Welcome to MST124 online

Unit 7 Methods of Differentiation

19:00 – 20:30

You will need

- paper and pencil
- scientific calculator
- MST124 Handbook

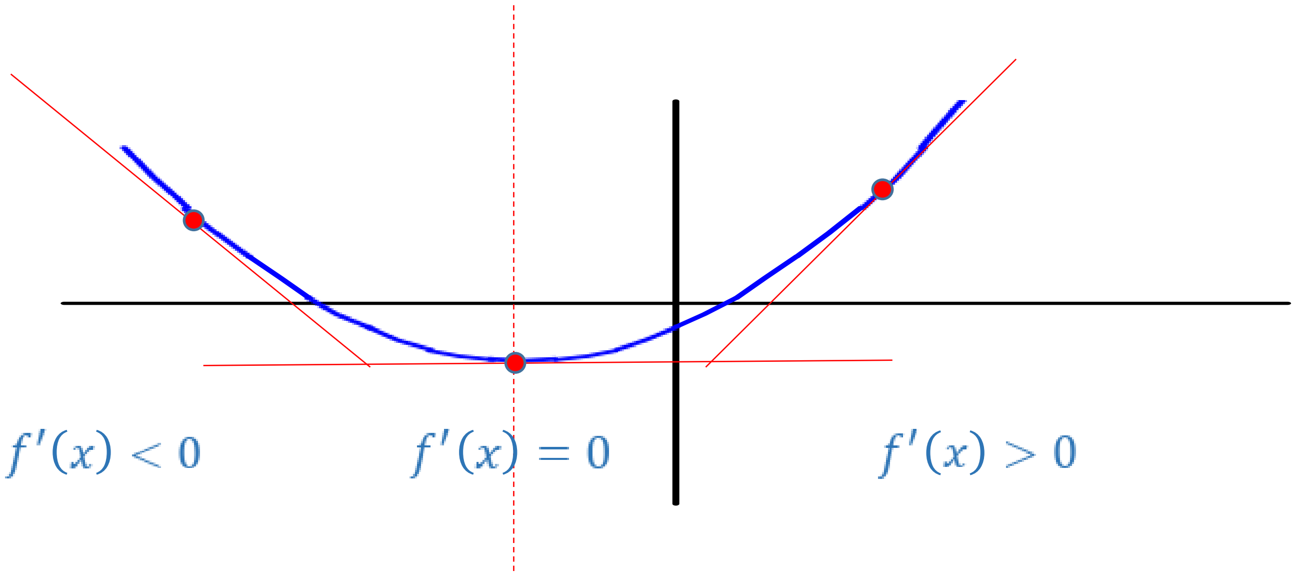
Unit 6 reminder

The gradient of a function $f(x)$ is given by its derivative $f'(x)$ which is obtainable from $f(x)$ by differentiation (provided $f(x)$ is differentiable).

$f(x)$ is increasing over the interval $[a, b]$ if, for all x in $[a, b]$, $f'(x) > 0$

$f(x)$ is decreasing over the interval $[a, b]$ if, for all x in $[a, b]$, $f'(x) < 0$

$f(x)$ is stationary when $f'(x) = 0$



Unit 6 reminder

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$f(x)$ is decreasing over the interval $[a, b]$ if, for all x in $[a, b]$, $f'(x) < 0$

$f(x)$ is stationary when $f'(x) = 0$

$f''(x)$ is the second derivative of $f(x)$

If $f'(a) = 0$ and $f''(x) > 0$ then $f(x)$ has a local minimum at $x = a$

If $f'(a) = 0$ and $f''(x) < 0$ then $f(x)$ has a local maximum at $x = a$

Unit 6 reminder

An alternative (Leibniz) notation when using $y = f(x)$ is $\frac{dy}{dx} = f'(x)$ and $\frac{d^2y}{dx^2} = f''(x)$

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Unit 7

What's in this unit:

More Calculus

- Derivatives of some standard functions like $\sin(x)$.......
- How to differentiate products and quotients of functions.......
- How to differentiate composite functions.........
- More applications of differentiation such as optimisation

- An introduction to Integration

Standard derivatives – Trig functions

Standard derivatives – Trig functions

$y = \sin x \qquad \frac{dy}{dx} = \cos x$

$y = \cos x \qquad \frac{dy}{dx} = -\sin x$

$y = \tan x \qquad \frac{dy}{dx} = \sec^2 x \quad \left(= \frac{1}{\cos^2 x}\right)$

$y = \operatorname{cosec} x \qquad \frac{dy}{dx} = -\operatorname{cosec} x \cot x$

$y = \sec x \qquad \frac{dy}{dx} = \sec x \tan x$

$y = \cot x \qquad \frac{dy}{dx} = -\operatorname{cosec}^2 x$

You must use radians

These are all in your Handbook (p7) so no need to memorise them

Standard derivatives – Trig functions

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Example:

Find the stationary points of $y = \sin x$ in the interval $[0,2\pi]$

Standard derivatives – Trig functions

$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$
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You must use radians

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Example:

Find the stationary points of $y = \sin x$ in the interval $[0,2\pi]$

$$\frac{dy}{dx} = \cos x$$

For stationary points: $\frac{dy}{dx} = 0$

So $\cos x = 0$ giving $x = \cos^{-1} 0$

So stationary points are at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

Also $\frac{d^2y}{dx^2} = -\sin x$

At $x = \frac{\pi}{2}$ $\frac{d^2y}{dx^2} = -1$ so this is a local maximum

At $x = \frac{3\pi}{2}$ $\frac{d^2y}{dx^2} = 1$ so this is a local minimum

(*by the second derivative rule*)

Standard derivatives – Trig functions

$y = \sin x \qquad \frac{dy}{dx} = \cos x$

$y = \cos x \qquad \frac{dy}{dx} = -\sin x$

$y = \tan x \qquad \frac{dy}{dx} = \sec^2 x \quad \left(= \frac{1}{\cos^2 x}\right)$

$y = \operatorname{cosec} x \qquad \frac{dy}{dx} = -\operatorname{cosec} x \cot x$

$y = \sec x \qquad \frac{dy}{dx} = \sec x \tan x$

$y = \cot x \qquad \frac{dy}{dx} = -\operatorname{cosec}^2 x$

You must use radians

These are all in your Handbook (p7) so no need to memorise them

Find the stationary points of $y = \sin x$ in the interval $[0,2\pi]$

$\frac{dy}{dx} = \cos x$, and for stationary points $\frac{dy}{dx} = 0$

So $\quad \cos x = 0 \quad$ giving $\quad x = \cos^{-1}(0)$

So stationary points are at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

Also $\frac{d^2y}{dx^2} = -\sin x$

At $x = \frac{\pi}{2}$ $\frac{d^2y}{dx^2} = -1$ so this is a local maximum

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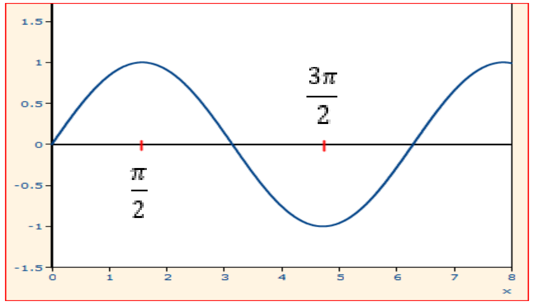

Standard derivatives – Trig functions

Find the stationary points for this function in the interval $[0,2\pi]$ and determine their nature:

$$f(x) = \sin x + \cos x$$

$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x \quad (= \frac{1}{\cos^2 x})$

Standard derivatives – Trig functions

Find the stationary points for this function in the interval $[0,2\pi]$ and determine their nature:

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

For stationary points $f'(x) = 0$

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = \tan x = 1 \quad (\cos x \neq 0)$$

$y = \sin x \qquad \frac{dy}{dx} = \cos x$

$y = \cos x \qquad \frac{dy}{dx} = -\sin x$

$y = \tan x \qquad \frac{dy}{dx} = \sec^2 x \quad (= \frac{1}{\cos^2 x})$

Standard derivatives – Trig functions

Find the stationary points for this function in the interval $[0,2\pi]$ and determine their nature:

$$f(x) = \sin x + \cos x$$

$f'(x) = \cos x - \sin x = 0$ for stationary points

$$\sin x = \cos x$$

$$\tan x = 1 \quad (\cos x \neq 0)$$

Solving: $x = \tan^{-1}(1)$

$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ since $\tan x$ is positive in 1st and 3rd quadrants

$y = \sin x$	$\frac{dy}{dx} = \cos x$
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Standard derivatives – Trig functions

Find the stationary points for this function in the interval $[0,2\pi]$ and determine their nature:

$$f(x) = \sin x + \cos x$$

$f'(x) = \cos x - \sin x = 0$ for stationary points

$$\sin x = \cos x$$

$$\tan x = 1 \quad (\cos x \neq 0)$$

$$x = \tan^{-1}(1)$$

$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ since $\tan x$ is positive in 1st and 3rd quadrants

$$f''(x) = -\sin x - \cos x$$

For $x = \frac{\pi}{4}$, $f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$ so maximum

For $x = \frac{5\pi}{4}$, $f''\left(\frac{3\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} > 0$ so minimum

$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x \quad (= \frac{1}{\cos^2 x})$

Standard derivatives – Trig functions

Find the stationary points for this function in the interval $[0,2\pi]$ and determine their nature:

$f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x = 0$ for stationary points

$$\sin x = \cos x$$

$$\tan x = 1 \quad (\cos x \neq 0)$$

$$x = \tan^{-1} 1$$

$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ since $\tan x$ is positive in 1st and 3rd quadrants

Also $\quad f''(x) = -\sin x - \cos x$

For $x = \frac{\pi}{4}$, $f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$ so maximum

For $x = \frac{5\pi}{4}$, $f''\left(\frac{3\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} > 0$ so minimum

$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$
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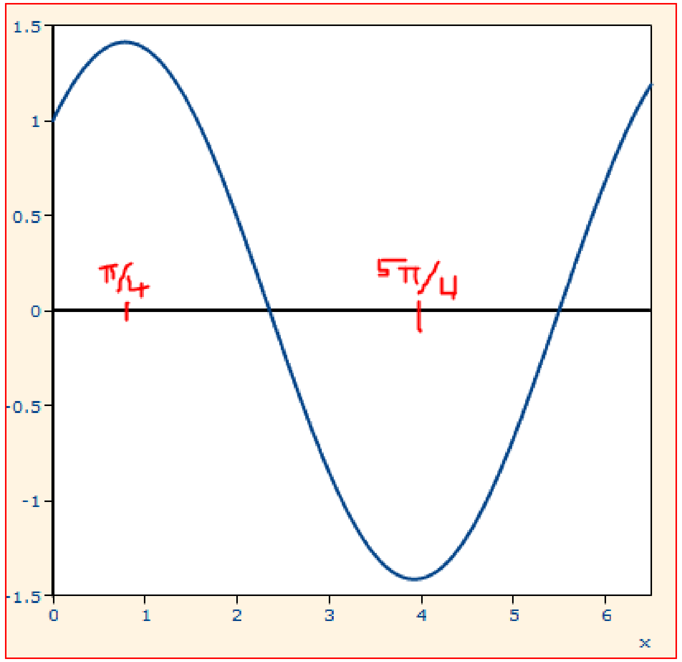
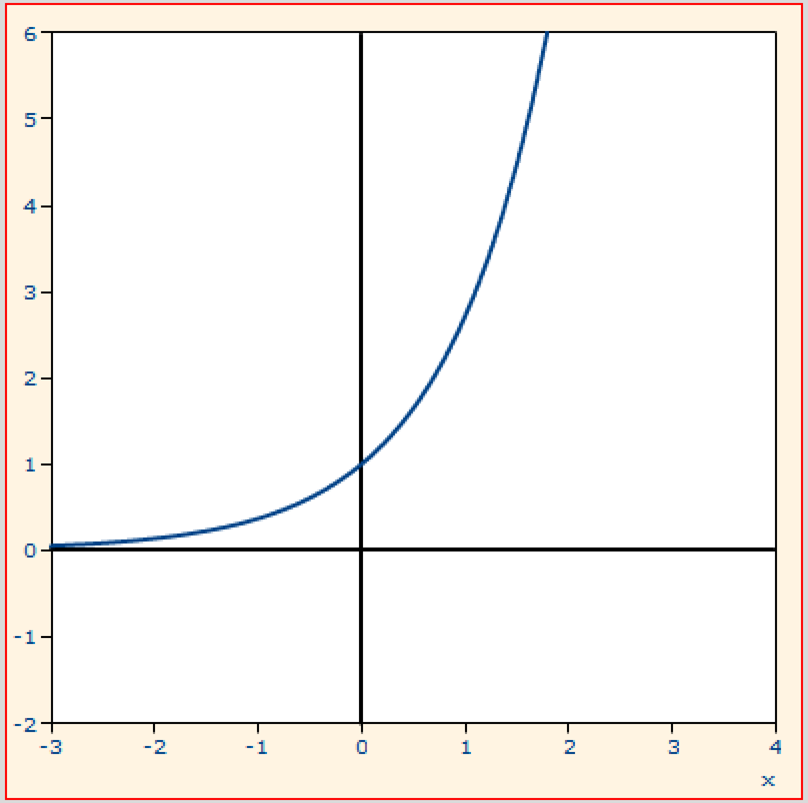


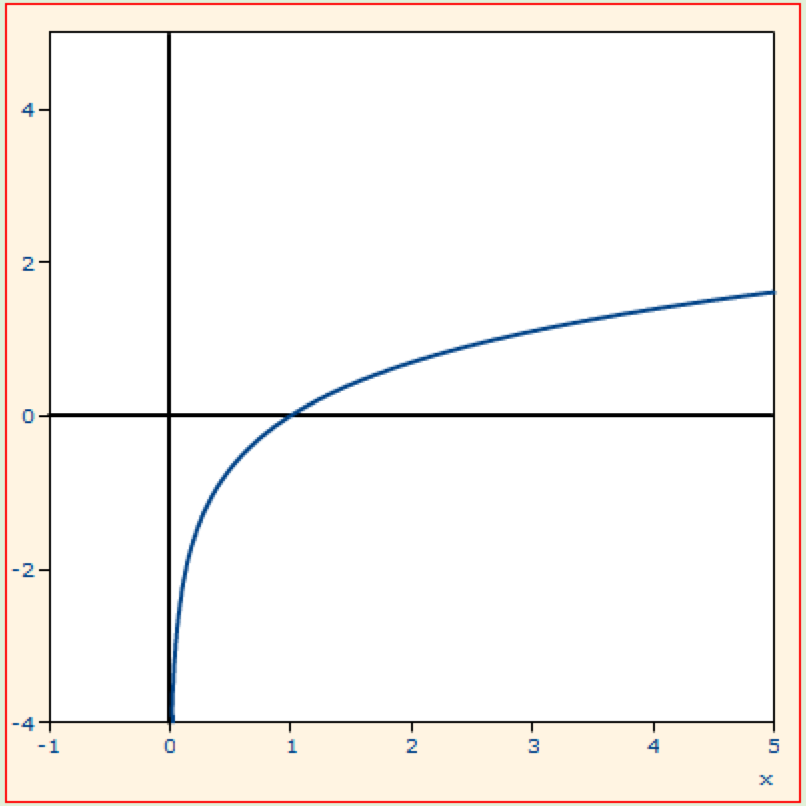
Standard derivatives – Exponential and Log

Standard derivatives – Exponential and Log

$$y = e^x \qquad \frac{dy}{dx} = e^x$$



$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x}$$



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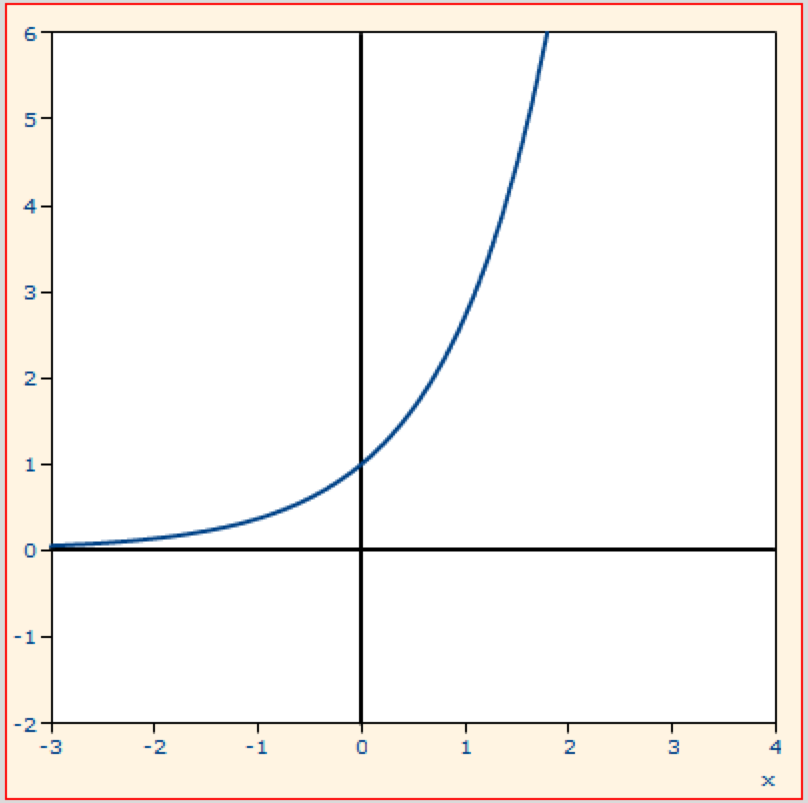
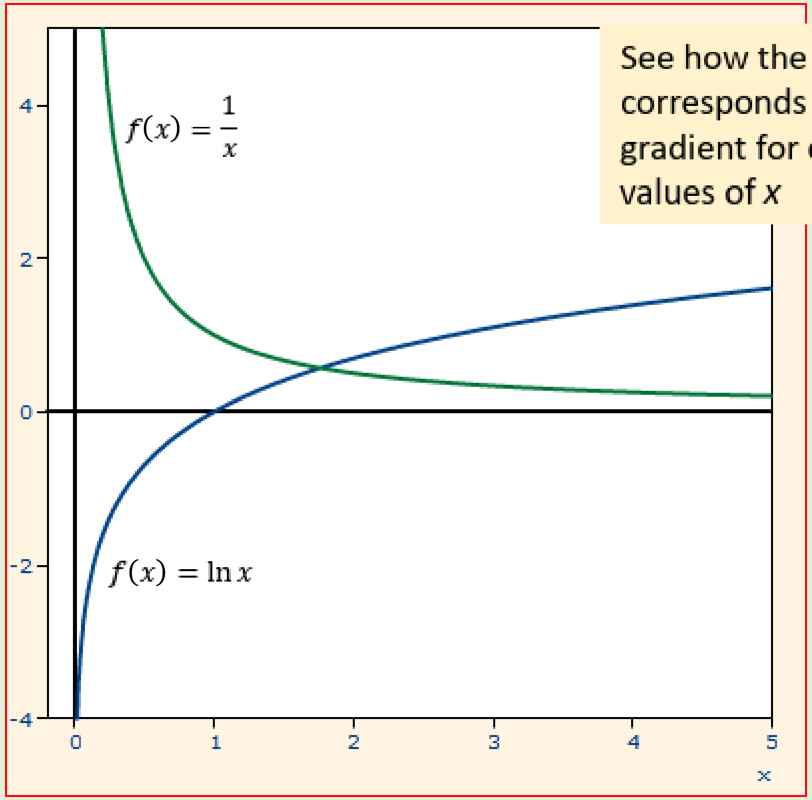
Standard derivatives – Exponential and Log

$$y = e^x \qquad \frac{dy}{dx} = e^x$$



$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x}$$



See how the function corresponds with its gradient for different values of *x*

These are all in your Handbook (p7) so no need to memorise them

Differentiating other types of functions

Differentiating Products

A product is in the form $y = f(x) \times g(x)$

These are examples of products: $y = x^2 \sin x$ $\quad y = \frac{1}{x} e^x$ $\quad y = \ln x \tan x$

In your Handbook (p48)

If $k(x) = f(x)g(x)$ then $k'(x) = f(x)g'(x) + f'(x)g(x)$

Or if $y = uv$ where u and v are functions of x then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Differentiating Products

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Method, using the example

$$k(x) = x^2 \sin x$$

$$k'(x) = 2x \sin x + x^2 \cos x$$

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Method, using the example

$$k(x) = x^2 \sin x$$

$$k'(x) = 2x \sin x + x^2 \cos x$$

Try this:

$$h(x) = x^3 \cos x$$

Differentiating Products

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Method, using the example

$$k(x) = x^2 \sin x$$

$$k'(x) = 2x \sin x + x^2 \cos x$$

Try this: $$h(x) = x^3 \cos x$$

$$h'(x) = 3x^2 \cos x + x^3 (-\sin x)$$

Differentiating Products

A product is in the form $y = f(x) \times g(x)$

These are examples of products: $y = x^2 \sin x$ $\quad y = \frac{1}{x} e^x$ $\quad y = \ln x \tan x$

Method, using the example

$$k(x) = x^2 \sin x$$

$$k'(x) = 2x \sin x + x^2 \cos x$$

Try this:

$$h(x) = x^3 \cos x$$

$$h'(x) = 3x^2 \cos x - x^3 \sin x$$

Differentiating Products

Differentiate:

$$y = \frac{1}{x}e^{x}$$

$$y = \ln x \tan x$$

Differentiating Products

Differentiate:

$y = \frac{1}{x}e^x$ $\quad\quad$ $\frac{dy}{dx} = -\frac{1}{x^2}e^x + \frac{1}{x}e^x = \frac{1}{x^2}e^x(x-1)$

$y = \ln x \tan x$

Differentiating Products

Differentiate:

$y = \frac{1}{x}e^x$ $\quad \frac{dy}{dx} = -\frac{1}{x^2}e^x + \frac{1}{x}e^x = \frac{1}{x^2}e^x(x-1)$

$y = \ln x \tan x$ $\quad \frac{dy}{dx} = \frac{1}{x}\tan x + \ln x \sec^2 x$

In your Handbook (p48)

If $k(x) = f(x)g(x)$ then $k'(x) = f(x)g'(x) + f'(x)g(x)$

Or if $y = uv$ where u and v are functions of x then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Differentiating Quotients

A quotient is in the form $y = \frac{f(x)}{g(x)}$

These are examples of quotients: $y = \frac{x^2}{\sin x}$ $\quad y = \frac{e^x}{x}$ $\quad y = \frac{\ln x}{\tan x}$

In your Handbook (p48)

If $k(x) = f(x)/g(x)$ then $k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Or if $y = \frac{u}{v}$ where u and v are functions of x then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiating Quotients

A quotient is in the form $y = \frac{f(x)}{g(x)}$

These are examples of quotients: $y = \frac{x^2}{\sin x}$ $y = \frac{e^x}{x}$ $y = \frac{\ln x}{\tan x}$

Method, using an example

$$k(x) = \frac{x^2}{\sin x}$$

$$k'(x) = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$

Square the denominator

Try this:

$$h(x) = \frac{x^3}{\cos x}$$

Differentiating Quotients

A quotient is in the form $y = \frac{f(x)}{g(x)}$

These are examples of products: $y = \frac{x^2}{\sin x}$ $y = \frac{e^x}{x}$ $y = \frac{\ln x}{\tan x}$

Method, using an example

$$k(x) = \frac{x^2}{\sin x}$$

$$k'(x) = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$

Square the denominator

Try this: $h(x) = \frac{x^3}{\cos x}$

$$h'(x) = \frac{3x^2 \cos x - x^3 (-\sin x)}{\cos^2 x}$$

Square the denominator

Differentiating Quotients

A quotient is in the form $y = \frac{f(x)}{g(x)}$

These are examples of products: $y = \frac{x^2}{\sin x}$ $\quad y = \frac{e^x}{x}$ $\quad y = \frac{\ln x}{\tan x}$

Method, using an example

$$k(x) = \frac{x^2}{\sin x}$$

$$k'(x) = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$

Square the denominator

Try this:

$$h(x) = \frac{x^3}{\cos x}$$

$$h'(x) = \frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x}$$

Differentiating Quotients

Differentiate:

$$y = \frac{e^x}{x}$$

$$y = \frac{\ln x}{\tan x}$$

Differentiating Quotients

Differentiate:

$$y = \frac{e^x}{x}$$

$$\frac{dy}{dx} = \frac{xe^x - 1 \times e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

$$y = \frac{\ln x}{\tan x}$$

Differentiating Quotients

Differentiate:

$y = \frac{e^x}{x}$ $\frac{dy}{dx} = \frac{xe^x - 1 \times e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

$y = \frac{\ln x}{\tan x}$ $\frac{dy}{dx} = \frac{\tan(x) \times \frac{1}{x} - \ln x \sec^2 x}{\tan^2 x}$

In your Handbook (p48)

If $k(x) = f(x)/g(x)$ then $k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Or if $y = \frac{u}{v}$ where u and v are functions of x then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiating composite functions (chain rule)

These are composite functions: $y = e^{\sin x}$ $\quad y = \sin(x^2)$ $\quad y = (2x^5 + 3x - 1)^3$

In your Handbook (p48)

If $k(x) = g\big(f(x)\big)$ then $k'(x) = g'\big(f(x)\big)f'(x)$

Or if y is a function of u, where u is a function of x then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Differentiating composite functions (chain rule)

These are composite functions: $y = e^{\sin x}$ $y = \sin(x^2)$ $y = (2x^5 + 3x - 1)^3$

Method:
Start with x and think about the order the functions were built up then differentiate in the opposite order

Taking $f(x) = \sin(x^2)$

- first function is x^2, call it u. The next function is then $\sin u$
- so differentiate $\sin u$ first to get $\cos u$ then multiply this by the derivative of u which, here, is $2x$

$$f'(x) = \underbrace{\cos(x^2)}_{u} \times \underbrace{2x}_{\frac{du}{dx}}$$

Differentiating composite functions (chain rule)

These are composite functions: $y = e^{\sin x}$ $y = \sin(x^2)$ $y = (2x^5 + 3x - 1)^3$

Method:
Start with x and think about the order the functions were built up then differentiate in the opposite order

Taking $f(x) = \sin(x^2)$

- first function is x^2, call it u. The next function is then $\sin u$
- so differentiate $\sin u$ first to get $\cos u$ then multiply this by the derivative of u which is x^2

$$f'(x) = \cos(x^2) \times 2x$$

Try this: $g(x) = e^{\sin x}$

Differentiating composite functions (chain rule)

These are composite functions: $y = e^{\sin x}$ $\quad y = \sin(x^2)$ $\quad y = (2x^5 + 3x - 1)^3$

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Start with x and think about the order the functions were built up then differentiate in the opposite order

Taking $f(x) = \sin(x^2)$

- first function is x^2, call it u. The next function is then $\sin u$
- so differentiate $\sin u$ first to get $\cos u$ then multiply this by the derivative of x^2

So $f'(x) = \cos(x^2) \times 2x$

Try this: $g(x) = e^{\sin x}$

$$g'(x) = e^{\underbrace{\sin x}_{u}} \times \underbrace{\cos x}_{\frac{du}{dx}}$$

or $g'(x) = \cos x e^{\sin x}$

Differentiating composite functions (chain rule)

Differentiate:

$$y = (2x^5 + 3x - 1)^3$$

$$y = \sin^2 x$$

Differentiating composite functions (chain rule)

Differentiate:

$y = (2x^5 + 3x - 1)^3$

$$\frac{dy}{dx} = 3(2x^5 + 3x - 1)^2 \times (10x^4 + 3)$$
$$= 3(10x^4 + 3)(2x^5 + 3x - 1)^2$$

$y = \sin^2 x$

Differentiating composite functions (chain rule)

Differentiate:

$y = (2x^5 + 3x - 1)^3$

$$\frac{dy}{dx} = 3(2x^5 + 3x - 1)^2 \times (10x^4 + 3)$$
$$= 3(10x^4 + 3)(2x^5 + 3x - 1)^2$$

$y = \sin^2 x$

$$\frac{dy}{dx} = 2\sin x \times \cos x$$
$$= 2\sin x \cos x \quad (= \sin 2x)$$

In your Handbook (p48)

If $k(x) = g\big(f(x)\big)$ then $k'(x) = g'\big(f(x)\big)f'(x)$

Or if y is a function of u, where u is a function of x then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Differentiating composite functions (chain rule)

What happens when the first (or inner) function is a simple linear expression like:

$$y = \sin(2x)$$

$2x$ is linear

$$y = e^{3x+1}$$

$3x + 1$ is linear

We can use the chain rule but since <u>the derivative of the linear expression is a constant</u>, we can just write it down directly:

Differentiating composite functions (chain rule)

What happens when the first (or inner) function is a simple linear expression like

$y = \sin(2x)$ then $\frac{dy}{dx} = 2\cos(2x)$

$y = e^{3x+1}$ then $\frac{dy}{dx} = 3e^{3x+1}$

In your Handbook (p48)

If $k(x) = f(ax+b)$ then $k'(x) = af'(ax+b)$

Or if $k(x) = f(ax)$ then $k'(x) = af'(ax)$

Differentiating composite functions (chain rule)

What happens when the first (or inner) function is a simple linear expression like

$y = \sin(2x)$ then $\frac{dy}{dx} = 2\cos(2x)$

$y = e^{3x+1}$ then $\frac{dy}{dx} = 3e^{3x+1}$

In your Handbook (p48)

If $k(x) = f(ax + b)$ then $k'(x) = af'(ax + b)$

Or if $k(x) = f(ax)$ then $k'(x) = af'(ax)$

Note that if $y = \ln(3x)$

then $\frac{dy}{dx} = \frac{1}{x}$

Can you see why?

Good idea to add this to your Handbook

Differentiation – summary exercises

Identify what method applies to these and differentiate them

$$f(x) = e^{2x} \ln x$$

$$g(x) = \sqrt{\cos(3x)}$$

$$h(x) = \frac{\cos x}{1 + \ln x}$$

Differentiation – summary exercises

Identify what method applies to these and differentiate them

$f(x) = e^{2x} \ln x$

(product)

$f'(x) = 2e^{2x} \ln x + e^{2x} \times \frac{1}{x} = e^{2x}(2 \ln x + \frac{1}{x})$

$g(x) = \sqrt{\cos(3x)}$

$h(x) = \frac{\cos x}{1 + \ln x}$

Differentiation – summary exercises

Identify what method applies to these and differentiate them

$f(x) = e^{2x} \ln x$

(product)

$f'(x) = 2e^{2x} \ln x + e^{2x} \times \frac{1}{x} = e^{2x}(2 \ln x + \frac{1}{x})$

$g(x) = \sqrt{\cos(3x)}$

(chain rule)

$g'(x) = \frac{1}{2}(\cos(3x))^{-\frac{1}{2}} \times (-3\sin(3x)) = -\frac{3 \sin 3x}{2\sqrt{\cos 3x}}$

$h(x) = \frac{\cos x}{1 + \ln x}$

Differentiation – summary exercises

Identify what method applies to these and differentiate them

$f(x) = e^{2x} \ln x$

(product)

$f'(x) = 2e^{2x} \ln x + e^{2x} \times \frac{1}{x} = e^{2x}(2 \ln x + \frac{1}{x})$

$g(x) = \sqrt{\cos(3x)}$

(chain rule)

$g'(x) = \frac{1}{2}(\cos(3x))^{-\frac{1}{2}} \times (-3 \sin(3x)) = -\frac{3 \sin 3x}{2\sqrt{\cos 3x}}$

$h(x) = \frac{\cos x}{1+\ln x}$

(quotient)

$h'(x) = \frac{-\sin x(1+\ln x)-(\cos x)\times\left(\frac{1}{x}\right)}{(1+\ln x)^2} = -\frac{(1+\ln x)\sin x+\frac{1}{x}\cos x}{(1+\ln x)^2}$

Differentiating inverse functions

For an inverse function, we use the fact that:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

For example: $y = \cos^{-1} x$ (see activity 24)

This can be written as $x = \cos y$

Now differentiate with respect to y: $\frac{dx}{dy} = -\sin y$ therefore $\frac{dy}{dx} = -\frac{1}{\sin y}$ $(\sin y \neq 0)$

We then have to convert back in terms of x:

Using $\sin^2 y + \cos^2 y = 1$ and re-arranging: $\sin y = \pm\sqrt{1 - \cos^2 y} = \pm\sqrt{1 - x^2}$

Since y takes values in $[0, \pi]$ we take the positive root. Therefore $\frac{dx}{dy} = -\sqrt{1 - x^2}$ and so $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

Optimisation problems

The length of fencing available to enclose an area of land is 200m. Assuming that the area enclosed must be a quadrilateral with right-angled corners, find the dimensions that will maximise the area enclosed.

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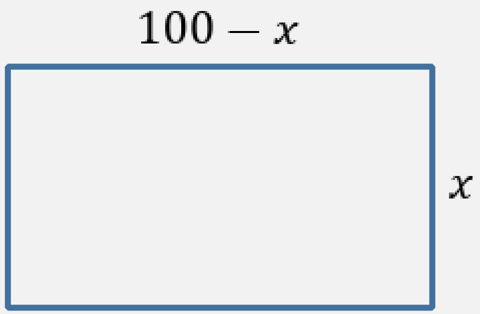

If the total perimeter is 200m then a pair of adjacent sides will have total length 100m. Let the width of the rectangle be x metres then the length will be $(100 - x)$ metres

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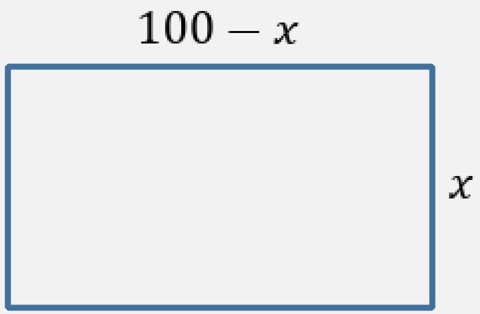

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The area A of the field is given by

$$
\begin{aligned}
A &= x(100 - x) \\
&= 100x - x^2
\end{aligned}
$$

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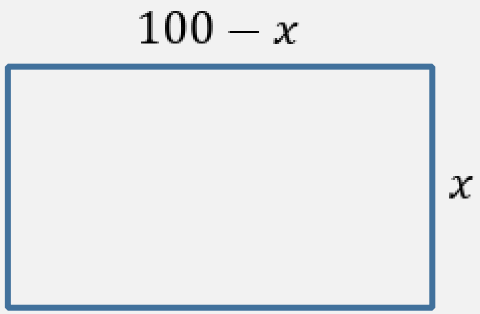

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Differentiate A with respect to x and put this equal to 0: $\frac{dA}{dx} = 100 - 2x = 0$

Solving gives $x = 50$ m

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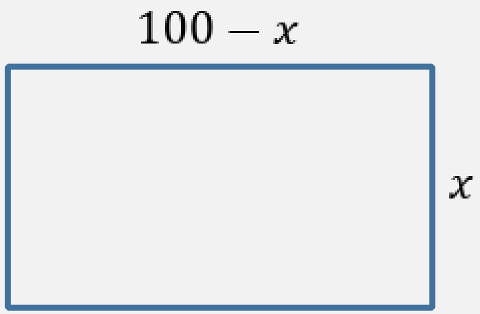

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So the maximum area enclosed by a 200m fence is a square of side 50m.

The maximum area possible is $50 \times 50 = 2500$ m^2

Anti-differentiation or **integration**

Integration

In this unit we start integration by regarding it as *anti-differentiation*

So if $f(x) = 2x$ then the antiderivative is the function $F(x)$ that gives $f(x)$ when we differentiate it.

Here $F(x) = x^2 + c$ where c is a constant (an arbitrary constant)

Check: $F'(x) = 2x = f(x)$

Getting from $f(x)$ to $F(x)$ is called indefinite integration and later on (in Unit 8) we will use this notation to denote it:

$$F(x) = \int f(x)\,dx$$

If we consider the ideas of displacement, velocity and acceleration as in TMA02, then integration enables you to go from acceleration to velocity and then to displacement.......

Integration

Power functions:

Arbitrary constant

If $f(x) = x^3$ then the antiderivative is $F(x) = \frac{x^4}{4} + \mathrm{c}$ [check: $F'(x) = \frac{4x^3}{4} = x^3 = f(x)$]

In general, if

$$f(x) = x^n$$

then the antiderivative is

$$F(x) = \frac{x^{n+1}}{n+1} + c$$

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In general, if $f(x) = x^n$ then the antiderivative $F(x) = \frac{x^{n+1}}{n+1} + c$

Try integrating these:

$f(x) = 3x^2$

$f(x) = \frac{1}{x^2}$

$f(x) = \sqrt{x}$

Integration

Power functions:

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$f(x) = \frac{1}{x^2} = x^{-2}$ $F(x) = -x^{-1} + c$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $F(x) = \frac{2x^{\frac{3}{2}}}{3} + c$

Integration – constant multiples

If $f(x) = 3x^2 + x - 5$ then $F(x) = x^3 + \frac{x^2}{2} - 5x + c$

Integration – constant multiples

If $f(x) = 3x^2 + x - 5$ then $F(x) = x^3 + \frac{x^2}{2} - 5x + c$

Note the antiderivative of the constant -5

Think of -5 as $-5x^0$
And then apply the usual method

Check by differentiating $-5x$

And remember the arbitrary constant

Integration – constant multiples

If $f(x) = 3x^2 + x - 5$ then $F(x) = x^3 + \frac{x^2}{2} - 5x + c$

Try integrating this:

$$f(x) = \frac{1}{\sqrt{x}} + 3x^4 - \frac{1}{x^3} + 2$$

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First put into index form: $f(x) = x^{-\frac{1}{2}} + 3x^4 - x^{-3} + 2$

Then integrate, term by term $F(x) = 2x^{\frac{1}{2}} + \frac{3x^5}{5} + \frac{x^{-2}}{2} + 2x + c$

$$F(x) = 2\sqrt{x} + \frac{3x^5}{5} + \frac{1}{2x^2} + 2x + c$$

Integration – finding c

If we have some information about a particular value of $F(x)$ we can use it to find c

Example: When a cyclist sets off, her displacement from home is 2km. She cycles with velocity $v = 3 + 2t$

Find a function for her displacement (s) after time t

Since v is the derivative of s with respect to t $\left(\text{or } \frac{ds}{dt}\right)$ we need to integrate v to obtain the function for s:

$$s = 3t + t^2 + c$$

We are told that at $t = 0$ the cyclist is at 2km, so we can substitute these values into the function for s:

$$2 = 3 \times 0 + 0^2 + c \qquad \text{giving } c = 2$$

Therefore $s = 3t + t^2 + 2$

Standard integrals

Standard integrals

As with the derivatives of the special functions, the equivalent integrals are given alongside on p7 of the Handbook

Standard indefinite integrals

Function	Indefinite integral
a (constant)	$ax + c$
$x^n \ (n \neq -1)$	$\dfrac{1}{n+1}\, x^{n+1} + c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + c$ or $\ln x + c$, for $x > 0$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\operatorname{cosec}^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$ or $-\cos^{-1} x + c$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + c$

These are the two we've looked at already

These you will use often in MST124

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Try integrating:

$$f(x) = \sin x + \frac{1}{x} - e^x + \frac{1}{1+x^2}$$

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Try integrating:

$$f(x) = \sin x + \frac{1}{x} - e^x + \frac{1}{1+x^2}$$

$$F(x) = -\cos x + \ln|x| - e^x + \tan^{-1} x + c$$

It is very important that you do as many exercises as you can on differentiating and integrating to get used to spotting what approach to take. In Unit 8 you will learn how to integrate more complicated functions such as products and composites so it will be assumed that you can do the easy ones fluently.

iCMA due Tuesday 22nd February

It covers the work in Units 5, 6 and 7 so you can start now!