AN INTRODUCTION TO MICHOR-MUMUFORD FRAMEWORK

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ABSTRACT. As known to all, [2, 3] are famous for their succinctness as well as insightfulness. Here is a cheat sheet...

We want to study the closed planar curves up to reparameterization. A natural way of studying is to modulo out this equivalence relation among the collection of all closed planar curves and see how the quotient space looks like.

(1) Figure out the representation of all closed planar curve up to reparameterization in terms of a quotient space.
(2) What is an appropriate metric on this space? It is too hard and unnatural to define metric directly on quotient spaces.
(3) Let's investigate the larger space first, how does it look like? (Here [2] seems to make an excursion to study Imm instead of the Emb under the same quotient. It turns out that it has richer structure than Emb BUT it has a natural induced metric on Emb because the Imm differs from Emb only on the rank of the mapping so it is a simple restriction if we want to jump from Imm down to Emb.) My guess their excursion here is to follow the classical differential topology, where immersion is extensively studied while the embedding is treated as a sub-case. Also, the immersion space just have richer covering space.
(4) It seems that up to path-homotopy it suffices to investigate $Imm_a^k, k \neq 0$, therefore we simplify the problem of defining a metric further.

[2]'s idea:Figure out the representation of closed planar curves up It is relatively hard to define

(1) Two Representations of Closed Planar Curves
    (a) Immersion representation
    This representation is firstly adopted [2] because its intuition is clear and differential topological approach is well-suited on such a space.
        (i) The space of closed planar curves $B_e \approxeq Emb(S^1, \mathbb{R}^2)/Diff(S^1)$ where $Diff(\bullet)$ is the Lie group of diffeomorphism on $\bullet$.
        A curve in B_e consisting of curves $c(t) \subset \mathbb{R}^2$ is called a **path**. e is usually parameterized as $e(t, \varphi(t, \theta)), \varphi \in Diff(S^1)$[1] where t is the **index** variable and θ is the **trace** variable[2].
        The tangential field of a path e is split into **vertical part** $T_e \oplus$**horizontal part** N_e.
        Vertical part is the tangential direction of the path e

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[1]We should understand as φ_θ acting on c to form a path e in B_e.

[2]The index variable indicates which curve in c we are studying at; the trace variable indicates a what point of $c(t, \cdot)$ curve we are studying at.

Horizontal part is the normal direction of the path e
If you wish, you can regard them as two vector fields along the path c providing decomposition all along the c. But there is no simple expression using inner product on B_e because the inner product here must be compatible with the quotient structure and hence hard to define directly using their representatives. So a better way of doing this is to define on the "top spaces" Emb or even Imm first, and then require them to be invariant, and then induce a metric on the quotient space.
We investigate the larger space $B_i \approxeq Imm(S^1, \mathbb{R}^2)/Diff(S^1)$, the difference is that immersion might not be *free* [2]2.4-2.5.

(A) Free immersions $Imm_f(S^1, \mathbb{R}^2)$ makes $Diff(S^1)$ act freely on it so the $B_{i,f}$ is a smooth manifold.
Principal $Diff(S^1)$-bundle and associated connection exists, SO horizontal path still exists.

(B) Non-free immersions $Imm(S^1, \mathbb{R}^2)$ may cause B_i to be an orbifold.
Principal $Diff(S^1)$-bundle and associated connection might not exists, BUT horizontal path still exists.

(ii) Horizontal path $e^{\perp}$ of a path e
No matter the immersion itself is free or not, the corresponding horizontal path $e^{\perp}$ of a path c exists and $\left\langle e_t^{\perp}, e_\theta^{\perp} \right\rangle$ in the tangent space of B_e while $e^{\perp}$ and e coincide in B_e. The horizontal path is like the Jacobi field of c in a lower sense, we are using this horizontal path to characterize the planar curve homotopy from c_1 to c_2.

(iii) The tangent space $T_c B_i \approxeq Imm(S^1, \mathbb{R}^2) \times C^{\infty}(S^1, \mathbb{R}^2), \forall e \subset B_e$

(b) Lie group representation
This representation is investigated extensively in [3] because its computational easiness.

(i) The space of closed planar curves $B_e \approxeq Diff(\mathbb{R}^2)/Diff(\mathbb{R}^2, \Delta)$ where Δ is $S^1 \hookrightarrow \mathbb{R}^2$ in the canonical way $\iota : \theta \mapsto (cos\theta, sin\theta)$

(2) Two Approaches of Studying Metrics

(a) Differential Topology [2]
The idea here is that we have a detail investigation of $Imm_a(S^1, \mathbb{R}^2)$, and try to argue that

(i) The path with constant speed.
$Imm_a(S^1, \mathbb{R}^2)$ is the collection of those paths such that its trace variable is arclength l. By using arclength parameter l the term $|c'(\theta)|$ in the metric will vanish and $Imm_a(S^1, \mathbb{R}^2)$ is a smooth manifold.

(ii) Reduction to $Imm_a(S^1, \mathbb{R}^2)$
(1)$Imm(S^1, \mathbb{R}^2) = Imm_a(S^1, \mathbb{R}^2) \times Diff_1^+(S^1)$ where $Diff_1^+(S^1)$ is the normal subgroup of $Diff(S^1)$ that preserves S^1 and orientation of all paths.
(2)$B_i(S^1, \mathbb{R}^2) \coloneqq Imm(S^1, \mathbb{R}^2)/Diff(S^1) = Imm_a(S^1, \mathbb{R}^2)/(S^1 \rtimes \mathbb{Z}_2)$ where $(S^1 \rtimes \mathbb{Z}_2)$ is the semi-direct product, flipping of S^1. [2]2.6.

The Lie group structure is very clear here and that is why [3] is introducing it when dealing with calculation, partly because it is nature.

(iii) Contraction to a strong kernel at each layer of the topological covering of $Imm(S^1, \mathbb{R}^2)$
Pretty common a technique which is used since Hopf, $Imm(S^1, \mathbb{R}^2) = \amalg Imm^k(S^1, \mathbb{R}^2) \sim \amalg Imm^k_a(S^1, \mathbb{R}^2) = Imm_a(S^1, \mathbb{R}^2)$ where $\sim$ is homotopic equivalence. So up to homotopy it suffices to study $Imm^k_a(S^1, \mathbb{R}^2)$.

(iv) Reduction to S^1
S^1 is a strong deformation retract kernel of $Imm^k_a(S^1, \mathbb{R}^2), k \neq 0$

(b) Hamiltonian Topological System [3]
The advantage of this approach is that we can regard the index variable as a time variable and the trace variable as phase variable. By doing this, the geodesic equation can not only be derived as minimizer flow of energy function but also as the G-gradient vanishing flow which somehow easier to compute(?)

(3) Comparison
(a) Two kinds of metrics[2] p.3
(i) H^0-metric $G^0_c \coloneqq \int_{S^1} \langle h(\theta), k(\theta) \rangle |c'(\theta)| d\theta$
(ii) A-metric $G^A_c \coloneqq \int_{S^1} \left(1 + A\kappa_c(\theta)^2\right) \langle h(\theta), k(\theta) \rangle |c'(\theta)| d\theta$ where κ_c is the curvature of path c.
The A-metric is better than H_0 in following ways, so [2, 3] devoted to studying variants of A-metric.
(A) Infinmum of arclengths of paths connecting different curves is > 0.
(B) The length function of a curve have a Lipschitz bound $\sqrt{l(C_1)} - \sqrt{l(C_2)} \leq \frac{1}{\sqrt{4A}} d_{G^A_c}(C_1, C_2)$[3][4].

(b) The invariance requirement.
Any metric on the spaces $Imm(S^1, \mathbb{R}^2)$ must be invariant under $Diff(S^1)$ in order to induce a legal metric on the quotient group B_e. Because defining a metric on $Imm(S^1, \mathbb{R}^2)$ is relatively easy due to the availability of a direct sum decomposition of its tangent bundle without even taking quotient.
$T_c Imm(S^1, \mathbb{R}^2) = T_c(c \circ Diff^+(S^1)) \oplus N_c, h = \frac{\langle h, c_\theta \rangle}{\langle c_\theta, c_\theta \rangle} c_\theta + \frac{\langle h, c_\theta^\perp \rangle}{\langle c_\theta, c_\theta \rangle} c_\theta^\perp$.
The only requirement is that we choose the inner product $\langle \cdot, \cdot \rangle$ invariant w.r.t. $Diff(S^1)$ which we planned to quotient out.

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[3]The formula on p.3 of [2] is wrong

[4]They use $c \subset B_i$ to denote the path and use $C \in B_i$ to denote the curve which is very confusing

References

[1] Michor, Peter W., et al. "A metric on shape space with explicit geodesics." arXiv preprint arXiv:0706.4299 (2007).

[2] Michor, Peter W., and David Mumford. "Riemannian geometries on spaces of plane curves." arXiv preprint math/0312384 (2003).

[3] Michor, Peter W., and David Mumford. "An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach." Applied and Computational Harmonic Analysis 23.1 (2007): 74-113.