Locally Stationary Processes

Rainer Dahlhaus Institut für Angewandte Mathematik Universität Heidelberg Im Neuenheimer Feld 294 69120 Heidelberg Germany

Abstract

The article contains an overview over locally stationary processes. At the beginning time varying autoregressive processes are discussed in detail - both as as a deep example and an important class of locally stationary processes. In the next section a general framework for time series with time varying finite dimensional parameters is discussed with special emphasis on nonlinear locally stationary processes. Then the paper focusses on linear processes where a more general theory is possible. First a general definition for linear processes is given and time varying spectral densities are discussed in detail. Then the Gaussian likelihood theory is presented for locally stationary processes. In the next section the relevance of empirical spectral processes for locally stationary time series is discussed. Empirical spectral processes play a major role in proving theoretical results and provide a deeper understanding of many techniques. The article concludes with an overview of other results for locally stationary processes.

Keywords: locally stationary process, time varying parameter parameter, local likelihood, derivative process, time varying autoregressive process, shape curve, empirical spectral process, time varying spectral density

Acknowledgement: I am grateful to Suhasini Subba Rao for helpful comments on an earlier version which lead to significant improvements.

1 Introduction

Stationarity has played a major role in time series analysis for several decades. For stationary processes there exist a large variety of models and powerful methods, such as bootstrap methods or methods based on the spectral density. Furthermore, there are important mathematical tools such as the ergodic theorem or several central limit theorems. As an example we mention the likelihood theory for Gaussian processes which is well developed.

During recent years the focus has turned to nonstationary time series. Here the situation is more difficult: First, there exists no natural generalization from stationary to nonstationary time series and second, it is often not clear how to set down a meaningful asymptotics for nonstationary processes. An exception are nonstationary models which are generated by a time invariant generation mechanism – for examples integrated or cointegrated models. These models have attracted a lot of attention during recent years. For general nonstationary processes ordinary asymptotic considerations are often contradictory to the idea of nonstationarity since future observations of a nonstationary process may not contain any information at all on the probabilistic structure of the process at present. For this reason the theory of locally stationary processes is based on infill asymptotics originating from nonparametric statistics.

As a consequence valuable asymptotic concepts such as consistency, asymptotic normality, efficiency, LAN-expansions, neglecting higher order terms in Taylor expansions, etc. can be used in the theoretical treatment of statistical procedures for such processes. This leads to several meaningful results also for the original non-rescaled case such as the comparison of different estimates, the approximations for the distribution of estimates and bandwidth selection (for a detailed example see Remark 2.3).

The type of processes which can be described with this infill asymptotics are processes which locally at each time point are close to a stationary process but whose characteristics (covariances, parameters, etc.) are gradually changing in an unspecific way as time evolves. The simplest example for such a process may be an AR(p)-process whose parameters are varying in time. The infill asymptotic approach means that time is rescaled to the unit interval. For time varying AR-processes this is explained in detail in the next section. Another example are GARCH-processes which have recently been investigated by several authors – see Section 3.

The idea of having locally approximately a stationary process was also the starting point of Priestley's theory of processes with evolutionary spectra (Priestley (1965) – see also Priestley (1988), Granger and Hatanaka (1964), Tjøstheim (1976) and Mélard and Herteleer-

de-Schutter (1989) among others). Priestley considered processes having a time varying spectral representation

$$X_t = \int_{-\pi}^{\pi} \exp(i\lambda t) \, \tilde{A}_t(\lambda) \, d\xi(\lambda), \quad t \in \mathbf{Z}$$

with an orthogonal increment process $\xi(\lambda)$ and a time varying transfer function $\tilde{A}_t(\lambda)$. (Priestley mainly looked at continuous time processes, but the theory is the same). Also within this approach asymptotic considerations (e.g. for judging the efficiency of a local covariance estimator) are not possible or meaningless from an applied view. Using the above mentioned infill asymptotics means in this case basically to replace $\tilde{A}_t(\lambda)$ with some function $A(t/T, \lambda)$ – see (78).

Beyond the above cited references on processes with evolutionary spectra there has also been work on processes with time varying parameters which does not use the infill asymptotics discussed in this paper (cf. Subba Rao (1970); Hallin (1986) among others). Furthermore, there have been several papers on inference for processes with time varying parameters — mainly within the engineering literature (cf. Grenier (1983), Kayhan et.al. (1994) among others).

The paper is organized as follows: In Section 2 we start with time varying autoregressive processes as a deep example and an important class of locally stationary processes. There we mark many principles and problems addressed at later stages with higher generality. In Section 3 we present a more general framework for time series with time varying finite dimensional parameters and show how nonparametric inference can be done and theoretically handled. We also introduce derivative processes which play a major role in the derivations. The results cover in particular nonlinear processes such as GARCH-processes with time varying parameters.

If one restrict to linear processes or even more to Gaussian processes then a much more general theory is possible which is developed in the subsequent sections. In Section 4 we give a general definition for linear processes and discuss time varying spectral densities in detail. Section 5 then contains the Gaussian likelihood theory for locally stationary processes. In Section 6 we discuss the relevance of empirical spectral processes for locally stationary time series. Empirical spectral processes play a major role in proving theoretical results and provide a deeper understanding of many techniques.

2 Time varying autoregressive processes – a deep example

We now discuss time varying autoregressive processes in detail. In particular we mark many principles and problems addressed at later stages with higher generality. Consider the time varying AR(1) process

$$X_t + \alpha_t X_{t-1} = \sigma_t \,\varepsilon_t \qquad \text{with } \varepsilon_t \text{ iid } \mathcal{N}(0, 1).$$
 (1)

We now apply infill asymptotics that is we rescale the parameter curves α_t and σ_t to the unit interval. This means that we replace them by $\alpha(\frac{t}{T})$ and $\sigma(\frac{t}{T})$ with curves $\alpha(\cdot):[0,1] \to (-1,1)$ and $\sigma(\cdot):[0,1] \to (0,\infty)$ leading in the general AR(p)-case to the definition given in (2) below. Formally this results in replacing X_t by a triangular array of observations $(X_{t,T}; t = 1, \ldots, T; T \in \mathbb{N})$ where T is the sample size.

We now indicate again the reason for this rescaling. Suppose we fit the parameteric model $\alpha_{\theta,t} := b + ct + dt^2$ to the nonrescaled model (1) which we assume to be observed for t=1,...,T. It is easy to construct different estimators for the parameters (e.g. the least squares estimator, the maximum likelihood estimator or a moment estimator) but it is nearly impossible to derive the finite sample properties of these estimators. On the other hand classical non-rescaled asymptotic considerations for comparing these estimators make no sense since with $t \to \infty$ also $\alpha_{\theta,t} \to \infty$ while e.g. $|\alpha_t|$ may be less than one within the observed segment – i.e. the resulting asymptotic results are without any relevance for the observed stretch of data. By rescaling α_t and σ_t to the unit interval as described above we overcome these problems. As T tends to infinity more and more observations of each local structure become available and we obtain a reasonable framework for a meaningful asymptotic analysis of statistical procedures allowing to retain such powerful tools as consistency, asymptotic normality, efficiency, LAN-expansions, etc. for nonstationary processes. For example the results on asymptotic normality of an estimator obtained in this framework may be used to approximate the distribution of the estimator in the finite sample situation. It is important to note that classical asymptotics for stationary processes arises as a special case of this infill asymptotics in case where all parameter curves are constant.

Unfortunately infill asymptotics does not describe the physical behavior of the process as $T \to \infty$. This may be unusual for time series analysis but it has been common in other branches of statistics for many years. We remark that all statistical methods and procedures stay the same or can easily be translated from the rescaled processes to the original non-rescaled processes. A more complicated example on how the results of the rescaled case transfer to the non-rescaled case is given in Remark 2.3.

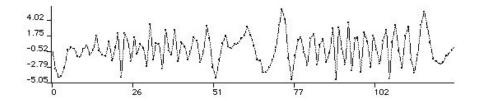


Figure 1: T=128 realizations of a time varying AR(2)-model

In the following we therefore consider time varying autoregressive (tvAR(p)) processes defined by

$$X_{t,T} + \sum_{j=1}^{p} \alpha_j(\frac{t}{T}) X_{t-j,T} = \sigma(\frac{t}{T}) \varepsilon_t, \ t \in \mathbf{Z}$$
 (2)

where the ε_t are independent random variables with mean zero and variance 1. We assume $\sigma(u) = \sigma(0)$, $\alpha_j(u) = \alpha_j(0)$ for u < 0 and $\sigma(u) = \sigma(1)$, $\alpha_j(u) = \alpha_j(1)$ for u > 1. In addition we usually assume some smoothness conditions on $\sigma(\cdot)$ and the $\alpha_j(\cdot)$. In addition one may include a time varying mean by replacing $X_{t-j,T}$ in (2) by $X_{t-j,T} - \mu(\frac{t-j}{T})$ – see Section 7.6.

In some neighborhood of a fixed time point $u_0 = t_0/n$ the process $X_{t,T}$ can be approximated by the stationary process $\tilde{X}_t(u_0)$ defined by

$$\tilde{X}_t(u_0) + \sum_{j=1}^p \alpha_j(u_0) \ \tilde{X}_{t-j}(u_0) = \sigma(u_0) \, \varepsilon_t, \ t \in \mathbf{Z}.$$
(3)

It can be shown (see Section 3) that we have under suitable regularity conditions

$$\left|X_{t,T} - \tilde{X}_t(u_0)\right| = O_p\left(\left|\frac{t}{T} - u_0\right| + \frac{1}{T}\right) \tag{4}$$

which justifies the notation "locally stationary process". $X_{t,T}$ has an unique time varying spectral density which is locally the same as the spectral density of $\tilde{X}_t(u)$, namely

$$f(u,\lambda) := \frac{\sigma^2(u)}{2\pi} \left| 1 + \sum_{j=1}^p \alpha_j(u) \exp(-ij\lambda) \right|^{-2}$$
 (5)

(see Example 4.2). Furthermore it has locally in some sense the same autocovariance

$$c(u,j) := \int_{-\pi}^{\pi} e^{ij\lambda} f(u,\lambda) d\lambda, \quad j \in \mathbf{Z}$$

since $cov(X_{[uT],T}, X_{[uT]+k,T}) = c(u,k) + O(T^{-1})$ uniformly in u and k (cf.(73)). This justifies to term c(u,k) the local covariance function of $X_{t,T}$ at time u = t/T.

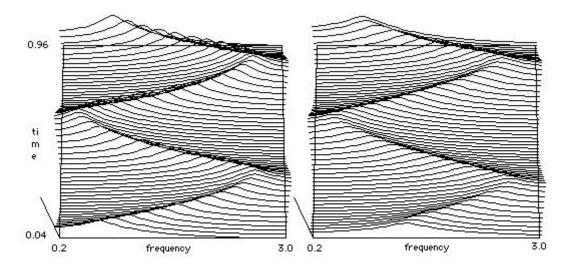


Figure 2: True and estimated time varying spectrum of a tvAR(2)-process

As an example Figure 1 shows T = 128 observations of a tvAR(2)-process with mean 0 and parameters $\sigma(u) \equiv 1$, $\alpha_1(u) \equiv -1.8\cos(1.5 - \cos 4\pi u)$, $\alpha_2(u) = 0.81$ and Gaussian innovations ε_t . The parameters are chosen in a way such that for fixed u the complex roots of the characteristic polynomial are $\frac{1}{0.9} \exp[\pm i(1.5 - \cos 4\pi u)]$, that is they are close to the unit circle and their phase varies cyclically with u. As could be expected from these roots the observations show a periodic behavior with time varying period-length. The left picture of Figure 2 shows the true time varying spectrum of the process. One clearly sees that the location of the peak is also time varying (it is located at frequency $1.5 - \cos 4\pi u$).

1. Local estimation by stationary methods on segments

An ad-hoc method which works in nearly all cases for locally stationary processes is to do inference via stationary methods on segments. The idea is that the process $X_{t,T}$ is almost stationary on a reasonably small segment $\{t: |t/T - u_0| \leq b/2\}$. The parameter of interest (or the correlation, spectral density, etc) is estimated by some classical method and the resulting estimate is assigned to the midpoint u_0 of the segment. By shifting the segment this finally leads to an estimate of the unknown parameter curve (time varying correlation, time varying spectral density, etc). An important modification of this method is obtained when more weight is put on data in the center of the interval than at the edges. This can often be achieved by using a data taper on the segment or by using a kernel type estimate. Since we use observations from the process $X_{t,T}$ (instead of $\tilde{X}_t(u_0)$) the procedure causes a

Since we use observations from the process $X_{t,T}$ (instead of $\tilde{X}_t(u_0)$) the procedure causes a bias which depends on the degree of non-stationarity of the process on the segment. It is

possible to evaluate this bias and to use the resulting expression for an optimal choice of the segment length. To demonstrate this we now discuss the estimation of the AR coefficient functions by classical Yule-Walker estimates on segments. Since the approximating process $\tilde{X}_t(u_0)$ is stationary we obtain from (3) that the Yule-Walker equations hold locally at time u_0 , that is we have with $\alpha(u_0) := (\alpha_1(u_0), ..., \alpha_p(u_0))'$

$$\alpha(u_0) = -R(u_0)^{-1} r(u_0)$$
 and $\sigma^2(u_0) = c(u_0, 0) + \alpha(u_0)' r(u_0)$ (6)

where $r(u_0) := (c(u_0, 1), ..., c(u_0, p))'$ and $R(u_0) := \{c(u_0, i - j)\}_{i,j=1,...,p}$.

To estimate $\alpha(u_0)$ we use the classical Yule Walker estimator on the segment $[u_0T] - N/2 + 1, \ldots, [u_0T] + N/2$ (ordinary time) or on $[u_0 - b_T/2, u_0 + b_T/2]$ (rescaled time with bandwidth $b_T := N/T$), that is

$$\hat{\alpha}_T(u_0) = -\hat{R}_T(u_0)^{-1} \hat{r}_T(u_0) \quad \text{and} \quad \hat{\sigma}_T^2(u_0) = \hat{c}_T(u_0, 0) + \hat{\alpha}_T(u_0)' \hat{r}_T(u_0) \quad (7)$$

where $\hat{r}_T(u_0) := (\hat{c}_T(u_0, 1), ..., \hat{c}_T(u_0, p))'$ and $\hat{R}_T(u_0) := \{\hat{c}_T(u_0, i - j)\}_{i,j=1,...,p}$ with some covariance estimator $\hat{c}_T(u_0, j)$.

Before we discuss the properties of this estimator we first discuss different covariance estimates and their properties.

2. Local covariance estimation

The covariance estimate with data taper on the segment $[u_0T]-N/2+1,\ldots,[u_0T]+N/2$ is

$$\hat{c}_T(u_0, k) := \frac{1}{H_N} \sum_{\substack{s,t=1\\s-t=i}}^N h(\frac{s}{N}) h(\frac{t}{N}) X_{[u_0T] - \frac{N}{2} + s, T} X_{[u_0T] - \frac{N}{2} + t, T}.$$
(8)

where $h:[0,1]\to \mathbf{R}$ is a data taper with h(x)=h(1-x), $H_N:=\sum_{j=0}^{N-1}h^2(\frac{j}{N})\sim N\int_0^1h^2(x)\,dx$ is the normalizing factor. The data taper usually is largest at x=1/2 and decays slowly to 0 at the edges. For $h(x)=\chi_{(0,1]}(x)$ we obtain the classical non-tapered covariance estimate.

An asymptotically equivalent (and from a certain viewpoint more intuitive estimator) is the kernel density estimator

$$\tilde{c}_T(u_0, k) := \frac{1}{b_T T} \sum_t K\left(\frac{u_0 - (t + k/2)/T}{b_T}\right) X_{t,T} X_{t+k,T} \tag{9}$$

where $K: \mathbf{R} \to [0, \infty)$ is a kernel with K(x) = K(-x), $\int K(x)dx = 1$, K(x) = 0 for $x \notin [-1/2, 1/2]$ and b_T is the bandwidth. Also equivalent is

$$\tilde{\tilde{c}}_T(u_0, i, j) := \frac{1}{b_T T} \sum_{t} K\left(\frac{u_0 - t/T}{b_T}\right) X_{t-i, T} X_{t-j, T}$$
(10)

with i-j=k which appears in least square regression – cf. Example 3.1(i). If $K(x)=h(x)^2$ all three estimators are equivalent in the sense that they lead to the same asymptotic bias, variance and mean squared error. For reasons of clarity a few remarks are in order:

- 1) The classical stationary method on a segment is in this case the estimator without data taper which is the same as the kernel estimator with a rectangular kernel.
- 2) A first step towards a better estimate (as it is proved below) is to put higher weights in the middle and lower weights at the edges of the observation domain in order to cope in a better way with the nonstationarity of $X_{t,T}$ on the segment. In this context this may be either achieved by using a kernel estimate or a data-taper which is asymptotically equivalent. This is straightforward for local covariance estimates and local Yule-Walker estimates and can usually also be applied to other estimation problems.
- 3) Data-tapers have also been used for stationary time series (in particular in spectral estimation, but also with Yule Walker estimates and covariance estimation where they give positive definite autocovariances with a lower bias). Thus the reason for using data-tapers for segment estimates is twofold: reducing the bias due to nonstationarity on the segment and reducing the (classical) bias of the procedure as a stationary method.

We now determine the mean squared error of the above estimators. Furthermore, we determine the optimal segment length N and show that weighted estimates are better than ordinary estimates.

Theorem 2.1 Suppose $X_{t,T}$ is locally stationary with mean 0. Under suitable regularity conditions (in particular second order smoothness of $c(\cdot, k)$) we have for $\hat{c}_T(u_0, k)$, $\tilde{c}_T(u_0, k)$ and $\tilde{c}_T(u_0, i, j)$ with $K(x) = h(x)^2$ and $b_T = N/T$

(i)
$$\mathbf{E}\,\hat{c}_T(u_0,k) = c(u_0,k) + \frac{1}{2}\,b_T^2 \int x^2 K(x)\,dx \left[\frac{\partial^2}{\partial^2 u}\,c(u_0,k)\right] + o(b_T^2) + O\left(\frac{1}{b_T T}\right)$$

and

(ii)
$$\operatorname{var}(\hat{c}_T(u_0, k)) = \frac{1}{b_T T} \int_{-1/2}^{1/2} K(x)^2 dx \sum_{\ell = -\infty}^{\infty} c(u_0, \ell) \left[c(u_0, \ell) + c(u_0, \ell + 2k) \right] + o\left(\frac{1}{b_T T}\right).$$

PROOF. (i) see Dahlhaus (1996c), (ii) is omitted (the form of the asymptotic variance is the same as in the stationary case).

Note that the above bias of order b_T^2 is solely due to nonstationarity which is measured by $\frac{\partial^2}{\partial u^2}c(u_0,k)$. If the process is stationary this second derivative is zero and the bias disappears. The bandwidth b_T may now be chosen to minimize the mean squared error.

Remark 2.2 (Minimizing the mean squared error)

Let $\mu(u_0) := \frac{\partial^2}{\partial^2 u_0} c(u_0, k)$, $\tau(u_0) := \sum_{\ell=-\infty}^{\infty} c(u_0, \ell) \left[c(u_0, \ell) + c(u_0, \ell+2k) \right]$, $d_K := \int x^2 K(x) dx$ and $v_K := \int K(x)^2 dx$. Then we have for the mean squared error

$$\mathbf{E} \left| \hat{c}_T(u_0, k) - c(u_0, k) \right|^2 = \frac{b^4}{4} d_K^2 \mu(u_0)^2 + \frac{1}{bT} v_K \tau(u_0) + o(b^4 + \frac{1}{bT}). \tag{11}$$

It can be shown (cf. Priestley, 1981, Chapter 7.5) that this MSE gets minimal for

$$K(x) = K_{out}(x) = 6x(1-x), \quad 0 \le x \le 1$$
 (12)

and

$$b = b_{opt}(u_0) = C(K_{opt})^{1/5} \left[\frac{\tau(u_0)}{\mu(u_0)^2} \right]^{1/5} T^{-1/5}$$
(13)

where $C(K) = v_K/d_K^2$. In this case we have with $c(K) = v_K d_K^{1/2}$

$$T^{4/5} \mathbf{E} \left| \hat{c}_T(u_0, k) - c(u_0, k) \right|^2 = \frac{5}{4} c(K_{opt})^{4/5} \mu(u_0)^{2/5} \tau(u_0)^{4/5} + o(1).$$
 (14)

 $\mu(u_0) = \frac{\partial^2}{\partial^2 u_0} c(u_0, k)$ measures the "degree of nonstationarity" while $\tau(u_0)$ measures the variability of the estimate at time u_0 . The segment length $N_{opt} = b_{opt}T$ gets larger if $\mu(u_0)$ gets smaller, i.e. if the process is closer to stationarity (in this case: if the k-th order covariance is more constant/more linear in time). At the same time the mean squared error decreases. The results are similar to kernel estimation in nonparametric regression. A yet unsolved problem is how to adaptively determine the bandwidth from the observed process.

3. Segment selection and asymptotic mean squared-error for local Yule-Walker estimates

For the local Yule-Walker estimates from (7) with the covariances $\hat{c}_T(u_0, k)$ as defined in (8) Dahlhaus and Giraitis (1998) have proved (see also Example 3.7)

$$\mathbf{E}\,\hat{\boldsymbol{\alpha}}_{T}(u_{0}) = \boldsymbol{\alpha}(u_{0}) - \frac{b^{2}}{2}\,d_{K}\,\boldsymbol{\mu}(u_{0}) + o(b^{2})$$

with

$$\boldsymbol{\mu}(u_0) = R(u_0)^{-1} \left[\left(\frac{\partial^2}{\partial u^2} R(u) \right) \boldsymbol{\alpha}(u_0) + \left(\frac{\partial^2}{\partial u^2} r(u) \right) \right]_{u=u_0}$$

and

$$\operatorname{var}(\hat{\alpha}_{T}(u_{0})) = \frac{1}{bT} v_{K} \sigma^{2}(u_{0}) R(u_{0})^{-1} + o\left(\frac{1}{bT}\right).$$

Thus, we obtain for $\mathbf{E} \|\hat{\boldsymbol{\alpha}}_T(u_0) - \boldsymbol{\alpha}(u_0)\|^2$ the same expression as in (11) with $\tau(u_0) = \sigma^2(u_0) \operatorname{tr}\{R(u_0)^{-1}\}$ and $\mu(u_0)^2$ replaced by $\|\boldsymbol{\mu}(u_0)\|^2$. With these changes the optimal bandwidth is given by (13) and the optimal mean squared error by (14).

Remark 2.3 (Implications for non-rescaled processes) Suppose that we observe data from a (non-rescaled) tvAR(p)-process

$$X_t + \sum_{j=1}^p \alpha_{tj} \ X_{t-j} = \sigma_t \ \varepsilon_t, \ t \in \mathbf{Z}.$$
 (15)

In order to estimate α_t at some time t_0 we may use the segment Yule-Walker estimator as given in (7). The theoretically optimal segment length is given by (13) as

$$N_{opt}(u_0) = C(K_{opt})^{1/5} \left[\frac{\tau(u_0)}{\|\boldsymbol{\mu}(u_0)\|^2} \right]^{1/5} T^{4/5}$$
(16)

which at first sight depends on T and the rescaling.

Suppose that we have parameter functions $\tilde{a}_j(\cdot)$ and some $T > t_0$ with $\tilde{a}_j(\frac{t_0}{T}) = \alpha_j(t_0)$ (i.e. the original function has been rescaled to the unit interval) and we denote by \tilde{R} , \tilde{r} and $\tilde{\alpha}$ the corresponding parameters in the rescaled world (i.e. $\tilde{R}(u_0) = R(t_0)$ etc.). Then

$$\tau(u_0) = \tilde{\sigma}^2(u_0) \operatorname{tr} \{ \tilde{R}(u_0)^{-1} \} = \sigma^2(t_0) \operatorname{tr} \{ R(t_0)^{-1} \}$$

and (with the second order difference as an approximation of the second derivative)

$$\mu(u_0) = \tilde{R}(u_0)^{-1} \left[\left(\frac{\partial^2}{\partial u^2} \tilde{R}(u) \right) \tilde{\boldsymbol{\alpha}}(u_0) + \left(\frac{\partial^2}{\partial u^2} \tilde{r}(u) \right) \right]_{u=u_0}$$

$$\approx R(t_0)^{-1} \left[\frac{R(t_0) - 2R(t_0 - 1) + R(t_0 - 2)}{1/T^2} \boldsymbol{a}(t_0) + \frac{r(t_0) - 2r(t_0 - 1) + r(t_0 - 2)}{1/T^2} \right]$$

Plugging this into (16) reveals that T drops out completely and the optimal segment length can completely be determined in terms of the original non-rescaled process. This is a nice example on how the asymptotic considerations in the rescaled world can be transferred with benefit to the original non-rescaled world.

These considerations justify the asymptotic approach of this paper: While it is not possible to set down a meaningful asymptotic theory for the non-rescaled model (1) an approach using the rescaled model (2) leads to meaningful results also for the model (1). Another example for this relevance is the construction of confidence intervals for the local Yule-Walker estimates from the central limit theorem in Dahlhaus and Giraitis (1998), Theorem 3.2.

4. Parametric Whittle-type estimates – a first approach

We now assume that the p+1-dimensional parameter curve $\boldsymbol{\theta}(\cdot) = (\alpha_1(\cdot), \dots, \alpha_p(\cdot), \sigma^2(\cdot))'$ is parameterized by a finite dimensional parameter $\eta \in \mathbf{R}^q$ that is $\boldsymbol{\theta}(\cdot) = \boldsymbol{\theta}_{\eta}(\cdot)$. An example

studied below is where the AR-coefficients are modeled by polynomials. Another example is where the AR-coefficients are modeled by a parametric transition curve as in Section 2.6(iv). In particular when the length of the time series is short this may be a proper choice. We now show how the stationary Whittle likelihood can be generalized to the locally stationary case (another generalization is given in (89)).

If we were looking for a <u>nonparametric</u> estimate for the parameter curve $\theta(\cdot)$ we could apply the stationary Whittle estimate on a segment leading to

$$\hat{\boldsymbol{\theta}}_{T}^{W}(u_{0}) := \underset{\boldsymbol{\theta} \in \Theta}{\operatorname{argmin}} \mathcal{L}_{T}^{W}(u_{0}, \boldsymbol{\theta})$$
(17)

with the Whittle likelihood

$$\mathcal{L}_{T}^{W}(u_{0}, \boldsymbol{\theta}) := \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^{2} f_{\boldsymbol{\theta}}(\lambda) + \frac{I_{T}(u_{0}, \lambda)}{f_{\boldsymbol{\theta}}(\lambda)} \right\} d\lambda$$
 (18)

with the tapered periodogram on a segment about u_0 , that is

$$I_T(u_0, \lambda) := \frac{1}{2\pi H_N} \left| \sum_{s=1}^N h\left(\frac{s}{N}\right) X_{[u_0T]-N/2+s,T} \exp\left(-i\lambda s\right) \right|^2.$$
 (19)

Here $h(\cdot)$ is a data taper as in (8). For $h(x) = \chi_{(0,1]}(x)$ we obtain the non-tapered periodogram. The properties of this nonparametric estimate are discussed later - in particular in Example 3.6 and at the end of Example 6.6. In case of a tvAR(p)-process $\hat{\theta}_T(u_0)$ is exactly the local Yule-Walker estimate defined in (7) with the covariance-estimate given in (8).

Suppose now that we want to fit globally the parametric model $\theta(\cdot) = \theta_{\eta}(\cdot)$ to the data, that is we have the time varying spectrum $f_{\eta}(u,\lambda) := f_{\theta_{\eta}(u)}(\lambda)$. Since $\mathcal{L}_{T}^{W}(u,\theta)$ is an approximation of the Gaussian log-likelihood on the segment $\{[uT]-N/2+1,\ldots,[uT]+N/2\}$ a reasonable approach is to use

$$\hat{\eta}_T^{BW} := \underset{\eta \in \Theta_\eta}{\operatorname{argmin}} \mathcal{L}_T^{BW}(\eta) \tag{20}$$

with the block Whittle likelihood

$$\mathcal{L}_{T}^{BW}(\eta) := \frac{1}{4\pi} \frac{1}{M} \sum_{j=1}^{M} \int_{-\pi}^{\pi} \left\{ \log 4\pi^{2} f_{\eta}(u_{j}, \lambda) + \frac{I_{T}(u_{j}, \lambda)}{f_{\eta}(u_{j}, \lambda)} \right\} d\lambda.$$
 (21)

Here $u_j := t_j/T$ with $t_j := S(j-1) + N/2$ (j=1,...,M) i.e. we calculate the likelihood on overlapping segments which we shift each time by S. Furthermore T = S(M-1) + N. A better justification of the form of the likelihood is provided by the asymptotic Kullback-Leibler information divergence derived in Theorem 4.4.

\mathbf{K}_{2}	4	5	6	7	8	9
0	0.929	0.888	0.669	0.685	0.673	0.689
1	0.929	0.901	0.678	0.694	0.682	0.698
2	0.916	0.888	0.694	0.709	0.697	0.712

Table 1: Values for AIC for p = 2 and different polynomial orders

As discussed above the reason for using data-tapers is twofold: they reduce the bias due to nonstationarity on the segment and they reduce the leakage (already known from the stationary case). It is remarkable that the taper in this case does not lead to an increase of the asymptotic variance if the segments are overlapping (cf. Dahlhaus (1997), Theorem 3.3).

The properties of the above estimate are discussed in Dahlhaus (1997) including consistency, asymptotic normality, model selection and the behavior if the model is misspecified. The estimate is asymptotically efficient if $S/N \to 0$.

As an example we now fit a tvAR(p)-model to the data from Figure 1 and estimate the parameters by minimizing $\mathcal{L}_{T}^{BW}(\eta)$. The AR-coefficients are modeled as polynomials with different orders. Thus, we fit the model

$$\alpha_j(u) = \sum_{k=0}^{K_j} b_{jk} u^k \quad (j = 1, \dots, p) \quad \text{and} \quad \sigma(u) \equiv c$$

to the data. The model orders p, K_1, \ldots, K_p are chosen by minimizing the AIC-criterion

$$AIC(p, K_1, \dots, K_p) = \log \hat{\sigma}^2(p, K_1, \dots, K_p) + 2(p+1 + \sum_{j=1}^p K_j) / T.$$

Table 1 shows these values for p=2 and different K_1 and K_2 . The values for other p turned out to be larger. Thus, a model with $p=2, K_1=6, K_2=0$ is fitted. The function $\alpha_1(u)$ and its estimate are plotted in Figure 3. For $\hat{a}_2(u)$ we obtain 0.71 (a constant is fitted because of $K_2=0$) while the true $\alpha_2(u)$ is 0.81. Furthermore, $\hat{\sigma}^2=1.71$ while $\sigma^2=1.0$. The corresponding (parametric) estimate of the spectrum is the right picture of Figure 2 and the difference to the true spectrum is plotted in Figure 4.

Given the small sample size the quality of the fit is remarkable. Two negative effects can be observed. First, the fit of $\alpha_1(u)$ becomes rather bad outside $u_1 = 0.063$ and $u_M = 0.938$. This is not surprising, due to the behavior of a polynomial and the fact that the use of $\mathcal{L}_T^{BW}(\eta)$ as a distance only punishes bad fits inside the interval $[u_1, u_M]$. This end effect improves if one chooses $K_1 = 8$ instead of $K_1 = 6$. A better way seems to modify $\mathcal{L}_T^{BW}(\eta)$

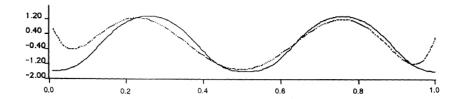


Figure 3: True and estimated parameter curve $\alpha_1(\cdot)$

and to include periodograms of shorter lengths at the edges. The second effect is that the peak in the spectrum is underestimated. This bias is in part due to the non-stationarity of the process on intervals $(u_j - N/(2T), u_j + N/(2T))$ where $I_T(u_j, \lambda)$ is calculated.

We mention that the above estimates can be written in closed form and calculated without an optimization routine. More generally this holds for tvAR(p)-models if σ^2 is constant and $\alpha_j(u) = \sum_{k=1}^K b_{jk} f_k(u)$ with some functions $f_1(u), \ldots, f_K(u)$ (in the above case $f_k(u) = u^{k-1}$). For details see Dahlhaus (1997), Section 4.

A closer look at the above estimate reveals that it is somehow the outcome of a two step procedure where in the first step the periodogram is calculated on segments (which implicitly includes some smoothing with bandwidth b = N/T) and afterwards the AR(p)-process with the above polynomials is fitted to the outcome (instead of a direct fit of the AR(p)-model and the polynomials to the data). We now make this more precise.

With the above form of the spectrum $f_{\eta}(u,\lambda)$ (cf.(5)) and Kolmogorov's formula, (cf. Brockwell and Davis, 1991, Theorem 5.8.1) we obtain with $\hat{R}_T(u_j)$ and $\hat{r}_T(u_j)$ as defined in (7) after some straightforward calculations

$$\mathcal{L}_{T}^{BW}(\eta) = \frac{1}{2} \frac{1}{M} \sum_{j=1}^{M} \left[\log 4\pi^{2} \sigma_{\eta}^{2}(u_{j}) + \frac{1}{\sigma_{\eta}^{2}(u_{j})} \left(\hat{c}_{T}(u_{j}, 0) - \hat{r}_{T}(u_{j})' \hat{R}_{T}(u_{j})^{-1} \hat{r}_{T}(u_{j}) \right) \right]$$

$$+ \frac{1}{2} \frac{1}{M} \sum_{j=1}^{M} \frac{1}{\sigma_{\eta}^{2}(u_{j})} \left[\left(\hat{R}_{T}(u_{j}) \boldsymbol{\alpha}_{\eta}(u_{j}) + \hat{r}_{T}(u_{j}) \right)' \hat{R}_{T}(u_{j})^{-1} \left(\hat{R}_{T}(u_{j}) \boldsymbol{\alpha}_{\eta}(u_{j}) + \hat{r}_{T}(u_{j}) \right) \right].$$

We now plug in the Yule-Walker estimate $\hat{\alpha}_T(u) = -\hat{R}_T(u)^{-1} \hat{r}_T(u)$ with asymptotic variance proportional to $\sigma^2(u) R(u)^{-1}$ and $\hat{\sigma}_T^2(u) = \hat{c}_T(u,0) - \hat{r}_T(u)' \hat{R}_T(u)^{-1} \hat{r}_T(u)$ with asymptotic variance proportional to $\sigma^2(u) R(u)^{-1}$ and $\hat{\sigma}_T^2(u) = \hat{c}_T(u,0) - \hat{r}_T(u)' \hat{R}_T(u)^{-1} \hat{r}_T(u)$ with asymptotic variance proportional to $\sigma^2(u) R(u)^{-1}$ and $\hat{\sigma}_T^2(u) = \hat{c}_T(u,0) - \hat{r}_T(u)' \hat{R}_T(u)^{-1} \hat{r}_T(u)$

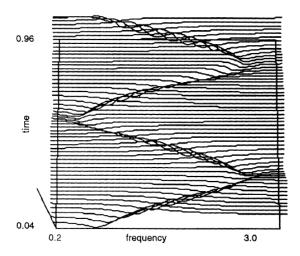


Figure 4: Difference of estimated and true spectrum

totic variance $2\sigma^2(u)$. Since $\log x = (x-1) - \frac{1}{2}(x-1)^2 + o((x-1)^2)$ we obtain

$$\begin{split} \mathcal{L}_{T}^{BW}(\eta) &= \frac{1}{2} \frac{1}{M} \sum_{j=1}^{M} \frac{1}{2 \, \sigma_{\eta}^{4}(u_{j})} \Big[\sigma_{\eta}^{2}(u_{j}) - \hat{\sigma}_{T}^{2}(u_{j}) \Big]^{2} \\ &+ \frac{1}{2} \frac{1}{M} \sum_{j=1}^{M} \Big[\Big(\boldsymbol{\alpha}_{\eta}(u_{j}) - \hat{\boldsymbol{\alpha}}_{T}(u_{j}) \Big)' \sigma_{\eta}^{2}(u_{j})^{-1} \, \hat{R}_{T}(u_{j}) \Big(\boldsymbol{\alpha}_{\eta}(u_{j}) - \hat{\boldsymbol{\alpha}}_{T}(u_{j}) \Big) \Big] \\ &+ \frac{1}{2} \frac{1}{M} \sum_{j=1}^{M} \log 4\pi^{2} \hat{\sigma}_{T}^{2}(u_{j}) + \frac{1}{2} + o\Big(\Big(\frac{\sigma_{\eta}^{2}(u_{j}) - \hat{\sigma}_{T}^{2}(u)}{\sigma_{\eta}^{2}(u_{j})} \Big)^{2} \Big). \end{split}$$

If the model is correctly specified then we have for η close to the minimum: $\sigma_{\eta}^2(u_j)^{-1} \hat{R}_T(u_j) \approx \sigma^2(u_j)^{-1} R(u_j)$ and $2 \sigma_{\eta}^4(u_j) \approx 2 \sigma^2(u_j)$ which means that $\hat{\eta}_T$ is approximately obtained by a weighted least squares fit of $\alpha_{\eta}(u)$ and $\sigma_{\eta}^2(u)$ to the Yule-Walker estimates on the segments. The method works in this case since the (parametric!) model fitted in the second step is somehow 'smoother' than the first smoothing implicitly induced by using the periodogram on a segment. However, we would clearly run into problems if the fitted polynomials were of high order or if even $K_j = K_j(T) \to \infty$ as $T \to \infty$.

A good alternative seems to use the quasi-likelihood $\mathcal{L}_{T}^{GW}(\eta)$ from (89) or (in particular for AR(p)-models) the conditional likelihood estimate from (30) with $\ell_{t,T}(\cdot)$ as in (23) for which the estimator can explicitly be calculated if $\sigma(\cdot) \equiv c$. For $\sigma_0(\cdot) \neq c$ iterative or approximative solutions are needed. The properties of this estimator have not been investigated yet. In any case the benefit of the likelihood $\mathcal{L}_{T}^{BW}(\eta)$ and even more of the improved likelihood $\mathcal{L}_{T}^{GW}(\eta)$ are their generality because they can be applied to arbitrary parametric models which can

be identified from the second order spectrum.

Furthermore, algorithmic issues, such as in-order algorithms (e.g. generalizations of the Levinson-Durbin algorithm) need to be developed.

5. Inference for nonparametric tvAR-models – an overview

In the last section we studied parametric estimates for tvAR(p)-models. This is an important option if the length of the time series is short or if we have specific parametric models in mind. In general however one would prefer nonparametric models. For nonparametric statistics a large variety of different estimates are available (local polynomial fits, estimation under shape restrictions, wavelet methods etc) and it turns out that it is not too difficult to apply such methods to tvAR(p)-models and moreover also to other possibly nonlinear models (while the derivation of the corresponding theory may be very challenging). A key role is played by the conditional likelihood at time t which in the tvAR(p)-case is

$$\ell_{t,T}(\boldsymbol{\theta}) := -\log f_{\boldsymbol{\theta}}(X_{t,T}|X_{t-1,T},\dots,X_{1,T})$$
(22)

$$= \frac{1}{2} \log(2\pi \sigma^2) + \frac{1}{2\sigma^2} \left(X_{t,T} + \sum_{j=1}^p \alpha_j \ X_{t-j,T} \right)^2$$
 (23)

where $\boldsymbol{\theta} = (\alpha_1, \dots, \alpha_p, \sigma^2)'$ and its approximation $\ell_{t,T}^*(\boldsymbol{\theta})$ defined in (96). As a simple example consider the estimation of the curve $\alpha_1(\cdot)$ of a tvAR(1)-process by a local linear fit given by $\hat{\alpha}_1(\cdot) = \hat{c}_0$ where

$$(\hat{c}_0, \hat{c}_1) = \underset{c_0, c_1}{\operatorname{argmin}} \frac{1}{bT} \sum_{t=1}^T K\left(\frac{u_0 - t/T}{b}\right) \left(X_{t,T} + \left[c_0 + c_1\left(\frac{t}{T} - u_0\right)\right] X_{t-1,T}\right)^2$$
(24)

or more generally (with vectors c_0 and c_1) given by $\hat{\theta}(u_0) = \hat{c}_0$ with

$$(\hat{\boldsymbol{c}}_0, \hat{\boldsymbol{c}}_1) = \operatorname*{argmin}_{\boldsymbol{c}_0, \boldsymbol{c}_1} \frac{1}{bT} \sum_{t=1}^T K\left(\frac{u_0 - t/T}{b}\right) \ell_{t,T} \left(\boldsymbol{c}_0 + \boldsymbol{c}_1\left(\frac{t}{T} - u_0\right)\right). \tag{25}$$

Besides this local linear estimate many other estimates can be constructed based on the conditional likelihood $\ell_{t,T}(\boldsymbol{\theta})$ from above:

1. A kernel estimate defined by

$$\hat{\boldsymbol{\theta}}(u_0) = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \frac{1}{bT} \sum_{t=1}^{T} K\left(\frac{u_0 - t/T}{b}\right) \ell_{t,T}(\boldsymbol{\theta}). \tag{26}$$

This estimate is studied in Section 3. We are convinced that it is equivalent to the local Yule-Walker estimate from (7) with $K(x) = h(x)^2$, b = N/T and that all results from 3. are exactly the same for this estimate.

2. A local polynomial fit defined by $\hat{\boldsymbol{\theta}}(u_0) = \hat{\boldsymbol{c}}_0$ with

$$(\hat{\boldsymbol{c}}_0, \dots, \hat{\boldsymbol{c}}_d)' = \operatorname*{argmin}_{\boldsymbol{c}_0, \dots, \boldsymbol{c}_d} \frac{1}{bT} \sum_{t=1}^T K\left(\frac{u_0 - t/T}{b}\right) \ell_{t,T}\left(\sum_{j=0}^d \boldsymbol{c}_j (\frac{t}{T} - u_0)^j\right). \tag{27}$$

Local polynomial fits for tvAR(p)-models have been investigated by Kim (2001) and Jentsch (2006).

3. An orthogonal series estimate (e.g. a <u>wavelet estimate</u>) defined by

$$\bar{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^{T} \ell_{t,T} \left(\sum_{i=1}^{J(T)} \boldsymbol{\beta}_{j} \psi_{j} \left(\frac{t}{T} \right) \right)$$
 (28)

together with some shrinkage of $\bar{\beta}$ to obtain $\hat{\beta}$ and $\hat{\theta}(u_0) = \sum_{j=1}^{J(T)} \hat{\beta}_j \psi_j(u_0)$. Usually $J(T) \to \infty$ as $T \to \infty$. Such an estimate has been investigated for a truncated wavelet expansion for tvAR(p)-models in Dahlhaus, Neumann and von Sachs (1999).

4. A nonparametric maximum likelihood estimate defined by

$$\hat{\boldsymbol{\theta}}(\cdot) = \underset{\boldsymbol{\theta}(\cdot) \in \Theta}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^{T} \ell_{t,T} \left(\boldsymbol{\theta} \left(\frac{t}{T} \right) \right)$$
 (29)

where Θ is an adequate function space, for example a space of curves under shape restrictions such as monotonicity constraints. In Dahlhaus and Polonik (2006) the estimation of a monotonic variance function in a tvAR-model is studied, including explicit algorithms involving isotonic regression.

5. A parametric fit for the curves $\theta(\cdot) = \theta_{\eta}(\cdot)$ with $\eta \in \mathbf{R}^q$ defined by

$$\hat{\eta} = \underset{\eta}{\operatorname{argmin}} \frac{1}{T} \sum_{t=1}^{T} \ell_{t,T} \left(\boldsymbol{\theta}_{\eta} \left(\frac{t}{T} \right) \right)$$
(30)

The resulting estimate has not been investigated yet. It is presumably very close to the exact MLE studied in Theorem 5.1.

Remark 2.4 (i) In the tvAR(p)-case the situation simplifies a lot if $\sigma^2(\cdot) \equiv c$. In that case the estimates for $\alpha(\cdot)$ and σ^2 "split" and $\ell_{t,T}(\boldsymbol{\theta})$ can in all cases be replaced by $(X_{t,T} + \sum_{j=1}^p \alpha_j X_{t-j,T})^2$ leading to least squares type estimates.

(ii) All estimates from above can be transferred to other models by using the conditional likelihood (22) for the specific model. The kernel estimate will be investigated in Section 3.

(iii) As mentioned above an alternative choice is to replace $\ell_{t,T}(\boldsymbol{\theta})$ by the local generalized Whittle likelihood $\ell_{t,T}^*(\boldsymbol{\theta})$ from (96). With that likelihood several estimates from above have been investigated – see the detailed discussion at the end of Section 5. In that case the d-dimensional parameter curve $\boldsymbol{\theta}(\cdot) = (\theta_1(\cdot), \dots, \theta_d(\cdot))'$ must be uniquely identifiable from the time varying spectrum $f(u, \lambda) = f_{\boldsymbol{\theta}(u)}(\lambda)$.

6. Shape- and transition curves

There exist several alternative models for tvAR-processes – in particular models where specific characteristics of the time series are modeled by a curve. Below we give 4 examples where we restrict ourselves to tvAR(2)-models. Suppose we have a stationary AR(2)-model with complex roots $\frac{1}{r} \exp(i\phi)$ and $\frac{1}{r} \exp(-i\phi)$, that is with parameters $a_1 = -2r\cos(\phi)$, $a_2 = r^2$, and variance σ^2 . The corresponding process shows a quasi-periodic behavior with period of length $\frac{2\pi}{\phi}$, that is with frequency ϕ . The more r gets closer to 1 the more the shape of the process gets closer to a sine-wave. The amplitude is proportional to σ (if σ (say in (2)) is replaced by $c \cdot \sigma$, then X_t is replaced by $c \cdot X_t$).

In the specific tvAR(2)-case we can now consider the following shape- and transition-models for quasi-periodic processes:

(i) Model with a time varying amplitude curve:

$$a_1(\cdot), a_2(\cdot)$$
 constant; $\sigma(\cdot)$ time varying.

Chandler and Polonik (2006) use this model with a unimodal $\sigma(\cdot)$ and a nonparametric maximum likelihood estimate for the discrimination of earthquakes and explosions. The properties of the estimator have been investigated in Dahlhaus and Polonik (2006).

(ii) Model with a time varying frequency curve:

$$a_1(\cdot) = -2r\cos(\phi(\cdot)), \ a_2(\cdot) = r^2 \text{ with } r \text{ constant and } \phi(\cdot) \text{ time varying, } \sigma(\cdot) \text{ constant.}$$

The model in Figure 1 is of this form with r = 0.9 and $\phi(u) = 1.5 - \cos 4\pi u$.

(iii) Model with a time varying period-distinctiveness:

$$a_1(\cdot) = -2 r(\cdot) \cos(\phi), \ a_2(\cdot) = r(\cdot)^2$$
 with $r(\cdot)$ time varying and ϕ constant, $\sigma(\cdot)$ constant.

(iv) Transition models: Amado and Teräsvirta (2011) have recently used the logistic transition function to model parameter transitions in GARCH-models. The simplest transition function is

$$G\Big(\frac{t}{T};\gamma,c\Big) := \Big[1 + \exp\Big\{-\gamma\left(\frac{t}{n} - c\right)\Big\}\Big]^{-1}.$$

Since $G(0; \gamma, c) \approx 0$ and $G(1; \gamma, c) \approx 1$ the model

$$a_1(u) = a_1^{\text{start}} + G(u; \gamma, c) \left(a_1^{\text{end}} - a_1^{\text{start}} \right), \ a_2(u) = a_2^{\text{start}} + G(u; \gamma, c) \left(a_2^{\text{end}} - a_2^{\text{start}} \right)$$

is a parametric model for a smooth transition from the AR-model with parameters $(a_1^{\text{start}}, a_2^{\text{start}})$ at u=0 to the model with parameters $(a_1^{\text{end}}, a_2^{\text{end}})$ at u=1. Here c and γ are the location and the 'smoothness' of transition respectively. More general transition models (in particular with more states) may be found in Amado and Teräsvirta (2011). $G(\cdot; \gamma, c)$ may also be replaced by a (nonparametric) function $G(\cdot)$ with G(0)=0 and G(1)=0.

It is obvious that all methods from subsection 5 can be applied in cases (i)-(iv) to estimate the constant parameters and the shape- and transition-curves. We mention that the theoretical results for local Whittle estimates of Dahlhaus and Giraitis (1998) apply to these models (cf. Example 3.6), the uniform convergence result for the local generalized Whittle estimate in Theorem 6.9, the asymptotic results of Dahlhaus and Neumann (2001) where the parameter curves are estimated by a nonlinear wavelet method, the results of Dahlhaus and Polonik (2006) on nonparametric maximum likelihood estimates under shape constraints, and the results for parametric models in Theorem 5.1 on the MLE and the generalized Whittle estimator, and in Dahlhaus (1997) on the block Whittle estimator.

3 Local likelihoods, derivative processes and nonlinear models with time varying parameters

In this section we present a more general framework for time series with time varying finite dimensional parameters $\theta(\cdot)$ and show how nonparametric inference can be done and theoretically handled. Typically such models result from the generalization of classical parametric models to the time varying case. If we restrict ourselves to linear processes or even more to Gaussian processes then a much more general theory is possible which is developed in the subsequent sections. Large parts of the present section are based on the ideas presented in Dahlhaus and Subba Rao (2006) where time varying ARCH-models have been investigated.

The key idea is to use at each time point $u_0 \in (0,1)$ the stationary approximation $X_t(u_0)$ to the original process $X_{t,T}$ and to calculate the bias resulting from the use of this approximation. This will end in Taylor-type expansions of $X_{t,T}$ in terms of so-called derivative processes. These expansions play a major role in the theoretical derivations.

Suppose for example that we estimate the multivariate parameter curve $\theta(\cdot)$ by minimizing

the (negative) local conditional log-likelihood, that is

$$\hat{\boldsymbol{\theta}}_{T}^{C}(u_{0}) := \operatorname*{argmin}_{\boldsymbol{\theta} \in \Theta} \mathcal{L}_{T}^{C}(u_{0}, \boldsymbol{\theta})$$

with

$$\mathcal{L}_{T}^{C}(u_{0},\boldsymbol{\theta}) := \frac{1}{T} \sum_{t=1}^{T} \frac{1}{b} K\left(\frac{u_{0} - t/T}{b}\right) \ell_{t,T}(\boldsymbol{\theta})$$
(31)

and

$$\ell_{t,T}(\boldsymbol{\theta}) := -\log f_{\boldsymbol{\theta}}(X_{t,T}|X_{t-1,T},\ldots,X_{1,T})$$

where K is symmetric, has compact support $[-\frac{1}{2},\frac{1}{2}]$ and fulfills $\int_{-1/2}^{1/2} K(x) dx = 1$. We assume that $b = b_T \to 0$ and $bT \to \infty$ as $T \to \infty$. Two examples for this likelihood are given below.

We approximate $\mathcal{L}_{T}^{C}(u_0, \boldsymbol{\theta})$ with $\tilde{\mathcal{L}}_{T}^{C}(u_0, \boldsymbol{\theta})$ which is the same function but with $\ell_{t,T}(\boldsymbol{\theta})$ replaced by

$$\tilde{\ell}_t(u_0, \boldsymbol{\theta}) := -\log f_{\boldsymbol{\theta}} \big(\tilde{X}_t(u_0) \big| \tilde{X}_{t-1}(u_0), \dots, \tilde{X}_1(u_0) \big),$$

which means that $X_{t,T}$ is replaced by its stationary approximation $\tilde{X}_t(u_0)$. Usually this is the local conditional likelihood for the process $\tilde{X}_t(u_0)$.

Example 3.1 (i) Consider the tvAR(p) process defined in (2) together with its stationary approximation at time u_0 given by (3). Under suitable regularity conditions it can be shown that $X_{t,T} = \tilde{X}_t(u_0) + O_p(\left|\frac{t}{T} - u_0\right| + \frac{1}{T})$ (cf.(51)). In case where the ε_t are Gaussian the conditional likelihood at time t is given by

$$\ell_{t,T}(\boldsymbol{\theta}) = \frac{1}{2} \log(2\pi \,\sigma^2) + \frac{1}{2\,\sigma^2} \left(X_{t,T} + \sum_{j=1}^p \alpha_j \, X_{t-j,T} \right)^2 \tag{32}$$

where $\boldsymbol{\theta} = (\alpha_1, \dots, \alpha_p, \sigma^2)'$. It is easy to show that the resulting estimate is the same as in (7) but with $\hat{r}_T(u_0) := (\tilde{\tilde{c}}_T(u_0, 0, 1), \dots, \tilde{\tilde{c}}_T(u_0, 0, p))'$ and $\hat{R}_T(u_0) := {\tilde{\tilde{c}}_T(u_0, i, j)}_{i,j=1,\dots,p}$ with the local covariance estimator $\tilde{\tilde{c}}_T(u, i, j)$ as defined in (10).

(ii) A tvARCH(p) model where $\{X_{t,T}\}$ is assumed to satisfy the representation

$$X_{t,T} = \sigma_{t,T} Z_t$$
where
$$\sigma_{t,T}^2 = \alpha_0(\frac{t}{T}) + \sum_{j=1}^p \alpha_j(\frac{t}{T}) X_{t-j,N}^2 \quad \text{for } t = 1, \dots, N$$
(33)

with Z_t being independent, identically distributed random variables with $\mathbf{E}Z_t = 0$, $\mathbf{E}Z_t^2 = 1$.

The corresponding stationary approximation $\tilde{X}_t(u_0)$ at time u_0 is given by

$$\tilde{X}_{t}(u_{0}) = \sigma_{t}(u_{0}) Z_{t}$$
where
$$\sigma_{t}(u_{0})^{2} = \alpha_{0}(u_{0}) + \sum_{j=1}^{p} \alpha_{j}(u_{0}) \tilde{X}_{t-j}(u_{0})^{2} \quad \text{for } t \in \mathbf{Z}.$$
(34)

It is shown in Dahlhaus and Subba Rao (2006) that $\{X_{t,T}^2\}$ as defined above has an almost surely well-defined unique solution in the set of all causal solutions and $X_{t,T}^2 = \tilde{X}_t(u_0)^2 + O_p(|\frac{t}{T} - u_0| + \frac{1}{N})$. In case where the Z_t are Gaussian the conditional likelihood is given by

$$\ell_{t,T}(\boldsymbol{\theta}) = \frac{1}{2} \log w_{t,T}(\boldsymbol{\theta}) + \frac{X_{t,T}^2}{2 w_{t,T}(\boldsymbol{\theta})} \text{ with } w_{t,T}(\boldsymbol{\theta}) = \alpha_0 + \sum_{j=1}^p \alpha_j X_{t-j,T}^2$$
 (35)

where $\theta = (\alpha_0, \dots, \alpha_p)'$. Dahlhaus and Subba Rao (2006) prove consistency of the resulting estimate also in case where the true process is not Gaussian. As an alternative Fryzlewicz et.al. (2008) propose a kernel normalized-least-squares estimator which has a closed form and thus has some advantages over the above kernel estimate for small samples.

We now discuss the derivation of the asymptotic bias, mean squared error, consistency and asymptotic normality of $\hat{\boldsymbol{\theta}}_T(u_0)$ for an "arbitrary" local minimum distance function $\mathcal{L}_T(u_0, \boldsymbol{\theta})$ (keeping in mind the above local conditional likelihood). The results are obtained by approximating $\mathcal{L}_T(u_0, \boldsymbol{\theta})$ with $\tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta})$ which is the same function but with $X_{t,T}$ replaced by its stationary approximation $\tilde{X}_t(u_0)$. Typically both, $\mathcal{L}_T(u_0, \boldsymbol{\theta})$ and $\tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta})$ will converge to the same limit-function which we denote by $\mathcal{L}(u_0, \boldsymbol{\theta})$. Let

$$\boldsymbol{\theta}_0(u_0) := \operatorname*{argmin}_{\boldsymbol{\theta} \in \Theta} \mathcal{L}(u_0, \boldsymbol{\theta}).$$

If the model is correctly specified then typically $\theta_0(u_0)$ is the true curve. Furthermore, let

$$\mathcal{B}_T(u_0, \boldsymbol{\theta}) := \mathcal{L}_T(u_0, \boldsymbol{\theta}) - \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}).$$

The following two results describe how the asymptotic properties of $\hat{\boldsymbol{\theta}}_T(u_0)$ can be derived. They should be regarded as a general roadmap and the challenge is to prove the conditions in a specific situation which may be quite difficult.

Theorem 3.2 (i) Suppose that Θ is compact with $\theta_0(u_0) \in Int(\Theta)$, the function $\mathcal{L}(u_0, \theta)$ is continuous in θ and the minimum $\theta_0(u_0)$ is unique. If

$$\sup_{\boldsymbol{\theta} \in \Theta} \left| \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}) - \mathcal{L}(u_0, \boldsymbol{\theta}) \right| \stackrel{P}{\to} 0, \tag{36}$$

and

$$\sup_{\boldsymbol{\theta} \in \Theta} \left| \mathcal{B}_T(u_0, \boldsymbol{\theta}) \right| \stackrel{P}{\to} 0 \tag{37}$$

then

$$\hat{\boldsymbol{\theta}}_T(u_0) \stackrel{P}{\to} \boldsymbol{\theta}_0(u_0). \tag{38}$$

(ii) Suppose in addition that $\mathcal{L}(u, \boldsymbol{\theta})$ and $\boldsymbol{\theta}_0(u)$ are uniformly continuous in u and $\boldsymbol{\theta}$ and the convergence in (36) and (37) is uniformly in $u_0 \in [0, 1]$. Then

$$\sup_{u_0 \in [0,1]} \left| \hat{\boldsymbol{\theta}}_T(u_0) - \boldsymbol{\theta}_0(u_0) \right| \stackrel{P}{\to} 0. \tag{39}$$

PROOF. The proof of (i) is standard – cf. the proof of Theorem 2 in Dahlhaus and Subba Rao (2006). The proof of (ii) is a straightforward generalization.

Note that in (i) all conditions apart from (37) are conditions on the stationary process $\tilde{X}_t(u_0)$ with (fixed) parameter $\boldsymbol{\theta}(u_0)$ and the stationary likelihood / minimum-distance function $\tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta})$. These properties are usually known from existing results on stationary processes. It only remains to verify the condition (37) which can be done by using the expansion (51) in terms of derivative processes (see the discussion below). (ii) contains a little pitfall: Usually the estimate $\hat{\boldsymbol{\theta}}_T(u_0)$ is defined for $u_0 = 0$ or $u_0 = 1$ in a different way due to edge-effects. This means that also $\tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta})$ looks different, that is one would usually prefer a uniform convergence result for $u_0 \in (0, 1)$ which is more difficult to prove.

Even more interesting and challenging is a uniform convergence result with a rate of convergence. For time varying AR(p)-processes this is stated for a different likelihood in Theorem 6.9. We mention that such a result usually requires an exponential bound and maximal inequalities which need to be tailored to the specific model at hand.

We now state the corresponding result on asymptotic normality in case of second order smoothness. ∇ denotes the derivatives with respect to the θ_i , i.e. $\nabla := \left(\frac{\partial}{\partial \theta_i}\right)_{i=1,\dots,d}$.

Theorem 3.3 Let $\theta_0 := \theta_0(u_0)$. Suppose that $\mathcal{L}_T(u_0, \theta)$, $\tilde{\mathcal{L}}_T(u_0, \theta)$ and $\mathcal{L}(u_0, \theta)$ are twice continuously differentiable in θ with nonsingular matrix $\Gamma(u_0) := \nabla^2 \mathcal{L}(u_0, \theta_0)$. Let further

$$\sqrt{bT} \nabla \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}_0) \stackrel{\mathcal{D}}{\rightarrow} \mathcal{N}(0, V(u_0))$$

with some sequence $b=b_T$ where $b\to 0$ and $bT\to \infty$ (the definition of b is part of the definition of the likelihood – it is usually some bandwidth) and

$$\sup_{\boldsymbol{\theta} \in \Theta} \left| \nabla^2 \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}) - \nabla^2 \mathcal{L}(u_0, \boldsymbol{\theta}) \right| \stackrel{P}{\to} 0.$$

If in addition

$$\sqrt{bT} \left(\Gamma(u_0)^{-1} \nabla \mathcal{B}_T(u_0, \boldsymbol{\theta}_0) - \frac{b^2}{2} \boldsymbol{\mu}^0(u_0) \right) = o_p(1)$$
(40)

with some $\mu^0(\cdot)$ (to be specified below – cf.(47)) and

$$\sup_{\boldsymbol{\theta} \in \Theta} \left| \nabla^2 \mathcal{B}_T(u_0, \boldsymbol{\theta}) \right| \stackrel{P}{\to} 0 \tag{41}$$

then

$$\sqrt{bT} \left(\hat{\boldsymbol{\theta}}_T(u_0) - \boldsymbol{\theta}_0(u_0) + \frac{b^2}{2} \boldsymbol{\mu}^0(u_0) \right) \stackrel{\mathcal{D}}{\to} \mathcal{N} \left(0, \Gamma(u_0)^{-1} V(u_0) \Gamma(u_0)^{-1} \right). \tag{42}$$

PROOF. The usual Taylor-expansion of $\nabla \mathcal{L}_T(u_0, \boldsymbol{\theta})$ around $\boldsymbol{\theta}_0$ yields

$$\sqrt{bT} \left(\hat{\boldsymbol{\theta}}_T(u_0) - \boldsymbol{\theta}_0 + \Gamma(u_0)^{-1} \nabla \mathcal{B}_T(u_0, \boldsymbol{\theta}_0) \right) = -\sqrt{bT} \Gamma(u_0)^{-1} \nabla \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}_0) + o_p(1). \tag{43}$$

The result then follows immediately.

Remark 3.4 (i) Again the first two conditions are conditions on the stationary process $\tilde{X}_t(u_0)$ with (fixed) parameter $\theta(u_0)$ and the stationary likelihood / minimum-distance function $\tilde{\mathcal{L}}_T(u_0, \theta)$ which are usually known from existing results on stationary processes.

- (ii) Of course an analogous result also holds under different smoothness conditions and with other rates than b^2 in (40) and (42).
- (iii) Under additional regularity conditions one can usually prove that the same expansion as in (43) also holds for the moments, leading to

$$\mathbf{E}\,\hat{\boldsymbol{\theta}}_T(u_0) = \boldsymbol{\theta}_0(u_0) - \frac{b^2}{2}\,\boldsymbol{\mu}^0(u_0) + o(b^2) \tag{44}$$

and

$$\operatorname{var}(\hat{\boldsymbol{\theta}}_{T}(u_{0})) = \frac{1}{bT} \Gamma(u_{0})^{-1} V(u_{0}) \Gamma(u_{0})^{-1} + o\left(\frac{1}{bT}\right)$$
(45)

(note that (43) is a stochastic expansion which does not automatically imply these moment relations). The proof of these properties is usually not easy.

Example 3.5 (Kernel-type local likelihoods) We now return to the local conditional likelihood (31) as a special case and provide some heuristics on how to calculate the above terms (in particular the bias $\mu^0(u_0)$). We stress that in the concrete situation where a specific model is given the exact proof usually goes along the same lines but the details may be quite challenging.

Suppose that the local likelihood of the stationary process $\tilde{X}_t(u_0)$ converges in probability to

$$\mathcal{L}(u_0, \boldsymbol{\theta}) := \lim_{T \to \infty} \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}) = \lim_{t \to \infty} \mathbf{E} \,\tilde{\ell}_t(u_0, \boldsymbol{\theta}).$$

Usually we have $X_{t,T} = \tilde{X}_t(t/T) + O_p(T^{-1})$ and

$$\mathbf{E} \nabla \ell_{t,T}(\boldsymbol{\theta}) = \mathbf{E} \nabla \tilde{\ell}_t(\frac{t}{T}, \boldsymbol{\theta}) + o((bT)^{-1/2}) = \nabla \mathcal{L}(\frac{t}{T}, \boldsymbol{\theta}) + o((bT)^{-1/2})$$

uniformly in t. A Taylor-expansion then leads in the case $b^3 = o((bT)^{-1/2})$ with the symmetry of the kernel K to

$$\mathbf{E} \, \nabla \mathcal{L}_{T}(u_{0}, \boldsymbol{\theta}) = \frac{1}{bT} \sum_{t=1}^{T} K\left(\frac{u_{0} - t/T}{b}\right) \nabla \mathcal{L}\left(\frac{t}{T}, \boldsymbol{\theta}\right) + o\left((bT)^{-1/2}\right)$$

$$= \nabla \mathcal{L}(u_{0}, \boldsymbol{\theta}) + \left[\frac{\partial}{\partial u} \nabla \mathcal{L}(u_{0}, \boldsymbol{\theta})\right] \frac{1}{bT} \sum_{t=1}^{T} K\left(\frac{u_{0} - t/T}{b}\right) \left(\frac{t}{T} - u_{0}\right) + \frac{1}{2} \left[\frac{\partial^{2}}{\partial u^{2}} \nabla \mathcal{L}(u_{0}, \boldsymbol{\theta})\right] \frac{1}{bT} \sum_{t=1}^{T} K\left(\frac{u_{0} - t/T}{b}\right) \left(\frac{t}{T} - u_{0}\right)^{2} + o\left((bT)^{-1/2}\right)$$

$$= \nabla \mathcal{L}(u_{0}, \boldsymbol{\theta}) + \frac{1}{2} b^{2} d_{K} \frac{\partial^{2}}{\partial u^{2}} \nabla \mathcal{L}(u_{0}, \boldsymbol{\theta}) + o\left((bT)^{-1/2}\right)$$

$$(46)$$

with $d_K := \int x^2 K(x) dx$. Since $\mathbf{E} \nabla \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}) = \nabla \mathcal{L}(u_0, \boldsymbol{\theta}) + o((bT)^{-1/2})$ this leads with (40) to the bias term

$$\boldsymbol{\mu}^{0}(u_{0}) = d_{K} \Gamma(u_{0})^{-1} \frac{\partial^{2}}{\partial u^{2}} \nabla \mathcal{L}\left(u, \boldsymbol{\theta}_{0}(u_{0})\right) \rfloor_{u=u_{0}} =: d_{k} \boldsymbol{\mu}(u_{0})$$

$$\tag{47}$$

Let $\theta_0 := \theta_0(u_0)$. If the model is correctly specified it usually can be shown that $\nabla \tilde{\ell}_t(u_0, \theta_0)$ is a martingale difference sequence and the condition of the Lindeberg martingale central limit theorem are fulfilled leading to

$$\sqrt{bT} \nabla \tilde{\mathcal{L}}_T(u_0, \boldsymbol{\theta}_0) \stackrel{\mathcal{D}}{\to} \mathcal{N} \Big(0, v_K \mathbf{E} \big(\nabla \tilde{\ell}_t(u_0, \boldsymbol{\theta}_0) \big) \big(\nabla \tilde{\ell}_t(u_0, \boldsymbol{\theta}_0) \big)' \Big)$$

with $v_K = \int K(x)^2 dx$. Furthermore, if the model is correctly specified we usually have

$$\mathbf{E}(\nabla \tilde{\ell}_t(u_0, \boldsymbol{\theta}_0)) (\nabla \tilde{\ell}_t(u_0, \boldsymbol{\theta}_0))' = \nabla^2 \mathcal{L}(u_0, \boldsymbol{\theta}_0) = \Gamma(u_0)$$

that is

$$\sqrt{bT} \left(\hat{\boldsymbol{\theta}}_T(u_0) - \boldsymbol{\theta}_0(u_0) + \frac{b^2}{2} d_K \Gamma(u_0)^{-1} \frac{\partial^2}{\partial u^2} \nabla \mathcal{L}(u_0, \boldsymbol{\theta}_0) \right) \stackrel{\mathcal{D}}{\to} \mathcal{N} \left(0, v_K \Gamma(u_0)^{-1} \right). \tag{48}$$

If we are able to prove in addition the formulas (44) and (45) on the asymptotic bias and variance we obtain the same formula for the asymptotic mean squared error as in (11) with $\tau(u_0) = \operatorname{tr}\{\Gamma(u_0)^{-1}\}$ and $\mu(u_0)^2$ replaced by $\|\mu(u_0)\|^2$ where $\mu(u_0) = \Gamma(u_0)^{-1} \frac{\partial^2}{\partial u^2} \nabla \mathcal{L}(u_0, \boldsymbol{\theta}_0)$. As in Remark 2.2 this leads to the optimal segment length and the optimal mean squared error. The implications for non-rescaled processes are the same as in Remark 2.3.

We now present three examples where the above results have been proved explicitly.

Example 3.6 (Local Whittle estimates) The first example are local Whittle estimates on segments $\hat{\boldsymbol{\theta}}_{T}^{W}(u_0)$ obtained by minimizing $\mathcal{L}_{T}^{W}(u_0,\boldsymbol{\theta})$ (cf.(18)). In case of a tvAR(p)-process $\hat{\boldsymbol{\theta}}_{T}^{W}(u_0)$ is exactly the local Yule-Walker estimate defined in (7) with the covariance-estimates given in (8). $\mathcal{L}_{T}^{W}(u,\boldsymbol{\theta})$ is not exactly a local conditional likelihood as defined in (31) but approximately (in the same sense as $\hat{c}_{T}(u_0,k)$ from (8) is an approximation to the kernel covariance estimate). For that reason the above heuristics also applies to this estimate and can be made rigorous.

In Dahlhaus and Giraitis (1998), Theorem 3.1 and 3.2, bias and asymptotic normality of $\hat{\boldsymbol{\theta}}_T^W(u_0)$ have been derived rigorously including a derivation of the variance and the mean squared error as given in (44) and (45) (i.e. not only the stochastic expansion in (43)). We mention that therefore also the results on the optimal kernel and bandwidth in (12) and (13) apply to this situation.

In the present situation we have (cf. Dahlhaus and Giraitis (1998), (3.7)))

$$\mathcal{L}(u, \boldsymbol{\theta}) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^2 f_{\boldsymbol{\theta}}(\lambda) + \frac{f(u, \lambda)}{f_{\boldsymbol{\theta}}(\lambda)} \right\} d\lambda.$$

Therefore

$$\frac{\partial^2}{\partial u^2} \nabla \mathcal{L}(u_0, \boldsymbol{\theta}) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \nabla f_{\boldsymbol{\theta}}(\lambda)^{-1} \frac{\partial^2}{\partial u^2} f(u_0, \lambda) d\lambda$$

and in the correctly specified case where $f(u, \lambda) = f_{\theta_0(u)}(\lambda)$

$$\Gamma(u_0) = \nabla^2 \mathcal{L}(u_0, \boldsymbol{\theta}_0) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\nabla \log f_{\boldsymbol{\theta}_0} \right) \left(\nabla \log f_{\boldsymbol{\theta}_0} \right)' d\lambda$$

leading to the asymptotic bias $\mu(u_0)$ in (47) and the asymptotic variance in the central limit theorem (48). A uniform convergence result for $\hat{\boldsymbol{\theta}}_T^W(u_0)$ is stated in Theorem 6.9.

Example 3.7 (tvAR(p)-processes) In the special case of a Gaussian tvAR(p)-process the exact results for the local Yule-Walker estimates (7) follow as a special case from the above results on local Whittle estimates (see also Section 2 in Dahlhaus and Giraitis, 1998,

where tvAR(p)-processes are discussed separately). In that case we have with R(u) and r(u) as in (6) that $\Gamma(u) = \frac{1}{\sigma^2(u)}R(u)$. Furthermore

$$\nabla \mathcal{L}(u, \boldsymbol{\theta}) = \frac{1}{\sigma^2} [R(u) \boldsymbol{\alpha} + r(u)]$$

which implies

$$\boldsymbol{\mu}(u_0) = R(u_0)^{-1} \left[\left(\frac{\partial^2}{\partial u^2} R(u) \right) \boldsymbol{\alpha}(u_0) + \left(\frac{\partial^2}{\partial u^2} r(u) \right) \right]_{u=u_0}.$$

We conjecture that exactly the same asymptotic results hold for the conditional likelihood estimate obtained by minimizing

$$\mathcal{L}_{T}^{C}(u_{0}, \boldsymbol{\theta}) := \frac{1}{T} \sum_{t=1}^{T} \frac{1}{b} K\left(\frac{u_{0} - t/T}{b}\right) \left[\frac{1}{2} \log(2\pi \sigma^{2}) + \frac{1}{2\sigma^{2}} \left(X_{t,T} + \sum_{j=1}^{p} \alpha_{j} X_{t-j,T}\right)^{2}\right]. \quad \Box$$

We now introduce derivative processes. The key idea in the proofs of Dahlhaus and Giraitis (1998) is to use at time $u_0 \in (0,1)$ the stationary approximation $\tilde{X}_t(u_0)$ (there denoted by Y_t) to the original process $X_{t,T}$ and to calculate the bias resulting from the use of this approximation. As in Dahlhaus and Subba Rao (2006) we now extend this idea leading to the Taylor-type expansion (51) which is an expansion of the original process in terms of (usually ergodic) stationary processes called derivative processes. This expansion is a powerful tool since all techniques for stationary processes including the ergodic theorem may be applied for the local investigation of the nonstationary process $X_{t,T}$. The use of this expansion and of derivative processes in general leads to a general structure of the proofs and simplifies the derivations a lot.

We start with the simple example of a tvAR(1)-process since in this case everything can be calculated directly. Then $X_{t,T}$ is defined by $X_{t,T} + \alpha_1(t/T)X_{t-1,T} = \varepsilon_t, t \in \mathbf{Z}$ and the stationary approximation $\tilde{X}_t(u_0)$ at time $u_0 = t_0/n$ by $\tilde{X}_t(u_0) + \alpha_1(u_0)\tilde{X}_t(u_0) = \varepsilon_t, t \in \mathbf{Z}$. Repeated plug-in yields under suitable regularity conditions (for a rigorous argument see the proof of Theorem 2.3 in Dahlhaus (1996a))

$$X_{t,T} = \sum_{j=0}^{\infty} (-1)^j \left[\prod_{k=0}^{j-1} \alpha_1 \left(\frac{t-k}{T} \right) \right] \varepsilon_{t-j} = \sum_{j=0}^{\infty} (-1)^j \alpha_1 \left(\frac{t}{T} \right)^j \varepsilon_{t-j} + O_p \left(\frac{1}{T} \right)$$
(49)

$$= \tilde{X}_t \left(\frac{t}{T}\right) + O_p \left(\frac{1}{T}\right) = \tilde{X}_t(u_0) + \left(\frac{t}{T} - u_0\right) \frac{\partial X_t(u)}{\partial u} \rfloor_{u=u_0} + O_p \left(\frac{1}{T}\right). \tag{50}$$

We have in the present situation

$$\frac{\partial \tilde{X}_t(u)}{\partial u} = \sum_{j=0}^{\infty} (-1)^j \frac{\partial \alpha_1(u)^j}{\partial u} \, \varepsilon_{t-j} = \sum_{j=0}^{\infty} (-1)^j \left[j \, \alpha_1(u)^{j-1} \alpha_1(u)' \right] \varepsilon_{t-j}$$

that is $\frac{\partial \tilde{X}_t(u)}{\partial u}$ is a stationary ergodic process in t with $\left|\frac{\partial \tilde{X}_t(u)}{\partial u}\right| \leq \sum_{j=1}^{\infty} j \rho^{j-1} |\varepsilon_{t-j}|$ where $|\rho| < 1$. In the same way we have

$$X_{t,T} = \tilde{X}_t(u_0) + \left(\frac{t}{T} - u_0\right) \frac{\partial \tilde{X}_t(u)}{\partial u} \Big|_{u=u_0} + \frac{1}{2} \left(\frac{t}{T} - u_0\right)^2 \frac{\partial^2 \tilde{X}_t(u)}{\partial u^2} \Big|_{u=u_0} + O_p\left(\left(\frac{t}{T} - u_0\right)^3 + \frac{1}{T}\right)$$

$$(51)$$

with the second order derivative process $\frac{\partial^2 \tilde{X}_t(u)}{\partial u^2} \rfloor_{u=u_0}$ which is defined analogously. It is not difficult to prove existence and uniqueness in a rigorous sense.

For general tvAR(p)-processes the same results holds – however, it is difficult in that case to write the derivative process in explicit form. It is interesting to note that the derivative process fulfills the equation

$$\frac{\partial \tilde{X}_t(u)}{\partial u} + \sum_{i=1}^p \left(\alpha_j(u) \frac{\partial \tilde{X}_{t-j}(u)}{\partial u} + \alpha'_j(u) \, \tilde{X}_{t-j}(u) \right) = \frac{\partial \sigma(u)}{\partial u} \, \varepsilon_t.$$

where $\alpha'_{j}(u)$ denotes the derivative of $\alpha_{j}(u)$ with respect to u. This is formally obtained by differentiating both sides of equation (3). Furthermore, it can be shown that this equation system uniquely defines the derivative process.

We are convinced that the expansion (51) and equation systems like (52) can be established for several other locally stationary time series models. As mentioned above the important point is that (51) is an expansion in terms of stationary processes.

In the next example we show how derivative processes are used for deriving the properties of local likelihood estimates.

Example 3.8 (tvARCH-processes) The definition of the processes $X_{t,T}$ and $\tilde{X}_t(u_0)$ has been given above in (33) and (34) and of the local likelihood in (35) and (31). In Dahlhaus and Subba Rao (2006), Theorem 2 and 3, consistency and asymptotic normality have been established for the resulting estimate and in particular (48) has been proved. Derivative processes play a major role in the proofs and we briefly indicate how they are used. First, existence and uniqueness of the derivative processes have been proved including the Taylor-type expansion for the process $X_{t,T}^2$:

$$X_{t,T}^{2} = \tilde{X}_{t}(u_{0})^{2} + \left(\frac{t}{T} - u_{0}\right) \frac{\partial \tilde{X}_{t}(u)^{2}}{\partial u} \rfloor_{u=u_{0}} + \frac{1}{2} \left(\frac{t}{T} - u_{0}\right)^{2} \frac{\partial^{2} \tilde{X}_{t}(u)^{2}}{\partial u^{2}} \rfloor_{u=u_{0}} + O_{p}\left(\left(\frac{t}{T} - u_{0}\right)^{3} + \frac{1}{T}\right)$$
(52)

(in this model we are working with $X_{t,T}^2$ rather than $X_{t,T}$ since $X_{t,T}^2$ is uniquely determined). Furthermore, $\frac{\partial \tilde{X}_t(u)^2}{\partial u}$ is almost surely the unique solution of the equation

$$\frac{\partial \tilde{X}_t(u)^2}{\partial u} = \left(\alpha_0'(u) + \sum_{i=1}^{\infty} \alpha_j'(u) \,\tilde{X}_{t-j}(u)^2 + \sum_{i=1}^{\infty} \alpha_j(u) \,\frac{\partial \tilde{X}_{t-j}(u)^2}{\partial u}\right) Z_t^2 \tag{53}$$

which can formally be obtained by differentiating (34). By taking the second derivative of this expression we obtain a similar expression for the second derivative $\frac{\partial^2 \tilde{X}_t(u)^2}{\partial u^2}$ etc.

A key step in the above proofs is the derivation of (40) and of the bias term $\mu^0(\cdot)$ in this situation. We briefly sketch this. We have with $\theta_0 = \theta_0(u_0)$

$$\nabla \mathcal{B}_T(u_0, \boldsymbol{\theta}_0) = \frac{1}{bT} \sum_{t=1}^T K\left(\frac{u_0 - t/T}{b}\right) \left(\nabla \ell_{t,T}(\boldsymbol{\theta}_0) - \nabla \tilde{\ell}_t(u_0, \boldsymbol{\theta}_0)\right).$$

First $\nabla \ell_{t,T}(\boldsymbol{\theta}_0)$ is replaced by $\nabla \tilde{\ell}_t(t/T,\boldsymbol{\theta}_0)$ where we omit details (this works since $X_{t,T}^2$ is approximately the same as $\tilde{X}_t^2(t/T)$). Then a Taylor-expansion is applied:

$$\nabla \tilde{\ell}_t(\frac{t}{T}, \boldsymbol{\theta}_0) - \nabla \tilde{\ell}_t(u_0, \boldsymbol{\theta}_0) = \left(\frac{t}{T} - u_0\right) \frac{\partial \nabla \tilde{\ell}_t(u, \boldsymbol{\theta}_0)}{\partial u} \rfloor_{u=u_0}$$

$$+ \frac{1}{2} \left(\frac{t}{T} - u_0\right)^2 \frac{\partial^2 \nabla \tilde{\ell}_t(u, \boldsymbol{\theta}_0)}{\partial u^2} \rfloor_{u=u_0} + \frac{1}{6} \left(\frac{t}{T} - u_0\right)^3 \frac{\partial^3 \nabla \tilde{\ell}_t(u, \boldsymbol{\theta}_0)}{\partial u^3} \rfloor_{u=\tilde{U}_t}$$
 (54)

with a random variable $\tilde{U}_t \in (0,1]$. The breakthrough now is that $\frac{\partial \nabla \tilde{\ell}_t(u,\theta_0)}{\partial u}$ can be written explicitly in terms of the derivative process $\frac{\partial \tilde{X}_t(u)^2}{\partial u}$ and of the process $\tilde{X}_t(u)^2$, that is we obtain with the formula for the total derivative

$$\frac{\partial \nabla \tilde{\ell}_t(u, \boldsymbol{\theta}_0)}{\partial u} = \sum_{j=0}^p \left(\frac{\partial}{\partial \tilde{X}_{t-j}(u)^2} \left[\frac{\nabla w_t(u, \boldsymbol{\theta}_0)}{w_t(u, \boldsymbol{\theta}_0)} - \frac{\tilde{X}_t(u)^2 \nabla w_t(u, \boldsymbol{\theta}_0)}{w_t(u, \boldsymbol{\theta}_0)^2} \right] \times \frac{\partial \tilde{X}_{t-j}(u)^2}{\partial u} \right),$$

where $w_t(u, \boldsymbol{\theta}) = c_0(\boldsymbol{\theta}_0) + \sum_{j=1}^{\infty} c_j(\boldsymbol{\theta}) \tilde{X}_{t-j}(u)^2$ (the same holds true for the higher order terms). In particular $\frac{\partial \nabla \tilde{\ell}_t(u, \boldsymbol{\theta}_0)}{\partial u}$ is a stationary process with constant mean. Due to the symmetry of the kernel we therefore obtain after some lengthy but straightforward calculations

$$\sqrt{bT} \Big(\Gamma(u_0)^{-1} \nabla \mathcal{B}_T(u_0, \boldsymbol{\theta}_0) - \frac{b^2}{2} d_K \Gamma(u_0)^{-1} \frac{\partial^2}{\partial u^2} \nabla \mathcal{L}(u, \boldsymbol{\theta}_0) \big|_{u=u_0} \Big) = o_p(1).$$
 (55)

A very simple example is the tvARCH(0) process

$$X_{t,T} = \sigma_{t,T} Z_t, \quad \sigma_{t,T}^2 = \alpha_0(\frac{t}{T}).$$

In this case $\frac{\partial \tilde{X}_t(u)^2}{\partial u} = \alpha_0'(u) Z_t^2$ and we have

$$\frac{\partial^2 \nabla \mathcal{L}(u, \boldsymbol{\alpha}_{u_0})}{\partial u^2} \rfloor_{u=u_0} = -\frac{1}{2} \frac{\alpha_0''(u_0)}{\alpha_0(u_0)^2} \quad \text{ and } \quad \Sigma(u_0) = \frac{1}{2 \alpha_0(u_0)^2}$$

that is $\mu(u_0) = -\alpha_0''(u_0)$. This is another example which illustrates how the bias is linked to the nonstationarity of the process - if the process were stationary the derivatives of $\alpha_0(\cdot)$ would be zero causing the bias also to be zero. The formula (13) for the optimal bandwidth leads in this case to

$$b_{opt}(u_0) = \left[\frac{2v_K}{d_K^2}\right]^{1/5} \left[\frac{\alpha_0(u_0)}{\alpha_0''(u_0)}\right]^{2/5} T^{-1/5}$$

leading to a large bandwidth if $\alpha_0''(u_0)$ is small and vice versa. As in Remark 2.3 this can be "translated" to the non-rescaled case.

Example 3.9 (tvGARCH-processes) A tvGARCH(p, q)-process satisfies the following representation

$$X_{t,T} = \sigma_{t,T} Z_t$$
where $\sigma_{t,T}^2 = \alpha_0(\frac{t}{T}) + \sum_{j=1}^p \alpha_j(\frac{t}{T}) X_{t-j,T}^2 + \sum_{i=1}^q \beta_i(\frac{t}{T}) \sigma_{t-i,T}^2,$ (56)

where $\{Z_t\}$ are iid random variables with $\mathbf{E}Z_t = 0$ and $\mathbf{E}Z_t^2 = 1$. The corresponding stationary approximation at time u_0 is given by

$$\tilde{X}_{t}(u_{0}) = \sigma_{t}(u_{0}) Z_{t} \quad \text{for } t \in \mathbf{Z}$$
where
$$\sigma_{t}(u_{0})^{2} = \alpha_{0}(u_{0}) + \sum_{j=1}^{p} \alpha_{j}(u_{0}) \tilde{X}_{t-j}(u_{0})^{2} + \sum_{i=1}^{q} \beta_{i}(u_{0}) \sigma_{t-i}(u_{0})^{2}.$$
(57)

Under the condition that $\sup_{u} \left(\sum_{j=1}^{p} \alpha_{j}(u) + \sum_{i=1}^{q} \beta_{i}(u) \right) < 1$ Subba Rao (2006), Section 5, has shown that $X_{t,T}^{2} = \tilde{X}_{t}(u_{0})^{2} + O_{p}(|\frac{t}{T} - u_{0}| + \frac{1}{T})$. To obtain estimators of the parameters $\{\alpha_{j}(\cdot)\}$ and $\{\beta_{i}(\cdot)\}$ an approximation of the conditional quasi-likelihood is used, which is constructed as if the innovations $\{Z_{t}\}$ were Gaussian. As the infinite past is unobserved, an observable approximation of the conditional quasi-likelihood is

$$\ell_{t,T}(\boldsymbol{\theta}) = \frac{1}{2} \log w_{t,T}(\boldsymbol{\theta}) + \frac{X_{t,T}^2}{2 w_{t,T}(\boldsymbol{\theta})} \text{ with } w_{t,T}(\boldsymbol{\theta}) = c_0(\boldsymbol{\theta}) + \sum_{j=1}^{t-1} c_j(\boldsymbol{\theta}) X_{t-j,T}^2,$$
 (58)

where a recursive formula for $c_j(\boldsymbol{\theta})$ in terms of the parameters of interest, $\{\alpha_j\}$ and $\{\beta_i\}$, can be found in Berkes et.al. (2003). Given that the derivatives of the time varying GARCH

parameters exist we can formally differentiate (57) to obtain

$$\begin{split} \frac{\partial \tilde{X}_{t}(u)^{2}}{\partial u} &= \frac{\partial \sigma_{t}(u)^{2}}{\partial u} Z_{t}^{2} \\ \frac{\partial \sigma_{t}(u)^{2}}{\partial u} &= \alpha'_{0}(u) + \sum_{j=1}^{p} \left(\alpha'_{j}(u) \, \tilde{X}_{t-j}(u)^{2} + \alpha_{j}(u) \, \frac{\partial \tilde{X}_{t-j}(u)^{2}}{\partial u} \right) \\ &+ \sum_{i=1}^{q} \left(\beta'_{i}(u) \, \sigma_{t-i}(u)^{2} + \beta_{i}(u) \, \frac{\partial \sigma_{t-i}(u)^{2}}{\partial u} \right). \end{split}$$

Subba Rao (2006) has shown that one can represent the above as a state-space representation which almost surely has a unique solution which is the derivative of $\tilde{X}_t(u)^2$ with respect to u. Thus $X_{t,T}^2$ satisfies the expansion in (52). Moreover, Fryzlewicz and Subba Rao (2011) show geometric α -mixing of the tvGARCH process. Using these results and under some technical assumptions it can be shown that Theorem 3.2(i) and Theorem 3.3 hold for the local approximate conditional quasi-likelihood estimator. In particular, a result analogous to (55) holds true, where

$$\mathcal{L}(u,\boldsymbol{\theta}) = \mathbf{E} \bigg(\log \big(c_0(\boldsymbol{\theta}) + \sum_{j=1}^{\infty} c_j(\boldsymbol{\theta}) \tilde{X}_{t-j}(u) \big) \bigg) + \mathbf{E} \bigg(\frac{\tilde{X}_t(u)^2}{c_0(\boldsymbol{\theta}) + \sum_{j=1}^{\infty} c_j(\boldsymbol{\theta}) \tilde{X}_{t-j}(u)^2} \bigg).$$

Amado and Teräsvirta (2011) investigate parametric tvGARCH-models where the time varying parameters are modeled with the logistic transition function – see Section 2.6. \Box

Similar methods as described in this section have also been applied in Koo and Linton (2010) who investigate semiparametric estimation of locally stationary diffusion models. They also prove a central limit theorem with a bias term as in (42). In their proofs they use the stationary approximation $\tilde{X}_t(u_0)$ and the Taylor-type expansion (51). Vogt (2011) investigates nonlinear nonparametric models allowing for locally stationary regressors and a regression function that changes smoothly over time.

4 A general definition, linear processes and time varying spectral densities

The intuitive idea for a general definition is to require that locally around each rescaled time point u_0 the process $\{X_{t,T}\}$ can be approximated by a stationary process $\{\tilde{X}_t(u_0)\}$ in a stochastic sense by using the property (4) (cf. Dahlhaus and Subba Rao, 2006). Vogt (2011) has formalized this by requiring that for each u_0 there exists a stationary process

$$\tilde{X}_t(u_0)$$
 with

$$||X_{t,T} - \tilde{X}_t(u_0)| \le \left(\left|\frac{t}{T} - u_0\right| + \frac{1}{T}\right) U_{t,T}(u_0)$$
 (59)

where $U_{t,T}(u_0)$ is a positive stochastic process fulfilling some uniform moment conditions. However up to now no general theory exists based on such a general definition.

In the following we move on towards a general theory for linear locally stationary processes. In some cases we even assume Gaussianity or use Gaussian likelihood methods and their approximations. In this situation a fairly general theory can be derived in which parametric and nonparametric inference problems, goodness of fit tests, bootstrap procedures etc can be treated in high generality. We use a general definition tailored for linear processes which however implies (59).

Definition of linear locally stationary processes: We give this definition in terms of the time varying $MA(\infty)$ representation

$$X_{t,T} = \mu(\frac{t}{T}) + \sum_{j=-\infty}^{\infty} a_{t,T}(j) \,\varepsilon_{t-j}$$
 where $a_{t,T}(j) \approx a(\frac{t}{T}, j)$

with coefficient functions $a(\cdot, j)$ which need to fulfill additional regularity function (dependent on the result to be proved – details are provided below). In several papers of the author instead the time varying spectral representation

$$X_{t,T} = \mu\left(\frac{t}{T}\right) + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp(i\lambda t) A_{t,T}(\lambda) d\xi(\lambda) \quad \text{where} \quad A_{t,T}(\lambda) \approx A(\frac{t}{T}, \lambda) \quad (60)$$

with the time varying transfer function $A(\cdot, \lambda)$ was used. Both representations are basically equivalent – see the derivation of (78). In the results presented below we will always use the formulation "Under suitable regularity conditions . . ." and refer the reader to the original paper. We conjecture however that all results can be reproved under Assumption 4.1. We emphasize that this is not an easy task since in most situations it means to transfer the proof from the frequency to the time domain. In that case it would be worthwhile to require only martingale differences ε_t since also some nonlinear processes admit such a representation.

Let

$$V(g) = \sup \left\{ \sum_{k=1}^{m} \left| g(x_k) - g(x_{k-1}) \right| : 0 \le x_o < \dots < x_m \le 1, \ m \in \mathbf{N} \right\}$$
 (61)

be the total variation of g.

Assumption 4.1 The sequence of stochastic processes $X_{t,T}$ has a representation

$$X_{t,T} = \mu\left(\frac{t}{T}\right) + \sum_{j=-\infty}^{\infty} a_{t,T}(j) \,\varepsilon_{t-j} \tag{62}$$

where μ is of bounded variation and the ε_t are iid with $E\varepsilon_t = 0$, $E\varepsilon_s\varepsilon_t = 0$ for $s \neq t$, $E\varepsilon_t^2 = 1$. Let

$$\ell(j) := \begin{cases} 1, & |j| \le 1\\ |j| \log^{1+\kappa} |j|, & |j| > 1 \end{cases}$$

for some $\kappa > 0$ and

$$\sup_{t} |a_{t,T}(j)| \le \frac{K}{\ell(j)} \quad \text{(with } K \text{ indep. of } T\text{)}. \tag{63}$$

Furthermore we assume that there exist functions $a(\cdot,j):(0,1]\to\mathbf{R}$ with

$$\sup_{u} |a(u,j)| \le \frac{K}{\ell(j)},\tag{64}$$

$$\sup_{j} \sum_{t=1}^{T} \left| a_{t,T}(j) - a(\frac{t}{T}, j) \right| \le K, \tag{65}$$

$$V(a(\cdot,j)) \le \frac{K}{\ell(j)}.$$
(66)

The above assumptions are weak in the sense that only bounded variation is required for the coefficient functions. In particular for local results stronger smoothness assumptions have to be imposed – for example in addition for some i

$$\sup_{u} \left| \frac{\partial^{i} \mu(u)}{\partial u^{i}} \right| \le K, \tag{67}$$

$$\sup_{u} \left| \frac{\partial^{i} a(u,j)}{\partial u^{i}} \right| \le \frac{K}{\ell(j)} \quad \text{for } j = 0, 1, \dots$$
 (68)

and instead of (65) the stronger assumption

$$\sup_{t,T} \left| a_{t,T}(j) - a(\frac{t}{T}, j) \right| \le \frac{K}{T\ell(j)}. \tag{69}$$

The construction with $a_{t,T}(j)$ and $a(\frac{t}{T},j)$ looks complicated at first glance. The function $a(\cdot,j)$ is needed for rescaling and to impose necessary smoothness conditions while the additional use of $a_{t,T}(j)$ makes the class rich enough to cover interesting cases such as tvAR-models (the reason for this in the AR(1)-case can be understood from (49)). Cardinali and Nason (2010) created the term close pair for $(a(\frac{t}{T},j),a_{t,T}(j))$. Usually, additional moment conditions on ε_t are required.

It is straightforward to construct the stationary approximation and the derivative processes. We have

$$\tilde{X}_t(u) := \mu(u) + \sum_{j=-\infty}^{\infty} a(u,j) \, \varepsilon_{t-j}$$

and

$$\frac{\partial^{i} \tilde{X}_{t}(u)}{\partial u^{i}} = \frac{\partial^{i} \mu(u)}{\partial u^{i}} + \sum_{j=-\infty}^{\infty} \frac{\partial^{i} a(u,j)}{\partial u^{i}} \, \varepsilon_{t-j}$$

and it is easy to prove (59) and more general the expansion (51). We define the time varying spectral density by

$$f(u,\lambda) := \frac{1}{2\pi} |A(u,\lambda)|^2 \tag{70}$$

where

$$A(u,\lambda) := \sum_{j=-\infty}^{\infty} a(u,j) \exp(-i\lambda j), \tag{71}$$

and the time varying covariance of $\log k$ at rescaled time u by

$$c(u,k) := \int_{-\pi}^{\pi} f(u,\lambda) \exp(i\lambda k) d\lambda = \sum_{j=-\infty}^{\infty} a(u,k+j) a(u,j).$$
 (72)

 $f(u,\lambda)$ and c(u,k) are the spectral density and the covariance function of the stationary approximation $\tilde{X}_t(u)$. Under Assumption 4.1 and (69) it can be shown that

$$cov(X_{[uT],T}, X_{[uT]+k,T}) = c(u,k) + O(T^{-1})$$
(73)

uniformly in u and k – therefore we call c(u,k) also the time varying covariance of the processes $X_{t,T}$. In Theorem 4.3 we show that $f(u,\lambda)$ is the uniquely defined time varying spectral density of $X_{t,T}$.

Example 4.2 (i) A simple example of a process $X_{t,T}$ which fulfills the above assumptions is $X_{t,T} = \mu(\frac{t}{T}) + \phi(\frac{t}{T})Y_t$ where $Y_t = \Sigma_j \, a(j) \, \varepsilon_{t-j}$ is stationary with $|a(j)| \leq K/\ell(j)$ and μ and ϕ are of bounded variation. If Y_t is an AR(2)-process with complex roots close to the unit circle then Y_t shows a periodic behavior and $\phi(\cdot)$ may be regarded as a time varying amplitude modulating function of the process $X_{t,T}$. $\phi(\cdot)$ may either be parametric or nonparametric.

(ii) The tvARMA(p,q) process

$$\sum_{i=0}^{p} \alpha_j(\frac{t}{T}) X_{t-j,T} = \sum_{k=0}^{q} \beta_k(\frac{t}{T}) \sigma(\frac{t-k}{T}) \varepsilon_{t-k}$$
 (74)

where ε_t are iid with $E\varepsilon_t = 0$ and $E\varepsilon_t^2 < \infty$ and all $\alpha_j(\cdot)$, $\beta_k(\cdot)$ and $\sigma^2(\cdot)$ are of bounded variation with $\alpha_0(\cdot) \equiv \beta_0(\cdot) \equiv 1$ and $\sum_{j=0}^p \alpha_j(u)z^j \neq 0$ for all u and all $|z| \leq 1 + \delta$ for some

 $\delta > 0$, fulfills Assumption 4.1. If the parameters are differentiable with bounded derivatives then also (67)-(69) are fulfilled (for i=1). The time varying spectral density is

$$f(u,\lambda) = \frac{\sigma^2(u)}{2\pi} \frac{\left|\sum_{k=0}^q \beta_k(u) \exp(i\lambda k)\right|^2}{\left|\sum_{j=0}^p \alpha_j(u) \exp(i\lambda j)\right|^2}.$$
 (75)

This is proved in Dahlhaus and Polonik (2006). $\alpha_j(\cdot)$ and $\beta_k(\cdot)$ may either be parametric or nonparametric.

The time varying MA(∞)-representation (62) can easily be transformed into a time varying spectral representation as used e.g. in Dahlhaus (1997, 2000). If the ε_t are assumed to be stationary then there exists a Cramér representation (cf. Brillinger (1981))

$$\varepsilon_t = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp(i\lambda t) \, d\xi(\lambda) \tag{76}$$

where $\xi(\lambda)$ is a process with mean 0 and orthonormal increments. Let

$$A_{t,T}(\lambda) := \sum_{j=-\infty}^{\infty} a_{t,T}(j) \exp(-i\lambda j).$$
 (77)

Then

$$X_{t,T} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp(i\lambda t) A_{t,T}(\lambda) d\xi(\lambda).$$
 (78)

(69) now implies

$$\sup_{t,\lambda} \left| A_{t,T}(\lambda) - A(\frac{t}{T}, \lambda) \right| \le KT^{-1} \tag{79}$$

which was assumed in the above cited papers. Conversely, if we start with (78) and (79) then we can conclude from adequate smoothness conditions on $A(u, \lambda)$ to the conditions of Assumption 4.1.

We now state a uniqueness property of our spectral representation. The Wigner-Ville spectrum for fixed T (cf. Martin and Flandrin (1985)) is

$$f_T(u,\lambda) := \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \text{cov}(X_{[uT-s/2],T}, X_{[uT+s/2],T}) \exp(-i\lambda s)$$

with $X_{t,T}$ as in (62) (either with the coefficient extended as constants for $u \notin [0,1]$ or set to 0). Below we prove that $f_T(u,\lambda)$ tends in squared mean to $f(u,\lambda)$ as defined in (70). Therefore it is justified to call $f(u,\lambda)$ the time varying spectral density of the process.

Theorem 4.3 If $X_{t,T}$ is locally stationary and fulfills Assumption 4.1 and (68) for all j then we have for all $u \in (0,1)$

$$\int_{-\pi}^{\pi} |f_T(u,\lambda) - f(u,\lambda)|^2 d\lambda = o(1).$$

PROOF. The result was proved in Dahlhaus (1996b) under a different set of conditions. It is not very difficult to prove the result also under the present conditions. \Box

As a consequence the time varying spectral density $f(u, \lambda)$ is uniquely defined. If in addition the process $X_{t,T}$ is non-Gaussian, then even $A(u, \lambda)$ and therefore also the coefficients a(u, j)are uniquely determined which may be proved similarly by considering higher-order spectra. Since $\mu(t/T)$ is the mean of the process it is also uniquely determined. This is remarkable since in the non-rescaled case time varying processes do not have a unique spectral density or a unique time varying spectral representation (cf. Priestley (1981), Chapter 11.1; Mélard and Herteleer-de Schutter (1989)). $f(u, \lambda)$ from Theorem 4.3 has been called instantaneous spectrum (in particular for tvAR-process – c.f. Kitagawa and Gersch (1985)). The above theorem gives a theoretical justification for this definition.

There is a huge benefit from having a unique time varying spectral density. We now give an example for this. We derive the limit of the Kullback-Leibler information for Gaussian processes and show that it depends on $f(u, \lambda)$. Replacing this by a spectral estimate will lead to a quasi likelihood for parametric models similar to the Whittle likelihood for stationary processes. Without a unique spectral density such a construction were not possible.

Consider the exact Gaussian maximum likelihood estimate

$$\hat{\eta}_T^{ML} := \operatorname*{argmin}_{\eta \in \Theta_{\eta}} \mathcal{L}_T^E(\eta)$$

where η is a finite-dimensional parameter (as in (20)) and

$$\mathcal{L}_T^E(\eta) = \frac{1}{2}\log(2\pi) + \frac{1}{2T}\log\det\Sigma_{\eta} + \frac{1}{2T}(\mathbf{X} - \mu_{\eta})'\Sigma_{\eta}^{-1}(\mathbf{X} - \mu_{\eta})$$
(80)

with $\mathbf{X} = (X_{1,T}, \dots, X_{T,T})'$, $\mu_{\eta} = (\mu_{\eta}(1/T), \dots, \mu_{\eta}(T/T))'$ and Σ_{η} being the covariance matrix of the model. Under certain regularity conditions $\hat{\eta}_T^{ML}$ will converge to

$$\eta_0 := \underset{\eta \in \Theta_{\eta}}{\operatorname{argmin}} \mathcal{L}(\eta) \tag{81}$$

where

$$\mathcal{L}(\eta) := \lim_{T \to \infty} \mathbf{E} \, \mathcal{L}_T^E(\eta).$$

If the model is correct, then typically η_0 is the true parameter value. Otherwise it is some "projection" onto the parameter space. It is therefore important to calculate $\mathcal{L}(\eta)$ which is equivalent to the calculation of the Kullback-Leibler information divergence.

Theorem 4.4 Let $X_{t,T}$ be a locally stationary process with true mean- and spectral density curves $\mu(\cdot)$, $f(u,\lambda)$ and model curves $\mu_{\eta}(\cdot)$, $f_{\eta}(u,\lambda)$ respectively. Under suitable regularity conditions we have

$$\mathcal{L}(\eta) = \lim_{T \to \infty} \mathbf{E} \mathcal{L}_T^E(\eta)$$

$$= \frac{1}{4\pi} \int_0^1 \int_{-\pi}^{\pi} \left\{ \log 4\pi^2 f_{\eta}(u, \lambda) + \frac{f(u, \lambda)}{f_{\eta}(u, \lambda)} \right\} d\lambda du + \frac{1}{4\pi} \int_0^1 \frac{\left(\mu_{\eta}(u) - \mu(u)\right)^2}{f_{\eta}(u, 0)} du.$$

PROOF. See Dahlhaus (1996b), Theorem 3.4.

The Kullback-Leibler information divergence for stationary processes is obtained from this as a special case (cf. Parzen (1983)).

Example 4.5 Suppose that the model is stationary, i.e. $f_{\eta}(\lambda) := f_{\eta}(u, \lambda)$ and $m := \mu_{\eta}(u)$ do not depend on u. Then

$$\mathcal{L}(\eta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^2 f_{\eta}(\lambda) + \frac{\int_{0}^{1} f(u,\lambda) du}{f_{\eta}(\lambda)} \right\} d\lambda + \frac{1}{4\pi} f_{\eta}(0)^{-1} \int_{0}^{1} (m - \mu(u))^2 du$$

i.e. $m_0 = \int_0^1 \mu(u) du$, and $f_{\eta_0}(\lambda)$ give the best approximation to the time integrated true spectrum $\int_0^1 f(u,\lambda) du$. These are the values which are "estimated" by the MLE or a quasi-MLE if a stationary model is fitted to locally stationary data.

Given the form of $\mathcal{L}(\eta)$ as in Theorem 4.4 we can now suggest a quasi-likelihood criterion

$$\mathcal{L}_{T}^{QL}(\eta) = \frac{1}{4\pi} \int_{0}^{1} \int_{-\pi}^{\pi} \left\{ \log 4\pi^{2} f_{\eta}(u,\lambda) + \frac{\hat{f}(u,\lambda)}{f_{\eta}(u,\lambda)} \right\} d\lambda \, du + \frac{1}{4\pi} \int_{0}^{1} \frac{\left(\mu_{\eta}(u) - \hat{\mu}(u)\right)^{2}}{f_{\eta}(u,0)} \, du$$

where $\hat{f}(u, \lambda)$ and $\hat{\mu}(u)$ are suitable nonparametric estimates of $f(u, \lambda)$ and $\mu(u)$ respectively. The block Whittle likelihood $\mathcal{L}_{T}^{BW}(\eta)$ in (21) and the generalized Whittle likelihood $\mathcal{L}_{T}^{GW}(\eta)$ in (89) are of this form.

We now calculate the Fisher information matrix

$$\Gamma := \lim_{T \to \infty} T \mathbf{E}_{\eta_0} \left(\nabla \mathcal{L}_T^E(\eta_0) \right) \left(\nabla \mathcal{L}_T^E(\eta_0) \right)'$$

in order to study efficiency of parameter estimates (see also Theorem 5.1).

Theorem 4.6 Let $X_{t,T}$ be a locally stationary process with correctly specified mean curve $\mu_{\eta}(u)$ and time varying spectral density $f_{\eta}(u,\lambda)$. Under suitable regularity conditions we have

$$\Gamma = \frac{1}{4\pi} \int_0^1 \int_{-\pi}^{\pi} \left(\nabla \log f_{\eta_0} \right) \left(\nabla \log f_{\eta_0} \right)' d\lambda \, du + \frac{1}{2\pi} \int_0^1 \left(\nabla \mu_{\eta_0}(u) \right) \left(\nabla \mu_{\eta_0}(u) \right)' f_{\eta_0}^{-1}(u,0) \, du.$$

PROOF. See Dahlhaus (1996b), Theorem 3.6.

We now briefly discuss how the time varying spectral density can be estimated. Following the discussion in the last section we start with a classical "stationary" smoothed periodogram estimate on a segment. Let $I_T(u,\lambda)$ be the tapered periodogram on a segment of length N about u as defined in (19). Even in the stationary case $I_T(u,\lambda)$ is not a consistent estimate of the spectrum and we have to smooth it over neighboring frequencies. Let therefore

$$\hat{f}_T(u,\lambda) := \frac{1}{b_f} \int K_f\left(\frac{\lambda - \mu}{b_f}\right) I_T(u,\mu) d\mu \tag{82}$$

where K_f is a symmetric kernel with $\int K_f(x) dx = 1$ and b_f is the bandwidth in frequency direction. Theorem 5.5 below shows that the estimate is implicitly also a kernel estimate in time direction with kernel

$$K_t(x) := \left\{ \int_0^1 h(x)^2 dx \right\}^{-1} h(x+1/2)^2, \qquad x \in [-1/2, 1/2]$$
 (83)

and bandwidth $b_t := N/T$, that is the estimate behaves like a kernel estimates with two convolution kernels in frequency and time direction. We mention that an asymptotically equivalent estimate is the kernel estimate

$$\tilde{f}_T(u,\lambda) := \frac{2\pi}{T^2} \sum_{t=1}^T \sum_{j=1}^T \int \frac{1}{b_t} K_t \left(\frac{u - t/T}{b_t}\right) \frac{1}{b_f} K_f \left(\frac{\lambda - \lambda_j}{b_f}\right) J_T \left(\frac{t}{T}, \lambda_j\right) \tag{84}$$

with the pre-periodogram $J_T(u, \lambda)$ as defined in (88). One may also replace the integral in frequency direction in (82) by a sum over the Fourier frequencies.

Theorem 4.7 Let $X_{t,T}$ be a locally stationary process with $\mu(\cdot) \equiv 0$. Under suitable regu-

larity conditions we have

(i)
$$\mathbf{E}I_T(u,\lambda) = f(u,\lambda) + \frac{1}{2}b_t^2 \int_{-1/2}^{1/2} x^2 K_t(x) dx \frac{\partial^2}{\partial u^2} f(u,\lambda) + o(b_t^2) + O\left(\frac{\log(b_t T)}{b_t T}\right);$$

(ii)
$$\mathbf{E}\hat{f}_{T}(u,\lambda) = f(u,\lambda) + \frac{1}{2}b_{t}^{2} \int_{-1/2}^{1/2} x^{2} K_{t}(x) dx \frac{\partial^{2}}{\partial u^{2}} f(u,\lambda) + \frac{1}{2}b_{f}^{2} \int_{-1/2}^{1/2} x^{2} K_{f}(x) dx \frac{\partial^{2}}{\partial \lambda^{2}} f(u,\lambda) + o\left(b_{t}^{2} + b_{f}^{2} + \frac{\log(b_{t}T)}{b_{t}T}\right);$$

(iii)
$$\operatorname{var}(\hat{f}_T(u,\lambda)) = (b_t b_f T)^{-1} 2\pi f(u,\lambda)^2 \int_{-1/2}^{1/2} K_t(x)^2 dx \int_{-1/2}^{1/2} K_f(x)^2 dx (1 + \delta_{\lambda 0}).$$

PROOF. A sketch of the proof can be found in Dahlhaus (1996c), Theorem 2.2.

Note, that the first bias term of \hat{f} is due to nonstationarity while the second is due to the variation of the spectrum in frequency direction.

As in Remark 2.2 one may now minimize the relative mean squared error RMSE(\hat{f}) := $E(\hat{f}(u,\lambda)/f(u,\lambda)-1)^2$ with respect to b_f , b_t (i.e. N), K_f and K_t (i.e. the data taper h). This has been done in Dahlhaus (1996c), Theorem 2.3. The result says that with

$$\Delta_u := \frac{\partial^2}{\partial u^2} f(u, \lambda) / f(u, \lambda)$$
 and $\Delta_\lambda := \frac{\partial^2}{\partial \lambda^2} f(u, \lambda) / f(u, \lambda)$

the optimal RMSE is obtained with

$$b_t^{\text{opt}} = T^{-1/6} (576\pi)^{1/6} \left(\frac{\Delta_{\lambda}}{\Delta_u^5}\right)^{1/12}, \qquad b_f^{\text{opt}} = T^{-1/6} (576\pi)^{1/6} \left(\frac{\Delta_u}{\Delta_{\lambda}^5}\right)^{1/12}$$

and optimal kernels $K_t^{\text{opt}}(x) = K_f^{\text{opt}}(x) = 6(1/4 - x^2)$ with optimal rate $T^{-2/3}$.

The relations $b_t = N/T$ and (83) immediately lead to the optimal segment length and the optimal data taper h. The result of Theorem 5.5 is quite reasonable: If the degree of nonstationarity is small, then Δ_u is small and b_t^{opt} gets large. If the variation of f is small in frequency direction, then Δ_{λ} is small and b_t^{opt} gets smaller (more smoothing is put in frequency direction than in time direction). This is another example, how the bias due to nonstationarity can be quantified with the approach of local stationarity and balanced with another bias term and a variance term. Of course the data-adaptive choice of the bandwidth parameters remains to be solved. Asymptotic normality of the estimates can be derived from Theorem 6.3 (cf. Dahlhaus (2009), Example 4.2).

Rosen et.al. (2009) estimate the logarithm of the local spectrum by using a Bayesian mixture of splines. They assume that the log spectrum on a partition of the data is a mixture of individual log spectra and use a mixture of smoothing splines with time varying mixing weights to estimate the evolutionary log spectrum. Guo et.al. (2003) use a smoothing spline ANOVA to estimate the time varying log spectrum.

5 Gaussian likelihood theory for locally stationary processes

The basics of the likelihood theory for univariate stationary processes were laid by Whittle (1953, 1954). His work was much later taken up and continued by many others. Among the large number of papers we mention the results of Dzhaparidze (1971) and Hannan (1973) for univariate time series, Dunsmuir (1979) for multivariate time series and e.g. Hosoya and Taniguchi (1982) for misspecified multivariate time series. A general overview over this likelihood theory and in particular Whittle estimates for stationary models may be found in the monographs Dzhaparidze (1986) and Taniguchi and Kakizawa (2000).

From a practical point of view the most famous outcome of this theory is the Whittle likelihood

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^2 f_{\eta}(\lambda) + \frac{I_T(\lambda)}{f_{\eta}(\lambda)} \right\} d\lambda \tag{85}$$

as an approximation of the negative log Gaussian likelihood (80) where $I_T(\lambda)$ is the periodogram. This likelihood has been used also beyond the classical framework – for example by Mikosch et al. (1995) for linear processes where the innovations have heavy tailed distributions, by Fox and Taqqu (1986) for long range dependent processes and by Robinson (1995) to construct semiparametric estimates for long range dependent processes.

The outcome of this likelihood theory goes far beyond the construction of the Whittle likelihood. Its technical core is the theory of Toeplitz matrices and in particular the approximation of the inverse of a Toeplitz matrix by the Toeplitz matrix of the inverse function. It is essentially this approximation which leads from the ordinary Gaussian likelihood to the Whittle likelihood. Beyond that the theory can be used to derive the convergence of experiments for Gaussian stationary processes in the Hájek-Le Cam sense, construct the properties of many tests and derive the properties of the exact MLE and the Whittle estimate (cf. Dzhaparidze (1986); Taniguchi and Kakizawa (2000)).

For locally stationary processes it turns out that this likelihood theory can be generalized in a nice way such that the classical likelihood theory for stationary processes arises as a special case. Technically speaking this is achieved by a generalization of Toeplitz matrices tailored especially for locally stationary processes (the matrix $U_T(\phi)$ defined in (92)).

Some results coming from this theory have already been stated in Section 4, namely the limit of the Kullback-Leibler information divergence in Theorem 4.4 and the limit of the Fischer information in Theorem 4.6. We now describe further results. We start with a decomposition of the periodogram leading to a Whittle-type likelihood. We have

$$I_{T}(\lambda) = \frac{1}{2\pi T} \left| \sum_{r=1}^{T} X_{r} \exp(-i\lambda r) \right|^{2}$$

$$= \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \left(\frac{1}{T} \sum_{t=1}^{T-|k|} X_{t} X_{t+|k|} \right) \exp(-i\lambda k)$$

$$= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{2\pi} \sum_{1 \le [t+0.5+k/2], [t+0.5-k/2] \le T} X_{[t+0.5+k/2], T} X_{[t+0.5-k/2], T} \exp(-i\lambda k)$$

$$= \frac{1}{T} \sum_{t=1}^{T} J_{T}(\frac{t}{T}, \lambda), \tag{87}$$

where the so-called pre-periodogram

$$J_T(u,\lambda) := \frac{1}{2\pi} \sum_{1 \le [uT+0.5+k/2], [uT+0.5-k/2] \le T} X_{[uT+0.5+k/2], T} X_{[uT+0.5-k/2], T} \exp(-i\lambda k)$$
 (88)

may be regarded as a local version of the periodogram at time t. While the ordinary periodogram $I_T(\lambda)$ is the Fourier transform of the covariance estimator of lag k over the whole segment (see (86)) the pre-periodogram just uses the pair $X_{[t+0.5+k/2]}X_{[t+0.5-k/2]}$ as a kind of "local estimator" of the covariance of lag k at time t (note that [t+0.5+k/2]-[t+0.5-k/2]=k). The pre-periodogram was introduced by Neumann and von Sachs (1997) as a starting point for a wavelet estimate of the time varying spectral density. The above decomposition means that the periodogram is the average of the pre-periodogram over time.

If we replace $I_T(\lambda)$ in (85) by the above average of the pre-periodogram and afterwards replace the model spectral density $f_{\eta}(\lambda)$ by the time varying spectral density $f_{\eta}(u,\lambda)$ of a nonstationary model, we obtain the generalized Whittle likelihood

$$\mathcal{L}_{T}^{GW}(\eta) := \frac{1}{T} \sum_{t=1}^{T} \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^{2} f_{\eta} \left(\frac{t}{T}, \lambda \right) + \frac{J_{T}(\frac{t}{T}, \lambda)}{f_{\eta}(\frac{t}{T}, \lambda)} \right\} d\lambda. \tag{89}$$

If the fitted model is stationary, i.e. $f_{\eta}(u,\lambda) = f_{\eta}(\lambda)$ then (due to (87)) the above likelihood is identical to the Whittle likelihood and we obtain the classical Whittle estimator. Thus the above likelihood is a true generalization of the Whittle likelihood to nonstationary processes.

In Theorem 5.4 we show that this likelihood is a very close approximation to the Gaussian log likelihood for locally stationary processes. In particular (we conjecture that) it is a better approximation than the block Whittle likelihood $\mathcal{L}_{T}^{BW}(\eta)$ from (21).

We now briefly state the asymptotic normality result in the parametric case. An example is the tvAR(2)-model with polynomial parameter curves from Section 2.4. Let

$$\hat{\eta}_T^{GW} := \underset{\eta \in \Theta_n}{\operatorname{argmin}} \mathcal{L}_T^{GW}(\eta) \tag{90}$$

be the corresponding quasi likelihood estimate, $\hat{\eta}_T^{ML}$ be the Gaussian MLE defined in (80), and η_0 as in (81) i.e. the model may be misspecified.

Theorem 5.1 Let $X_{t,T}$ be a locally stationary process. Under suitable regularity conditions we have in the case $\mu(\cdot) = \mu_{\eta}(\cdot) = 0$

$$\sqrt{T}(\hat{\eta}_T^{GW} - \eta_0) \stackrel{\mathcal{D}}{\to} \mathcal{N}(0, \Gamma^{-1}V\Gamma^{-1}) \quad and \quad \sqrt{T}(\hat{\eta}_T^{ML} - \eta_0) \stackrel{\mathcal{D}}{\to} \mathcal{N}(0, \Gamma^{-1}V\Gamma^{-1})$$

with

$$\Gamma_{ij} = \frac{1}{4\pi} \int_0^1 \int_{-\pi}^{\pi} (f - f_{\eta_0}) \nabla_{ij} f_{\eta_0}^{-1} d\lambda du + \frac{1}{4\pi} \int_0^1 \int_{-\pi}^{\pi} (\nabla_i \log f_{\eta_0}) (\nabla_j \log f_{\eta_0}) d\lambda du$$

and

$$V_{ij} = \frac{1}{4\pi} \int_0^1 \int_{-\pi}^{\pi} f\left(\nabla_i f_{\eta}^{-1}\right) f\left(\nabla_j f_{\eta}^{-1}\right) d\lambda du.$$

If the model is correctly specified then $V = \Gamma$ and Γ is the same as in Theorem 4.6 – that is both estimates are asymptotically Fisher-efficient. Even more the sequence of experiments is locally asymptotically normal (LAN) and both estimates are locally asymptotically minimax.

PROOF. See Dahlhaus (2000), Theorem 3.1. LAN and LAM has been proved for the MLE in Dahlhaus (1996b), Theorem 4.1 and 4.2 – these results together with the LAM-property of the generalized Whittle estimate also follow from the technical lemmas in Dahlhaus (2000) (cf. Remark 3.3 in that paper).

The corresponding result in the multivariate case and in the case $\mu(\cdot) \neq 0$ or $\mu_{\eta}(\cdot) \neq 0$ can be found in Dahlhaus (2000), Theorem 3.1.

A deeper investigation of $\mathcal{L}_{T}^{GW}(\eta)$ reveals that it can be derived from the Gaussian loglikelihood by an approximation of the inverse of the covariance matrix. Let $\underline{X} = (X_{1,T}, \dots, X_{T,T})'$, $\mu = (\mu(\frac{1}{T}), \dots, \mu(\frac{T}{T}))'$, and $\Sigma_{T}(A, B)$ and $U_{T}(\phi)$ be $T \times T$ matrices with (r, s)-entry

$$\Sigma_T(A,B)_{r,s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(i\lambda(r-s)\right) A_{r,T}(\lambda) B_{s,T}(-\lambda) d\lambda \tag{91}$$

and

$$U_T(\phi)_{r,s} = \int_{-\pi}^{\pi} \exp\left(i\lambda(r-s)\right) \phi\left(\frac{1}{T} \left[\frac{r+s}{2}\right]^*, \lambda\right) d\lambda \tag{92}$$

(r, s = 1, ... T) where the functions $A_{r,T}(\lambda)$, $B_{r,T}(\lambda)$, $\phi(u, \lambda)$ fulfill certain regularity conditions $(A_{r,T}(\lambda) B_{r,T}(\lambda))$ are transfer functions or their derivatives as defined in (77)). $[x]^* = [x]$ denotes the largest integer less or equal to x (we have added the * to discriminate the notation from brackets). Direct calculation shows that

$$\mathcal{L}_{T}^{GW}(\eta) = \frac{1}{4\pi} \frac{1}{T} \sum_{t=1}^{T} \int_{-\pi}^{\pi} \log\left[4\pi^{2} f_{\eta}(\frac{t}{T}, \lambda)\right] d\lambda + \frac{1}{8\pi^{2} T} (\underline{X} - \underline{\mu}_{\eta})' U_{T}(f_{\eta}^{-1}) (\underline{X} - \underline{\mu}_{\eta}). \tag{93}$$

Furthermore, the exact Gaussian likelihood is

$$\mathcal{L}_{T}^{E}(\eta) := \frac{1}{2}\log(2\pi) + \frac{1}{2T}\log\det\Sigma_{\eta} + \frac{1}{2T}(\underline{X} - \underline{\mu}_{\eta})'\Sigma_{\eta}^{-1}(\underline{X} - \underline{\mu}_{\eta})$$
(94)

where $\Sigma_{\eta} = \Sigma_T(A_{\eta}, A_{\eta}).$

Proposition 5.2 below states that $U_T(\frac{1}{4\pi^2}f_\eta^{-1})$ is an approximation of Σ_η^{-1} . Together with the generalization of the Szegö identity in Proposition 5.3 this implies that \mathcal{L}_T^{GW} is an approximation of \mathcal{L}_T^E (see Theorem 5.4). If the model is stationary, then A_η is constant in time and $\Sigma_\eta = \Sigma_T(A_\eta, A_\eta)$ is the Toeplitz matrix of the spectral density $f_\eta(\lambda) = \frac{1}{2\pi}|A_\eta|^2$ while $U_T(\frac{1}{4\pi^2}f_\eta^{-1})$ is the Toeplitz matrix of $\frac{1}{4\pi^2}f_\eta^{-1}$. This is the classical matrix-approximation leading to the Whittle likelihood (cf. Dzhaparidze, 1986).

Proposition 5.2 Under suitable regularity conditions we have for each $\varepsilon > 0$ for the Euclidean norm

$$\frac{1}{T} \left\| \Sigma_T(A, A)^{-1} - U_T(\{2\pi A\bar{A}'\}^{-1}) \right\|_2^2 = O(T^{-1+\varepsilon})$$
(95)

and

$$\frac{1}{T} \| U_T(\phi)^{-1} - U_T(\{4\pi^2 \phi\}^{-1}) \|_2^2 = O(T^{-1+\varepsilon}).$$

PROOF. See Dahlhaus (2000), Proposition 2.4.

By using the above approximation it is possible to prove the following generalization of the Szegö identity (cf. Grenander and Szegö (1958), Section 5.2) to locally stationary processes.

Proposition 5.3 Under suitable regularity conditions we have with $f(u, \lambda) = \frac{1}{2\pi} |A(u, \lambda)|^2$ for each $\varepsilon > 0$

$$\frac{1}{T}\log\det\Sigma_T(A,A) = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} \log\left[2\pi f(u,\lambda)\right] d\lambda \, du + O(T^{-1+\varepsilon}).$$

If $A = A_{\eta}$ depends on a parameter η then the $O(T^{-1+\varepsilon})$ term is uniform in η .

PROOF. See Dahlhaus (2000), Proposition 2.5.

In certain situations the right hand side can be written in the form $\int_0^1 \log(2\pi\sigma^2(u)) du$ where $\sigma^2(u)$ is the one step prediction error at time u.

The mathematical core of the above results consists of the derivation of properties of products of matrices $\Sigma_T(A, B)$, $\Sigma_T(A, A)^{-1}$ and $U_T(\phi)$. These properties are derived in Dahlhaus (2000) in Lemma A.1, A.5, A.7 and A.8. These results are generalizations of corresponding results in the stationary case proved by several authors before.

We now state the properties of the different likelihoods.

Theorem 5.4 Under suitable regularity conditions we have for k = 0, 1, 2

(i)
$$\sup_{\eta \in \Theta_{\eta}} \left| \nabla^{k} \left\{ \mathcal{L}_{T}^{GW}(\eta) - \mathcal{L}_{T}^{E}(\eta) \right\} \right| \stackrel{P}{\to} 0,$$

(ii)
$$\sup_{\theta \in \Theta_{\eta}} \left| \nabla^{k} \left\{ \mathcal{L}_{T}^{GW}(\eta) - \mathcal{L}(\eta) \right\} \right| \stackrel{P}{\to} 0,$$

(iii)
$$\sup_{\eta \in \Theta_{\eta}} \left| \nabla^{k} \left\{ \mathcal{L}_{T}^{E}(\eta) - \mathcal{L}(\eta) \right\} \right| \stackrel{P}{\to} 0.$$

PROOF. See Dahlhaus (2000), Theorem 3.1.

Under stronger assumptions one may also conclude that $\hat{\eta}_T^{GW} - \hat{\eta}_T^{ML} = O_p(T^{-1+\varepsilon})$ which means that $\hat{\eta}_T^{GW}$ is a close approximation of the MLE. A sketch of the proof is given in Dahlhaus (2000), Remark 3.4.

Remark 5.5 It is interesting to compare the generalized Whittle estimate $\hat{\eta}_T^{GW}$ and its underlying approximation $U_T(\frac{1}{4\pi^2}f_\eta^{-1})$ of Σ_η^{-1} with the block Whittle estimate $\hat{\eta}_T^{BW}$ defined in (21). There some overlapping block Toeplitz matrices are used as an approximation which we regard as worse. A similar result as in Proposition 5.2 has been proved in Lemma 4.7 of Dahlhaus (1996a) for this approximation. We conjecture that also a similar result as in Theorem 5.4 with $\mathcal{L}_T^{BW}(\eta)$ can be proved and even more that $\hat{\eta}_T^{BW} - \hat{\eta}_T^{ML} = O_p(\frac{N}{T^{1-\varepsilon}} + \frac{1}{N})$ (this is more a vague guess than a solid conjecture) which means that the latter approximation and presumably also the estimate $\hat{\eta}_T^{BW}$ are worse. It would be interesting to have more rigorous results and a careful simulation study with a comparison of both estimates.

We now remember the generalized Whittle likelihood from (89) which was

$$\mathcal{L}_{T}^{GW}(\eta) = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^{2} f_{\eta} \left(\frac{t}{T}, \lambda \right) + \frac{J_{T}(\frac{t}{T}, \lambda)}{f_{\eta}(\frac{t}{T}, \lambda)} \right\} d\lambda.$$

Contrary to the true Gaussian likelihood this is a sum over time and the summands can be interpreted as a local log likelihood at time point t. We therefore define

$$\ell_{t,T}^*(\boldsymbol{\theta}) := \frac{1}{4\pi} \int_{-\pi}^{\pi} \left\{ \log 4\pi^2 f_{\boldsymbol{\theta}}(\lambda) + \frac{J_T(\frac{t}{T}, \lambda)}{f_{\boldsymbol{\theta}}(\lambda)} \right\} d\lambda. \tag{96}$$

(to avoid confusion we mention that we use the notation η for a finite dimensional parameter which determines the whole curve, that is $\boldsymbol{\theta}(\cdot) = \boldsymbol{\theta}_{\eta}(\cdot)$ and $f_{\eta}(u,\lambda) = f_{\boldsymbol{\theta}_{\eta}(u)}(\lambda)$). We now can construct all nonparametric estimates (26)–(30) with $\ell_{t,T}(\boldsymbol{\theta})$ replaced by $\ell_{t,T}^*(\boldsymbol{\theta})$ leading in each of the 5 cases to an alternative local quasi likelihood estimate.

The parametric estimator (30) with this local likelihood is the estimate $\hat{\eta}_T^{GW}$ from above. The orthogonal series estimator (28) with $\ell_{t,T}^*(\boldsymbol{\theta})$ has been investigated for a truncated wavelet series expansion together with nonlinear thresholding in Dahlhaus and Neumann (2001). The method is fully automatic and adapts to different smoothness classes. It is shown that the usual rates of convergence in Besov classes are attained up to a logarithmic factor. The nonparametric estimator (29) with $\ell_{t,T}^*(\boldsymbol{\theta})$ is studied in Dahlhaus and Polonik (2006). Rates of convergence, depending on the metric entropy of the function space, are derived. This includes in particular maximum likelihood estimates derived under shape restriction. The main tool for deriving these results is the so called empirical spectral processes discussed in the next section. The kernel estimator (26) with $\ell_{t,T}^*(\boldsymbol{\theta})$ has been investigated in Dahlhaus (2009), Example 3.6. Uniform convergence has been proved in Dahlhaus and Polonik (2009), Section 4 (see also Example 6.6 and Theorem 6.9 below). The local polynomial fit (27) has not been investigated yet in combination with this likelihood.

The whole topic needs a more careful investigation – both theoretically and from a practical point including simulations and data-examples.

6 Empirical spectral processes

We now emphasize the relevance of the empirical spectral process for linear locally stationary time series. The theory of empirical processes does not only play a major role in proving theoretical results for statistical methods but also provides a deeper understanding of many techniques and the arising problems. The theory was first developed for stationary processes (c.f. Dahlhaus (1988), Mikosch and Norvaisa (1997), Fay and Soulier (2001)) and then extended to locally stationary processes in Dahlhaus and Polonik (2006,2009) and Dahlhaus (2009). The empirical spectral process is indexed by classes of functions. Basic results that later lead to several statistical applications are a functional central limit theorem, a maximal exponential inequality and a Glivenko-Cantelli type convergence result. All results use conditions based on the metric entropy of the index class. Many results stated earlier in this article have been proved by using these techniques.

The empirical spectral process is defined by

$$E_T(\phi) := \sqrt{T} \left(F_T(\phi) - F(\phi) \right)$$

where

$$F(\phi) := \int_0^1 \int_{-\pi}^{\pi} \phi(u, \lambda) f(u, \lambda) d\lambda du$$
 (97)

is the generalized spectral measure and

$$F_T(\phi) := \frac{1}{T} \sum_{t=1}^T \int_{-\pi}^{\pi} \phi(\frac{t}{T}, \lambda) J_T(\frac{t}{T}, \lambda) d\lambda$$
 (98)

the empirical spectral measure with the pre-periodogram as defined in (88).

We first give an overview of statistics that can be written in the form $F_T(\phi)$ - several of them have already been discussed earlier in this article (K_T always denotes a kernel function).

- 1. $\phi(u,\lambda) = K_T(u_0 u) \cos(\lambda k)$ local covariance estimator (9) a.s.; Remark 6.7
- 2. $\phi(u,\lambda) = K_T(u_0 u) K_T(\lambda_0 \lambda)$ spectral density estimator (84) a.s.; Remark 6.7
- 3. $\phi(u,\lambda) = K_T(u_0 u) \nabla f_{\theta_0}(u,\lambda)^{-1} \nabla \mathcal{L}_T^{GW}(u_0,\theta_0), \theta_0 = \theta_0(u_0)$ Example 6.6
- 4. $\phi(u,\lambda) \approx K_T(u_0-u) \nabla f_{\theta_0}(u,\lambda)^{-1}$ local least squares Ex. 3.1; Rem. 6.7
- 5. $\phi(u,\lambda) = \nabla f_{\eta_0}(u,\lambda)^{-1}$ param. Whittle estimator Example 3.7 in Dahlhaus and Polonik (2009)
- 6. $\phi(u,\lambda) = (I_{[0,u_0]}(u) u_0)I_{[0,\lambda_0]}(\lambda)$ testing stationarity Example 6.10
- 7. $\phi(u,\lambda) = \cos(\lambda k)$ stationary covariance Remark 6.2
- 8. $\phi(u,\lambda) = \nabla f_{\eta_0}(\lambda)^{-1}$ stat. Whittle estimator Remark 6.2
- 9. $\phi(u,\lambda) = K_T(\lambda_0 \lambda)$ stationary spectral density Remark 6.2

Examples 1-4 and 9 are examples with index functions ϕ_T depending on T. More complex examples are nonparametric maximum likelihood estimation under shape restrictions

(Dahlhaus and Polonik, 2006), model selection with a sieve estimator (Van Bellegem and Dahlhaus, 2006) and wavelet estimates (Dahlhaus and Neumann, 2001). Moreover $F_T(\phi)$ occurs with local polynomial fits (Kim, 2001; Jentsch, 2006) and several statistics suitable for goodness of fit testing. These applications are quite involved.

However, applications are limited to quadratic statistics, that is the empirical spectral measure is usually of no help in dealing with nonlinear models. Furthermore, for linear processes the empirical process only applies without further modification to the (score function and the Hessian of the) likelihood $\mathcal{L}_T^{GW}(\eta)$ and its local variant $\mathcal{L}_T^{GW}(u,\boldsymbol{\theta})$ and the local Whittle likelihood $\mathcal{L}_T^W(u,\boldsymbol{\theta})$. It also applies to the exact likelihood $\mathcal{L}_T^E(\eta)$ after proving $\nabla \mathcal{L}_T^{GW}(\eta_0) - \nabla \mathcal{L}_T^E(\eta_0) = o_p(T^{-1/2})$ (see also Theorem 5.4 (i)) and the conditional likelihoods $\mathcal{L}_T^C(\eta)$ and $\mathcal{L}_T^C(u,\boldsymbol{\theta})$ in the tvAR-case (see Remark 6.7 - in the general case this is not clear yet). For the block Whittle likelihood $\mathcal{L}_T^{BW}(\eta)$ it may also be applied after establishing $\nabla \mathcal{L}_T^{GW}(\eta_0) - \nabla \mathcal{L}_T^{BW}(\eta_0) = o_p(T^{-1/2})$. However, this is also not clear yet.

We first state a central limit theorem for $E_T(\phi)$ with index functions ϕ that do not vary with T. We use the assumption of bounded variation in both components of $\phi(u, \lambda)$. Besides the definition in (61) we need a definition in two dimensions. Let

$$V^{2}(\phi) = \sup \Big\{ \sum_{j,k=1}^{\ell,m} |\phi(u_{j}, \lambda_{k}) - \phi(u_{j-1}, \lambda_{k}) - \phi(u_{j}, \lambda_{k-1}) + \phi(u_{j-1}, \lambda_{k-1})| :$$

$$0 \le u_{0} < \dots < u_{\ell} \le 1; \ -\pi \le \lambda_{0} < \dots < \lambda_{m} \le \pi; \ \ell, m \in \mathbf{N} \Big\}.$$

For simplicity we set

$$\|\phi\|_{\infty,V} := \sup_{u} V(\phi(u,\cdot)), \quad \|\phi\|_{V,\infty} := \sup_{\lambda} V(\phi(\cdot,\lambda)),$$
$$\|\phi\|_{V,V} := V^{2}(\phi) \quad \text{and} \quad \|\phi\|_{\infty,\infty} := \sup_{u,\lambda} |\phi(u,\lambda)|.$$

Theorem 6.1 Suppose Assumption 4.1 holds and let ϕ_1, \ldots, ϕ_d be functions with $\|\phi_j\|_{\infty,V}$, $\|\phi_j\|_{V,\infty}$, $\|\phi_j\|_{V,V}$ and $\|\phi_j\|_{\infty,\infty}$ being finite $(j=1,\ldots,d)$. Then

$$(E_T(\phi_j))_{j=1,\dots,d} \xrightarrow{\mathcal{D}} (E(\phi_j))_{j=1,\dots,d}$$

where $(E(\phi_j))_{j=1,...,d}$ is a Gaussian random vector with mean 0 and

$$cov(E(\phi_j), E(\phi_k)) = 2\pi \int_0^1 \int_{-\pi}^{\pi} \phi_j(u, \lambda) \left[\phi_k(u, \lambda) + \phi_k(u, -\lambda)\right] f^2(u, \lambda) d\lambda du$$

$$+ \kappa_4 \int_0^1 \left(\int_{-\pi}^{\pi} \phi_j(u, \lambda_1) f(u, \lambda_1) d\lambda_1\right) \left(\int_{-\pi}^{\pi} \phi_k(u, \lambda_2) f(u, \lambda_2) d\lambda_2\right) du.$$
(99)

PROOF. See Dahlhaus and Polonik (2009), Theorem 2.5.

Remark 6.2 (Stationary processes/Model misspecification by stationary models)

The classical central limit theorem for the weighted periodogram in the stationary case can be obtained as a corollary: If $\phi(u,\lambda) = \tilde{\phi}(\lambda)$ is time-invariant then

$$F_T(\phi) = \int_{-\pi}^{\pi} \tilde{\phi}(\lambda) \frac{1}{T} \sum_{t=1}^{T} J_T(\frac{t}{T}, \lambda) d\lambda = \int_{-\pi}^{\pi} \tilde{\phi}(\lambda) I_T(\lambda) d\lambda$$
 (100)

(see(87)) that is $F_T(\phi)$ is the classical spectral measure in the stationary case with the following applications:

- (i) $\phi(u,\lambda) = \tilde{\phi}(\lambda) = \cos \lambda k$ is the empirical covariance estimator of lag k;
- (ii) $\phi(u,\lambda) = \tilde{\phi}(\lambda) = \frac{1}{4\pi} \nabla f_{\theta}^{-1}(\lambda)$ is the score function of the Whittle likelihood.

Theorem 6.1 gives the asymptotic distribution for these examples - both in the stationary case and in the misspecified case where the true underlying process is only locally stationary. If $\phi(u,\lambda) = \tilde{\phi}(\lambda)$ is a kernel we obtain an estimate of the spectral density whose asymptotic distribution is a special case of Theorem 6.3 below (also in the misspecified case).

We now state a central limit theorem for $F_T(\phi_T) - F(\phi_T)$ with index functions ϕ_T depending on T. In addition we extend the hitherto definitions to tapered data

$$X_{t,T}^{(h_T)} := h_T \left(\frac{t}{T}\right) \cdot X_{t,T}$$

where $h_T:(0,1]\to[0,\infty)$ is a data taper (with $h_T(\cdot)=I_{(0,1]}(\cdot)$ being the nontapered case). The main reason for introducing data-tapers is to include segment estimates - see the discussion below. As before the empirical spectral measure is defined by

$$F_T(\phi) = F_T^{(h_T)}(\phi) := \frac{1}{T} \sum_{t=1}^T \int_{-\pi}^{\pi} \phi(\frac{t}{T}, \lambda) J_T^{(h_T)}(\frac{t}{T}, \lambda) d\lambda$$
 (101)

now with the tapered pre-periodogram

$$J_T^{(h_T)}\left(\frac{t}{T},\lambda\right) = \frac{1}{2\pi} \sum_{k:1 < [t+1/2 \pm k/2] < T} X_{[t+1/2 + k/2],T}^{(h_T)} X_{[t+1/2 - k/2],T}^{(h_T)} \exp(-i\lambda k)$$
 (102)

(we mention that in some cases a rescaling may be necessary for $J_T^{(h_T)}(u,\lambda)$ to become a pre-estimate of $f(u,\lambda)$ - an obvious example for this is $h_T(u) = (1/2) I_{(0,1]}(u)$).

 $F(\phi)$ is the theoretical counterpart of $F_T(\phi)$

$$F(\phi) = F^{(h_T)}(\phi) := \int_0^1 h_T^2(u) \int_{-\pi}^{\pi} \phi(u, \lambda) f(u, \lambda) d\lambda du.$$
 (103)

Note that (87) also holds with a data-taper, that is

$$\frac{1}{T} \sum_{t=1}^{T} J_T^{(h_T)} \left(\frac{t}{T}, \lambda \right) = \frac{H_{2,T}}{T} I_T^{(h_T)} (\lambda)$$

with the tapered periodogram

$$I_T^{(h_T)}(\lambda) := \frac{1}{2\pi H_{2,T}} \left| \sum_{s=1}^T X_s^{(h_T)} \exp(-i\lambda s) \right|^2 \quad \text{where} \quad H_{2,T} := \sum_{t=1}^T h_T \left(\frac{t}{T}\right)^2. \tag{104}$$

An important special case is $h_T^{(u_0)}(\frac{t}{T}) := k(\frac{u_0 - t/T}{b_T})$ with bandwidth b_T and k having compact support on $[-\frac{1}{2}, \frac{1}{2}]$. If $b_T := N/T$ then $I_T^{(h_T)}(\lambda) = I_T(u_0, \lambda)$ with $I_T(u_0, \lambda)$ as in (19). If in addition $\phi(u, \lambda) = \psi(\lambda)$ we obtain

$$F_T(\phi) = \int_{-\pi}^{\pi} \psi(\lambda) \left(\frac{1}{T} \sum_{t=1}^{T} J_T^{(h_T)} \left(\frac{t}{T}, \lambda \right) \right) d\lambda = \frac{H_{2,T}}{T} \int_{-\pi}^{\pi} \psi(\lambda) I_T^{(h_T)}(\lambda) d\lambda.$$

For example for $\psi(\lambda) := \exp i\lambda k$ this is exactly $\frac{H_{2,T}}{T} \hat{c}_T(u_0,k)$ with the tapered covariance estimate from (8). In this case $\frac{H_{2,T}}{T}$ is proportional to b_T .

The last example suggests to use $\frac{1}{H_{2,T}}$ instead of $\frac{1}{T}$ in (101) as a norming constant. However, this is not always the right choice (as can be seen from case (ii) in Remark 6.5).

It turns out that in the above situation the rate of converge of the empirical spectral measure becomes $\sqrt{T}/\rho_2^{(h_T)}(\phi_T)$ where

$$\rho_2^{(h_T)}(\phi) := \left(\int_0^1 h_T^4(u) \int_{-\pi}^{\pi} \phi(u, \lambda)^2 d\lambda \ du \right)^{1/2}.$$

Therefore we can embed this case into the situation treated in the last section by studying the convergence of

$$E_T^{(h_T)} \left(\frac{\phi_T}{\rho_2^{(h_T)}(\phi_T)} \right) = \frac{\sqrt{T}}{\rho_2^{(h_T)}(\phi_T)} \left(F_T(\phi_T) - F^{(h_T)}(\phi_T) \right).$$

Furthermore, let

$$c_{E}^{(h_{T})}(\phi_{j},\phi_{k}) := 2\pi \int_{0}^{1} h_{T}^{4}(u) \int_{-\pi}^{\pi} \phi_{j}(u,\lambda) \left[\phi_{k}(u,\lambda) + \phi_{k}(u,-\lambda) \right] f^{2}(u,\lambda) d\lambda du$$

$$+ \kappa_{4} \int_{0}^{1} h_{T}^{4}(u) \left(\int_{-\pi}^{\pi} \phi_{j}(u,\lambda_{1}) f(u,\lambda_{1}) d\lambda_{1} \right) \left(\int_{-\pi}^{\pi} \phi_{k}(u,\lambda_{2}) f(u,\lambda_{2}) d\lambda_{2} \right) du.$$
(105)

Theorem 6.3 Suppose that $X_{t,T}$ is a locally stationary process and suitable regularity conditions hold. If the limit

$$\Sigma_{j,k} := \lim_{T \to \infty} \frac{c_E^{(h_T)}(\phi_{Tj}, \phi_{Tk})}{\rho_2^{(h_T)}(\phi_{Tj}) \, \rho_2^{(h_T)}(\phi_{Tk})} \tag{106}$$

exists for all j, k = 1, ..., d then

$$\left(\frac{\sqrt{T}}{\rho_2^{(h_T)}(\phi_{Tj})} \left(F_T(\phi_{Tj}) - F^{(h_T)}(\phi_{Tj})\right)\right)_{j=1,\dots,d} \stackrel{\mathcal{D}}{\to} \mathcal{N}(0,\Sigma). \tag{107}$$

Remark 6.4 (Bias) In addition we have the bias term

$$\frac{\sqrt{T}}{\rho_2^{(h_T)}(\phi_T)} \left(F^{(h_T)}(\phi_T) - \lim_{T \to \infty} F^{(h_T)}(\phi_T) \right).$$

The magnitude of this bias depends on the smoothness of the time varying spectral density. In this section we usually require conditions such that this bias is of lower order. This is different in Section 3 where the bias has explicitly been investigated. \Box

Remark 6.5 (Typical applications) A typical application of this result is the case of kernel type local estimators which can be constructed by using kernels, data-tapers or a combination of both:

(i)
$$\phi_T(u,\lambda) = \frac{1}{b_T} K(\frac{u_0 - u}{b_T}) \ \psi(\lambda) \qquad h_T(\cdot) = I_{(0,1]}(\cdot)$$

(ii)
$$\phi_T(u,\lambda) = \frac{1}{b_T} K(\frac{u_0 - u}{b_T}) \psi(\lambda)$$
 $h_T(u) = I_{[u_0 - b_T/2, u_0 + b_T/2]}(u)$

(iii)
$$\phi_T(u,\lambda) = \psi(\lambda)$$
 $h_T(\frac{t}{T}) = k(\frac{u_0 - t/T}{b_T})$

where $K(\cdot)$ and $k(\cdot)$ are kernel functions and b_T is the bandwidth. If $K(\cdot) = k(\cdot)^2$ then the resulting estimates all have the same asymptotic properties - see below. Dependent on the function $\psi(\lambda)$ this leads to different applications: If we set $\psi(\lambda) = \cos(\lambda k)$ the estimate (iii) is the estimate $\hat{c}_T(u_0, k)$ from (8) and (i) is "almost" the estimate $\tilde{c}_T(u_0, k)$ from (9) (for k even it is exactly the same, for k odd the difference can be treated with the methods mentioned in Remark 5.5).

We now show how Theorem 6.3 leads to the asymptotic distribution for these estimates:

(i) If $K(\cdot)$ and $\psi(\cdot)$ are of bounded variation and $b_T \to 0$, $b_T T \to \infty$ then the regularity conditions of Theorem 5.5 are fulfilled (see Dahlhaus (2009), Remark 3.4). Furthermore,

$$\rho_2^{(h_T)}(\phi_T) = \rho_2(\phi_T) = \left(\frac{1}{b_T} \int K^2(x) \, dx \int |\psi(\lambda)|^2 \, d\lambda\right)^{1/2} \approx b_T^{-1/2}.$$
 (108)

For $f(\cdot, \lambda)$ continuous at u_0 we have

$$c_E^{(h_T)}(\phi_{Tj}, \phi_{Tk}) \sim \frac{1}{b_T} \int K^2(x) dx \left[2\pi \int_{-\pi}^{\pi} \psi_j(\lambda) \left[\psi_k(\lambda) + \psi_k(-\lambda) \right] f^2(u_0, \lambda) d\lambda \right]$$
$$+ \kappa_4 \left(\int_{-\pi}^{\pi} \psi_j(\lambda_1) f(u_0, \lambda_1) d\lambda_1 \right) \left(\int_{-\pi}^{\pi} \psi_k(\lambda_2) f(u_0, \lambda_2) d\lambda_2 \right) =: \frac{1}{b_T} \Gamma_{jk}$$

that is (106) is also fulfilled and we obtain from Theorem 6.3

$$\sqrt{b_T T} \left(F_T(\phi_{Tj}) - F^{(h_T)}(\phi_{Tj}) \right)_{j=1,\dots,d} \stackrel{\mathcal{D}}{\to} \mathcal{N}(0,\Gamma). \tag{109}$$

(ii) The additional taper $h_T(u) = I_{[u_0 - b_T/2, u_0 + b_T/2]}(u)$ implies that we use only data from the interval $[u_0 - b_T/2, u_0 + b_T/2]$. We obtain in this case

$$\rho_2^{(h_T)}(\phi_T) = \left(\int_0^1 \frac{1}{b_T^2} K\left(\frac{u_0 - u}{b_T}\right)^2 du \int_{-\pi}^{\pi} |\psi(\lambda)|^2 d\lambda\right)^{1/2},$$

i.e. we have the same $\rho_2^{(h_T)}(\phi_T)$ as above. Furthermore, $c_E^{(h_T)}(\phi_T, \phi_T)$ is the same. Thus we obtain the same asymptotic distribution and the same rate of convergence.

(iii) If $K(\cdot) = k(\cdot)^2$ we obtain in this case

$$\frac{1}{b_T} \rho_2^{(h_T)}(\phi_T) = \left(\int_0^1 \frac{1}{b_T^2} K\left(\frac{u_0 - u}{b_T}\right)^2 du \int_{-\pi}^{\pi} |\psi(\lambda)|^2 d\lambda \right)^{1/2}$$

i.e. we obtain again the same expression. Furthermore, $\frac{1}{b_T^2}c_E^{(h_T)}(\phi_{Tj},\phi_{Tk})$ is the same as $c_E^{(h_T)}(\phi_{Tj},\phi_{Tk})$ above. Thus we have again the same asymptotic distribution and the same rate of convergence.

Example 6.6 (Curve estimation by local quasi likelihood estimates)

Local Whittle estimates on a segment where defined in (17) and discussed in Example 3.6 (the bias was heuristically derived in Example 3.5). We now consider the presumably equivalent local quasi likelihood estimate defined by

$$\hat{\boldsymbol{\theta}}_{T}^{GW}(u_0) := \underset{\boldsymbol{\theta} \in \Theta}{\operatorname{argmin}} \mathcal{L}_{T}^{GW}(u_0, \boldsymbol{\theta})$$
(110)

with

$$\mathcal{L}_{T}^{GW}(u_0, \boldsymbol{\theta}) := \frac{1}{4\pi} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{b_T} K\left(\frac{u_0 - t/T}{b_T}\right) \int_{-\pi}^{\pi} \left\{ \log 4\pi^2 f_{\boldsymbol{\theta}}(\lambda) + \frac{J_T(\frac{t}{T}, \lambda)}{f_{\boldsymbol{\theta}}(\lambda)} \right\} d\lambda. \quad (111)$$

(this is a combination of (26) and (96)). The asymptotic normality of the estimate $\hat{\boldsymbol{\theta}}_{T}^{GW}(u_0)$ is derived in Dahlhaus (2009), Example 3.6. Key steps in the proof are the fact that both the score function and the Hessian matrix can be written in terms of the empirical spectral process leading to a rather simple proof. For example

$$\sqrt{b_T T} \nabla_i \mathcal{L}_T \left(u_0, \boldsymbol{\theta}_0(u_0) \right) = \sqrt{b_T T} \left(F_T(\phi_{T, u_0, i}) - F(\phi_{T, u_0, i}) \right) + o_p(1)$$
(112)

where $\phi_{T,u_0,i}(v,\lambda) := \frac{1}{b_T}K(\frac{u_0-v}{b_T})\frac{1}{4\pi}\nabla_i f_{\boldsymbol{\theta}}^{-1}(\lambda)_{|\boldsymbol{\theta}=\boldsymbol{\theta}_0(u_0)}$. Theorem 6.3 then immediately gives the asymptotic normality of the score function and after some additional considerations also asymptotic normality of $\hat{\boldsymbol{\theta}}_T^{GW}(u_0)$. For details see Dahlhaus (2009), Example 3.6.

The above estimate corresponds to case (i) in Remark 6.5. Case (iii) in Remark 6.5 leads instead to the tapered Whittle estimate $\hat{\boldsymbol{\theta}}_T^W(u_0)$ on the segment, since for $h_T^{(u_0)}(\frac{t}{T}) := k\left(\frac{u_0-t/T}{b_T}\right)$ we have $I_T^{(h_T)}(\lambda) = I_T(u_0,\lambda)$ with $I_T(u_0,\lambda)$ as in (19). This estimate has the same asymptotic properties provided $k(\cdot)^2 = K(\cdot)$. It's asymptotic properties can now also be derived by using Theorem 6.3.

Remark 6.7 (Related estimates) Many estimates are only approximately of the form discussed above - for example the sum statistic

$$F_T^{\Sigma}(\phi) := \frac{2\pi}{T^2} \sum_{t=1}^T \sum_{j=1}^T \phi\left(\frac{t}{T}, \lambda_j\right) J_T^{(h_T)}\left(\frac{t}{T}, \lambda_j\right)$$
(113)

where $\lambda_j = \frac{2\pi j}{T}$ - or representations in terms of the Fourier-coefficients. Important examples of related estimates are the spectral density estimate (84), the covariance estimates (9) and (10) and the score function of the local least squares tvAR(p)-estimate from Example 3.1. We mention that the central limit theorem in Theorem 6.3 also holds for several modified estimators. Details and proofs can be found in Dahlhaus (2009), Section 4.

We now briefly mention the exponential inequality. Since this is a non-asymptotic result it holds regardless whether ϕ depends on T. Let $\rho_{2,T}(\phi) := \left(\frac{1}{T} \sum_{t=1}^{T} \int_{-\pi}^{\pi} \phi(\frac{t}{T}, \lambda)^2 d\lambda\right)^{1/2}$.

Theorem 6.8 (Exponential inequality) Under suitable regularity conditions we have for all $\eta > 0$

$$P\left(\left|\sqrt{T}\left(F_T(\phi) - \mathbf{E}F_T(\phi)\right)\right| \ge \eta\right) \le c_1 \exp\left(-c_2\sqrt{\frac{\eta}{\rho_{2,T}(\phi)}}\right)$$
 (114)

with some constants c_1 , $c_2 > 0$ independent of T.

This result is proved in Dahlhaus and Polonik (2009), Theorem 2.7. There exist several versions of this result - for example in the Gaussian case it is possible to omit the $\sqrt{\cdot}$ in

(114) or to prove a Bernstein-type inequality which is even stronger (cf. Dahlhaus and Polonik, 2006, Theorem 4.1).

Subsequently, a maximal inequality, i.e. an exponential inequality for $\sup_{\phi \in \Phi} |E_T(\phi)|$ has been proved in Dahlhaus and Polonik (2009), Theorem 2.9 under conditions on the metric entropy of the corresponding function class Φ . We refer to that paper for details.

With the maximal inequality tightness of the empirical spectral process can be proved leading to a functional central limit theorem for the empirical spectral process indexed by a function class (cf. Dahlhaus and Polonik (2009), Theorem 2.11). Furthermore a Glivenko Cantelli type result for the empirical spectral process can be obtained (Theorem 2.12).

Other applications of the maximal inequality are for example uniform rates of convergence for different estimates. As an example we now state a uniform convergence result for the local quasi likelihood estimate $\hat{\boldsymbol{\theta}}_{T}^{GW}(u_0)$ from (110).

Theorem 6.9 Let $X_{t,T}$ be a locally stationary process with $\mu(\cdot) \equiv 0$. Under suitable regularity conditions (in particular under the assumption that $f_{\theta}(\lambda)$ is twice differentiable in θ with uniformly Lipschitz continuous derivatives in λ) we have for $b_T T >> (\log T)^6$

$$\sup_{u_0 \in [b_T/2, 1-b_T/2]} \|\hat{\boldsymbol{\theta}}_T^{GW}(u_0) - \boldsymbol{\theta}_0(u)\|_2 = O_p \left(\frac{1}{\sqrt{b_T T}} + b_T^2\right),$$

that is for $b_T \sim T^{-1/5}$ we obtain the uniform rate $O_p(T^{-2/5})$.

PROOF. The result has been proved in Dahlhaus and Polonik (2009), Theorem 4.1.

Example 6.10 (Testing for stationarity) Another application of the maximal inequality is the derivation of a functional central limit for the empirical spectral process. A possible application is a test for stationarity. We briefly present the idea - although we clearly mention that the construction below is finally not successful. The idea for a test of stationarity is to test whether the time varying spectral density $f(u, \lambda)$ is constant in u. This is for example achieved by the test statistic

$$\sqrt{T} \sup_{u \in [0,1]} \sup_{\lambda \in [0,\pi]} \left| F_T(u,\lambda) - u F_T(1,\lambda) \right| \tag{115}$$

where

$$F_T(u,\lambda) := \frac{1}{T} \sum_{t=1}^{[uT]} \int_0^{\lambda} J_T(\frac{t}{T},\mu) d\mu$$

is an estimate of the integrated time frequency spectral density $F(u,\lambda) := \int_0^u \int_0^\lambda f(v,\mu) \, d\mu dv$, and

 $u F_T(1,\lambda) = u \int_0^{\lambda} I_T(\mu) d\mu$

is the corresponding estimate of $F(u, \lambda)$ under the hypothesis of stationarity where $f(v, \mu) = f(\mu)$. Under the hypothesis of stationarity we have

$$F(u,\lambda) - u F(1,\lambda) = \int_0^1 \int_0^\lambda (I_{[0,u]}(v) - u) f(\mu) \, d\mu \, dv = 0$$

and therefore

$$\sqrt{T}\Big(F_T(u,\lambda) - u F_T(1,\lambda)\Big) = E_T(\phi_{u,\lambda})$$

with $\phi_{u,\lambda}(v,\mu) = \left(I_{[0,u]}(v) - u\right)I_{[0,\lambda]}(\mu)$. We now need functional convergence of $E_T(\phi_{u,\lambda})$. Convergence of the finite dimensional distributions follows from Theorem 6.1 above. Tightness and therefore the functional convergence follows from Theorem 2.11 of Dahlhaus and Polonik (2009). As a consequence we obtain under the null hypothesis

$$\sqrt{T} \left(F_T(u,\lambda) - u \, F_T(1,\lambda) \right)_{u \in [0,1], \lambda \in [0,\pi]} \stackrel{\mathcal{D}}{\to} E(u,\lambda)_{u \in [0,1], \lambda \in [0,\pi]}$$

where $E(u, \lambda)$ is a Gaussian process. If $\kappa_4 = 0$ (Gaussian case) and $f(\mu) = c$ it can be shown that this is the Kiefer-Müller process. However, for general f it is a difficult and unsolved task to calculate or estimate the limit distribution and in particular the distribution of the test statistic in (115). This may be done by transformations (like U_p - or T_p - type transforms) and/or by finding an adequate bootstrap method.

We mention that Paparoditis (2009, 2010) has given two different solutions of this testing problem. \Box

7 Additional topics and further references

This section gives an overview over additional topics with further references. We concentrate on work which uses the infill asymptotic approach of local stationarity. Even in this case it is not possible to give a complete overview.

1. Locally stationary wavelet processes: There exists a large number of papers on the use of wavelets for modeling locally stationary processes. The first type of application is to estimate the parameter curves via the use of wavelets. This has been mentioned a few times in the above presentation (cf. (28)).

A breakthrough for the application of wavelets to nonstationary processes was the introduction of "locally stationary wavelet processes" by Nason et.al. (2000). This class is somehow the counterpart to the representation (60) for locally stationary processes. It also uses a rescaling argument - thus making all methods for these processes accessible to a meaningful asymptotic theory. Locally stationary wavelet processes are processes with the wavelet representation

$$X_{t,T} = \mu(\frac{t}{T}) + \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} w_{j,k;T} \,\psi_{j,k-t} \,\xi_{j,k}$$
(116)

where $\{\xi_{j,k}\}$ are a collection of uncorrelated random variables with mean 0 and variance 1, the $\{\psi_{j,t}\}$ are a set of discrete nondecimated wavelets (compactly supported oscillatory vectors with support proportional to 2^j), and $\{w_{j,k;T}\}$ are a collection of amplitudes that are smooth in a particular way as a function of k. The smoothness of $\{w_{j,k;T}\}$ controls the degree of local stationarity of $X_{t,T}$. The spectrum is linked to the process by $\{w_{j,k;T}\} \approx S_j(\frac{k}{T})$. Nason et.al. (2000) also define the "evolutionary wavelet spectrum" and show how this can be estimated by a smoothed wavelet periodogram. In addition this leads to an estimate of the local covariance. An introduction to LSW-processes and an overview on early results for such processes can be found in Nason and von Sachs (1999). Fryzlewicz and Nason (2006) suggest the use of a Haar-Fisz method for the estimation of evolutionary wavelet spectra by combining Haar wavelets and the variance stabilizing Fisz transform. Van Bellegem and von Sachs (2008) consider wavelet processes whose spectral density function changes very quickly in time. By using a wavelet-type transform of the autocovariance function with respect to so-called autocorrelation wavelets they propose a pointwise adaptive estimator of the time varying autocovariance and the time varying spectrum.

Furthermore, several papers mentioned below use the framework of LSW-processes.

2. Multivariate locally stationary processes: We first mention that in particular the Gaussian likelihood theory for locally stationary processes from Section 5 also holds for multivariate processes – see Dahlhaus (2000).

Beyond that Chiann and Morettin (1999, 2005) investigate the estimation of time varying coefficients of a linear system where the input and output are locally stationary processes. They study different estimation techniques in the frequency- and time domain.

Ombao et al. (2001) analyze bivariate nonstationary time series. They use SLEX functions (time-localized generalization of the Fourier waveform) and propose a method that automatically segments the time series into approximately stationary blocks and selects the span to be used to obtain the smoothed estimates of the time varying spectra and coherence. Om-

bao et.al. (2005) use the SLEX framework to build a family of multivariate models that can explicitly characterize the time varying spectral and coherence properties of a multivariate time series. Ombao and Van Bellegem (2008) estimate the time varying coherence by using time-localized linear filtering. Their method automatically selects via tests of homogeneity the optimal window width for estimating local coherence.

Motta et.al. (2011) propose a locally stationary factor model for large cross-section and time dimensions. Factor loadings are estimated by the eigenvectors of a nonparametrically estimated covariance matrix. Eichler et. al. (2011) investigate dynamic factor modeling of locally stationary processes. They estimate the common components of the dynamic factor model by the eigenvectors of an estimator of the time varying spectral density matrix. This can also be seen as a time varying principal components approach in the frequency domain.

Cardinali and Nason (2010) introduce the concept of costationary of two locally stationary time series where some linear combination of the two processes is stationary. They show that costationarity imply a error-correction type of formula in which changes in the variance of one series are reflected by simultaneous balancing changes in the other. Sanderson et.al. (2010) propose a new method of measuring the dependence between non-stationary time series based on a wavelet coherence between two LSW-processes.

3. Testing of locally stationary processes – in particular tests for stationarity: Among the large literature on testing there is a considerable part devoted to testing of stationarity. Tests of stationarity have already been proposed and theoretically investigated before the framework of local stationarity was created. In that cases the theoretical investigations mainly consisted in the investigation of the asymptotic distribution of the test statistics under the hypothesis of stationarity.

Priestley and Subba Rao (1969) proposed testing the homogeneity of a set of evolutionary spectra evaluated at different points of time. For Gaussian processes and for the purpose of a change point detection Picard (1985) developed a test based on the difference between spectral distribution functions estimated on different parts of the series and evaluated using a supremum type statistic. Giraitis and Leipus (1992) generalized this approach to the case of linear processes. Von Sachs and Neumann (2000) developed a test of stationarity based on empirical wavelet coefficients estimated using localized versions of the periodogram. Paparoditis (2009) developed a nonparametric test for stationarity against the alternative of a smoothly time varying spectral structure based on a local spectral density estimate. He also investigated the power under the fixed alternative of a locally stationary processes. Paparoditis (2010) tested the assumption of stationarity by evaluating the supremum over

time of an L_2 -distance between the local periodogram over a rolling segment and an estimator of the spectral density obtained using the entire time series at hand. The critical values of a supremum type test are obtained using a stationary bootstrap procedure. Dwivedi and Subba Rao (2011) construct a Portmanteau type test statistic for testing stationarity of a time series by using the property that the discrete Fourier transforms of a time series at the canonical frequencies are asymptotically uncorrelated if and only if the time series is second-order stationary.

Tests of general hypothesis are derived in Sakiyama and Taniguchi (2003) who test parametric composite hypothesis by the Gaussian likelihood ratio test, the Wald test and the Lagrange multiplier test. Sergides and Paparoditis (2009) develop tests of the hypothesis that the time varying spectral density has a semiparametric structure. The test introduced is based on a L_2 -distance measure in the spectral domain. As a special case they test for the presence of a tvAR model. A bootstrap procedure is applied to approximate more accurately the distribution of the test statistic under the null hypothesis. Preuß et. al. (2011) also test semiparametric hypotheses. Their method is based on an empirical version of the L_2 -distance between the true time varying spectral density and its best approximation under the null hypothesis.

Zhou and Wu (2010) construct simultaneous confidence tubes for time varying regression coefficients in functional linear models. Using a Gaussian approximation result for non-stationary multiple time series, they show that the constructed simultaneous confidence tubes have asymptotically correct nominal coverage probabilities.

4. Bootstrap methods for locally stationary processes: Bootstrap methods are in particular needed to derive the asymptotic distribution of test statistics. A time domain local block bootstrap procedure for locally stationary processes has been proposed by Paparoditis and Politis (2002) and by Dowla et al. (2003). Sergides and Paparoditis (2008) develop a method to bootstrap the local periodogram. Their method generates pseudo local periodogram ordinates by combining a parametric time and nonparametric frequency domain bootstrap approach. They first fit locally a time varying autoregressive model to capture the essential characteristics of the underlying process. A locally calculated non-parametric correction in the frequency domain is then used so as to improve upon the locally parametric autoregressive fit. Kreiss and Paparoditis (2011) propose a nonparametric bootstrap method by generating pseudo time series which mimic the local second and fourth order moment structure of the underlying process. They prove a bootstrap central limit theorem for a general class of preperiodogram based statistics.

5. Model misspecification and model selection: Model selection criteria have been heuristically suggested many times for time varying processes – c.f. Ozaki and Tong (1975); Kitagawa and Akaike (1978) and Dahlhaus (1996b, 1997) among others – in all papers AIC-type criteria have been suggested for different purposes.

Van Bellegem and Dahlhaus (2006) consider semiparametric estimation and estimate the Kullback-Leibler distance between the semiparametric model and the true process. They use this estimate then as a model selection criterion. Hirukawa et.al. (2008) propose a generalized information criterion based on nonlinear functionals of the time varying spectral density. Chandler (2010) investigates how time varying parameters affect order selection.

Another interesting aspect is that many results of this paper also hold under model - misspecification – for example Theorem 5.1 and the corresponding result for the Block Whittle estimate from (20). An important example is the case where a stationary model is fitted and the underlying process in truth is only locally stationary - see Example 4.5 and the more detailed discussion for stationary Yule-Walker estimates in Dahlhaus (1997), Section 5.

6. Likelihood theory and large deviations: Local asymptotic normality (LAN) is derived in the parametric Gaussian case in Dahlhaus (1996b) and Dahlhaus (2000) (cf. Remark 3.3 in that paper). A nonparametric LAN-result is derived in Sakiyama and Taniguchi (2003) and a LAN result under non-Gaussianity in Hirukawa and Taniguchi (2006). In both papers the results are applied to asymptotically optimal estimation and testing. For some statistics also the asymptotic distribution under contiguous alternatives is derived. Tamaki (2009) studies second order asymptotic efficiency of appropriately modified maximum likelihood estimators for Gaussian locally stationary processes.

Large deviations principles for quadratic forms of locally stationary processes are derived in Zani (2002) including applications to local spectral density and covariance estimation. Wu and Zhou (2011) obtain an invariance principle for non-stationary vector-valued stochastic processes. They show that the partial sums of non-stationary processes can be approximated on a richer probability space by sums of independent Gaussian random vectors.

7. Recursive estimation: Recursive estimation algorithms are of the form

$$\widehat{\theta}_t = \widehat{\theta}_{t-1} + \lambda_t \ \psi(\boldsymbol{X}_t, \widehat{\theta}_{t-1}) \tag{117}$$

where $X_t = (X_1, ..., X_t)'$. The recursive structure yields an update of the estimate as soon as the next observation becomes available and the estimate therefore is particularly of importance in an online situation. For stationary processes the algorithm is used with

 $\lambda_t \sim 1/t$ while in nonstationary situations one uses a nondecreasing λ (constant stepsize case) that is the estimate puts stronger weights on the last observations.

Adaptive estimates of the above type have been investigated over the last 30 years in different scientific communities: by system theorists under the name "recursive identification methods" (cf. Ljung (1977); Ljung and Söderström (1983)), in the stochastic approximation community (cf. Benveniste, Métivier and Priouret (1990); Kushner and Yin (1997)), in the neural network community under the name "back-propagation algorithm" (cf. White (1992) or by Haykin (1994)), and in applied sciences, particularly for biological and medical applications (cf. Schack and Grieszbach (1994)).

The properties of recursive estimation algorithms have rigorously been investigated in many papers under the premise that the underlying true process is stationary. However, for nonstationary processes and the constant stepsize case there did not exist for a long time a reasonable framework to study theoretically the properties of these algorithms. This has changed with the concept of locally stationary processes with it's infill asymptotics which now allows for theoretical investigations of these algorithms.

In Moulines et.al. (2005) the properties of recursive estimates of tvAR-processes have been investigated in the framework of locally stationary processes. The asymptotic properties of the estimator have been proved including a minimax result. In Dahlhaus and Subba Rao (2007) a recursive algorithm for estimating the parameters of a tvARCH-process has been proposed. Again the asymptotic properties of the estimator have been proved.

8. Inference for the mean curve: Modeling the time varying mean of a locally stationary process is an important task which has not been discussed in this overview. In principle nearly all known techniques from nonparametric regression may be used such as kernel estimates, local polynomial fits, wavelet estimates or others. The situation is however much more challenging since the "residuals" are in this case a locally stationary process which usually is modeled at the same time.

In general the topic needs more investigation. Dahlhaus (1996a, 1996b, 1997, 2000) and Dahlhaus and Neumann (2001) contain also results where the mean is time varying and/or estimated. A more detailed investigation is contained in Tunyavetchakit (2010) in the context of time varying AR(p)-processes where the mean curve is estimated in parallel and the optimal segment length is determined similar to (16).

9. Piecewise constant models: Davis et.al. (2005) consider the problem of modeling a class of nonstationary time series using piecewise constant AR-processes. The number and locations

of the piecewise AR segments, as well as the orders of the respective AR processes, are determined by the minimum description length principle. The best combination is then determined by a genetic algorithm. In Davis et.al. (2008) to general parametric time series models for the segments and illustrate the method with piecewise GARCH-models, stochastic volatility and generalized state space models.

Locally constant parametric models have also been considered in a non-asymptotic approach by Mercurio and Spokoiny (2004) and others where the so-called small modeling bias condition is used to determine the length of the interval of time homogeneity and to fit the parameters – for more details see also Spokoiny (2010).

- 10. Long memory processes: Beran (2009) and Palma and Olea (2010) have extended the concept of local stationarity to long-range dependent processes. While Beran (2009) uses a nonparametric approach with a local least squares estimate similar to (26) Palma and Olea (2010) use a parametric approach and use the block Whittle likelihood from (21). Both papers then investigate the asymptotic properties. Roueff and von Sachs (2011) use a local log-regression wavelet estimator of the time-dependent long memory parameter and study it's asymptotic properties.
- 11. Locally stationary random fields: Fuentes (2001) studies different methods for locally stationary isotropic random fields with parameters varying across space. In particular she uses local Whittle estimates. Eckley et.al. (2010) propose the modeling and analysis of image texture by using an extension of a locally stationary wavelet process model for lattice processes. They construct estimates of a spatially localized spectrum and a localized autocovariance which are then used to characterize textures in a multiscale and spatially adaptive way. Anderes and Stein (2011) develop a weighted local likelihood estimate for the parameters that govern the local spatial dependency of a locally stationary random field.
- 12. Discrimination Analysis: Discrimination Analysis for locally stationary processes based on the Kullback-Leibler divergence as a classification criterion has been investigated in Sakiyama and Taniguchi (2004) and for multivariate processes in Hirukawa (2004). Huang et.al. (2004) propose a discriminant scheme based on the SLEX-library and a discriminant criterion that is also related to the Kullback-Leibler divergence. Chandler and Polonik (2006) develop methods for the discrimination of locally stationary processes based on the shape of different features. In particular they use shape measures of the variance function as a criterion for discrimination and apply their method to the discrimination of earthquakes and explosions. Fryzlewicz and Ombao (2009) use a bias-corrected non-decimated wavelet transform for classification in the framework of LSW-processes.

13. Prediction: Fryzlewicz et.al. (2003) address the problem of how to forecast non-stationary time series by means of non-decimated wavelets. Using the class of LSW-processes they introduce a new predictor based on wavelets and derive the prediction equations as a generalization of the Yule-Walker equations. Van Bellegem and von Sachs (2004) apply locally stationary processes to the forecasting of several economic data sets such as returns and exchange rates.

14. Finance: There is a growing interest in finance for models with time varying parameters. An overview on locally stationary volatility models is given in Van Bellegem (2011). A general discussion on local stationary in different areas of finance can be found in Guegan (2007) – see also Taniguchi et. al. (2008). For example, many researchers are convinced that the observed slow decay of the sample autocorrelation function of absolute stock returns is not a long memory effect but due to nonstationary changes in the unconditional variance (c.f. Mikosch and Stărică (2004), Stărică and Granger (2005), Fryzlewicz et. al. (2006)) leading for example to GARCH-models with time varying parameters.

References for work on tvGARCH-models have been given in Section 3. Other work on applications of locally stationary processes in finance is for example the work on optimal portfolios with locally stationary returns of assets by Shiraishi and Taniguchi (2007). Hirukawa (2006) uses locally stationary processes for a clustering problem of stock returns. Fryzlewicz (2005) models some stylized facts of financial log returns by LSW-processes. Fryzlewicz et. al. (2006) consider a locally stationary model for financial log-returns and propose a wavelet thresholding algorithm for volatility estimation, in which Haar wavelets are combined with the variance-stabilizing Fisz transform.

15. Further topics: Robinson (1989) uses also the infill asymptotics approach in his work on nonparametric regression with time varying coefficients. Orbe et al. (2000) estimate nonparametrically a time varying coefficients model allowing for seasonal and smoothness constraints. Orbe et.al. (2005) estimate the time varying coefficients under shape restrictions over and for locally stationary regressors. Chiann and Morettin (2005) investigate the estimation of coefficient curves in time varying linear systems.

Estimation of time varying <u>quantile curves</u> for nonstationary processes has been done in Draghicescu et.al. (2009) and Zhou and Wu (2009). Specification tests of time varying quantile curves have been investigated in Zhou (2010).

References

- Amado, C., and Teräsvirta, T. (2011). Modelling volatility with variance decomposition. CREATES Research Paper 2011-1, Aarhus University.
- Anderes, E.B. and Stein, M.L. (2011). Local likelihood estimation for nonstationary random fields. *Journal of Multivariate Analysis* **102**, 506–520.
- Benveniste, A., Métivier, M. and Priouret, P. (1990). Adaptive Algorithms and Stochastic Approximations. Springer Verlag, Berlin.
- Beran, J. (2009). On parameter estimation for locally stationary long-memory processes. J. Statist. Plann. Inference 139, 900–915.
- Berkes, I., Horváth, L. and Kokoskza, P. (2003). GARCH processes: structure and estimation. *Bernoulli* 9, 201–207.
- Brillinger, D.R. (1981). Time Series: Data Analysis and Theory. Holden Day, San Francisco.
- Brockwell P.J., and Davis R.A. (1991). *Time Series: Theory and Methods*, 2nd ed. Springer-Verlag, New York.
- Cardinali, A. and Nason, G. (2010). Costationarity of locally stationary time series. *Journal of Time Series Econometrics* 2, No. 2, Article 1. DOI: 10.2202/1941-1928.1074
- Chandler, G. (2010). Order selection for heteroscedastic autoregression: A study on concentration. *Statistics and Probability Letters* **80**, 1904–1910.
- Chandler, G. and Polonik, W. (2006). Discrimination of locally stationary time series based on the excess mass functional. *J. Amer. Statist. Assoc.* **101**, 240–253.
- Chiann, C. and Morettin, P. (1999). Estimation of time varying linear systems. *Statist. Inference Stoch. Proc.* **2**, 253–285.
- Chiann, C. and Morettin, P. (2005). Time-domain estimation of time-varying linear systems. J. Nonpar. Statist. 17, 365–383.
- Dahlhaus, R. (1988). Empirical spectral processes and their applications to time series analysis. Stoch. Proc. Appl. 30, 69–83.
- Dahlhaus, R. (1996a). On the Kullback-Leibler information divergence for locally stationary processes. *Stoch. Proc. Appl.* **62**, 139–168.
- Dahlhaus, R. (1996b). Maximum likelihood estimation and model selection for locally stationary processes. J. Nonpar. Statist. 6, 171–191.
- Dahlhaus, R. (1996c). Asymptotic statistical inference for nonstationary processes with evolutionary spectra. In: *Athens Conference on Applied Probability and Time Series Vol II.* (P.M. Robinson and M. Rosenblatt, eds.), 145–159, Lecture Notes in Statistics 115, Springer, New York.
- Dahlhaus, R. (1997). Fitting time series models to nonstationary processes. *Ann. Statist.* **25**, 1–37.
- Dahlhaus, R. (2000). A likelihood approximation for locally stationary processes. *Ann. Statist.* **28**, 1762–1794.

- Dahlhaus, R. (2009). Local inference for locally stationary time series based on the empirical spectral measure. *J. Econometrics* **151**, 101–112.
- Dahlhaus, R. and Giraitis, L. (1998). On the optimal segment length for parameter estimates for locally stationary time series. J. Time Series Anal. 19, 629–655.
- Dahlhaus, R., Neumann, M.H. and von Sachs, R. (1999). Nonlinear wavelet estimation of time-varying autoregressive processes *Bernoulli* 5, 873–906.
- Dahlhaus, R. and Neumann, M.H. (2001). Locally adaptive fitting of semiparametric models to nonstationary time series. *Stoch. Proc. and Appl.* **91**, 277–308.
- Dahlhaus, R. and Polonik, W. (2006). Nonparametric quasi maximum likelihood estimation for Gaussian locally stationary processes. *Ann. Statist.* **34**, 2790–2824.
- Dahlhaus, R. and Polonik, W. (2009). Empirical spectral processes for locally stationary time series. *Bernoulli* 15, 1–39.
- Dahlhaus, R. and Subba Rao, S. (2006). Statistical inference for locally stationary ARCH models. *Ann. Statist.* **34**, 1075–1114.
- Dahlhaus R. and Subba Rao S. (2007). A recursive online algorithm for the estimation of time-varying ARCH parameters. *Bernoulli* 13, 389–422.
- Davis, R.A. and Lee, T., and Rodriguez-Yam, G. (2005). Structural break estimation for nonstationary time series models. *J. Amer. Statist. Assoc.* **101**, 223–239.
- Davis, R.A., Lee, T., and Rodriguez-Yam, G. (2008). Break detection for a class of non-linear time series models. *J. Time Ser. Anal.* **29**, 834–867.
- Dowla, A., Paparoditis, E. and Politis, D.N. (2003). Locally stationary processes and the local bootstrap. In: *Recent Advances and Trends in Nonparametric Statistics* (Eds. M. G. Akritas and D. N. Politis). Elsevier Science B.V., Amsterdam, 437–445.
- Draghicescu, D., Guillas, S. and Wu, W.B. (2009). Quantile curve estimation and visualization for non-stationary time series. *J. Comput. Graph. Statist.* **18**, 1–20.
- Dunsmuir, W. (1979). A central limit theorem for parameter estimation in stationary vector time series and its application to models for a signal observed with noise. *Ann. Statist.* 7, 490–506.
- Dzhaparidze, K. (1971). On methods for obtaining asymptotically efficient spectral parameter estimates for a stationary Gaussian process with rational spectral density. *Theory Probab. Appl.* **16**, 550–554.
- Dzhaparidze, K. (1986). Parameter Estimation and Hypothesis Testing in Spectral Analysis of Stationary Time Series. Springer Verlag, New York.
- Dwivedi, Y. and Subba Rao, S. (2011). A test for second-order stationarity of a time series based on the discrete Fourier transform. J. Time Series Anal. 32 68–91.
- Eckley, I.A., Nason, G.P. and Treloar, R.L. (2010). Locally stationary wavelet fields with application to the modelling and analysis of image texture. *Appl. Statist.* **59**, 595–616.
- Eichler, M., Motta, G. and von Sachs, R. (2011). Fitting dynamic factor models to non-stationary time series. *J. Econometrics* **163**, 51-70.
- Fay, G. and Soulier, P. (2001) The periodogram of an i.i.d. sequence. *Stoch. Proc. Appl.* **92**, 315–343.

- Fox, R. and Taqqu, M.S. (1986). Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. *Ann. Statist.* **14**, 517–532.
- Fryzlewicz, P. (2005) Modelling and forecasting financial log-returns as locally stationary wavelet processes. *J. Appl. Statist.* **32**, 503–528.
- Fryzlewicz, P. and Nason, G. P. (2006). Haar-Fisz estimation of evolutionary wavelet spectra. J. R. Statist. Soc. B 68, 611–634.
- Fryzlewicz, P. and Ombao, H. (2009). Consistent classification of nonstationary time series using stochastic wavelet representations. J. Amer. Statist. Assoc. 104, 299-312.
- Fryzlewicz, P., Sapatinas, T. and Subba Rao, S. (2006). A Haar-Fisz technique for locally stationary volatility estimation. *Biometrika* **93**, 687–704.
- Fryzlewicz, P., Sapatinas, T. and Subba Rao, S. (2008). Normalised least-squares estimation in time-varying ARCH models. *Ann. Statist.* **36**, 742–786.
- Fryzlewicz, P. and Subba Rao, S. (2011). On mixing properties of ARCH and time-varying ARCH processes. *Bernoulli* 17, 320–346.
- Fryzlewicz, P., Van Bellegem, S. and von Sachs, R. (2003). Forecasting non-stationary time series by wavelet process modeling. *Ann. Inst. Statist. Math.* **55**, 737–764.
- Fuentes, M. (2001). A high frequency kriging approach for non-stationary environmental processes. *Environmetrics* **12**, 469–483.
- Giraitis, L. and Leipus, R. (1992). Testing and estimating in the change-point problem of the spectral function. *Lithuanian Mathematical Journal* **32**, 15–29.
- Granger, C.W.J. and Hatanaka, M. (1964). Spectral Analysis of Economic Time Series. Princeton University Press. Princeton, New Jersey.
- Grenander, U. and Szegö, G. (1958). *Toeplitz Forms and their Applications*. University of California Press, Berkeley.
- Grenier, Y. (1983). Time dependent ARMA modelling of nonstationary signals. *IEEE Trans. Acoust. Speech Signal Process.* **31**, 899–911.
- Guégan D. (2007). Global and local stationary modelling in finance: Theory and empirical evidence. Preprint, Centre dŠEconomique de la Sorbonne.
- Guo, W., Dai, M., Ombao, H.C. and von Sachs, R. (2003). Smoothing spline ANOVA for time-dependent spectral analysis. *J. Amer. Statist. Assoc.* **98**, 643–652.
- Hannan, E.J. (1973). The asymptotic theory of linear time series models. *J. Appl. Prob.* **10**, 130–145.
- Hallin, M. (1986). Nonstationary q-dependent processes and time-varying moving average models: invertibility properties and the forecasting problem. Adv. Appl. Probab. 18, 170–210.
- Hirukawa, J. (2004). Discriminant analysis for multivariate non-Gaussian locally stationary processes. *Scientiae Mathematicae Japonicae Online* **10**, 235–258.
- Hirukawa, J. (2006). Cluster analysis for non-Gaussian locally stationary processes. *Intern. J. Theor. Appl. Finance* **9**, 113–132.

- Hirukawa, J., Kato, H.S., Tamaki, K. and Taniguchi, M. (2008). Generalized information criteria in model selection for locally stationary processes. *J. Japan Statist. Soc.* **38**, 157–171.
- Hirukawa, J. and Taniguchi, M. (2006). LAN theorem for non-Gaussian locally stationary processes and its applications. *J. Statist. Planning Infer.* **136**, 640–688.
- Hosoya, Y. and Taniguchi, M. (1982). A central limit theorem for stationary processes and the parameter estimation of linear processes *Ann. Statist.* **10**, 132–153.
- Huang, H.-Y. Ombao, H.C. and Stoffer, D.S. (2004). Discrimination and Classification of Nonstationary Time Series Using the SLEX Model. J. Amer. Statist. Assoc. 99, 763–774.
- Jentsch, C. (2006). Asymptotik eines nicht-parametrischen Kernschätzers für zeitvariable autoregressive Prozesse. Diploma thesis, University of Braunschweig.
- Kayhan, A., El-Jaroudi, A. and Chaparro, L. (1994). Evolutionary periodogram for non-stationary signals. *IEEE Trans. Signal Process.* **42**, 1527–1536.
- Kim, W. (2001). Nonparametric kernel estimation of evolutionary autoregressive processes. Discussion paper 103. Sonderforschungsbereich 373, Berlin.
- Kitagawa, G. and Akaike, H. (1978). A Procedure for The Modeling of Non-Stationary Time Series. *Ann. Inst. Statist. Math.* **30 B**, 351–363.
- Kitagawa, G. and Gersch, W. (1985). A smoothness priors time-varying AR coefficient modeling of the nonstationary covariance time series. *IEEE Trans. Automat. Control.* **30**, 48-56.
- Koo, B. and Linton, O. (2010). Semiparametric estimation of locally stationary diffusion models. LSE STICERD Research Paper No. EM/2010/551.
- Kreiss, J.-P. and Paparoditis, E. (2011). Bootstrapping Locally Stationary Processes. Technical report.
- Kushner, H. J. and Yin, G.G. (1997). Stochastic Approximation Algorithms and Applications. Springer Verlag, New York.
- Ljung, L. (1977). Analysis of recursive stochastic algorithms. *IEEE Trans. Automatic Control* 22, 551–575.
- Ljung, L. and Söderström, T. (1983). Theory and Practice of Recursive Identification. MIT Press, Cambridge, MA.
- Martin, W. and Flandrin, P. (1985). Wigner-Ville spectral analysis of nonstationary processes. *IEEE Trans. Acoust. Speech Signal Process.* **33**, 1461–1470.
- Mélard, G. and A. Herteleer-de-Schutter, A. (1989). Contributions to evolutionary spectral theory. J. Time Series Anal. 10 41–63.
- Mercurio, D. and Spokoiny, V. (2004). Statistical inference for time-inhomogenous volatility models. *Ann. Statist.* **32**, 577–602.
- Mikosch, T., Gadrich, T., Klüppelberg, C., Adler, R.J. (1995). Parameter estimation for ARMA models with infinite variance innovations. *Ann. Statist.* **23**, 305–326.
- Mikosch, T. and Norvaisa, R. (1997). Uniform convergence of the empirical spectral distribution function. Stoch. Proc. Appl. 70, 85–114.

- Mikosch, T., and C. Stărică, C. (2004). Nonstationarities in financial time series, the long-range dependence, and the IGARCH effects. *The Review of Economics and Statistics* **86**, 378–390.
- Motta, G., Hafner, C.M. and von Sachs, R. (2011). Locally stationary factor models: Identification and nonparametric estimation. *Econometric Theory* **27**, 1279–1319 doi:10.1017/S026646661100005
- Moulines, E., Priouret, P. and Roueff, F. (2005). On recursive estimation for locally stationary time varying autoregressive processes. *Ann. Statist.* **33**, 2610–2654.
- Nason, G.P. and von Sachs, R. (1999). Wavelets in time series analysis. *Phil. Trans. R. Soc. Lond. A* **357**, 2511–2526.
- Nason, G.P., von Sachs, R. and Kroisandt, G. (2000). Wavelet processes and adaptive estimation of evolutionary wavelet spectra. J. Royal Statist. Soc. B 62, 271–292.
- Neumann, M.H. and von Sachs, R. (1997). Wavelet thresholding in anisotropic function classes and applications to adaptive estimation of evolutionary spectra. *Ann. Statist.* **25**, 38–76.
- Ombao, H.C., Raz, J.A., von Sachs, R. and Malow, B.A. (2001). Automatic statistical analysis of bivariate nonstationary time series. *J. Amer. Statist. Assoc.* **96**, 543–560.
- Ombao, H.C., von Sachs, R. and Guo, W. (2005). The SLEX analysis of multivariate non-stationary time series. J. Amer. Statist. Assoc. 100, 519–531.
- Ombao, H.C. and Van Bellegem, S. (2008). Evolutionary Coherence of Nonstationary Signals. *IEEE Transactions on Signal Processing* **56**, 2259–2266.
- Orbe, S., Ferreira, E., Rodriguez-Poo, R.M. (2000). A nonparametric method to estimate time varying coefficients. *J. Nonparam. Statist.* **12**, 779–806.
- Orbe, S., Ferreira, E., Rodriguez-Poo, R.M. (2005). Nonparametric estimation of time varying parameters under shape restrictions. *Journal of Econometrics* **126**, 53–77.
- Ozaki, T. and Tong, H. (1975). On the fitting of non-stationary autoregressive models in time series analysis. *Proceedings of the 8-th Hawaii International Conference on System Sciences*. Western Periodical Company, North Hollywood, California.
- Palma, W. and Olea, R. (2010). An efficient estimator for locally stationary Gaussian long-memory processes. *Ann. Statist.*. **38**, 2958–2997.
- Paparoditis, E. (2009). Testing temporal constancy of the spectral structure of a time series. *Bernoulli* 15, 1190–1221.
- Paparoditis, E. (2010). Validating stationarity assumptions in time series analysis by rolling local periodograms. J. Amer. Statist. Assoc. 105, 839–851.
- Paparoditis, E. and Politis, D.N. (2002). Local block bootstrap. C. R. Acad. Sci. Paris, Ser. I 335, 959–962.
- Parzen, E. (1983). Autoregressive spectral estimation. In: *Handbook of Statistics* (D.R. Brillinger and P.R. Krishnaiah, eds.), **3**, 221–247, North-Holland, Amsterdam.
- Picard, D. (1985). Testing and estimating change-points in time series. Advances in Applied Probability 17, 841–867.

- Preuß, P., Vetter, M., and Dette, H. (2011). Testing semiparametric hypotheses in locally stationary processes. Discussion paper 13/11. SFB 823, TU Dortmund.
- Priestley, M.B. (1965). Evolutionary spectra and non-stationary processes. *J. Roy. Statist.* Soc. Ser. B 27, 204–237.
- Priestley, M. B. and Subba Rao, T. (1969). A test for non-stationarity of time series. Journal of the Royal Statistical Society B 31, 140–149.
- Priestley, M.B. (1981). Spectral Analysis and Time Series, Academic Press, London.
- Priestley, M.B. (1988). *Nonlinear and Nonstationary Time Series Analysis*, Academic Press, London.
- Robinson, P.M., (1989). Nonparametric estimation of time varying parameters. In: Hackl, P. (Ed.), Statistics Analysis and Forecasting of Economic Structural Change. Springer, Berlin, 253–264.
- Robinson, P.M. (1995). Gaussian semiparametric estimation of long range dependence. *Ann. Statist.* **23**, 1630–1661.
- Rosen, O., Stoffer, D.S. and Wood, S. (2009). Local Spectral Analysis via a Bayesian Mixture of Smoothing Splines. *Journal of the American Statistical Association* **104**, 249–262.
- Roueff, F. and von Sachs, R. (2011). Locally stationary long memory estimation. *Stoch. Proc. Appl.* **121**, 813–844.
- von Sachs, R. and Neumann, M. (2000). A wavelet-based test for stationarity. *J. Time Ser. Anal.* **21**, 597–613.
- Sanderson, J., Fryzlewicz, P. and Jones, M. (2010). Estimating linear dependence between nonstationary time series using the locally stationary wavelet model. *Biometrika* **97**, 435-446.
- Sakiyama, K. and Taniguchi, M. (2003). Testing composite hypotheses for locally stationary processes. J. Time Ser. Anal. 24, 483–504.
- Sakiyama, K. and Taniguchi, M. (2004). Discriminant analysis for locally stationary processes. J. Multiv. Anal. **90**, 282–300.
- Schack, B. and Grieszbach, G. (1994). Adaptive methods of trend detection and their application in analyzing biosignals. *Biom. J.* **36**, 429–452.
- Sergides, M. and Paparoditis, E. (2008). Bootstrapping the Local Periodogram of Locally Stationary Processes. J. Time Ser. Anal. 29, 264–299. Corrigendum: J. Time Ser. Anal. 30, 260–261.
- Sergides, M. and Paparoditis, E. (2009). Frequency domain tests of semiparametric hypotheses for locally stationary processes. *Scandin. J. Statist.* **36**, 800–821.
- Shiraishi, H. and Taniguchi, M. (2007). Statistical estimation of optimal portfolios for locally stationary returns of assets. *Int. J. Theor. Appl. Finance.* **10**, 129–154.
- Spokoiny, V. (2010). Local parametric methods in nonparametric estimation. Springer-Verlag, Berlin Heidelberg New York.
- Stărică, C. and Granger, C. (2005). Nonstationarities in stock returns. The Review of Economics and Statistics 87, 503–522.

- Subba Rao, S. (2006). On some nonstationary, nonlinear random processes and their stationary approximations. Advances in Applied Probability 38, 1155–1172.
- Subba Rao, T. (1970). The fitting of non-stationary time series models with time-dependent parameters. J. Roy Stat Soc B 32, 312–322.
- Tamaki, K. (2009). Second order properties of locally stationary processes. *J. Time Ser. Anal.* **30**, 145–166.
- Taniguchi, M., and Kakizawa, Y. (2000). Asymptotic Theory of Statistical Inference for Time Series. Springer Verlag, New York.
- Taniguchi, M., Hirukawa, J. and Tamaki, K. (2008). Optimal Statistical Inference in Financial Engineering. Chapman and Hall/CRC. Boca Raton, Florida.
- Tjøstheim, D. (1976). Spectral generating operators for non-stationary processes. Adv. Appl. Probab. 8, 831–846.
- Tunyavetchakit, S. (2010). On the optimal segment length for tapered Yule-Walker estimates for time-varying autoregressive processes. Diploma Thesis, Heidelberg.
- Van Bellegem, S. and Dahlhaus, R. (2006). Semiparametric estimation by model selection for locally stationary processes. J. Roy. Statist. Soc. B 68, 721–764.
- Van Bellegem, S. and von Sachs, R. (2004). Forecasting economic time series with unconditional time varying variance. *International Journal of Forecasting* **20**, 611–627.
- Van Bellegem, S. and von Sachs, R. (2008). Locally adaptive estimation of evolutionary wavelet spectra. *Ann. Statist.* **36**, 1879–1924.
- Van Bellegem, S. (2011). Locally stationary volatility models, in L. Bauwens, C. Hafner and S. Laurent (eds), Wiley Handbook in Financial Engineering and Econometrics: Volatility Models and Their Applications, Wiley, New York.
- Vogt, M. (2011). Nonparametric regression for locally stationary time series. Preprint, University of Mannheim.
- White, H. (1992). Artificial Neural Networks. Blackwell, Oxford.
- Whittle, P. (1953). Estimation and information in stationary time series. Ark. Mat. 2, 423–434.
- Whittle, P. (1954). Some recent contributions to the theory of stationary processes. Appendix to A study in the analysis of stationary time series, by H. Wold, 2nd ed. 196–228. Almqvist and Wiksell, Uppsala.
- Wu, W.B. and Zhou, Z. (2011). Gaussian approximations for non-stationary multiple time series. *Statistica Sinica* **21**, 1397–1413.
- Zani, M. (2002). Large deviations for quadratic forms of locally stationary processes. *J. Multivar. Anal.* **81**, 205–228.
- Zhou, Z. (2010). Nonparametric inference of quantile curves for nonstationary time series. *Ann. Statist.* **38**, 2187–2217.
- Zhou, Z. and Wu, W.B. (2009). Local linear quantile estimation for non-stationary time series. *Ann. Statist.* **37**, 2696–2729.
- Zhou, Z. and Wu, W.B. (2010). Simultaneous inference of linear models with time varying coefficients. J. R. Statist. Soc. B 72, 513–531.