# APPLICATIONS OF FREDHOLM ALTERNATIVE IN HYPOTHESIS TESTING

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## 1. Introduction and Terminologies

We adopt the convention in [1] and define the Fredholm operator as following:

**Definition 1.** (Fredholm operator) Let $X, Y$ be two Hilbert spaces. An operator $A : X \to Y$ is called Fredholm if and only if $dimN(A) = dimN(A^*) := n < \infty$ and both $R(A), R(A^*)$ are closed subspaces in $Y$.

and the Fredholm alternative is the following result Theorem 1.1 in [1]

**Theorem 2.** *([1] Theorem 1.1)*

*If $A = B + F$, where $B : X \to Y$ is an isomorphism and $F : X \to Y$ is a finite rank operator, then $A$ is a Fredholm operator.*

*For any Fredholm operator $A$ the following Fredholm alternative holds:*

*Either*

*(I) $Au_0 = 0$ has only the trivial solution $u_0 = 0_X$ and $A^*v_0 = 0$ has only the trivial solution $v_0 = 0_Y$*

*$Au = f$ and $A^*v = g$ have unique solutions for any $f \in Y, g \in X$*

*or*

*(II) $Au_0 = 0$ has exactly $n > 0$ linearly independent solutions $\{\phi_i, 1 \le i \le n\} \subset X$ and $A^*v_0 = 0$ has also $n > 0$ linearly independent solutions $\{\psi_j, 1 \le j \le n\} \subset Y$*

*$Au = f$ and $A^*v = g$ are solvable iff $\langle f, \psi_j \rangle_Y, 1 \le j \le n$, and correspondingly $\langle g, \phi_i \rangle_X, 1 \le i \le n$.*

*In case that the latter equations are solvable, their solutions are not unique and their general solutions are given respectively by:*

*$u = u_p + \sum_{i=1}^n a_i\phi_i$ and $v = v_p + \sum_{j=1}^n b_j\psi_j$ , where $a_i$ and $b_j$ are constants in the base field of $X, Y$ respectively.*

*$u_p$ and $v_p$ are some particular solutions to $Au = f$ and $A^*v = g$ respectively.*

## 2. Complete Sufficient Statistics for Parametric Families

**Definition 3.** (Complete Sufficient Statistics) Assume $P_\theta, \theta \in \Theta$ is a family of probability measures defined on $X$ and they are absolutely continuous with respect to a $\sigma$-finite measure $\mu$ defined on $X$. Therefore we can talk about the densities $p_\theta(x)$ associated with these probability measures.

**Definition 4.** (Neymann structure) A test function $\phi$ for a family of distributions $P_\theta$ are said to have Neymann structure with respect to $T$ if for any value $t \in R(T)$ such that $E(\phi \mid T = t) = \alpha$ for a fixed $\alpha \in [0, 1]$.

Now we state and prove the main result of this article.

**Theorem 5.** *Given that $T$ is a complete sufficient statistic for the family of distribution $P_\theta$, $A : h(x) \mapsto \boldsymbol{E}(h \mid T = t)$ is a Fredholm operator from $L^2(X)$ to $L^2(T) \simeq L^2(X)$* [1].

*Proof.* For any given $h \in L^2(X)$, we have that $Ah = \boldsymbol{E}(h(X) \mid T = t) = \int_X h(x)dP_{X|t}(x) \leq \left(\int_X h^2(x)d\mu(x)\right)\left(\int_X \left(\frac{dP_{X|t}(x)}{d\mu(x)}\right)^2 d\mu(x)\right) < \infty$, and linearity follows from the linearity of the integration. Therefore $A$ is a linear bounded operator.

For any $h \in L^2(X)$, we can define $a(t) = \boldsymbol{E}(h \mid T = t) < \infty \in L^2(T)$ and consider the following decomposition given by

$$Ah = \boldsymbol{E}(h - a + a \mid T = t) = \boldsymbol{E}(h - a \mid T = t) + \boldsymbol{E}(a \mid T = t)$$

and denote two new operators

$$B : L^2(X) \to L^2(T), h(x) \mapsto \boldsymbol{E}(h(x) - a(t) \mid T = t)$$

and

$$F : L^2(X) \to L^2(T), h(x) \mapsto \boldsymbol{E}(a(t) \mid T = t) = a(t)$$

. However if we notice that $\boldsymbol{E}(a \mid T = t) = a(t) \in \mathbb{R}$ then by definition the range of $F$ is a subset of $\mathbb{R}$ and the rank of operator $F$ is less than 1. Therefore $F$ is a finite rank operator.

The other operator $B$ is also linearly bounded since it is dominated by a linearly bounded operator $A$. $B$ also has the subspace spanned by $\{\mathbf{1}\}$ as $KerB$ (constant function being the unit element in $L^2(X)$), since the definition of complete sufficient statistics $T$ said that $\boldsymbol{E}(h - a \mid T = t) = 0$ iff $h(x) - a(t) \equiv 0$ a.e.

$B$ is surjective since for any $a(t) \in L^2(T)$, we have $\boldsymbol{E}(\mathbf{1} \cdot a(t) \mid T = t) = a(t)$ if we let $h(x) = \mathbf{1} \cdot a(t)$; $B$ is also injective since if $h_1(x) \neq h_2(x)$ on a nonzero measure set $S$ yet $Bh_1 = Bh_2$, then $\boldsymbol{E}([h_1 - a] - [h_2 - a] \mid T = t) = 0$ which yields $[h_1 - a] - [h_2 - a] = h_1 - h_2 = 0$ a.e. by the assumption of completeness, therefore $B$ is an isomorphism.

By Theorem 2 above, we assert that $A$ is a Fredholm operator and Fredholm alternative applies. □

From the proof above we also see that the completeness of $T$, a statistic we condition on, plays a role in ensuring that the operator $A$ is Fredholm. To examine the definition of Neymann structure closely, we now view the following result almost trivial and $\boldsymbol{E}(h \mid T = t) = 0 \Rightarrow h \equiv 0$ only have to hold for $h \in L^2(X)$.

**Definition 6.** ($L^2$complete) A statistics $T$ is called $L^2$ complete if $\boldsymbol{E}(h \mid T = t) = 0$ implies $h = 0$ a.e. for any $h \in L^2(X)$.

**Corollary 7.** *([2] Theorem 4.3.2, improved)A family has Neymann structure with respect to $T$ if $T$ is complete for $L^2(X)$.*

Note that we cannot extend this result to "$L^p$ complete" case since $L^2$ is proven to be the only natural Hilbert space among all $L^p$ spaces. Another reason is that $L^2$ is a reflexive space which allows us to argue Fredholm alternative more easily [1, 3].

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[1] The isomorphism is guaranteed by the fact that $T(X), X$ have ranges in the same $\mathbb{R}$, so $h(x) \longleftrightarrow h(t)$ is a natural isomorphism.

On the other hand, Neymann structure allows us to derive UMP (uniformly most powerful) test at significance level $\alpha$ even if there exists nuisance parameter as indicated in Sec 4.4 of [2]. From our perspective described above, we can see that the elimination of nuisance parameters requires complete sufficient statistics in order to preserve the significance level. If we condition on some sufficient (yet not complete) statistic $T$ of nuisance parameter $\vartheta$, then Neymann structure is not guaranteed. An example in Sec 4.5 [2] is the UMP unbiased test for Poisson rate $H : \frac{\lambda}{\mu} < 1$. We must condition on $(X_1 + X_2)$ since it is complete sufficient for the nuisance parameter $\mu + \lambda$ but not complete for $\frac{\lambda}{\mu}$; if we condition on $X_2$ which is complete sufficient for nuisance parameter $\mu$ then we cannot have Neymann structure for a hypothesis involving $\mu$.

The second application of Fredholm property is that the generalized Neymann-Pearson Lemma can be derived as a consequence of convexity of $M^2$ in $L^2(X)$.

## References

[1] Ramm, Alexander G. "A simple proof of the Fredholm alternative and a characterization of the Fredholm operators." The American Mathematical Monthly 108.9 (2001): 855-860.

[2] Lehmann, Erich L., and Joseph P. Romano. Testing statistical hypotheses. Springer Science & Business Media, 2006.

[3] Rudin, Walter. "Functional analysis. International series in pure and applied mathematics." (1991).

[2] This can be regarded as the risk set if we treat the hypothesis testing problem as a decisional thoretic problem with 0-1 loss function.