Unit 8

Mathematical modelling

Introduction

Applied mathematics is concerned not only with the development of *mathematical methods* but also with the application of these methods in *mathematical models*, which are idealised representations of aspects of the real world. This text aims to develop your appreciation of the role of mathematics in understanding and predicting the behaviour of the real world (as opposed to the world of mathematical theories). This process is called *mathematical modelling*.

For example, some mathematical methods are dealt with in Unit 1, whereas established mathematical models are the subject of Units 2 and 3.

There was a time when the application of mathematics was restricted more or less to physics and engineering. Those days are long gone, and now the application of mathematics is widespread. This unit introduces skills that will enable you to develop your own mathematical models for simple real-world situations. The mathematical modelling process starts with a problem in the real world. This problem is translated into a mathematical model, whose solution may provide insights into the original real-world problem. The mathematical model may also help to predict what will happen in the real world if changes are made. The key stages in the mathematical modelling process are as follows.

We may indicate these stages in abbreviated form in the margin, as we have done for procedures elsewhere.

1. **Specify the purpose of the model:**
   define the problem;
   decide which aspects of the problem to investigate.
2. **Create the model:**
   state assumptions;
   choose variables and parameters;
   formulate mathematical relationships.
3. **Do the mathematics:**
   solve equations;
   draw graphs;
   derive results.
4. **Interpret the results:**
   collect relevant data;
   describe the mathematical solution in words;
   decide what results to compare with reality.
5. **Evaluate the model:**
   test the model by comparing its predictions with reality;
   criticise the model.

The essential difference between a variable and a parameter is that a variable is a quantity whose values change during the situation described by the model, while a parameter is a constant of the model (for a given situation).

The mathematical modelling process is sometimes referred to as the *mathematical modelling cycle*.

The diagram in Figure 1 may help you to remember the five key stages in the mathematical modelling process.

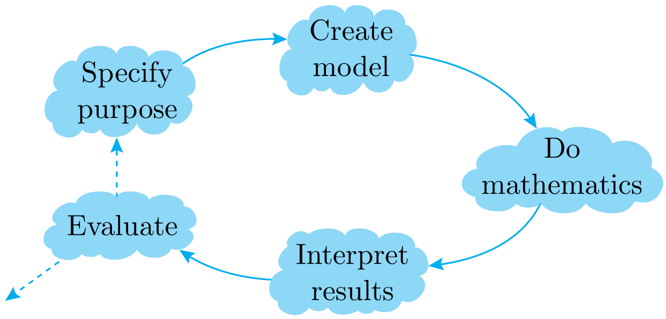


Figure 1 Mathematical modelling process

In developing a mathematical model, you may need to go around the loop in Figure 1 several times, improving your model each time. When you start to create a mathematical model for a real problem, begin with a simple model, in order to obtain a feel for the problem. Often a simple model that gives a reasonable approximation to the real world is more useful than a complicated model. A more complex model may give a better fit to the available data, but the key processes that are being modelled may be more difficult to identify, and the mathematical problem may be harder to solve. If, in evaluating a model, you find that it is not satisfactory for its purpose, then identify why it is deficient and try to include additional features that address those deficiencies in your next pass through the modelling loop.

As you become more experienced as a mathematical modeller, you will probably find that the above schema for mathematical modelling is insufficiently flexible for the mathematical model that you are developing. The purpose of specifying the modelling process clearly is to try to bring some structure to the subject, but the schema should not be seen as a straitjacket. However, it is recommended that you use this schema when tackling the modelling problems that you will encounter in this module.

Mathematical modelling is unlike other branches of mathematics. The mathematical models that you develop depend critically on the original problem and on the simplifying assumptions that you make about the system being modelled. Consequently, there are many ways of tackling any modelling problem and none can be expected to yield an exact solution. When you attempt a modelling exercise in this text, your mathematical model may differ from the one suggested in our solution, but provided that your model is justified by reasoned arguments based on reasonable assumptions, it may be just as valid as the one developed here.

1 Pollution in the Great Lakes

This section explores a real-world system where mathematical modelling has been used to aid understanding of what is happening and to predict what will happen if changes are made. The system concerned is extremely complex, but by keeping things as simple as possible, we will extract sufficient information to obtain a mathematical model of the system. Refinements to this simple model will be made in Unit 18.

The Great Lakes of North America are shown in Figure 2. Lake Superior and Lake Michigan drain into Lake Huron, which drains via Lake Erie into Lake Ontario, and this lake drains into the St Lawrence River and thence into the Atlantic Ocean.

Figure 2 The Great Lakes of North America

The Great Lakes provide drinking water for tens of millions of people who live in the surrounding area. They also provide a source of food, transport and recreation. In the first half of the twentieth century they were used for dumping sewage and other pollutants. Sources of pollution include industrial waste, agricultural chemicals, acid rain, and oil and chemical spills. Our case study concerns the construction of a mathematical model of how the pollution level in a lake varies over time.

The information for this case study comes from two main sources: R.H. Rainey (1967) 'Natural displacement of pollution from the Great Lakes', *Science*, **155**, 1242–3; R.V. Thomann and J.A. Mueller (1987) *Principles of Surface Water Quality Modeling and Control*, Harper and Row.

The mathematical modelling process can be applied to the problem of predicting future pollution levels in the Great Lakes; the details are given below.

◀ Specify purpose ▶

1 Specify the purpose of the model

Define the problem

The problem is to predict how long it will take for the level of pollution in a lake to reduce to a target level if all sources of pollution are eliminated. It is intended to use a mathematical model to investigate pollution levels in any one of the Great Lakes, although the model could also be used for any other polluted lake.

Decide which aspects of the problem to investigate

We want to investigate how the pollution level varies with time as clean water flows into the lake and polluted water flows out.

◀ Create model ▶

2 Create the model

State assumptions

The model makes the following assumptions.

(a) All sources of pollution have been removed.

(b) A pollutant does not biodegrade in the lake or decay through any other biological, chemical or physical process.

(c) A pollutant is evenly dispersed within the lake at all times.

(d) Water flows into and out of the lake at the same constant rate (so all seasonal effects can be ignored).

(e) All other water gains and losses (e.g. rainfall, evaporation, extraction and seepage) can be ignored.

(f) The volume of water in the lake is constant.

The statement that the volume is constant is a consequence of the previous two assumptions, but it is included as an assumption in its own right because of its importance in the modelling process.

(g) If the mathematical model is to be used for the downstream lakes (Huron, Erie and Ontario), then negligible pollution is flowing into them from the upstream lakes.

Choose variables and parameters

The variables in the model are:

t the time, in seconds, since all sources of pollution were removed;

$m(t)$ the mass, in kilograms, of pollutant in the lake at time t;

$c(t)$ the concentration, in kilograms per cubic metre, of pollutant in the lake at time t.

The parameters in the model are:

c_{target}	the target concentration level, in kilograms per cubic metre, of pollutant in the lake;
T	the time taken, in seconds, to reduce the concentration of pollutant to the target level c_{target} (i.e. $c(T) = c_{\text{target}}$);
V	the volume, in cubic metres, of water in the lake;
r	the water flow rate, in cubic metres per second, into and out of the lake;
$k = r/V$	the **proportionate flow rate**, in seconds^{-1}.

Formulate mathematical relationships

By Assumption (c), the pollution concentration is uniform, so the relationship between concentration and mass is given by

$$c(t) = \frac{m(t)}{V}.$$

The **input–output principle**,

$$\boxed{\text{accumulation}} = \boxed{\text{input}} - \boxed{\text{output}},$$

is applied to the mass of pollutant during the time interval $[t, t+\delta t]$. This principle is often used in mathematical modelling.

The input–output principle was introduced in Unit 1.

By Assumptions (a) and (g), the mass of pollutant entering the lake is zero, so the *input* is zero. By Assumption (b), the pollutant leaves the lake only through the outflow of water. In the time interval $[t, t+\delta t]$, the volume of water leaving the lake is $r\,\delta t$, where r, the flow rate, is constant because of Assumptions (d), (e) and (f). Multiplying this by the concentration of pollutant, which is uniform by Assumption (c), gives the mass of pollutant that leaves the lake in that time interval as $r\,c(t)\,\delta t$ or $(r/V)\,m(t)\,\delta t$. This is the *output*.

Note how the assumptions are identified while the mathematical models is developed.

The *accumulation* of the mass of pollutant within the lake, over the time interval $[t, t+\delta t]$, is the difference between $m(t+\delta t)$ and $m(t)$, that is, $m(t+\delta t) - m(t)$. Applying the input–output principle gives

$$m(t+\delta t) - m(t) \simeq 0 - \frac{r}{V}\,m(t)\,\delta t.$$

It is sometimes useful to replace a group of parameters by a single parameter, particularly when the same group of parameters is often repeated; this is called **reparametrisation**. Replacing r/V by k, dividing by δt and then letting $\delta t \to 0$, leads to the differential equation

$$\frac{dm}{dt} = -k\,m(t), \quad \text{where } k = \frac{r}{V}. \tag{1}$$

◀ Do mathematics ▶

3 Do the mathematics

Solve equations

The solution of the differential equation for m is

$$m(t) = m(0)\,e^{-kt}, \quad \text{where } m(0) \text{ is the initial mass of pollutant.}$$

Since $c(t) = m(t)/V$, the pollutant concentration c is given by

$$c(t) = c(0)\,e^{-kt}, \quad \text{where } c(0) \text{ is the initial concentration of pollutant.}$$

The time T taken for the pollutant concentration to reduce to the target level $c_{\text{target}} = c(T) = c(0)\,e^{-kT}$ is given by

$$T = -\frac{1}{k}\ln\left(\frac{c_{\text{target}}}{c(0)}\right).$$

Draw graphs

Typical graphs of concentration against time are shown below.

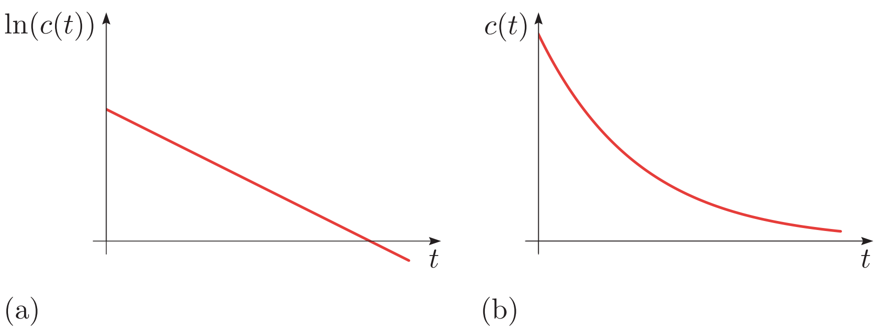


Figure 3 (a) Log concentration versus time; (b) concentration versus time

Derive results

Suppose that a political decision has been taken to reduce the level of pollution to a tenth of its initial level, so that $c_{\text{target}}/c(0) = \frac{1}{10}$. The corresponding length of time required is

$$T = -\frac{1}{k}\ln\left(\tfrac{1}{10}\right) = \frac{\ln 10}{k} \simeq \frac{2.30}{k},$$

so T is inversely proportional to the proportionate flow rate k.

◀ Interpret results ▶

4 Interpret the results

Collect relevant data

The only data required are the values of k. These can be determined for each lake from its volume V and water flow rate r, since $k = r/V$. The data are given in Table 1.

Table 1 Data for the Great Lakes

Lake	Volume V ($10^{12}\,\mathrm{m}^3$)	Water flow rate r ($10^3\,\mathrm{m}^3\,\mathrm{s}^{-1}$)	$k = r/V$ ($10^{-9}\,\mathrm{s}^{-1}$)
Superior	12.10	2.01	0.166
Michigan	4.92	1.57	0.319
Huron	3.54	5.10	1.441
Erie	0.48	5.90	12.292
Ontario	1.64	6.78	4.134

The units given mean, for example, that the volume of Lake Superior is $12.10 \times 10^{12}\,\mathrm{m}^3$, its water flow rate is $2.01 \times 10^3\,\mathrm{m}^3\,\mathrm{s}^{-1}$, and its proportionate flow rate is $0.166 \times 10^{-9}\,\mathrm{s}^{-1}$.

Describe the mathematical solution in words

For Lake Superior, for example, the model predicts that a reduction in the level of pollution to a tenth of its initial level will take a time (in seconds)

$$T = \frac{\ln 10}{k} = 13.9 \times 10^9.$$

Dividing T by $60 \times 60 \times 24 \times 365.24$ to convert it into years, the model predicts that it will take 440 years for the pollution level in Lake Superior to drop by a factor of ten. A similar calculation for Lake Michigan predicts that it will take 7.22×10^9 seconds or 229 years for the pollution level to drop by a factor of ten.

According to Assumption (g), only clean water enters the downstream lakes Huron, Erie and Ontario from the upstream lakes, so the model can be used for these lakes as well. Similar calculations then predict that the times required to reduce the pollution level to a tenth of its initial level are 51 years for Lake Huron, 6 years for Lake Erie and 18 years for Lake Ontario.

However, it may be necessary to revisit this assumption later.

The reason for the long time periods that are predicted for purging the two upper lakes is their rather low proportionate flow rates. The quantity $V/r = 1/k$ is the *average retention time* for water in the lake, that is, the length of time that it takes for an amount of water equal to the volume of water in the lake to flow out of the lake, and hence the time that any molecule of water would expect to remain in the lake. For Lake Superior, the average retention time is 191 years, and for Lake Michigan it is 99 years. For the downstream lakes the average retention times are much shorter.

Decide what results to compare with reality

It is perhaps fortunate that the two most polluted lakes in the 1960s were Lake Erie and Lake Ontario, which have relatively small volumes of water and fairly high water flow rates. It was thus possible, as predicted by the model, to clean up these lakes fairly quickly.

You will see a model of the cumulative effects of pollution levels on the whole system in Unit 18.

The model predicts that if significant amounts of pollution are allowed to enter Lake Superior or Lake Michigan, then it will take hundreds of years before the lakes can recover. It could be argued that the mathematical model is quite conservative in its estimation, since it assumes that all sources of pollution have stopped and that the pollutant is completely dispersed in the water, rather than in the flora and fauna or in the sediment at the bottom of the lake. However, since most pollutants biodegrade, it could also be argued that the estimate is rather pessimistic. Even so, a major contamination of either of these lakes, particularly by a pollutant that does not biodegrade, would be a catastrophe for which there would, apparently, be no short-term solution.

It is salutary to note that in recent years there have been fish consumption advisory warnings in place for many areas, and in particular for Lake Michigan. People are advised not to eat some fish because of the high levels of toxic chemicals in them. Although these chemicals are present in small quantities in the lakes, they tend to accumulate in the fish and then on through the ecosystem. The battle to control pollution in these lakes has not yet been won.

Summary

The development of the pollution model for the Great Lakes has highlighted a number of important aspects of mathematical modelling. These include the following.

This concludes the summary of the first four modelling stages for this problem. The fifth stage, 'Evaluate the model', will be addressed in Unit 18.

- By representing quantities using symbols, it is possible to develop models that can be applied in a variety of different situations.
- The simplifying assumptions must be relevant to the model, and should be linked to the mathematical relationships between the variables and parameters.
- Data for the model should be collected after it has been established what data are required.

Exercise 1

In your own words, and without using any equations or symbols, give an outline, or description, of the formulation of the model in this section. (As a guide the outline should be between 50 and 100 words.)

Exercise 2

(a) Suppose that pollution continues to enter a lake at a constant rate q (in $\mathrm{kg\,s^{-1}}$), and assume that this pollutant instantaneously diffuses throughout the lake. How would the mathematical model given by equation (1) change?

(b) Show that the mass of pollution in the lake is now modelled by

$$m(t) = \frac{q}{k} + \left(m(0) - \frac{q}{k}\right) e^{-kt}.$$

What is the long-term effect on the mass of pollution in the lake?

(c) What happens if initially there is no pollution in the lake?

(d) What is the corresponding mathematical model for pollutant concentration?

(e) Sketch the graph of $m(t)$ against t if (i) $m(0) > q/k$, (ii) $m(0) < q/k$.

2 Analysing a model: skid marks

The Introduction described the stages of the mathematical modelling process, and in Section 1 you saw how some of those stages can be applied in the context of modelling pollution in the Great Lakes. In this section you are asked to relate the stages of the mathematical modelling process to a previously formulated mathematical model. This model was not developed using the structure for modelling described in the Introduction, and consequently some of the modelling ideas have been obscured. This example is typical of accounts of modelling that you may see in books, or produced in the workplace. The aim of this section is to help you to draw out and clarify mathematical modelling ideas by considering the example.

The solution in the following example includes all of the elements of mathematical modelling, but you may have to search a little to find them. The basic mathematical idea used is an equation for motion in one dimension with constant acceleration.

First read the whole description of the mathematical model. Then turn to the exercises, which consist of a number of questions about the model, drawing out the main modelling points. You will need to refer back to the modelling description in the example as you work through the exercises.

As you read through the description of the model, bear in mind the points made in the previous section. Look out for the purpose of the model, the system that is being modelled, the simplifying assumptions, definitions of the variables and the derivation of relations between them, and the conclusions that are drawn from the model.

Example 1

When the police are investigating a road accident, skid marks made by a car can be very informative. From the length of the skid marks, the police can estimate the speed at which the car was travelling before the wheels locked and the car went into the skid.

To assist in this estimation, the police sometimes drive a similar 'test' car with similar tyres and under similar road conditions, and cause it to skid at the same place, but at a lower speed. They then compare the skid marks produced by the test car with the original ones.

Solution

The police assume that the frictional force between the wheels and the road is dependent not on the speed of the car, but only on the mass of the car, the condition of the road and the type of surface. The frictional force is assumed to be proportional to the mass of the car, as are any accelerating or decelerating forces due to gravity if the car is going up or down a hill. The decelerations of the original car and the test car are the same according to these assumptions.

So it is necessary to use the test car's results to calculate the deceleration and then to use the calculated deceleration to estimate the speed of the original car.

For the test car, let us call the initial speed u_{test} and the distance that the car travels when skidding x_{test}. The final speed v is 0, so

$$u = u_{\text{test}}, \quad v = 0, \quad x = x_{\text{test}}, \quad x_0 = 0,$$

a_{test} is to be found.

The equation linking u, v, x, x_0 and constant acceleration a is

$$v^2 = u^2 + 2ax, \tag{2}$$

This equation is derived in Unit 3 as

$$v^2 = v_0^2 + 2a_0(x - x_0),$$

where x_0 is the initial position, v_0 is the initial velocity and a_0 is the constant acceleration.

so

$$0 = u_{\text{test}}^2 + 2a_{\text{test}}x_{\text{test}},$$

that is,

$$a_{\text{test}} = -\frac{u_{\text{test}}^2}{2x_{\text{test}}}.$$

For the original car, if it skidded to rest over a distance x_{car}, assuming that its deceleration a_{car} is the same as that for the test car a_{test}, then

$$a = a_{\text{car}} = a_{\text{test}} = -\frac{u_{\text{test}}^2}{2x_{\text{test}}}, \quad v = 0, \quad x = x_{\text{car}},$$

$u = u_{\text{car}}$ is to be found.

Using the constant acceleration equation again gives

$$0 = u_{\text{car}}^2 - \frac{2u_{\text{test}}^2 x_{\text{car}}}{2x_{\text{test}}},$$

that is,

$$u_{\text{car}} = u_{\text{test}}\sqrt{\frac{x_{\text{car}}}{x_{\text{test}}}}. \tag{3}$$

So the speed of the car before the accident can be estimated. Bear in mind that the speed of the original car would be greater than u_{car} if it had crashed, rather than stopped, at the end of the skid.

Exercise 3

(a) State in your own words the purpose of this model, and say when it may be useful.

◀ Specify purpose ▶

(b) What is the role of the test car? Why could the police not just be issued with tables giving the speed in terms of the length of the skid marks?

(c) The basic concern of mathematical modelling is with finding relationships between variables that specify the system under consideration; in other words, to find formulas that enable you to calculate something if you know the value of something else, or that tell you how something varies with something else.

What are the appropriate 'something' and 'something else' in this model (in words)? (The second sentence of the description of the model may be of help here.) On the basis of what you know about skids, say what you can, in the simplest terms, about the nature of the variation.

Exercise 4

(a) There are several symbols appearing in the description of the model, but according to part (c) of Exercise 3 we are really concerned with the relationship between just two key variables. The symbols conveniently form four groups: symbols for the two key variables; symbols for the data; symbols introduced to make the calculations easier; and symbols used in a general formula employed in the model. Classify the symbols according to this scheme.

◀ Create model ▶

(b) No units of measurement are given anywhere in the definitions of the symbols. This is not the practice adopted in this module, and you are usually advised not to follow it. But does it matter in this instance?

Exercise 5

(a) What is the basic model that underlies the whole discussion?

(b) The basic model identified in your answer to part (a) assumes particle motion with constant acceleration in a straight line. Are these assumptions mentioned in the example? What other assumptions are mentioned or implicit in the example?

(c) Using Newton's second law and the properties of sliding friction, justify the assumption of constant acceleration (i.e. constant deceleration for a skidding car) for the case where the road is flat.

You may find it helpful to draw a force diagram. Sliding friction is discussed in Unit 3.

Exercise 6

(a) Justify the 'assumption' that the decelerations of the two cars are the same.

(b) The possibility that the accident took place on a slope is alluded to in the description of the model, but it is not stated explicitly whether the model applies to such a situation. Does it?

(c) Decide whether the model would apply to the following situations.
   (i) A crashed Rolls-Royce
   (ii) A crash into a strong headwind
   (iii) A crash in a shower of rain, if the road dries out before the police arrive on the scene

Exercise 7

(a) The model, so far as the original car is concerned, might be summarised as follows.

> We want to determine the initial speed u_{car} in terms of the length of the skid x_{car} (which is the distance that the car travels from the initial point of the skid mark before coming to rest). Under the assumption of constant deceleration in a straight line, this is given by equation (2) as
>
> $$u_{\text{car}}^2 + 2a_{\text{car}}x_{\text{car}} = 0, \quad \text{or equivalently,} \quad u_{\text{car}} = \sqrt{-2a_{\text{car}}x_{\text{car}}}.$$

The car's acceleration a_{car} is negative because the car is decelerating, so $-2a_{\text{car}}x_{\text{car}} > 0$.

Confirm that this result agrees with intuitive expectations, as given in the answer to part (c) of Exercise 3. Explain why this is not the end of the story, and how the test car is involved.

◀ Do mathematics ▶

(b) By combining the formulas for the original car and the test car, the final formula (equation (3))

$$u_{\text{car}} = u_{\text{test}}\sqrt{\frac{x_{\text{car}}}{x_{\text{test}}}}$$

is obtained. From this formula, the example says, 'the speed of the car before the accident can be estimated'. If the actual skid marks are four times as long as those of the test car, and the test car was going at 40 mph when it skidded, was the original car breaking the 70 mph speed limit?

Exercise 8

◀ Interpret results ▶

(a) Suppose that you are a police instructor. Explain to a police officer, who is about to go out on an accident investigation for the first time, how to estimate the speed of the car before the accident.

(b) Explain the final statement of the example: 'the speed of the original car would be greater than u_{car} if it had crashed, rather than stopped, at the end of the skid'.

Reading and interpreting descriptions of mathematical models can be hard work: there is usually a lot of information to absorb, and it can be difficult to focus on what is really important. For example, in the Great Lakes model, the names and volumes of the lakes do not help our understanding of the mathematics (though they are of crucial importance to a geographer). The process of coming to terms with a mathematical model is one of digging away until you find what lies at the root of it all. In the Great Lakes model it is the differential equation $dm/dt = -k\,m(t)$ that lies at the root, whereas in the skid marks model it is the formula relating initial and final velocity to distance, for motion with constant acceleration in a straight line.

See equation (2).

One of the basic skills of mathematical modelling (as you will find when you come to construct models for yourself) is to formulate the fundamental equation or relationship that describes the process in which you are interested. There are several points in the two accounts seen so far that show how this can be done.

- It is important to pick appropriate variables and parameters, and to define them carefully.
- It is necessary to simplify matters in order to make progress. For example, it was recognised in creating the Great Lakes model that the problem of seasonal variations in water level could be postponed, if not ignored.
- It is good practice to record the assumptions that you make in deriving the model. This certainly helps the reader, but also the written record is then clearer and suggests ways of developing the model later.
- It is important to collect relevant data, both to check the predictions of the model and to furnish the values of any parameters that are needed to apply the model to any particular situation. The same data cannot be used for parameter estimation and the validation of the model.

Exercise 9

This exercise is based on the skid marks model discussed above.

(a) Discuss the advantages and disadvantages of using compound symbols for variables, such as u_{test} and u_{car}, instead of single symbols such as u and U.

(b) Why is it useful to classify parameters and variables as distinct categories?

(c) Do you consider the following statement to be a useful assumption? ‘The value of the coefficient of friction is not greater than 1.0.’

(d) Data are necessary both to provide the values of parameters and to validate the model. What data do you think will be required to provide values of the parameters, and to validate the model?

(e) Do you consider the skid marks modelling report (Example 1) to be easy to follow?

3 The skills of modelling

Mathematical modelling involves many different skills. To be good at mathematical modelling you have to be able to do some mathematics, but you need to be capable of doing other things as well. If you tend to think of mathematics as a set of procedures (such as the procedures for solving differential equations), then you may not regard these additional mathematical modelling skills as being relevant to mathematics. Most practising mathematicians, concerned with pure as well as with applied mathematics, would disagree with that interpretation. One of the most powerful motivations for studying mathematics is the desire to solve *new* problems for which there is no known solution procedure. The skills required in mathematical modelling include many general problem-solving skills. To be able to deal with mathematical modelling problems is more generally useful, and more difficult, than (say) being able to solve first-order differential equations by the integrating factor method. It calls on skills of creativity, analysis and interpretation that apply to all sorts of problems, not just mathematical ones.

Here is a list of skills that may be required in the solution of a modelling problem, placed in the order of the modelling framework introduced earlier. You need to be able to:

- specify the purpose of the model, by defining or interpreting the problem that you are investigating
- create the model by
    - simplifying the problem (by means of appropriate assumptions)
    - choosing appropriate variables and parameters
    - formulating relationships between the variables
- use mathematics to find a solution from the relationships
- interpret the results by describing them in words (or otherwise) so that they can be understood by a possible user
- evaluate the model by
    - checking that the mathematical relationships and the solution make sense
    - comparing the results with reality
    - checking their sensitivity to changes in the data (this will be discussed in Unit 18).

For the problem to be considered shortly, the 'Do mathematics' stage of modelling is very brief and is therefore included in the subsection for 'Create the model'.

When tackling a modelling problem in earnest, you have to call on these skills repeatedly, in a complicated and interactive way. It seems wise, therefore, to practise the skills individually at first, and this is the aim of the current section. Each subsection deals with a subgroup of skills from the list above, and is illustrated by reference to the two models considered in Sections 1 and 2. You are also asked to try out the individual skills by applying them to the modelling problem in the following example.

Example 2

What shape should a tin can be? Most cans are cylindrical, so suppose that the best tin can is cylindrical. What shape of cylinder is best: short and fat, long and thin, or somewhere in between? Think of the standard kind of can in which soup, beans and other foodstuffs are preserved. Many of the cans in supermarkets contain about 400 grams of food and are rather taller than they are wide.

Although cans are made from a variety of materials, they are usually called *tin cans* because they were first made from tinned steel.

There is a degree of uniformity in the general shape of such cans, which is quite surprising when there are so many different brands and they contain so many different things; imported cans seem to have a shape similar to those that are manufactured in the UK. Is this traditional, or is it because they all conform to some ideal shape, or because the same machine makes most of them? And if the shape is not ideal, could there be advantages in changing it?

In this section you are asked to investigate the best shape for a cylindrical tin can. This is a modelling problem for you to do, but you are not left entirely on your own to tackle it: the problem has been broken down into steps, corresponding to the modelling skills listed above.

There is no unique correct answer in modelling; if you reach a different answer to the one given, that does not mean that your answer is wrong. However, for the purposes of moving the story forward at each stage, the model will be developed in the text from the solution given to the preceding exercise.

3.1 Specify the purpose of the model

◄ Specify purpose ►

In mathematical modelling, problems are rarely posed in a way that can be translated directly into mathematical form. For example, from the description of the treatment of pollution in the account of the Great Lakes model, you might have thought that in order to construct a mathematical model, you would need to know quite a lot about pollution. However, it was possible to produce a straightforward non-technical statement of the underlying problem that could serve as the basis for a model, namely:

> to investigate how the pollution level varies with time as clean water flows into the lake and polluted water flows out.

It is important to establish at the outset a clear statement of the purpose of a model. For example, the purpose of the Great Lakes model is to discover the time that it will take for the pollution level in a lake to reduce to a given proportion of its initial pollution level. Such a statement is typical of the approach that is required in order to start a modelling problem.

Although a clear statement of the problem is necessary, it may change as the model develops, and the final statement may be different from that conceived at the outset. However, it is important to have a target at which to aim, even if this target changes during the process.

It is worth bearing in mind that some models, created for a specific purpose, may be applicable in other situations. It may also be possible to save time by modifying a model that has been used for one situation so that it can be applied to another. For example, the mathematical model developed to predict how long it would take for pollution in the Great Lakes to reduce to a target level could be adapted for use in drug therapy, for the cleaning of milk churns by running water through them, or for the distillation of whisky if a stream feeding the distillery is becoming polluted.

Exercise 10

Consider the problem of finding the best shape for a cylindrical tin can that is to contain a specified quantity of baked beans.

(a) The idea of 'best' occurs frequently in mathematical modelling, and its meaning needs to be made precise. What should the word 'best' mean in the phrase 'the best shape for a cylindrical tin can'?

(b) Try to formulate a clear statement of a suitable modelling problem, based on your answer to part (a).

3.2 Create the model

◀ Create model ▶

There are a number of skills that are needed when building a sensible model that approximates a real situation. In creating a model, these skills may be required at a variety of stages and not necessarily in the order in which they are presented here.

Simplify the problem

The skid marks model depended on the results that the deceleration of a skidding car is constant and that, for given conditions, different cars have the same deceleration while skidding. These follow from assumptions that underpin a well-established theory of sliding friction, but hold only if (for example) air resistance is ignored. To ignore air resistance is justified on two counts: first, its effects are probably small compared with those of sliding friction; second, the resulting model is relatively easy to analyse, and may provide some insight into the problem. In modelling you should always look for as simple a model as possible, consistent with the principal features of the problem. (To have ignored the effects of friction would obviously have been counter-productive.)

In Unit 18, you will see how we can check whether simplifying assumptions are justifiable.

It is important to be clear about the simplifying assumptions that have been made in order to arrive at the model. Recording an explicit list of the assumptions makes it easier for the reader to follow the development of the model, and should you need to improve your model, you then have an obvious place to start: review the assumptions, and ask which should be modified or relaxed.

Exercise 11

Continuing with the tin can problem as specified in Exercise 10(b), what simplifying assumptions, if any, need to be made?

Choose appropriate variables and parameters

Identification of the key variables is of paramount importance. If you can summarise the problem in terms of describing roughly how one quantity varies with another, or several others, then you should have no difficulty in identifying the key variables. Once these key variables have been identified, it should be possible to obtain relationships between them, which may throw up other variables and/or parameters. It is good practice to keep a list of all the variables and parameters, adding to it as necessary, to ensure that all of them have been consistently defined and used. It pays to be careful in defining variables and parameters, to avoid confusion later. For example, 'time since all pollution ceased' is clearer, and less likely to be misinterpreted, than just 'time'.

It is possible that not all the variables and parameters are identified before the relationships are formed, and you may need to add more to your list as the model progresses. You may also find it convenient to replace a group of parameters with a symbol in the mathematical formulation of the model; then the symbol must be added to the list.

In the case of the Great Lakes model, there are three key variables: the time (in seconds) since all sources of pollution were removed, the mass (in kilograms) of pollutant in the lake, and the concentration (in kilograms per cubic metre) of pollutant in the lake. The creation of the mathematical model involves identifying relationships between these key variables. In writing down the relationships, five parameters were identified: the target concentration level (in kilograms per cubic metre) of pollutant in the lake; the time taken (in seconds) to reduce the concentration of pollutant to the target level; the volume (in cubic metres) of water in the lake; the water flow rate (in cubic metres per second); and the proportionate flow rate (in seconds^{-1}).

In the skid marks model, there are two key variables: the length of the skid for the original car, and its initial speed, each in appropriate units. In writing down the relationship between these two key variables, two further variables and a parameter were identified: the length of the skid for the test car, its initial speed, and the common deceleration of the two cars. The units for these quantities need to match the units chosen for the key variables. (The final speeds of the two cars could be regarded as additional parameters.)

Exercise 12

Continuing with the tin can problem, define the variables and parameters that you think will be needed, giving appropriate units.

Formulate relationships

The use of the input–output principle to formulate relationships in creating the Great Lakes model is a good illustration of a quite common modelling technique. Another common technique for formulating relationships, in the case of mechanics problems, is to make use of Newton's laws of motion, although their use may be hidden, as in the skid marks model.

Often it is helpful to draw a diagram. Not only does a diagram help in the definition of the variables and parameters, but it tends to help in gathering together some key factors.

A picture is said to be worth a thousand words.

Exercise 13

Draw a diagram to help with the tin can problem.

Exercise 14

(a) Write down formulas that relate the variables in the tin can problem. You should explain on which assumptions any formula is based.

(b) Derive a formula that relates the area A of the can to its radius r, where the volume V is a parameter.

(c) Are there any assumptions that have not been used in the formulation? Are they needed?

(d) Are there any variables or parameters that have not been used in the derivation?

Find a solution

◀ Do mathematics ▶

Typical mathematical techniques used in simple models are solving algebraic equations, solving differential equations and finding an optimum value.

Once the formulas that relate variables have been derived, some mathematics will probably be needed to find a solution to the model. In the skid marks model, the solution for u_{car} of the equation

$$0 = u_{\text{car}}^2 - \frac{2u_{\text{test}}^2 x_{\text{car}}}{2x_{\text{test}}}$$

was required. In the Great Lakes model a differential equation was solved, an initial condition was used, and an algebraic equation had to be solved to find the target time.

For the tin can problem, Exercise 14 shows that we need to find the value(s) of r for which

$$A = 2\left(\frac{V}{r} + \pi r^2\right) \tag{4}$$

is a minimum (as specified in the solution to Exercise 10(b)).

Exercise 15

Use formula (4) to find 'the best shape for a cylindrical tin can'.

3.3 Interpret the results, evaluate the model

Obtaining a mathematical solution to a modelling problem is not the end of the modelling process. The solution needs to be interpreted in terms of the original problem posed, and a number of checks should be made. This subsection outlines some of the techniques used to interpret the solution and to check its reliability.

Check that the model and solution make sense

◀ Interpret results ▶

It is sometimes possible to take some particular values for the variables, and so make a quick check on the correctness of the model. In the skid marks example, it makes sense that the longer the skid mark of the car, the faster the speed that it was travelling beforehand, and this is borne out by the solution. Also, if the length of the skid mark is zero, then the model and intuition both give the same value, zero, for the speed of the car. In the Great Lakes model, the time taken to reach the target pollution level is increased if the target level is decreased, and this makes sense. It is always worth investigating the solution with checks such as these in mind.

Exercise 16

(a) How would you expect the surface area of the tin can to change as the radius becomes very small? Is this what the model predicts?

(b) How would you expect the surface area of the tin can to change as the radius becomes very large? Is this what the model predicts?

(c) Is there any other test that can be applied to check whether the solution is reasonable?

Compare the results with reality

◀ Evaluate model ▶

A check that the model predicts the kind of results that one would expect from common sense and from experience, as described above, is one type of comparison with reality. Beyond that, if possible, one should validate the model by comparing its predictions with data from an experiment or other reliable source. It is good practice to try to reformulate the results so that this check turns into something simple such as drawing a straight line. You may also require some data to give an explicit numerical solution to the problem, such as the values of lake volume and water flow rate in the Great Lakes problem. The practicality of the skid marks model lies in the use of the test car to provide these data.

Exercise 17

(a) Summarise the solution obtained for the tin can problem descriptively, giving the optimal shape of the can in terms of the ratio of height to radius.

(b) Measure the radius and volume of some tin cans. Does the shape of can predicted by your solution correspond to the actual shapes of tin cans?

The volume can be an estimate based on measurements of the diameter and height.

Exercise 18

In your own words, and without using any equations or symbols, give an outline of the formulation of the model.

---

4 Dimensions and units

The choice of unit of measurement does not affect the physical quantity being measured. For example, the width of this page is the same irrespective of whether we measure it in millimetres or inches. Similarly, no matter what units we use, speed must always be expressed as length divided by time. To measure speed we could use metres per second, centimetres per hour or yards per day, but it has to be a unit-of-length per unit-of-time.

4.1 Dimensional consistency

In modelling, we attempt to establish relationships between variables by using assumptions to develop equations. In this subsection we investigate a technique used to verify that these equations make sense, in that they are *dimensionally consistent*. Such an analysis can never establish that an equation is correct, but it provides a useful check and can sometimes tell us that an equation is wrong.

For example, suppose that you are cooking a meringue using egg whites and sugar. In order to determine the mass m of the meringue, you would first need to determine the mass m_1 of an egg white and the mass m_2 of the sugar, all measured in the same units. Then, assuming that no mass is lost during the cooking process, the mass of a meringue using n egg whites is given by $m = nm_1 + m_2$. Each additive term on the right-hand side of this equation (nm_1 and m_2) is a mass, and the equation makes sense only if each of these terms is measured in the same units, giving the overall mass m in those units. This is the essence of dimensional consistency.

In reality, the mass of an egg white varies slightly from egg to egg.

In terms of dimensional consistency, the important property of volume is that it is length cubed, and the important property of speed is that it is length divided by time. In these examples, we can think of length and time as being fundamental properties, referred to as *base dimensions*.

Mass, length and time are fundamental properties and are referred to as **base dimensions**. The base dimension mass is denoted by M, length by L, and time by T.

The dimensions of all physical quantities (in this module) can be expressed in terms of these three base dimensions. For example, the dimensions of speed, namely length divided by time, are $\mathrm{L\,T^{-1}}$. To write this more succinctly, we introduce square brackets to mean ‘the dimensions of’, and write

$$[\text{speed}] = \mathrm{L\,T^{-1}}.$$

This is read as 'the dimensions of speed are $\mathrm{L\,T^{-1}}$'. Similarly,

$$[\text{volume}] = \mathrm{L}^3.$$

The dimensions of other physical quantities can be expressed in terms of M, L and T by working from their units of measurement.

Example 3

Find the dimensions of the magnitude of acceleration and of the magnitude of force.

Solution

The SI unit for the magnitude of acceleration is $\mathrm{m\,s^{-2}}$, so

$$[\text{acceleration}] = \mathrm{L\,T^{-2}}.$$

The SI unit for the magnitude of force is the newton, where $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$, therefore

$$[\text{force}] = \mathrm{M\,L\,T^{-2}}.$$

As Example 3 has illustrated, we can multiply, divide and take powers of dimensions using the usual rules of arithmetic and algebra. However, dimensions cannot be added or subtracted.

Note that some quantities and units are **dimensionless**. A number is dimensionless, and we write $[\text{number}] = 1$.

Exercise 19

(a) What are the dimensions of area?

(b) Given that in SI units density is measured in $\mathrm{kg\,m^{-3}}$, what are the dimensions of density?

(c) The size of an angle θ, measured in radians, can be determined by assuming that the angle lies at the centre of a circle and that the two lines defining the angle are radii of the circle, as shown in Figure 4. If the radii have length r and the arc defined by θ has length l, then $\theta = l/r$. What are the dimensions of angle?

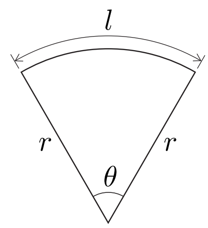


Figure 4 Size of an angle

The dimensions of some of the physical quantities used in this module are given in Table 2. For vector quantities, we give the dimensions of the *magnitude* of the vector.

Table 2 Dimensions

Physical quantity	Dimensions
Number	1
Angle	1
Mass	M
Length	L
Time	T
Area	L^2
Volume	L^3
Speed	$\mathrm{L\,T}^{-1}$
Angular speed	T^{-1}
Acceleration	$\mathrm{L\,T}^{-2}$
Angular acceleration	T^{-2}
Force	$\mathrm{M\,L\,T}^{-2}$
Energy	$\mathrm{M\,L^2\,T}^{-2}$
Pressure	$\mathrm{M\,L^{-1}\,T}^{-2}$
Torque	$\mathrm{M\,L^2\,T}^{-2}$
Density	$\mathrm{M\,L}^{-3}$

Note that energy and torque have the same dimensions although they are different physical quantities. For example, a torque is a vector, and energy is a scalar quantity.

In modelling, any equation must be **dimensionally consistent**: the dimensions must be the same for each of the additive terms on either side of the equation. If they are, then our confidence in the model is increased; if they are not, then we know that we have made a mistake somewhere in deriving the equation.

Example 4

The period τ of small oscillations of a particle of mass m suspended from a fixed point by a light inextensible string of length l is given by

$$\tau = 2\pi\sqrt{\frac{l}{g}}.$$

Show that this equation is dimensionally consistent.

Solution

The period τ has dimensions T. For the right-hand side of the equation we have

$$\left[2\pi\sqrt{l/g}\right] = [2\pi]\,([l]/[g])^{1/2} = 1 \times (\mathrm{L}/(\mathrm{L\,T}^{-2}))^{1/2} = (\mathrm{T}^2)^{1/2} = \mathrm{T}.$$

So both sides of the equation have dimensions T, and the equation is dimensionally consistent.

Example 5

In equation (4) the area of the tin plate used in making a tin can was given as

$$A = 2\left(\frac{V}{r} + \pi r^2\right).$$

Establish that this equation is dimensionally consistent.

Solution

All three terms in the equation, A, $2V/r$ and $2\pi r^2$, must have the same dimensions if the equation is to be dimensionally consistent. We have

$$[A] = \mathrm{L}^2, \quad [2V/r] = [2]\,[V]/[r] = \mathrm{L}^3/\mathrm{L} = \mathrm{L}^2 \quad \text{and} \quad [2\pi r^2] = \mathrm{L}^2,$$

so dimensional consistency is assured.

Example 5 illustrates that when checking for dimensional consistency, each additive term must be checked separately: although we can multiply, divide and take powers of dimensions, we can add or subtract only terms that have the same dimensions.

Exercise 20

Show that the equation

$$v^2 = v_0^2 + 2a_0 x,$$

which relates the speed of something moving with constant acceleration a_0 to the distance x travelled, is dimensionally consistent.

Exercise 21

The mathematical model of pollution in a lake in Section 1 led to the differential equation

$$\frac{dm}{dt} = -k\,m(t),$$

where $m(t)$ is the mass of pollutant in the lake at time t, and $k = r/V$, where r is the volume flow rate of water through the lake, and V is the volume of water in the lake.

Show that this equation is dimensionally consistent.

(*Hint*: A derivative is essentially a ratio, or fraction, so its dimensions are those of its numerator divided by those of its denominator.)

Exercise 22

One term in the following equation is in error:

$$\frac{p}{\rho} + \tfrac{1}{2}u - gz = \text{constant},$$

where p is the pressure of a fluid, ρ is its density, u is its speed, g is the magnitude of the acceleration due to gravity, and z is the height above some given level. By checking the dimensions of each term of the equation, find which term is in error, and suggest how it might be made dimensionally correct. (The dimensions of pressure are $\mathrm{M\,L^{-1}\,T^{-2}}$.)

Dimensional consistency can also be used to determine the units of parameters of models.

Exercise 23

See Unit 3, Subsection 4.2.

The magnitude R of the air resistance force on an object can be modelled, in certain circumstances, as

$$R = c_2 D^2 v^2,$$

where c_2 is a constant, D is the effective diameter of the object, and v is its speed.

Use dimensional consistency to determine the dimensions of the constant c_2. What SI units would be used to measure c_2?

We have seen how to check for dimensional consistency in equations where the additive terms involve multiplying, dividing or taking powers. But many of the models that you have seen in the module involve functions such as exp, ln, sin, cos, and so on. How does one check for dimensional consistency in such cases? For example, the solution of the differential equation $dm/dt = -k\,m(t)$ in the lake pollution model is

$$m(t) = m(0)\,e^{-kt}, \tag{5}$$

where $m(0)$ is the initial mass of pollutant. To be able to check for dimensional consistency in this equation, we need to be able to find the dimensions of e^{-kt}. Now we know that exp, ln, sin, cos, and so on, all have real numbers as their domains and image sets. Therefore for dimensional consistency, the argument x of the function must be dimensionless, and we can take e^x, $\ln x$, $\sin x$, $\cos x$, and so on, to be dimensionless.

So the key to checking for dimensional consistency in equations involving such functions is to ensure that their arguments are dimensionless. In the above case, we know from the equation $k = r/V$ given in Exercise 21 that

$$[k] = [r/V] = [r][V]^{-1} = (\mathrm{L^3\,T^{-1}}) \times (\mathrm{L^3})^{-1} = \mathrm{T^{-1}},$$

so $[-kt] = 1$. Hence $[e^{-kt}] = 1$, and equation (5) is dimensionally consistent, since $[m(t)] = [m(0)] = \mathrm{M}$.

Sometimes, however, the arguments of functions are *not* dimensionless. For example, in obtaining equation (5), one might have begun by using the separation of variables method to solve the differential equation $dm/dt = -k\,m(t)$ and obtained

$$\ln(m(t)) = -kt + C, \tag{6}$$

where C is an arbitrary constant. Now $m(t)$ is not dimensionless (it has dimensions M), so we cannot determine the dimensions of $\ln(m(t))$. However, putting $t = 0$ into equation (6) gives $C = \ln(m(0))$, and this enables us to rewrite equation (6) as

$$\ln\left(\frac{m(t)}{m(0)}\right) = -kt, \tag{7}$$

where $[m(t)/m(0)] = \mathrm{M}/\mathrm{M} = 1$. So now the argument of ln *is* dimensionless, and it is an easy matter to check for dimensional consistency (both sides of equation (7) have dimensions 1).

Very often, if the argument of a function in an equation is not dimensionless, some simple manipulation of the equation can render it so and hence enable dimensional consistency to be checked.

Exercise 24

The motion of a particle in a certain problem can be modelled as

$$x(t) = A\cos(\omega t + \phi),$$

where $x(t)$ measures the displacement at time t, and ϕ is a constant. What dimensions for A, ϕ and ω will ensure dimensional consistency?

4.2 Change of units

There is now almost universal acceptance of the SI system in professional scientific and engineering circles. This system is based on seven **base units**, of which the following four are used in this module.

Table 3 Base units

Physical quantity	Unit	Abbreviation
Length	metre	m
Mass	kilogram	kg
Time	second	s
Temperature	kelvin	K

The other three base units are the ampere (unit of electric current), the candela (unit of luminous intensity) and the mole (unit of the amount of substance).

In addition, there are various **derived units** in common use, which can be formed by combining the above base units. For example, we have already introduced the *newton* (N) as the unit of force, and the *joule* (J) is the unit of energy. In terms of the base units, $1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$ and $1\,\mathrm{J} = 1\,\mathrm{kg\,m^2\,s^{-2}}$.

To avoid very large or very small numbers, we also use multiple or fractional units. For example, the distance between London and Edinburgh is more conveniently expressed as 665 km (kilometres) rather than 6.65×10^5 m. Similarly, the thickness of copper used in a central heating pipe is usually stated to be 1 mm (millimetres) rather than 1×10^{-3} m. The most important prefixes for forming these units are given in Table 4.

Table 4 Prefixes

Multiplication factor	Prefix	Symbol
$10^9 = 1\,000\,000\,000$	giga	G
$10^6 = 1\,000\,000$	mega	M
$10^3 = 1\,000$	kilo	k
$10^{-2} = 0.01$	centi	c
$10^{-3} = 0.001$	milli	m
$10^{-6} = 0.000\,001$	micro	μ
$10^{-9} = 0.000\,000\,001$	nano	n

For example, the pressure of the atmosphere, which is about $10^5\,\mathrm{N\,m^{-2}}$, is sometimes written as $100\,\mathrm{kN\,m^{-2}}$, that is, 100 kilonewtons per square metre. The conversion between such alternative units is quite straightforward but does need some care. For example, to convert 6 kilometres into metres, we know that

$$1\,\mathrm{km} = 10^3\,\mathrm{m},$$

so the conversion factor from kilometres to metres is

$$\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}} \quad (= 1).$$

This example may seem trivial, but the method used here can be used in more complicated problems.

Using this conversion factor, we have

$$6\,\mathrm{km} = 6\,\mathrm{km} \times \left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right) = 6 \times 10^3\,\mathrm{m}.$$

In this calculation we have, in essence, treated the units in a similar way to algebraic quantities and 'cancelled' the 'km' between the numerator and denominator.

Example 6

The speed of a car is $50\,\mathrm{km\,h^{-1}}$. Express this speed in the SI units of $\mathrm{m\,s^{-1}}$.

Solution

We have

$$1\,\mathrm{km} = 10^3\,\mathrm{m}$$

and

$$1\,\mathrm{h} = 60\,\mathrm{min} = 60 \times 60\,\mathrm{s}.$$

So

$$\begin{aligned} 50\,\mathrm{km\,h^{-1}} &= 50\,\mathrm{km\,h^{-1}} \times \left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right) \times \left(\frac{60 \times 60\,\mathrm{s}}{1\,\mathrm{h}}\right)^{-1} \\ &= 50 \times 10^3 \times 60^{-1} \times 60^{-1}\,\mathrm{m\,s^{-1}} \\ &\simeq 13.89\,\mathrm{m\,s^{-1}}. \end{aligned}$$

So the speed of the car is $13.9\,\mathrm{m\,s^{-1}}$ to one decimal place.

Exercise 25

The density of a metal is $13.546\,\mathrm{g\,cm^{-3}}$. Express this density in the SI units of $\mathrm{kg\,m^{-3}}$.

Unfortunately, the metric system of units is not the one in everyday use everywhere, particularly in the UK and the USA, where lengths may be measured in inches (in), feet (ft) and miles, masses in ounces (oz), pounds (lb) and tons, and temperatures in degrees Fahrenheit. The method that we have used above can equally be used to convert between such units and SI units. However, when converting between Imperial and metric systems of units, the conversion factors are not ‘nice round numbers’. For example,

$$1\,\mathrm{lb} = 0.453\,592\,\mathrm{kg}.$$

Some of the common conversion factors between Imperial and SI units are given in Table 5.

Table 5 Imperial units

Imperial unit	SI unit
1 in	$2.54 \times 10^{-2}\,\mathrm{m}$
1 ft	$0.304\,8\,\mathrm{m}$
1 mile	$1.609\,344 \times 10^3\,\mathrm{m}$
1 pint	$5.682\,613 \times 10^{-4}\,\mathrm{m^3}$ (= 0.568 litres)
1 gallon	$4.546\,09 \times 10^{-3}\,\mathrm{m^3}$ (= 4.546 litres)
1 oz	$2.834\,952 \times 10^{-2}\,\mathrm{kg}$
1 lb	$0.453\,592\,\mathrm{kg}$
1 ton	$1.016\,047 \times 10^3\,\mathrm{kg}$

Example 7

The speed of a car is 30 miles per hour (mph). Express this speed in SI units.

Solution

We have

$$1 \text{ mile} = 1.609\,344 \times 10^3\,\text{m}$$

and

$$1\,\text{h} = 60 \times 60\,\text{s}.$$

So

$$\begin{aligned} 30\,\text{mph} &= 30\,\text{miles h}^{-1} \times \left(\frac{1.609\,344 \times 10^3\,\text{m}}{1 \text{ mile}}\right) \times \left(\frac{60 \times 60\,\text{s}}{1\,\text{h}}\right)^{-1} \\ &= 30 \times 1.609\,344 \times 10^3 \times 60^{-1} \times 60^{-1}\,\text{m}\,\text{s}^{-1} \\ &\simeq 13.41\,\text{m}\,\text{s}^{-1}. \end{aligned}$$

So 30 mph is equivalent to $13.4\,\text{m}\,\text{s}^{-1}$ to one decimal place.

Exercise 26

The pressure on a piston is 61 lb per square inch. Express this pressure (to 4 s.f.) in the SI units of $\text{kg}\,\text{m}^{-2}$.

(In the SI system, pressure is *force* per unit area, not *mass* per unit area. To convert our answer here to the correct units, we would have to multiply it by g, the magnitude of the acceleration due to gravity.)

Exercise 27

Before a recent holiday to South America, I was unable to buy local currency in the UK. So I bought US dollars in the UK, at the exchange rate £1 = $1.75. In Bolivia, I exchanged my dollars for the local currency, at the rate of exchange $1 = 7.75 boliviano. How much was 1 boliviano worth in £ ?

Learning outcomes

After studying this unit, you should be able to:

- create a simple model, given a clear statement of a problem
- write down the simplifying assumptions that underpin a model
- identify the key variables and the parameters of a model
- apply the input–output principle to obtain a mathematical model, where appropriate
- obtain mathematical relationships between variables, based on or linking back to the simplifying assumptions
- interpret the mathematical solution to a modelling problem in terms of the original statement of the problem
- understand the processes involved in evaluating a model by comparing with reality
- appreciate the role of data in testing the model and, if necessary, in providing parameter values for the model
- understand that the purpose of a model is the measure used to judge the suitability of the model
- appreciate that simple models can be as useful as more complex models, if they serve their intended purpose
- check that mathematical relationships between variables and parameters are dimensionally consistent
- convert the values of physical quantities from one system of units to another.

Solutions to exercises

Solution to Exercise 1

In describing the formulation of the model, it may be a help to write down some words or phrases first, and then to combine them into a paragraph.

Possible key words or phrases are: pollution has ceased; the water flowing from the lake is polluted; concentration of the pollutant is uniform; small time interval; mass of pollutant; one lake; no change in volume of the lake; input–output principle.

A possible description might read:

> This model considers a polluted lake to which no further pollutant is added. The lake is of constant volume. The water flowing from the lake is polluted, and this is the only way that the pollutant leaves the lake. The input–output principle is applied to the mass of pollutant within the lake, based on the change over a small time interval.

In a modelling report, the description of the formulation should be placed before the actual formulation, in order to guide the reader. However, it could be written (or revised) *last*, after the formulation has been completed.

Solution to Exercise 2

(a) The input–output principle is applied to the mass of pollutant within the lake. In the time interval $[t, t + \delta t]$, the input (mass entering the lake) is $q\,\delta t$, while the output is $(r/V)\,m(t)\,\delta t$, as shown in the text. The accumulation over this time interval is $m(t + \delta t) - m(t)$. Hence the change in the mass of pollutant in this interval is given by

$$m(t + \delta t) - m(t) \simeq q\,\delta t - \frac{r}{V}\,m(t)\,\delta t.$$

Putting $k = r/V$, this leads to the differential equation

$$\frac{dm}{dt} = q - k\,m(t).$$

(b) The integrating factor method gives the general solution of the above equation as

$$m(t) = \frac{q}{k} + Ce^{-kt},$$

where C is an arbitrary constant. Rearranging the solution, and using the initial condition at $t = 0$ to evaluate C, gives the required result:

$$m(t) = \frac{q}{k} + \left(m(0) - \frac{q}{k}\right) e^{-kt}.$$

In the long term, the mass of pollution tends to q/k.

(c) If there is initially no pollution in the lake, then $m(0) = 0$ and we have

$$m(t) = \frac{q}{k}\left(1 - e^{-kt}\right).$$

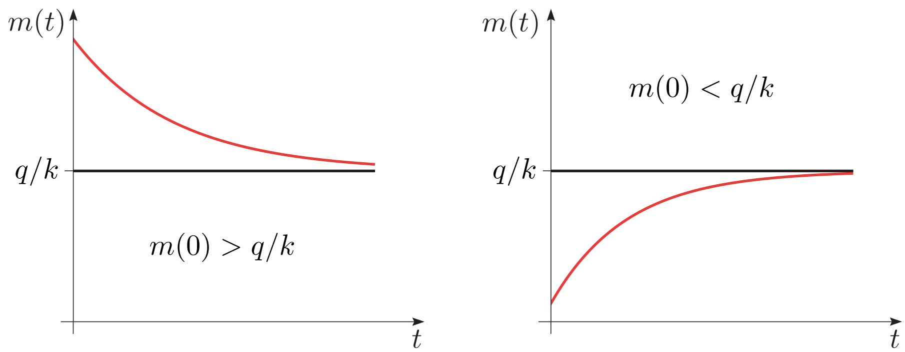
The mass of pollution increases from zero up to the steady-state level q/k.

(d) Using the equation found in part (b), with $m(t) = V\,c(t)$, and putting $kV = r$, we have

$$c(t) = \frac{q}{k} + \left(c(0) - \frac{q}{r}\right) e^{-kt}.$$

This is the corresponding mathematical model for pollutant concentration.

(e) Typical graphs are shown below.



Solution to Exercise 3

(a) When a car skids, its tyres may leave marks on the road. The purpose of the model is to find a method of using the lengths of such skid marks to work out how fast the car was travelling when it began to skid. This information may be useful to the police after a road accident, when they try to find out what happened.

(b) The test car provides data in addition to the length of the skid marks in the accident. These data allow a direct comparison to be made between the length of the skid marks in the accident, for which the speed of the car is unknown, and the length of skid marks made by a car whose speed at the onset of the skid is known. In effect, the data for the test car are used to estimate the deceleration of the car involved in the accident while it was skidding. This is possible because the conditions under which the test takes place are similar to those for the accident, apart from the speed of the car.

The gradient and state of the road surface, the condition of the car's tyres, and other circumstances of the accident have a significant effect on the skid. They may vary widely from accident to accident. It is more reliable to reproduce the conditions in a test than it is to make allowance for them in a table.

(c) The relationship required is one that gives the speed of the vehicle at the onset of the skid in terms of the length of the skid marks. The longer the skid marks, the faster the car was travelling when it began to skid (all other things being equal), so the speed is an increasing function of the length of the skid marks.

Solution to Exercise 4

(a) The key variables are the original car's initial speed u_{car}, and the length of its skid x_{car}. The data, which are the corresponding quantities for the test car, are represented by the symbols u_{test} (initial speed) and x_{test} (length of skid). All of the symbols appearing in the final formula (3) have now been identified, but there are still others. Two that are used to make the calculations easier are a_{car}, the acceleration of the original car, and a_{test}, the acceleration of the test car. (The final speed is zero in each case, so there was no need to introduce symbols for the final speed of either car.) There are also the four symbols v, u, a and x that are used in the general equation (2) for constant acceleration.

(b) No, it does not matter in this instance. One way of seeing why is to rewrite formula (3) as

$$\frac{u_{\text{car}}}{u_{\text{test}}} = \sqrt{\frac{x_{\text{car}}}{x_{\text{test}}}}.$$

The left-hand side of this equation is the *ratio* of two speeds, while the right-hand side is the square root of the *ratio* of two distances. Now the ratio of two speeds is the same value whether the speeds are measured in $\mathrm{m\,s^{-1}}$, mph or leagues per century (provided that both speeds are measured in the same units). Likewise, the ratio of two distances takes the same value whatever unit of measurement is used (again provided that both distances are measured in the same units).

The omission of units in the description of the variables is therefore not important in this case, although you are usually encouraged to state units of measurement when defining variables.

Solution to Exercise 5

(a) The model that underlies the whole discussion is that of the motion of a particle moving in a straight line with constant acceleration (which you met in Unit 3). The formula required here is that relating the initial and final velocity, acceleration and position, namely

$$v^2 = v_0^2 + 2a_0x,$$

which becomes the same as equation (2) when u is substituted for v_0 and a for a_0.

(b) The assumption of particle motion in a straight line is not mentioned. (It may be reasonable, based on the police's knowledge of skids and the record left by skid marks, but it is not mentioned explicitly as an assumption.)

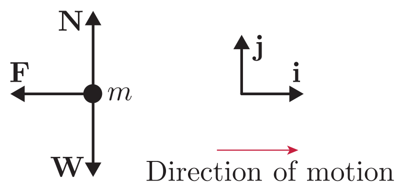
The assumption of constant acceleration is not mentioned either.

Two other assumptions are mentioned explicitly. The first is that 'the frictional force between the wheels and the road is dependent not on the speed of the car, but only on the mass of the car, the condition of the road and the type of surface'. The second is that the frictional force is 'proportional to the mass of the car, as are any accelerating or decelerating forces due to gravity'. From these two assumptions it is deduced that 'The decelerations of the original car and the test car are the same' (although this is restated as an assumption shortly after equation (2)).

Implicit in what follows, though not obvious from any part of the example, is the assumption that the only motive or resistive forces acting on a car are friction and 'accelerating and decelerating forces due to gravity' (i.e. weight). Thus, for example, it has been assumed that air resistance can be ignored. It could be argued that the lack of mention of forces other than friction and weight amounts to an assumption that no other motive or resistive forces are acting. In sum, then, the only forces assumed to be acting on a car are friction, its weight and, of course, the normal reaction.

Another assumption, implicit in carrying out the test skid, is that the road and tyre conditions are the same for both cars. (It is not clear whether the masses of the cars are assumed to be the same.)

(c) In the light of the answer to part (b), the only forces that are assumed to be acting on a car are its weight $\mathbf{W}$, the friction $\mathbf{F}$, and the normal reaction $\mathbf{N}$. Since the motion is assumed to be in a straight line, we need only two axes. The direction of motion is horizontal, because the road is flat. Modelling the car as a particle of mass m produces the force diagram in the figure in the margin.



Expressing the forces in components gives

$$\mathbf{F} = |\mathbf{F}|(-\mathbf{i}), \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{W} = mg(-\mathbf{i}).$$

Newton's second law gives

$$\mathbf{F} + \mathbf{W} + \mathbf{N} = m\mathbf{a},$$

where $\mathbf{a} = a\mathbf{i}$ is the acceleration of the car. Resolving in the $\mathbf{i}$-direction gives

$$-|\mathbf{F}| = ma.$$

Resolving in the $\mathbf{j}$-direction gives

$$-|\mathbf{W}| + |\mathbf{N}| = 0.$$

Now $\mathbf{W} = -mg\mathbf{j}$, so $|\mathbf{W}| = mg$. Hence we obtain $|\mathbf{N}| = |\mathbf{W}| = mg$. Since the car is skidding (sliding), we have

$$|\mathbf{F}| = \mu'|\mathbf{N}|,$$

where μ' is the coefficient of sliding friction. Hence $|\mathbf{F}| = \mu' mg$. Using the equation for friction, it follows that $ma = -\mu' mg$, leading to

$$a = -\mu' g.$$

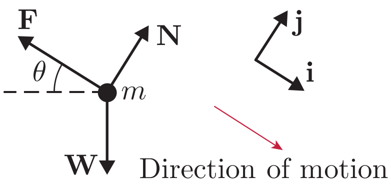
For any given set of road and tyre conditions, μ' is constant. This equation justifies the assumption of constant acceleration (i.e. deceleration).

(Note that if air resistance, or any other force that depends on velocity, were included, then the acceleration a would not be constant. Hence the assumption of constant acceleration implies that all such forces can be ignored.)

Solution to Exercise 6

(a) Since the road and tyre conditions are assumed to be the same for both cars, μ' is the same for both. Hence, by the equation for acceleration $a = -\mu' g$ derived in Exercise 5(c), a is also the same for both, so the 'assumption' that the decelerations of the two cars are the same is justified.

(Note that m does not appear in the equation for determining the acceleration. Hence the conclusion of equal decelerations is independent of the masses of the two cars, which do not therefore need to be taken into account.)



(b) The model does apply to an accident on a slope, provided that the slope does not vary. To see this, consider a car skidding down a constant slope that is at an angle θ to the horizontal, for which the force diagram is shown in the margin. (The analysis for a car skidding up a slope is similar, and results in similar conclusions, but it is not given here.)

Expressing the forces in components gives

$$\mathbf{F} = |\mathbf{F}|(-\mathbf{i}), \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j},$$
$$\mathbf{W} = mg\,(\sin\theta\,\mathbf{i} + \cos\theta\,(-\mathbf{j})) = mg(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}).$$

Newton's second law gives

$$\mathbf{F} + \mathbf{W} + \mathbf{N} = m\mathbf{a},$$

where $\mathbf{a} = a\mathbf{i}$. Resolving in the $\mathbf{i}$- and $\mathbf{j}$-directions gives

$$-|\mathbf{F}| + mg\sin\theta = ma, \quad -mg\cos\theta + |\mathbf{N}| = 0.$$

Using $|\mathbf{F}| = \mu'|\mathbf{N}|$, since the car is slipping, we obtain

$$ma = mg\sin\theta - \mu' mg\cos\theta,$$

so

$$a = (\sin\theta - \mu'\cos\theta)g.$$

Hence if the slope is constant (so that θ is constant) and the road and tyre conditions are constant (so that μ' is constant), then a is constant and is the same for both cars. So the model does apply to a skid down a constant slope.

(We must assume here that $\tan\theta < \mu'$, so that $a < 0$. Otherwise the car will never stop!)

(c) (i) The model will apply to a Rolls-Royce provided that the test car has similar tyres. The size and weight of the crashed car (let alone the price) are not relevant, so long as the coefficient of sliding friction can be duplicated in the test.

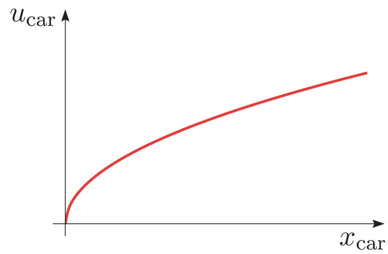
(ii) A crash into a headwind will be covered by the model provided that we continue to ignore the effects of air resistance (this assumption was referred to in part (b) of Exercise 5). Otherwise, the headwind will increase the air resistance force, which depends on the speed of the car relative to the air.

If we include air resistance in the model, then the assumption of constant deceleration will no longer apply.

(iii) If the road is wet at the time of the crash, but dries before the test, then the model will not apply. The assumption that the acceleration in the test is the same as that in the crash will not be valid. The coefficient of sliding friction for a wet road is different from (less than) that for a dry road.

Solution to Exercise 7

(a) The formula $u_{\text{car}} = \sqrt{-2a_{\text{car}}x_{\text{car}}}$ does predict that u_{car} is an increasing function of x_{car}, as expected (see the figure in the margin).



However, by itself this is not enough to solve the problem, because there is no direct way of finding the value of a_{car}. The test overcomes this difficulty, and yields $a_{\text{car}} = a_{\text{test}}$, provided that the assumptions are satisfied.

(b) If the skid marks of the original car are four times as long as those of the test car, then $x_{\text{car}}/x_{\text{test}} = 4$, so $u_{\text{car}} = 2u_{\text{test}}$. This means that the original car was travelling at 80 mph when it began to skid, thus *was* exceeding the 70 mph speed limit.

Solution to Exercise 8

(a) The instructions given to a police officer who is going to the scene of an accident for the first time might take the following form.

- If there has been a skid that has left a mark on the road, measure the length of this mark.
- Check that the tyres on the car involved in the accident are in a comparable state of wear to those on the test car, and that the slipperiness of the road has not altered significantly since the accident (e.g. due to a change in the weather).
- With a clear road, drive the test car (at a safe speed) towards the spot where the accident took place, and induce a skid. Note the initial speed of the test car and measure the length of its skid.
- To calculate the speed of the original car when it began to skid, divide the length of the original skid by the length of the test skid. Then take the square root of the result, and multiply this by the initial speed of the test car (in mph). The answer will be an estimate of the speed, in mph, of the original car when it started to skid.

(b) The model has been based on the assumption that the final speed of the original car was $v_{\text{car}} = 0$. If this was not the case, then provided that the test still gives the correct value for the deceleration, we have

$$u_{\text{car}} = \sqrt{v_{\text{car}}^2 - 2a_{\text{car}}x_{\text{car}}} > \sqrt{-2a_{\text{car}}x_{\text{car}}}.$$

Hence the actual speed of the crashed car at the start of its skid would have been greater than that estimated using the model.

Solution to Exercise 9

(a) Compound symbols for variables, such as u_{test} and u_{car}, immediately indicate what they represent. It is also useful to have the same base symbol for all the variables of one type, as in this case where u stands for speed. However, compound symbols are more cumbersome to write and to manipulate.

Single symbols for variables, such as u and U, are easier to work with, but it may be difficult to remember what they represent.
A compromise would be to use single-letter suffices, for example u_{t} and u_{c}; there is some danger of forgetting what these represent, but less so than with symbols such as u and U.

(The use of compound symbols without suffices, such as uc or $ucar$, is not recommended, because it is unclear whether this is intended to represent a single variable or whether it stands for a product of variables, each of which is represented by a single symbol.)

(b) Parameters do not change their value during the process.
Mathematical models often involve differential equations, and knowing which symbols are constant in such equations is important when considering their solution.

It *is* possible to have a coefficient of friction greater than 1; this is the case for racing tyres on tarmac.

(c) Although in practice there seems to be an upper limit of 1 for the coefficient of friction between tyre and road surface, there is no point in stating this as an assumption. One of the reasons for the model is to avoid determining the coefficient of friction. This is in any case a statement about a data value and not an assumption.

(d) There are four variables in the final formula: u_{car}, u_{test}, x_{car} and x_{test}. The last three are measured, and their values are used to estimate u_{car}.

To validate the model, it would be necessary to measure corresponding values for all four variables, and to check that the predicted value for u_{car} is a reasonable approximation to the measured value.

There are no parameters in the final formula. However, data values for u_{test} and x_{test} are required in order to obtain a value for the parameter $a = -u_{\text{test}}^2/(2x_{\text{test}})$, which is constant for the given road conditions. This parameter value and the measured value of x_{car} are then used to find $u_{\text{car}} = \sqrt{-2ax_{\text{car}}}$.

(e) This is a subjective assessment and depends on the reader. The report could have been improved by the inclusion of a diagram, a table of variables and parameters (including units of measurements), and a list of the assumptions on which the model is based. The final sentence is intuitively correct, but is it confirmed by the mathematics? On the whole, this report sets out what it wants to do fairly well, but requires a better format.

Solution to Exercise 10

The objective may be different for different parties, and the purpose of the objective needs to be included.

(a) Features such as convenience of handling and stacking must, presumably, be taken into account. However, the most important thing, from the manufacturer's point of view, is surely to minimise the cost of making each can.

In terms of the shape of a cylindrical can, this means that the 'best' shape will be the one that uses as little tin plate as possible, while meeting the requirement that the can is to hold a specified quantity of baked beans.

(b) A possible statement is as follows.

> The problem is first to find a formula for the area of tin plate needed to make a cylindrical can for containing a specified volume of food, where the formula is in terms of the height and radius of the can. This formula will then be used to find the dimensions of the can for which the area of tin plate required is a minimum.

Note that volume, rather than weight (or, more properly, mass) is used to specify the quantity of food here because, as mentioned in Example 2, there is a degree of uniformity in the sizes of cans containing a range of different weights of foodstuff. The '400 gram' cans mentioned, on the evidence of supermarket shelves, may contain anything between 385 and 425 grams of food.

Solution to Exercise 11

We suggest the following assumptions.

- The circular ends fit exactly at the top and bottom of the cylinder – there is no lip.
- The seams use a negligible amount of tin plate (as it will be difficult to estimate the amount of tin plate used in the seams – at least for a first model).
- There is no wastage of tin plate: any 'leftover bits', resulting from cutting circular pieces for the ends of a can from a sheet of tin plate, are assumed to be recycled at negligible cost.
- The cans are perfectly smooth cylinders; there are no corrugations on the surface, and no ring pulls.

You might have thought of assuming that the cost of a tin can is proportional to the area of tin plate used, and that the manufacturing cost is negligible. These would have been appropriate if the purpose of the model had been stated in terms of minimising a financial cost, which is the ultimate aim, but they are essentially built into the problem statement in Exercise 10(b).

Solution to Exercise 12

In SI units, the linear dimensions would be in metres. However, the metre is rather too large a unit here, while the millimetre is too small, so the centimetre seems to be the best choice.

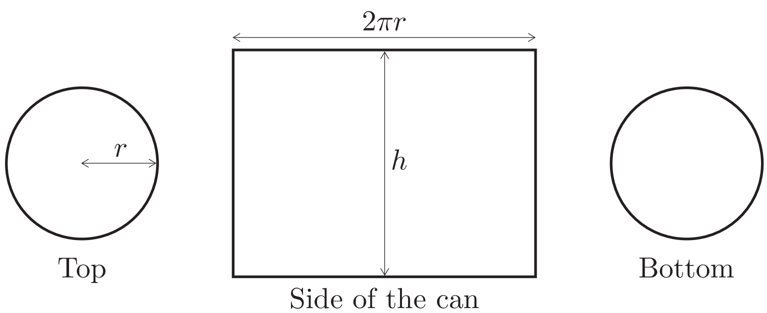
The key variables are as follows.

Symbol	Definition	Units
A	Area of tin plate used in manufacture of a can	cm^2
h	Height of can	cm
r	Radius of each end	cm

The specified volume V (in cm^3) is a parameter of the model.

Solution to Exercise 13

A possible diagram is as follows.



Solution to Exercise 14

(a) The can is made up of one rectangle (which forms the curved body of the can), of sides h and $2\pi r$, and two circular ends, each of area πr^2. Hence the area of tin plate used to make a can is

$$A = 2\pi rh + 2\pi r^2.$$

This formula relies on the assumptions that no tin plate is used for the seams, that there is no wastage of tin plate, and that the surfaces of the can are perfectly smooth.

(b) The volume of the can is given by $V = \pi r^2 h$. To express A in terms of r (and V, but not h), write $h = V/(\pi r^2)$ then eliminate h from the equation in part (a), to obtain

$$A = 2\pi r\left(\frac{V}{\pi r^2}\right) + 2\pi r^2 = 2\left(\frac{V}{r} + \pi r^2\right).$$

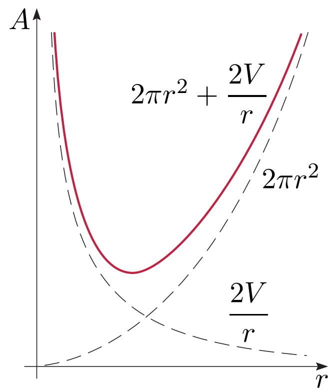
(Note that the area will always be positive.)

(c) All of the assumptions have been used in the formulation. This is a useful check to perform; if any assumptions are not used in the derivation of the model, then their presence is questionable.

(d) All of the variables and parameters that were defined have been used in the derivation. This is a useful check to perform; if any defined parameters or variables are not used in the derivation of the model, then their presence is questionable.

Solution to Exercise 15

It is easy to sketch the graph of A against r by noting that it is the sum of a quadratic function and a multiple of $1/r$; the graph is given in the margin.



There appears to be just one local minimum, which will provide the solution to the problem.

From equation (4), we have

$$A = 2\left(\frac{V}{r} + \pi r^2\right),$$

where V is a constant. Thus

$$\frac{dA}{dr} = 2\left(-\frac{V}{r^2} + 2\pi r\right),$$

so $dA/dr = 0$ where

$$r^3 = \frac{V}{2\pi}.$$

(It is clear from the graph that this gives a minimum; but to confirm this, note that

$$\frac{d^2A}{dr^2} = 4\left(\frac{V}{r^3} + \pi\right),$$

which is positive for all $r > 0$.)

Thus the minimum area, for a specified volume V, is obtained when

$$r = \left(\frac{V}{2\pi}\right)^{1/3} \quad \text{and} \quad h = \frac{V}{\pi r^2} = \left(\frac{4V}{\pi}\right)^{1/3}.$$

The corresponding expression for the minimum area is

$$A_{\text{min}} = 2V\left(\frac{2\pi}{V}\right)^{1/3} + 2\pi\left(\frac{V}{2\pi}\right)^{2/3} = 3(2\pi V^2)^{1/3}.$$

Solution to Exercise 16

(a) With V fixed, we expect that $A \to \infty$ as $r \to 0$ (i.e. the can becomes long and thin). The model (4) predicts this.

(b) We expect that $A \to \infty$ as $r \to \infty$ (i.e. the can becomes short and fat). This also is predicted by the model.

(c) The solution to the model predicts that the minimum area increases as the volume is increased, which agrees with common sense.

Solution to Exercise 17

(a) The minimum area of a cylindrical can, of specified volume V, is given by the solution to Exercise 15 as $3(2\pi V^2)^{1/3}$. In this case its height is $h = (4V/\pi)^{1/3}$ and the radius of either end is $r = (V/(2\pi))^{1/3}$, for which the shape of the can satisfies $h/r = 2$. This means that the height of the can is equal to the diameter of its ends, so if it is seen from the side, it will appear square.

(b) In practice, cans are rarely the shape indicated in part (a). Most of them are taller than they are wide.

Solution to Exercise 18

Possible key phrases are: cylindrical can of fixed volume; area of tin plate; only cost is the tin plate used; minimise the area.

A possible description might read as follows.

> This model considers a cylindrical can of fixed volume and the area of tin plate used in its construction. This area, assumed to be proportional to the total cost of manufacture, is expressed as a function of the radius. This function is differentiated to find the radius for which the area is a minimum.

Solution to Exercise 19

(a) $[\text{area}] = \mathrm{L}^2$.

(b) $[\text{density}] = \mathrm{M\,L^{-3}}$.

(c) $[\theta] = [l/r] = [l]\,[r]^{-1} = \mathrm{L} \times \mathrm{L}^{-1} = 1$.

So angle is a dimensionless quantity, and radians are dimensionless units.

Solution to Exercise 20

The symbol v represents a speed, so

$$[v^2] = [v]^2 = (\mathrm{L\,T^{-1}})^2 = \mathrm{L^2\,T^{-2}}.$$

Similarly, v_0 is a speed, so v_0^2 has dimensions $\mathrm{L^2\,T^{-2}}$.

The final term in the equation has dimensions

$$[2a_0x] = [2]\,[a_0]\,[x] = 1 \times (\mathrm{L\,T^{-2}}) \times \mathrm{L} = \mathrm{L^2\,T^{-2}}$$

also.

Hence the equation is dimensionally consistent.

Solution to Exercise 21

The left-hand side of the equation, dm/dt, has the same dimensions as m/t, namely $\mathrm{M\,T^{-1}}$. The right-hand side of the equation has dimensions

$$\begin{aligned}[-km] &= \left[-\frac{r}{V}m\right] \\ &= [-1]\,[r]\,[V]^{-1}\,[m] \\ &= 1 \times (\mathrm{L^3\,T^{-1}}) \times (\mathrm{L^3})^{-1} \times \mathrm{M} = \mathrm{M\,T^{-1}}.\end{aligned}$$

Hence the equation is dimensionally consistent.

Solution to Exercise 22

Consider the dimensions of the terms on the left-hand side of the equation one by one:

$$\left[\frac{p}{\rho}\right] = \frac{[p]}{[\rho]} = \frac{\mathrm{M\,L^{-1}\,T^{-2}}}{\mathrm{M\,L^{-3}}} = \mathrm{L^2\,T^{-2}},$$

$$\left[\tfrac{1}{2}u\right] = [u] = \mathrm{L\,T^{-1}},$$

$$[gz] = [g]\,[z] = (\mathrm{L\,T^{-2}}) \times \mathrm{L} = \mathrm{L^2\,T^{-2}}.$$

The equation is dimensionally inconsistent because the second term has different dimensions from the other two. It therefore seems likely that the second term is incorrect, and that its dimensions should be $\mathrm{L^2\,T^{-2}}$. One form that would be consistent is $\frac{1}{2}u^2$ rather than $\frac{1}{2}u$.

Note that the 'constant' on the right-hand side of this equation cannot be dimensionless. It must have the same dimensions as the terms on the left-hand side.

Solution to Exercise 23

We require the dimensions of

$$c_2 = \frac{R}{D^2 v^2}.$$

R is the magnitude of a force thus has dimensions $\mathrm{M\,L\,T^{-2}}$. D is a length and has dimensions L. v is a speed and has dimensions $\mathrm{L\,T^{-1}}$. For dimensional consistency we must have

$$\begin{aligned}[c_2] &= \left[\frac{R}{D^2 v^2}\right] \\ &= [R]\,[D]^{-2}\,[v]^{-2} \\ &= (\mathrm{M\,L\,T^{-2}}) \times \mathrm{L^{-2}} \times (\mathrm{L\,T^{-1}})^{-2} = \mathrm{M\,L^{-3}}.\end{aligned}$$

Hence in SI units, c_2 would be measured in $\mathrm{kg\,m^{-3}}$ (the same units as density).

Solution to Exercise 24

First consider the argument $\omega t + \phi$ of the cosine function. Since ϕ is an argument of a cosine, $[\phi] = 1$. For dimensional consistency, $[\omega t] = 1$. Since $[t] = \mathrm{T}$, we must have $[\omega] = \mathrm{T}^{-1}$.

The result of the cosine function is dimensionless, so $[\cos(\omega t + \phi)] = 1$. Now $[x] = \mathrm{L}$, so the equation is dimensionally consistent if $[A] = \mathrm{L}$.

Solution to Exercise 25

We have

$$1\,\mathrm{kg} = 10^3\,\mathrm{g} \quad \text{and} \quad 1\,\mathrm{cm} = 10^{-2}\,\mathrm{m}.$$

Hence

$$\begin{aligned} 13.546\,\mathrm{g\,cm^{-3}} &= 13.546\,\mathrm{g\,cm^{-3}} \times \left(\frac{1\,\mathrm{kg}}{10^3\,\mathrm{g}}\right) \times \left(\frac{10^{-2}\,\mathrm{m}}{1\,\mathrm{cm}}\right)^{-3} \\ &= 13.546 \times 10^{-3} \times 10^{6}\,\mathrm{kg\,m^{-3}} \\ &= 1.3546 \times 10^{4}\,\mathrm{kg\,m^{-3}}. \end{aligned}$$

So the density of the metal is $1.3546 \times 10^4\,\mathrm{kg\,m^{-3}}$.

Solution to Exercise 26

We have

$$1\,\mathrm{lb} = 0.453\,592\,\mathrm{kg} \quad \text{and} \quad 1\,\mathrm{in} = 2.54 \times 10^{-2}\,\mathrm{m}.$$

Hence

$$\begin{aligned} 61\,\mathrm{lb\,in^{-2}} &= 61\,\mathrm{lb\,in^{-2}} \times \left(\frac{0.453\,592\,\mathrm{kg}}{1\,\mathrm{lb}}\right) \times \left(\frac{2.54 \times 10^{-2}\,\mathrm{m}}{1\,\mathrm{in}}\right)^{-2} \\ &= \frac{61 \times 0.453\,592}{(2.54 \times 10^{-2})^2}\,\mathrm{kg\,m^{-2}} \\ &\simeq 4.2887 \times 10^{4}\,\mathrm{kg\,m^{-2}}. \end{aligned}$$

So the pressure on the piston is $42\,890\,\mathrm{kg\,m^{-2}}$ (to 4 s.f.).

As mentioned in the question, in the SI system, pressure is *force* per unit area, not *mass* per unit area – this arises because of the difference between mass and weight. So we should multiply the above answer by g, the magnitude of the acceleration due to gravity, to give a pressure of $4.289 \times 9.81 \times 10^4 = 420\,700\,\mathrm{kg\,m^{-1}\,s^{-2}}$.

Solution to Exercise 27

We have

$$1 \text{ boliviano} = 1 \text{ boliviano} \times \left(\frac{\$1}{7.75 \text{ boliviano}}\right) \times \left(\frac{£1}{\$1.75}\right) = £0.074.$$

I approximated this exchange rate as 10 boliviano $= £\frac{3}{4}$.

For example, I bought a memory card for my camera for 830 boliviano, and I approximated this cost in £ as $84 \times \frac{3}{4} = £63$ (using 84 because it is divisible by 4). (The cost on my credit card bill was £61.59.)