Unit 11

Eigenvalues and eigenvectors

Introduction

This unit develops the study of geometric transformations that you began in Unit 6. Much of our attention will be on finding the position vectors of points that are mapped by a linear transformation to scalar multiples of themselves. Here a scalar multiple of a point (x, y) is another point of the form (kx, ky), for some real number k.

Consider, for example, the linear transformation

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This computation shows that f maps the point $(1, 1)$ with position vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to a scalar multiple of itself. More precisely, it scales this point and the corresponding position vector by the factor 2, as illustrated in Figure 1.

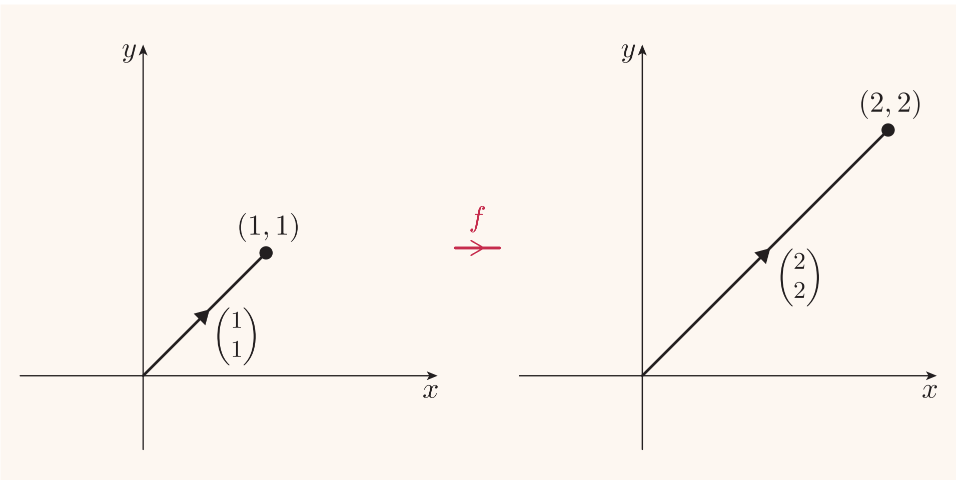


Figure 1 A transformation mapping $(1, 1)$ to $(2, 2)$

A non-zero vector that is scaled by a linear transformation in this sort of way is called an *eigenvector* of the matrix representing the tranformation, and the scale factor is called an *eigenvalue* of the matrix corresponding to this eigenvector. Here you'll learn how to find the eigenvalues and eigenvectors of 2×2 matrices. In fact, you'll see that finding the eigenvalues of a 2×2 matrix comes down to solving a quadratic equation, and finding eigenvectors corresponding to these eigenvalues is a matter of solving pairs of simultaneous equations.

Later in the unit you'll get a taste of some of the uses of eigenvalues and eigenvectors in modelling real-world situations. For example, you'll see how the eigenvalues and eigenvectors of matrices can be used in an efficient method for calculating powers of matrices, which has applications to modelling the sizes of human or wildlife populations in a subject called *population dynamics*. You'll also find out how the eigenvalues and eigenvectors of a matrix are employed in solving systems of differential equations; systems that have important applications to the biological and physical sciences.

1 Eigenvalues and eigenvectors of matrices

In this section you'll find out what *eigenvalues* and *eigenvectors* of matrices are, and learn strategies for finding them.

1.1 What are eigenvalues and eigenvectors?

In Unit 6 you learned that a **linear transformation** of the plane $\mathbb{R}^2$ is a transformation f whose rule is of the form

$$f(x,y) = (ax+by, cx+dy),$$

where a, b, c and d are real constants. You also learned how to represent a linear transformation by a 2×2 matrix, as follows. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The vector $\mathbf{x}$ is the position vector of the point (x,y), in column vector form. Since

$$\mathbf{Ax} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix},$$

it follows that the rule for f can be expressed in the column vector form

$$f(\mathbf{x}) = \mathbf{Ax}.$$

As you saw in Unit 6, one-to-one linear transformations map lines to lines. You didn't learn about the images of lines under linear transformations that are not one-to-one in Unit 6, but in fact such transformations map lines to either lines or single points, though we won't prove that here. We say that a line is an **invariant line** of a linear transformation if its image under the transformation is the original line, or just a point on the original line. In this unit we're particularly interested in finding invariant lines of linear transformations that pass through the origin. Since linear transformations fix the origin, a linear transformation that has an invariant line that passes through the origin maps that line either to itself, or to the origin.

For example, consider the linear transformation f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

You'll recall from Unit 6 that this transformation is called a $(2,3)$-**scaling** because it scales horizontal distances by the factor 2 and scales vertical distances by the factor 3. In particular, it fixes the x-axis and the y-axis, so both these lines are invariant lines of the transformation, and both lines pass through the origin.

Since the x-axis is an invariant line of the $(2,3)$-scaling f, any point on the x-axis is mapped to a scalar multiple of itself. Here are some examples, using matrix and column vector notation:

$$\mathbf{A}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix},\quad \mathbf{A}\begin{pmatrix}3\\0\end{pmatrix}=\begin{pmatrix}6\\0\end{pmatrix}\quad\text{and}\quad \mathbf{A}\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}-2\\0\end{pmatrix}.$$

The blue arrows and points on the x-axes in Figure 2 illustrate the first of these equations. (The other arrows and points in Figure 2 will be discussed shortly.)

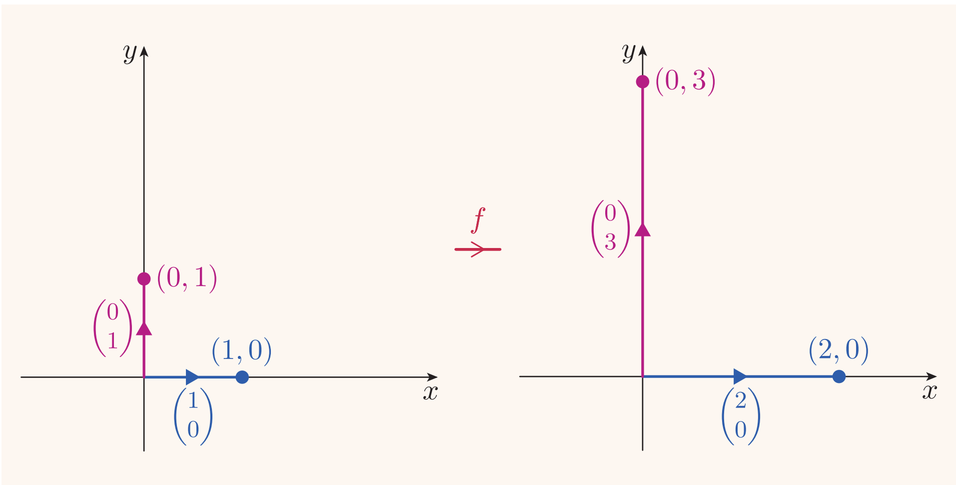


Figure 2 The effect of the $(2,3)$-scaling

These examples indicate that f scales any point on the x-axis by the factor 2. To verify this, choose any point $(k,0)$ on the x-axis, and observe that

$$\mathbf{A}\begin{pmatrix}k\\0\end{pmatrix}=\begin{pmatrix}2k\\0\end{pmatrix}=2\begin{pmatrix}k\\0\end{pmatrix},$$

so f does indeed scale $(k,0)$ by the factor 2.

The y-axis is also an invariant line of f, so points on the y-axis are also mapped to scalar multiples of themselves, as the following calculations demonstrate:

$$\mathbf{A}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix},\quad \mathbf{A}\begin{pmatrix}0\\3\end{pmatrix}=\begin{pmatrix}0\\9\end{pmatrix}\quad\text{and}\quad \mathbf{A}\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}0\\-3\end{pmatrix}.$$

The purple arrows and points on the y-axes in Figure 2 illustrate the first of these equations. The examples indicate that f scales any point on the y-axis by the factor 3. You'll carry out a calculation to check this assertion in the next activity.

Activity 1 *Investigating the $(2,3)$-scaling*

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Show that if $\mathbf{x} = \begin{pmatrix} 0 \\ k \end{pmatrix}$, for some number k, then $\mathbf{Ax} = 3\mathbf{x}$.

To summarise, we've found that for the linear transformation $f(x) = \mathbf{Ax}$, where $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, the equation $\mathbf{Ax} = 2\mathbf{x}$ is satisfied whenever $\mathbf{x}$ is the position vector of a point on the x-axis, and the equation $\mathbf{Ax} = 3\mathbf{x}$ is satisfied whenever $\mathbf{x}$ is the position vector of a point on the y-axis. The first equation tells us that the x-axis is an invariant line of the transformation f, and each point on the x-axis is scaled under the transformation by the factor 2. Likewise, the second equation tells us that the y-axis is an invariant line of f, and each point on the y-axis is scaled by the factor 3.

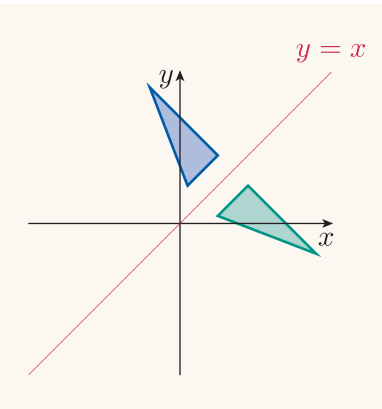


Figure 3 Reflection in the line $y = x$

Let's find a similar pair of equations and invariant lines for another linear transformation that you met in Unit 6, namely the linear transformation f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This transformation is the reflection in the line $y = x$, illustrated in Figure 3 by a triangle and its image. This line is an invariant line of the reflection; in fact, each point on the line is fixed by the transformation. For example, the points with position vectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

all lie on the line $y = x$, and

$$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{A}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

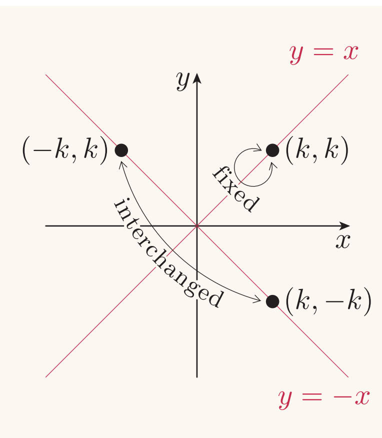


Figure 4 Images of points under the reflection in the line $y = x$

The line $y = -x$, which is perpendicular to the line $y = x$, is also an invariant line of f that passes through the origin. The effect of f on points on these two lines is illustrated in Figure 4.

Each point on the line $y = -x$ is scaled by f by the factor -1. For example, the points with position vectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

both lie on the line $y = -x$, and

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} = -\begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

In the next activity you are asked to perform calculations that confirm that f fixes each point on the line $y = x$, and f scales each point on the line $y = -x$ by the factor -1.

Activity 2 *Investigating the reflection in the line $y = x$*

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Show that if $\mathbf{x} = \begin{pmatrix} k \\ k \end{pmatrix}$, for some number k, then $\mathbf{Ax} = \mathbf{x}$.

(b) Show that if $\mathbf{x} = \begin{pmatrix} k \\ -k \end{pmatrix}$, for some number k, then $\mathbf{Ax} = -\mathbf{x}$.

In each of the examples you've seen so far there has been a 2×2 matrix $\mathbf{A}$, a vector $\mathbf{x}$ and a number λ such that

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

More generally, we have the following definitions, which apply to square matrices of any size (not just 2×2 matrices).

Eigenvalues and eigenvectors

An **eigenvalue** of a square matrix $\mathbf{A}$ is a number λ such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

for some non-zero vector $\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of $\mathbf{A}$ corresponding to λ.

For each eigenvalue λ of $\mathbf{A}$, the equation $\mathbf{Ax} = \lambda\mathbf{x}$ is called the **eigenvector equation** of $\mathbf{A}$ for the eigenvalue λ.

For example, if

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then, as you have seen, $\mathbf{Ax} = 2\mathbf{x}$, so 2 is an eigenvalue of $\mathbf{A}$, and $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2. In fact, any non-zero scalar multiple of $\mathbf{x}$ is also an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2. This is because if we scale $\mathbf{x}$ by the factor k then we obtain the vector $\begin{pmatrix} k \\ 0 \end{pmatrix}$, and

$$\mathbf{A}\begin{pmatrix} k \\ 0 \end{pmatrix} = 2\begin{pmatrix} k \\ 0 \end{pmatrix},$$

as you saw earlier.

More generally, suppose that $\mathbf{x}$ is an eigenvector corresponding to an eigenvalue λ of some matrix $\mathbf{A}$ (so that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$). Now choose *any* non-zero scalar multiple of $\mathbf{x}$, say $k\mathbf{x}$, for some non-zero real number k. Then

$$\mathbf{A}(k\mathbf{x}) = k(\mathbf{A}\mathbf{x}) = k(\lambda\mathbf{x}) = \lambda(k\mathbf{x}),$$

which shows that $k\mathbf{x}$ is also an eigenvector of $\mathbf{A}$ corresponding to λ. These vectors $k\mathbf{x}$ are the position vectors of the points on a line through the origin. This line is an invariant line of the linear transformation represented by the matrix $\mathbf{A}$.

We record this important observation about scalar multiples of eigenvectors in the box below.

Scalar multiples of eigenvectors

If $\mathbf{x}$ is an eigenvector of a square matrix $\mathbf{A}$ corresponding to an eigenvalue λ of $\mathbf{A}$, then any non-zero scalar multiple of $\mathbf{x}$ is also an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

You've just seen that there are infinitely many eigenvectors corresponding to a given eigenvalue. In contrast, given an eigenvector $\mathbf{x}$ of a matrix $\mathbf{A}$, there can be only one number λ that satisfies the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. So for a given eigenvector there is only one corresponding eigenvalue.

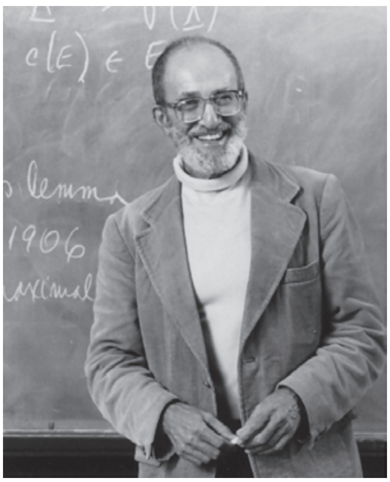
You'll notice from the definition of eigenvectors that an eigenvector is a non-zero vector. This means that an eigenvector cannot equal the zero vector $\mathbf{0}$, which in two dimensions is

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The zero vector is excluded because the equation

$$\mathbf{A}\mathbf{0} = \lambda\mathbf{0},$$

where $\mathbf{0}$ is the zero vector in n-dimensions, is true for *any* $n \times n$ matrix $\mathbf{A}$ and *any* number λ, as both sides of the equation equal $\mathbf{0}$.



Paul Halmos (1916–2006)

Origin of the terms 'eigenvalue' and 'eigenvector'

The word 'eigen' is German, meaning 'own' or 'self', and was brought into major use in the early twentieth century by the German mathematician David Hilbert (1862–1943). However, in English the term 'proper value' was used instead of 'eigenvalue' and remained in use until the 1960s. Indeed, in 1967 the Hungarian-born mathematician Paul Halmos finally admitted defeat, as you can see from the following extract from the preface to his text *A Hilbert Space Problem Book*.

For many years I have battled for proper values, and against the one and a half times translated German–English hybrid that is often used to refer to them. I have now become convinced that the war is over, and eigenvalues have won it; in this book I use them.

(Halmos, P. (1967) *A Hilbert Space Problem Book*, D. Van Nostrand Co., p. x)

Example 1 *Finding an eigenvalue corresponding to a given eigenvector*

Show that the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix

$$\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix},$$

and find the corresponding eigenvalue.

Solution

You need to show that $\mathbf{Ax} = \lambda\mathbf{x}$ for the given matrix $\mathbf{A}$ and vector $\mathbf{x}$, and for some number λ, which you must find. First calculate $\mathbf{Ax}$.

We have

$$\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Write the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ in the form $\lambda\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ for some number λ.

$$= -2\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

So $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2.

Activity 3 *Finding eigenvalues corresponding to given eigenvectors*

Show that the vectors $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix

$$\begin{pmatrix} -3 & -1 \\ 4 & 2 \end{pmatrix},$$

and find the corresponding eigenvalues.

1.2 Finding eigenvalues

In the previous subsection you saw that if

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

then the eigenvector equation $\mathbf{Ax} = 2\mathbf{x}$ is satisfied when $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the eigenvector equation $\mathbf{Ax} = 3\mathbf{x}$ is satisfied when $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This shows that 2 and 3 are eigenvalues of $\mathbf{A}$.

However, it's often the case that you won't know (or you won't be able to guess) eigenvectors of a matrix in advance. Here you'll learn a systematic way to find the eigenvalues of a 2×2 matrix directly, without first knowing what the eigenvectors are. You'll see that a 2×2 matrix has either one or two eigenvalues, which might be real numbers, or they might be complex numbers that are not real numbers. In the next subsection, you'll use the eigenvalues of a matrix to find corresponding eigenvectors. Finally, at the end of this section, we'll discuss how to find eigenvalues and eigenvectors of square matrices of size larger than 2×2.

The method we use to find the eigenvalues of a 2×2 matrix $\mathbf{A}$ is to obtain a quadratic equation whose solutions are the eigenvalues. To this end, let's begin by writing the eigenvector equation

$$\mathbf{Ax} = \lambda\mathbf{x},$$

for a given eigenvalue λ in an alternative form. First subtract $\lambda\mathbf{x}$ from both sides to give

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}. \tag{1}$$

The right-hand side is the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

It would now be helpful to factorise the left-hand side of this equation by taking out the common factor $\mathbf{x}$. However, we can't do this directly, as this would leave the factor '$\mathbf{A} - \lambda$' which doesn't make sense: we can't subtract a number from a 2×2 matrix.

Instead, we can make use of the 2×2 identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix satisfies $\mathbf{Ix} = \mathbf{x}$, so we can write equation (1) as

$$\mathbf{Ax} - \lambda\mathbf{Ix} = \mathbf{0}.$$

Since $\mathbf{A}$ and $\lambda\mathbf{I}$ are both 2×2 matrices, we can use standard properties of matrices to write the left-hand side as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}$. This gives an alternative, equivalent form of the eigenvector equation.

Equivalent form of the eigenvector equation

An equivalent form of the eigenvector equation $\mathbf{Ax} = \lambda\mathbf{x}$ is

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

This equivalent form of the eigenvector equation is valid when $\mathbf{A}$ is an $n \times n$ matrix, for any positive integer n, provided that $\mathbf{I}$ is interpreted as the $n \times n$ identity matrix, and $\mathbf{x}$ and $\mathbf{0}$ are n-dimensional vectors.

We can use the equivalent form of the eigenvector equation to explain how to find the eigenvalues of a 2×2 matrix. To demonstrate the process, let's write

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and let's suppose that λ is an eigenvalue of $\mathbf{A}$ with a corresponding eigenvector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

As $\mathbf{A}$ and $\lambda\mathbf{I}$ are 2×2 matrices, the expression $\mathbf{A} - \lambda\mathbf{I}$ also represents a 2×2 matrix. For convenience, let's denote this matrix by $\mathbf{B}$, so that the equivalent form of the eigenvector equation for the eigenvalue λ is

$$\mathbf{Bx} = \mathbf{0}.$$

If $\mathbf{B}$ were an invertible matrix, then you could multiply both sides of this equation on the left by $\mathbf{B}^{-1}$ to give

$$\mathbf{B}^{-1}\mathbf{Bx} = \mathbf{B}^{-1}\mathbf{0}.$$

Since $\mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$, the resulting equation simplifies to give

$$\mathbf{x} = \mathbf{0}.$$

But this cannot be so, because $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, so it's not the zero vector. It follows that $\mathbf{B}$ is not an invertible matrix after all, and so it has determinant zero. Now,

$$\mathbf{B} = \mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix},$$

so

$$\det \mathbf{B} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc.$$

Since $\det \mathbf{B} = 0$, we obtain the quadratic equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0.$$

This quadratic equation is called the *characteristic equation* of $\mathbf{A}$.

We've shown that any eigenvalue of $\mathbf{A}$ is a solution of the characteristic equation of $\mathbf{A}$. Let's now reverse this argument to show that any solution of the characteristic equation of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$. Suppose then that μ is a solution of the characteristic equation, so that

$$\det(\mathbf{A} - \mu\mathbf{I}) = \mu^2 - (a + d)\mu + ad - bc = 0.$$

Let's write

$$\mathbf{A} - \mu\mathbf{I} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Since the determinant of $\mathbf{A} - \mu\mathbf{I}$ is 0, we have $ps - qr = 0$. Now we define

$$\mathbf{x} = \begin{pmatrix} s \\ -r \end{pmatrix}.$$

Then

$$(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} s \\ -r \end{pmatrix} = \begin{pmatrix} ps - qr \\ rs - sr \end{pmatrix} = \begin{pmatrix} ps - qr \\ 0 \end{pmatrix}.$$

But we've just seen that $ps - qr = 0$, so

$$(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = \mathbf{0}.$$

It follows that if $\mathbf{x}$ isn't the zero vector, then it's an eigenvector of $\mathbf{A}$ with corresponding eigenvalue μ.

If $\mathbf{x}$ is the zero vector, then we can define

$$\mathbf{y} = \begin{pmatrix} q \\ -p \end{pmatrix},$$

and use a similar argument to see that if $\mathbf{y}$ isn't the zero vector, then it's an eigenvector of $\mathbf{A}$ with corresponding eigenvalue μ.

In the exceptional case when both $\mathbf{x}$ and $\mathbf{y}$ are equal to the zero vector, all four numbers p, q, r and s are 0, so

$$\mathbf{A} - \mu\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, we have

$$(\mathbf{A} - \mu\mathbf{I})\mathbf{v} = \mathbf{0}$$

for *any* vector $\mathbf{v}$, so any non-zero vector is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue μ.

To summarise, we saw earlier that any eigenvalue of $\mathbf{A}$ is a solution of the characteristic equation of $\mathbf{A}$, and now we've seen that any solution of the characteristic equation of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$. These important observations are encapsulated in the box below.

Characteristic equation of a 2×2 matrix

The **characteristic equation** of a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the quadratic equation in λ given by

$$\lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The eigenvalues of $\mathbf{A}$ are the solutions of this equation.

For example, the characteristic equation of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is

$$\lambda^2 - 5\lambda + 6 = 0.$$

As you have seen, the eigenvalues – which are the solutions of the characteristic equation – are 2 and 3.

In the next example and activity you'll practise finding characteristic equations. Later on, you'll not only find characteristic equations, but also solve them to give the eigenvalues.

Example 2 *Finding the characteristic equation of a 2×2 matrix*

Find the characteristic equation of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Solution

The characteristic equation of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

Substitute values of a, b, c and d into this formula.

The characteristic equation is

$$\lambda^2 - (2+2)\lambda + 2 \times 2 - 1 \times 1 = 0.$$

That is,

$$\lambda^2 - 4\lambda + 3 = 0.$$

Activity 4 *Finding the characteristic equations of 2×2 matrices*

Find the characteristic equation of each of the following matrices.

(a) $\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$

You may have noticed that the constant term of the characteristic equation of a 2×2 matrix is equal to the determinant of the matrix. That is, the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has characteristic equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

and the constant term is the number $ad - bc$, which is the determinant of $\mathbf{A}$.

The coefficient of λ in the characteristic equation is $-(a+d)$. You'll recall that the elements a and d are said to lie on the **leading diagonal** of the matrix (the diagonal that starts at the top-left element and ends at the bottom-right element). Their sum $a+d$ is known as the *trace* of the matrix, so the coefficient of λ in the characteristic equation of $\mathbf{A}$ is equal to minus the trace of $\mathbf{A}$.

Trace of a 2×2 matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **trace** of $\mathbf{A}$ is the number $a+d$, written $\operatorname{tr}\mathbf{A}$.

It is the sum of the elements on the leading diagonal of $\mathbf{A}$.

With the determinant and trace, there is now an alternative, equivalent form of the characteristic equation, which you may find easier to remember.

Equivalent form of the characteristic equation

The characteristic equation of a 2×2 matrix $\mathbf{A}$ is the quadratic equation in λ given by

$$\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = 0.$$

Example 3 *Using the trace and determinant to find the characteristic equation of a 2×2 matrix*

Find the trace and determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix},$$

and hence find its characteristic equation.

Solution

The trace and determinant are

$$\operatorname{tr}\mathbf{A} = a+d \quad \text{and} \quad \det\mathbf{A} = ad-bc.$$

We have

$$\begin{aligned} \operatorname{tr}\mathbf{A} &= 1+5 = 6,\\ \det\mathbf{A} &= 1\times 5 - (-2)\times 2 = 9. \end{aligned}$$

The characteristic equation is $\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = 0$.

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda + 9 = 0.$$

Activity 5 *Using the trace and determinant to find the characteristic equations of 2×2 matrices*

For each of the following matrices, find the trace and determinant and hence find the characteristic equation.

(a) $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$

Once you've found the characteristic equation of a matrix, you can then find the eigenvalues of the matrix by solving the characteristic equation. This is a quadratic equation, so it either has one repeated solution or two distinct solutions. Each of these solutions is a complex number that may or may not be a real number. Since the solutions are the eigenvalues of the matrix, we have the following observation about the number of eigenvalues of a 2×2 matrix.

Number of eigenvalues of a 2×2 matrix

Every 2×2 matrix has either one or two eigenvalues, each of which is a complex number that may or may not be a real number.

If the matrix has only one eigenvalue, then because it is a repeated solution of the characteristic equation, it is said to be a **repeated eigenvalue** of the matrix.

For example, consider the characteristic equation $\lambda^2 - 6\lambda + 9 = 0$ that was obtained in Example 3. The quadratic expression can be factorised to give

$$(\lambda - 3)^2 = 0.$$

The number 3 is a repeated solution of this equation, so 3 is a repeated eigenvalue of the matrix.

Let's now practise finding eigenvalues of 2×2 matrices. The strategy you've learned for doing so is summarised in the next box.

Strategy: To find the eigenvalues of a 2×2 matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. Find the trace and determinant of $\mathbf{A}$,
$$\operatorname{tr} \mathbf{A} = a + d \quad \text{and} \quad \det \mathbf{A} = ad - bc.$$
2. Write down the characteristic equation
$$\lambda^2 - (\operatorname{tr} \mathbf{A})\lambda + \det \mathbf{A} = 0.$$
3. Solve the characteristic equation to obtain the eigenvalues of $\mathbf{A}$.

For example, you saw in Example 2 that the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

has characteristic equation $\lambda^2 - 4\lambda + 3 = 0$. You can solve this equation by factorising the quadratic expression to give

$$(\lambda - 1)(\lambda - 3) = 0.$$

The eigenvalues are the solutions of this equation, namely 1 and 3.

There is a useful way to check whether the values you obtain for the eigenvalues are correct. To understand how this check works, multiply out the expression $(\lambda - 1)(\lambda - 3)$ on the left-hand side of the equation above to obtain

$$\lambda^2 - (1 + 3)\lambda + (-1) \times (-3).$$

As you can see, the coefficient of λ in this quadratic expression is equal to minus the sum of the eigenvalues. You learned earlier that the coefficient of λ is also equal to minus the trace of the matrix. Therefore the sum of the eigenvalues is equal to the trace.

You can use this observation to check that the eigenvalues you've found are correct: add them together and check that you obtain the trace.

A useful way to check your eigenvalues

The sum of the eigenvalues of a 2×2 matrix is equal to the trace of the matrix.

If the matrix has only a single, repeated eigenvalue, then it must be counted twice in the sum.

For instance, you saw in Example 3 that the matrix

$$\begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

has characteristic equation $\lambda^2 - 6\lambda + 9 = 0$, and earlier we solved this equation to see that the matrix has a single, repeated eigenvalue 3. Since it is repeated, we count it twice when we work out the sum of the eigenvalues, to give

$$3 + 3 = 6,$$

which is equal to the trace of the matrix.

This check doesn't guarantee that the eigenvalues you've found are correct; after all, it may be that you've found two incorrect numbers for the eigenvalues that happen to add up to the trace. However, it's still a useful quick way of testing for numerical mistakes.

There is another way to check whether the values you obtain for the eigenvalues are correct: their *product* must equal the determinant of the matrix. For example, the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

from Example 2 are 1 and 3, and their product is 3, which is equal to the determinant of the matrix. This check is less useful than the trace check, because it's often more difficult to work out a product than a sum, and more difficult to work out a determinant than a trace. In this unit we use the trace test, but not the determinant test.

Example 4 *Finding the eigenvalues of 2×2 matrices*

Find the eigenvalues of the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$

Solution

(a) Find the trace and determinant of the matrix.

We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= 1 + 2 = 3, \\ \det \mathbf{A} &= 1 \times 2 - 2 \times 3 = -4. \end{aligned}$$

Write down the characteristic equation $\lambda^2 - (\operatorname{tr} \mathbf{A})\lambda + \det \mathbf{A} = 0$.

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 3\lambda - 4 = 0.$$

Solve the characteristic equation to find the eigenvalues. This one can be solved by factorising the quadratic expression.

Hence

$$(\lambda + 1)(\lambda - 4) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -1 and 4.

Check that the sum of the eigenvalues is equal to the trace.

(Check: $(-1) + 4 = 3 = \operatorname{tr} \mathbf{A}$.)

(b) Find the trace and determinant of the matrix.

We have

$$\begin{aligned} \operatorname{tr}\mathbf{A} &= 3+1=4,\\ \det\mathbf{A} &= 3\times 1 - 1\times(-2) = 5. \end{aligned}$$

Write down the characteristic equation $\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = 0$.

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda + 5 = 0.$$

Solve the characteristic equation to find the eigenvalues. Use the quadratic formula.

Hence

$$\begin{aligned} \lambda &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4\times 1\times 5}}{2\times 1}\\ &= \frac{4\pm\sqrt{-4}}{2}\\ &= \frac{4\pm 2i}{2}\\ &= 2\pm i. \end{aligned}$$

So the eigenvalues of $\mathbf{A}$ are $2-i$ and $2+i$.

Check that the sum of the eigenvalues is equal to the trace.

(Check: $(2-i)+(2+i) = 4 = \operatorname{tr}\mathbf{A}$.)

Activity 6 *Finding the eigenvalues of 2×2 matrices*

Find the eigenvalues of the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} -1 & 2\\ 1 & 0 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} -5 & -3\\ 3 & 1 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 1 & -1\\ 1 & 1 \end{pmatrix}$

1.3 Finding eigenvectors

After you've found the eigenvalues of a 2×2 matrix by solving the characteristic equation, you can then find the corresponding eigenvectors by using the eigenvector equations. In this subsection you'll see how to do this for **real eigenvalues** (eigenvalues that are real numbers). The strategy for finding eigenvectors is similar when the eigenvalues are not real numbers.

Let's demonstrate the strategy using the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

In Example 4(a) you learned that $\mathbf{A}$ has two eigenvalues, -1 and 4. Near the start of the previous subsection you saw that the eigenvector equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ can be stated in the equivalent form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. In the example here, the equivalent form of the eigenvector equation is

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where λ is an eigenvalue of $\mathbf{A}$, and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

When $\lambda = -1$, we obtain

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can write this matrix equation as a pair of simultaneous equations,

$$2x + 2y = 0,$$
$$3x + 3y = 0.$$

By dividing both sides of the first equation by 2, and dividing both sides of the second equation by 3, you see that each of the two equations is equivalent to the single equation

$$x + y = 0; \quad \text{that is,} \quad y = -x.$$

This final equation $y = -x$ is the equation of the line shown in Figure 5.

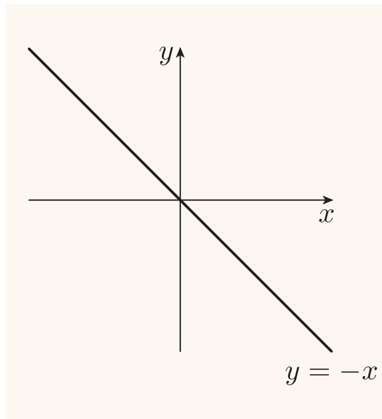


Figure 5 The line $y = -x$

The position vector $\mathbf{x}$ of any point on this line satisfies $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, with $\lambda = -1$, so any such vector other than the zero vector is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1. For example, choosing $x = 1$ on the line $y = -x$ gives $y = -1$, so

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1. Similarly, choosing x equal to 5 and -2 in turn on the line $y = -x$ gives vectors

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 2 \end{pmatrix},$$

each an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1. More generally, any eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1 has the form

$$\begin{pmatrix} x \\ -x \end{pmatrix}$$

for some non-zero number x.

You can check that such vectors are indeed eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue -1 by verifying that the eigenvector equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix} = - \begin{pmatrix} x \\ -x \end{pmatrix}$$

is satisfied. Geometrically, this equation demonstrates that each point on the line $y = -x$ is scaled by the linear transformation represented by $\mathbf{A}$ by the factor -1, so the line $y = -x$ is an invariant line of the transformation.

This method for finding eigenvectors is summarised in the following strategy box.

Strategy:
To find an eigenvector of a 2×2 matrix corresponding to a given eigenvalue

To find an eigenvector of a 2×2 matrix $\mathbf{A}$ corresponding to an eigenvalue λ of $\mathbf{A}$, carry out the following steps.

1. Write down the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector to be determined.
2. Rewrite this equation as a pair of simultaneous equations in x and y.
3. These simultaneous equations are equivalent to a single equation in x and y. Any solution of this equation other than $x = y = 0$ gives an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

Example 5 *Finding another eigenvector of* $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Find an eigenvector of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ corresponding to the eigenvalue 4.

Solution

Write down the eigenvector equations of the matrix in the form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 4$, we obtain

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Write this matrix equation as a pair of simultaneous equations.

This gives

$$-3x + 2y = 0,$$
$$3x - 2y = 0.$$

Find a solution of these equations.

This pair of equations is equivalent to the single equation

$$2y = 3x.$$

Choose a value of x that gives a simple solution, preferably without fractions. Choosing $x = 1$ would give $y = \frac{3}{2}$, which involves a fraction. It's better to choose $x = 2$ because then $y = 3$, and neither x nor y is a fraction.

If $x = 2$, then $2y = 3 \times 2$, so $y = 3$. Hence $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 4.

Check your answer by making sure that the eigenvector equation $\mathbf{Ax} = \lambda\mathbf{x}$ is satisfied.

(Check: $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.)

Activity 7 *Finding eigenvectors of a 2×2 matrix*

The matrix $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ has eigenvalues -2 and 5. For each of these eigenvalues, find a corresponding eigenvector.

As you've seen, to find an eigenvector of a 2×2 matrix $\mathbf{A}$ corresponding to an eigenvalue λ of $\mathbf{A}$, you must find a solution other than $x = y = 0$ of a pair of simultaneous equations

$$px + qy = 0,$$
$$rx + sy = 0,$$

where p, q, r and s are real numbers (they are the elements of the matrix $\mathbf{A} - \lambda\mathbf{I}$). For instance, in Example 5 this pair of simultaneous equations is

$$-3x + 2y = 0,$$
$$3x - 2y = 0.$$

If p and q are not both 0, then the equation $px + qy = 0$ represents a line in the plane that passes through the origin. If p and q *are* both 0, then the equation is

$$0x + 0y = 0.$$

This equation is satisfied by *every* pair of values of x and y, so it represents the entire plane. Of course, similar comments apply to the line $rx + sy = 0$.

Often when calculating eigenvectors by solving the pair of simultaneous equations $px + qy = 0$ and $rx + sy = 0$, you'll find that neither equation is $0x + 0y = 0$, so both equations represent lines through the origin. In fact, they must represent the *same* line through the origin because otherwise the two lines would intersect only at the origin, which would imply that the only solution of the simultaneous equations is $x = y = 0$. If you discover that the only solution is $x = y = 0$, then you've made a mistake, because you should be able to find a (non-zero) eigenvector from one of the solutions.

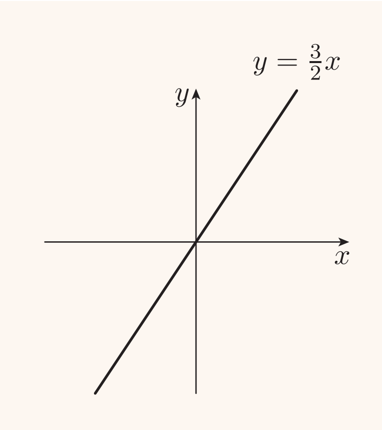


Figure 6 The line $y = \frac{3}{2}x$

As an example of when both equations represent the same line, consider again the simultaneous equations

$$-3x + 2y = 0,$$
$$3x - 2y = 0,$$

from Example 5. Each of these equations is equivalent to the single equation $2y = 3x$; that is, $y = \frac{3}{2}x$. The line given by this equation is shown in Figure 6.

Sometimes you'll find that one of the two equations $px + qy = 0$ and $rx + sy = 0$ represents a line, but the coefficients of the other equation are both 0, so it represents the entire plane. For example, the pair of simultaneous equations

$$2x + y = 0,$$
$$0x + 0y = 0,$$

is of this type. In this case, the solutions of the simultaneous equations are given by the equation representing a line, because all values of x and y satisfy the equation with 0 coefficients.

There is one final very special pair of simultaneous equations that we haven't yet considered, when *all* the coefficients of the simultaneous equations are 0. To understand this case, let's suppose that we are trying to find an eigenvector of a 2×2 matrix $\mathbf{A}$ corresponding to an eigenvalue λ of $\mathbf{A}$. In this very special case, when we write the eigenvector equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

as a pair of simultaneous equations, all the coefficients are 0. It follows that $\mathbf{A} - \lambda\mathbf{I}$ must be the zero matrix; that is

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Adding $\lambda\mathbf{I}$ to both sides of this equation gives

$$\mathbf{A} = \lambda\mathbf{I}.$$

So $\mathbf{A}$ is a scalar multiple of the identity matrix. As you saw in Unit 6, matrices of this type represent linear transformations called **dilations**. For example, the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

represents a dilation. This matrix has an eigenvalue 5 because the eigenvector equation

$$\mathbf{Ax} = 5\mathbf{x}$$

is satisfied *for any vector* $\mathbf{x}$ *whatsoever*. What is more, because this equation is satisfied for every vector $\mathbf{x}$, the number 5 is the only eigenvalue of $\mathbf{A}$.

This discussion about the eigenvectors corresponding to a real eigenvalue is summarised below.

Eigenvectors corresponding to a real eigenvalue

Suppose that $\mathbf{A}$ is a 2×2 matrix with a real eigenvalue λ.

- If $\mathbf{A} \neq \lambda\mathbf{I}$, then the eigenvectors of $\mathbf{A}$ corresponding to λ are the position vectors of the points on a single line through the origin, other than the position vector of the origin itself.
- If $\mathbf{A} = \lambda\mathbf{I}$, then every non-zero vector is an eigenvector of $\mathbf{A}$ corresponding to λ.

Let's now see an example of finding eigenvalues and eigenvectors of a matrix.

Example 6 *Finding eigenvalues and eigenvectors of a 2×2 matrix*

Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},$$

and for each eigenvalue find a corresponding eigenvector.

Solution

To find the eigenvalues of the matrix, first find the trace and determinant.

We have

$$\operatorname{tr} \mathbf{A} = 1 + 3 = 4,$$
$$\det \mathbf{A} = 1 \times 3 - 1 \times (-1) = 4.$$

Write down the characteristic equation $\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = 0$.

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda + 4 = 0.$$

Solve the characteristic equation to find the eigenvalues. This one can be solved by factorising the quadratic expression.

Hence

$$(\lambda - 2)^2 = 0,$$

so $\mathbf{A}$ has only a single, repeated eigenvalue 2.

You should now carry out a quick mental check that the sum of the eigenvalues is equal to the trace of $\mathbf{A}$. Later on, after you've found an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2, you'll check both this eigenvalue and eigenvector using the eigenvector equation for 2.

Find the eigenvectors. To do this, first write down the eigenvector equations of the matrix in the form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $\lambda = 2$ gives

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Write this matrix equation as a pair of simultaneous equations.

This gives

$$-x + y = 0,$$
$$-x + y = 0.$$

Find a solution of these equations.

This pair of equations is equivalent to the single equation

$$y = x.$$

Choose a value of x that gives a simple solution.

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

Check your answer by making sure that the eigenvector equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is satisfied.

(Check: $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.)

Activity 8 *Finding eigenvalues and eigenvectors of 2×2 matrices*

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

(a) $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$

(d) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

All the matrices that you'll meet in this unit are **real matrices**, which means that their elements are real numbers. As you've seen already, a real matrix may have eigenvalues that are not real numbers. For example, you saw in Example 4(b) that the matrix

$$\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$$

has eigenvalues $2 - i$ and $2 + i$. There are eigenvectors corresponding to these eigenvalues, but the components of those eigenvectors are complex numbers that are not real numbers. Here we restrict our attention to **real eigenvectors** (eigenvectors whose components are real numbers) and do not consider eigenvectors corresponding to eigenvalues that are not real numbers.

1.4 Matrices larger than 2×2

Many of the techniques that you've learned for finding eigenvalues and eigenvectors of 2×2 matrices can also be applied to bigger square matrices. In particular, the method used to find the characteristic equation of a 2×2 matrix can be adapted for $n \times n$ matrices. The result of applying this method to an $n \times n$ matrix $\mathbf{A}$ is a polynomial equation, again called the **characteristic equation** of $\mathbf{A}$, whose solutions are the eigenvalues of $\mathbf{A}$.

The degree of the characteristic equation of an $n \times n$ matrix is n. A polynomial equation of degree n has at least 1 and at most n solutions. It follows that an $n \times n$ matrix has at least 1 and at most n eigenvalues.

For example, the characteristic equation of a 2×2 matrix is a quadratic equation, as you've seen already, so a 2×2 matrix has 1 or 2 eigenvalues. As another example, consider the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix}.$$

Using the computer algebra system (see Activity 10) you can show that the characteristic equation of $\mathbf{A}$ is the cubic equation

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0.$$

In general, cubic equations are difficult to solve; however, this particular equation happens to have exactly three solutions, namely 1, 2 and 3 (which you can check yourself). These are the eigenvalues of $\mathbf{A}$.

You can find eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues 1, 2 and 3 by using similar methods to those that you used for 2×2 matrices. The difference this time is that you have to solve *three* simultaneous equations in *three* unknowns, rather than two simultaneous equations in two unknowns. Although you can do this by hand, the procedure takes a lot longer than it does for 2×2 matrices.

In the next activity, you are asked to check that three vectors are eigenvectors of $\mathbf{A}$, and to find the eigenvalues corresponding to each of them.

Activity 9 *Finding the eigenvalues corresponding to given eigenvectors*

Show that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix},$$

and for each eigenvector find the corresponding eigenvalue.

Usually the most practical way of finding the eigenvalues and eigenvectors of an $n \times n$ matrix, where $n \geq 3$, is to use the computer algebra system.

Activity 10 *Finding eigenvalues and eigenvectors with a computer*

Work through Section 8 of the *Computer algebra guide*, where you will learn how to find eigenvalues, eigenvectors and characteristic equations of matrices using the computer algebra system.

The matrices in the remainder of this unit are all of size 2×2. However, many (but not all) of the techniques that you'll meet can also be applied to bigger square matrices.

Olga Taussky-Todd: a torchbearer for matrix theory

The mathematician Olga Taussky-Todd was born in 1906 in the city of Olomouc, which at the beginning of the twentieth century was part of the Austro-Hungarian Empire, but now belongs to the Czech Republic. She moved to England in 1935, and during the Second World War she worked at the National Physical Laboratory in Teddington, Middlesex. Part of her work there involved using matrices to analyse the stability of aircraft.

One of her major mathematical achievements was to develop a result about eigenvalues known as the *Gershgorin circle theorem*, and utilise it in her work on aeronautics. Later in life she wrote that on first meeting the result, she 'immediately tinkered with the theorem, applying it to a very nasty looking 6×6 matrix'. She published her findings in a report to the Aeronautical Research Council in 1947.

Because of her contributions to the theory of matrices, the British-American mathematician Hans Schneider wrote that Tausky-Todd's 'work has altered the consciousness of several generations of matrix theorists'.

(Source: Schneider, H. (1977) 'Olga Taussky-Todd's influence on matrix theory and matrix theorists', *Linear and Multilinear Algebra*, vol. 5, no. 3, pp. 197–224)

Olga Taussky-Todd (1906–95)

2 Eigenvalues and eigenvectors of special types of matrices

Using the strategies of the previous section, you can find the eigenvalues and eigenvectors of any 2×2 matrix. However, for some types of matrices there are far quicker ways of identifying the eigenvalues and eigenvectors. Here you'll learn about some of these special types of matrices, and you'll also study the eigenvalues and eigenvectors of matrices representing many of the standard types of linear transformations that you met in Unit 6, such as flattenings, reflections, rotations, scalings and shears.

2.1 Triangular matrices

In this subsection you'll learn about a class of matrices called the *triangular matrices*. The eigenvalues of matrices from this class are particularly easy to identify.

Triangular matrices

A 2×2 **triangular matrix** is a matrix of one of the forms

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

Matrices of the first type are called **upper triangular matrices** and matrices of the second type are called **lower triangular matrices**.

For example,

$$\begin{pmatrix} 2 & 3 \\ 0 & 9 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 7 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

are all triangular matrices. The first matrix is an upper triangular matrix, the second is a lower triangular matrix, and the third is both an upper and a lower triangular matrix. In contrast,

$$\begin{pmatrix} 0 & 2 \\ 3 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 7 \\ 4 & 0 \end{pmatrix}$$

are not triangular matrices.

Notice that 2×2 **diagonal matrices**, that is, matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

are triangular matrices.

The eigenvalues of a triangular matrix are easy to identify using the following observation.

Eigenvalues of 2×2 triangular matrices

The eigenvalues of a 2×2 triangular matrix are the elements on the leading diagonal.

For example, the eigenvalues of the triangular matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 9 \end{pmatrix}$$

are 2 and 9. To see why this is so, remember that the eigenvalues of $\mathbf{A}$ are the solutions of the characteristic equation of $\mathbf{A}$, and the characteristic equation is obtained by writing the left-hand side of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

as a quadratic expression in λ. In this case

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2-\lambda & 3 \\ 0 & 9-\lambda \end{pmatrix},$$

and the determinant of this matrix is $(2-\lambda)(9-\lambda)$. Therefore the characteristic equation is

$$(2-\lambda)(9-\lambda)=0.$$

This equation has solutions 2 and 9, which are the elements on the leading diagonal of $\mathbf{A}$.

You can also quickly identify an eigenvector of a 2×2 triangular matrix, using the following observation.

Eigenvectors of 2×2 triangular matrices

If $\mathbf{A}=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue a.

If $\mathbf{A}=\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, then $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue d.

Let's see why this is true for upper triangular matrices; you can reason in a similar way for lower triangular matrices. Observe that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} a \\ 0 \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue a. To find an eigenvector of the matrix corresponding to the other eigenvalue d you should use the strategy given in the previous section.

As a 2×2 diagonal matrix is both an upper and a lower triangular matrix, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are both eigenvectors of such a matrix.

Eigenvalues and eigenvectors of 2×2 diagonal matrices

Let $\mathbf{A}=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. The eigenvalues of $\mathbf{A}$ are the elements a and d on the leading diagonal, and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues a and d, respectively.

Example 7 *Finding eigenvalues and eigenvectors of 2×2 triangular matrices*

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

(a) $\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

(a) This matrix is a triangular matrix, so the eigenvalues are the elements on the leading diagonal.

Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 1 and 4.

Next find the eigenvectors. The matrix is a lower triangular matrix, so the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue given by the bottom-right element of the matrix.

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 4.

Find an eigenvector corresponding to the other eigenvalue in the usual way, using the corresponding eigenvector equation.

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Write this matrix equation as a pair of simultaneous equations.

This gives

$$0x + 0y = 0,$$
$$2x + 3y = 0.$$

This pair of equations is equivalent to the single equation

$$2x + 3y = 0; \quad \text{that is,} \quad 3y = -2x.$$

Choose a value of x that gives a simple solution.

If $x = 3$, then $3y = -2 \times 3$, so $y = -2$. Hence $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

Check your answer by making sure that the eigenvector equation $\mathbf{Ax} = \lambda\mathbf{x}$ is satisfied.

(Check: $\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} = 1\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.)

(b) This matrix is a diagonal matrix, so you can immediately write down the eigenvalues and some corresponding eigenvectors.

Since the matrix is a diagonal matrix, the eigenvalues are the elements on the leading diagonal, namely -2 and 1. The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

Activity 11 *Finding eigenvalues and eigenvectors of 2×2 triangular matrices*

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

(a) $\begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 0 \\ 0 & -7 \end{pmatrix}$

In Unit 6 you learned about a class of linear transformations called **shears**. In particular, you saw that **horizontal shears** and **vertical shears** are transformations represented by triangular matrices of the forms

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix},$$

respectively.

Activity 12 *Finding eigenvalues and eigenvectors of a matrix representing a horizontal shear*

Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

and for each eigenvalue find a corresponding eigenvector.

Suppose that $\mathbf{A}$ is the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

from the preceding activity, and let f be the horizontal shear represented by $\mathbf{A}$. Let $\mathbf{i}$ and $\mathbf{j}$ be the usual unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and let $\mathbf{i}' = \mathbf{Ai}$ and $\mathbf{j}' = \mathbf{Aj}$. The effect of f on the unit square is shown in Figure 7.

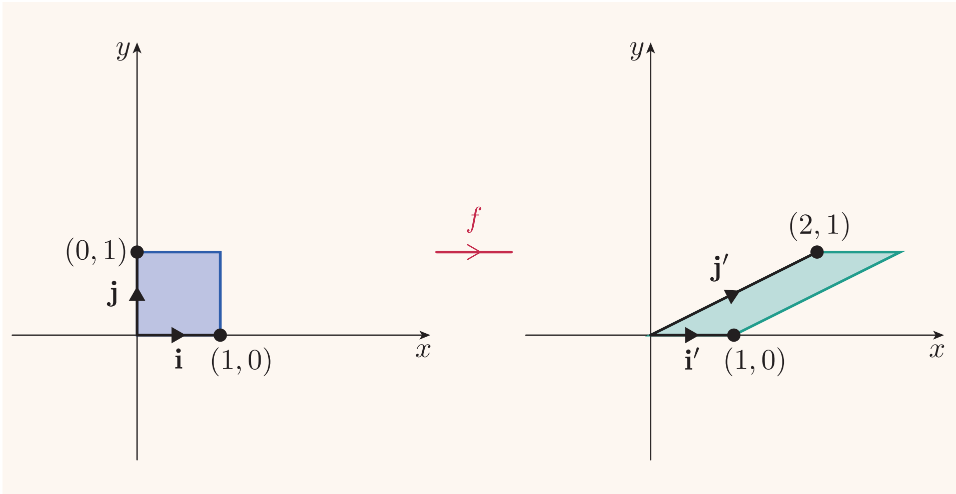


Figure 7 The effect of a horizontal shear

You learned in Unit 6 that a horizontal shear fixes every point on the x-axis, and moves points that are not on the x-axis horizontally. In particular, the shear f fixes the point $(1, 0)$, as you can see from Figure 7. Using matrix notation, we can write the observation that $f(1, 0) = (1, 0)$ as

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This gives us a geometric way of seeing that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1, a fact that you may already have verified in Activity 12.

However, the matrix $\mathbf{A}$ has no eigenvectors other than scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This is because any point that doesn't lie on the x-axis is moved horizontally by f, so the image point cannot be a scalar multiple of the original point.

Similar remarks about eigenvalues and eigenvectors can be made for other horizontal and vertical shears. We summarise them below.

Eigenvalues and eigenvectors of matrices representing horizontal and vertical shears

The matrices

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \qquad \text{where } k \neq 0,$$

which represent a horizontal shear and a vertical shear, respectively, each have only a single, repeated eigenvalue 1. The corresponding eigenvectors are all non-zero scalar multiples of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively.

2.2 Flattenings

In Unit 6 you learned that a linear transformation is a **flattening** if its image set is either a line through the origin, or, in the case of the zero transformation, the set containing the origin alone. You also saw that to test whether a linear transformation is a flattening you can check whether the determinant of the matrix representing the transformation is zero. Here we investigate the eigenvalues and eigenvectors of matrices representing flattenings.

Suppose that the 2×2 matrix $\mathbf{A}$ represents a flattening, so $\det \mathbf{A} = 0$. The characteristic equation of $\mathbf{A}$ then becomes

$$\lambda^2 - (\operatorname{tr} \mathbf{A})\lambda = 0.$$

The quadratic expression on the left-hand side factorises to give

$$\lambda(\lambda - \operatorname{tr} \mathbf{A}) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and $\operatorname{tr} \mathbf{A}$.

Eigenvalues of 2×2 matrices with determinant zero

The eigenvalues of a 2×2 matrix $\mathbf{A}$ with determinant zero are 0 and $\operatorname{tr} \mathbf{A}$.

If $\operatorname{tr} \mathbf{A} = 0$, then $\mathbf{A}$ has only a single, repeated eigenvalue 0.

For example, the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$$

has determinant zero, and $\operatorname{tr} \mathbf{A} = 5$, so its eigenvalues are 0 and 5.

As the determinant of this matrix $\mathbf{A}$ is zero, the linear transformation f represented by $\mathbf{A}$ is a flattening. It's not the zero transformation, so the image set of f is a line through the origin. We can find out what that line is by looking at the images of points under f.

For example, under f the images of the points with position vectors

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

must both be points on the line. But $\mathbf{Ai}$ and $\mathbf{Aj}$ are just the first and second columns of $\mathbf{A}$, respectively, namely

$$\begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Each of these vectors is a scalar multiple of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, so we see that f flattens the plane onto the line that passes through the origin and the point $(1, 3)$. This is the line $y = 3x$ shown in Figure 8.

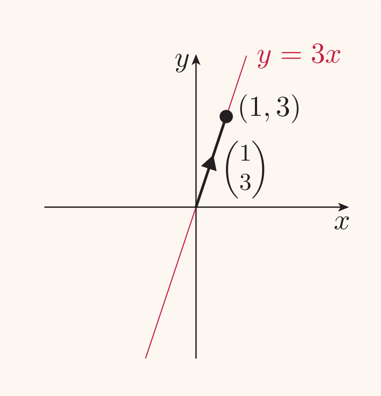


Figure 8 The transformation f flattens the plane onto the line $y = 3x$

The line $y = 3x$ is mapped to itself by f, so it's an invariant line of f. It follows that any point on this line is mapped by f to a scalar multiple of itself, so the position vector of any non-zero point on the line is an eigenvector of $\mathbf{A}$. In particular, the two columns of $\mathbf{A}$ are eigenvectors of $\mathbf{A}$. Let's check this for the second column:

$$\mathbf{A}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Therefore $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 5 (and so is $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$).

Notice that the eigenvectors given by the columns of $\mathbf{A}$ both correspond to the eigenvalue $\operatorname{tr}\mathbf{A} = 5$, rather than to the other eigenvalue 0. This is because they are the position vectors of points on the line $y = 3x$, which is mapped to itself by f, rather than to the origin.

With similar reasoning we can obtain the following observations about the eigenvectors of a 2×2 matrix with determinant zero.

Eigenvectors of 2×2 matrices with determinant zero

Let $\mathbf{A}$ be a 2×2 matrix with determinant zero. Each column of $\mathbf{A}$, if non-zero, is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\operatorname{tr}\mathbf{A}$.

For example, the determinant and trace of the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

are both zero, so it has only a single, repeated eigenvalue 0.

Each column of the matrix is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is an eigenvector of the matrix corresponding to the eigenvalue 0.

In this example, the observations you've seen about eigenvalues and eigenvectors of 2×2 matrices with determinant zero allow you to quickly find the eigenvalue and a corresponding eigenvector of the matrix. This matrix is special though, because both the determinant *and* the trace are zero, so there is only one eigenvalue, namely 0.

A 2×2 matrix $\mathbf{A}$ whose determinant is zero but trace is not zero has two eigenvalues, 0 and $\operatorname{tr} \mathbf{A}$, and the observations that you've met so far in this subsection don't help you to find an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0. So in many cases, the observations won't save you much time, and rather than memorising them you may prefer to stick with the general strategies for finding eigenvalues and eigenvectors given in the previous section (and use these observations as additional checks).

Example 8 *Finding eigenvalues and eigenvectors of a 2×2 matrix with determinant zero*

Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix},$$

and for each eigenvalue find a corresponding eigenvector.

Solution

To find the eigenvalues of the matrix, first find the trace and determinant.

We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= 4 + 3 = 7, \\ \det \mathbf{A} &= 4 \times 3 - 6 \times 2 = 0. \end{aligned}$$

Since the determinant is zero, the eigenvalues are equal to 0 and $\operatorname{tr} \mathbf{A}$. You don't have to remember this observation though; it doesn't take long to go through the usual strategy of finding eigenvalues by solving the characteristic equation.

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 7\lambda = 0.$$

Hence

$$\lambda(\lambda - 7) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and 7.

Next find the eigenvectors. Write down the eigenvector equations of the matrix in the form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

The eigenvector equations have the form

$$\begin{pmatrix} 4-\lambda & 6 \\ 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substitute $\lambda = 0$ and solve the resulting equations to find an eigenvector corresponding to the eigenvalue 0.

When $\lambda = 0$, we obtain

$$\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$4x + 6y = 0,$$
$$2x + 3y = 0.$$

This pair of equations is equivalent to the single equation

$$2x + 3y = 0; \quad \text{that is,} \quad 3y = -2x.$$

If $x = 3$, then $3y = -2 \times 3$, so $y = -2$. Hence $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

(Check: $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.)

Solve the eigenvector equation with $\lambda = 7$ to find an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 7.

When $\lambda = 7$, the eigenvector equation is

$$\begin{pmatrix} -3 & 6 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-3x + 6y = 0,$$
$$2x - 4y = 0.$$

This pair of equations is equivalent to the single equation

$$x - 2y = 0; \quad \text{that is,} \quad x = 2y.$$

If $y = 1$, then $x = 2$. Hence $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 7.

(Check: $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.)

Remember that the two columns of $\mathbf{A}$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue $\operatorname{tr}\mathbf{A} = 7$. As both these vectors are scalar multiples of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, this gives us an additional way of checking that this vector is indeed an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 7.

The matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

from the preceding example represents a flattening f onto a line that passes through the origin. Since the columns of $\mathbf{A}$ are position vectors of points in the image of f, and both columns are scalar multiples of the eigenvector

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

we see that the image line passes through the point $(2, 1)$. This line has equation $y = \frac{1}{2}x$.

The other eigenvector we found in the example was $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ corresponding to the eigenvalue 0. Since non-zero scalar multiples of this eigenvector are also eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 0, we see that f maps the whole of the line $y = -\frac{2}{3}x$ that passes through the origin and the point $(3, -2)$ to the origin, as shown in Figure 9.

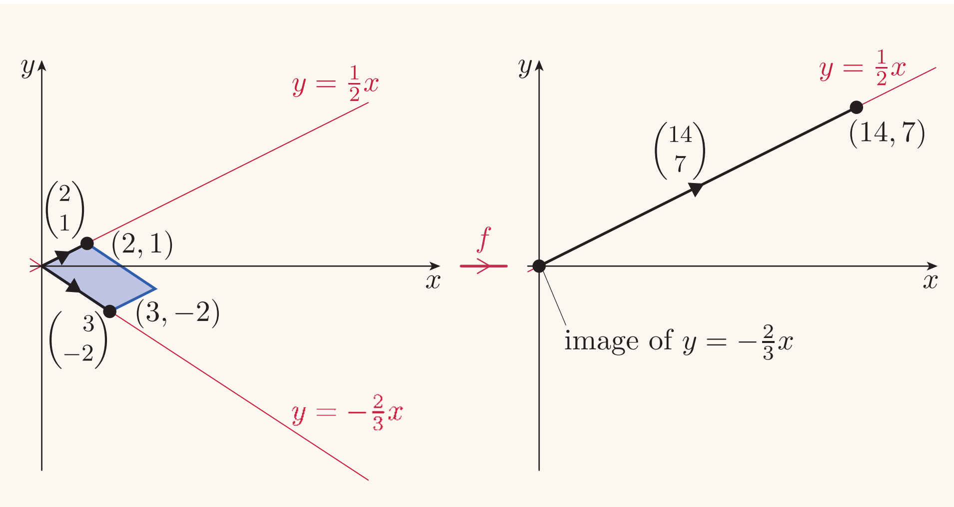


Figure 9 The transformation f flattens the plane onto the line $y = \frac{1}{2}x$

Activity 13 *Finding eigenvalues and eigenvectors of 2×2 matrices with determinant zero*

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

(a) $\mathbf{A} = \begin{pmatrix} -1 & -3 \\ 3 & 9 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

2.3 Rotations and reflections

In this subsection, you'll study the eigenvalues and eigenvectors of 2×2 matrices that represent rotations and reflections. You'll see that matrices representing rotations about the origin typically have no real eigenvalues, whereas matrices representing reflections in lines through the origin each have two real eigenvalues.

Rotations

In Unit 6 you learned that a rotation through an angle θ about the origin is represented by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For example, the rotation through an angle $\pi/2$ about the origin is represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

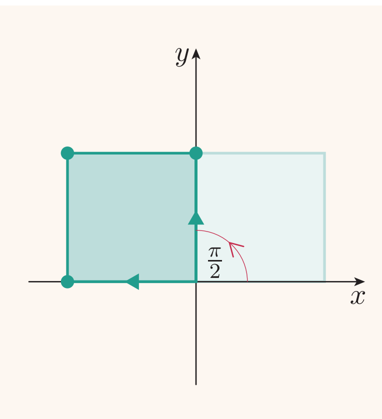


Figure 10 Rotation through an angle of $\pi/2$ about the origin

This rotation is illustrated in Figure 10, which shows the image of the unit square after applying the rotation.

In the next activity you'll see that this matrix has no real eigenvalues.

Activity 14 *Finding the eigenvalues of a matrix representing a rotation*

Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix representing a rotation from the preceding activity has no real eigenvalues, so it also has no real eigenvectors. This is to be expected, for if there was a real eigenvector, then it would be the position vector of a point that lies on an invariant line of the rotation that passes through the origin. But there is no such invariant line for a rotation by $\pi/2$.

Reasoning in a similar way, you can see that almost any matrix that represents a rotation about the origin has no real eigenvalues. There are two exceptions to this, namely the matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad -\mathbf{I} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which represent rotations about the origin through angles 0 and π, respectively. As you can see, both matrices are scalar multiples of the identity matrix. The matrix $\mathbf{I}$ has only a single, repeated eigenvalue 1, and $-\mathbf{I}$ has only a single, repeated eigenvalue -1, and *every* non-zero vector is an eigenvector of both matrices.

Eigenvalues and eigenvectors of 2×2 matrices representing rotations

With the exception of the matrices $\mathbf{I}$ and $-\mathbf{I}$, every 2×2 matrix that represents a rotation about the origin has no real eigenvalues or real eigenvectors.

Reflections

In Unit 6 you saw that a reflection in a line ℓ through the origin is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix},$$

where α is the angle of inclination of ℓ. Figure 11 shows the effect of this reflection on the unit square.

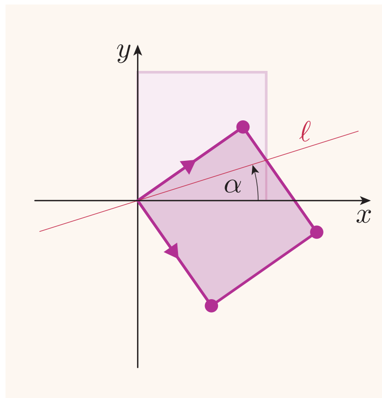


Figure 11 Reflection in the line ℓ through the origin

The matrices of such reflections, unlike those of rotations about the origin, always have real eigenvalues and eigenvectors. To see why this is so, notice that any point on the line ℓ is fixed by the reflection, so the position vector of that point must be an eigenvector of $\mathbf{A}$ with eigenvalue 1. For instance, the point $(\cos\alpha, \sin\alpha)$ lies on ℓ, as you can see from Figure 12, so we should expect that

$$\mathbf{A}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

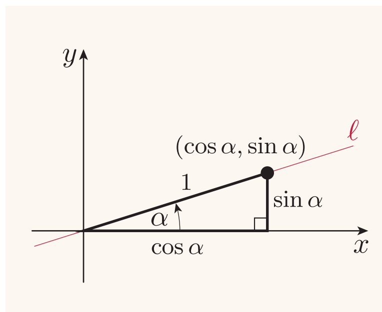


Figure 12 The point $(\cos\alpha, \sin\alpha)$ lies on ℓ

To verify this, we need to recall some trigonometric identities.

Angle difference identities

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Now

$$\mathbf{A}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} = \begin{pmatrix}\cos 2\alpha\cos\alpha + \sin 2\alpha\sin\alpha\\ \sin 2\alpha\cos\alpha - \cos 2\alpha\sin\alpha\end{pmatrix},$$

and using the angle difference identities this vector becomes

$$\begin{pmatrix}\cos(2\alpha-\alpha)\\ \sin(2\alpha-\alpha)\end{pmatrix} = \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},$$

as expected.

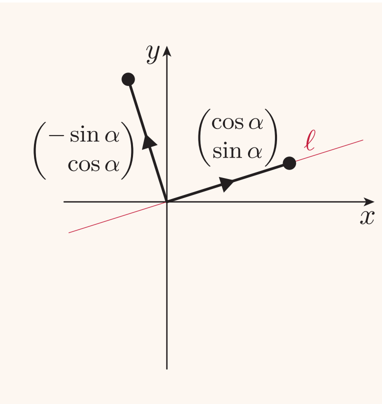


Figure 13 Eigenvectors of a reflection

Next consider a vector that is perpendicular to $\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$ such as $\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}$. You know that these two vectors are perpendicular because their scalar product is 0. This new vector is the position vector of a point on a line through the origin that is perpendicular to ℓ. After a reflection in ℓ, such a point is scaled by the factor -1, so we should expect that

$$\mathbf{A}\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix} = -\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}.$$

Again, you can check that this is true using the angle difference identities.

The two eigenvectors of $\mathbf{A}$ that we've found are shown in Figure 13.

Eigenvalues and eigenvectors of 2×2 matrices representing reflections

The matrix

$$\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix},$$

which represents a reflection in the line through the origin with angle of inclination α, has eigenvalues -1 and 1, and corresponding eigenvectors

$$\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},$$

respectively.

In part (a) of the next activity you'll revisit a matrix representing a reflection that you met right at the start of this unit.

Activity 15 *Finding eigenvalues and eigenvectors of matrices representing reflections*

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

(a) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (b) $\begin{pmatrix}1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{pmatrix}$

2.4 Generalised scalings

Let's begin by revisiting another matrix that you met at the start of this unit, namely

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

This represents a $(2,3)$-scaling, which scales horizontal distances by the factor 2 and vertical distances by the factor 3. You saw that

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2,

and

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

These two eigenvectors are perpendicular to one another. Here you'll meet a more general class of scalings, in which each scaling is represented by a matrix that has two eigenvectors that, although not necessarily perpendicular, are not scalar multiples of one another.

To introduce you to this more general class of scalings, we'll consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}.$$

Since

$$\mathbf{A}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

you can see that $\mathbf{A}$ has eigenvalues 1 and 2, with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. Let f be the linear transformation represented by $\mathbf{A}$. The effect of f on the two eigenvectors is shown in Figure 14.

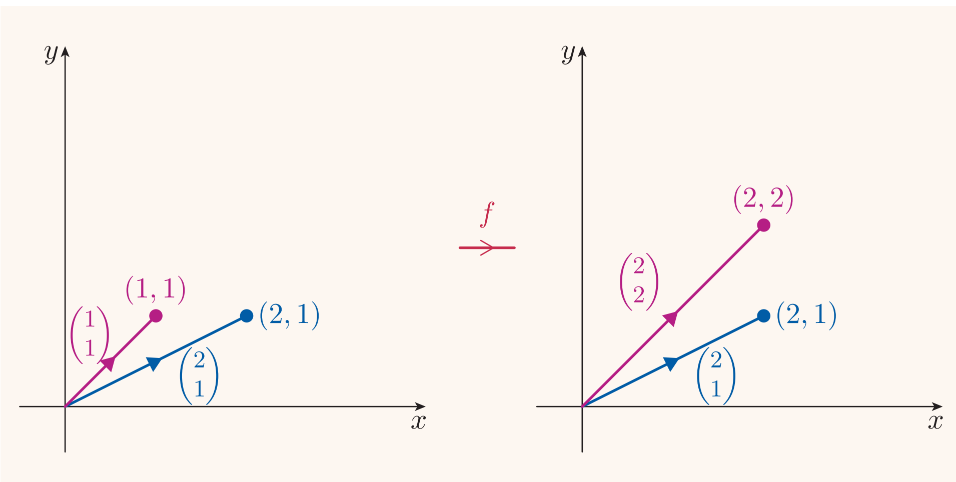


Figure 14 The effect of f on two eigenvectors

As you know, any point on the line that passes through the origin and the point $(1, 1)$ is scaled under f by the factor 2, because the position vector of any point on that line (other than the origin) is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2. With similar reasoning, you can see that every point on the line that passes through the origin and the point $(2, 1)$ is scaled under f by the factor 1; in other words, any such point is fixed by f.

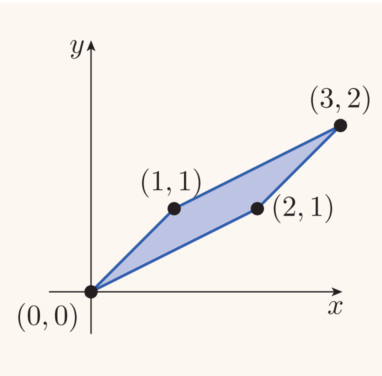


Figure 15 Parallelogram with vertices $(0, 0)$, $(2, 1)$, $(3, 2)$ and $(1, 1)$

Let's now try to work out how f transforms points that don't lie on one of these two lines. To do this, consider the parallelogram with vertices

$$(0, 0),\ (2, 1),\ (3, 2) \text{ and } (1, 1).$$

The position vectors of the second and fourth of these vertices are the two eigenvectors that we found. The parallelogram is shaded blue in Figure 15.

Now 'tile' the rest of the plane using this base parallelogram to give a grid of parallelograms, as shown in Figure 16. Each line of the grid is parallel to one of the sides of the parallelogram.

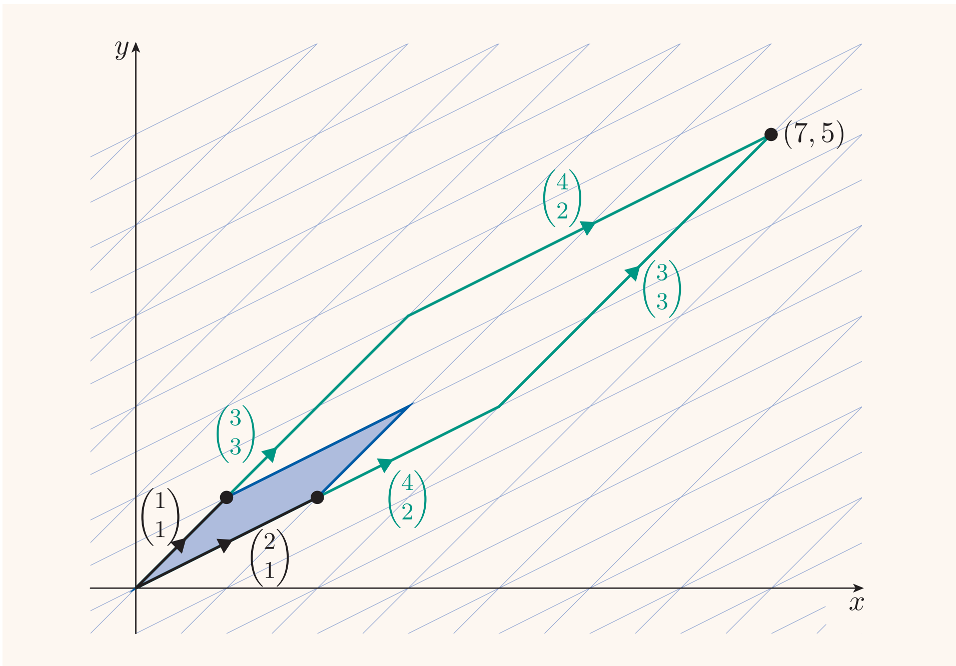


Figure 16 Grid of parallelograms

Consider a vertex of this grid, such as the point $(7, 5)$ shown in Figure 16. You can write the position vector of this vertex as a sum of integer multiples of the eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In this case,

$$\begin{aligned} \begin{pmatrix} 7 \\ 5 \end{pmatrix} &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ &= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

We can use this expression for $\begin{pmatrix}7\\5\end{pmatrix}$ in terms of the eigenvectors of $\mathbf{A}$ to determine $\mathbf{A}\begin{pmatrix}7\\5\end{pmatrix}$, as follows. First, using standard properties of matrices, we have

$$\mathbf{A}\begin{pmatrix}7\\5\end{pmatrix} = \mathbf{A}\left(2\begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix}\right) = 2\mathbf{A}\begin{pmatrix}2\\1\end{pmatrix} + 3\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix}.$$

But we already know that

$$\mathbf{A}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix}1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix},$$

so

$$\mathbf{A}\begin{pmatrix}7\\5\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix} + 3\times 2\begin{pmatrix}1\\1\end{pmatrix}.$$

Simplifying the right-hand side gives

$$\mathbf{A}\begin{pmatrix}7\\5\end{pmatrix} = \begin{pmatrix}10\\8\end{pmatrix}.$$

So f maps the point $(7,5)$ to the point $(10,8)$.

In summary, to find the image of the point $(7,5)$ under f you write

$$\begin{pmatrix}7\\5\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix},$$

and then observe that

$$\mathbf{A}\begin{pmatrix}7\\5\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix} + 6\begin{pmatrix}1\\1\end{pmatrix}.$$

The first of these equations shows how to write $\begin{pmatrix}7\\5\end{pmatrix}$ as the sum of a vector with direction $\begin{pmatrix}2\\1\end{pmatrix}$ and a vector with direction $\begin{pmatrix}1\\1\end{pmatrix}$.

The second of these equations shows that the image vector $\mathbf{A}\begin{pmatrix}7\\5\end{pmatrix}$ is obtained by scaling the part of the sum with direction $\begin{pmatrix}2\\1\end{pmatrix}$ by the factor 1, and scaling the part with direction $\begin{pmatrix}1\\1\end{pmatrix}$ by the factor 2.

You can find the image of any vertex of the grid under f in this way, by writing the position vector of that vertex as a sum of integer multiples of the eigenvectors

$$\begin{pmatrix}2\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\1\end{pmatrix},$$

in that order, and then scaling the first part of the sum by the factor 1 and scaling the second part of the sum by the factor 2. In fact, you can find the image of *any* point in the plane under f by writing the position vector of that point as a sum of real number (not necessarily integer) multiples of the two eigenvectors, and then scaling each part separately.

Try this strategy for working out the images of points in the next activity.

Activity 16 *Finding images of points under a linear transformation*

Find the images of the following points under the linear transformation f from above, which is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}.$$

(a) $(4, 4)$ (b) $(-2, -1)$ (c) $(1, -2)$ [Hint: $\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.]

Figure 17 shows the effect of the transformation

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

that we've been discussing on the base parallelogram and the grid of parallelograms.

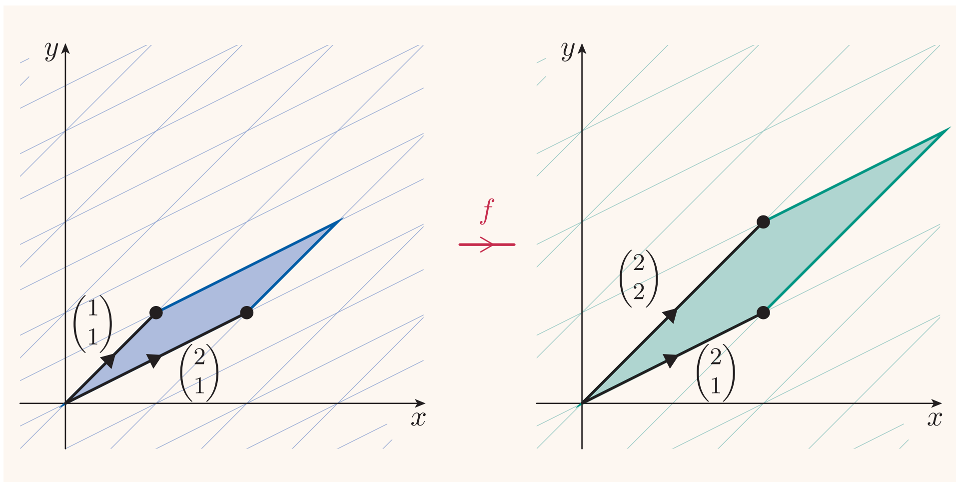


Figure 17 The effect of a generalised scaling

'Obviously the architect did some tweaking but the basic idea was mine.'

This transformation f is called a *generalised scaling* because of the way you can work out images of points by scaling in the directions of the two eigenvectors. In fact, you can find images of points in a similar way for any linear transformation of the plane that is represented by a matrix that has two real eigenvectors that are not scalar multiples of one another, so all such transformations are referred to as generalised scalings.

Generalised scalings

A **generalised scaling** is a linear transformation represented by a 2×2 matrix that has two real eigenvectors that are not scalar multiples of one another.

For example, a matrix with two distinct real eigenvalues represents a generalised scaling because any eigenvector corresponding to one eigenvalue is not a scalar multiple of any eigenvector corresponding to the other eigenvalue. Also, dilations are generalised scalings because every non-zero vector is an eigenvector of a matrix representing a dilation.

In contrast, a horizontal shear with non-zero shear factor is not a generalised scaling because all eigenvectors of the matrix representing this shear are scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Activity 17 *Investigating generalised scalings*

Open the *Visualising generalised scalings* applet.

Set the first eigenvector to be $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the second to be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with corresponding eigenvalues 1 and 2, respectively. Check that the diagram obtained matches Figure 17 (although the colours may differ).

Change the eigenvectors and eigenvalues to investigate the behaviour of generalised scalings. In particular, investigate the effects of positive, negative and zero values of the eigenvalues.

3 Diagonalising matrices

By now you've probably come to see that diagonal matrices have a number of properties that make them easy to work with. For example, the eigenvalues of a 2×2 diagonal matrix are the elements on the leading diagonal, and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are eigenvectors of the matrix corresponding to these eigenvalues. Also, multiplying two diagonal matrices is straightforward, and results in another diagonal matrix.

Here you'll learn about a procedure called *diagonalising* a matrix, which allows you to represent a matrix that has two distinct real eigenvalues in terms of a diagonal matrix, and thereby make use of some of the helpful properties of diagonal matrices. You'll see that diagonalising a matrix gives you a useful method for finding powers of the matrix. Then at the end of the section this method will be applied in modelling the populations of certain groups of predators and prey.

3.1 Diagonalising a matrix

Suppose that $\mathbf{A}$ is a 2×2 matrix that has two distinct real eigenvalues. In this subsection you'll learn how to diagonalise matrices of this type, which involves finding a 2×2 diagonal matrix $\mathbf{D}$ and a 2×2 invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. Later in the section you'll see why writing $\mathbf{A}$ in this way can help you to perform matrix calculations.

Let's demonstrate the method for diagonalising a 2×2 matrix using the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},$$

which you saw in the examples of Section 1 has eigenvalues -1 and 4, and corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, respectively. We can check that these are indeed eigenvalues and eigenvectors of $\mathbf{A}$ without referring to the earlier examples by checking the corresponding eigenvector equations:

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Next consider the matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},$$

whose columns are the two eigenvectors of $\mathbf{A}$ that we found. Two eigenvectors that are scalar multiples of one another must correspond to the same eigenvalue, and we can see that neither of the columns of $\mathbf{P}$ is a scalar multiple of the other. Now, as you saw in Unit 6, this implies that the determinant of $\mathbf{P}$ is not zero, so $\mathbf{P}$ is invertible. In this case, as

$$\det \mathbf{P} = 1 \times 3 - 2 \times (-1) = 5,$$

the inverse matrix is

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}.$$

We'll use the inverse matrix $\mathbf{P}^{-1}$ shortly.

If you multiply $\mathbf{A}$ on the right by $\mathbf{P}$, then you'll notice that the columns of the resulting matrix $\mathbf{AP}$ are the vectors $\mathbf{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$:

$$\mathbf{AP} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 8 \\ 1 & 12 \end{pmatrix}.$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \mathbf{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} & \mathbf{A}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array}$$

Let's now introduce another matrix,

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix},$$

which is a diagonal matrix with the eigenvalues -1 and 4 of $\mathbf{A}$ on the leading diagonal. The eigenvalue -1 in the first column of $\mathbf{D}$ corresponds to the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ of $\mathbf{A}$ in the first column of $\mathbf{P}$, and the eigenvalue 4 in the second column of $\mathbf{D}$ corresponds to the eigenvector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ of $\mathbf{A}$ in the second column of $\mathbf{P}$.

If you multiply $\mathbf{P}$ on the right by $\mathbf{D}$, then you'll notice that the columns of the resulting matrix $\mathbf{PD}$ are the vectors $-\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $4\begin{pmatrix} 2 \\ 3 \end{pmatrix}$:

$$\mathbf{PD} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} -1 & 8 \\ 1 & 12 \end{pmatrix}. \\ \begin{array}{cc} \uparrow & \uparrow \\ -\begin{pmatrix} 1 \\ -1 \end{pmatrix} & 4\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array} \end{array}$$

As you can see, the matrices $\mathbf{AP}$ and $\mathbf{PD}$ are equal. This is because the columns of the first matrix are, from left to right,

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

and the columns of the second matrix are, from left to right,

$$-\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad 4\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

and the eigenvector equations for $\mathbf{A}$ tell us that the first pair of vectors is the same as the second pair of vectors.

If you multiply each side of the equation

$$\mathbf{AP} = \mathbf{PD}$$

on the right by $\mathbf{P}^{-1}$, the inverse of $\mathbf{P}$, then you obtain

$$\mathbf{APP}^{-1} = \mathbf{PDP}^{-1}.$$

Since $\mathbf{PP}^{-1} = \mathbf{I}$, the 2×2 identity matrix, it follows that

$$\mathbf{A} = \mathbf{PDP}^{-1}.$$

You've now seen how to write $\mathbf{A}$ as a product of three matrices, one of them a diagonal matrix. The same procedure can be applied to any 2×2 matrix that has two distinct real eigenvalues.

Diagonalising a 2×2 matrix

A 2×2 matrix $\mathbf{A}$ is said to be **diagonalisable** if there is a 2×2 diagonal matrix $\mathbf{D}$ and a 2×2 invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{PDP}^{-1}.$$

The process of finding matrices $\mathbf{D}$ and $\mathbf{P}$ that satisfy this equation is called **diagonalising** the matrix $\mathbf{A}$.

If a 2×2 matrix $\mathbf{A}$ has only one real eigenvalue, and it's not already a diagonal matrix, then it's not diagonalisable; that is, you cannot find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

Strategy: To diagonalise a 2×2 matrix

Let $\mathbf{A}$ be a 2×2 matrix with two distinct real eigenvalues.

1. Find the eigenvalues λ_1 and λ_2 of $\mathbf{A}$, and corresponding eigenvectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, respectively.
2. Form the 2×2 matrix
$$\mathbf{P} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$
3. Find the inverse $\mathbf{P}^{-1}$ of $\mathbf{P}$.
4. Write down the equation $\mathbf{A} = \mathbf{PDP}^{-1}$, where
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

It's important to remember that the order in which you place the eigenvectors to form the matrix $\mathbf{P}$, and the order in which you place the eigenvalues to form the matrix $\mathbf{D}$, must correspond. That is, the first column of $\mathbf{P}$ must be the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue in the first column of $\mathbf{D}$ (the top-left element), and the second column of $\mathbf{P}$ must be the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue in the second column of $\mathbf{D}$ (the bottom-right element). If you switch the elements on the leading diagonal of $\mathbf{D}$, then you must switch the columns of $\mathbf{P}$ (and then recalculate $\mathbf{P}^{-1}$).

You should also remember that each eigenvalue of a matrix has many corresponding eigenvectors. You can choose any of these eigenvectors to place in the columns of $\mathbf{P}$. Just make sure that the two columns of $\mathbf{P}$ are eigenvectors corresponding to *different* eigenvalues.

Example 9 *Diagonalising a 2×2 matrix*

Diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

Solution

First find the eigenvalues of $\mathbf{A}$.

We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= (-1) + 5 = 4, \\ \det \mathbf{A} &= (-1) \times 5 - 4 \times (-2) = 3. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda + 3 = 0.$$

Hence

$$(\lambda - 1)(\lambda - 3) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 1 and 3.

Find eigenvectors corresponding to these eigenvalues.

The eigenvector equations have the form

$$\begin{pmatrix} -1-\lambda & 4 \\ -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} -2x + 4y &= 0, \\ -2x + 4y &= 0. \end{aligned}$$

Hence $2x = 4y$, so $x = 2y$. If $y = 1$, then $x = 2$. Hence $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

There's no need to check this now as you'll carry out a check at the end.

When $\lambda = 3$, the eigenvector equation is

$$\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} -4x + 4y &= 0, \\ -2x + 2y &= 0. \end{aligned}$$

This pair of equations is equivalent to the single equation

$$-x + y = 0; \quad \text{that is,} \quad y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

Form a matrix whose columns are the two eigenvectors.

Let

$$\mathbf{P} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}.$$

Find the inverse of $\mathbf{P}$.

Then

$$\det \mathbf{P} = 2 \times 1 - 1 \times 1 = 1,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix} = \begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}.$$

Write down the equation $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on the leading diagonal. Make sure that the eigenvalue of $\mathbf{A}$ in the first column of $\mathbf{D}$ corresponds to the eigenvector of $\mathbf{A}$ in the first column of $\mathbf{P}$, and similarly for the second columns.

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}.$$

Check that the equation $\mathbf{A} = \mathbf{PDP}^{-1}$ is indeed satisfied. To do this, multiply the matrices in the expression $\mathbf{PDP}^{-1}$, and check that you obtain $\mathbf{A}$.

(Check:

$$\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 3\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix} = \begin{pmatrix}-1 & 4\\-2 & 5\end{pmatrix}.)$$

We could have switched the order of the eigenvalues and eigenvectors in the preceding example to give different choices for $\mathbf{D}$ and $\mathbf{P}$, namely

$$\mathbf{P} = \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}.$$

In the activity that follows, and in other activities that require you to diagonalise a matrix, you may find that you order the eigenvalues in the opposite way to that given in the solution. You may also find that you choose different eigenvectors to those given in the solution, affecting the matrix $\mathbf{P}$ and its inverse. To be sure that the matrices you've obtained are correct, always check that the equation $\mathbf{A} = \mathbf{PDP}^{-1}$ is satisfied at the end of your working.

Activity 18 *Diagonalising 2×2 matrices*

Diagonalise the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$

Jacobi eigenvalue algorithm

Carl Jacobi was a German mathematician whose research covered many mathematical subjects, including calculus, dynamics and number theory. One of his numerous contributions to mathematics, which came in 1846 while he was working on celestial mechanics, was to develop an algorithm to calculate the eigenvalues and eigenvectors of a particular type of matrix.

The method was rediscovered in 1949 by the mathematicians Herman Goldstine (1913–2004), Francis Murray (1911–1996) and John von Neumann (1903–1957). In his book *The Computer from Pascal to von Neumann*, Goldstine describes how he presented the method at a conference in 1951, only to be told that Jacobi had discovered it 100 years earlier!

Carl Jacobi (1804–51)

3.2 Powers of matrices

One of the uses of diagonalisation is to calculate powers of matrices. To see how this is done, let's begin by investigating powers of diagonal matrices, then move on to looking at powers of diagonalisable matrices.

Suppose that you wish to work out powers of the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

You will find that

$$\mathbf{D}^2 = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 \\ 0 & 2^2 \end{pmatrix},$$

$$\mathbf{D}^3 = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 5^2 & 0 \\ 0 & 2^2 \end{pmatrix} = \begin{pmatrix} 5^3 & 0 \\ 0 & 2^3 \end{pmatrix},$$

$$\mathbf{D}^4 = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 5^3 & 0 \\ 0 & 2^3 \end{pmatrix} = \begin{pmatrix} 5^4 & 0 \\ 0 & 2^4 \end{pmatrix},$$

and so on. More generally, we have the following observation.

Powers of 2×2 diagonal matrices

Let

$$\mathbf{D} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Then

$$\mathbf{D}^n = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix} \quad \text{for } n = 1, 2, 3, \ldots.$$

Activity 19 *Finding powers of 2×2 diagonal matrices*

Find the following powers of matrices.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}^3$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^5$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{10}$

You've just seen that there is a straightforward method for calculating powers of diagonal matrices; now we'll use that method to help us find powers of *diagonalisable* matrices. Suppose then that $\mathbf{A}$ is a 2×2 diagonalisable matrix. This means that there is a 2×2 diagonal matrix $\mathbf{D}$ and a 2×2 invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{PDP}^{-1}.$$

It follows that

$$\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1}.$$

Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, the 2×2 identity matrix, we obtain

$$\mathbf{A}^2 = \mathbf{PDIDP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}.$$

Similarly, we have

$$\mathbf{A}^3 = (\mathbf{PD}\underbrace{\mathbf{P}^{-1})(\mathbf{P}}_{\mathbf{P}^{-1}\mathbf{P}=\mathbf{I}}\mathbf{D}\underbrace{\mathbf{P}^{-1})(\mathbf{P}}_{\mathbf{P}^{-1}\mathbf{P}=\mathbf{I}}\mathbf{DP}^{-1}) = \mathbf{PD}^3\mathbf{P}^{-1},$$

and in general the following observation holds.

Powers of 2×2 diagonalisable matrices

Let $\mathbf{A}$ be a 2×2 diagonalisable matrix that satisfies

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where $\mathbf{D}$ is a 2×2 diagonal matrix and $\mathbf{P}$ is a 2×2 invertible matrix. Then

$$\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} \quad \text{for } n = 1, 2, 3, \ldots.$$

This gives us a strategy to find powers of diagonalisable matrices.

Strategy: To find powers of 2×2 diagonalisable matrices

Suppose that you want to find $\mathbf{A}^n$, where $\mathbf{A}$ is a 2×2 diagonalisable matrix and n is a positive integer.

1. Diagonalise $\mathbf{A}$; that is, find a diagonal matrix $\mathbf{D} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and a 2×2 invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.
2. Calculate $\mathbf{D}^n = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix}$.
3. Apply the formula $\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$.

Remember that you can only apply this strategy to diagonalisable matrices, so that excludes those 2×2 matrices that are not diagonal matrices and have only one real eigenvalue.

When you apply the strategy, the final calculation to find $\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ may involve large numbers, so you might want to check your answer at the end. One way to do this is to use the module computer algebra system. Here we give an alternative method to check your solution, by hand. This method is based on a useful identity about the trace of the product of two matrices, which you'll establish in the next activity.

Activity 20 *Establishing a trace identity*

Given any pair of matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

show that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.

Suppose now that $\mathbf{P}$ is an invertible 2×2 matrix. If you replace $\mathbf{A}$ by $\mathbf{P}^{-1}$ and $\mathbf{B}$ by $\mathbf{PB}$ in the trace identity $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$, then you obtain

$$\operatorname{tr}(\mathbf{P}^{-1}(\mathbf{PB})) = \operatorname{tr}((\mathbf{PB})\mathbf{P}^{-1}).$$

Since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, the identity matrix, the matrix expression on the left of this equation simplifies to give $\mathbf{B}$. So we obtain the following useful formula.

Trace formula for 2×2 matrices

If $\mathbf{B}$ is a 2×2 matrix, and $\mathbf{P}$ is an invertible 2×2 matrix, then

$$\operatorname{tr}\mathbf{B} = \operatorname{tr}(\mathbf{PBP}^{-1}).$$

We can apply this formula to our strategy for working out matrix powers, as follows. Suppose that $\mathbf{A}$ is a 2×2 diagonalisable matrix, so that there is a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$$

for any positive integer n. If we substitute $\mathbf{D}^n$ for $\mathbf{B}$ in the formula from the box above, and let the matrix $\mathbf{P}$ in the box be the same as the matrix $\mathbf{P}$ used to diagonalise $\mathbf{A}$, then we obtain

$$\operatorname{tr}(\mathbf{D}^n) = \operatorname{tr}(\mathbf{PD}^n\mathbf{P}^{-1}).$$

But $\mathbf{PD}^n\mathbf{P}^{-1}$ is equal to $\mathbf{A}^n$, so we conclude that $\mathbf{A}^n$ and $\mathbf{D}^n$ have the same trace. This gives us a handy way of checking matrix powers, summarised below.

Trace test for checking 2×2 matrix powers

Suppose that the 2×2 matrices $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{P}$ satisfy $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. Then

$$\operatorname{tr}(\mathbf{A}^n) = \operatorname{tr}(\mathbf{D}^n)$$

for any positive integer n.

For example, you saw in Example 9 that if you diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix},$$

then you obtain the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Now, you can check that

$$\mathbf{A}^3 = \begin{pmatrix} -25 & 52 \\ -26 & 53 \end{pmatrix} \quad \text{and} \quad \mathbf{D}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}.$$

Then

$$\operatorname{tr}(\mathbf{A}^3) = (-25) + 53 = 28 \qquad \text{and} \qquad \operatorname{tr}(\mathbf{D}^3) = 1 + 27 = 28,$$

so the traces of $\mathbf{A}^3$ and $\mathbf{D}^3$ agree, as the trace test predicts.

Of course, checking that $\mathbf{A}^3$ and $\mathbf{D}^3$ have the same trace doesn't ensure that $\mathbf{A}^3 = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}$, but nonetheless this trace test is a useful way of picking up numerical errors.

Here's an example in which the strategy for finding powers of 2×2 diagonalisable matrices is applied, and the trace test is carried out at the end.

Example 10 *Finding a power of a 2×2 diagonalisable matrix*

Diagonalise the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$, and hence find $\mathbf{A}^6$.

Solution

Diagonalise $\mathbf{A}$. To do this, first find the eigenvalues of $\mathbf{A}$.

Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely -2 and 1.

Find eigenvectors corresponding to these eigenvalues.

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 1 \\ 0 & -2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -2$, we obtain

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$3x + y = 0,$$
$$0x + 0y = 0.$$

This pair of equations is equivalent to the single equation

$$3x + y = 0; \quad \text{that is,} \quad y = -3x.$$

If $x = 1$, then $y = -3$. Hence $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -2.

Find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}.$$

Then $\det \mathbf{P} = -3$, so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} -3 & -1 \\ 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

After so many calculations, it's worth checking that the equation $\mathbf{A} = \mathbf{PDP}^{-1}$ is satisfied, even though you'll carry out another check at the end.

(Check:

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 3 & 3 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}.) \end{aligned}$$

Calculate $\mathbf{D}^6$.

Then

$$\mathbf{D}^6 = \begin{pmatrix} 1^6 & 0 \\ 0 & (-2)^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}.$$

Apply the formula $\mathbf{A}^6 = \mathbf{PD}^6\mathbf{P}^{-1}$.

So

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{PD}^6\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 64 \\ 0 & -192 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 3 & -63 \\ 0 & 192 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -21 \\ 0 & 64 \end{pmatrix}. \end{aligned}$$

Check that the traces of $\mathbf{A}^6$ and $\mathbf{D}^6$ are equal.

(Check: $\operatorname{tr}(\mathbf{A}^6) = 1 + 64 = 65$ and $\operatorname{tr}(\mathbf{D}^6) = 1 + 64 = 65$.)

Activity 21 *Finding powers of 2×2 diagonalisable matrices*

(a) Diagonalise the matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, and hence find $\mathbf{A}^9$.

(b) Diagonalise the matrix $\mathbf{A} = \begin{pmatrix} -2 & -6 \\ 2 & 5 \end{pmatrix}$, and hence find $\mathbf{A}^5$.

You may have found that it's quicker to work out the matrix powers in the preceding activity by multiplying the matrices directly rather than using the diagonalisation strategy. The benefits in speed for calculating powers of matrices using diagonalisation really only become apparent for much larger powers. This is because far more computations are needed to work out $\mathbf{A}^n$ than to work out $\mathbf{D}^n$ when n is a large positive integer, and you are more likely to make errors in the former calculation than in the latter.

Another point to consider here is accuracy. Each time a calculation is carried out by a scientific calculator, the accuracy of the answer is limited by the number of digits stored in the memory. The more calculations that are done, the more errors creep in due to rounding. Calculation of matrix powers using diagonalisation involves fewer calculations and hence gives more accurate answers.

3.3 A predator–prey system

In this subsection you'll see an application of diagonalisation in modelling a wildlife population. Although the model is unrealistically simplistic, and the numbers used are made-up, it nevertheless exhibits many of the features of more sophisticated models employed by biologists.

On the Canadian tundra there is a population of Arctic wolves and a population of Arctic hares. The wolves rely on the hares to form a major part of their diet, so the populations of wolves and hares are related.

An Arctic wolf

Starting from some fixed time, the populations of wolves and hares are recorded after each year passes. Let's write w_n for the number of wolves after exactly n years have passed, and h_n for the numbers of hares after exactly n years. In particular, w_0 and h_0 are the numbers of wolves and hares, respectively, at the very start of the process. We'll assume that the initial number of hares is at least twice the initial number of wolves; that is, $h_0 \geq 2w_0$.

A biologist studying these animals proposes the following system of equations to model the populations:

$$\begin{aligned} w_n &= 0.94w_{n-1} + 0.02h_{n-1} \\ h_n &= -0.03w_{n-1} + 1.01h_{n-1} \end{aligned} \qquad (n = 1, 2, 3, \ldots).$$

Each of the coefficients on the right-hand side of these equations indicates how much the population of wolves or hares after $n-1$ years will affect the population after n years.

For example, the terms $0.94w_{n-1}$ and $0.02h_{n-1}$ in the first equation tell us that there must be a large number of hares relative to wolves in order for the wolf population to grow. Similarly, the terms $-0.03w_{n-1}$ and $1.01h_{n-1}$ in the second equation tell us that if there are a large number of wolves relative to hares, then the hare population will shrink.

We can represent the system of equations in matrix form as

$$\begin{pmatrix} w_n \\ h_n \end{pmatrix} = \begin{pmatrix} 0.94 & 0.02 \\ -0.03 & 1.01 \end{pmatrix} \begin{pmatrix} w_{n-1} \\ h_{n-1} \end{pmatrix} \quad (n = 1, 2, 3, \dots).$$

Let's write

$$\mathbf{A} = \begin{pmatrix} 0.94 & 0.02 \\ -0.03 & 1.01 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_n = \begin{pmatrix} w_n \\ h_n \end{pmatrix},$$

then the matrix equations tell us that

$$\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} \quad (n = 1, 2, 3, \dots).$$

Now, as $\mathbf{x}_{n-1} = \mathbf{A}\mathbf{x}_{n-2}$ provided $n \geq 2$, we obtain

$$\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} = \mathbf{A}(\mathbf{A}\mathbf{x}_{n-2}) = \mathbf{A}^2\mathbf{x}_{n-2}.$$

Reasoning in a similar way, using $\mathbf{x}_{n-2} = \mathbf{A}\mathbf{x}_{n-3}$ and so on, we find that

$$\begin{aligned} \mathbf{x}_n &= \mathbf{A}^2\mathbf{x}_{n-2} \\ &= \mathbf{A}^3\mathbf{x}_{n-3} \\ &= \mathbf{A}^4\mathbf{x}_{n-4} \\ &\;\;\vdots \\ &= \mathbf{A}^n\mathbf{x}_0. \end{aligned}$$

To summarise, we have the following formula for calculating the populations of wolves and hares after n years.

Formula for populations of wolves and hares

The populations of wolves and hares after n years (given by $\mathbf{x}_n$) is related to the initial populations of wolves and hares (given by $\mathbf{x}_0$) according to the formula

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0.$$

Let's use this formula to predict the populations of wolves and hares after 10 years starting from some initial populations that we'll specify later. The first task is to find $\mathbf{A}^{10}$, and you can do this using the strategy of the previous subsection. Here are the results of applying that strategy, in brief. You should find that 1 and 0.95 are eigenvalues of $\mathbf{A}$, and that $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1, and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.95. Of course, you can check these eigenvalues and eigenvectors using the eigenvector equations. Next, let

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

Since $\det \mathbf{P} = -5$, the inverse of $\mathbf{P}$ is

$$\mathbf{P}^{-1} = -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -0.2 & 0.4 \\ 0.6 & -0.2 \end{pmatrix}.$$

Finally, diagonalising $\mathbf{A}$ we obtain $\mathbf{A} = \mathbf{PDP}^{-1}$ and

$$\mathbf{A}^{10} = \mathbf{PD}^{10}\mathbf{P}^{-1},$$

where

$$\mathbf{D}^{10} = \begin{pmatrix} 1^{10} & 0 \\ 0 & 0.95^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.95^{10} \end{pmatrix}.$$

We could now work out the elements of the matrix $\mathbf{A}^{10}$, but we can actually avoid doing so here (remember that it's $\mathbf{x}_{10}$ that we wish to calculate, not $\mathbf{A}^{10}$). As $\mathbf{x}_{10} = \mathbf{A}^{10}\mathbf{x}_0$, it follows that

$$\mathbf{x}_{10} = \mathbf{PD}^{10}\mathbf{P}^{-1}\mathbf{x}_0,$$

so to find $\mathbf{x}_{10}$ you can start by calculating $\mathbf{P}^{-1}\mathbf{x}_0$, the result of which is a vector. You can then multiply this vector on the left by $\mathbf{D}^{10}$ to give another vector, and then multiply this vector on the left by $\mathbf{P}$ to give $\mathbf{x}_{10}$. The next example works through the details for a particular value of $\mathbf{x}_{10}$.

Example 11 *Predicting the populations after 10 years*

Use the model above to predict the populations of wolves and hares after 10 years given that there are initially 200 wolves and 1000 hares.

Solution

We have $\mathbf{x}_0 = \begin{pmatrix} 200 \\ 1000 \end{pmatrix}$.

Write this as a scalar multiple of a simpler vector to simplify the calculations.

So $\mathbf{x}_0 = 200\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Ignore the scale factor 200 for the moment; you can reintroduce it at the end.

Calculate $\mathbf{A}^{10}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$; that is, calculate $\mathbf{PD}^{10}\mathbf{P}^{-1}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$. To do this, first find $\mathbf{P}^{-1}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Now

$$\mathbf{P}^{-1}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}-0.2 & 0.4\\0.6 & -0.2\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}1.8\\-0.4\end{pmatrix}.$$

Next find $\mathbf{D}^{10}\mathbf{P}^{-1}\begin{pmatrix}1\\5\end{pmatrix}$. To avoid rounding errors, leave the number 0.95^{10} (the bottom-right element of $\mathbf{D}$) in that form for now, and use a calculator to compute numerical values at the end.

So

$$\mathbf{D}^{10}\mathbf{P}^{-1}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0.95^{10}\end{pmatrix}\begin{pmatrix}1.8\\-0.4\end{pmatrix} = \begin{pmatrix}1.8\\-0.4\times 0.95^{10}\end{pmatrix}.$$

Now find $\mathbf{P}\mathbf{D}^{10}\mathbf{P}^{-1}\begin{pmatrix}1\\5\end{pmatrix}$.

Hence

$$\begin{aligned}\mathbf{A}^{10}\begin{pmatrix}1\\5\end{pmatrix} &= \mathbf{P}\mathbf{D}^{10}\mathbf{P}^{-1}\begin{pmatrix}1\\5\end{pmatrix}\\ &= \begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}\begin{pmatrix}1.8\\-0.4\times 0.95^{10}\end{pmatrix}\\ &= \begin{pmatrix}1.8-0.8\times 0.95^{10}\\5.4-0.4\times 0.95^{10}\end{pmatrix}.\end{aligned}$$

Reintroduce the factor 200 that was dropped earlier.

So

$$\begin{aligned}\mathbf{A}^{10}\mathbf{x}_0 &= \mathbf{A}^{10}\left(200\begin{pmatrix}1\\5\end{pmatrix}\right)\\ &= 200\mathbf{A}^{10}\begin{pmatrix}1\\5\end{pmatrix}\\ &= 200\begin{pmatrix}1.8-0.8\times 0.95^{10}\\5.4-0.4\times 0.95^{10}\end{pmatrix}\\ &= \begin{pmatrix}360-160\times 0.95^{10}\\1080-80\times 0.95^{10}\end{pmatrix}\end{aligned}$$

Work out these values using a calculator.

$$= \begin{pmatrix}264.20\ldots\\1032.10\ldots\end{pmatrix}.$$

You can't have 0.2 of a wolf or 0.1 of a hare, so round the final answers to the nearest integer. It's important to round only at the end of the calculation.

The model predicts that, to the nearest integers, there will be 264 wolves and 1032 hares after 10 years.

Activity 22 *Exploring other initial conditions*

Use the model above to predict the populations of wolves and hares after 10 years given that there are initially 600 wolves and 2100 hares.

Models for predator–prey systems

The system you've seen for modelling populations is rather crude. A much more widely used system is provided by the Lotka–Volterra equations, which are,

$$\frac{dx}{dt} = x(a - by),$$
$$\frac{dy}{dt} = -y(c - dx).$$

Here x and y denote the populations of some prey and predator species, respectively, at a certain time t (they are functions of t), and a, b, c and d are real constants.

The Lotka–Volterra equations are named after two scientists who independently investigated the equations in the early twentieth century. The American biophysicist Alfred Lotka first used them in the context of chemical reactions in 1910, and the equations were also discovered later, in 1926, by the Italian mathematician and physicist Vito Volterra, who was carrying out analysis of fish catches in the Adriatic Sea.

Alfred Lotka (1880–1949)

Vito Volterra (1860–1940)

4 Systems of differential equations

In Unit 8 you learned methods for solving differential equations such as

$$\frac{dy}{dt} = 2y.$$

The general solution of this particular equation is $y = Ce^{2t}$, where C is an arbitrary constant. This section is about solving pairs of differential equations, such as

$$\frac{dx}{dt} = 2x + 3y,$$
$$\frac{dy}{dt} = -x - 2y.$$

In this pair of equations, x and y are both functions of t, and the solution (to follow later in the section, in Example 13) consists of an expression for x in terms of t and an expression for y in terms of t. You'll see how the strategies you've learned for finding eigenvalues and eigenvectors can help you to solve such differential equations.

4.1 Matrix differential equations

A pair of differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy,$$

where x and y are functions of a variable t, and a, b, c and d are real number parameters, is called a **system of differential equations**, or sometimes just a **system**. In further mathematics modules, a system of differential equations may refer to a more complicated set of equations than those considered here (involving, for example, higher derivatives and more complex expressions in x and y).

To save space we will in future write $\dot{x}$ and $\dot{y}$ for the derivatives of x and y, respectively, with respect to t. With this notation, the system of differential equations has the simpler form

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy.$$

Using matrix notation we can give another representation of these equations. Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Both x and y are functions of t, so $\mathbf{x}$ is a vector function of t. We'll write the derivative of $\mathbf{x}$ with respect to t as $\dot{\mathbf{x}}$. It's given by

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

We can now write the system of differential equations using vectors and matrices.

Matrix form of a system of differential equations

The system of differential equations

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

can be written in matrix notation as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

or more briefly as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

Systems of differential equations written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are called **matrix differential equations**.

Activity 23 *Writing systems of differential equations in matrix notation*

Write the following systems of differential equations in matrix notation.

(a) $\begin{aligned} \dot{x} &= -x + 4y \\ \dot{y} &= 7x - 3y \end{aligned}$ (b) $\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \end{aligned}$

A **solution** of a system of differential equations consists of an equation for x in terms of t and an equation for y in terms of t that simultaneously satisfy the system. The two equations can be written together as a single vector equation. The **general solution** of the system is a solution containing arbitrary constants, such that any solution of the system is obtained by choosing particular values for the arbitrary constants. **Solving** a system of differential equations means finding its general solution.

The remainder of this section is concerned with solving systems of differential equations.

4.2 Decoupled systems

Consider the system of differential equations,

$$\dot{x} = -x,$$
$$\dot{y} = 4y.$$

This is said to be a *decoupled system* because the first differential equation involves the function x but not the function y, and the second differential equation involves y but not x.

The matrix form of this system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

As you can see, the matrix is a diagonal matrix. In general, you can determine whether a system of differential equations is a decoupled system by checking whether the matrix in the matrix representation is a diagonal matrix.

Decoupled systems of differential equations

A **decoupled system of differential equations**, or more briefly a **decoupled system**, is a system of the form

$$\dot{x} = ax,$$
$$\dot{y} = dy.$$

In matrix notation, a decoupled system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

A decoupled system consists of a differential equation in x and a differential equation in y, which you can solve separately. Let's solve the system

$$\dot{x} = -x,$$
$$\dot{y} = 4y,$$

in this way.

In Unit 8, you learned that differential equations such as $\dot{x} = -x$ can be solved by the method of separating variables. Applying this method (with $x \neq 0$) gives

$$\int \frac{1}{x}\,\mathrm{d}x = \int -1\,\mathrm{d}t.$$

Now carry out both integrations to obtain

$$\ln|x| = -t + c,$$

where c is an arbitrary constant. Hence

$$|x| = e^{-t+c} = e^{c}e^{-t}.$$

It follows that

$$x = e^c e^{-t} \quad (\text{if } x > 0) \quad \text{and} \quad x = -e^c e^{-t} \quad (\text{if } x < 0).$$

Since e^c is a constant, the general solution is

$$x = Ce^{-t},$$

where C is an arbitrary constant. (When $C = 0$ we obtain the particular solution $x = 0$, which we discounted before applying the method of separating variables to avoid division by zero.)

Similarly, the general solution of the differential equation $\dot{y} = 4y$ is

$$y = De^{4t},$$

where D is an arbitrary constant.

Using similar methods we can find the general solution of any decoupled system.

Solution of a decoupled system of differential equations

The general solution of the decoupled system of differential equations

$$\dot{x} = ax,$$
$$\dot{y} = dy,$$

is

$$x = Ce^{at},$$
$$y = De^{dt},$$

where C and D are arbitrary constants.

Notice that the coefficients of t in the exponents of e^{at} and e^{dt} are equal to the diagonal elements of the matrix

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Of course, these are also the eigenvalues of the matrix, and this observation will be important later, when we consider more general systems of differential equations.

Example 12 *Solving a decoupled system*

Find the general solution of the following system of differential equations.

$$\dot{x} = x,$$
$$\dot{y} = 2y.$$

Solution

The general solution of a decoupled system of differential equations has the form

$$x = Ce^{at},$$
$$y = De^{dt}.$$

This is a decoupled system of differential equations, so the general solution is

$$x = Ce^{t},$$
$$y = De^{2t},$$

where C and D are arbitrary constants.

Activity 24 *Solving decoupled systems*

Find the general solutions of the following systems of differential equations.

(a) $\begin{aligned}\dot{x} &= -2x\\ \dot{y} &= 3y\end{aligned}$ (b) $\begin{aligned}\dot{x} &= 0.2x\\ \dot{y} &= -0.5y\end{aligned}$

4.3 Coupled systems

A system of differential equations such as

$$\dot{x} = -x + 4y,$$
$$\dot{y} = -2x + 5y,$$

is said to be a *coupled system* because at least one of the two equations involves both functions x and y. That is, a coupled system is a system that isn't decoupled.

The matrix form of this system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix is not a diagonal matrix, which tells you that the system is a coupled system.

Coupled systems of differential equations

A **coupled system of differential equations**, or more briefly a **coupled system**, is a system of the form

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

where b and c are not both 0. In matrix notation, a coupled system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not a diagonal matrix.

Let's discuss how to solve coupled systems. To do this, we recall the decoupled system

$$\dot{x} = x,$$
$$\dot{y} = 2y,$$

from Example 12. The matrix form of this decoupled system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

and we know that

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is an eigenvector of } \mathbf{A} \text{ corresponding to the eigenvalue } \lambda_1 = 1,$$

and

$$\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is an eigenvector of } \mathbf{A} \text{ corresponding to the eigenvalue } \lambda_2 = 2.$$

You saw in Example 12 that the general solution to this decoupled system is

$$x = Ce^t,$$
$$y = De^{2t},$$

where C and D are arbitrary constants. This solution can be written using column vectors as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ce^t \\ De^{2t} \end{pmatrix} = Ce^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + De^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So you can see that the solution of the decoupled system is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ce^{\lambda_1 t}\mathbf{x}_1 + De^{\lambda_2 t}\mathbf{x}_2,$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2 of $\mathbf{A}$, respectively.

In fact, the general solution of any system of differential equations given by a matrix with two distinct real eigenvalues has a similar form.

Solution of a system given by a matrix with two distinct real eigenvalues

The system

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with two distinct real eigenvalues λ_1 and λ_2, has general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ce^{\lambda_1 t}\mathbf{x}_1 + De^{\lambda_2 t}\mathbf{x}_2,$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2 of $\mathbf{A}$, respectively, and C and D are arbitrary constants.

Let's demonstrate why the general solution given in the box above is true, using the coupled system

$$\dot{x} = -x + 4y,$$
$$\dot{y} = -2x + 5y,$$

which was considered at the start of this subsection. The matrix form of this system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}.$$

In Example 9 you saw that

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1,

and

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3,

and you saw that you can diagonalise $\mathbf{A}$ to give

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

After diagonalising, the coupled system can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{PDP}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then multiplying both sides of this equation on the left by $\mathbf{P}^{-1}$ gives

$$\mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{DP}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

Let's now define

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

Then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix}.$$

Since x and y are both functions of t, so are u and v. Differentiating both sides of this equation with respect to t gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{x} - \dot{y} \\ -\dot{x} + 2\dot{y} \end{pmatrix};$$

that is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

Substituting this formula and formula (3) into equation (2) gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \mathbf{D} \begin{pmatrix} u \\ v \end{pmatrix};$$

that is,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We now have a *decoupled* system in u and v, rather than a coupled system in x and y. You learned earlier how to find the general solution of decoupled systems; in this case the general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Ce^t \\ De^{3t} \end{pmatrix} = Ce^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + De^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where C and D are arbitrary constants.

All that remains is to write the solution for u and v as a solution for x and y. Recall that $\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$, which implies that $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P} \begin{pmatrix} u \\ v \end{pmatrix}$.

Using the solution for $\begin{pmatrix} u \\ v \end{pmatrix}$ given above, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P}(Ce^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + De^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = Ce^t\mathbf{P} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + De^{3t}\mathbf{P} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\mathbf{P} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{P} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the first and second columns of $\mathbf{P}$, respectively, which are the eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of $\mathbf{A}$, we obtain the desired solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ce^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + De^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let's finish by writing this solution in the same form that the system was given. To do this, add the vectors on the right-hand side to give

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2Ce^t + De^{3t} \\ Ce^t + De^{3t} \end{pmatrix},$$

so

$$x = 2Ce^t + De^{3t},$$
$$y = Ce^t + De^{3t}.$$

All this might seem like a lot of hard work, but that's because we explained why the formula for the general solution of this coupled system is true; you needn't go through all these steps each time you wish to solve a coupled system. Instead just follow the strategy below.

Strategy:
To find the general solution of a system of differential equations given by a matrix with two distinct real eigenvalues

Consider the system of differential equations

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

where a, b, c and d are real numbers such that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real eigenvalues. To solve the system, proceed as follows.

1. Write down the matrix form of the system of differential equations.
2. Find the eigenvalues λ_1 and λ_2 of the matrix, and find corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
3. Write down the general solution, which is
$$\mathbf{x} = Ce^{\lambda_1 t}\mathbf{x}_1 + De^{\lambda_2 t}\mathbf{x}_2,$$
where C and D are arbitrary constants.
4. State the solution as an equation for x and an equation for y.

Example 13 *Solving a coupled system*

Find the general solution of the following system of differential equations

$$\dot{x} = 2x + 3y,$$
$$\dot{y} = -x - 2y.$$

Solution

Write down the matrix form of the system of differential equations.

The matrix form of the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

Find eigenvalues and eigenvectors of the matrix.

We have

$$\operatorname{tr} \mathbf{A} = 2 + (-2) = 0,$$
$$\det \mathbf{A} = 2 \times (-2) - 3 \times (-1) = -1.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 1 = 0.$$

Hence $\lambda^2 = 1$, so the eigenvalues of $\mathbf{A}$ are -1 and 1.

The eigenvector equations have the form

$$\begin{pmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -1$, we obtain

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$3x + 3y = 0,$$
$$-x - y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0; \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1.

(Check: $\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

When $\lambda = 1$, the eigenvector equation is

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$x + 3y = 0,$$
$$-x - 3y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 3y = 0; \quad \text{that is,} \quad x = -3y.$$

If $y = 1$, then $x = -3$. Hence $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

(Check: $\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.)

Write down the general solution.

The general solution is

$$\mathbf{x} = Ce^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + De^{t} \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

where C and D are arbitrary constants.

Give the solution in the same form that the system was given. In this case, write it as two equations.

Hence

$$x = Ce^{-t} - 3De^{t},$$
$$y = -Ce^{-t} + De^{t}.$$

Activity 25 *Solving coupled systems*

Find the general solutions of the following systems of differential equations.

(a) $\begin{aligned} \dot{x} &= -2x + 2y \\ \dot{y} &= -2x + 3y \end{aligned}$ (b) $\begin{aligned} \dot{x} &= 7x + 3y \\ \dot{y} &= 3x - y \end{aligned}$

You've now seen how to solve systems of differential equations given by matrices with two distinct real eigenvalues. In further mathematics modules you can learn how to solve systems given by matrices that don't have two distinct real eigenvalues, as well as how to solve more complex systems such as those involving second derivatives.

Learning outcomes

After studying this unit, you should be able to:

- give a geometric interpretation of eigenvalues and eigenvectors
- find eigenvalues and eigenvectors of 2×2 matrices
- find eigenvalues and eigenvectors of matrices larger than 2×2 using the computer algebra system
- use efficient strategies to find eigenvalues and eigenvectors of special types of matrices such as triangular matrices and matrices representing flattenings, rotations and reflections
- understand the geometric interpretation of special types of matrices such as matrices representing shears, flattenings, rotations, reflections and generalised scalings
- diagonalise 2×2 matrices that have two distinct real eigenvalues
- find powers of matrices by diagonalising
- write systems of differential equations as matrix differential equations, and solve those equations given by matrices with two distinct real eigenvalues.

Solutions to activities

Solution to Activity 1

We have

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 3k \end{pmatrix} = 3\begin{pmatrix} 0 \\ k \end{pmatrix}.$$

Hence

$$\mathbf{Ax} = 3\mathbf{x}.$$

Solution to Activity 2

(a) We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}.$$

Hence

$$\mathbf{Ax} = \mathbf{x}.$$

(b) We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k \\ -k \end{pmatrix} = \begin{pmatrix} -k \\ k \end{pmatrix} = -\begin{pmatrix} k \\ -k \end{pmatrix}.$$

Hence

$$\mathbf{Ax} = -\mathbf{x}.$$

Solution to Activity 3

We have

$$\begin{pmatrix} -3 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

so $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

Also,

$$\begin{pmatrix} -3 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2.

Solution to Activity 4

(a) The characteristic equation is

$$\lambda^2 - ((-2) + 0)\lambda + (-2) \times 0 - 1 \times (-1) = 0.$$

That is,

$$\lambda^2 + 2\lambda + 1 = 0.$$

(b) The characteristic equation is

$$\lambda^2 - ((-3) + (-1))\lambda + (-3) \times (-1) - 1 \times 4 = 0.$$

That is,

$$\lambda^2 + 4\lambda - 1 = 0.$$

Solution to Activity 5

(a) We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= (-1) + (-2) = -3, \\ \det \mathbf{A} &= (-1) \times (-2) - 1 \times 0 = 2. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 3\lambda + 2 = 0.$$

(b) We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= 4 + \tfrac{1}{2} = \tfrac{9}{2}, \\ \det \mathbf{A} &= 4 \times \tfrac{1}{2} - 1 \times 1 = 1. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - \tfrac{9}{2}\lambda + 1 = 0.$$

Solution to Activity 6

(a) We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= (-1) + 0 = -1, \\ \det \mathbf{A} &= (-1) \times 0 - 2 \times 1 = -2. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + \lambda - 2 = 0.$$

Hence

$$(\lambda + 2)(\lambda - 1) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -2 and 1.

(Check: $(-2) + 1 = -1 = \operatorname{tr} \mathbf{A}$.)

(b) We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= (-5) + 1 = -4, \\ \det \mathbf{A} &= (-5) \times 1 - (-3) \times 3 = 4. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 4\lambda + 4 = 0.$$

Hence

$$(\lambda + 2)^2 = 0,$$

so $\mathbf{A}$ has a single, repeated eigenvalue -2.

(Check: $(-2) + (-2) = -4 = \operatorname{tr} \mathbf{A}$.)

(c) We have

$$\begin{aligned} \operatorname{tr} \mathbf{A} &= 1 + 1 = 2, \\ \det \mathbf{A} &= 1 \times 1 - (-1) \times 1 = 2. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 2\lambda + 2 = 0.$$

Hence

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 2}}{2 \times 1}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2i}{2}$$
$$= 1 \pm i.$$

So the eigenvalues of $\mathbf{A}$ are $1 - i$ and $1 + i$.

(Check: $(1 - i) + (1 + i) = 2 = \operatorname{tr} \mathbf{A}$.)

Solution to Activity 7

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -2$, we obtain

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$3x + 3y = 0,$$
$$4x + 4y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0; \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2.

(Check: $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

When $\lambda = 5$, the eigenvector equation is

$$\begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-4x + 3y = 0,$$
$$4x - 3y = 0.$$

This pair of equations is equivalent to the single equation

$$3y = 4x.$$

If $x = 3$, then $3y = 4 \times 3$, so $y = 4$. Hence $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 5.

(Check: $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.)

Solution to Activity 8

(a) We have

$$\operatorname{tr} \mathbf{A} = 6 + 2 = 8,$$
$$\det \mathbf{A} = 6 \times 2 - 1 \times 5 = 7.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 8\lambda + 7 = 0.$$

Hence

$$(\lambda - 1)(\lambda - 7) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 1 and 7.

The eigenvector equations have the form

$$\begin{pmatrix} 6-\lambda & 1 \\ 5 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$5x + y = 0,$$
$$5x + y = 0.$$

This pair of equations is equivalent to the single equation

$$y = -5x.$$

If $x = 1$, then $y = -5$. Hence $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

(Check: $\begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.)

When $\lambda = 7$, the eigenvector equation is

$$\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-x + y = 0,$$
$$5x - 5y = 0.$$

This pair of equations is equivalent to the single equation

$$x - y = 0; \quad \text{that is,} \quad y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 7.

(Check: $\begin{pmatrix} 6 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.)

(b) We have

$$\operatorname{tr}\mathbf{A} = 1 + 3 = 4,$$
$$\det\mathbf{A} = 1 \times 3 - 0 \times 3 = 3.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda + 3 = 0.$$

Hence

$$(\lambda - 1)(\lambda - 3) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 1 and 3.

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 3 \\ 0 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$3y = 0,$$
$$2y = 0.$$

This pair of equations is equivalent to the single equation

$$y = 0.$$

Choose $x = 1$. Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

(Check: $\begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.)

When $\lambda = 3$, the eigenvector equation is

$$\begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-2x + 3y = 0,$$
$$0x + 0y = 0.$$

This pair of equations is equivalent to the single equation

$$-2x + 3y = 0; \quad \text{that is,} \quad 3y = 2x.$$

If $x = 3$, then $3y = 2 \times 3$, so $y = 2$. Hence $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

(Check: $\begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.)

(c) We have

$$\mathbf{Ax} = -4\mathbf{x}$$

for every vector $\mathbf{x}$. So $\mathbf{A}$ has only a single, repeated eigenvalue -4, and any non-zero vector, such as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, is an eigenvector of $\mathbf{A}$ corresponding to this eigenvalue.

(d) We have

$$\operatorname{tr}\mathbf{A} = 2 + 0 = 2,$$
$$\det\mathbf{A} = 2 \times 0 - 1 \times (-1) = 1.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 2\lambda + 1 = 0.$$

Hence

$$(\lambda - 1)^2 = 0,$$

so $\mathbf{A}$ has only a single, repeated eigenvalue 1.

The eigenvector equations have the form

$$\begin{pmatrix} 2-\lambda & 1 \\ -1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $\lambda = 1$ gives

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$x + y = 0,$$
$$-x - y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0; \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

(Check: $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

Solution to Activity 9

We have

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of the matrix, and the corresponding eigenvalues are 2, 3 and 1, respectively.

Solution to Activity 11

(a) Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 1 and 5.

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

The eigenvector equations have the form

$$\begin{pmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 5$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$0x + 0y = 0,$$
$$2x - 4y = 0.$$

This pair of equations is equivalent to the single equation

$$2x - 4y = 0; \quad \text{that is,} \quad x = 2y.$$

If $y = 1$, then $x = 2$. Hence $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 5.

(Check: $\begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.)

(b) Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, so there is only a single, repeated eigenvalue -2.

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2.

(c) Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 1 and 3.

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 3.

The eigenvector equations have the form

$$\begin{pmatrix} 3-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$2x + y = 0,$$
$$0x + 0y = 0.$$

This pair of equations is equivalent to the single equation

$$2x + y = 0; \quad \text{that is,} \quad y = -2x.$$

If $x = 1$, then $y = -2$. Hence $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

(Check: $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.)

(d) Since the matrix is a diagonal matrix, the eigenvalues are the elements on the leading diagonal, namely -7 and 7. The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -7, and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 7.

Solution to Activity 12

Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, so there is only a single, repeated eigenvalue 1.

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

Solution to Activity 13

(a) We have

$$\operatorname{tr}\mathbf{A} = (-1) + 9 = 8,$$
$$\det\mathbf{A} = (-1)\times 9 - (-3)\times 3 = 0.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 8\lambda = 0.$$

Hence

$$\lambda(\lambda - 8) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and 8.

The eigenvector equations have the form

$$\begin{pmatrix} -1-\lambda & -3 \\ 3 & 9-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 0$, we obtain

$$\begin{pmatrix} -1 & -3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-x - 3y = 0,$$
$$3x + 9y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 3y = 0; \quad \text{that is,} \quad x = -3y.$$

If $y = 1$, then $x = -3$. Hence $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

(Check: $\begin{pmatrix} -1 & -3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.)

When $\lambda = 8$, the eigenvector equation is

$$\begin{pmatrix} -9 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-9x - 3y = 0,$$
$$3x + y = 0.$$

This pair of equations is equivalent to the single equation

$$3x + y = 0; \quad \text{that is,} \quad y = -3x.$$

If $x = 1$, then $y = -3$. Hence $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 8.

(Check:

$$\begin{pmatrix} -1 & -3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -24 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -3 \end{pmatrix}.)$$

(Here is an alternative way to find the eigenvalues and one of the eigenvectors. Since $\det\mathbf{A} = 0$, the matrix represents a flattening, so the eigenvalues are 0 and $\operatorname{tr}\mathbf{A}$, and here $\operatorname{tr}\mathbf{A} = 8$. Next, since $\mathbf{A}$ has determinant zero, the first column of $\mathbf{A}$, namely

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 8. This eigenvector is not the same as the eigenvector

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

found using the previous method, but both vectors are acceptable answers: they are scalar multiples of each other using the scale factor -1.)

(b) We have

$$\operatorname{tr}\mathbf{A} = 1 + 1 = 2,$$
$$\det\mathbf{A} = 1\times 1 - 1\times 1 = 0.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 2\lambda = 0.$$

Hence

$$\lambda(\lambda - 2) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and 2.

The eigenvector equations have the form

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 0$, we obtain

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$x + y = 0,$$
$$x + y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0; \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

(Check: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

When $\lambda = 2$, the eigenvector equation is

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-x + y = 0,$$
$$x - y = 0.$$

This pair of equations is equivalent to the single equation

$$x - y = 0; \quad \text{that is,} \quad y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

(Check: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.)

(Alternatively, you can find the eigenvalues and one of the eigenvectors using a similar strategy to that shown at the end of the solution to part (a).)

Solution to Activity 14

We have

$$\operatorname{tr} \mathbf{A} = 0 + 0 = 0,$$
$$\det \mathbf{A} = 0 \times 0 - (-1) \times 1 = 1.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 1 = 0.$$

Hence

$$\lambda^2 = -1,$$

so the eigenvalues of $\mathbf{A}$ are $-i$ and i.

(Check: $(-i) + i = 0 = \operatorname{tr} \mathbf{A}$.)

Solution to Activity 15

(a) This matrix represents a reflection in a line through the origin because it is of the form

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix},$$

with $\alpha = \pi/4$.

Therefore it has eigenvalues -1 and 1, and corresponding eigenvectors

$$\begin{pmatrix} -\sin(\pi/4) \\ \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

and

$$\begin{pmatrix} \cos(\pi/4) \\ \sin(\pi/4) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

respectively. Multiplying each of these eigenvectors by $\sqrt{2}$ gives the simpler pair of eigenvectors

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

respectively.

(Check: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.)

(b) This matrix represents a reflection in a line through the origin because it is of the form

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix},$$

with $\alpha = \pi/6$. Therefore it has eigenvalues -1 and 1, and corresponding eigenvectors

$$\begin{pmatrix} -\sin(\pi/6) \\ \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos(\pi/6) \\ \sin(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix},$$

respectively. Multiplying each of these eigenvectors by 2 gives the simpler pair of eigenvectors

$$\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix},$$

respectively.

(Check:

$$\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = -\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

and

$$\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = 1 \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}.)$$

Solution to Activity 16

(a) Since

$$\begin{pmatrix}4\\4\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix},$$

we have

$$\mathbf{A}\begin{pmatrix}4\\4\end{pmatrix}=4\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix}=4\times 2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}8\\8\end{pmatrix}.$$

So f maps the point $(4,4)$ to the point $(8,8)$.

(b) Since

$$\begin{pmatrix}-2\\-1\end{pmatrix}=-\begin{pmatrix}2\\1\end{pmatrix},$$

we have

$$\mathbf{A}\begin{pmatrix}-2\\-1\end{pmatrix}=-\mathbf{A}\begin{pmatrix}2\\1\end{pmatrix}=-\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}.$$

So f maps the point $(-2,-1)$ to the point $(-2,-1)$.

(c) Since

$$\begin{pmatrix}1\\-2\end{pmatrix}=3\begin{pmatrix}2\\1\end{pmatrix}-5\begin{pmatrix}1\\1\end{pmatrix},$$

we have

$$\begin{aligned}\mathbf{A}\begin{pmatrix}1\\-2\end{pmatrix}&=\mathbf{A}\left(3\begin{pmatrix}2\\1\end{pmatrix}-5\begin{pmatrix}1\\1\end{pmatrix}\right)\\&=3\mathbf{A}\begin{pmatrix}2\\1\end{pmatrix}-5\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix}\\&=3\begin{pmatrix}2\\1\end{pmatrix}-5\times 2\begin{pmatrix}1\\1\end{pmatrix}\\&=\begin{pmatrix}-4\\-7\end{pmatrix}.\end{aligned}$$

So f maps the point $(1,-2)$ to the point $(-4,-7)$.

Solution to Activity 17

For each eigenvector of the matrix, the transformation scales distances in the direction of the eigenvector by a factor equal to the magnitude of the corresponding eigenvalue.

If each eigenvalue of the matrix is non-zero, then the image of the grid of parallelograms is another grid of parallelograms. Each line in the original grid is parallel to its image line.

If the matrix has one positive eigenvalue and one negative eigenvalue, then the transformation reverses orientation. If the eigenvalues of the matrix are both positive or both negative, then the transformation preserves orientation.

If one eigenvalue of the matrix is zero, and the transformation is not the zero transformation, then the image grid collapses onto the line through the origin in the direction of a vector given by any non-zero column of the matrix. In this case the transformation is a flattening.

Solution to Activity 18

(a) Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely -1 and 2.

The vector $\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

The eigenvector equations have the form

$$\begin{pmatrix}2-\lambda & 6\\0 & -1-\lambda\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

When $\lambda=-1$, we obtain

$$\begin{pmatrix}3 & 6\\0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

This gives

$$\begin{aligned}3x+6y&=0,\\0x+0y&=0.\end{aligned}$$

This pair of equations is equivalent to the single equation

$$3x+6y=0;\quad\text{that is,}\quad x=-2y.$$

If $y=1$, then $x=-2$. Hence $\begin{pmatrix}-2\\1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1.

Let

$$\mathbf{P}=\begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}.$$

Then

$$\det\mathbf{P}=1\times 1-(-2)\times 0=1,$$

so

$$\mathbf{P}^{-1}=\frac{1}{\det\mathbf{P}}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}.$$

We have

$$\mathbf{A}=\mathbf{PDP}^{-1},$$

where

$$\mathbf{D}=\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}.$$

(Check:

$$\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}2 & 2\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}2 & 6\\ 0 & -1\end{pmatrix}.)$$

(b) We have

$$\operatorname{tr}\mathbf{A} = 1 + 2 = 3,$$
$$\det\mathbf{A} = 1\times 2 - 2\times 1 = 0.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 3\lambda = 0.$$

Hence

$$\lambda(\lambda - 3) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and 3.

The eigenvector equations have the form

$$\begin{pmatrix}1-\lambda & 2\\ 1 & 2-\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

When $\lambda = 0$, we obtain

$$\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

This gives

$$x + 2y = 0,$$
$$x + 2y = 0.$$

Hence $x = -2y$. If $y = 1$, then $x = -2$. Hence $\begin{pmatrix}-2\\ 1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

When $\lambda = 3$, the eigenvector equation is

$$\begin{pmatrix}-2 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

This gives

$$-2x + 2y = 0,$$
$$x - y = 0.$$

This pair of equations is equivalent to the single equation

$$x - y = 0; \quad \text{that is,} \quad y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix}1\\ 1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

Let

$$\mathbf{P} = \begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}.$$

Then

$$\det\mathbf{P} = (-2)\times 1 - 1\times 1 = -3,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det\mathbf{P}}\begin{pmatrix}1 & -1\\ -1 & -2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix}0 & 0\\ 0 & 3\end{pmatrix}.$$

(Check:

$$\begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 3\end{pmatrix}\frac{1}{3}\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}$$
$$= \begin{pmatrix}-2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}$$
$$= \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}$$
$$= \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}.)$$

(Alternatively, you can find the eigenvalues and one of the eigenvectors of $\mathbf{A}$ by using the observations about eigenvalues and eigenvectors of matrices with determinant zero from Subsection 2.2. In this case, as $\det\mathbf{A} = 0$, the eigenvalues of $\mathbf{A}$ are 0 and $\operatorname{tr}\mathbf{A} = 3$. Furthermore, the vector $\begin{pmatrix}1\\ 1\end{pmatrix}$ given by the first column of $\mathbf{A}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.)

(c) We have

$$\operatorname{tr}\mathbf{A} = 2 + (-6) = -4,$$
$$\det\mathbf{A} = 2\times(-6) - 7\times(-1) = -5.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 4\lambda - 5 = 0.$$

Hence

$$(\lambda + 5)(\lambda - 1) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -5 and 1.

The eigenvector equations have the form

$$\begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -5$, we obtain

$$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$7x + 7y = 0,$$
$$-x - y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0; \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -5.

When $\lambda = 1$, the eigenvector equation is

$$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$x + 7y = 0,$$
$$-x - 7y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 7y = 0; \quad \text{that is,} \quad x = -7y.$$

If $y = 1$, then $x = -7$. Hence $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

Let

$$\mathbf{P} = \begin{pmatrix} 1 & -7 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\det \mathbf{P} = 1 \times 1 - (-7) \times (-1) = -6,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}}\begin{pmatrix} 1 & 7 \\ 1 & 1 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 1 & 7 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Check:

$$\begin{aligned} &\begin{pmatrix} 1 & -7 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}\left(-\frac{1}{6}\begin{pmatrix} 1 & 7 \\ 1 & 1 \end{pmatrix}\right) \\ &= -\frac{1}{6}\begin{pmatrix} -5 & -7 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 7 \\ 1 & 1 \end{pmatrix} \\ &= -\frac{1}{6}\begin{pmatrix} -12 & -42 \\ 6 & 36 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}.) \end{aligned}$$

Solution to Activity 19

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}^3 = \begin{pmatrix} 3^3 & 0 \\ 0 & (-4)^3 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & -64 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^5 = \begin{pmatrix} (-1)^5 & 0 \\ 0 & 2^5 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 32 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{10} = \begin{pmatrix} 1^{10} & 0 \\ 0 & 0^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Solution to Activity 20

We have

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

and

$$\mathbf{BA} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa+qc & pb+qd \\ ra+sc & rb+sd \end{pmatrix}.$$

So

$$\operatorname{tr}(\mathbf{AB}) = ap + br + cq + ds$$

and

$$\operatorname{tr}(\mathbf{BA}) = pa + qc + rb + sd.$$

The expressions on the right-hand sides of these two equations are equal, so $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.

Solution to Activity 21

(a) Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely -1 and 1.

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

The eigenvector equations have the form

$$\begin{pmatrix} -1-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -1$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$0x + 0y = 0,$$
$$2x + 2y = 0.$$

This pair of equations is equivalent to the single equation

$$2x + 2y = 0; \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1.

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then $\det \mathbf{P} = 1$, so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Check:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.)$$

Then

$$\mathbf{D}^9 = \begin{pmatrix} (-1)^9 & 0 \\ 0 & 1^9 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is, $\mathbf{D}^9 = \mathbf{D}$, so

$$\mathbf{A}^9 = \mathbf{PD}^9\mathbf{P}^{-1} = \mathbf{PDP}^{-1} = \mathbf{A}.$$

(You'll notice that we chose the matrix $\mathbf{P}$ to be

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \text{rather than} \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

(with the columns interchanged). This is because the left-hand matrix is a lower triangular matrix, as is the matrix $\mathbf{A}$. Multiplying two lower triangular matrices gives another lower triangular matrix, and the inverse of a lower triangular matrix is also a lower triangular matrix. So choosing $\mathbf{P}$ to be a lower triangular matrix simplifies the computations in the solution.)

(b) We have

$$\operatorname{tr} \mathbf{A} = (-2) + 5 = 3,$$
$$\det \mathbf{A} = (-2) \times 5 - (-6) \times 2 = 2.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 3\lambda + 2 = 0.$$

Hence

$$(\lambda - 1)(\lambda - 2) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 1 and 2.

The eigenvector equations have the form

$$\begin{pmatrix} -2-\lambda & -6 \\ 2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} -3 & -6 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-3x - 6y = 0,$$
$$2x + 4y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 2y = 0; \quad \text{that is,} \quad x = -2y.$$

If $y = 1$, then $x = -2$. Hence $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

When $\lambda = 2$, the eigenvector equation is

$$\begin{pmatrix} -4 & -6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-4x - 6y = 0,$$
$$2x + 3y = 0.$$

This pair of equations is equivalent to the single equation

$$2x + 3y = 0; \quad \text{that is,} \quad 3y = -2x.$$

If $x = 3$, then $3y = -2 \times 3$, so $y = -2$. Hence $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

Let

$$\mathbf{P} = \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}.$$

Then $\det \mathbf{P} = 1$, so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} -2 & -3 \\ -1 & -2 \end{pmatrix} = -\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(Check:

$$\begin{aligned} &\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \left(-\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right) \\ &\quad = -\begin{pmatrix} -2 & 6 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \\ &\quad = \begin{pmatrix} -2 & -6 \\ 2 & 5 \end{pmatrix}.) \end{aligned}$$

Then

$$\mathbf{D}^5 = \begin{pmatrix} 1^5 & 0 \\ 0 & 2^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 32 \end{pmatrix}.$$

So

$$\begin{aligned} \mathbf{A}^5 &= \mathbf{PD}^5\mathbf{P}^{-1} \\ &= \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 32 \end{pmatrix} \left(-\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right) \\ &= -\begin{pmatrix} -2 & 96 \\ 1 & -64 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -92 & -186 \\ 62 & 125 \end{pmatrix}. \end{aligned}$$

(Check: $\operatorname{tr}(\mathbf{A}^5) = (-92) + 125 = 33$ and $\operatorname{tr}(\mathbf{D}^5) = 1 + 32 = 33$.)

Solution to Activity 22

We have $\mathbf{x}_0 = \begin{pmatrix} 600 \\ 2100 \end{pmatrix} = 300 \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Now

$$\mathbf{P}^{-1} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -0.2 & 0.4 \\ 0.6 & -0.2 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 2.4 \\ -0.2 \end{pmatrix}.$$

So

$$\begin{aligned} \mathbf{D}^{10}\mathbf{P}^{-1} \begin{pmatrix} 2 \\ 7 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0.95^{10} \end{pmatrix} \begin{pmatrix} 2.4 \\ -0.2 \end{pmatrix} \\ &= \begin{pmatrix} 2.4 \\ -0.2 \times 0.95^{10} \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{A}^{10} \begin{pmatrix} 2 \\ 7 \end{pmatrix} &= \mathbf{PD}^{10}\mathbf{P}^{-1} \begin{pmatrix} 2 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2.4 \\ -0.2 \times 0.95^{10} \end{pmatrix} \\ &= \begin{pmatrix} 2.4 - 0.4 \times 0.95^{10} \\ 7.2 - 0.2 \times 0.95^{10} \end{pmatrix}. \end{aligned}$$

So

$$\begin{aligned} \mathbf{A}^{10}\mathbf{x}_0 &= \mathbf{A}^{10} \left(300 \begin{pmatrix} 2 \\ 7 \end{pmatrix} \right) \\ &= 300\mathbf{A}^{10} \begin{pmatrix} 2 \\ 7 \end{pmatrix} \\ &= 300 \begin{pmatrix} 2.4 - 0.4 \times 0.95^{10} \\ 7.2 - 0.2 \times 0.95^{10} \end{pmatrix} \\ &= \begin{pmatrix} 720 - 120 \times 0.95^{10} \\ 2160 - 60 \times 0.95^{10} \end{pmatrix} \\ &= \begin{pmatrix} 648.15\ldots \\ 2124.07\ldots \end{pmatrix}. \end{aligned}$$

The model predicts that, to the nearest integers, there will be 648 wolves and 2124 hares after 10 years.

Solution to Activity 23

(a) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 7 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution to Activity 24

(a) This is a decoupled system of differential equations, so the general solution is

$$x = Ce^{-2t},$$
$$y = De^{3t},$$

where C and D are arbitrary constants.

(b) This is a decoupled system of differential equations, so the general solution is

$$x = Ce^{0.2t},$$
$$y = De^{-0.5t},$$

where C and D are arbitrary constants.

Solution to Activity 25

(a) The matrix form of the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}.$$

We have

$$\operatorname{tr} \mathbf{A} = (-2) + 3 = 1,$$
$$\det \mathbf{A} = (-2) \times 3 - 2 \times (-2) = -2.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - \lambda - 2 = 0.$$

Hence

$$(\lambda + 1)(\lambda - 2) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -1 and 2.

The eigenvector equations have the form

$$\begin{pmatrix} -2-\lambda & 2 \\ -2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -1$, we obtain

$$\begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-x + 2y = 0,$$
$$-2x + 4y = 0.$$

This pair of equations is equivalent to the single equation

$$-x + 2y = 0; \quad \text{that is,} \quad x = 2y.$$

If $y = 1$, then $x = 2$. Hence $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1.

(Check: $\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.)

When $\lambda = 2$, the eigenvector equation is

$$\begin{pmatrix} -4 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-4x + 2y = 0,$$
$$-2x + y = 0.$$

This pair of equations is equivalent to the single equation

$$-2x + y = 0; \quad \text{that is,} \quad y = 2x.$$

If $x = 1$, then $y = 2$. Hence $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

(Check: $\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.)

The general solution is

$$\mathbf{x} = Ce^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + De^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where C and D are arbitrary constants. Hence

$$x = 2Ce^{-t} + De^{2t},$$
$$y = Ce^{-t} + 2De^{2t}.$$

(b) The matrix form of the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}.$$

We have

$$\operatorname{tr} \mathbf{A} = 7 + (-1) = 6,$$
$$\det \mathbf{A} = 7 \times (-1) - 3 \times 3 = -16.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda - 16 = 0.$$

Hence

$$(\lambda + 2)(\lambda - 8) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -2 and 8.

The eigenvector equations have the form

$$\begin{pmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -2$, we obtain

$$\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$9x + 3y = 0,$$
$$3x + y = 0.$$

This pair of equations is equivalent to the single equation

$$3x + y = 0; \quad \text{that is,} \quad y = -3x.$$

If $x = 1$, then $y = -3$. Hence $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -2.

(Check: $\begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.)

When $\lambda = 8$, the eigenvector equation is

$$\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-x + 3y = 0,$$
$$3x - 9y = 0.$$

This pair of equations is equivalent to the single equation

$$-x + 3y = 0; \quad \text{that is,} \quad x = 3y.$$

If $y = 1$, then $x = 3$. Hence $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 8.

(Check: $\begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \end{pmatrix} = 8\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.)

The general solution is

$$\mathbf{x} = Ce^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + De^{8t}\begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

where C and D are arbitrary constants. Hence

$$x = Ce^{-2t} + 3De^{8t},$$
$$y = -3Ce^{-2t} + De^{8t}.$$

Acknowledgements

Grateful acknowledgement is made to the following sources:

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