Simply-Typed Lambda Calculus

(Slides mostly follow Dan Grossman's teaching materials)

Review of untyped λ-calculus

- Syntax: notation for defining functions

(Terms) M, N ::= x | λx. M | M N

- Semantics: reduction rules

$$\frac{}{(\lambda x. M)N \;\to\; M[N/x]}\,(\beta)$$

$$\frac{M \to M'}{M\ N \to M'N} \qquad \frac{N \to N'}{M\ N \to M\ N'} \qquad \frac{M \to M'}{\lambda x. M \to \lambda x. M'}$$

$$\frac{}{(\lambda x. M)N \;\to\; M[N/x]}\ (\beta)$$

(λf. λz. f (f z)) (λy. y+x)

→ λz. (λy. y+x) ((λy. y+x) z)

→ λz. (λy. y+x) (z+x)

→ λz. z+x+x

$$\frac{M \to M'}{M\,N \to M'N}$$

$$\frac{N \to N'}{M\,N \to M\,N'}$$

$$\frac{M \to M'}{\lambda x. M \to \lambda x. M'}$$

Review of untyped λ-calculus

$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$

$\to (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$

$\to \ldots$

This class: adding a type system

(We will see that well-typed terms in STLC always terminate.)

Why types

- Type checking catches “simple” mistakes early
  - Example: 2 + true + “a”
- (**Type safety**) Well-typed programs will not go wrong
  - Ensure execution never reach a “meaningless” state
  - But “meaningless” depends on the semantics (each language typically defines some as type errors and others run-time errors)
- Typed programs are easier to analyze and optimize
  - Compilers can generate better code (e.g. access components of structures by known offset)

Cons: impose constraints on the programmer

- Some valid programs might be rejected

Why formal type systems

- Many typed languages have informal descriptions of the type systems (e.g., in language reference manuals)
- A fair amount of careful analysis is required to avoid false claims of type safety
- A formal presentation of a type system is a precise specification of the type checker
- And allows formal proofs of type safety

What we will study about types

- Type system
  - Typing rules: assign types to terms
  - Type safety (soundness of typing rules): well-typed terms cannot go wrong

- Connection to constructive propositional logic
  - Curry-Howard isomorphism: “Propositions are Types”, “Proofs are Programs”

Adding types to λ-calculus – wrong attempt

base type
(e.g. int, bool)

function type

(Types) τ, σ ::= T | **fun**

Adding types to λ-calculus – wrong attempt

- Typing judgment (to assign types to terms)

$$\vdash M : \tau$$

M is of type τ

Judgment

- A statement J about certain formal properties
- Has a derivation $\vdash J$ (i.e. "a proof")
- Has a meaning ("judgment semantics") $\vDash J$

- Typing rules (to derive the typing judgment)

Adding types to λ-calculus – wrong attempt

Typing rules

$$\frac{}{\vdash (\lambda x. M) : \mathbf{fun}}$$

$$\frac{\vdash M : \mathbf{fun} \qquad \vdash N : \mathrm{T}}{\vdash M\ N : \mathrm{T}}$$

Not type safe, since well-typed terms may go wrong (reduce to a "meaningless" state)

e.g. ((λf. f 1) 3) will go "wrong", though ⊢ (λf. f 1) 3 $: \mathrm{int}$

Adding types to λ-calculus – getting it right

- Classify functions using argument and result types
  - (λx. x) and (λf. f 1) should be of different types: ((λx. x) 3) is acceptable, but ((λf. f 1) 3) is not
  - Explicitly specify argument types in function syntax
- Type-check function bodies, which have free variables
  - Types of free variables are the context: type of (f 1) depends on the type of f

Simply-typed λ-calculus (STLC)

base type
(e.g. int, bool)

function type

(Types) $\tau, \sigma \quad ::= \ T \mid \sigma \to \tau$

An infinite number of types:

$\text{int} \to \text{int}, \ \text{int} \to (\text{int} \to \text{int}), \ (\text{int} \to \text{int}) \to \text{int}, \ \ldots$

$\to$ is right-associative: $\tau \to \tau \to \tau$ is $\tau \to (\tau \to \tau)$

Simply-typed λ-calculus (STLC)

(Types) τ, σ ::= T | $\sigma \to \tau$

(Terms) M, N ::= x | $\lambda x : \tau.\ M$ | M N

Reduction rules

$$\frac{}{(\lambda x{:}\,\tau.\,M)N \;\rightarrow\; M[N/x]}\;(\beta)$$

$$\frac{M \rightarrow M'}{M\ N \rightarrow M'N}$$

$$\frac{N \rightarrow N'}{M\ N \rightarrow M\ N'}$$

$$\frac{M \rightarrow M'}{\lambda x{:}\,\tau.\,M \rightarrow \lambda x{:}\,\tau.\,M'}$$

Same as untyped λ-calculus

Typing judgment

M is of type τ in context Γ

$$\Gamma \vdash M : \tau$$

- Typing context (a set of typing assumptions)

$$\Gamma ::= \cdot \mid \Gamma, x : \tau$$

  - Include types of all the free variables in M (each free variable x is of type τ)
  - Empty context $\cdot$ is for closed terms

- Under Γ, M is a well-typed term of type τ

Typing rules

$$\frac{}{\Gamma,\ x : \tau \ \vdash\ x : \tau}\ \text{(var)}$$

$$\frac{\Gamma \vdash M :\ \sigma \to \tau \qquad\qquad \Gamma \vdash N :\ \sigma}{\Gamma \vdash M\ N : \tau}\ \text{(app)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x{:}\sigma.\, M) : \sigma \to \tau}\ \text{(abs)}$$

Typing derivation examples

$$\frac{}{\Gamma,\ x : \tau \ \vdash\ x : \tau}\ \text{(var)}$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}\ \text{(app)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x{:}\sigma. M) : \sigma \to \tau}\ \text{(abs)}$$

$$\dfrac{\dfrac{}{x{:}\tau \ \vdash x{:}\tau}\ \text{(var)}}{\cdot \vdash (\lambda x{:}\tau. x) : \tau \to \tau}\ \text{(abs)}$$

Typing derivation examples

$$\frac{}{\Gamma,\ x : \tau \ \vdash\ x : \tau}\ \text{(var)}$$

$$\frac{\Gamma \vdash M :\ \sigma \to \tau \qquad \Gamma \vdash N :\ \sigma}{\Gamma \vdash M\ N : \tau}\ \text{(app)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x{:}\sigma.M) : \sigma \to \tau}\ \text{(abs)}$$

$$\cfrac{\cfrac{\cfrac{}{x{:}\,\tau, y{:}\,\sigma\ \vdash x : \tau}\ \text{(var)}}{x{:}\,\tau\ \vdash (\lambda y{:}\,\sigma.x) : \sigma \to \tau}\ \text{(abs)}}{\cdot \vdash (\lambda x{:}\,\tau.\lambda y{:}\,\sigma.x) : \tau \to \sigma \to \tau}\ \text{(abs)}$$

Typing derivation examples

$$\frac{}{\Gamma,\ x : \tau \vdash x : \tau}\ \text{(var)}$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}\ \text{(app)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x{:}\sigma. M) : \sigma \to \tau}\ \text{(abs)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}\tau \to \tau, y{:}\tau \vdash x{:}\tau \to \tau}\ \text{(var)} \qquad \dfrac{}{x{:}\tau \to \tau, y{:}\tau \vdash y{:}\tau}\ \text{(var)}}{x{:}\tau \to \tau, y{:}\tau \vdash x\ y : \tau}\ \text{(app)}}{x{:}\tau \to \tau \vdash (\lambda y{:}\tau. x\ y) : \tau \to \tau}\ \text{(abs)}}{\cdot \vdash (\lambda x{:}\tau \to \tau. \lambda y{:}\tau. x\ y) : (\tau \to \tau) \to \tau \to \tau}\ \text{(abs)}$$

Soundness and completeness

- A sound type system never accepts a program that can go wrong
  - No false negatives
  - The language is type-safe
- A complete type system never rejects a program that can't go wrong
  - No false positives
- However, for any Turing-complete PL, the set of programs that may go wrong is undecidable
  - Type system cannot be sound and complete
  - Choose soundness, try to reduce false positives in practice

Soundness – well-typed terms in STLC never go wrong

Theorem (Type Safety):

If $\cdot \vdash M : \tau$ and $M \rightarrow^* M'$, then

$\cdot \vdash M' : \tau$, and either $M' \in$ Values or $\exists M''. M' \rightarrow M''$

Defined in language semantics (e.g. λ-abstraction, constants)

That is, the reduction of a well-typed term either diverges, or terminates in a value of the expected type.

Follows from two key lemmas (next page).

Soundness – well-typed terms in STLC never go wrong

- Preservation (subject reduction): well-typed terms reduce only to well-typed terms of the same type

$$\text{If } \cdot \vdash M : \tau \text{ and } M \to M', \text{then } \cdot \vdash M' : \tau$$

- Progress: a well-typed term is either a value or can be reduced

$$\text{If } \cdot \vdash M : \tau, \text{then either } M \in \text{Values or } \exists M'. M \to M'$$

Not complete – the type system may reject terms that do not go wrong

- (λx. (x (λy. y)) (x 3)) (λz. z)

  Cannot find σ, τ such that

  $$x{:}\,\sigma \ \vdash \ \big(x\ (\lambda y.\, y)\big)(x\ 3) \ : \ \tau$$

  because we have to pick one type for x

- But (λx. (x (λy. y)) (x 3)) (λz. z)

  $\rightarrow$ ((λz. z) (λy. y)) ((λz. z) 3)

  $\rightarrow$ (λy. y) 3 $\rightarrow$ 3

Well-typed terms in STLC always terminate (strong normalization theorem)

- Recall $(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$
  $\to (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$
  $\to \ldots$

- $(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$ cannot be assigned a type

Expect σ to be in the form of σ → τ, which is impossible!

$$\frac{x{:}\,\sigma\ \vdash x{:}\,? \qquad x{:}\,\sigma\ \vdash x{:}\,\sigma}{x{:}\,\sigma\ \vdash\ x\ x\ :\ ?}$$

Main points of STLC

(Types) $\tau, \sigma \ ::= \ T \mid \sigma \to \tau$

(Terms) $M, N \ ::= \ x \mid \lambda x : \tau.\, M \mid M\, N$

Reduction rules

$$\frac{}{(\lambda x{:}\tau.M)N \ \to \ M[N/x]} \ (\beta)$$

Typing rules

$$\frac{}{\Gamma,\ x : \tau \ \vdash \ x : \tau} \ (\text{var}) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x{:}\sigma.M) : \sigma \to \tau} \ (\text{abs})$$

$$\frac{\Gamma \vdash M : \ \sigma \to \tau \qquad \Gamma \vdash N : \ \sigma}{\Gamma \vdash M\ N : \tau} \ (\text{app})$$

Soundness (type safety)

Adding stuff

Use STLC as a foundation for understanding other common language constructs

- Extend the syntax (types & terms)
- Extend the operational semantics (reduction rules)
- Extend the type system (typing rules)
- Extend the soundness proof (new proof cases)

Adding product type

(Types) τ, σ ::= … | $\sigma \times \tau$

(Terms) M, N ::= … | <M, N> | proj1 M | proj2 M

Consider structures in C:

```
struct date{
    int year;
    int month;
    int day;
}
```

Product type

(Types) τ, σ ::= … | $\sigma \times \tau$

(Terms) M, N ::= … | <M, N> | proj1 M | proj2 M

Reduction rules

$$\frac{}{\text{proj1} \langle M, N \rangle \to M} \qquad \frac{}{\text{proj2} \langle M, N \rangle \to N}$$

$$\frac{M \to M'}{\langle M, N \rangle \to \langle M', N \rangle} \qquad \frac{N \to N'}{\langle M, N \rangle \to \langle M, N' \rangle}$$

$$\frac{M \to M'}{\text{proj1}\ M \to \text{proj1}\ M'} \qquad \frac{M \to M'}{\text{proj2}\ M \to \text{proj2}\ M'}$$

Product type

(Types) τ, σ ::= … | $\sigma \times \tau$

(Terms) M, N ::= … | <M, N> | proj1 M | proj2 M

Typing rules

$$\frac{\Gamma \vdash \mathrm{M}\colon \sigma \qquad \Gamma \vdash \mathrm{N}\colon \tau}{\Gamma \vdash \langle \mathrm{M}, \mathrm{N} \rangle : \sigma \times \tau} \text{ (pair)}$$

$$\frac{\Gamma \vdash \mathrm{M}\colon \sigma \times \tau}{\Gamma \vdash \text{proj1 } \mathrm{M} : \sigma} \text{ (proj1)} \qquad \frac{\Gamma \vdash \mathrm{M}\colon \sigma \times \tau}{\Gamma \vdash \text{proj2 } \mathrm{M} : \tau} \text{ (proj2)}$$

Typing derivation example

$$
\dfrac{
\dfrac{\dfrac{}{x : \sigma \times \tau \ \vdash x : \sigma \times \tau}\,(\text{var})}{x : \sigma \times \tau \ \vdash \text{proj2}\ x : \tau}\,(\text{proj2})
\qquad
\dfrac{\dfrac{}{x : \sigma \times \tau \ \vdash x : \sigma \times \tau}\,(\text{var})}{x : \sigma \times \tau \ \vdash \text{proj1}\ x : \sigma}\,(\text{proj1})
}{
\dfrac{x : \sigma \times \tau \ \vdash \langle \text{proj2}\ x, \text{proj1}\ x \rangle : \tau \times \sigma}{\cdot \vdash (\lambda x : \sigma \times \tau . \langle \text{proj2}\ x, \text{proj1}\ x \rangle) : (\sigma \times \tau) \to (\tau \times \sigma)}\,(\text{abs})
}\,(\text{pair})
$$

Soundness theorem (type safety)

- Preservation:

$$\text{If } \cdot \vdash M : \tau \text{ and } M \to M', \text{then} \cdot \vdash M' : \tau$$

- Progress:

$$\text{If } \cdot \vdash M : \tau, \text{then either } M \in \text{Values or } \exists M'. M \to M'$$

Include <v1, v2> now

Adding sum type

(Types) τ, σ ::= … | $\sigma + \tau$

(Terms) M, N ::= … | left M | right M | case M do M1 M2

Consider unions in C:

```
union data{
    int i;
    float f;
    char c;
}
```

Using the same location
for multiple data.
Can contain only one value
at any given time.

Adding sum type

(Types) τ, σ ::= … | $\sigma + \tau$

(Terms) M, N ::= … | left M | right M | case M do M1 M2

Subclasses in Java:

```
abstract class t {abstract t' m();}
class A extends t { t1 x; t' m(){...}}
class B extends t { t2 x; t' m(){...}}
...
e.m();
```

case e do A.m B.m

Adding sum type

(Types) τ, σ ::= … | $\sigma + \tau$

(Terms) M, N ::= … | left M | right M | case M do M1 M2

In Coq:

```
Inductive bool : Set :=
  | true : bool
  | false : bool.
Definition not (b : bool) : bool :=
  match b with
    | true => false
    | false => true
  end.
```

Sum type: reduction rules

$$\frac{}{\text{case (left M) do M1 M2} \rightarrow \text{M1 M}}$$

$$\frac{}{\text{case (right M) do M1 M2} \rightarrow \text{M2 M}}$$

$$\frac{\text{M} \rightarrow \text{M'}}{\text{case M do M1 M2} \rightarrow \text{case M' do M1 M2}}$$

$$\frac{\text{M1} \rightarrow \text{M1'}}{\text{case M do M1 M2} \rightarrow \text{case M do M1' M2}}$$

$$\frac{\text{M2} \rightarrow \text{M2'}}{\text{case M do M1 M2} \rightarrow \text{case M do M1 M2'}}$$

$$\frac{\text{M} \rightarrow \text{M'}}{\text{left M} \rightarrow \text{left M'}}$$

$$\frac{\text{M} \rightarrow \text{M'}}{\text{right M} \rightarrow \text{right M'}}$$

Sum type: typing rules

$$\frac{\Gamma\ \vdash \mathrm{M}\colon\ \sigma}{\Gamma\ \vdash \mathrm{left\ M} : \sigma + \tau}\ \text{(left)} \qquad \frac{\Gamma\ \vdash \mathrm{M}\colon \tau}{\Gamma\ \vdash \mathrm{right\ M} : \sigma + \tau}\ \text{(right)}$$

$$\frac{\Gamma\ \vdash \mathrm{M}\colon\ \sigma + \tau \qquad \Gamma\ \vdash \mathrm{M1}\colon\ \sigma \to \rho \qquad \Gamma\ \vdash \mathrm{M2}\colon\ \tau \to \rho}{\Gamma\ \vdash \mathrm{case\ M\ do}\ M1\ M2\colon \rho}\ \text{(case)}$$

Typing derivation examples

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\tau \ \vdash x:\tau}\,(\text{var})}{x:\tau \ \vdash \text{left}\ x:\tau+\sigma}\,(\text{left}) \qquad \dfrac{\dfrac{}{x:\tau \ \vdash x:\tau}\,(\text{var})}{x:\tau \ \vdash \text{left}\ x:\tau+\rho}\,(\text{left})}{x:\tau \ \vdash \text{<left}\ x,\ \text{left}\ x\text{>} : (\tau+\sigma)\times(\tau+\rho)}\,(\text{pair})}{\cdot \vdash (\lambda x:\tau.\text{<left}\ x,\ \text{left}\ x\text{>}) : \tau \to (\tau+\color{red}{\sigma}\color{black}{)} \times (\tau+\color{red}{\rho}\color{black}{)}}\,(\text{abs})$$

other side can be anything

Soundness theorem (type safety)

- Preservation:

$$\text{If } \cdot \vdash M : \tau \text{ and } M \to M', \text{then } \cdot \vdash M' : \tau$$

- Progress:

$$\text{If } \cdot \vdash M : \tau, \text{then either } M \in \text{Values or } \exists M'. M \to M'$$

Include “left v” and “right v” now

Products vs. sums

- "logical duals" (more on this later)
  - To make a $\sigma \times \tau$, we need a σ and a τ
  - To make a $\sigma + \tau$, we need a σ or a τ
  - Given a $\sigma \times \tau$, we can get a σ or a τ or both (our "choice")
  - Given a $\sigma + \tau$, we must be prepared for either a σ or a τ (the value's "choice")

Main points till now

- STLC extended with products and sums:

(Types) $\tau, \sigma \quad ::= T \ | \ \sigma \to \tau \ | \ \sigma \times \tau \ | \ \sigma + \tau$

(Terms) M, N ::= x | λx : τ. M | M N
| <M, N> | proj1 M | proj2 M
| left M | right M | case M do M1 M2

- Next: recursion

Recursion

- Recall in untyped λ-calculus, every term has a fixpoint
    - Fixpoint combinator is a higher-order function h satisfying

      for all f, (h f) gives a fixpoint of f

      i.e. $h\ f = f\ (h\ f)$

    - Turing's fixpoint combinator Θ

      Let $A = \lambda x.\ \lambda y.\ y\ (x\ x\ y)$ and $\Theta = A\ A$

    - Church's fixpoint combinator **Y**

      Let $\mathbf{Y} = \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))$

Recursion

- Recall “strong normalization theorem”: well-typed terms in STLC always terminate
  - Extensions so far (products & sums) preserve termination
- Recursion is not allowed by the typing rules: it is impossible to find types for fixed-point combinators
- So we add an explicit construct for recursion

(Terms) M, N ::= … | **fix** M

(Types) τ, σ ::= … (no new types)

Reduction rules for fix

$$\frac{}{\mathbf{fix}\,\lambda x.\,M \;\to\; M[\mathbf{fix}\,\lambda x.\,M/x]} \qquad \frac{M \to M'}{\mathbf{fix}\,M \to \mathbf{fix}\,M'}$$

(**fix** λf. λn. if (n == 0) then 1 else n * f(n-1)) 3

→ (λn. if (n == 0) then 1 else n*((**fix** λf. λn. if (n == 0) then 1 else n * f(n-1))(n-1))) 3

→ if (3 == 0) then 1 else 3 * ((**fix** λf. λn. if (n == 0) then 1 else n * f(n-1))(3-1))

→ 3 * ((**fix** λf. λn. if (n == 0) then 1 else n * f(n-1))(3-1))

→ …

Typing fix

$$\frac{\Gamma \ \vdash M \ : \ \tau \to \tau}{\Gamma \ \vdash \mathbf{fix}\, M \ : \tau} \text{ (fix)}$$

- Math explanation: If M is a function from τ to τ , then **fix** M, the fixed-point of M, is some τ with the fixed-point property
- Operational explanation: **fix** λx.M’ reduces to M’[**fix** λx.M’/x].
  - The substitution means x and **fix** λx.M’ need the same type
  - The result means M’ and **fix** λx.M’ need the same type
- Soundness (type safety) is straightforward
- But strong normalization is eliminated

Main points till now

- STLC with products and sums:

(Types) $\tau, \sigma \quad ::= T \mid \sigma \to \tau \mid \sigma \times \tau \mid \sigma + \tau$

(Terms) $M, N \quad ::= x \mid \lambda x : \tau.\ M \mid M\ N$

$\mid \langle M, N \rangle \mid \text{proj1 } M \mid \text{proj2 } M$

$\mid \text{left } M \mid \text{right } M \mid \text{case } M \text{ do } M1\ M2$

- We can also add recursion

- Next: Curry-Howard isomorphism

Curry-Howard Isomorphism

- What we did:
  - Define a programming language
  - Define a type system to rule out “bad” programs
- What logicians do:
  - Define logic propositions
    - E.g. $p, q ::= B \mid p \wedge q \mid p \vee q \mid p \Rightarrow q$
  - Define a proof system to prove “good” propositions
- Turn out to be related
  - Propositions are Types
  - Proofs are Programs

Curry-Howard Isomorphism

- Slogans
  - Propositions are Types
  - Proofs are Programs

In this class, we will show correspondence between formulas of constructive propositional logic

(Prop) $p, q \quad ::= \; B \mid p \Rightarrow q \mid p \wedge q \mid p \vee q$

and types of STLC with products and sums

(Types) $\tau, \sigma \quad ::= \; T \mid \sigma \rightarrow \tau \mid \sigma \times \tau \mid \sigma + \tau$

Examples of terms and types

$\lambda x: \tau.\ x$

has type

$\tau \rightarrow \tau$

Examples of terms and types

$$\lambda x\colon \tau.\ \lambda f\colon \tau \to \sigma.\ f\ x$$

has type

$$\tau \to (\tau \to \sigma) \to \sigma$$

Examples of terms and types

$\lambda f: \tau \to \sigma \to \rho.\ \lambda x: \sigma.\ \lambda y: \tau.\ f\ y\ x$

has type

$(\tau \to \sigma \to \rho) \to \sigma \to \tau \to \rho$

Examples of terms and types

$$\lambda x\colon \tau.\ <\text{left } x, \text{left } x>$$

has type

$$\tau \rightarrow ((\tau + \sigma) \times (\tau + \rho))$$

Examples of terms and types

$\lambda f\colon \tau \to \rho.\ \lambda g\colon \sigma \to \rho.\ \lambda x\colon \tau + \sigma.$ (case x do f g)

has type

$(\tau \to \rho) \to (\sigma \to \rho) \to (\tau + \sigma) \to \rho$

Examples of terms and types

$\lambda x: \tau \times \sigma.\ \lambda y: \rho.$ < <y, proj1 x>, proj2 x >

has type

$$(\tau \times \sigma) \to \rho \to ((\rho \times \tau) \times \sigma)$$

Empty and nonempty types

Have seen several "nonempty" types (closed terms of that type)

$$\tau \to \tau$$
$$\tau \to (\tau \to \sigma) \to \sigma$$
$$(\tau \to \sigma \to \rho) \to \sigma \to \tau \to \rho$$
$$\tau \to ((\tau + \sigma) \times (\tau + \rho))$$
$$(\tau \to \rho) \to (\sigma \to \rho) \to (\tau + \sigma) \to \rho$$
$$(\tau \times \sigma) \to \rho \to ((\rho \times \tau) \times \sigma)$$

There're also lots of "empty" types (no closed terms of that type)

$$\tau \qquad \tau \to \sigma \qquad \tau + (\tau \to \sigma) \qquad \tau \to (\sigma \to \tau) \to \sigma$$

How to know whether a type is nonempty?

How to know whether a type is nonempty?

Let's replace $\to$ with $\Rightarrow$, $\times$ with $\wedge$, + with $\vee$:

$$\left.\begin{array}{l}
\tau \Rightarrow \tau \\
\tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma \\
(\tau \Rightarrow \sigma \Rightarrow \rho) \Rightarrow \sigma \Rightarrow \tau \Rightarrow \rho \\
\tau \Rightarrow ((\tau \vee \sigma) \wedge (\tau \vee \rho)) \\
(\tau \Rightarrow \rho) \Rightarrow (\sigma \Rightarrow \rho) \Rightarrow (\tau \vee \sigma) \Rightarrow \rho \\
(\tau \wedge \sigma) \Rightarrow \rho \Rightarrow ((\rho \wedge \tau) \wedge \sigma)
\end{array}\right\}$$

Can be proved in propositional logic

(corresponding to nonempty types – have closed terms)

$$\left.\begin{array}{l}
\tau \\
\tau \Rightarrow \sigma \\
\tau \vee (\tau \Rightarrow \sigma) \\
\tau \Rightarrow (\sigma \Rightarrow \tau) \Rightarrow \sigma
\end{array}\right\}$$

Cannot be proved in propositional logic

(corresponding to empty types – no closed terms)

Example – propositional-logic proof

$\boxed{\Gamma \ \vdash p}$

assumptions

$$\frac{\dfrac{\dfrac{\tau, \tau \Rightarrow \sigma \vdash \tau \Rightarrow \sigma \qquad \tau, \tau \Rightarrow \sigma \vdash \tau}{\tau, \tau \Rightarrow \sigma \vdash \sigma}}{\tau \ \vdash (\tau \Rightarrow \sigma) \Rightarrow \sigma}}{\cdot \vdash \tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma}$$

Propositional logic (natural deduction)

(Prop) p, q ::= B | p $\Rightarrow$ q | p $\wedge$ q | p $\vee$ q

(Ctxt) Γ ::= $\cdot$ | Γ, p

$$\frac{}{\Gamma, p \vdash p}\text{(axiom)} \qquad \frac{\Gamma, p \vdash q}{\Gamma \vdash p \Rightarrow q}(\Rightarrow\text{-intro}) \qquad \frac{\Gamma \vdash p \Rightarrow q \quad \Gamma \vdash p}{\Gamma \vdash q}(\Rightarrow\text{-elim})$$

$$\frac{\Gamma \vdash p \quad \Gamma \vdash q}{\Gamma \vdash p \wedge q}(\wedge\text{-intro}) \qquad \frac{\Gamma \vdash p \wedge q}{\Gamma \vdash p}(\wedge\text{-elim-l}) \qquad \frac{\Gamma \vdash p \wedge q}{\Gamma \vdash q}(\wedge\text{-elim-r})$$

$$\frac{\Gamma \vdash p}{\Gamma \vdash p \vee q}(\vee\text{-intro-l}) \qquad \frac{\Gamma \vdash q}{\Gamma \vdash p \vee q}(\vee\text{-intro-r})$$

$$\frac{\Gamma \vdash p \vee q \quad \Gamma \vdash p \Rightarrow r \quad \Gamma \vdash q \Rightarrow r}{\Gamma \vdash r}(\vee\text{-elim})$$

This is exactly our type system, erasing terms, replacing $\rightarrow$ with $\Rightarrow$, $\times$ with $\wedge$, + with $\vee$

$$\frac{}{\Gamma,\ x : \tau \vdash x : \tau}\ \text{(var)} \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x{:}\sigma. M) : \sigma \to \tau}\ \text{(abs)}$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}\ \text{(app)} \qquad \frac{\Gamma \vdash \mathrm{M}: \sigma \qquad \Gamma \vdash \mathrm{N}: \tau}{\Gamma \vdash \langle \mathrm{M}, \mathrm{N} \rangle : \sigma \times \tau}\ \text{(pair)}$$

$$\frac{\Gamma \vdash \mathrm{M}: \sigma \times \tau}{\Gamma \vdash \text{proj1 } \mathrm{M} : \sigma}\ \text{(proj1)} \qquad \frac{\Gamma \vdash \mathrm{M}: \sigma \times \tau}{\Gamma \vdash \text{proj2 } \mathrm{M} : \tau}\ \text{(proj2)}$$

$$\frac{\Gamma \vdash \mathrm{M}: \sigma}{\Gamma \vdash \text{left } M : \sigma + \tau}\ \text{(left)} \qquad \frac{\Gamma \vdash \mathrm{M}: \tau}{\Gamma \vdash \text{right } M : \sigma + \tau}\ \text{(right)}$$

$$\frac{\Gamma \vdash \mathrm{M}: \sigma + \tau \qquad \Gamma \vdash \mathrm{M1}: \sigma \to \rho \qquad \Gamma \vdash \mathrm{M2}: \tau \to \rho}{\Gamma \vdash \text{case } M \text{ do } M1\ M2 : \rho}\ \text{(case)}$$

Curry-Howard isomorphism

- Given a well-typed closed term, take the typing derivation, erase the terms, and have a propositional-logic proof
- Given a propositional-logic proof, there exists a closed term with that type
- A term that type-checks is a proof — it tells you exactly how to derive the logic formula corresponding to its type
- Constructive *(hold that thought)* propositional logic and simply-typed lambda-calculus with pairs and sums are the same thing.
  - Computation and logic are deeply connected
  - λ is no more or less made up than implication

Revisit our examples: “terms are proofs”

$\lambda x: \tau.\ x$

is a proof that

$\tau \Rightarrow \tau$

Revisit our examples: “terms are proofs”

$\lambda x\colon \tau.\ \lambda f\colon \tau \to \sigma.\ f\ x$

is a proof that

$\tau \Rightarrow (\tau \Rightarrow \sigma) \Rightarrow \sigma$

Revisit our examples: “terms are proofs”

$\lambda f\colon \tau \to \sigma \to \rho.\ \lambda x\colon \sigma.\ \lambda y\colon \tau.\ f\ y\ x$

is a proof that

$(\tau \Rightarrow \sigma \Rightarrow \rho) \Rightarrow \sigma \Rightarrow \tau \Rightarrow \rho$

Revisit our examples: “terms are proofs”

λx: τ. <left x, left x>

is a proof that

$$\tau \Rightarrow ((\tau \vee \sigma) \wedge (\tau \vee \rho))$$

Revisit our examples: “terms are proofs”

$\lambda f\colon \tau \to \rho.\ \lambda g\colon \sigma \to \rho.\ \lambda x\colon \tau + \sigma.\ (\text{case } x \text{ do } f\ g)$

is a proof that

$(\tau \Rightarrow \rho) \Rightarrow (\sigma \Rightarrow \rho) \Rightarrow (\tau \vee \sigma) \Rightarrow \rho$

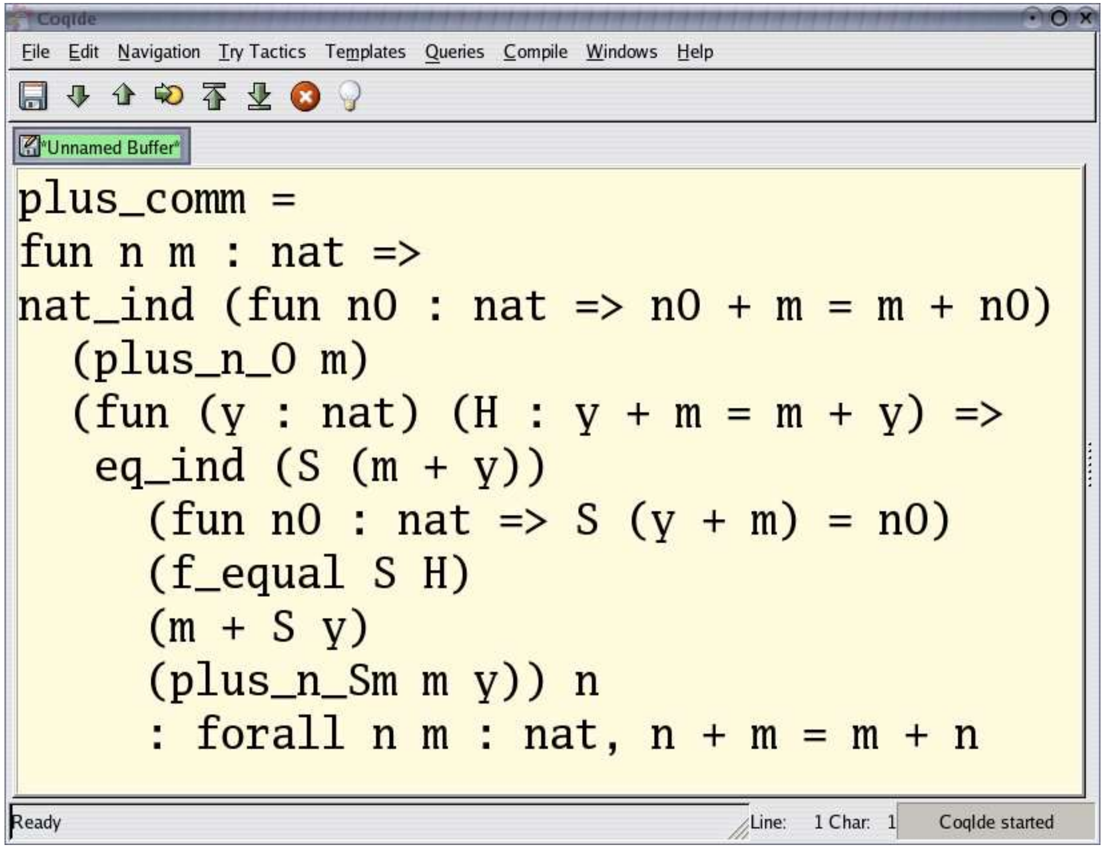
Revisit our examples: “terms are proofs”

$\lambda x\colon \tau \times \sigma.\ \lambda y\colon \rho.\ \langle \langle y, \text{proj1 } x\rangle, \text{proj2 } x \rangle$

is a proof that

$$(\tau \wedge \sigma) \Rightarrow \rho \Rightarrow ((\rho \wedge \tau) \wedge \sigma)$$

Coq example: proof can be written as functional program



Proof of commutativity of addition on nat in Coq.

Why care?

Because:

- This is just fascinating
- Don’t think of logic and computing as distinct fields
- Thinking “the other way” can help you know what’s possible/impossible
- Can form the basis for automated theorem provers
- Type systems should not be *ad hoc* piles of rules!

So, every typed λ-calculus is a proof system for some logic...

Is STLC with pairs and sums a *complete* proof system for propositional logic? Almost...

Classical vs. constructive

- Classical propositional logic has the "law of the excluded middle":

$$\frac{}{\Gamma \ \vdash p \vee (p \Rightarrow q)}$$

Think "$p \ \vee \neg p$"

- STLC does not support it: e.g. no closed term has type $\rho + (\rho \rightarrow \sigma)$

- Logics without this rule are called "constructive" or "intuitionistic".
  - Formulae are *only* considered "true" when we have direct evidence ("proofs produce examples")

Example classical proof

- Theorem: There exist two irrational numbers a and b such that a^b is rational.
- Can be proved using “the law of exclusive middle”.
    - It’s known that $\sqrt{2}$ is irrational.
    - Consider the number ${\sqrt{2}}^{\sqrt{2}}$.
        - If it is rational, the proof is complete, and $a = b = \sqrt{2}$.
        - If it is irrational, then let $a = {\sqrt{2}}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then $a^b = ({\sqrt{2}}^{\sqrt{2}})^{\sqrt{2}} = {\sqrt{2}}^{\sqrt{2}\times\sqrt{2}} = (\sqrt{2})^2 = 2$, and the proof is complete.
- Constructive logics would not accept this argument

Classical vs. constructive

- In constructive logics, "branch on possibilities" by making "excluded middle" an explicit assumption:

$$(p \lor (p \Rightarrow q)) \land (p \Rightarrow r) \land \big((p \Rightarrow q) \Rightarrow r\big) \Rightarrow r$$

- "if any number is either rational or irrational, then there exist two irrational numbers a and b such that a^b is rational"

What about “fix”?

- A “non-terminating proof” is no proof at all
- Remember the typing rule

$$\frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \mathbf{fix}\, M : \tau} \text{ (fix)}$$

- It lets us prove anything! E.g. **fix** $\lambda x{:}\tau.\ x$ has type τ
- So the “logic” is inconsistent

Last word on Curry-Howard

- Not just constructive propositional logic & STLC
- ***Every*** logic has a corresponding typed system
  - Classical logics
  - Inconsistent logics

- If you remember one thing:

$$\frac{\Gamma \vdash M\colon \sigma \to \tau \qquad \Gamma \vdash N\colon\ \sigma}{\Gamma \vdash M\ N\ :\ \tau}\text{(app)} \quad \Leftrightarrow \quad \frac{\Gamma \vdash p \Rightarrow q \quad \Gamma \vdash p}{\Gamma \vdash q}\text{(}\Rightarrow\text{-elim)}$$