Convex Optimization
Project 3

Spring 1402
Due date: 25th of Ordibehesht

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1. *Computing market-clearing prices.* We consider n commodities or goods with $p \in R^n_{++}$ the vector of prices (per unit quantity) of them. The (nonnegative) demand for the products is a function function of the prices, which we denote $D : R^n \to R^n$, so $D(p)$ is the demand when the product prices are p. The (nonnegative) supply of the products (*i.e.*, the amounts that manufacturers are willing to produce) is also a function of the prices, which we denote $S : R^n \to R^n$, so $S(p)$ is the supply when the product prices are p. We say that the market *clears* if $S(p) = D(p)$, *i.e.,* supply equals demand, and we refer to p in this case as a set of *market-clearing prices.*

   Elementary economies courses consider the special case $n = 1$, *i.e.,* a single commodity, so supply and demand can be plotted (vertically) against the price (on the horizontal axis). It is assumed that demand decreases with increasing price, and supply increases; the market clearing price can be found 'graphically', as the point where the supply and demand curves intersect. In this problem we examine some cases which market-clearing prices (for the general case $n > 1$) can be computed using convex optimization.

   We assume that the demand function is *Hicksian*, which means it has the form $D(p) = \nabla E(p)$,where $E : R^n \to R$ is a differentiable function that is concave and increasing in each argument, called the *expenditure function.* (While not relevant in this problem, Hicksian demand arises from a model in which consumers make purchases by maximizing a concave utility function.)

   We will assume that the producers are independent, so $S(p)_i = S_i(p_i), i = 1, ..., n$, where $S_i : R \to R$ is the supply function for good i. We will assume that the supply functions are positive and increasing on their domain R_+.

   (a) Explain how to use convex optimization to find market-clearing prices under the assumptions given above. (You do not need to worry about technical details like zero prices, or cases in which there are no market-clearing prices.)

   (b) Compute market-clearing prices for the specific case with $n = 4$,

   $$E(p) = (\textstyle\prod_i^4 p_i)^{1/4},$$
   $$S(p) = (0.2p_1 + 0.5, 0.02p_2 + 0.1, 0.04p_3, 0.1p_4 + 0.2).$$

   Give the market-clearing prices and the demand and supply (which should match) at those prices.

   Hint: In CVX and CVXPY, **geo_mean** gives the geometric mean of the entries of a vector argument. Julia does not yet have a vector argument **geo_mean** function, but you can get the geometric mean of 4 variables a,b,c,d using $\mathbf{geomean}(\mathbf{geomean}(\mathbf{a}, \mathbf{b}), \mathbf{geomean}(\mathbf{c}, \mathbf{d}))$.

Good Luck!