Convex Optimization

Homework 1

Winter 1401
Due date: 5th of Esfand

1. For each of the following statements, either show that it is true, or give a (specific) counterexample.

   - If AB is full rank then A and B are full rank.
   - If A and B are full rank then AB is full rank.
   - If A and B have zero nullspace, then so does AB.
   - If A and B are onto, then so is AB.

   You can assume that $A \in R^{m\times n}$ and $B \in R^{n\times p}$. Some of the false statements above become true under certain assumptions on the dimensions of A and B. As a trivial example, all of the statements above are true when A and B are scalars, i.e., $n = m = p = 1$. For each of the statements above, find conditions on n, m, and p that make them true. Try to find the most general conditions you can. You can give your conditions as inequalities involving n, m, and p, or you can use more informal language such as "A and B are both skinny."

2. Determine if the following statements are true or false. What we mean by "true" is that the statement is true for all values of the matrices and vectors given. You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A+B = B+A$" is true, because no matter what the dimensions of A and B (which must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, e.g., an identity matrix. But that doesn't make the statement true.)

   (a) If all coefficients (i.e., entries) of the matrices A and B are nonnegative, and both A and B are onto, then $A + B$ is onto.

   (b) $\mathcal{N}\left(\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix}\right) = \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C)$.

   (c) $\mathcal{N}\left(\begin{bmatrix} A \\ AB \\ ABC \end{bmatrix}\right) = \mathcal{N}(A) \cap \mathcal{N}(B) \cap \mathcal{N}(C)$.

   (d) $\mathcal{N}(B^TA^TAB + B^TB) = \mathcal{N}(B)$.

   (e) If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is fullrank, then so are the matrices A and B.

   (f) If $\begin{bmatrix} A & 0 \end{bmatrix}$ is onto, then A is full rank.

   (g) If A^2 is onto, then A is onto.

   (h) If A^TA is onto, then A is onto.

   (i) Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ are nonzero vectors such that $u_i^T u_j \geq 0$ for all i, j. Then the vectors are *nonnegative independent*, which means if $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ are nonnegative scalars, and , $\sum_{i=1}^k \alpha_i u_i = 0$ then $\alpha_j = 0$ for $i = 1, \ldots, k$.

   (j) Suppose $A \in \mathbb{R}^{N\times k}$ and $B \in \mathbb{R}^{n\times m}$ are skinny, full rank matrices that satisfy $A^TB = 0$. Then $\begin{bmatrix} A & B \end{bmatrix}$ is skinny and full rank.

3. Are the following statements true or false?
   (a) Least squares is a special case of convex optimization.
   (b) By and large, convex optimization problems can be solved efficiently.
   (c) Almost any problem you'd like to solve in practice is convex.
   (d) Convex optimization problems are attractive because they always have a unique solution.

4. The exponential cone $K_{\text{exp}} \subseteq \mathbb{R}^3$ is defined as

$$K_{\text{exp}} = \{(x, y, z) \mid y > 0,\ ye^{x/y} \le z\}.$$

   Find the dual cone K^*_{exp}.

   We are not worried here about the fine details of what happens on the boundaries of these cones, so you really needn't worry about it. But we make some comments here for those who do care about such things.

   The cone K_{exp} as defined above is not closed. To obtain its closure, we need to add the points

$$\{(x, y, z) \mid x \le 0,\ y = 0,\ z \ge 0\}.$$

   (This makes no difference, since the dual of a cone is equal to the dual of its closure.)

5. Describe the dual cone for each of the following cones.
   (a) $K = \{0\}$.
   (b) $K = \mathbb{R}^2$.
   (c) $K = \{(x_1, x_2) \mid |x_1| \le x_2\}$.
   (d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

6. Determine if each set below is convex.
   (a) $\{(x, y) \in \mathbb{R}^2_{++} \;\; x/y \le 1\}$
   (b) $\{(x, y) \in \mathbb{R}^2_{++} \;\; x/y \ge 1\}$
   (c) $\{(x, y) \in \mathbb{R}^2_{+} \;\; xy \le 1\}$
   (d) $\{(x, y) \in \mathbb{R}^2_{+} \;\; xy \ge 1\}$

7. Consider the set $S = \{(0, 2), (1, 1), (2, 3), (1, 2), (4, 0)\}$. Are the following statements true or false?
   (a) $(0, 2)$ is the minimum element of S.
   (b) $(0, 2)$ is a minimal element of S.
   (c) $(2, 3)$ is a minimal element of S.
   (d) $(1, 1)$ is a minimal element of S.

Good Luck!