LOCALLY STATIONARY PROCESS AND ITS APPLICATIONS: A BRIEF TUTORIAL

HENGRUI LUO AND KORI KHAN

1. Introduction to Locally Stationary Processes

Nonstationary data occurred in various scenarios like spatial statistics, geology and time series [3, 10]. Among other approaches that can be used for modeling a seemingly nonstationary time series dataset, locally stationary process method is a natural extension from the vast literature existing for stationary processes method in time series data. There are two ways of defining a locally stationary process, let us consider a typical stationary process $AR(1)$ first. Suppose the $AR(1)$ process is defined by

$$\phi(B)X_t = X_t - \phi_1 X_{t-1} = Z_t$$

where $Z_t \sim WN(0, \sigma^2)$ as $t \in \{1, \cdots, T\}$. Then the time range $t \in \{1, \cdots, T\}$ defines a domain within which the process X_t can be described by an $AR(1)$ process. This specification is different from the usual autoregressive process where the $t \in \mathbb{Z}$ is usually assumed. Therefore locality is introduced in the model definition.

We can imagine the following definition for a time series X_t on different time domain ($\phi_1 \neq \phi_2$):

$$\phi_1(B)X_t = X_t - \phi_1 X_{t-1} = Z_t t = 1, 2, \cdots, T$$
$$\phi_2(B)X_t = X_t - \phi_2 X_{t-1} = Z_t t = T+1, T+2, \cdots$$

which is a conjunction of two time series over the integer time domain $t \in \mathbb{Z}^+$. It is no longer stationary because $\gamma_X(1, 2) = Cov(X_1, X_2) = \phi_1$ is clearly different from $\gamma_X(T+1, T+2) = Cov(X_{T+1}, X_{T+2}) = \phi_2$. However, on $[1, T]$ and $[T+1, \infty)$ the time series X_t are separately $AR(1)$ time series and hence stationary. Therefore we can call such a time series a "locally stationary" process, "local" means on each of the time domains above. In a greater generality, we could let the defining polynomial of $AR(1)$ processes vary and get an even more general locally stationary process in following form

$$\phi(B)X_t = X_t - \phi(t)X_{t-1} = Z_t, t = 1, \cdots, T$$

and our example above can be written in this form by specifying the coefficient function (as a function of time index t)

$$\phi(t) = \begin{cases} \phi_1 & t = 1, 2, \cdots, T \\ \phi_2 & t = T+1, T+2, \cdots \end{cases}$$

Therefore a locally stationary $AR(1)$ process can be defined through a time-varying polynomial $\phi(B, t)$. An alternative perspective is to investigate the spectral representation of the $AR(1)$ process. It is known that $AR(1)$ process $\phi(B)X_t = X_t - \phi_1 X_{t-1} = Z_t$ has a spectral density as follows, we use $S_X(\bullet)$ for spectral density function and f for frequency notation in the discussion hereafter.

$$S_X(f) = \frac{\sigma^2}{|1-\phi_1 e^{-i2\pi f}|^2}, |f| < \frac{1}{2}$$

And with the same calculation as above we know a general locally stationary $AR(1)$ can be defined as a process with spectral density in form of

$$S_X(f) = S_X(f,t) = \frac{\sigma^2}{|1-\phi(t) e^{-i2\pi f}|^2}, |f| < \frac{1}{2}$$

We should notice that although these two ways of describing a locally stationary $AR(1)$ process are equivalent. In more generality, the spectral representation seems to be preferred [3, 2].

We focus on the contrast and comparison between likelihoods of stationary and local stationary processes. For simplicity we restrict our attention to the unit interval. We write down our **locally stationary** time series model

$$X_{t,T} + \sum_{j=1}^{p} \alpha_j \left(\frac{t}{T}\right) \cdot X_{t-j,T} = \sigma\left(\frac{t}{T}\right) \cdot Z_t$$

$$Z_t \sim WN(0,1), t = 1, \cdots, T$$

with corresponding spectral density $S_X(f)$ as defined above.

However, it is also convenient to use following local approximation by the **stationary** process around a fixed time point $u_0 = \frac{t_0}{T}$,

$$\tilde{X}_t(u_0) + \sum_{j=1}^{p} \tilde{\alpha_j}\left(u_0\right) \cdot \tilde{X}_{t-j}(u_0) = \sigma\left(u_0\right) \cdot Z_t$$

$$Z_t \sim WN(0,1), t = 1, \cdots, T$$

2. Stationary Approximations

Intuitively speaking, this idea of analyzing is parallel to the method of using *Taylor expansion* around a certain point on a certain function. In Taylor expansion, we use the $f(x_0) + (x - x_0)f'(x_0) + \cdots$ to approximate $f(x)$ around $x = x_0$. In following procedure we use the derivative processes $\tilde{X}_t(u_0)$ at each time point $u_0 \in (0,1)$ to approximate the $X_{t,T}$ and control the error term. Here we construct a stationary time series $\tilde{X}_t$ and try to approximate $X_{t,T}$ around the time point u_0. To see this idea better, we write down the likelihoods of these two different models for $p = 1$. i.e.

$$X_{t,T} + \alpha_1\left(\frac{t}{T}\right) \cdot X_{t-1,T} = \sigma\left(\frac{t}{T}\right) \cdot Z_t$$

$$\tilde{X}_t(u_0) + \tilde{\alpha_1}\left(u_0\right) \cdot \tilde{X}_{t-1}(u_0) = \sigma\left(u_0\right) \cdot Z_t$$

$$Z_t \sim WN(0,1), t = 1, \cdots, T$$

The locally stationary likelihood can be written as $L_T(u_0, \theta)$ and the stationary likelihood can be written as $\tilde{L}_T(u_0, \theta)$. Assume that X_0 is already known for simplicity. $X_{T,i} = \tilde{X}_i = X_i$

in the following likelihoods,

$$L_T(\frac{t}{T},\theta)=\prod_{i=1}^{T}\frac{1}{\sqrt{2\pi\sigma\left(\frac{t}{T}\right)^2}}\cdot exp\left(-\frac{(X_{T,i}-\alpha_1\left(\frac{t}{T}\right)\cdot X_{T,i-1})^2}{2\sigma\left(\frac{t}{T}\right)^2}\right)$$

$$\tilde{L}_T(u_0,\theta)=\prod_{i=1}^{T}\frac{1}{\sqrt{2\pi\sigma\left(u_0\right)^2}}\cdot exp\left(-\frac{(\tilde{X}_i-\tilde{\alpha_1}\left(u_0\right)\cdot \tilde{X}_{i-1})^2}{2\sigma\left(u_0\right)^2}\right)$$

For the same value of $u_0=\frac{t}{T}$, these two likelihoods are essentially the same with probability one (with respect to the product measure of $X_1,\cdots,X_T$ in the case of T samples). Therefore they converge to the same limiting likelihood function $L(u_0,\theta)=L_T(u_0,\theta)$ (because the likelihood function is actually defined for different T's).
Then strictly speaking, we have following theoretical result that supports our application of approximating **local stationarity** using **global stationarity**.

Theorem 1. *(Stationary approximation, [2]) Assume that $X_{t,T}$ has a representation*

$$X_{t,T}=\mu(\frac{t}{T})+\sum_{j=-\infty}^{\infty}a_{t,T}(j)Z_{t-j}$$

with a bounded variation μ and $Z_t\sim WN(0,1)$. Let

$$\ell(j)=\begin{cases}1 & |j|\leq 1\\ |j|\log^{1+\kappa}|j| & |j|>1\end{cases}$$

for some $\kappa>0$ and $\sup_t|a_{t,T}(j)|\leq\frac{K}{\ell(j)}$ with K independent of T. Also we need a bounded variation function $a(\cdot,j):(0,1]\to\mathbb{R}$ with

$$\sup_u|a(u,j)|\leq\frac{K}{\ell(j)}$$

$$\sup_j\sum_{t=1}^{T}|a_{t,T}(j)-a(\frac{t}{T},j)|\leq K$$

$$\|a(\cdot,j)\|_{TV}\leq\frac{K}{\ell(j)}$$

*i.e. the coefficients that defines the time-varying local stationary time series are all bounded variation. Then the **derivative stationary process** for $X_{t,T}$ around x_0 is*

$$\tilde{X}_t(u):=\mu(u)+\sum_{j=-\infty}^{\infty}a(u,j)Z_{t-j}$$

The proof of the theorem is a direct check that the error term between the likelihoods of $X_t,\tilde{X}_t$ is bounded by $O\left((\frac{t}{T}-u_0)+\frac{1}{T}\right)$. The idea of constructing $\tilde{X}_t$ is to approximate the j-th time-varying coefficients $a_{t,T}(j)$ at u_0 using a fixed coefficient $a(u_0,j)$, these two $a_{t,T}(j),a(u_0,j)$ are called *closed pairs* [2]. There are multiple ways of constructing these close pairs, one analysis technique is to use a partition of unity over time domain to join these derivative stationary processes; any interpolation technique with bounded variation basis functions can also be used since the time domain is discrete. The reason why this

stationary approximation will not extend to continuous time series is the basic difficulty of doing a uniformly bounded function approximation on a continuous time domain.
As for consistency of the estimates from local likelihoods we introduce following result, whose proof is the classic two-step procedure, we first assert $\sup_{\theta\in\Theta}|L_T(u_0,\theta)-\tilde{L}_T(u_0,\theta)|\overset{PT}{\to}0$ the stationary approximation is asymptotically good; and then $\sup_{\theta\in\Theta}|L(u_0,\theta)-\tilde{L}_T(u_0,\theta)|\overset{PT}{\to}0$ the stationary likelihood is somehow consistent, then the MLE from local likelihood is consistent by first going to global stationary likelihood estimates. Here we again make use of the locality to assemble the results we gain from the global stationary likelihood approximation. This idea of **approximating locally and glue them together to get a global approximation** perpetuates throughout the literature about local stationary processes even beyond the scope of time series.

Theorem 2. *(Dahlhaus-Rao, Theorem 3.2 in* [3]*) Suppose that* Θ *is compact with* $\theta_0(u_0):=\arg\min_{\theta\in\Theta}L(u_0,\theta)\in Int\Theta$*, the function* $L(u_0,\theta)$ *is continuous at* $\theta_0(u_0)$ *and the minimum* $\theta_0(u_0)$ *is unique. If*

$$\sup_{\theta\in\Theta}|L_T(u_0,\theta)-\tilde{L}_T(u_0,\theta)|\overset{PT}{\to}0$$

$$\sup_{\theta\in\Theta}|L(u_0,\theta)-\tilde{L}_T(u_0,\theta)|\overset{PT}{\to}0$$

then

$$\hat{\theta}_T(u_0)-\theta_0(u_0)\overset{PT}{\to}0$$

A very perceptual view is to view $L(\bullet,\theta)$ as a functional defined on the product sample space where θ are the *autoregressive coefficients and variance parameters* of the locally stationary process at time point $u_0\in(0,1)$ that we defined for. For example in locally stationary AR(1) $\theta=(\phi_1,\sigma^2)$ at time point u_0. These parameters will obviously vary as the time parameters u_0 changes. that maps a time point u_0 to a function $L(u_0,\theta):\Theta\to\mathbb{R}$. With some nontrivial efforts, then $L(\bullet,\theta)$ can be regarded as a Banach valued random variable whose law of probability can be determined by $p_{X_1}\times\cdots\times p_{X_T}$ for a fixed T. The asymptotic techniques which can be applied onto the Banach valued random variables like Banach law of large numbers and Banach CLT [8].
The two theorems above reduces the work of doing inference on a locally stationary time series to doing inference on an approximating stationary time series, which we are familiar with. For example, we can obtain a Yule-Walker estimate for autoregessive coefficient for each fixed time point u_0 by solving the Yule-Walker equations based on the stationary approximation $\tilde{X}_t$ and then denote this Yule-Walker estimate $\alpha(u_0)$ and $\sigma^2(u_0)$ since it depends on the stationary approximation constructed at this specific point u_0.
If we think that $\alpha(u_0),\sigma^2(u_0)$ as a function as u_0 varies, then a natural question to ask is that how close $\alpha(u_0)$ is to the real coefficient $a_1(u_0)$, since now we are comparing two curves in the time domain. This consideration also justifies our assumption on the bounded variation assumption on the coefficients of the locally stationary time series from another angle.
As in the introduction section, we end our discussion with a result about the spectral density of the stationary approximation.

Theorem 3. *(Spectral density of stationary approximation, Theorem 4.3 in* [3]*) Assume that* $X_{t,T}$ *satisfies the assumptions in Theorem 1 and in addition for some* $i = 1, 2, \cdots$

$$\sup_{u\in(0,1)} \left|\frac{\partial^i \mu(u)}{\partial u^i}\right| \leq K$$

$$\sup_{u\in(0,1)} \left|\frac{\partial^i a(u,j)}{\partial u^i}\right| \leq \frac{K}{\ell(j)}$$

$$\sup_{t,T} \left|a_{t,T}(j) - a(\frac{t}{T}, j)\right| \leq \frac{K}{T\cdot \ell(j)}$$

the corresponding (true) spectral density $\tilde{S}(u,f)$ *of the stationary approximation process at time point* u *(remember that the stationary process approximation carries out at each of these time points)*

$$\tilde{X}_t(u) := \mu(u) + \sum_{j=-\infty}^{\infty} a(u,j) Z_{t-j}$$

will approximate the spectral density $S_T(u,f)$ *of the locally stationary process* $X_{t,T}$ *in*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |S_T(u,f) - \tilde{S}(u,f)|^2 du = o(1)$$

Besides stationary approximation proposed by Dahlhaus [5] whose fundamental idea is approximating the defining polynomial of the process, an alternative way of approximating the locally stationary process is to approximate the time-varying spectral density function $S_T(u,f)$ using a pre-specified collection of basis functions in the spectral spaces. A widely accepted basis in practice is wavelet basis proposed by Nason and his collaborators [9]. These two equivalent ways of dealing with nonstationarity are usually equivalent, yet sometimes one way is easier to work with than the other.
The example we work with is the time-varying $AR(p)$ process, one remark to make is that $AR(p)$ process has more significance than pedagogic purpose. By rational approximation of spectral density (for example, Theorem 4.4.3 in [1]), any stationary process can be approximated by an $AR(p)$ for p sufficiently large. If we have already known the local domains $I_1, I_2, \cdots$ upon which the X_t are stationary, then we can fit a time-varying $AR(p_j)$ process on I_j for sufficiently large p_j on each domain to get a satisfying approximation. This idea of approximating the local stationary part using $AR(p)$ processes reduces our discussion of approximating locally stationary processes to approximating locally stationary $AR(p)$ processes.

3. Gaussian-Whittle Likelihood Theory

In this section, we will discuss a generalization of the Whittle likelihood method for the estimation of parametric models for Gaussian locally stationary processes. The stationary Whittle likelihood results discussed in class will turn out to be a special case of this generalization. Likelihood based approaches of estimation have historically been encouraged

in statistics as a means by which to avoid arbitrariness. In the stationary case, the Whittle likelihood offered a more efficient way to compute an approximation for the likelihood. The estimates obtained were asymptotically equivalent to those obtained from the exact likelihood.
As we depart from the stationary case, we face two key obstacles in generalizing these results (for the purpose of this paper). First, for a locally stationary process, the traditional "increasing domain" asymptotics are often no longer appropriate. Here, we will rely on infill asymptotics (hence our use of re-scaling to the unit interval). "Increasing domain" and infill asymptotics need not, in general, agree. It turns out they do for stationary processes. [1] Second, we need an approximation for both the log of the determinant of the "covariance" as well as the "precision" of the Gaussian process which will be asymptotically equivalent to the true values.
We will focus on the second of these two problems. First, we introduce the definition of the Gaussian locally stationary process adapted from [4] and [6].

Definition 4. A sequence of Gaussian multivariate stochastic processes $X_{t,T} = (X^1_{t,T}, \ldots, X^d_{t,T})^T, t = 1, \ldots, T, T \geq 1$ is called a *Gaussian locally stationary process* with *transfer function matrix* A° and *trend function* μ if there exists a representation such that,

$$X_{t,T} = \mu(\frac{t}{T}) + \int_{-1/2}^{1/2} \exp(i2\pi f t)\ A^\circ_{t,T}(f)\ d\xi(f)$$

where the following conditions holds for all $a, b = 1, \ldots, d$.

(1) $\xi(f) = (\xi_1(f), \cdots, \xi_d(f))$ is a complex vector valued Gaussian vector process on the spectral domain $[-\frac{1}{2}, \frac{1}{2}]$ with $\overline{\xi_a(f)} = \xi_a(-f)$, $E(\xi_a(f)) = 0$, and

$$E\left\{d\xi_a(f) d\xi_b(\mu)\right\} = \delta_{ab} \cdot \eta(f+\mu) df d\mu$$

where $\eta(f) = \sum_{j=-\infty}^{\infty} \delta(f+j)$ is the period 1 extension of the Dirac delta function.

(2) There exists a constant $K > 0$ and an 1-periodic matrix valued function $A : [0, 1] \times \mathbb{R} \to \mathbb{C}^{d\times d}$ with $\overline{A(u, f)} = A(u, -f)$ and

$$\sup_{t,f} |A^\circ_{t,T}(f)_{a,b} - A(\frac{t}{T}, f)_{a,b}| \leq \frac{K}{T}$$

for all $T \in \mathbb{N}$. $A(u, f)$ and $\mu(u)$ are assumed to be continuous in u.

We call $S_X(u, f) = A(u, f)\overline{A(u, f)}^T$ the time-varying spectral density decomposition of the process. Local stationarity is a property of the whole triangular array $X_{t,T}$; however, here we focus on the case in which we observe a realization $X_{1,T}, \ldots, X_{T,T}$ with T fixed.
We assume this realization is from a locally stationary Gaussian process with true mean function μ and transfer function A. We fit a class of locally stationary Gaussian processes indexed by parameter $\theta \in \Theta$ with mean function μ_θ and transfer function A_θ, where the parameter space Θ is compact. We will call $S_{X,\theta} = |A_\theta(u, f)|^2$ the time varying spectral density of the model process and μ_θ the model mean function.

[1] Not all regularity assumptions are explicitly written in this paper. However, based on assumptions in [6] and [4] the two methods will yield equivalent results for stationary processes

For mean-zero univariate stationary processes, the Whittle likelihood employs the periodogram to approximate the negative Gaussian likelihood.

$$\tilde{l}_T(\theta) = -2l_T(\theta) \approx \int_{-1/2}^{1/2} \left\{ \log(2\pi S_{X,\theta}(f)) + \frac{\hat{S}_X^{(p)}(f)}{S_{X,\theta}(f)} \right\} df$$

For locally stationary processes, the periodogram is no longer appropriate. The periodogram is a "global" measure of dependence, in some respects; giving us a measure of the covariance of lag h over the entirety of the spectral domain. In the notation of class, for univariate case we will denote this as,

$$\hat{S}_X^{(p)}(f) = \sum_{h=-(T-1)}^{T-1} \left(\frac{1}{T} \sum_{t=1}^{T-|h|} X_t X_{t+|h|} \right) e^{-i2\pi f h}$$

But,for locally stationary processes, our covariance structures will now vary as function of t. So, if we instead fixed the time at t and considered the covariances at lag h centered at t, $t \leq T$, we would have "local" measure of covariance. The *pre-periodogram* is such a "local" measure of dependence, and we define it as follows for the univariate case.

$$J_T(u,f) = \sum_{\substack{h \\ 1 \leq [uT + \frac{1+h}{2}], [uT + \frac{1-h}{2}] \leq T}} X_{[uT+\frac{1+h}{2}],T} X_{[uT+\frac{1-h}{2}],T}$$

The periodogram can be decomposed into an average of the pre-periodograms at each of the observations.

Returning to the Whittle likelihood approximation to exact Gaussian likelihood, for the locally stationary process, we will replace $S_{X,\theta}(f)$, the spectral density, with the time varying spectral density $S_{X,\theta}(\frac{t}{T}, f)$. We will then replace the periodogram $\hat{S}_X^{(p)}(f)$ with the pre-periodogram $\tilde{J}_T(\frac{t}{T}, f)$ to obtain the generalized Whittle likelihood:

$$l_T^{GW} := \frac{1}{T} \sum_{t=1}^{T} \int_{-1/2}^{1/2} \left\{ \log(2\pi S_{X,\theta}(\frac{t}{T}, f)) + \frac{\tilde{J}_T(\frac{t}{T}, f)}{S_{X,\theta}(\frac{t}{T}, f)} \right\} df$$

Note, that if the process were stationary (i.e, $S_{X,\theta}(\frac{t}{T}, f) = S_{X,\theta}(f)$ and $A(u,f) = A(f)$), the classic Whittle likelihood would be obtained. More generally, for the multivariate case where we relax the assumption that $\mu = \mu_\theta = 0$ but still assume that Θ is compact, this approximation is:

$$l_T^{GW} := \frac{1}{T} \sum_{t=1}^{T} \int_{-1/2}^{1/2} \left\{ \log\left((2\pi)^d \det(S_{X,\theta}(t/T, f)) \right) + \mathrm{tr}\left[\frac{\tilde{J}_T^{\mu_\theta}(\frac{t}{T}, f)}{S_{X,\theta}(\frac{t}{T}, f)} \right] \right\} df$$

where

$$\tilde{J}_T(u,f)_{a,b}^{\mu} = \sum_{\substack{h \\ 1 \leq [uT + \frac{1+h}{2}], [uT + \frac{1-h}{2}] \leq T}} \left\{ X^a_{[uT+\frac{1+h}{2}],T} - \mu^a \left(\frac{[uT + \frac{1+h}{2}]}{T} \right) \right\}$$
$$\times \left\{ X^b_{[uT+\frac{1-h}{2}],T} - \mu^b \left(\frac{[uT + \frac{1-h}{2}]}{T} \right) \right\} \exp\{-i2\pi f h\}$$

At the heart of the classical likelihood approximation (i.e., in the stationary case) is a technical observation that uses the theory of Toeplitz matrices to approximate the inverse of the covariance matrix with the Toeplitz matrix of the inverse spectral density $S_X(f)$.

Additionally, the Szegö identity allows for the approximation of the log determinant of the Toeplitz covariance matrix via an integral of the log of the spectral density. What is driving the above generalization of the likelihood approximation for the locally stationary case is a corresponding generalization of the Toeplitz matrix theory. Since the approximation of the inverse of the covariance matrix is central to both the approximation and the asymptotic results discussed next, we will take a moment to investigate it further here.
For $\underset{-}{X} = (X_{1,T}, \ldots, X_{T,T})^T$, we can express the covariance matrix $\Sigma_\theta = \Sigma_T(A_\theta, A_\theta)$ to be the $T \times T$ block matrix with corresponding (j,k) block for $j,k = 1,\ldots,T$:

$$\Sigma_T(A_\theta, A_\theta)_{j,k} = \int_{-1/2}^{1/2} \exp\{i2\pi f(j-k)\} A^\circ_{\theta,j,T}(f) A^\circ_{\theta,k,T}(-f)^T df$$

We also introduce the $T \times T$ block matrix $U_t(\phi)$ with (j,k) block:

$$U_T(\phi)_{j,k} = \int_{-1/2}^{1/2} \exp\{i2\pi f(j-k)\} \phi\left(\frac{1}{T}\left\lfloor \frac{j+k}{2} \right\rfloor, f\right) df$$

where $\phi(u,f)$ is a $d \times d$ matrix and the integral is calculated entry-wise in the matrix. In a result analogous to the classical case, we see that an approximation of Σ_θ^{-1} can be done with $U_T(S_X^{-1}(u,f))$.

Theorem 5. *(Proposition 2.4 in* [6]*) Suppose the matrices A and ϕ fulfill the smoothness conditions denoted in Assumptions A.3(i)-(iii) in the Appendix of* [6] *with existing and bounded derivatives $\frac{\partial^2}{\partial u^2}\frac{\partial}{\partial f}A(u,f)_{a,b}$ and eigenvalues of $\phi(u,f)$ which are bounded from below uniformly in u and f. Then for all $\epsilon > 0$, we have the following:*

$$\frac{1}{T}|\Sigma_T(A,A)^{-1} - U_T(\{A\overline{A}^T\}^{-1})|^2 = O(T^{-1+\epsilon})$$

$$\frac{1}{T}|U_T(\phi)^{-1} - U_T(\{\phi\}^{-1})|^2 = O(T^{-1+\epsilon})$$

This approximation in turn leads to a generalization of the Szegö identity.

Theorem 6. *(Generalization of Szegö Identity to multivariate locally stationary processes, Proposition 2.5 in* [6]*) Suppose A fulfills Assumptions A.3(i) in* [6]*, with bounded derivatives $\frac{\partial^2}{\partial u^2}\frac{\partial}{\partial f}A(u,f)_{a,b}$. Then for $S_X(u,f) = A(u,f)A(u,-f)^T$ for each $\epsilon > 0$*

$$\frac{1}{T}\log\det\Sigma_T(A,A) = \int_0^1 \int_{-1/2}^{1/2} \log[(2\pi)^d \det S_X(u,f)] df du + O(T^{-1+\epsilon})$$

If $A = A_\theta$ depends on a parameter θ and fulfills the smoothness conditions of Assumption 2.6(iii),(iv), then $O(T^{-1+\epsilon})$ is uniform in θ.

Now that we have the generalized Whittle likelihood, we are interested in the asymptotic properties of the estimates obtained from that approximate likelihood. For the following discussion, we make a series of technical assumptions regarding regularity conditions on the true mean function vector μ, the transfer function matrix A, and the covariance matrix Σ_T; as well as on the corresponding model functions μ_θ, A_θ, and Σ_θ. We let θ_0 denote the true parameter. Returning for a moment to the univariate stationary case (under a number of assumptions), we know that the estimate, $\hat{\theta}$, found by minimizing the $\tilde{l}_T$ is asymptotically equivalent to MLE estimate. Happily, a corresponding result is possible here. For the following result, we let $\hat{\theta}^{GW} = \arg\min_{\theta\in\Theta} l_T^{GW}$ and $\tilde{\theta}_T$ denote the MLE of θ.

Theorem 7. *(Asymptotic Equivalence to MLE, Adapted from Theorem 3.1 in* [6]*) Suppose that Assumption 2.6 holds and that $\mu = \mu_{\theta_0}$ and $A_{\theta_0}(u, f) = A(u, f)$ for some $\theta_0 \in \Theta$ (i.e., the model is correctly specified), then*

$$\sqrt{T}(\hat{\theta}_T^{GW} - \theta_0) \xrightarrow{d} N(0, \Gamma^{-1})$$
$$\sqrt{T}(\tilde{\theta}_T - \theta_0) \xrightarrow{d} N(0, \Gamma^{-1})$$

where

$$\Gamma_{ij} = \frac{1}{2}\int_0^1 \int_{-1/2}^{1/2} tr\{S_{X,\theta_0}(\nabla_i S_{X,\theta_0}^{-1}) S_{X,\theta_0}(\nabla_i S_{X,\theta_0}^{-1})\} df\, du + \int_0^1 (\nabla_i \mu_{\theta_0}(u))^T S_{X,\theta_0}^{-1}(u, 0)(\nabla_j \mu_{\theta_0}(u)) du$$

Here, both estimates are asymptotically efficient. We stress again, that if the process is indeed stationary (and mean zero), the above result yields the same result discussed in class. Hence, for locally stationary Gaussian processes, we have arrived at an approximation of the likelihood that will allow us to derive estimates for the parameters that are asymptotically efficient.

4. Simulation Studies

In this section, we simulate realizations of a stationary $AR(1)$ process as well as for a time-varying $AR(1)$ process to explore the differences between the fits of stationary processes and locally stationary processes. We consider the implications of naively using methods for stationary processes to fit the time varying $AR(1)$ process.

We first generate a realization from the stationary $AR(1)$ process ($n = 256$)

$$X_t = .8 \cdot X_{t-1} + Z_t$$
$$Z_t \overset{iid}{\sim} N(0, 1)$$

Next, we consider a simple form of a realization from a "time varying" $AR(1)$ model:

$$X_t = .9X_{t-1} + Z_t \; t \leq 128$$
$$X_t = -.5X_{t-1} + Z_t \; 128 < t \leq 256$$
$$Z_t \overset{iid}{\sim} N(0, 1)$$

Below, we plot the realizations, and we note that in this case we can see what appears to be a change in behavior for the locally stationary process at about $t = 128$.

FIGURE 4.1. Sample path for stationary $AR(1)$ process

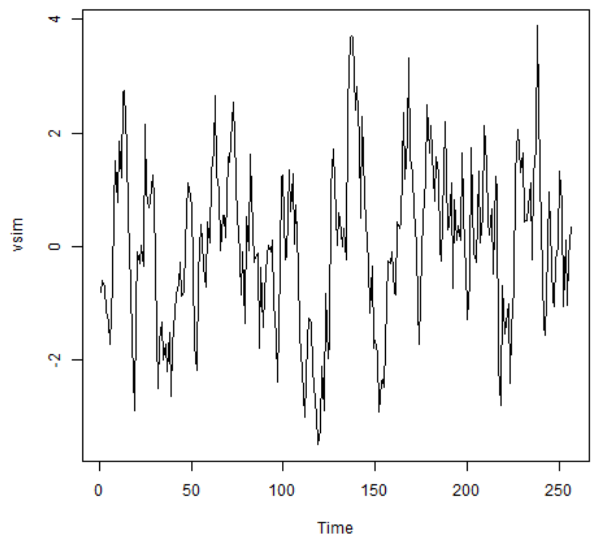
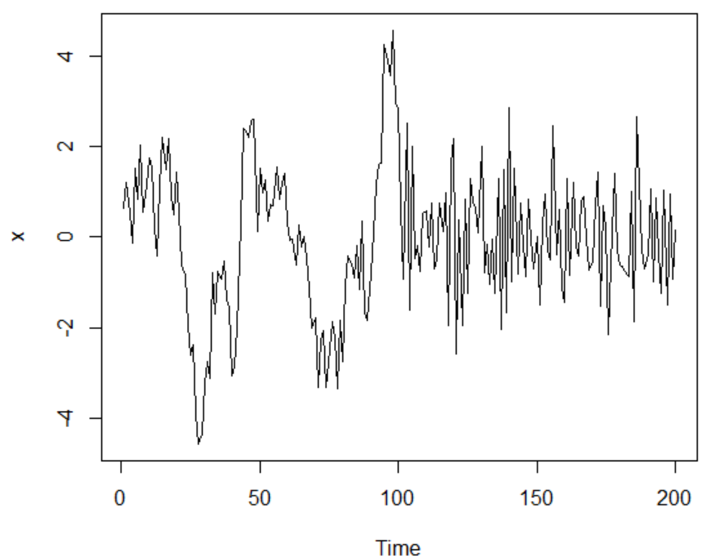


FIGURE 4.2. Sample path for time-varying piecewise $AR(1)$ process



Now, let us consider if we assumed that both are stationary processes, and we fit an $AR(1)$. If we consider the residuals for the fitted models, we see that the residuals from the time-varying $AR(1)$ model show trends (particularly for $t < 128$) suggesting the model is a poor fit. Also the Ljung-Box statistics shows that there is a significant nonstationarity in the model.

FIGURE 4.3. Diagnostics for stationary $AR(1)$ process

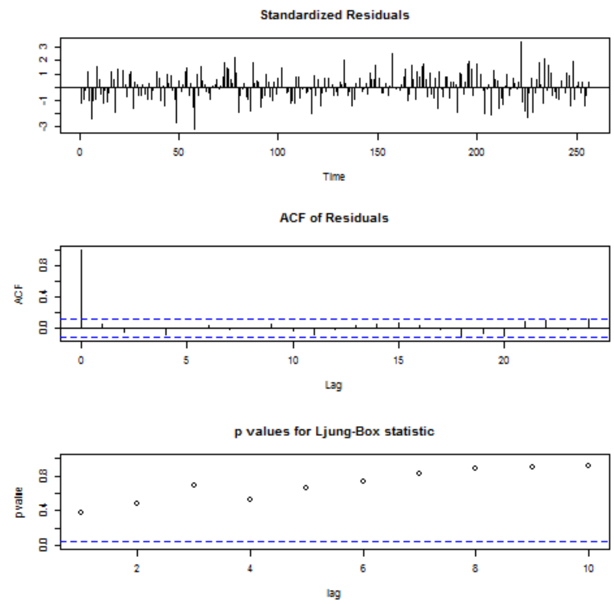
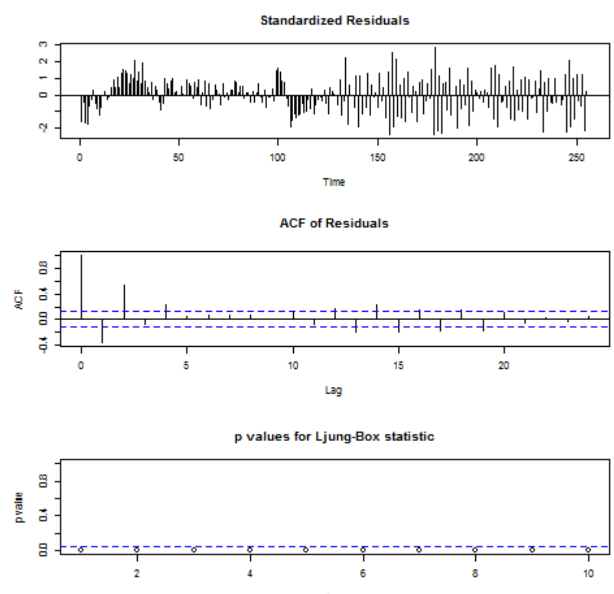
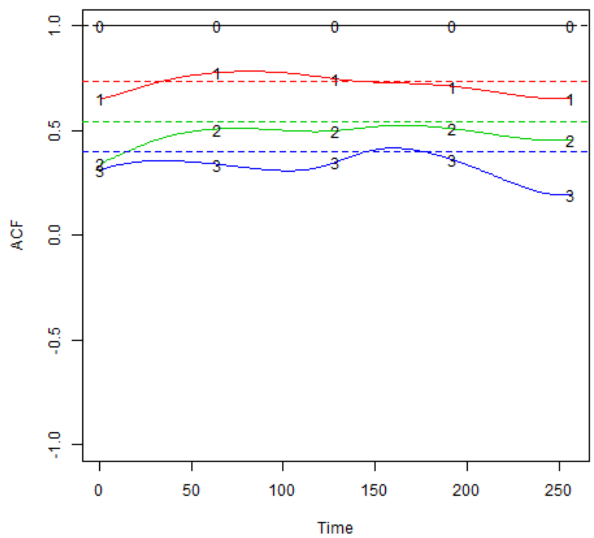


FIGURE 4.4. Diagnostics for time-varying piecewise $AR(1)$ process



Now, we estimate the localized autocovariance using a wavelet filter for both realizations of $AR(1)$ and piecewise $AR(1)$. We then plot the ACF curves of the fitted locally stationary process (LSP) at lags 1,2, and 3 and compare them to the estimated ACF from a stationary model. The ACF curves of the fitted LSP is shown in solid lines while the ACF from a stationary model is straight dashed lines since they are not varying over time. Different

FIGURE 4.5. Fitted ACF based on data from stationary $AR(1)$ process

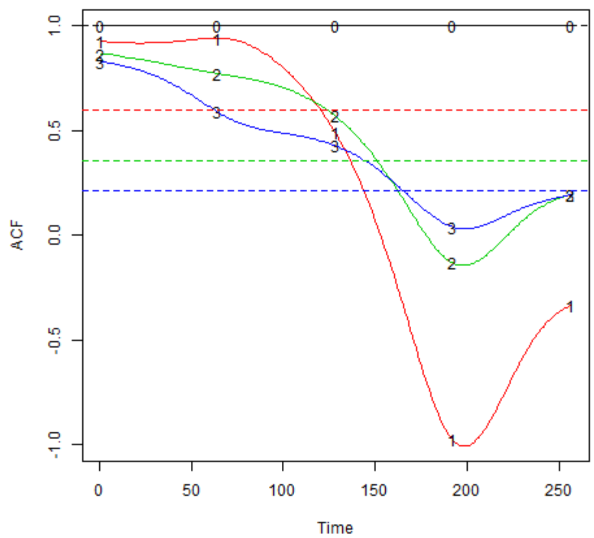


colors of curves means the different lags global ACF $C_t(h)$, different numbers denotes the lag. Clearly, for the time-varying models these values are vastly different.
Here we used the `costat` package [9] to estimate and analyze the ACF using `DaubExPhase` wavelet basis containing 6 filters. For the stationary $AR(1)$ process the ACF curve of fitted LSP does not wiggle much and is almost the same as the estimated ACF of fitted $AR(1)$ process. But for the time-varying $AR(1)$ process the ACF curve of fitted LSP shows significant difference from the estimated ACF of fitted $AR(1)$ process. There the $AR(1)$ is actually misspecified and we can see from the diagnostics as well as the data generating mechanism that LSP actually provides a far more better fit that $AR(1)$.
The reason is due to our data generating mechanism is of locally stationary nature. From the corresponding residual plots and the Ljung-Box testing statistics for stationarity we can see that the locally stationary process performs better in such a nonstationary scenarios. This is a simulation example yet we also expect that for nonstationary data sets we can actually have a better fit from the locally stationary processes. This can be explored in future works.

5. Discussions and Remarks

A more general theory can be built using Banach space theory by extending the idea of using stationary derivative processes approximating locally stationary process, the benefit is unification of the ergodicity behaviors (mainly consistency and central limit theorems) as mentioned. The cost is obvious that very limited possibilities of nonlinearity they can handle with certain complexity of the processes.

FIGURE 4.6. Fitted ACF based on data from time-varying piecewise $AR(1)$ process



For Gaussian process with nonstationarity, the generalized Whittle likelihood allows for parametric estimation and inference therein. This generalized Whittle likelihood is a sum of local likelihoods over time, and the elements of the summation can be viewed as a "local" (log) likelihood for a fixed time point.

In short, we believe that when piecewise sample paths are observed and with appropriate division of stationary time domain, the locally stationary process is a powerful modeling tool that allows us to capture some of these nonstationary behavior exhibited in the data set in terms of their spectrum or defining equations. Moreover, with additional Gaussian likelihood assumption we can perform an parametric estimation and inference. Nonparametric estimators could also be used on these summands to arrive at alternative estimators. At the time of the writing of [3], there was still work to be done in this area and Bayesian version of LSP is still an unexplored area [7].

References

[1] Brockwell, Peter J., and Richard A. Davis. Time series: theory and methods. Springer Science & Business Media, 2013.

(document)

[2] Cardinali, Alessandro, and Guy P. Nason. "Costationarity of locally stationary time series." Journal of Time Series Econometrics 2.2 (2010). (document), 1

[3] Dahlhaus, Rainer. "Locally stationary processes." Handbook of statistics. Vol. 30. Elsevier, 2012. 351-413. (document), 2, 3

[4] Dahlhaus, Rainer. "Maximum Likelihood Estimation and Model Selection for Local Stationary Processes. " Nonparametric Statistics. 6 (1996): 171-191. (document), 1

[5] Dahlhaus, Rainer. "Fitting time series models to nonstationary processes." The annals of Statistics 25.1 (1997): 1-37. (document)

[6] Dahlhaus, Rainer. "A Likelihood Approximation for Locally Stationary Processes. " The annals of statistics. 38.6 (2000): 1762-1794. (document), 1, 5, 6, 7

[7] Dahlhaus, Rainer, Stefan Richter, and Wei Biao Wu. "Towards a general theory for non-linear locally stationary processes." arXiv preprint arXiv:1704.02860 (2017).

(document)

[8] Ledoux, Michel, and Michel Talagrand. Probability in Banach Spaces: isoperimetry and processes. Springer Science & Business Media, 2013. (document)

[9] Nason, Guy P., Rainer Von Sachs, and Gerald Kroisandt. "Wavelet processes and adaptive estimation of the evolutionary wavelet spectrum." Journal of the Royal Statistical Society: Series B (Statistical Methodology) 62.2 (2000): 271-292. (document)

[10] Paciorek, Christopher Joseph. Nonstationary Gaussian processes for regression and spatial modeling Diss. PhD thesis, Carnegie Mellon University, Pittsburgh, Pennsylvania, 2003.

(document)