the directions of crossings and paths in the neighbourhoods of equilibrium points.

The other nullcline for $\dot{y}$, namely the line $x = -2y$, has no equilibrium points on it. So the direction of crossing must be the same along the whole length of this line. So we need to evaluate $\dot{x}$ at only one point to determine this sign. In fact, we do not even have to do one evaluation, as this nullcline crosses the other nullcline for $\dot{y}$, and we know that the sign at the point of crossing is negative. So the sign of $\dot{y}$ must be negative along the whole length of this nullcline. This is marked in Figure 25(b).

We also know that the point $(4,2)$ is a source and the point $(-2,-1)$ is a spiral sink. So the solution paths will flow outwards from the point $(4,2)$ and spiral inwards to the point $(-2,-1)$. This information has also been added to Figure 25(b) as small sketches of the phase paths in the neighbourhoods of the equilibrium points. It helps to delay adding these sketches in the neighbourhoods of the equilibrium points until the nullcline crossings are known, as this helps to determine the sketch; for example, it indicates whether a spiral sink is a clockwise or anticlockwise spiral.

◀ Nullcline regions ▶

We now have a lot of information about the directions of the solution paths on the nullclines, and we can extend this to the regions of the plane with the nullclines as boundaries. Consider the region between the two branches of the red hyperbola. As the red hyperbola corresponds to the equation $\dot{x} = 0$, we know that $\dot{x}$ cannot change sign in this region, so all solution paths must progress leftwards. Similarly, we know that all solution paths in the top-right and bottom-left regions must progress to the right. Also, the plane is divided into four regions by the pair of blue nullclines. The solution paths must be generally pointing upwards in the north and south regions, and generally pointing downwards in the west and east regions. These are represented by arrows in the regions in Figure 26.

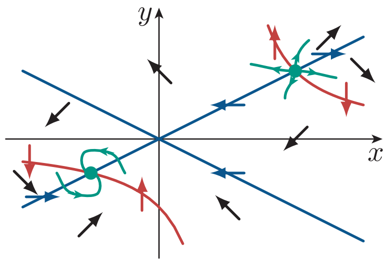


Figure 26 Collected information about the system of differential equations ready for sketching the phase portrait

◀ Complete paths ▶

Now that we have gathered all the information about the system of differential equations, we can sketch the phase portrait, as shown in Figure 27. You should not try to be too precise when drawing a sketch phase portrait. The aim is to convey the main features of the paths rather than precise details.

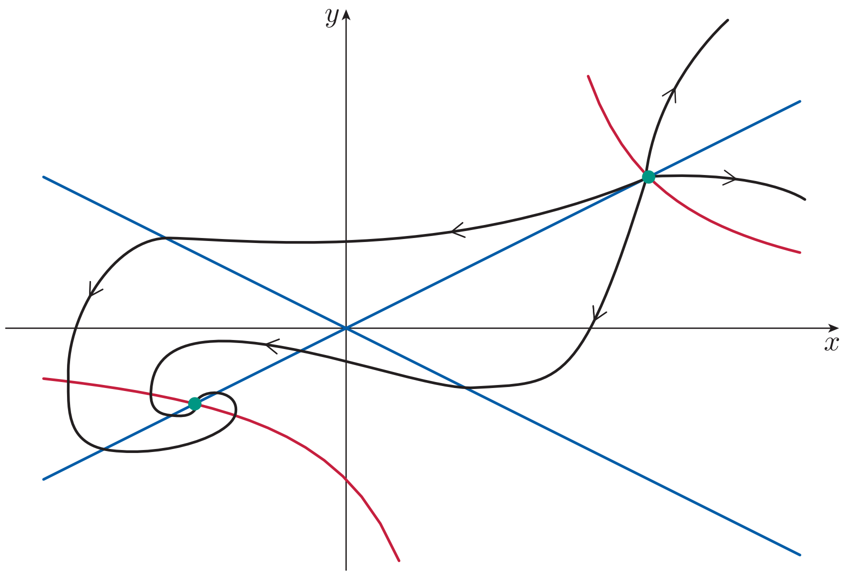


Figure 27 Phase portrait of the differential equations

The following exercise leads you through Procedure 5 step by step.

Exercise 25

Exercise 23 considered the system of differential equations

$$\dot{x} = (1 + x - 2y)x, \quad \dot{y} = (x - 1)y,$$

and found that this system has three equilibrium points, namely a saddle point at $(0, 0)$, a sink at $(-1, 0)$ and a spiral source at $(1, 1)$.

◀ Equilibrium points ▶

(a) Mark the equilibrium points on a sketch, and draw short paths in the neighbourhood of each equilibrium point.

(b) Find the nullclines, and add these to your sketch. ◀ Nullclines ▶

(c) Find the signs of the paths crossing the nullclines, and mark these on your sketch. ◀ Nullcline crossings ▶

(d) Add NW, NE, SW or SE arrows to each region of the phase plane separated by the nullclines on your sketch. ◀ Nullcline regions ▶

(e) Sketch the phase portrait of this system of equations. ◀ Complete sketch ▶

This section concludes by looking ahead to phenomena that you may meet in your studies after completing this module. The first of these phenomena is when a solution tends to a repeating pattern rather than tending to an equilibrium point.

Recall that periodic solutions are represented by closed curves in the phase plane. Here we name a particular type of periodic solution.

A **limit cycle** of a system of differential equations is a closed solution curve to which nearby solution curves tend (either forwards or backwards in time).

An example will make this definition clearer, so consider the system of differential equations given by

$$\dot{x} = x(1 - x^2 - y^2) - y, \quad \dot{y} = y(1 - x^2 - y^2) + x. \tag{27}$$

The vector field plot for this system is shown in Figure 28.

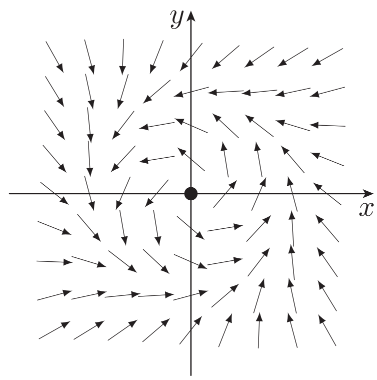


Figure 28 A vector field plot of a system of differential equations that contains a limit cycle

This system of equations has a single equilibrium point at the origin, which is a spiral source. What is more interesting is what happens away from the origin. As the origin is a spiral source, the paths near the origin spiral outwards. In the figure, the arrows near the origin spiral anticlockwise and away from the origin.

However, near the edge of Figure 28 it can clearly be seen that the arrows correspond to paths that spiral inwards. This behaviour is because of the presence of the factor $(1 - x^2 - y^2)$ in both equations. For x and y small, this factor is positive, and this gives rise to paths spiralling outwards. For x and y large, the factor is negative, and this in turn gives rise to paths spiralling inwards. This factor is zero when $x^2 + y^2 = 1$, which corresponds to points on a circle of radius 1. For these points the system reduces to the simple linear system

$$\dot{x} = -y, \quad \dot{y} = x,$$

which has the solution $x(t) = \cos(t + \phi)$, $y(t) = \sin(t + \phi)$ that satisfies the initial condition $x = \cos\phi$, $y = \sin\phi$. So the circle $x^2 + y^2 = 1$ is a path in the phase plane. We have already seen that paths starting inside this circle spiral outwards towards the circle, and paths starting outside the circle spiral inwards towards the circle. Another way of saying the previous two sentences is to say that the circle $x^2 + y^2 = 1$ is a limit cycle.

Another way of looking at equations (27) that is in some ways simpler is to consider transforming the equations into polar coordinates (r, θ) using the equations $x = r\cos\theta$ and $y = r\sin\theta$. In polar coordinates, equations (27) simplify to

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$

Now the solution curves can be simply derived. The equation for θ can be integrated to give $\theta = t + C$ for some constant C, which means that the solution curves will spiral anticlockwise (the positive direction for polar coordinates) at a constant rate. The sign of $\dot{r}$ from the first equation shows that r is increasing for $r < 1$ and decreasing for $r > 1$. So the solutions of the r differential equation tend to $r = 1$ as t tends to infinity. This shows that the curve $r = 1$ (i.e. the circle with radius 1) is a limit cycle.

Methods of establishing the existence of limit cycles and determining their equations are beyond the scope of this module. However, when investigating systems of non-linear differential equations, you should be aware that limit cycles can exist.

For systems of two differential equations involving the time derivatives of two variables, the only features in phase portraits are equilibrium points and limit cycles (this celebrated result is known as the Poincaré–Bendixson theorem). In systems with three or more variables, other possibilities can occur, such as so-called chaotic motion. The study of chaos is a fascinating topic in any deeper study of systems of differential equations, but now seems to be a good point at which to finish this introduction to the topic.

Learning outcomes

After studying this unit, you should be able to:

- use a vector field to describe a pair of first-order non-linear differential equations, and use paths in the phase plane to represent the solutions
- understand how systems of differential equations arise from mathematical modelling, and in particular the modelling of populations of predators and prey
- convert a higher-order differential equation to a system of first-order differential equations
- find the equilibrium points of a system of non-linear differential equations
- find linear equations that approximate the behaviour of a system of non-linear differential equations near an equilibrium point, by calculating the Jacobian matrix
- determine whether an equilibrium point is stable or unstable
- use the eigenvalues and eigenvectors of the Jacobian matrix at an equilibrium point to classify an equilibrium point as a source, a sink, a star source, a star sink, an improper source, an improper sink, a spiral source, a spiral sink, a saddle, or a centre
- use nullclines to sketch the phase portrait of a system of differential equations.

Solutions to exercises

Solution to Exercise 1

The differential equation $\dot{x} = kx$ can be solved using the method of separation of variables to get, as $x > 0$,

$$\int \frac{dx}{x} = \int k\,dt.$$

Evaluating the integrals (and noting that x is positive) gives

$$\ln x = kt + A,$$

where A is a constant.

Using the given initial condition determines the value of the constant A:

$$\ln x_0 = k \times 0 + A.$$

Substituting for A and simplifying gives the required particular solution

$$\ln x = kt + \ln x_0.$$

We can write this as

$$\ln x - \ln x_0 = kt \quad \text{or} \quad \ln(x/x_0) = kt,$$

so

$$x = x_0 e^{kt}.$$

Solution to Exercise 2

The system of differential equations is

$$\dot{x} = x, \quad \dot{y} = -y.$$

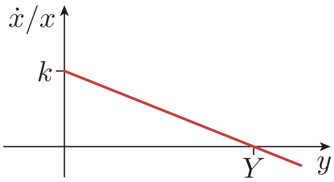
These equations can be solved applying separation of variables to get the two equations

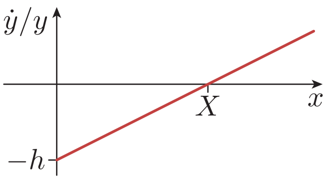
$$\int \frac{1}{x}\,dx = \int 1\,dt, \quad \int \frac{1}{y}\,dy = \int -1\,dt.$$

Performing the integrations and rearranging gives the general solution as

$$x(t) = Ce^{t}, \quad y(t) = De^{-t},$$

where C and D are constants.





Solution to Exercise 3

Rearranging equation (8), we obtain

$$\frac{\dot{x}}{x} = k - \frac{k}{Y}\,y,$$

which is the equation of a straight line, as shown in the margin.

The proportionate growth rate $\dot{x}/x$ of rabbits decreases as the population y of foxes increases, becoming zero when $y = Y$. The population x of rabbits will increase if the population y of foxes is less than Y, but will decrease if y is greater than Y.

Similarly rearranging equation (9), we have

$$\frac{\dot{y}}{y} = -h + \frac{h}{X}\,x,$$

so the graph of $\dot{y}/y$ as a function of x is as shown in the margin.

The proportionate growth rate $\dot{y}/y$ of foxes increases linearly as the population x of rabbits increases. The population y of foxes will decrease if the population x of rabbits is less than X, but will increase if x is greater than X.

Solution to Exercise 4

(a) The Lotka–Volterra equations can be written as

$$\dot{\mathbf{x}} = \mathbf{u}(x, y),$$

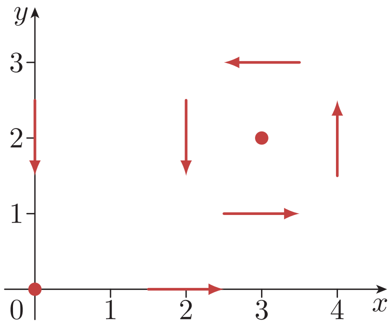
where $\dot{\mathbf{x}} = (\dot{x} \quad \dot{y})^T$ and the vector field $\mathbf{u}(x, y)$ is given by

$$\mathbf{u}(x, y) = \begin{pmatrix} kx\left(1 - \dfrac{y}{Y}\right) \\ -hy\left(1 - \dfrac{x}{X}\right) \end{pmatrix}.$$

(b) The completed table is shown below.

x	y	$\mathbf{u}(x, y)$
0	0	$\mathbf{0}$
0	2	$(0 \quad -1)^T$
2	0	$(2 \quad 0)^T$
2	2	$(0 \quad -1/3)^T$
3	1	$(3/2 \quad 0)^T$
3	2	$\mathbf{0}$
3	3	$(-3/2 \quad 0)^T$
4	2	$(0 \quad 1/3)^T$

(c) The vectors are plotted in the following figure.



Solution to Exercise 5

(a) Let $y = \dot{x}$, so $\dot{y} = \ddot{x}$. Then the given equation becomes $\dot{y} + \sin x = 0$. This leads to the system of first-order equations

$$\dot{x} = y, \quad \dot{y} = -\sin x.$$

(b) As before, let $y = \dot{x}$, so $\dot{y} = \ddot{x}$. Then the given equation becomes $\dot{y} - 2y + x^2 = 0$, and the equivalent first-order system is

$$\dot{x} = y, \quad \dot{y} = 2y - x^2.$$

(c) Here we have different variables and a third-order equation, but the basic idea is still the same – that is, we introduce new variables for the derivatives of the independent variable. So let

$$v = \frac{du}{dx} \quad \text{and} \quad w = \frac{d^2u}{dx^2} = \frac{dv}{dx}.$$

In terms of these variables the given equation becomes

$$\frac{dw}{dx} = 6uv.$$

So the equivalent system of first-order equations is

$$\frac{du}{dx} = v, \quad \frac{dv}{dx} = w, \quad \frac{dw}{dx} = 6uv.$$

Solution to Exercise 6

Procedure 1 leads to the pair of simultaneous equations

$$x(20 - y) = 0,$$
$$y(10 - y)(10 - x) = 0.$$

From the first equation, either $x = 0$ or $y = 20$.

Substituting $x = 0$ into the second equation gives $y(10 - y) \times 10 = 0$, which gives $y = 0$ or $y = 10$. So $(0, 0)$ and $(0, 10)$ are equilibrium points.

Substituting $y = 20$ into the second equation gives $20 \times (-10) \times (10 - x) = 0$, which has solution $x = 10$. So $(10, 20)$ is an equilibrium point.

Hence the complete list of equilibrium points is $(0, 0)$, $(0, 10)$ and $(10, 20)$.

Solution to Exercise 7

Using Procedure 1, we must solve the pair of simultaneous equations

$$2x^2y + 7xy^2 + 2y + 1 = 0,$$
$$xy - x = 0.$$

In order to reduce the effort needed, we start by considering solutions of the second equation as it is much simpler. The second equation factorises as $x(y-1) = 0$, which gives $x = 0$ or $y = 1$.

Substituting $x = 0$ into the first equation gives $2y + 1 = 0$, which has solution $y = -\frac{1}{2}$. So $(0, -\frac{1}{2})$ is an equilibrium point.

Substituting $y = 1$ into the first equation gives $2x^2 + 7x + 3 = 0$, which factorises as $(2x+1)(x+3) = 0$. So $x = -\frac{1}{2}$ or $x = -3$, which gives the equilibrium points $(-\frac{1}{2}, 1)$ and $(-3, 1)$.

So the complete list of equilibrium points is $(0, -\frac{1}{2})$, $(-\frac{1}{2}, 1)$ and $(-3, 1)$.

Solution to Exercise 8

The paths move towards the equilibrium point A, so it is stable.

The paths move away from the equilibrium point B, so it is unstable.

The paths move around the equilibrium point C thus stay in the neighbourhood of the point, so it is stable.

Solution to Exercise 9

We evaluate the various partial derivatives found in Example 3. At the equilibrium point $(0, 0)$, we obtain

$$u_x(0,0) = k, \quad u_y(0,0) = 0,$$
$$v_x(0,0) = 0, \quad v_y(0,0) = -h.$$

Thus the required linear approximation is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

giving the pair of equations

$$\dot{p} = kp, \quad \dot{q} = -hq.$$

Solution to Exercise 10

Here we have

$$u(x,y) = x(20-y), \quad v(x,y) = y(10-y)(10-x),$$

giving partial derivatives

$$u_x(x,y) = 20 - y, \quad u_y(x,y) = -x,$$
$$v_x(x,y) = -y(10-y), \quad v_y(x,y) = (10-y)(10-x) - y(10-x).$$

So the Jacobian matrix of the vector field $\mathbf{u}(x,y)$ is

$$\mathbf{J}(x,y) = \begin{pmatrix} 20 - y & -x \\ -y(10-y) & (10-y)(10-x) - y(10-x) \end{pmatrix}.$$

At the equilibrium point $(10, 20)$ we have

$$\mathbf{J}(10,20) = \begin{pmatrix} 0 & -10 \\ 200 & 0 \end{pmatrix},$$

so the linear approximation is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -10 \\ 200 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(This is equivalent to the pair of equations

$$\dot{p} = -10q, \quad \dot{q} = 200p.)$$

Solution to Exercise 11

Solving the equations $3x + 2y - 8 = 0$ and $x + 4y - 6 = 0$, we obtain the equilibrium point $(2, 1)$.

Putting $x = 2 + p$ and $y = 1 + q$, we obtain the matrix equation

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

(Note that this linear approximation near $(2, 1)$ is exact, since the original system is linear.)

Solution to Exercise 12

(a) To find the equilibrium points, we solve the simultaneous equations

$$0.5x - 0.000\,05x^2 = 0,$$
$$-0.1y + 0.0004xy - 0.01y^2 = 0.$$

Factorising these equations gives

$$0.5x(1 - 0.0001x) = 0,$$
$$-0.1y(1 - 0.004x + 0.1y) = 0.$$

The first equation gives

$$x = 0 \quad \text{or} \quad x = 10\,000.$$

If $x = 0$, the second equation is

$$-0.1y(1 + 0.1y) = 0,$$

which gives $y = 0$ or $y = -10$. As $y \geq 0$, only the first solution is possible. This leads to the equilibrium point $(0, 0)$.

If $x = 10\,000$, the second equation is

$$-0.1y(-39 + 0.1y) = 0,$$

which gives $y = 0$ or $y = 390$. So we have found two more equilibrium points, namely $(10\,000, 0)$ and $(10\,000, 390)$.

So this system has three equilibrium points, namely $(0, 0)$, $(10\,000, 0)$ and $(10\,000, 390)$.

(b) We have

$$u(x,y) = 0.5x - 0.000\,05x^2,$$
$$v(x,y) = -0.1y + 0.0004xy - 0.01y^2.$$

So the Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} 0.5 - 0.0001x & 0 \\ 0.0004y & -0.1 + 0.0004x - 0.02y \end{pmatrix}.$$

(c) At the equilibrium point $(0,0)$,

$$\mathbf{J}(0,0) = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.1 \end{pmatrix},$$

and the linearised approximations to the differential equations near this equilibrium point are

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

At the equilibrium point $(10\,000, 0)$,

$$\mathbf{J}(10\,000, 0) = \begin{pmatrix} -0.5 & 0 \\ 0 & 3.9 \end{pmatrix},$$

and the linearised approximations to the differential equations near this equilibrium point are

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ 0 & 3.9 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Finally, at the equilibrium point $(10\,000, 390)$,

$$\mathbf{J}(10\,000, 390) = \begin{pmatrix} -0.5 & 0 \\ 0.156 & -3.9 \end{pmatrix},$$

and the linearised approximations to the differential equations near this equilibrium point are

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ 0.156 & -3.9 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Solution to Exercise 13

(a) As the given matrix is lower triangular, the eigenvalues can be read off the leading diagonal, so the eigenvalues are 1 and 3.

(b) As the eigenvalues are positive and distinct, the equilibrium point is a source.

Solution to Exercise 14

(a) The characteristic equation of the matrix of coefficients is

$$-\lambda(-3 - \lambda) + 2 = 0,$$

which simplifies to

$$\lambda^2 + 3\lambda + 2 = 0,$$

which factorises to give

$$(\lambda + 1)(\lambda + 2) = 0,$$

so the eigenvalues are $\lambda = -1$ and $\lambda = -2$.

(b) As the eigenvalues are negative and distinct, the equilibrium point is a sink.

Solution to Exercise 15

(a) The characteristic equation of the matrix of coefficients is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0,$$

or $\lambda^2 + \lambda - 6 = 0$, which factorises to give

$$(\lambda - 2)(\lambda + 3) = 0,$$

so the eigenvalues are $\lambda = 2$ and $\lambda = -3$.

The eigenvectors $(a \quad b)^T$ corresponding to $\lambda = 2$ satisfy the equations

$$\begin{aligned} -a + 2b &= 0, \\ 2a - 4b &= 0. \end{aligned}$$

So an eigenvector corresponding to the positive eigenvalue $\lambda = 2$ is $(2 \quad 1)^T$, and all the eigenvectors are along the line $q = \frac{1}{2}p$.

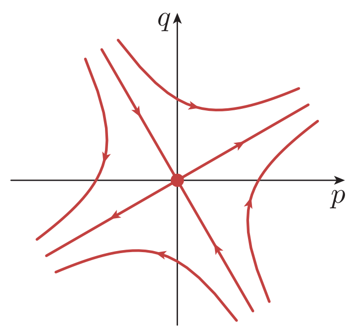
The eigenvectors $(a \quad b)^T$ corresponding to $\lambda = -3$ satisfy the equations

$$\begin{aligned} 4a + 2b &= 0, \\ 2a + b &= 0. \end{aligned}$$

So an eigenvector corresponding to the negative eigenvalue $\lambda = -3$ is $(1 \quad -2)^T$, and all the eigenvectors are along the line $q = -2p$.

(b) The matrix of coefficients has a positive eigenvalue and a negative eigenvalue, so the equilibrium point is a saddle.

(c) There are two straight-line paths, namely $q = \frac{1}{2}p$ and $q = -2p$. On the line $q = \frac{1}{2}p$, the point $(p(t), q(t))$ moves away from the origin as t increases, because the corresponding eigenvalue is *positive*. On the line $q = -2p$, the point approaches the origin as t increases, because the corresponding eigenvalue is *negative*. This information, together with the knowledge that the equilibrium point is a saddle, allows us to sketch the phase portrait shown in the margin.



Solution to Exercise 16

(a) The characteristic equation of the matrix of coefficients is

$$(2 - \lambda)(-2 - \lambda) + 5 = 0,$$

that is, $\lambda^2 + 1 = 0$, so the eigenvalues are $\lambda = i$ and $\lambda = -i$.

(b) As the eigenvalues are imaginary, the equilibrium point is a centre.

Solution to Exercise 17

(a) The characteristic equation of the matrix of coefficients is

$$(1-\lambda)^2 + 1 = 0,$$

that is, $\lambda^2 - 2\lambda + 2 = 0$, which has complex roots $\lambda = 1 + i$ and $\lambda = 1 - i$.

(b) As the eigenvalues are complex with positive real part, the equilibrium point is a spiral source.

Solution to Exercise 18

(a) As the matrix is diagonal, the eigenvalues can be read off the leading diagonal. So the eigenvalue is 2 (repeated).

In fact, the matrix is twice the identity matrix, so any vector transforms to twice itself. So any non-zero vector is an eigenvector. So we can choose $(1 \quad 0)^T$ and $(0 \quad 1)^T$ to be two linearly independent eigenvectors.

(b) The differential equations are

$$\dot{p} = 2p, \quad \dot{q} = 2q,$$

which have general solution

$$p(t) = Ce^{2t}, \quad q(t) = De^{2t},$$

where C and D are arbitrary constants.

(c) Eliminating t from the general solution, the equations of the paths are

$$q = \frac{D}{C}p = Kp \quad (C \neq 0),$$

where $K = D/C$ is also an arbitrary constant. So the paths are all straight lines passing through the origin.

The above analysis has neglected the possibility $C = 0$. In this case the path is $p = 0$, which is also a straight line passing through the origin, namely the q-axis.

(d) Both $p(t)$ and $q(t)$ are increasing functions of time, so the point $(p(t), q(t))$ moves away from the origin as t increases. So the equilibrium point is unstable.

Solution to Exercise 19

(a) The characteristic equation is

$$(2-\lambda)^2 = 0,$$

so the matrix has the repeated eigenvalue $\lambda = 2$.

The eigenvectors $(a \quad b)^T$ corresponding to this repeated eigenvalue satisfy the equations

$$0 = 0, \quad a = 0,$$

so all the eigenvectors take the form $(0 \quad k)^T$, where k is a (non-zero) constant. There is only one independent eigenvector, an obvious choice being $\mathbf{v} = (0 \quad 1)^T$.

(b) Using the solution to part (a) and Procedure 2 of Unit 6, we need to find the vector $\mathbf{b} = (c \quad d)^T$ that satisfies the equation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

that is, $0 = 0$, $c = 1$.

So $\mathbf{b} = (1 \quad 0)^T$, and the general solution of the system of differential equations is

$$\begin{pmatrix} p \\ q \end{pmatrix} = C\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^{2t} + D \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t},$$

that is,

$$p(t) = Ce^{2t}, \quad q(t) = Cte^{2t} + De^{2t},$$

where C and D are constants.

Solution to Exercise 20

The characteristic equation of the matrix of coefficients is

$$\lambda^2 + hk = 0.$$

Because h, k are positive, the eigenvalues are $\lambda = \pm i\sqrt{hk}$, so the equilibrium point is a centre.

Solution to Exercise 21

The eigenvalues of the matrix of coefficients are $\lambda = k$ and $\lambda = -h$, where h, k are positive. So the equilibrium point is a saddle. (In fact, in this case we have to restrict p and q to non-negative values, but this does not affect our conclusion.)

Solution to Exercise 22

(a) The characteristic equation is

$$\lambda^2 - 4\lambda + 13 = 0,$$

so the eigenvalues are $2 + 3i$ and $2 - 3i$, corresponding to the eigenvectors $(1 \quad -i)^T$ and $(1 \quad i)^T$, respectively.

As the eigenvalues are complex with a positive real component, the equilibrium point is a spiral source.

(b) As the equilibrium point of the linear approximation is not a centre, the corresponding equilibrium point of the non-linear system is also a spiral source.

Solution to Exercise 23

(a) The equilibrium points are given by

$$(1 + x - 2y)x = 0,$$
$$(x - 1)y = 0.$$

The second equation gives

$$x = 1 \quad \text{or} \quad y = 0.$$

When $x = 1$, substituting into the first equation gives

$$2 - 2y = 0,$$

which leads to $y = 1$. So $(1, 1)$ is an equilibrium point.

When $y = 0$, substituting into the first equation gives

$$(1 + x)x = 0,$$

hence $x = 0$ or $x = -1$. So we have found two further equilibrium points, namely $(0, 0)$ and $(-1, 0)$.

Thus we have three equilibrium points: $(1, 1)$, $(0, 0)$ and $(-1, 0)$.

(b) With the usual notation,

$$u(x, y) = (1 + x - 2y)x = x + x^2 - 2xy,$$
$$v(x, y) = (x - 1)y = xy - y.$$

So the Jacobian matrix is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 + 2x - 2y & -2x \\ y & x - 1 \end{pmatrix}.$$

(c) At the point $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so the linearised system is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

The eigenvalues of the matrix of coefficients are $\lambda = 1$ and $\lambda = -1$. As one of the eigenvalues is positive and the other is negative, the equilibrium point of the linearised system is a saddle.

At the point $(-1, 0)$, the Jacobian matrix is

$$\begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix},$$

so the linearised system is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

The eigenvalues of the matrix of coefficients are $\lambda = -1$ and $\lambda = -2$. As the eigenvalues are negative and distinct, the equilibrium point of the linearised system is a sink.

At the point $(1, 1)$, the Jacobian matrix is

$$\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix},$$

so the linearised system is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

The characteristic equation of the matrix of coefficients is

$$\lambda^2 - \lambda + 2 = 0.$$

The roots of this quadratic equation are

$$\lambda = \tfrac{1}{2}(1 \pm i\sqrt{7}),$$

so the eigenvalues are complex with a positive real component.

Hence the equilibrium point of the linearised system is a spiral source.

(d) As none of the equilibrium points of the linearised systems found in part (c) are centres, the behaviour of the original non-linear system near the equilibrium points is the same as that of the linear approximations. In other words,

$(0, 0)$ is a saddle,
$(-1, 0)$ is a sink,
$(1, 1)$ is a spiral source.

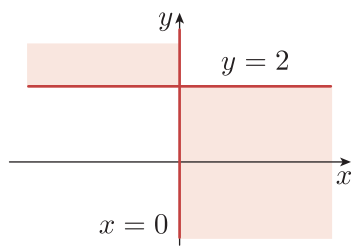
Solution to Exercise 24

(a) To answer this part of the exercise we use the equation

$$\dot{x} = x\left(1 - \frac{y}{2}\right).$$

As this equation is already factorised, we deduce that $\dot{x} = 0$ when $x = 0$ or when $y = 2$.

For $\dot{x}$ to be positive, either both terms in the product must be positive or both terms must be negative. Both terms are positive when $x > 0$ and $y < 2$. Both terms are negative when $x < 0$ and $y > 2$. This gives two regions where the growth rate $\dot{x}$ is positive.



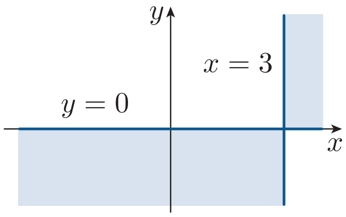
The figure in the margin is a sketch of these lines and regions. The two lines where $\dot{x}$ is zero are marked, and the two regions where $\dot{x}$ is positive are shaded.

(b) To answer this part of the exercise we use the equation

$$\dot{y} = -\frac{1}{2}y\left(1 - \frac{x}{3}\right).$$

Again, this equation is already factorised, so $\dot{y} = 0$ has solutions $y = 0$ or $x = 3$.

For $\dot{y}$ to be positive, the terms in the product $y(1 - x/3)$ must have opposite signs. This occurs when $y > 0$ and $x > 3$ and also when $y < 0$ and $x < 3$.



The figure in the margin is a sketch of these lines and regions. The two lines where $\dot{y}$ is zero are marked, and the two regions where $\dot{y}$ is positive are shaded.

Solution to Exercise 25

(a) The equilibrium points are marked in the figure in the margin.

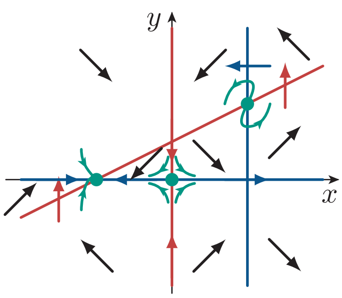
As described in Example 8, it is easiest to defer marking the paths near equilibrium until after the nullcline crossings have been determined. In this case the nullcline crossings indicate whether the spiral source at $(1, 1)$ is a clockwise or anticlockwise spiral.

(b) The nullclines for $\dot{x}$ are $x = 0$ and $1 + x - 2y = 0$ (which is the line $y = (1 + x)/2$). These are shown as the two red lines in the figure in the margin.

The nullclines for $\dot{y}$ are $y = 0$ and $x = 1$, which are shown as the two blue lines in the figure in the margin.

(c) The nullcline $x = 0$ has one equilibrium point on it, so to determine the direction of the crossings, we evaluate $\dot{y}$ at two points.

(x, y)	$(0, 1)$	$(0, -1)$
$\dot{y}$	-1	1



The signs of these values are marked by the red arrows along the y-axis in the figure in the margin.

The nullcline $y = (1 + x)/2$ has two equilibrium points on it, so to determine the direction of the crossings, we evaluate $\dot{y}$ at three points.

(x, y)	$(-2, -1/2)$	$(0, 1/2)$	$(3, 2)$
$\dot{y}$	$3/2$	$-1/2$	4

These are represented by up and down arrows marked in the figure in the margin.

The nullcline $y = 0$ has two equilibrium points on it, so we need to evaluate $\dot{x}$ at three points.

(x, y)	$(-2, 0)$	$(-1/2, 0)$	$(1, 0)$
$\dot{x}$	2	$-1/4$	2

These are represented by left and right arrows marked in the figure in the margin.

The nullcline $x = 1$ has one equilibrium point, so we need to evaluate $\dot{x}$ at two points.

(x, y)	$(1, 0)$	$(1, 2)$
$\dot{x}$	2	-2

These are represented by left and right arrows marked in the figure in the margin.

(d) Arrows indicating the general directions in which the phase paths curve in each region are shown in the diagram in the margin.

(e) The phase portrait for these equations is shown below.