Boole's Rule

Numerical Integration Method

Harold R. Mansilla[1]

[1]Department of Physical Sciences and Mathematics
University of the Philippines Manila

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Outline I

Outline

1. Background
   - Newton-Cotes Formulas
   - Closed Newton-Cotes Formulas
2. Boole's Rule
   - History
   - Derivation [4]
   - Example Comparison [3]

Newton-Cotes Formulas

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- They are also known as *Quadrature formulas*

Langrange Polynomial Definition

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- Given a function $f(x)$ to be integrated over some interval $[a, b]$

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- Given a function $f(x)$ to be integrated over some interval $[a, b]$
- Divide the function into n equal parts such that $f_n = f(x_n)$ and $h = \frac{b-a}{n}$
  - f_n is the function value at the point x_n
  - h (called the *step size*) is the length of each interval such that $x_{n+1} = x_n + h$

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$$\begin{aligned}\int_a^b f(x)\,dx &\approx \int_a^b P(x)\,dx \\ &\approx \int_a^b \left(\sum_{i=0}^{n} f(x_i)L_i(x)\right) dx \\ &\approx \sum_{i=0}^{n} f(x_i)\int_a^b L_i(x)\,dx\end{aligned} \tag{1}$$

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$$\approx \int_a^b \left(\sum_{i=0}^{n} f(x_i)L_i(x)\right) dx \tag{1}$$
$$\approx \sum_{i=0}^{n} f(x_i)\int_a^b L_i(x)\,dx$$

- *Note:* $\int_a^b L_i(x)\,dx$ can be expressed as w_i or the *weights*

Langrange Polynomial Definition

- Compute the value of the summation.

$$\sum_{i=0}^{n} f(x_i)w_i \tag{2}$$

General Quadrature Formula [2]

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$$\begin{aligned}\int_a^b f(x)\,dx &\approx \int_a^b P(x)\,dx \\ &= \int_a^b \sum_{k=0}^{n} \binom{i}{k} \Delta^k y_0\,dx\end{aligned}$$

General Quadrature Formula [2]

$$
\begin{aligned}
\int_a^b f(x)\,dx = \int_a^b \Big[y_0 &+ \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1) \\
&+ \frac{\Delta^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) \\
&+ \frac{\Delta^4 y_0}{4!h^4}(x-x_0)(x-x_1)(x-x_2)(x-x_3) + \dots \Big]\,dx
\end{aligned}
\tag{3}
$$

General Quadrature Formula [2]

Using the transformation $x = x_0 + hu$ or $u = \dfrac{x - x_0}{h}$ and $du = \dfrac{x}{h}\, dx$

$$\int_a^b f(x)\, dx = h \int_0^n \left[y_0 + u\Delta y_0 + \frac{\Delta^2 y_0}{2!} u(u-1) + \frac{\Delta^3 y_0}{4!} u(u-1)(u-2) + \frac{\Delta^4 y_0}{4!} u(u-1)(u-2)(u-3) + \ldots \right] du$$

General Quadrature Formula [2]

Integrating the right-hand side, we have:

$$\int_a^b f(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\Delta^2 y_0}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\right.$$
$$+ \frac{\Delta^3 y_0}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right) \tag{4}$$
$$\left. + \frac{\Delta^4 y_0}{4!}\left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right) + \dots\right]$$

Newton-Cotes Formulas

Closed Newton-Cotes Formulas

- A Newton-Cotes formula is **closed** if it uses the function value at all points
  - i.e. The interval may be $[x_0, x_{n-1}]$

Newton-Cotes Formulas

Open Newton-Cotes Formulas

- A Newton-Cotes formula is **open** if it does not use function values at the endpoints
  - i.e. The interval may be $[x_1, x_{n-2}]$
- Will not be discussed further in this report

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Examples of Closed Newton-Cotes Formulas
Trapezoid Rule

2-point closed Newton-Cotes formula $(n = 1)$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f_0 + f_1\right) \tag{5}$$

Examples of Closed Newton-Cotes Formulas

Simpson's 1/3 Rule

3-point closed Newton-Cotes formula ($n = 2$)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f_0 + 4f_1 + f_2\right) \tag{6}$$

Examples of Closed Newton-Cotes Formulas

Simpson's 3/8 Rule

4-point closed Newton-Cotes formula ($n = 3$)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right) \tag{7}$$

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History [1]

- Named after **George Boole**, an English mathematician
- Mistakenly called *Bode's Rule* due to a propagation of a typographical error

Outline

Derivation

Consider the function $y = f(x)$ over the interval $[x_0, \ldots, x_4]$. Boole's rule is an approximation to the integral of $f(x)$ over $[x_0, x_4]$.

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From Equation 4 with $a = x_0$ and $b = x_4$
Let $n = 4$, such that $f(x)$ can be approximated by a 4th degree polynomial

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Let $n = 4$, such that $f(x)$ can be approximated by a 4th degree polynomial

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_4} f(x)\,dx$$
$$= \int_a^{a+4h} f(x)\,dx$$

Derivation

$$= h\left[4f(a) + 8\Delta f_a + \left(\frac{64}{3} - 8\right)\frac{\Delta^2 f(a)}{2!} + (64 - 64 + 16)\frac{\Delta^3 f(a)}{3!}\right.$$

$$\left. + \left(\frac{1024}{5} - 384 + \frac{704}{3} - 48\right)\frac{\Delta^4 f(a)}{4!}\right]$$

$$= h\left[4f(a) + 8\{f(a+h) - f(a)\} + \frac{20}{3}\{f(a+2h) - 2f(a+h)\right.$$

$$+ f(a)\} + \frac{8}{3}\{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)\}$$

$$\left. + \frac{14}{45}\{f(a+4h) - 4f(a+3h) + 6f(a+2h) - 4f(a+h) + f(a)\}\right]$$

Derivation

$$
\begin{aligned}
&= \frac{h}{45}\left[14f(a) + 64f(a+h) + 24(a+2h) + 64f(a+3h) + 14f(a+4h)\right] \\
&= \frac{2h}{45}\left[7f(a) + 32f(a+h) + 12f(a+2h) + 32f(a+3h) + 7f(a+4h)\right] \\
&= \frac{2h}{45}\left[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\right]
\end{aligned}
$$

Derivation

Definition

Boole's Rule

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}\left[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\right] \tag{8}$$

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Solve the following using analytic method, Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule, and Boole's Rule:

- $\int_0^1 \cos x \, dx$
- $\int_2^4 \ln x \, dx$
- $\int_{-3}^1 6x^2 - 5x + 2 \, dx$
- $\int_1^2 x^4 + 4x^3 + 9x^2 + 2x + 10 \, dx$

Example Comparison [3]

Table: Results

Method	**Solution**			
Analytic	0.84147	2.15888	84	55.2
Trapezoidal	0.77015	2.07944	148	62
Simpson's 1/3	0.84177	2.15796	84	55.20833
Simpson's 3/8	0.84160	2.15846	84	55.20370
Boole's	**0.84147**	**2.15887**	**84**	**55.2**

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