Algebra

Welcome to MST124

In this module you'll learn the essential ideas and techniques that underpin university-level study in mathematics and mathematical subjects such as physics, engineering and economics. You'll also develop your skills in communicating mathematics.

Here are some of the topics that you'll meet.

Vectors are quantities that have both a size and a direction. For example, a ship on the ocean moves not only with a particular speed, but also in a particular direction. Speed in a particular direction is a vector quantity known as *velocity*.

Calculus is a fundamental topic in mathematics that's concerned with quantities that change continuously. If you know that an object is moving at a constant speed, then it's straightforward to work out how much distance it covers in any given period of time. It's not so easy to do this if the object's speed is *changing* – for example, if it's accelerating, as a falling object does. Calculus can be used to deal with situations like this.

Matrices are rectangular arrays of numbers – for example, any rectangular table of numbers forms a matrix. Matrices have many applications, which involve performing operations on them that are similar to the operations that you perform on individual numbers. For example, you can add, subtract and multiply matrices.

Sequences are lists of numbers. Sequences whose numbers have a connecting mathematical relationship arise in many different contexts. For example, if you invest £100 at a 5% rate of interest paid annually, then the value in pounds of your investment at the beginning of each year forms the sequence $100, 105, 110.25, 115.76, 121.55, \ldots$.

The **complex numbers** include all the real numbers that you know about already, and also many 'imaginary' numbers, such as the square root of -1. Amazingly, they provide a simple way to deal with some types of complicated mathematics that arise in practical problems.

You'll see that not only do the topics above have important practical applications, but they're also intriguing areas of study in their own right.

One of the main aims of the first few units of MST124 is to make sure that you're confident with the basic skills in algebra, graphs, trigonometry, indices and logarithms that you'll need. The mathematics in the later units of the module depends heavily on these basic skills, and you'll find it much easier and much quicker to study and understand if you can work with all the basic skills fluently and correctly.

To help you attain confidence with these skills, the first few units of the module include many revision topics, as well as some new ones. Which parts, and how much, of the revision material you'll need to study will depend on your mathematical background – different students start MST124 with widely differing previous mathematical experiences. When you're deciding which revision topics you need to study, remember that
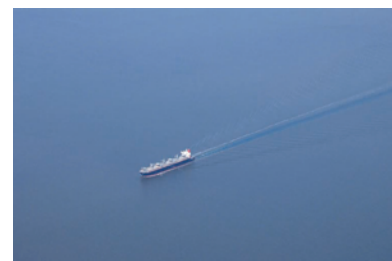
The motion of a ship consists of speed and direction
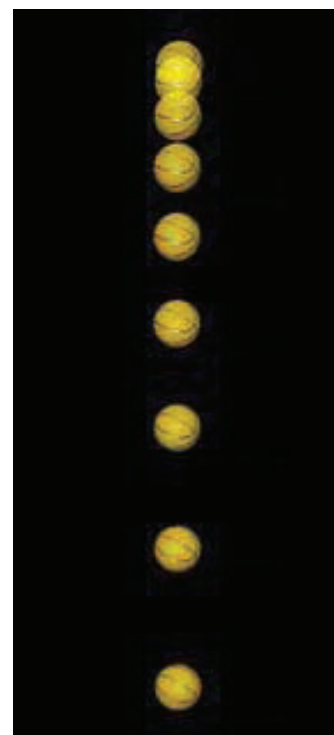
The speed of a falling ball increases as it falls

even though you'll have met most of them before, you won't necessarily have acquired the 'at your fingertips' fluency in working with them that you'll need. Where that's the case, you'll benefit significantly from working carefully through the revision material.

Information for joint MST124 and MST125 students

If you are studying *Essential mathematics 2* (MST125) with the same start date as MST124, then you should *not* study the MST124 units on the dates shown on the main MST124 study planner. Instead, you should follow the MST124 and MST125 joint study planner, which is available from the MST124 and MST125 websites. This is important because you will not be prepared to study many of the topics in MST125 if you have not already studied the related topics in MST124. The MST124 and MST125 joint study planner ensures that you study the units of the two modules interleaved in the correct order.

The MST124 assignment cut-off dates shown in the MST124 and MST125 joint study planner are the same as those shown on the MST124 study planner.

Introduction

The main topic of this first unit is basic algebra, the most important of the essential mathematical skills that you'll need. You'll find it difficult to work through many of the units in the module, particularly the calculus units, if you're not able to manipulate algebraic expressions and equations fluently and accurately. So it's worth spending some time now practising your algebra skills. This unit gives you the opportunity to do that.

The unit covers a lot of topics quite rapidly, in the expectation that you'll be fairly familiar with much of the material. You should use it as a resource to help you make sure that your algebra skills are as good as they can be. You may not need to study all the topics – you should concentrate on those in which you need practice. For many students these will be the topics in Sections 3 to 6. A good strategy might be to read through the whole unit, doing the activities on the topics in which you know you need practice. For the topics in which you think you *don't* need practice, try one or two of the later parts of each activity to make sure – there may be gaps and rustiness in your algebra skills of which you're unaware. Remember to check all your answers against the correct answers provided (these are at the end of the unit in the print book, and can be obtained by pressing the 'show solution' buttons in some screen versions).

As with all the units in this module, further practice questions are available in both the online practice quiz and the exercise booklet for the unit.

Working through the revision material in this unit should also help you to clarify your thinking about algebra. For example, you might know *what to do* with a particular type of algebraic expression or equation, but you might not know, or might have forgotten, *why* this is a valid thing to do. If you can clearly understand the 'why', then you'll be in a much better position to decide whether you can apply the same sort of technique to a slightly different situation, which is the sort of thing that you'll need to do as you study more mathematics.

Some of the topics in the unit may seem very easy – basic algebra is revised starting from the simplest ideas. Others may seem quite challenging – some of the algebraic expressions and equations that you're asked to manipulate may be more complicated than those that you've dealt with before, particularly the ones involving algebraic fractions and indices.

The final section of the unit, Section 6, describes some basic principles of communicating mathematics in writing. This will be important throughout your study of this module and in any further mathematical modules that you study.

If you find that much of the content of this unit (and/or Unit 2) is unfamiliar to you, then contact your tutor and/or Learner Support Team as soon as possible, to discuss what to do.

The word 'algebra' is derived from the title of the treatise *al-Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa'l-muqābala* (Compendium on calculation by completion and reduction), written by the Islamic mathematician Muḥammad ibn Mūsā al-Khwārizmī in around 825. This treatise deals with solving linear and quadratic equations, but it doesn't use algebra in the modern sense, as no letters or other symbols are used to represent numbers. Modern, symbolic algebra emerged in the 1500s and 1600s.

1 Numbers

In this section you'll revise different types of numbers, and some basic skills associated with working with numbers.

1.1 Types of numbers

We'll make a start by briefly reviewing some different types of numbers.

Remember that all the definitions given here, and all the other definitions and important facts and techniques given in the module, are also set out in the *Handbook*, so you can refer to them easily.

The **natural numbers**, also known as the **positive integers**, are the counting numbers,

  1, 2, 3, ...

(The symbol '...' here is called an *ellipsis* and is used when something has been left out. You can read it as 'dot, dot, dot'. In some texts the natural numbers are defined to be $0, 1, 2, 3, ...$)

The natural numbers, together with their negatives and zero, form the **integers**:

  ..., −3, −2, −1, 0, 1, 2, 3, ...

> The Latin word *integer* consists of the prefix *in*, meaning 'not', attached to the root of *tangere*, meaning 'to touch'. So it literally means 'untouched', in the sense of 'whole'.

The **rational numbers** are the numbers that can be written in the form

$$\frac{\text{integer}}{\text{integer}};$$

that is, as an integer divided by an integer.

For example, all the following numbers are rational numbers:

  $\frac{3}{4}$, $2\frac{1}{3}$,   4,   −4, $-\frac{8}{7}$,   0.16,   7.374.

You can check this by writing them in the form above, as follows:

$$\frac{3}{4}, \quad \frac{7}{3}, \quad \frac{4}{1}, \quad \frac{-4}{1}, \quad \frac{-8}{7}, \quad \frac{16}{100}, \quad \frac{7374}{1000}.$$

The **real numbers** include all the rational numbers, and many other numbers as well. A useful way to think of the real numbers is to envisage them as lying along a straight line that extends infinitely far in each direction, called the **number line** or the **real line**. Every point on the number line corresponds to a real number, and every real number corresponds to a point on the line. Some points on the line correspond to rational numbers, while others correspond to numbers that are not rational, which are known as **irrational** numbers. Figure 1 shows some numbers on the number line.
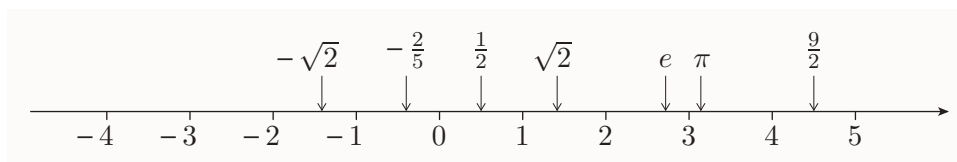
Figure 1 Some numbers on the number line

Four of the numbers marked in Figure 1 are irrational, namely $-\sqrt{2}$, $\sqrt{2}$, e and π. The number $\sqrt{2}$ is the positive square root of 2, that is, the positive number that when multiplied by itself gives the answer 2. Its value is approximately 1.41. The number $-\sqrt{2}$ is the negative of this number. The numbers e and π are two important constants that occur frequently in mathematics. You probably know that π is the number obtained by dividing the circumference of any circle by its diameter (see Figure 2). Its value is approximately 3.14. (The symbol π is a lower-case Greek letter, pronounced 'pie'.) The constant e has value approximately 2.72, and you'll learn more about it in this module, starting in Unit 3.

To check that the numbers $-\sqrt{2}$, $\sqrt{2}$, e and π are irrational, you have to prove that they can't be written as an integer divided by an integer. If you'd like to see how this can be done for $\sqrt{2}$, then look at the document *A proof that $\sqrt{2}$ is irrational* on the module website. Proving that e and π can't be written as an integer divided by an integer is more difficult, and outside the scope of this module.

Figure 2 The circumference and diameter of a circle

Every *rational* number can be written as a decimal number. To do this, you divide the top number of the fraction of the form $\dfrac{\text{integer}}{\text{integer}}$ by the bottom number. For example,

$\frac{1}{8} = 0.125,$

$\frac{2}{3} = 0.666\,666\,666\ldots,$

$\frac{83}{74} = 1.1\,216\,216\,216\,216\ldots.$

As you can see, the decimal form of $\frac{1}{8}$ is **terminating**: it has only a finite number of digits after the decimal point. The decimal forms of both $\frac{2}{3}$ and $\frac{83}{74}$ are **recurring**: each of them has a block of one or more digits after the decimal point that repeats indefinitely. There are two alternative notations for indicating a recurring decimal: you can either put a dot above the first and last digit of the repeating block, or you can put a line above the whole repeating block. For example,

$\frac{2}{3} = 0.666\,666\,666\ldots = 0.\dot{6} = 0.\overline{6}, \quad \text{and}$

$\frac{83}{74} = 1.1\,216\,216\,216\,216\ldots = 1.1\dot{2}1\dot{6} = 1.1\overline{216}.$

In fact, the decimal form of *every* rational number is either terminating or recurring. Also, every terminating or recurring decimal can be written as an integer divided by an integer and is therefore a rational number. If you'd like to know why these facts hold, then look at the document *Decimal forms of rational numbers* on the module website.
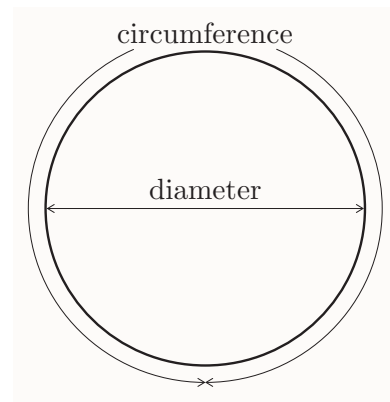
The decimal numbers that are neither terminating nor recurring – that is, those that are infinitely long but have no block of digits that repeats indefinitely – are the irrational numbers. This gives you another way to distinguish between the rational and irrational numbers, summarised below.

> ### Decimal forms of rational and irrational numbers
>
> The rational numbers are the decimal numbers that terminate or recur.
>
> The irrational numbers are the decimal numbers with an infinite number of digits after the decimal point but with no block of digits that repeats indefinitely.

So, for example, the irrational number π has a decimal expansion that is infinitely long and has no block of digits that repeats indefinitely. Here are its first 40 digits:

$$\pi = 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\,502\,884\,197\ldots.$$

You might like to watch the one-minute video clip entitled *The decimal expansion of π*, available on the module website.

> In 2006, a Japanese retired engineer and mental health counsellor, Akira Haraguchi, recited the first $100\,000$ digits of π from memory. It took him 16 hours.
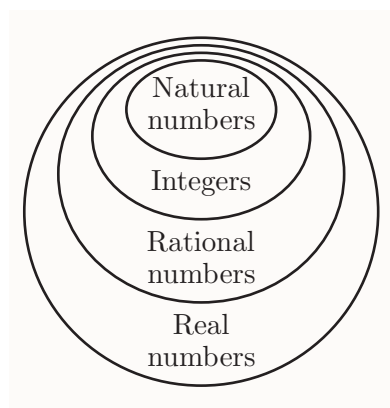

Figure 3 Types of numbers

Figure 3 is a summary of the types of numbers mentioned in this subsection. It illustrates that all the natural numbers are also integers, all the integers are also rational numbers, and all the rational numbers are also real numbers.

In Unit 12 you'll learn about yet another type of number. The *complex numbers* include all the numbers in Figure 3, and also many 'imaginary' numbers, such as the square root of -1. The idea of imaginary numbers might seem strange, but these numbers are the foundation of a great deal of interesting and useful mathematics. They provide a natural, elegant way to work with seemingly complicated mathematics, and have many practical applications.

1.2 Working with numbers

In this subsection, you'll revise some basic skills associated with working with numbers. It's easy to make mistakes with these particular skills, and people often do! So you should find it helpful to review and practise them.

Before doing so, notice the label '(1)' on the right of the next paragraph. It's used later in the text to refer back to the contents of the line in which it appears. Labels like this are used throughout the module.

The BIDMAS rules

When you *evaluate* (find the value of) an expression such as

$$200 - 3 \times (1 + 5 \times 2^3) + 7, \tag{1}$$

it's important to remember the following convention for the order of the operations, so that you get the right answer.

Order of operations: BIDMAS

Carry out mathematical operations in the following order.

$$
\begin{array}{ll}
\textbf{B} & \text{brackets} \\
\textbf{I} & \text{indices (powers and roots)} \\
\textbf{D} & \text{divisions} \\
\textbf{M} & \text{multiplications}
\end{array}
\left.\right\} \text{same precedence}
$$

$$
\begin{array}{ll}
\textbf{A} & \text{additions} \\
\textbf{S} & \text{subtractions}
\end{array}
\left.\right\} \text{same precedence}
$$

Where operations have the same precedence, work from left to right.

As you can see, the I in the BIDMAS rules refers to 'indices (powers and roots)'. Remember that raising a number to a **power** means multiplying it by itself a specified number of times. For example, 2^3 (2 to the power 3) means three 2s multiplied together:

$$2^3 = 2 \times 2 \times 2.$$

In particular, **squaring** and **cubing** a number mean raising it to the powers 2 and 3, respectively. When you write an expression such as 2^3, you're using **index notation**. Taking a *root* of a number means taking its square root, for example, or another type of root. Roots are revised in Subsection 4.1.

If you type an expression like expression (1) into a calculator of the type recommended in the *MST124 guide*, then it will be evaluated according to the BIDMAS rules. However, it's essential that you understand and remember the rules yourself. For example, you'll need to use them when you work with algebra.

Example 1 reminds you how to use the BIDMAS rules. It also illustrates another feature that you'll see throughout the module. Some of the worked examples include lines of blue text, marked with the following icons.

This text tells you what someone doing the mathematics might be thinking, but wouldn't write down. It should help you understand how you might do a similar calculation yourself.

Example 1   *Using the BIDMAS rules*

Evaluate the expression

$$200 - 3 \times (1 + 5 \times 2^3) + 7$$

without using your calculator.

Solution

🔍 The brackets have the highest precedence, so start by evaluating what's inside them. Within the brackets, first deal with the power, then do the multiplication, then the addition. 💬

$$
\begin{aligned}
200 - 3 \times (1 + 5 \times 2^3) + 7 &= 200 - 3 \times (1 + 5 \times 8) + 7 \\
&= 200 - 3 \times (1 + 40) + 7 \\
&= 200 - 3 \times 41 + 7
\end{aligned}
$$

🔍 Now do the multiplication, then the addition and subtraction from left to right. 💬

$$
\begin{aligned}
&= 200 - 123 + 7 \\
&= 77 + 7 \\
&= 84
\end{aligned}
$$

You can practise using the BIDMAS rules in the next activity. Remember that where division is indicated using fraction notation, the horizontal line not only indicates division but also acts as brackets for the expressions above and below the line. For example,

$$\frac{1+2}{1+3^2} \quad \text{means} \quad \frac{(1+2)}{(1+3^2)}, \quad \text{that is,} \quad (1+2) \div (1+3^2).$$

In a line of text, this expression would normally be written as $(1+2)/(1+3^2)$, with a slash replacing the horizontal line. The brackets are needed here because $1 + 2/1 + 3^2$ would be interpreted as $1 + (2/1) + 3^2$.

Part (b) of the activity involves algebraic expressions. Remember that multiplication signs are usually omitted when doing algebra – quantities that are multiplied are usually just written next to each other instead (though, for example, 3×4 can't be written as 34).

Activity 1   *Using the BIDMAS rules*

(a) Evaluate the following expressions without using your calculator.

(i) $23 - 2 \times 3 + (4 - 2)$ (ii) $2 - \frac{1}{2} \times 4$ (iii) 4×3^2

(iv) $2 + 2^2$ (v) $\dfrac{1 + 2}{1 + 3^2}$ (vi) $1 - 2/3^2$

(b) Evaluate the following expressions when $a = 3$ and $b = 5$, without using your calculator.

(i) $3(b - a)^2$ (ii) $a + b(2a + b)$ (iii) $a + 9\left(\dfrac{b}{a}\right)$ (iv) $30/(ab)$

Rounding

When you use your calculator to carry out a calculation, you often need to round the result. There are various ways to round a number. Sometimes it's appropriate to round to a particular number of **decimal places** (often abbreviated to 'd.p.'). The decimal places of a number are the positions of the digits to the right of the decimal point. You can also round to the nearest whole number, or to the nearest 10, or to the nearest 100, for example. More often, it's appropriate to round to a particular number of **significant figures** (often abbreviated to 's.f.' or 'sig. figs.'). The first significant figure of a number is the first non-zero digit (from the left), the next significant figure is the next digit along (whether zero or not), and so on.

Once you've decided where to round a number, you need to look at the digit immediately after where you want to round. You round up if this digit is 5 or more, and round down otherwise. When you round a number, you should state how it's been rounded, in brackets after the rounded number, as illustrated in the next example.

Notice the 'play button' icon next to the following example. It indicates that the example has an associated *tutorial clip* – a short video in which a tutor works through the example and explains it. You can watch the clip, which is available on the module website, instead of reading through the worked example. Many other examples in the module have tutorial clips, indicated by the same icon.

Example 2   *Rounding numbers*

Round the following numbers as indicated.

(a) 0.0238 to three decimal places

(b) 50 629 to three significant figures

(c) 0.002 958 2 to two significant figures

Solution

(a) Look at the digit after the first three decimal places: 0. 023̲ **8**. It's 8, which is 5 or more, so round up.

$0.0238 = 0.024$ (to 3 d.p.)

(b) Look at the digit after the first three significant figures: 506̲ **2**9. It's 2, which is less than 5, so round down.

$50\,629 = 50\,600$ (to 3 s.f.)

(c) Look at the digit after the first two significant figures: 0.00 29̲ **5**82. It's 5, which is 5 or more, so round up.

$0.002\,958\,2 = 0.0030$ (to 2 s.f.)

Notice that in Example 2(c), a 0 is included after the 3 to make it clear that the number is rounded to *two* significant figures. You should do likewise when you round numbers yourself.

Activity 2   *Rounding numbers*

Round the following numbers as indicated.

(a) 41.394 to one decimal place

(b) 22.325 to three significant figures

(c) 80 014 to three significant figures

(d) 0.056 97 to two significant figures

(e) 0.006 996 to three significant figures

(f) 56 311 to the nearest hundred

(g) 72 991 to the nearest hundred

The use of the digit 0 to indicate an empty place in the representation of a number seems essential nowadays. For example, the digit 0 in 3802 distinguishes it from 382. However, many civilisations managed to use place-value representations of numbers for hundreds of years with no symbol for the digit zero. Instead, they distinguished numbers by their context. Evidence from surviving clay tablets shows that the Babylonians used place-value representations of numbers from at least 2100 BC, and used a place-holder for zero from around 600 BC.

A Babylonian clay tablet from around 1700 BC

When you need to round a negative number, you should round the part after the minus sign in the same way that you would round a positive number. For example,

$$-0.25 = -0.3 \text{ (to 1 d.p.).}$$

When you're rounding an answer obtained from your calculator, it's often useful to write down a more precise version of the answer before you round it. You can do this by using the '...' symbol, like this:

$$9.869\,604\,40\ldots = 9.87 \text{ (to 2 d.p.).}$$

Also, as an alternative to writing in brackets how you rounded a number, you can replace the equals sign by the symbol $\approx$, which means, and is read as, 'is approximately equal to'. For example, you can write

$$9.869\,604\,40\ldots \approx 9.87.$$

The activities and TMA questions in this module will sometimes tell you what rounding to use in your answers. In other situations where you need to round answers, a useful rule of thumb is to round to the number of significant figures in the *least* precise number used in your calculation. For example, suppose that you're asked to calculate how long it would take to travel 11 400 metres at a speed of 8.9 metres per second. The first and second numbers here seem to be given to three and two significant figures, respectively, so you would round your answer to two significant figures. Note, however, that there are situations where this rule of thumb is not appropriate. Note also that if an activity or TMA question includes a number with no units, such as '120', then you should usually assume that this number is exact, whereas if it includes a number with units, such as '120 cm', then you should usually assume that this is a measurement and has been rounded.

Now try the following activity. Don't skip it: it might look easy, but it illustrates an important point about rounding – one that's a common source of errors. To do the activity, you need to use the facts that the radius r, circumference c and area A of any circle (see Figure 4) are linked by the formulas

$$c = 2\pi r \quad \text{and} \quad A = \pi r^2.$$

Remember to use the π button on your calculator to obtain a good approximation for π.
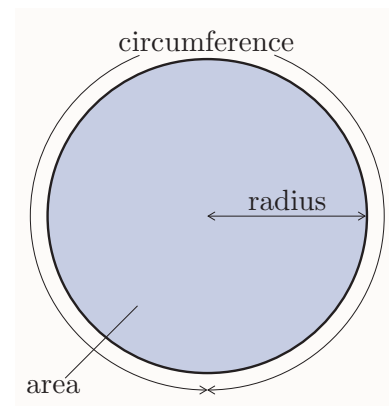
Figure 4   The radius, circumference and area of a circle

Activity 3   *Rounding in a multi-stage calculation*

The circumference of a circle is $77.2\,\text{cm}$.

(a) Find the radius of the circle, in cm to three significant figures.

(b) Find the area of the circle, in cm^2 to two significant figures.

The correct answer to part (b) of Activity 3 is $470\,\text{cm}^2$. If you obtained the answer $480\,\text{cm}^2$, then this was probably because you carried out your calculation in part (b) using the rounded answer for the radius that you found in part (a). To obtain the correct answer in part (b), you need to use a more precise value for the radius, such as the value that you obtained on your calculator before you rounded it. (Alternatively, you could combine the two formulas $c = 2\pi r$ and $A = \pi r^2$ to obtain the formula $A = c^2/(4\pi)$ for A in terms of c, and use that to obtain the answer to part (b).)

Errors of this sort are known as **rounding errors**. To avoid them, whenever you carry out a calculation using an answer that you found earlier, you should use the full-calculator-precision version of the earlier answer. One way to do this is to write down the full value and re-enter it in your calculator, but a more convenient way is to store it in your calculator's memory. Another convenient way to avoid rounding errors is to carry out your calculations using a computer algebra system. You'll start to learn how to do this in Unit 2.

Sometimes people who work with numbers, such as statisticians, use slightly different rounding conventions to those described above. These alternative conventions usually differ only in how they deal with cases in which the digit immediately after where you want to round is the last non-zero digit of the unrounded number, and it's a 5. For example, with some conventions, $3.65 = 3.6$ (to 1 d.p.), and there are conventions for which $-0.25 = -0.2$ (to 1 d.p.).

Negative numbers

Negative numbers occur frequently in mathematics, so it's important that you're confident about working with them.

When you're carrying out calculations that involve negative numbers, it's sometimes helpful to mention the **sign** of a number. This is either $+$ or $-$, that is, plus or minus, according to whether the number is positive or negative, respectively. The number zero doesn't have a sign.

Here's a reminder of how to deal with addition and subtraction when negative numbers are involved.

When you have a number (positive, negative or zero), and you want to add or subtract a *positive* number, you simply increase or decrease the number that you started with by the appropriate amount. For example, as shown in Figure 5, to add 3 to -5 you increase -5 by 3, and to subtract 3 from -5 you decrease -5 by 3. This gives:

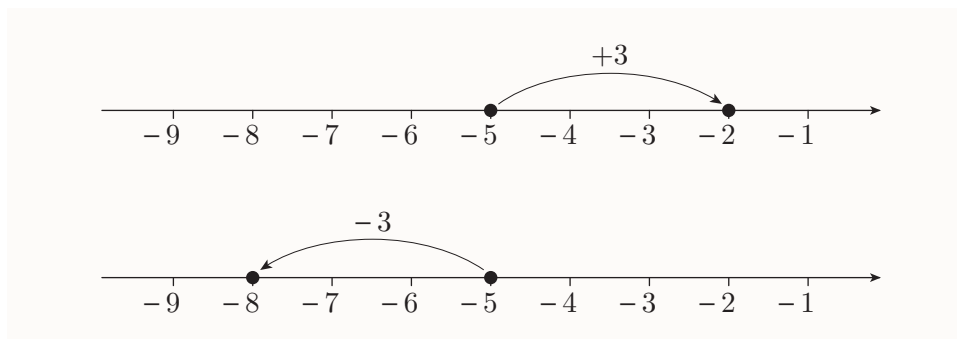$$-5 + 3 = -2 \quad \text{and} \quad -5 - 3 = -8.$$

Figure 5   Increasing -5 by 3 and decreasing -5 by 3

When you have a number (positive, negative or zero), and you want to add or subtract a *negative* number, you use the rules below.

Adding and subtracting negative numbers

Adding a negative number is the same as subtracting the corresponding positive number.

Subtracting a negative number is the same as adding the corresponding positive number.

For example,

$$7 + (-3) = 7 - 3 = 4 \quad \text{and} \quad -9 - (-3) = -9 + 3 = -6.$$

Now here's a reminder of how to deal with multiplication and division when you're working with negative numbers.

To multiply or divide two negative or positive numbers, you multiply or divide them without their signs in the usual way, and use the rules below to find the sign of the answer.

Multiplying and dividing negative numbers

When two numbers are multiplied or divided:

- if the signs are *different*, then the answer is *negative*
- if the signs are *the same*, then the answer is *positive*.

For example,

$$2 \times (-3) = -6,$$
$$\frac{-8}{-2} = 4,$$
$$2 \times (-3) \times (-5) = (-6) \times (-5) = 30.$$

Notice that some of the negative numbers in the calculations above are enclosed in brackets. This is because no two of the mathematical symbols $+$, $-$, $\times$ and $\div$ should be written next to each other, as that would look confusing (and is often meaningless). So if you want to show that you're adding -2 to 4, for example, then you shouldn't write $4 + -2$, but instead you should put brackets around the -2 and write $4 + (-2)$.

If you'd like to know more about *why* negative numbers are added, subtracted, multiplied and divided in the way that they are, then have a look at the document *Arithmetic of negative numbers* on the module website.

You can practise working with negative numbers in the next activity. Remember that the BIDMAS rules apply in the usual way.

Activity 4   *Working with negative numbers*

Evaluate the following expressions without using a calculator.

(a) $-3 + (-4)$ (b) $2 + (-3)$ (c) $2 - (-3)$ (d) $-1 - (-5)$

(e) $5 \times (-4)$ (f) $\dfrac{-15}{-3}$ (g) $(-2) \times (-3) \times (-4)$

(h) $6(-3 - (-1))$ (i) $20 - (-5) \times (-2)$

(j) $-5 + (-3) \times (-1) - 2 \times (-2)$ (k) $\dfrac{-2 - (-1) \times (-2)}{-8}$

When you're working with negative numbers, there's an extra operation that you have to deal with, as well as the usual operations of addition, subtraction, multiplication and division. When you put a minus sign in front of a number, the new number that you get is called the **negative** of the original number. For example, the negative of 4 is -4. The operation of putting a minus sign in front of a number is called **taking the negative** of the number.

You can take the negative of a number that's already negative. This changes its sign to plus. For example,

$$-(-7) = +7 = 7.$$

To see why this is, notice that taking the negative of a *positive* number is the same as subtracting it from zero: for example, $-3 = 0 - 3$. It's just the same for negative numbers: $-(-7) = 0 - (-7) = 0 + 7 = 7$.

You can also take the negative of zero. This leaves it unchanged: $-0 = 0 - 0 = 0$.

In general, taking the negative of a positive or negative number changes its sign to the opposite sign. Taking the negative of zero leaves it unchanged. Another helpful way to think about negatives is that a number and its negative always add up to zero.

The operation of taking a negative has the same precedence in the BIDMAS rules as subtraction. For example, in the expression -3^2, the operation of taking the power is done *before* the operation of taking the negative, by the BIDMAS rules. So -3^2 is equal to -9, not 9, as you might have expected. However, $(-3)^2$ is equal to 9.

Activity 5   *Practice with taking negatives*

Evaluate the following expressions without using a calculator. Then check that your calculator gives the same answers.

(a) -5^2 (b) $(-5)^2$ (c) $-(-8)$ (d) $-(-8)^2$ (e) $-2^2 + 7$

(f) $-(-5) - (-1)$ (g) $-4^2 - (-4)^2$ (h) $-3 \times (-2^2)$

When you substitute a negative number into an algebraic expression, you usually have to enclose it in brackets, to make sure that you evaluate the expression correctly. For example, to evaluate the expression $x^2 - 2x$ when $x = -3$, you proceed as follows:

$$\begin{aligned} x^2 - 2x &= (-3)^2 - 2 \times (-3) \\ &= 9 - (-6) \\ &= 9 + 6 = 15. \end{aligned}$$

Activity 6   *Substituting negative numbers into algebraic expressions*

Evaluate the following expressions when $a = -2$ and $b = -3$, without using a calculator.

(a) $-b$ (b) $-a - b$ (c) $-b^2$ (d) $a^2 + ab$ (e) $\dfrac{3 - a^2}{b}$

(f) $a^2 - 2a + 5$ (g) $(6 - a)(2 + b)$ (h) a^3 (i) $-b^3$

Units of measurement

Most of the units of measurement used in this module come from the standard metric system known as the Système International d'Unités (SI units). This system is used by the scientific community generally and is the main system of measurement in nearly every country in the world.

There are seven SI **base units**, from which all the other units are derived. The base units (and their abbreviations) used in this module are the metre (m), the kilogram (kg) and the second (s). Prefixes are used to indicate smaller or larger units. Some common prefixes are shown in Table 1.

Table 1   Some prefixes for SI units

Prefix	Abbreviation	Meaning	Example
nano	n	a billionth $\left(\frac{1}{10^9}\right)$	1 nanometre (nm) $= \frac{1}{10^9}$ metre
micro	μ	a millionth $\left(\frac{1}{10^6}\right)$	1 micrometre (μm) $= \frac{1}{10^6}$ metre
milli	m	a thousandth $\left(\frac{1}{1000}\right)$	1 millimetre (mm) $= \frac{1}{1000}$ metre
centi	c	a hundredth $\left(\frac{1}{100}\right)$	1 centimetre (cm) $= \frac{1}{100}$ metre
kilo	k	a thousand (1000)	1 kilometre (km) $= 1000$ metres
mega	M	a million (10^6)	1 megametre (Mm) $= 10^6$ metres

The metric system was founded in France after the French Revolution. Its modern form, the SI, was founded in 1960 and continues to evolve, keeping pace with the increasing precision of measurement. At the time of writing, only three countries have chosen not to adopt SI units as their sole/primary system of measurement.

Four mathematical words

Finally in this subsection, here's a reminder of four standard mathematical words that are used frequently throughout the module.

- The **sum** of two or more numbers is the result of adding them.
- The **product** of two or more numbers is the result of multiplying them.
- A **difference** of two numbers is the result of subtracting one from the other.
- A **quotient** of two numbers is the result of dividing one by the other.

Each pair of numbers has *two* differences and (provided neither of the numbers in the pair is zero) *two* quotients. For example, the numbers 2

and 8 have the two differences $8 - 2 = 6$ and $2 - 8 = -6$, and the two quotients $\frac{8}{2} = 4$ and $\frac{2}{8} = \frac{1}{4}$. When we say *the* difference of a pair of positive numbers, we mean their positive difference. For example, the difference of 2 and 8 is 6.

1.3 Integers

In this subsection, you'll revise some properties of the integers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.$$

If an integer a divides exactly into another integer b, then we say that

- b is a **multiple** of a, or
- b is **divisible** by a, or
- a is a **factor** or **divisor** of b.

For example, 15 is a multiple of 5; also 15 is divisible by 5; and 5 is a factor of 15. Similarly, -15 is a multiple of 5, and so on.

Notice that every integer is both a multiple and a factor of itself: for example, 5 is a multiple of 5, and 5 is a factor of 5.

A **factor pair** of an integer is a pair of its factors that multiply together to give the integer. For example, the factor pairs of 12 are

$$1, 12; \quad 2, 6; \quad 3, 4; \quad -1, -12; \quad -2, -6; \quad -3, -4;$$

and the factor pairs of -4 are

$$1, -4; \quad 2, -2; \quad -1, 4; \quad -2, 2.$$

(The order of the two numbers within a factor pair doesn't matter – for example, the factor pair 1, 12 is the same as the factor pair 12, 1.)

A **positive factor pair** of a positive integer is a factor pair in which both factors are positive. For example, the positive factor pairs of 12 are

$$1, 12; \quad 2, 6; \quad 3, 4.$$

An integer that has a factor pair in which the same factor is repeated – in other words, an integer that can be written in the form a^2 for some integer a – is called a **square number** or a **perfect square**. For example, 9 is a square number, since $9 = 3^2$. Here are the first few square numbers.

> **The square numbers up to 15^2**
>
> 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225

You can use the following strategy to help you find all the positive factor pairs (and hence all the positive factors) of a positive integer.

Strategy:
To find the positive factor pairs of a positive integer

- Try dividing the integer by each of the numbers $1, 2, 3, 4, \ldots$ in turn. Whenever you find a factor, write it down along with the other factor in the factor pair.

- Stop when you get a factor pair that you have already.

Activity 7 *Finding factor pairs of integers*

(a) For each of the following positive integers, find all its positive factor pairs and then list all its positive factors in increasing order.

(i) 28   (ii) 25   (iii) 36

(b) Use your answer to part (a)(i) to find *all* the factor pairs of the following integers.

(i) 28   (ii) -28

An integer greater than 1 whose only factors are 1 and itself is called a prime number, or just a **prime**. For example, 3 is a prime number because its only factors are 1 and 3, but 4 is not a prime number because its factors are 1, 2 and 4. Note that 1 is not a prime number.

Here are the first 25 prime numbers.

The prime numbers under 100

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

An integer greater than 1 that isn't a prime number is called a **composite number**. The first ten composite numbers are

4, 6, 8, 9, 10, 12, 14, 15, 16, 18.

You can break down any composite number into a product of prime factors. For example, here's how you can break down the composite number 504:

$$504 = 2 \times 252$$
$$= 2 \times 2 \times 126$$
$$= 2 \times 2 \times 2 \times 63$$
$$= 2 \times 2 \times 2 \times 3 \times 21$$
$$= 2 \times 2 \times 2 \times 3 \times 3 \times 7.$$

The product of prime numbers in the last line of this working is the *only* product of prime numbers that's equal to 504 (except that you can change the order of the primes in the product, of course: for example, $504 = 2 \times 7 \times 2 \times 2 \times 3 \times 3$). In general, the following important fact holds.

> ### The fundamental theorem of arithmetic
>
> Every integer greater than 1 can be written as a product of prime factors in just one way (except that you can change the order of the factors).

The **prime factorisation** of an integer greater than 1 is any expression that shows it written as a product of prime factors. For example, the working above shows that the prime factorisation of 504 is

$$504 = 2^3 \times 3^2 \times 7.$$

Here the prime factors are written using index notation, with the prime factors in increasing order. This is the usual way that prime factorisations are written.

Notice that the fundamental theorem of arithmetic applies to all integers greater than 1, not just composite numbers. The prime factorisation of a prime number is just the prime number itself (a 'product of one prime'!).

The strategy that was used above to find the prime factorisation of 504 is summarised below. You can use it to find the prime factorisation of any integer greater than 1 (though for a large integer it can take a long time).

> **Strategy:**
> **To find the prime factorisation of an integer greater than 1**
>
> - Repeatedly 'factor out' the prime 2 until you obtain a number that isn't divisible by 2.
>
> - Repeatedly factor out the prime 3 until you obtain a number that isn't divisible by 3.
>
> - Repeatedly factor out the prime 5 until you obtain a number that isn't divisible by 5.
>
> Continue this process with each of the successive primes, 2, 3, 5, 7, 11, 13, 17, ..., in turn. Stop when you have a product of primes.
>
> Write out the prime factorisation with the prime factors in increasing order, using index notation for any repeated factors.

Your calculator may be able to find prime factorisations of integers, but it's useful to know how to do this yourself, to improve your understanding of numbers and algebra.

Activity 8   *Finding prime factorisations*

Find the prime factorisations of the following integers, without using your calculator.

(a) 594   (b) 525   (c) 221   (d) 223

We finish this section with a reminder about *common multiples* and *common factors*.

A **common multiple** of two or more integers is a number that is a multiple of all of them. For example, the common multiples of 4 and 6 are $\ldots, -36, -24, -12, 0, 12, 24, 36, \ldots$.

The **lowest** (or **least**) **common multiple** (**LCM**) of two or more integers is the *smallest positive* integer that is a multiple of all of them. For example, the lowest common multiple of 4 and 6 is 12.

Similarly, a **common factor** of two or more integers is a number that is a factor of all of them. For example, the common factors of 24 and 30 are $-6, -3, -2, -1, 1, 2, 3$ and 6.

The **highest common factor** (**HCF**)(or **greatest common divisor** (**GCD**)) of two or more integers is the *largest positive* number that is a factor of all of them. For example, the highest common factor of 24 and 30 is 6.

Notice that the lowest common multiple and highest common factor of two or more integers are the same as the lowest common multiple and highest common factor of the corresponding *positive* integers. For example, the

lowest common multiple of −4 and 6 is the same as the lowest common multiple of 4 and 6, which is 12. So you only ever need to find the lowest common multiple and the highest common factor of two or more *positive* integers.

You can often find the lowest common multiple or highest common factor of two or more fairly small positive integers simply by thinking about their common multiples or common factors. For larger integers, or more tricky cases, you can use their prime factorisations, as illustrated in the next example.

Example 3   *Using prime factorisations to find LCMs and HCFs*

Find the lowest common multiple and highest common factor of 594 and 693.

Solution

💬 Find and write out the prime factorisations, lining up each prime with the same prime in the other factorisation(s), where possible. 💬

$$594 = 2 \times 3^3 \quad\ \times 11$$
$$693 = \quad\ \ 3^2 \times 7 \times 11$$

💬 To find the lowest common multiple, identify the *highest* power of the prime in each column, and multiply all these numbers together. 💬

The LCM of 594 and 693 is

$$2 \times 3^3 \times 7 \times 11 = 4158.$$

💬 To find the highest common factor, identify the *lowest* power of the prime in each column, and multiply all these numbers together. (Omit any primes, such as 2 and 7 here, that are missing from one or more rows.) 💬

The HCF of 594 and 693 is

$$3^2 \times 11 = 99.$$

$$②\times ③^3 \qquad \times ⑪$$
$$3^2 \times ⑦\times 11$$

Figure 6   Highest powers of primes in columns

$$2 \times 3^3 \qquad \times ⑪$$
$$③^2\times 7 \times 11$$

Figure 7   Lowest powers of primes in columns

The methods used in Example 3 can be summarised as follows.

> **Strategy:**
> **To find the lowest common multiple or highest common factor of two or more integers greater than 1**
> - Find the prime factorisations of the numbers.
> - To find the LCM, multiply together the highest power of each prime factor occurring in any of the numbers.
> - To find the HCF, multiply together the lowest power of each prime factor common to all the numbers.

Activity 9   *Using prime factorisations to find LCMs and HCFs*

Find the prime factorisations of 9, 18 and 24, and use them to find the lowest common multiple and highest common factor of each of the following sets of numbers.

(a) 18 and 24   (b) 9, 18 and 24   (c) -18 and -24

There's an efficient method, known as *Euclid's algorithm*, for finding lowest common multiples and highest common factors, without having to find prime factorisations first. You can learn about it in the module *Essential Mathematics 2* (MST125).

2 Algebraic expressions

In this section you'll revise some basic skills that you need when working with algebraic expressions.

2.1 Algebraic terminology

In mathematics, an **expression** is an arrangement of letters, numbers and/or mathematical symbols (such as $+$, $-$, $\times$, $\div$, brackets, and so on), which is such that if numbers are substituted for any letters present, then you can work out the value of the arrangement. So, for example, $3x + 4$ is an expression, but $3x + \div 4$ isn't, because '$+ \div$' doesn't make sense.

An expression that contains letters (usually as well as numbers and mathematical symbols) is an **algebraic expression**. An expression, such as $10 - 2 \times 4$, that contains only numbers and mathematical symbols is a **numerical expression**. An expression that's part of a larger expression, and can be enclosed in brackets without affecting the meaning of the larger expression, is called a **subexpression** of the larger expression. For example, the expression $3x$ is a subexpression of the expression $3x + 4$, but $x + 4$ is not, because $3(x + 4)$ has a different meaning.

Letters representing numbers in algebraic expressions can be any of three different types. A letter may be a **variable**: this means that it represents any number, or any number of a particular kind, such as any positive number, or any integer. Alternatively, it may be an **unknown**: this means that it represents a particular number that you don't know, but usually you want to discover, perhaps by solving an equation. Or it may represent a **constant**, a particular number whose value is specified, or regarded as unchanging for a particular calculation.

For example, as you saw earlier, there's a mathematical constant, whose value is approximately 2.72, that's denoted by the letter e. The word 'variable' is sometimes used as a catch-all for both variables and unknowns. When you're working with algebraic expressions, you don't usually need to think about which types of letter they contain – you treat all letters that represent numbers in a similar way.

As you know, multiplication signs are usually omitted in algebraic expressions (unless they are between two numbers) – things that are multiplied are just written next to each other instead. For example, $3 \times x$ is written as $3x$. Similarly, division signs are not normally used – fraction notation is used instead. For example, $3 \div x$ is usually written as

$$\frac{3}{x},$$

or as $3/x$, within a line of text. However, sometimes it's helpful to include multiplication signs or division signs in algebraic expressions.

You use equals signs when you're working with expressions, but expressions don't *contain* equals signs. For example, the statement

$$x + 2x = 3x$$

isn't an expression – it's an equation. An **equation** is made up of *two* expressions, with an equals sign between them.
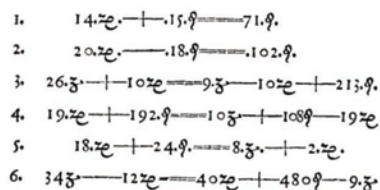
Figure 8 The first equations written using equals signs, from Robert Recorde's *The Whetstone of Witte* (1577)

> The equals sign was introduced by the Tudor mathematician Robert Recorde. It first appeared in his algebra book *The Whetstone of Witte* of 1577, where he described his invention as a useful abbreviation: 'And to auoide the tedioufe repetition of thefe woordes: is equalle to: I will fette as I doe often in woorke vfe, a paire of paralleles, or Gemowe [twin] lines of one lenghte, thus: =, bicaufe noe .2. thynges can be moare equalle.' This statement is immediately followed by the first equations to be written using his new notation, which are reproduced in Figure 8.

If an expression is a list of quantities that are all added or subtracted, then it's often helpful to think of it as a list of quantities that are all *added*. For example, the expression

$$5ab - 2a^2 + 3b + 6\sqrt{a} - 4 \tag{2}$$

means the same as

$$5ab + (-2a^2) + 3b + 6\sqrt{a} + (-4).$$

The quantities that are added are called the **terms** of the expression. For example, the terms of expression (2) are

$$5ab, \quad -2a^2, \quad 3b, \quad 6\sqrt{a} \quad \text{and} \quad -4.$$

A term like -4, that doesn't contain any variables, is known as a **constant term** (because its value doesn't change when the values of the variables in the expression are changed). On the other hand, if a term has the form

a fixed value $\times$ a combination of variables,

then the fixed value is called the **coefficient** of the term, and we say that the term is a term *in* whatever the combination of variables is. For example, in expression (2),

$5ab$ has coefficient 5 and is a term in ab;

$-2a^2$ has coefficient -2 and is a term in a^2;

$3b$ has coefficient 3 and is a term in b;

$6\sqrt{a}$ has coefficient 6 and is a term in $\sqrt{a}$;

-4 is a constant term.

It's possible for a term to have coefficient 1 or -1. For example, in the expression $x^2 - x$, the two terms x^2 and $-x$ have coefficients 1 and -1, respectively. This is because these terms can also be written as $1x^2$ and $-1x$, though normally we wouldn't write them like that, as the forms x^2 and $-x$ are simpler. It's also possible for a coefficient to include constants. For example, in the expression $8r^3 - \frac{4}{3}\pi r^3$ the term $-\frac{4}{3}\pi r^3$ is a term in r^3 and its coefficient is $-\frac{4}{3}\pi$.

Sometimes two different expressions 'mean the same thing'. For example, $x + x$ and $2x$ are two ways of saying that there are 'two lots of x'. Expressions like these, that mean the same, are said to be *equivalent*.

More precisely, we say that two expressions are **equivalent**, or different **forms** of the same expression, if they have the same value as each other whatever values are chosen for their variables. We usually indicate this by writing an equals sign between them. (In some texts, the symbol $\equiv$ is used instead.) We also say that either expression can be **rearranged**, **manipulated** or **rewritten** to give the other. **Simplifying** an expression means rewriting it as a simpler expression.

One way to change an expression into an equivalent expression is to change the order of the terms. This doesn't change the meaning of the expression, because the order in which you add quantities doesn't affect the overall result. For example, the expressions

$$5ab - 2a^2 + 3b + 6\sqrt{a} - 4 \quad \text{and} \quad -2a^2 - 4 + 5ab + 6\sqrt{a} + 3b$$

are equivalent, because they are each the sum of the same five terms

$$5ab, \quad -2a^2, \quad 3b, \quad 6\sqrt{a} \quad \text{and} \quad -4.$$

2.2   Simplifying algebraic expressions

When you're working with an algebraic expression, you should usually try to write it in as simple a form as you can. One way in which some expressions can be simplified is by **collecting like terms**. If two or more terms of an expression differ only in the value of their coefficients, then you can combine them into a single term by adding the coefficients. For example:

$$3h + 9h - h \qquad \text{can be simplified to} \quad 11h;$$
$$a^2 - 3ab - 5a^2 + ab \quad \text{can be simplified to} \quad -4a^2 - 2ab.$$

You can also often simplify the individual terms in an expression. For example, the term

$$ab(-a) \quad \text{can be simplified to} \quad -a^2b.$$

Usually a term in an expression consists of a product of numbers and letters (and possibly other items like square roots). If such a term isn't in its simplest form, then you can simplify it by using the following strategy.

> **Strategy:**
> **To simplify a term**
>
> 1. Find the overall sign and write it at the front.
> 2. Simplify the rest of the coefficient and write it next.
> 3. Write any remaining parts of the term in an appropriate order; for example, letters are usually ordered alphabetically. Use index notation to avoid writing letters (or other items) more than once.

In Step 1 of the strategy, you find the overall sign of the term by using the following rules, which you saw for numbers on page 16.

When multiplying or dividing:

two signs the same give a plus sign;

two different signs give a minus sign.

Remember that a plus sign or a minus sign at the start of a term has the same effect as multiplying the term by 1 or -1, respectively, so you can apply the rule in the box above to such signs.

Example 4 *Simplifying single terms*

Simplify the following single-term expressions.

(a) $+(-2x)$ (b) $-(-7y)$ (c) $-3a^3b \times (-2a^2b)$

(d) $-c \times (-c) \times d \times (-d)$ (e) $(2x)^3$

Solution

(a) A positive times a negative gives a negative, so the overall sign is minus.

$+(-2x) = -2x$

(b) A negative times a negative gives a positive, so the overall sign is plus.

$-(-7y) = +7y = 7y$

(c) First find the overall sign. A negative times a negative gives a positive, so the overall sign is plus. The rest of the coefficient is $3 \times 2 = 6$. The last part of the term is $a^3b \times a^2b = a^5b^2$.

$-3a^3b \times (-2a^2b) = +6a^5b^2 = 6a^5b^2$

(d) First find the overall sign. A negative times a negative gives a positive, then this positive times a positive gives a positive, then this positive times a negative gives a negative. So the overall sign is minus. The rest of the term is $c \times c \times d \times d = c^2d^2$.

$-c \times (-c) \times d \times (-d) = -c^2d^2$

(e) Use the fact that the cube of a number is three copies of the number multiplied together. Then proceed in the same way as in the earlier parts.

$(2x)^3 = (2x) \times (2x) \times (2x) = 8x^3$

Parts (a) and (b) of Example 4 illustrate the following rules, which you saw for numbers on page 15.

- Adding the negative of something is the same as subtracting the something.

- Subtracting the negative of something is the same as adding the something.

Notice also that in Example 4(c) the final power of a was found by calculating $a^3 \times a^2 = a^5$. That's because

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5.$$

You can see that, in general, for any natural numbers m and n, we have $a^m \times a^n = a^{m+n}$. This is an example of an *index law*. There's much more about index laws in Subsection 4.3.

Activity 10   *Simplifying single terms*

Simplify the following single-term expressions.

(a) $-(-uv)$ (b) $+(-9p)$ (c) $-(-4r^2)$ (d) $-(8z)$

(e) $2x^2y^2 \times 5xy^4$ (f) $-P(-PQ)$ (g) $5m \times (-\frac{2}{5}n)$

(h) $(-a^3)(-2b^3)(-2a^3)$ (i) $(cd)^2$ (j) $(-3x)^2$ (k) $-(3x)^2$

(l) $-(-3x)^2$ (m) $(-2x)^3$ (n) $-(-2x)^3$

You can simplify an expression that has more than one term by using the strategy below. In the first step you have to identify where each term begins and ends. You can do that by scanning through the expression from left to right – each time you come across a plus or minus sign that isn't inside brackets, that's the start of the next term.

Strategy:
To simplify an expression with more than one term

1. Identify the terms. Each term after the first starts with a plus or minus sign that isn't inside brackets.

2. Simplify each term, using the strategy on page 27. Include the sign (plus or minus) at the start of each term (except the first term, of course, if it has a plus sign).

3. Collect any like terms.

Example 5 *Simplifying an expression with more than one term*

Simplify the expression $3x \times 2x - 4y(-3x) - 10x^2$.

Solution

🔍 First identify the terms: you might find it helpful to mark them in the way shown below. Then simplify each term individually. Finally, collect any like terms. 💭

$$\underline{3x \times 2x} \ \underline{- 4y(-3x)} \ \underline{- 10x^2} = 6x^2 + 12xy - 10x^2$$
$$= 12xy - 4x^2$$

🔍 The answer could also be written as $-4x^2 + 12xy$, but $12xy - 4x^2$ is slightly shorter and tidier. 💭

Activity 11 *Simplifying expressions with more than one term*

Simplify the following expressions.

(a) $3a \times 3b - 2b \times 3b$ (b) $5x \times 8x - 3x(-3x)$

(c) $3x^2 - (-3y^2) + (-x^2) + (2y^2)$ (d) $-3cd + (-5c \times 2d^2) - (-cd^2)$

(e) $-6p(-\frac{1}{3}p) + (-5p \times p) - 2(-\frac{1}{2}p^2)$

(f) $A(-B) + (-AB) - (-AB) + (-A)(-B)$

2.3 Multiplying out brackets

In this subsection you'll revise how to rewrite an expression that contains a pair of brackets, such as

$$3b(1 + 2a),$$

as an expression that doesn't contain brackets. This process is called **multiplying out the brackets**, **expanding the brackets**, or simply **removing the brackets**. The new form of the expression, with no brackets, is called the **expansion** of the original expression.

In an expression like the one above, the subexpression that multiplies the brackets is called the **multiplier**. For example, the multiplier in the expression above is $3b$. The basic rule for multiplying out brackets is as follows.

> **Strategy:**
> **To multiply out brackets**
>
> Multiply each term inside the brackets by the multiplier.

For example,

$$3b(1 + 2a) = 3b \times 1 + 3b \times 2a$$
$$= 3b + 6ab.$$

Remember that you have to multiply *each* term inside the brackets by the multiplier. Multiplying only the first term is a common mistake!

When you multiply out brackets, it's usually best *not* to write down an expression that contains multiplication signs, and then simplify it, as was done above. Instead, you should simplify the terms in your head as you multiply out. This leads to tidier expressions and fewer errors, and it's particularly helpful when minus signs are involved. Here's an example.

Example 6   *Multiplying out brackets*

Multiply out the brackets in the expression

$$-x(2x - y + 5).$$

Solution

The first term in the expanded form of the expression is $-x \times 2x$. Simplify this in your head (using the strategy of first finding the overall sign, then the rest of the coefficient, then the letters) and write it down. Do likewise for the other terms, which are $-x \times (-y)$ and $-x \times 5$.

$$-x(2x - y + 5) = -2x^2 + xy - 5x$$

When you multiply out brackets, it doesn't matter whether the multiplier is before or after the brackets. Here's an example of multiplying out where the multiplier is after the brackets:

$$(5g - 1)h = 5gh - h.$$

Activity 12 *Multiplying out brackets*

Multiply out the brackets in the following expressions.

(a) $a(a^4 + b)$ (b) $-x(6x - x^2)$ (c) $3pq(2p + 3q - 1)$

(d) $(C^3 - C^2 - C)C^2$ (e) $-\frac{1}{2}x\left(\frac{1}{3}x^2 + \frac{2}{3}x\right)$

Sometimes you have to remove brackets that have just a minus or plus sign in front, such as

$$-(p + 5q - s) \text{ or } +(a - b).$$

You do this by using the following strategy.

Strategy:
To remove brackets with a plus or minus sign in front

- If the sign is plus, keep the sign of each term inside the brackets the same.

- If the sign is minus, change the sign of each term inside the brackets.

For example,

$$-(p + 5q - s) = -p - 5q + s \text{ and } +(a - b) = a - b.$$

The strategy comes from the fact that a minus sign in front of brackets is just the same as multiplying by -1, and a plus sign in front is just the same as multiplying by 1.

Activity 13 *Plus and minus signs in front of brackets*

Remove the brackets in the following expressions.

(a) $-(-2x^2 + x - 1)$ (b) $+(2x - 3y + z)$ (c) $-(p - 2q)$

Notice that, for any expressions A and B,

$$-(A - B) = -A + B = B - A.$$

So we have the following useful fact.

For any expressions A and B,
     the negative of $A - B$ is $B - A$.

It's helpful to remember this fact. For example, it tells you immediately that

$$-(x - y) = y - x \quad \text{and} \quad -(n - 3n^2) = 3n^2 - n,$$

and so on.

The strategy for multiplying out brackets leads to two further useful facts. You've seen how to use the strategy to write, for example,

$$(a - b + 2)c = ac - bc + 2c.$$

In effect this says that when you want to multiply the expression $a - b + 2$ by c, you simply multiply each of its terms by c to obtain $ac - bc + 2c$.

Similarly, since dividing by c is the same as multiplying by $1/c$,

$$\frac{a - b + 2}{c} = \frac{1}{c}(a - b + 2) = \frac{1}{c} \times a - \frac{1}{c} \times b + \frac{1}{c} \times 2 = \frac{a}{c} - \frac{b}{c} + \frac{2}{c}.$$

So when you want to divide the expression $a - b + 2$ by c, you simply divide each of its terms by c to obtain the result $(a/c) - (b/c) + (2/c)$.

In general, you can use the following facts.

> Multiplying an expression that contains several terms by a second expression is the same as multiplying *each term* of the first expression by the second expression.
>
> Similarly, dividing an expression that contains several terms by a second expression is the same as dividing *each term* of the first expression by the second expression.

Brackets in expressions with more than one term

Sometimes you have to multiply out brackets in an expression that contains more than one term. For example, the expression

$$2c(c + d) + 5c^2 - d(c - d)$$

has three terms, two of which contain brackets, as follows:

$$\underline{2c(c + d)} + \underline{5c^2} - \underline{d(c - d)}.$$

If you multiply out the brackets in both the first and last terms, then you obtain a new expression with five terms. You can then collect any like terms. The approach is summarised in the following strategy.

Strategy:
To multiply out brackets in an expression with more than one term

1. Identify the terms. Each term after the first starts with a plus or minus sign that isn't inside brackets.

2. Multiply out the brackets in each term. Include the sign (plus or minus) at the start of each resulting term.

3. Collect any like terms.

Example 7   *Multiplying out brackets when there's more than one term*

Multiply out the brackets in the expression

$$2c(c + d) + 5c^2 - d(c - d),$$

and simplify your answer.

Solution

Identify the terms; you might find it helpful to mark them. Multiply out the brackets in each term individually to obtain a new expression with five terms. Finally, collect any like terms.

$$\underline{2c(c + d)} \; \underline{+ 5c^2} \; \underline{- d(c - d)} = 2c^2 + 2cd + 5c^2 - cd + d^2$$

$$= 7c^2 + cd + d^2$$

Activity 14   *Multiplying out brackets when there's more than one term*

Multiply out the brackets in the following expressions, simplifying where possible.

(a) $x + x^2(1 + 3x)$ (b) $7ab - b(a + 2b)$ (c) $-6(c + d) + 3(c - d)$

(d) $2X - 5Y(-4X + 2Y)$ (e) $(1 - p^4)p + p^2 - p$

Multiplying out two pairs of brackets

Some expressions, such as the one in Example 8 below, contain two pairs of brackets multiplied together. You can multiply out brackets like these by choosing one of the two bracketed expressions to be the multiplier, and multiplying out the other pair of brackets in the usual way. This gives you an expression with several terms each containing a pair of brackets, which again you can multiply out in the usual way.

It doesn't matter which of the two bracketed expressions you choose to be the multiplier to start with, but it's probably slightly easier to go for the second one, as illustrated in the next example.

Example 8   *Multiplying out two pairs of brackets*

Multiply out the brackets in the expression

$$(x - xy + 3y)(x - 2y),$$

and simplify your answer.

Solution

🔍 Take the second bracketed expression to be the multiplier. Multiply each term in the first pair of brackets by this multiplier. 💬

$$(x - xy + 3y)(x - 2y) = x(x - 2y) - xy(x - 2y) + 3y(x - 2y)$$

🔍 Multiply out the brackets in each term, then collect any like terms. 💬

$$= x^2 - 2xy - x^2y + 2xy^2 + 3xy - 6y^2$$
$$= x^2 + xy - x^2y + 2xy^2 - 6y^2$$

Activity 15   *Multiplying out two pairs of brackets*

Use the method described above to multiply out the brackets in the following expressions. Simplify your answers where possible.

(a) $(a + b)(c + d + e)$ (b) $(x + 3)(x + 5)$

(c) $(x^2 - 2x + 3)(3x^2 - x - 1)$

You can see from Example 8 and Activity 15 that the effect of multiplying out two pairs of brackets is that *each* term in the first pair of brackets is multiplied by *each* term in the second pair of brackets, and the resulting terms are added. So, for example, if there are three terms in the first pair of brackets and two terms in the second pair (as there are in Example 8,

for instance), then altogether there will be $3 \times 2 = 6$ multiplications, which gives six terms in the multiplied-out expression before any like terms are collected.

You can use this fact to give you an alternative method for multiplying out two pairs of brackets, though you have to be careful not to miss out any of the multiplications! It's a good way to multiply out two pairs of brackets that each contain only two terms. In this case there are only four multiplications to be done, and you can use the acronym FOIL (first terms, outer terms, inner terms, last terms) to help you remember them. Here's an example.
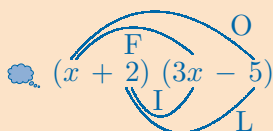
Example 9   *Using FOIL to multiply out brackets*

Multiply out the brackets in the expression

$$(x + 2)(3x - 5),$$

and simplify your answer.

Solution

$$(x + 2)(3x - 5)$$

First: $x \times 3x = 3x^2$.
Outer: $x \times (-5) = -5x$.
Inner: $(+2) \times 3x = +6x$.
Last: $(+2) \times (-5) = -10$.

$$(x + 2)(3x - 5) = 3x^2 - 5x + 6x - 10 = 3x^2 + x - 10$$

You can practise using FOIL in the next activity.

Activity 16   *Multiplying out brackets containing two terms each*

Multiply out the brackets in the following expressions, and simplify your answers.

(a) $(x + 5)(x - 7)$ (b) $(x - 3)(x - 1)$ (c) $(2x - 1)(8x + 3)$

(d) $(2 - 5x)(x - 9)$ (e) $(c - 2d)(1 + c)$ (f) $(A - B)(2A - 3B^2)$

(g) $(a - 1)(a + 1)$ (h) $(2 + 3x)(2 - 3x)$

(i) $x(1 + x) + (x - 1)(2 - x)$

Make sure that you've done parts (g) and (h) of Activity 16, in particular, as they illustrate a useful fact that's discussed next.

Differences of two squares

In each of parts (g) and (h) of Activity 16, you'll have noticed that the product of the inner terms and the product of the outer terms added to zero. This happens whenever you multiply out an expression of the form

$$(A + B)(A - B),$$

where A and B are subexpressions. In fact, as you can check by multiplying out the brackets, the following holds.

> ### Difference of two squares
>
> For any expressions A and B,
> $$(A + B)(A - B) = A^2 - B^2.$$

Any expression that has the form of the right-hand side of the equation in the box above is known as a **difference of two squares**. The fact in the box will be useful in Subsection 4.2, and also in Unit 2.

Squared brackets

Sometimes you have to multiply out squared brackets, such as $(x - 5)^2$. You can do this by first writing the squared brackets as two pairs of brackets multiplied together, and then multiplying them out in the usual way. For example,

$$\begin{aligned}(x - 5)^2 &= (x - 5)(x - 5) \\ &= x^2 - 5x - 5x + 25 \\ &= x^2 - 10x + 25.\end{aligned}$$

Notice that $(x - 5)^2$ is *not* equal to $x^2 - 5^2$.

Activity 17   *Multiplying out squared brackets*

Multiply out the brackets in the following expressions, and simplify your answers.

(a) $(x + 1)^2$ (b) $(3x - 2)^2$ (c) $(2p + 3q)^2$

A quick way to multiply out squared brackets that contain two terms, like those in Activity 17, is to use the fact that they all follow a similar pattern.

To see this pattern, notice what happens when you multiply out the expressions $(A + B)^2$ and $(A - B)^2$:

$$(A + B)^2 = (A + B)(A + B)$$
$$= A^2 + AB + AB + B^2$$
$$= A^2 + 2AB + B^2;$$

$$(A - B)^2 = (A - B)(A - B)$$
$$= A^2 - AB - AB + B^2$$
$$= A^2 - 2AB + B^2.$$

So the following useful facts hold.

Squaring brackets

For any expressions A and B,
$$(A + B)^2 = A^2 + 2AB + B^2, \quad \text{and}$$
$$(A - B)^2 = A^2 - 2AB + B^2.$$

Here's an example of how you can use these facts to quickly multiply out squared brackets.

Example 10 *Multiplying out squared brackets efficiently*

Multiply out the brackets in the following expressions, and simplify your answers.

(a) $(x + 3y)^2$ (b) $(4h - 1)^2$

Solution

(a) The answer is the square of x, plus twice the product of x and $3y$, plus the square of $3y$.
$$(x + 3y)^2 = x^2 + 2 \times x \times (3y) + (3y)^2 = x^2 + 6xy + 9y^2$$

(b) The answer is the square of $4h$, minus twice the product of $4h$ and 1, plus the square of 1.
$$(4h - 1)^2 = (4h)^2 - 2 \times (4h) \times 1 + 1^2 = 16h^2 - 8h + 1$$

Activity 18 *Multiplying out squared brackets, again*

Use the facts in the box above to multiply out the brackets in the following expressions, and simplify your answers.

(a) $(x + 6)^2$ (b) $(x - 2)^2$ (c) $(1 + m)^2$ (d) $(1 - 2u)^2$

(e) $(2x - 3)^2$ (f) $(3c + d)^2$

Simplest forms of expressions

As you know, you should usually write expressions in the simplest way you can. For example, you should write

$$x + 2x + 3x \quad \text{as} \quad 6x.$$

The second form of this expression is clearly simpler than the first:

- it's shorter and easier to understand, and
- it's easier to evaluate for any particular value of x.

These are the attributes to aim for when you try to write an expression in a simpler way.

However, sometimes it's not so clear that one way of writing an expression is better than another. For example,

$$x(x + 1) \quad \text{is equivalent to} \quad x^2 + x.$$

Both of these forms are reasonably short, and both are reasonably easy to evaluate. So this expression doesn't have a simplest form.

The same is true of many other expressions. You should try to write each expression that you work with in a reasonably simple way, but often there's no 'right answer' for the simplest form. One form might be better for some purposes, and a different form might be better for other purposes.

In particular, multiplying out the brackets in an expression doesn't always simplify it. You should multiply out brackets only if you think that this is likely to make the expression simpler, or if you think that it will help you with the problem that you're working on.

They tell me it's important to simplify expressions!

3 Algebraic factors, multiples and fractions

In this section you'll revise what are meant by factors and multiples of algebraic expressions, how to take out common factors and how to work with algebraic fractions.

3.1 Factors and multiples of algebraic expressions

Algebraic expressions have *factors* and *multiples* in a similar way to integers, though these words are often used a little more loosely for algebraic expressions than they are for integers. Roughly speaking, if an algebraic expression can be written in the form

something $\times$ something,

then both 'somethings' are **factors** of the expression, and the expression is a **multiple** of both 'somethings'. For example, the equation

$$a^2b = a \times ab$$

shows that both a and ab are factors of a^2b, and it also shows that a^2b is a multiple of both a and ab.

Every algebraic expression is both a factor and a multiple of itself. For example, the equation

$$a^2b = 1 \times a^2b$$

shows that a^2b is both a factor and a multiple of itself.

Two or more algebraic expressions also have *common factors* and *common multiples*, in a similar way to two or more integers.

As you'd expect, a **common factor** of two or more algebraic expressions is an expression that's a factor of all of them. For example, the expression a is a common factor of the two expressions

$$a^2b \quad \text{and} \quad abc,$$

because

$$a^2b = a \times ab \quad \text{and} \quad abc = a \times bc.$$

Similarly, a **common multiple** of two or more algebraic expressions is an expression that's a multiple of all of them. For example, the expression $abcd$ is a common multiple of the two expressions

$$ab \quad \text{and} \quad bc,$$

because

$$abcd = ab \times cd \quad \text{and} \quad abcd = bc \times ad.$$

Activity 19   *Checking common factors and common multiples*

Show that:

(a) (i) $2a$ is a common factor of $4a^2$ and $2ab$;

   (ii) x^2 is a common factor of x^3 and x^5;

   (iii) $6z$ is a common factor of $18z^2$, $6z^2$ and $6z$;

(b) (i) $10cd^2$ is a common multiple of $5c$ and $2cd$;

   (ii) p^7 is a common multiple of p^2 and p^3;

   (iii) $9y^2$ is a common multiple of 3, $9y^2$ and $3y$.

Two or more algebraic expressions with integer coefficients have a *highest common factor* and a *lowest common multiple*, just as two or more integers do, though again these words are often used more loosely for algebraic expressions than they are for integers. In the context of algebraic expressions, **highest common factor** means a common factor that is a multiple of all other common factors. Similarly, **lowest common multiple** means a common multiple that is a factor of all other common multiples.

The next example shows you how to find highest common factors and lowest common multiples of algebraic expressions.

Example 11   *Finding HCFs and LCMs of algebraic expressions*

Consider the expressions

$$10a^6 \quad \text{and} \quad 15a^8b^3.$$

(a) Find the highest common factor of the expressions, and write each expression in the form

   highest common factor × something.

(b) Find the lowest common multiple of the expressions, and, for each expression, write the lowest common multiple in the form

   the expression × something.

Solution

(a)

🔍 First consider the coefficients. The largest integer that is a factor of both 10 and 15 (that is, their HCF) is 5.

Then consider the powers of a. The largest power of a that is a factor of both a^6 and a^8 is a^6. (Note that a^6 is a factor of both a^6 and a^8 because $a^6 = a^6 \times 1$ and $a^8 = a^6 \times a^2$.)

Finally, consider the powers of b. There is no power of b in $10a^6$, so there is no power of b in the highest common factor. 💬

The highest common factor of the two terms is

$$5a^6.$$

The expressions can be written as

$$10a^6 = 5a^6 \times 2 \quad \text{and} \quad 15a^8b^3 = 5a^6 \times 3a^2b^3.$$

(b)

🔍 First consider the coefficients. The smallest positive integer that is a multiple of both 10 and 15 (that is, their LCM) is 30.

Then consider the powers of a. The smallest power of a that is a multiple of both a^6 and a^8 is a^8.

Finally, consider the powers of b. The smallest power of b that is a multiple of both 'no power of b' and b^3 is b^3. 💬

The lowest common multiple of the two terms is

$$30a^8b^3.$$

It can be written as

$$30a^8b^3 = 10a^6 \times 3a^2b^3 \quad \text{and} \quad 30a^8b^3 = 15a^8b^3 \times 2.$$

Activity 20   *Finding HCFs and LCMs of algebraic expressions*

(a) Consider the expressions

$$3x^2 \quad \text{and} \quad 9xy.$$

 (i)   Find the highest common factor of the expressions, and write each expression in the form

   highest common factor × something.

(ii)   Find the lowest common multiple of the expressions, and, for each expression, write the lowest common multiple in the form

the expression $\times$ something.

(b) Repeat part (a) for the expressions

$$6p^2q^3, \quad 4pq^2 \quad \text{and} \quad 2pq.$$

3.2 Taking out common factors

Factorising an expression means writing it as the *product* of two or more expressions, neither of which is 1 (and, usually, neither of which is -1).

If all of the terms of an expression have a common factor other than 1, then the expression can be factorised. For example, consider the expression

$$x^3y + xy.$$

The terms of this expression, x^3y and xy, have xy as a common factor. So the expression can be written as

$$xy \times x^2 + xy \times 1.$$

From your work on multiplying out brackets, you know that this is the same as

$$xy(x^2 + 1).$$

The original expression has now been factorised. We say that we have **taken out the common factor** xy. This example illustrates the thinking behind the following general strategy for taking out common factors.

> **Strategy:**
> **To take out a common factor from an expression**
>
> 1.   Find a common factor of the terms (usually the highest common factor).
> 2.   Write the common factor in front of a pair of brackets.
> 3.   Write what's left of each term inside the brackets.

Example 12   *Taking out a common factor*

Factorise the expression $-8g^5 + 4g^2h^2 - 2g^2$.

Solution

🔍 The highest common factor of the three terms is $2g^2$. 💬

$$-8g^5 + 4g^2h^2 - 2g^2 = 2g^2(-4g^3 + 2h^2 - 1)$$

Notice that if you're taking out a common factor, and it's the same as, or the negative of, one of the terms, then 'what's left' of the term is 1 or -1. This is illustrated in Example 12 above, and also in the example in the text at the beginning of this subsection.

Remember that you can check whether a factorisation that you've carried out is correct by just multiplying out the brackets again. It's often a good idea to carry out some sort of check on an answer that you've found to a mathematical problem, where this is possible, especially when you're learning a new technique, or when you're answering an assignment question.

If you're trying to take out the *highest* common factor from an expression, then it's worth checking the expression inside the brackets when you've finished, to make sure that you haven't missed any common factors.

Activity 21   *Taking out common factors*

Factorise the following expressions. Take out the highest common factor in each case.

(a) $pq + 12qr$ (b) $14cd - 7cd^2$ (c) $m^3 - m^7 - 8m^2$

(d) $-6AB + 3A^2B - 12A^3B^2$ (e) $\sqrt{T} - s\sqrt{T}$

(f) $5x^2 - 10x$ (g) $18y^2 + 6$

Sometimes it's convenient to take out a common factor with a minus sign. When you do this, 'what's left' of each term will have the opposite sign to the original sign. For example, an alternative way to factorise the expression in Example 12 is

$$-8g^5 + 4g^2h^2 - 2g^2 = -2g^2(4g^3 - 2h^2 + 1).$$

You can also take out *just* a minus sign from an expression. For example,

$$-a - b + c = -(a + b - c).$$

It can be helpful to do these things when all or most of the terms in an expression have minus signs.

Factorise each of the following expressions by taking out the negative of the highest common factor of the terms. (If the highest common factor of the terms is 1, then take out just a minus sign.)

(a) $-2x^3 + 3x^2 - x - 5$ (b) $-ab - a - b$ (c) $5cd^2 - 10c^2d - 5cd$

If the coefficients of the terms of an expression aren't integers, then you can often still factorise the expression in a helpful way. For example, sometimes you can simplify an expression by taking out a fraction as a 'common factor', leaving terms with integer coefficients inside the brackets. To achieve this, usually the fraction that you take out must have a denominator that is a common multiple of the denominators of the coefficients. Here's an example:

$$\tfrac{1}{2}n^2 - \tfrac{2}{3}n + 1 = \tfrac{1}{6}(3n^2 - 4n + 6).$$

Sometimes the expression in the brackets has further common factors that can also be taken out, as in Activity 23(a) and (d) below.

Activity 23   *Working with non-integer coefficients*

Simplify each of the following expressions by factorising them.

(a) $\tfrac{1}{2}a^2 + \tfrac{3}{2}a$ (b) $\tfrac{1}{3}x - \tfrac{1}{6}$ (c) $2x^2 - \tfrac{1}{2}x + \tfrac{1}{4}$ (d) $\tfrac{2}{3}u^2v^2 + \tfrac{1}{2}u^3v$

3.3 Algebraic fractions

As you know, in algebraic expressions, division is normally indicated by fraction notation rather than by division signs. For example, the expression $2x \div (x + 1)$ is written as

$$\frac{2x}{x + 1}.$$

An algebraic expression written in the form of a fraction, such as the expression above, is called an **algebraic fraction**. As with a numerical fraction, the top and bottom of the algebraic fraction are called the **numerator** and the **denominator**, respectively.

When you want to write an algebraic fraction as part of a line of text, you can replace the horizontal line by a slash symbol, but remember that you may need to enclose the numerator and/or denominator in brackets to make it clear what's divided by what. For example, you can write the algebraic fraction above as $(2x)/(x + 1)$ or as $2x/(x + 1)$, but not as $2x/x + 1$, which means $(2x/x) + 1$.

There are other special situations where you may find the slash notation helpful. For example, it's convenient when you're dealing with a fraction whose numerator and denominator are themselves fractions, or when a fraction appears in a power, such as $2^{x/5}$. However, in most situations, and especially when you're manipulating fractions, you should use the horizontal line notation, as it's simpler to work with and helps you avoid mistakes.

You can manipulate algebraic fractions using the same rules that you use for numerical fractions. However, there's an extra issue that you need to take into account: an algebraic fraction is valid only for the values of the variables that make the denominator non-zero. This is because division by zero isn't possible. For example, the algebraic fraction $2x/(x+1)$ is valid for every value of x except -1.

> On 21 September 1997, the computer system of the US Navy cruiser USS Yorktown failed because of a division by zero error, and the ship was incapacitated.

In the rest of this subsection you'll revise and practise manipulating algebraic fractions.

Equivalence of algebraic fractions

Two numerical fractions are equal or *equivalent* if one can be obtained from the other by multiplying or dividing both the numerator and the denominator by the same (non-zero) number. For example,

$$\frac{2}{10} \quad \text{and} \quad \frac{1}{5} \quad \text{are equivalent, because} \quad \frac{2}{10} = \frac{1 \times 2}{5 \times 2}.$$

Similarly, two algebraic fractions are equal or *equivalent* if one can be obtained from the other by multiplying or dividing both the numerator and the denominator by the same expression. For example,

$$\frac{a(b-2)}{b(b-2)} \quad \text{and} \quad \frac{a}{b} \quad \text{are equivalent.}$$

Notice that the first of these two algebraic fractions is valid for all values of its variables except $b = 0$ and $b = 2$, whereas the second is valid for all values of its variables except just $b = 0$. So the two fractions aren't *exactly* the same. This sort of thing doesn't usually matter, but sometimes it does. For example, it can occasionally cause problems when you're solving equations. There's more about this in Subsections 5.3 and 5.4.

Simplifying algebraic fractions

A numerical fraction whose numerator and denominator have a common factor can be simplified by dividing both the numerator and the denominator by this factor. For example, 2 is a common factor of 2 and 10, so

$$\frac{2}{10} = \frac{1 \times 2}{5 \times 2} = \frac{1}{5}.$$

You can indicate this sort of working by crossing out the numbers on the numerator and denominator of the fraction, and replacing them with the results of the divisions, like this:

$$\frac{2}{10} = \frac{\overset{1}{\cancel{2}}}{\underset{5}{\cancel{10}}} = \frac{1}{5}.$$

In the same way, an algebraic fraction whose numerator and denominator have a common factor can be simplified by dividing both the numerator and the denominator by the factor. For example, x is a common factor of xy and x^3, so

$$\frac{xy}{x^3} = \frac{y \times x}{x^2 \times x} = \frac{y}{x^2},$$

which you can write as

$$\frac{xy}{x^3} = \frac{\overset{1}{\cancel{x}}y}{\underset{x^2}{\cancel{x^3}}} = \frac{y}{x^2}.$$

Notice that this calculation uses the fact that $x^3 \div x = x^2$. This is because $x^3 = x \times x \times x$, so dividing x^3 by x gives x^2. You can see that for any natural numbers m and n with m larger than n, we have $a^m \div a^n = a^{m-n}$. This is another *index law*. It's discussed more thoroughly in Subsection 4.3.

The process of dividing the numerator and denominator of a fraction, whether numerical or algebraic, by a common factor is known as **cancelling** the factor, or **cancelling down** the fraction.

Usually it's best to simplify a numerical or algebraic fraction as much as possible. To do this, you need to cancel the *highest* common factor of the numerator and denominator. It's often easiest to do this by cancelling in stages: first you cancel one common factor, then another, and so on, until eventually the overall effect is that you've cancelled the highest common factor. This is illustrated in the next example.

Example 13   *Simplifying algebraic fractions*

Simplify the following algebraic fractions.

(a) $\dfrac{12a^2b^4}{9a^3b}$ (b) $\dfrac{x^3(x+1)}{x(x+1)^2}$

Solution

(a)   Divide top and bottom by 3 (the highest common factor of the coefficients), then by a^2 (the highest common factor of the powers of a), and finally by b (the highest common factor of the powers of b).

$$\frac{12a^2b^4}{9a^3b} = \frac{\overset{4}{\cancel{12}}a^2b^4}{\underset{3}{\cancel{9}}a^3b} = \frac{\overset{4}{\cancel{12}}\overset{1}{\cancel{a^2}}b^4}{\underset{3}{\cancel{9}}\underset{a}{\cancel{a^3}}b} = \frac{\overset{4}{\cancel{12}}\overset{1}{\cancel{a^2}}\overset{b^3}{\cancel{b^4}}}{\underset{3}{\cancel{9}}\underset{a}{\cancel{a^3}}\underset{1}{\cancel{b}}} = \frac{4b^3}{3a} \quad (b \neq 0)$$

(b)   Divide top and bottom by x (the highest common factor of the powers of x), then by $x+1$ (the highest common factor of the powers of $x+1$).

$$\frac{x^3(x+1)}{x(x+1)^2} = \frac{\overset{x^2}{\cancel{x^3}}(x+1)}{\underset{1}{\cancel{x}}(x+1)^2} = \frac{\overset{x^2}{\cancel{x^3}}\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{x}}\underset{x+1}{\cancel{(x+1)^2}}} = \frac{x^2}{x+1} \quad (x \neq 0)$$

In order to show the cancelling-down process clearly, Example 13 displays each cancellation on a fresh copy of the original fraction. Normally, however, you should be able to show all stages of the cancellation on a single copy of the fraction.

Another feature of Example 13 is that restrictions, $b \neq 0$ and $x \neq 0$, are used to indicate that the original and the final fraction in each part are equivalent only if the restriction holds. (The symbol '$\neq$' means, and is read as, 'is not equal to'.) Often we won't note such restrictions explicitly, but you should keep in mind that they apply.

Activity 24   *Simplifying algebraic fractions*

Simplify the following algebraic fractions.

(a) $\dfrac{8xy^3}{6x^2y^2}$
(b) $\dfrac{2(3x-1)}{10(3x-1)^3}$
(c) $\dfrac{(x-1)^2(x-2)}{(x-1)(x-2)^2}$

As you become familiar with cancelling down algebraic fractions, you can make your working tidier by not actually crossing out factors. Instead, you can just write down the original form of the fraction, followed by an equals sign and then the simplified form, as you would with other types of algebraic simplification. If you do want to cross out factors in a fraction, perhaps because it's a complicated one, then you should make sure that your working also includes a copy of the unsimplified fraction with nothing crossed out.

The next example illustrates that even if there aren't any obvious common factors of the numerator and the denominator of an algebraic fraction, you may still be able to find some by factorising the numerator and/or the denominator.

Example 14   *Simplifying more algebraic fractions*

Simplify the following algebraic fractions.

(a) $\dfrac{x^2+2x}{5x^3-3x}$
(b) $\dfrac{u^2-2u}{3u-6}$

Solution

🔍 Factorise the numerator and denominator to check for common factors. Cancel any common factors. 💬

(a) $\dfrac{x^2+2x}{5x^3-3x}=\dfrac{x(x+2)}{x(5x^2-3)}=\dfrac{x+2}{5x^2-3}$ $(x\neq 0)$

(b) $\dfrac{u^2-2u}{3u-6}=\dfrac{u(u-2)}{3(u-2)}=\dfrac{u}{3}$ $(u\neq 2)$

In part (a) of Example 14, you can avoid having to write down the middle expression in the working if you notice that *each term in both the numerator and denominator* of the original fraction has x as a common factor. In general, if all the terms in both the numerator and the denominator of a fraction have a common factor, then you can simplify the fraction by dividing each of these terms individually by this common factor. That's because, as stated in a box on page 33, if an expression is a

sum of several terms, then dividing it by a second expression is the same as dividing *each term* of the expression by the second expression.

For example, each term in both the numerator and denominator of the fraction below has c^2 as a common factor, so you can simplify it by dividing each term individually by c^2:

$$\frac{c^4 + c^3 + c^2}{2c^3 + c^2} = \frac{c^2 + c + 1}{2c + 1}.$$

Activity 25 *Simplifying more algebraic fractions*

Simplify the following algebraic fractions.

(a) $\dfrac{ab + a}{a^2 + a}$ (b) $\dfrac{3}{9 + 6y^2}$ (c) $\dfrac{u^2 - u^3}{u^5}$

(d) $\dfrac{2x^2 - 4x^3}{6x^4 - 2x^2}$ (e) $\dfrac{x^3 + x^2}{2x + 2}$ (f) $\dfrac{1 - n}{n^2 - n}$

Adding and subtracting algebraic fractions

As you know, if the denominators of two or more numerical fractions are the same, then to add or subtract them you just add or subtract the numerators. For example,

$$\frac{5}{7} - \frac{2}{7} = \frac{3}{7}.$$

If the denominators are different, then before you can add or subtract the fractions, you need to write them with the same denominator. This denominator has to be a common multiple of the denominators of all the fractions to be added or subtracted. Here's an example:

$$\frac{5}{4} + \frac{2}{3} - \frac{1}{6} = \frac{15}{12} + \frac{8}{12} - \frac{2}{12} = \frac{21}{12} = \frac{7}{4}.$$

In this calculation, a common multiple of the original denominators 4, 3 and 6 is 12. So each fraction was written with denominator 12, by multiplying top and bottom by an appropriate number (namely 3, 4 and 2, respectively). Then the numerators were added and subtracted in the usual way, and the resulting answer was simplified by cancelling.

You can use the same method to add or subtract algebraic fractions, as follows.

Strategy:
To add or subtract algebraic fractions

1. Make sure that the fractions have the same denominator – if necessary, rewrite each fraction as an equivalent fraction to achieve this.

2. Add or subtract the numerators.

3. Simplify the answer if possible.

Example 15 *Adding and subtracting algebraic fractions*

Write each of the following expressions as a single algebraic fraction.

(a) $\dfrac{x+2}{x^2} - \dfrac{2}{x^2}$
(b) $\dfrac{1}{ab} + \dfrac{2}{a} - \dfrac{3}{b}$
(c) $\dfrac{x}{1-x} - \dfrac{1}{x}$
(d) $\dfrac{5}{d} + c$

Solution

(a) The denominators are the same, so subtract the second numerator from the first. Simplify the answer.
$$\frac{x+2}{x^2} - \frac{2}{x^2} = \frac{x+2-2}{x^2} = \frac{x}{x^2} = \frac{1}{x}$$

(b) First write the fractions with the same denominator, by multiplying the top and bottom of each fraction by an appropriate expression. Then add and subtract the numerators.
$$\frac{1}{ab} + \frac{2}{a} - \frac{3}{b} = \frac{1}{ab} + \frac{2b}{ab} - \frac{3a}{ab} = \frac{1+2b-3a}{ab}$$

(c) Proceed in a similar way to part (b). Simplify the answer.
$$\frac{x}{1-x} - \frac{1}{x} = \frac{x^2}{x(1-x)} - \frac{1-x}{x(1-x)} = \frac{x^2 - (1-x)}{x(1-x)} = \frac{x^2+x-1}{x(1-x)}$$

(d) First write c as a fraction, then proceed as before.
$$\frac{5}{d} + c = \frac{5}{d} + \frac{c}{1} = \frac{5}{d} + \frac{cd}{d} = \frac{5+cd}{d}$$

Activity 26   *Adding and subtracting algebraic fractions*

Write each of the following expressions as a single algebraic fraction, simplifying your answer if possible.

(a) $\dfrac{1}{x} + \dfrac{y}{x}$ 　　(b) $\dfrac{c+2}{c^2+c} - \dfrac{1}{c^2+c}$ 　　(c) $\dfrac{1}{ab} - \dfrac{1}{bc}$ 　　(d) $\dfrac{2}{3a} + \dfrac{1}{2a}$

(e) $\dfrac{1}{x^2} - \dfrac{2}{x} + 3$ 　　(f) $5 - \dfrac{1}{x} + \dfrac{2}{y}$ 　　(g) $A - \dfrac{A^2-1}{2A}$ 　　(h) $\dfrac{3}{2u-1} + u$

(i) $\dfrac{x+2}{x+1} + \dfrac{1}{x}$ 　　(j) $\dfrac{2}{p+2} - \dfrac{1}{p-3}$ 　　(k) $\dfrac{x+1}{x(x-1)} + \dfrac{1}{x} - x$

Expanding algebraic fractions

You saw on page 33 that if an expression is a sum of several terms, then dividing it by a second expression is the same as dividing *each term* of the expression by the second expression.

This means that if you have an algebraic fraction whose *numerator* is a sum of terms, then you can rewrite it so that each term is individually divided by the denominator. For example,

$$\frac{2x^2 - y + 3}{x} = \frac{2x^2}{x} - \frac{y}{x} + \frac{3}{x} = 2x - \frac{y}{x} + \frac{3}{x}.$$

This procedure is known as **expanding** the algebraic fraction. Like expanding brackets, it's sometimes a useful thing to do, and sometimes not. Expanding an algebraic fraction is essentially the reverse procedure to adding or subtracting algebraic fractions.

Activity 27   *Expanding algebraic fractions*

Expand the following algebraic fractions.

(a) $\dfrac{a^2 + a^5 - 1}{a^2}$ 　　(b) $\dfrac{2 - 5cd}{c}$ 　　(c) $\dfrac{2a + 3a^2}{6}$

Remember that an algebraic fraction can be expanded only if it has a sum of terms in the *numerator*. The following fraction, which has a sum of terms in the denominator, can't be expanded:

$$\frac{a^2}{a^2 + a^5 - 1}.$$

Multiplying and dividing algebraic fractions

To *multiply* numerical fractions, you multiply the numerators together and multiply the denominators together. For example,

$$\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2}.$$

The rule for *dividing* numerical fractions can be conveniently described using the idea of the *reciprocal* of a number. A number and its **reciprocal** multiply together to give 1. So, for example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, since $\frac{2}{3} \times \frac{3}{2} = 1$. Similarly, the reciprocal of $\frac{1}{2}$ is $\frac{2}{1} = 2$, and the reciprocal of $-\frac{3}{5}$ is $-\frac{5}{3}$. Every number except 0 has a reciprocal.

Since a number and its reciprocal multiply together to give 1, another way to think about the reciprocal of a number is that it is 1 divided by the number.

To find the reciprocal of a fraction, you just swap the numerator and the denominator. For example, the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$, since $\frac{3}{5} \times \frac{5}{3} = \frac{15}{15} = 1$.

The rule for dividing numerical fractions is as follows: to divide by a numerical fraction, you multiply by its reciprocal. For example,

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8}.$$

If you don't know *why* fractions are multiplied and divided using the rules described here, have a look at the document *Fraction arithmetic* on the module website.

Algebraic fractions are multiplied and divided in the same way as numerical ones.

> **Strategy:**
> **To multiply or divide algebraic fractions**
>
> - To multiply two or more algebraic fractions, multiply the numerators together and multiply the denominators together.
> - To divide by an algebraic fraction, multiply by its reciprocal.
>
> Simplify the answer if possible.

Here's an example of multiplying algebraic fractions:

$$\frac{4}{7a^3} \times \frac{7a}{2b^3} = \frac{28a}{14a^3b^3} = \frac{2}{a^2b^3}.$$

In this manipulation the two numerators were multiplied and the two denominators were multiplied, and then the answer was simplified by cancelling common factors. However, it's often quicker and easier to cancel

any common factors *before* you multiply the numerators and denominators of fractions, like this:

$$\frac{4}{7a^3} \times \frac{7a}{2b^3} = \frac{\overset{2}{\cancel{4}}}{7a^3} \times \frac{7a}{\cancel{2}b^3} = \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{7}a^3}} \times \frac{\overset{1}{\cancel{7}}a}{\underset{1}{\cancel{2}}b^3} = \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{7}}\underset{a^2}{\cancel{a^3}}} \times \frac{\overset{1}{\cancel{7}}\overset{1}{\cancel{a}}}{\underset{1}{\cancel{2}}b^3} = \frac{2}{a^2} \times \frac{1}{b^3} = \frac{2}{a^2 b^3}.$$

This technique is known as **cross-cancelling**. As with ordinary cancelling, there is no need to show every, or indeed any, stage of the cancelling process.

Example 16 *Multiplying and dividing algebraic fractions*

Simplify the following expressions.

(a) $\dfrac{2x-5}{(x-1)^2} \times \dfrac{x-1}{4}$ (b) $\dfrac{9P^2}{Q^8} \div \dfrac{3P^3}{Q^9}$

Solution

(a) To multiply the fractions, multiply the numerators and multiply the denominators. Cross-cancel any common factors first.

$$\frac{2x-5}{(x-1)^2} \times \frac{x-1}{4} = \frac{2x-5}{\underset{x-1}{\cancel{(x-1)^2}}} \times \frac{\overset{1}{\cancel{x-1}}}{4} = \frac{2x-5}{4(x-1)}$$

(b) To divide by a fraction, multiply by the reciprocal. Cross-cancel any common factors before doing the multiplication.

$$\frac{9P^2}{Q^8} \div \frac{3P^3}{Q^9} = \frac{9P^2}{Q^8} \times \frac{Q^9}{3P^3} = \frac{\overset{3}{\cancel{9}}P^2}{Q^8} \times \frac{Q^9}{\underset{1}{\cancel{3}}P^3}$$

$$= \frac{\overset{3}{\cancel{9}}\overset{1}{P^2}}{Q^8} \times \frac{Q^9}{\underset{1}{\cancel{3}}\underset{P}{P^3}} = \frac{\overset{3}{\cancel{9}}\overset{1}{P^2}}{\cancel{Q^8}} \times \frac{\overset{Q}{\cancel{Q^9}}}{\underset{1}{\cancel{3}}\underset{P}{P^3}} = \frac{3Q}{P}$$

You can practise multiplying and dividing algebraic fractions in the next activity. Notice that division is indicated in three different ways in this activity, namely with a division sign, with a horizontal line and with a slash symbol.

> **Activity 28**   *Multiplying and dividing algebraic fractions*
>
> Simplify the following expressions.
>
> (a) $\dfrac{40A}{B} \times \dfrac{BC}{16A^4}$ (b) $\dfrac{b}{c^2} \div c^3$ (c) $\left(\dfrac{6y}{x^7}\right) \Big/ \left(\dfrac{15y^{10}}{x^4}\right)$
>
> (d) $\dfrac{a/(a+1)}{a^6/(a+1)^2}$ (e) $\dfrac{3x}{y} \div \dfrac{6}{y^2}$ (f) $g \times \dfrac{5}{k}$

4 Roots and powers

In this section you'll revise how to work with roots and powers of numbers and algebraic expressions.

4.1 Roots of numbers

As you know, a **square root** of a number is a number that when squared (raised to the power 2) gives the original number. For example, both 6 and -6 are square roots of 36, since

$$6^2 = 6 \times 6 = 36 \quad \text{and} \quad (-6)^2 = (-6) \times (-6) = 36.$$

Similarly, a **cube root** of a number is a number that when cubed (raised to the power 3) gives the original number. For example, 4 is a cube root of 64, and -4 is a cube root of -64, because

$$4^3 = 4 \times 4 \times 4 = 64 \quad \text{and} \quad (-4)^3 = (-4) \times (-4) \times (-4) = -64.$$

As you'd expect, a **fourth root** of a number is a number that when raised to the power 4 gives the original number, and **fifth roots**, **sixth roots** and so on are defined in a similar way.

Every positive number has two square roots, a positive one and a negative one. For example, as you saw above, the two square roots of 36 are 6 and -6. When we say *the* square root of a positive number, we mean the positive square root. Every negative number has no real square roots – in other words, it has no square roots that are real numbers. For example, there's no real number that when squared gives -36. (You'll see in Unit 12 that negative numbers do have 'imaginary' square roots.) The number 0 has one square root, namely itself.

The situation for cube roots is simpler: every real number, whether positive, negative or zero, has exactly one real cube root.

The situation for fourth roots, sixth roots and all even-numbered roots is similar to that for square roots, and the situation for fifth roots, seventh roots and all odd-numbered roots is similar to that for cube roots. That is,

the following facts hold. For every even natural number n, every positive number has exactly two real nth roots (a positive one and a negative one), every negative number has no real nth roots, and the number 0 has exactly one nth root (itself). For every *odd* natural number n, every real number has exactly one real nth root. For example,

64 has two real sixth roots, 2 and -2;
-64 has no real sixth roots;
243 has one real fifth root, 3;
-243 has one real fifth root, -3.

You can use the symbol '$\pm$', which means 'plus or minus', to write a positive and a negative root together. For example, the two square roots of 36 are ± 6, and the two real sixth roots of 64 are ± 2.

The symbol $\sqrt{\ }$ is used to denote the positive square root of a positive real number, or the square root of zero. For example,

$$\sqrt{36} = 6 \quad \text{and} \quad \sqrt{0} = 0.$$

Because $\sqrt{\ }$ means a positive or zero square root, it's incorrect to write, for example,

'if $x^2 = 4$, then $x = \sqrt{4} = \pm 2$'.

What you should write is

'if $x^2 = 4$, then $x = \pm\sqrt{4} = \pm 2$'.

Similarly, the symbol $\sqrt[3]{\ }$ denotes the cube root of a positive or zero real number, the symbol $\sqrt[4]{\ }$ denotes the positive or zero fourth root of a positive or zero real number, and so on. For example, $\sqrt[4]{81} = 3$ and $\sqrt[4]{0} = 0$. Note in particular that the number under any of the symbols $\sqrt{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots$ must be positive or zero, and the resulting root is always positive or zero.

The symbol $\sqrt{\ }$ for roots was introduced by René Descartes, an influential French philosopher and mathematician.

René Descartes (1596–1650)

Most roots of numbers are irrational numbers. In particular, the square root of any natural number that isn't a perfect square is irrational. So, for example, $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and $\sqrt{6}$ are all irrational numbers.

Because numbers like these can't be written down exactly as fractions or terminating decimals, we often leave them just as they are in calculations and in the answers to calculations. For example, we might say that the answer to a calculation is $1 - 2\sqrt{5}$. The advantage of this approach is that it allows us to work with exact numbers, rather than approximations. This can help to simplify calculations.

An expression such as $1 - 2\sqrt{5}$ is called a *surd*. That is, a **surd** is a numerical expression that contains one or more irrational roots of numbers. Here are some more surds:

$$\sqrt{2}, \quad -\sqrt{2}, \quad 2 + \sqrt[3]{5}, \quad \sqrt{6} + \sqrt{7}, \quad \frac{\sqrt{7}}{3}.$$

Note that in some texts, the word 'surd' means 'irrational root' rather than 'numerical expression containing one or more irrational roots'.

Surds are usually written concisely, in a similar way to algebraic expressions. Multiplication signs are usually omitted – for example, we write $2\sqrt{5}$ rather than $2 \times \sqrt{5}$. However, sometimes it's necessary or helpful to include multiplication signs in a surd. Also, where a number and an irrational root are multiplied together, the number is written first – for example, we write $2\sqrt{5}$ rather than $\sqrt{5}\,2$. This is because, for example, $\sqrt{5}\,2$ could easily be misread as $\sqrt{52}$.

> The word 'surd' is derived from the same Latin word as 'absurd'. The original Latin word is 'surdus', which means deaf or silent.

4.2 Manipulating surds

You sometimes have to manipulate surds, in a similar way to algebraic expressions. In particular, you sometimes have to simplify them. Many calculators can simplify surds, but you should also know how to do this yourself. This is because it is useful to be able to simplify straightforward surds without having to resort to a calculator, and because you'll have to use similar methods to simplify algebraic expressions involving roots.

You can manipulate surds using the usual rules of algebra – you treat the irrational roots in the same way that you treat variables. For example, you can collect like terms. In the expression below, there are two terms with $\sqrt{3}$ in them, which can be collected, and two terms with $\sqrt{7}$ in them, which can also be collected:

$$\begin{aligned}\sqrt{3} + 2\sqrt{7} + 4\sqrt{3} - \sqrt{7} &= (1 + 4)\sqrt{3} + (2 - 1)\sqrt{7} \\ &= 5\sqrt{3} + \sqrt{7}.\end{aligned}$$

There are also several ways to manipulate surds that don't involve treating irrational roots in the same way that you treat variables. In particular, you should keep in mind that an expression of the form $(\sqrt{a})^2$ can be simplified to a. For example,

$$\sqrt{8}\sqrt{8} = (\sqrt{8})^2 = 8.$$

Two further useful rules are given below. The reason why they hold is explained in the next subsection. These rules apply to all appropriate numbers – for example, in the second rule b must be non-zero, because division by zero isn't possible.

The square root of a product of numbers is the same as the product of the square roots of the numbers, and similarly for a quotient:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Analogous rules apply to cube roots, fourth roots and so on.

For example,

$$\sqrt{5}\sqrt{3} = \sqrt{5 \times 3} = \sqrt{15} \quad \text{and} \quad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}.$$

Similarly,

$$\sqrt[3]{7}\sqrt[3]{2} = \sqrt[3]{7 \times 2} = \sqrt[3]{14} \quad \text{and} \quad \frac{\sqrt[3]{8}}{\sqrt[3]{2}} = \sqrt[3]{\frac{8}{2}} = \sqrt[3]{4}.$$

It's *not true* that, in general, $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$. For example, $\sqrt{9 + 16} = \sqrt{25} = 5$, whereas $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

Activity 29   *Simplifying surds*

Simplify the following surds.

(a) $\sqrt{5}\sqrt{6}$ (b) $\dfrac{\sqrt{75}}{\sqrt{15}}$ (c) $3 + \sqrt{10}\sqrt{10}$ (d) $\sqrt{8}\sqrt{2}$ (e) $\dfrac{\sqrt[3]{5}}{\sqrt[3]{15}}$

Here are some more complicated examples.

Example 17   *Manipulating surds*

In parts (a) and (b), multiply out the brackets. In part (c), write the expression as a single fraction. Simplify your answers.

(a) $\sqrt{7}(\sqrt{7} - \sqrt{2})$ (b) $(3 + \sqrt{2})(1 + 5\sqrt{2})$ (c) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$

Solution

Use the usual rules of algebra, combining them with the extra rules for manipulating surds. In part (a), start by multiplying each term in the brackets by the multiplier. In part (b), start by using FOIL. In part (c), start by writing the fractions with the same denominator.

(a) $\sqrt{7}(\sqrt{7} - \sqrt{2}) = (\sqrt{7})^2 - \sqrt{7}\sqrt{2} = 7 - \sqrt{14}$

(b) $(3 + \sqrt{2})(1 + 5\sqrt{2}) = 3 + 15\sqrt{2} + \sqrt{2} + 5(\sqrt{2})^2$

$$= 3 + 16\sqrt{2} + 10$$

$$= 13 + 16\sqrt{2}$$

(c) $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{\sqrt{2}\sqrt{3}} + \dfrac{\sqrt{2}}{\sqrt{2}\sqrt{3}} = \dfrac{\sqrt{3}}{\sqrt{6}} + \dfrac{\sqrt{2}}{\sqrt{6}} = \dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{6}}$

Activity 30   *Manipulating surds*

In parts (a)–(e), multiply out the brackets. In parts (f) and (g), write the expression as a single fraction. Simplify your answers.

(a) $\sqrt{3}(2\sqrt{2} + \sqrt{3})$ (b) $\sqrt{2}(1 + \sqrt{3}) + 9\sqrt{2}$ (c) $(1 - \sqrt{5})(6 - 2\sqrt{5})$

(d) $(6 - \sqrt{7})(6 + \sqrt{7})$ (e) $(\sqrt{3} + 2\sqrt{2})(\sqrt{3} - 2\sqrt{2})$

(f) $3 - \dfrac{1}{\sqrt{2}}$ (g) $\dfrac{4}{\sqrt{5}} + \dfrac{\sqrt{3}}{\sqrt{2}}$

Here's another useful way to simplify surds. If, in an irrational square root (such as $\sqrt{48}$ or $\sqrt{10}$) in a surd, the number under the square root sign has a factor that's a perfect square, then you can simplify this square root.

To do this, you write the number under the square root sign as the product of the perfect square and another number, then you use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$. This is illustrated in the next example.

Example 18 *Simplifying square roots in surds*

Simplify the following surds, where possible.

(a) $\sqrt{48}$ (b) $\sqrt{10}$

Solution

(a) 👁 48 has a factor that's a perfect square, namely 16. 💭

$$\sqrt{48} = \sqrt{16 \times 3}$$

👁 Use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$. 💭

$$= \sqrt{16}\sqrt{3} = 4\sqrt{3}.$$

(b) $\sqrt{10}$ doesn't have a factor that's a perfect square, so it can't be simplified.

Note that, as mentioned earlier, the first 15 perfect squares are:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225.$$

Activity 31 *Simplifying square roots in surds*

(a) Simplify the following surds, where possible.

(i) $\sqrt{8}$ (ii) $\sqrt{150}$ (iii) $\sqrt{22}$ (iv) $\sqrt{32}$ (v) $5 + \sqrt{108}$

(vi) $\sqrt{12} + \sqrt{4}$ (vii) $\sqrt{27} - \sqrt{3}$

(b) Multiply out the brackets in the following surds, and simplify your answers.

(i) $\sqrt{6}(\sqrt{3} + \sqrt{2})$ (ii) $(\sqrt{10} - \sqrt{5})(2\sqrt{5} + 1)$

Here's a final way to simplify surds. If a surd contains an irrational root in the denominator of a fraction, then it's sometimes useful to rewrite it so that the denominator no longer contains this irrational root. This is called **rationalising** the denominator, and it can make the surd easier to work with. It can often be achieved by multiplying the top and bottom of the fraction by a suitable surd. (As you know, multiplying the top and bottom of a fraction by the same number doesn't change the value of the fraction.)

For example, you can rationalise the denominator of the surd $1/\sqrt{2}$ by multiplying the top and bottom by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

In more complicated cases, the surd might include a fraction such as

$$\frac{5}{4\sqrt{7} + \sqrt{2}} \quad \text{or} \quad \frac{\sqrt{2}}{5 - 2\sqrt{3}},$$

in which the denominator is a sum of two terms, either or both of which is a rational number multiplied by an irrational square root. You can rationalise a denominator like this by multiplying the top and bottom of the fraction by a **conjugate** of the expression in the denominator. This is the expression that you get when you change the sign of one of the two terms – it's usual to choose the second term.

For example, the expressions in the denominators of the two fractions above have conjugates

$$4\sqrt{7} - \sqrt{2} \quad \text{and} \quad 5 + 2\sqrt{3},$$

respectively.

Part (b) of the next example shows you how to rationalise a denominator by multiplying top and bottom by a conjugate of the denominator. To see why this method works, notice that when you multiply top and bottom of the fraction by the conjugate, in the denominator you obtain a product of the form

$$(A + B)(A - B),$$

where A and B are the terms of the denominator of the original fraction.

As you saw on page 37, when you multiply out this product you get the expression

$$A^2 - B^2.$$

The squaring of the terms A and B gets rid of the irrational square roots, as you can see in the example.

Example 19   *Rationalising denominators*

Rationalise the denominators of the following surds.

(a) $\dfrac{1}{3\sqrt{2}}$ (b) $\dfrac{\sqrt{2}}{5 - 2\sqrt{3}}$

Solution

(a) $\dfrac{1}{3\sqrt{2}} = \dfrac{1}{3\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{3 \times 2} = \dfrac{\sqrt{2}}{6}$

(b) A conjugate of $5 - 2\sqrt{3}$ is $5 + 2\sqrt{3}$.

$$\dfrac{\sqrt{2}}{5 - 2\sqrt{3}} = \dfrac{\sqrt{2}}{5 - 2\sqrt{3}} \times \dfrac{5 + 2\sqrt{3}}{5 + 2\sqrt{3}}$$

$$= \dfrac{\sqrt{2}(5 + 2\sqrt{3})}{(5 - 2\sqrt{3})(5 + 2\sqrt{3})}$$

$$= \dfrac{5\sqrt{2} + 2\sqrt{3}\sqrt{2}}{5^2 - (2\sqrt{3})^2}$$

$$= \dfrac{5\sqrt{2} + 2\sqrt{6}}{5^2 - 2^2(\sqrt{3})^2}$$

$$= \dfrac{5\sqrt{2} + 2\sqrt{6}}{25 - 4 \times 3}$$

$$= \dfrac{5\sqrt{2} + 2\sqrt{6}}{13}$$

Activity 32   *Rationalising denominators*

Rationalise the denominators of the following surds.

(a) $\dfrac{3}{\sqrt{7}}$ (b) $\dfrac{\sqrt{2}}{\sqrt{6}}$ (c) $\dfrac{5}{\sqrt{5}}$ (d) $\dfrac{2}{1 + \sqrt{17}}$ (e) $\dfrac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$

In general, if the final answer that you've found to a question is a surd, then you should use the methods in this subsection to simplify it as much as possible, except that it isn't always necessary to rationalise a denominator that contains an irrational root. For example, it's acceptable to leave the surd $1/\sqrt{2}$ as it is. However, many calculators rationalise denominators: for example, they display $1/\sqrt{2}$ as $\sqrt{2}/2$.

4.3 Working with powers

As you know, *raising a number to a power* means multiplying the number by itself a specified number of times. For example, raising 2 to the power 3 gives

$$2^3 = 2 \times 2 \times 2 = 8.$$

Here the number 2 is called the **base number** or just **base**, and the superscript 3 is called the **power**, **index** or **exponent**. The word 'power' is also used to refer to the *result* of raising a number to a power – for example, we say that 2^3, or 8, is a **power** of 2. These two alternative meanings of the word 'power' don't normally cause any confusion, as it's usually clear from the context which of them is meant.

Note that the plural of 'index', in the context here, is 'indices'. (With some other meanings of 'index', such as the type of index that you get at the back of a book, the plural is 'indexes'.)

Activity 33   *Calculating powers*

Evaluate the following expressions without using your calculator.

(a) 2^4 (b) $(-2)^4$ (c) -2^4 (d) $(-3)^3$ (e) $\left(\frac{1}{2}\right)^2$ (f) $\left(\frac{1}{3}\right)^3$

There are some useful rules that you can use to manipulate expressions that contain indices. These are known as **index laws**. In this subsection you'll revise these rules, and practise using them.

We'll start with the three most basic index laws. To see where the first of them comes from, suppose that you want to simplify the expression $a^5 \times a^2$.

You could do this as follows:

$$a^5 \times a^2 = \underbrace{a \times a \times a \times a \times a}_{5 \text{ copies of } a} \times \underbrace{a \times a}_{2 \text{ copies of } a}$$

$$= \underbrace{a \times a \times a \times a \times a \times a \times a}_{7 \text{ copies of } a} = a^7.$$

You can see that, in general, to *multiply* two powers with the same base, you *add* the indices. This gives the first index law in the box below.

Now suppose that you want to simplify the expression a^5/a^2. You could do this as follows:

$$\frac{a^5}{a^2} = \frac{\overbrace{a \times a \times a \times a \times a}^{5 \text{ copies of } a}}{\underbrace{a \times a}_{2 \text{ copies of } a}} = \frac{\cancel{a} \times \cancel{a} \times a \times a \times a}{\cancel{a} \times \cancel{a}} = \underbrace{a \times a \times a}_{3 \text{ copies of } a} = a^3.$$

You can see that, in general, to *divide* a power by another power with the same base, you *subtract* the 'bottom' index from the 'top' one. This gives the second index law in the box below.

Finally, suppose that you want to simplify the expression $(a^2)^3$. You could do this as follows:

$$(a^2)^3 = a^2 \times a^2 \times a^2 = (a \times a) \times (a \times a) \times (a \times a) = a^6.$$

You can see that, in general, to *raise* a power *to a power* you *multiply* the indices. This gives the third index law in the box below.

Index laws for a single base

To multiply two powers with the same base, add the indices:

$$a^m a^n = a^{m+n}.$$

To divide two powers with the same base, subtract the indices:

$$\frac{a^m}{a^n} = a^{m-n}.$$

To find a power of a power, multiply the indices:

$$(a^m)^n = a^{mn}.$$

If you think about where the first index law comes from, you'll see that it extends to products of more than two powers of the same base. For example,

$$a^m a^n a^r = a^{m+n+r}.$$

Another way to think about such extensions of the first index law is as two or more applications of the law. For example,

$$a^m a^n a^r = (a^m a^n)a^r = a^{m+n}a^r = a^{m+n+r}.$$

Activity 34   *Using the three basic index laws*

Simplify the following expressions.

(a) $\dfrac{a^{20}}{a^5}$ (b) $(y^4)^5$ (c) $\dfrac{b^4 b^7}{b^3}$ (d) $p^3 p^5 p^3 p$ (e) $(x^3)^3 x^2$

(f) $(m^3 m^2)^5$ (g) $\left(\dfrac{c^5}{c^2}\right)^2$

Activity 35   *Using the three basic index laws again*

Multiply out the brackets or expand the fraction, as appropriate, and simplify your answers.

(a) $(x^3 - 1)(2x^3 + 5)$ (b) $(a^4 + b^4)^2$ (c) $\dfrac{p^8 + p^2}{p^2}$

All the indices in Activities 34 and 35 are positive integers, but you can also have zero or negative indices. It might not be immediately clear to you what it means to raise a number to the power 0, or to the power -3, for example, but if you think about the fact that we want the basic index laws in the box above to work for powers like these, then it soon becomes clear what they must mean.

For example, consider the power 2^0. If the first index law in the box above works for this power, then, for instance,

$$2^3 2^0 = 2^{3+0} = 2^3.$$

This calculation tells you that if you multiply 2^3 by 2^0, then you get 2^3 again. So the value of 2^0 must be 1. You can see that, in general, a non-zero number raised to the power 0 must be 1. Usually we don't give a meaning to 0^0, but in some contexts it's convenient to take it to be equal to 1.

There's no obvious, correct meaning for 0^0. This is because on the one hand you'd expect that if you raise a number to the power 0 then the answer will be 1, but on the other hand you'd expect that if you raise 0 to a power then the answer will be 0. Mathematicians have debated the meaning of 0^0 for several centuries, and in 1821 Augustin-Louis Cauchy included it in a list of undefined forms, along with expressions like $0/0$. Cauchy was a French mathematician, one of the pioneers of mathematical *analysis*, the theory that underlies calculus.

Augustin-Louis Cauchy
(1789–1857)

Now let's consider 2^{-3}, for example. Again, if the first index law in the box above works for this power, then, for example,

$$2^3 2^{-3} = 2^{3+(-3)} = 2^0 = 1.$$

This calculation tells you that if you multiply 2^3 by 2^{-3} then you get 1. So 2^{-3} must be the reciprocal of 2^3; that is, $2^{-3} = 1/2^3$. You can see that, in general, a non-zero number raised to a negative index must be the reciprocal of the number raised to the corresponding positive index.

In summary, the meanings of zero and negative indices are given by the following index laws.

> ### More index laws for a single base
>
> A number raised to the power 0 is 1:
>
> $$a^0 = 1.$$
>
> A number raised to a negative power is the reciprocal of the number raised to the corresponding positive power:
>
> $$a^{-n} = \frac{1}{a^n}.$$

These index laws hold for all appropriate numbers. So, for example, in the second law above, a can be any number except 0; it can't be 0 because division by 0 isn't possible.

The second index law in the box above tells you that, in particular,

$$a^{-1} = \frac{1}{a}.$$

So raising a number to the power -1 is the same as finding its reciprocal. For example,

$$2^{-1} = \tfrac{1}{2}, \quad \left(\tfrac{1}{2}\right)^{-1} = 2 \quad \text{and} \quad \left(\tfrac{2}{3}\right)^{-1} = \tfrac{3}{2}.$$

This fact can help you to evaluate numerical expressions that contain negative indices. For example,

$$\left(\tfrac{1}{2}\right)^{-3} = \left(\tfrac{1}{2}\right)^{-1 \times 3} = \left(\left(\tfrac{1}{2}\right)^{-1}\right)^3 = 2^3 = 8.$$

Activity 36   *Understanding zero and negative indices*

Evaluate the following powers without using your calculator.

(a) 4^{-2} (b) 3^{-1} (c) 5^0 (d) $\left(\frac{2}{7}\right)^{-1}$ (e) $\left(\frac{1}{3}\right)^{-1}$ (f) $\left(\frac{1}{3}\right)^{-2}$

Indices can also be fractions, or any real numbers at all – the meaning of such indices is discussed later in this subsection. All the index laws given in this subsection hold for indices and base numbers that are any real numbers (except that the numbers must be appropriate for the operations – for example, you can't divide by zero).

The index laws that you've seen so far in this subsection involve just one base number, but there are also two index laws that involve two different base numbers. To see where they come from, consider the following algebraic manipulations:

$$(ab)^3 = ab \times ab \times ab = ababab = aaabbb = a^3b^3,$$
$$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{aaa}{bbb} = \frac{a^3}{b^3}.$$

In general, the following rules hold.

> ## Index laws for two bases
>
> A power of a product of numbers is the same as the product of the same powers of the numbers, and similarly for a power of a quotient:
>
> $$(ab)^n = a^n b^n,$$
> $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

The first index law above extends to a product of any number of factors. For example,

$$(abc)^n = a^n b^n c^n.$$

You can see why this is if you think about where the law comes from, or you can think of it as more than one application of the law:

$$(abc)^n = (ab)^n c^n = a^n b^n c^n.$$

The next example illustrates how you can use all the index laws that you've seen so far in this subsection to simplify expressions that contain indices. There are several alternative ways to do this. Whichever you use, you should check that you have combined all powers of the same base. For example, you should change $a^2 a^3$ to a^5, and a^2/a^6 to $1/a^4$. It's often best for your final version to include only positive indices, especially if the expression has a denominator. You can change negative indices into

positive indices by using the rule $a^{-n} = 1/a^n$. For example, you can change a^{-3} to $1/a^3$, and $1/a^{-3}$ to a^3.

Example 20 *Simplifying expressions containing indices*

Simplify the following expressions, ensuring that the simplified versions contain no negative indices.

(a) $\dfrac{d^2}{d^{-4}}$ (b) $\dfrac{b^{-3}}{b^2 c^{-4}}$ (c) $(2h^{-3}g)^2$

Solution

(a) Use the law $a^{-n} = 1/a^n$ to change the negative index into a positive index, then use the law $a^m a^n = a^{m+n}$ to combine the powers.

$$\frac{d^2}{d^{-4}} = d^2 \times \frac{1}{d^{-4}} = d^2 d^4 = d^6$$

Alternatively, use the law $a^m/a^n = a^{m-n}$.

$$\frac{d^2}{d^{-4}} = d^{2-(-4)} = d^6$$

(b) Use the law $a^{-n} = 1/a^n$ to change the negative indices into positive indices, then use the law $a^m a^n = a^{m+n}$ to combine the powers of b.

$$\frac{b^{-3}}{b^2 c^{-4}} = \frac{c^4}{b^2 b^3} = \frac{c^4}{b^5}$$

Alternatively, use the law $a^m/a^n = a^{m-n}$ to combine the powers of b, then use the law $a^{-n} = 1/a^n$ to change the negative indices into positive indices.

$$\frac{b^{-3}}{b^2 c^{-4}} = \frac{b^{-5}}{c^{-4}} = \frac{c^4}{b^5}$$

(c) Remove the brackets by using the law $(ab)^n = a^n b^n$, then the law $(a^m)^n = a^{mn}$. Then use the law $a^{-n} = 1/a^n$ to change the negative index into a positive index.

$$(2h^{-3}g)^2 = 2^2(h^{-3})^2 g^2 = 4h^{-6}g^2 = \frac{4g^2}{h^6}$$

Alternatively, use the law $a^{-n} = 1/a^n$ to change the negative index into a positive index, then use the law $(a/b)^n = a^n/b^n$, then the laws $(ab)^n = a^n b^n$ and $(a^m)^n = a^{mn}$.

$$(2h^{-3}g)^2 = \left(\frac{2g}{h^3}\right)^2 = \frac{(2g)^2}{(h^3)^2} = \frac{2^2 g^2}{h^6} = \frac{4g^2}{h^6}$$

As mentioned above and illustrated in Example 20, when you're simplifying an expression that contains indices, it's often a good idea to aim for a final version that contains no negative indices. However, sometimes a final form that contains negative indices is simpler, or more useful. As with many algebraic expressions, there's often no 'right answer' for the simplest form of an expression that contains indices. One form might be better for some purposes, and a different form might be better for other purposes.

Activity 37   *Simplifying expressions containing indices*

Simplify the following expressions, ensuring that the simplified versions contain no negative indices. (In parts (l), (n) and (o) you're not expected to multiply out any brackets.)

(a) $5g^{-1}$

(b) $\dfrac{1}{y^{-1}}$

(c) $\dfrac{2}{3x^{-5}}$

(d) $\dfrac{a^{-3}}{b^{-4}}$

(e) $\dfrac{P^2}{Q^{-5}}$

(f) $(3h^2)^2$

(g) $(3h^2)^{-2}$

(h) $\dfrac{(b^{-4})^3}{(3c)^2}$

(i) $\dfrac{(A^{-1}B)^2}{(B^{-3})^3}$

(j) $\left(\dfrac{2y^{-1}}{z^2}\right)^5$

(k) $\dfrac{x^{-5}}{x}$

(l) $\dfrac{(x-1)^{-3}}{(x-1)^2}$

(m) $\left(\dfrac{3}{z^2}\right)^{-3}$

(n) $\dfrac{x}{(2x-3)^3} \times x^{-2}$

(o) $\left(\dfrac{(x+2)^2}{x^5}\right) \Big/ (x+2)^{-4}$

Now let's consider indices that are fractions or any real numbers. To understand what's meant by a fractional index, first consider the power $2^{1/3}$. If the index law for raising a power to a power, $(a^m)^n = a^{mn}$, is to work for fractional indices, then, for example,

$$(2^{1/3})^3 = 2^{(1/3)\times 3} = 2^1 = 2.$$

This calculation tells you that if you raise $2^{1/3}$ to the power 3, then you get 2. So $2^{1/3}$ must be the cube root of 2; that is, $2^{1/3} = \sqrt[3]{2}$. In general, you can see that raising a positive number to the power $1/n$, say, is the same as taking the nth root of the number. So $5^{1/2} = \sqrt{5}$, and $12^{1/4} = \sqrt[4]{12}$, and so on. This is the first rule in the following box.

To understand what's meant by a fractional index when the numerator of the fraction isn't 1, consider, for example, the power $2^{5/3}$. If the index law for raising a power to a power is to work for fractional indices, then we must have

$$2^{5/3} = 2^{(1/3)\times 5} = \left(2^{1/3}\right)^5 = \left(\sqrt[3]{2}\right)^5.$$

So $2^{5/3}$ must be the cube root of 2, raised to the power 5. However, by the same index law we must also have

$$2^{5/3} = 2^{5\times(1/3)} = \left(2^5\right)^{1/3} = \sqrt[3]{2^5}.$$

So $2^{5/3}$ must be the cube root of 2^5. Luckily these two alternative definitions give the same answer, both in this case and for other fractional powers of a positive real number. This gives the second, more general rule in the following box.

Converting between fractional indices and roots

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Notice that since the notation $\sqrt[n]{a}$ is defined only when a is positive or zero, only positive numbers and zero can be raised to fractional powers. So the notation a^n has no meaning if a is negative and n is not an integer.

Activity 38   *Converting fractional indices to root signs*

Rewrite the following expressions so that they contain root signs (such as $\sqrt{}$ and $\sqrt[3]{}$) instead of fractional indices.

Hint for parts (d), (f) and (g): first rewrite the expression to change the negative index into a positive index.

(a) $t^{1/2}$ (b) $x^{1/3}$ (c) $p^{2/3}$ (d) $x^{-1/2}$ (e) $(2x - 3)^{1/2}$

(f) $(1 + x^2)^{-1/2}$ (g) $(1 + x)x^{-1/2}$

Although you were asked to convert several different fractional indices to root signs in Activity 38, it's often best to avoid using any root signs other than the square root sign $\sqrt{}$ in algebraic expressions. This is because the small numbers in root signs such as $\sqrt[3]{}$ and $\sqrt[4]{}$ can be easily misread, especially when they're handwritten. For example,

$$p\sqrt[3]{q} \quad \text{could be misread as} \quad p^3\sqrt{q},$$

so it is better written as $pq^{1/3}$.

Activity 39   *Converting root signs to fractional indices*

Rewrite the following expressions so that they contain fractional indices instead of root signs.

(a) $\sqrt[4]{y}$ (b) $\sqrt[5]{1-2x}$ (c) $\dfrac{1}{\sqrt[3]{x}}$ (d) $\left(\sqrt[5]{u}\right)^2$ (e) $\dfrac{1}{\sqrt[3]{x^4}}$

(f) $\sqrt[4]{(y+2)^3}$

You should simplify an algebraic expression that contains fractional indices in the same ways as an expression that contains only integer indices – you should combine powers of the same base where possible, and so on. When you've done that, if the expression contains the index $\frac{1}{2}$ or $-\frac{1}{2}$, then you might consider rewriting it so that it contains the square root sign instead. For example, you might prefer to write

$$x^{1/2} \text{ as } \sqrt{x}, \quad \text{and} \quad x^{-1/2} \text{ as } \frac{1}{\sqrt{x}}.$$

You should usually leave any other fractional indices as they are (except that it may be helpful to convert negative indices to positive ones). Here's an example.

Example 21   *Simplifying expressions containing fractional indices*

Simplify the following expressions.

(a) $\dfrac{c^{3/4}}{c^{5/4}}$ (b) $(2h^{1/6})^2$

Solution

(a) 🔍 Use the index laws in the same ways as for integer indices. 💭

$$\frac{c^{3/4}}{c^{5/4}} = c^{(3/4)-(5/4)} = c^{-2/4} = c^{-1/2}$$

🔍 Usually, change the negative index to a positive one, and perhaps use a square root sign instead of the index $\frac{1}{2}$. 💭

$$= \frac{1}{c^{1/2}} = \frac{1}{\sqrt{c}}$$

(b) 🔍 Proceed as in part (a). Leave the final fractional index as it is, since it's not $\frac{1}{2}$ or $-\frac{1}{2}$. 💭

$$(2h^{1/6})^2 = 2^2(h^{1/6})^2 = 4h^{(1/6)\times 2} = 4h^{1/3}$$

Activity 40   *Simplifying expressions containing fractional indices*

Simplify the following expressions. (In part (f) you're not expected to multiply out any brackets.)

(a) $x^{1/3}x^{1/3}$ (b) $\dfrac{a}{a^{1/3}}$ (c) $\dfrac{a}{a^{1/2}}$ (d) $\dfrac{x}{x^{-1/2}}$ (e) $(2x^{1/5})^3$

(f) $\dfrac{(1+x)^2}{\sqrt{1+x}}$ (g) $\left(\dfrac{1}{u}\right)^{1/3}$ (h) $\dfrac{A^{5/2}}{A^3}$ (i) $\dfrac{x^{1/2}y^2}{x^3y^{1/2}}$ (j) $\sqrt{4x}$

It follows from the meaning of fractional indices that the two rules for square roots that you used in the last subsection,

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \quad \text{and} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$

are just particular cases of the two index laws in the box on page 67. They're these index laws with the index n taken to be $\frac{1}{2}$.

So far in this subsection you've seen what's meant by indices that are rational numbers. However, as mentioned earlier, an index can be any real number. For example, you can raise any positive number to the irrational index $\sqrt{2}$. A precise definition of the meaning of an irrational index is beyond the scope of this module, but the basic idea is that you can work out the value of any positive number raised to any irrational index as accurately as you like, by using as many decimal places of the decimal form of the irrational index as you like. For example, suppose that you're interested in the value of $3^{\sqrt{2}}$. Since

$$\sqrt{2} = 1.414\,213\,562\,373\,09\ldots,$$

one approximation to $3^{\sqrt{2}}$ is

$$3^{1.414} = 4.727\,695\,035\,268\,53\ldots,$$

and a more accurate one is

$$3^{1.414\,213} = 4.728\,801\,466\,241\,14\ldots,$$

and so on. The indices here, 1.414 and $1.414\,213$, and so on, are rational, as they are terminating decimals. It's possible to show that the closer a rational index r is to $\sqrt{2}$, the closer 3^r is to some fixed real number. The value of $3^{\sqrt{2}}$ is defined to be this number.

So an index can be any real number. All the index laws that you've met in this subsection hold for indices and base numbers that are any real numbers (except that the numbers must be appropriate for the operations – for example, you can't divide by zero or apply $\sqrt{}$ to a negative number). Here's a summary of them all.

Index laws

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

$$(ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^{1/n} = \sqrt[n]{a} \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Here's an activity involving indices that contain variables as well as numbers. You can deal with these using the index laws in the usual way.

Activity 41   *Simplifying indices that contain variables*

Simplify the following expressions.

(a) $a^{2p} a^{5p}$ (b) $\dfrac{b^{7k}}{b^{4k}}$ (c) $(g^n)^k$ (d) $(2y^{2t})^2 y^{-t}$ (e) $\dfrac{m^{3x}}{m^{-x}}$

(f) $\dfrac{c^{3y}}{c^{5y}}$ (g) $\dfrac{(a^{-3t}b^{3t})^2}{a^{3t}b^t}$ (h) $(d^{1/r})^{2r}$ (i) $(3h)^{2p}(9h)^p$

Scientific notation

One use of indices is in **scientific notation** (also known as **standard form**). This is a way of writing numbers that's particularly helpful when you're dealing with very large or very small numbers. To express a number in scientific notation, you write it in the form

(a number between 1 and 10, but not including 10)

$\times$ (an integer power of ten).

Here are some examples.

$$
\begin{aligned}
427 &= 4.27 \times 100 &= 4.27 \times 10^2 \\
42.7 &= 4.27 \times 10 &= 4.27 \times 10^1 \\
4.27 &= 4.27 \times 1 &= 4.27 \times 10^0 \\
0.427 &= 4.27 \times 0.1 &= 4.27 \times 10^{-1} \\
0.0427 &= 4.27 \times 0.01 &= 4.27 \times 10^{-2}
\end{aligned}
$$

> **Activity 42** *Using scientific notation*
>
> (a) Express the following numbers in scientific notation.
>
>   (i) 38 800 000   (ii) 4237   (iii) 0.0973   (iv) 1.303
>
>   (v) 0.000 000 028
>
> (b) Express the following numbers in ordinary notation.
>
>   (i) 2.8×10^4 (ii) 5.975×10^{-1} (iii) 2.78×10^{-7}
>
>   (iv) 3.43×10^7

In computer output, scientific notation is sometimes represented with the power of 10 indicated by the letter E (for exponent). For example, 4.27×10^{-2} would be represented as $4.27\text{E}{-2}$.

5 Equations

In this section you'll revise how to rearrange and solve equations.

5.1 Terminology for equations

You saw earlier that to indicate that two expressions are equivalent, you put an equals sign between them. However, there's another use of equals signs. You can place an equals sign between *any* two expressions, to form an **equation**. Here's an example:

$$3(d + 1) = 7d - 5. \tag{3}$$

Usually you form an equation like this when you're interested in the values of the variables in the equation that make the equation true. These values are said to **satisfy** the equation, and are called **solutions** of the equation. For example, the value $d = 2$ satisfies equation (3), as shown in Example 22 below, but the equation isn't satisfied by any other value of d.

Notice that equation (3) contains the variable d and no other variable. This fact is expressed by saying that it's an **equation in** d. Similarly, an equation that contains the variables x and y and no other variables is an equation in x and y, and so on.

The next example illustrates how you should set out your working when you're checking whether an equation is satisfied. You should evaluate the left- and right-hand sides *separately*, and check whether each side gives the same answer. If you wish, you can use the abbreviations LHS and RHS for the left-hand side and right-hand side of the equation.

Example 22   *Checking whether an equation is satisfied*

Show that $d = 2$ satisfies the equation

$$3(d + 1) = 7d - 5.$$

Solution

If $d = 2$, then

$$\text{LHS} = 3(2 + 1) = 3 \times 3 = 9$$

and

$$\text{RHS} = 7 \times 2 - 5 = 14 - 5 = 9.$$

Since LHS = RHS, $d = 2$ satisfies the equation.

In the next activity you're asked to check whether another equation is satisfied for particular values of its variables. The right-hand side of this equation is just a number, so to check whether it's satisfied you just need to evaluate the other side to see whether you get this number.

Activity 43   *Checking whether an equation is satisfied*

Show that the equation $2x + 3y = 5$ is

(a) satisfied by $x = 4$ and $y = -1$;

(b) not satisfied by $x = 3$ and $y = -2$.

It's traditional to use letters from near the end of the alphabet, such as x, y and z, to represent unknowns, and letters from near the start of the alphabet, such as a, b and c, to represent known constants. This tradition originates from René Descartes (see page 56). It's thought that x was preferred to y and z because printers tended to have more type for the letter x than for the letters y and z, due to the frequency of occurrence of these letters in the French and Latin languages.

An equation that is satisfied by *all* possible values of its variables is called an **identity**. For example, the equation

$$x(x + y) = x^2 + xy$$

is an identity: it's true no matter what the values of x and y are. However, in this section we'll mostly be concerned with equations that are satisfied

by only some values of their variables. (In some texts, an identity is indicated using the symbol $\equiv$, rather than $=$.)

Sometimes the variables in an equation are restricted to numbers of a particular type. For example, if an equation contains a variable that represents the length of an object, then this variable would take only positive values.

This means that the solutions of an equation depend not only on the equation itself, but also on the possible values that its variables can take. For example, if x can be any number, then the equation $x^2 = 4$ has two solutions, namely $x = 2$ and $x = -2$, but if you know that x is a positive number, then the same equation has only one solution, $x = 2$.

It's often clear from the context of an equation what type of values its variables can take. If it isn't, then you should assume that the variables can be any real numbers.

The process of finding the solutions of an equation is called **solving** the equation. When we're trying to solve an equation, the variables in the equation are often referred to as **unknowns**, since they represent particular numbers whose values aren't yet known, rather than any numbers at all. The key to solving equations is the technique of *rearranging* them, which you'll revise in the next subsection.

5.2   Rearranging equations

It's often helpful to transform an equation into a different equation that contains the same variables, and is satisfied by the same values of these variables. This is called **rearranging**, **manipulating** or simply **rewriting** the equation. When you rearrange an equation, the original equation and the new one are said to be **equivalent**, or different **forms** of the same equation. Rewriting an equation as a simpler equation is called **simplifying** the equation.

There are three main ways to transform an equation into an equivalent equation – these are summarised below.

Rearranging equations

Carrying out any of the following operations on an equation gives an equivalent equation.

- Rearrange the expressions on one or both sides.
- Swap the sides.
- Do the same thing to both sides.

Here are some examples.

- The equations $y + y = 2$ and $2y = 2$ are equivalent, because the second equation is the same as the first but with the expression on the left-hand side rearranged.

- The equations $2x = 1$ and $1 = 2x$ are equivalent, because the second equation is the same as the first but with its sides swapped.

- The equations $y = 3x$ and $2y = 6x$ are equivalent, because the second equation is the same as the first but with both sides multiplied by 2.

Some of the things that you can do to both sides of an equation to obtain an equivalent equation are set out below. Note that a **non-negative** number is one that's either positive or zero.

> **Doing the same thing to both sides of an equation**
>
> Doing any of the following things to *both sides* of an equation gives an equivalent equation.
>
> - Add something.
> - Subtract something.
> - Multiply by something (provided that it is non-zero).
> - Divide by something (provided that it is non-zero).
> - Raise to a power (provided that the power is non-zero, and that the expressions on each side of the equation can take only non-negative values).

To understand why doing any of the things in the box above to both sides of an equation gives an equivalent equation, consider, for example, adding 2 to each side of the equation

$$2x - 3 = 9 - x.$$

The value $x = 4$ satisfies the original equation, because with this value of x, each side of the equation is equal to 5. Adding 2 to both sides of the equation gives the new equation

$$2x - 3 + 2 = 9 - x + 2,$$

and the value $x = 4$ also satisfies this new equation, because with this value of x, each side of the new equation is equal to $5 + 2$, that is, 7. Similarly, the value $x = 3$ *doesn't* satisfy the original equation, because it gives LHS $= 3$ and RHS $= 6$, and this value doesn't satisfy the new equation either, because it gives LHS $= 3 + 2 = 5$ and RHS $= 6 + 2 = 8$. In general, doing any of the things in the box above to both sides of an equation doesn't change the values of the variables that satisfy the equation.

Notice that there are some restrictions given in brackets in the box above. For example, the restriction on the third item in the box tells you that multiplying both sides of an equation by something is guaranteed to give an equivalent equation *only if the something is non-zero*. To see why this

restriction is needed, notice that if you multiply both sides of any equation by zero, then you obtain the equation $0 = 0$, which will usually not be equivalent to the original equation. There's a more detailed discussion of the restrictions in the next subsection.

There are 'shortcuts' for adding or subtracting something on both sides of an equation, and for multiplying or dividing both sides, which can be useful in particular, common situations. These are described below. Note that some tutors recommend that it's best *not* to use the first shortcut, 'change the side, change the sign', since it can easily lead to mistakes, as detailed below.

Change the side, change the sign

Suppose, for example, that you have the equation

$$7x = 3 - 2y \tag{4}$$

and you want to rewrite it as an equation in which all the variables appear on the left-hand side only. You can achieve this by adding the term $2y$ to each side. This gives the equivalent equation

$$7x + 2y = 3 - 2y + 2y,$$

which can be simplified to

$$7x + 2y = 3. \tag{5}$$

If you compare equations (4) and (5), then you can see that the overall effect on the original equation is that the term $2y$ has been moved to the other side, and its sign has been changed. In general, the technique of adding or subtracting on both sides of an equation leads to the following rule.

> Moving a term of one side of an equation to the other side, and changing its sign, gives an equivalent equation.

This rule is sometimes summarised as 'change the side, change the sign'. Some people find it useful, while others find it just as convenient to stick with thinking about adding or subtracting on both sides. If you do use the rule, then you need to do so carefully, as it can easily lead to mistakes of the kinds explained next.

When you use the rule, you must make sure that the term you're moving is a term *of the whole expression* on one side of the equation, not just a term of a subexpression. For example, the equations

$$3(a + b + c) = d \quad \text{and} \quad 3(a + b) = d - c$$

are *not* equivalent, because the term that's been moved, c, is not a term of the whole expression on the left-hand side, but only of the subexpression $a + b + c$. (The first equation *is* equivalent to $3(a + b) = d - 3c$, which is obtained by moving the term $3c$ that's obtained when you multiply out the left-hand side of the original equation.)

You must also make sure that the term you're moving becomes a term of the whole expression on the other side, not just a term of a subexpression. For example, the equations

$$a + b = \frac{c + d}{2} \quad \text{and} \quad a = \frac{c + d - b}{2}$$

are *not* equivalent, because the term that's been moved, b, has not become a term of the whole expression on the right-hand side, but only of the subexpression $c + d - b$. (The first equation *is* equivalent to $a = \frac{c + d}{2} - b$.)

If you find that you tend to make these types of errors, then it might be better to avoid using the 'change the side, change the sign' rule, and instead think about adding or subtracting on both sides.

Activity 44   *Adding or subtracting on both sides correctly*

Which of the following are pairs of equivalent equations?

(a) $y = \sqrt{x - 3}$ and $y + 3 = \sqrt{x}$

(b) $u^2 = \frac{1}{2}v + 7$ and $u^2 - 7 = \frac{1}{2}v$

(c) $\dfrac{x^2 + x}{3} = \dfrac{x + 9}{2}$ and $\dfrac{x^2}{3} = \dfrac{9}{2}$

Multiplying or dividing through

Consider the equation

$$a + b = c + d + e,$$

and suppose that you want to multiply both sides by 3. You know (by the box on page 33) that multiplying the left-hand side by 3 is the same as multiplying each individual term of the left-hand side by 3, and the same goes for the right-hand side. So multiplying both sides of the equation by 3 gives

$$3a + 3b = 3c + 3d + 3e.$$

You can see that the overall effect on the original equation is that each individual term on both sides has been multiplied by 3.

Similarly, if you *divide* both sides of the equation by 3, then (by the same box on page 33) the overall effect is that each individual term on both sides is divided by 3:

$$\frac{a}{3} + \frac{b}{3} = \frac{c}{3} + \frac{d}{3} + \frac{e}{3}.$$

It's useful to remember the following rules.

- Multiplying each term on both sides of an equation by something (provided that it is non-zero) gives an equivalent equation.
- Dividing each term on both sides of an equation by something (provided that it is non-zero) gives an equivalent equation.

Multiplying each term on both sides of an equation by something is known as **multiplying through** by the something, and similarly dividing each term on both sides of an equation by something is known as **dividing through** by the something. For example, the equation above was multiplied through by 3, and also divided through by 3.

When you use the rules in the box above, you must make sure that you're multiplying or dividing terms *of the whole expression* on a side, not just terms of a subexpression. For example, multiplying the equation

$$(a + b)^2 = 5$$

through by 3 does *not* give

$$(3a + 3b)^2 = 15.$$

This is because a and b are not terms of the whole expression on the left-hand side of the first equation, but only of the subexpression $a + b$. (Multiplying the first equation through by 3 gives $3(a + b)^2 = 15$.)

One situation where it's often useful to multiply through an equation is when it contains fractions. You can often obtain an equivalent equation with no fractions by multiplying through by an appropriate number or expression. For example, if you have the equation

$$\frac{x}{4} + 1 = x,$$

then you can multiply through by 4 to obtain

$$x + 4 = 4x.$$

This is known as **clearing**, or simply *removing*, the fractions in the equation. When you clear fractions in this way, you should keep in mind that multiplying an equation through by an expression is guaranteed to give an equivalent equation *only if the expression doesn't take the value zero*. There's more about this issue in Subsection 5.3.

Activity 45   *Multiplying and dividing through correctly*

Which of the following are pairs of equivalent equations?

(a) $\dfrac{a}{3} = \dfrac{a^2}{6} - 3$ and $2a = a^2 - 3$

(b) $\dfrac{x^2 - 1}{2x} + x = \dfrac{1}{2x}$ and $x^2 - 1 + x = 1$ $(x \neq 0)$

(c) $hy = x + 2$ and $y = \dfrac{x}{h} + \dfrac{2}{h}$ $(h \neq 0)$

(d) $\sqrt{\dfrac{x}{2y}} = 1$ and $\sqrt{\dfrac{x}{y}} = 2$ $(y \neq 0)$

Cross-multiplying

Suppose that you want to clear the fractions in the equation

$$\frac{x}{2} = \frac{x+1}{3}.$$

You can do this by multiplying through by 2, and then multiplying through by 3. This gives the equation

$$3x = 2(x + 1).$$

You can see that the overall effect on the original equation is that you have 'multiplied diagonally across the equals sign', like this:

$$\frac{x}{2} \bigtimes \frac{x+1}{3} \quad \text{gives} \quad 3x = 2(x + 1).$$

This technique is called **cross-multiplying**. You can use it as a shortcut for multiplying through, whenever you have an equation of the form fraction = fraction. The general rule is summarised in the box below.

Cross-multiplying

If A, B, C and D are any expressions, then the equations

$$\frac{A}{B} = \frac{C}{D} \quad \text{and} \quad AD = BC$$

are equivalent (provided that B and D are never zero).

If only one side of an equation is a fraction, then you can still cross-multiply (the other side can be thought of as a fraction with denominator 1).

For example,

$$\frac{x}{2} = x + 1 \quad \text{is equivalent to} \quad x = 2(x+1).$$

Activity 46   *Cross-multiplying correctly*

Which of the following are pairs of equivalent equations?

(a) $3 = \dfrac{1}{x+4}$ and $3(x+4) = 1$ $(x \neq -4)$

(b) $b + \dfrac{a}{a-3} = 6$ and $b + a = 6(a-3)$ $(a \neq 3)$

(c) $\dfrac{y}{2y-1} = \dfrac{y+1}{y}$ and $y^2 = (y+1)(2y-1)$ $(y \neq \frac{1}{2}, 0)$

(d) $2 + x = \dfrac{9}{2x+5}$ and $2 + x(2x+5) = 9$ $(x \neq -\frac{5}{2})$

5.3 Solving linear equations in one unknown

In this subsection you'll revise how to solve *linear equations in one unknown*. Equations of this type occur frequently, and are straightforward to solve.

A **linear** equation is one in which, after you've expanded any brackets and cleared any fractions, each term is either a constant term or a constant value times a variable. As you'd expect, a linear equation *in one unknown* is a linear equation that contains just one unknown.

For example,

$$2x - 5 = 8x$$

is a linear equation in the single unknown x.

Usually, a linear equation in one unknown has exactly one solution, which can be found by using the following strategy.

Strategy:
To solve a linear equation in one unknown

Use the rules for rearranging equations to obtain successive equivalent equations. Aim to obtain an equation in which the unknown is alone on one side, with only a number on the other side. To achieve that, do the following, in order.

1.   Clear any fractions and multiply out any brackets. To clear fractions, multiply through by a suitable expression.

2.   Add or subtract terms on both sides to get all the terms in the unknown on one side, and all the other terms on the other side. Collect like terms.

3.   Divide both sides by the coefficient of the unknown.

This strategy is illustrated in the next example.

Example 23   *Solving linear equations*

Solve the following equations.

(a) $\dfrac{x}{5} - 4 = 3(4 - x)$ (b) $\dfrac{1}{a} - 1 = \dfrac{1}{7a}$

Solution

(a) $\dfrac{x}{5} - 4 = 3(4 - x)$

There is a fraction with denominator 5, so multiply through by 5 to clear it.

$$x - 20 = 15(4 - x)$$

Multiply out the brackets.

$$x - 20 = 60 - 15x$$

Get all the terms in the unknown on one side, and all the other terms on the other side. Collect like terms.

$$x + 15x = 60 + 20$$
$$16x = 80$$

Divide both sides by 16, the coefficient of the unknown.

$$x = 5$$

The solution is $x = 5$.

If you wish, check the answer by substituting into the original equation, as follows.

Check: if $x = 5$,

$$\text{LHS} = \tfrac{5}{5} - 4 = 1 - 4 = -3,$$

and

$$\text{RHS} = 3(4 - 5) = 3 \times (-1) = -3.$$

Since LHS = RHS, $x = 5$ satisfies the equation.

(b) $\dfrac{1}{a} - 1 = \dfrac{1}{7a}$

💬 To clear the fractions, multiply through by a common multiple of the denominators, such as the lowest common multiple, $7a$. For this to be guaranteed to give an equivalent equation, you have to assume that $7a \neq 0$, that is, $a \neq 0$. 💬

Assume that $a \neq 0$.

$$\dfrac{7a}{a} - 7a = \dfrac{7a}{7a}$$

💬 Simplify, then proceed as in part (a). 💬

$$7 - 7a = 1$$
$$7 - 1 = 7a$$
$$6 = 7a$$
$$\tfrac{6}{7} = a$$

The value $a = \tfrac{6}{7}$ satisfies the assumption $a \neq 0$, so it is the solution.

Notice that before you can multiply the equation in Example 23(b) through by $7a$ to clear the fractions, you have to make the assumption that a never takes the value 0. In other words, you have to assume that the solutions of the equation are restricted to non-zero numbers. It's fine to make this assumption, because it's clear that $a = 0$ isn't a solution of the equation anyway, because it makes the fractions undefined. But because the rearrangements of the equation are based on this assumption, you have to check the solution that you obtain: if you get $a = 0$, then this doesn't count as a valid solution.

Notice also that in Example 23(b) the solution was left as $a = \tfrac{6}{7}$. It wasn't converted to a rounded decimal, such as $a = 0.857$ to three decimal places. In general, in mathematics you should always try to use *exact* numbers, where it's reasonably straightforward to do so. (However, it's often helpful to give answers to practical problems as rounded decimals.)

You can practise solving linear equations in the next activity. When you're solving a linear equation, you don't need to rigidly follow the steps of the strategy above if you can see a better way to proceed. You can do

whatever you think will be helpful, as long as you're following the rules for rearranging equations given in the boxes in Subsection 5.2.

Activity 47   *Solving linear equations*

Solve the following equations.

(a) $12 - 5q = q + 3$ (b) $3(x - 1) = 4(1 - x)$ (c) $\dfrac{t - 3}{4} + 5t = 1$

(d) $\dfrac{3}{b} = \dfrac{2}{3b} - \dfrac{1}{3}$ (e) $2(A + 2) = \dfrac{A}{3} + 1$ (f) $\dfrac{1}{z + 1} + \dfrac{1}{5(z + 1)} = 1$

In the next activity, each of the equations is of a form that allows you to clear the fractions by cross-multiplying.

Activity 48   *Solving more linear equations*

Use cross-multiplication to help you solve the following equations.

(a) $h + 1 = \dfrac{4h}{5}$ (b) $\dfrac{2}{y} = \dfrac{3}{y + 2}$ (c) $\dfrac{3}{2x - 1} = \dfrac{2}{3 - x}$

Finally in this subsection, let's have a more detailed look at why the restrictions in the box 'Doing the same thing to both sides of an equation' on page 77 are needed. Remember that this box sets out the things that you can do to both sides of an equation to obtain an equivalent equation, and the restrictions appear in brackets. Take a look back at the box before you read on.

The restrictions don't normally cause problems when you're solving linear equations in one unknown, but they can be an issue when you manipulate other types of equations.

For example, suppose that you want to solve the equation

$$x^2 = x.$$

You might think that you could simplify this equation by dividing both sides by x, to give

$$x = 1.$$

However, something has gone wrong here, because the first equation has two solutions, namely $x = 0$ and $x = 1$, whereas the second equation has only one solution, $x = 1$. So dividing the first equation by x has *not* given an equivalent equation. The problem here is that dividing by the variable x is guaranteed to give an equivalent equation only if you know that x can't take the value 0 (this is the restriction on the fourth item in the box).

If you do know that x can't take the value 0, then the first equation has only one solution, $x = 1$, and dividing the first equation by x does indeed give an equivalent second equation.

As another example, consider the simple equation

$$x = 2.$$

You might think that you could obtain an equivalent equation by raising both sides to the power 2 (that is, squaring both sides). This gives

$$x^2 = 4.$$

Again, something has gone wrong here, because when the variable x has the value -2, it doesn't satisfy the first equation, and yet it does satisfy the second equation. So raising both sides of the first equation to the power 2 has *not* given an equivalent equation. The problem this time is that raising both sides of an equation to a power is guaranteed to give an equivalent equation only if you know that the expression on each side of the equation can take only non-negative values (this is a restriction on the fifth item in the box).

By contrast, if you know that x is positive, then negative values of x such as -2 cannot arise, and squaring both sides of the first equation does give an equivalent second equation (since $x = -2$ no longer counts as a solution).

Similar issues can arise when you're solving an equation involving algebraic fractions. For example, consider the equation

$$\frac{x^2}{x} = 0.$$

This equation has no solutions. To see this, notice that the only way that a fraction can be equal to zero is for its numerator to be equal to zero. So the only possible solution of the equation is $x = 0$, but this value isn't a solution, because it makes the fraction undefined. Now consider what happens when you cancel down the left-hand side of the equation. You obtain the equation

$$x = 0,$$

which has one solution, namely $x = 0$. So cancelling down the fraction on the left-hand side has *not* given an equivalent equation. This is because the fraction x^2/x is valid for all values of x except zero, whereas its cancelled-down version, x, is valid for all values of x, with no exceptions. The two expressions aren't quite the same – this issue was mentioned on page 46. Essentially, the problem is that cancelling down an algebraic fraction in an equation is guaranteed to give an equivalent equation only if the factor that you cancel can't take the value 0.

Activity 49 *Thinking about the restrictions*

What's wrong with the following 'proof' that $1 = 2$?

Suppose that a and b are non-zero numbers such that $a = b$. Then we have:

$a = b$

$a^2 = ab$ (by multiplying through by a)

$a^2 - b^2 = ab - b^2$ (by subtracting b^2 from both sides)

$(a + b)(a - b) = b(a - b)$ (by using the facts that $a^2 - b^2$ is a difference of two squares, and $ab - b^2$ has a common factor)

$a + b = b$ (by dividing through by $a - b$)

$2b = b$ (since $a = b$)

$2 = 1$ (by dividing through by b)

5.4 Making a variable the subject of an equation

If an equation contains more than one variable, and one side of the equation is just a single variable that doesn't appear at all on the other side, then that variable is called the **subject** of the equation. For example, the equation

$$a = c(c + b) \tag{6}$$

has a subject, namely a. The subject of an equation is usually written on the left-hand side.

An equation with a subject is called a **formula**. We say that it's a formula *for* whatever the subject is; for example, equation (6) is a formula for a. The word *formula* is often used for the expression to which the subject is equal, as well as for the whole equation. For example, if a, b and c are related by equation (6), then we say that $c(c + b)$ is a formula for a.

The purpose of a formula is usually to allow you to find the value of the subject when you know the values of the other variables. For example, if a, b and c are related by equation (6), and you know that $b = 16$ and $c = 3$, then you can substitute these values of b and c into the equation to find the value of a:

$$a = 3(3 + 16) = 3 \times 19 = 57.$$

If you have an equation relating two or more variables, then it's often useful to rearrange it so that a particular variable becomes its subject. This isn't always possible, but for many equations you can do it by using essentially the same method that you use to solve linear equations, treating the variable that you want to be the subject (which we'll call the *required subject*) as the unknown. This method is summarised in the following strategy.

> **Strategy:**
> **To make a variable the subject of an equation**
> **(this works for some equations but not all)**
>
> Use the rules for rearranging equations to obtain successive equivalent equations. Aim to obtain an equation in which the required subject is alone on one side. To achieve this, do the following, in order.
>
> 1.   Clear any fractions and multiply out any brackets. To clear fractions, multiply through by a suitable expression.
>
> 2.   Add or subtract terms on both sides to get all the terms containing the required subject on one side, and all the other terms on the other side. Collect like terms.
>
> 3.   If more than one term contains the required subject, then take it out as a common factor.
>
> 4.   Divide both sides by the expression that multiplies the required subject.

This strategy works provided that the equation is 'linear in the required subject'. That is, its form must be such that if you replace every variable other than the required subject by a suitable number (one that doesn't lead to multiplication or division by zero) then the result is a linear equation in the required subject.

As when you're solving equations, you don't need to follow the steps of the strategy rigidly if you can see a better way to proceed. You just have to make sure that you're following the rules for rearranging equations given in the boxes in Subsection 5.2.

Example 24   *Making a variable the subject of an equation*

Rearrange the equation

$$t(h-1) = \frac{2}{t} + hr$$

to make h the subject.

Solution

$$t(h-1) = \frac{2}{t} + hr$$

🗨 Multiply through by t to clear the fraction (assume that $t \neq 0$). 🗨

$$t^2(h-1) = 2 + thr \quad \text{(assuming } t \neq 0)$$

🔍 Multiply out the brackets. 💬

$$t^2h - t^2 = 2 + thr$$

🔍 Get all the terms in the required subject, h, on one side, and all the other terms on the other side. Check for like terms – there are none. 💬

$$t^2h - thr = 2 + t^2$$

🔍 Take the required subject, h, out as a common factor. 💬

$$h(t^2 - tr) = 2 + t^2$$

🔍 Divide both sides by the expression that multiplies the required subject h (assume that $t \neq r$, to ensure that $t^2 - tr \neq 0$). 💬

$$h = \frac{2 + t^2}{t^2 - tr} \quad \text{(assuming } t \neq r\text{)}$$

The working in the example above tells you that, provided the variable t does not take the value 0 or the same value as r, then the initial and final equations are equivalent. It tells you nothing about what happens when t does take these values. In the next activity, you should note the assumptions that you make as you carry out the manipulations, as in the example above.

Activity 50 *Making variables the subjects of equations*

(a) Make m the subject of the equation $am = 2a + 3m$.

(b) Make d the subject of the equation $c = \frac{1}{3}(2 + 5d)$.

(c) Make X the subject of the equation $\dfrac{Y}{X} = \dfrac{3Y + 2}{X + 1}$.

(d) Make c the subject of the equation $a = \dfrac{b}{1 - 2c}$.

(e) Make B the subject of the equation $A = \dfrac{2}{B} + AC$.

In the next example, the strategy given earlier for making a variable the subject of an equation doesn't apply, but you can make the required variable the subject simply by raising both sides of the equation to an appropriate power, and simplifying.

Example 25   *Raising both sides of an equation to a power*

Rearrange the equation

$$a^6 = 64bc^6$$

to make a the subject. All the variables in this equation take only positive values.

Solution

$$a^6 = 64bc^6$$

🔍 Raise both sides to the power $\frac{1}{6}$ (that is, take the sixth root of both sides). This is okay, because the expressions on both sides of the original equation are always positive – this follows from the fact that all the variables in the equation take only positive values. 💬

$$a = (64bc^6)^{1/6}$$
$$a = 2b^{1/6}c$$

Activity 51   *Raising both sides of equations to powers*

In this activity, all the variables take only positive values.

(a) Make Q the subject of the equation $Q^3 = PR^2$.

(b) Make h the subject of the equation $h^{1/4} = \dfrac{2k^{1/2}}{m}$.

(c) Make u the subject of the equation $u^2 = v + w$.

You can often adapt the strategy given earlier for making a variable the subject of an equation to allow you to deal with equations for which the strategy doesn't quite work. In the next example, the required subject, p, is raised to the power 4 in the original equation, rather than just appearing as p. The strategy doesn't apply directly to situations like this, but you can make p the subject by first 'making p^4 the subject', then raising both sides to the power $\frac{1}{4}$.

Example 26   *Making a variable the subject of an equation, again*

Rearrange the equation

$$\frac{3p^4 - D}{c} = 1$$

to make p the subject. All the variables in this equation take only positive values.

Solution

🔍 First 'make p^4 the subject'. 💭

$$\frac{3p^4 - D}{c} = 1$$
$$3p^4 - D = c$$
$$3p^4 = c + D$$
$$p^4 = \frac{c + D}{3}$$

🔍 Now raise both sides to the power $\frac{1}{4}$. This is fine, because both sides are always positive – this follows from the fact that all the variables in the equation take only positive values. 💭

$$p = \left(\frac{c + D}{3}\right)^{1/4}$$

Activity 52   *Making variables the subjects of equations, again*

In this activity, all the variables take only positive values.

(a) Make a the subject of the equation $a^2 - 2b^2 = 3c^2$.

(b) Make t the subject of the equation $a^2 = \dfrac{bt^2}{N}$.

(c) Make r the subject of the equation $\left(\dfrac{r}{3s}\right)^5 = \sqrt{d}$.

(d) Make y the subject of the equation $(4y)^{1/3} = x$.

6 Writing mathematics

An important part of studying mathematics at university level is learning how to communicate it effectively. In this section you'll make a start on that, by learning how to write good solutions to TMA questions.

Usually you won't be able to write down an answer to a TMA question, or part of a TMA question, immediately. You'll need to carry out some working to find the answer, and it's important that you include this working as part of the solution that you write out to send to your tutor. However, it's not enough to *simply write down your working*. Instead, what you need to do is *clearly explain how you reached your answer*.

To illustrate the difference, let's look at a sample TMA question. This particular question can be answered by using Pythagoras' theorem, so before you see it, here's a reminder about that.

You may remember that Pythagoras' theorem allows you to work out the length of one of the sides of a right-angled triangle, if you know the lengths of the other two sides. The side opposite the right angle in a right-angled triangle is called the **hypotenuse** – this is always the longest side.

> **Pythagoras' theorem**
>
> For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

For example, applying Pythagoras' theorem to the right-angled triangle in Figure 9 gives

$$c^2 = a^2 + b^2.$$

So if you know the side lengths a and b, for example, then you can substitute them into this equation, and solve it to find the side length c.

Now here's the sample TMA question.

Figure 9 A right-angled triangle

Question 1

A symmetrical, circular circus dais, shown below, has diameter $1.1\,\text{m}$ at the top and $1.7\,\text{m}$ at the bottom, and is $40\,\text{cm}$ high. Find its slant height.
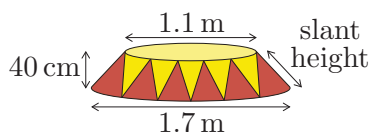
Here are two different solutions to the question.

Poorly-written solution
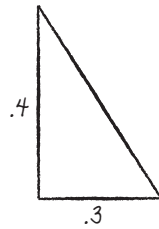
$1.7 - 1.1 = 0.6 \times \frac{1}{2} \Rightarrow 0.3$

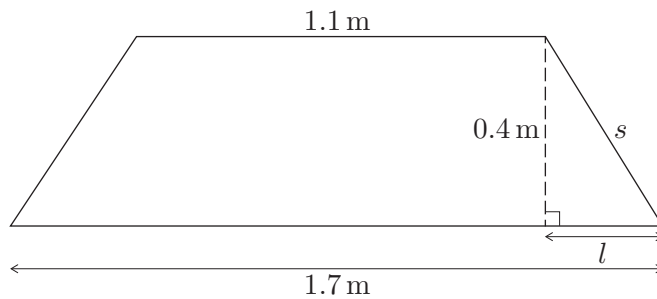$40 \Rightarrow 0.4$

Pythagoras' theorem

$$a^2 + b^2 = c^2$$
$$= 0.3^2 + 0.4^2 = c^2$$
$$= 0.25 = c^2$$
$$= c = \sqrt{0.25} = \pm 0.5$$

Length $\Rightarrow$ positive.

$\therefore \underline{c = 0.5}$

Well-written solution

A cross-section of the dais is shown below.

The length l (in metres) shown in the diagram is given by

$l = \frac{1}{2}(1.7 - 1.1) = \frac{1}{2} \times 0.6 = 0.3.$

So, by Pythagoras' theorem, the slant height s (in metres) is given by

$$s^2 = 0.4^2 + l^2$$
$$= 0.4^2 + 0.3^2$$
$$= 0.16 + 0.09$$
$$= 0.25.$$

Therefore, since s is positive,

$s = \sqrt{0.25} = 0.5.$

That is, the slant height of the dais is $0.5\,\text{m}$.

Each of the two solutions above uses a correct method, obtains the correct answer and shows all the working. However, the author of the poorly-written solution has done little more than just jot down some working (and has also used notation incorrectly, as explained later), whereas the author of the well-written solution has written out a clear

explanation of how he or she has reached the answer. The result is that the well-written solution is much easier for a reader to understand.

Writing mathematics so that it can be easily understood by a reader is an important skill. It will be useful not only when you write TMA solutions for your tutor, but in other situations too. For example, it will be valuable if you need to write mathematical reports, or if you teach mathematics to others, or even when you make notes for yourself to be read at a later date. Also, explaining your thinking clearly can help you to deepen your understanding of the mathematics, identify any errors and remember the techniques.

Here are some things that you should try to do when you write mathematics, to make it easier for a reader to understand. The author of the well-written solution has done all of these things, whereas the author of the poorly-written solution has done few of them.

Write in sentences.

    The well-written solution is written as a sequence of sentences. In contrast, the poorly-written solution is just a collection of fragments of English and mathematics, such as 'Pythagoras' theorem' and '$40 \Rightarrow 0.4$'.

    You should always aim to write your mathematics in sentences, though these sentences will often consist mostly of mathematical notation. However, if you're answering a question that simply asks you to carry out a mathematical manipulation, such as rearranging an expression, then you don't need to include any extra words.

Explain your reasoning.

    In the well-written solution, the author has explained how each claim that he or she makes follows from something that was calculated earlier in the solution, or is given in the question, or is a known fact (such as Pythagoras' theorem). Notice in particular how he or she has used the link words 'so', 'therefore' and 'that is' to indicate what follows from what. Other examples of useful link words and phrases are 'hence', 'it follows that' and 'since'.

    In your answers to TMA questions, you can use any of the facts that are stated in the *Handbook* or in the boxes in the units. You do not need to prove these facts, or state where you saw them.

Use equals signs only between two numbers, quantities or algebraic expressions that are equal.

    In the first line of the poorly-written solution, the author has written '$1.7 - 1.1 = 0.6 \times \frac{1}{2}$'. This is incorrect! Whenever you write an equals sign, whatever is on the left of the sign must be equal to whatever is on the right.

    There's one situation in which it's sometimes helpful to relax this rule, however. When you're carrying out a calculation that includes units, it's acceptable to omit the units until the end of the calculation.

For example, an alternative solution to the TMA question above might involve the calculation

$$\sqrt{0.3^2 + 0.4^2}\,\text{m} = \sqrt{0.25}\,\text{m} = 0.5\,\text{m}.$$

(You can see such a solution on page 98.) It's acceptable to write this calculation as

$$\sqrt{0.3^2 + 0.4^2} = \sqrt{0.25} = 0.5\,\text{m}.$$

Although strictly it's incorrect to write this, omitting units in this way can help to prevent calculations looking unnecessarily complicated.

Don't use equals signs to link equations.

In the poorly-written solution, the author has solved the equation $0.3^2 + 0.4^2 = c^2$ by writing down a sequence of equivalent equations, ending with $c = \pm 0.5$. This is fine, but he or she has linked the equations in the sequence by writing equals signs between them. Don't do this! Whenever you manipulate an equation in this way, the only equals signs should be the equals signs in the equations themselves.

You can link just two or three equivalent equations by using link words such as 'that is'. When you want to link a longer sequence of equivalent equations, you can begin by making it clear that you're manipulating an equation, and then just write the equivalent equations underneath each other, with no linking symbols, like this:

Solving the equation $0.3^2 + 0.4^2 = c^2$ gives:

$$0.3^2 + 0.4^2 = c^2$$
$$0.25 = c^2$$
$$c = \pm\sqrt{0.25}$$
$$c = \pm 0.5.$$

It's important to appreciate that this point applies to *equations* (which contain equals signs), not *expressions* (which don't). (The definitions of an expression and an equation are given on pages 24 and 74, respectively, and in the *Handbook*.) When you manipulate an *expression*, you *should* link the equivalent expressions with equals signs (often aligned vertically below each other). For example, you can see this done in the well-written solution, at the end of the sentence beginning 'So, by Pythagoras' theorem, . . .'.

If you introduce new variables, explain what they are.

The poorly-written solution includes the line '$a^2 + b^2 = c^2$', with no indication of what a, b and c represent. Don't do this: if you introduce a variable that isn't mentioned in the question, then explain what it represents.

If you introduce a variable that represents a physical quantity, then specify its units. For example, the author of the well-written solution has specified near the beginning that l and s are lengths *in metres*.

Finish with a conclusion that clearly answers the question.

The poorly-written solution finishes with '$c = 0.5$'. This is not a good conclusion, because the TMA question didn't ask, for example, 'What is the value of c?'. Also, it's not clear whether the final answer, 0.5, is in metres, or centimetres, for example, as the author hasn't specified units. The final line of the well-written solution is much more helpful: it's a clear answer to the question that was asked, with units included.

Include enough detail, but no more than is needed.

You need to give enough detail to enable your reader to easily understand what you've done, but try to give no more than this. Excess detail can waste your time and your reader's time, and it can actually make your solution harder to understand.

Sometimes you might be unsure about how much detail you need to include, and indeed as your mathematical experience grows it's appropriate to include less detail of routine procedures, such as solving linear equations. As a general guideline, you should include enough detail to make your solution clear to a reader whose mathematical experience is about the same as yours. Another useful guideline is that the amount of detail should be similar to that in the worked examples and activity solutions in the module units. (Remember that any blue 'thinks' text is not part of the solution, but any other text is.) But, if in doubt, include the extra detail!

If there's a worked example or an activity in the module that's similar to a question that you're answering, then it's fine to use the format of its solution as a guide for the format of your solution. This isn't plagiarism, as it's the format that you're copying, rather than the solution itself.

Make sure that the mathematical symbols you use are appropriate.

Often it's better to use words instead. For example, the author of the poorly-written solution has written

Length $\Rightarrow$ positive.

It would be clearer, and more mathematically correct, to write something like

The variable c represents a length, so its value must be positive.

Similarly, rather than writing

$40 \Rightarrow 0.4$,

it would be better to write, for example,

The height of the dais is $40\,\text{cm} = 0.4\,\text{m}$.

The symbol $\Rightarrow$, which means 'implies', is often used incorrectly, and it's probably best not to use it at all in this module. It isn't used in any of the materials provided (except here!). You can learn about its correct use in the module *Essential mathematics 2* (MST125).

The symbol $\therefore$ means 'therefore' and can be useful when you're short of time, such as in an examination, or in rough or informal work. Usually, however, it's better to use a word, such as 'so', 'hence' or 'therefore', as this makes your mathematical writing more pleasant to read. The symbol $\therefore$ isn't much used in university-level mathematics.

Display larger pieces of mathematical notation on separate lines.

Notice how the author of the well-written solution has done this. For example, the part of the solution immediately below the diagram would be less easy to read if it were written like this:

> The length l (in metres) shown in the diagram is given by $l = \frac{1}{2}(1.7 - 1.1) = \frac{1}{2} \times 0.6 = 0.3$. So, by Pythagoras' theorem, ...

Include the digit 0 in numbers such as 0.3.

In the poorly-written solution, the numbers 0.3 and 0.4 are written as .3 and .4 (on the diagram). Don't do this. Whenever you write a decimal point, there should always be a digit on each side of it. This is because, for example, .3 could easily be read as 3 by mistake.

Make sure that diagrams are clear and helpful.

If you include a diagram – and it's often useful to do so, even if the question doesn't explicitly ask for one – then make sure that it's neat, clearly labelled and not too cluttered, and that it's clear how it relates to the question. It's a good idea to use a ruler to draw straight lines. There's usually no need to use graph paper for diagrams or graphs.

A geometric diagram, such as the one in the well-written solution, doesn't need to be an exact scaled-down version of the situation that it represents – an approximate representation is fine.

Any well-written TMA solution will comply with the points above, but there's no single 'correct way' to write a solution to a TMA question. For example, the following alternative solution to the TMA question on page 92 looks quite different from the well-written solution you saw on page 93, but it is just as acceptable.

Notice in particular that the final sentence of this solution consists of some explanation, some mathematical working and a clear final answer, including units, all within a single sentence. This can be appropriate for a straightforward calculation.

Alternative well-written solution

The radius of the top of the dais is

$\frac{1}{2} \times 1.1\,\text{m} = 0.55\,\text{m}$,
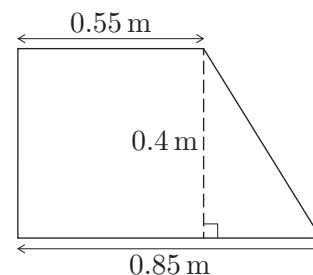
and the radius of the bottom is

$\frac{1}{2} \times 1.7\,\text{m} = 0.85\,\text{m}$.

The diagram on the right shows half of a cross-section of the dais. The base of the right-angled triangle shown is

$0.85\,\text{m} - 0.55\,\text{m} = 0.3\,\text{m}$.

Hence, by Pythagoras' theorem, the slant height of the dais is

$\sqrt{0.3^2 + 0.4^2}\,\text{m} = \sqrt{0.25}\,\text{m} = 0.5\,\text{m}$.

Activity 53   *Improving a TMA solution*

Consider the following TMA question and poorly-written solution.

Question 2

A cold frame is constructed by vertically fixing a pane of glass that is 1.15 m high at a distance of 98 cm from a wall, and then fixing another pane of glass of the same height at an angle between the top of the first pane and the wall, as shown. (Two trapezium-shaped sides are also attached.) Find the height of the cold frame where it meets the wall, to the nearest centimetre.
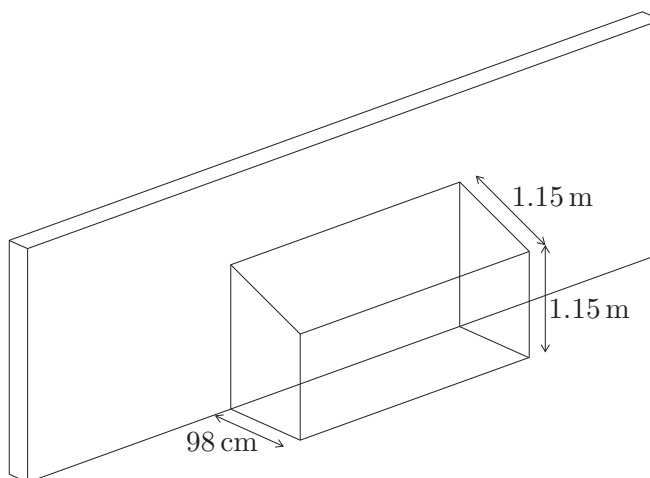
Poorly-written solution

$$h^2 + 98^2 = 1.15^2$$
$$\Rightarrow h^2 + 0.98^2 = 1.15^2$$
$$\Rightarrow h^2 = 1.15^2 - 0.98^2 = 0.3621$$
$$\Rightarrow h = \sqrt{0.3621} = 0.6$$
$$\therefore h = 0.6 + 1.15 \Rightarrow 1.75$$

1.15

98
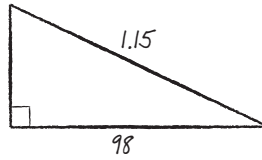
(a) Criticise the solution: list the features that contribute to its being poorly-written and difficult to follow, and describe how it could be improved.

(b) Write out a better solution.

In many of the TMAs for this module, a few marks are allocated for 'good mathematical communication'. The two well-written solutions that you've seen in this subsection (on pages 93 and 98) would merit full marks for their mathematical communication, as well as full marks for their mathematics. On the other hand, although the poorly-written solutions that you've seen (on page 93 and in Activity 53) would achieve some marks for their mathematics, they wouldn't be awarded any marks for mathematical communication. They would also lose an additional mark or half-mark because their final answers don't include units and hence are incomplete.

When you're working on a TMA question, you might find it helpful to first write out a rough solution, and then work on a better version to send to your tutor. This may be especially helpful near the start of the module, while you're learning the basics of mathematical communication. It's also helpful to read through your TMA solutions after you've written them, to try to judge for yourself how easy they are to follow. Then you can improve them where this seems appropriate. If you have time, try to leave them aside for a few days, or even longer, between writing them and reading them through.

You'll find that the feedback that your tutor provides on your TMA solutions will help you to improve your mathematical writing as you progress through the module.

You might be wondering whether you should handwrite your TMA solutions, or type them. The answer is that it's your choice: either is just as acceptable as the other. Remember, though, that you're expected to handwrite your answers in many mathematics exams, so you might find it useful to practise in preparation for that. Another thing to think about is that it can take time to learn to type mathematics, and it can take time to check typed mathematics for typing errors. If you're short of time, then it would be better to concentrate on learning and practising the mathematics in the module rather than typing your TMA solutions.

If you do plan to type your TMA solutions, then you should use an equation editor to format most of the mathematics. The ordinary text features of a word processor aren't adequate for the mathematics that you'll learn in this module. The module *Essential mathematics 2* (MST125) teaches you how to typeset mathematics, using your choice from three different typesetting programs, so if your study programme includes this module, then you may wish to delay learning to type mathematics until you take it. Alternatively, you can access the MST125 teaching materials on this topic via the MST124 website.

Another thing that you might be wondering about is how you should write a solution to a mathematics question in a written examination. The short answer is that you should try to use many of the same writing skills that you use for TMA questions, but you need to adapt them so that you can get the questions done as quickly as possible. For example, you might use abbreviated forms of sentences, and the 'therefore' symbol.

Learning outcomes

After studying this unit, you should be able to:

- work fluently and accurately with different types of numbers, including negative numbers
- understand how to avoid some common types of errors, such as rounding errors, errors arising from incorrect use of the BIDMAS rules and errors in algebraic manipulation
- manipulate and simplify algebraic expressions fluently and accurately, including those involving brackets, algebraic fractions and indices
- manipulate and simplify surds
- solve linear equations, including those involving algebraic fractions
- rearrange equations fluently and accurately
- appreciate some principles of writing mathematics, and begin to apply them.

Solutions to activities

Solution to Activity 1

(a) (i) $\quad 23 - 2 \times 3 + (4 - 2)$
$\quad\quad = 23 - 2 \times 3 + 2$
$\quad\quad = 23 - 6 + 2$
$\quad\quad = 19$

(ii) $\quad 2 - \frac{1}{2} \times 4 = 2 - 2 = 0$

(iii) $\quad 4 \times 3^2 = 4 \times 9 = 36$

(iv) $\quad 2 + 2^2 = 2 + 4 = 6$

(v) $\quad \dfrac{1 + 2}{1 + 3^2} = \dfrac{3}{1 + 9} = \dfrac{3}{10}$

(vi) $\quad 1 - 2/3^2 = 1 - 2/9 = 7/9$

(b) (i) $\quad 3(b - a)^2 = 3(5 - 3)^2$
$\quad\quad = 3 \times 2^2 = 3 \times 4 = 12$

(ii) $\quad a + b(2a + b) = 3 + 5(2 \times 3 + 5)$
$\quad\quad = 3 + 5(6 + 5)$
$\quad\quad = 3 + 5 \times 11$
$\quad\quad = 3 + 55$
$\quad\quad = 58$

(iii) $\quad a + 9\left(\dfrac{b}{a}\right) = 3 + 9 \times \dfrac{5}{3}$
$\quad\quad = 3 + 15$
$\quad\quad = 18$

(iv) $\quad 30/(ab) = 30/(3 \times 5)$
$\quad\quad = 30/15$
$\quad\quad = 2$

Solution to Activity 2

(a) $41.394 = 41.4$ (to 1 d.p.)

(b) $22.325 = 22.3$ (to 3 s.f.)

(c) $80\,014 = 80\,000$ (to 3 s.f.)

(d) $0.056\,97 = 0.057$ (to 2 s.f.)

(e) $0.006\,996 = 0.007\,00$ (to 3 s.f.)

(f) $56\,311 = 56\,300$ (to the nearest 100)

(g) $72\,991 = 73\,000$ (to the nearest 100)

(Notice that rounding $80\,014$ gives the same answer, $80\,000$, whether you're rounding to 1, 2 or 3 significant figures. Also, rounding $0.006\,996$ gives the same answer whether you're rounding it to 3 or 4 significant figures, but in the first case you'd express the answer as $0.007\,00$, and in the second case you'd express it as $0.007\,000$.)

Solution to Activity 3

(a) Let the radius of the circle be r (in cm). Then
$$77.2 = 2\pi r,$$
so
$$r = \frac{77.2}{2\pi} = 12.286\,761\,60\ldots = 12.3 \text{ (to 3 s.f.)}.$$
That is, the radius of the circle is $12.3\,\text{cm}$ (to 3 s.f.).

(b) Let the area of the circle be A (in cm^2). Then
$$A = \pi r^2$$
$$= \pi \times (12.286\,761\,60\ldots)^2$$
$$= 474.268\,998\ldots$$
$$= 470 \text{ (to 2 s.f.)}.$$
That is, the area of the circle is $470\,\text{cm}^2$ (to 2 s.f.).

Solution to Activity 4

(a) $-3 + (-4) = -3 - 4 = -7$

(b) $2 + (-3) = 2 - 3 = -1$

(c) $2 - (-3) = 2 + 3 = 5$

(d) $-1 - (-5) = -1 + 5 = 4$

(e) $5 \times (-4) = -20$

(f) $\dfrac{-15}{-3} = 5$

(g) $(-2) \times (-3) \times (-4) = 6 \times (-4) = -24$

(h) $6(-3 - (-1)) = 6(-3 + 1) = 6 \times (-2) = -12$

(i) $20 - (-5) \times (-2) = 20 - 10 = 10$

(j) $-5 + (-3) \times (-1) - 2 \times (-2) = -5 + 3 + 4 = 2$

(k) $\dfrac{-2 - (-1) \times (-2)}{-8} = \dfrac{-2 - 2}{-8} = \dfrac{-4}{-8} = \frac{1}{2}$

Solution to Activity 5

(a) $-5^2 = -25$

(b) $(-5)^2 = 25$

(c) $-(-8) = 8$

(d) $-(-8)^2 = -64$

(e) $-2^2 + 7 = -4 + 7 = 3$

(f) $-(-5) - (-1) = 5 + 1 = 6$

(g) $-4^2 - (-4)^2 = -16 - 16 = -32$

(h) $-3 \times (-2^2) = -3 \times (-4) = 12$

Solution to Activity 6

(a) $-b = -(-3) = 3$

(b) $-a - b = -(-2) - (-3) = 2 + 3 = 5$

(c) $-b^2 = -(-3)^2 = -9$

(d) $a^2 + ab = (-2)^2 + (-2) \times (-3) = 4 + 6 = 10$

(e) $\dfrac{3 - a^2}{b} = \dfrac{3 - (-2)^2}{-3} = \dfrac{3 - 4}{-3} = \dfrac{-1}{-3} = \frac{1}{3}$

(f) $a^2 - 2a + 5 = (-2)^2 - 2 \times (-2) + 5$
$$= 4 + 4 + 5 = 13$$

(g) $(6 - a)(2 + b) = (6 - (-2)) \times (2 + (-3))$
$$= (6 + 2) \times (2 - 3)$$
$$= 8 \times (-1) = -8$$

(h) $a^3 = (-2)^3 = -8$

(i) $-b^3 = -(-3)^3 = -(-27) = 27$

Solution to Activity 7

(a) (i)   The positive factor pairs of 28 are
$$1, 28; \quad 2, 14; \quad 4, 7.$$
The positive factors of 28 are
$$1,\ 2,\ 4,\ 7,\ 14,\ 28.$$

(ii)   The positive factor pairs of 25 are
$$1, 25; \quad 5, 5.$$
The positive factors of 25 are
$$1,\ 5,\ 25.$$

(iii)   The positive factor pairs of 36 are
$$1, 36; \quad 2, 18; \quad 3, 12; \quad 4, 9; \quad 6, 6.$$
The positive factors of 36 are
$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36.$$

(b) (i)   The factor pairs of 28 are
$$1, 28; \quad 2, 14; \quad 4, 7;$$
$$-1, -28; \quad -2, -14; \quad -4, -7.$$

(ii)   The factor pairs of -28 are
$$1, -28; \quad 2, -14; \quad 4, -7;$$
$$-1, 28; \quad -2, 14; \quad -4, 7.$$

Solution to Activity 8

(a) $594 = 2 \times 3^3 \times 11$

(b) $525 = 3 \times 5^2 \times 7$

(c) $221 = 13 \times 17$

(d) $223 = 223$
(The number 223 is prime.)

Solution to Activity 9

The prime factorisations are
$$9 = 3^2$$
$$18 = 2 \times 3^2$$
$$24 = 2^3 \times 3$$

(a) The LCM of 18 and 24 is $2^3 \times 3^2 = 72$.
The HCF of 18 and 24 is $2 \times 3 = 6$.

(b) The LCM of 9, 18 and 24 is $2^3 \times 3^2 = 72$.
The HCF of 9, 18 and 24 is 3.

(c) The LCM and HCF of -18 and -24 are the same as the LCM and HCF of 18 and 24 found in part (a). That is, the LCM is 72 and the HCF is 6.

Solution to Activity 10

(a) $-(-uv) = +uv = uv$

(b) $+(-9p) = -9p$

(c) $-(-4r^2) = +4r^2 = 4r^2$

(d) $-(8z) = -8z$
(The expression in part (d) can be simplified by deleting the brackets, as shown. These brackets aren't necessary, as the multiplication is done before taking the negative, anyway, by the BIDMAS rules.)

(e) $2x^2y^2 \times 5xy^4 = 10x^3y^6$

(f) $-P(-PQ) = P^2Q$

(g) $5m \times (-\frac{2}{5}n) = -2mn$

(h) $(-a^3)(-2b^3)(-2a^3) = -4a^6b^3$

(i) $(cd)^2 = cd \times cd = c^2d^2$

(j) $(-3x)^2 = (-3x) \times (-3x) = 9x^2$

(k) $-(3x)^2 = -(3x) \times (3x) = -9x^2$

(l) $-(-3x)^2 = -(-3x) \times (-3x) = -9x^2$

(m) $(-2x)^3 = (-2x) \times (-2x) \times (-2x) = -8x^3$

(n) $-(-2x)^3 = -(-2x) \times (-2x) \times (-2x) = 8x^3$

Solution to Activity 11

(a) $3a \times 3b - 2b \times 3b = 9ab - 6b^2$

(b) $\quad 5x \times 8x - 3x(-3x)$
$\quad = 40x^2 + 9x^2 = 49x^2$

(c) $\quad 3x^2 - (-3y^2) + (-x^2) + (2y^2)$
$\quad = 3x^2 + 3y^2 - x^2 + 2y^2 = 2x^2 + 5y^2$

(d) $\quad -3cd + (-5c \times 2d^2) - (-cd^2)$
$\quad = -3cd - 10cd^2 + cd^2$
$\quad = -3cd - 9cd^2$

(e) $\quad -6p(-\frac{1}{3}p) + (-5p \times p) - 2(-\frac{1}{2}p^2)$
$\quad = 2p^2 - 5p^2 + p^2$
$\quad = -2p^2$

(f) $\quad A(-B) + (-AB) - (-AB) + (-A)(-B)$
$\quad = -AB - AB + AB + AB = 0$

Solution to Activity 12

(a) $a(a^4 + b) = a^5 + ab$

(b) $-x(6x - x^2) = -6x^2 + x^3 = x^3 - 6x^2$

(c) $3pq(2p + 3q - 1) = 6p^2q + 9pq^2 - 3pq$

(d) $(C^3 - C^2 - C)C^2 = C^5 - C^4 - C^3$

(e) $-\frac{1}{2}x\left(\frac{1}{3}x^2 + \frac{2}{3}x\right) = -\frac{1}{6}x^3 - \frac{1}{3}x^2$

Solution to Activity 13

(a) $-(-2x^2 + x - 1) = 2x^2 - x + 1$

(b) $+(2x - 3y + z) = 2x - 3y + z$

(c) $-(p - 2q) = -p + 2q$

Solution to Activity 14

(a) $\underline{x} + \underline{x^2(1 + 3x)} = x + x^2 + 3x^3$

(b) $\underline{7ab} - \underline{b(a + 2b)} = 7ab - ab - 2b^2$
$\quad\quad\quad\quad\quad\quad\quad = 6ab - 2b^2$

(c) $\quad \underline{-6(c + d)} + \underline{3(c - d)}$
$\quad = -6c - 6d + 3c - 3d$
$\quad = -3c - 9d$

(d) $\underline{2X} - \underline{5Y(-4X + 2Y)} = 2X + 20XY - 10Y^2$

(e) $\quad \underline{(1 - p^4)p} + \underline{p^2} - \underline{p} = p - p^5 + p^2 - p$
$\quad\quad\quad\quad\quad\quad\quad = -p^5 + p^2$
$\quad\quad\quad\quad\quad\quad\quad = p^2 - p^5$

Solution to Activity 15

(a) $\quad (a + b)(c + d + e)$
$\quad = a(c + d + e) + b(c + d + e)$
$\quad = ac + ad + ae + bc + bd + be$

(b) $\quad (x + 3)(x + 5)$
$\quad = x(x + 5) + 3(x + 5)$
$\quad = x^2 + 5x + 3x + 15$
$\quad = x^2 + 8x + 15$

(c) $\quad (x^2 - 2x + 3)(3x^2 - x - 1)$
$\quad = x^2(3x^2 - x - 1) - 2x(3x^2 - x - 1)$
$\quad\quad + 3(3x^2 - x - 1)$
$\quad = 3x^4 - x^3 - x^2 - 6x^3 + 2x^2 + 2x$
$\quad\quad + 9x^2 - 3x - 3$
$\quad = 3x^4 - 7x^3 + 10x^2 - x - 3$

Solution to Activity 16

(a) $\quad (x + 5)(x - 7) = x^2 - 7x + 5x - 35$
$\quad\quad\quad\quad\quad\quad\quad = x^2 - 2x - 35$

(b) $\quad (x - 3)(x - 1) = x^2 - x - 3x + 3$
$\quad\quad\quad\quad\quad\quad\quad = x^2 - 4x + 3$

(c) $\quad (2x - 1)(8x + 3) = 16x^2 + 6x - 8x - 3$
$\quad\quad\quad\quad\quad\quad\quad = 16x^2 - 2x - 3$

(d) $\quad (2 - 5x)(x - 9) = 2x - 18 - 5x^2 + 45x$
$\quad\quad\quad\quad\quad\quad\quad = -5x^2 + 47x - 18$

(e) $\quad (c - 2d)(1 + c) = c + c^2 - 2d - 2cd$

(f) $(A - B)(2A - 3B^2)$
$$= 2A^2 - 3AB^2 - 2AB + 3B^3$$

(g) $(a - 1)(a + 1) = a^2 + a - a - 1 = a^2 - 1$

(h) $(2 + 3x)(2 - 3x) = 4 - 6x + 6x - 9x^2$
$$= 4 - 9x^2$$

(i) $x(1 + x) + (x - 1)(2 - x)$
$$= x + x^2 + 2x - x^2 - 2 + x$$
$$= 4x - 2$$

Solution to Activity 17

(a) $(x + 1)^2 = (x + 1)(x + 1)$
$$= x^2 + x + x + 1$$
$$= x^2 + 2x + 1$$

(b) $(3x - 2)^2 = (3x - 2)(3x - 2)$
$$= 9x^2 - 6x - 6x + 4$$
$$= 9x^2 - 12x + 4$$

(c) $(2p + 3q)^2 = (2p + 3q)(2p + 3q)$
$$= 4p^2 + 6pq + 6pq + 9q^2$$
$$= 4p^2 + 12pq + 9q^2$$

Solution to Activity 18

(a) $(x + 6)^2 = x^2 + 2 \times x \times 6 + 6^2$
$$= x^2 + 12x + 36$$

(b) $(x - 2)^2 = x^2 - 2 \times x \times 2 + 2^2$
$$= x^2 - 4x + 4$$

(c) $(1 + m)^2 = 1^2 + 2 \times 1 \times m + m^2$
$$= 1 + 2m + m^2$$

(d) $(1 - 2u)^2 = 1^2 - 2 \times 1 \times (2u) + (2u)^2$
$$= 1 - 4u + 4u^2$$

(e) $(2x - 3)^2 = (2x)^2 - 2 \times (2x) \times 3 + 3^3$
$$= 4x^2 - 12x + 9$$

(f) $(3c + d)^2 = (3c)^2 + 2 \times (3c) \times d + d^2$
$$= 9c^2 + 6cd + d^2$$

Solution to Activity 19

(a) (i) $4a^2 = 2a \times 2a$ and $2ab = 2a \times b$

  (ii) $x^3 = x^2 \times x$ and $x^5 = x^2 \times x^3$

  (iii) $18z^2 = 6z \times 3z$, $6z^2 = 6z \times z$ and $6z = 6z \times 1$

(b) (i) $10cd^2 = 5c \times 2d^2$ and $10cd^2 = 2cd \times 5d$

  (ii) $p^7 = p^2 \times p^5$ and $p^7 = p^3 \times p^4$

  (iii) $9y^2 = 3 \times 3y^2$, $9y^2 = 9y^2 \times 1$ and $9y^2 = 3y \times 3y$

Solution to Activity 20

(a) (i)   The highest common factor is $3x$.
$3x^2 = 3x \times x$ and $9xy = 3x \times 3y$

  (ii)   The lowest common multiple is $9x^2y$.
$9x^2y = 3x^2 \times 3y$ and $9x^2y = 9xy \times x$

(b) (i)   The highest common factor is $2pq$.
$6p^2q^3 = 2pq \times 3pq^2$, $4pq^2 = 2pq \times 2q$ and $2pq = 2pq \times 1$

  (ii)   The lowest common multiple is $12p^2q^3$.
$12p^2q^3 = 6p^2q^3 \times 2$, $12p^2q^3 = 4pq^2 \times 3pq$ and $12p^2q^3 = 2pq \times 6pq^2$

Solution to Activity 21

(a) $pq + 12qr = q(p + 12r)$

(b) $14cd - 7cd^2 = 7cd(2 - d)$

(c) $m^3 - m^7 - 8m^2 = m^2(m - m^5 - 8)$

(d) $-6AB + 3A^2B - 12A^3B^2$
$$= 3AB(-2 + A - 4A^2B)$$
$$= 3AB(A - 4A^2B - 2)$$

(e) $\sqrt{T} - s\sqrt{T} = \sqrt{T}(1 - s)$

(f) $5x^2 - 10x = 5x(x - 2)$

(g) $18y^2 + 6 = 6(3y^2 + 1)$

Solution to Activity 22

(a) $-2x^3 + 3x^2 - x - 5 = -(2x^3 - 3x^2 + x + 5)$

(b) $-ab - a - b = -(ab + a + b)$

(c) $5cd^2 - 10c^2d - 5cd$
$$= -5cd(-d + 2c + 1)$$
$$= -5cd(2c - d + 1)$$

Solution to Activity 23

(a) $\frac{1}{2}a^2 + \frac{3}{2}a = \frac{1}{2}(a^2 + 3a) = \frac{1}{2}a(a+3)$

(b) $\frac{1}{3}x - \frac{1}{6} = \frac{1}{6}(2x - 1)$

(c) $2x^2 - \frac{1}{2}x + \frac{1}{4} = \frac{1}{4}(8x^2 - 2x + 1)$

(d) $\frac{2}{3}u^2v^2 + \frac{1}{2}u^3v = \frac{1}{6}(4u^2v^2 + 3u^3v)$
$$= \frac{1}{6}u^2v(4v + 3u)$$

Solution to Activity 24

(a) $\dfrac{8xy^3}{6x^2y^2} = \dfrac{4y}{3x}$

(b) $\dfrac{2(3x-1)}{10(3x-1)^3} = \dfrac{1}{5(3x-1)^2}$

(c) $\dfrac{(x-1)^2(x-2)}{(x-1)(x-2)^2} = \dfrac{x-1}{x-2}$

(The fraction in part (c) can't be cancelled down any further. There's an x in both the numerator and the denominator, but x isn't a *factor* of either the numerator or the denominator.)

Solution to Activity 25

(a) $\dfrac{ab+a}{a^2+a} = \dfrac{b+1}{a+1}$

(b) $\dfrac{3}{9+6y^2} = \dfrac{1}{3+2y^2}$

(c) $\dfrac{u^2 - u^3}{u^5} = \dfrac{1-u}{u^3}$

(d) $\dfrac{2x^2 - 4x^3}{6x^4 - 2x^2} = \dfrac{x^2 - 2x^3}{3x^4 - x^2} = \dfrac{1-2x}{3x^2-1}$

(e) $\dfrac{x^3 + x^2}{2x+2} = \dfrac{x^2(x+1)}{2(x+1)} = \dfrac{x^2}{2}$

(f) $\dfrac{1-n}{n^2-n} = \dfrac{1-n}{n(n-1)} = \dfrac{-(n-1)}{n(n-1)} = \dfrac{-1}{n} = -\dfrac{1}{n}$

Solution to Activity 26

(a) $\dfrac{1}{x} + \dfrac{y}{x} = \dfrac{1+y}{x}$

(b) $\dfrac{c+2}{c^2+c} - \dfrac{1}{c^2+c} = \dfrac{c+2-1}{c^2+c} = \dfrac{c+1}{c(c+1)} = \dfrac{1}{c}$

(c) $\dfrac{1}{ab} - \dfrac{1}{bc} = \dfrac{c}{abc} - \dfrac{a}{abc} = \dfrac{c-a}{abc}$

(d) $\dfrac{2}{3a} + \dfrac{1}{2a} = \dfrac{4}{6a} + \dfrac{3}{6a} = \dfrac{7}{6a}$

(e) $\dfrac{1}{x^2} - \dfrac{2}{x} + 3 = \dfrac{1}{x^2} - \dfrac{2x}{x^2} + \dfrac{3x^2}{x^2}$
$$= \dfrac{1 - 2x + 3x^2}{x^2}$$

(f) $5 - \dfrac{1}{x} + \dfrac{2}{y} = \dfrac{5}{xy} - \dfrac{y}{xy} + \dfrac{2x}{xy}$
$$= \dfrac{5 - y + 2x}{xy}$$

(g) $A - \dfrac{A^2 - 1}{2A} = \dfrac{2A^2}{2A} - \dfrac{A^2 - 1}{2A}$
$$= \dfrac{2A^2 - (A^2 - 1)}{2A}$$
$$= \dfrac{A^2 + 1}{2A}$$

(h) $\dfrac{3}{2u-1} + u = \dfrac{3}{2u-1} + \dfrac{u(2u-1)}{2u-1}$
$$= \dfrac{3 + u(2u-1)}{2u-1}$$
$$= \dfrac{2u^2 - u + 3}{2u-1}$$

(i) $\dfrac{x+2}{x+1} + \dfrac{1}{x} = \dfrac{x(x+2)}{x(x+1)} + \dfrac{x+1}{x(x+1)}$
$$= \dfrac{x(x+2) + x + 1}{x(x+1)}$$
$$= \dfrac{x^2 + 2x + x + 1}{x(x+1)}$$
$$= \dfrac{x^2 + 3x + 1}{x(x+1)}$$

(j) $\dfrac{2}{p+2} - \dfrac{1}{p-3}$

$= \dfrac{2(p-3)}{(p+2)(p-3)} - \dfrac{p+2}{(p+2)(p-3)}$

$= \dfrac{2(p-3)-(p+2)}{(p+2)(p-3)}$

$= \dfrac{2p-6-p-2}{(p+2)(p-3)}$

$= \dfrac{p-8}{(p+2)(p-3)}$

(k) $\dfrac{x+1}{x(x-1)} + \dfrac{1}{x} - x$

$= \dfrac{x+1}{x(x-1)} + \dfrac{x-1}{x(x-1)} - \dfrac{x^2(x-1)}{x(x-1)}$

$= \dfrac{x+1+(x-1)-x^2(x-1)}{x(x-1)}$

$= \dfrac{x+1+x-1-x^3+x^2}{x(x-1)}$

$= \dfrac{-x^3+x^2+2x}{x(x-1)}$

$= \dfrac{-x^2+x+2}{x-1}$

$= \dfrac{x^2-x-2}{1-x}$

$= \dfrac{(x-2)(x+1)}{1-x}$

(Equivalent answers such as

$-\dfrac{(x-2)(x+1)}{x-1}$

or

$\dfrac{2+x-x^2}{x-1}$

are fine.)

Solution to Activity 27

(a) $\dfrac{a^2+a^5-1}{a^2} = \dfrac{a^2}{a^2} + \dfrac{a^5}{a^2} - \dfrac{1}{a^2}$

$= 1 + a^3 - \dfrac{1}{a^2}$

(b) $\dfrac{2-5cd}{c} = \dfrac{2}{c} - \dfrac{5cd}{c} = \dfrac{2}{c} - 5d$

(c) $\dfrac{2a+3a^2}{6} = \dfrac{2a}{6} + \dfrac{3a^2}{6} = \dfrac{a}{3} + \dfrac{a^2}{2}$

Solution to Activity 28

(a) $\dfrac{40A}{B} \times \dfrac{BC}{16A^4} = \dfrac{5C}{2A^3}$

(b) $\dfrac{b}{c^2} \div c^3 = \dfrac{b}{c^2} \times \dfrac{1}{c^3} = \dfrac{b}{c^5}$

(c) $\left(\dfrac{6y}{x^7}\right) \Big/ \left(\dfrac{15y^{10}}{x^4}\right) = \dfrac{6y}{x^7} \times \dfrac{x^4}{15y^{10}} = \dfrac{2}{5x^3y^9}$

(d) $\dfrac{a/(a+1)}{a^6/(a+1)^2} = \dfrac{a}{a+1} \div \dfrac{a^6}{(a+1)^2}$

$= \dfrac{a}{a+1} \times \dfrac{(a+1)^2}{a^6} = \dfrac{a+1}{a^5}$

(e) $\dfrac{3x}{y} \div \dfrac{6}{y^2} = \dfrac{3x}{y} \times \dfrac{y^2}{6} = \dfrac{xy}{2}$

(f) $g \times \dfrac{5}{k} = \dfrac{g}{1} \times \dfrac{5}{k} = \dfrac{5g}{k}$

Solution to Activity 29

(a) $\sqrt{5}\sqrt{6} = \sqrt{5 \times 6} = \sqrt{30}$

(b) $\dfrac{\sqrt{75}}{\sqrt{15}} = \sqrt{\dfrac{75}{15}} = \sqrt{5}$

(c) $3 + \sqrt{10}\sqrt{10} = 3 + 10 = 13$

(d) $\sqrt{8}\sqrt{2} = \sqrt{8 \times 2} = \sqrt{16} = 4$

(e) $\dfrac{\sqrt[3]{5}}{\sqrt[3]{15}} = \sqrt[3]{\dfrac{5}{15}} = \sqrt[3]{\dfrac{1}{3}} = \dfrac{\sqrt[3]{1}}{\sqrt[3]{3}} = \dfrac{1}{\sqrt[3]{3}}$

Solution to Activity 30

(a) $\sqrt{3}(2\sqrt{2} + \sqrt{3}) = 2\sqrt{2}\sqrt{3} + \sqrt{3}\sqrt{3}$

$= 2\sqrt{6} + 3$

$= 3 + 2\sqrt{6}$

(It's slightly tidier to write the term containing the irrational root as the last term.)

(b) $\sqrt{2}(1+\sqrt{3}) + 9\sqrt{2} = \sqrt{2} + \sqrt{2}\sqrt{3} + 9\sqrt{2}$

$= \sqrt{6} + 10\sqrt{2}$

(c) $(1-\sqrt{5})(6-2\sqrt{5}) = 6 - 2\sqrt{5} - 6\sqrt{5} + 2\sqrt{5}\sqrt{5}$

$= 6 - 8\sqrt{5} + 10$

$= 16 - 8\sqrt{5}$

(d) $(6-\sqrt{7})(6+\sqrt{7}) = 36 + 6\sqrt{7} - 6\sqrt{7} - \sqrt{7}\sqrt{7}$

$= 36 - 7$

$= 29$

(You can shorten your working in part (d) by noticing that the given expression has the form that

gives a difference of two squares when multiplied out. So
$$(6 - \sqrt{7})(6 + \sqrt{7}) = 6^2 - (\sqrt{7})^2 = 36 - 7 = 29.)$$

(e) $(\sqrt{3} + 2\sqrt{2})(\sqrt{3} - 2\sqrt{2})$
$$= \sqrt{3}\sqrt{3} - 2\sqrt{3}\sqrt{2} + 2\sqrt{3}\sqrt{2} - 4\sqrt{2}\sqrt{2}$$
$$= 3 - 4 \times 2$$
$$= -5$$

(Again, you can shorten your working in part (e) by noticing that the given expression has the form that gives a difference of two squares when multiplied out. So $(\sqrt{3} + 2\sqrt{2})(\sqrt{3} - 2\sqrt{2}) = (\sqrt{3})^2 - (2\sqrt{2})^2$ $= 3 - 4 \times 2 = -5$.)

(f) $3 - \dfrac{1}{\sqrt{2}} = \dfrac{3\sqrt{2}}{\sqrt{2}} - \dfrac{1}{\sqrt{2}} = \dfrac{3\sqrt{2} - 1}{\sqrt{2}}$

(g) $\dfrac{4}{\sqrt{5}} + \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{4\sqrt{2}}{\sqrt{2}\sqrt{5}} + \dfrac{\sqrt{3}\sqrt{5}}{\sqrt{2}\sqrt{5}} = \dfrac{4\sqrt{2} + \sqrt{15}}{\sqrt{10}}$

Solution to Activity 31

(a) (i) $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

(ii) $\sqrt{150} = \sqrt{25 \times 6} = \sqrt{25} \times \sqrt{6} = 5\sqrt{6}$

(iii) $\sqrt{22}$ can't be simplified.

(iv) $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

(v) $5 + \sqrt{108} = 5 + \sqrt{36 \times 3}$
$$= 5 + \sqrt{36}\sqrt{3}$$
$$= 5 + 6\sqrt{3}$$

(vi) $\sqrt{12} + \sqrt{4} = \sqrt{4 \times 3} + 2$
$$= 2\sqrt{3} + 2$$
$$= 2 + 2\sqrt{3}$$
$$= 2(1 + \sqrt{3})$$

(It's slightly tidier to write the term containing the irrational root as the last term. It's not essential to take out the common factor 2.)

(vii) $\sqrt{27} - \sqrt{3} = \sqrt{9 \times 3} - \sqrt{3}$
$$= \sqrt{9}\sqrt{3} - \sqrt{3}$$
$$= 3\sqrt{3} - \sqrt{3}$$
$$= 2\sqrt{3}$$

(b) (i) $\sqrt{6}(\sqrt{3} + \sqrt{2}) = \sqrt{3}\sqrt{6} + \sqrt{2}\sqrt{6}$
$$= \sqrt{3}\sqrt{3 \times 2} + \sqrt{2}\sqrt{2 \times 3}$$
$$= \sqrt{3}\sqrt{3}\sqrt{2} + \sqrt{2}\sqrt{2}\sqrt{3}$$
$$= 3\sqrt{2} + 2\sqrt{3}$$

(ii) $(\sqrt{10} - \sqrt{5})(2\sqrt{5} + 1)$
$$= 2\sqrt{5}\sqrt{10} + \sqrt{10} - 2\sqrt{5}\sqrt{5} - \sqrt{5}$$
$$= 2\sqrt{5}\sqrt{5}\sqrt{2} + \sqrt{10} - 10 - \sqrt{5}$$
$$= 10\sqrt{2} + \sqrt{10} - 10 - \sqrt{5}$$

Solution to Activity 32

(a) $\dfrac{3}{\sqrt{7}} = \dfrac{3}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{3\sqrt{7}}{\sqrt{7}\sqrt{7}} = \dfrac{3\sqrt{7}}{7}$

(b) $\dfrac{\sqrt{2}}{\sqrt{6}} = \dfrac{\sqrt{2}}{\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}}$
$$= \dfrac{\sqrt{2}\sqrt{6}}{\sqrt{6}\sqrt{6}}$$
$$= \dfrac{\sqrt{2}\sqrt{2}\sqrt{3}}{6}$$
$$= \dfrac{2\sqrt{3}}{6}$$
$$= \dfrac{\sqrt{3}}{3}$$

(c) $\dfrac{5}{\sqrt{5}} = \dfrac{5}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{5\sqrt{5}}{\sqrt{5}\sqrt{5}} = \dfrac{5\sqrt{5}}{5} = \sqrt{5}$

(d) $\dfrac{2}{1 + \sqrt{17}} = \dfrac{2}{1 + \sqrt{17}} \times \dfrac{1 - \sqrt{17}}{1 - \sqrt{17}}$
$$= \dfrac{2(1 - \sqrt{17})}{1^2 - (\sqrt{17})^2}$$
$$= \dfrac{2(1 - \sqrt{17})}{1 - 17}$$
$$= \dfrac{2(1 - \sqrt{17})}{-16}$$
$$= \dfrac{1 - \sqrt{17}}{-8}$$
$$= \dfrac{\sqrt{17} - 1}{8}$$

(Remember that $\sqrt{17} - 1$ is the negative of $1 - \sqrt{17}$; see page 32.)

(e) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

$\qquad = \dfrac{(\sqrt{5}+\sqrt{3})(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}$

$\qquad = \dfrac{(\sqrt{5})^2 + 2\sqrt{3}\sqrt{5} + (\sqrt{3})^2}{5-3}$

$\qquad = \dfrac{5 + 2\sqrt{15} + 3}{5-3}$

$\qquad = \dfrac{8 + 2\sqrt{15}}{2}$

$\qquad = 4 + \sqrt{15}$

Solution to Activity 33

(a) $2^4 = 2 \times 2 \times 2 \times 2 = 16$

(b) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$

(c) $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$

(d) $(-3)^3 = (-3) \times (-3) \times (-3) = -27$

(e) $\left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(f) $\left(\frac{1}{3}\right)^3 = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

Solution to Activity 34

(a) $\dfrac{a^{20}}{a^5} = a^{15}$

(b) $(y^4)^5 = y^{20}$

(c) $\dfrac{b^4 b^7}{b^3} = \dfrac{b^{11}}{b^3} = b^8$

(d) $p^3 p^5 p^3 p = p^{12}$

(e) $(x^3)^3 x^2 = x^9 x^2 = x^{11}$

(f) $(m^3 m^2)^5 = (m^5)^5 = m^{25}$

(g) $\left(\dfrac{c^5}{c^2}\right)^2 = (c^3)^2 = c^6$

Solution to Activity 35

(a) $\quad (x^3-1)(2x^3+5) = 2x^6 + 5x^3 - 2x^3 - 5$

$\qquad\qquad\qquad\qquad = 2x^6 + 3x^3 - 5$

(b) $\quad (a^4+b^4)^2 = (a^4+b^4)(a^4+b^4)$

$\qquad\qquad\qquad = a^8 + 2a^4 b^4 + b^8$

(c) $\dfrac{p^8 + p^2}{p^2} = p^6 + 1$

Solution to Activity 36

(a) $4^{-2} = \dfrac{1}{4^2} = \frac{1}{16}$

(b) $3^{-1} = \frac{1}{3}$

(c) $5^0 = 1$

(d) $\left(\frac{2}{7}\right)^{-1} = \frac{7}{2}$

(e) $\left(\frac{1}{3}\right)^{-1} = 3$

(f) $\left(\frac{1}{3}\right)^{-2} = \left(\left(\frac{1}{3}\right)^{-1}\right)^2 = 3^2 = 9$

$\left(\text{or } \left(\frac{1}{3}\right)^{-2} = \dfrac{1}{\left(\frac{1}{3}\right)^2} = \dfrac{1}{\left(\frac{1}{9}\right)} = 9\right)$

Solution to Activity 37

(a) $5g^{-1} = \dfrac{5}{g}$

(b) $\dfrac{1}{y^{-1}} = y$

(c) $\dfrac{2}{3x^{-5}} = \dfrac{2x^5}{3}$

(d) $\dfrac{a^{-3}}{b^{-4}} = \dfrac{b^4}{a^3}$

(e) $\dfrac{P^2}{Q^{-5}} = P^2 Q^5$

(f) $(3h^2)^2 = 3^2(h^2)^2 = 9h^4$

(g) $(3h^2)^{-2} = \dfrac{1}{(3h^2)^2} = \dfrac{1}{3^2(h^2)^2} = \dfrac{1}{9h^4}$

(h) $\dfrac{(b^{-4})^3}{(3c)^2} = \dfrac{b^{-12}}{3^2 c^2} = \dfrac{1}{9b^{12}c^2}$

(i) $\dfrac{(A^{-1}B)^2}{(B^{-3})^3} = \dfrac{(A^{-1})^2 B^2}{(B^{-3})^3}$

$\qquad = \dfrac{A^{-2}B^2}{B^{-9}} = \dfrac{B^2 B^9}{A^2} = \dfrac{B^{11}}{A^2}$

(j) $\left(\dfrac{2y^{-1}}{z^2}\right)^5 = \dfrac{(2y^{-1})^5}{(z^2)^5} = \dfrac{2^5 y^{-5}}{z^{10}} = \dfrac{32}{y^5 z^{10}}$

(k) $\dfrac{x^{-5}}{x} = \dfrac{1}{x^5 x} = \dfrac{1}{x^6}$

(l) $\dfrac{(x-1)^{-3}}{(x-1)^2} = \dfrac{1}{(x-1)^3(x-1)^2} = \dfrac{1}{(x-1)^5}$

(m) $\left(\dfrac{3}{z^2}\right)^{-3} = \left(\dfrac{3}{z^2}\right)^{-1\times 3}$

$= \left(\dfrac{z^2}{3}\right)^3 = \dfrac{(z^2)^3}{3^3} = \dfrac{z^6}{27}$

$\left(\text{or } \left(\dfrac{3}{z^2}\right)^{-3} = \dfrac{1}{(3/z^2)^3}\right.$

$\left. = \dfrac{1}{3^3/(z^2)^3} = \dfrac{1}{27/z^6} = \dfrac{z^6}{27}\right)$

(n) $\dfrac{x}{(2x-3)^3} \times x^{-2} = \dfrac{x}{x^2(2x-3)^3} = \dfrac{1}{x(2x-3)^3}$

(o) $\left(\dfrac{(x+2)^2}{x^5}\right)\Big/ (x+2)^{-4}$

$= \dfrac{(x+2)^2}{x^5} \times \dfrac{1}{(x+2)^{-4}}$

$= \dfrac{(x+2)^2}{x^5(x+2)^{-4}}$

$= \dfrac{(x+2)^2(x+2)^4}{x^5}$

$= \dfrac{(x+2)^6}{x^5}$

Solution to Activity 38

(a) $t^{1/2} = \sqrt{t}$

(b) $x^{1/3} = \sqrt[3]{x}$

(c) $p^{2/3} = \sqrt[3]{p^2}$ $\left(\text{or } \left(\sqrt[3]{p}\right)^2\right)$

(d) $x^{-1/2} = \dfrac{1}{x^{1/2}} = \dfrac{1}{\sqrt{x}}$

(e) $(2x-3)^{1/2} = \sqrt{2x-3}$

(f) $(1+x^2)^{-1/2} = \dfrac{1}{(1+x^2)^{1/2}} = \dfrac{1}{\sqrt{1+x^2}}$

(g) $(1+x)x^{-1/2} = \dfrac{1+x}{x^{1/2}} = \dfrac{1+x}{\sqrt{x}}$

Solution to Activity 39

(a) $\sqrt[4]{y} = y^{1/4}$

(b) $\sqrt[5]{1-2x} = (1-2x)^{1/5}$

(c) $\dfrac{1}{\sqrt[3]{x}} = \dfrac{1}{x^{1/3}}$

(d) $\left(\sqrt[5]{u}\right)^2 = u^{2/5}$

(e) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}}$

(f) $\sqrt[4]{(y+2)^3} = (y+2)^{3/4}$

Solution to Activity 40

(a) $x^{1/3}x^{1/3} = x^{2/3}$

(b) $\dfrac{a}{a^{1/3}} = a^{2/3}$

(c) $\dfrac{a}{a^{1/2}} = a^{1/2} = \sqrt{a}$
(It's fine to leave the answer as $a^{1/2}$.)

(d) $\dfrac{x}{x^{-1/2}} = x^{3/2}$

(e) $(2x^{1/5})^3 = 8x^{3/5}$

(f) $\dfrac{(1+x)^2}{\sqrt{1+x}} = \dfrac{(1+x)^2}{(1+x)^{1/2}} = (1+x)^{3/2}$

(g) $\left(\dfrac{1}{u}\right)^{1/3} = \dfrac{1}{u^{1/3}}$
(It's fine to leave the answer as $u^{-1/3}$.)

(h) $\dfrac{A^{5/2}}{A^3} = A^{(5/2)-3} = A^{-1/2} = \dfrac{1}{A^{1/2}} = \dfrac{1}{\sqrt{A}}$
(It's fine to write the answer as $\dfrac{1}{A^{1/2}}$, or as $A^{-1/2}$.)

(i) $\dfrac{x^{1/2}y^2}{x^3y^{1/2}} = \dfrac{y^{3/2}}{x^{5/2}}$ $\left(\text{or } \sqrt{\dfrac{y^3}{x^5}}\right)$

(j) $\sqrt{4x} = \sqrt{4}\sqrt{x} = 2\sqrt{x}$

Solution to Activity 41

(a) $a^{2p}a^{5p} = a^{7p}$

(b) $\dfrac{b^{7k}}{b^{4k}} = b^{3k}$

(c) $(g^n)^k = g^{nk}$

(d) $(2y^{2t})^2 y^{-t} = 2^2 (y^{2t})^2 y^{-t}$
$$= 4y^{4t}y^{-t}$$
$$= 4y^{3t}$$

(e) $\dfrac{m^{3x}}{m^{-x}} = m^{3x}m^x$
$$= m^{4x}$$

(f) $\dfrac{c^{3y}}{c^{5y}} = \dfrac{1}{c^{2y}}$

(g) $\dfrac{(a^{-3t}b^{3t})^2}{a^{3t}b^t} = \dfrac{(a^{-3t})^2(b^{3t})^2}{a^{3t}b^t}$
$$= \dfrac{a^{-6t}b^{6t}}{a^{3t}b^t}$$
$$= \dfrac{b^{5t}}{a^{9t}}$$

(h) $(d^{1/r})^{2r} = d^2$

(i) $(3h)^{2p}(9h)^p = 3^{2p}h^{2p}9^p h^p$
$$= 3^{2p}h^{3p}(3^2)^p$$
$$= 3^{2p}h^{3p}3^{2p}$$
$$= 3^{4p}h^{3p}$$

Solution to Activity 42

(a) (i) $38\,800\,000 = 3.88 \times 10^7$

(ii) $4237 = 4.237 \times 10^3$

(iii) $0.0973 = 9.73 \times 10^{-2}$

(iv) $1.303 = 1.303 \times 10^0$

(v) $0.000\,000\,028 = 2.8 \times 10^{-8}$

(b) (i) $2.8 \times 10^4 = 28\,000$

(ii) $5.975 \times 10^{-1} = 0.5975$

(iii) $2.78 \times 10^{-7} = 0.000\,000\,278$

(iv) $3.43 \times 10^7 = 34\,300\,000$

Solution to Activity 43

(a) If $x = 4$ and $y = -1$, then
$$\text{LHS} = 2 \times 4 + 3 \times (-1) = 8 - 3 = 5 = \text{RHS}.$$
So the equation is satisfied.

(b) If $x = 3$ and $y = -2$, then
$$\text{LHS} = 2 \times 3 + 3 \times (-2) = 6 - 6 = 0 \neq \text{RHS}.$$
So the equation is not satisfied.

Solution to Activity 44

(a) These equations are not equivalent. The term in the first equation that's been moved, -3, isn't a term of the whole expression on the right-hand side, but only of the subexpression $x - 3$.

(b) These equations are equivalent.

(c) These equations are not equivalent. The second equation is obtained from the first by subtracting $x/3$ from the left-hand side and $x/2$ from the right-hand side, but this does not give an equivalent equation.

Solution to Activity 45

(a) These equations are not equivalent. Multiplying the first equation through by 6 gives
$$2a = a^2 - 18.$$

(b) These equations are not equivalent. Multiplying the first equation through by $2x$ gives
$$x^2 - 1 + 2x^2 = 1.$$

(c) These equations are equivalent.

(d) These equations are not equivalent. Multiplying the first equation through by 2 gives $\sqrt{\dfrac{2x}{y}} = 2.$

Solution to Activity 46

(a) These equations are equivalent.

(b) These equations are not equivalent. Multiplying both sides of the first equation by $a - 3$ gives
$$b(a - 3) + a = 6(a - 3).$$
Alternatively, you can rearrange the first equation to give
$$\dfrac{a}{a - 3} = 6 - b,$$
and cross-multiply to give
$$a = (6 - b)(a - 3).$$

(c) These equations are equivalent.

(d) These equations are not equivalent. Cross-multiplying in the first equation gives
$$(2 + x)(2x + 5) = 9.$$

Solution to Activity 47

(a)　$12 - 5q = q + 3$

　　$12 - 3 = q + 5q$

　　$9 = 6q$

　　$\frac{3}{2} = q$

　The solution is $q = \frac{3}{2}$.

(b)　$3(x - 1) = 4(1 - x)$

　　$3x - 3 = 4 - 4x$

　　$3x + 4x = 4 + 3$

　　$7x = 7$

　　$x = 1$

　The solution is $x = 1$.

(c)　$\dfrac{t - 3}{4} + 5t = 1$

　　$\dfrac{4(t - 3)}{4} + 4 \times 5t = 4$

　　$t - 3 + 20t = 4$

　　$t + 20t = 4 + 3$

　　$21t = 7$

　　$t = \frac{1}{3}$

　The solution is $t = \frac{1}{3}$.

(d)　$\dfrac{3}{b} = \dfrac{2}{3b} - \dfrac{1}{3}$

　　$\dfrac{3 \times 3b}{b} = \dfrac{2 \times 3b}{3b} - \dfrac{3b}{3}$　　(assuming $b \neq 0$)

　　$9 = 2 - b$

　　$7 = -b$

　　$-7 = b$

　The solution is $b = -7$.

(e)　$2(A + 2) = \dfrac{A}{3} + 1$

　　$6(A + 2) = \dfrac{3A}{3} + 3$

　　$6A + 12 = A + 3$

　　$6A - A = 3 - 12$

　　$5A = -9$

　　$A = -\frac{9}{5}$

　The solution is $A = -\frac{9}{5}$.

(f)　$\dfrac{1}{z + 1} + \dfrac{1}{5(z + 1)} = 1,$

　so assuming that $z \neq -1$,

　　$\dfrac{5(z + 1)}{z + 1} + \dfrac{5(z + 1)}{5(z + 1)} = 5(z + 1)$

　　$5 + 1 = 5z + 5$

　　$1 = 5z$

　　$\frac{1}{5} = z$

　The solution is $z = \frac{1}{5}$.

　(The fourth equation above is obtained by subtracting 5 from each side of the third equation.)

Solution to Activity 48

(a)　$h + 1 = \dfrac{4h}{5}$

　　$5(h + 1) = 4h$

　　$5h + 5 = 4h$

　　$5h - 4h = -5$

　　$h = -5$

　The solution is $h = -5$.

(b)　$\dfrac{2}{y} = \dfrac{3}{y + 2}$

　　$2(y + 2) = 3y$　　(assuming $y \neq -2, 0$)

　　$2y + 4 = 3y$

　　$4 = 3y - 2y$

　　$4 = y$

　The solution is $y = 4$.

(c)　$\dfrac{3}{2x - 1} = \dfrac{2}{3 - x}$

　　$3(3 - x) = 2(2x - 1)$　　(assuming $x \neq \frac{1}{2}, 3$)

　　$9 - 3x = 4x - 2$

　　$9 + 2 = 4x + 3x$

　　$11 = 7x$

　　$\frac{11}{7} = x$

　The solution is $x = \frac{11}{7}$.

Solution to Activity 49

The problem with the 'proof' is that it involves dividing by $a - b$, which is equal to zero, since $a = b$.

Solution to Activity 50

(a) $am = 2a + 3m$

$am - 3m = 2a$

$m(a - 3) = 2a$

$m = \dfrac{2a}{a - 3}$ (assuming $a \neq 3$)

(b) $c = \frac{1}{3}(2 + 5d)$

$3c = 2 + 5d$

$3c - 2 = 5d$

$5d = 3c - 2$

$d = \frac{1}{5}(3c - 2)$

(The alternative form $d = \dfrac{3c - 2}{5}$ is fine.)

(c) $\dfrac{Y}{X} = \dfrac{3Y + 2}{X + 1}$

$Y(X + 1) = X(3Y + 2)$ (assuming $X \neq 0, -1$)

$XY + Y = 3XY + 2X$

$Y = 3XY - XY + 2X$

$Y = 2XY + 2X$

$Y = 2X(Y + 1)$

$\dfrac{Y}{2(Y + 1)} = X$ (assuming $Y \neq -1$)

$X = \dfrac{Y}{2(Y + 1)}$

(d) $a = \dfrac{b}{1 - 2c}$

$a(1 - 2c) = b$ (assuming $c \neq \frac{1}{2}$)

$a - 2ac = b$

$a - b = 2ac$

$2ac = a - b$

$c = \dfrac{a - b}{2a}$ (assuming $a \neq 0$)

(e) $A = \dfrac{2}{B} + AC$

$A - AC = \dfrac{2}{B}$

$B(A - AC) = 2$ (assuming $B \neq 0$)

$AB(1 - C) = 2$

$B = \dfrac{2}{A(1 - C)}$ (assuming $A \neq 0, C \neq 1$)

Solution to Activity 51

(a) $Q^3 = PR^2$

$Q = (PR^2)^{1/3}$

$Q = P^{1/3}R^{2/3}$

(The forms $Q = (PR^2)^{1/3}$ and $Q = \sqrt[3]{PR^2}$ are also suitable final answers.)

(b) $h^{1/4} = \dfrac{2k^{1/2}}{m}$

$h = \left(\dfrac{2k^{1/2}}{m}\right)^4$

$h = \dfrac{16k^2}{m^4}$

(c) $u^2 = v + w$

$u = \sqrt{v + w}$

Solution to Activity 52

(a) $a^2 - 2b^2 = 3c^2$

$a^2 = 2b^2 + 3c^2$

$a = \sqrt{2b^2 + 3c^2}$

(b) $a^2 = \dfrac{bt^2}{N}$

$a^2 N = bt^2$

$\dfrac{a^2 N}{b} = t^2$

$t^2 = \dfrac{a^2 N}{b}$

$t = \sqrt{\dfrac{a^2 N}{b}}$

$t = a\sqrt{\dfrac{N}{b}}$

(c) $\left(\dfrac{r}{3s}\right)^5 = \sqrt{d}$

$\left(\dfrac{r}{3s}\right)^5 = d^{1/2}$

$\dfrac{r}{3s} = (d^{1/2})^{1/5}$

$r = 3sd^{1/10}$

(d) $(4y)^{1/3} = x$

$4y = x^3$

$y = \frac{1}{4}x^3$

(The alternative form $y = \dfrac{x^3}{4}$ is fine.)

Solution to Activity 53

(a) Here are some ways in which the solution could have been improved.

- It should be written in sentences.

- It needs some words of explanation.

- The three $\Rightarrow$ symbols at the left should be omitted.

- The fourth $\Rightarrow$ symbol (the one just before the number 1.75) should be replaced by an equals sign. Here, the author of the solution has worked out that $0.6 + 1.15$ is *equal* to 1.75, so an equals sign is the correct symbol.

- The equation
  $$\sqrt{0.3621} = 0.6$$
  should be changed to
  $$\sqrt{0.3621} = 0.60 \text{ (to 2 d.p.)}$$
  or
  $$\sqrt{0.3621} \approx 0.60.$$

  The reason why either '(to 2 d.p.)' should be appended or the equals sign changed to '$\approx$' is that, without such a change, the equals sign is used incorrectly. This is because $\sqrt{0.3621}$ is *not equal* to 0.6. These numbers are only approximately equal.

  The reason why the rounded answer 0.6 should be written as 0.60 is to indicate that it has been rounded to *two* decimal places.

- The solution should explain what the variable h represents.

- The first equation, $h^2 + 98^2 = 1.15^2$, should not have been written at all. It is incorrect, because it has been obtained from Pythagoras' theorem by substituting in one length expressed in centimetres and another expressed in metres. Any lengths substituted into Pythagoras' theorem must be in the same units.

- Ideally the letter h should not be used to represent two different quantities, and it should certainly not be used in this way without explanation. (In most of the solution the letter h seems to represent the vertical height of the sloping pane of glass,

but in the final line it seems to represent the total height of the cold frame. This makes the solution difficult to understand.)

- The solution should finish with a clear answer to the question, including units.

- The lengths in the diagram should include units (or they should be expressed in the same units and it should be made clear what the units are).

- The diagram should relate more clearly to the question.

- Ideally, the $\therefore$ symbol should be replaced by a word, such as 'So'.

(b) Here is a better solution.

A side view of the cold frame is shown below.
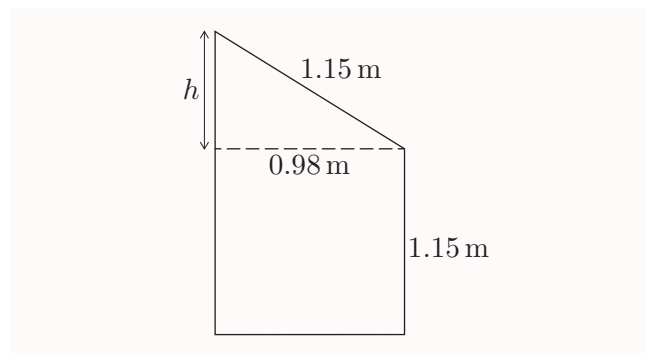
By Pythagoras' theorem, the vertical height h (in metres) of the slant pane of glass is given by
$$h^2 + 0.98^2 = 1.15^2.$$
Solving this equation (and using the fact that h is positive) gives
$$h^2 = 1.15^2 - 0.98^2$$
$$h^2 = 0.3621$$
$$h = \sqrt{0.3621}$$
$$h = 0.601\,747\ldots$$
Hence the height of the cold frame where it meets the wall is
$$(1.15 + 0.601\,747\ldots)\,\text{m}$$
$$= 1.751\,747\ldots\text{m}$$
$$= 1.75\,\text{m (to the nearest cm)}.$$

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