

Boole's Rule

Numerical Integration Method

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Outline I

1 Background

- Newton-Cotes Formulas
- Closed Newton-Cotes Formulas

2 Boole's Rule

- History
- Derivation [4]
- Example Comparison [3]

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Newton-Cotes Formulas

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- They are also known as *Quadrature formulas*

Langrange Polynomial Definition

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- Given a function $f(x)$ to be integrated over some interval $[a, b]$

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- Given a function $f(x)$ to be integrated over some interval $[a, b]$
- Divide the function into n equal parts such that $f_n = f(x_n)$ and $h = \frac{b-a}{n}$
 - f_n is the function value at the point x_n
 - h (called the *step size*) is the length of each interval such that $x_{n+1} = x_n + h$



Langrange Polynomial Definition

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 \end{aligned} \tag{1}$$

- Note: $\int_a^b L_i(x) dx$ can be expressed as w_i or the *weights*



Langrange Polynomial Definition

- Compute the value of the summation.

$$\sum_{i=0}^n f(x_i)w_i \quad (2)$$



General Quadrature Formula [2]

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Let $f(x_k) = y_k$ be the nodal value at point x_k for $k = 0, 1, \dots, x_n$ where $x_0 = a$ and $x_n = x_0 + nh = b$



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$$\int_a^b f(x) dx \approx \int_a^b P(x) dx$$



General Quadrature Formula [2]

$$\int_a^b f(x) dx \approx \int_a^b \left[y_0 + \frac{\Delta y_0}{h} (x - x_0) + \frac{\Delta^2 y_0}{2! h^2} (x - x_0) (x - x_1) \right. \\ \left. + \frac{\Delta^3 y_0}{3! h^3} (x - x_0) (x - x_1) (x - x_2) \right. \\ \left. + \frac{\Delta^4 y_0}{4! h^4} (x - x_0) (x - x_1) (x - x_2) (x - x_3) + \dots \right] dx$$



General Quadrature Formula [2]

Using the transformation $x = x_0 + hu$ or letting $u = \frac{x - x_0}{h}$ and $du = \frac{1}{h} dx$

$$\begin{aligned} \int_a^b f(x) dx \approx h \int_0^n \left[y_0 + u\Delta y_0 + \frac{\Delta^2 y_0}{2!} u(u-1) \right. \\ \left. + \frac{\Delta^3 y_0}{4!} u(u-1)(u-2) + \frac{\Delta^4 y_0}{4!} u(u-1)(u-2)(u-3) \right. \\ \left. + \dots \right] du \end{aligned}$$



General Quadrature Formula [2]

Integrating the right-hand side, we have:

$$\begin{aligned} \int_a^b f(x) dx \approx h \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{\Delta^2 y_0}{2!} \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \right. \\ \left. + \frac{\Delta^3 y_0}{3!} \left(\frac{n^4}{4} - n^3 + n^2 \right) \right. \\ \left. + \frac{\Delta^4 y_0}{4!} \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2 \right) + \dots \right] \end{aligned}$$



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Definition

General Quadrature Formula

$$\begin{aligned}
 \int_a^b f(x) dx \approx h & \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{\Delta^2 y_0}{2!} \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \right. \\
 & + \frac{\Delta^3 y_0}{3!} \left(\frac{n^4}{4} - n^3 + n^2 \right) \\
 & \left. + \frac{\Delta^4 y_0}{4!} \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2 \right) + \dots \right]
 \end{aligned} \tag{3}$$



Newton-Cotes Formulas

Closed Newton-Cotes Formulas

- A Newton-Cotes formula is **closed** if it uses the function value at all points
 - i.e. The interval may be $[x_0, x_{n-1}]$



Newton-Cotes Formulas

Open Newton-Cotes Formulas

- A Newton-Cotes formula is **open** if it does not use function values at the endpoints
 - i.e. The interval may be $[x_1, x_{n-2}]$
- Will not be discussed further in this report



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Examples of Closed Newton-Cotes Formulas

Trapezoid Rule

2-point closed Newton-Cotes formula ($n = 1$)

$$\int_a^b f(x) dx = \frac{h}{2} (f_0 + f_1) \quad (4)$$



Examples of Closed Newton-Cotes Formulas

Simpson's 1/3 Rule

3-point closed Newton-Cotes formula ($n = 2$)

$$\int_a^b f(x) dx = \frac{h}{3} (f_0 + 4f_1 + f_2) \quad (5)$$



Examples of Closed Newton-Cotes Formulas

Simpson's 3/8 Rule

4-point closed Newton-Cotes formula ($n = 3$)

$$\int_a^b f(x) dx = \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) \quad (6)$$



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History [1]

- Named after **George Boole**, an English mathematician
- Mistakenly called *Bode's Rule* due to propagation of a typographical error

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Derivation

Consider the function $y = f(x)$ over the interval $[x_0, \dots, x_4]$.
Boole's rule is an approximation to the integral of $f(x)$ over $[x_0, x_4]$.



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From Equation 3 with $a = x_0$ and $b = x_4$

Let $n = 4$, such that $f(x)$ can be approximated by a 4th degree polynomial

$$\int_a^b f(x) dx = \int_{x_0}^{x_4} f(x) dx$$



Derivation

Consider the function $y = f(x)$ over the interval $[x_0, \dots, x_4]$.
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From Equation 3 with $a = x_0$ and $b = x_4$

Let $n = 4$, such that $f(x)$ can be approximated by a 4th degree polynomial

$$\begin{aligned}\int_a^b f(x) dx &= \int_{x_0}^{x_4} f(x) dx \\ &= \int_a^{a+4h} f(x) dx\end{aligned}$$



Derivation

$$\begin{aligned}
 &= h \left[4f(a) + 8\Delta f(a) + \left(\frac{64}{3} - 8 \right) \frac{\Delta^2 f(a)}{2!} + (64 - 64 + 16) \frac{\Delta^3 f(a)}{3!} \right. \\
 &+ \left. \left(\frac{1024}{5} - 384 + \frac{704}{3} - 48 \right) \frac{\Delta^4 f(a)}{4!} \right] \\
 &= h \left[4f(a) + 8 \{f(a+h) - f(a)\} + \frac{20}{3} \{f(a+2h) - 2f(a+h) \right. \\
 &+ f(a)\} + \frac{8}{3} \{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)\} \\
 &+ \left. \frac{14}{45} \{f(a+4h) - 4f(a+3h) + 6f(a+2h) - 4f(a+h) + f(a)\} \right]
 \end{aligned}$$



Derivation

$$\begin{aligned} &= \frac{h}{45} [14f(a) + 64f(a+h) + 24f(a+2h) + 64f(a+3h) + 14f(a+4h)] \\ &= \frac{2h}{45} [7f(a) + 32f(a+h) + 12f(a+2h) + 32f(a+3h) + 7f(a+4h)] \\ &= \frac{2h}{45} [7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4] \end{aligned}$$



Derivation

Definition

Boole's Rule

$$\int_{x_0}^{x_4} f(x) dx = \frac{2h}{45} [7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4] \quad (7)$$



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Solve the following using analytic method, Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule, and **Boole's Rule**:

- $\int_0^1 \cos x \, dx$
- $\int_2^4 \ln x \, dx$
- $\int_{-3}^1 6x^2 - 5x + 2 \, dx$
- $\int_1^2 x^4 + 4x^3 + 9x^2 + 2x + 10 \, dx$



Table: Results

Method	Solution			
Analytic	0.84147	2.15888	84	55.2
Trapezoidal	0.77015	2.07944	148	62
Simpson's 1/3	0.84177	2.15796	84	55.20833
Simpson's 3/8	0.84160	2.15846	84	55.20370
Boole's	0.84147	2.15887	84	55.2



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