Writing Assignment 2

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Setup

```r
library(pracma)
library(MASS)
library(ggplot2)
```

Part 1

Set up a linear system and replicate the above process to determine the equation of the line y = mx+b passing through points (−1, 2) and (1, 12). (Show and explain all computations, including your rref.) Plot a graph of the points and the resulting line using your own variation of the code above.

We want the point (-1, 2) on the line, so

$$-m + b = 2$$

We also want the point (1, 12) in the line, so

$$m + b = 12$$

Thus, our system is

$$\begin{cases} -m + b = 2 \\ m + b = 12 \end{cases}$$

Then we can write the above system as an augmented matrix as

```r
Linear_System <- matrix(
  c(-1, 1, 2, 1, 1, 12),
  nrow = 2,
  ncol = 3,
  byrow = TRUE,
)

Linear_System
```

```
     [,1] [,2] [,3]
[1,]   -1    1    2
[2,]    1    1   12
```

The rref can then be calculated

```r
rref(Linear_System)
```

```
     [,1] [,2] [,3]
[1,]    1    0    5
[2,]    0    1    7
```

We see that there is a unique solution given by $m = 5$ and $b = 7$. The unique line through these two points is therefore

$$y = 5x + 7$$

This equation can be graphed as

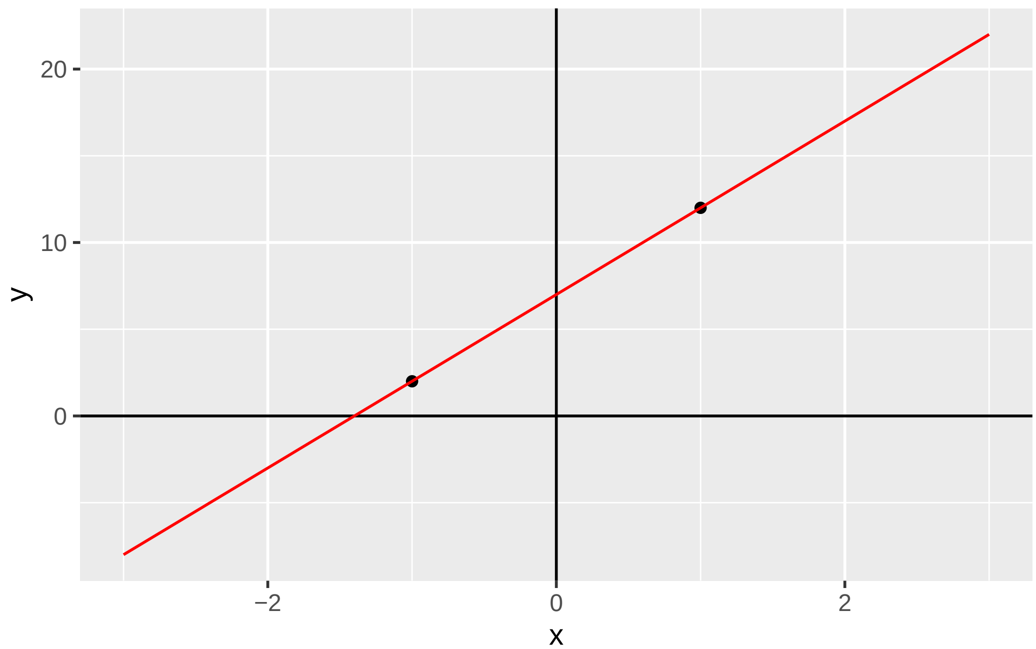
```r
p1 <- c(-1, 2)
p2 <- c(1, 12)

points <- matrix(
  c(p1, p2),
  nrow = 2,
  ncol = 2,
  byrow = TRUE,
  dimnames = list(rows = c("p1", "p2"), cols = c("x", "y")))

eq <- function(x) {
```

```
  return(5*x + 7)
}

ggplot(data = data.frame(points)) +
  xlim(-3, 3) +
  geom_point(aes(x = x, y = y)) +
  geom_hline(yintercept = 0) +
  geom_vline(xintercept = 0) +
  geom_function(fun = eq, color = "red")
```



Part 2

Choose three points in the xy-plane that are not colinear and have different x coordinates. Set up a linear system as above to determine a line $y = mx + b$ that passes through your three chosen points. Use an rref to show that this system is inconsistent.

I chose the points (3, 4), (5, 12), (-2, 3).

We want the point (3, 4) on the line, so

$$3m + b = 4$$

We also want the point (5, 12) in the line, so

$$5m + b = 12$$

Finally, we want the point (-2, 3) in the line, so

$$-2m + b = 3$$

Thus, our system is

$$\begin{cases} 3m + b = 4 \\ 5m + b = 12 \\ -2m + b = 3 \end{cases}$$

Then we can write the above system as an augmented matrix as

```r
Linear_System <- matrix(
  c(3, 1, 4, 5, 1, 12, -2, 1, 3),
  nrow = 3,
  ncol = 3,
  byrow = TRUE
)

Linear_System
```

```
     [,1] [,2] [,3]
[1,]    3    1    4
[2,]    5    1   12
[3,]   -2    1    3
```

The rref can then be calculated

```r
rref(Linear_System)
```

```
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

Since there is a pivot in the rightmost column in row 3, the system is inconsistent.

Part 3

Set up and solve a linear system to find a parabola of the form $y = ax^2 + bx + c$ that passes through the three points that you chose in #2. Compute the required rref, so that you can give exact (fraction, not decimal) answers. Plot a graph of this parabola along with the points.

We want the point (3, 4) on the line, so

$$3^2a + 3b + c = 4$$

$$9a + 3b + c = 4$$

We also want the point (5, 12) in the line, so

$$5^2a + 5b + c = 12$$

$$25a + 5b + c = 12$$

Finally, we want the point (-2, 3) in the line, so

$$(-2)^2a - 2b + c = 3$$

$$4a - 2b + c = 3$$

Thus, our system is

$$\begin{cases} 9a + 3b + c = 4 \\ 25a + 5b + c = 12 \\ 4a - 2b + c = 3 \end{cases}$$

Then we can write the above system as an augmented matrix,

```
Linear_System <- matrix(
  c(9, 3, 1, 4, 25, 5, 1, 12, 4, -2, 1, 3),
  nrow = 3,
  ncol = 4,
  byrow = TRUE
)

Linear_System
```

```
     [,1] [,2] [,3] [,4]
[1,]    9    3    1    4
[2,]   25    5    1   12
[3,]    4   -2    1    3
```

The rref can then be calculated,

```
fractions(rref(Linear_System))
```

```
     [,1]   [,2]   [,3]   [,4]  
[1,]      1      0      0  19/35
[2,]      0      1      0 -12/35
[3,]      0      0      1    1/7
```

Notice we now have a consistent system.

We see that there is a solution given by $a = \frac{19}{35}$, $b = \frac{-12}{35}$, and $c = \frac{1}{7}$. The parabola through these three points can therefore be expressed by the equation,

$$y = \frac{19}{35}x^2 - \frac{12}{35}x + \frac{1}{7}$$

This equation can be graphed as,

```
p1 <- c(3, 4)
p2 <- c(5, 12)
p3 <- c(-2, 3)

points <- matrix(
  c(p1, p2, p3),
  nrow = 3,
  ncol = 2,
```

```r
  byrow = TRUE,
  dimnames = list(rows = c("p1", "p2", "p3"), cols = c("x", "y")))

eq <- function(x) {
  return(19/35*x^2 - 12/35*x + 1/7)
}

ggplot(data = data.frame(points)) +
  xlim(-10, 10) +
  geom_point(aes(x, y)) +
  geom_hline(yintercept = 0) +
  geom_vline(xintercept = 0) +
  geom_function(fun = eq, color = "red")
```

Part 4

Set up and solve a linear system to find a degree 3 polynomial function of the form $y = ax^3 + bx^2 + cx + d$ with $a \neq 0$ whose graph passes through the three points that you chose in #2. How many solutions are there to this problem? Plot a graph of at least three cubic curves that work (on the same graph) along with your three points. Include at least one example with a $>$ 0 and at least one example with a

$<$ 0.

We will take our previous system from #4,

$$\begin{cases} 9a + 3b + c = 4 \\ \\ 25a + 5b + c = 12 \\ \\ 4a - 2b + c = 3 \end{cases}$$

and modify it to a degree 3 polynomial system of the form $y = ax^3 + bx^2 + cx + d$,

$$\begin{cases} 27a + 9b + 3c + d = 4 \\ \\ 125a + 25b + 5c + d = 12 \\ \\ -8a + 4b - 2c + d = 3 \end{cases}$$

We can convert this into an augmented matrix,

```r
Linear_System <- matrix(
  c(27, 9, 3, 1, 4, 125, 25, 5, 1, 12, -8, 4, -2, 1, 3),
  nrow = 3,
  ncol = 5,
  byrow = TRUE
)

Linear_System
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]   27    9    3    1    4
[2,]  125   25    5    1   12
[3,]   -8    4   -2    1    3
```

And the RREF can be calculated to be,

```r
fractions(rref(Linear_System))
```

```
     [,1]  [,2]  [,3]   [,4]    [,5]
[1,]    1     0     0  -1/30  -1/210
[2,]    0     1     0    1/5     4/7
[3,]    0     0     1   1/30 -71/210
```

Notice that we have a free variable. This means that we can generate infinite solutions to this system, each solution based on the value we choose for the free variable. From here, we can solve for each variable, a, b, and c in terms of d,

$$a - \frac{1}{30}d = \frac{-1}{210}a = \frac{1}{30}d - \frac{1}{210}$$

$$b + \frac{1}{5}d = \frac{4}{7}b = -\frac{1}{5}d + \frac{4}{7}$$

$$c + \frac{1}{30}d = \frac{-71}{210}c = -\frac{1}{30}d - \frac{71}{210}$$

$$d = d$$

Thus the degree 3 polynomial function can be written as,

$$y = (\frac{1}{30}d - \frac{1}{210})x^3 + (-\frac{1}{5}d + \frac{4}{7})x^2 - (-\frac{1}{30}d - \frac{71}{210})x + d$$

Let's choose three different values for d and plot the results. I am going to choose $d = -5$, $d = 0$, and $d = 5$. Notice that $a = \frac{1}{30}(0) - \frac{1}{210} = \frac{-1}{210} < 0$ and $a = \frac{1}{30}(5) - \frac{1}{210} = \frac{1}{6} - \frac{1}{210} > 0$, so both examples are present. Here are three different graphs that can be generated with our 3 points

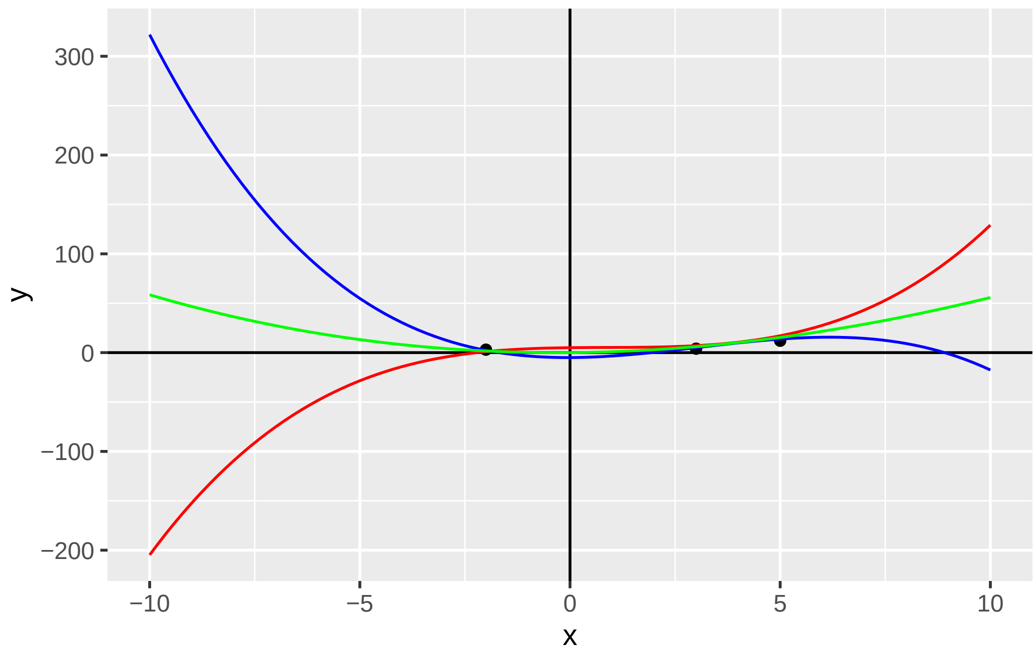
```r
p1 <- c(3, 4)
p2 <- c(5, 12)
p3 <- c(-2, 3)

points <- matrix(
  c(p1, p2, p3),
  nrow = 3,
  ncol = 2,
  byrow = TRUE,
  dimnames = list(rows = c("p1", "p2", "p3"), cols = c("x", "y")))

eq <- function(x, d) {
```

```
  return(((1/30*d)-(1/210))*x^3 + ((-1/5)*d+(4/7))*x^2 - ((-1/30)*d-(71/210))*x + d)
}

ggplot(data = data.frame(points)) +
  xlim(-10, 10) +
  geom_point(aes(x, y)) +
  geom_hline(yintercept = 0) +
  geom_vline(xintercept = 0) +
  geom_function(fun = eq, args = list(d = 5), color = "red") +
  geom_function(fun = eq, args = list(d = -5), color = "blue") +
  geom_function(fun = eq, args = list(d = 0), color = "green")
```



Part 5

Continue to use your set of 3 points from #2. Add a 4th point whose x-coordinate is not the same as the x-coordinates of any of your other points and which doesn't lie on the parabola you found in #3. Show that now, there is a unique cubic $y = ax^3 + bx^2 + cx + d$ that passes through your 4 points (and use rref to find it). Once again, include a graph of the cubic and your 4 points.

We will take our previous system from #4,

$$
\begin{cases}
9a + 3b + c = 4 \\
25a + 5b + c = 12 \\
4a - 2b + c = 3
\end{cases}
$$

and modify it to a degree 3 polynomial system,

$$
\begin{cases}
27a + 9b + 3c + d = 4 \\
125a + 25b + 5c + d = 12 \\
-8a + 4b - 2c + d = 3
\end{cases}
$$

and add a 4th point (1, -2) to it,

$$
a + b + c + d = -2
$$

and finally,

$$
\begin{cases}
27a + 9b + 3c + d = 4 \\
125a + 25b + 5c + d = 12 \\
-8a + 4b - 2c + d = 3 \\
a + b + c + d = -2
\end{cases}
$$

Then we can write the above system as an augmented matrix as

```r
Linear_System <- matrix(
  c(27, 9, 3, 1, 4, 125, 25, 5, 1, 12, -8, 4, -2, 1, 3, 1, 1, 1, 1, -2),
  nrow = 4,
  ncol = 5,
  byrow = TRUE
)

Linear_System
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]   27    9    3    1    4
[2,]  125   25    5    1   12
[3,]   -8    4   -2    1    3
[4,]    1    1    1    1   -2
```

The RREF can then be calculated,

```
fractions(rref(Linear_System))
```

```
     [,1]      [,2]      [,3]      [,4]      [,5]     
[1,]         1         0         0         0  -41/420
[2,]         0         1         0         0    79/70
[3,]         0         0         1         0 -103/420
[4,]         0         0         0         1   -39/14
```

Since all of the coefficient columns are pivots, there is only one unique solution to this problem.

We see that there is a solution given by a = -41/420, b = 79/70, c = -103/420, and d = -39/14. The cubic curve through these four points is therefore

$$y = \frac{-41}{420}x^3 + \frac{79}{70}x^2 - \frac{103}{420}x - \frac{39}{14}$$

Finally, we can plot the graph,

```
p1 <- c(3, 4)
p2 <- c(5, 12)
p3 <- c(-2, 3)

points <- matrix(
  c(p1, p2, p3),
  nrow = 3,
  ncol = 2,
  byrow = TRUE,
  dimnames = list(rows = c("p1", "p2", "p3"), cols = c("x", "y")))

eq <- function(x) {
  return(-41/420*x^3 + 79/70*x^2 - 103/420*x - 39/14)
}
```

```r
ggplot(data = data.frame(points)) +
  xlim(-10, 10) +
  geom_point(aes(x, y)) +
  geom_hline(yintercept = 0) +
  geom_vline(xintercept = 0) +
  geom_point(aes(x = 1, y = -2), color = "red") +
  geom_function(fun = eq, color = "red")
```