Weddle's Rule

Numerical Integration Method

Harold R. Mansilla[1]

[1]Department of Physical Sciences and Mathematics
University of the Philippines Manila

December 5, 2018

Outline I

Newton-Cotes Formulas

- The Newton-Cotes formulas are a family of **numerical integration** techniques.

Newton-Cotes Formulas

- The Newton-Cotes formulas are a family of **numerical integration** techniques.
- They are also known as *Quadrature formulas*

Langrange Polynomial Definition

The general process of approximating the definite integral is as follows:

- Given a function $f(x)$ to be integrated over some interval $[a, b]$

Langrange Polynomial Definition

The general process of approximating the definite integral is as follows:

- Given a function $f(x)$ to be integrated over some interval $[a, b]$
- Divide the function into n equal parts such that $f_n = f(x_n)$ and $h = \frac{b-a}{n}$
  - f_n is the function value at the point x_n
  - h (called the *step size*) is the length of each interval such that $x_{n+1} = x_n + h$

Langrange Polynomial Definition

- Express the function as a **Langrange Interpolating Polynomial** $P(x)$

Langrange Polynomial Definition

- Express the function as a **Langrange Interpolating Polynomial** $P(x)$

$$\int_a^b f(x)dx \approx \int_a^b P(x)dx$$

Langrange Polynomial Definition

- Express the function as a **Langrange Interpolating Polynomial** $P(x)$

$$\int_a^b f(x)dx \approx \int_a^b P(x)dx$$
$$\approx \int_a^b \left(\sum_{i=0}^{n} f(x_i)L_i(x) \right) dx$$

Langrange Polynomial Definition

- Express the function as a **Langrange Interpolating Polynomial** $P(x)$

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b P(x)dx \\ &\approx \int_a^b \left(\sum_{i=0}^{n} f(x_i)L_i(x)\right) dx \\ &\approx \sum_{i=0}^{n} f(x_i)\int_a^b L_i(x)dx\end{aligned} \tag{1}$$

Langrange Polynomial Definition

- Express the function as a **Langrange Interpolating Polynomial** $P(x)$

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b P(x)dx \\ &\approx \int_a^b \left(\sum_{i=0}^{n} f(x_i)L_i(x)\right) dx \\ &\approx \sum_{i=0}^{n} f(x_i) \int_a^b L_i(x)dx\end{aligned} \tag{1}$$

- *Note:* $\int_a^b L_i(x)dx$ can be expressed as w_i or the *weights*

Langrange Polynomial Definition

- Compute the value of the summation.

$$\sum_{i=0}^{n} f(x_i)w_i \tag{2}$$

General Quadrature Formula [1]

Alternatively, we can express the approximation of the definite integral by *Newton's forward difference interpolating polynomial*

- Compute the value of the summation.

$$\sum_{i=0}^{n} f(x_i)w_i \tag{3}$$

Newton-Cotes Formulas

Closed Newton-Cotes Formulas

- A Newton-Cotes formula is **closed** if it uses the function value at all points
  - i.e. The interval may be $[x_1, x_n]$

Newton-Cotes Formulas

Open Newton-Cotes Formulas

- A Newton-Cotes formula is **open** if it does not use function values at the endpoints
  - i.e. The interval may be $[x_2, x_{n-1}]$
- Will not be discussed further in this report

Examples of Closed Newton-Cotes Formulas

Trapezoid Rule

2-point closed Newton-Cotes formula $(n = 1)$

$$\int_a^b f(x)dx = \frac{h}{2}(f_1 + f_2) \tag{4}$$

Examples of Closed Newton-Cotes Formulas

Simpson's Rule

3-point closed Newton-Cotes formula $(n = 2)$

$$\int_a^b f(x)dx = \frac{h}{3}(f_1 + 4f_2 + f_3) \tag{5}$$

Examples of Closed Newton-Cotes Formulas

Simpson's 3/8 Rule

4-point closed Newton-Cotes formula ($n = 3$)

$$\int_a^b f(x)dx = \frac{3h}{8}(f_1 + 3f_2 + 3f_3 + f_4) \tag{6}$$

Examples of Closed Newton-Cotes Formulas

Boole's Rule

5-point closed Newton-Cotes formula ($n = 4$)

$$\int_a^b f(x)dx = \frac{2h}{45}(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5) \qquad (7)$$

Thomas Weddle (1817-1853) [2]

- Mathematician who introduced the Weddle surface (unrelated to this report).
- Mathematics professor at the Royal Military College at Sandhurst

Derivation I

Let $n = 6$
From Equation 1 with $a = x_0$ and $b = x_n$

$$\begin{aligned}
\int_{x_0}^{x_6} &\approx \int_{x_0}^{x_6} P_6(x)dx \\
&\approx \int_{x_0}^{x_6} \left(\sum_{i=0}^{n} f(x_i)L_i(x) \right) dx \\
&\approx \int_{x_0}^{x_6} f_0 L_0(x) + f_1 L_1(x) + \cdots + f_6 L_6(x) dx
\end{aligned}$$

Solving for $L_0, \ldots, L_6$

$$L_0 = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)(x_0-x_6)}$$

References I

- *A General Quadrature Formula*. URL: https://nptel.ac.in/courses/122104018/node119.html.
- *Thomas Weddle*. URL: https://en.wikipedia.org/wiki/Thomas_Weddle.