Unit 2

Vector algebra and statics

Introduction

This module discusses *Newtonian mechanics*. Isaac Newton (Figure 1) was the great English mathematician whose name is given to this subject. His *Philosophiae Naturalis Principia Mathematica* of 1687 (*Mathematical Principles of Natural Philosophy*, known as the *Principia* for short) incorporates one of the most celebrated examples of mathematical modelling. It was in the *Principia* that Newton laid down the foundations of Newtonian mechanics. This great book, which showed for the first time how earthly and heavenly movements obey the same laws, is cast in the form of a set of propositions all deriving from three axioms, or *laws of motion*. It is these, here translated into modern English from the original Latin, that still provide the basis for Newtonian mechanics.

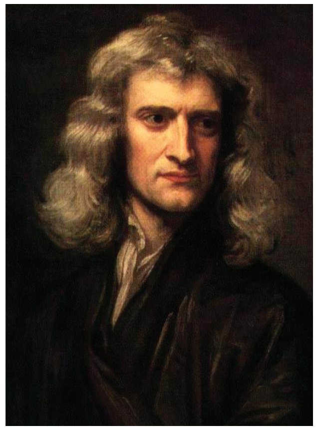


Figure 1 Sir Isaac Newton (1642–1727)

Law I Every body continues in a state of rest, or moves with constant velocity in a straight line, unless a force is applied to it.

Law II The rate of change of velocity of a body is proportional to the resultant force applied to the body, and happens in the direction of the resultant force.

Law III To every action (i.e. force) by one body on another there is always opposed an equal reaction (i.e. force) – that is, the actions of two bodies on each other are always equal in magnitude and opposite in direction.

One of the central concepts in Newtonian mechanics is that of a *force*. The word ‘force’ is used in everyday conversation in a variety of ways: he *forced* his way in; the *force* of destiny; to put into *force*; the labour *force*. In mathematics and science, the word ‘force’ has a precise definition. However, this definition relies on the movement of objects thus is deferred until the next unit. Essentially, though, this definition says that a **force** either changes the shape of the object on which it acts, or causes movement of the object. When we experience a force, in the mathematical sense of the word, we feel it through contact: pulling on a rope, lifting a shopping bag, pushing against a car, holding a child aloft. In each case, the force that we experience has a magnitude and a direction, so we model a force as a *vector quantity*.

We often need to represent physical quantities – such as mass, force, velocity, acceleration, time – mathematically. Most of the physical quantities that we need can be classified into two types: *scalars* and *vectors*. *Scalar quantities* are quantities, like mass, temperature, energy, volume and time, that can be represented by a single real number. Other quantities, like force, velocity and acceleration, possess *magnitude* and *direction* in space, and cannot be represented by a single real number; they are called *vector quantities*.

The definitive vector quantity is *displacement*. The displacement of a point specifies the position of the point in space relative to some reference point. We use the concept of displacement whenever we want to describe spatial relationships.

Section 1 defines a vector and discusses ways of representing vectors in two dimensions. Section 2 discusses another way of representing vectors, one that easily generalises from two to three (or more) dimensions, and considers ways of operating on and combining vectors – that is, the fundamentals of *vector algebra*.

Mathematical representation of a force

A force is represented mathematically by a vector. The magnitude of the vector represents the magnitude of the force, and the direction of the vector specifies the direction in which the force is applied.

Sometimes we can see the effect of a force, such as when a mattress depresses under the weight of someone sitting on it, a washing line sags under the weight of the washing, a rubber band is stretched over some packages, a door is pulled open, or a bag of shopping is lifted. In each case there is an obvious deformation or movement that indicates that a force is present. Sometimes, however, the presence of a force is not so obvious, for example in situations such as a ladder leaning against a wall (though you could appreciate the presence of a force in this situation were *you* to replace the wall and hold the ladder steady yourself), a box resting on a shelf, or a cable holding up a ceiling lamp (consider holding up the lamp by the cable yourself).

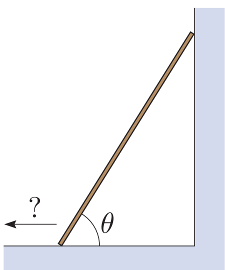


Figure 2 A ladder leaning against a wall

The second half of this unit considers the conditions under which objects remain stationary when subjected to forces, which is a topic known as **statics**. For example, what is the minimum angle θ that a ladder leaning against a wall can make with the ground before the ladder slides to the ground (see Figure 2)? Cases where forces cause motion are discussed in Unit 3 and elsewhere in the module. The study of motion is called **dynamics**.

Before forces and their effects can be analysed mathematically, they and the objects on which they act must be modelled mathematically. In Section 3, objects are modelled as particles, and we show how various forces such as the forces of gravity, tension and friction can be modelled. We also show how to analyse one-particle systems in equilibrium, and extend the ideas to systems involving two or more particles. Section 4 considers situations where an object needs to be modelled as a solid body rather than as a particle. It also discusses the turning effect of a force (known as a *torque*), which happens only if a force acts on a solid body rather than a particle. Section 5 describes the application to statics problems of the concepts and principles described in the earlier sections.

1 Describing, representing and combining vectors

Subsection 1.1 explains what scalars and vectors are. Subsection 1.2 explains how to denote vectors symbolically (i.e. algebraically) and how to show them in diagrams. It then explains what is meant by saying that two vectors are equal to one another, which is a necessary first step in the development of an algebra for vectors. Subsection 1.3 describes how a vector can be scaled by multiplying it by a scalar, and how vectors can be added together.

1.1 Scalars and vectors

A **scalar** is any quantity, such as mass, time, volume or temperature, that can be represented mathematically by a single real number (and often a unit of measurement). Real numbers themselves are examples of scalars, and you can regard the terms *scalar* and *real number* as synonymous. Examples of scalar quantities, quoted to some convenient degree of accuracy, are:

- the mass of the Earth, 5.975×10^{24} kilograms
- the temperature of melting ice, 0 degrees Celsius
- my current bank account balance, -153.12 pounds sterling
- pi (π), $3.141\,592\,653\,589\,79\ldots$

A real number x is defined by two properties: its *modulus* $|x|$ and its *sign*. Thus the **magnitude** of a scalar x is $|x|$. For example, the magnitude of my current account balance is $|-153.12|$ pounds $= 153.12$ pounds, which sounds a lot better since it doesn't remind me that I'm in debt. Note that magnitudes are always non-negative (i.e. positive or zero).

The modulus of a real number is also called its *magnitude*.

A **vector** is any quantity, such as force, velocity or displacement, that has a magnitude *and* a direction in space (or, in two dimensions, a direction in a plane). An example is the velocity of a motor car travelling on the M4 motorway from London to Bristol with a speed of 95 km per hour in a westerly direction. The magnitude of the velocity vector is 95 (dropping units for convenience), and the direction of the velocity vector is due west. Thus the specification of a vector consists of:

The familiar term *speed* is used to mean the magnitude of velocity. Speed is a non-negative scalar.

- a non-negative real number, called its modulus or **magnitude**
- a **direction** in space.

Vectors are denoted in printed text by bold letters, e.g. $\mathbf{v}$, $\mathbf{F}$. In handwritten work, you should denote vectors by drawing either a straight line or a wavy line under the letter, e.g. $\underline{\mathrm{v}}$, $\underline{\mathrm{F}}$ or $\underset{\sim}{\mathrm{v}}$, $\underset{\sim}{\mathrm{F}}$. Thus if a symbol is used to represent the velocity of an object, then it must be handwritten by you as either $\underline{\mathrm{v}}$ or $\underset{\sim}{\mathrm{v}}$ (but will be printed in the text as $\mathbf{v}$). An exception to this rule is that a vector representing the displacement of the point Q from the point P is often written as $\overrightarrow{PQ}$, and if $\mathbf{x}$ represents this displacement vector, then we can write $\mathbf{x} = \overrightarrow{PQ}$.

It is *important* that you learn to write vectors using underlining: if you do not do so, then someone reading your work may not be able to tell that you are referring to a vector. In particular, you may lose marks in assessed work!

We read $|\mathbf{v}|$ as 'the modulus of v' or 'the magnitude of v', or simply 'mod v'.

The modulus or magnitude of the vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$, or sometimes, where there is no possibility of ambiguity, by v; $|\mathbf{v}|$ is a non-negative scalar.

1.2 Using arrows to represent vectors

A vector can be conveniently represented in a diagram by an arrow, that is, a straight line with an arrowhead on it. The tail of the arrow may be placed at some fixed origin, its direction is chosen to represent that of the vector, and its length is chosen to be proportional to the magnitude of the vector. In Figure 3, which uses the origin of the Cartesian coordinate system as the fixed origin, the shorter arrow represents a vector of magnitude 1 in the positive x-direction, and the longer arrow represents a vector of magnitude $2\sqrt{2}$ in a direction at $\pi/4$ radians (45°) to the positive x-direction. (Note that we use the convention that positive angles are measured anticlockwise.) If we decide to denote these vectors by letters $\mathbf{a}$ and $\mathbf{b}$, then we can also put this information on the diagram, by writing $\mathbf{a}$ and $\mathbf{b}$ near the arrowheads.

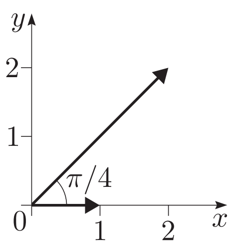


Figure 3 Examples of vectors

Exercise 1

Represent the following two vectors on a diagram by arrows:

- vector $\mathbf{a}$ has magnitude 3 units and points in the positive y-direction
- vector $\mathbf{b}$ has magnitude 4 units and points in the direction at $\pi/3$ radians (60°) to the positive x-direction.

What about a vector whose magnitude is zero? Clearly its length is zero, but what is its direction? The answer is that it does not have one! We make the following definition.

Zero vector

The **zero vector** is the unique vector with magnitude zero and no direction. It is denoted by $\mathbf{0}$.

Be particularly careful to underline the zero vector ($\underline{0}$ or $\underset{\sim}{0}$) in your written work, and be aware that the vector $\mathbf{0}$ is different from the scalar 0!

A constant *velocity* is defined by a magnitude and a direction. For instance, in a weather forecast, a typical wind velocity might be 35 mph from the north-west. It is not sufficient to say that 'the wind velocity is 35 mph'; the obvious question about such a statement would be 'from which direction?'. The vector $\mathbf{v}$ representing this velocity has magnitude 35 and direction from the north-west and towards the south-east (since the air is travelling in the south-easterly direction).

It can be represented on a diagram like Figure 4. The length of the arrow represents a wind speed of 35 mph.

Note that the *direction* of a vector consists of two attributes:

- an *orientation*, represented by the *slope* of the arrow in diagrams like Figures 3 and 4
- a *sense*, represented by the *arrowhead*.

For instance, the arrow representing the velocity 35 mph from the north-west in Figure 4 is a line making an angle of 45° anticlockwise from the south direction (the orientation) and an arrowhead pointing towards south-east as opposed to north-west (the sense).

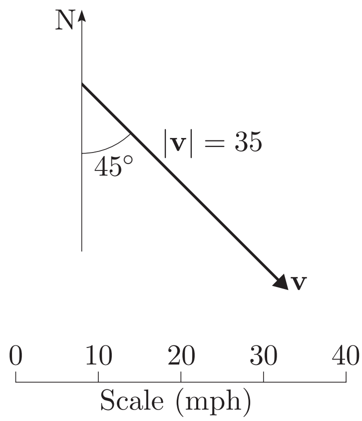


Figure 4 A vector representing wind speed and direction

Exercise 2

A car travelling from London along the M1 with speed 70 mph heads in the direction N 60° W near Junction 14.

Represent the velocity of the car by an arrow, drawn to a suitable scale.

You have seen how to represent a vector by an arrow. Here is a definition of equality of vectors.

> Two vectors are said to be **equal** if they have the same magnitude *and* the same direction.

This definition tells us that the two features needed to define a vector uniquely are its magnitude and direction. This means that any two arrows that are drawn at different places on the page but are equal in length, are parallel and have the same sense, can be used to represent the same vector. For instance, the two arrows in Figure 5 are each of length 2 units and point in the positive x-direction. They represent two equal vectors, and we write $\mathbf{b} = \mathbf{d}$. In other words, the arrow representing a vector does not have to be drawn so that its tail is at any particular point.

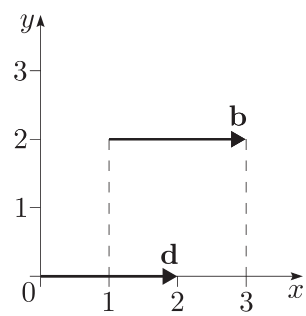


Figure 5 Two equal vectors

1.3 Vector algebra

If $\mathbf{v}$ is a vector and m is a positive number, then the product $m\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$ but with magnitude $m|\mathbf{v}|$, that is, m times the magnitude of $\mathbf{v}$. This multiplication of a vector by a scalar is called *scaling* or *scalar multiplication*, and $m\mathbf{v}$ is called a *scalar multiple* of $\mathbf{v}$. For example, if $\mathbf{v}$ has magnitude 4 and points in the positive x-direction, then $3\mathbf{v}$ has magnitude 12 and points in the positive x-direction also. This is illustrated in Figure 6.

Note that there is no multiplication sign between the m and the $\mathbf{v}$. In vector algebra, the dot and cross symbols are reserved for other products, to be discussed in Section 2.

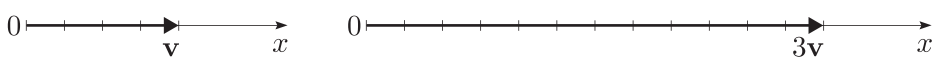


Figure 6 Scaling a vector

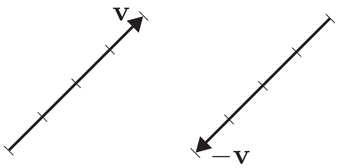


Figure 7 Reversing the sign of a vector

We can also scale a vector $\mathbf{v}$ by a negative number. When m is negative, the vector $m\mathbf{v}$ has magnitude $|m|\,|\mathbf{v}|$ and points in the *opposite* direction to $\mathbf{v}$. A special case is when $m = -1$. Then the vector $(-1)\mathbf{v}$ has the same magnitude as $\mathbf{v}$ but points in the opposite direction; see Figure 7. We normally write $(-1)\mathbf{v}$ simply as $-\mathbf{v}$, that is, $(-1)\mathbf{v} = -\mathbf{v}$.

For any vector $\mathbf{v}$ and any real number m, the **scalar multiple** $m\mathbf{v}$ is the vector with magnitude $|m|\,|\mathbf{v}|$ that is:

- in the same direction as $\mathbf{v}$ if $m > 0$
- in the opposite direction to $\mathbf{v}$ if $m < 0$
- the zero vector (i.e. with unspecified direction) if $m = 0$.

The multiplication of $\mathbf{v}$ by m is called **scaling** or **scalar multiplication**.

Exercise 3

(a) If $\mathbf{v}$ represents the velocity of a wind of 35 mph from the north-east, what vector represents a wind of 35 mph from the south-west?

(b) Relate the direction and magnitude of $-1.5\mathbf{v}$ to those of $\mathbf{v}$, where $\mathbf{v}$ is any given non-zero vector. Do the same for $-k\mathbf{v}$, where k is an arbitrary positive number.

(c) If $\mathbf{v}$ is any non-zero vector, what are the magnitude and direction of the vector $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$?

The vector $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$ in Exercise 3(c) is a vector that has magnitude 1 and points in the direction of $\mathbf{v}$. It is called the *unit vector* in the direction of $\mathbf{v}$ and is often denoted by the symbol $\widehat{\mathbf{v}}$ (read as 'v hat').

For any non-zero vector $\mathbf{v}$, the **unit vector** in the direction of $\mathbf{v}$ is the vector

$$\widehat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}. \tag{1}$$

Unit vectors are often used to denote directions in the plane or in space.

A particular example is provided by the unit vectors in the positive directions of the x- and y-axes in the plane Cartesian coordinate system. (We will develop the Cartesian representation of vectors in Section 2.) These unit vectors are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively, and are called **Cartesian unit vectors**. Note that we do not denote these unit vectors as $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ as they are widely used without the 'hat' in mathematics.

Figure 8 shows these Cartesian unit vectors and two other vectors, $\mathbf{a}$ and $\mathbf{b}$. The vector $\mathbf{a}$ has magnitude 2 and points in the positive x-direction; $\mathbf{b}$ has magnitude 3.5 and points in the positive y-direction. The unit vector $\mathbf{i}$ has magnitude 1 and points in the same direction as $\mathbf{a}$. Thus we can write $\mathbf{a}$ in terms of $\mathbf{i}$ by a scaling:

$$\mathbf{a} = 2\,\mathbf{i}.$$

Similarly, we can write $\mathbf{b}$ in terms of $\mathbf{j}$:

$$\mathbf{b} = 3.5\,\mathbf{j}.$$

Any vector parallel to the x- or y-axis can be written as a scaling of $\mathbf{i}$ or $\mathbf{j}$.

Note that although $\mathbf{i}$ and $\mathbf{j}$ are shown in Figure 8 with their tails at the origin, this is not necessary. They can be drawn at any convenient position, provided only that they are of unit magnitude and point in the positive x- and y-directions, respectively.

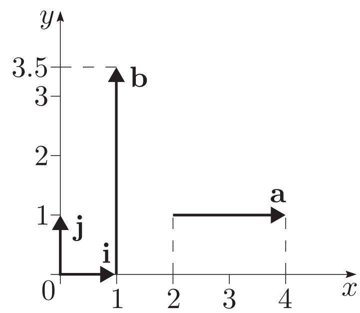


Figure 8 Vectors $\mathbf{a}$ and $\mathbf{b}$, and two unit vectors, $\mathbf{i}$ and $\mathbf{j}$

Exercise 4

Let the unit vectors $\mathbf{i}$ and $\mathbf{j}$ denote the directions of east and north, respectively. Specify the following vectors as scalings of $\mathbf{i}$ and $\mathbf{j}$.

(a) A wind velocity of 35 km per hour due south

(b) The displacement of Bristol from London (112 miles due west)

Let us consider what is meant by the addition of vectors. Suppose that we make a journey from Bristol to Leeds, and then another journey from Leeds to Norwich. The first journey produces a displacement of $\mathbf{d}_1$ and the second a displacement of $\mathbf{d}_2$. The net result of the two journeys is a displacement of $\mathbf{d}_3$ from Bristol to Norwich. This is illustrated by the triangle of displacements shown in Figure 9. Displacements are said to add by the *triangle rule*, and we write $\mathbf{d}_3 = \mathbf{d}_1 + \mathbf{d}_2$. The vector $\mathbf{d}_3$ is called the *resultant* of $\mathbf{d}_1$ and $\mathbf{d}_2$.

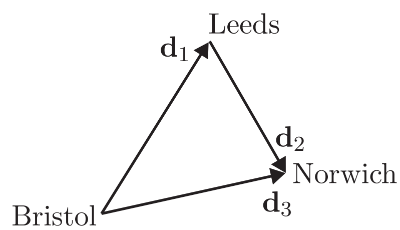


Figure 9 A triangle of displacements

Velocities also add by the triangle rule, and so do forces, accelerations and all other vector quantities. Thus the triangle rule is also called the *vector addition rule*.

Triangle rule or vector addition rule

To add any two vectors $\mathbf{a}$ and $\mathbf{b}$: choose an origin O; draw the line OP in the direction of $\mathbf{a}$ and with length equal to the magnitude of $\mathbf{a}$; and draw the line PQ in the direction of $\mathbf{b}$ and with length equal to the magnitude of $\mathbf{b}$ (as in Figure 10). Then $\mathbf{a} + \mathbf{b}$ is the vector with magnitude equal to the length of OQ and with direction from O to Q. The vector $\mathbf{a} + \mathbf{b}$ is called the **sum** or **resultant** of $\mathbf{a}$ and $\mathbf{b}$.

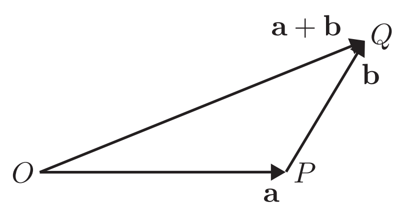


Figure 10 The triangle rule or vector addition rule

Now recall that when discussing displacements, we mentioned the zero vector $\mathbf{0}$ (representing no displacement). Once addition of vectors is introduced, we *need* the zero vector in order to answer questions such as 'What is $\mathbf{i} + (-1)\mathbf{i}$?'. Geometrically, no construction is needed when adding the zero vector, which obeys the rather obvious rule

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

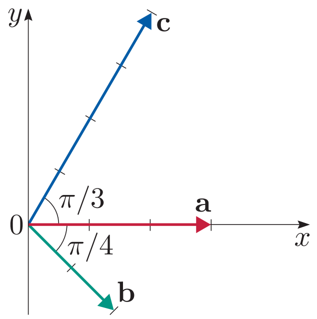


Figure 11

Exercise 5

Three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of magnitudes 3, 2 and 4 are shown in Figure 11.

(a) Draw a rough sketch to show the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$.

(b) Sketch the vector $-\mathbf{b}$, and draw a rough sketch to show the sum of $\mathbf{a}$ and $-\mathbf{b}$.

Exercise 5(b) suggests a definition of **vector subtraction**. To subtract the vector $\mathbf{b}$ from the vector $\mathbf{a}$, we *add* the vectors $\mathbf{a}$ and $-\mathbf{b}$ by the triangle rule of vector addition; that is, in symbols,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Exercise 6

A vector $\mathbf{a}$ has magnitude 3 units and points in the positive x-direction. A vector $\mathbf{b}$ has magnitude 4 units and points in the positive y-direction. Draw a diagram showing the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

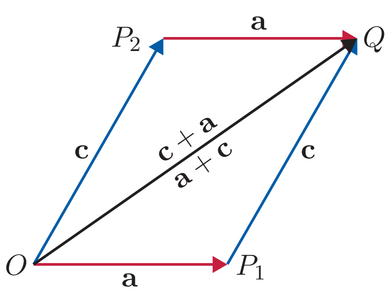


Figure 12 The parallelogram rule

Vector addition is *commutative*, that is, the order in which we add two vectors does not matter. This can be illustrated by reference to vectors $\mathbf{a}$ and $\mathbf{c}$ of Exercise 5 (see Figure 12). The triangle OP_1Q illustrates the addition $\mathbf{a} + \mathbf{c}$, while triangle OP_2Q illustrates $\mathbf{c} + \mathbf{a}$. The same resultant $\overrightarrow{OQ}$ is obtained in both cases. Thus

$$\mathbf{a} + \mathbf{c} = \mathbf{c} + \mathbf{a}.$$

An alternative geometric construction for adding two vectors can be seen from Figure 12. It is called the **parallelogram rule**. Draw the two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same beginning point O. Complete the parallelogram OP_1QP_2. Then the resultant vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is the vector on the diagonal of the parallelogram. The parallelogram rule gives the same resultant as the triangle rule.

The algebraic rules are summarised below.

Algebraic rules for scaling and adding vectors

- A vector has magnitude and direction.
- Two vectors can be added by the triangle rule.
- A vector can be scaled by a real number. For example, any non-zero vector $\mathbf{a}$ can be written as $\mathbf{a} = |\mathbf{a}|\,\widehat{\mathbf{a}}$, where $|\mathbf{a}|$ is the magnitude of $\mathbf{a}$, and $\widehat{\mathbf{a}}$ is a unit vector in the direction of $\mathbf{a}$.

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors, and let m, m_1 and m_2 be scalars.

- Addition is commutative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
- Addition is associative: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.
- $m\mathbf{a}$ is a vector with magnitude $|m|\,|\mathbf{a}|$, in the same direction as $\mathbf{a}$ when $m > 0$, and in the opposite direction when $m < 0$.
- Scaling is associative: $m_1(m_2\mathbf{a}) = (m_1 m_2)\mathbf{a}$.
- Scaling is distributive: $(m_1 + m_2)\mathbf{a} = m_1\mathbf{a} + m_2\mathbf{a}$.
- Scaling is distributive over vector addition: $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$.
- Addition and scaling involving the zero vector are as expected: $\mathbf{0} + \mathbf{a} = \mathbf{a}$ and $0\mathbf{a} = \mathbf{0}$.
- Subtraction is defined by $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$.

These rules allow us to manipulate algebraic expressions involving scalings and vector addition in a familiar way.

Example 1

Simplify the expression $2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c})$.

Solution

We have

$$\begin{aligned} &2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c}) \\ &\quad = 2\mathbf{a} + 2\mathbf{b} + 3\mathbf{b} + 3\mathbf{c} - 5\mathbf{a} - 5\mathbf{b} + 5\mathbf{c} \\ &\quad = 8\mathbf{c} - 3\mathbf{a}. \end{aligned}$$

Exercise 7

Simplify the expression $4(\mathbf{a} - \mathbf{c}) + 3(\mathbf{c} - \mathbf{b}) + 2(2\mathbf{a} - \mathbf{b} - 3\mathbf{c})$.

2 Cartesian components and products of vectors

So far we have approached vectors, and the rules of vector addition and scaling, geometrically. To add vectors geometrically requires drawing diagrams representing the vectors by arrows. An alternative, and sometimes more convenient, *algebraic* approach to representing three-dimensional vectors is developed in this section.

So far in this unit we have defined two algebraic operations: vector addition (by the triangle rule) and scaling a vector. The addition of vectors can be usefully applied only to two vectors representing the same type of physical quantity. For example, the addition of a displacement and a velocity has no physical meaning. However, vectors representing the same or different types of physical quantity can be combined in operations that are called the *dot product* and the *cross product*. They are called products because in some respects they behave like 'multiplications' in the algebra of real numbers. Dot products and cross products of vectors have numerous applications in geometry, mechanics and electromagnetism.

In Subsections 2.2 and 2.3 the dot product and cross product are defined geometrically and also in terms of components of vectors. The dot product of two vectors is interpreted in terms of projecting a shadow of one vector onto another, and is applied to the problem of finding the angle between two vectors or lines. The cross product of two vectors is interpreted as a vector whose magnitude is an area. Both dot and cross products can be used in problems involving finding the areas of plane figures and the volumes of solid objects.

2.1 Vectors in three dimensions

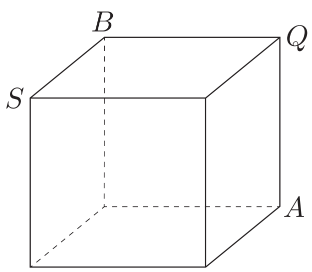


Figure 13 A cube

In previous modules you may have discussed vectors in the plane. However, the world is three-dimensional, and few real problems are restricted to a plane surface. For example, starting at point A at one corner of the cube shown in Figure 13, you can reach the opposite corner S by three successive displacements: $\overrightarrow{AQ} + \overrightarrow{QB} + \overrightarrow{BS}$. In order to work with such addition of displacements in three dimensions, it is necessary to introduce a three-dimensional coordinate system.

A three-dimensional Cartesian coordinate system

Consider a two-dimensional Cartesian coordinate system Oxy. Draw a third axis, the z-axis, through the origin O, perpendicular to both the x- and y-axes of the two-dimensional system. This produces a coordinate system with three mutually perpendicular axes, the x-, y- and z-axes (see Figure 14), intersecting at O.

The z-axes shown in Figure 14 are meant to point *out* of the plane of the page.

Alternatively, the coordinate system can be characterised by three planes:

- the **(x, y)-plane**, which contains the x- and y-axes and is perpendicular to the z-axis
- the **(x, z)-plane**, analogously defined
- the **(y, z)-plane**, again analogously defined.

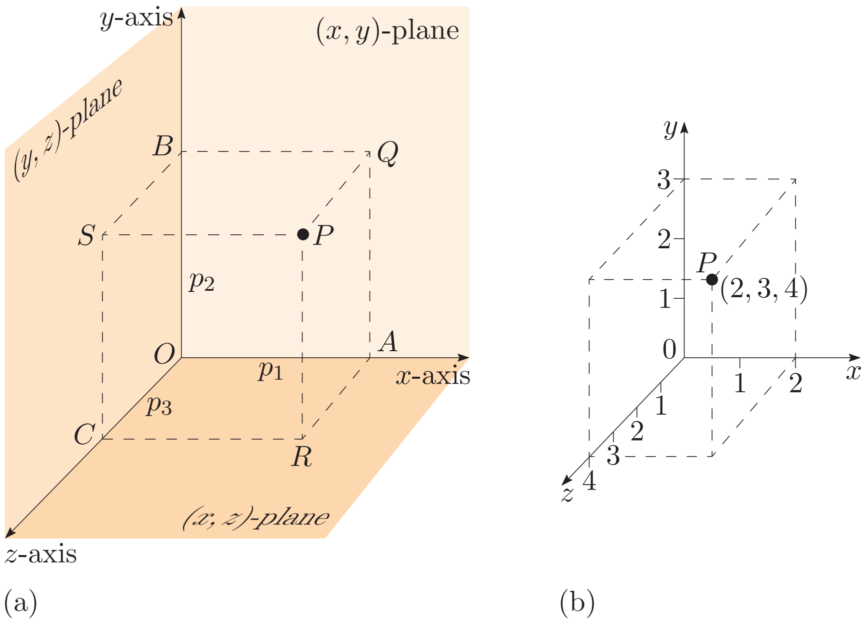


Figure 14 The Cartesian coordinates of a point in space

Any point P can be represented uniquely by its perpendicular distances from the (x, y)-, (x, z)- and (y, z)-planes. These distances, called the (**Cartesian**) **coordinates** of P, are shown in Figure 14(a). QP, RP and SP are perpendicular to the (x, y)-plane, (x, z)-plane and (y, z)-plane, respectively.

We denote the point P by the ordered triple of coordinates (p_1, p_2, p_3), where

$$p_1 = SP = OA,$$
$$p_2 = RP = OB,$$
$$p_3 = QP = OC.$$

For example, the point $(2, 3, 4)$ is shown in Figure 14(b).

When drawing Figure 14 it was necessary to choose one of two possible ways for the positive z-direction to be defined; these are shown in Figure 15, where the z-axis is meant to point out of the plane of the page in the top diagram, and into the plane of the page in the bottom diagram.

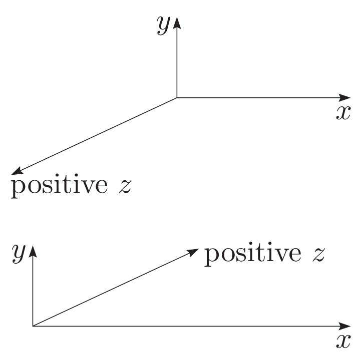


Figure 15 Choosing the direction of positive z

The usual convention for relating the positive directions of x, y and z is given by the following rule, called the **right-hand rule**. The right hand is held with the middle finger, first finger and thumb placed (roughly) perpendicular to each other, and the other two fingers closed (see Figure 16). If the thumb and first finger are pointing in the directions of the positive x- and y-axes, respectively, then the middle finger is pointing in the direction of the positive z-axis.

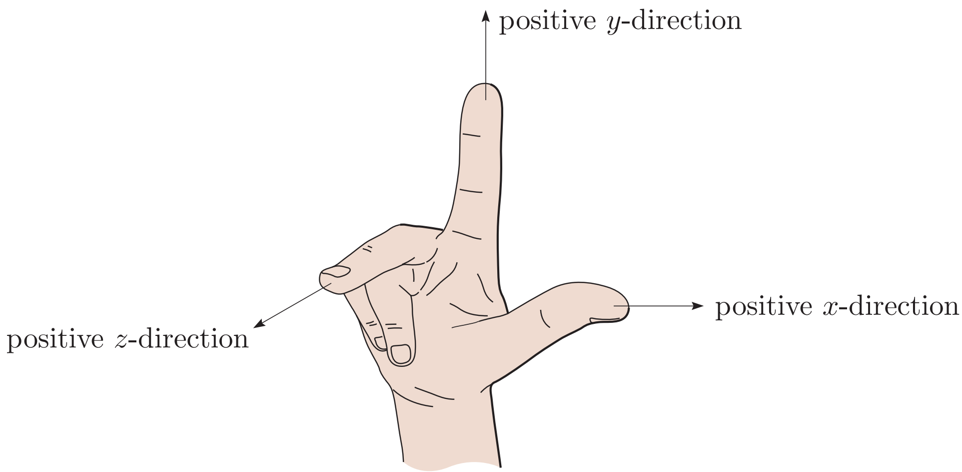


Figure 16 The right-hand rule

Alternatively, you can think of Figure 14(a) as showing a corner of a room (with the y-axis pointing upwards). If you are standing in the corner facing outwards, then the left-hand edge of the floor is the x-axis, and the right-hand edge is the z-axis. A coordinate system defined in this way is called a **right-handed system**. Only right-handed systems will be used in this module. The systems drawn in Figure 14 and the top of Figure 15 are right-handed systems.

An alternative definition of the same positive z-direction is given by the **right-hand grip rule**, stated as follows (see Figure 17). If you hold your right hand in a fist, so that your fingers rotate from the positive x-axis towards the positive y-axis, then your thumb points in the positive z-direction of a right-handed coordinate system.

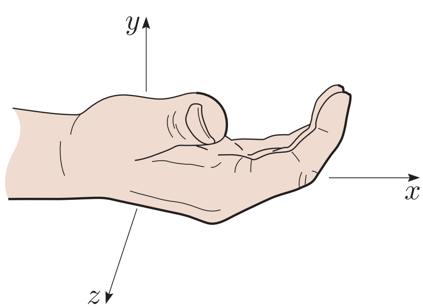


Figure 17 The right-hand grip rule

You should use whichever rule you find easier to apply.

Exercise 8

Decide which of the sets of perpendicular axes in Figure 18 define right-handed coordinate systems.

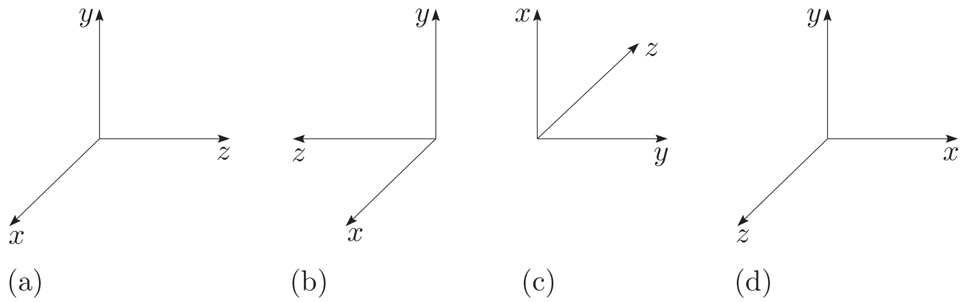


Figure 18

(The x-axis points out of the plane of the page in (a) and (b). The z-axis points into and out of the plane of the page in (c) and (d), respectively.)

Component form of three-dimensional vectors

The algebraic representation of vectors can be extended to vectors in three dimensions, such as in Figure 19. The vector $\mathbf{a}$, drawn from the origin O, is the *position vector* of point A with three-dimensional Cartesian coordinates (a_1, a_2, a_3). A third Cartesian unit vector $\mathbf{k}$ is introduced to represent the positive z-direction. We now have three Cartesian unit vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, which are perpendicular to each other. The vector $\mathbf{a}$ may thus be written in *component form* as

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \quad \text{or} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{or} \quad \mathbf{a} = (a_1 \quad a_2 \quad a_3)^T. \qquad (2)$$

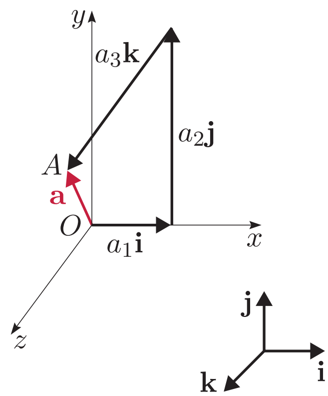


Figure 19 Expressing a vector in component form

The third form above is often used in the text, to save space. The *transpose symbol* T indicates that a column vector has been written 'on its side' as a row vector.

The **position vector** of a point A relative to the origin O of three-dimensional space is the displacement of A from O, that is, the vector

$$\mathbf{a} = \overrightarrow{OA}.$$

The **i**-, **j**- and **k**-components of the position vector $\mathbf{a}$ are the coordinates a_1, a_2 and a_3 of the point A, respectively.

These may sometimes be referred to as x-, y- and z-components.

The components of vectors not based at the origin are defined similarly, as follows.

Note that this component form may also be written as in (2).

A vector $\mathbf{a} = \overrightarrow{PQ}$ in three-dimensional space, where P is the point (p_1, p_2, p_3) and Q is the point (q_1, q_2, q_3), has **component form**

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}, \tag{3}$$

where $a_1 = q_1 - p_1$, $a_2 = q_2 - p_2$, $a_3 = q_3 - p_3$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the Cartesian unit vectors. The numbers a_1, a_2, a_3 are the (**Cartesian**) **components** of $\mathbf{a}$.

For example, in Figure 14(b), we have $\overrightarrow{OP} = 2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$.

As in two dimensions, the operations of vector algebra can be expressed in terms of components.

Adding and scaling three-dimensional vectors in component form

If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ and m is a scalar, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \tag{4}$$

and

$$m\mathbf{a} = (ma_1)\mathbf{i} + (ma_2)\mathbf{j} + (ma_3)\mathbf{k}. \tag{5}$$

The magnitude of a vector in terms of its components a_1, a_2, a_3 can be found using Pythagoras' theorem (see Figure 20).

The length ON is $\sqrt{a_1^2 + a_2^2}$, and $OA^2 = ON^2 + NA^2$. But $OA = |\mathbf{a}|$, thus

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

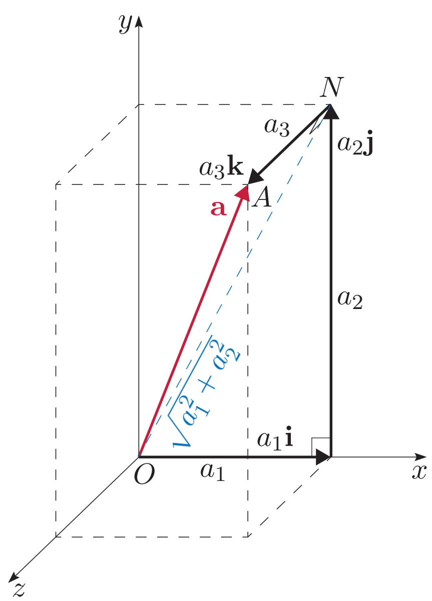


Figure 20 Finding the magnitude of a vector

This can be summarised as follows.

Magnitude of a three-dimensional vector in component form

If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, then its magnitude is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{6}$$

A unit vector in the direction of $\mathbf{a}$ is

$$\widehat{\mathbf{a}} = \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\,\mathbf{i} + \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\,\mathbf{j} + \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\,\mathbf{k}. \tag{7}$$

Exercise 9

Consider the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{k}$.

(a) Express $\mathbf{d} = 2\mathbf{a} - 3\mathbf{b}$ and $\mathbf{e} = \mathbf{a} - 2\mathbf{b} + 4\mathbf{c}$ in component form.

(b) Find the magnitudes of the vectors $\mathbf{d}$ and $\mathbf{e}$.

(c) Evaluate $|\mathbf{a}|$, and write down a unit vector in the direction of $\mathbf{a}$.

(d) Find the components of a vector $\mathbf{x}$ such that $\mathbf{a} + \mathbf{x} = \mathbf{b}$.

Exercise 10

Write the vectors $\mathbf{0}$, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as column vectors in three dimensions.

Vector equation of a straight line

One useful application of position vectors (in two or three dimensions) is in obtaining a vector equation of a straight line.

Example 2

Find the position vector of a point P lying on a straight-line segment AB in terms of the position vectors of A and B.

Solution

Let P be any point on AB (see Figure 21). The position vector $\overrightarrow{OP}$ of P relative to the origin can also be written, using the triangle rule, as

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}.$$

Now $\overrightarrow{AP} = s\,\overrightarrow{AB}$, for some number s, and the point P traces out the line segment AB as s varies from 0 to 1. Thus the straight-line segment AB is described by the vector equation

$$\overrightarrow{OP} = \overrightarrow{OA} + s\,\overrightarrow{AB} \quad (0 \le s \le 1).$$

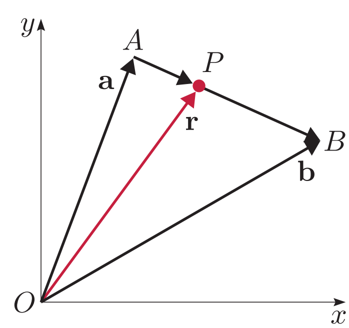


Figure 21 Finding a general point on a straight line

Writing $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{r} = \overrightarrow{OP}$, and noting (using the triangle rule) that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$, this equation can also be written as

$$\mathbf{r} = \mathbf{a} + s(\mathbf{b} - \mathbf{a}) = (1 - s)\mathbf{a} + s\mathbf{b} \quad (0 \le s \le 1).$$

Note that if the parameter s in Example 2 is allowed to range over all the real numbers ($-\infty < s < \infty$), then the point P traces out the entire straight line of which AB is a segment. Also note that the ideas in Example 2 are easily extended to three dimensions.

Vector equation of a straight line

If A and B are any two distinct points on a straight line in space, with position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, with respect to some given origin, then the **vector equation of the straight line** is

$$\mathbf{r} = (1 - s)\mathbf{a} + s\mathbf{b} \quad (-\infty < s < \infty), \tag{8}$$

where $\mathbf{r}$ represents the position vector of any point on the line.

If $0 \le s \le 1$, then the equation represents only the line segment AB.

In Cartesian coordinate form, with $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, the equation of the line is

$$x = a_1 + s(b_1 - a_1), \quad y = a_2 + s(b_2 - a_2), \quad z = a_3 + s(b_3 - a_3).$$

This is called the **parametric form of the straight line**, where the coordinates are expressed in terms of the parameter s. We can recover a more familiar form of the straight line by writing everything in terms of s as

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} (= s).$$

Exercise 11

(a) Write down, in component form, the vector equation of the straight line on which lie the points with Cartesian coordinates $(1, 1, 2)$ and $(2, 3, 1)$.

(b) Find the coordinates of the point where the line cuts the (x, z)-plane.

Exercise 12

Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(a) Find the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, and describe the direction of $\mathbf{a}$.

(b) Find the vectors $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} - \mathbf{b}$ and $\mathbf{c} + 2\mathbf{b} - 3\mathbf{a}$ in component form.

(c) What is the endpoint Q of the displacement represented by the vector $2\mathbf{a} - \mathbf{b}$ if $(0, 2, 3)$ is its beginning point P?

Exercise 13

(a) A straight line in Cartesian form is expressed as

$$\frac{x+1}{3} = \frac{y-8}{-2} = 1 - z.$$

Express this equation in parametric and vector form. If A is the point corresponding to $s = 0$, and B is the point corresponding to $s = 1$, determine the component form of $\overrightarrow{AB}$ and the distance between A and B.

(b) (i) Determine the vector position of M, the point midway between A and B.

(ii) Determine the vector position of N, the point that divides the line AB in the ratio $3:2$.

(c) Find the vector equation of the straight line through the points $(-1,-2,0)$ and $(2,1,-1)$.

(d) Determine whether the lines in parts (a) and (c) intersect. If so, determine the coordinates of the point of intersection.

2.2 The dot product

We make the following definition.

The **dot product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta, \qquad (9)$$

where θ $(0 \le \theta \le \pi)$ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$ (see Figure 22).

The product $\mathbf{a} \cdot \mathbf{b}$ is read as 'a dot b'.

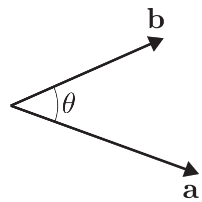


Figure 22 The angle between two vectors

The dot product of two vectors is a *scalar quantity*, that is, it is a real number: $\mathbf{a} \cdot \mathbf{b}$ is the product of the three scalars $|\mathbf{a}|$, $|\mathbf{b}|$ and $\cos\theta$. So the operation of the dot product combines two vectors to define a scalar, and for this reason the dot product is also called the **scalar product**. The angle θ lies in the range $0 \le \theta \le \pi$, and the value of $\mathbf{a} \cdot \mathbf{b}$ is:

- positive for $0 \le \theta < \frac{\pi}{2}$, i.e. when θ is an acute angle
- negative for $\frac{\pi}{2} < \theta \le \pi$, i.e. when θ is obtuse
- zero for $\theta = \frac{\pi}{2}$, i.e. when θ is a right angle.

It is important, when writing a dot product, to make sure that the dot between the vectors is clear.

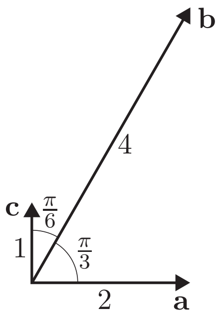


Figure 23

Exercise 14

Three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of magnitudes 2, 4 and 1 units, respectively, lying in the same plane, are represented by arrows as shown in Figure 23. The angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$ radians, and that between the vectors $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{6}$ radians. Use the definition of the dot product to find the values of $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \cdot \mathbf{c}$, $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{b}$.

This exercise demonstrates two important properties of the dot product.

- If two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other (i.e. the angle between them is $\frac{\pi}{2}$ radians), then since $\cos \frac{\pi}{2} = 0$,

  $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \tfrac{\pi}{2} = 0.$$

- The dot product of a vector with itself gives the square of the magnitude of the vector, that is,

  $$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|\,|\mathbf{a}| \cos 0 = |\mathbf{a}|^2.$$

The converse of the first property also holds: if $\mathbf{a}$ and $\mathbf{b}$ are two *non-zero* vectors such that $\mathbf{a} \cdot \mathbf{b} = 0$, then the definition of the dot product tells us that $\cos\theta = 0$; therefore $\theta = \frac{\pi}{2}$ and the vectors are perpendicular.

In a product of real numbers, $xy = 0$ implies that either x or y (or both) is zero. In contrast, for the dot product, $\mathbf{a} \cdot \mathbf{b} = 0$ gives an extra possibility: either $\mathbf{a}$ or $\mathbf{b}$ (or both) is the zero vector, or the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{2}$ radians.

Properties of the dot product

The following are some important properties of the dot product of two vectors. They include the rules for manipulating dot products in algebraic expressions.

These properties can all be derived from the definition of the dot product, but the derivations are not given here.

Properties of the dot product

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors, and let m be a scalar.

- $\mathbf{a} \cdot \mathbf{b}$ is a scalar.
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, i.e. the dot product is commutative.
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$, i.e. the dot product is distributive over vector addition.
- $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$, i.e. a scalar can be 'moved through' a dot product.
- If neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector, then $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ is perpendicular to $\mathbf{b}$.
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

The following example shows how these properties can be used to simplify expressions.

Example 3

Expand the expression $\mathbf{x} \cdot \mathbf{y}$, given that $\mathbf{x} = 2\mathbf{u} + \mathbf{v}$ and $\mathbf{y} = \mathbf{u} - 5\mathbf{v}$. Calculate its value when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors.

Solution

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= (2\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 5\mathbf{v}) \\ &= 2(\mathbf{u} \cdot \mathbf{u}) - 10(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{u} - 5(\mathbf{v} \cdot \mathbf{v}) \\ &= 2(\mathbf{u} \cdot \mathbf{u}) - 9(\mathbf{u} \cdot \mathbf{v}) - 5(\mathbf{v} \cdot \mathbf{v}).\end{aligned}$$

Now $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$ and $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = 1$ when $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Furthermore, $\mathbf{u} \cdot \mathbf{v} = 0$ when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors. So when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors, we have

$$\mathbf{x} \cdot \mathbf{y} = 2 - 0 - 5 = -3.$$

Exercise 15

(a) Expand the expression $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$.

(b) Expand the expression $|\mathbf{a} + \mathbf{b}|^2$.

Recall that $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

Exercise 16

Suppose that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular unit vectors.

(a) Find the value of m such that the two vectors $2\mathbf{a} + 3\mathbf{b}$ and $m\mathbf{a} + \mathbf{b}$ are perpendicular.

(b) Find the value of $|\mathbf{c}|$ if $\mathbf{c} = 3\mathbf{a} + 5\mathbf{b}$.

A word of caution: $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is not in general the same as $\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$. The vector $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is a scaling of $\mathbf{c}$ by the number $\mathbf{a} \cdot \mathbf{b}$, whereas $\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ is a scaling of $\mathbf{a}$ by the number $\mathbf{b} \cdot \mathbf{c}$. Clearly these two vectors are not generally even parallel, let alone equal. For example, if $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$, then

$$(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (\mathbf{i} \cdot \mathbf{i})\mathbf{j} = \mathbf{j} \quad \text{but} \quad \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = \mathbf{i}(\mathbf{i} \cdot \mathbf{j}) = \mathbf{0}.$$

In general, if m is a scalar and $\mathbf{a}$ is a vector, we can write $m\mathbf{a}$ or $\mathbf{a}m$ as is convenient, although $m\mathbf{a}$ is more usual. Thus $\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ means the same as $(\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.

Component form of the dot product

We saw in Section 2 that an arbitrary vector $\mathbf{a}$ in three dimensions may be expressed in terms of the Cartesian unit vectors as

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

With this representation, vector addition and scaling become simple algebraic operations without any reference to diagrams. The definition of the dot product was expressed in terms of the magnitudes of two vectors and the angle between them. We will now see how to express the dot product in terms of components of vectors.

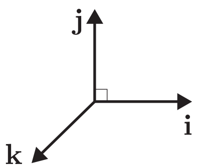


Figure 24 The Cartesian unit vectors ($\mathbf{k}$ points out of the plane of the page)

First, observe that by definition, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors and are perpendicular to one another (see Figure 24). Thus

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{i}=0,\quad \mathbf{i}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0,\quad \mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}=0,$$
$$\mathbf{i}\cdot\mathbf{i}=1,\quad \mathbf{j}\cdot\mathbf{j}=1,\quad \mathbf{k}\cdot\mathbf{k}=1.$$

If two vectors $\mathbf{a}$ and $\mathbf{b}$ have component forms $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$ and $\mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ may be written as

$$(a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k})\cdot(b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}).$$

We can now apply properties of the dot product and the above rules for combining $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to this expression to obtain a very simple formula for the dot product of vectors in component form. Specifically, we have

$$\begin{aligned}&(a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k})\cdot(b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k})\\&\quad=a_1b_1(\mathbf{i}\cdot\mathbf{i})+a_1b_2(\mathbf{i}\cdot\mathbf{j})+a_1b_3(\mathbf{i}\cdot\mathbf{k})\\&\qquad+a_2b_1(\mathbf{j}\cdot\mathbf{i})+a_2b_2(\mathbf{j}\cdot\mathbf{j})+a_2b_3(\mathbf{j}\cdot\mathbf{k})\\&\qquad+a_3b_1(\mathbf{k}\cdot\mathbf{i})+a_3b_2(\mathbf{k}\cdot\mathbf{j})+a_3b_3(\mathbf{k}\cdot\mathbf{k})\\&\quad=a_1b_1+a_2b_2+a_3b_3.\end{aligned}$$

This extremely important formula is worth remembering.

Component form of the dot product

If $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$ and $\mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is a scalar, given by

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3. \tag{10}$$

In terms of vectors,

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3. \tag{11}$$

Exercise 17

If $\mathbf{a}=4\mathbf{i}+\mathbf{j}-5\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-3\mathbf{j}+\mathbf{k}$, show that $\mathbf{a}\cdot\mathbf{b}=-4$. What does the negative sign tell us?

Angle between two vectors

The component form of the dot product has an important application in calculating the angle between two vectors. You have already seen that if $\mathbf{a}\cdot\mathbf{b}=0$ and neither $\mathbf{a}$ nor $\mathbf{b}$ is zero, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. For instance, if $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$, then $\mathbf{a}\cdot\mathbf{b}=(2\times2)+(-1\times3)+(1\times-1)=0$, so the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/2$ radians. In general, the equation defining the dot product of $\mathbf{a}$ and $\mathbf{b}$, that is, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta$, gives the following simple expression for finding the angle between $\mathbf{a}$ and $\mathbf{b}$.

Angle between two vectors

The angle θ between any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}, \tag{12}$$

where $0 \le \theta \le \pi$.

Example 4

Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \sqrt{2}\mathbf{k}$.

(a) Find the angle between the vector $\mathbf{a}$ and the x-axis.

(b) Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

(c) Show that $\mathbf{c}$ is perpendicular to $\mathbf{a}$.

Solution

(a) The direction of the x-axis is the same as the direction of $\mathbf{i}$, and the angle θ between $\mathbf{a}$ and $\mathbf{i}$ is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}|\,|\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} = \frac{1}{\sqrt{1+1+2}} = \frac{1}{2}.$$

Thus the angle between $\mathbf{a}$ and the x-axis is $\pi/3$ radians.

(b) We have $|\mathbf{a}| = \sqrt{1+1+2} = 2$, $|\mathbf{b}| = \sqrt{4+1+1} = \sqrt{6}$ and

$$\mathbf{a}\cdot\mathbf{b} = (1\times 2) + (1\times 1) + (\sqrt{2}\times 1) = 3 + \sqrt{2}.$$

Therefore the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\theta = \frac{3+\sqrt{2}}{2\times\sqrt{6}} = 0.901\,05,$$

so $\theta = 0.4486$ (radians).

(c) To test whether $\mathbf{a}$ and $\mathbf{c}$ are perpendicular, we calculate their dot product:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{c} &= (\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k})\cdot(\mathbf{i} + \mathbf{j} - \sqrt{2}\mathbf{k})\\ &= (1\times 1) + (1\times 1) + (\sqrt{2}\times\sqrt{2})\\ &= 0.\end{aligned}$$

Since $\mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{a}$ and $\mathbf{c}$ are non-zero vectors, $\mathbf{c}$ is perpendicular to $\mathbf{a}$.

Exercise 18

Consider the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

Find the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, and the angle between them.

Resolving a vector into components

The dot product has a useful geometric interpretation.

Exercise 19

If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, find the values of $\mathbf{a} \cdot \mathbf{i}$, $\mathbf{a} \cdot \mathbf{j}$ and $\mathbf{a} \cdot \mathbf{k}$.

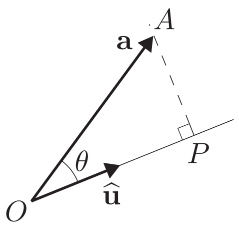


Figure 25 Finding the component of a vector in an arbitrary direction

The solution to Exercise 19 shows the important fact that the **i**-component of any vector **a** may be found by taking the dot product $\mathbf{a} \cdot \mathbf{i}$. The **j**- and **k**-components can be found similarly (by taking dot products with **j** and **k**, respectively).

We can also find the components of a vector in other directions. Suppose that a vector **a**, represented by $\overrightarrow{OA}$, makes an angle θ with a *unit vector* $\widehat{\mathbf{u}}$ (see Figure 25). Draw the line AP perpendicular to the direction of $\widehat{\mathbf{u}}$. Then the distance OP is seen from simple trigonometry to be $|\mathbf{a}|\cos\theta$. Now observe that the dot product of **a** and $\widehat{\mathbf{u}}$ is

$$\mathbf{a} \cdot \widehat{\mathbf{u}} = |\mathbf{a}|\,|\widehat{\mathbf{u}}|\cos\theta = \pm|OP| \quad (\text{since } |\widehat{\mathbf{u}}| = 1).$$

Note that $\mathbf{a} \cdot \widehat{\mathbf{u}}$ will be negative if $\theta > \frac{\pi}{2}$, i.e. if P and $\widehat{\mathbf{u}}$ lie on opposite sides of O.

The signed distance $\pm|OP|$ represents the *component* of **a** in the direction of $\widehat{\mathbf{u}}$.

The **component** of a vector **a** in the direction of an arbitrary *unit vector* $\widehat{\mathbf{u}}$ is $\mathbf{a} \cdot \widehat{\mathbf{u}}$.

Exercise 20

Consider the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

(a) Which of the vectors

$$\mathbf{c} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{d} = -2\mathbf{i} + \mathbf{k}, \quad \mathbf{e} = -\mathbf{i} - \mathbf{j} - \mathbf{k},$$

is perpendicular to **a**?

(b) Find the component of the vector $\mathbf{a} + 2\mathbf{b}$ in the direction of the line joining the origin to the point $(1, 1, 1)$.

Resolving vectors will be a vital technique in this and subsequent units, and sometimes you will need to be able to resolve a vector into components in directions other than horizontal and vertical.

The dot product method of obtaining components always works, but a geometric view is also useful. This follows because the component of a vector **a** in the direction of a unit vector $\widehat{\mathbf{u}}$ is

$$\mathbf{a} \cdot \widehat{\mathbf{u}} = |\mathbf{a}|\cos\theta,$$

where θ is the angle between **a** and $\widehat{\mathbf{u}}$. We summarise the method as a procedure.

Procedure 1 Resolving a vector into components

Given a vector $\mathbf{a}$ and a unit vector $\widehat{\mathbf{u}}$, to find the component of $\mathbf{a}$ in the direction of $\widehat{\mathbf{u}}$, carry out the following steps.

1. Find (usually from a diagram) the angle θ between $\mathbf{a}$ and $\widehat{\mathbf{u}}$ (with $0 \le \theta \le \pi$).
2. The component of the vector $\mathbf{a}$ in the direction of the unit vector $\widehat{\mathbf{u}}$ is $|\mathbf{a}| \cos\theta$.
3. If necessary (e.g. if $\theta > \pi/2$), use trigonometric formulas from the Handbook to simplify the result.

Example 5

Figure 26 shows a line inclined at an angle α to the x-axis, and unit vectors $\mathbf{i}$ and $\mathbf{j}$ aligned along and perpendicular to the line. The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ have magnitudes 1, 1.5, 1.5 and 2, respectively.

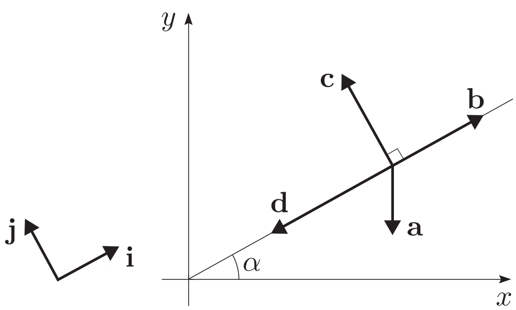


Figure 26

Resolve each of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ into their $\mathbf{i}$- and $\mathbf{j}$-components.

Solution

We must find the angles between $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and the unit vectors $\mathbf{i}$ and $\mathbf{j}$. First notice that $\mathbf{b}$ points in the direction $\mathbf{i}$, so $\mathbf{b} = 1.5\mathbf{i}$. Similarly, $\mathbf{d}$ points in the direction $-\mathbf{i}$, so $\mathbf{d} = -2\mathbf{i}$. Also, $\mathbf{c}$ points in the direction $\mathbf{j}$, so $\mathbf{c} = 1.5\mathbf{j}$.

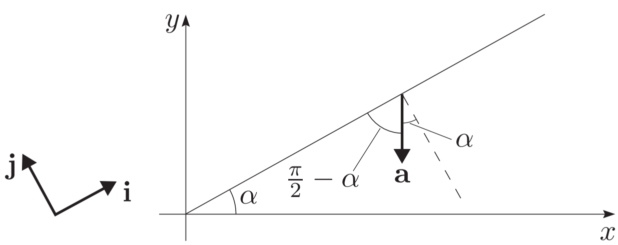


Figure 27

The remaining vector, $\mathbf{a}$, makes an angle $\frac{\pi}{2} - \alpha$ with $-\mathbf{i}$, and an angle α with $-\mathbf{j}$ (see Figure 27). Hence the $\mathbf{i}$-component of $\mathbf{a}$ is

$$-|\mathbf{a}| \cos\left(\tfrac{\pi}{2} - \alpha\right) = -\sin\alpha,$$

and the $\mathbf{j}$-component of $\mathbf{a}$ is

$$-|\mathbf{a}| \cos\alpha = -\cos\alpha.$$

Therefore

$$\mathbf{a} = -\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j}.$$

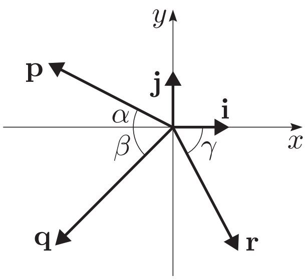


Figure 28

Exercise 21

The vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in Figure 28 have magnitudes 2.5, 3 and 2.5, respectively. Resolve $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ into their $\mathbf{i}$- and $\mathbf{j}$-components.

2.3 The cross product

You have seen that the dot product of two vectors is a scalar (i.e. a real number). In contrast, the *cross product* of two vectors is a *vector*, whose direction is perpendicular to both. The cross product has numerous applications in geometry and mechanics, as you will see later in the module.

The product $\mathbf{a} \times \mathbf{b}$ is read as 'a cross b'.

The **cross product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\,|\mathbf{b}| \sin\theta)\,\widehat{\mathbf{n}},$$

where θ ($0 \leq \theta \leq \pi$) is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$, and $\widehat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose sense is given by the right-hand grip rule as shown in Figure 29(a).

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ lies in the range $0 \leq \theta \leq \pi$, so $\sin\theta \geq 0$ and hence $|\mathbf{a}|\,|\mathbf{b}| \sin\theta \geq 0$. So the cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector with magnitude $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$ and direction defined by $\widehat{\mathbf{n}}$. The direction of $\widehat{\mathbf{n}}$ is the direction in which the fingers in the grip in Figure 29(a) would advance when turned from $\mathbf{a}$ towards $\mathbf{b}$ through the angle θ. Notice that $\widehat{\mathbf{n}}$ is not defined if $\mathbf{a}$ and $\mathbf{b}$ are parallel or if $\mathbf{a}$ or $\mathbf{b}$ is the zero vector; but in these cases $|\mathbf{a}|\,|\mathbf{b}| \sin\theta = 0$, so we take $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. The cross product is also called the **vector product**, which stresses the fact that $\mathbf{a} \times \mathbf{b}$ is a vector.

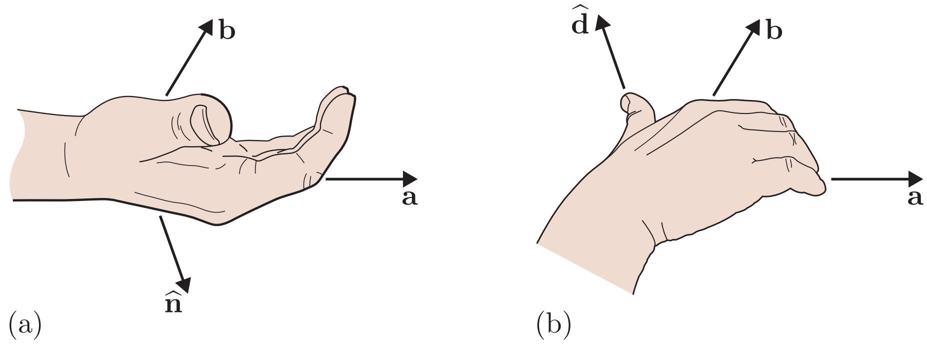


Figure 29 Using the right-hand grip rule for the cross product: (a) $\mathbf{a} \times \mathbf{b}$; (b) $\mathbf{b} \times \mathbf{a}$

The order of writing down $\mathbf{a}$ and $\mathbf{b}$ is very important. According to the right-hand grip rule, $\mathbf{b} \times \mathbf{a}$ is a vector in the direction opposite to $\mathbf{a} \times \mathbf{b}$ (see Figure 29(b)):

$$\mathbf{b} \times \mathbf{a} = (|\mathbf{b}|\,|\mathbf{a}| \sin\theta)\, \widehat{\mathbf{d}} = -(|\mathbf{b}|\,|\mathbf{a}| \sin\theta)\, \widehat{\mathbf{n}} = -(\mathbf{a} \times \mathbf{b}).$$

Exercise 22

Three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the (x, y)-plane. Their magnitudes are 2, 3 and 4 units, respectively, their directions make angles $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{6}$ radians, respectively, with the positive x-axis, and they have positive $\mathbf{j}$-components. Use the definition of the cross product to find the vectors $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \times \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$.

Exercise 22 illustrates an important property of the cross product. We know that if two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then the angle θ between their directions is zero or π radians, so the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the zero vector, because the magnitude of the vector, that is, $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$, is zero. The converse also holds: if $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then the definition of the cross product tells us that $\sin\theta = 0$; therefore $\theta = 0$ or $\theta = \pi$, and the vectors are parallel. We can also deduce that

$$\mathbf{a} \times \mathbf{a} = \mathbf{0} \quad \text{for any vector } \mathbf{a}.$$

So we can test for perpendicular vectors by using the dot product and for parallel vectors by using the cross product.

Properties of the cross product

The following are some important properties of the cross product of two vectors. They include the rules for manipulating cross products in algebraic expressions.

Properties of the cross product

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors, and let m be a scalar.

- $\mathbf{a} \times \mathbf{b}$ is a vector.
- $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$, i.e. the cross product is *not* commutative – the order does matter.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$, i.e. the cross product is distributive over vector addition.
- $(m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (m\mathbf{b})$, i.e. a scalar can be 'moved through' a cross product.
- If neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- In general, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

These properties can all be derived from the definition of the cross product, but the derivations are not given here.

Component form of the cross product

If two vectors $\mathbf{a}$ and $\mathbf{b}$ have component forms $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, what is the component form of $\mathbf{a} \times \mathbf{b}$? We begin with an exercise.

Exercise 23

(a) Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

(b) Calculate $\mathbf{j} \times \mathbf{i}$, $\mathbf{k} \times \mathbf{j}$ and $\mathbf{i} \times \mathbf{k}$.

(c) Calculate $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$ and $\mathbf{k} \times \mathbf{k}$.

(d) Expand and simplify

$$(\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad (\mathbf{i} \times (\mathbf{i} + \mathbf{k})) - ((\mathbf{i} + \mathbf{j}) \times \mathbf{k}).$$

(e) For two non-zero non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$, simplify

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b}) \quad \text{and} \quad (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}).$$

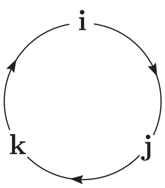


Figure 30 The cyclic pattern for the cross product

The cyclic pattern of the products $\mathbf{i} \times \mathbf{j}$, $\mathbf{j} \times \mathbf{k}$, $\mathbf{k} \times \mathbf{i}$ and of the products $\mathbf{i} \times \mathbf{k}$, $\mathbf{k} \times \mathbf{j}$, $\mathbf{j} \times \mathbf{i}$, as demonstrated in Exercise 23, can be remembered using Figure 30. For example, if we go round the circle clockwise starting at $\mathbf{i}$, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

However, if we go in an anticlockwise direction, the cross products are negative:

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}.$$

Given $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, the cross product $\mathbf{a} \times \mathbf{b}$ may be written as

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}) \times (b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.\end{aligned}$$

We highlight this important formula.

Component form of the cross product

If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \qquad (13)$$

Alternatively,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}. \qquad (14)$$

This formula is not easy to remember or use in this form. Another quick way to evaluate cross products is to use determinants. This method will be introduced in Unit 4 when we discuss determinants. If you already know this method, then we suggest that you continue to use it.

The determinant form of the cross-product $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, can be written as

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} \\ &\quad + (a_3b_1 - a_1b_3)\mathbf{j} \\ &\quad + (a_1b_2 - a_2b_1)\mathbf{k}.\end{aligned}$$

Exercise 24

If

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{c} = -4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k},$$

find $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$. From your results, what can you say about $\mathbf{a}$ and $\mathbf{c}$?

Exercise 25

If

$$\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k},$$

find a unit vector whose direction is perpendicular to the directions of both $\mathbf{a}$ and $\mathbf{b}$.

Geometric applications

We close this subsection with some useful geometric applications of the cross product. The following example is the first step.

Example 6

Any two non-zero and non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ define a parallelogram, as shown in Figure 31. Express the area of the parallelogram in terms of $\mathbf{a} \times \mathbf{b}$.

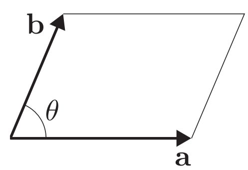


Figure 31 A parallelogram defined by two vectors

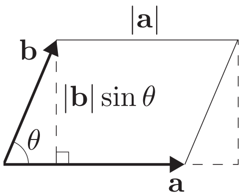


Figure 32 Finding the area of a parallelogram

Solution

The area A of the parallelogram defined by the two vectors $\mathbf{a}$ and $\mathbf{b}$ is the same as the area of the rectangle of height $|\mathbf{b}|\sin\theta$ and width $|\mathbf{a}|$ (see Figure 32). Thus $A = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, and this is the magnitude of $\mathbf{a}\times\mathbf{b}$. So

$$A = |\mathbf{a}\times\mathbf{b}|.$$

Area of a parallelogram

The area of a parallelogram with sides defined by vectors $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.

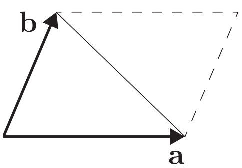


Figure 33 Finding the area of a triangle

This idea is easily extended for the area of a triangle. Any two non-zero non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ define a triangle (see Figure 33). The area of this triangle is half that of the corresponding parallelogram, so it is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

Area of a triangle

The area of a triangle with sides defined by vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

A *parallelepiped* is like a distorted brick. All of its faces are parallelograms.

Using the formula for the area of a parallelogram, we can find the volume of a parallelepiped (see Figure 34).

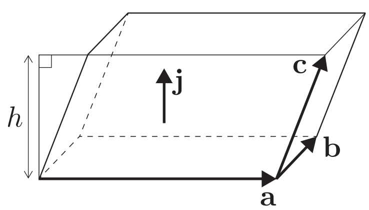


Figure 34 Finding the volume of a parallelepiped

The base is a parallelogram (assumed to be in the (x,z)-plane) defined by the vectors $\mathbf{a}$ and $\mathbf{b}$. The base therefore has an area equal to the magnitude of $\mathbf{a}\times\mathbf{b}$. Now the vertical height h is the component of the vector $\mathbf{c}$ in the direction of the Cartesian unit vector $\mathbf{j}$ pointing vertically upwards, that is, it is the y-component of $\mathbf{c}$, given by $\mathbf{c}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{c}$. So the volume of the parallelepiped is $|\mathbf{a}\times\mathbf{b}|(\mathbf{j}\cdot\mathbf{c})$. But the vector product $\mathbf{a}\times\mathbf{b}$ points vertically upwards and can therefore be expressed as $|\mathbf{a}\times\mathbf{b}|\,\mathbf{j}$. Hence we have

$$|\mathbf{a}\times\mathbf{b}|(\mathbf{j}\cdot\mathbf{c}) = (|\mathbf{a}\times\mathbf{b}|\,\mathbf{j})\cdot\mathbf{c} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.$$

Of course, the scalar $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ can be negative if one of the defining vectors $\mathbf{a}$ or $\mathbf{b}$ is chosen to be in the direction opposite to the one chosen in Figure 34, or if the order of the cross product is reversed. We use modulus signs to ensure that the volume comes out positive:

$$\begin{aligned}\text{volume of parallelepiped} &= \text{base area} \times \text{vertical height } h\\ &= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|.\end{aligned} \tag{15}$$

The scalar quantity $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is an example of a *scalar triple product*.

Exercise 26

Suppose that the vectors $\mathbf{r}$ and $\mathbf{s}$ are directed towards north and north-east, respectively, and define $\mathbf{r} \times \mathbf{s} = \mathbf{t}$.

(a) What is the direction of $\mathbf{t}$?

(b) In what direction is $\mathbf{s} \times \mathbf{r}$?

(c) In what direction is $\mathbf{t} \times \mathbf{r}$?

(d) If $|\mathbf{r}| = |\mathbf{s}| = 1$, what is $|\mathbf{t}|$?

(e) Calculate the vector $\mathbf{t} \times (\mathbf{r} \times \mathbf{s})$.

(f) If $|\mathbf{r}| = |\mathbf{s}| = 1$, what is the value of $\mathbf{r} \cdot \mathbf{s}$?

(g) If $|\mathbf{r}| = |\mathbf{s}| = 1$, what is the value of $\mathbf{s} \cdot (\mathbf{t} \times \mathbf{r})$?

Exercise 27

Find the area of the triangle ABC, where the coordinates of A, B and C are $(2, 1, -3)$, $(1, 0, 2)$ and $(4, -2, -1)$.

3 Modelling forces

This section shows how four common types of force can be modelled: the force of gravity, the force exerted by a surface on an object in contact with it, the tension force due to a string, and the friction force between two surfaces. These forces and the situations in which they occur are modelled and analysed in Subsections 3.2–3.5. In Subsection 3.6 we look at the forces on two particle systems. First, however, we look at one way of modelling the objects on which forces act.

3.1 Particles

When we create a mathematical model, the aim is to simplify the real situation being modelled so that only the essential features are included. This enables us to analyse the situation mathematically. In mechanics, the most important things to model are the forces acting on objects, and throughout this unit and the other mechanics units you will see how to do this. However, we also need to model the objects on which the forces act. The simplest model for an object is a *particle*.

A **particle** is a material object whose size and internal structure may be neglected. It has mass but no size, thus occupies a single point in space. A particle is often represented in diagrams by a black dot.

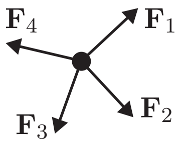


Figure 35 A force diagram

Observation has shown that each force acting on an object can be modelled as acting *at a particular point* on the object, this point being referred to as the **point of action** of the force. In situations where a particle model is appropriate, all the forces acting on the object are modelled as acting through the point in space occupied by the particle. It is conventional to show these forces in diagrams – known as **force diagrams** – by vector arrows whose tails coincide with the particle and whose directions correspond to the directions in which the forces act, as Figure 35 illustrates.

Note that in force diagrams, arrows are usually drawn with arbitrary lengths. This contrasts with the usual convention for vectors, where length indicates magnitude.

When several forces are acting on a particle, observation has shown that the overall effect of these forces can be represented by a single vector given by the sum of the vectors representing the individual forces. In this unit, we deal with objects that do not move, that is, objects in **equilibrium**. For a particle in equilibrium, the forces acting on it must balance each other (or else it would move), so we have the following important condition.

This condition was first stated by Isaac Newton as part of his first law of motion.

We often say that the sum of the forces is zero, with the implication that this means the zero vector.

Equilibrium condition for a particle

A particle subjected to forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ is in equilibrium if the forces sum to the zero vector, that is,

$$\sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0}. \qquad (16)$$

3.2 Weight

When you hold a shoe, your fingers experience a force. The shoe, like all objects, has a force associated with it, and if you do not provide opposition to this force in holding the shoe, it will fall to the ground. But what is the source of the force exerted by the shoe?

This force is due to the attraction of the shoe to the Earth. The force of attraction of objects to the Earth is called the **force of gravity** or the **gravitational force**. The gravitational force acting on a particular object is not constant, but depends on the position of the object relative to the Earth: there is a small variation of this force with height above ground (or depth below ground), and there is an even smaller variation with latitude and longitude. When applied to a particular object, this force is called the *weight* of the object. In this module, we assume that the weight of a particular object is constant near the Earth's surface.

In everyday speech, the words *mass* and *weight* are interchangeable. Mathematically, however, they are different. The **mass** of an object is the amount of matter in the object and is independent of the object's position in the Universe; it is a *scalar* quantity, measured in *kilograms* (kg) in the SI system. The **weight** of an object is the gravitational force on the object, and is dependent on where the object is situated; it is a *vector* quantity, whose magnitude is measured in *newtons* (N) in the SI system and whose direction is downwards towards the centre of the Earth.

Mass and weight are, however, related in that an object of mass m has weight of magnitude mg, where g is a constant known as the **magnitude of the acceleration due to gravity**. Near the Earth's surface, g has the value approximately $9.81\,\mathrm{m\,s^{-2}}$, and we assume this value for g throughout this module. If the Cartesian unit vector $\mathbf{j}$ points vertically upwards from the surface of the Earth, then the weight $\mathbf{W}$ of an object of mass m is $mg(-\mathbf{j})$ (where we need the negative sign because the force of gravity acts vertically downwards, that is, the weight acts vertically downwards).

The relationship between mass and weight is based on Newton's second law of motion, which is discussed in Unit 3.

An object of mass m has weight $\mathbf{W}$ of magnitude $|\mathbf{W}| = mg$, where g is the magnitude of the acceleration due to gravity, with direction towards the centre of the Earth. If the object is modelled as a particle, the force of gravity on the object can be illustrated by the force diagram in Figure 36.

Figure 36 A force diagram illustrating the weight of an object

Recall that a vector can be represented by its magnitude times a unit vector in the direction of the vector, that is, $\mathbf{a} = |\mathbf{a}|\,\widehat{\mathbf{a}}$.

Exercise 28

What is the weight (in newtons) of a particle of mass 3 kg in a coordinate system where the $\mathbf{k}$-direction is vertically downwards?

When modelling forces acting on objects, it is often convenient to define Cartesian unit vectors and to express the force vectors in component form, that is, to resolve the vectors into their components. These Cartesian unit vectors define the directions of the axes in a Cartesian coordinate system, so we often refer to the process of defining Cartesian unit vectors as **choosing axes**.

Exercise 29

Later in this unit we will find it convenient to use axes that are not horizontal and vertical. Express the weight $\mathbf{W}$ of a particle of mass 15 kg in terms of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ both lie in a vertical plane and are oriented as shown in Figure 37.

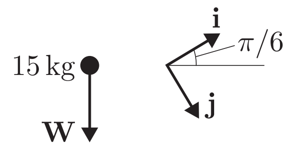


Figure 37 Using axes that are not horizontal and vertical

In Exercise 29, 'nice' angles (i.e. multiples of $\frac{\pi}{6}$ (30°)) were chosen in order to help you to evaluate the sine and cosine involved without having to use a calculator. The sines and cosines of some 'convenient' angles are as follows:

These values are given in the Handbook.

$$\sin 0 = 0, \quad \sin\tfrac{\pi}{6} = \tfrac{1}{2}, \quad \sin\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}, \quad \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}, \quad \sin\tfrac{\pi}{2} = 1,$$
$$\cos 0 = 1, \quad \cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2}, \quad \cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}, \quad \cos\tfrac{\pi}{3} = \tfrac{1}{2}, \quad \cos\tfrac{\pi}{2} = 0.$$

Sometimes, obtuse angles are used; sines and cosines of such angles can be derived from the addition formulas given in the Handbook. For example,

$$\cos\tfrac{2\pi}{3} = \cos\left(\tfrac{\pi}{2} + \tfrac{\pi}{6}\right) = \cos\tfrac{\pi}{2}\cos\tfrac{\pi}{6} - \sin\tfrac{\pi}{2}\sin\tfrac{\pi}{6} = -\tfrac{1}{2}.$$

3.3 Normal reaction

Consider an empty coffee mug resting on a table. Let us model the mug as a particle and consider the forces acting on it. We know that one force, the mug's weight, is acting on the mug. But since the mug is at rest (i.e. not moving), the equilibrium condition for particles tells us that some other force(s) must be acting on the mug (so that all the forces acting on the mug sum to zero). The only possible source for another force on the mug is the table. The force exerted by the table on the mug, and indeed exerted by any surface on an object in contact with it, is called the **normal reaction force** or simply the **normal reaction**.

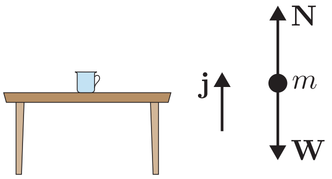


Figure 38 The forces on a mug resting on a table

The situation is illustrated in Figure 38, which shows not only the mug and table, but the corresponding force diagram (plus the Cartesian unit vector $\mathbf{j}$ pointing vertically upwards). The normal reaction force is denoted by $\mathbf{N}$, and the weight of the mug is denoted by $\mathbf{W}$. Using the equilibrium condition for particles, we have

$$\mathbf{W} + \mathbf{N} = \mathbf{0}.$$

If the mug has mass m, then $\mathbf{W} = mg\,(-\mathbf{j})$, hence

$$\mathbf{N} = -\mathbf{W} = mg\,\mathbf{j},$$

that is, the normal reaction is a force acting vertically upwards with the same magnitude as the weight of the mug.

The normal reaction force is remarkable in that it adjusts itself to the magnitude required. For example, if the coffee mug is replaced by a full pot of coffee, then the normal reaction increases (unless the weight of the coffee pot is too much for the table, in which case the table collapses and the pot is no longer at rest). Contrast this with the weight of an object, which is fixed and constant, regardless of what is happening to the object. Our basic modelling assumption is that the magnitude of the normal reaction force is potentially unlimited.

There is a normal reaction force whenever one object (e.g. a mug) presses on another (e.g. a table). Observation has shown that this force acts *normally* (i.e. at a right angle) to the surface at the point of contact between the objects. It therefore does not always act vertically upwards.

This explains the name *normal* reaction force.

For example, if the table on which the mug is resting is on an uneven floor, so that the table top makes an angle θ with the horizontal, then the normal reaction force makes an angle θ with the vertical, as shown in Figure 39. (In such a case there must be other forces acting on the mug if it is to remain in equilibrium. These other forces are discussed later.)

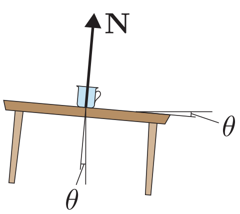


Figure 39 The normal reaction force acts at right angles to the surface

3.4 Tension

Consider a lamp hanging from a ceiling on an electric cable. Let us model the lamp as a particle. As in the case of the mug and table in the previous subsection, we know that there is a weight associated with the lamp, and that since the lamp is at rest, by the equilibrium condition some other force(s) must be acting on it. The only possible source for another force is the cable, so the cable must exert a force on the lamp. The force exerted by the cable on the lamp is a vector quantity called the **tension force**.

Tension forces occur whenever objects are tautly joined, for example by cables, ropes, strings or threads. These cables and ropes can be modelled in different ways. For example, if we want to model the ceiling lamp and are interested only in the force in the cable, then we can model the cable as a **model string**, defined as an object possessing length, but no area, volume or mass, and which does not stretch (i.e. it is inextensible). On the other hand, if we are interested in how much the cable stretches under the weight of the lamp, then we can model the cable as a *model spring*, which has properties similar to those of a model string (i.e. it has no area, volume or mass), but allows extension. In this unit we consider only strings.

Springs are discussed in Unit 9.

The ceiling lamp example is illustrated in Figure 40. The tension force due to the model string is denoted by $\mathbf{T}$, and the weight of the lamp is denoted by $\mathbf{W}$. In a manner similar to the case of normal reaction forces, the equilibrium condition for particles gives

$$\mathbf{W} + \mathbf{T} = \mathbf{0}.$$

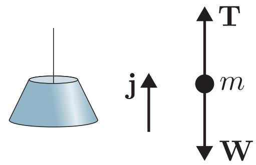


Figure 40 Modelling a lamp hanging from a ceiling

If the lamp has mass m, then $\mathbf{W} = mg\,(-\mathbf{j})$, hence

$$\mathbf{T} = -\mathbf{W} = mg\,\mathbf{j},$$

that is, tension is a force acting vertically upwards (along the length of the model string) with the same magnitude as the weight of the lamp.

We assume that the tension force due to a model string acts along the length of the string and away from the point of its attachment to an object. As in the case of a normal reaction, the magnitude of this force (a scalar quantity, often referred to as the *tension in the string*) depends on the requirements necessary to maintain equilibrium, so it is potentially unlimited. (In reality, a string can exert only a certain tension force before it breaks, but a *model* string supports an unlimited tension force.)

A **model string** is an object with a fixed finite length, and no area, volume or mass, that exerts a force at the point of attachment.

The **tension force due to a string** is a vector quantity that acts along the length of the string away from the point of attachment.

As in the case of normal reaction forces, the tension force due to a string need not be vertically upwards, as the following example illustrates.

Example 7

A hanging flower basket of mass 4 kg is suspended by one cord from a porch and tied by another cord to the wall, as shown in Figure 41. Model the basket as a particle and the cords as model strings. What are the magnitudes of the tension forces due to the cords?

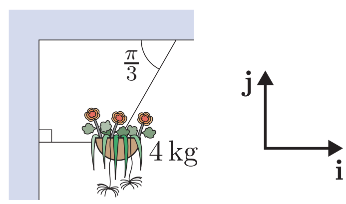


Figure 41 A flower basket hanging in a porch

Solution

We choose axes as shown in Figure 41. Note that we need choose only two axes because all the forces act in the same vertical plane. Denoting the tension forces by $\mathbf{T}_1$ and $\mathbf{T}_2$, and the weight of the basket by $\mathbf{W}$, we have the force diagram shown in Figure 42.

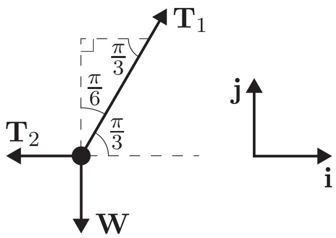


Figure 42 The force diagram for the hanging basket

In the diagram, the angle between the vector $\mathbf{T}_1$ and the unit vector $\mathbf{j}$ is calculated by imagining the right-angled triangle shown, and using the fact that the angles of a triangle sum to π radians. The angle between the vectors $\mathbf{T}_1$ and $\mathbf{i}$ is $\frac{\pi}{3}$.

The equilibrium condition for particles tells us that

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{0}. \tag{17}$$

To progress further, we need to express the three forces in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. To do this, we apply Procedure 1, the technique for resolving vectors. Starting with the weight $\mathbf{W}$, where $|\mathbf{W}| = 4g$ and $\mathbf{W}$ acts in the direction $-\mathbf{j}$, we have

$$\mathbf{W} = -4g\,\mathbf{j}.$$

Similarly, the tension force $\mathbf{T}_1$ has magnitude $|\mathbf{T}_1|$ in the direction $\widehat{\mathbf{u}} = \cos\frac{\pi}{3}\,\mathbf{i} + \sin\frac{\pi}{3}\,\mathbf{j}$ and can be expressed as

A vector $\mathbf{u} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ has magnitude

$$|\mathbf{u}| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$$

and is thus a unit vector.

$$\mathbf{T}_1 = |\mathbf{T}_1|\cos\tfrac{\pi}{3}\,\mathbf{i} + |\mathbf{T}_1|\sin\tfrac{\pi}{3}\,\mathbf{j} = \tfrac{1}{2}|\mathbf{T}_1|\,\mathbf{i} + \tfrac{\sqrt{3}}{2}|\mathbf{T}_1|\,\mathbf{j}.$$

Finally, the tension force $\mathbf{T}_2$ has magnitude $|\mathbf{T}_2|$ in the direction $-\mathbf{i}$ and can be written as

$$\mathbf{T}_2 = -|\mathbf{T}_2|\,\mathbf{i}.$$

Substituting in equation (17), the equilibrium condition is

$$\tfrac{1}{2}|\mathbf{T}_1|\,\mathbf{i} + \tfrac{\sqrt{3}}{2}|\mathbf{T}_1|\,\mathbf{j} - |\mathbf{T}_2|\,\mathbf{i} - 4g\,\mathbf{j} = \mathbf{0}.$$

Either by separating out the $\mathbf{i}$- and $\mathbf{j}$-components, or equivalently, taking the dot product with $\mathbf{i}$ and $\mathbf{j}$ in turn, gives the two scalar equations

$$\tfrac{1}{2}|\mathbf{T}_1| - |\mathbf{T}_2| = 0, \tag{18}$$

$$\tfrac{\sqrt{3}}{2}|\mathbf{T}_1| - 4g = 0. \tag{19}$$

Equation (19) gives

$$|\mathbf{T}_1| = 8g/\sqrt{3} \simeq 45.31.$$

Substituting this into equation (18) gives

$$|\mathbf{T}_2| = 4g/\sqrt{3} \simeq 22.66.$$

So the model predicts that the tension force due to the cord from the porch has magnitude about 45.3 N, and the tension force due to the cord from the wall has magnitude about 22.7 N, both correct to one decimal place.

The procedure that was used in Example 7 can be used to solve many problems in statics, and may be summarised as follows.

Procedure 2 Solving statics problems for particles

Given a statics problem, perform some or all of the following steps.

1. Draw a sketch of the physical situation, and annotate it with any relevant information. ◀ Draw picture ▶
2. Choose axes, and mark them on your sketch. ◀ Choose axes ▶
3. Draw a force diagram for each particle. ◀ Draw force diagram(s) ▶
4. Use the equilibrium condition on each particle and any other appropriate law(s) to obtain equation(s). ◀ Apply law(s) ▶
5. Solve the equation(s). ◀ Solve equation(s) ▶
6. Interpret the solution in terms of the original problem. ◀ Interpret solution ▶

In this unit, the steps in this procedure will often be identified (using the marginal abbreviations shown above) in the solutions to examples and exercises. However, the procedure is intended to be a guide rather than a rigid set of rules. For example, if it is not obvious which set of axes to choose, then draw the force diagram first, and the best choice may become more apparent. Try using the procedure in the following exercise.

Exercise 30

During December, a large plastic Christmas tree of mass 10 kg is suspended by its apex using two ropes attached to buildings either side of the high street of Treppendorf. The ropes make angles of $\pi/6$ and $\pi/4$ with the horizontal. Model the Christmas tree as a particle and the ropes as model strings. What are the magnitudes of the tension forces due to the two ropes?

3.5 Friction

Consider a book resting on a horizontal surface such as a table. There are two forces acting on the book: the weight downwards and the normal reaction upwards. Suppose that you push the book gently sideways (see Figure 43). If you do not push hard enough, the book will not move; it will remain in equilibrium. The force opposing motion along the surface is known as the **friction force**. It is considered to act parallel to the surface, that is, at right angles to the normal reaction, and in a direction that opposes any (possible) motion along that surface. Modelling the book as a particle, and denoting the pushing force by $\mathbf{P}$, the friction force by $\mathbf{F}$, the weight by $\mathbf{W}$ and the normal reaction by $\mathbf{N}$, the force diagram for this example is shown in Figure 44.

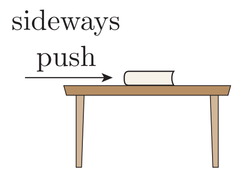


Figure 43 Pushing a book resting on a table

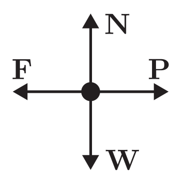


Figure 44 The force diagram for the book being pushed

Friction forces are caused by the roughness of even seemingly very smooth surfaces – a roughness that serves to inhibit the smooth movement of one surface over another. So friction forces are present only where there is movement or the possibility of movement. There is no friction force present when an object is resting on a horizontal surface, where the only two forces acting on the object are its weight and the normal reaction. But when an object is being pushed or pulled, or is resting on a sloping surface, then a friction force is present (see Figure 45).

In this unit we consider only cases where objects remain at rest, so that there is only the *possibility* of movement. Friction in cases where there *is* movement is considered in Unit 3.

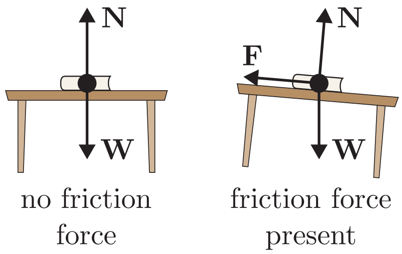


Figure 45 The possible forces acting on the book

Unlike the normal reaction, which is potentially unlimited in magnitude, there is a limit to the magnitude of the friction force; if this limit is reached, then slipping occurs. The limiting value of the magnitude of the friction force depends almost entirely on the materials of the two surfaces and on the magnitude of the normal reaction force between them. It does not usually depend on the area of contact between the two surfaces, or on the angle at which the two surfaces are inclined to the horizontal. Experiments show that the limiting value of the magnitude of the friction force $\mathbf{F}$ (which *just* prevents slipping for two given surfaces) is approximately proportional to the magnitude of the normal reaction force $\mathbf{N}$ between the two surfaces. So on the verge of slipping, we have the scalar equation

$$|\mathbf{F}| = \mu|\mathbf{N}|,$$

where μ is the **coefficient of static friction**, which depends on the materials of the two surfaces. Some approximate values of μ for different materials are given in Table 1.

Table 1 Approximate coefficients of static friction

Surface	μ
Steel on steel (dry)	0.74
Steel on steel (oiled)	0.14
Plastic on plastic	0.35
Rubber on tarmac	0.85
Steel on wood	0.55
Wood on wood	0.42

Example 8

A steel fork of mass 0.05 kg rests on a horizontal wooden table. Model the fork as a particle. What is the maximum sideways force that can be applied before the fork starts to move?

Solution

The situation is illustrated in Figure 46. Since all the forces act in a vertical plane, we can choose axes as shown. The force diagram is also shown in the figure, where **F** is the friction force, **P** is the sideways force, **W** is the weight, and **N** is the normal reaction. The obvious choice for the directions of the unit vectors **i** and **j** is horizontal and vertical, respectively, as all the forces act in these directions.

◀ Draw picture ▶

◀ Choose axes ▶

◀ Draw force diagram(s) ▶

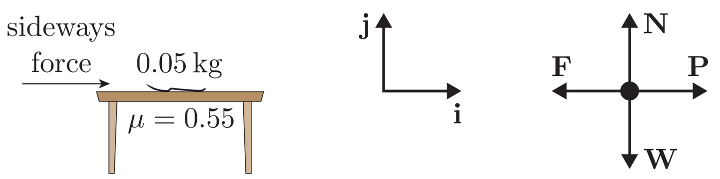


Figure 46 A force acting on a fork on a table, and the force diagram

The equilibrium condition for particles gives

◀ Apply law(s) ▶

$$\mathbf{F} + \mathbf{N} + \mathbf{P} + \mathbf{W} = \mathbf{0}. \tag{20}$$

To be able to use equation (20), we need to express the forces in terms of the unit vectors **i** and **j**. Looking at Figure 46, the forces can be written in component form as

◀ Solve equation(s) ▶

$$\mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}), \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{P} = |\mathbf{P}|\,\mathbf{i}, \quad \mathbf{W} = |\mathbf{W}|\,(-\mathbf{j}).$$

Resolving equation (20) in the **i**-direction gives

$$-|\mathbf{F}| + 0 + |\mathbf{P}| + 0 = 0,$$

so (as expected)

$$|\mathbf{F}| = |\mathbf{P}|.$$

Resolving equation (20) in the **j**-direction gives

$$|\mathbf{N}| = |\mathbf{W}|.$$

When the fork is on the point of moving (slipping), we have

$$|\mathbf{F}| = \mu|\mathbf{N}|,$$

where $\mu = 0.55$ is the coefficient of static friction for steel on wood.

Therefore when the fork is on the point of moving,

$$|\mathbf{P}| = |\mathbf{F}| = \mu|\mathbf{N}| = \mu|\mathbf{W}| = 0.55 \times 0.05g = 0.0275g \simeq 0.27.$$

So the model predicts that a sideways force of magnitude 0.27 N, correct to two decimal places, can be applied without moving the fork.

◀ Interpret solution ▶

Here is a summary of how we go about modelling problems that involve static friction, that is, problems involving friction but no motion.

Modelling static friction

Consider two surfaces in contact.

- The friction force $\mathbf{F}$ acts in a direction perpendicular to the normal reaction $\mathbf{N}$ between the surfaces and opposite to any possible motion along the common tangent to the surfaces.
- $|\mathbf{F}| \leq \mu|\mathbf{N}|$, where μ is a constant called the coefficient of static friction for the two surfaces involved.
- $|\mathbf{F}| = \mu|\mathbf{N}|$ when the object is on the verge of slipping. This equality is sometimes referred to as describing a situation of **limiting friction**.
- If one of the surfaces is designated as being **smooth**, it may be assumed that there is no friction present when this surface is in contact with another, regardless of the roughness of the other surface.

$|\mathbf{F}|$ cannot exceed its limiting value $\mu|\mathbf{N}|$. Slipping occurs if a friction force of magnitude greater than $\mu|\mathbf{N}|$ would be needed to prevent it.

Let us now apply these ideas to some examples, in which we will also apply the steps of Procedure 2. In most of the situations that we investigate, we will be concerned with limiting friction.

Exercise 31

A steel block of mass 0.5 kg rests on a horizontal dry steel surface (with coefficient of static friction $\mu = 0.74$) and is pulled by a horizontal force of 2 N. Model the block as a particle. Use Procedure 2 to determine whether the block will move. What is the magnitude of the friction force?

Exercise 32

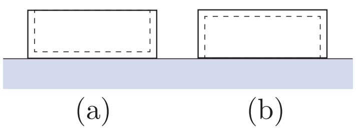


Figure 47 A box without a lid: (a) with its base in contact with a table; (b) with its open top in contact with the table

A shallow box made of a uniform material and without a lid can be placed on a horizontal table in two possible ways (as shown in Figure 47):

(a) with its base in contact with the table surface

(b) with its open top in contact with the table surface.

Which of these two positions requires the smaller sideways force to start the box slipping?

Inclined planes

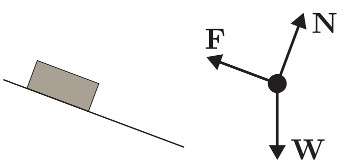


Figure 48 An object on an inclined plane, with its force diagram

Consider now an object resting on a sloping plane surface, often referred to as an **inclined plane**, such as the one shown in Figure 48. Provided that the angle of inclination is not large, the object can remain at rest and does not slide down the slope. The forces acting on the object are its weight, the normal reaction and friction. The weight **W** acts vertically downwards. The normal reaction **N** acts normally to the surface between the object and the slope. The friction force **F** is perpendicular to the normal reaction and hence parallel to the slope, and it acts up the slope to counteract the natural tendency of the object to move down the slope.

Example 9

A crate of empty bottles of total mass 30 kg is to be hauled by a rope up a ramp. The rope is parallel to the ramp, and the ramp makes an angle of $\pi/6$ radians with the horizontal. The coefficient of static friction between the plastic crate and the wooden ramp is 0.2.

What is the tension force due to the rope when the crate is on the point of moving upwards?

Solution

The situation is illustrated in Figure 49. ◀ Draw picture ▶

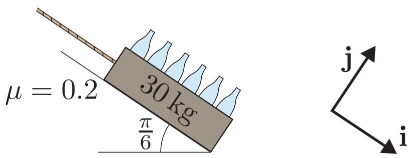


Figure 49 A crate of bottles being pulled up an inclined ramp

All the forces act in a vertical plane, so we need only two axes. We could choose **i** to be horizontal and **j** vertical as before, but it makes calculations easier if we choose **i** to be parallel to the slope and **j** perpendicular to it, as shown in Figure 49. This is because when we come to resolve the forces in the **i**- and **j**-directions, three of the four forces (all except **W**) will then act along one or other of the axes, making resolving them much simpler. ◀ Choose axes ▶

Modelling the crate as a particle and the rope as a model string, the force diagram is as shown in Figure 50, where **W** is the weight, **N** is the normal reaction, **F** is the friction force, and **T** is the tension force. ◀ Draw force diagram(s) ▶

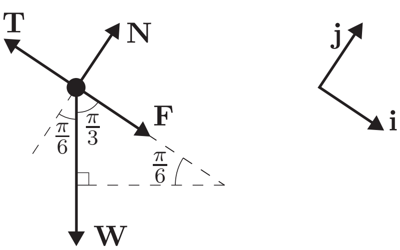


Figure 50 The force diagram for the crate

The equilibrium condition for particles gives ◀ Apply law(s) ▶

$$\mathbf{T} + \mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}. \tag{21}$$

When the crate is on the point of moving, we have

$$|\mathbf{F}| = \mu|\mathbf{N}|,$$

where $\mu = 0.2$ is the coefficient of static friction.

◀ Solve equation(s) ▶

As before, the first step in solving the equations involves resolving the force vectors into components. In this case, three of the force vectors are aligned with the axes and can be written down immediately:

$$\mathbf{F} = |\mathbf{F}|\,\mathbf{i}, \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{T} = |\mathbf{T}|\,(-\mathbf{i}).$$

Note that $\mathbf{W} = |\mathbf{W}|\,\widehat{\mathbf{u}}$, where $\widehat{\mathbf{u}}$ is a unit vector. A unit vector in the plane is always of the form $\widehat{\mathbf{u}} = \cos\alpha\,\mathbf{i} \pm \sin\alpha\,\mathbf{j}$.

To find the weight of the crate in component form, we resolve:

$$\mathbf{W} = |\mathbf{W}|\cos\tfrac{\pi}{3}\,\mathbf{i} - |\mathbf{W}|\sin\tfrac{\pi}{3}\,\mathbf{j} = \tfrac{1}{2}|\mathbf{W}|\,\mathbf{i} - \tfrac{\sqrt{3}}{2}|\mathbf{W}|\,\mathbf{j}.$$

Now equation (21) can easily be resolved in the **i**-direction, giving

$$-|\mathbf{T}| + 0 + |\mathbf{F}| + \tfrac{1}{2}|\mathbf{W}| = 0,$$

so

$$|\mathbf{T}| = |\mathbf{F}| + \tfrac{1}{2}|\mathbf{W}|. \tag{22}$$

Resolving equation (21) in the **j**-direction gives

$$0 + |\mathbf{N}| + 0 - \tfrac{\sqrt{3}}{2}|\mathbf{W}| = 0,$$

so

$$|\mathbf{N}| = \tfrac{\sqrt{3}}{2}|\mathbf{W}|. \tag{23}$$

At the point of moving, $|\mathbf{F}| = 0.2|\mathbf{N}|$, and equations (22) and (23) give

$$|\mathbf{T}| = 0.2|\mathbf{N}| + \tfrac{1}{2}|\mathbf{W}| = 0.2 \times \tfrac{\sqrt{3}}{2}|\mathbf{W}| + \tfrac{1}{2}|\mathbf{W}|.$$

Thus, since $|\mathbf{W}| = 30g$,

$$|\mathbf{T}| = \left(\tfrac{\sqrt{3}}{10} + \tfrac{1}{2}\right) \times 30g \simeq 198.$$

◀ Interpret solution ▶

Therefore when the crate is on the point of moving, the model predicts that the tension force due to the rope is 198 N, to the nearest whole number, up the ramp.

Mathematically, different choices of axes make no difference to the final solution obtained to a mechanics problem. However, a sensible choice of axes, as in Example 9, can reduce the amount of calculation. You will find that with experience, you will be able to choose axes that reduce the work involved.

Choice of axes is discussed again in Unit 3.

Exercise 33

A full crate of bottles of mass 60 kg is at the top of the ramp described in Example 9, ready to be lowered down it. What force needs to be applied to the rope to keep the crate from sliding down the ramp?

Exercise 34

(a) A box of mass m is resting on a surface inclined at an angle α to the horizontal. If the box is on the point of slipping, what is the coefficient of static friction?

(b) Two identical mugs are placed on a tray. One mug is half full of coffee, while the other is empty. The tray is tilted slowly. Use your answer to part (a) to determine which mug will start to move first.

The result of Exercise 34(a) provides us with a technique for estimating the coefficient of static friction μ for two surfaces. Put the two surfaces in contact, and increase the angle of inclination from the horizontal until slipping begins. The tangent of the angle at which this happens is the required value of μ.

3.6 Two-particle systems

In the previous subsections we considered the action of forces on one particle and introduced the equilibrium condition for particles. In this subsection we extend these ideas to situations involving two or more particles. We will model such situations by considering the forces acting on each particle separately. Each particle is aware of only the forces acting on it, and is unaware of any forces acting on other particles.

We begin by introducing a new modelling device – the model pulley – then consider friction in the two-particle case.

Pulleys

The pulley is a common device with which you are probably familiar. You may have seen pulleys in use on building sites, for example, as an aid to raising or lowering heavy loads. The idea of a pulley is useful in modelling mechanics problems, as it enables us to model a change in direction of a tension force.

In diagrams, we use an idealised pulley as shown in Figure 51. In order to keep the model simple, we make simplifying assumptions, which are formally stated in the following definition.

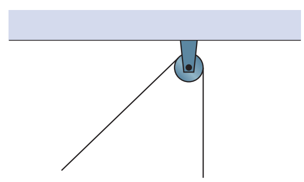


Figure 51 An idealised pulley

A **model pulley** is an object with no mass or size, over which a model string may pass without any resistance to motion. The magnitude of the tension in a string passing over a model pulley is the same either side of the pulley.

The point to remember is that the result of these assumptions implies that the tension forces due to the string on either side of the model pulley are equal in magnitude, that is, the magnitude of the tension in the string is the same on either side of the pulley (although the tension forces act in different directions).

A model pulley provides a reasonable model of an actual pulley, provided that its dimensions are small compared with the length of the rope or cable passing over it and that its weight is small compared with the other forces involved. Model pulleys can also be used to model a variety of situations that do not involve pulleys at all, but merely involve a change in direction of a tension force (such as when a rope is hanging over the edge of a building). Their use is illustrated by the following example and exercise.

Example 10

A sack of flour of mass 50 kg is lying on the floor of a mill, ready to be loaded into a cart. To help with the loading process, a light rope is attached to the sack, passes over a pulley fixed to the ceiling immediately above the sack, and is attached at its other end to a stone of mass 15 kg that hangs without touching the floor. The system is shown in Figure 52.

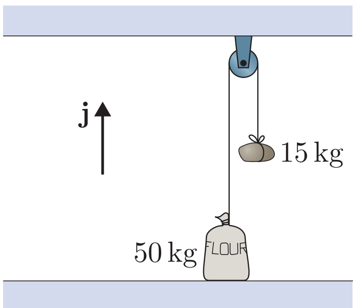


Figure 52 Raising a sack of flour using a pulley

Model the sack and the stone as particles, the pulley as a model pulley, and the rope as a model string, and consider the forces acting on each particle separately.

(a) Calculate the normal reaction of the floor on the sack.

(b) What force does the pulley exert on the ceiling?

Solution

◀ Choose axes ▶

(a) All the forces are vertical, so we need only one axis, as shown in Figure 52.

◀ Draw force diagram(s) ▶

The force diagrams for the sack and the stone are shown in Figure 53, where $\mathbf{W}_1$ and $\mathbf{W}_2$ represent the weights, $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension forces, and $\mathbf{N}$ is the normal reaction of the floor on the sack.

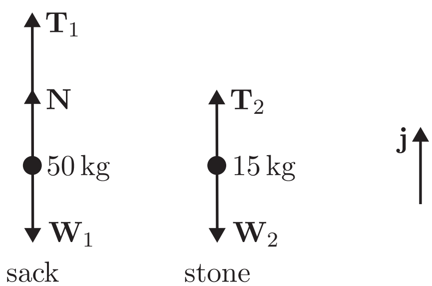


Figure 53 Force diagrams for the sack and the stone

◀ Apply law(s) ▶

The equilibrium condition for particles gives

$$\mathbf{W}_1 + \mathbf{N} + \mathbf{T}_1 = \mathbf{0}, \tag{24}$$

$$\mathbf{W}_2 + \mathbf{T}_2 = \mathbf{0}. \tag{25}$$

Since the tension forces on either side of a model pulley have the same magnitude, we have

$$|\mathbf{T}_1| = |\mathbf{T}_2|.$$

To solve the equations, the first step is to write the forces in component form as

◀ Solve equation(s) ▶

$$\mathbf{W}_1 = |\mathbf{W}_1|\,(-\mathbf{j}), \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{T}_1 = |\mathbf{T}_1|\,\mathbf{j},$$
$$\mathbf{T}_2 = |\mathbf{T}_2|\,\mathbf{j}, \quad \mathbf{W}_2 = |\mathbf{W}_2|\,(-\mathbf{j}).$$

Then resolving equations (24) and (25) in the $\mathbf{j}$-direction gives

$$-|\mathbf{W}_1| + |\mathbf{N}| + |\mathbf{T}_1| = 0,$$
$$-|\mathbf{W}_2| + |\mathbf{T}_2| = 0.$$

Therefore $|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{W}_2|$, so

$$|\mathbf{N}| = |\mathbf{W}_1| - |\mathbf{T}_1| = |\mathbf{W}_1| - |\mathbf{W}_2| = 50g - 15g = 35g \simeq 343.$$

So the normal reaction of the floor on the sack has magnitude about 343 N ($35g$) and is directed upwards. In the absence of the stone and the pulley, the magnitude of the normal reaction would have been equal to the magnitude of the weight of the sack ($50g$). The effect of the stone, transmitted via the pulley, is as if the magnitude of the sack's weight were reduced by the magnitude of the weight of the stone ($15g$).

◀ Interpret solution ▶

Note that the sack remains in contact with the floor if $|\mathbf{N}| > 0$, and is on the point of losing contact if $|\mathbf{N}|$ approaches zero. The model breaks down if the sack loses contact with the floor with $|\mathbf{N}| = 0$.

(b) To answer this question, we need to model the forces on the pulley. We can consider the pulley as a particle of no mass. Let $\mathbf{T}_3$ and $\mathbf{T}_4$ represent the tension forces in the left- and right-hand ropes, respectively. Modelling the short piece of metal that attaches the pulley to the ceiling as a model string, with tension force $\mathbf{T}_5$, we have the force diagram shown in Figure 54.

◀ Draw force diagram(s) ▶

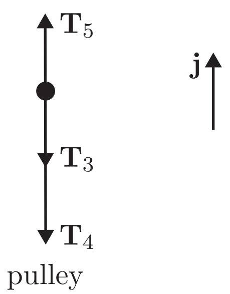


Figure 54 Force diagram for the pulley

The equilibrium condition for particles gives

◀ Apply law(s) ▶

$$\mathbf{T}_3 + \mathbf{T}_4 + \mathbf{T}_5 = \mathbf{0}. \qquad (26)$$

Since the tension in a string around a model pulley remains constant, we have

$$|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{T}_3| = |\mathbf{T}_4|.$$

◀ Solve equation(s) ▶

Resolving equation (26) in the $\mathbf{j}$-direction gives

$$-|\mathbf{T}_3| - |\mathbf{T}_4| + |\mathbf{T}_5| = 0.$$

Using the result from part (a) that $|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{W}_2|$, we have

$$|\mathbf{T}_5| = |\mathbf{T}_3| + |\mathbf{T}_4| = 2|\mathbf{W}_2| = 30g \simeq 294.$$

◀ Interpret solution ▶

So the model predicts that the force exerted by the short piece of metal (shown in Figure 52) on the pulley is about 294 N ($30g$) upwards. Hence, by Newton's third law, there is an equal and opposite force exerted by this short piece of metal on the ceiling – that is, the force exerted by the pulley on the ceiling is about 294 N ($30g$) downwards. This force (which is twice the weight of the stone) balances the weights of the stone ($15g$) and the sack ($50g$), less the normal reaction ($35g$) of the floor on the sack.

Exercise 35

Suppose that the pulley in Example 10 is no longer immediately above the sack, so that the rope attached to the sack makes an angle of $\pi/4$ to the vertical, as shown in Figure 55.

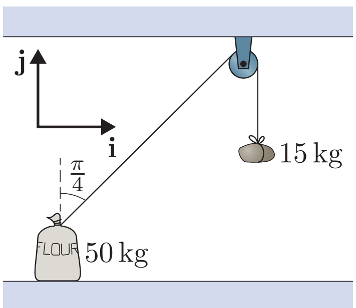


Figure 55

(a) Set up the problem using Procedure 2.

(b) Find the magnitude of the normal reaction of the floor on the sack.

(c) Find the magnitude of the friction force on the sack.

(d) Find the smallest value of the coefficient of static friction that would allow the system to remain in equilibrium.

(e) Find the weight of stone such that the sack just remains in contact with the floor.

Slipping

You have already investigated slipping in the case of a one-particle system. In this subsection we examine the phenomenon in systems of more than one particle.

Example 11

Consider a scarf draped over the edge of a table. Model the scarf as two particles, one of mass m_1 hanging over the edge, and the other of mass m_2 resting on the table, with the masses joined by a model string passing over the edge of the table, which is modelled as a model pulley. Assume that the scarf's mass is uniformly distributed along its length, so that the masses of the two particles are proportional to the corresponding lengths of scarf.

If the coefficient of static friction between the scarf and the table surface is μ, what proportion of the scarf's length can hang over the edge of the table before the scarf slips off the table?

Solution

We can answer this question if we can find the ratio of m_1 (the mass of scarf hanging over the edge) to $m_1 + m_2$ (the total mass of scarf) when the scarf is on the verge of slipping.

◀ Draw picture ▶

◀ Choose axes ▶

The situation is illustrated in Figure 56, which also shows a suitable choice of axes.

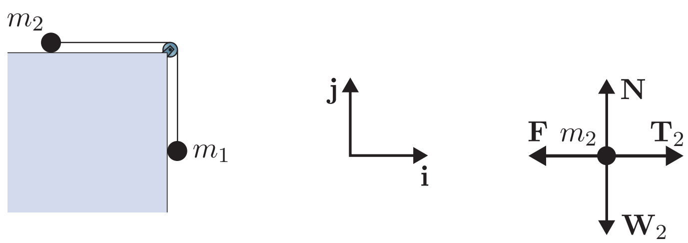


Figure 56 Modelling a scarf draped over the edge of a table, with force diagrams

◀ Draw force diagram(s) ▶

There are two forces acting on the hanging particle: its weight $\mathbf{W}_1$ and the tension force $\mathbf{T}_1$. There are four forces acting on the particle on the table: its weight $\mathbf{W}_2$, the tension force $\mathbf{T}_2$, the normal reaction force $\mathbf{N}$, and the friction force $\mathbf{F}$. The force diagrams are shown in Figure 56.

◀ Apply law(s) ▶

While the scarf does not slip, we can apply the equilibrium condition for particles to each particle in turn. For the hanging particle, we have

$$\mathbf{T}_1 + \mathbf{W}_1 = \mathbf{0}. \tag{27}$$

For the particle on the table, we have

$$\mathbf{F} + \mathbf{N} + \mathbf{T}_2 + \mathbf{W}_2 = \mathbf{0}. \tag{28}$$

The assumption of a model pulley gives

$$|\mathbf{T}_1| = |\mathbf{T}_2|. \tag{29}$$

When the particle is on the verge of slipping, we have

$$|\mathbf{F}| = \mu|\mathbf{N}|. \tag{30}$$

◀ Solve equation(s) ▶

From Figure 56, the component forms of the force vectors can immediately be written down:

$$\mathbf{T}_1 = |\mathbf{T}_1|\,\mathbf{j}, \quad \mathbf{W}_1 = |\mathbf{W}_1|\,(-\mathbf{j}),$$
$$\mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}), \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{T}_2 = |\mathbf{T}_2|\,\mathbf{i}, \quad \mathbf{W}_2 = |\mathbf{W}_2|\,(-\mathbf{j}).$$

Resolving equation (27) in the $\mathbf{j}$-direction gives

$$|\mathbf{T}_1| - |\mathbf{W}_1| = 0,$$

so

$$|\mathbf{T}_1| = |\mathbf{W}_1| = m_1 g.$$

Resolving equation (28) in the **i**-direction gives

$$-|\mathbf{F}| + 0 + |\mathbf{T}_2| + 0 = 0,$$

so, using equation (29),

$$|\mathbf{F}| = |\mathbf{T}_2| = |\mathbf{T}_1| = m_1 g.$$

Resolving equation (28) in the **j**-direction gives

$$0 + |\mathbf{N}| + 0 - |\mathbf{W}_2| = 0,$$

so

$$|\mathbf{N}| = |\mathbf{W}_2| = m_2 g.$$

Using equation (30), we have

$$m_1 g = \mu m_2 g,$$

so

$$m_1 = \mu m_2.$$

◀ Interpret solution ▶

Therefore the model predicts that when the scarf is on the verge of slipping, the proportion of its length that hangs over the edge is

$$\frac{m_1}{m_1 + m_2} = \frac{\mu m_2}{\mu m_2 + m_2} = \frac{\mu}{\mu + 1}.$$

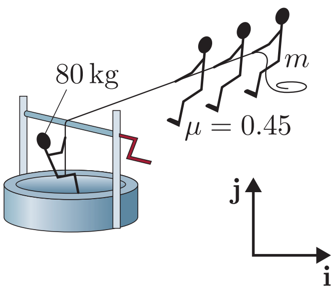


Figure 57 Lowering a man into a well

Exercise 36

A man of mass 80 kg is about to be lowered into a well from a rope that passes over a horizontal rotating axle above the well. The other end of the rope is held by several men each of mass 80 kg, as shown in Figure 57, with the rope between the men and the axle horizontal. Assume that n men can be represented by a single particle of mass $80n$ kg, the rope by a model string and the axle by a model pulley.

If the coefficient of static friction between the men's boots and the ground is 0.45, how many men are required to hold the man at the end of the rope before he is lowered into the well?

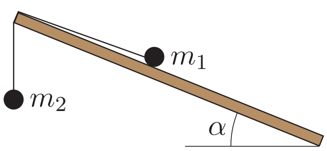


Figure 58 Two connected particles on an inclined plane

Exercise 37

An object of mass m_1, resting on a board inclined at an angle α to the horizontal, is attached to an object of mass m_2 by a string hanging over the edge of the board, as shown in Figure 58.

Assuming that the objects can be modelled as particles, the string as a model string and the edge of the board as a model pulley, find the condition on the coefficient of static friction μ between the first object and the board for this system to remain in equilibrium.

(*Hint*: There are two ways in which the equilibrium can be disturbed.)

4 Torques

This section looks at solid bodies, and in particular at a phenomenon that does not apply to particles: the turning effect of forces. We begin in Subsection 4.1 by introducing ways of modelling objects when their size is important, as it is when the turning effects of forces are considered. Subsection 4.2 goes on to explain what is meant by the turning effect of a force, and to provide a mathematical description of such an effect.

4.1 Extended and rigid bodies

Consider, for example, a tall thin box of cereal on a breakfast table. If you push it near its base, it slides across the table. But if you push it near its top, it tips over. The *position* at which the pushing force acts is important here, so the particle model – which allows forces to act at only one point – is inadequate. In this situation, we model the cereal box as an **extended body**, which is defined to be a material object that has one or more of length, breadth and depth, but whose internal structure may be neglected. So like a particle, it has mass, but unlike a particle, it has size of some sort and occupies more than a single point in space. For a particle, all the forces act at a point. For a rigid body, the forces can act at different points on the body.

The shape of an extended body may be one-, two- or three-dimensional (i.e. it may have a length, an area or a volume), depending on the situation.

Extended bodies are complicated objects because they can flex or vibrate. To model the slipping or tipping behaviour of the cereal packet, this generality is not needed. So we restrict our attention to *rigid bodies*. A **rigid body** is defined to be an extended body that does not change its shape (so it does not flex or vibrate).

The formal definition of a rigid body states that the position vector of a point on the body relative to any other point on the body is constant.

For a particle, all forces are applied at the point represented by the particle; we say that the forces *act* at this point. For extended bodies, we need to be more careful to specify where a force acts. For example, the weight of a body always acts through the **centre of mass** of the body. In this unit we consider only symmetric bodies made of uniform material. For such a body, as you would intuitively expect, the centre of mass is at its centre of symmetry (or geometric centre). For example, consider a coin, which can be modelled as a disc. Using symmetry, we can state that the centre of mass of the coin is along the axis of the disc (which runs between the centres of the flat circular faces of the disc), halfway between the flat circular faces.

A definition of the centre of mass and ways of finding it for a general rigid body are given in Units 19 and 21.

4.2 Turning effect of a force

Suppose that you try to balance a ruler on a horizontal extended finger. When the centre of the ruler is over your finger, the ruler should balance. If the centre of the ruler is not over your finger, then the weight of the ruler will cause it to turn about your finger. Although the finger can provide a normal reaction force that is equal in magnitude to the weight of the ruler, if the two forces are not in line, then turning occurs. The weight provides a turning effect if the vertical line of its action (through the centre of mass) does not pass through your finger.

How do we measure the turning effect of a force in terms of what we already know, namely the magnitude and direction of the force, and the point on the object at which the force acts? In order to begin to answer this question, try a simple experiment. Balance a 30-centimetre (12-inch) ruler on a pencil rubber or a hexagonal pencil, or some other object that is not too wide and so will act as a pivot. Place two identical small coins on either side of the pivot so that each is 10 cm from the pivot (see Figure 59).

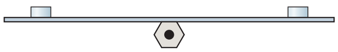


Figure 59 A ruler balanced on a pencil, with coins at each end

Then experiment with moving one of the coins in steps of 2 cm from its initial position, and see how the other coin has to be moved in order to re-establish balance. The conclusion from this experiment is not, perhaps, a surprising one: coins of equal mass have to be placed at equal distances on either side of the pivot for the ruler to remain balanced.

Next place the two coins together at a point on the ruler, say at 6 cm from the pivot. Where does a third identical coin have to be placed to achieve balance?

You should find that two identical coins placed together at 6 cm from the pivot are balanced by another identical coin placed on the other side of the ruler at 12 cm from the pivot. If you continue to experiment with varying numbers of coins placed at various pairs of positions along the ruler, you will find in each case that if the masses of the two sets of coins are unequal, then in order to achieve balance, the greater mass has to be placed nearer to the pivot than the smaller one. The turning effect due to the weight of the coins acting at a point depends not only on the mass of the coins, but also on the distance of the point of action from the pivot.

A long symmetrical object, such as the ruler in this experiment, can often be modelled as a rigid body with length, but no breadth or depth. Such a rigid body is known as a **model rod** and is often drawn as a straight line. The pivot on which the object rests is often modelled as a **model pivot**, which has a single point of contact with the rod and is often drawn as a triangle. Using these notions, the above experiments should allow you to believe the following result.

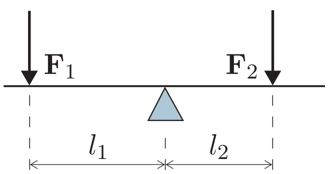


Figure 60 A balanced rod

Balanced rod

A horizontal rod, pivoted at its centre, will remain horizontal under the action of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting vertically downwards at distances l_1 and l_2, respectively, on either side of the pivot (see Figure 60), provided that

$$|\mathbf{F}_1|\, l_1 = |\mathbf{F}_2|\, l_2. \qquad (31)$$

In other words, the horizontal rod will remain in equilibrium provided that the distances of the forces from the pivot are in inverse ratio to the magnitudes of the forces.

Exercise 38

Jack and Jill are sitting on opposite ends of a seesaw. Jill is 1.2 m from the pivot, and Jack is 1 m from the pivot. Jack's mass is 60 kg. If the seesaw is at rest and horizontal, what is Jill's mass?

In the situation described in Exercise 38 and in the ruler example, there was an obvious way to measure each distance, namely along the seesaw or ruler. We need to generalise this to situations where there is not such an obvious way to measure distance. In these two examples the distances along the ruler and seesaw happen to be distances measured perpendicular to the direction of the force and from its point of action, that is, perpendicular to the *line of action* of the force.

The **line of action of a force** is a straight line in the direction of the force and through the point of action of the force.

So the turning effect of a force about a fixed point needs to encompass a measure of the force itself and the perpendicular distance of its line of action from the fixed point. It also needs to increase in magnitude if either the force or the distance increases in magnitude, and vice versa.

Consider a force $\mathbf{F}$ with line of action AB, as shown in Figure 61, and some fixed point O. Let R be any point on AB with position vector $\mathbf{r}$ with respect to O. Then the cross product is

$$\mathbf{r} \times \mathbf{F} = (|\mathbf{r}|\,|\mathbf{F}|\sin\theta)\,\widehat{\mathbf{n}},$$

Here $\mathbf{F}$ is used to denote a general force rather than a friction force.

where $\widehat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{r}$ and $\mathbf{F}$, and with direction out of the page (as given by the right-hand grip rule). This cross product satisfies all the above requirements for the turning effect of a force. It includes a measure $|\mathbf{F}|$ of the force and the perpendicular distance $|\mathbf{r}|\sin\theta$ of its line of action AB from the fixed point O. The direction of the cross product, represented by $\widehat{\mathbf{n}}$, corresponds to the direction of the turning effect. In this example, the turning effect of $\mathbf{F}$ about O is anticlockwise, which corresponds to the anticlockwise motion of the fingers in the right-hand grip rule. We refer to the cross product $\mathbf{r} \times \mathbf{F}$ as the *torque* of the force $\mathbf{F}$ relative to the origin O, and we use it as our measure of the turning effect of the force.

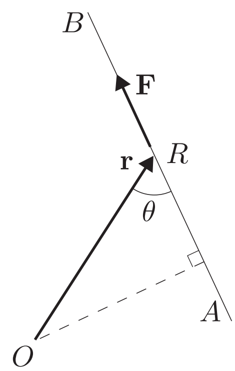


Figure 61 Calculating the turning effect of force $\mathbf{F}$ relative to O

The **torque** $\mathbf{\Gamma}$ of a force $\mathbf{F}$ about a fixed point O is the cross product

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F},$$

where $\mathbf{r}$ is the position vector, relative to O, of a point on the line of action of the force.

In SI units, the units of torque are newton metres, written as N m (or $\mathrm{kg\,m^2\,s^{-2}}$).

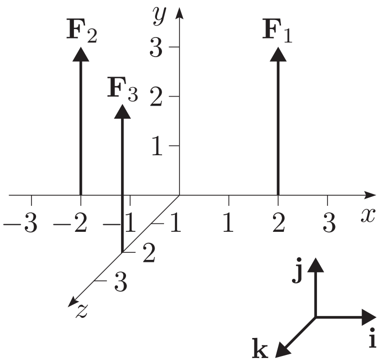


Figure 62

Note that a torque is a vector quantity, but it does not have the same units as a force, which is measured in newtons.

Exercise 39

Find the torque of each of the forces in Figure 62 relative to the origin, where each force is of magnitude $3\,\mathrm{N}$. Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ each have their point of action on the x-axis, and $\mathbf{F}_3$ has its point of action on the z-axis.

Exercise 40

(a) Show that if O is any point on the line of action of a force $\mathbf{F}$, and $\mathbf{r}$ is the position vector, relative to O, of any other point on the line of action, then $\mathbf{r} \times \mathbf{F} = \mathbf{0}$. Deduce that the torque of a force about a point on its line of action is zero.

(b) Suppose now that O is *not* on the line of action of $\mathbf{F}$, and let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors, relative to O, of two points on the line of action. Show that $\mathbf{r}_1 - \mathbf{r}_2$ is parallel to $\mathbf{F}$, and hence that $\mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F}$. Deduce that the torque of a force about a fixed point O is independent of the choice of the position of the point on the line of action.

Let us now convince ourselves that the definition of torque makes sense in terms of the examples of turning forces that we have seen so far. To do this, we need an equilibrium condition for rigid bodies that extends the equilibrium condition for particles. You will not be surprised that it requires not only that all the forces on a rigid body sum to zero, but also that all the torques sum to zero.

Equilibrium condition for a rigid body

A rigid body subjected to forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ is in **equilibrium** if the forces sum to the zero vector and the torques $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \ldots, \boldsymbol{\Gamma}_n$ corresponding to the forces, relative to the same fixed point O, also sum to the zero vector, that is,

$$\sum_{i=1}^{n} \mathbf{F}_i = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{n} \boldsymbol{\Gamma}_i = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0}, \tag{32}$$

where $\mathbf{r}_i$ is the position vector, relative to O, of a point on the line of action of $\mathbf{F}_i$.

5 Applying the principles

This section contains some examples and exercises that use the principles developed so far to solve more complicated statics problems. These examples are more complicated than those in the previous section because the forces and positions are not conveniently aligned. This is not a fundamental difficulty; it merely makes the calculation of the vector products more complicated.

Example 12

A ladder of mass M and length l stands on rough horizontal ground, and rests against a smooth vertical wall (see Figure 63). The ladder can be modelled as a model rod. Find the minimum angle θ between the ladder and the ground for which the ladder can remain static, if the coefficient of static friction μ between the ladder and the ground is 0.5.

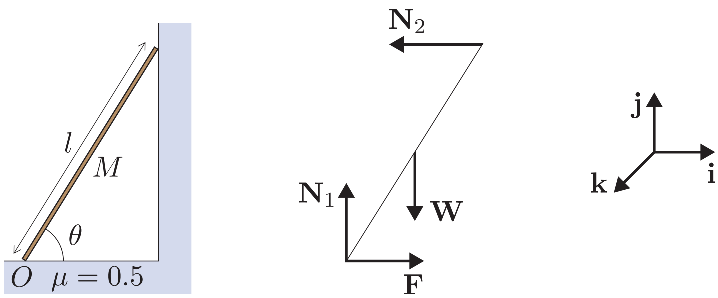


Figure 63 A ladder resting against a wall

Solution

The forces acting on the rod are shown in the force diagram in Figure 63, where $\mathbf{W}$ is the weight of the ladder, $\mathbf{N}_1$ is the normal reaction from the ground, $\mathbf{N}_2$ is the normal reaction from the wall, and $\mathbf{F}$ is the friction force at the bottom of the ladder. (The wall is smooth, so there is no friction force at the top of the ladder.)

◀ Draw force diagram(s) ▶

As two of the three unknown forces act at the bottom of the ladder, this is a convenient point for the origin O. The axes are chosen as shown in Figure 63.

◀ Choose axes ▶

The equilibrium condition for rigid bodies gives

◀ Apply law(s) ▶

$$\mathbf{N}_1 + \mathbf{N}_2 + \mathbf{F} + \mathbf{W} = \mathbf{0}, \tag{33}$$

$$\boldsymbol{\Gamma}_{\mathrm{N}_1} + \boldsymbol{\Gamma}_{\mathrm{N}_2} + \boldsymbol{\Gamma}_{\mathrm{F}} + \boldsymbol{\Gamma}_{\mathrm{W}} = \mathbf{0}, \tag{34}$$

where $\boldsymbol{\Gamma}_{\mathrm{N}_1}$, $\boldsymbol{\Gamma}_{\mathrm{N}_2}$, $\boldsymbol{\Gamma}_{\mathrm{F}}$ and $\boldsymbol{\Gamma}_{\mathrm{W}}$ are the torques with respect to O of $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{F}$ and $\mathbf{W}$, respectively.

If the ladder is not going to slip, we must have

$$|\mathbf{F}| \leq \mu |\mathbf{N}_1|. \tag{35}$$

◀ Solve equation(s) ▶

The position vectors $\mathbf{r}_{\mathrm{N}_1}$ and $\mathbf{r}_{\mathrm{F}}$ are both zero, hence the corresponding torques $\boldsymbol{\Gamma}_{\mathrm{N}_1}$ and $\boldsymbol{\Gamma}_{\mathrm{F}}$ (relative to O) are also zero. To calculate the non-zero torques, we need the position vectors of the points of application of the forces. These are given by

$$\mathbf{r}_{\mathrm{N}_2} = l\cos\theta\,\mathbf{i} + l\sin\theta\,\mathbf{j},$$
$$\mathbf{r}_{\mathrm{W}} = \tfrac{1}{2}l\cos\theta\,\mathbf{i} + \tfrac{1}{2}l\sin\theta\,\mathbf{j}.$$

All of the forces in this example are aligned with the coordinate axes, so the components can be written down by inspection:

$$\mathbf{N}_1 = |\mathbf{N}_1|\,\mathbf{j}, \quad \mathbf{N}_2 = |\mathbf{N}_2|\,(-\mathbf{i}), \quad \mathbf{F} = |\mathbf{F}|\,\mathbf{i}, \quad \mathbf{W} = Mg\,(-\mathbf{j}).$$

Now we can calculate the two non-zero torques:

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathrm{N}_2} &= (l\cos\theta\,\mathbf{i} + l\sin\theta\,\mathbf{j}) \times (-|\mathbf{N}_2|\,\mathbf{i})\\ &= -|\mathbf{N}_2|\,l\sin\theta\,\mathbf{j}\times\mathbf{i}\\ &= |\mathbf{N}_2|\,l\sin\theta\,\mathbf{k},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathrm{W}} &= (\tfrac{1}{2}l\cos\theta\,\mathbf{i} + \tfrac{1}{2}l\sin\theta\,\mathbf{j}) \times (-Mg\,\mathbf{j})\\ &= -\tfrac{1}{2}Mgl\cos\theta\,\mathbf{i}\times\mathbf{j}\\ &= -\tfrac{1}{2}Mgl\cos\theta\,\mathbf{k}.\end{aligned}$$

Substituting these torques into equation (34) gives

$$|\mathbf{N}_2|\,l\sin\theta\,\mathbf{k} - \tfrac{1}{2}Mgl\cos\theta\,\mathbf{k} = \mathbf{0},$$

thus

$$|\mathbf{N}_2| = \tfrac{1}{2}Mg\cot\theta.$$

Resolving equation (33) in the **i**- and **j**-directions in turn gives

$$-|\mathbf{N}_2| + |\mathbf{F}| = 0, \quad |\mathbf{N}_1| - Mg = 0.$$

Therefore

$$|\mathbf{F}| = |\mathbf{N}_2| = \tfrac{1}{2}Mg\cot\theta, \quad |\mathbf{N}_1| = Mg.$$

Substituting these into inequality (35) gives

Note that since M and g are positive, we can safely divide through by them without reversing the inequality.

$$\tfrac{1}{2}Mg\cot\theta \le \mu Mg,$$

which, on rearrangement and using $\mu = 0.5$, gives

$$\cot\theta \le 1.$$

◀ Interpret results ▶

Therefore the model predicts that the minimum angle that the ladder can make with the ground before slipping is $\pi/4$ radians (45°).

The second example involves a *hinge*, where a body is constrained to rotate about a fixed point. For such problems we need a new type of force, a **reaction force R**, which models the force acting on a hinge.

Example 13

During the erection of a marquee, a heavy pole OA of mass m and length l must be held in place by a rope AB, as shown in Figure 64. The angle between the pole and the ground is $\pi/4$, and the angle between the rope and the ground is $\pi/6$. Model the pole as a model rod, and the rope as a model string. Assume that the pole is freely hinged at O, that is, the end of the model rod is fixed at O and is free to pivot about O.

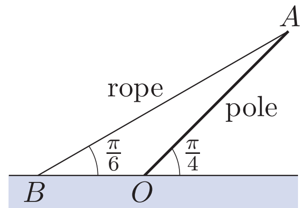


Figure 64 A rope holding up a pole

If the pole is in equilibrium, find the magnitude of the tension in the rope. Is the magnitude of the tension in the rope larger or smaller than it would be if the pole were hanging freely on the end of the rope?

Solution

The best choice of axes is not obvious for this problem. So in this case we proceed by drawing the force diagram first, as shown in Figure 65.

◀ Draw force diagram(s) ▶

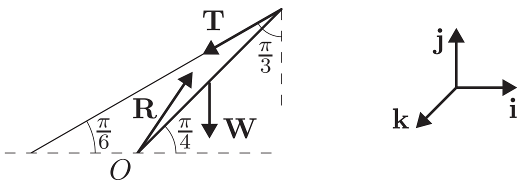


Figure 65 The force diagram for the rope and pole

From Figure 65 we see that there are three forces acting on the rod: its weight $\mathbf{W}$, the tension force in the string $\mathbf{T}$, and the reaction force $\mathbf{R}$ at the hinge. These forces act in three different directions, so we choose to orient the axes as shown in the figure. In this problem, the choice of origin O is vital, since we are going to take torques about the origin. Choosing the origin at the base of the pole eliminates the reaction force from the torque equation, which simplifies the calculations.

◀ Choose axes ▶

The equilibrium condition for rigid bodies gives

◀ Apply law(s) ▶

$$\mathbf{R} + \mathbf{W} + \mathbf{T} = \mathbf{0}, \tag{36}$$

$$\mathbf{\Gamma}_{\mathrm{R}} + \mathbf{\Gamma}_{\mathrm{W}} + \mathbf{\Gamma}_{\mathrm{T}} = \mathbf{0}, \tag{37}$$

where $\mathbf{\Gamma}_{\mathrm{R}}$, $\mathbf{\Gamma}_{\mathrm{W}}$ and $\mathbf{\Gamma}_{\mathrm{T}}$ are the torques with respect to O of $\mathbf{R}$, $\mathbf{W}$ and $\mathbf{T}$, respectively.

In this case, equation (36) is not very useful since it contains two forces of unknown magnitude, namely $\mathbf{R}$ and $\mathbf{T}$. However, by taking torques about O, one of the torques appearing in equation (37) is zero, namely $\mathbf{\Gamma}_{\mathrm{R}}$. Now we proceed by calculating the other two torques.

◀ Solve equation(s) ▶

The position vectors of the points of application of the forces $\mathbf{W}$ and $\mathbf{T}$ (at the centre and end of the pole, respectively) are

$$\mathbf{r}_{\mathrm{W}} = \tfrac{1}{2}l\cos\tfrac{\pi}{4}\,\mathbf{i} + \tfrac{1}{2}l\sin\tfrac{\pi}{4}\,\mathbf{j} = \frac{l}{2\sqrt{2}}(\mathbf{i}+\mathbf{j}),$$

$$\mathbf{r}_{\mathrm{T}} = l\cos\tfrac{\pi}{4}\,\mathbf{i} + l\sin\tfrac{\pi}{4}\,\mathbf{j} = \frac{l}{\sqrt{2}}(\mathbf{i}+\mathbf{j}).$$

The weight of the pole can be written in component form by inspection of the diagram, as $\mathbf{W} = mg\,(-\mathbf{j})$. The angle between the direction of the tension force and the vertical is shown in Figure 65 to be $\pi/3$. We have

There are many ways of computing the components of vectors. It is up to you to choose the way that you feel most comfortable with.

$$\mathbf{T} = |\mathbf{T}| \sin\tfrac{\pi}{3}\,(-\mathbf{i}) + |\mathbf{T}| \cos\tfrac{\pi}{3}\,(-\mathbf{j}) = -|\mathbf{T}| \left(\tfrac{\sqrt{3}}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j}\right).$$

Now that the forces and position vectors are written in component form, we can proceed to calculate the torques:

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathrm{W}} = \mathbf{r}_{\mathrm{W}} \times \mathbf{W} &= \frac{l}{2\sqrt{2}}(\mathbf{i}+\mathbf{j}) \times (-mg\,\mathbf{j}) \\ &= -\frac{lmg}{2\sqrt{2}}\,\mathbf{k}, \\ \boldsymbol{\Gamma}_{\mathrm{T}} = \mathbf{r}_{\mathrm{T}} \times \mathbf{T} &= \frac{l}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \times \left(-|\mathbf{T}| \left(\tfrac{\sqrt{3}}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j}\right)\right) \\ &= -\frac{l\,|\mathbf{T}|}{\sqrt{2}} \left(\tfrac{1}{2}\,\mathbf{i} \times \mathbf{j} + \tfrac{\sqrt{3}}{2}\,\mathbf{j} \times \mathbf{i}\right) \\ &= \frac{l\,|\mathbf{T}|}{2\sqrt{2}}(\sqrt{3}-1)\mathbf{k}.\end{aligned}$$

Substituting these torques into equation (37) gives

$$-\frac{lmg}{2\sqrt{2}}\,\mathbf{k} + \frac{l\,|\mathbf{T}|}{2\sqrt{2}}(\sqrt{3}-1)\mathbf{k} = \mathbf{0}.$$

Rearranging gives

$$|\mathbf{T}| = \frac{mg}{\sqrt{3}-1}.$$

◀ Interpret results ▶

So the magnitude of the tension in the rope is $mg/(\sqrt{3}-1) \simeq 1.4mg$, which is greater than the magnitude of the weight of the pole, mg. So, rather counter-intuitively, a stronger rope is needed to erect a pole in this way than is needed to lift the pole.

Exercise 41

A model rod OA of length l and mass m is fixed to a wall by a free hinge, as shown in Figure 66. The rod is free to turn in a vertical plane about the hinge, which is assumed to be smooth (i.e. there is no friction force associated with it). The rod is supported in a horizontal position by a string AB inclined at an angle θ to the horizontal.

Find the reaction force at the hinge and the tension force acting on the rod due to the string. Comment on the magnitudes and directions of these two forces.

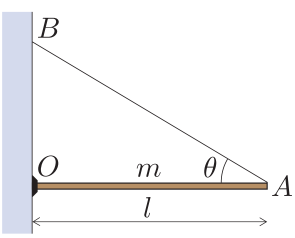


Figure 66 A rod fixed to a wall by a free hinge and supported by a string

Exercise 42

A light ladder of length 3.9 m stands on a horizontal floor and rests against a smooth vertical wall, as shown in Figure 67.

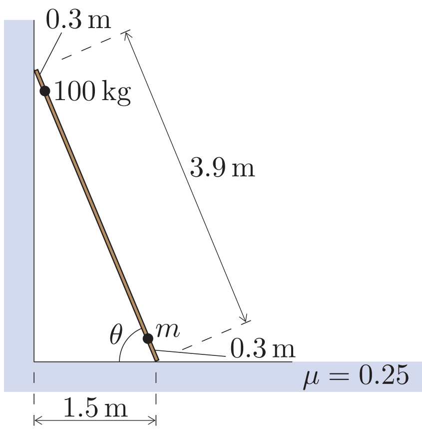


Figure 67 A ladder with people standing on the top and bottom rungs

The base of the ladder is 1.5 m from the base of the wall. The coefficient of static friction between the ladder and the floor is 0.25. The end rungs are each 0.3 m from an end of the ladder. The ladder may be modelled as a rod, and its mass may be neglected (as it is a light ladder, so its mass is negligible compared with the masses of any people standing on it).

What is the minimum mass of a person standing on the bottom rung that prevents the ladder from slipping when a person of mass 100 kg stands on the top rung?

Exercise 43

At a building site, a plank OB of length $2l$ and mass m is resting against a large smooth pipe of radius r, as shown in Figure 68. The pipe is fixed to the ground, or else it would slide or roll to the left. The angle between the plank and the horizontal is $\pi/3$, so by symmetry the angle between the horizontal and the line between O and the centre of the pipe is $\pi/6$, as shown. In the figure, the distance OA is greater than the distance AB, so the centre of mass of the plank is between O and A.

What is the least coefficient of static friction between the plank and the ground that will ensure equilibrium?

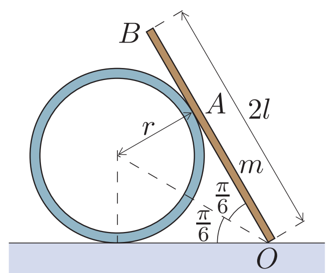


Figure 68 A plank resting against a large smooth pipe

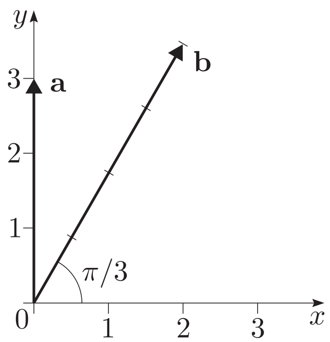
Learning outcomes

After studying this unit, you should be able to:

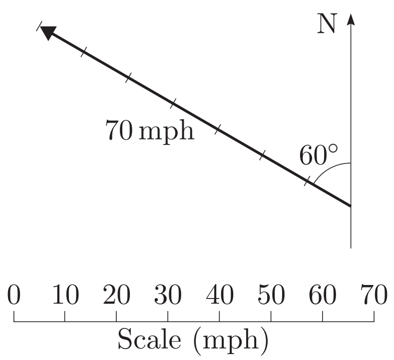
- understand the meaning of the terms scalar, vector, displacement vector, unit vector and position vector, and know what it means to say that two vectors are equal
- use vector notation and represent vectors as arrows on diagrams
- scale a vector by a number, and add two vectors geometrically using the triangle rule (or the parallelogram rule)
- resolve a vector into its Cartesian components, and scale and add vectors given in Cartesian component form
- calculate the dot product and cross product of two vectors
- determine whether or not two given vectors are perpendicular or parallel to one another
- determine the magnitude of a vector and the angle between the directions of two vectors
- write down the vector equation of a given straight line
- resolve a vector in a given direction
- manipulate vector expressions and equations involving the scaling, addition, dot product and cross product of vectors
- use the cross product to determine the area of a parallelogram or triangle
- appreciate the concept of a force
- understand and model forces of weight, normal reaction, tension and friction
- recognise and model the forces that act on an object in equilibrium
- model objects as particles or as rigid bodies
- use model strings, model rods, model pulleys and model pivots in modelling systems involving forces
- draw force diagrams, and choose appropriate axes and an origin
- use the equilibrium conditions for particles and for rigid bodies
- understand and use torques
- model and solve a variety of problems involving systems in equilibrium and systems on the verge of leaving equilibrium.

Solutions to exercises

Solution to Exercise 1



Solution to Exercise 2



Solution to Exercise 3

(a) $-\mathbf{v}$ represents a wind of 35 mph from the south-west.

(b) The vector $-1.5\mathbf{v}$ has magnitude $1.5|\mathbf{v}|$ and direction opposite to $\mathbf{v}$.

The vector $-k\mathbf{v}$ (k positive) has magnitude $k|\mathbf{v}|$ and direction opposite to $\mathbf{v}$.

(c) $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$ is a scaling of $\mathbf{v}$ by the positive scalar $m = \dfrac{1}{|\mathbf{v}|}$. The direction of $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$ is thus the same as that of $\mathbf{v}$, and its magnitude is

$$m|\mathbf{v}| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1.$$

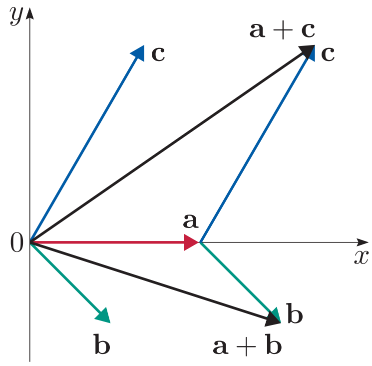
Solution to Exercise 4

(a) $-35\,\mathbf{j}$ (where $|\mathbf{j}|$ represents 1 km per hour).

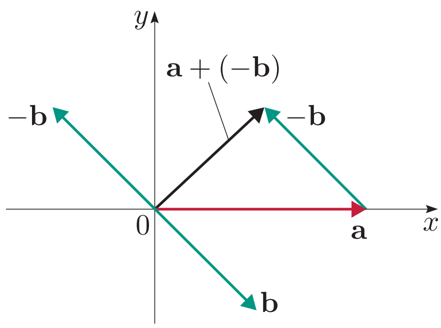
(b) $-112\,\mathbf{i}$ (where $|\mathbf{i}|$ represents 1 mile).

Solution to Exercise 5

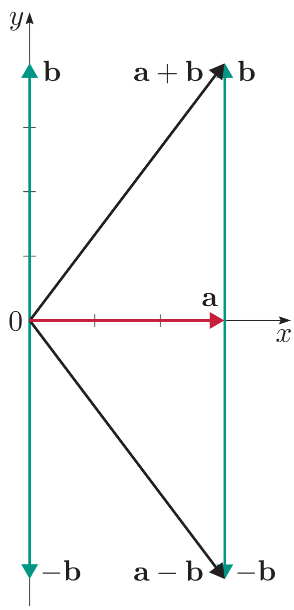
(a) The figure below shows $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}+\mathbf{c}$.



(b) The figure below shows $-\mathbf{b}$ and the sum of $\mathbf{a}$ and $-\mathbf{b}$.



Solution to Exercise 6



Solution to Exercise 7

$$\begin{aligned}&4(\mathbf{a}-\mathbf{c})+3(\mathbf{c}-\mathbf{b})+2(2\mathbf{a}-\mathbf{b}-3\mathbf{c})\\&=4\mathbf{a}-4\mathbf{c}+3\mathbf{c}-3\mathbf{b}+4\mathbf{a}-2\mathbf{b}-6\mathbf{c}\\&=8\mathbf{a}-5\mathbf{b}-7\mathbf{c}.\end{aligned}$$

Solution to Exercise 8

Systems (b), (c) and (d) are right-handed.

Solution to Exercise 9

(a) $\mathbf{d}=2(\mathbf{i}+\mathbf{j}+\mathbf{k})-3(2\mathbf{i}-3\mathbf{j}-\mathbf{k})=-4\mathbf{i}+11\mathbf{j}+5\mathbf{k}$,

$\mathbf{e}=(\mathbf{i}+\mathbf{j}+\mathbf{k})-2(2\mathbf{i}-3\mathbf{j}-\mathbf{k})+4(3\mathbf{i}+\mathbf{k})=9\mathbf{i}+7\mathbf{j}+7\mathbf{k}$.

(b) $|\mathbf{d}|=\sqrt{(-4)^2+11^2+5^2}=\sqrt{162}=9\sqrt{2}$,

$|\mathbf{e}|=\sqrt{9^2+7^2+7^2}=\sqrt{179}$.

(c) $|\mathbf{a}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$.

A unit vector in the direction of $\mathbf{a}$ is

$$\frac{1}{|\mathbf{a}|}\mathbf{a}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

(d) If $\mathbf{a}+\mathbf{x}=\mathbf{b}$, then

$$\mathbf{x}=\mathbf{b}-\mathbf{a}=(2\mathbf{i}-3\mathbf{j}-\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})=\mathbf{i}-4\mathbf{j}-2\mathbf{k}.$$

Thus the components of $\mathbf{x}$ are 1, -4 and -2.

Solution to Exercise 10

$\mathbf{0}=(0\quad 0\quad 0)^T,\quad \mathbf{i}=(1\quad 0\quad 0)^T,\quad \mathbf{j}=(0\quad 1\quad 0)^T,\quad \mathbf{k}=(0\quad 0\quad 1)^T.$

Solution to Exercise 11

(a) Relative to the origin of the Cartesian coordinate system, the two points have position vectors $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$. Thus the vector equation of the line is

$$\begin{aligned}\mathbf{r}&=(1-s)(\mathbf{i}+\mathbf{j}+2\mathbf{k})+s(2\mathbf{i}+3\mathbf{j}+\mathbf{k})\\&=(1+s)\mathbf{i}+(1+2s)\mathbf{j}+(2-s)\mathbf{k},\end{aligned}$$

where $-\infty<s<\infty$.

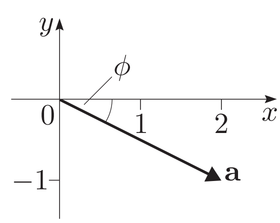
(b) The line cuts the (x,z)-plane when $y=0$, that is, $1+2s=0$, giving $s=-\frac{1}{2}$. Hence $\mathbf{r}=\frac{1}{2}\mathbf{i}+\frac{5}{2}\mathbf{k}$.

Solution to Exercise 12

(a) $|\mathbf{a}|=\sqrt{2^2+(-1)^2}=\sqrt{5}$,

$|\mathbf{b}|=\sqrt{1^2+3^2+5^2}=\sqrt{35}$.

The vector $\mathbf{a}$ lies in the (x,y)-plane, and the angle ϕ that it makes with the x-axis (see the figure in the margin) is given by $\cos\phi=2/\sqrt{5}$ and $\sin\phi=-1/\sqrt{5}$. Hence $\phi\simeq-0.4636$ (radians).



(b) $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$,

$2\mathbf{a} - \mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$,

$\mathbf{c} + 2\mathbf{b} - 3\mathbf{a} = -3\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}$.

(c) The vector $\overrightarrow{PQ}$ is equal to $2\mathbf{a} - \mathbf{b}$. The point Q is the end of the vector $\overrightarrow{OQ}$, which is given by

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = (2\mathbf{j} + 3\mathbf{k}) + (3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}) = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k},$$

so Q is the point $(3, -3, -2)$.

Solution to Exercise 13

(a) Writing

$$\frac{x+1}{3} = \frac{y-8}{-2} = 1 - z = s$$

gives

$$x = 3s - 1, \quad y = 8 - 2s, \quad z = 1 - s.$$

This gives the vector equation of the line as

$$\begin{aligned}\mathbf{r} &= (3s-1)\mathbf{i} + (8-2s)\mathbf{j} + (1-s)\mathbf{k}\\ &= (-\mathbf{i} + 8\mathbf{j} + \mathbf{k}) + s(3\mathbf{i} - 2\mathbf{j} - \mathbf{k}).\end{aligned}$$

Hence A has position vector $\mathbf{a} = -\mathbf{i} + 8\mathbf{j} + \mathbf{k}$, B has position vector $\mathbf{b} = 2\mathbf{i} + 6\mathbf{j}$, and $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, with length $\sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$.

(b) (i) The midpoint between A and B has $s = \frac{1}{2}$, so M has position vector

$$\mathbf{m} = \tfrac{1}{2}\mathbf{i} + 7\mathbf{j} + \tfrac{1}{2}\mathbf{k}.$$

(ii) The point N has $s = 0.6$, so N has position vector

$$\mathbf{n} = 0.8\mathbf{i} + 6.8\mathbf{j} + 0.4\mathbf{k}.$$

(c) Let $\mathbf{c} = -\mathbf{i} - 2\mathbf{j}$ and $\mathbf{d} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Then $\mathbf{d} - \mathbf{c} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, so the vector equation of the line (using a different parameter t) is

$$\begin{aligned}\mathbf{r} = \mathbf{c} + t(\mathbf{d} - \mathbf{c}) &= (-\mathbf{i} - 2\mathbf{j}) + t(3\mathbf{i} + 3\mathbf{j} - \mathbf{k})\\ &= (3t-1)\mathbf{i} + (3t-2)\mathbf{j} - t\mathbf{k}.\end{aligned}$$

(d) The lines intersect if there are values of s and t such that

$$(3t-1)\mathbf{i} + (3t-2)\mathbf{j} - t\mathbf{k} = (3s-1)\mathbf{i} + (8-2s)\mathbf{j} + (1-s)\mathbf{k}.$$

This gives rise to the three equations

$$3t - 1 = 3s - 1, \quad 3t - 2 = 8 - 2s, \quad -t = 1 - s.$$

Since there are no values of s and t that satisfy all three equations, the lines do not intersect.

Solution to Exercise 14

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = 2 \times 4 \times \cos\tfrac{\pi}{3} = 4,$$

$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}|\,|\mathbf{c}| \cos\theta = 4 \times 1 \times \cos\tfrac{\pi}{6} = 2\sqrt{3},$$

$\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}|\,|\mathbf{c}| \cos\theta = 2 \times 1 \times \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = 2\cos\frac{\pi}{2} = 0,$

$\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|\,|\mathbf{b}| \cos\theta = 4 \times 4 \times \cos 0 = 16.$

Solution to Exercise 15

(a)
$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}.\end{aligned}$$

(b)
$$\begin{aligned}|\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}.\end{aligned}$$

Solution to Exercise 16

(a) If $2\mathbf{a} + 3\mathbf{b}$ and $m\mathbf{a} + \mathbf{b}$ are perpendicular, then

$$(2\mathbf{a} + 3\mathbf{b}) \cdot (m\mathbf{a} + \mathbf{b}) = 0.$$

Expanding this expression,

$$2m\mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + 3m\mathbf{b}\cdot\mathbf{a} + 3\mathbf{b}\cdot\mathbf{b} = 0.$$

Now $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, so $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = 0$, and they are unit vectors, so $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = 1$. Thus

$$2m + 3 = 0,$$

so $m = -1.5$.

(b)
$$\begin{aligned}|\mathbf{c}|^2 &= \mathbf{c}\cdot\mathbf{c}\\ &= (3\mathbf{a} + 5\mathbf{b})\cdot(3\mathbf{a} + 5\mathbf{b})\\ &= 9\mathbf{a}\cdot\mathbf{a} + 15\mathbf{a}\cdot\mathbf{b} + 15\mathbf{b}\cdot\mathbf{a} + 25\mathbf{b}\cdot\mathbf{b}.\end{aligned}$$

Thus, since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular unit vectors,

$$|\mathbf{c}|^2 = 9 + 25 = 34,$$

so $|\mathbf{c}| = \sqrt{34} \simeq 5.831$.

Solution to Exercise 17

$\mathbf{a}\cdot\mathbf{b} = (4 \times 1) + (1 \times -3) + (-5 \times 1) = -4.$

The negative sign tells us that the angle between $\mathbf{a}$ and $\mathbf{b}$ is between $\pi/2$ and π radians, that is, it is an obtuse angle.

Solution to Exercise 18

$|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14},$

$|\mathbf{b}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21}.$

Also,

$$\mathbf{a}\cdot\mathbf{b} = (2 \times -1) + (-3 \times 2) + (1 \times 4) = -4,$$

so if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{-4}{\sqrt{14}\times\sqrt{21}} = -\frac{4}{7\sqrt{6}}.$$

The negative sign means that θ is obtuse; we have $\theta \simeq 1.806$ (radians).

Solution to Exercise 19

$$\begin{aligned}\mathbf{a}\cdot\mathbf{i} &= (a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k})\cdot\mathbf{i}\\ &= a_1\,\mathbf{i}\cdot\mathbf{i}+a_2\,\mathbf{j}\cdot\mathbf{i}+a_3\,\mathbf{k}\cdot\mathbf{i}\\ &= a_1.\end{aligned}$$

Similarly,

$$\mathbf{a}\cdot\mathbf{j} = a_2 \quad\text{and}\quad \mathbf{a}\cdot\mathbf{k} = a_3.$$

(Notice that this means that the components of a vector are given by the dot products of the vector with the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.)

Solution to Exercise 20

(a) $\mathbf{a}\cdot\mathbf{c} = -2$, $\mathbf{a}\cdot\mathbf{d} = -3$, $\mathbf{a}\cdot\mathbf{e} = 0$.

Thus only $\mathbf{e}$ is perpendicular to $\mathbf{a}$.

(b) First,

$$\mathbf{a}+2\mathbf{b} = \mathbf{j}+9\mathbf{k}.$$

Now a suitable vector along the line joining the origin to the point $(1,1,1)$ is $\mathbf{i}+\mathbf{j}+\mathbf{k}$. The corresponding unit vector is $\widehat{\mathbf{u}} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$. The component of $\mathbf{a}+2\mathbf{b}$ in the direction of this line is

$$\begin{aligned}(\mathbf{a}+2\mathbf{b})\cdot\widehat{\mathbf{u}} &= (\mathbf{j}+9\mathbf{k})\cdot\tfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})\\ &= \tfrac{10}{\sqrt{3}}.\end{aligned}$$

Solution to Exercise 21

The angle between $-\mathbf{i}$ and $\mathbf{p}$ is α, and the angle between $\mathbf{j}$ and $\mathbf{p}$ is $\frac{\pi}{2}-\alpha$. Therefore the $\mathbf{i}$-component of $\mathbf{p}$ is

$$-|\mathbf{p}|\cos(\alpha) = -2.5\cos\alpha,$$

and the $\mathbf{j}$-component of $\mathbf{p}$ is

$$|\mathbf{p}|\cos\left(\tfrac{\pi}{2}-\alpha\right) = 2.5\sin\alpha.$$

Hence

$$\mathbf{p} = -2.5\cos\alpha\,\mathbf{i}+2.5\sin\alpha\,\mathbf{j}.$$

The angle between $-\mathbf{i}$ and $\mathbf{q}$ is β, and the angle between $-\mathbf{j}$ and $\mathbf{q}$ is $\frac{\pi}{2}-\beta$. Therefore the $\mathbf{i}$-component of $\mathbf{q}$ is

$$-|\mathbf{q}|\cos(\beta) = -3\cos\beta,$$

and the $\mathbf{j}$-component of $\mathbf{q}$ is

$$-|\mathbf{q}|\cos\left(\tfrac{\pi}{2}-\beta\right) = -3\sin\beta.$$

Hence

$$\mathbf{q} = -3\cos\beta\,\mathbf{i}-3\sin\beta\,\mathbf{j}.$$

Finally, the angle between $\mathbf{i}$ and $\mathbf{r}$ is γ, and the angle between $-\mathbf{j}$ and $\mathbf{r}$ is $\frac{\pi}{2} - \gamma$. Therefore the $\mathbf{i}$-component of $\mathbf{r}$ is

$$|\mathbf{r}| \cos\gamma = 2.5\cos\gamma,$$

and the $\mathbf{j}$-component of $\mathbf{r}$ is

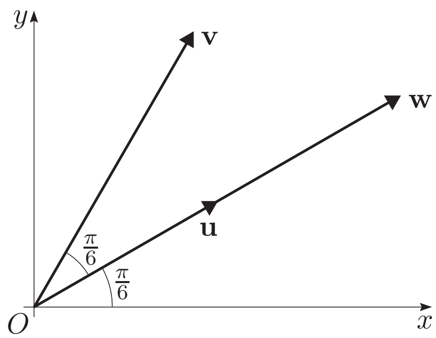
$$-|\mathbf{r}| \cos\left(\tfrac{\pi}{2} - \gamma\right) = -2.5\sin\gamma.$$

Hence

$$\mathbf{r} = 2.5\cos\gamma\,\mathbf{i} - 2.5\sin\gamma\,\mathbf{j}.$$

Solution to Exercise 22

For the sake of clarity, here is a diagram showing $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ (where all three vectors start at O) drawn in the (x, y)-plane. (The z-axis points out of the page.)



The cross products are all perpendicular to the (x, y)-plane.

A unit vector in the direction of $\mathbf{u} \times \mathbf{v}$ is $\mathbf{k}$, so

$$\mathbf{u} \times \mathbf{v} = \left(|\mathbf{u}|\,|\mathbf{v}| \sin\tfrac{\pi}{6}\right)\mathbf{k} = \left(2 \times 3 \times \tfrac{1}{2}\right)\mathbf{k} = 3\mathbf{k}.$$

The angle between $\mathbf{u}$ and $\mathbf{w}$ is zero, so

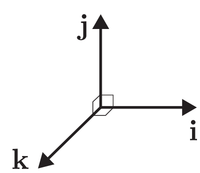
$$\mathbf{u} \times \mathbf{w} = \mathbf{0}.$$

A unit vector in the direction of $\mathbf{v} \times \mathbf{w}$ is $-\mathbf{k}$, so

$$\mathbf{v} \times \mathbf{w} = \left(|\mathbf{v}|\,|\mathbf{w}| \sin\tfrac{\pi}{6}\right)(-\mathbf{k}) = \left(3 \times 4 \times \tfrac{1}{2}\right)(-\mathbf{k}) = -6\mathbf{k}.$$

Solution to Exercise 23

(a) $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors forming a right-handed system (see the figure in the margin).



Thus, using the definition of the cross product,

$$\mathbf{i} \times \mathbf{j} = \left(|\mathbf{i}|\,|\mathbf{j}| \sin\tfrac{\pi}{2}\right)\mathbf{k} = \mathbf{k}.$$

Similarly,

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

(b) Since $(\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a})$ for any vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \text{and} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

(c) Since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$, we have

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

(d)
$$\begin{aligned}(\mathbf{i}+\mathbf{k}) \times (\mathbf{i}+\mathbf{j}+\mathbf{k}) &= (\mathbf{i} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})) + (\mathbf{k} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})) \\ &= (\mathbf{0}+\mathbf{k}+(-\mathbf{j})) + (\mathbf{j}+(-\mathbf{i})+\mathbf{0}) \\ &= -\mathbf{i}+\mathbf{k},\end{aligned}$$

$$(\mathbf{i} \times (\mathbf{i}+\mathbf{k})) - ((\mathbf{i}+\mathbf{j}) \times \mathbf{k}) = (\mathbf{0}+(-\mathbf{j})) - (-\mathbf{j}+\mathbf{i}) = -\mathbf{i}.$$

(e)
$$\begin{aligned}(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}+2\mathbf{b}) &= \mathbf{a} \times \mathbf{a} + \mathbf{a} \times 2\mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times 2\mathbf{b} \\ &= \mathbf{0} + 2\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} + \mathbf{0} \\ &= \mathbf{a} \times \mathbf{b},\end{aligned}$$

$$\begin{aligned}(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) &= \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} \\ &= \mathbf{0} + \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} + \mathbf{0} = \mathbf{0}.\end{aligned}$$

Solution to Exercise 24

First, we list the components of the vectors:

$$\begin{aligned}&a_1 = 2,\ a_2 = -3,\ a_3 = 1,\\ &b_1 = -1,\ b_2 = 2,\ b_3 = 4,\\ &c_1 = -4,\ c_2 = 6,\ c_3 = -2.\end{aligned}$$

To compute $\mathbf{a} \times \mathbf{b}$, we need

$$\begin{aligned}a_2b_3 - a_3b_2 &= -3 \times 4 - 1 \times 2 = -14,\\ a_3b_1 - a_1b_3 &= 1 \times (-1) - 2 \times 4 = -9,\\ a_1b_2 - a_2b_1 &= 2 \times 2 - (-3) \times (-1) = 1.\end{aligned}$$

Hence $\mathbf{a} \times \mathbf{b} = -14\mathbf{i} - 9\mathbf{j} + \mathbf{k}$.

Similarly, for $\mathbf{a} \times \mathbf{c}$ we need

$$\begin{aligned}a_2c_3 - a_3c_2 &= -3 \times (-2) - 1 \times 6 = 0,\\ a_3c_1 - a_1c_3 &= 1 \times (-4) - 2 \times (-2) = 0,\\ a_1c_2 - a_2c_1 &= 2 \times 6 - (-3) \times (-4) = 0.\end{aligned}$$

Hence $\mathbf{a} \times \mathbf{c} = \mathbf{0}$.

Finally, for $\mathbf{b} \times \mathbf{c}$ we need

$$\begin{aligned}b_2c_3 - b_3c_2 &= 2 \times (-2) - 4 \times 6 = -28,\\ b_3c_1 - b_1c_3 &= 4 \times (-4) - (-1) \times (-2) = -18,\\ b_1c_2 - b_2c_1 &= (-1) \times 6 - 2 \times (-4) = 2.\end{aligned}$$

Hence $\mathbf{b} \times \mathbf{c} = -28\mathbf{i} - 18\mathbf{j} + 2\mathbf{k}$.

Since $\mathbf{a} \times \mathbf{c} = \mathbf{0}$, and neither vector is zero, the vectors $\mathbf{a}$ and $\mathbf{c}$ are parallel. In fact, $\mathbf{c} = -2\mathbf{a}$.

Solution to Exercise 25

A vector perpendicular to **a** and **b** is $\mathbf{a} \times \mathbf{b}$. The components of **a** and **b** are

$$a_1 = 2,\ a_2 = 2,\ a_3 = 1,$$
$$b_1 = 4,\ b_2 = 4,\ b_3 = -7.$$

So we have

$$a_2b_3 - a_3b_2 = 2 \times (-7) - 1 \times 4 = -18,$$
$$a_3b_1 - a_1b_3 = 1 \times 4 - 2 \times (-7) = 18,$$
$$a_1b_2 - a_2b_1 = 2 \times 4 - 2 \times 4 = 0.$$

Hence $\mathbf{a} \times \mathbf{b} = -18\mathbf{i} + 18\mathbf{j}$.

We are asked for a unit vector, so the obvious choice is

$$\begin{aligned}\frac{1}{|\mathbf{a} \times \mathbf{b}|}(\mathbf{a} \times \mathbf{b}) &= \frac{-18\mathbf{i} + 18\mathbf{j}}{18\sqrt{2}} \\ &= \tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}).\end{aligned}$$

(Note that $\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ is also a unit vector perpendicular to **a** and **b**. This can be obtained by considering $\mathbf{b} \times \mathbf{a}$ rather than $\mathbf{a} \times \mathbf{b}$.)

Solution to Exercise 26

(a) **t** is perpendicular to both **r** and **s**, and its sense is vertically down, that is, into the ground.

(b) Conversely, the sense of $\mathbf{s} \times \mathbf{r}$ is vertically up.

(c) $\mathbf{t} \times \mathbf{r}$ is perpendicular to **t** (thus in the horizontal plane) and perpendicular to **r**, and by the right-hand grip rule its sense is due east.

(d) $|\mathbf{t}| = |\mathbf{r}|\,|\mathbf{s}| \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

(e) $\mathbf{t} \times (\mathbf{r} \times \mathbf{s}) = \mathbf{t} \times \mathbf{t} = \mathbf{0}$.

(f) $\mathbf{r} \cdot \mathbf{s} = |\mathbf{r}|\,|\mathbf{s}| \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

(g)
$$\begin{aligned}\mathbf{s} \cdot (\mathbf{t} \times \mathbf{r}) &= |\mathbf{s}|\,|\mathbf{t} \times \mathbf{r}| \cos \tfrac{\pi}{4} \quad \text{(by part (c))} \\ &= |\mathbf{s}| \left(|\mathbf{t}|\,|\mathbf{r}| \sin \tfrac{\pi}{2}\right) \cos \tfrac{\pi}{4} \\ &= 1 \times \tfrac{1}{\sqrt{2}} \times 1 \times 1 \times \tfrac{1}{\sqrt{2}} = \tfrac{1}{2}.\end{aligned}$$

Solution to Exercise 27

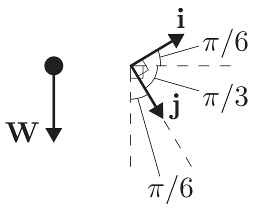
Two sides of the triangle are $\overrightarrow{AB} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{AC} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. So the area of the triangle is

$$\begin{aligned}\tfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| &= \tfrac{1}{2}\big|((-1) \times 2 - 5 \times (-3))\mathbf{i} + (5 \times 2 - (-1) \times 2)\mathbf{j} \\ &\qquad + ((-1) \times (-3) - (-1) \times 2)\mathbf{k}\big| \\ &= \tfrac{1}{2}|13\mathbf{i} + 12\mathbf{j} + 5\mathbf{k}| \\ &= \tfrac{1}{2}\sqrt{169 + 144 + 25} \\ &= \tfrac{13}{2}\sqrt{2}.\end{aligned}$$

Solution to Exercise 28

The weight is $3g\,\mathbf{k}$.

Solution to Exercise 29



Since the weight acts vertically downwards, its direction is in the vertical plane defined by $\mathbf{i}$ and $\mathbf{j}$, so $\mathbf{W}$ has no $\mathbf{k}$-component. To find the $\mathbf{i}$- and $\mathbf{j}$-components, the first step is to draw a diagram and work out the angles involved. In this case we use the two right angles marked in the diagram in the margin to work out the required angles.

A unit vector in the direction of $\mathbf{W}$ is $\widehat{\mathbf{u}} = -\sin\frac{\pi}{6}\,\mathbf{i} + \cos\frac{\pi}{6}\,\mathbf{j}$.

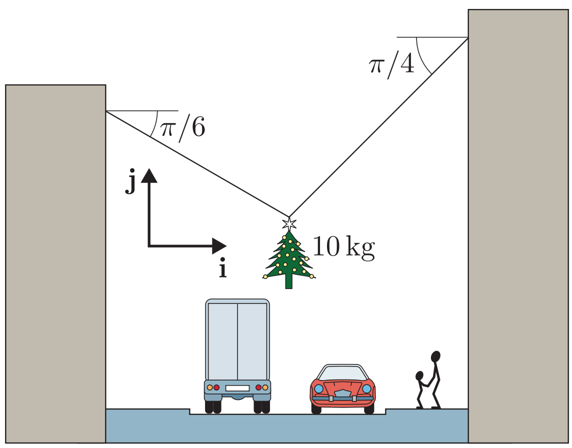
Resolving the weight gives

$$\begin{aligned}
\mathbf{W} = |\mathbf{W}|\,\widehat{\mathbf{u}} &= \left(-|\mathbf{W}|\sin\tfrac{\pi}{6}\right)\mathbf{i} + \left(|\mathbf{W}|\cos\tfrac{\pi}{6}\right)\mathbf{j}\\
&= -15g \times -\tfrac{1}{2}\,\mathbf{i} + 15g \times \tfrac{\sqrt{3}}{2}\,\mathbf{j}\\
&= -\tfrac{15}{2}g\,\mathbf{i} + \tfrac{15\sqrt{3}}{2}g\,\mathbf{j}.
\end{aligned}$$

Solution to Exercise 30

◀ Draw picture ▶

In this and other solutions, you may find that your diagrams and chosen axes are different from those given. You should still be able to check the validity of your solution against the given one, as the basic concepts are unchanged by these differences. Any choice of axes should lead to the same final answers as those given.

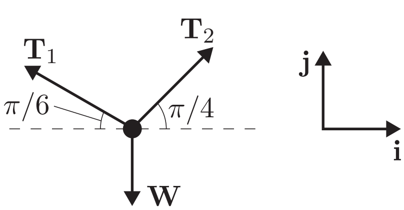


◀ Choose axes ▶

The forces all lie in a vertical plane, so we need only two axes. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are shown in the diagram.

◀ Draw force diagram(s) ▶

The force diagram, where the tension forces are denoted by $\mathbf{T}_1$ and $\mathbf{T}_2$, and the weight of the tree is denoted by $\mathbf{W}$, is as follows.



The equilibrium condition for particles gives ◀ Apply law(s) ▶

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{0}.$$

From the force diagram we have ◀ Solve equation(s) ▶

$$\mathbf{W} = -|\mathbf{W}|\,\mathbf{j} = -10g\,\mathbf{j}.$$

The other forces can be expressed in terms of components:

$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos\tfrac{\pi}{6}\,\mathbf{i} + |\mathbf{T}_1|\sin\tfrac{\pi}{6}\,\mathbf{j} = -\tfrac{\sqrt{3}}{2}|\mathbf{T}_1|\,\mathbf{i} + \tfrac{1}{2}|\mathbf{T}_1|\,\mathbf{j},$$
$$\mathbf{T}_2 = |\mathbf{T}_2|\cos\tfrac{\pi}{4}\,\mathbf{i} + |\mathbf{T}_2|\sin\tfrac{\pi}{4}\,\mathbf{j} = \tfrac{1}{\sqrt{2}}|\mathbf{T}_2|\,\mathbf{i} + \tfrac{1}{\sqrt{2}}|\mathbf{T}_2|\,\mathbf{j}.$$

Resolving the equilibrium equation in the **i**-direction gives

$$-\tfrac{\sqrt{3}}{2}|\mathbf{T}_1| + \tfrac{1}{\sqrt{2}}|\mathbf{T}_2| + 0 = 0,$$

and resolving in the **j**-direction gives

$$\tfrac{1}{2}|\mathbf{T}_1| + \tfrac{1}{\sqrt{2}}|\mathbf{T}_2| - 10g = 0.$$

Subtracting these two equations gives

$$\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\right)|\mathbf{T}_1| - 10g = 0,$$

so

$$|\mathbf{T}_1| = \frac{20}{1+\sqrt{3}}\,g \simeq 71.81.$$

Substituting this value of $|\mathbf{T}_1|$ into the **i**-direction equation gives

$$|\mathbf{T}_2| = \frac{20\sqrt{3}}{\sqrt{2}+\sqrt{6}}\,g \simeq 87.95.$$

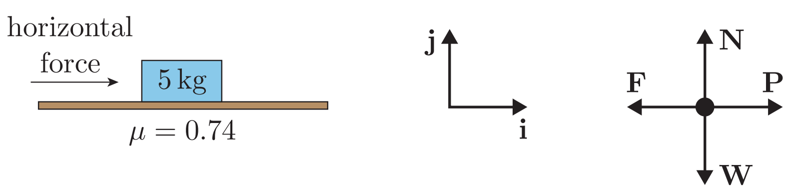
The model predicts that the magnitudes of the tension forces due to the ropes are 72 N and 88 N to the nearest whole number. ◀ Interpret solution ▶

Solution to Exercise 31

Draw the picture, choose axes and draw the force diagram. ◀ Draw picture ▶

◀ Choose axes ▶

◀ Draw force diagram(s) ▶



If the object does not move, the equilibrium condition for particles holds, so ◀ Apply law(s) ▶

$$\mathbf{F} + \mathbf{N} + \mathbf{P} + \mathbf{W} = \mathbf{0}.$$

If the block does not move,

$$|\mathbf{F}| \le \mu|\mathbf{N}|,$$

where μ is the coefficient of static friction.

◀ Solve equation(s) ▶

Resolving the equilibrium equation in the **i**-direction gives

$$|\mathbf{F}| = |\mathbf{P}|.$$

Resolving in the **j**-direction gives

$$|\mathbf{N}| = |\mathbf{W}|.$$

Therefore if the block does not move,

$$|\mathbf{P}| = |\mathbf{F}| \leq \mu|\mathbf{N}| = \mu|\mathbf{W}|.$$

Since $|\mathbf{P}| = 2$ and $|\mathbf{W}| = 0.5g$, the block does not move provided that

$$\mu \geq \frac{2}{0.5g} \simeq 0.41.$$

◀ Interpret solution ▶

The coefficient of static friction for steel on dry steel is $\mu = 0.74$, so the model predicts that the block will not move.

Since we have $|\mathbf{F}| = |\mathbf{P}|$, the magnitude of the friction force is $2\,\text{N}$.

Solution to Exercise 32

Since the friction force does not depend on the area of contact, the sideways force required to start the box slipping is the same in both cases.

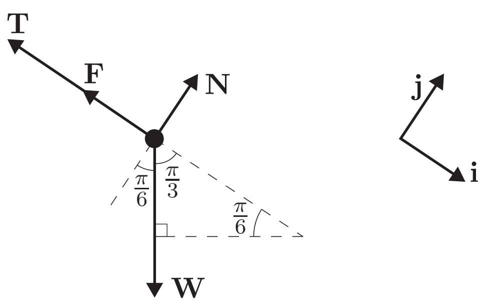
Solution to Exercise 33

◀ Choose axes ▶

Choose the same axes as in Example 9.

◀ Draw force diagram(s) ▶

Modelling the crate as a particle and the rope as a model string, the force diagram is as shown below, where the notation is as in Example 9. Both **F** and **T** act in the same direction (on the implied assumption that the crate will slide down the ramp if nothing holds it back).



(It is conventional to draw two forces acting in the same direction on a particle as overlapping arrows of different lengths.)

◀ Apply law(s) ▶

The equilibrium condition for particles gives

$$\mathbf{F} + \mathbf{T} + \mathbf{N} + \mathbf{W} = \mathbf{0}.$$

If the crate does not move, then

$$|\mathbf{F}| \leq \mu|\mathbf{N}| = 0.2|\mathbf{N}|,$$

where μ is the coefficient of static friction.

From the diagram, we have ◀ Solve equations ▶

$$\mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{T} = |\mathbf{T}|\,(-\mathbf{i}) \quad \text{and} \quad \mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}).$$

Resolving gives

$$\mathbf{W} = |\mathbf{W}| \cos \tfrac{\pi}{3}\,\mathbf{i} - |\mathbf{W}| \sin \tfrac{\pi}{3}\,\mathbf{j} = \tfrac{1}{2}|\mathbf{W}|\,\mathbf{i} - \tfrac{\sqrt{3}}{2}|\mathbf{W}|\,\mathbf{j}.$$

Resolving the equilibrium equation in the **i**-direction gives

$$-|\mathbf{F}| - |\mathbf{T}| + 0 + \tfrac{1}{2}|\mathbf{W}| = 0,$$

so

$$|\mathbf{F}| = \tfrac{1}{2}|\mathbf{W}| - |\mathbf{T}|.$$

Resolving in the **j**-direction gives

$$0 + 0 + |\mathbf{N}| - \tfrac{\sqrt{3}}{2}|\mathbf{W}| = 0,$$

so

$$|\mathbf{N}| = \tfrac{\sqrt{3}}{2}|\mathbf{W}|.$$

Substituting for $|\mathbf{F}|$ and $|\mathbf{N}|$ into the friction condition gives

$$\tfrac{1}{2}|\mathbf{W}| - |\mathbf{T}| \leq 0.2 \times \tfrac{\sqrt{3}}{2}|\mathbf{W}|,$$

which, on rearrangement and using $|\mathbf{W}| = 60g$, gives

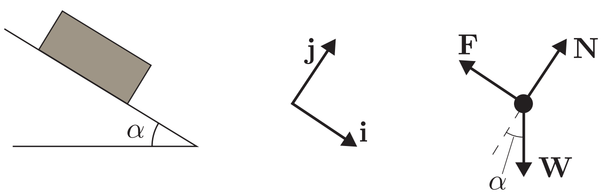
$$|\mathbf{T}| \geq 30g - 6\sqrt{3}g \simeq 192.$$

Therefore the model predicts that a force of at least 192 N up the slope needs to be applied to the rope to keep the crate from sliding down the ramp. ◀ Interpret solution ▶

Solution to Exercise 34

(a) The situation is illustrated below. ◀ Draw picture ▶

◀ Draw force diagram(s) ▶



Choose axes so that **i** points down the slope and **j** is in the direction of the normal reaction. ◀ Choose axes ▶

The equilibrium condition for particles gives ◀ Apply law(s) ▶

$$\mathbf{F} + \mathbf{N} + \mathbf{W} = \mathbf{0}.$$

Since the box is on the point of slipping,

$$|\mathbf{F}| = \mu|\mathbf{N}|,$$

where μ is the coefficient of static friction.

◀ Solve equation(s) ▶

Two of the forces are aligned with the axes and can be written down immediately:

$$\mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}) \quad \text{and} \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}.$$

The third force is inclined to the axes, so resolving gives

$$\begin{aligned}\mathbf{W} &= |\mathbf{W}|\cos\left(\tfrac{\pi}{2}-\alpha\right)\mathbf{i} - |\mathbf{W}|\sin\left(\tfrac{\pi}{2}-\alpha\right)\mathbf{j}\\ &= mg\sin\alpha\,\mathbf{i} - mg\cos\alpha\,\mathbf{j}.\end{aligned}$$

Resolving the equilibrium equation in the **i**-direction gives

$$-|\mathbf{F}| + 0 + mg\sin\alpha = 0,$$

so

$$|\mathbf{F}| = mg\sin\alpha.$$

Resolving in the **j**-direction gives

$$0 + |\mathbf{N}| - mg\cos\alpha = 0,$$

so

$$|\mathbf{N}| = mg\cos\alpha.$$

Substituting $|\mathbf{F}|$ and $|\mathbf{N}|$ into the friction condition gives

$$\mu = \frac{|\mathbf{F}|}{|\mathbf{N}|} = \tan\alpha.$$

◀ Interpret solution ▶

(b) The result in part (a) tells us that for an object on an inclined plane, the angle at which the object starts to slip depends only on the two surfaces in contact (i.e. on the coefficient of static friction μ). The mass of the object is irrelevant – so the half-full mug will start to slip at the same angle as the empty one.

Solution to Exercise 35

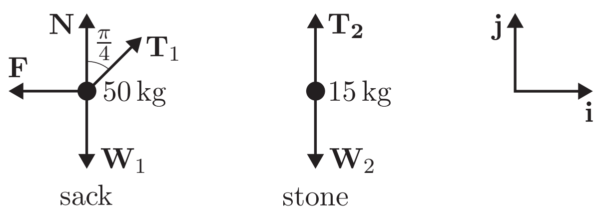
(a) The sack, stone, pulley and rope are modelled as in Example 10.

◀ Choose axes ▶

This time we need two axes, as shown in Figure 55.

◀ Draw force diagram(s) ▶

The force diagrams for the sack and the stone, using the usual notation, are as follows.



◀ Apply law(s) ▶

The equilibrium condition for particles gives

$$\mathbf{W}_1 + \mathbf{F} + \mathbf{N} + \mathbf{T}_1 = \mathbf{0},$$
$$\mathbf{W}_2 + \mathbf{T}_2 = \mathbf{0}.$$

The fact that we have a model pulley tells us that

$$|\mathbf{T}_1| = |\mathbf{T}_2|.$$

From the diagram we have ◀ Solve equation(s) ▶

$$\mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}), \quad \mathbf{W}_1 = |\mathbf{W}_1|\,(-\mathbf{j}),$$
$$\mathbf{T}_2 = |\mathbf{T}_2|\,\mathbf{j}, \quad \mathbf{W}_2 = |\mathbf{W}_2|\,(-\mathbf{j}).$$

Resolving for $\mathbf{T}_1$ gives

$$\mathbf{T}_1 = |\mathbf{T}_1|\cos\tfrac{\pi}{4}\,\mathbf{i} + |\mathbf{T}_1|\sin\tfrac{\pi}{4}\,\mathbf{j} = \tfrac{1}{\sqrt{2}}|\mathbf{T}_1|\,\mathbf{i} + \tfrac{1}{\sqrt{2}}|\mathbf{T}_1|\,\mathbf{j}.$$

Resolving the first equilibrium equation in the $\mathbf{i}$-direction gives

$$0 - |\mathbf{F}| + 0 + \tfrac{1}{\sqrt{2}}|\mathbf{T}_1| = 0,$$

so

$$|\mathbf{F}| = \tfrac{1}{\sqrt{2}}|\mathbf{T}_1|.$$

Resolving both equilibrium equations in the $\mathbf{j}$-direction gives

$$-|\mathbf{W}_1| + 0 + |\mathbf{N}| + \tfrac{1}{\sqrt{2}}|\mathbf{T}_1| = 0,$$
$$-|\mathbf{W}_2| + |\mathbf{T}_2| = 0,$$

so

$$|\mathbf{N}| = |\mathbf{W}_1| - \tfrac{1}{\sqrt{2}}|\mathbf{T}_1| \quad \text{and} \quad |\mathbf{W}_2| = |\mathbf{T}_2|.$$

(b) We have ◀ Interpret solution ▶

$$|\mathbf{W}_1| = 50g, \quad |\mathbf{W}_2| = 15g, \quad |\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{W}_2|,$$

so

$$|\mathbf{N}| = |\mathbf{W}_1| - \tfrac{1}{\sqrt{2}}|\mathbf{T}_1| = 50g - \tfrac{15}{\sqrt{2}}g \simeq 386.$$

So the model predicts that the magnitude of the normal reaction is about 386 N.

(c) We have

$$|\mathbf{F}| = \tfrac{1}{\sqrt{2}}|\mathbf{T}_1| = \tfrac{1}{\sqrt{2}}|\mathbf{W}_2| = \tfrac{15}{\sqrt{2}}g \simeq 104,$$

so the model predicts that the magnitude of the friction force is about 104 N.

(d) For equilibrium, we must have $|\mathbf{F}| \leq \mu|\mathbf{N}|$, that is,

$$\mu \geq |\mathbf{F}|/|\mathbf{N}| \simeq 104/386 \simeq 0.27.$$

So the model predicts that the smallest value of μ that allows the system to remain in equilibrium is about 0.27.

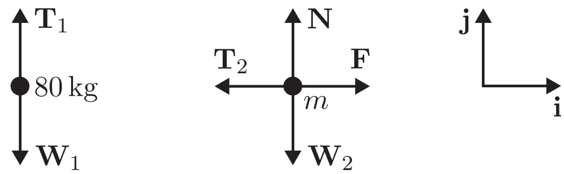
(e) If $|\mathbf{N}|$ approaches zero, then $|\mathbf{W}_1| - \frac{1}{\sqrt{2}}|\mathbf{T}_1| = |\mathbf{W}_1| - \frac{1}{\sqrt{2}}|\mathbf{W}_2|$ approaches zero, so $|\mathbf{W}_2|$ tends to $|\sqrt{2}\,\mathbf{W}_1|$. Thus the stone would have to weigh approximately 70.7 kg.

Solution to Exercise 36

Model the man in the well as a particle of mass 80 kg, and all the men on the ground as a single particle of mass $m = 80n$, where n is the number of men.

◀ Choose axes ▶
◀ Draw force diagram(s) ▶

A suitable choice of axes is shown in Figure 57. A force diagram for each particle, using the usual notation, is shown below.



man in well   men on ground

◀ Apply law(s) ▶

The equilibrium condition for particles gives

$$\mathbf{T}_1 + \mathbf{W}_1 = \mathbf{0},$$
$$\mathbf{T}_2 + \mathbf{N} + \mathbf{F} + \mathbf{W}_2 = \mathbf{0}.$$

The assumption of a model pulley gives

$$|\mathbf{T}_1| = |\mathbf{T}_2|.$$

If the men are not to slip, we must have

$$|\mathbf{F}| \leq \mu |\mathbf{N}|,$$

where μ is the coefficient of static friction.

◀ Solve equation(s) ▶

Resolving the second equilibrium equation in the **i**- and **j**-directions in turn gives

$$-|\mathbf{T}_2| + 0 + |\mathbf{F}| + 0 = 0,$$
$$0 + |\mathbf{N}| + 0 - |\mathbf{W}_2| = 0.$$

Resolving the first equilibrium equation in the **j**-direction gives

$$|\mathbf{T}_1| - |\mathbf{W}_1| = 0.$$

Therefore

$$|\mathbf{F}| = |\mathbf{T}_2| = |\mathbf{T}_1| = |\mathbf{W}_1| = 80g,$$
$$|\mathbf{N}| = |\mathbf{W}_2| = mg = 80ng,$$

hence

$$\mu|\mathbf{N}| = 0.45 \times mg = 0.45mg.$$

So, for the friction condition to be satisfied, we need $0.45 \times 80ng \geq 80g$, which gives $n \geq 1/0.45$. As n is an integer, the minimum value of n is 3.

◀ Interpret solution ▶

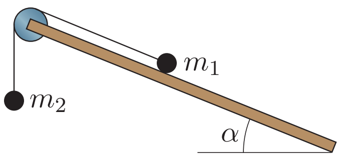
The model predicts that three men each of mass 80 kg are required to hold the man above the well.

Solution to Exercise 37

Equilibrium can be disturbed either by the object of mass m_1 sliding down the board and pulling up the object of mass m_2, or by the object of mass m_2 dropping down and pulling the object of mass m_1 up the board.

◀ Draw picture ▶

Model the masses as particles joined by a model string hanging over a model pulley representing the edge of the board.



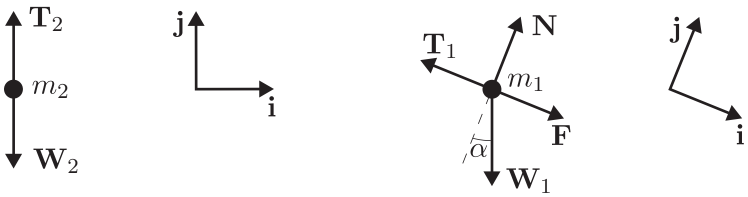
We need to consider the two cases of possible movement separately, as they lead to different force diagrams and hence to different results.

In this problem it makes sense to use different sets of unit vectors for the two particles, as indicated below.

◀ Choose axes ▶

◀ Draw force diagram(s) ▶

Consider first the case where m_2 is likely to drop down and pull m_1 up the board. The force diagrams are as follows.



◀ Apply law(s) ▶

The equilibrium condition for particles gives

$$\mathbf{T}_1 + \mathbf{N} + \mathbf{F} + \mathbf{W}_1 = \mathbf{0},$$
$$\mathbf{T}_2 + \mathbf{W}_2 = \mathbf{0}.$$

The use of a model pulley gives

$$|\mathbf{T}_1| = |\mathbf{T}_2|.$$

If the system is to remain in equilibrium, we require

$$|\mathbf{F}| \le \mu |\mathbf{N}|.$$

◀ Solve equation(s) ▶

For the particle of mass m_2, the forces are $\mathbf{T}_2 = |\mathbf{T}_2|\,\mathbf{j}$ and $\mathbf{W}_2 = -|\mathbf{W}_2|\,\mathbf{j}$. From the second equilibrium equation we have $|\mathbf{T}_2|\,\mathbf{j} - |\mathbf{W}_2|\,\mathbf{j} = \mathbf{0}$, therefore

$$|\mathbf{T}_2| = |\mathbf{W}_2| = m_2 g.$$

So by the model pulley equation,

$$|\mathbf{T}_1| = m_2 g.$$

For the particle of mass m_1, the forces acting are

$$\mathbf{F} = |\mathbf{F}|\,\mathbf{i}, \quad \mathbf{N} = |\mathbf{N}|\,\mathbf{j}, \quad \mathbf{T}_1 = -|\mathbf{T}_1|\,\mathbf{i} = -m_2 g\,\mathbf{i}$$

and $\mathbf{W}_1$, which can be resolved as

$$\mathbf{W}_1 = |\mathbf{W}_1| \sin\alpha\, \mathbf{i} + |\mathbf{W}_1| \cos\alpha\, (-\mathbf{j}).$$

Resolving the first equilibrium equation in the **i**- and **j**-directions in turn gives

$$-|\mathbf{T}_1| + 0 + |\mathbf{F}| + |\mathbf{W}_1| \sin\alpha = 0,$$
$$0 + |\mathbf{N}| + 0 - |\mathbf{W}_1| \cos\alpha = 0.$$

Therefore

$$|\mathbf{F}| = |\mathbf{T}_1| - |\mathbf{W}_1| \sin\alpha = m_2 g - m_1 g \sin\alpha,$$
$$|\mathbf{N}| = |\mathbf{W}_1| \cos\alpha = m_1 g \cos\alpha.$$

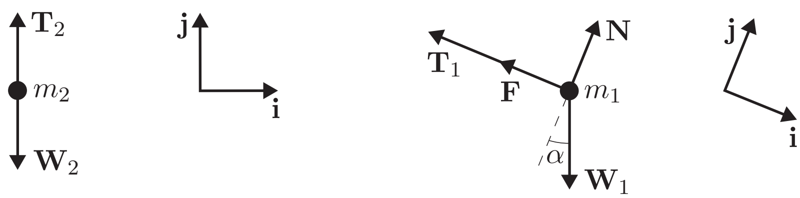
◀ Interpret solution ▶

For the friction condition to be satisfied, the model predicts that

$$\mu \geq \frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{m_2 g - m_1 g \sin\alpha}{m_1 g \cos\alpha} = \frac{m_2 - m_1 \sin\alpha}{m_1 \cos\alpha}.$$

◀ Draw force diagram(s) ▶

Now consider the other case where equilibrium could be disturbed, that is, where m_1 may slide down the board and pull m_2 up. The force diagrams are as follows.



◀ Apply law(s) ▶

The equilibrium condition and the use of a model pulley lead to the same equilibrium equations as before.

◀ Solve equation(s) ▶

The only difference is that the friction force is in the opposite direction, that is, $\mathbf{F} = -|\mathbf{F}|\, \mathbf{i}$. This means that when the first equilibrium equation is resolved in the **i**-direction, it now gives

$$-|\mathbf{T}_1| + 0 - |\mathbf{F}| + |\mathbf{W}_1| \sin\alpha = 0,$$

leading to

$$|\mathbf{F}| = |\mathbf{W}_1| \sin\alpha - |\mathbf{T}_1| = m_1 g \sin\alpha - m_2 g.$$

◀ Interpret solution ▶

For the friction condition to be satisfied, the model predicts that

$$\mu \geq \frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{m_1 \sin\alpha - m_2}{m_1 \cos\alpha}.$$

Both conditions for μ must hold for the system to be in equilibrium. Since μ is non-negative and the right-hand sides are of opposite signs, we can say

the first condition applies when $m_2 > m_1 \sin\alpha$,
the second condition applies when $m_2 < m_1 \sin\alpha$.

If there is no friction, then $\mu = 0$, and the conditions can both be satisfied only if the right-hand sides are also zero, that is, $m_2 = m_1 \sin\alpha$.

Solution to Exercise 38

We can model the seesaw as a model rod. Let Jill's mass be m. Then, assuming that neither Jack's nor Jill's feet are on the ground, we can use equation (31) to obtain

$$mg \times 1.2 = 60g \times 1.$$

Therefore $m = 50$, so Jill's mass is 50 kg.

Solution to Exercise 39

The simplest choice of position vector for a point on the line of action of $\mathbf{F}_1$ is $\mathbf{r}_1 = 2\mathbf{i}$, the position vector of the point $(2, 0, 0)$ relative to the origin. Similarly, suitably simple choices for $\mathbf{F}_2$ and $\mathbf{F}_3$ are $\mathbf{r}_2 = -2\mathbf{i}$ and $\mathbf{r}_3 = 2\mathbf{k}$. Hence the torques of the three forces relative to the origin are

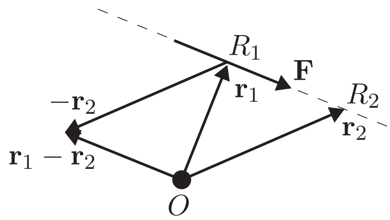
$$\mathbf{r}_1 \times \mathbf{F}_1 = 2\mathbf{i} \times 3\mathbf{j} = 6\mathbf{k},$$
$$\mathbf{r}_2 \times \mathbf{F}_2 = -2\mathbf{i} \times 3\mathbf{j} = -6\mathbf{k},$$
$$\mathbf{r}_3 \times \mathbf{F}_3 = 2\mathbf{k} \times 3\mathbf{j} = -6\mathbf{i}.$$

Solution to Exercise 40

(a) If O is on the line of action of $\mathbf{F}$, and $\mathbf{r}$ is the position vector relative to O of another point on that line of action, then $\mathbf{r}$ is parallel to $\mathbf{F}$, so $\mathbf{r} \times \mathbf{F} = \mathbf{0}$.

Thus when O is on the line of action of a force $\mathbf{F}$, the formula $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$ gives $\mathbf{0}$ whatever the choice of $\mathbf{r}$.

(b) If $\mathbf{r}_1$ and $\mathbf{r}_2$ are position vectors, relative to O, of two points R_1 and R_2 on the line of action of $\mathbf{F}$ (where O is not on this line of action), then the triangle rule for the addition of vectors shows that $\mathbf{r}_1 - \mathbf{r}_2$ is parallel to $\mathbf{F}$ (as shown in the diagram in the margin).



Therefore $(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F} = \mathbf{0}$, and since

$$(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F} = \mathbf{r}_1 \times \mathbf{F} - \mathbf{r}_2 \times \mathbf{F},$$

we have

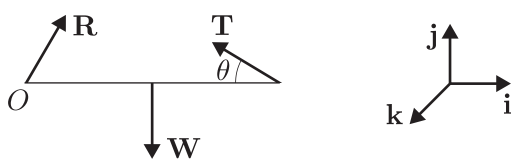
$$\mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F}.$$

Thus the formula $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}$ gives the same vector, whatever the choice of $\mathbf{r}$.

Solution to Exercise 41

◀ Draw force diagram(s) ▶

There are three forces acting on the rod: the weight $\mathbf{W}$ (at the centre of the rod), the tension force $\mathbf{T}$ due to the string, and the reaction force $\mathbf{R}$ at the hinge. The direction of $\mathbf{R}$ is not known, except that it is in the plane of $\mathbf{W}$ and $\mathbf{T}$. The force diagram is shown below.



◀ Choose axes ▶ Choose the origin to be at O and axes as shown above.

◀ Apply law(s) ▶ The equilibrium condition for rigid bodies gives

$$\mathbf{R} + \mathbf{T} + \mathbf{W} = \mathbf{0},$$
$$\boldsymbol{\Gamma}_{\mathrm{R}} + \boldsymbol{\Gamma}_{\mathrm{T}} + \boldsymbol{\Gamma}_{\mathrm{W}} = \mathbf{0}.$$

◀ Solve equation(s) ▶ Position vectors of the points of application are

$$\mathbf{r}_{\mathrm{R}} = \mathbf{0}, \quad \mathbf{r}_{\mathrm{T}} = l\,\mathbf{i}, \quad \mathbf{r}_{\mathrm{W}} = \tfrac{1}{2}l\,\mathbf{i}.$$

From the force diagram,

$$\mathbf{W} = mg(-\mathbf{j}).$$

The force $\mathbf{R}$ is unknown except that it is in the plane of $\mathbf{W}$ and $\mathbf{T}$, that is, the plane defined by $\mathbf{i}$ and $\mathbf{j}$. For convenience we write $\mathbf{R}$ in component form as

$$\mathbf{R} = R_i\,\mathbf{i} + R_j\,\mathbf{j},$$

where R_i and R_j are the $\mathbf{i}$- and $\mathbf{j}$-components of $\mathbf{R}$.

The remaining force can be resolved as

$$\mathbf{T} = |\mathbf{T}|\cos\theta\,(-\mathbf{i}) + |\mathbf{T}|\sin\theta\,\mathbf{j} = -|\mathbf{T}|\cos\theta\,\mathbf{i} + |\mathbf{T}|\sin\theta\,\mathbf{j}.$$

The torques are given by

$$\boldsymbol{\Gamma}_{\mathrm{R}} = \mathbf{0},$$
$$\boldsymbol{\Gamma}_{\mathrm{T}} = \mathbf{r}_{\mathrm{T}} \times \mathbf{T} = l\,\mathbf{i} \times |\mathbf{T}|(-\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = l\,|\mathbf{T}|\sin\theta\,\mathbf{k},$$
$$\boldsymbol{\Gamma}_{\mathrm{W}} = \mathbf{r}_{\mathrm{W}} \times \mathbf{W} = \tfrac{1}{2}l\,\mathbf{i} \times mg(-\mathbf{j}) = -\tfrac{1}{2}lmg\,\mathbf{k}.$$

Resolving the second equilibrium equation in the $\mathbf{k}$-direction gives

$$l\,|\mathbf{T}|\sin\theta - \tfrac{1}{2}lmg = 0,$$

so

$$|\mathbf{T}| = \tfrac{1}{2}mg\operatorname{cosec}\theta.$$

Resolving the first equilibrium equation in the $\mathbf{i}$- and $\mathbf{j}$-directions in turn gives

$$R_i - |\mathbf{T}|\cos\theta = 0,$$
$$R_j + |\mathbf{T}|\sin\theta - mg = 0,$$

so

$$R_i = |\mathbf{T}|\cos\theta = \tfrac{1}{2}mg\cot\theta,$$
$$R_j = mg - |\mathbf{T}|\sin\theta = mg - \tfrac{1}{2}mg = \tfrac{1}{2}mg.$$

So the model predicts that the reaction force at the hinge is

$$\mathbf{R} = \tfrac{1}{2}mg\cot\theta\,\mathbf{i} + \tfrac{1}{2}mg\,\mathbf{j}.$$

From the first equilibrium equation, the tension force due to the string is

$$\mathbf{T} = -\mathbf{R} - \mathbf{W} = -\tfrac{1}{2}mg\cot\theta\,\mathbf{i} + \tfrac{1}{2}mg\,\mathbf{j}.$$

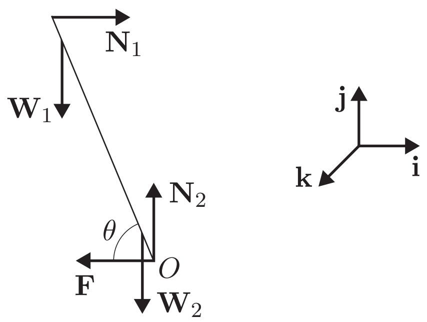
The **j**-components of **R** and **T** are the same. The **i**-components are of equal magnitude, but with opposite signs. Thus **R** and **T** have equal magnitude and make the same angle θ with the horizontal, where θ is measured anticlockwise for **R** and clockwise for **T**.

◀ Interpret solution ▶

Solution to Exercise 42

Since the mass of the ladder may be neglected, the forces acting on the ladder are the weights $\mathbf{W}_1$ and $\mathbf{W}_2$ of the two people, the normal reaction forces $\mathbf{N}_1$ and $\mathbf{N}_2$ at the wall and at the floor, and the friction force $\mathbf{F}$ at the floor. (There is no friction force at the wall, as it is smooth.) The force diagram is shown below.

◀ Draw force diagram(s) ▶



The origin O is best chosen to be at the foot of the ladder. The axes are shown above.

◀ Choose axes ▶

The equilibrium condition for rigid bodies gives

◀ Apply law(s) ▶

$$\mathbf{N}_1 + \mathbf{N}_2 + \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{F} = \mathbf{0},$$
$$\boldsymbol{\Gamma}_{\mathrm{N}_1} + \boldsymbol{\Gamma}_{\mathrm{N}_2} + \boldsymbol{\Gamma}_{\mathrm{W}_1} + \boldsymbol{\Gamma}_{\mathrm{W}_2} + \boldsymbol{\Gamma}_{\mathrm{F}} = \mathbf{0}.$$

At the point of slipping,

$$|\mathbf{F}| = \mu|\mathbf{N}_2|,$$

where $\mu = 0.25$ is the coefficient of static friction.

Let l denote the length of the ladder (i.e. 3.9 m), and let d denote the distance of the end rungs from the ends of the ladder (i.e. 0.3 m). Relative to the origin O at the bottom of the ladder, two position vectors are zero, namely $\mathbf{r}_{\mathrm{N}_2}$ and $\mathbf{r}_{\mathrm{F}}$. The remaining position vectors are

◀ Solve equation(s) ▶

$$\mathbf{r}_{\mathrm{N}_1} = -l\cos\theta\,\mathbf{i} + l\sin\theta\,\mathbf{j},$$
$$\mathbf{r}_{\mathrm{W}_1} = -(l-d)\cos\theta\,\mathbf{i} + (l-d)\sin\theta\,\mathbf{j},$$
$$\mathbf{r}_{\mathrm{W}_2} = -d\cos\theta\,\mathbf{i} + d\sin\theta\,\mathbf{j}.$$

The forces are aligned with the axes, and can be put into component form by inspection:

$$\mathbf{N}_1 = |\mathbf{N}_1|\,\mathbf{i}, \quad \mathbf{N}_2 = |\mathbf{N}_2|\,\mathbf{j},$$
$$\mathbf{W}_1 = Mg\,(-\mathbf{j}), \quad \mathbf{W}_2 = mg\,(-\mathbf{j}), \quad \mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}),$$

where m is the mass of the person on the lower rung, and M is the mass of the person on the upper rung (i.e. 100 kg).

Two of the torques are zero by choice of origin. The remaining three torques can be calculated as

$$\begin{aligned}
\boldsymbol{\Gamma}_{N_1} &= (-l\cos\theta\,\mathbf{i} + l\sin\theta\,\mathbf{j}) \times (|\mathbf{N}_1|\,\mathbf{i}) = -l\,|\mathbf{N}_1|\sin\theta\,\mathbf{k},\\
\boldsymbol{\Gamma}_{W_1} &= (-(l-d)\cos\theta\,\mathbf{i} + (l-d)\sin\theta\,\mathbf{j}) \times (-Mg\,\mathbf{j}) = M(l-d)g\cos\theta\,\mathbf{k},\\
\boldsymbol{\Gamma}_{W_2} &= (-d\cos\theta\,\mathbf{i} + d\sin\theta\,\mathbf{j}) \times (-mg\,\mathbf{j}) = mdg\cos\theta\,\mathbf{k}.
\end{aligned}$$

Substituting these into the second equilibrium equation and resolving in the **k**-direction gives

$$-l\,|\mathbf{N}_1|\sin\theta + M(l-d)g\cos\theta + mdg\cos\theta = 0.$$

Now we need to find the magnitude of the normal reaction $|\mathbf{N}_1|$, which we do by resolving the first equilibrium equation in the **i**- and **j**-directions in turn, to obtain

$$\begin{aligned}
&|\mathbf{N}_1| - |\mathbf{F}| = 0,\\
&|\mathbf{N}_2| - Mg - mg = 0.
\end{aligned}$$

From these equations and the friction condition, we have

$$|\mathbf{N}_1| = |\mathbf{F}| = \mu|\mathbf{N}_2| = \mu(Mg + mg).$$

Substituting for $|\mathbf{N}_1|$ in the **k**-direction equation gives

$$-l\mu(Mg + mg)\sin\theta + M(l-d)g\cos\theta + mdg\cos\theta = 0,$$

thus

$$-Ml\mu\tan\theta - ml\mu\tan\theta + M(l-d) + md = 0,$$

so

$$m = M\,\frac{l\mu\tan\theta - l + d}{d - l\mu\tan\theta}.$$

◀ Interpret solution ▶

To use this formula, we need the value of $\tan\theta$, which by trigonometry is equal to $\sqrt{3.9^2 - 1.5^2}/1.5 = 2.4$. Now we substitute in the known values, namely $\mu = 0.25$, $d = 0.3$, $l = 3.9$ and $M = 100$, to find

$$m = 100 \times \frac{3.9 \times 0.25 \times 2.4 - 3.9 + 0.3}{0.3 - 3.9 \times 0.25 \times 2.4} \simeq 61.8.$$

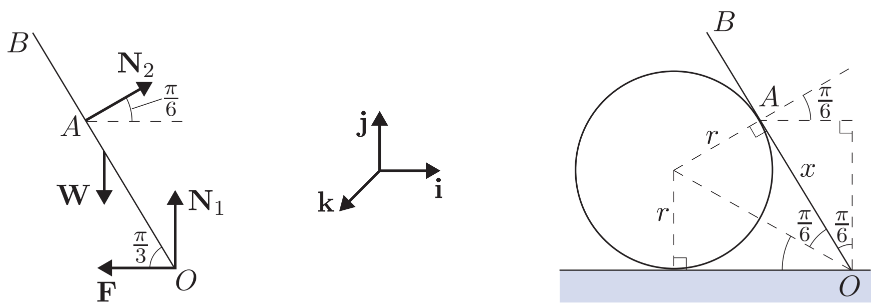
So to prevent the ladder from slipping, the model predicts that the person on the bottom rung must have a mass of at least 61.8 kg.

Solution to Exercise 43

◀ Choose axes ▶

◀ Draw force diagram(s) ▶

Take the origin at O at the base of the plank, and axes as shown below. The force diagram, using the usual notation, is also shown below.



To calculate the torque of the force $\mathbf{N}_2$ about O, we need the angle between $\mathbf{N}_2$ and $\mathbf{i}$, which from the diagram is equal to $\pi/6$. We also need the distance OA along the plank to the point of contact of the pipe and the plank: call this distance x. From the diagram, we have $\tan\frac{\pi}{6} = r/x$, which gives $x = \sqrt{3}r$. Now that we know the distances and angles involved, we can continue with Procedure 2.

The equilibrium condition for rigid bodies gives ◀ Apply law(s) ▶

$$\mathbf{N}_1 + \mathbf{N}_2 + \mathbf{F} + \mathbf{W} = \mathbf{0},$$
$$\boldsymbol{\Gamma}_{\mathrm{N}_1} + \boldsymbol{\Gamma}_{\mathrm{N}_2} + \boldsymbol{\Gamma}_{\mathrm{F}} + \boldsymbol{\Gamma}_{\mathrm{W}} = \mathbf{0}.$$

In equilibrium,

$$|\mathbf{F}| \leq \mu|\mathbf{N}_1|,$$

where μ is the coefficient of static friction.

By the choice of origin, two of the position vectors are zero. The other position vectors are ◀ Solve equation(s) ▶

$$\mathbf{r}_{\mathrm{W}} = -l\cos\tfrac{\pi}{3}\,\mathbf{i} + l\sin\tfrac{\pi}{3}\,\mathbf{j} = -\tfrac{1}{2}l\,\mathbf{i} + \tfrac{\sqrt{3}}{2}l\,\mathbf{j},$$
$$\mathbf{r}_{\mathrm{N}_2} = -\sqrt{3}r\cos\tfrac{\pi}{3}\,\mathbf{i} + \sqrt{3}r\sin\tfrac{\pi}{3}\,\mathbf{j} = -\tfrac{\sqrt{3}}{2}r\,\mathbf{i} + \tfrac{3}{2}r\,\mathbf{j}.$$

The forces are given by

$$\mathbf{N}_1 = |\mathbf{N}_1|\,\mathbf{j},$$
$$\mathbf{N}_2 = |\mathbf{N}_2|\cos\tfrac{\pi}{6}\,\mathbf{i} + |\mathbf{N}_2|\sin\tfrac{\pi}{6}\,\mathbf{j} = \tfrac{\sqrt{3}}{2}|\mathbf{N}_2|\,\mathbf{i} + \tfrac{1}{2}|\mathbf{N}_2|\,\mathbf{j},$$
$$\mathbf{F} = |\mathbf{F}|\,(-\mathbf{i}),$$
$$\mathbf{W} = mg\,(-\mathbf{j}).$$

So the non-zero torques are

$$\boldsymbol{\Gamma}_{\mathrm{W}} = \left(-\tfrac{1}{2}l\,\mathbf{i} + \tfrac{\sqrt{3}}{2}l\,\mathbf{j}\right) \times (-mg\,\mathbf{j}) = \tfrac{1}{2}mgl\,\mathbf{k},$$

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathrm{N}_2} &= \left(-\tfrac{\sqrt{3}}{2}r\,\mathbf{i} + \tfrac{3}{2}r\,\mathbf{j}\right) \times \left(\tfrac{\sqrt{3}}{2}|\mathbf{N}_2|\,\mathbf{i} + \tfrac{1}{2}|\mathbf{N}_2|\,\mathbf{j}\right)\\ &= -\tfrac{\sqrt{3}}{4}r\,|\mathbf{N}_2|\,\mathbf{i}\times\mathbf{j} + \tfrac{3\sqrt{3}}{4}r\,|\mathbf{N}_2|\,\mathbf{j}\times\mathbf{i}\\ &= -\sqrt{3}r\,|\mathbf{N}_2|\,\mathbf{k}.\end{aligned}$$

Substituting into the second equilibrium equation and resolving in the $\mathbf{k}$-direction gives

$$-\sqrt{3}r\,|\mathbf{N}_2| + \tfrac{1}{2}mgl = 0.$$

Therefore

$$|\mathbf{N}_2| = \frac{\frac{1}{2}mgl}{\sqrt{3}r} = \frac{mgl}{2\sqrt{3}r}.$$

Resolving the first equilibrium equation in the $\mathbf{i}$-direction gives

$$\tfrac{\sqrt{3}}{2}|\mathbf{N}_2| - |\mathbf{F}| = 0,$$

so

$$|\mathbf{F}| = \frac{\sqrt{3}}{2} \times \frac{mgl}{2\sqrt{3}r} = \frac{mgl}{4r}.$$

Resolving in the **j**-direction gives

$$|\mathbf{N}_1| + \tfrac{1}{2}|\mathbf{N}_2| - mg = 0,$$

so

$$|\mathbf{N}_1| = mg - \frac{1}{2} \times \frac{mgl}{2\sqrt{3}r} = mg\left(1 - \frac{l}{4\sqrt{3}r}\right).$$

Hence, in equilibrium, from the friction condition we have

$$|\mathbf{F}| = \frac{mgl}{4r} \leq \mu|\mathbf{N}_1| = \mu mg\left(1 - \frac{l}{4\sqrt{3}r}\right),$$

so

$$\frac{l}{4r} \leq \mu\left(1 - \frac{l}{4\sqrt{3}r}\right) = \mu\left(\frac{4\sqrt{3}r - l}{4\sqrt{3}r}\right).$$

Now, we are given that the distance OA is greater than l, and as we saw above, the distance OA is $x = \sqrt{3}r$. So we have $\sqrt{3}r > l$. Therefore we certainly have $4\sqrt{3}r - l > 0$, hence we can rearrange the equation above as

$$\mu \geq \frac{\sqrt{3}l}{4\sqrt{3}r - l}.$$

◀ Interpret solution ▶

The model predicts that the least coefficient of static friction that ensures equilibrium is $\sqrt{3}l/(4\sqrt{3}r - l)$.