Weddle's Rule Numerical Integration Method

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Outline I

- Background
 - Newton-Cotes Formulas
 - Closed Newton-Cotes Formulas

- 2 Weddle's Rule
 - History
 - Derivation

3 References





Newton-Cotes Formulas

 The Newton-Cotes formulas are a family of numerical integration techniques.





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- They are also known as Quadrature formulas





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- Given a function f(x) to be integrated over some interval [a, b]
- Divide the function into n equal parts such that $f_n = f(x_n)$ and $h = \frac{b-a}{n}$
 - f_n is the function value at the point x_n
 - h (called the *step size*) is the length of each interval such that $x_{n+1} = x_n + h$





Langrange Polynomial Definition

• Express the function as a Langrange Interpolating Polynomial P(x)





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Background

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• Note: $\int_a^b L_i(x) dx$ can be expressed as w_i or the weights





• Compute the value of the summation.

$$\sum_{i=0}^{n} f(x_i) w_i \tag{2}$$





Alternatively, we can express the approximation of the definite integral by Newton's forward difference interpolating polynomial

• Compute the value of the summation.

$$\sum_{i=0}^{n} f(x_i) w_i \tag{3}$$





Newton-Cotes Formulas Closed Newton-Cotes Formulas

- A Newton-Cotes formula is closed if it uses the function value at all points
 - i.e. The interval may be $[x_1, x_n]$





- A Newton-Cotes formula is open if it does not use function values at the endpoints
 - i.e. The interval may be $[x_2, x_{n-1}]$
- Will not be discussed further in this report





2-point closed Newton-Cotes formula (n = 1)

$$\int_{a}^{b} f(x)dx = \frac{h}{2}(f_1 + f_2) \tag{4}$$

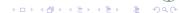




3-point closed Newton-Cotes formula (n = 2)

$$\int_{3}^{b} f(x)dx = \frac{h}{3} (f_1 + 4f_2 + f_3)$$
 (5)



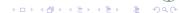


Examples of Closed Newton-Cotes Formulas Simpson's 3/8 Rule

4-point closed Newton-Cotes formula (n = 3)

$$\int_{3}^{b} f(x)dx = \frac{3h}{8} (f_1 + 3f_2 + 3f_3 + f_4)$$
 (6)





Examples of Closed Newton-Cotes Formulas Boole's Rule

5-point closed Newton-Cotes formula (n = 4)

$$\int_{3}^{b} f(x)dx = \frac{2h}{45} \left(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5\right) \tag{7}$$



- Mathematician who introduced the Weddle surface (unrelated to this report).
- Mathematics professor at the Royal Military College at Sandhurst





Derivation I

Let n=6From Equation 1 with $a = x_0$ and $b = x_0$

$$\int_{x_0}^{x_6} \approx \int_{x_0}^{x_6} P_6(x) dx$$

$$\approx \int_{x_0}^{x_6} \left(\sum_{i=0}^n f(x_i) L_i(x) \right) dx$$

$$\approx \int_{x_0}^{x_6} f_0 L_0(x) + f_1 L_1(x) + \dots + f_6 L_6(x) dx$$

Solving for L_0, \ldots, L_6

$$L_0 = \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)(x_0 - x_5)(x_0 - x_6)}$$





References I



Thomas Weddle. URL: https://en.wikipedia.org/wiki/Thomas_Weddle.

