Q 1. The general solution of an inhomogenous second order differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b, and c are constants, is given by $y = y_c + y_p$, where y_c, the complementary function, is the general solution of the associated homogenous equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

and y_p is any particular solution of the original inhomogenous equation (MST210 Handbook, page 30). The associated homogenous equation for the differential equation to be solved is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 0$$

with auxiliary equation

$$\begin{aligned} \lambda^2 + 6\lambda + 8 &= 0 \\ (\lambda+2)(\lambda+4) &= 0. \end{aligned}$$

As the auxiliary equation has two real roots $\lambda_1 = -2$ and $\lambda_2 = -4$,

$$\begin{aligned} y_c &= Ce^{\lambda_1 x} + De^{\lambda_2 x} \\ &= Ce^{-2x} + De^{-4x}, \end{aligned}$$

where C and D are arbitrary constants.

To find y_p, we note that $f(x)$ is the sum of an exponential function and a polynomial function. We apply Procedure 7 (MST210 Book A, page 53) to find particular integrals for

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 3e^{-x} \tag{1.1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 8x + 22 \tag{1.2}$$

separately, then apply the principle of superposition (MST210 Book A, page 36) to find y_p. To solve (1.1) we try a solution of the form $y = pe^{-x}$, where p is a coefficient to be determined. Differentiating $y = pe^{-x}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -pe^{-x} \quad , \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = pe^{-x}$$

Substituting these derivatives into (1.1) gives

$$\begin{aligned} pe^{-x} - 6pe^{-x} + 8pe^{-x} &= 3e^{-x} \\ 3pe^{-x} &= 3e^{-x} \\ p &= 1 \end{aligned}$$

Therefore a particular solution of (1.1) is $y_{p1} = e^{-x}$.

To solve (1.2) we try a solution of the form $y = p_1 x + p_0$, where p_1 and p_0 are coefficients to be determined. Differentiating $y = p_1 x + p_0$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p_1 \quad , \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

Substituting these derivatives into (1.2) gives

$$6p_1 + 8(p_1 x + p_0) = 8x + 22$$

Comparing terms in x and the constant terms gives $p_1 = 1$ and $p_0 = 2$. Therefore a particular solution of (1.2) is $y_{p2} = x + 2$.

Then, by the principle of superposition

$$\begin{aligned} y_p &= y_{p1} + y_{p2} \\ &= e^{-x} + x + 2 \end{aligned}$$

and the general solution to the original inhomogenous differential equation is

$$\begin{aligned} y &= y_c + y_p \\ &= Ce^{-2x} + De^{-4x} + e^{-x} + x + 2 \end{aligned}$$

where C and D are arbitrary constants.

Q 2. The vectors **a** and **b** in component form are

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ &= 2\mathbf{i} + \mathbf{j} + -3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \\ &= -1\mathbf{j} + \mathbf{k} \end{aligned}$$

where **i**, **j**, and **k** are Cartesian unit vectors.

As shown in MST210 Handbook page 35, in component form, the cross product of vectors **a** and **b** is

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= (1 - 3)\mathbf{i} + (0 - 2)\mathbf{j} + (-2 - 0)\mathbf{k} \\ &= -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \end{aligned}$$

Q 3.

(a) Figure 1 shows a diagram of the situation.

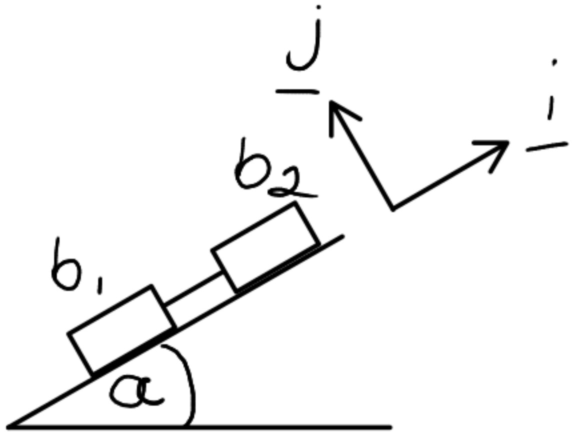

Figure 1: The **i** and **j** coordinate axes have been chosen to point in the direction of the plane and perpendicular to it, respectively. b_1 and b_2 indicate the blocks, and α indicates the angle between the plane and the horizontal.

(b) Figure 2 shows a force diagram for each block.

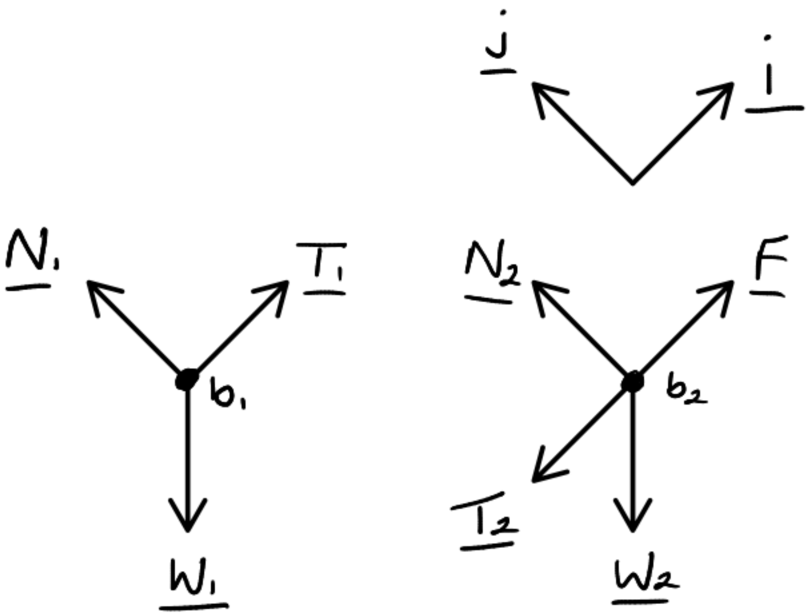

Figure 2: Each block, b_1 and b_2, is represented as a particle with arrows indicating the forces acting on it. W, N, T, and F indicate the weight, normal reaction, tension, and friction forces, respectively, with subscripts indicating whether each force is acting on b_1 or b_2 (F acts only on b_2 and is not given a subscript). The force of gravity acts vertically downwards, the tension and friction forces act parallel to the **i** direction and the normal reaction acts in the positive **j** direction.

(c) We model the blocks b_1 and b_2 as particles with masses m_1 and m_2, respectively, and the string connecting them as a model string. We also make the following definitions:

$\mathbf{W_1}$, $\mathbf{N_1}$, and $\mathbf{T_1}$ are the weight, normal reaction, and tension, respectively, acting on b_1. $\mathbf{W_2}$, $\mathbf{N_2}$, $\mathbf{T_2}$, and $\mathbf{F}$ are the weight, normal reaction, tension, and friction, respectively, acting on b_2.

Resolving each force into it's $\mathbf{i}$ and $\mathbf{j}$ components gives

$$\begin{aligned}
\mathbf{N_1} &= |\mathbf{N_1}|\mathbf{j} \\
\mathbf{T_1} &= |\mathbf{T_1}|\mathbf{i} \\
\mathbf{W_1} &= m_1 g \cos\left(\frac{\pi}{2}+\alpha\right)\mathbf{i} + m_1 g \sin\left(\frac{\pi}{2}+\alpha\right)\mathbf{j} \\
&= -m_1 g \sin\alpha \mathbf{i} + m_1 \cos\alpha \mathbf{j} \\
\mathbf{N_2} &= |\mathbf{N_2}|\mathbf{j} \\
\mathbf{T_2} &= -|\mathbf{T_1}|\mathbf{i} \\
\mathbf{F} &= |\mathbf{F}|\mathbf{i} \\
\mathbf{W_2} &= m_2 g \cos\left(\frac{\pi}{2}+\alpha\right)\mathbf{i} + m_2 g \sin\left(\frac{\pi}{2}+\alpha\right)\mathbf{j} \\
&= -m_2 g \sin\alpha \mathbf{i} + m_2 \cos\alpha \mathbf{j}
\end{aligned}$$

where g is the acceleration due to gravity. The equilibrium condition tells us that

$$\begin{aligned}
\mathbf{W_1} + \mathbf{T_1} + \mathbf{N_1} &= \mathbf{0} \\
(|\mathbf{T_1}| - m_1 g \sin\alpha)\mathbf{i} + (|\mathbf{N_1}| + m_1 g \cos\alpha)\mathbf{j} &= \mathbf{0} \qquad (3.1) \\
&\text{and} \\
\mathbf{W_2} + \mathbf{T_2} + \mathbf{N_2} + \mathbf{F} &= \mathbf{0} \\
(|\mathbf{F}| - |\mathbf{T_2}| - m_2 g \sin\alpha)\mathbf{i} + (|\mathbf{N_2}| + m_2 g \cos\alpha)\mathbf{j} &= \mathbf{0} \qquad (3.2)
\end{aligned}$$

and also that $|\mathbf{F}| \leq \mu|\mathbf{N_2}|$. As the string is taut, $|\mathbf{T_1}| = |\mathbf{T_2}|$.

(d) Resolving equations (3.1) and (3.2) into their $\mathbf{i}$ and $\mathbf{j}$ components gives $|\mathbf{T_1}| = m_1 g \sin\alpha$ and $|\mathbf{N_2}| = -m_2 g \cos\alpha$. Then, taking the $\mathbf{i}$ component of (3.2), and using the fact that $|\mathbf{T_1}| = |\mathbf{T_2}|$:

$$\begin{aligned}
|\mathbf{F}| - |\mathbf{T_2}| - m_2 g \sin\alpha &= 0 \\
|\mathbf{F}| &= m_1 g \sin\alpha + m_2 g \sin\alpha
\end{aligned}$$

Then, as $|\mathbf{F}| \leq \mu|\mathbf{N_2}|$:

$$\begin{aligned}
(m_1+m_2) g \sin\alpha &\geq -\mu m_2 g \cos\alpha \\
(m_1+m_2) g \sin\alpha &\leq \mu m_2 g \cos\alpha \\
(m_1+m_2) \tan\alpha &\leq \mu m_2
\end{aligned}$$

as required.

(e) Substituting $m_1 = 2$, $\mu = \frac{2}{3}$ and $\tan\alpha = \frac{1}{2}$ into the solution from part (d) gives

$$\begin{aligned}
\frac{1}{2}(2 + m_2) &\leq \frac{2}{3}m_2 \\
\frac{1}{2}m_2 + 1 &\leq \frac{2}{3}m_2 \\
1 &\leq \frac{1}{6}m_2 \\
6 \leq m_2 &
\end{aligned}$$

Therefore, the system will remain in equilibrium when m_2 is greater than or equal to 6kg.

Q 4. There are three forces acting on the pole, $\mathbf{W}$, $\mathbf{T}$, and $\mathbf{P}$.

The equilibrium condition for rigid bodies (MST210 Book A, page 144) tells us that

$$\mathbf{W} + \mathbf{T} + \mathbf{P} = \mathbf{0}$$
$$\mathbf{\Gamma}_W + \mathbf{\Gamma}_T + \mathbf{\Gamma}_P = \mathbf{0}$$

where $\mathbf{\Gamma}_W$, $\mathbf{\Gamma}_T$, and $\mathbf{\Gamma}_P$ are the torques with respect to the origin O of $\mathbf{W}$, $\mathbf{T}$, and $\mathbf{P}$, respectively. By choosing O to be at the hinge, $\mathbf{\Gamma}_P$ can be eliminated to give

$$\mathbf{\Gamma}_W + \mathbf{\Gamma}_T = \mathbf{0} \tag{4.1}$$

The position vectors of the points of application of $\mathbf{W}$ and $\mathbf{T}$ are

$$\mathbf{r}_W = L\mathbf{i}$$
$$\mathbf{r}_T = 2L\mathbf{i}$$

The weight of the pole in component form is $\mathbf{W} = -Mg\mathbf{j}$. As the wire can be modelled as a model string, $|\mathbf{T}| = mg$ and the tension is

$$\begin{aligned}\mathbf{T} &= |\mathbf{T}|\cos\left(\frac{\pi}{2} - \theta\right)\mathbf{i} + |\mathbf{T}|\sin\left(\frac{\pi}{2} - \theta\right)\mathbf{j} \\ &= |\mathbf{T}|\sin\theta\mathbf{i} + |\mathbf{T}|\cos\theta\mathbf{j} \\ &= mg\sin\theta\mathbf{i} + mg\cos\theta\mathbf{j}\end{aligned}$$

Torque is the cross product of a force and the position vector of its point of application (MST210 Book A, page 143), so $\mathbf{\Gamma}_W$ and $\mathbf{\Gamma}_T$ are given by

$$\begin{aligned}\mathbf{\Gamma}_W &= \mathbf{r}_W \times \mathbf{W} \\ &= L\mathbf{i} \times (-Mg)\mathbf{j} \\ &= -MgL\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{\Gamma}_T &= \mathbf{r}_T \times \mathbf{T} \\ &= 2L\mathbf{i} \times mg\sin\theta\mathbf{i} + mg\cos\theta\mathbf{j} \\ &= 2Lmg\cos\theta\mathbf{k}\end{aligned}$$

Substituting these torques into (4.1) gives

$$\begin{aligned}(2Lmg\cos\theta - MgL)\mathbf{k} &= \mathbf{0} \\ 2Lmg\cos\theta - MgL &= 0 \\ 2Lmg\cos\theta &= MgL \\ \cos\theta &= \frac{M}{2m} \\ \theta &= \cos^{-1}\left(\frac{M}{2m}\right)\end{aligned}$$

Q 5.

(a) Figure 3 shows a picture of the situation.

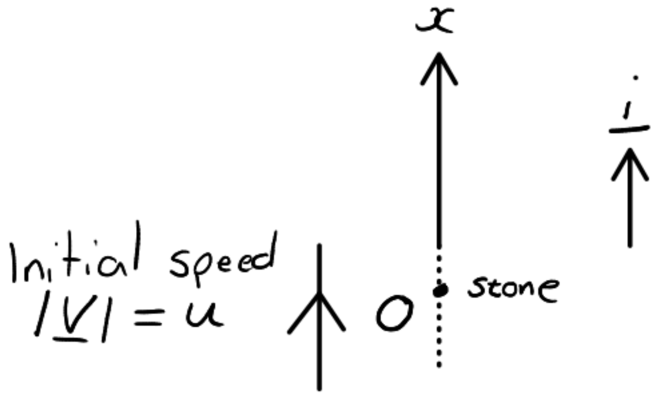

Figure 3: The x axis points vertically upwards (in the same direction as the unit vector $\mathbf{i}$) with the origin O at the position of the stone. The initial speed of the stone is u.

(b) The x axis is chosen to point vertically upwards (in the same direction as the unit vector $\mathbf{i}$) with the origin O at the position of the stone.

(c)

Figure 4 shows a force diagram of the situation.

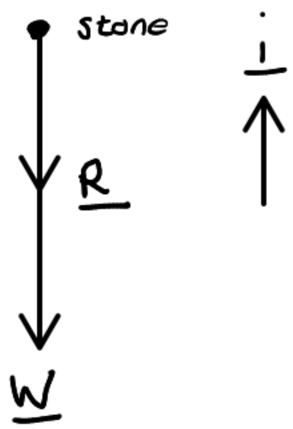

Figure 4: The two forces acting on the stone in flight, its weight $\underline{W}$ and air resistance $\underline{R}$, act vertically downwards in the negative $\mathbf{i}$ direction.

(d) We model the stone as a particle and assume the only forces acting on it in flight are its weight and air resistance, both acting downwards against its initial direction of travel.

(e) Newton's second law gives

$$m\mathbf{a} = \mathbf{W} + \mathbf{R} \tag{5.1}$$

From the force diagram we can say that $\mathbf{W} = -|\mathbf{W}|\mathbf{i} = mg\mathbf{i}$.

The quadratic model of air resistance (MST210 Book A, page 206) tells us that $\mathbf{R} = -c_2 D^2 |\mathbf{v}|\mathbf{v}$, where $c_2 = 0.2$ and D is the effective diameter of the particle. Since the motion is in the positive $\mathbf{i}$ direction, $\mathbf{v} = v\mathbf{i}$ where $v = |\mathbf{v}|$ and $\mathbf{R} = -c_2 D^2 v^2 \mathbf{i}$.

Resolving (5.1) in the $\mathbf{i}$ direction gives

$$ma = -mg - c_2 D^2 v^2$$

Using the substitutions $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$ and $k = \frac{c_2 D^2}{m}$ gives

$$\begin{aligned} mv\frac{\mathrm{d}v}{\mathrm{d}x} &= -mg - c_2 D^2 v^2 \\ v\frac{\mathrm{d}v}{\mathrm{d}x} &= -(g + kv^2) \end{aligned}$$

as required.

(f) Using the method of separation of variables gives

$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}x} &= -(g + kv^2) \\ -\int \frac{v}{g + kv^2} dv &= \int 1 dx \\ -\frac{1}{2k}\int \frac{1}{g + kv^2} dv &= \int 1 dx \\ -\frac{1}{2k}\ln(g + kv^2) &= x - A \end{aligned}$$

where A is an arbitrary constant. Therefore the distance travelled by the stone is given by

$$x = A - \frac{\ln(g + kv^2)}{2k}$$

(g) Note to tutor: I couldn't derive the target equation but present my thinking here in the hope of some feedback.

When $x = 0$, $v = u$ and substituting these initial values into the equation for x from part (f) gives

$$0 = A - \frac{\ln(g + ku^2)}{2k}$$
$$A = \frac{\ln(g + ku^2)}{2k}$$

At the stone's maximum height, $v = 0$. Substituting $v = 0$ and the expression for A into the equation for x gives

$$x_{\text{max}} = -\frac{\ln(g) + \ln(g + ku^2)}{2k}$$
$$- \frac{1}{2k}\big(\ln(g) + \ln(g + ku^2)\big)$$

Q 6.

(a) Figure 5 shows a force diagram of the situation.

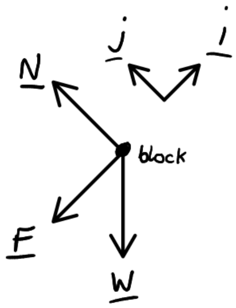

Figure 5: The three forces acting on the block, its weight W, normal reaction N, and friction F.

(b) The three forces in component form are

$$\begin{aligned}
\mathbf{N} &= |\mathbf{N}|\mathbf{j} \\
\mathbf{F} &= -|\mathbf{F}|\mathbf{i} \\
\mathbf{W} &= |\mathbf{W}|\cos\left(\frac{\pi}{2}+\alpha\right)\mathbf{i} + |\mathbf{W}|\sin\left(\frac{\pi}{2}+\alpha\right)\mathbf{j} \\
&= -mg\sin\alpha\mathbf{i} + mg\cos\alpha\mathbf{j}
\end{aligned}$$

(c) Substituting $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$ into the equation for the acceleration of the block and solving the differential equation gives

$$\begin{aligned}
v\frac{\mathrm{d}v}{\mathrm{d}x} &= -g(\sin\alpha + \mu'\cos\alpha) \\
-\int \frac{v}{g(\sin\alpha+\mu'\cos\alpha)}dv &= \int 1dx \\
x &= -\frac{v^2}{2g(\sin\alpha+\mu'\cos\alpha)} + C \qquad (6.1)
\end{aligned}$$

where C is an arbitrary constant.

When $x = 0$:

$$C = \frac{u^2}{2g(\sin\alpha+\mu'\cos\alpha)}$$

When the block comes to rest, $v = 0$, and so substituting this and the expression for C into (6.1) gives

$$\begin{aligned} x &= -\frac{0^2}{2g(\sin\alpha + \mu' \cos\alpha)} + \frac{u^2}{2g(\sin\alpha + \mu' \cos\alpha)} \\ &= \frac{u^2}{2g(\sin\alpha + \mu' \cos\alpha)} \end{aligned}$$

as required.

(d) As shown in figure 6, the angle that the slope needs to have to slow the block to rest at the top of the slope is 5° (to the nearest degree).

```
(%i1) kill(all);
(%o0) done

(%i1) x:0.75;
(%o1) 0.75

(%i2) u:3.5;
(%o2) 3.5

(%i3) g:9.81;
(%o3) 9.81

(%i4) mu:3/4;
(%o4) 3/4

(%i7) rads:find_root(x=u^2/(2·g·(sin(a)+mu·cos(a))), a, 0, %pi/2);
(%o7) 0.08531516831887809

(%i9) float(rads·(180/%pi));
(%o9) 4.888199073119945
```

Figure 6: Maxima worksheet showing the solution to the problem.