Revise and refresh for MST124:

Welcome to Session 5

Wednesday 22nd September 2021

We'll start at 7.00pm and aim to finish by 9.00pm

Please check your Audio levels:

Speaker and Microphone setup

This session will cover the topics in

Trigonometry

Please feel free to use the chat box while waiting

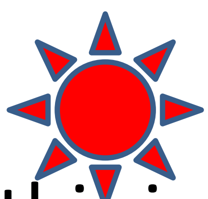
Have paper, pen and your calculator to hand.

What we aim to cover tonight

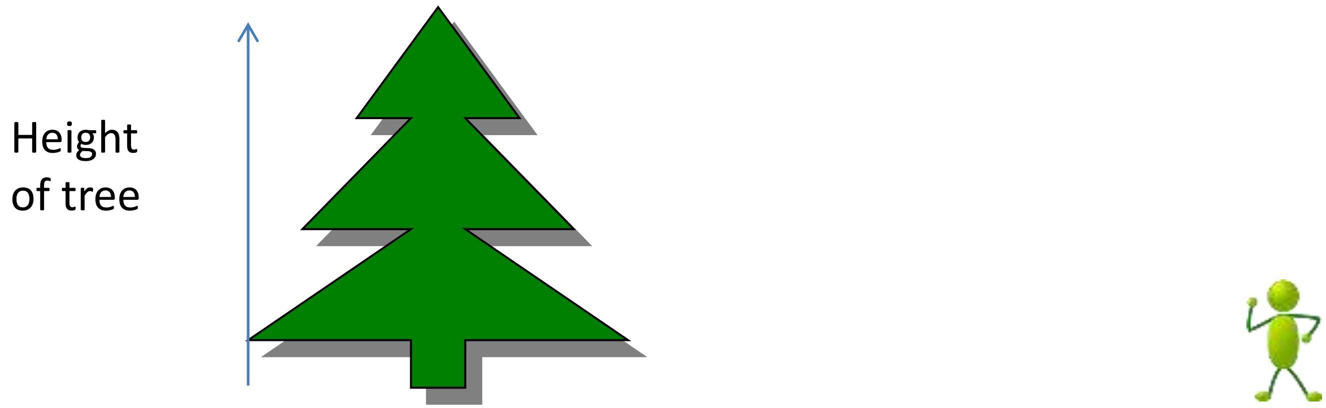
- Right angled triangles: sine, cosine and tangent.
- Finding unknown lengths.
- Finding unknown angles.
- Useful trigonometrical ratios and identities.
- Sine and Cosine Rules
- The area of a triangle.
- Angles > 90 degrees.
- Graphs of sine, cosine and tangent.
- Solving obtuse-angled triangles.
- Radians.
- Solving simple trigonometrical equations.

See MU123, Unit 12, and MST124, Unit 4.

What to do

- Do the exercises as we go along.
- I can't give you much time to do them, so if necessary do them afterwards.
- If you took MU123, go back through Unit 12.
- The subject is covered again in Unit 4 of MST124.
- If you see a  in the top left-hand corner of a frame, this is an issue that students frequently confuse, and lose marks on.

Measuring the height of a tree



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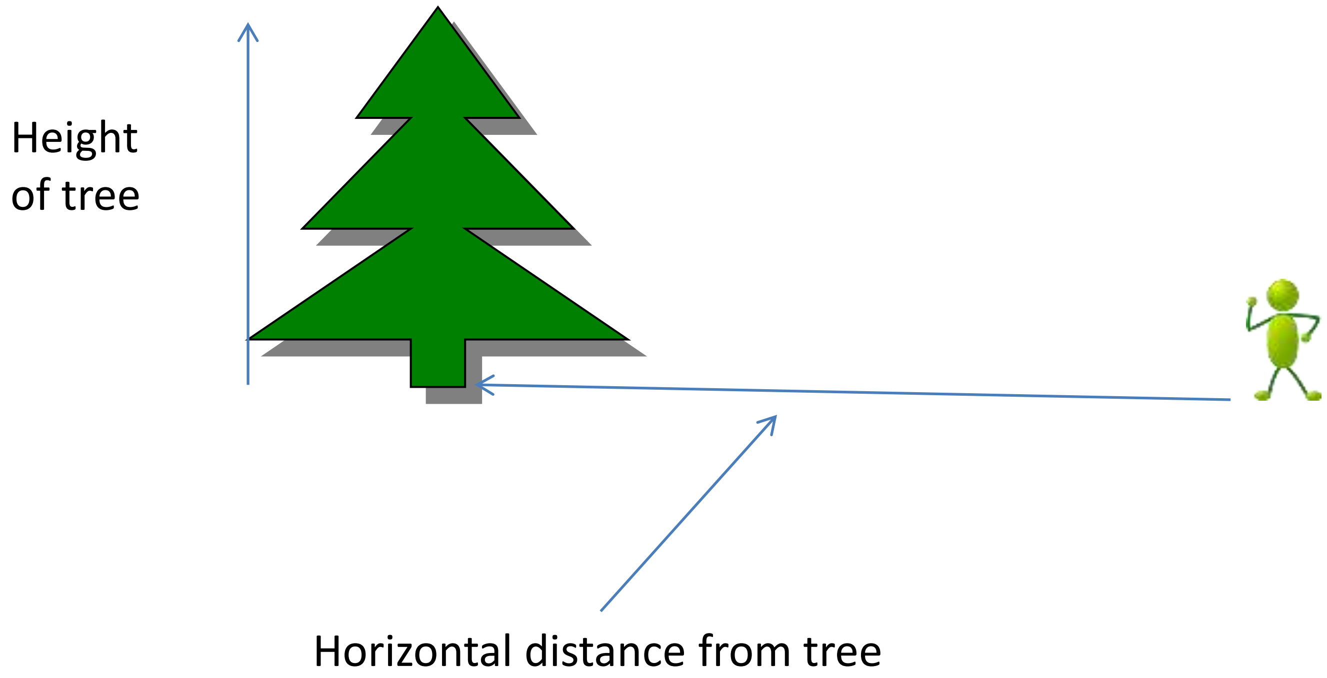
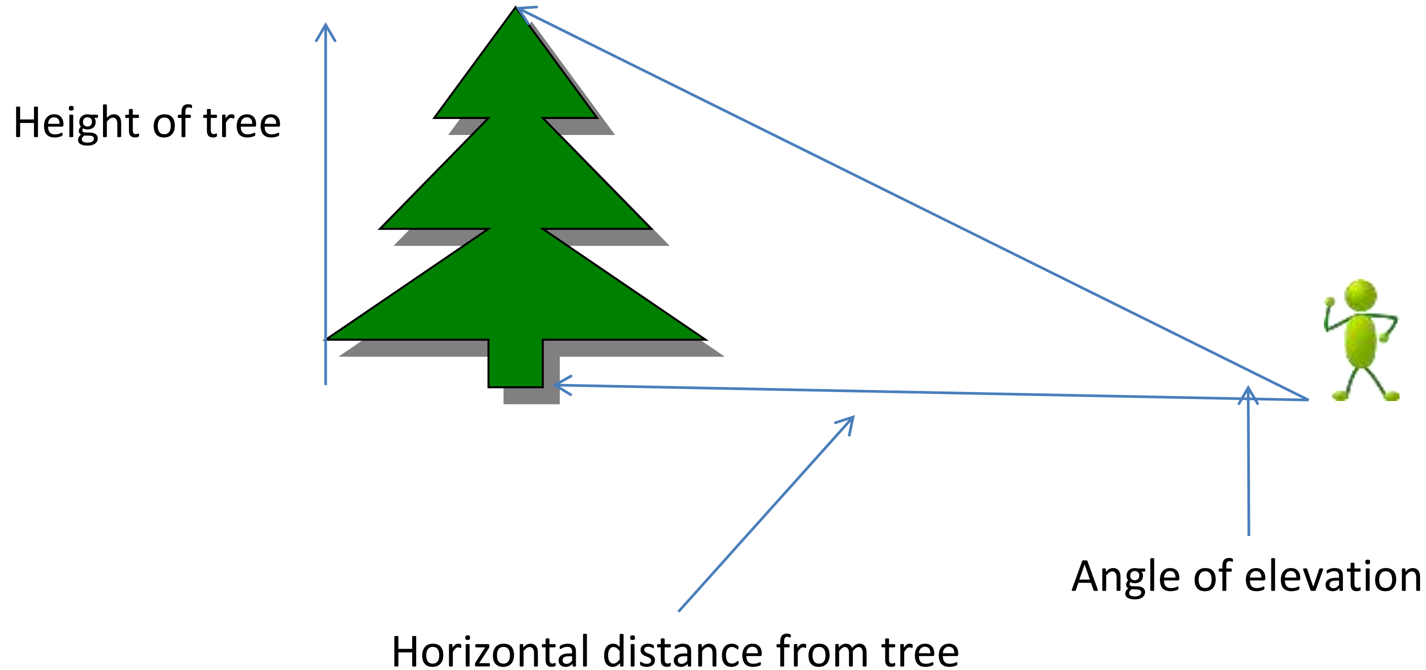


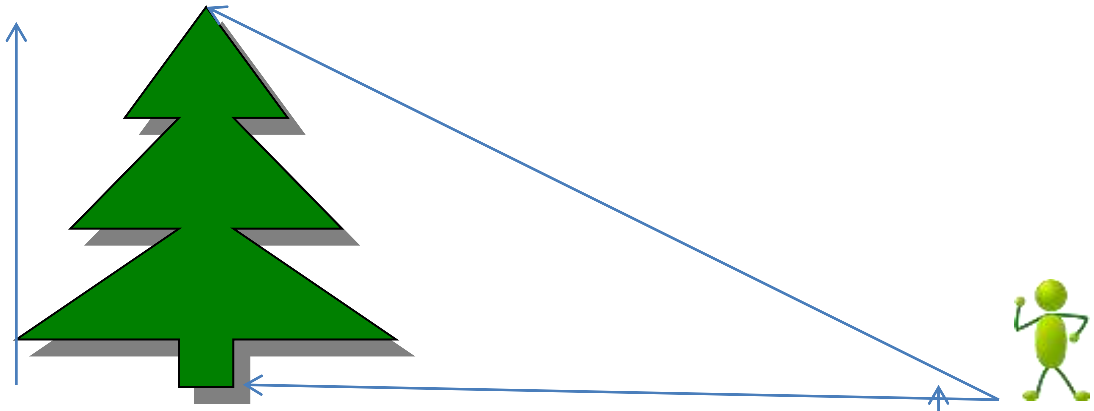
Measuring the height of a tree



Measuring the height of a tree

Height of tree

We'll call it h metres



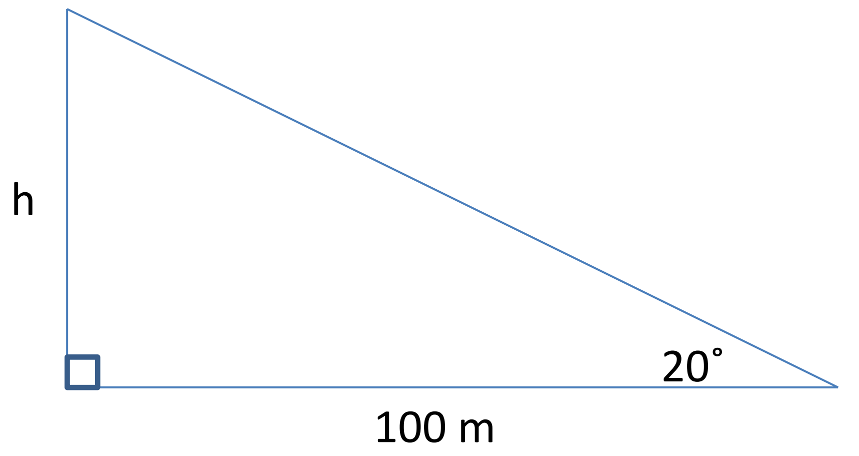
Horizontal distance from tree

Let's say 100 metres

Angle of elevation

Let's say 20°

Measuring the height of a tree

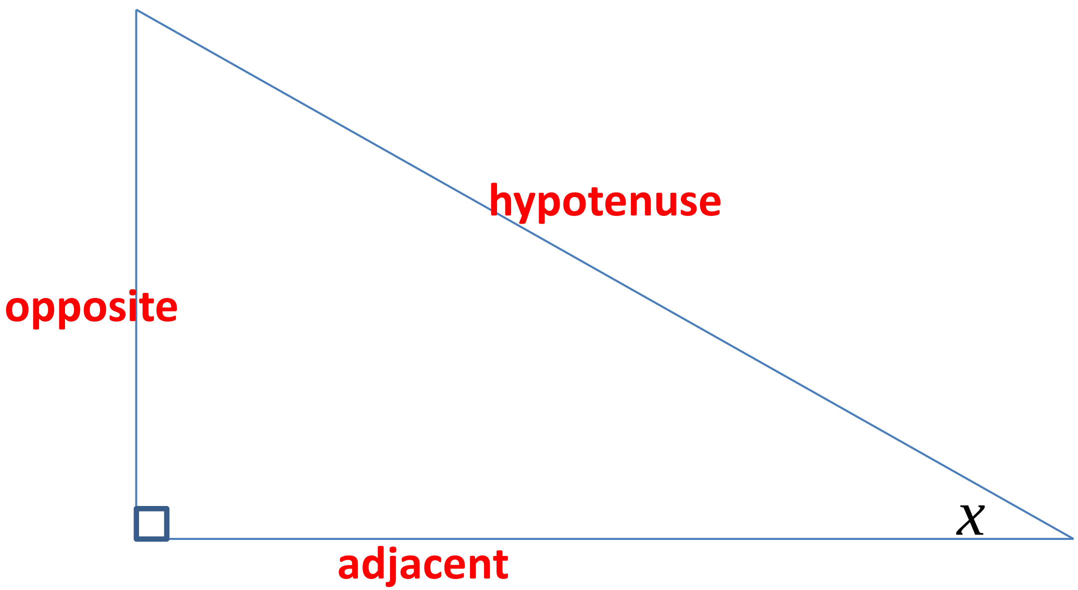


Trigonometry

Basic trigonometry is about the relationships between the lengths of the sides and the angles in right-angled triangles.

Terminology

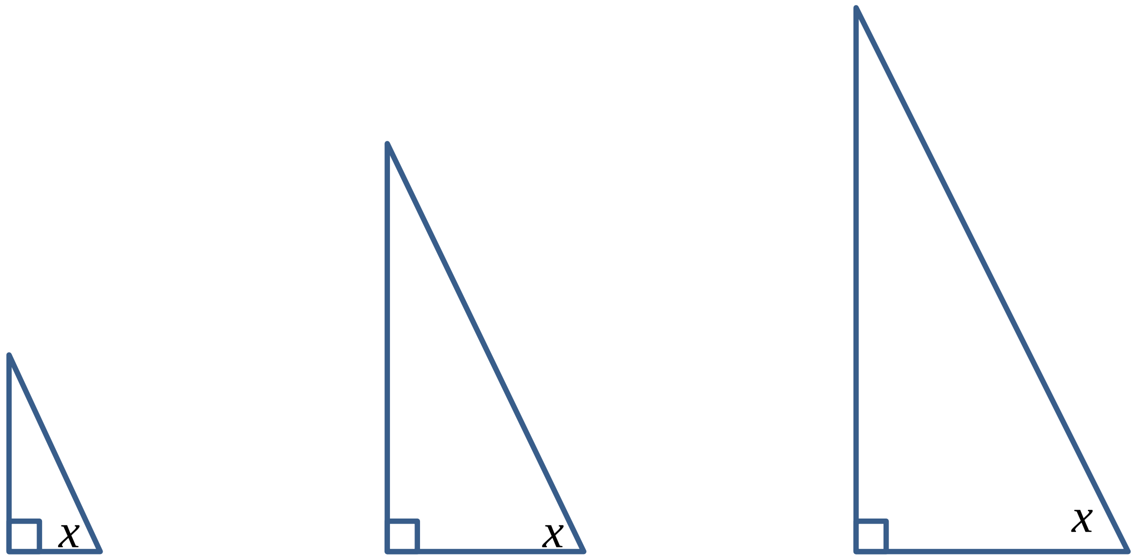
These definitions refer to the angle x in a right-angled triangle.



You should remember that in any triangle, the longest side is opposite the biggest angle, and the shortest side is opposite the smallest angle.

- The side opposite the right-angle is called the **hypotenuse**.
- The side opposite the angle x is called the **opposite**.
- The remaining side, the side next to x, is called the **adjacent**.

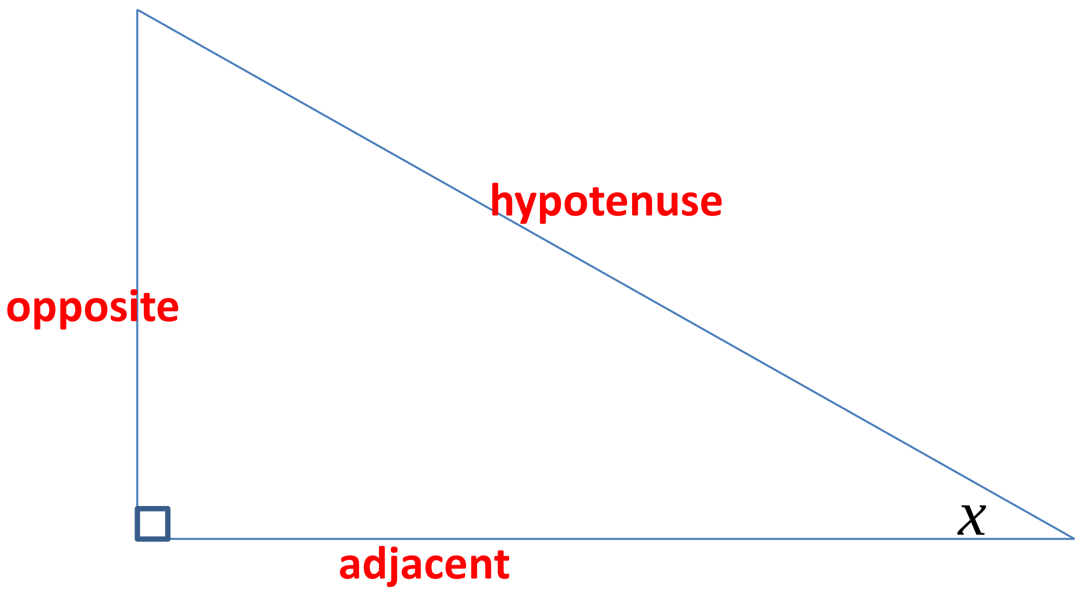
Similar right-angled triangles



$\boxed{\dfrac{opposite}{hypotenuse}}$ is the same for all three triangles, as are $\boxed{\dfrac{adjacent}{hypotenuse}}$ and $\boxed{\dfrac{opposite}{adjacent}}$

This means that if we know the value of x, we know the value of these ratios – regardless of the size of the triangle. It is this fact that underpins all trigonometry.

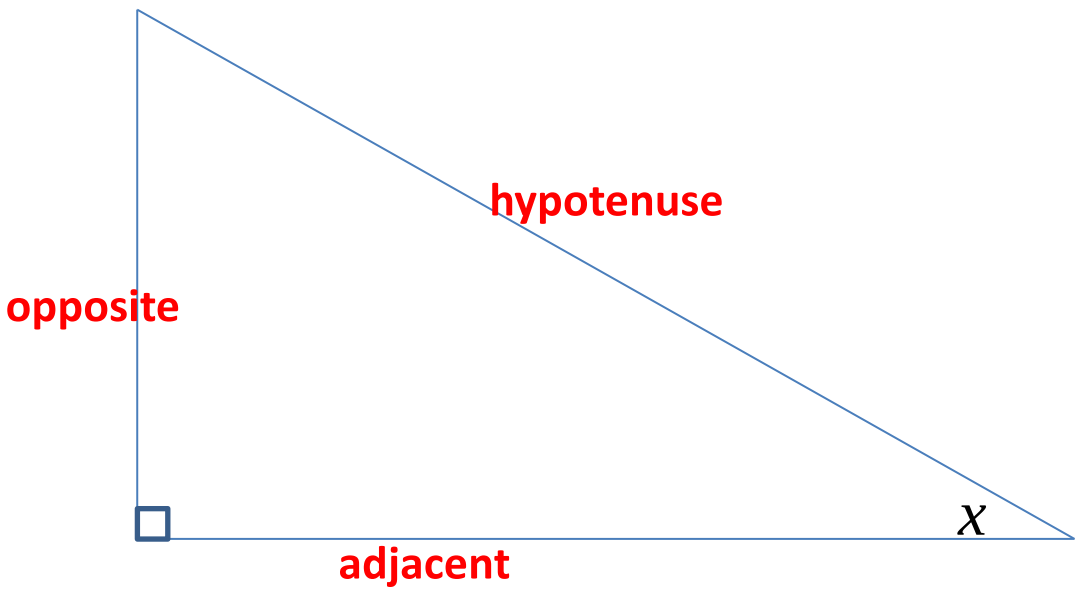
Terminology



For example, $\boxed{\dfrac{opposite}{hypotenuse}}$ is called the sine of x, which we abbreviate to sin x or sin(x).

(Note that “sin” is pronounced to rhyme with “wine”, not “bin”.)

Terminology



Other ratios are $\boxed{\dfrac{adjacent}{hypotenuse}}$ which is the cosine of x, which we write as cos x or cos(x)

and $\boxed{\dfrac{opposite}{adjacent}}$ which is the tangent of x, which we write as tan x or tan(x)

A useful mnemonic

- Remember: **SOHCAHTOA**

(pronounced “sock-ah-tow-ah”)

- If you prefer a more graphic mnemonic, you could try:

Some **O**ld **H**ouses **C**an **A**lways **H**ide **T**heir **O**ld **A**ge.

Trigonometric tables

The first table of sines was produced in 499 CE. Once they attained their modern form, they didn't change a lot until Hewlett Packard produced the first scientific calculator in 1972, thus rendering all such tables redundant.

55°6′ 5490 2

NATURAL COSINES $\cos x^{\circ}$

x°	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Δ_m –	1′ SUBTRACT	2′	3′	4′	5′
50°	0·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	14	2	5	7	9	12
51	0·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	14	2	5	7	9	12
52	0·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	14	2	5	7	9	12
53	0·6018	6004	$\boxed{5990}$	5976	5962	5948	5934	5920	5906	5892	14	2	5	7	9	$\boxed{12}$
54	0·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	14	2	5	7	9	12
55	0·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	14	2	5	7	9	12
56	0·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	15	2	5	7	10	12
57	0·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	15	2	5	7	10	12
58	0·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	15	2	5	7	10	12
59	0·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	15	3	5	8	10	13
60	0·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	15	3	5	8	10	13
61	0·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	15	3	5	8	10	13
62	0·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	16	3	5	8	11	13
63	0·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	16	3	5	8	11	13
64	0·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	16	3	5	8	11	13
65	0·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	16	3	5	8	11	13
66	0·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	16	3	5	8	11	13
67	0·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	16	3	5	8	11	13
68	0·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	16	3	5	8	11	13
69	0·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	16	3	5	8	11	13
70	0·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	16	3	5	8	11	13
71	0·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	17	3	6	8	11	14
72	0·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	17	3	6	8	11	14
73	0·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	17	3	6	8	11	14
74	0·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	17	3	6	8	11	14
75	0·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	17	3	6	8	11	14
76	0·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	17	3	6	8	11	14
77	0·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	17	3	6	9	11	14
78	0·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	17	3	6	9	11	14
79	0·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	17	3	6	9	11	14
80	0·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	17	3	6	9	11	14
81	0·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	17	3	6	9	11	14
82	0·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	17	3	6	9	11	14
83	0·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	17	3	6	9	11	14
84	0·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	17‡	3	6	9	11	14
85	0·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	17‡	3	6	9	11	14
86	0·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	17‡	3	6	9	11	14
87	0·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	17‡	3	6	9	11	14
88	0·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	17‡	3	6	9	11	14
89	0·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	18‡	3	6	9	12	15

* For interpolation to tenths (0°·01) use PPs on p. 45 for the difference Δ between successive tabular values.

‡ For 4 significant figures see footnote ‡, p. 17.

§ tan *and* cot *of angles near* 0° *and* 90°

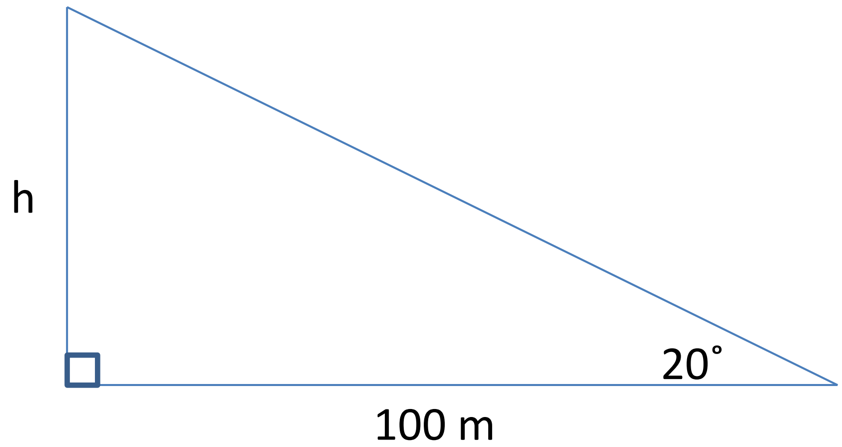
For angles $x^{\circ} = 90^{\circ} - y^{\circ} \leqslant 8^{\circ}$:

$$\cot y^{\circ} = \tan x^{\circ} = (1/\tau)\, x$$
$$x = \tau \tan x^{\circ} \qquad y = 90 - \tau \cot y^{\circ}$$
$$\tan y^{\circ} = \cot x^{\circ} = \tau / x$$
$$x = \tau / \cot x^{\circ} \qquad y = 90 - \tau / \tan y^{\circ}$$

If $x' \leqslant 60' = 1^{\circ}$: $\tan x' \doteqdot 0{\cdot}0002909x$

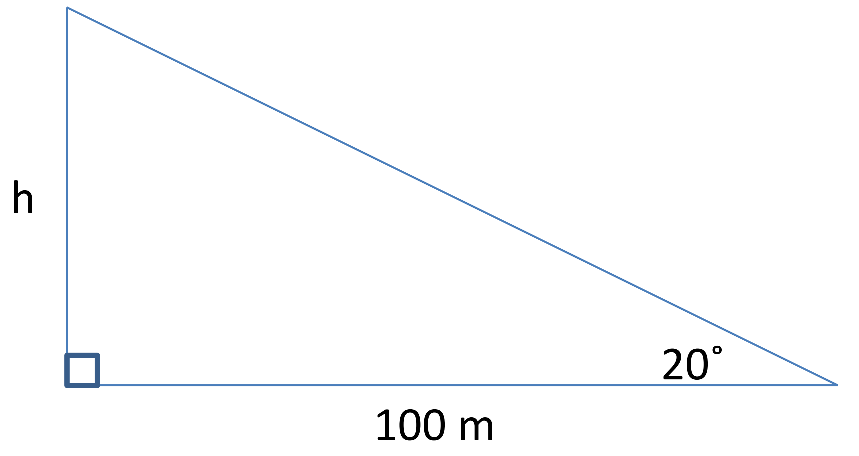
x°	τ	$1/\tau$	$\tan x^{\circ}$	$\cot x^{\circ}$
0°	57·30	·017453	·00000	∞
1	57·29	·017455	·01746	57·29
2	57·27	·017460	·03492	28·64
3	57·24	·017469	·05241	19·08
4	57·20	·017482	·06993	14·30
5	57·15	·017498	·08749	11·430
6	57·09	·017517	·10510	9·514
7	57·01	·017541	·12278	8·144
8	56·92	·017568	·14054	7·115

Measuring the height of a tree



$$\text{We know that tangent} = \frac{\text{opposite}}{\text{adjacent}} \text{ so we can say}$$

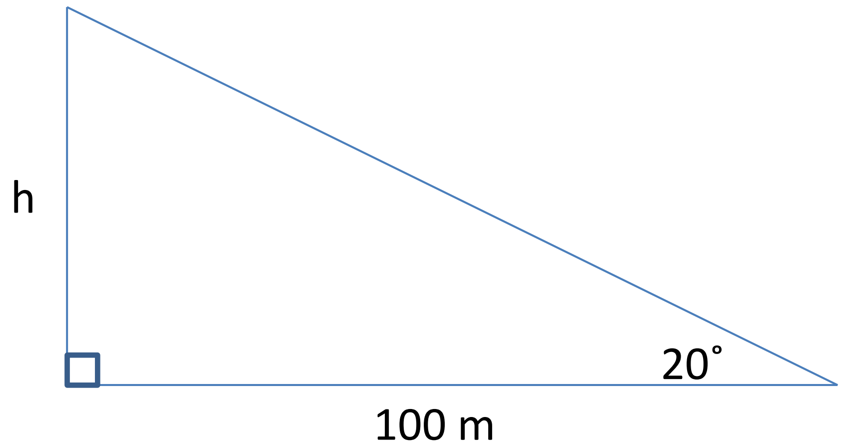
Measuring the height of a tree



$$\text{We know that tangent} = \frac{\text{opposite}}{\text{adjacent}} \text{ so we can say}$$

$$\tan 20° = \frac{h}{100}$$

Measuring the height of a tree

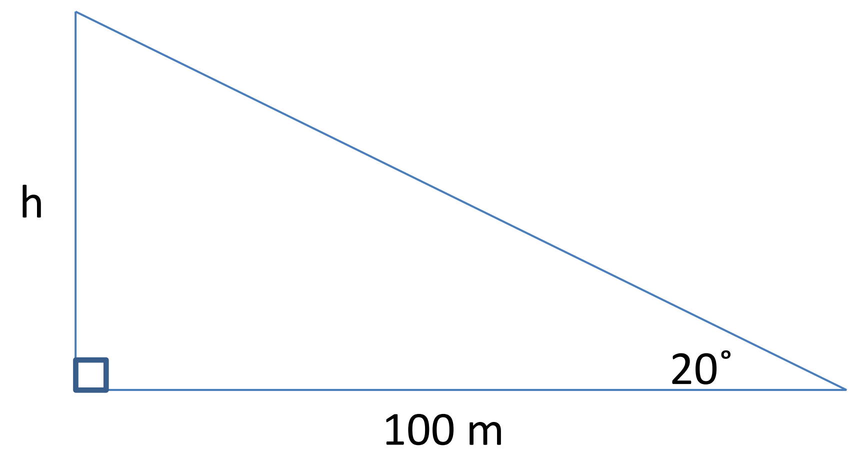


$$\text{We know that tangent} = \frac{\text{opposite}}{\text{adjacent}} \text{ so we can say}$$

$$\tan 20° = \frac{h}{100}$$

$$\therefore h = 100 \tan 20°$$

Measuring the height of a tree



$$\text{We know that tangent} = \frac{\text{opposite}}{\text{adjacent}} \text{ so we can say}$$

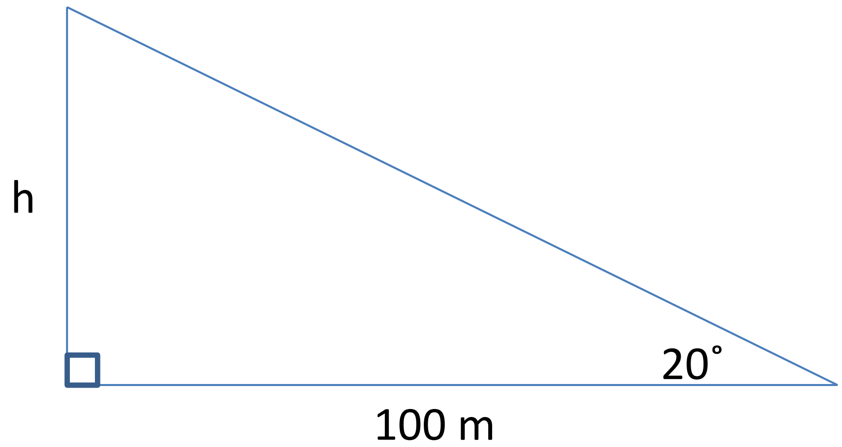
$$\tan 20° = \frac{h}{100}$$

$$\therefore h = 100 \tan 20°$$

$$= 100 \times 0.36397...$$

Remember: ***always*** work to full calculator accuracy, which is 10 significant figures. This is ***especially*** important in calculations involving trigonometry, where tiny errors in the input numbers can make a huge difference to the result.

Measuring the height of a tree



$$\text{We know that tangent} = \frac{\text{opposite}}{\text{adjacent}} \text{ so we can say}$$

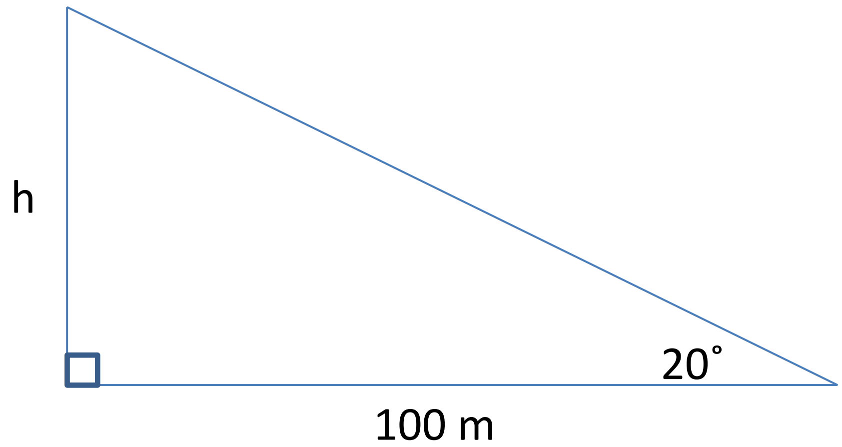
$$\tan 20° = \frac{h}{100}$$

$$\therefore h = 100 \tan 20°$$

$$= 100 \times 0.36397...$$

$$= 36.397... \text{ metres}$$

Measuring the height of a tree



$$\text{We know that tangent} = \frac{\text{opposite}}{\text{adjacent}} \text{ so we can say}$$

$$\tan 20^\circ = \frac{h}{100}$$

$$\therefore h = 100 \tan 20^\circ$$

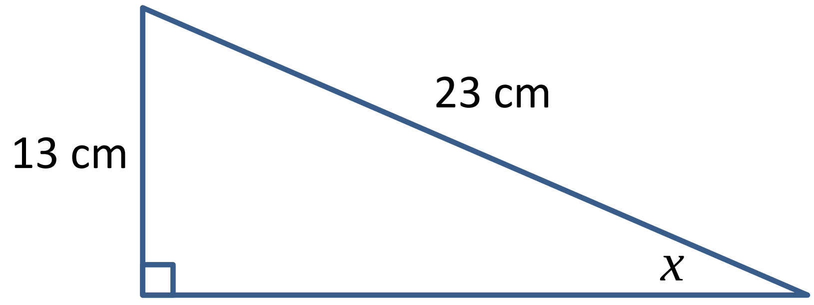
$$= 100 \times 0.36397...$$

$$= 36.397... \text{ metres}$$

So the height of the tree is 36 metres, to the nearest metre.

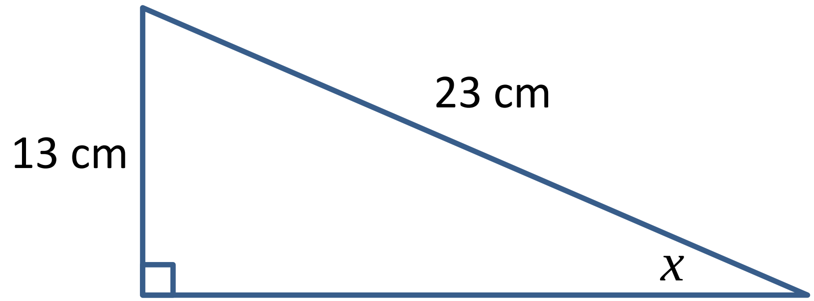
1. Never round until you have ***finished*** the calculation; then round appropriately for the conclusion.
2. Always write down the full calculator value as the last line of your working before you round it.

Finding angles



What is the value of x?

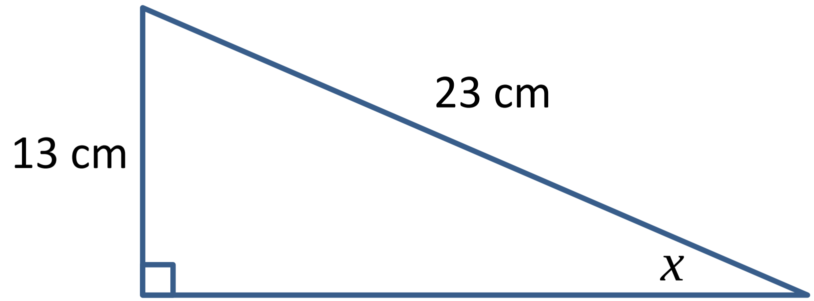
Finding angles



$$\sin x = \frac{13}{23}$$

We want the inverse of sin, just as subtraction is the inverse of addition, or square root is the inverse of square. Finding the inverse is the operation that gets you back to where you started.

Finding angles



$$\sin x = \frac{13}{23}$$

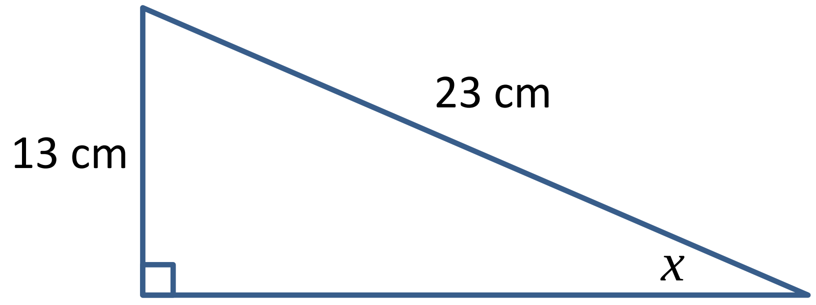
$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{13}{23}\right)$$

$$x = \sin^{-1}\left(\frac{13}{23}\right)$$

Sin^{-1} and sin cancel each other out, not because $\sin^{-1}$ is the reciprocal of sin – it isn't – but because it is the inverse.

Sin^{-1} is the generally accepted notation but it can also be written as arcsin or asin, and you may find any of these symbols on your calculator.

Finding angles



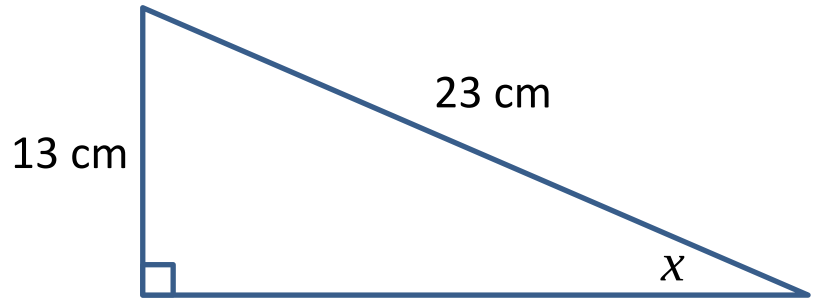
$$\sin x = \frac{13}{23}$$

$$x = \sin^{-1}\left(\frac{13}{23}\right)$$

$$= 34.4173...$$

Never convert a number to a decimal in the middle of a calculation; always work with the most accurate figure possible. Modern calculators can do the calculation in this form.

Finding angles



$$\sin x = \frac{13}{23}$$

$$x = \sin^{-1}\left(\frac{13}{23}\right)$$

$$= 34.4173\ldots$$

$$= 34.4° \text{ (to three sf)}$$

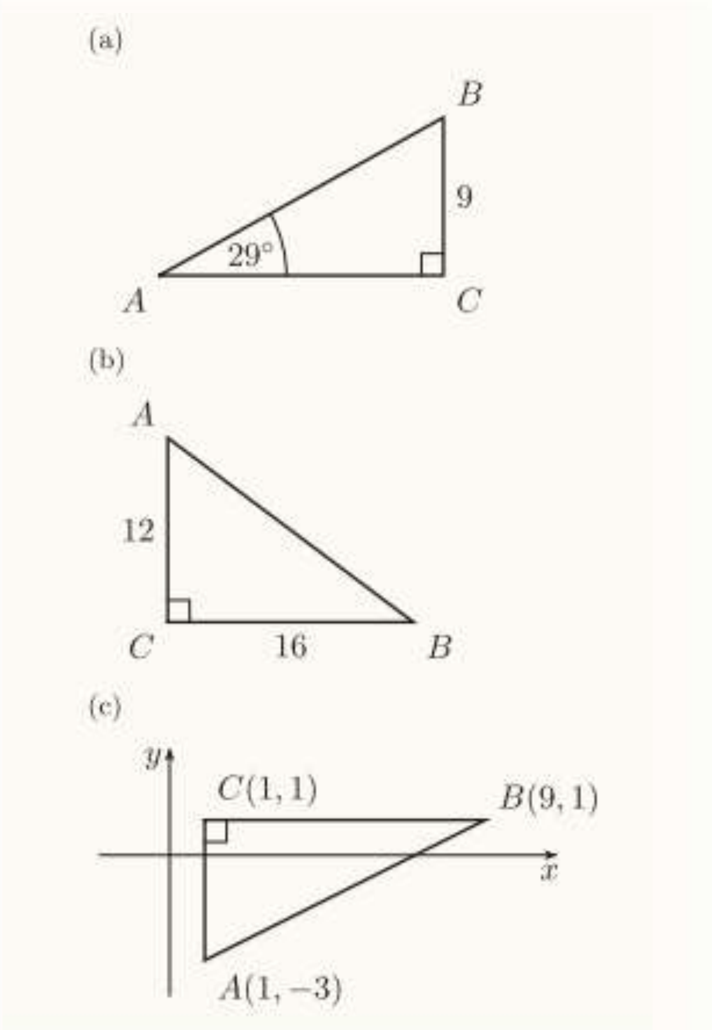
$\sin^{-1}(x)$ and $(\sin x)^{-1}$

Do not confuse these two operations:

the inverse of $\sin x$ is $\sin^{-1} x$

$$\text{the reciprocal of } \sin x = \frac{1}{\sin x} = (\sin x)^{-1}$$

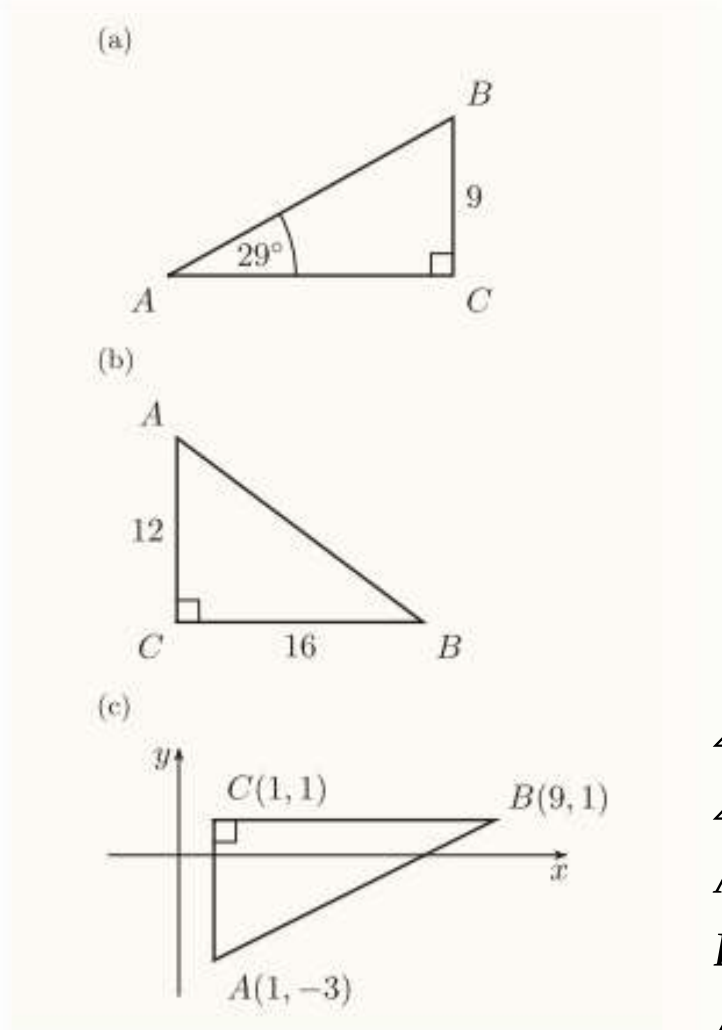
Finding lengths & angles



Find all the unknown lengths and angles.

These questions, like all those I have asked in this tutorial, are taken from the MST124, Unit 4 Exercise Book, which is available on the MST124 website.

Finding lengths & angles



$\angle B = 61°$

$AC = 16.2cm\ (to\ 3sf)$

$AB = 18.6cm\ (to\ 3sf)$

$\angle B = 36.9°\ (to\ 3sf)$

$\angle A = 53.1°\ (to\ 3sf)$

$AB = 20cm$

$\angle A = 63.4°\ (to\ 3sf)$

$\angle B = 26.6°\ (to\ 3sf)$

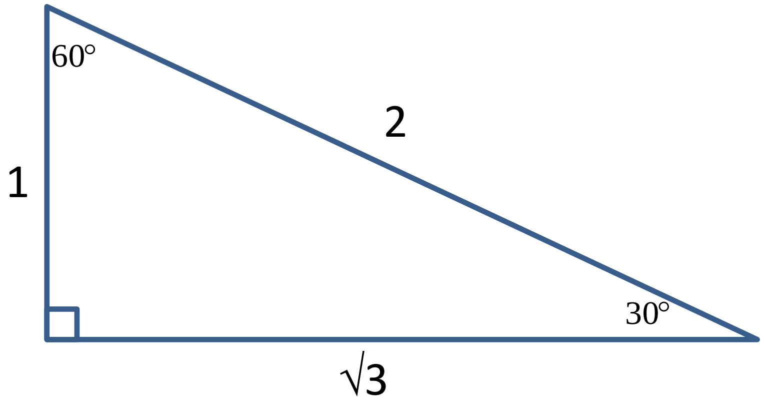
$AB = 8.94cm\ (to\ 3sf)$

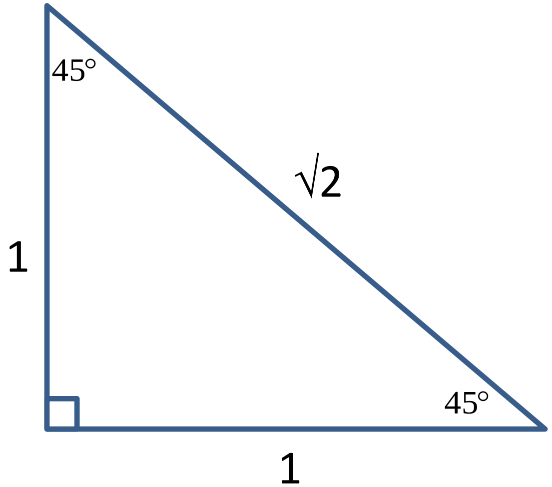
$BC = 8cm$

$AC = 4cm$

Two useful triangles

30,60,90 and 45,45,90.

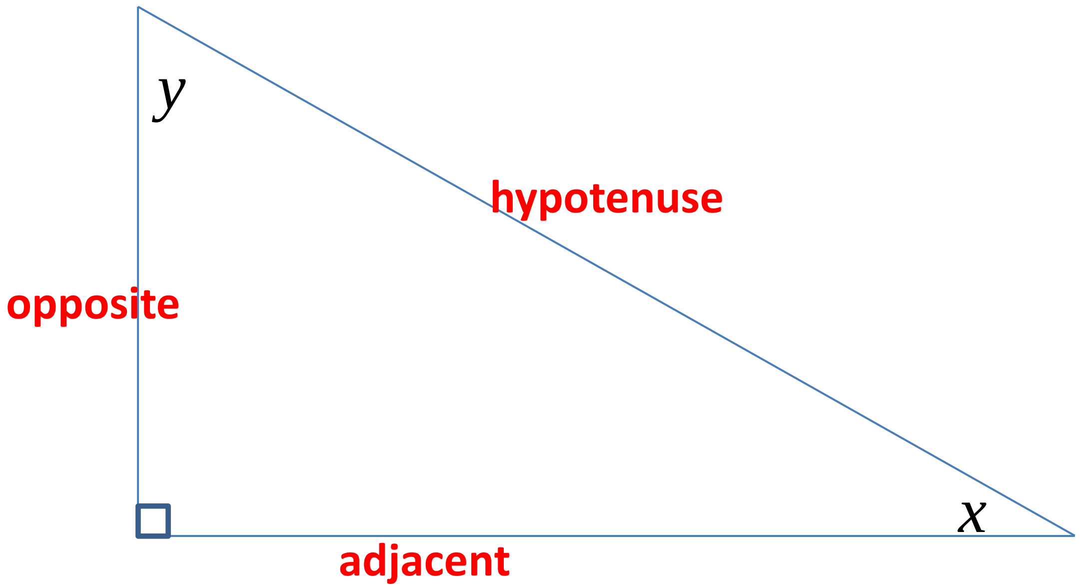




$$\text{So } \sin(30^\circ) = \frac{1}{2}, \sin(60^\circ) = \frac{\sqrt{3}}{2}, \tan(45^\circ) = 1, \text{ and so on.}$$

It is well worth remembering these triangles and their values. They occur a lot.

Relationship between sine and cosine



$$\sin(x) = \cos(90 - x)$$

$$\cos(x) = \sin(90 - x)$$

Relationship between sine, cosine and tan.

$$\frac{\sin x}{\cos x} = \frac{\dfrac{opposite}{hypotenuse}}{\dfrac{adjacent}{hypotenuse}} = \frac{opposite}{adjacent} = \tan x$$

$$\frac{\sin x}{\cos x} = \tan x$$

Relationship between sin and cos

$$\sin^2 x + \cos^2 x = \frac{opp^2}{hyp^2} + \frac{adj^2}{hyp^2} = \frac{opp^2 + adj^2}{hyp^2} = \frac{hyp^2}{hyp^2} = 1$$

This is a very important identity, so I'll repeat it

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

Note that, unlike $\sin^{-1}x$, $\sin^2 x$ <u>is</u> the same as $(\sin x)^2$, and by convention we always write $\sin^2 x$. This is an unfortunate confusion of notation when you are first learning, but you will soon become used to it.

Extending basic trigonometry

We need to be able to:

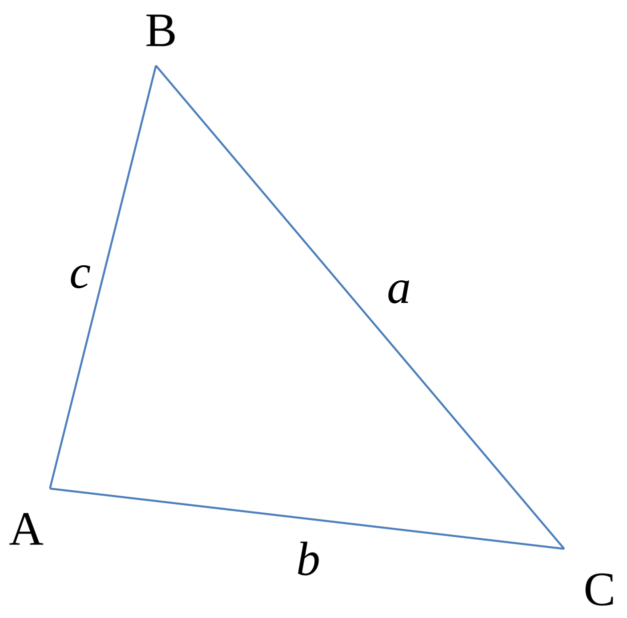
1. Solve ***<u>any</u>*** type of triangle, not just right-angled ones.

2. Work with angles of more than 90 degrees.

Sine Rule

In ***any*** triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



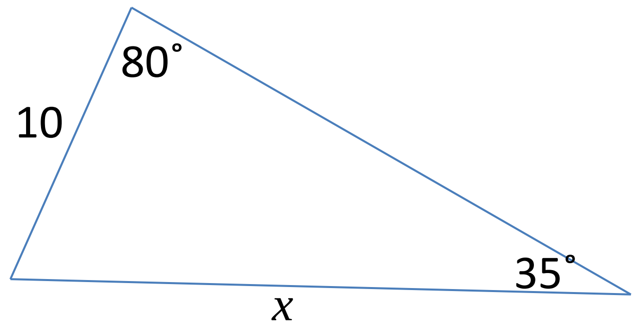
*By convention, if we label the sides of a triangle a, b and c, and the angles A, B and C, then a is the side **opposite** angle A, b is the side **opposite** angle B, and so on.*

The statements of the Sine and Cosine Rules assume that this convention is being used.

Sine Rule

In ***<u>any</u>*** triangle

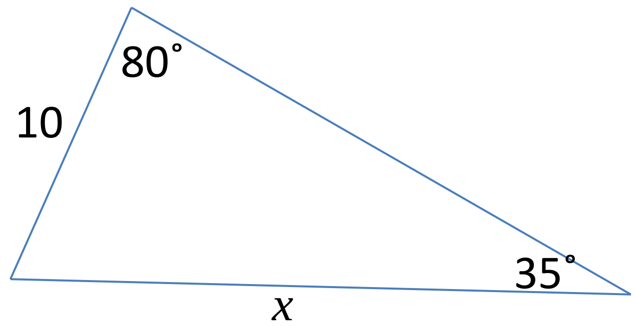
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Sine Rule

In ***<u>any</u>*** triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

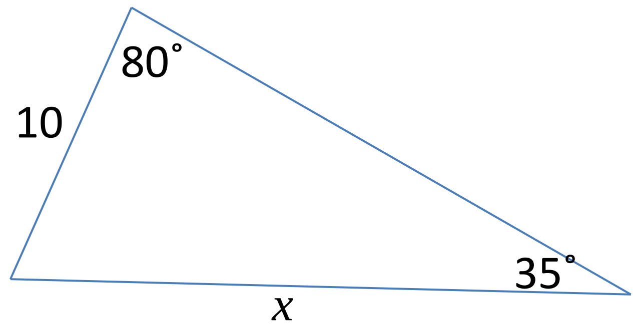


$$\frac{x}{\sin 80°} = \frac{10}{\sin 35°}$$

Sine Rule

In ***<u>any</u>*** triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



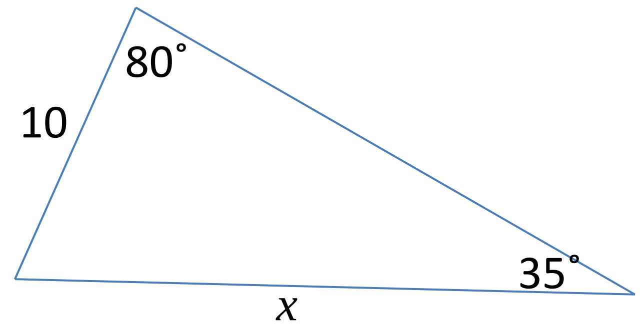
$$\frac{x}{\sin 80°} = \frac{10}{\sin 35°}$$

$$\therefore x = \frac{10 \sin 80°}{\sin 35°}$$

Sine Rule

In ***<u>any</u>*** triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$$\frac{x}{\sin 80°} = \frac{10}{\sin 35°}$$

$$\therefore x = \frac{10 \sin 80°}{\sin 35°}$$

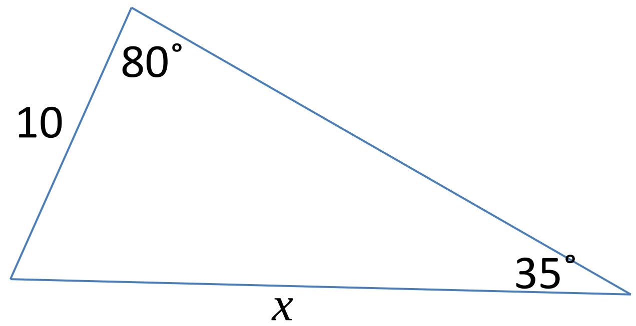
$$= 17.169...$$

There is no need to convert sin 80° and sin 35° to decimals. Just enter the whole expression into your calculator as it stands. It's quicker, it's more accurate, and it reduces the chances of making an error.

Sine Rule

In ***<u>any</u>*** triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$$\frac{x}{\sin 80°} = \frac{10}{\sin 35°}$$

$$\therefore x = \frac{10 \sin 80°}{\sin 35°}$$

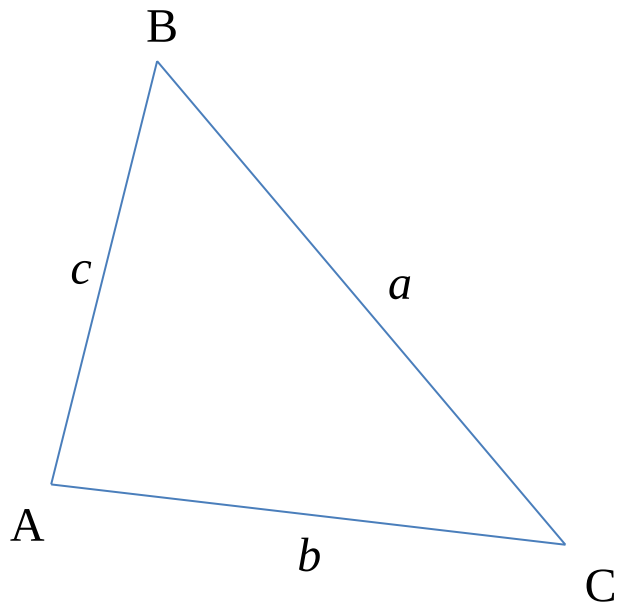
$$= 17.169...$$

Hence x = 17.2 (to 3 sf)

Cosine Rule

In ***<u>any</u>*** triangle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



Cosine Rule

In ***<u>any</u>*** triangle

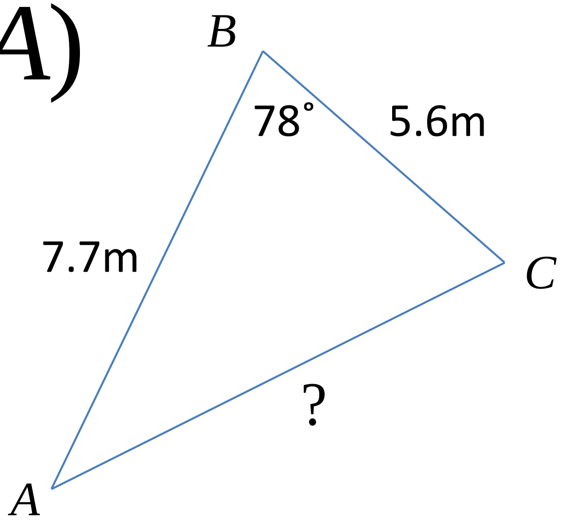
$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

For example, if you are interested in the angle at B, the formula becomes

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

Cosine Rule to find side length

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



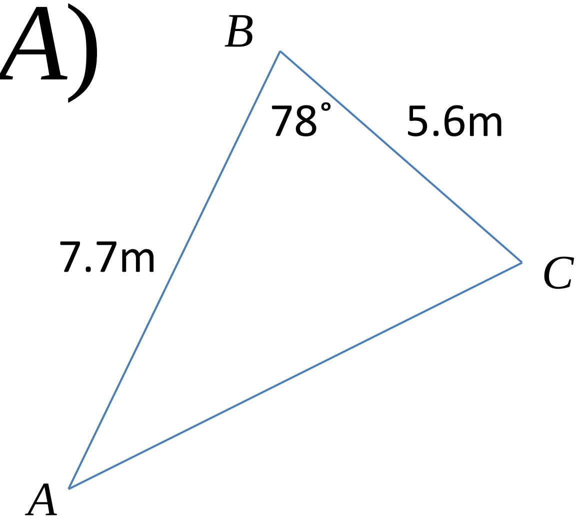
We are looking for the side opposite angle B, so we use the formula in the form

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

The side on the left-hand side of the equation ***must*** *be the side* ***opposite*** *the angle on the right-hand side. Getting this wrong is a very common mistake.*

Cosine Rule to find side length

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



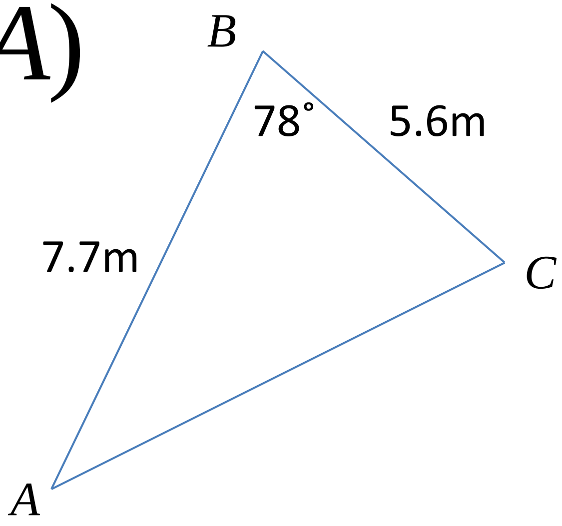
We are looking for side B, so we use the formula in the form

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

$$= 5.6^2 + 7.7^2 - 2 \times 5.6 \times 7.7 \times \cos 78°$$

Cosine Rule to find side length

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



We are looking for side B, so we use the formula in the form

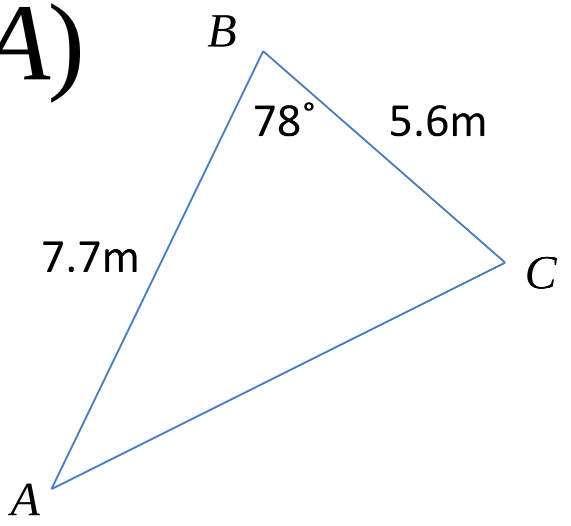
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

$$= 5.6^2 + 7.7^2 - 2 \times 5.6 \times 7.7 \times \cos 78°$$

$$= 72.719...$$

Cosine Rule to find side length

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



We are looking for side B, so we use the formula in the form

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

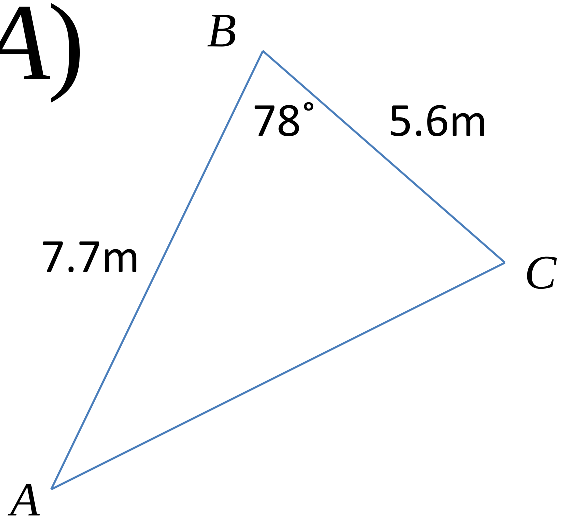
$$= 5.6^2 + 7.7^2 - 2 \times 5.6 \times 7.7 \times \cos 78°$$

$$= 72.719...$$

$$\therefore b = \sqrt{72.719...} = 8.52...$$

Cosine Rule to find side length

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



We are looking for side B, so we use the formula in the form

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

$$= 5.6^2 + 7.7^2 - 2 \times 5.6 \times 7.7 \times \cos 78°$$

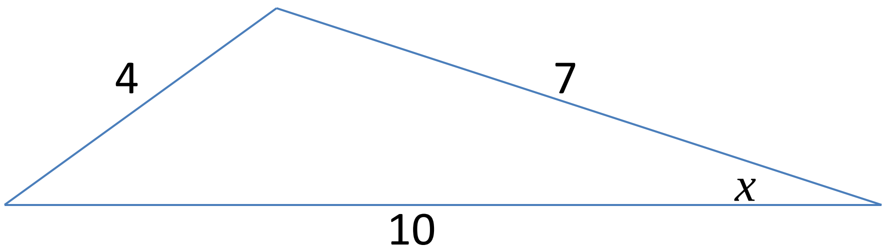
$$= 72.719...$$

$$\therefore b = \sqrt{72.719...} = 8.52...$$

Hence the length of AC is 8.5 m (to 2 sf)

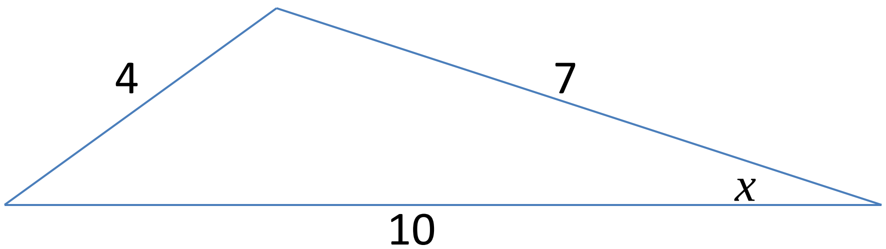
Cosine Rule to find angle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



Cosine Rule to find angle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

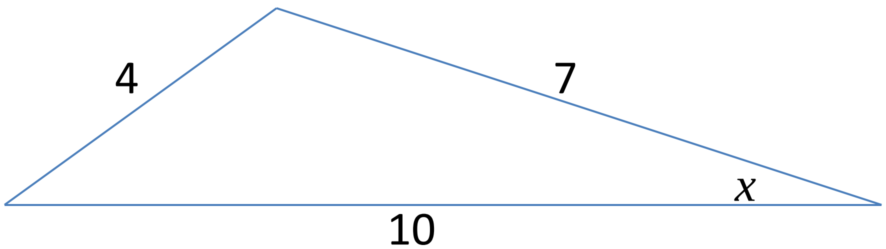


By the Cosine Rule

$$4^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos x$$

Cosine Rule to find angle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



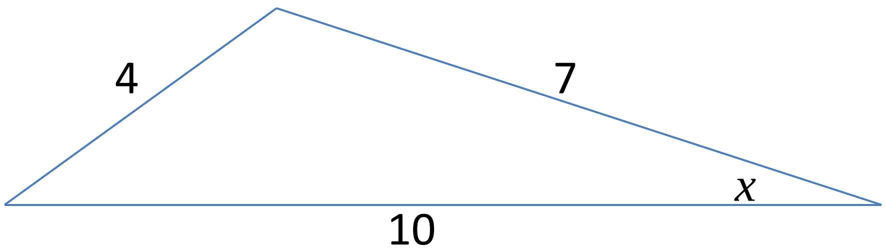
By the Cosine Rule

$$4^2 = 7^2 + 10^2 - 2\times 7\times 10\times \cos x$$

$$\therefore 16 = 149 - 140\cos x$$

Cosine Rule to find angle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



By the Cosine Rule

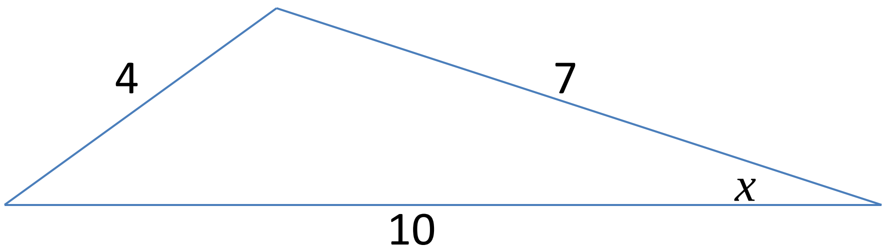
$$4^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos x$$

$$\therefore 16 = 149 - 140\cos x$$

$$\therefore 140\cos x = 133$$

Cosine Rule to find angle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



By the Cosine Rule

$$4^2 = 7^2 + 10^2 - 2\times 7\times 10\times \cos x$$

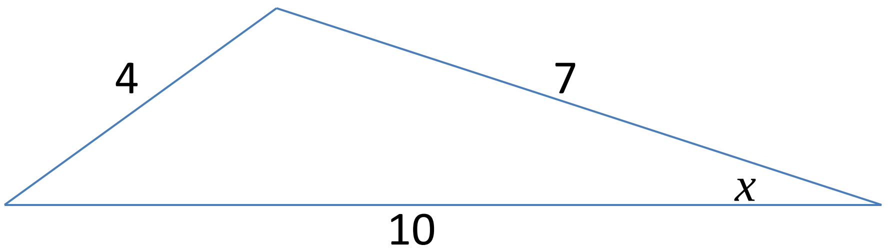
$$\therefore 16 = 149 - 140\cos x$$

$$\therefore 140\cos x = 133$$

$$\therefore \cos x = \frac{133}{140}$$

Cosine Rule to find angle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



By the Cosine Rule

$$4^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos x$$

$$\therefore 16 = 149 - 140\cos x$$

$$\therefore 140\cos x = 133$$

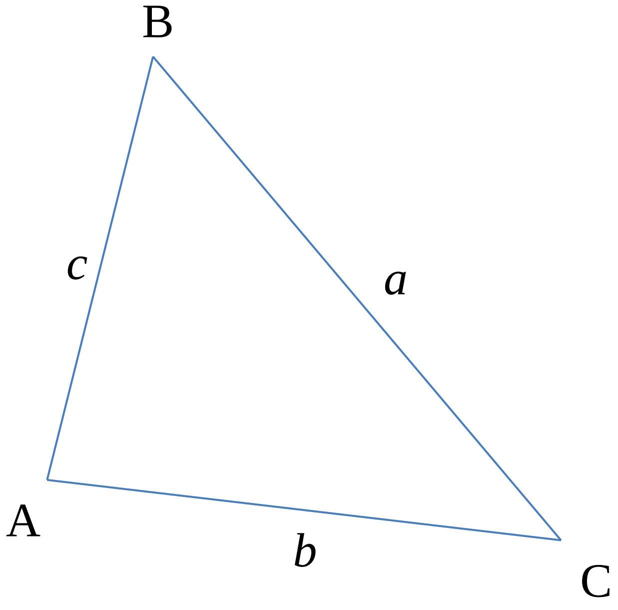
$$\therefore \cos x = \frac{133}{140}$$

$$\therefore x = \cos^{-1}\left(\frac{133}{140}\right) = 18.194...^\circ = 18^\circ \text{ (to the nearest degree)}$$

Cosine Rule

In ***<u>any</u>*** triangle

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$



What happens if angle A is equal to 90 degrees. What do you get?

Sine and Cosine Rules

Don't confuse the sine and cosine ***ratios*** *with the Sine and Cosine* ***Rules.***

- The sine, cosine and tangent ***ratios*** only hold for right-angled triangles

- The Sine and Cosine ***Rules*** work in non-right-angled triangles. They could be used in right-angled triangles, but they are the not the simplest tool for the job. It's a bit like using a calculator to add 4 and 5.

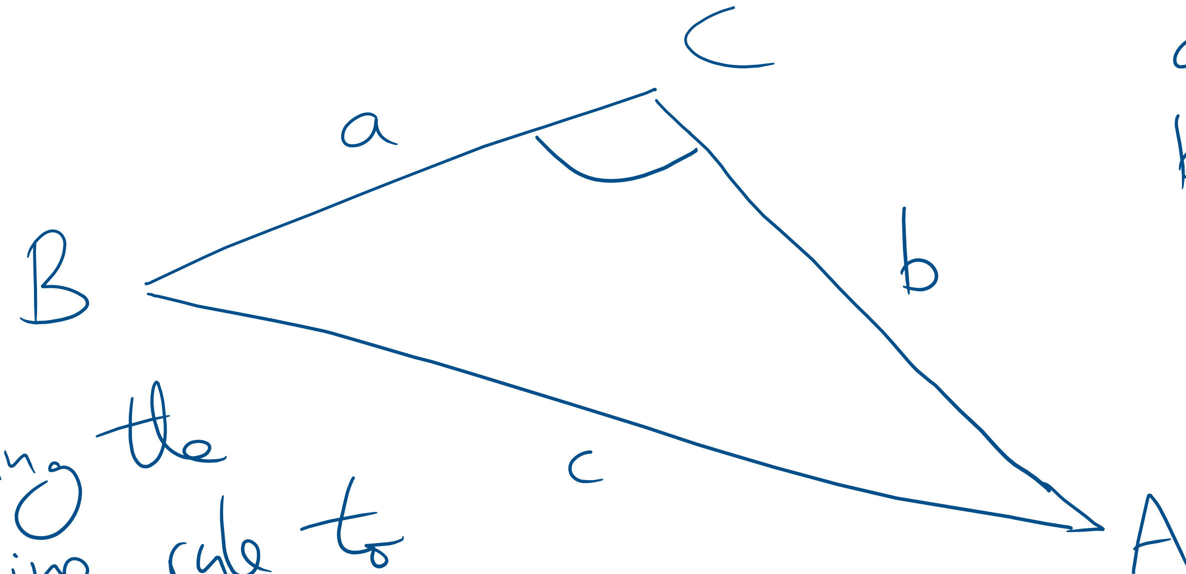
One to try ...

In this question give your answers to four decimal places.

A triangle ABC has $a = 4.6$, $b = 5.4$ and $C = 108°$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4).

(a) Use the cosine rule to find c.

(b) Use the sine rule to find the other two angles.



$a = 4.6$
$b = 5.4$
$c = 108°$

Using the cosine rule to find 'c'

Using the cosine rule in the form

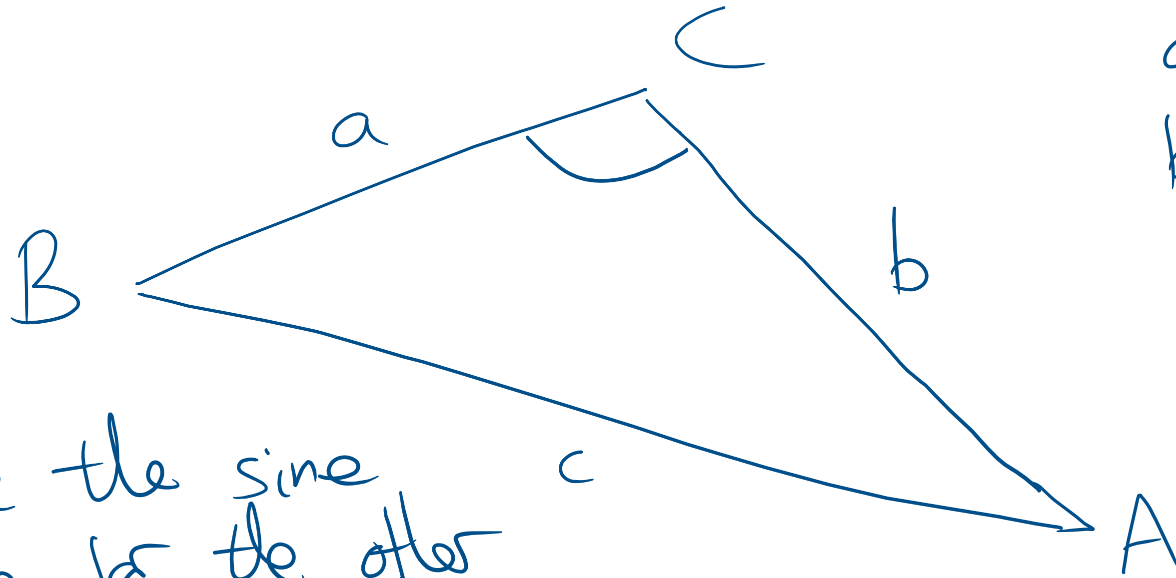
$$c^2 = a^2 + b^2 - 2ab\cos C$$

gives

$$c^2 = 4.6^2 + 5.4^2 - 2 \times 4.6 \times 5.4\cos 108°.$$

This gives $c = \sqrt{65.67196\ldots}$.

So $c = 8.10382\ldots = 8.1038$ (to 4 d.p.).



$a = 4.6$
$b = 5.4$
$c = 108°$

Use the sine rule for the other two angles

Since $C > 90°$, both A and B must be less than $90°$. The sine rule gives

$$\frac{\sin A}{4.6} = \frac{\sin B}{5.4} = \frac{\sin 108°}{8.1038\ldots}.$$

So

$$\sin A = \frac{4.6 \times \sin 108°}{8.1038\ldots}$$

which gives $A = 32.6735°$ (to 4 d.p.), and

$$\sin B = \frac{5.4 \times \sin 108°}{8.1038\ldots}$$

which gives $B = 39.3265°$ (to 4 d.p.).

(A quick check shows that the angles add up to $180°$.)

One to try ...

In this question give your answers to four decimal places.

A triangle ABC has $a = 4.6$, $b = 5.4$ and $C = 108°$ (where the sides and angles are labelled in accordance with Figure 44 in Subsection 3.1 of Unit 4).

(a) Use the cosine rule to find c.

(b) Use the sine rule to find the other two angles.

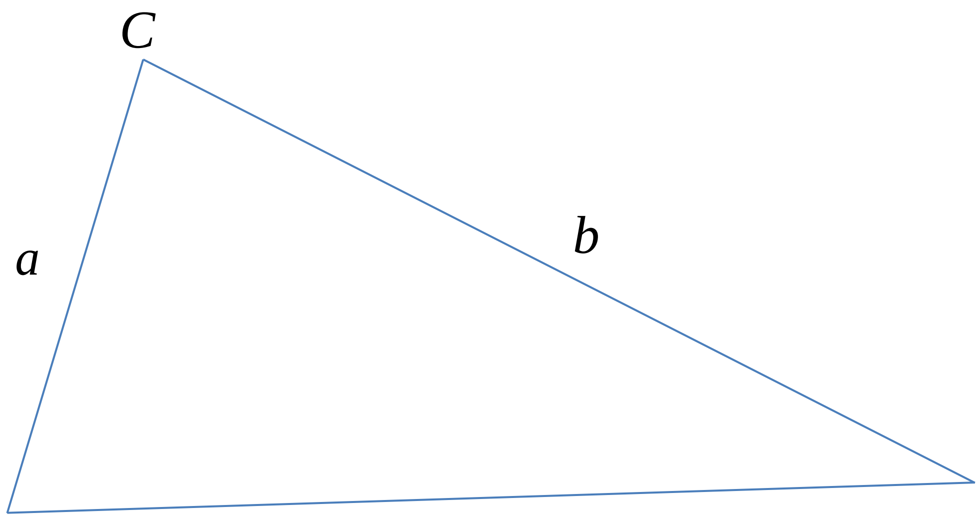
$c = 8.1038 \,(to\ 4dp)$

$A = 32.6735° \,(to\ 4dp)$

$B = 39.3265° \,(to\ 4dp)$

Check that the angles add up to $180°$

The area of a triangle



$$\text{Area} = \frac{1}{2} ab \sin(C)$$

If angle C is 90 degrees, this formula reduces to the familiar

A = ½ base x height

since sin(90) is equal to 1.

Other angles

What is the meaning of

$\sin(150^\circ)$

$\cos(1000^\circ)$

$\tan(-45^\circ)$

Other angles

Also, we want the Sine and Cosine Rules to work in obtuse-angled triangles.

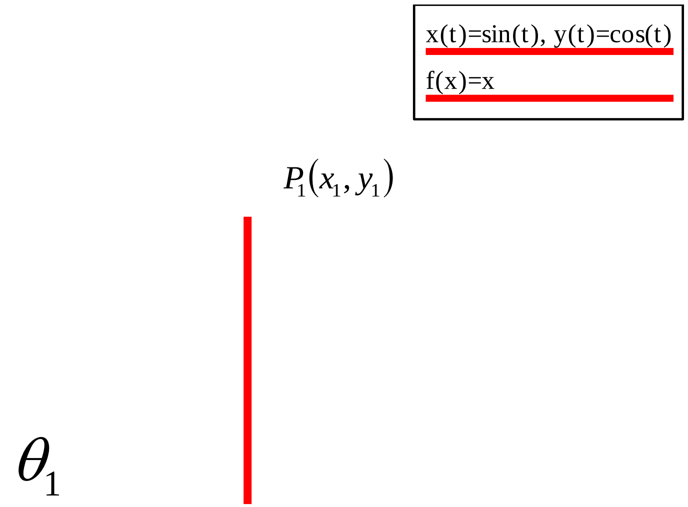
We need to change the definition of sine, cosine and tangent.

Terminology

A general angle is a measure of rotation round a point, measured in degrees. Positive angles give anticlockwise rotations, and negative angles give clockwise rotations.

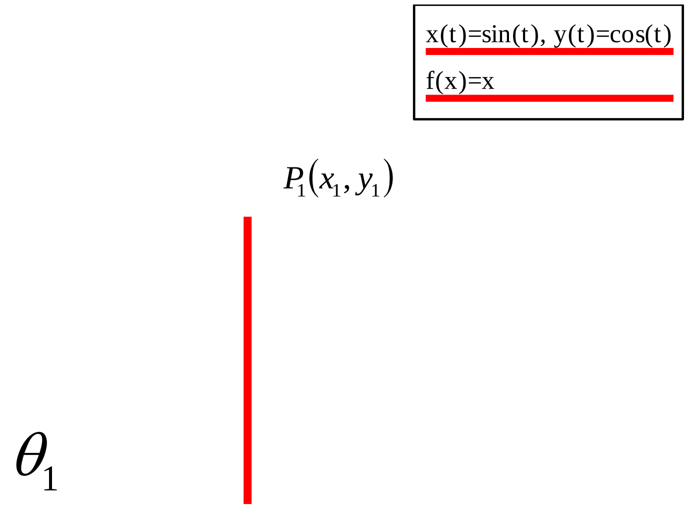
If we are going to change the definition of sine, cosine and tangent, we must make sure that the new definition gives the same results as the old one for right-angled triangles.

The ASTC diagram



Right round the circle would be 360°, and since we can go round the circle as many times as we like, an angle of 1000° does make sense, as does an angle of -45°.

Points on the unit circle: 1^{st} quadrant



By our original definition, $\sin(\theta_1) = y_1/r = y_1$, since $r = 1$, so $\sin(\theta_1)$ is the y-coordinate of the point P_1, and this is our new definition of $\sin(\theta_1)$.

Points on the unit circle: 1st quadrant

Similarly, $\cos(\theta_1)$ is defined as the x-coordinate of P_1.

x(t)=sin(t), y(t)=cos(t)

f(x)=x

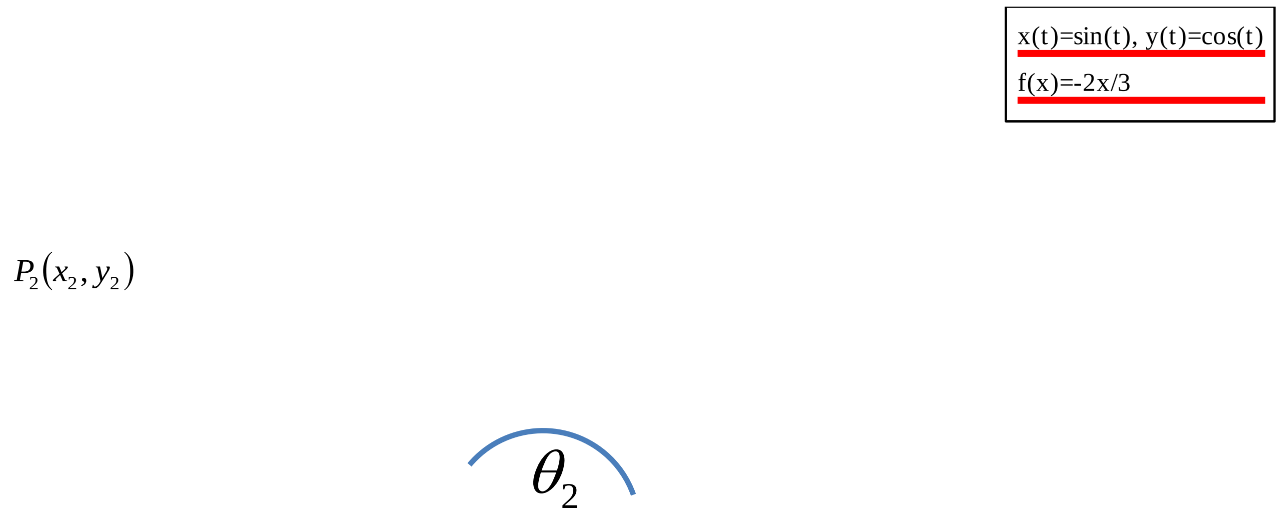
$P_1(x_1, y_1)$

θ_1

And $\tan(\theta_1)$ is defined as the y-coordinate of P_1 divided by the x-coordinate.

These definitions are identical to the original ones in the first quadrant, but they allow us to define sin, cos and tan for angles greater than 90°.

Points on the unit circle: 2^{nd} quadrant



Note that in the second quadrant, x is negative; whereas y is positive, so in this quadrant, sine is positive, but cosine and tangent are negative.

Implications for the Sine Rule

$\sin(x) = \sin(180 - x)$

so, for example

$\sin(140) = \sin(40)$

With the Sine Rule, there are two possible answers for angles, and you need to make sure you have the right angle.

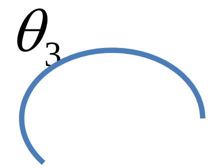
Sine Rule: getting the right angle

- *Make sure you know whether the triangle has two acute angles and an obtuse angle, or three acute angles. If necessary, draw a rough sketch of the triangle.*
- *Remember the rule that the biggest angle is opposite the longest side, and the smallest angle is opposite the shortest side.*
- *This problem doesn't arise with the Cosine Rule, because cosines are negative in the second quadrant.*

Points on the unit circle: 3rd quadrant

x(t)=sin(t), y(t)=cos(t)

f(x)=2x/3



$P_3(x_3, y_3)$

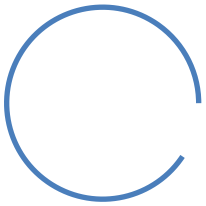
Tangent is positive.

Points on the unit circle: 4^{th} quadrant

x(t)=sin(t), y(t)=cos(t)

f(x)=-2x/3

θ_3



$P_4(x_4, y_4)$

Cosine is positive.

The ASTC diagram

- First quadrant: All positive
- Second quadrant: Sine positive
- Third quadrant: Tangent positive
- Fourth quadrant: Cosine positive

Note that, for example -45° and 315° are both in the fourth quadrant. They have the same sine, cosine and tangent.

Degrees and radians

Radians are units for measuring angles, which can be used as an alternative to degrees.

We have $1\,\text{radian} = \left(\dfrac{180}{\pi}\right)\text{degrees}$

or

1 radian = 57.296 degrees (to 5 sf)

Never use the decimal equivalent in calculations. Always use 180/π; it's more accurate. You will find conversions for principal angles on page 6 of MST124, Unit 4.

Degrees and radians

Calculator Health Warning!

If you work in both degrees and radians, always check your calculator setting every time you start work.

Round and round the circle (degrees)

f(x)=sin(x)

$$y = \sin(x)$$

Things to note:

1. We can have angles of any size. Twice round the circle is 720°.
2. Sin lies between -1 and 1.
3. Sin 90° = sin 450° = sin (-270°) and so on.
4. The graph is symmetrical over a given range so, for example, sin(30°) is equal to sin(150°) (which is 180 - 30)

Round and round the circle (radians)

f(x)=sin(x)

Sin(π/2) = sin(5π/2) = sin(-3π/2) and so on.

Round and round the circle (degrees)

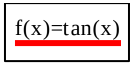
f(x)=cos(x)

$$y = \cos(x)$$

Things to note:

1. The cosine graph is the sine graph translated to the left by 90° or π/2 radians.
2. The symmetries are different, so whereas $\sin(x°) = \sin(180° - x°)$, we have $\cos(x°) = \cos(360° - x°)$.

Round and round the circle (degrees)



$$y = \tan(x)$$

There is no value for tan(90°), tan(270°) and so on. For these values, cos(x) is zero, and we cannot divide by zero.

Trigonometrical equations

Solve the equation sin(x)=0.5

If we put $x = \sin^{-1}(0.5)$ into a calculator, we get the answer $x = 30°$ $(\text{or } \frac{\pi}{6} \text{ radians})$

But this isn't the whole story

Trigonometrical equations

Solve the equation sin(x)=0.5

f(x)=sin(x)

f(x)=0.5

Solutions

$\frac{\pi}{6}$

$\frac{5\pi}{6}$

$\frac{13\pi}{6}$

Trigonometrical equations

- An equation like $\sin(x) = 0.5$ has infinitely many solutions.
- The principal solution is $x = 30°$, or $\pi/6$ radians.
- We restrict the number of solutions by giving a range, ie $-\pi < x < \pi$, or $0° \le x \le 360°$

Trigonometrical equations

From the graph, and using its symmetry, if $\sin(x) = 0.5$, we have

$$x = \frac{\pi}{6} \text{ or } \left(2\pi - \frac{\pi}{6}\right)$$

$$so$$

$$x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

We can also use the identity

$$\sin x = \sin(\pi - x)$$

Important identities for solving equations

Two values of sin, cos, tan between 0 and 2π.

$$\sin x = \sin(\pi - x), \quad \cos x = \cos(2\pi - x), \quad \tan x = \tan(\pi + x),$$

If you want values outside the range 0 to 2π radians (or 0° to 360°), you can also use these identities:

$$\sin x = \sin(2\pi n + x), \quad \cos x = \cos(2\pi n + x), \quad \tan x = \tan(2\pi n + x)$$

All these last three identities are saying is that if you add or subtract any multiple Of 2π radians or 360° to an angle, the sine, cosine and tangent will be the same.

*I strongly recommend using **both** methods: the graph and the identities, and checking between them, until you are completely comfortable with the process. This will help you to understand what is going on, and improve your grasp of the techniques. Using the graph will help you to make sure that you don't miss any solutions.*

Trigonometrical equations

Solve

$$\cos x = \frac{\sqrt{2}}{2} \text{ for} -\pi \le x \le \pi$$

Trigonometrical equations

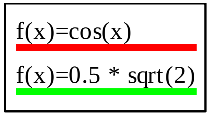
Solve

$$\cos x = \frac{\sqrt{2}}{2} \text{ for } -\pi \le x \le \pi$$

The principal value is

$$x = \frac{\pi}{4}$$

Trigonometrical equations



Trigonometrical equations

- We can see that there are two solutions between $-\pi$ and π.
- Cos$(2\pi - x)$ gives $7\pi/4$, which is outside the range, but you can always add or subtract 2π radians (or 360°), which is once round the circle, so we subtract 2π
- From this, we get $-\pi/4$, which is in the range.
- So the solution is $x = -\pi/4$ or $\pi/4$, which you can check with your calculator.

One to solve ...

Find all the solutions between -360° and 360° of the following equations. Give your answers both in degrees and radians.

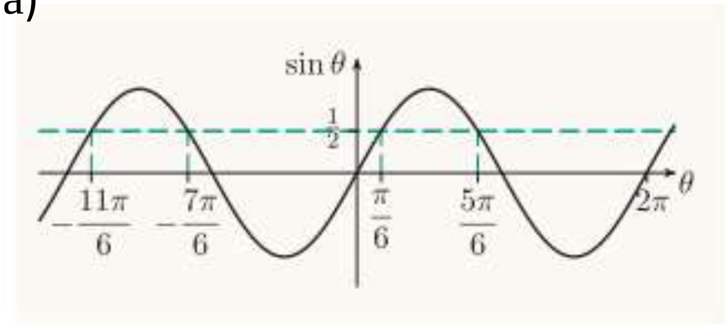
(a) $\sin\theta = \dfrac{1}{2}$ (b) $\cos\theta = -\dfrac{\sqrt{3}}{2}$

One to solve …

Find all the solutions between $-360°$ and $360°$ of the following equations. Give your answers both in degrees and radians.

(a) $\sin\theta = \dfrac{1}{2}$ (b) $\cos\theta = -\dfrac{\sqrt{3}}{2}$

(a)



One solution of the equation $\sin\theta = \dfrac{1}{2}$ is

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

By the symmetry of the graph, for θ between -2π and 2π, the solutions are

$$\theta = -\frac{11\pi}{6},\ -\frac{7\pi}{6},\ \frac{\pi}{6} \text{ and } \frac{5\pi}{6}.$$

That is, the solutions between $-360°$ and $360°$ are

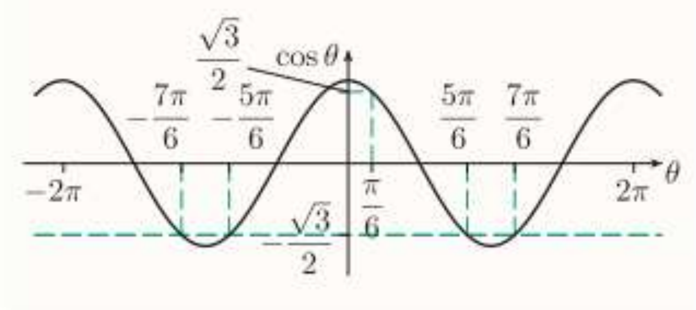
$$\theta = -330°,\ -210°,\ 30° \text{ and } 150°.$$

One to solve ...

Find all the solutions between $-360°$ and $360°$ of the following equations. Give your answers both in degrees and radians.

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\theta = -\frac{\sqrt{3}}{2}$

(b)



A solution of the equation

$$\cos\theta = \frac{\sqrt{3}}{2}$$

is

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

By the symmetry of the graph, for θ between -2π and 2π, the solutions of the equation

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

are

$$\theta = -\frac{7\pi}{6},\ -\frac{5\pi}{6},\ \frac{5\pi}{6} \text{ and } \frac{7\pi}{6}.$$

That is, the solutions between $-360°$ and $360°$ are

$$\theta = -210°,\ -150°,\ 150° \text{ and } 210°.$$

Preparing for MST124: Session 6 is on Friday 24th September

We'll start at 7.00pm and aim to finish at 9.00pm

This session will cover the topics in
Exponentials and Logarithms
that you will need to know

Have paper, pen and your calculator to hand.