Convex Optimization

Homework 2

Winter 1401

Due date: 15th of Esfand

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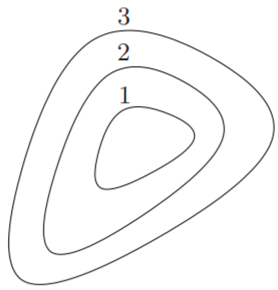
1. Level sets of convex, concave, quasiconvex, and quasiconcave functions. Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.



   Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.

2. For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

   (a) $f(x) = e^x - 1$ on $\mathbf{R}$.

   (b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

   (c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

   (d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

   (e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

   (f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbf{R}^2_{++}$.

3. show that the conjugate of $\mathbf{f}(\mathbf{X}) = \mathbf{tr}(\mathbf{X}^{-1})$ with $\mathbf{dom}\ \mathbf{f} = \mathbf{S}^n_{++}$ is given by

$$\mathbf{f}^*(\mathbf{Y}) = -2\mathbf{tr}(-\mathbf{Y})^{1/2}, \mathbf{domf}^* = -\mathbf{S}^n_{+}.$$

   Hint. The gradient of $\mathbf{f}$ is $\nabla\mathbf{f}(\mathbf{X}) = -\mathbf{X}^{-2}$.

4. *Maximum of a convex function over a polyhedron.* show that the maximum of a convex function **f** over the polyhedron $\wp = \mathbf{conv}\{\mathbf{v_1}, ..., \mathbf{v_k}\}$ is achieved at one of its vertices, *i.e.*,

$$\sup_{x\in\wp} f(x) \;=\; \max_{i=1,..,k} \,(\, f(v_i)\,)$$

(A stronger statement is: the maximum of a convex function over a closed bounded convex set is achieved at an extreme point, *i.e.*, a point in the set is not a convex combination of any other points in the set.) *Hint.* Assume the statement is false, and use Jensen's inequality.

5. *Convex hull of functions.* Suppose g and h are convex function, bounded below, with $\mathbf{dom\ g} = \mathbf{dom\ h} = \mathbf{R}^n$. The convex hull function of $\mathbf{g}$ and $\mathbf{h}$ is defined as

$$f(x) \;=\; \inf\left\{\theta g(y) + (1-\theta)\, h(z) \;\,|\theta y \,+\, (1-\theta)\, z = x,\, 0 \le \theta \le 1\right\} \;,$$

where the infimum is over θ, y, z. Show that the convex hull of h and g is convex. Describe $\mathbf{epi}f$ in terms of $\mathbf{epi}\ g$ and $\mathbf{epi}\ h$.

6. *Functions of a random variable with log-concave density.* Suppose the random variable X on $\mathbf{R}^n$ has log-concave density, and let $Y = g(X)$, where $g : \mathbf{R}^n \longrightarrow \mathbf{R}$. For each of the following statements, either give a counterexample, or show that the statement is true.

   (a) If g is affine and not constant, then Y has log-concave density.

   (b) If g is convex, then $\mathbf{Prob}(Y \le a)$ is a log-concave function of a.

   (c) If g is concave, then $\mathbf{E}\,((Y-a)_+)$ is a convex and log-concave function of a. (This quantity is called the tail expectation of Y; you can assume it exists. We define $(s)_+$ as $(s)_+ = max\{s, 0\}$.)

7. DCP rules. The function $f(x, y) = \sqrt{1 + x^4/y}$, with dom $f = \mathbf{R} \times \mathbf{R}_{++}$, is convex. Use disciplined convex programming (DCP) to express f so that it is DCP convex. You can use any of the following atoms

   inv_pos (u), which is $1/u$, with domain $\mathbf{R}_{++}$
   square (u), which is u^2, with domain $\mathbf{R}$
   sqrt(u), which is $\sqrt{u}$, with domain $\mathbf{R}_+$
   geo_mean (u,v), which is $\sqrt{uv}$, with domain $\mathbf{R}^2_+$
   quad_over_lin (u, v), which is u^2/v, with domain $\mathbf{R} \times \mathbf{R}_{++}$
   norm2(u, v), which is $\sqrt{u^2 + v^2}$, with domain $\mathbf{R}^2$.

   You may also use addition, subtraction, scalar multiplication, and any constant functions. Assume that DCP is sign-sensitive, e.g., square (u) is known to be increasing in u for $u \ge 0$.

8. *curvature of some functions.* Determine the curvature of the functions below. Your responses can be: affine, convex, concave, and none (meaning, neither convex nor concave).

   (a) $f(u, v) = uv$, with $\mathbf{dom}\ f = \mathbf{R}^2$.

   (b) $f(x, u, v) = log(v - x^T x/u), with$ $\mathbf{dom}\ f = \{(x, u, v)|uv > x^T x, u > 0\}$.

   (c) the 'exponential barrier' for a polyhedron,

   $$f(x) \;=\; \sum_{i=1}^{m} \exp\left(\frac{1}{b_i - a_i^T x}\right)$$

   with $\mathbf{dom}\ f = \{x|a_i^T x < b_i, i = 1, ..., m\}$, and $a_i \in \mathbf{R}^n, b \in \mathbf{R}^m$

Good Luck!