Neural Network 기초 Assignment1

이름: 14기 이혜린

Part 1. 함수 (20 points)

1. Sigmoid를 z에 대해 미분하세요. (2 points)

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

$$\frac{\partial \sigma}{\partial z} = \frac{(-1)\cdot(-e^{-z})}{(1+e^{-z})^2} = \frac{e^{-z}}{(1+e^{-z})^2} = \frac{1}{(1+e^{-z})}\left(1-\frac{1}{1+e^{-z}}\right)$$

2. Mean Square Error를 w_i에 대해 편미분하세요. (3 points)

$$MSE = J(w) = \frac{1}{2}\sum(y_k - o_k)^2, \qquad o_k = \boldsymbol{w}_k^T\boldsymbol{x} + b_k$$

$$\frac{\partial J}{\partial w_i} = \frac{\partial J}{\partial o_i}\cdot\frac{\partial o_i}{\partial w_i} = (y_i - o_i)\cdot(-1)\cdot x_i = -(y_i - o_i)x_i$$

3. Logistic Regression의 Log Likelihood를 w_i에 대해 편미분하세요. (3 points)

$$\log likelihood = J(w) = -\sum\{y_k \log p_k + (1-y_k)\log(1-p_k)\},$$

$$p_k = \sigma(z_k), \qquad z_k = \boldsymbol{w}_k^T\boldsymbol{x} + b_k$$

$$\frac{\partial J}{\partial w_i} = \frac{\partial J}{\partial p_i}\cdot\underbrace{\frac{\partial p_i}{\partial z_i}}_{\text{1번참고}}\cdot\frac{\partial z_i}{\partial w_i} = -\left(\frac{y_i}{p_i} - \frac{1-y_i}{1-p_i}\right)\cdot\frac{1}{1+e^{-z_i}}\left(1-\frac{1}{1+e^{-z_i}}\right)\cdot x_i$$

$$= -\left(\frac{y_i}{p_i} - \frac{1-y_i}{1-p_i}\right)p_i(1-p_i)x_i = \boxed{-(y_i - p_i)x_i}$$

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4. 다음 식이 올바른 이유를 증명하세요. (5 points)

$$-\sum_{k=1}^{K} y_k \log(p_k) = -\log p_i\,, \qquad 1 \le i \le k$$

$k=1$, $-y_1 \log p_1 = -\log p_1 \longrightarrow y_1 = 1$

$k=2$, $-y_1 \log p_1 - y_2 \log p_2 = -\log p_1$ or $-y_1 \log p_1 - y_2 \log p_2 = -\log p_2 \longrightarrow (y_1, y_2) = (1, 0)$, $(y_1, y_2) = (0, 1)$

$k=3$, $(y_1, y_2, y_3) = (1, 0, 0)$ or $(y_1, y_2, y_3) = (0, 1, 0)$ or $(y_1, y_2, y_3) = (0, 0, 1)$

k값에 관계없이 y_k 중 하나만 1이고 나머지는 0이다.
그러므로 위 식은 성립한다.

5. Softmax-Cross Entropy를 z_i에 대해 편미분하세요. (7 points)

$$CE = -\sum y_k \log(p_k)\,, \qquad p_i = \frac{e^{z_i}}{\sum e^{z_k}}$$

$$\frac{\partial CE}{\partial z_i} = \frac{\partial CE}{\partial p_i} \cdot \frac{\partial p_i}{\partial z_i}$$

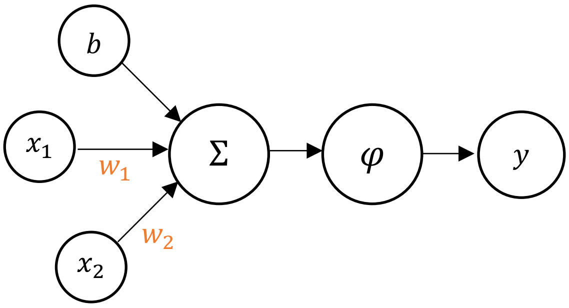
$$\frac{\partial p_i}{\partial z_i} = \partial \frac{e^{z_i}}{\Sigma e^{z_k}} \Big/ \partial z_i = \frac{e^{z_i} \Sigma e^{z_k} - e^{z_i} e^{z_i}}{(\Sigma e^{z_k})^2} = \frac{e^{z_i}}{\Sigma e^{z_k}} \left(\frac{\Sigma e^{z_k} - e^{z_i}}{\Sigma e^{z_k}} \right)$$

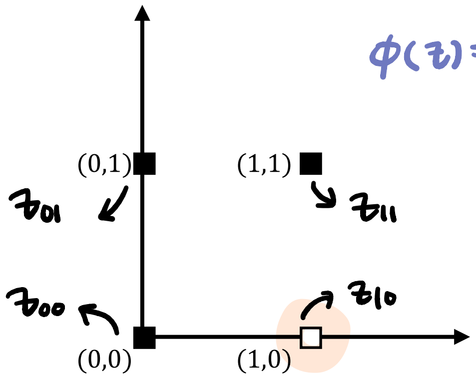
$$= \frac{e^{z_i}}{\Sigma e^{z_k}} \left(1 - \frac{e^{z_i}}{\Sigma e^{z_k}} \right) = p_i(1 - p_i)$$

$$\frac{\partial CE}{\partial z_i} = \frac{\partial CE}{\partial p_i} \cdot \frac{\partial p_i}{\partial z_i} = -\frac{y_i}{p_i} \cdot p_i(1-p_i) = \boxed{-y_i(1-p_i)}$$

Part 2. 퍼셉트론 (15 points)

다음과 같은 구조의 퍼셉트론과 ■(=1), □(=0)을 평면좌표상에 나타낸 그림이 있습니다.





$$\phi(z)=\begin{cases}0, & z<0\\ 1, & z\ge 0\end{cases}$$

1. ■, □를 분류하는 임의의 b, w를 선정하고 분류하는 과정을 보이세요. (5 points)

$(w_0, w_1, w_2) = (0.4,\ 0.1,\ -0.2)$ / $b=-1$

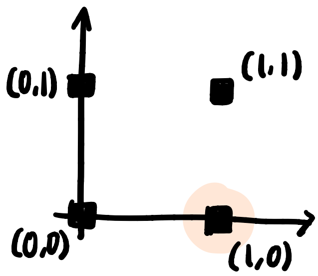
$z = w_0 + w_1x_1 + w_2x_2 = 0.4 + 0.1x_1 - 0.2x_2$

$z_{00} = 0.4 \rightarrow \phi(z_{00}) = 1$

$z_{01} = 0.2 \rightarrow \phi(z_{01}) = 1$

$z_{10} = 0.5 \rightarrow \phi(z_{10}) = 1$

$z_{11} = 0.3 \rightarrow \phi(z_{11}) = 1$



2. Perceptron 학습 규칙에 따라 임의의 학습률 η을 정하고 b, w를 한 번 업데이트해 주세요. (5 points)

$\hookrightarrow w_j \leftarrow w_j + \eta(\underbrace{y}_{\text{target}} - \phi)x$, $\eta = 0.05$

ex. $x = (-1, 1, 0)$

$x_0 = b$

$w_0 + \eta(y-0)x_0 = 0.4 + 0.05(0-1)\cdot(-1) = 0.45 \rightarrow w_0$

$w_1 + \eta(y-0)x_1 = 0.1 + 0.05(0-1)\cdot 1 = 0.05 \rightarrow w_1$

$w_2 + \eta(y-0)x_2 = -0.2 + 0.05(0-1)\cdot 0 = -0.2 \rightarrow w_2$

$\therefore (w_0, w_1, w_2) = (0.45,\ 0.05,\ -0.2)$

3. Adaline Gradient Descent에 따라 임의의 학습률 η을 정하고 b, w를 한 번 업데이트해 주세요. (5 points)

$\hookrightarrow w_j \leftarrow w_j + \eta(y^{(i)} - w^T x^{(i)})x_j^{(i)}$, $\eta = 0.05$

ex. $x = (-1, 1, 0)$

$w_0 + \eta(y^{(i)} - w^T x^{(i)})x_0^{(i)} = 0.4 + 0.05(0-(-0.3))(-1)$
$= 0.4 - 0.015 = 0.385 \rightarrow w_0$

$w_1 + \eta(y^{(i)} - w^T x^{(i)})x_1^{(i)} = 0.1 + 0.05(0-(-0.3))\cdot 1$
$= 0.1 + 0.015 = 0.115 \rightarrow w_1$

$w_2 + \eta(y^{(i)} - w^T x^{(i)})x_2^{(i)} = -0.2 + 0.05(0-(-0.3))\cdot 0$
$= -0.2 \rightarrow w_2$

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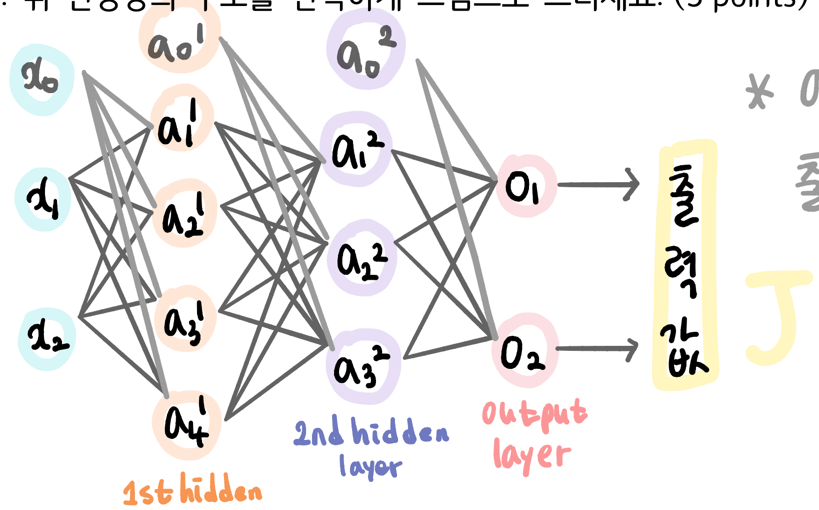
$\therefore (w_0, w_1, w_2) = (0.385,\ 0.115,\ -0.2)$

Part 3. 다층 퍼셉트론 (30 points)

Input Layer가 2차원, 첫 번째 Hidden Layer가 4차원 첫 번째 활성화 함수가 ReLU, 두 번째 Hidden Layer가 3차원, 두 번째 활성화 함수가 Sigmoid, Output Layer가 2차원인 다층 퍼셉트론 구조의 신경망이 있습니다.

1. 위 신경망의 구조를 간략하게 그림으로 그리세요. (5 points)



* a_0^1, x_0 는 bias 항.
출력층에는 bias항없음

2. Bias를 포함하여 각 Layer에 존재하는 Weight(Parameter)의 개수와 전체 Weight의 개수를 구하세요. (10 points)

1st hidden layer : 12개
2nd hidden layer : 15개
output layer : 6개

총 33개

3. 위 신경망을 식으로 나타날 때 필요한 함수, 벡터와 행렬을 정의하고 순전파 과정을 행렬식으로 표현하세요. (ex) input: $x^i = \left(x_1^{(i)}, x_2^{(i)}, \dots\right)^T$, x는 4x1차원) (15 points)

* bias항은 따로 'b' *
matrix로 표현

$$\underset{2\times1}{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \underset{4\times1}{b^1} = \begin{pmatrix} b_1^1 \\ b_2^1 \\ b_3^1 \\ b_4^1 \end{pmatrix}$$

$$\underset{4\times2}{W^1} = \begin{pmatrix} W_{11}^1 & W_{12}^1 \\ W_{21}^1 & W_{22}^1 \\ W_{31}^1 & W_{32}^1 \\ W_{41}^1 & W_{42}^1 \end{pmatrix}$$

$$\underset{4\times1}{z^1} = W^1 X + b^1$$

$$\underset{4\times1}{a^1} = \mathrm{Relu}(z^1)$$

$$\underset{3\times4}{W^2} = \begin{pmatrix} W_{11}^2 & W_{12}^2 & W_{13}^2 & W_{14}^2 \\ W_{21}^2 & W_{22}^2 & W_{23}^2 & W_{24}^2 \\ W_{31}^2 & W_{32}^2 & W_{33}^2 & W_{34}^2 \end{pmatrix}$$

$$\underset{3\times1}{b^2} = (b_1^2, b_2^2, b_3^2)^T$$

$$\underset{3\times1}{z^2} = W^2 a^1 + b^2$$

$$\underset{3\times1}{a^2} = \sigma(z^2)$$

$$\underset{2\times3}{W^3} = \begin{pmatrix} W_{11}^3 & W_{12}^3 & W_{13}^3 \\ W_{21}^3 & W_{22}^3 & W_{23}^3 \end{pmatrix}$$

$$\underset{2\times1}{z^3} = W^3 a^2$$

g: output activation fn

$$\underset{2\times1}{O} = g(z^3)$$

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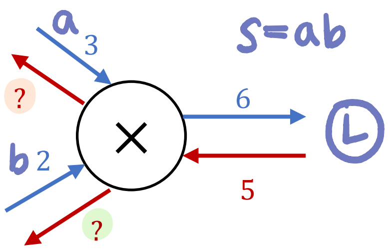
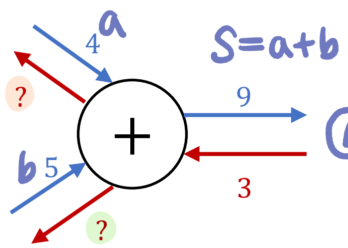
Part 4. 역전파 (35 points)

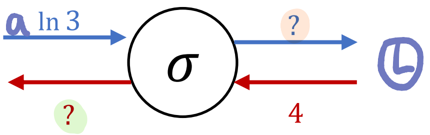
1. 다음 그림들의 물음표에 들어갈 숫자를 구하세요. (5 points)

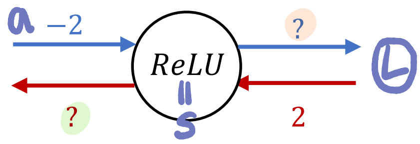




$$\frac{\partial L}{\partial a} = \frac{\partial L}{\partial s}\cdot\frac{\partial s}{\partial a} = 5\cdot b = 5\cdot 2 = 10 \qquad \frac{\partial L}{\partial a} = \frac{\partial L}{\partial s}\cdot\frac{\partial s}{\partial a} = 3\cdot 1 = 3$$

$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial s}\cdot\frac{\partial s}{\partial b} = 5\cdot a = 5\cdot 3 = 15 \qquad \frac{\partial L}{\partial b} = \frac{\partial L}{\partial s}\cdot\frac{\partial s}{\partial b} = 3\cdot 1 = 3$$





$$\sigma(\ln 3) = \frac{1}{1+e^{-\ln 3}} = \frac{3}{4} \qquad s(-2) = 0$$

$$\frac{\partial L}{\partial a} = \frac{\partial L}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial a} = 4\cdot\left[\frac{1}{1+e^{-a}}\left(1-\frac{1}{1+e^{-a}}\right)\Big|_{a=\ln 3}\right] = \frac{3}{4} \qquad \frac{\partial L}{\partial a} = \frac{\partial L}{\partial s}\cdot\frac{\partial s}{\partial a} = 2\cdot 0 = 0$$

2. 3-3에서 정의한 함수, 벡터와 행렬로 각 Layer에 존재하는 Bias값들의 업데이트를 행렬 식으로 표현하세요. (15 points)

$$\underset{2\times1}{\delta^3} = \frac{\partial J}{\partial z^3} = \begin{pmatrix} \partial J/\partial z_1^3 \\ \partial J/\partial z_2^3 \end{pmatrix} = \begin{pmatrix} \delta_1^3 \\ \delta_2^3 \end{pmatrix}, \quad \delta^3(a^2)^T = \begin{pmatrix} \delta_1^3 \\ \delta_2^3 \end{pmatrix}\begin{pmatrix} a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

$$\underset{2\times3}{W^3} \leftarrow W^3 - \eta\, \underset{2\times1}{\delta^3}\, \underset{1\times3}{(a^2)^T}$$

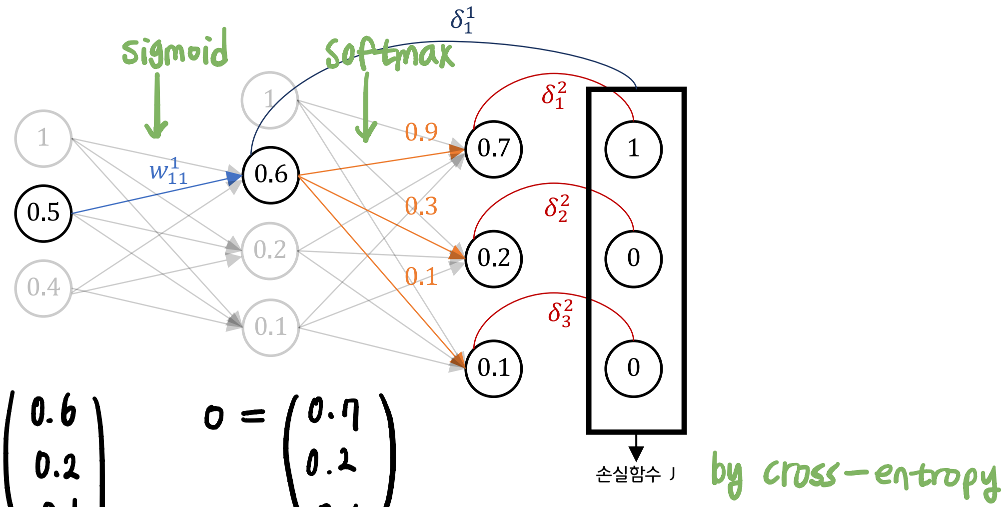
$$\underset{3\times1}{\delta^2} = \frac{\partial J}{\partial z^2} = \frac{\partial a^2}{\partial z^2}\frac{\partial z^3}{\partial a^2}\cdot\frac{\partial J}{\partial z^3} = \underset{3\times1}{\begin{pmatrix} \sigma'(a_1^2) \\ \sigma'(a_2^2) \\ \sigma'(a_3^2) \end{pmatrix}} \odot \underbrace{\begin{pmatrix} W_{11}^3 & W_{21}^3 \\ W_{12}^3 & W_{22}^3 \\ W_{13}^3 & W_{23}^3 \end{pmatrix}}_{(W^3)^T} \begin{pmatrix} \delta_1^3 \\ \delta_2^3 \end{pmatrix} = \begin{pmatrix} \delta_1^2 \\ \delta_2^2 \\ \delta_3^2 \end{pmatrix}$$

$$\underset{3\times4}{W^2} \leftarrow W^2 - \eta\, \underset{3\times1}{\delta^2}\, \underset{1\times4}{(a^1)^T}$$

$$\underset{4\times1}{\delta^1} = \frac{\partial J}{\partial z^1} = \frac{\partial a^1}{\partial z^1}\cdot\frac{\partial z^2}{\partial a^1}\cdot\frac{\partial J}{\partial z^2} = \begin{pmatrix} \mathrm{ReLU}'(a_1^1) \\ \mathrm{ReLU}'(a_2^1) \\ \mathrm{ReLU}'(a_3^1) \\ \mathrm{ReLU}'(a_4^1) \end{pmatrix} \odot \begin{pmatrix} W_{11}^2 & W_{21}^2 & W_{31}^2 \\ W_{12}^2 & W_{22}^2 & W_{32}^2 \\ W_{13}^2 & W_{23}^2 & W_{33}^2 \\ W_{14}^2 & W_{24}^2 & W_{34}^2 \end{pmatrix} \begin{pmatrix} \delta_1^2 \\ \delta_2^2 \\ \delta_3^2 \end{pmatrix} = \begin{pmatrix} \delta_1^1 \\ \delta_2^1 \\ \delta_3^1 \\ \delta_4^1 \end{pmatrix}$$

$$\underset{4\times2}{W^1} \leftarrow \underset{4\times2}{W^1} - \eta\, \underset{4\times1}{\delta^1}\, \underset{1\times2}{X^T}$$

3. 다음 그림에서 Loss Function은 Cross Entropy, Output Layer의 Activation Function은 Softmax, Hidden Layer의 Activation Function은 Sigmoid이고 Learning Rate는 0.05일 때, 각 δ의 값과 w_{11}^{1}의 변화량을 구하세요. (각 노드의 숫자는 활성화 이후의 숫자입니다.) (15 points)



$$X = \begin{pmatrix} 0.5 \\ 0.4 \end{pmatrix} \qquad a^1 = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.1 \end{pmatrix} \qquad o = \begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix}$$

$$w_{\cdot 1}^2 - \eta \delta^2 (a_1^1)$$

$$(w_{11}^2 \ w_{21}^2 \ w_{31}^2)^T = (0.9 \ \ 0.3 \ \ 0.1)^T = w_{\cdot 1}^2$$

$$\delta^2 = \frac{\partial J}{\partial z^2} = \begin{pmatrix} -1(1-0.7) \\ 0(1-0.2) \\ 0(1-0.1) \end{pmatrix} = \begin{pmatrix} -0.3 \\ 0 \\ 0 \end{pmatrix}$$

$$w_{\cdot 1}^2 - \eta \delta^2 a_1^1 = \begin{pmatrix} 0.9 \\ 0.3 \\ 0.1 \end{pmatrix} - 0.05 \begin{pmatrix} -0.3 \\ 0 \\ 0 \end{pmatrix} 0.6 = \begin{pmatrix} 0.909 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow w_{\cdot 1}^2$$

$$\delta^1 = \frac{\partial J}{\partial z^1} = \frac{\partial a^1}{\partial z^1} \cdot \frac{\partial z^2}{\partial a^1} \cdot \frac{\partial o}{\partial z^2} \cdot \frac{\partial J}{\partial o} = \begin{pmatrix} \sigma'(z_1^1) \\ \sigma'(z_2^1) \\ \sigma'(z_3^1) \end{pmatrix} \odot (w_{\cdot 1}^2)^T \delta^2 = \begin{pmatrix} \sigma'(z_1^1) \cdot (-0.27027) \\ 0 \\ 0 \end{pmatrix}$$

$$\delta_1^1 = -0.27027\, \sigma'(z_1^1)$$

$$\begin{aligned} w_{11}^1 \leftarrow w_{11}^1 + \Delta w_{11}^1 &= w_{11}^1 - \eta \delta_1^1 x_1 \\ &= w_{11}^1 - 0.05 \times (-0.27027\, \sigma'(z_1^1)) \times 0.5 \\ &\approx w_{11}^1 + 0.00682\, \sigma'(z_1^1) \end{aligned}$$

$$\therefore \ \Delta w_{11}^1 = 0.00682\, \sigma'(z_1^1)$$

고생하셨습니다~

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