Unit 11

Taylor polynomials

Introduction

You may have wondered how a calculator or mathematical software package finds an approximate numerical value for $\ln 3$, $e^{1/2}$ or $\sin(0.2)$, for example. There are various ways in which this can be done, but one common method involves approximating functions such as the natural logarithm function, the exponential function or the sine function by *polynomial functions*.

Recall from Unit 3 that a **polynomial function** has the form

$$\begin{aligned} f(x) = {} & \text{a sum of terms, each of the form } cx^k, \text{ where } k \text{ is a} \\ & \text{non-negative integer and } c \text{ is a constant.} \end{aligned}$$

In other words, it has the form

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n,$$

where n is a non-negative integer and $c_0, c_1, c_2, \ldots, c_n$ are constants (possibly 0). The phrase 'polynomial function' is often abbreviated to **polynomial**, and we'll use this abbreviation frequently in this unit. If $c_n \neq 0$, then the polynomial has **degree** n. For example,

$$f(x) = 6 - 3x + 2x^2 + x^3$$

is a polynomial of degree 3.

To find the value of a polynomial function at a particular input value, the only operations that you have to use are addition, subtraction and multiplication. For example, the value of the polynomial function f above when $x = 4$ is

$$\begin{aligned} f(4) &= 6 - 3 \times 4 + 2 \times 4^2 + 4^3 \\ &= 6 - 12 + 32 + 64 = 90. \end{aligned}$$

By approximating the natural logarithm function by a polynomial function, a computer can evaluate $\ln 3$, for example, to the accuracy of the computer, using just the operations of addition, subtraction and multiplication. Similarly, by approximating the exponential function and the sine function by polynomial functions, it can evaluate $e^{1/2}$ and $\sin(0.2)$.

In this unit you'll study a particular way of approximating functions by polynomials, called *Taylor polynomials*. By using suitable Taylor polynomials, you can approximate many functions to any required level of accuracy. In fact, calculators and software packages don't use Taylor polynomials to approximate functions, since more efficient (though also more complicated) polynomial methods exist, but by studying Taylor polynomials you'll learn about the basic ideas of polynomial approximation.

In most cases a function can't be approximated by a polynomial function over the whole of its domain. What we'll be interested in throughout the unit is the approximation of a function by a polynomial function close to a particular point (value) in its domain.

Another reason why polynomial approximations are important is that it is straightforward to multiply polynomials together, and to differentiate and integrate them. Also, polynomial approximations allow complex problems to be described by simple mathematical models, making these problems easier to understand and to solve.

Taylor polynomials are also of theoretical importance. They lead naturally to a way of representing functions by infinite series, at least for some points in their domains. Such representations are called *Taylor series*.

In Section 1 you'll study the approximation of functions by linear and quadratic polynomial functions. This is extended to approximation by Taylor polynomials of higher degree in Section 2, and then to Taylor series in Section 3. Finally, in Section 4, you'll see various methods for using known Taylor series to derive Taylor series for further functions.

Brook Taylor (1685–1731)

James Gregory (1638–1675)

Taylor series and Taylor polynomials are named after the English mathematician Brook Taylor, who was educated at home and then at St John's College, Cambridge. In 1715 he published *Methodus Incrementorum Directa et Inversa*, which includes the work on which this unit is based, as well as the technique for integration by parts, which you studied in Unit 8. The importance of Taylor polynomials remained largely unrecognised until much later in the eighteenth century. Taylor was elected as a Fellow of the Royal Society in 1712, and was appointed to the committee for adjudicating the claims of Newton and Leibniz to have invented the calculus. He also wrote works on perspective and was a talented musician and artist.

Taylor was not, in fact, the first person to discover Taylor series. The Scottish mathematician James Gregory discovered them more than forty years before Taylor, and several other mathematicians, including Newton and Leibniz, also independently discovered versions of them before Taylor published his work. However, Taylor was the first to appreciate their fundamental significance and applicability.

1 Taylor polynomials of small degree

In this section you'll look at how you can approximate many functions by polynomial functions of degrees 0, 1 and 2.

Here, and throughout the unit, we'll usually use f to denote a function that is to be approximated, and p to denote an approximating polynomial function. We'll usually use a to denote a point in the domain of f close to which we want to approximate f. Note that we'll usually refer to numbers in the domain of a function f as *points*. You met this use of the word 'point' in Unit 6.

1.1 Constant Taylor polynomials

Let's start by considering how you could approximate a function f, close to a particular point a in its domain, by the simplest type of polynomial function, namely, a *constant* function. Remember from Unit 3 that a **constant function** is a function of the form $p(x) = c$, where c is a constant. Its graph is a horizontal line. Approximating a function by a constant function is rarely useful, but it illustrates the ideas, and it's the first step in obtaining better approximating polynomials, as you'll see.

For example, suppose that you want to approximate the function $f(x) = \sin x$, close to the point $\pi/6$ in its domain, by a constant function p. The best constant function to choose is the one whose graph is the horizontal line through the point with x-coordinate $\pi/6$ on the graph of f, as illustrated in Figure 1. Since $f(\pi/6) = \sin(\pi/6) = \frac{1}{2}$, this constant function is the function $p(x) = \frac{1}{2}$.

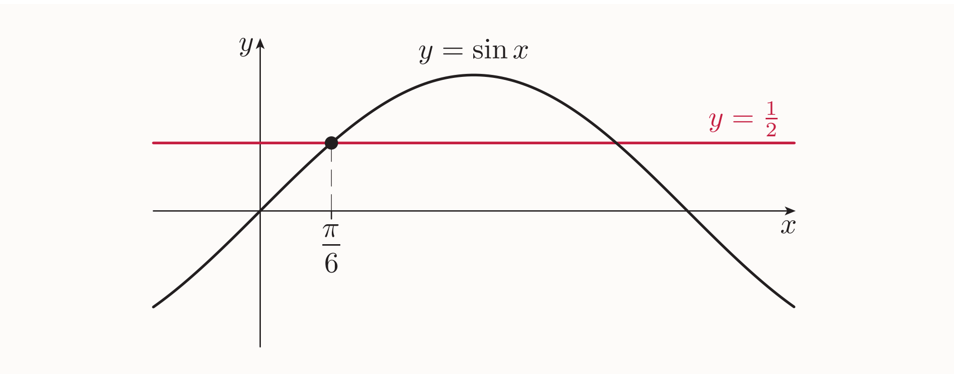


Figure 1 The graph of the function $f(x) = \sin x$ approximated by a constant function near $x = \pi/6$

You can see that, for values of x close to $\pi/6$, the value of $p(x)$ is close to the value of $f(x)$. So the value of $p(x)$ can be used as an approximation to the value of $f(x)$. The approximation is better when x is closer to $\pi/6$ than when it is further away.

For example, the point $\pi/4$ is fairly close to $\pi/6$, and the approximating polynomial $p(x) = \frac{1}{2}$ gives the following approximation for $\sin(\pi/4)$:

$$p\left(\frac{\pi}{4}\right) = \frac{1}{2} = 0.5.$$

The true value of $\sin(\pi/4)$ is

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = 0.707 \text{ (to 3 d.p.).}$$

In general, consider any function f that is **continuous** at a point a in its domain. Informally, this means that you can draw the part of the graph of f that corresponds to values of x slightly less than a to values of x slightly greater than a without taking your pen tip off the paper. Then the constant function p that best approximates f close to a is the constant function whose graph is the horizontal line though the point $(a, f(a))$, as illustrated in Figure 2. In other words, it's the constant function $p(x) = c$ where $c = f(a)$. We say that this function p is the **constant Taylor polynomial about a for f.**

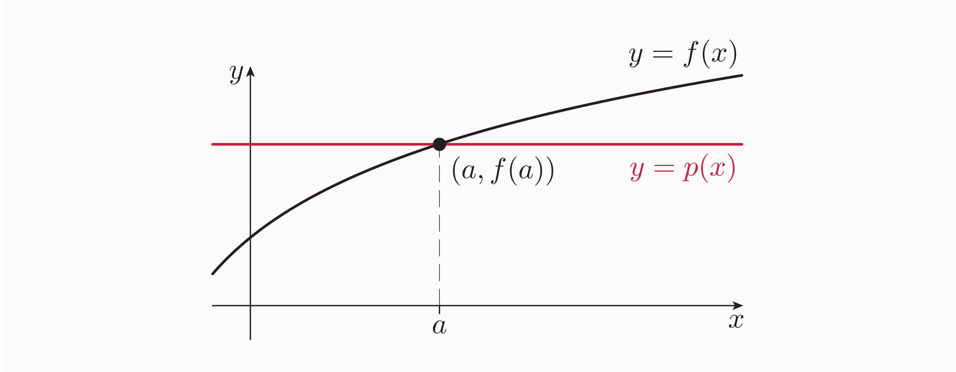


Figure 2 The graph of a function f approximated by a constant function near $x = a$

For example, you have just seen that the constant Taylor polynomial about $\pi/6$ for $f(x) = \sin x$ is $p(x) = \frac{1}{2}$.

If f is any function and p is any approximating polynomial for f, then you can obtain an indication of how good the approximation is at any particular value of x by subtracting the approximating value $p(x)$ from the actual value $f(x)$. The resulting value is known as the **remainder** at x. For example, consider again the function $f(x) = \sin x$ and the approximating polynomial $p(x) = \frac{1}{2}$. If $x = \pi/4$, then the remainder is

$$f(x) - p(x) = \sin\left(\frac{\pi}{4}\right) - \frac{1}{2} = \frac{1}{\sqrt{2}} - \frac{1}{2} = 0.207\ldots.$$

A remainder can be positive, negative or zero, depending on whether $f(x)$ is larger than, smaller than, or equal to $p(x)$. Essentially, the remainder is the size of the vertical gap between $(x, f(x))$ and $(x, p(x))$, with the appropriate sign, as illustrated in Figure 3. The smaller the magnitude of the remainder, the better the approximation.

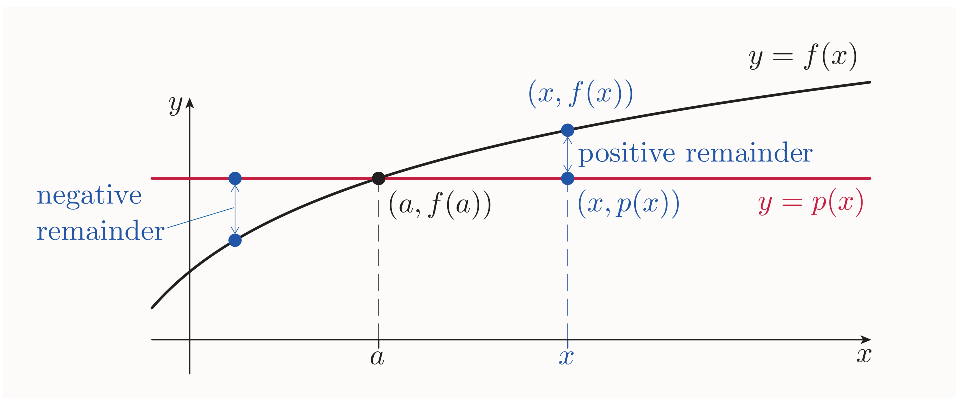


Figure 3 Remainders for a function f and an approximating constant function p

Example 1 *Finding and using a constant Taylor polynomial*

(a) Find the constant Taylor polynomial about 0 for the function $f(x) = e^x$.

(b) Use this constant Taylor polynomial to write down approximations for $e^{0.01}$ and $e^{0.1}$. In each case, use your calculator to find the value of the associated remainder to five decimal places.

Solution

(a) Since $f(0) = e^0 = 1$, the constant Taylor polynomial about 0 for $f(x) = e^x$ is $p(x) = 1$.

(b) The approximation for $e^{0.01}$ given by p is

$$p(0.01) = 1,$$

with remainder

$$f(0.01) - p(0.01) = e^{0.01} - 1 = 0.010\,05 \quad \text{(to 5 d.p.)}.$$

Similarly, the approximation for $e^{0.1}$ given by p is

$$p(0.01) = 1,$$

with remainder

$$f(0.1) - p(0.1) = e^{0.1} - 1 = 0.105\,17 \quad \text{(to 5 d.p.)}.$$

The constant Taylor polynomial found in part (a) of Example 1 is shown in Figure 4. The results of part (b) of the example illustrate the fact that the approximation provided by the constant Taylor polynomial is better (has a remainder of smaller magnitude) for values of x closer to 0 than for those further from 0. You can also see this from the graph in Figure 4, since the gap between the two graphs decreases in magnitude as the value of x moves towards 0.

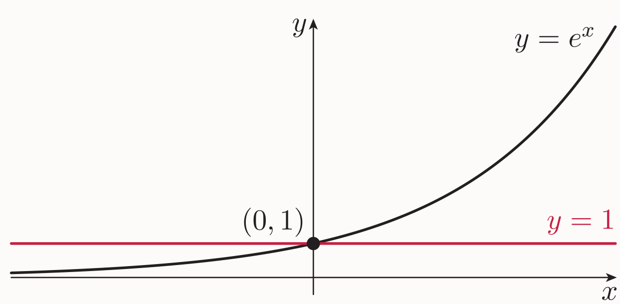


Figure 4 The graph of $f(x) = e^x$ and its constant Taylor polynomial about 0

In the next activity, and in the later activities in this unit, your calculator should be in radian mode when you're calculating the values of trigonometric functions. We'll use radians, rather than degrees, throughout the unit. This is because we'll be working with derivatives, and the standard formulas for the derivatives of trigonometric functions hold only when angles are measured in radians.

Activity 1 *Finding and using constant Taylor polynomials*

(a) Find the constant Taylor polynomial about 0 for the function $f(x) = \cos x$. Use this polynomial to write down approximations for $\cos(0.01)$ and $\cos(0.1)$. In each case, use your calculator to find the value of the associated remainder to five decimal places.

(b) Find the constant Taylor polynomial about 1 for the function $f(x) = \ln x$. Use this polynomial to write down approximations for $\ln(1.01)$ and $\ln(1.1)$. In each case, use your calculator to find the value of the associated remainder to five decimal places.

The constant Taylor polynomials from Activity 1 are shown in Figure 5.

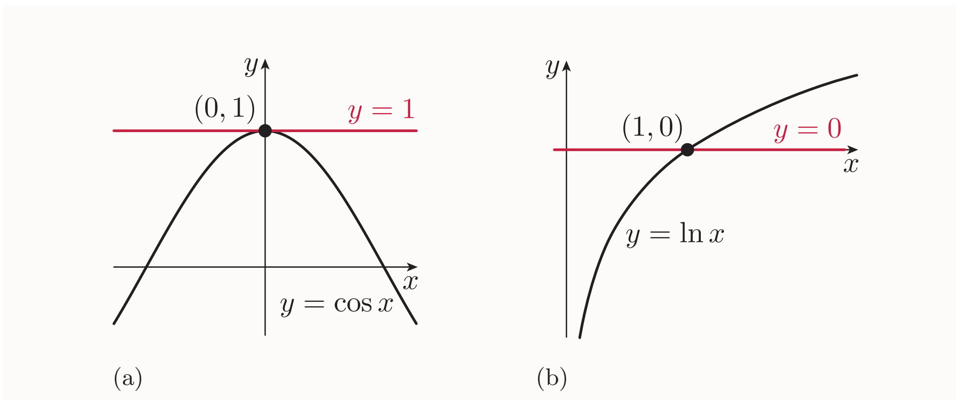


Figure 5 The graphs of two constant Taylor polynomials

As mentioned earlier, approximating a function f, close to a point a in its domain, by a constant Taylor polynomial p is usually not very useful. Unless x is extremely close to a, the accuracy of $p(x)$ as an approximation for $f(x)$ is not impressive!

This is illustrated by the fact that none of the approximating constant Taylor polynomials in Figures 1, 4 and 5 look particularly close to the original functions, as the value of x moves away to either side of the point a. However, it could be claimed that the Taylor polynomial in Figure 5(a) appears to be a better approximating polynomial than the others. In this case the function f has a local maximum at $x = a$, and hence the tangent to the graph of f at $(a, f(a))$ is horizontal, and so coincides with the graph of the constant Taylor polynomial, $p(x)$. So in this case the approximating polynomial p not only has the same *value* as the function f at $x = a$, but also its graph has the same *gradient* at $x = a$.

We'll use this idea to obtain better approximating polynomials in the next subsection.

1.2 Linear Taylor polynomials

In this subsection we'll continue to look at how we can approximate a function f, close to a particular point a in its domain, by a simple polynomial function p. In the previous subsection we chose the approximating polynomial p to be a constant function; that is, a function of the form

$$p(x) = c,$$

where c is a constant. We chose the value of c to be $f(a)$, to ensure that the function f and the approximating polynomial p have the same value as each other at $x = a$.

Here we'll choose the approximating polynomial p to be a **linear function**, that is, a function of the form

$$p(x) = mx + c,$$

where m and c are constants. As you know, the graph of such a function is a straight line. We'll choose p to have the property that not only do the function f and the approximating polynomial p have the same value at $x = a$, but also their *first derivatives* have the same value at $x = a$. The second condition ensures that the graphs of f and p have the same gradient at $x = a$.

In order for this to be possible, the function f must not only be continuous at a, but also *differentiable* at a. If you know that f is differentiable at a, then you don't have to check separately that it's continuous at a, as that follows automatically. This is because, as you saw in Unit 6, if a function has a discontinuity at a, then it isn't differentiable at a.

The function p obtained as described above is called the **linear Taylor polynomial** about a for f. Its graph is the tangent to f at a, as illustrated in Figure 6. Usually a linear Taylor polynomial gives better approximations than a constant Taylor polynomial. In some texts, linear Taylor polynomials are called *tangent approximations*.

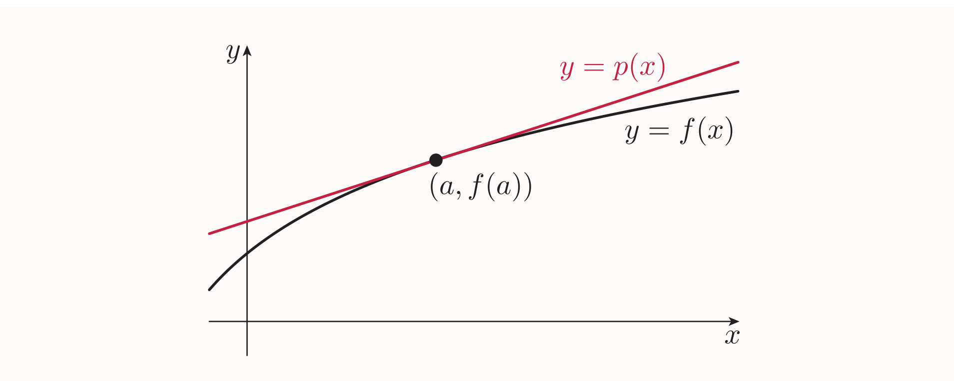


Figure 6 The graph of a function f and its linear Taylor polynomial about a

To illustrate the ideas, let's find the linear Taylor polynomial about 0 for the exponential function $f(x) = e^x$. We'll denote this approximating polynomial by p, as usual.

The graph of p is the straight line that passes through the point with x-coordinate 0 on the graph of the function $f(x) = e^x$, and has the same gradient at that point, as shown in Figure 7.

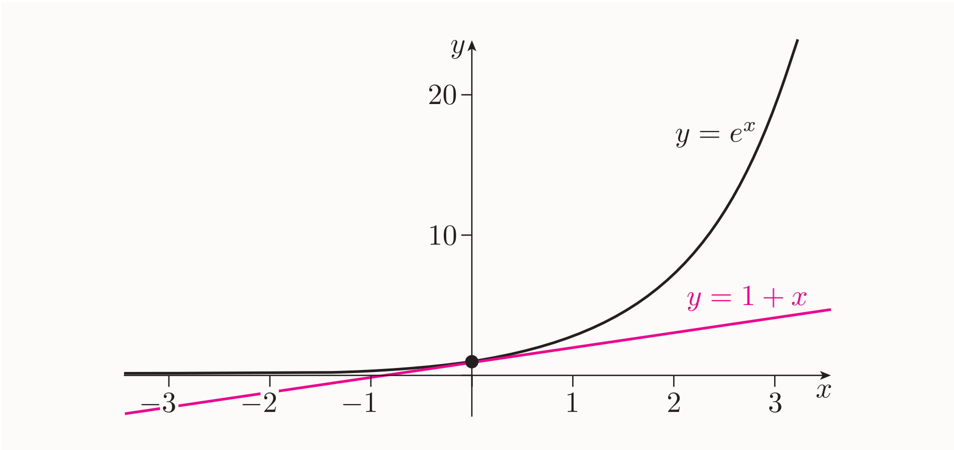


Figure 7 The linear Taylor polynomial about 0 for $f(x) = e^x$

The point with x-coordinate 0 on the graph of $f(x) = e^x$ is $(0, e^0) = (0, 1)$. Also, the derivative of the function $f(x) = e^x$ is $f'(x) = e^x$, so the gradient of the graph of f at the point $(0, 1)$ is $f'(0) = e^0 = 1$. So the graph of the approximating polynomial p is the straight line that passes through the point $(0, 1)$ and has gradient 1. From your work in Section 2 of Unit 2, you know that the straight line with gradient m that passes through the point (x_1, y_1) has equation

$$y - y_1 = m(x - x_1),$$

so the straight line required here has equation

$$(y - 1) = 1(x - 0),$$

which can be simplified to

$$y = 1 + x.$$

Thus the linear Taylor polynomial about 0 for $f(x) = e^x$ is

$$p(x) = 1 + x.$$

You can see from Figure 7 that, as you'd expect, the graph of $p(x) = 1 + x$ approximates that of $f(x) = e^x$ near $x = 0$ more closely than was the case for the graph of the constant Taylor polynomial for $f(x) = e^x$ about 0 in Figure 4.

As for constant Taylor polynomials, the approximation to $f(x)$ provided by $p(x)$ is better for values of x close to 0 than for values of x further away from 0, since the gap between the graphs of f and p increases as the value of x moves away from 0 on either side.

You can use the method above to work out the linear Taylor polynomial for any function f about any point a at which its graph has a gradient. However, a better way to proceed is to apply the method to a *general* function f and a *general* point a. This will give a general formula that you can use to work out a linear Taylor polynomial in any particular case.

To do this, let's suppose that f is a function and a is a point in its domain at which it's differentiable. The point on the graph of f with x-coordinate a is $(a, f(a))$. Also, the gradient of the graph of f at the point $(a, f(a))$ is $f'(a)$. So the graph of p is the straight line that passes through the point $(a, f(a))$ and has gradient $f'(a)$. This straight line has equation

$$y - f(a) = f'(a)(x - a),$$

which can be rearranged as

$$y = f(a) + f'(a)(x - a).$$

So we have the following general formula.

Linear Taylor polynomials

Let f be a function that is differentiable at a. The **linear Taylor polynomial about a for f** is

$$p(x) = f(a) + f'(a)(x - a).$$

When $a = 0$, this becomes

$$p(x) = f(0) + f'(0)x.$$

The particular case when $a = 0$ is stated separately in the box because this case occurs commonly and is simpler to work with.

In the next example the linear Taylor polynomial about 0 for the exponential function is worked out again, but this time directly using the formula above. The linear Taylor polynomial is also used to find an approximation for $e^{0.1}$.

Example 2 *Finding a linear Taylor polynomial about 0*

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = e^x$.

(b) Use this polynomial to find an approximation for $e^{0.1}$. Use your calculator to find the value of the associated remainder to five decimal places.

Solution

(a) Differentiate f to find f', and hence find the values of $f(0)$ and $f'(0)$.

We have $f(x) = e^x$, so

$$f'(x) = e^x.$$

Hence

$$f(0) = e^0 = 1 \quad \text{and} \quad f'(0) = e^0 = 1.$$

Apply the second formula in the box above, since in this case $a = 0$.

The linear Taylor polynomial about 0 for $f(x) = e^x$ is

$$p(x) = f(0) + f'(0)x;$$

Substitute in the values of $f(0)$ and $f'(0)$.

that is,

$$p(x) = 1 + x.$$

(b) The approximation for $e^{0.1}$ given by the linear Taylor polynomial p is

$$p(0.1) = 1 + 0.1 = 1.1.$$

The remainder for this approximation is

$$e^{0.1} - 1.1 = 0.005\,17 \quad \text{(to 5 d.p.)}.$$

The remainder found in Example 2(b) is about 20 times smaller than the remainder 0.105 17 found in Example 1(b), where $e^{0.1}$ was approximated by a constant Taylor polynomial. So the linear Taylor polynomial about 0 for $f(x) = e^x$ provides a much more accurate approximation for the value of $e^{0.1}$ than was obtained using a constant Taylor polynomial. The same applies when you approximate e^x for any other value of x close to 0.

In the next activity you're asked to find a linear Taylor polynomial for the sine function.

Activity 2 *Finding a linear Taylor polynomial about 0*

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = \sin x$.

(b) Use this polynomial to find approximations for $\sin(0.25)$ and $\sin(0.5)$, each to four decimal places. By comparing these approximations with the values obtained from your calculator, show that the magnitude of the remainder at $x = 0.5$ is much larger than that at $x = 0.25$.

Notice from Activity 2(a) that the linear Taylor polynomial about 0 for the sine function contains no constant term; it is $p(x) = x$. This happens because the graph of the sine function passes through the origin. The graphs of $f(x) = \sin x$ and $p(x) = x$ are shown in Figure 8.

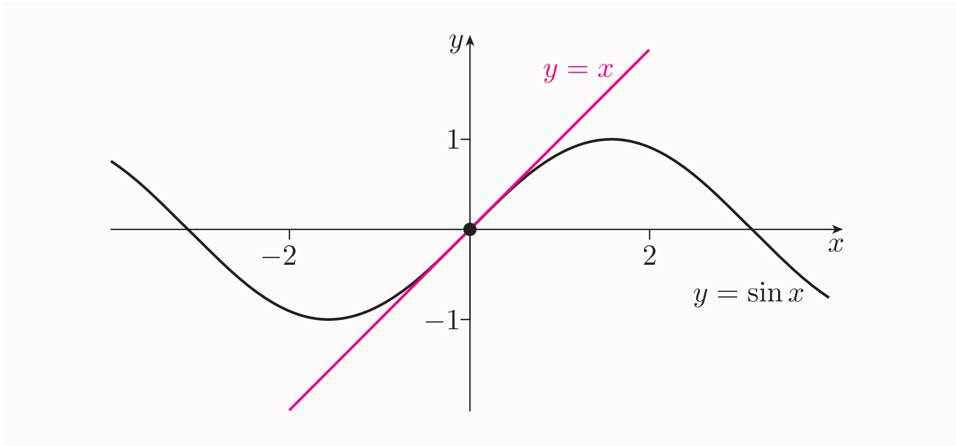


Figure 8 The linear Taylor polynomial about 0 for $f(x) = \sin x$

Activity 3 *Finding another linear Taylor polynomial about 0*

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = \cos x$.

(b) Use this polynomial to find an approximation for $\cos(0.2)$, and use your calculator to find the value of the associated remainder to four decimal places.

Notice from Activity 3(a) that the linear Taylor polynomial about 0 for the cosine function contains no term in x and is therefore a constant function; it is $p(x) = 1$. This happens because the graph of the function $f(x) = \cos x$ has gradient zero at the point where $x = 0$. The graphs of $f(x) = \cos x$ and $p(x) = 1$ are shown in Figure 9.

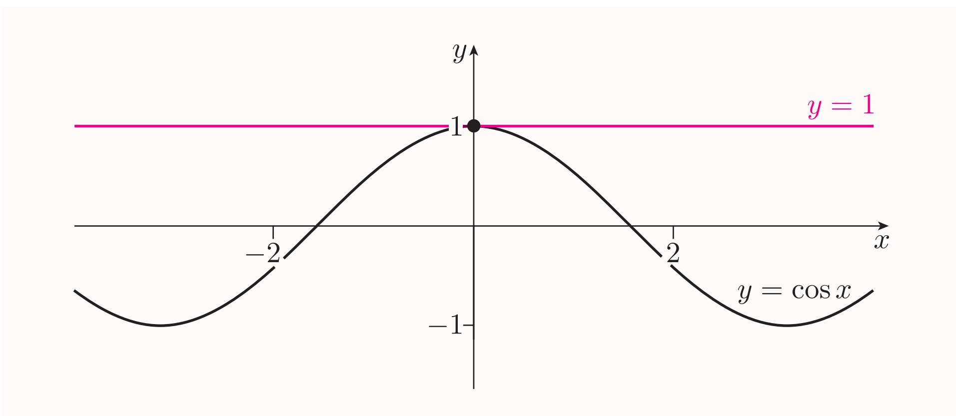


Figure 9 The linear Taylor polynomial about 0 for $f(x) = \cos x$

In fact, the linear Taylor polynomial p in this case is the same as the constant Taylor polynomial about 0 for the cosine function, which you found in Activity 1(a) and whose graph is in Figure 5(a). In Activity 1(a), you started by looking for (and finding) a constant function, whereas in Activity 3(a) you started by looking for a linear function, which turned out to be one in which the coefficient of x is zero.

In the next activity you're asked to use a linear Taylor polynomial to approximate a square root.

Activity 4 *Finding and using another linear Taylor polynomial*

(a) Show that the linear Taylor polynomial about 0 for the function

$$f(x) = (1+x)^{1/2},$$

is

$$p(x) = 1 + \tfrac{1}{2}x.$$

(b) Use the polynomial p from part (a), with $x = 0.01$, to find an approximate value for $\sqrt{1.01}$. Use your calculator to find, to six decimal places, the value of the associated remainder.

Linear Taylor polynomials about $a \neq 0$

All the linear Taylor polynomials that you've seen so far in this subsection have been about 0. The next example and activity relate to linear Taylor polynomials about another point.

Example 3 *Finding a linear Taylor polynomial about a point other than 0*

Find the linear Taylor polynomial about 1 for the function $f(x) = \ln x - 1/x$.

Solution

Use the first formula in the box on page 105. So start by differentiating f to find f', and then find the values of $f(1)$ and $f'(1)$.

We have $f(x) = \ln x - \dfrac{1}{x}$, so

$$f'(x) = \frac{1}{x} + \frac{1}{x^2}.$$

Hence

$$f(1) = \ln 1 - \frac{1}{1} = -1 \quad \text{and} \quad f'(1) = \frac{1}{1} + \frac{1}{1^2} = 2.$$

Now apply the formula. Remember that in this case $a = 1$.

Thus the linear Taylor polynomial about 1 for $f(x) = \ln x - 1/x$ is

$$p(x) = f(1) + f'(1)(x - 1);$$

Substitute in the values of $f(1)$ and $f'(1)$.

that is,

$$p(x) = -1 + 2(x - 1),$$

which can be simplified to

$$p(x) = -3 + 2x.$$

The graphs of the function $f(x) = \ln x - 1/x$ and the linear Taylor polynomial $p(x) = -3 + 2x$ that was found in Example 3 are shown in Figure 10. You can see that $p(x)$ is an approximation to $f(x)$ for values of x close to 1.

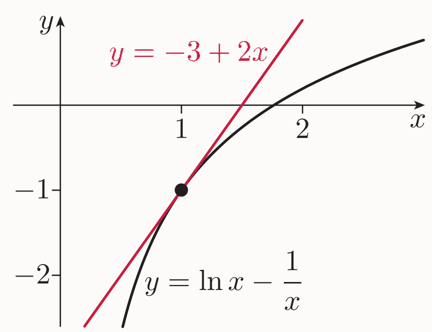


Figure 10 The linear Taylor polynomial about 1 for $f(x) = \ln x - 1/x$

Activity 5 *Finding a linear Taylor polynomial about a point other than 0*

Find the linear Taylor polynomial about 1 for the function $f(x) = e^x$.

In Activity 5 you were asked to obtain the linear Taylor polynomial about 1 for the function $f(x) = e^x$, while in Example 2(a) on page 105 the linear Taylor polynomial about 0 was obtained for the same function. The graphs of these linear Taylor polynomials are shown in Figure 11. As you'd expect, it appears that the first of these polynomials approximates e^x for values of x close to 1, while the second approximates e^x for values of x close to 0. This illustrates that, in general, Taylor polynomials about different points are different polynomials.

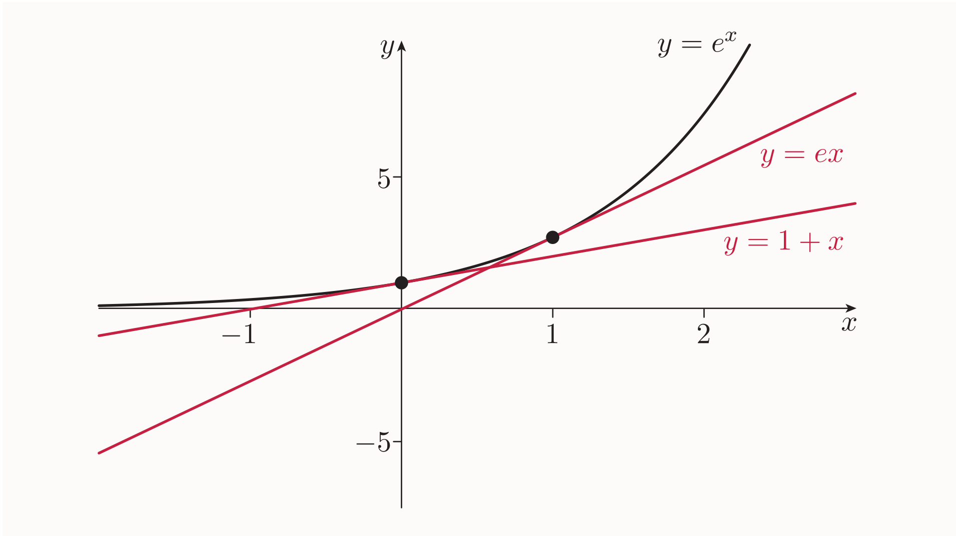


Figure 11 The linear Taylor polynomials about 0 and 1 for $f(x) = e^x$

In Subsection 1.1 we approximated functions by constant Taylor polynomials, and in this subsection we approximated them by linear Taylor polynomials. The linear Taylor polynomials usually provided better approximations than the constant Taylor polynomials. This suggests that we could obtain further improvements by increasing the degree of the approximating polynomial once more, and trying to approximate functions by polynomials of degree 2, which are *quadratic* functions. This is the topic of the next subsection.

1.3 Quadratic Taylor polynomials

We now look at approximating functions by quadratic functions. As you'd expect, this usually gives greater accuracy than approximating functions by linear functions.

Suppose that f is a function that's differentiable at a. In Subsection 1.2 you saw how to approximate f close to a by a linear function p. The particular linear function p was chosen to ensure that the following two conditions hold:

1. The values of the function and the approximating polynomial are equal at a; that is, $p(a) = f(a)$.
2. The values of the first derivatives of the function and the approximating polynomial are equal at a; that is, $p'(a) = f'(a)$.

Suppose that we now want to try to approximate f close to a by a *quadratic* function p. As you know, a quadratic function is a function of the form

$$p(x) = c_0 + c_1 x + c_2 x^2,$$

where c_0, c_1 and c_2 are constants. However, in the context of Taylor polynomials, it's more convenient to write the general form of an approximating quadratic function as

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2,$$

where c_0, c_1 and c_2 are constants, and a is the point about which we want to find the approximating function. This alternative form is the rule of a quadratic function, since if you multiply out the right-hand side then you obtain powers of x up to and including x^2, but no higher powers. You can see that the form above reduces to the more usual form in the case when $a = 0$.

It seems sensible to choose the approximating quadratic function p to ensure that conditions 1 and 2 above hold. Since there are now three constants to choose, we can also impose a third condition, and a natural one to choose is:

3. The values of the second derivatives of the function and the approximating polynomial are equal at a; that is, $p''(a) = f''(a)$.

We can impose this condition provided that $f''(a)$ exists; that is, provided that f is twice differentiable at a. This is the case for many functions and many points in their domains. In fact, many functions can be differentiated as many times as you wish at all points in their domains. Such functions include all polynomial, rational, trigonometric, exponential and logarithmic functions, and all constant multiples, sums, differences, products, quotients and composites of these.

As you've seen, condition 1 means that the graphs of the function f and the approximating polynomial p both pass through the same point $(a, f(a))$, and condition 2 means that the graphs of the function and the approximating polynomial both have the same gradient at that point. Condition 3 means that the function and the approximating polynomial also have the same rate of change of gradient at that point. Roughly speaking, this means that their graphs have the same 'curvature' at that point.

The polynomial p of the form above that satisfies conditions 1, 2 and 3 is called the **quadratic Taylor polynomial about** a **for** f. For any point x close to a, the value of $p(x)$ is an approximation for $f(x)$.

In some cases, the polynomial $p(x)$ that satisfies conditions 1, 2 and 3 has $c_2 = 0$ and so is *not* a quadratic polynomial, but has degree 1 or less. If this happens, then we still refer to the approximating polynomial as the *quadratic* Taylor polynomial about 0 for f. This means that a quadratic Taylor polynomial is not necessarily a quadratic polynomial! You'll see an example of this later in this section. You saw in Subsection 1.2 that a similar situation arises with linear Taylor polynomials (a linear Taylor polynomial can be a constant function).

As for linear Taylor polynomials, there's a general formula that you can use to find quadratic Taylor polynomials. It's given in the box on page 114. If you're not interested in knowing where the formula comes from, then you can skip ahead to this box. Otherwise, keep reading!

To illustrate the ideas of how the formula is derived, let's start by finding a particular quadratic Taylor polynomial, namely the quadratic Taylor polynomial about 0 for the exponential function $f(x) = e^x$. As discussed above, we can take it to be of the form

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2,$$

where c_0, c_1 and c_2 are constants. In this case, since $a = 0$, it reduces to

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

We have to determine what the values of the constants c_0, c_1 and c_2 must be to ensure that the value of p at 0, and the values of the first and second derivatives of p at 0, are the same as those of f.

The first and second derivatives of the function $f(x) = e^x$ are $f'(x) = e^x$ and $f''(x) = e^x$. Hence

$$f(0) = e^0 = 1, \quad f'(0) = e^0 = 1 \quad \text{and} \quad f''(0) = e^0 = 1.$$

So we have to choose the values of the constants c_0, c_1 and c_2 to ensure that

$$p(0) = 1, \quad p'(0) = 1 \quad \text{and} \quad p''(0) = 1.$$

Here's how we can do that.

First we ensure that $p(0) = 1$. We have

$$p(x) = c_0 + c_1 x + c_2 x^2,$$

so $p(0) = c_0$. Thus to ensure that $p(0) = 1$ we must have $c_0 = 1$.

Next we ensure that $p'(0) = 1$. Differentiating the formula for p gives

$$p'(x) = c_1 + 2c_2 x,$$

so $p'(0) = c_1$. Thus to ensure that $p'(0) = 1$ we must have $c_1 = 1$.

Finally we ensure that $p''(0) = 1$. Differentiating the formula for p' gives

$$p''(x) = 2c_2,$$

so $p''(0) = 2c_2$. Thus to ensure that $p''(0) = 1$ we must have $2c_2 = 1$; that is, $c_2 = \frac{1}{2}$.

So the quadratic Taylor polynomial p for $f(x) = e^x$ about 0 is

$$p(x) = 1 + x + \tfrac{1}{2}x^2.$$

The graphs of $f(x) = e^x$ and the approximating polynomial $p(x) = 1 + x + \frac{1}{2}x^2$ found above are shown in Figure 12. You can see that the quadratic function p appears to be a more accurate approximating polynomial for $f(x) = e^x$ for values of x close to 0 than the linear function found earlier, which is shown in Figure 7 on page 104.

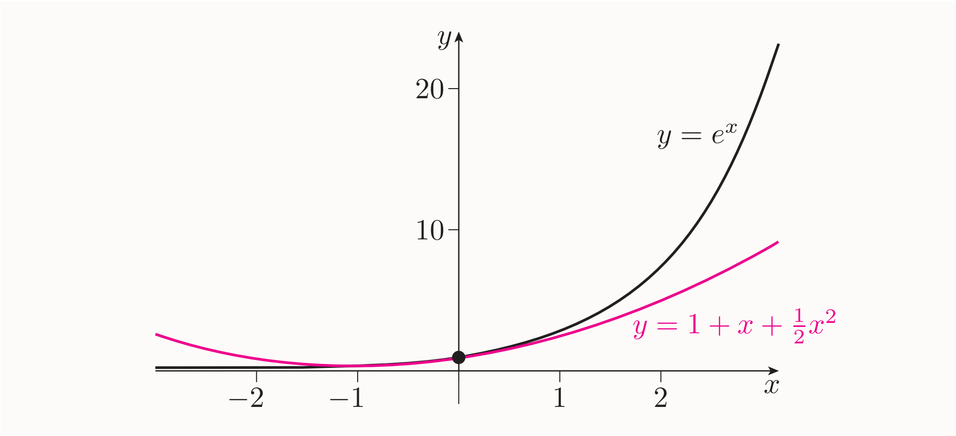


Figure 12 The quadratic Taylor polynomial about 0 for $f(x) = e^x$

As with the constant and linear Taylor polynomials about 0 for $f(x) = e^x$ that you met earlier, which are $p(x) = 1$ and $p(x) = 1 + x$, respectively, the quadratic Taylor polynomial $p(x) = 1 + x + \frac{1}{2}x^2$ provides better approximations for values of x close to 0 than for values of x further away.

You could use the method demonstrated above to work out the quadratic Taylor polynomial for any function f about any point a at which it is twice differentiable. However, let's instead apply the method to a *general* function f and *general* point a. This will give a general formula that we can use to work out a quadratic Taylor polynomial in any particular case.

So let's suppose that f is a function and a is a point in its domain at which it is twice differentiable.

The quadratic Taylor polynomial about a for f is of the form

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2,$$

where c_0, c_1 and c_2 are constants. We have to determine what the values of the constants c_0, c_1 and c_2 must be to ensure that the value of p at a, and the values of the first and second derivatives of p at a, are the same as those of f.

To do this, we apply the method demonstrated above, the only difference being that we can't evaluate the quantities $f(a)$, $f'(a)$ and $f''(a)$, so instead we keep them in their general form throughout our working. They'll then appear in the final formula for p, ready to be evaluated for any particular function f and point a.

As before, we have to choose the values of c_0, c_1 and c_2 to ensure that

$$p(a) = f(a), \quad p'(a) = f'(a) \quad \text{and} \quad p''(a) = f''(a).$$

To do that, first we ensure that $p(a) = f(a)$. We have

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2,$$

so $p(a) = c_0$. Thus to ensure that $p(a) = f(a)$ we must have $c_0 = f(a)$.

Next we ensure that $p'(a) = f'(a)$. The first step here is to differentiate the formula for p,

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2.$$

The first term, c_0, is a constant, so its derivative is 0. To differentiate the second term, $c_1(x-a)$, we use the constant multiple rule, which gives c_1. To differentiate the third term, $c_2(x-a)^2$, we use the constant multiple rule and the chain rule (or the rule for differentiating a function of a linear expression), which gives $2c_2(x-a)$. The final answer is

$$p'(x) = c_1 + 2c_2(x-a).$$

Hence $p'(a) = c_1$. Thus to ensure that $p'(a) = f'(a)$ we must have $c_1 = f'(a)$.

Finally we ensure that $p''(a) = f''(a)$. Differentiating the formula for p', using a method similar to that used to differentiate the formula for p above, gives

$$p''(x) = 2c_2.$$

Hence $p''(a) = 2c_2$. Thus to ensure that $p''(a) = f''(a)$ we must have $2c_2 = f''(a)$; that is, $c_2 = \frac{1}{2}f''(a)$.

So we have the following general formula.

Quadratic Taylor polynomials

Let f be a function that is twice differentiable at a. The **quadratic Taylor polynomial about a for f** is

$$p(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2.$$

When $a = 0$, this becomes

$$p(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2.$$

This formula allows you to find the quadratic Taylor polynomial p about a for any suitable function f. To do this, you calculate the value of f at a, and the values of the first and second derivatives of f at a, and then substitute them into the formula.

In the next example the quadratic Taylor polynomial about 0 for the exponential function is worked out again, this time directly using the formula in the box above.

Example 4 *Finding a quadratic Taylor polynomial about 0*

Find the quadratic Taylor polynomial about 0 for the function $f(x) = e^x$.

Solution

Differentiate f twice to find f' and f'', and hence find the values of $f(0)$, $f'(0)$ and $f''(0)$.

We have $f(x) = e^x$, so

$$f'(x) = e^x \quad \text{and} \quad f''(x) = e^x.$$

Hence

$$f(0) = e^0 = 1, \quad f'(0) = e^0 = 1 \quad \text{and} \quad f''(0) = e^0 = 1.$$

Apply the second formula in the box on page 114, since in this case $a = 0$.

The quadratic Taylor polynomial about 0 for $f(x) = e^x$ is

$$p(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2;$$

Substitute in the values of $f(0)$, $f'(0)$ and $f''(0)$.

that is,

$$p(x) = 1 + x + \tfrac{1}{2}x^2.$$

You can see that the quadratic Taylor polynomial about 0 for the function $f(x) = e^x$ found in Example 4 is the same as that found earlier in this subsection, as expected.

In the next two activities you can use the second formula in the box on page 114 to find the quadratic Taylor polynomials about 0 for the cosine and sine functions.

Activity 6 *Finding a quadratic Taylor polynomial about 0*

(a) Find the quadratic Taylor polynomial about 0 for the function

$$f(x) = \cos x.$$

(b) Use this polynomial to find an approximation for $\cos(0.2)$, and use your calculator to find the value of the associated remainder to six decimal places. Compare this approximation for $\cos(0.2)$ with the one found in Activity 3(b). Which is better?

The outcome of Activity 6(b) demonstrates that increasing the degree of the approximating Taylor polynomial from 1 to 2 can significantly increase the accuracy of an approximation at a particular point.

The graphs of $f(x) = \cos x$ and the quadratic Taylor polynomial $p(x) = 1 - \frac{1}{2}x^2$ found in Activity 6(a) are shown in Figure 13. You can see that $p(x)$ appears to be a good approximation for $f(x)$ for values of x in quite a large interval around 0.

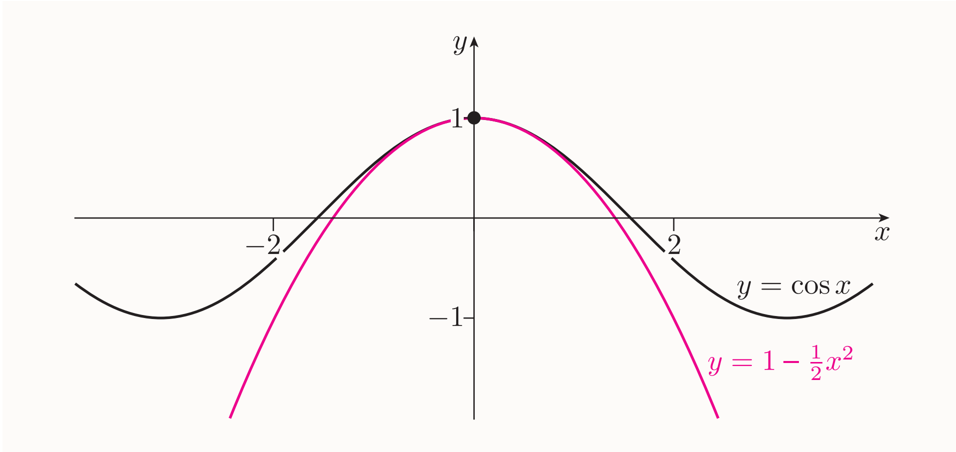


Figure 13 The quadratic Taylor polynomial about 0 for $f(x) = \cos x$

Activity 7 *Finding another quadratic Taylor polynomial about 0*

Find the quadratic Taylor polynomial about 0 for the function

$$f(x) = \sin x.$$

The quadratic Taylor polynomial about 0 for the sine function, $p(x) = x$, is an example of a quadratic Taylor polynomial that is not a quadratic polynomial. The quadratic Taylor polynomial about 0 for the sine function is the same as the linear Taylor polynomial about 0 for the sine function, whose graph is shown in Figure 8 on page 107.

We have now found constant, linear and quadratic Taylor polynomials about 0 for each of the functions $f(x) = e^x$, $f(x) = \sin x$ and $f(x) = \cos x$. These are listed in Table 1.

Table 1 Constant, linear and quadratic Taylor polynomials about 0

Function	Constant	Linear	Quadratic
e^x	1	$1 + x$	$1 + x + \frac{1}{2}x^2$
$\sin x$	0	x	x
$\cos x$	1	1	$1 - \frac{1}{2}x^2$

Notice that, for each of these three functions, the linear Taylor polynomial about 0 can be obtained from the constant Taylor polynomial about 0 by adding the appropriate term in x (this term is $0x$ in the case of cos). Similarly, the quadratic Taylor polynomial about 0 can be obtained from the linear Taylor polynomial about 0 by adding the appropriate term in x^2 (this term is $0x^2$ in the case of sin). You will see in Section 2 that these properties hold for every function f for which these Taylor polynomials can be found, and that similar properties hold for higher-degree Taylor polynomials about 0. When you're calculating Taylor polynomials, this is a very convenient feature.

Quadratic Taylor polynomials about $a \neq 0$

So far in this section, you've seen quadratic Taylor polynomials only about 0. The next example and activity involve quadratic Taylor polynomials about another point.

Example 5 *Finding a quadratic Taylor polynomial about a point other than 0*

Find the quadratic Taylor polynomial about 1 for the function $f(x) = \ln x$.

Solution

Differentiate f twice to find f' and f'', and find the values of $f(1)$, $f'(1)$ and $f''(1)$.

We have $f(x) = \ln x$, so

$$f'(x) = \frac{1}{x} \quad \text{and} \quad f''(x) = -\frac{1}{x^2}.$$

Hence

$$f(1) = \ln 1 = 0, \quad f'(1) = \frac{1}{1} = 1 \quad \text{and} \quad f''(1) = -\frac{1}{1^2} = -1.$$

Apply the first formula in the box on page 114. Remember that in this case $a = 1$.

The quadratic Taylor polynomial about 1 for $f(x) = \ln x$ is

$$p(x) = f(1) + f'(1)(x-1) + \tfrac{1}{2}f''(1)(x-1)^2;$$

Substitute in the values of $f(1)$, $f'(1)$ and $f''(1)$.

that is,

$$p(x) = 0 + 1(x-1) + \tfrac{1}{2}(-1)(x-1)^2,$$

which can be simplified to

$$p(x) = (x-1) - \tfrac{1}{2}(x-1)^2.$$

Usually, leave the answer in this form, rather than multiplying out.

You could simplify the quadratic Taylor polynomial found in Example 5 by multiplying out the squared brackets and collecting like terms. This gives

$$p(x) = -\tfrac{3}{2} + 2x - \tfrac{1}{2}x^2.$$

However, normally we don't simplify quadratic Taylor polynomials in this way. Instead, we leave them in the form

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2.$$

To understand why, remember that a convenient property of Taylor polynomials about 0 is that you can obtain the linear Taylor polynomial about 0 for a function f by adding the appropriate term in x to the constant Taylor polynomial about 0 for f, and similarly for higher-degree Taylor polynomials. Taylor polynomials about a point a other than 0 have the same property, as long as you use the form above for the Taylor polynomials; that is, as long as you consider terms in powers of $x - a$ rather than terms in powers of x. You'll see in the next section why this is true for any value of a.

The graphs of the function $f(x) = \ln x$ and the quadratic Taylor polynomial $p(x) = (x-1) - \frac{1}{2}(x-1)^2$ found in Example 5 are shown in Figure 14.

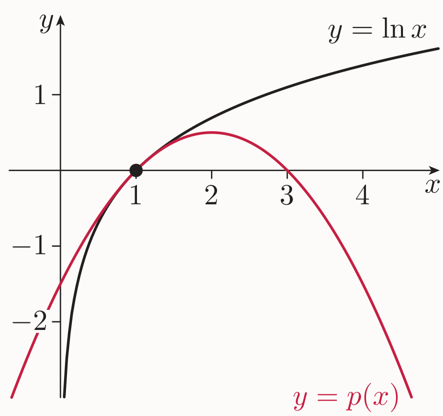


Figure 14 The quadratic Taylor polynomial about 1 for $f(x) = \ln x$

Activity 8 *Finding a quadratic Taylor polynomial about a point other than 0*

Find the quadratic Taylor polynomial about 1 for the function $f(x) = e^x$.

So far you've seen how you can use constant, linear and quadratic Taylor polynomials about a point a for a function f to approximate the values of f close to a. You saw that when we increased the degree of the Taylor polynomial the accuracy of the approximations was usually improved. In the next section you'll see that this fact generalises to approximating polynomials of higher degree.

2 Taylor polynomials of any degree

In this section you'll look at approximating functions by polynomials of any degree, and how you can use such polynomials to find approximate values for functions at particular points.

2.1 Taylor polynomials of degree n

From what you saw in Section 1, you might guess that for any suitable function f and any point a in its domain, you can obtain more and more accurate approximating polynomials for f close to a by taking polynomials of higher and higher degrees, and choosing the coefficients to ensure that the values of higher and higher derivatives of the polynomial at a are the same as those of the corresponding derivatives of f. Here 'suitable function f' means that f must be differentiable at a the required number of times.

For example, to try to improve on the approximations provided by the quadratic Taylor polynomial about a for f, you could attempt to approximate f by a cubic function, of the form

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3,$$

whose value is the same as that of f at a, and whose first, second and third derivatives have the same values at a as the corresponding derivatives of f.

More generally, for any chosen value of n, you could try to find a polynomial

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n, \quad (1)$$

whose value is the same as that of f at a, and whose first, second, third, $\dots$, nth derivatives have the same values at a as the corresponding derivatives of f. The polynomial that satisfies these conditions is called the **Taylor polynomial of degree n about a for f**.

There's a general formula for this Taylor polynomial, for any function f, any degree n, and any point a at which f is differentiable n times. As you'd expect, this formula involves the values of the first, second, third, $\dots$, nth derivatives of f at a. Remember that for $n \geq 3$ the nth derivative of the function f is denoted by $f^{(n)}$, so the value of the nth derivative of f at a is denoted by $f^{(n)}(a)$. (The third derivative $f^{(3)}$ is sometimes denoted by f'''.) The general formula is given in the box on page 121. If you're not interested in knowing how it's derived, then you can skip ahead to it. Otherwise, as before, keep reading.

To find the formula, we'll use the method that was used for quadratic polynomials in the previous section, but we'll continue with derivatives up to the nth, rather than just the second. Similarly to before, we can't evaluate the first, second, third, $\dots$, nth derivatives of f at a, because we don't know what f and a are, so throughout our working we'll denote them by $f'(a)$, $f''(a)$, $f^{(3)}(a)$, $\dots$, $f^{(n)}(a)$. These quantities will then appear in the final formula for p, ready to be evaluated for any particular function f and point a.

Here's how the working goes. We need to determine what the values of the constants c_0, c_1, c_2, $\dots$, c_n in equation (1) must be to ensure that the value of p at a, and the values of the first, second, third, $\dots$, nth derivatives of p at a, are the same as those of f.

First we ensure that $p(a) = f(a)$. We have

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n,$$

so $p(a) = c_0$. Thus to ensure that $p(a) = f(a)$ we must have

$$c_0 = f(a).$$

Next we ensure that $p'(a) = f'(a)$. We have

$$p'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1},$$

so $p'(a) = c_1$. Thus to ensure that $p'(a) = f'(a)$ we must have

$$c_1 = f'(a).$$

Then we ensure that $p''(a) = f''(a)$. We have

$$p''(x) = 2c_2 + 3 \times 2\,c_3(x-a) + 4 \times 3\,c_4(x-a)^2 + \cdots + n(n-1)c_n(x-a)^{n-2},$$

so $p''(a) = 2c_2$. Thus to ensure that $p''(a) = f''(a)$ we must have $2c_2 = f''(a)$; that is,

$$c_2 = \tfrac{1}{2}f''(a).$$

It's useful to write this as

$$c_2 = \frac{f''(a)}{2!},$$

where 2! denotes 2×1, that is, 2 factorial, as you saw in Unit 10. This allows the pattern in the next few calculations to be seen more easily. (Recall that if n is a positive integer, then, by definition,

$$n! = n(n-1) \times \cdots \times 2 \times 1.)$$

Then we ensure that $p^{(3)}(a) = f^{(3)}(a)$. We have

$$p^{(3)}(x) = 3 \times 2\,c_3 + 4 \times 3 \times 2\,c_4(x-a) + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3},$$

so $p^{(3)}(a) = 3!\,c_3$. Thus to ensure that $p^3(a) = f^3(a)$ we must have $3!\,c_3 = f^{(3)}(a)$; that is,

$$c_3 = \frac{f^{(3)}(a)}{3!}.$$

Continuing in this way, we find that we must have

$$c_4 = \frac{f^{(4)}(a)}{4!}, \qquad c_5 = \frac{f^{(5)}(a)}{5!}, \qquad \text{and so on,}$$

until finally, to ensure that $p^{(n)}(a) = f^{(n)}(a)$, we must have

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The resulting formula for the approximating polynomial is stated below. You can see that it generalises the formulas for constant, linear and quadratic Taylor polynomials given earlier. In the case where $a = 0$ the formula reduces to a simpler form, which is also stated below.

Taylor polynomials

Let f be a function that is n-times differentiable at a point a. The **Taylor polynomial of degree n about a for f** is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad (2)$$

The point a is called the **centre** of the Taylor polynomial.

When $a = 0$, the Taylor polynomial above becomes

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \qquad (3)$$

Formula (2) allows you to find the Taylor polynomial p of degree n about a for any suitable function f. To do this, you calculate the value of f at a, and the values of the first, second, third, ..., nth derivatives of f at a, and substitute them into the formula.

The formula also confirms that any Taylor polynomial about a for a function f can be obtained from a Taylor polynomial of lower degree about a for f by adding the appropriate further terms. For example, the Taylor polynomial of degree 1 about a for f is

$$p(x) = f(a) + f'(a)(x-a),$$

while the Taylor polynomial of degree 2 about 0 for f is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2.$$

The second polynomial is obtained from the first simply by adding the term in $(x-a)^2$.

For some functions f, the value of $f^{(n)}(a)$ is 0. In such a case, formula (2) gives a polynomial whose degree is less than n. If this happens, then the polynomial is still called the Taylor polynomial of degree n about a for f. This means that a Taylor polynomial of degree n is not necessarily a polynomial of degree n. You've seen examples of this already, in the cases $n = 1$ and $n = 2$, and you'll see more examples later in this section.

You've seen the terms *constant*, *linear* and *quadratic* used to describe Taylor polynomials of degrees 0, 1 and 2, respectively. The terms **cubic**, **quartic** and **quintic** are used to describe Taylor polynomials of degrees 3, 4 and 5, respectively.

The next example and the two activities that follow concern Taylor polynomials about 0. So in these we use formula (3) for a Taylor polynomial about 0, rather than the general formula (2).

Example 6 *Finding a quartic Taylor polynomial about 0*

Find the quartic Taylor polynomial about 0 for the function $f(x) = e^x$.

Solution

Repeatedly differentiate f to find f', f'', $f^{(3)}$ and $f^{(4)}$, and find the values of $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$.

Here $f(0) = e^0 = 1$. Also, for each positive integer n, the nth derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so

$$f^{(n)}(0) = e^0 = 1, \quad \text{for all positive integers } n.$$

Apply formula (3) in the box above, since in this case $a = 0$. Also, $2! = 2 \times 1 = 2$, $3! = 3 \times 2 \times 1 = 6$ and $4! = 4 \times 3 \times 2 \times 1 = 24$.

Hence, by formula (3), the Taylor polynomial of degree 4 about 0 for $f(x) = e^x$ is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4;$$

that is,

$$p(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4.$$

Notice that, as you'd expect, the first three terms found in Example 6 agree with those of the quadratic Taylor polynomial about 0 for the exponential function, which was found in Example 4.

The graphs of the function $f(x) = e^x$ and the quartic Taylor polynomial about 0 found in Example 6 are shown in Figure 15. Notice how much better this approximating polynomial is than the quadratic one shown in Figure 12 on page 113.

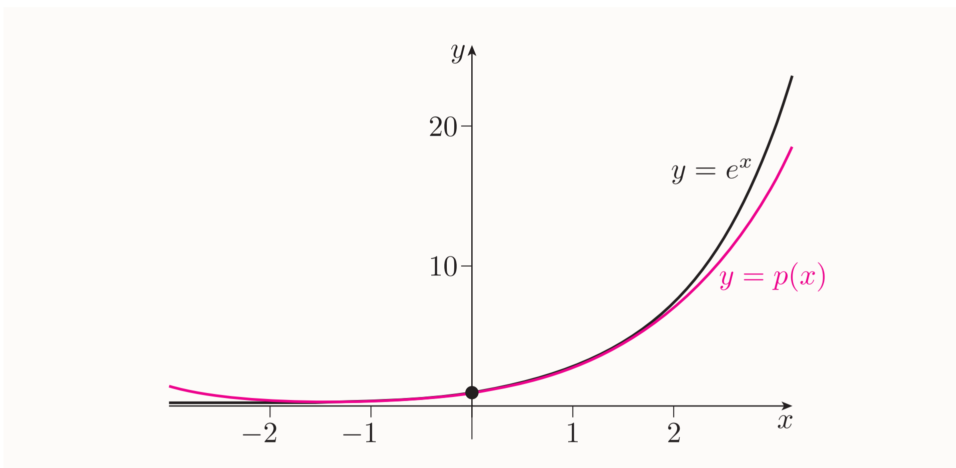


Figure 15 The quartic Taylor polynomial about 0 for $f(x) = e^x$

Since for the function $f(x) = e^x$ we have that $f^{(n)}(0) = 1$ for all positive integers n, it follows from formula (3) that, for any n, the Taylor polynomial of degree n about 0 for $f(x) = e^x$ is

$$p(x) = 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n.$$

In the next activity you're asked to find the quartic Taylor polynomials about 0 for the cosine and sine functions.

Activity 9 *Finding quartic Taylor polynomials about 0*

Find the quartic Taylor polynomial about 0 for each of the following functions.

(a) $f(x) = \cos x$ (b) $f(x) = \sin x$

Notice that the constant term, the term in x and the term in x^2 in the Taylor polynomials that you were asked to find in Activity 9 are the same as those in the quadratic Taylor polynomials about 0 for the cosine and sine functions, which you were asked to find in Activities 6 and 7.

The quartic Taylor polynomial about 0 for the sine function is an example of a Taylor polynomial of degree n whose polynomial degree is less than n.

The graphs of the cosine and sine functions, and the quartic Taylor polynomials for these functions that you were asked to find in Activity 9, are shown in Figure 16(a) and (b), respectively.

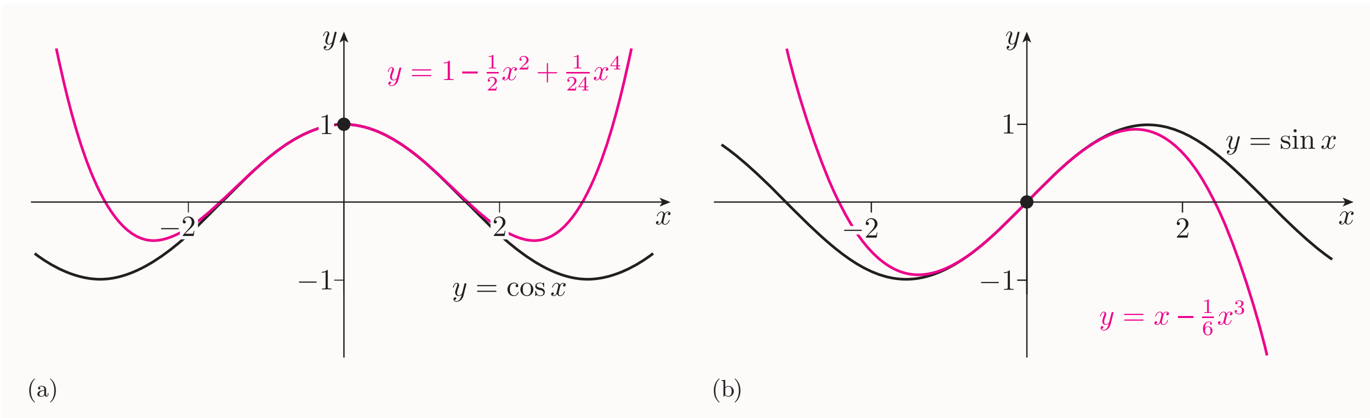


Figure 16 The quartic Taylor polynomials about 0 for cosine and sine

In the next activity you're asked to find a general formula for a Taylor polynomial of degree n about 0 for a particular function, by spotting how the pattern of derivatives develops.

Activity 10 *Finding a Taylor polynomial of degree n*

Consider the function

$$f(x) = \frac{1}{1-x}.$$

(a) (i) Use the chain rule (from Unit 7) to show that, if k is a constant, then

$$\frac{d}{dx}\left(\frac{1}{(1-x)^k}\right) = \frac{k}{(1-x)^{k+1}}.$$

(ii) Use the result of part (a)(i) to find formulas for f', f'' and $f^{(3)}$, and hence find the values of $f'(0)$, $f''(0)$ and $f^{(3)}(0)$.

(iii) Hence find the cubic Taylor polynomial about 0 for f.

(b) (i) Use the result of part (a)(i) to find successive derivatives of f beyond the third, until the pattern is clear and you can write down a formula for the nth derivative of f, in terms of x and n.

(ii) Hence write down a formula in terms of n for $f^{(n)}(0)$, the value of the nth derivative of f at 0.

(iii) Hence write down, using the $+\cdots+$ notation, a formula for the Taylor polynomial of degree n about 0 for f.

The graph of the function $f(x) = 1/(1-x)$, and the graph of the cubic Taylor polynomial about 0 for this function from Activity 10(a), are shown in Figure 17.

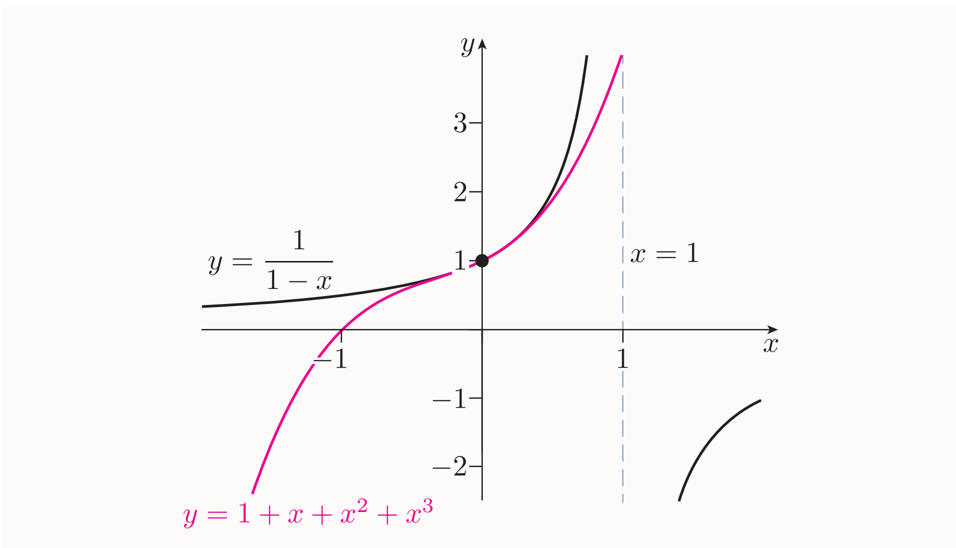


Figure 17 The cubic Taylor polynomial about 0 for $f(x) = 1/(1-x)$

Even and odd functions

In Activity 9 earlier in this subsection you saw that the quartic Taylor polynomials about 0 for the cosine and sine functions are

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 \quad \text{and} \quad x - \tfrac{1}{6}x^3,$$

respectively.

Notice that the quartic Taylor polynomial about 0 for the cosine function contains terms in even powers of x only, whereas that for the sine function contains terms in odd powers of x only. These observations are explained by the facts that the cosine function is an *even* function and the sine function is an *odd* function, as defined in general below.

A function f is said to be **even** if its graph is unchanged under reflection in the y-axis, as illustrated in Figure 18(a). Thus f is even if

$$f(-x) = f(x), \quad \text{for all } x \text{ in the domain of } f.$$

Similarly, a function f is said to be **odd** if its graph is unchanged by rotation through a half turn about the origin, as illustrated in Figure 18(b). Thus f is odd if

$$f(-x) = -f(x), \quad \text{for all } x \text{ in the domain of } f.$$

A rotation through a half turn about the origin has the same effect as a reflection in the y-axis followed by a reflection in the x-axis. Hence the graph of an odd function is the same as an 'upside down' reflection of itself in the y-axis.

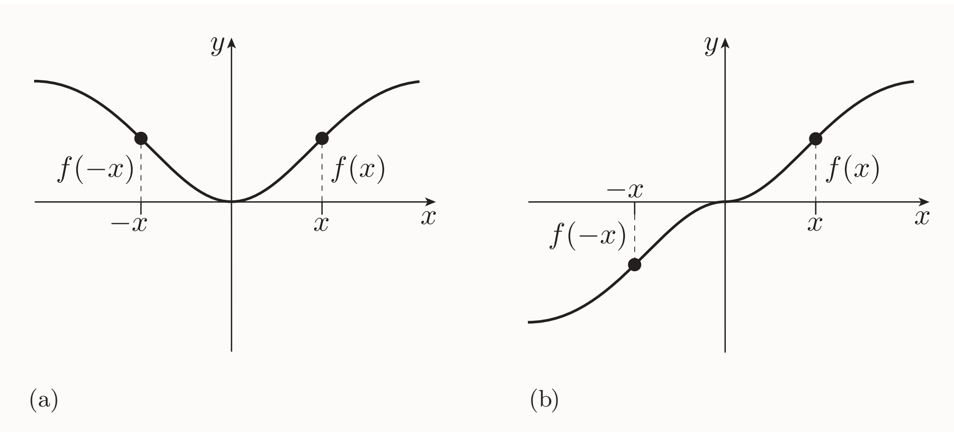


Figure 18 (a) An even function (b) An odd function

These definitions are summarised below.

Even and odd functions

A function f is

- **even** if $f(-x) = f(x)$ for all x in the domain of f
- **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

Here are some examples.

- $f(x) = x^2$ is an even function, since
  $$f(-x) = (-x)^2 = x^2 = f(x), \quad \text{for all } x \in \mathbb{R}.$$
- $f(x) = x^3$ is an odd function, since
  $$f(-x) = (-x)^3 = -x^3 = -f(x), \quad \text{for all } x \in \mathbb{R}.$$
- $f(x) = \sin x$ is an odd function, since
  $$f(-x) = \sin(-x) = -\sin x = -f(x), \quad \text{for all } x \in \mathbb{R}.$$
- $f(x) = \cos x$ is an even function, since
  $$f(-x) = \cos(-x) = \cos x = f(x), \quad \text{for all } x \in \mathbb{R}.$$
- $f(x) = e^x$ is neither even nor odd, since we can find a value of x, say $x = 1$, such that $e^{-x} \neq e^x$ and $e^{-x} \neq -e^x$. (In fact, $e^{-1} \approx 0.368$ and $e^1 \approx 2.718$.)

(Remember that the symbol $\in$ means 'in' or 'belongs to' and that the symbol $\mathbb{R}$ denotes the set of real numbers.)

Taylor polynomials about 0 for even and odd functions have the following properties.

Taylor polynomials about 0 for even and odd functions

A Taylor polynomial about 0 for an even function contains terms in even powers of x only.

A Taylor polynomial about 0 for an odd function contains terms in odd powers of x only.

Here's an explanation of why these facts are true.

By looking at the symmetry of the graph in Figure 18(a), and thinking about even functions in general, you can see that if f is an even function, then for all x in its domain the gradient of the graph of f at $-x$ has the same magnitude as the gradient at x, but the opposite sign (provided that the gradient exists). In other words,

$$\text{if } f(-x) = f(x), \text{ for all } x \text{ in the domain of } f,$$
$$\text{then } f'(-x) = -f'(x), \text{ for all } x \text{ in the domain of } f'.$$

This means that if f is an even function then f' is an odd function.

Similarly, by looking at the symmetry of the graph in Figure 18(b) and thinking about odd functions in general, you can see that if f is an odd function, then for all x in its domain the gradient of f is the same at $-x$ as at x (provided that the gradient exists). In other words,

$$\text{if } f(-x) = -f(x), \text{ for all } x \text{ in the domain of } f,$$
$$\text{then } f'(-x) = f'(x), \text{ for all } x \text{ in the domain of } f'.$$

This means that if f is an odd function then f' is an even function.

So differentiation turns any even function into an odd function, and vice versa.

Notice also that if f is an odd function whose domain contains 0, then, since $f(-x) = -f(x)$ for all x in its domain, we have in particular that $f(0) = -f(0)$, and it follows that $f(0) = 0$. Hence any odd function has the value 0 at 0.

Now let f be any even function that's differentiable infinitely many times at 0. It follows from the discussion above that f'', $f^{(4)}$, $f^{(6)}$, ..., are all even functions, and f', $f^{(3)}$, $f^{(5)}$, ..., are all odd functions. Since any odd function has the value 0 at the point 0, the values of $f'(0)$, $f^{(3)}(0)$, $f^{(5)}(0)$, ..., are all 0. Hence, from the general formula for a Taylor polynomial about 0, any Taylor polynomial about 0 for f contains only terms with even powers of x.

Similar reasoning applies in the case of an odd function.

By the facts in the box above, any Taylor polynomial about 0 for the cosine function contains terms in even powers of x only, and any Taylor polynomial about 0 for the sine function contains terms in odd powers of x only. For example, the Taylor polynomial of degree 9 about 0 for $f(x) = \cos x$ is

$$p(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8.$$

Similarly, the Taylor polynomial of degree 9 about 0 for $f(x) = \sin x$ is

$$p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9.$$

You can find these Taylor polynomials by extending the solution to Activity 9, and noting the patterns in the coefficients of the Taylor polynomials. These patterns continue indefinitely for higher-degree Taylor polynomials for the cosine and sine functions. You'll be asked to verify this later in the unit.

Taylor polynomials of degree n about $a \neq 0$

So far in this section, all the examples and activities have involved Taylor polynomials about 0. The next example and activity involve Taylor polynomials about another point a. So here we use the general formula for a Taylor polynomial given by equation (2) on page 121.

Example 7 *Finding a quartic Taylor polynomial about a point other than 0*

Find the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$.

Solution

Repeatedly differentiate f to find f', f'', $f^{(3)}$ and $f^{(4)}$, and find the values of $f(1)$, $f'(1)$, $f''(1)$, $f^{(3)}(1)$ and $f^{(4)}(1)$.

The first four derivatives of the function $f(x) = \ln x$ are as follows:

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3},$$
$$f^{(4)}(x) = -\frac{3 \times 2}{x^4} = -\frac{3!}{x^4}.$$

The product 3×2 has been simplified to $3!$ rather than 6 here to highlight the emerging pattern.

So

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f^{(3)}(1) = 2,$$
$$f^{(4)}(1) = -3!\,.$$

Apply formula (2) on page 121. Remember that in this case $a = 1$.

Hence the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$ is

$$p(x) = 0 + 1 \times (x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4;$$

that is,

$$p(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4.$$

As you saw earlier in the case of quadratic Taylor polynomials, we usually leave a Taylor polynomial about a point a as a sum of terms each of which is the product of a constant and a power of $x - a$, rather than multiplying it out. However, we might make an exception for a polynomial of low degree for which the multiplied-out form is simpler.

The graphs of the function $f(x) = \ln x$ and the quartic Taylor polynomial about 1 found in Example 7 are shown in Figure 19. As you'd expect, the quartic Taylor polynomial appears to be a better approximating polynomial than the quadratic Taylor polynomial about 1 for f shown in Figure 14 on page 118.

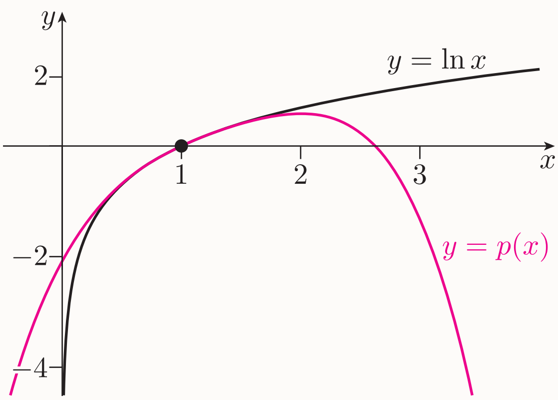


Figure 19 The quartic Taylor polynomial about 1 for $f(x) = \ln x$

If, in the solution to Example 7, you look at the patterns in the formulas for f, f', f'', $f^{(3)}$, …, and hence in the values of $f(1)$, $f'(1)$, $f''(1)$, $f^{(3)}(1)$, …, then you can see that the pattern of terms in the Taylor polynomial found in the example continues as the degree of the Taylor polynomial increases. So, for any positive integer n, the Taylor polynomial of degree n about 1 for $f(x) = \ln x$ is

$$p(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n.$$

The expression $(-1)^{n-1}$ in the final term here is, as you saw in Unit 10, a neat way to give the term a negative sign when n is even and a positive sign when n is odd. This expression can also be written as $-(-1)^n$, since

$$(-1)^n = (-1) \times (-1)^{n-1} = -(-1)^{n-1}.$$

Activity 11 *Finding a cubic Taylor polynomial about a point other than 0*

Find the cubic Taylor polynomial about $\pi/6$ for the function $f(x) = \sin x$.

The graphs of the function $f(x) = \sin x$ and the cubic Taylor polynomial about $\pi/6$ found in Activity 11 are shown in Figure 20.

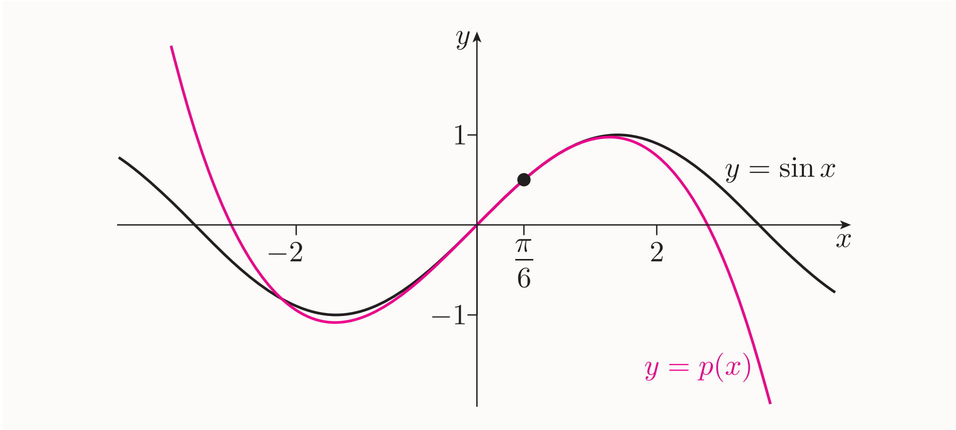


Figure 20 The cubic Taylor polynomial about $\pi/6$ for $f(x) = \sin x$

The cubic Taylor polynomial about $\pi/6$ for the function $f(x) = \sin x$, which you were asked to find in Activity 11, contains terms in $(x - \pi/6)^k$ with k even as well as with k odd. This is not surprising, since the earlier discussion about even and odd functions applies only to Taylor polynomials about 0.

Activity 12 *Investigating graphs of Taylor polynomials*

Open the *Graphs of Taylor polynomials* applet. Initially it shows the graphs of the function $f(x) = e^x$ and its Taylor polynomial of degree 1 about 0.

(a) Increase the degree n of the Taylor polynomial and observe the effect on its graph.

(b) Try changing the function f and then varying n.

(c) Try changing the centre a of the Taylor polynomial and then varying n.

(d) Try some other functions and other centres for the Taylor polynomial.

Using sigma notation for Taylor polynomials

Sigma notation for series, which you met in Unit 10, provides a concise way to write down Taylor polynomials. Formula (3) on page 121, for the Taylor polynomial of degree n about 0 for a function f, is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

This polynomial can be written in sigma notation as

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

(Here $f^{(0)}$ is interpreted to mean f itself. Also, by convention, 0^0 is taken to have the value 1 in series of this type. Recall also that $0! = 1$.)

For example, the quartic Taylor polynomial about 0 for the function $f(x) = e^x$, which was found in Example 6, is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 = \sum_{k=0}^{4} \frac{1}{k!}x^k.$$

You can often express particular Taylor polynomials concisely in this way, once the pattern of terms is clear.

2.2 Taylor polynomials for approximation

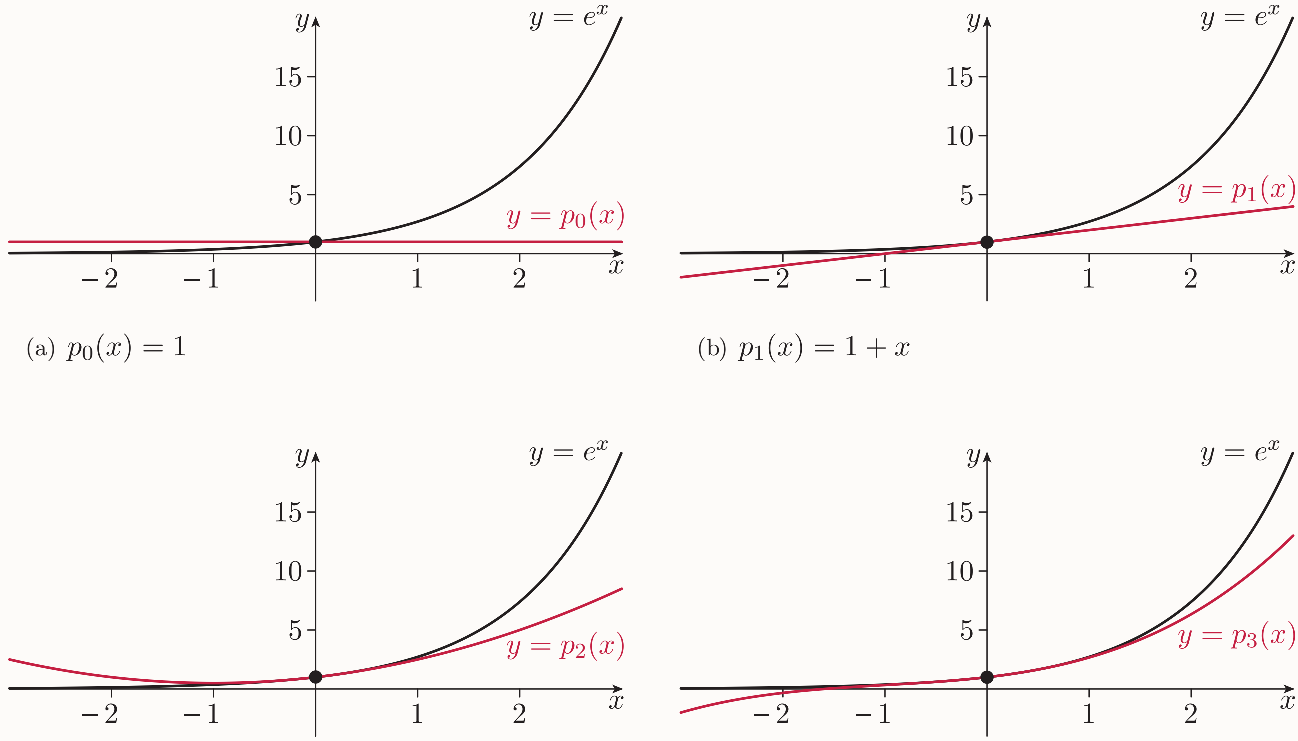
In this subsection we'll use Taylor polynomials to calculate approximations for values of functions at particular points. In doing so, we'll compare approximations obtained from Taylor polynomials of different degrees, and for clarity we need a notation that indicates the degree of each Taylor polynomial. The notation that we use is to denote a Taylor polynomial by p_n, where n is its degree, rather than by just p. Thus, for example, the Taylor polynomials of degrees 1 and 2 about 0 for the function $f(x) = e^x$ are $p_1(x) = 1 + x$ and $p_2(x) = 1 + x + \frac{1}{2}x^2$, respectively.

You've seen that usually the greater the degree of a Taylor polynomial p_n about a point a for a function f, the more accurate $p_n(x)$ is as an approximation to $f(x)$ for values of x close to a.

For example, consider the Taylor polynomials about 0 for the function $f(x) = e^x$. You saw earlier that, for any positive integer n, the Taylor polynomial of degree n about 0 for this function is

$$p_n(x) = 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n.$$

Figure 21 shows the graphs of the Taylor polynomials of degrees 0, 1, 2 and 3 about 0 for the function $f(x) = e^x$, together with the graph of $f(x) = e^x$ itself. As you'd expect, it appears that as the degree of the Taylor polynomial increases, its graph approximates the graph of f near 0 more and more closely.



(a) $p_0(x) = 1$

(b) $p_1(x) = 1 + x$

(c) $p_2(x) = 1 + x + \frac{1}{2!}x^2$

(d) $p_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$

Figure 21 Taylor polynomials about 0 for $f(x) = e^x$

Table 2 provides a numerical illustration for the same function, $f(x) = e^x$, for a particular value of x near 0, namely $x = 0.25$. The value of $e^{0.25}$ is 1.284 025 4167, to ten decimal places. For values of n from 0 to 8, the table gives the Taylor polynomial $p_n(x)$ about 0 for f, the value of this polynomial when $x = 0.25$, and the associated remainder. All the values are given to ten decimal places. You can see that as the degree n of the Taylor polynomial increases, the accuracy of $p_n(0.25)$ as an approximation for $e^{0.25}$ improves.

Table 2 Successive Taylor polynomial approximations for $e^{0.25}$

n	$p_n(x)$	$p_n(0.25)$	$e^{0.25} - p_n(0.25)$
0	1	1	0.284 025 4167
1	$1 + x$	1.25	0.034 025 4167
2	$1 + x + x^2/2!$	1.281 25	0.002 775 4167
3	$1 + x + x^2/2! + x^3/3!$	1.283 854 1667	0.000 171 2500
4	$1 + x + x^2/2! + \cdots + x^4/4!$	1.284 016 9271	0.000 008 4896
5	$1 + x + x^2/2! + \cdots + x^5/5!$	1.284 025 0651	0.000 000 3516
6	$1 + x + x^2/2! + \cdots + x^6/6!$	1.284 025 4042	0.000 000 0125
7	$1 + x + x^2/2! + \cdots + x^7/7!$	1.284 025 4163	0.000 000 0004
8	$1 + x + x^2/2! + \cdots + x^8/8!$	1.284 025 4167	0.000 000 0000

Table 2 shows that for $f(x) = e^x$ the Taylor polynomial of degree 8 about 0 for f gives a method of calculating the value of $e^{0.25}$ correct to 10 decimal places by using only the standard arithmetical operations of addition, subtraction and multiplication. (Raising to a power is just repeated multiplication.)

In general, Taylor polynomials can often be used to calculate approximations for values of functions to any desired accuracy. If f is a function and x is a particular value in the domain of f, then to find an approximation for $f(x)$ we calculate a Taylor polynomial for f about some suitable point a close to x, and then evaluate it at x using only the standard arithmetical operations.

Unfortunately, there's no easy method for determining a suitable degree for the Taylor polynomial in any individual case. However, there's a 'rule of thumb' that works in many cases, and in particular in most of the cases that you're likely to come across. If you want an approximation accurate to m decimal places, then you calculate approximations using Taylor polynomials of degree 1, 2, 3, and so on, until two successive approximations agree to $m + 2$ decimal places. (You start with degree 1 rather than 0 because constant Taylor polynomials rarely give useful approximations.) This method is illustrated in Example 8 below.

Note that when we say that two numbers agree to a particular number of decimal places, we mean that the values resulting from rounding them to that number of decimal places are equal. Thus, for example, 0.237 and 0.241 agree to two decimal places, since in each case rounding to two decimal places gives 0.24. However, 0.241 and 0.247 don't agree to two decimal places, since rounding to two decimal places gives 0.24 in the first case and 0.25 in the second.

Example 8 *Finding an approximate value of a function*

You saw on page 129 that, for each positive integer n, the Taylor polynomial of degree n about 1 for $f(x) = \ln x$ is

$$p_n(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n.$$

Use these Taylor polynomials to find the likely value of $\ln(1.1)$ to four decimal places.

Solution

Calculate $p_1(1.1)$, $p_2(1.1)$, $p_3(1.1)$, and so on, by repeatedly adding on extra terms. You want to find two successive values that agree to $4 + 2 = 6$ decimal places, so calculate values to 6 decimal places.

Calculating these values to six decimal places, we obtain

$$p_1(1.1) = 1.1 - 1 = 0.1$$
$$p_2(1.1) = p_1(1.1) - \tfrac{1}{2}(1.1-1)^2 = 0.095$$
$$p_3(1.1) = p_2(1.1) + \tfrac{1}{3}(1.1-1)^3 = 0.095\,333$$

$$p_4(1.1) = p_3(1.1) - \tfrac{1}{4}(1.1-1)^4 = 0.095\,308$$
$$p_5(1.1) = p_4(1.1) + \tfrac{1}{5}(1.1-1)^5 = 0.095\,310$$
$$p_6(1.1) = p_5(1.1) - \tfrac{1}{6}(1.1-1)^6 = 0.095\,310.$$

The values of $p_5(1.1)$ and $p_6(1.1)$ agree to six decimal places, so it is likely that

$$\ln(1.1) = 0.0953$$

to four decimal places.

You can check using your calculator that it is indeed true that $\ln(1.1) = 0.0953$ to four decimal places, as obtained in Example 8.

In Example 8, each successive approximation $p_n(1.1)$ was calculated by evaluating just the final term of $p_n(x)$ with $x = 1.1$, and then adding this value to $p_{n-1}(1.1)$, the previous approximation. This is an efficient way to proceed, but when working through a similar example yourself, you must make sure that each time you add an evaluated term to the previous approximation, you use the full-calculator-precision version of the previous approximation, rather than the rounded version that you just wrote down. Not doing so will cause errors in some cases.

If you have a modern calculator, then you should be able to carry out this procedure without having to write down the unrounded values. Each time you want to add a new term, you can access the previous answer and calculate and add the new term, all in one step. After each such addition you can round off the approximation and write it down.

Activity 13 *Finding an approximate value of a function*

You saw on page 123 that, for each positive integer n, the Taylor polynomial of degree n about 0 for $f(x) = e^x$ is

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Use these Taylor polynomials to find the likely value of $e^{-0.05}$ to four decimal places.

The next example is similar to Example 8 and Activity 13, but it involves Taylor polynomials about 0 for the sine function. Since this function is odd, its Taylor polynomials about 0 contain no even powers of x, as explained on page 126. Hence each Taylor polynomial of even degree is the same as the Taylor polynomial of degree one less; that is, $p_2(x) = p_1(x)$, $p_4(x) = p_3(x)$, and so on. You would therefore rapidly find two successive approximations that agree to any specified number of decimal places, but this would tell you nothing about the accuracy of the approximation! For this reason it makes sense to consider only the Taylor polynomials of odd degree for this function.

Example 9 *Finding an approximate value of an odd function*

The Taylor polynomials of odd degree about 0 for $f(x) = \sin x$ were discussed on page 127. They are

$$p_1(x) = x, \qquad p_3(x) = x - \frac{1}{3!}x^3, \qquad p_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$p_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7, \qquad \text{and so on.}$$

Use these Taylor polynomials to find the likely value of $\sin(0.2)$ to six decimal places.

Solution

Calculate $p_1(0.2)$, $p_3(0.2)$, $p_5(0.2)$, and so on. You want to find a pair of successive values that agree to $6 + 2 = 8$ decimal places, so calculate values to 8 decimal places. Remember that each successive polynomial is obtained by adding a new term to the previous polynomial.

Calculating values to eight decimal places, we obtain

$$p_1(0.2) = 0.2$$

$$p_3(0.2) = p_1(0.2) - \frac{1}{3!}(0.2)^3 = 0.198\,666\,67$$

$$p_5(0.2) = p_3(0.2) + \frac{1}{5!}(0.2)^5 = 0.198\,669\,33$$

$$p_7(0.2) = p_5(0.2) - \frac{1}{7!}(0.2)^7 = 0.198\,669\,33.$$

The values of $p_5(0.2)$ and $p_7(0.2)$ agree to eight decimal places, so it is likely that

$$\sin(0.2) = 0.198\,669$$

to six decimal places.

You can check using your calculator that it is indeed true that $\sin(0.2) = 0.198\,669$ to six decimal places, as obtained in the solution to Example 8.

Activity 14 *Finding an approximate value of an even function*

The Taylor polynomials of even degree about 0 for $f(x) = \cos x$ were discussed on page 127. They are

$$p_0(x) = 1, \qquad p_2(x) = 1 - \frac{1}{2!}x^2, \qquad p_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

$$p_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6, \qquad \text{and so on.}$$

Use these Taylor polynomials to find the likely value of $\cos(0.2)$ to six decimal places.

In the final activity of this section you can use an applet to see the graphs of a variety of functions and their Taylor polynomials.

3 Taylor series

In Section 2 you saw that, usually, the greater the degree of a Taylor polynomial about a point a for a function f, the more accurate the Taylor polynomial is as an approximating polynomial for f close to a. But what happens if we take a Taylor polynomial of 'infinite degree'; that is, if we add on all possible terms? We'll look at that in this section.

3.1 What is a Taylor series?

In Activity 10 on page 124 you saw that the Taylor polynomial of degree n about 0 for the function $f(x) = 1/(1 - x)$ is

$$p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n.$$

Let's now consider what happens to this expression if we include all possible terms. The result is an infinite series,

$$1 + x + x^2 + x^3 + \cdots . \tag{4}$$

From what you saw in Section 2, you'd expect that for any value of x close to 0, as you add on more and more terms to the infinite series above, the resulting sums will approach the value of $1/(1 - x)$ more and more closely.

As you saw in Unit 10, this is the same as saying that for any value of x close to 0, the infinite series has a *sum*, and the sum is given by $1/(1 - x)$.

But how close to 0 does x have to be for this to happen? The answer is that x must be in the range $-1 < x < 1$. For any value of x in this range, the infinite series has sum $1/(1 - x)$. For any other value of x the series doesn't have a sum at all.

These facts follow from a result that you met in Unit 10. Consider the infinite geometric series with first term a and common ratio r:

$$a + ar + ar^2 + ar^3 + \cdots .$$

You saw in Subsection 4.2 of Unit 10 that this series has

$$\begin{aligned} &\text{sum } \frac{a}{1-r}, && \text{if } -1 < r < 1; \\ &\text{no sum}, && \text{if } r \leq -1 \text{ or } r \geq 1. \end{aligned}$$

The infinite series (4) above is an infinite geometric series, with first term 1 and common ratio x. Hence it has

$$\begin{aligned} &\text{sum } \frac{1}{1-x}, && \text{if } -1 < x < 1; \\ &\text{no sum}, && \text{if } x \leq -1 \text{ or } x \geq 1, \end{aligned}$$

as stated above.

For example, if $x = 0.5$ then series (4) is

$$1 + 0.5 + (0.5)^2 + (0.5)^3 + \cdots = 1 + 0.5 + 0.25 + 0.125 + \cdots ,$$

and it has sum

$$\frac{1}{1-0.5} = 2.$$

In other words, for this infinite series, as more and more terms are added the resulting sum approaches $1/(1 - 0.5)$ more and more closely.

By contrast, if $x = 2$ then the function $f(x) = 1/(1 - x)$ has value $f(2) = 1/(1 - 2) = -1$, but series (4) is

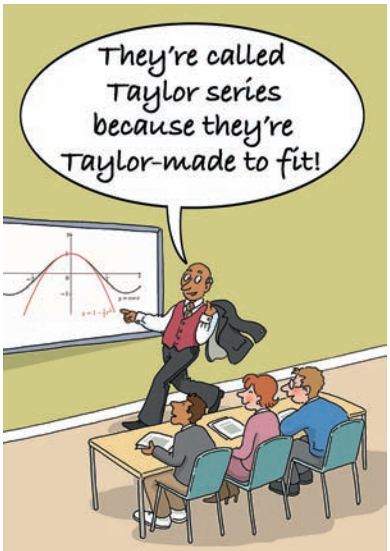
$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = 1 + 2 + 4 + 8 + 16 + \cdots ,$$

and this infinite series has no sum. As more and more terms are added, the resulting partial sums get larger and larger, without approaching any particular value.

In general, if f is a function that's differentiable infinitely many times at a point a in its domain, then you can form an infinite series in which, for any integer $n \geq 0$, the first $n + 1$ terms form the Taylor polynomial of degree n about a for f. This series is called the **Taylor series about a for f**.

For example, series (4) is the Taylor series about 0 for the function $f(x) = 1/(1 - x)$.

You can obtain formulas for Taylor series from those for Taylor polynomials of degree n, by taking infinitely many terms. The following general formulas for Taylor series follow from formulas (2) and (3) on page 121.



Taylor series about a

Let f be a function that is differentiable infinitely many times at a point a. The **Taylor series about a for f** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \quad (5)$$

The point a is called the **centre** of the Taylor series.

When $a = 0$, the Taylor series becomes

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \quad (6)$$

This series is also known as the **Maclaurin series** for f.

In this module, we usually refer to Taylor series about 0, rather than Maclaurin series.

Notice that in the box above the general term of each series, involving $(x-a)^n$ (or x^n, in the case $a = 0$), has been written down explicitly as part of the series. This helps to clarify the general pattern.


Colin Maclaurin (1698–1746)

Colin Maclaurin was born in Argyllshire, studied at the University of Glasgow and became professor of mathematics first at Marischal College, Aberdeen and then at the University of Edinburgh. In 1742 he published the two-volume *Treatise of Fluxions*, which was the first systematic exposition of Newton's methods in calculus. He wrote it as a reply to attacks made on calculus for its lack of rigorous foundations. In this treatise Maclaurin uses Taylor series about 0. Although he acknowledged Taylor, Maclaurin's name is now often used to describe these series.

Example 10 *Finding a Taylor series about 0*

Find the Taylor series about 0 for the function $f(x) = e^x$.

Solution

Repeatedly differentiate f to find f', f'', $f^{(3)}$, ..., and find the values of $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, ... Then apply formula (6).

Here $f(0) = e^0 = 1$. Also, the nth derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$ for all positive integers n.

Hence, by the formula for a Taylor series about 0, the required Taylor series is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots .$$

In the next activity you're asked to find the Taylor series about 0 for the cosine and sine functions.

Activity 15 *Finding Taylor series*

Find the Taylor series about 0 for each of the following functions, writing down enough terms to make the general pattern clear.

(a) $f(x) = \cos x$ (b) $f(x) = \sin x$

In each case you should be able to see a pattern in the values $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, ..., from your working for Activity 9.

It follows from what you saw earlier about Taylor polynomials for even and odd functions that the Taylor series about 0 for an even function contains terms in even powers of x only, and the Taylor series about 0 for an odd function contains terms in odd powers of x only. The Taylor series in Activity 15 are examples of this fact. Remember that the cosine function is an even function and the sine function is an odd function.

All the Taylor series that you've seen so far in this section have had centre 0. In the next activity you're asked to find a Taylor series with a different centre. So here you need to use formula (5), for a Taylor series about a point a, rather than formula (6), for a Taylor series about 0.

Activity 16 *Finding a Taylor series about a point other than 0*

Find the Taylor series about $\pi/2$ for the function $f(x) = \sin x$, writing down enough terms to make the general pattern clear.

Validity of Taylor series

You've seen that the Taylor series about a point a for a function f usually has sum $f(x)$ for values of x close to a, but may not have sum $f(x)$ for other values of x. If x is a point for which the Taylor series about a for f has sum $f(x)$, then we say that the Taylor series is **valid** at the point x.

For example, you saw at the beginning of this subsection that the Taylor series about 0 for the function $f(x) = 1/(1-x)$ is valid for all x in the interval $-1 < x < 1$, but isn't valid for values of x outside this interval.

You may be surprised to learn that the Taylor series about 0 for the exponential, sine and cosine functions (found in Example 10 and Activity 15) are all valid for *every* real number x. In other words, the equations

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots,$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots,$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots,$$

are true for all $x \in \mathbb{R}$.

When you remember that the coefficients of a Taylor series for a function are chosen by taking into account the value of the function and its derivatives at a *single* point a, it may seem amazing that the resulting series can turn out to be equal to the function for *every* real number x!

A Taylor series about a point a is always valid for $x = a$. This is because if we set $x = a$ in formula (5) on page 138, then all the terms except the first are equal to zero, so the sum of the series is just the first term $f(a)$, which is precisely the value of f at a. This is by design, since the first term of a Taylor polynomial is chosen to be $f(a)$ to ensure that the value of the polynomial at a is the same as the value of f at a.

Often, though not always, the largest set of points for which a Taylor series about a point a for a function is valid is either the whole set of real numbers $\mathbb{R}$, or an interval with two endpoints whose midpoint is a. Each endpoint may or may not be included in the interval.

Any interval of values of x for which a Taylor series is valid is called an **interval of validity** for the series, and the series is said to **represent** the function on any interval of validity. For example, $-1 < x < 1$ is an interval of validity for the Taylor series about 0 for the function $f(x) = 1/(1-x)$. Any interval that is contained within the interval $-1 < x < 1$, such as $-\frac{1}{2} < x < \frac{1}{2}$, is also an interval of validity for this series, but $-1 < x < 1$ is the largest such interval. This interval could also be denoted by $(-1, 1)$, using the usual notation for an open interval, but the inequality notation is more usual in the context of Taylor series, and it will be used throughout this unit.

You might wonder how it can be determined that $\mathbb{R}$ is an interval of validity for the Taylor series about 0 for the exponential, sine and cosine functions, and more generally, how the largest interval of validity for a Taylor series can be found. There's a method for doing this, as follows. You first find a formula for the remainder $r_n(x) = f(x) - p_n(x)$, which is the difference between the value of the function f at x and that of the Taylor polynomial of degree n about a for f. Then you have to decide for which values of x this remainder $r_n(x)$ tends to 0 as n tends to ∞. The techniques that you need to do this are taught in more advanced modules on pure mathematics. You won't be expected to find intervals of validity in this module, except in Section 4 where you'll be working from known results.

Using sigma notation for Taylor series

Formula (5) on page 138 for the Taylor series about a for a function f can be written concisely in sigma notation as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

You can often write Taylor series for particular functions concisely in sigma notation in a similar way. For example, the Taylor series about 0 for the function $f(x) = e^x$, given in Example 10, is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

In the cases of odd and even functions, it's more awkward to write down the Taylor series about 0 in sigma notation, because for even functions such series contain only even powers of x, and for odd functions they contain only odd powers of x. The Taylor series for the function $f(x) = \sin x$, given on page 140, is usually written as

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Here, for $n = 0, 1, 2, \ldots$, the expression x^{2n+1} equals x, x^3, x^5, $\ldots$, while the expression $(2n+1)!$ takes the values 1, $3!$, $5!$, $\ldots$, and the expression $(-1)^n$ deals with the alternating signs.

The Taylor series for the function $f(x) = \cos x$, given on page 140, is usually written as

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.$$

Here, for $n = 0, 1, 2, \ldots$, the expression x^{2n} equals 1, x^2, x^4, $\ldots$, while the expression $(2n)!$ takes the values $0! = 1$, $2!$, $4!$, $\ldots$, and again the expression $(-1)^n$ deals with the alternating signs.

Taylor series about 0 for other odd and even functions can be written in sigma notation in a similar way.

3.2 Some standard Taylor series

In Subsection 3.1 we obtained the Taylor series about 0 for the exponential, sine and cosine functions, and for the function $f(x) = 1/(1-x)$. For ease of reference, these series are stated in the following box, along with the Taylor series about 0 for two other standard functions. These Taylor series can be obtained by using the second formula in the box on page 138. The following box also gives intervals of validity for the Taylor series.

Standard Taylor series about 0

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots, \quad \text{for } x \in \mathbb{R}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots, \quad \text{for } x \in \mathbb{R}$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for } -1 < x < 1$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots,$$
$$\text{for } -1 < x < 1 \text{ (where } \alpha \text{ is any real number)}$$

The last series here is called the **binomial series**. You'll see shortly how it's linked to the binomial theorem, which you met in Unit 10.

The binomial series was discovered by Isaac Newton.

Notice that each of the intervals of validity given in the box is an open interval whose midpoint is the centre 0 of the series. These intervals of validity are the largest intervals for which the series are valid, with two exceptions. The series for $\ln(1+x)$ is also valid when $x = 1$, but the box gives the interval $-1 < x < 1$ because it's often convenient to work with open intervals. For example, this is the case when you're differentiating and integrating Taylor series, as you'll see later in the unit.

The other exception involves the binomial series. For $-1 < x < 1$, this series sums to $(1+x)^\alpha$ for any real number α, including negative and fractional numbers. For most values of α, the largest interval of validity is $-1 < x < 1$, but when α is a non-negative integer, the series is valid for *every* real number x. You'll see after the next example why this is so.

The box gives a Taylor series about 0 for the function $\ln(1+x)$, rather than for the standard function $\ln x$. This is because the function $\ln x$ has no Taylor series about 0, since its domain $(0, \infty)$ does not contain 0.

Note that it's often convenient to say 'the function $\ln(1+x)$', as in the paragraph above, as a shorthand for a more precise statement such as 'the function $f(x) = \ln(1+x)$'.

Example 11 *Using the binomial series*

Use the binomial series to find the Taylor series about 0 for each of the following functions. In each case state an interval of validity for the series.

(a) $f(x) = \dfrac{1}{1+x}$ (b) $f(x) = (1+x)^4$

Solution

(a) Since $1/(1+x) = (1+x)^{-1}$, we take $\alpha = -1$ in the binomial series, to give

$$\begin{aligned}\frac{1}{1+x} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + x^4 - \cdots .\end{aligned}$$

When writing down an infinite series, remember to include either $+\cdots$ or $-\cdots$ at the end, to indicate that there are further terms.

This Taylor series is valid for $-1 < x < 1$.

(b) Taking $\alpha = 4$ in the binomial series gives

$$\begin{aligned}(1+x)^4 &= 1 + 4x + \frac{4\times 3}{2!}x^2 + \frac{4\times 3\times 2}{3!}x^3 \\ &\quad + \frac{4\times 3\times 2\times 1}{4!}x^4 + \frac{4\times 3\times 2\times 1\times 0}{5!}x^5 \\ &\quad + \frac{4\times 3\times 2\times 1\times 0\times(-1)}{6!}x^6 + \cdots \\ &= 1 + 4x + 6x^2 + 4x^3 + x^4.\end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

In fact this Taylor series is valid for $x \in \mathbb{R}$, which would be an equally appropriate answer. (See the discussion below.)

Example 11(b) illustrates that it's possible for a Taylor series to have a *finite* number of terms. This occurs when the coefficients of all terms from some term onwards are zero. The series in this example may also look familiar to you: it is the binomial expansion of $(1+x)^4$.

If α is any positive integer, then all terms after the term in x^α in the binomial Taylor series for $(1+x)^\alpha$ contain the factor $\alpha - \alpha = 0$, and are therefore equal to 0. The series is then the same as the binomial expansion of $(1+x)^\alpha$, which is valid for all $x \in \mathbb{R}$. The binomial series therefore generalises the binomial expansion of $(1+x)^\alpha$ from cases where α is a positive integer to cases where α can be any real number.

Activity 17 *Using the binomial series*

Use the binomial series to find the Taylor series about 0 for the function $f(x) = 1/(1+x)^2$. (Write down enough terms to make the general pattern clear.) State an interval of validity for the series.

Activity 17 involved using the binomial series when the power of $1+x$ is a negative integer. The next example illustrates using the binomial series when the power of $1+x$ is a fraction.

Example 12 *Finding a binomial series for a fractional power of $1+x$*

Use the binomial series to find the Taylor series about 0 for the function $f(x) = (1+x)^{-1/2}$. (Write down enough terms to make the general pattern clear.) State an interval of validity for the series.

Solution

Taking $\alpha = -\frac{1}{2}$ in the binomial series gives

$$\begin{aligned}(1+x)^{-1/2} &= 1 + (-\tfrac{1}{2})x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 \\ &\qquad + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}x^4 + \cdots \\ &= 1 - \tfrac{1}{2}x + \frac{1\times 3}{2^2\times 2!}x^2 - \frac{1\times 3\times 5}{2^3\times 3!}x^3 \\ &\qquad + \frac{1\times 3\times 5\times 7}{2^4\times 4!}x^4 - \cdots .\end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

In Example 12, the coefficients in the Taylor series have been left in the form shown, rather than completely evaluated, so that the pattern involved is clear. It's usually a good idea to do this, but sometimes you may be asked to evaluate the first few coefficients of a Taylor series explicitly, in which case you should write each coefficient as a single integer or fraction.

Activity 18 *Finding a binomial series for a fractional power of $1+x$*

Use the binomial series to find the Taylor series about 0 for the function $f(x) = (1+x)^{1/2}$. (Write down enough terms to make the general pattern clear.) State an interval of validity for the series.

Once you know the Taylor series for a function f about a point a, you can find the corresponding Taylor polynomial of any degree n by truncating the series at the appropriate term. (To *truncate* a series at a term is to delete all subsequent terms.)

Activity 19 *Using a Taylor series to find a Taylor polynomial*

You saw earlier that the Taylor series about 0 for the function $f(x) = \ln(1+x)$ is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1.$$

Using this series, write down the cubic Taylor polynomial about 0 for this function f.

The Taylor polynomials obtained by truncating a Taylor series for a function f can, in principle, be used to find approximations for $f(x)$ for all values of x for which the series is valid. For example, you've seen that the Taylor series about 0 for the function $f(x) = \ln(1+x)$ is valid for all values of x in the interval $-1 < x < 1$. This means that, in principle, you can use Taylor polynomials about 0 to find an approximation for $\ln(1+x)$ for any value of x in this interval. However, the further x is from the centre 0 of the Taylor series, the greater is the degree of the Taylor polynomial that you need to provide the desired level of accuracy.

For instance, you can find an approximation for $\ln(1.1)$ by putting $x = 0.1$ in a Taylor polynomial about 0 for $\ln(1+x)$, and you can find an approximation for $\ln(1.5)$ by putting $x = 0.5$ in the same Taylor polynomial. However, to find an approximation for $\ln(1.1)$ correct to four decimal places, by using the method of Subsection 2.2 and obtaining the required Taylor polynomials by truncating the Taylor series at the appropriate terms, you have to evaluate six successive Taylor polynomials, whereas to find an approximation for $\ln(1.5)$ to the same level of accuracy you have to evaluate 17 successive Taylor polynomials. For $\ln(1.9)$ you need 92 successive Taylor polynomials!

In the next activity you're asked to use the method of Subsection 2.2 to find an approximation for a particular value $f(x)$ of a function f, obtaining the required Taylor polynomials by truncating the Taylor series for f about a point a close to x.

Activity 20 *Finding an approximate value for a function*

You saw in Activity 18 that the Taylor series about 0 for the function $f(x) = (1+x)^{1/2}$ is

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \frac{1}{2^2 \times 2!}\,x^2 + \frac{1 \times 3}{2^3 \times 3!}\,x^3 - \frac{1 \times 3 \times 5}{2^4 \times 4!}\,x^4 + \cdots,$$

and that it is valid for $-1 < x < 1$. By writing 1.1 as $1 + 0.1$, use this series to find the value of $\sqrt{1.1}$ to three decimal places.

(Notice that $x = 0.1$ lies within the interval of validity $-1 < x < 1$ for the Taylor series.)

3.3 Using a computer to find Taylor polynomials

In the following activity you'll learn how to use the module computer algebra system to find Taylor polynomials.

Activity 21 *Taylor polynomials on a computer*

Work through Section 12 of the *Computer algebra guide*.

4 Manipulating Taylor series

In this final section, you'll see some methods that allow you to obtain Taylor series for many functions from a few known Taylor series such as the standard ones in the box on page 142. This usually involves much less work than obtaining the required Taylor series by using one of the general formulas in the box on page 138.

When finding a Taylor series for a function, you can use any of the standard Taylor series. You're not expected to derive any of the standard series unless explicitly asked to do so.

4.1 Substituting for the variable in a Taylor series

You've seen that the Taylor series about 0 for the function $g(x) = 1/(1-x)$ is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Consider the effect of substituting $x = 2t$ in this equation. This gives

$$\begin{aligned}\frac{1}{1-2t} &= 1 + 2t + (2t)^2 + (2t)^3 + \cdots \\ &= 1 + 2t + 4t^2 + 8t^3 + \cdots .\end{aligned}$$

We have obtained a series equal to $1/(1-2t)$. Since the Taylor series for $g(x) = 1/(1-x)$ is valid for $-1 < x < 1$, the series in t above is equal to $1/(1-2t)$ for $-1 < 2t < 1$, that is, for $-\frac{1}{2} < t < \frac{1}{2}$.

Let's now replace t by x, since it's more usual to use x rather than t for the variable. This gives

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots , \quad \text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}. \tag{7}$$

The series in equation (7) is equal to the function $f(x) = 1/(1-2x)$ for $-\frac{1}{2} < x < \frac{1}{2}$, but is it a *Taylor series*? It's of the right form to be a Taylor series about 0, since each of its terms is a power of x multiplied by a constant. However, if we were to use formula (6) on page 138 to find the Taylor series about 0 for the function f, would we obtain the same series? The answer to this question is yes. This follows from the following fact, whose proof is beyond the scope of this module.

Uniqueness of Taylor series

Let f be a function. If you can by any means find a series

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

that is equal to $f(x)$ for all x in some open interval containing a, then this series is the Taylor series about a for f, and hence it is the *only* series of this form that is equal to $f(x)$ for all x in that interval.

You can assume this important fact throughout the rest of this unit.

You can find Taylor series for many functions by substituting for the variable in a Taylor series that you already know, and you can often deduce an interval of validity for the new series from an interval of validity for the original Taylor series. When substituting for the variable, it's quicker to avoid introducing a new variable t as was done above, and instead replace x in the original Taylor series by an expression involving x, as illustrated in the next example.

Example 13 *Substituting into a Taylor series*

Find the Taylor series about 0 for the function

$$f(x) = \frac{1}{1+x^2},$$

and determine an interval of validity for this series.

Solution

The expression $1/(1+x^2)$ is similar to $1/(1-x)$, for which there's a standard Taylor series (given on page 142). The first expression is obtained from the second expression by replacing x by $-x^2$.

The Taylor series about 0 for $1/(1-x)$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Replacing each occurrence of x by $-x^2$ gives

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots; \tag{8}$$

that is,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots.$$

This is the Taylor series about 0 for $1/(1+x^2)$.

Deduce an interval of validity for the new series from the interval of validity for the original series.

The Taylor series for $1/(1-x)$ is valid for $-1 < x < 1$, so the series for $1/(1-(-x^2))$ is valid for $-1 < -x^2 < 1$.

The double inequality $-1 < -x^2 < 1$ is equivalent to the two single inequalities $-1 < -x^2$ and $-x^2 < 1$.

The left-hand inequality is $-1 < -x^2$, which is equivalent to $1 > x^2$; that is, $-1 < x < 1$.

The right-hand inequality is $-x^2 < 1$, which is equivalent to $x^2 > -1$ and therefore does not place any restriction on x, since the square of any real number is non-negative.

Thus the Taylor series about 0 for $1/(1+x^2)$ is valid for $-1 < x < 1$.

In Example 13, the variable x was replaced by $-x^2$. This is equivalent to making the substitution $x = -t^2$ and then replacing t by x.

As illustrated in Example 13, when you deduce an interval of validity for a Taylor series from an interval of validity for another Taylor series, you usually have to rearrange inequalities. Rules for rearranging inequalities were given in Subsection 5.2 of Unit 3.

When you have to rearrange a double inequality like $-1 < -x^2 < 1$, it's often helpful to split it into two single inequalities and rearrange each independently, as was done in Example 13. With simple double inequalities, such as $-1 < 2x < 1$, you may be able to rearrange both single inequalities together; for example, in this case we simply multiply both inequalities by $\frac{1}{2}$ to obtain $-\frac{1}{2} < x < \frac{1}{2}$.

When you replace each occurrence of x in a Taylor series by an expression involving x, it's helpful to enclose the whole expression in brackets at each replacement, and then simplify the resulting terms, as illustrated in Example 13. Make sure that you enclose the *whole* expression in brackets. For instance, in equation (8) the third term is $(-x^2)^2 = x^4$, not $-(x^2)^2 = -x^4$.

Activity 22 *Substituting into a Taylor series*

By substituting for the variable in a standard Taylor series, find the Taylor series about 0 for each of the following functions. In each case determine an interval of validity for the series.

(a) $f(x) = 1/(1+2x)$ (b) $f(x) = \ln(1-x)$ (c) $f(x) = \ln(1+3x)$

(d) $f(x) = e^{x^3}$

You've seen that substituting for the variable in a Taylor series gives a Taylor series for another function. In each case so far, the new series has had the same centre as the original series. However, some substitutions lead to a new Taylor series with a different centre. For example, suppose that the Taylor series about 0 for a function g is

$$g(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots .$$

If you replace each occurrence of x by $x - a$, then you obtain

$$g(x-a) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

This is the Taylor series about a for the function f given by $f(x) = g(x-a)$. You're asked to use this fact in the following activity.

Activity 23 *Changing the centre of a Taylor series*

In Example 11(a), the binomial series was used to show that the Taylor series about 0 for $1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots , \quad \text{for } -1 < x < 1.$$

By replacing x by $x-1$ in this series, find the Taylor series about 1 for the function $f(x) = 1/x$. Determine an interval of validity for this series.

Remember that you can be sure that the series found in the solution to Activity 23 is indeed the Taylor series about 1 for $1/x$, because of the fact about the uniqueness of Taylor series in the box on page 147.

Sometimes you can find a Taylor series for a particular function by rewriting its rule to make it more similar to a function whose Taylor series you already know, and then replacing the variable x by a suitable expression in x. This is demonstrated in the following activity.

Activity 24 *Rearranging in order to find a Taylor series*

(a) Find the Taylor series about 0 for the function

$$f(x) = \frac{3}{3+2x},$$

by writing

$$f(x) = \frac{3}{3+2x} = \frac{1}{1+\frac{2}{3}x}.$$

Determine an interval of validity for this series.

(b) Find the Taylor series about -1 for the function

$$f(x) = \frac{1}{3+2x},$$

by writing

$$f(x) = \frac{1}{3+2x} = \frac{1}{1+2(x+1)}.$$

Determine an interval of validity for this series.

4.2 Adding, subtracting and multiplying Taylor series

Another way to find Taylor series for some functions is to apply standard arithmetical operations to known Taylor series, doing this term by term, as illustrated in the following example.

Example 14 *Adding Taylor series*

Find the Taylor series about 0 for the function

$$f(x) = e^x + \frac{1}{1-x},$$

explicitly evaluating the coefficients of the first five terms. Determine an interval of validity for this series.

Solution

There are standard Taylor series about 0 for e^x and for $1/(1-x)$ (given on page 142). Adding these two series together term by term will give the required Taylor series.

We have the following standard series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad \text{for } -1 < x < 1.$$

It follows that

$$\begin{aligned} e^x + \frac{1}{1-x} &= \left(1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \frac{1}{4!}\,x^4 + \cdots\right) \\ &\quad + \left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\ &= (1+1) + (1+1)x + \left(\frac{1}{2!} + 1\right)x^2 \\ &\quad + \left(\frac{1}{3!} + 1\right)x^3 + \left(\frac{1}{4!} + 1\right)x^4 + \cdots \\ &= 2 + 2x + \tfrac{3}{2}\,x^2 + \tfrac{7}{6}\,x^3 + \tfrac{25}{24}x^4 + \cdots . \end{aligned}$$

This is the Taylor series about 0 for the function $f(x) = e^x + 1/(1-x)$.

Deduce an interval of validity for the new Taylor series from the intervals of validity for the original Taylor series.

The series for e^x is valid for $x \in \mathbb{R}$, and the series for $f(x) = 1/(1-x)$ is valid for $-1 < x < 1$. The second interval is contained within the first, so both series are valid for $-1 < x < 1$. Hence $-1 < x < 1$ is an interval of validity for the Taylor series for $f(x) = e^x + 1/(1-x)$.

You can add or subtract any two Taylor series with the same centre term by term in the way demonstrated in Example 14. The resulting Taylor series is valid for all values of x for which *both* original Taylor series are valid, and possibly for a larger interval of values.

Activity 25 *Subtracting Taylor series*

The following standard Taylor series about 0 was stated earlier (page 142):

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots, \quad \text{for } -1 < x < 1.$$

In the solution to Activity 22(b) this series is used to deduce the following Taylor series about 0:

$$\ln(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots, \quad \text{for } -1 < x < 1.$$

By using the fact that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x),$$

use the two series above to find the Taylor series about 0 for

$$\ln\left(\frac{1+x}{1-x}\right).$$

Determine an interval of validity for this series.

The result from Activity 25 turns out to be useful for finding an approximation for the natural logarithm of any positive number. Any positive number t can be expressed in the form $t = (1+x)/(1-x)$ for some number x in the interval $-1 < x < 1$. For example, $3 = (1+\frac{1}{2})/(1-\frac{1}{2})$. You can see that this is possible for every positive number t by rearranging the equation above to make x the subject:

$$\begin{aligned} t &= \frac{1+x}{1-x} \qquad (x \neq 1) \\ (1-x)t &= 1+x \\ t - xt &= 1+x \\ t-1 &= xt+x \\ t-1 &= x(t+1) \\ x &= \frac{t-1}{t+1}. \end{aligned}$$

This final equation gives the value of x corresponding to a positive number t. For example, if $t = 3$, then $x = (3-1)/(3+1) = \frac{1}{2}$.

To see that the value of x always turns out to lie in the interval $-1 < x < 1$, note that the expression on the right-hand side of the final equation above can be rearranged as follows:

$$x = \frac{t-1}{t+1} = \frac{(t+1)-2}{t+1} = 1 - \frac{2}{t+1}.$$

Since t is positive, the value of $2/(t+1)$ is positive, and hence $x < 1$. Also, again since t is positive, the value of $2/(t+1)$ is less than 2, and hence $x > -1$.

Thus in principle you can use the Taylor series found in Activity 25 to find an approximation for $\ln t$ for any number t in the domain $(0, \infty)$ of the function ln.

In contrast, you can use the series for $\ln(1+x)$ to find an approximation for $\ln t$ only for $0 < t \leq 2$, since these are the only values of t that can be expressed in the form $t = 1 + x$ for some x in the interval $-1 < x \leq 1$ (see page 145).

For both series, the further x is from 0, the more terms of the series you have to evaluate to obtain the desired level of accuracy. However, to find the value of $\ln t$ for a number t for which you could use either series, you usually have to evaluate fewer terms of the series found in Activity 25 than of the series for $\ln(1+x)$ to obtain the desired level of accuracy.

Activity 26 *Finding the value of a particular logarithm*

Find the value of $x = (t-1)/(t+1)$ that corresponds to $t = 1.5$. Hence use the Taylor series found in Activity 25, namely

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \tfrac{2}{7}x^7 + \cdots,$$

to find the value of $\ln(1.5)$ to four decimal places.

In Activity 26, it was necessary to evaluate five successive Taylor polynomials in order to find the value of $\ln(1.5)$ correct to four decimal places. If you used the series for $\ln(1+x)$ for this task, with $x = 0.5$, then you would need to evaluate 17 successive Taylor polynomials, as mentioned earlier (page 145). This illustrates the comment above that you need to use fewer terms of the series found in Activity 25.

You've seen that you can add and subtract Taylor series. You can also multiply a Taylor series term by term by a non-zero constant. The resulting series is valid for every value of x for which the original Taylor series is valid.

For example, you can multiply the Taylor series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R},$$

by 3, to deduce that

$$3e^x = 3 + 3x + \frac{3}{2!}x^2 + \frac{3}{3!}x^3 + \frac{3}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

In the next activity you're asked to use the technique of multiplying a Taylor series by a constant, together with substitution, to find the Taylor series about 0 for the function $1/(3+x)^2$. Here it's useful to use the rearrangement

$$\frac{1}{(3+x)^2} = \frac{1}{(3(1+\frac{1}{3}x))^2} = \frac{1}{3^2(1+\frac{1}{3}x)^2}.$$

Activity 27 *Finding a Taylor series for* $1/(3+x)^2$

You were asked to show in Activity 17 that the Taylor series about 0 for the function $1/(1+x)^2$ is

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Use this Taylor series, together with the fact that

$$\frac{1}{(3+x)^2} = \frac{1}{3^2} \times \frac{1}{(1+\frac{1}{3}x)^2},$$

to find the Taylor series about 0 for $1/(3+x)^2$, and determine an interval of validity for this series.

You can find the Taylor series about 0 for any function of the form $(c+x)^\alpha$, where c is any positive constant and α is any real number, by deducing it from the series for $(1+x)^\alpha$ using the method of Activity 27. That is, you first express $(c+x)^\alpha$ as $c^\alpha(1+x/c)^\alpha$.

The next example and activity involve using the techniques that you've met in this section to find the Taylor series about 0 for two further standard mathematical functions. The function sinh (usually pronounced as 'shine' or 'sine-sh') is the **hyperbolic sine function**, and is given by

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

The function cosh (pronounced just as 'cosh') is the **hyperbolic cosine function**, and is given by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

The graphs of these two functions are shown in Figure 22.

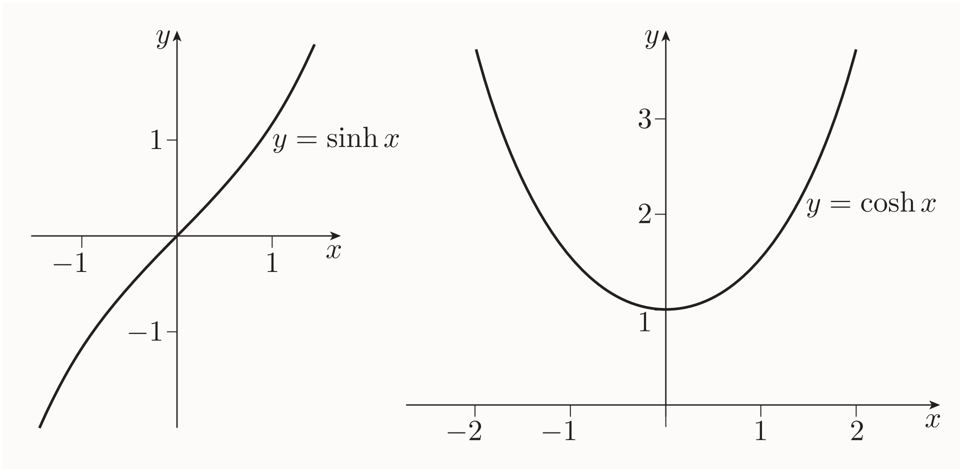


Figure 22 The graphs of the functions sinh and cosh

Although you might not expect it from their definitions or their graphs, the hyperbolic functions sinh and cosh have many properties analogous to those of the trigonometric functions sin and cos. For example, cosh, like cos, is an even function and sinh, like sin, is an odd function. Also, the derivative of sinh is cosh and

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

The two properties immediately above are exact analogues of properties of trigonometric functions. Some properties of sinh and cosh are similar to, but not exactly analogous to, those of sin and cos. For example, for sin and cos we have the trigonometric identity $\sin^2 x + \cos^2 x = 1$, whereas for sinh and cosh we have the identity $\cosh^2 x - \sinh^2 x = 1$. Also, the derivative of cos is $-\sin$, whereas the derivative of cosh is sinh.

You'll learn more about the functions sinh and cosh if you study the module MST125 *Essential mathematics 2*.

You can find the Taylor series about 0 for the functions sinh and cosh by using their definitions, together with the standard Taylor series about 0 for the exponential function. You have to use the techniques of adding and multiplying by a constant, and the technique of substitution. The series for cosh is found in the next example, and you're asked to find the series for sinh in the activity that follows.

Example 15 *Finding the Taylor series about 0 for* $\cosh x$

Find the Taylor series about 0 for the function $f(x) = \cosh x$ and determine an interval of validity for this series.

Solution

We use the formula

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Replacing each occurrence of x by $-x$ gives

$$e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots, \quad \text{for } x \in \mathbb{R}.$$

Therefore

$$\begin{aligned}\tfrac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\Bigg(&\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)\\ &+ \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right)\Bigg), \quad \text{for } x \in \mathbb{R};\end{aligned}$$

that is,

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Activity 28 *Finding the Taylor series about 0 for* $\sinh x$

Find the Taylor series about 0 for the function $f(x) = \sinh x$, and determine an interval of validity for this series.

You can see from the solutions to Example 15 and Activity 28 that the Taylor series about 0 for cosh has only even powers of x, while that for sinh has only odd powers of x. This is because cosh is an even function, while sinh is a odd function.

Notice that the Taylor series about 0 for sinh and cosh are similar to those for sin and cos, respectively. The only difference is that all the coefficients in the series for $\cosh x$ and $\sinh x$ are positive, whereas the coefficients in the series for $\cos x$ and $\sin x$ alternate in sign.

Multiplying Taylor series together

You've seen that Taylor series can be added, subtracted and multiplied by a non-zero constant. You can also multiply together two Taylor series with the same centre. The resulting Taylor series is valid for all values of x for which both original Taylor series are valid, and possibly for a larger interval of values.

The next example illustrates the multiplication of two Taylor series. It involves finding the Taylor series about 0 of the product of a polynomial and another function. Note that it's easy to write down the Taylor series about 0 of a polynomial,

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n,$$

because this is already in the form of the series in the box on page 147 (with $a = 0$, and the coefficients of all terms from x^{n+1} onwards equal to 0). It follows that the Taylor series about 0 of a polynomial is the polynomial itself! It's valid for all $x \in \mathbb{R}$.

Example 16 *Multiplying Taylor series*

Find the Taylor series about 0 for the function

$$f(x) = \frac{1-x}{1+x},$$

and determine an interval of validity for this series.

Solution

The Taylor series for $1 - x$ is $1 - x$, and the Taylor series for $1/(1 + x)$, found in Example 11(a), is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Therefore

$$\begin{aligned}\frac{1-x}{1+x} &= (1-x)(1 - x + x^2 - x^3 + \cdots)\\ &= 1(1 - x + x^2 - x^3 + \cdots) - x(1 - x + x^2 - x^3 + \cdots)\\ &= (1 - x + x^2 - x^3 + \cdots) - (x - x^2 + x^3 - \cdots)\\ &= 1 - 2x + 2x^2 - 2x^3 + \cdots.\end{aligned}$$

The Taylor series about 0 for $1 - x$ and for $1/(1 + x)$ are valid for $x \in \mathbb{R}$ and for $-1 < x < 1$, respectively. Hence the Taylor series for $(1 - x)/(1 + x)$ is valid for $-1 < x < 1$.

In the next activity you're asked to carry out two multiplications of Taylor series. In each case one of the two series is a polynomial.

Activity 29 *Multiplying Taylor series*

Find the Taylor series about 0 for each of the following functions. In each case determine an interval of validity for the series.

(a) $f(x) = x^2 \sin x$ (b) $f(x) = (1 + x)\cos x$

Earlier you saw that you can multiply a Taylor series by a non-zero constant. This is just a special case of the multiplication of two Taylor series. For example, you can obtain the Taylor series about 0 for $3e^x$ by multiplying together the Taylor series about 0 for the constant function $f(x) = 3$ and the Taylor series about 0 for e^x; the Taylor series about 0 for the function $f(x) = 3$ is simply 3, since 3 is a polynomial.

In each case where we've multiplied together two Taylor series, one of the series had only finitely many non-zero terms; that is, it was a polynomial. Multiplying together two Taylor series both of which have infinitely many non-zero terms is usually a difficult task, and you won't be asked to carry out any complete multiplications of this type in this module. However, it's fairly straightforward to multiply together the first few terms of two infinite Taylor series to find the first few terms of the product Taylor series; that is, to find a Taylor polynomial. This is illustrated in the next example.

Example 17 *Finding a Taylor polynomial by multiplication*

Find the cubic Taylor polynomial about 0 for the function $f(x) = e^x \cos x$.

Solution

Multiply the series in the usual way. At each stage, include explicitly only those terms that could eventually result in final terms with power 3 or less. Ignore any terms that can lead only to final terms with power 4 or more, but use the notation '$\cdots$' to indicate that further terms exist.

Using the Taylor series about 0 for e^x and for $\cos x$, we obtain

$$\begin{aligned}
e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \cdots\right) \\
&= 1\left(1 - \frac{x^2}{2!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \cdots\right) \\
&\quad + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \cdots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \cdots\right) + \cdots \\
&= (1 - \tfrac{1}{2}x^2 + \cdots) + (x - \tfrac{1}{2}x^3 + \cdots) + (\tfrac{1}{2}x^2 - \cdots) \\
&\quad + (\tfrac{1}{6}x^3 - \cdots) + \cdots \\
&= 1 + x + (-\tfrac{1}{2} + \tfrac{1}{2})x^2 + (-\tfrac{1}{2} + \tfrac{1}{6})x^3 + \cdots \\
&= 1 + x - \tfrac{1}{3}x^3 + \cdots .
\end{aligned}$$

Therefore the cubic Taylor polynomial about 0 for $f(x) = e^x \cos x$ is

$$p(x) = 1 + x - \tfrac{1}{3}x^3.$$

You can use a similar method in the next activity.

Activity 30 *Finding a Taylor polynomial by multiplication*

In Example 11(a), the binomial series was used to show that the Taylor series about 0 for $1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Use this Taylor series and the Taylor series for $\sin x$ to find the cubic Taylor polynomial about 0 for the function

$$f(x) = \frac{\sin x}{1+x}.$$

4.3 Differentiating and integrating Taylor series

You've seen that the Taylor series about 0 for the function $f(x) = \sin x$ is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

Let's consider the effect of differentiating this series term by term, in the way that we would if it had only finitely many terms and so was a polynomial. We obtain the series

$$1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 - \cdots = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

You may recognise this series as the Taylor series about 0 for $\cos x$. So by differentiating term by term the Taylor series about 0 for $f(x) = \sin x$, we obtained the Taylor series about 0 for its derivative, $f'(x) = \cos x$.

This observation suggests that term-by-term differentiation of a Taylor series about 0 for a function f gives the Taylor series about 0 for its derivative, f'. This conjecture can be verified as follows.

Let f be a function that's differentiable infinitely many times at 0, and let $g = f'$. The Taylor series about 0 for f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \tag{9}$$

Notice here that the general term, in x^n, has been written down explicitly as part of the series.

Differentiating series (9) term by term (keeping in mind that $f(0)$, $f'(0)$, $f''(0)$, ... are constants) gives the series

$$\begin{aligned} &0 + f'(0) + \frac{f''(0)}{2!}2x + \frac{f^{(3)}(0)}{3!}3x^2 + \cdots + \frac{f^{(n)}(0)}{n!}nx^{n-1} + \cdots \\ &= f'(0) + f''(0)x + \frac{f^{(3)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{(n-1)!}x^{n-1} + \cdots. \end{aligned}$$

Since $g = f'$, we have $g(0) = f'(0)$, $g'(0) = f''(0)$, $g''(0) = f^{(3)}(0)$, and so on. Therefore we can write the series above as

$$g(0) + g'(0)x + \frac{g''(0)}{2!}\,x^2 + \cdots + \frac{g^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \cdots .$$

This is the Taylor series about 0 for $g = f'$. (The general term is expressed in terms of $n-1$ instead of n.)

Taylor series can also be integrated term by term. If the Taylor series about 0 for a function f is integrated term by term, then the result is the Taylor series about 0 of an antiderivative of f.

These properties of Taylor series are summarised in the following box. Here c_0 is written for $f(0)$, c_1 for $f'(0)$, c_2 for $f''(0)/2!$, and so on, to simplify the notation.

Differentiating and integrating Taylor series about 0

Let f be a function that is differentiable infinitely many times at 0. If the Taylor series about 0 for f is

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots ,$$

then the Taylor series for f' is

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots + nc_nx^{n-1} + \cdots ,$$

and the Taylor series for any antiderivative of f is of the form

$$c + c_0x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots + \frac{c_n}{n+1}x^{n+1} + \cdots ,$$

where c is a constant.

Any interval of validity for the Taylor series for f that is an *open* interval is also an interval of validity for the Taylor series for f' and for any antiderivative of f.

The results in the box above can be extended to Taylor series with centres other than 0, but we won't need to use such series in this module.

Example 18 *Differentiating and integrating a Taylor series*

In Example 11(a), the binomial series was used to show that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \quad \text{for } -1 < x < 1.$$

Also, by either the chain rule or the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2},$$

and by a standard integral and the rule for integrating a function of a linear expression,

$$\int \frac{1}{1+x}\,\mathrm{d}x = \ln(1+x) + c \quad (x > -1),$$

where c is a constant. Use these facts to find the Taylor series about 0 for each of the following functions. In each case state an interval of validity for the series.

(a) $f(x) = \dfrac{1}{(1+x)^2}$ (b) $f(x) = \ln(1+x)$

Solution

(a) Differentiating the series for $1/(1+x)$ gives

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \cdots, \quad \text{for } -1 < x < 1.$$

Multiplying both sides by -1 gives the required Taylor series,

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

(b) Integrating the series for $1/(1+x)$ gives

$$\begin{aligned}&\ln(1+x)\\ &= c + x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1,\end{aligned}$$

where c is a constant.

To find the value of the constant c, put $x = 0$ in the equation above and solve the resulting equation for c.

Taking $x = 0$ gives $\ln 1 = c$, so $c = 0$. Therefore the required Taylor series is

$$\begin{aligned}&\ln(1+x)\\ &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1.\end{aligned}$$

An alternative way to find the Taylor series in Example 18(a) is to take $\alpha = -2$ in the binomial series, as was done in Activity 17.

The result of Example 18(b) is the standard Taylor series for $\ln(1+x)$, as you'd expect. This shows the connection between this standard series and the series for $1/(1+x)$.

The remaining activities in this subsection require you to differentiate or integrate Taylor series.

Activity 31 *Differentiating a Taylor series*

Verify that term-by-term differentiation of the Taylor series about 0 for the function $f(x) = e^x$ leaves the series unchanged.

(This result corresponds to the fact that the derivative of e^x is e^x.)

Activity 32 *Finding the Taylor series about 0 for* $\tan^{-1} x$

In Example 13 you saw that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots, \quad \text{for } -1 < x < 1.$$

In Unit 7 you saw that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$

Use integration to deduce the Taylor series about 0 for $\tan^{-1} x$, and state an interval of validity for this series.

Calculating π

In 1706 the mathematician John Machin used the Taylor series about 0 for $\tan^{-1}$, which you were asked to find in Activity 32, to calculate the first 100 digits of π. The series is

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots.$$

An interval of validity for this series is $-1 < x < 1$, but the series is also valid for $x = 1$. So, since $\tan^{-1} 1 = \frac{1}{4}\pi$, we obtain a representation for π as four times the sum of a series:

$$\pi = 4\tan^{-1} 1 = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots\right).$$

Unfortunately it's not practicable to use this particular series to calculate π accurately, because 1 is too far from the centre 0 of the series. However, Machin discovered the strange-looking formula

$$\pi = 16\tan^{-1}\left(\tfrac{1}{5}\right) - 4\tan^{-1}\left(\tfrac{1}{239}\right).$$

The values $\frac{1}{5}$ and $\frac{1}{239}$ are much closer to 0 than 1 is, so relatively few terms of the corresponding series need to be evaluated in order to calculate $\tan^{-1}\left(\frac{1}{5}\right)$ and $\tan^{-1}\left(\frac{1}{239}\right)$, and hence π, to 100 digits.

John Machin (1680–1751)

John Machin was at the University of Cambridge at the same time as Brook Taylor, having acted as a private tutor to Taylor beforehand. Machin was later a Fellow of the Royal Society and (for 38 years) Professor of Astronomy at Gresham College, London. He was also a member of the committee to adjudicate the claims of Newton and Leibniz to have invented the calculus.

In the final activity of this unit you're asked to use both substitution and integration to find the first few terms of the Taylor series about 0 for the inverse sine function.

Activity 33 *Finding the Taylor series about 0 for $f(x) = \sin^{-1} x$*

In Example 12, the binomial series was used to show that

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{1\times 3}{2^2\times 2!}\,x^2 - \frac{1\times 3\times 5}{2^3\times 3!}\,x^3 + \cdots;$$

that is,

$$\frac{1}{\sqrt{1+x}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

(a) By using substitution, find the first four terms of the Taylor series about 0 for the function $1/\sqrt{1-x^2}$. Determine an interval of validity for this series.

(b) You saw in Unit 7 that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}.$$

By using this fact and integrating the series in part (a) term by term, find the first three non-zero terms in the Taylor series about 0 for the function $f(x) = \sin^{-1} x$, explicitly evaluating the coefficients. State an interval of validity for this series.

Learning outcomes

After studying this unit, you should be able to:

- find Taylor polynomials about particular points for particular functions
- use Taylor polynomials to find approximations for values of functions, estimating such values to a particular accuracy
- find Taylor series about particular points for particular functions
- use known Taylor series to find further Taylor series by substitution, addition, subtraction, multiplication, differentiation and integration, and deduce intervals of validity for such series from intervals of validity for the original series.

Solutions to activities

Solution to Activity 1

(a) Since $f(0) = \cos 0 = 1$, the constant Taylor polynomial about 0 for $f(x) = \cos x$ is $p(x) = 1$.

The approximation is 1 in each case; that is,

$$p(0.01) = 1 \quad \text{and} \quad p(0.1) = 1.$$

The corresponding remainders are, to five decimal places,

$$\cos(0.01) - p(0.01) = \cos(0.01) - 1 = -0.000\,05,$$

$$\cos(0.1) - p(0.1) = \cos(0.1) - 1 = -0.005\,00.$$

(b) Since $f(1) = \ln 1 = 0$, the constant Taylor polynomial about 1 for $f(x) = \ln x$ is $p(x) = 0$.

The approximation is 0 in each case; that is,

$$p(1.01) = 0 \quad \text{and} \quad p(1.1) = 0.$$

The corresponding remainders are, to five decimal places,

$$\ln(1.01) - p(1.01) = \ln(1.01) - 0 = 0.009\,95,$$

$$\ln(1.1) - p(1.1) = \ln(1.1) - 0 = 0.095\,31.$$

Solution to Activity 2

(a) We have $f(x) = \sin x$, so

$$f'(x) = \cos x.$$

Hence

$$f(0) = \sin 0 = 0 \quad \text{and} \quad f'(0) = \cos 0 = 1.$$

Thus the linear Taylor polynomial about 0 for the sine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x \\ &= 0 + x \\ &= x. \end{aligned}$$

(b) The linear Taylor polynomial gives the approximations

$$p(0.25) = 0.25 \quad \text{and} \quad p(0.5) = 0.5.$$

A calculator gives (to 4 d.p.)

$$\sin(0.25) = 0.2474 \quad \text{and} \quad \sin(0.5) = 0.4794.$$

Hence the two remainders are

$$\sin(0.25) - 0.25 = -0.0026,$$

$$\sin(0.5) - 0.5 = -0.0206.$$

The magnitude of the remainder is about 8 times larger at $x = 0.5$ than it is at $x = 0.25$.

Solution to Activity 3

(a) We have $f(x) = \cos x$, so

$$f'(x) = -\sin x.$$

Hence

$$f(0) = \cos 0 = 1 \quad \text{and} \quad f'(0) = -\sin 0 = 0.$$

Thus the linear Taylor polynomial about 0 for the cosine function is

$$\begin{aligned} p(x) &= f(0) + f'(0) \\ &= 1 + 0x \\ &= 1. \end{aligned}$$

(b) The approximation for $\cos(0.2)$ given by the linear Taylor polynomial p is $p(0.2) = 1$. To four decimal places, the remainder is

$$\cos(0.2) - 1 = -0.0199.$$

Solution to Activity 4

(a) We have $f(x) = (1+x)^{1/2}$, so

$$f'(x) = \tfrac{1}{2}(1+x)^{-1/2}.$$

Hence

$$f(0) = (1+0)^{1/2} = 1$$

and

$$f'(0) = \tfrac{1}{2}(1+0)^{-1/2} = \tfrac{1}{2}.$$

Thus the linear Taylor polynomial about 0 for $f(x) = (1+x)^{1/2}$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x \\ &= 1 + \tfrac{1}{2}x, \qquad \text{as required.} \end{aligned}$$

(b) We have

$$\sqrt{1.01} = (1 + 0.01)^{1/2} = f(0.01).$$

The corresponding approximation for $\sqrt{1.01}$ is

$$p(0.01) = 1 + \tfrac{1}{2} \times 0.01 = 1.005.$$

To six decimal places, the remainder is

$$\begin{aligned} f(0.01) - p(0.01) &= 1.004\,988 - 1.005 \\ &= -0.000\,012. \end{aligned}$$

Solution to Activity 5

We have $f(x) = e^x$, so

$$f'(x) = e^x.$$

Hence

$$f(1) = e^1 = e \quad \text{and} \quad f'(1) = e^1 = e.$$

Thus the linear Taylor polynomial about 1 for $f(x) = e^x$ is

$$\begin{aligned} p(x) &= f(1) + f'(1)(x-1) \\ &= e + e(x-1) \\ &= ex. \end{aligned}$$

Solution to Activity 6

(a) We have $f(x) = \cos x$, so

$$f'(x) = -\sin x \quad \text{and} \quad f''(x) = -\cos x.$$

Hence

$$f(0) = \cos 0 = 1, \quad f'(0) = -\sin 0 = 0$$

and

$$f''(0) = -\cos 0 = -1.$$

Thus the quadratic Taylor polynomial about 0 for the cosine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 \\ &= 1 + 0x - \tfrac{1}{2}x^2 \\ &= 1 - \tfrac{1}{2}x^2. \end{aligned}$$

(b) The corresponding approximation to $\cos(0.2)$ is

$$p(0.2) = 1 - \tfrac{1}{2}(0.2)^2 = 0.98.$$

To six decimal places, the remainder is

$$\begin{aligned} \cos(0.2) - p(0.2) &= 0.980\,067 - 0.98 \\ &= 0.000\,067. \end{aligned}$$

This remainder has much smaller magnitude than that found in Activity 3(b), so the approximation to $\cos(0.2)$ by using the quadratic Taylor polynomial $p(x) = 1 - \frac{1}{2}x^2$ is much better than the approximation by using the linear Taylor polynomial $p(x) = 1$.

Solution to Activity 7

We have $f(x) = \sin x$, so

$$f'(x) = \cos x \quad \text{and} \quad f''(x) = -\sin x.$$

Hence

$$f(0) = \sin 0 = 0, \quad f'(0) = \cos 0 = 1$$

and

$$f''(0) = -\sin 0 = 0.$$

Thus the quadratic Taylor polynomial about 0 for the sine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 \\ &= 0 + x + 0x^2 \\ &= x. \end{aligned}$$

Solution to Activity 8

We have $f(x) = e^x$, so

$$f'(x) = e^x \quad \text{and} \quad f''(x) = e^x.$$

Hence

$$f(1) = e^1 = e, \quad f'(1) = e^1 = e$$

and

$$f''(1) = e^1 = e.$$

Thus the quadratic Taylor polynomial about 1 for $f(x) = e^x$ is

$$\begin{aligned} p(x) &= f(1) + f'(1)(x-1) + \tfrac{1}{2}f''(1)(x-1)^2 \\ &= e + e(x-1) + \tfrac{1}{2}e(x-1)^2. \end{aligned}$$

(By multiplying out the squared brackets and collecting like terms, this can also be written as

$$p(x) = \tfrac{1}{2}e\left(1 + x^2\right).)$$

Solution to Activity 9

(a) To find the quartic Taylor polynomial about 0 for $f(x) = \cos x$, we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$. We have:

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1; \\ f'(x) &= -\sin x, & f'(0) &= 0; \\ f''(x) &= -\cos x, & f''(0) &= -1; \\ f^{(3)}(x) &= \sin x, & f^{(3)}(0) &= 0; \\ f^{(4)}(x) &= \cos x, & f^{(4)}(0) &= 1. \end{aligned}$$

Hence the quartic Taylor polynomial about 0 for the cosine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &\qquad + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \\ &= 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4. \end{aligned}$$

(b) Similarly, to find the quartic Taylor polynomial about 0 for $f(x) = \sin x$, we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$. We have:

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0; \\ f'(x) &= \cos x, & f'(0) &= 1; \\ f''(x) &= -\sin x, & f''(0) &= 0; \\ f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1; \\ f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0. \end{aligned}$$

Hence the quartic Taylor polynomial about 0 for the sine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!}x^4 \\ &= x - \frac{1}{3!}x^3 \\ &= x - \tfrac{1}{6}x^3. \end{aligned}$$

(You may have noticed that the quartic Taylor polynomial about 0 for the cosine function contains terms in even powers of x only, whereas that for the sine function contains terms in odd powers of x only. This property is explained later in the subsection.)

Solution to Activity 10

(a) (i) Applying the chain rule gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(1-x)^k}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^{-k}\right) \\ &= (-k)(1-x)^{-k-1} \times \frac{\mathrm{d}}{\mathrm{d}x}(1-x) \\ &= (-k)(1-x)^{-(k+1)} \times (-1) \\ &= k(1-x)^{-(k+1)} \\ &= \frac{k}{(1-x)^{k+1}}, \end{aligned}$$

as required.

(ii) Applying the result from part (a)(i), with $k = 1, 2, 3$, gives

$$\begin{aligned} f(x) &= \frac{1}{1-x}, & f(0) &= 1; \\ f'(x) &= \frac{1}{(1-x)^2}, & f'(0) &= 1; \\ f''(x) &= \frac{2}{(1-x)^3}, & f''(0) &= 2; \\ f^{(3)}(x) &= \frac{3\times 2}{(1-x)^4} & & \\ &= \frac{3!}{(1-x)^4}, & f^{(3)}(0) &= 3!\,. \end{aligned}$$

(iii) Hence the cubic Taylor polynomial about 0 for f is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\quad + \frac{f^{(3)}(0)}{3!}x^3 \\ &= 1 + x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

(b) (i) The 4th derivative of $f(x)$ is

$$\begin{aligned} f^{(4)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3!}{(1-x)^4}\right) \\ &= \frac{4\times 3!}{(1-x)^5} = \frac{4!}{(1-x)^5}, \end{aligned}$$

the 5th derivative is

$$\begin{aligned} f^{(5)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4!}{(1-x)^5}\right) \\ &= \frac{5\times 4!}{(1-x)^6} = \frac{5!}{(1-x)^6}, \end{aligned}$$

and so on. The pattern is now clear; the nth derivative will be

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

(ii) By putting $x = 0$ in the formula for $f^{(n)}(x)$ above, we obtain

$$f^{(n)}(0) = \frac{n!}{(1-0)^{n+1}} = n!\,.$$

(iii) Hence the Taylor polynomial of degree n about 0 for f is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\quad + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{n!}{n!}x^n \\ &= 1 + x + x^2 + x^3 + \cdots + x^n. \end{aligned}$$

Solution to Activity 11

To find the cubic Taylor polynomial about $\pi/6$ for $f(x) = \sin x$, we evaluate $f(\pi/6)$, $f'(\pi/6)$, $f''(\pi/6)$ and $f^{(3)}(\pi/6)$, as follows:

$$\begin{aligned} f(x) &= \sin x, & f\left(\frac{\pi}{6}\right) &= \tfrac{1}{2}; \\ f'(x) &= \cos x, & f'\left(\frac{\pi}{6}\right) &= \tfrac{1}{2}\sqrt{3}; \\ f''(x) &= -\sin x, & f''\left(\frac{\pi}{6}\right) &= -\tfrac{1}{2}; \\ f^{(3)}(x) &= -\cos x, & f^{(3)}\left(\frac{\pi}{6}\right) &= -\tfrac{1}{2}\sqrt{3}. \end{aligned}$$

Hence the cubic Taylor polynomial about $\pi/6$ for the sine function is

$$\begin{aligned} p(x) &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) \\ &\quad + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3 \\ &= \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 \\ &\quad - \frac{1}{12}\sqrt{3}\left(x - \frac{\pi}{6}\right)^3. \end{aligned}$$

Solution to Activity 13

Using the given Taylor polynomials

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n,$$

and calculating values to six decimal places, we obtain

$$\begin{aligned} p_1(-0.05) &= 1 + (-0.05) = 0.95 \\ p_2(-0.05) &= p_1(-0.05) + \tfrac{1}{2}(-0.05)^2 \\ &= 0.951\,25 \\ p_3(-0.05) &= p_2(-0.05) + \tfrac{1}{6}(-0.05)^3 \\ &= 0.951\,229 \quad \text{(to 6 d.p.)} \\ p_4(-0.05) &= p_3(-0.05) + \tfrac{1}{24}(-0.05)^4 \\ &= 0.951\,229 \quad \text{(to 6 d.p.)}. \end{aligned}$$

The values of $p_3(-0.05)$ and $p_4(-0.05)$ agree to six decimal places, so it is likely that

$$e^{-0.05} = 0.9512$$

to four decimal places. (This is indeed the case.)

Solution to Activity 14

Using the given Taylor polynomials, and calculating values to eight decimal places, we obtain

$$\begin{aligned} p_0(0.2) &= 1 \\ p_2(0.2) &= 1 - \frac{1}{2!}(0.2)^2 = 0.98 \\ p_4(0.2) &= p_2(0.2) + \frac{1}{4!}(0.2)^4 = 0.980\,066\,67 \\ p_6(0.2) &= p_4(0.2) - \frac{1}{6!}(0.2)^6 = 0.980\,066\,58 \\ p_8(0.2) &= p_6(0.2) + \frac{1}{8!}(0.2)^8 = 0.980\,066\,58. \end{aligned}$$

The values of $p_6(0.2)$ and $p_8(0.2)$ agree to eight decimal places, so it is likely that

$$\cos(0.2) = 0.980\,067$$

to six decimal places. (This is indeed the case.)

Solution to Activity 15

(a) From the solution to Activity 9(a), we can see that the values of $f^{(n)}(0)$ form the repeating sequence

$$1, 0, -1, 0, 1, \ldots .$$

Hence the Taylor series about 0 for $f(x) = \cos x$ is

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots .$$

(b) Similarly, from the solution to Activity 9(b), we can see that the values of $f^{(n)}(0)$ form the repeating sequence

$$0, 1, 0, -1, 0, \ldots .$$

Hence the Taylor series about 0 for $f(x) = \sin x$ is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots .$$

Solution to Activity 16

We proceed initially in a similar way to the solution to Activity 9(b). To find the Taylor series about 0 for $f(x) = \sin x$, we need to evaluate $f(\pi/2)$, $f'(\pi/2)$, $f''(\pi/2)$, $f^{(3)}(\pi/2)$, $\ldots$. We have:

$$f(x) = \sin x, \qquad f\left(\frac{\pi}{2}\right) = 1;$$
$$f'(x) = \cos x, \qquad f'\left(\frac{\pi}{2}\right) = 0;$$
$$f''(x) = -\sin x, \qquad f''\left(\frac{\pi}{2}\right) = -1;$$
$$f^{(3)}(x) = -\cos x, \qquad f^{(3)}\left(\frac{\pi}{2}\right) = 0;$$
$$f^{(4)}(x) = \sin x, \qquad f^{(4)}\left(\frac{\pi}{2}\right) = 1.$$

The values of $f^{(n)}(\pi/2)$ form the repeating sequence

$$1, 0, -1, 0, 1, \ldots .$$

Hence, from formula (5), the Taylor series about $\pi/2$ for $f(x) = \sin x$ is

$$1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(x - \frac{\pi}{2}\right)^8 - \cdots .$$

Solution to Activity 17

Since $1/(1+x)^2 = (1+x)^{-2}$, we take $\alpha = -2$ in the binomial series, to give

$$\frac{1}{(1+x)^2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots .$$

This Taylor series is valid for $-1 < x < 1$.

Solution to Activity 18

Taking $\alpha = \frac{1}{2}$ in the binomial series gives

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4 + \cdots$$
$$= 1 + \tfrac{1}{2}x - \frac{1}{2^2 \times 2!}x^2 + \frac{1 \times 3}{2^3 \times 3!}x^3 - \frac{1 \times 3 \times 5}{2^4 \times 4!}x^4 + \cdots .$$

This Taylor series is valid for $-1 < x < 1$.

Solution to Activity 19

The cubic Taylor polynomial about 0 for the function $f(x) = \ln(1+x)$ is obtained from the Taylor series for $\ln(1+x)$ by deleting all the terms after $\frac{1}{3}x^3$, to give

$$p_3(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3.$$

Solution to Activity 20

Using the series from the solution to Activity 18, we obtain, to five decimal places,

$$p_1(0.1) = 1 + \tfrac{1}{2}(0.1) = 1.05$$
$$p_2(0.1) = p_1(0.1) - \frac{1}{2^2 \times 2!}(0.1)^2 = 1.048\,75$$
$$p_3(0.1) = p_2(0.1) + \frac{1 \times 3}{2^3 \times 3!}(0.1)^3 = 1.048\,81$$
$$p_4(0.1) = p_3(0.1) - \frac{1 \times 3 \times 5}{2^4 \times 4!}(0.1)^4 = 1.048\,81.$$

The values of $p_3(0.1)$ and $p_4(0.1)$ agree to five decimal places, so it is likely that

$$\sqrt{1.1} = 1.049$$

to three decimal places. (This is indeed the case.)

Solution to Activity 22

(a) The Taylor series about 0 for $1/(1-x)$, from page 142, is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

for $-1 < x < 1$. Replacing each occurrence of x by $-2x$ and using the fact that $1/(1-(-2x)) = 1/(1+2x)$ gives

$$\frac{1}{1+2x} = 1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots$$
$$= 1 - 2x + 4x^2 - 8x^3 + \cdots .$$

This is the Taylor series about 0 for $1/(1+2x)$. It is valid for $-1 < -2x < 1$; that is, for $-\frac{1}{2} < x < \frac{1}{2}$.

(This part could also be answered by taking the binomial series from page 142 for $(1+x)^\alpha$, with $\alpha = -1$, and then replacing each occurrence of x by $2x$.)

(b) The Taylor series about 0 for $\ln(1+x)$, from page 142, is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots ,$$

for $-1 < x < 1$.

Replacing each occurrence of x by $-x$ and using the fact that $\ln(1+(-x)) = \ln(1-x)$ gives

$$\ln(1-x)$$
$$= (-x) - \tfrac{1}{2}(-x)^2 + \tfrac{1}{3}(-x)^3 - \tfrac{1}{4}(-x)^4 + \cdots$$
$$= -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots .$$

This is the Taylor series about 0 for $\ln(1-x)$. It is valid for $-1 < -x < 1$; that is, for $-1 < x < 1$.

(c) The Taylor series about 0 for $\ln(1+x)$, from page 142, is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots,$$

for $-1 < x < 1$. Replacing each occurrence of x by $3x$ gives

$$\begin{aligned}&\ln(1+3x)\\ &= (3x) - \tfrac{1}{2}(3x)^2 + \tfrac{1}{3}(3x)^3 - \tfrac{1}{4}(3x)^4 + \cdots\\ &= 3x - \frac{3^2}{2}x^2 + \frac{3^3}{3}x^3 - \frac{3^4}{4}x^4 + \cdots.\end{aligned}$$

This is the Taylor series about 0 for $\ln(1+3x)$. It is valid for $-1 < 3x < 1$; that is, for $-\frac{1}{3} < x < \frac{1}{3}$.

(d) The Taylor series about 0 for e^x, from page 142, is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Replacing each occurrence of x by x^3 gives the Taylor series about 0 for e^{x^3}:

$$\begin{aligned}e^{x^3} &= 1 + (x^3) + \frac{1}{2!}(x^3)^2 + \frac{1}{3!}(x^3)^3 + \cdots\\ &= 1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \cdots, \quad \text{for } x \in \mathbb{R}.\end{aligned}$$

Solution to Activity 23

The Taylor series about 0 for $1/(1+x)$ is given as

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

for $-1 < x < 1$. Replacing each occurrence of x by $x-1$ in this equation, we obtain

$$\begin{aligned}&\frac{1}{1+(x-1)}\\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.\end{aligned}$$

But $1/(1+(x-1)) = 1/x$. Therefore the Taylor series about 1 for $1/x$ is

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.$$

This Taylor series is valid for $-1 < x-1 < 1$; that is, for $0 < x < 2$.

Solution to Activity 24

In each part, we use the Taylor series about 0 for $1/(1+x)$, from page 142, which is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

for $-1 < x < 1$.

(a) Replacing each occurrence of x by $\frac{2}{3}x$ in this equation, we obtain

$$\begin{aligned}\frac{1}{1+\frac{2}{3}x} &= 1 - \left(\tfrac{2}{3}x\right) + \left(\tfrac{2}{3}x\right)^2 - \left(\tfrac{2}{3}x\right)^3 + \cdots\\ &= 1 - \tfrac{2}{3}x + \tfrac{4}{9}x^2 - \tfrac{8}{27}x^3 + \cdots.\end{aligned}$$

It follows that the Taylor series about 0 for $g(x) = 3/(3+2x) = 1/\left(1+\frac{2}{3}x\right)$ is

$$\frac{3}{3+2x} = 1 - \tfrac{2}{3}x + \tfrac{4}{9}x^2 - \tfrac{8}{27}x^3 + \cdots.$$

This Taylor series is valid for $-1 < \frac{2}{3}x < 1$; that is, for $-\frac{3}{2} < x < \frac{3}{2}$.

(b) Replacing each occurrence of x by $2(x+1)$ in the series for $1/(1+x)$, we obtain

$$\begin{aligned}&\frac{1}{1+2(x+1)}\\ &= 1 - (2(x+1)) + (2(x+1))^2 - (2(x+1))^3 + \cdots\\ &= 1 - 2(x+1) + 4(x+1)^2 - 8(x+1)^3 + \cdots.\end{aligned}$$

It follows that the Taylor series about -1 for $1/(3+2x) = 1/(1+2(x+1))$ is

$$\begin{aligned}\frac{1}{3+2x} &= 1 - 2(x+1) + 4(x+1)^2\\ &\qquad - 8(x+1)^3 + \cdots.\end{aligned}$$

This Taylor series is valid for $-1 < 2(x+1) < 1$; that is, for $-\frac{1}{2} < x+1 < \frac{1}{2}$ or equivalently $-\frac{3}{2} < x < -\frac{1}{2}$.

Solution to Activity 25

Using the given Taylor series about 0 for $\ln(1+x)$ and $\ln(1-x)$, we obtain

$$\begin{aligned}&\ln(1+x) - \ln(1-x)\\ &= \left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots\right)\\ &\quad - \left(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \cdots\right)\\ &= 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \cdots;\end{aligned}$$

that is,

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \cdots.$$

The Taylor series for $\ln(1+x)$ and $\ln(1-x)$ are each valid for $-1 < x < 1$, so the Taylor series derived here is also valid for $-1 < x < 1$.

Solution to Activity 26

If $t = 1.5$, then the corresponding value of x is

$$x = \frac{1.5-1}{1.5+1} = \frac{0.5}{2.5} = \frac{1}{5} = 0.2.$$

Using the series given, we obtain, to six decimal places,

$$p_1(0.2) = 2 \times 0.2 = 0.4$$
$$p_3(0.2) = p_1(0.2) + \tfrac{2}{3}(0.2)^3 = 0.405\,333$$
$$p_5(0.2) = p_3(0.2) + \tfrac{2}{5}(0.2)^5 = 0.405\,461$$
$$p_7(0.2) = p_5(0.2) + \tfrac{2}{7}(0.2)^7 = 0.405\,465$$
$$p_9(0.2) = p_7(0.2) + \tfrac{2}{9}(0.2)^9 = 0.405\,465.$$

The values of $p_7(0.2)$ and $p_9(0.2)$ agree to six decimal places, so it is likely that

$$\ln(1.5) = \ln\left(\frac{1+0.2}{1-0.2}\right) = 0.4055$$

to four decimal places. (This is indeed the case.)

Solution to Activity 27

The given Taylor series is

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots,$$

for $-1 < x < 1$.

Using this series and replacing each occurrence of x by $\frac{1}{3}x$ gives

$$\begin{aligned}\frac{1}{(3+x)^2} &= \frac{1}{3^2} \times \frac{1}{(1+\frac{1}{3}x)^2}\\ &= \frac{1}{3^2}\left(1 - 2\left(\tfrac{1}{3}x\right) + 3\left(\tfrac{1}{3}x\right)^2 - 4\left(\tfrac{1}{3}x\right)^3 + \cdots\right)\\ &= \frac{1}{3^2}\left(1 - \frac{2}{3}x + \frac{3}{3^2}x^2 - \frac{4}{3^3}x^3 + \cdots\right)\\ &= \frac{1}{3^2} - \frac{2}{3^3}x + \frac{3}{3^4}x^2 - \frac{4}{3^5}x^3 + \cdots.\end{aligned}$$

This Taylor series is valid for $-1 < \frac{1}{3}x < 1$; that is, for $-3 < x < 3$.

Solution to Activity 28

We use the formula

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad \text{for } x \in \mathbb{R}.$$

On replacing each occurrence of x by $-x$, we obtain

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots, \quad \text{for } x \in \mathbb{R}.$$

Therefore

$$\begin{aligned}\tfrac{1}{2}(e^x - e^{-x}) &= \frac{1}{2}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\right.\\ &\qquad \left.- \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right)\\ &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots,\end{aligned}$$

for $x \in \mathbb{R}$; that is,

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Solution to Activity 29

(a) Using the Taylor series about 0 for $\sin x$, from page 142, we obtain

$$\begin{aligned}x^2 \sin x &= x^2\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right)\\ &= x^3 - \frac{1}{3!}x^5 + \frac{1}{5!}x^7 - \cdots, \quad \text{for } x \in \mathbb{R}.\end{aligned}$$

(b) Using the Taylor series about 0 for $\cos x$, from page 142, we obtain

$$\begin{aligned}(1+x)\cos x &= (1+x)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right)\\ &= 1\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right)\\ &\quad + x\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right)\\ &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right)\\ &\quad + \left(x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \cdots\right)\\ &= 1 + x - \frac{1}{2!}x^2 - \frac{1}{2!}x^3 + \frac{1}{4!}x^4 + \frac{1}{4!}x^5 - \cdots,\end{aligned}$$

for $x \in \mathbb{R}$.

Solution to Activity 30

Using the Taylor series about 0 for $1/(1+x)$ and $\sin x$, and ignoring all terms that lead to 4th or higher powers of x, we obtain

$$\begin{aligned}\frac{\sin x}{1+x} &= (1 - x + x^2 - \cdots)\left(x - \frac{1}{3!}x^3 + \cdots\right)\\ &= \left(x - \frac{1}{3!}x^3 + \cdots\right) - x(x - \cdots) + x^2(x - \cdots) - \cdots\\ &= (x - \tfrac{1}{6}x^3 + \cdots) - (x^2 - \cdots) + (x^3 - \cdots) - \cdots\\ &= x - x^2 + \tfrac{5}{6}x^3 - \cdots.\end{aligned}$$

Hence the cubic Taylor polynomial about 0 for $f(x) = (\sin x)/(1+x)$ is

$$p(x) = x - x^2 + \tfrac{5}{6}x^3.$$

Solution to Activity 31

The Taylor series about 0 for e^x is

$$1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \frac{1}{4!}\,x^4 + \cdots .$$

Differentiating this series gives

$$0 + 1 + \frac{2}{2!}\,x + \frac{3}{3!}\,x^2 + \frac{4}{4!}\,x^3 + \cdots$$
$$= 1 + x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots ,$$

which is the same series, as required.

Solution to Activity 32

We have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots ,$$

for $-1 < x < 1$. Integrating both sides of this equation gives

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int (1 - x^2 + x^4 - x^6 + \cdots)\,\mathrm{d}x;$$

that is,

$$\tan^{-1} x = c + x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots ,$$

for $-1 < x < 1$, where c is a constant. Taking $x = 0$ gives $\tan^{-1} 0 = c$, so $c = 0$. Therefore

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots ,$$

for $-1 < x < 1$.

Solution to Activity 33

(a) The given series is

$$\frac{1}{\sqrt{1+x}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots ,$$

for $-1 < x < 1$. Replacing each occurrence of x by $-x^2$ gives

$$\frac{1}{\sqrt{1-x^2}} = 1 - \tfrac{1}{2}(-x^2) + \tfrac{3}{8}(-x^2)^2 - \tfrac{5}{16}(-x^2)^3 + \cdots$$
$$= 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots .$$

This Taylor series is valid for $-1 < -x^2 < 1$. The left-hand inequality here is $-1 < -x^2$, which is equivalent to $1 > x^2$; that is, $-1 < x < 1$. The right-hand inequality is $-x^2 < 1$, which is equivalent to $x^2 > -1$ and therefore does not place any restriction on x, since the square of any real number is non-negative. Thus this Taylor series is valid for $-1 < x < 1$.

(b) Integrating both sides of the equation above gives

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int (1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \cdots)\,\mathrm{d}x;$$

that is,

$$\sin^{-1} x = c + x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \cdots ,$$

for $-1 < x < 1$, where c is a constant. Putting $x = 0$ gives $\sin^{-1} 0 = c$, so $c = 0$. Therefore

$$\sin^{-1} x = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \cdots ,$$

for $-1 < x < 1$.

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Page 145: Mark Hobbs, for the idea for the cartoon

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