NOTES ON THE SQUARE-ROOT OPERATORS AND SPECTRAL THEORY FOR SELF-ADJOINT LINEAR OPERATORS

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Abstract. The spectral theorem for self-adjoint linear operators is known to be one of the most important theorems in the spectral theory for linear operators, this note shows we can interpret it in many aspects. And the author provides some small corollary and proposition to refresh the view of readers on this topic.

1. The Abstract Algebraic Structure

The following theorem is a key step towards the proof of the spectral theorem of bounded self-adjoint operators, which is a crucial part consisting of the spectral theory of the linear operators:

> **Definition**(Positive square-root) Let$T : H \to H$ be a positive bounded self-adjoint linear operator on a complex Hilbert space H. Then a bounded self-adjoint linear operator A is a square root of T if $A^2 = T$, and if $A \geqq \theta$as an operational inequality, then the operator is also called a positive square-root of T.
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> **Theorem 1**([Kreyszig]P476) Every positive bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

In other words, we can introduce an algebraic structure to describe this theorem. Once we notice this structure, the last result is rather clear.

> **Corollary 1** The *commutative* polynomial ring $P_1 := \mathbb{C}(T) = \{B \in B(H) | B = a_0 + a_1T + ... + a_nT^n + ..., a_m \in \mathbb{C}, m \in \mathbb{N}\}$,and the positive square-root operator $A$of T falls into the center of the ring P_1.(REMARKS: The commutativity is due to the fact that T commutes with itself.)
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> **Theorem 2**(Kaplansky) Let R be an integral domain. Then$R$ is a UFD if and only if each nonzero prime ideal contains a nonzero principal prime ideal.
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> **Proposition 1** If T is not a nilpotent operator, then $Z(P_1)$ is a UFD.(PROOF:The P_1has $(T - aI)\,, a \in \mathbb{C}\backslash\sigma\,(T)$ as all of its nonzero prime ideals since T is not nilpotent, and itself is a principal ideal.)
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> **Proposition 2** The If T is not a nilpotent operator, Every $p(T) \in P_1$has one inverse or infinitively many. (PROOF:Due to the Kaplansky theorem saying that if an element in a ring has

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more than one right inverse, it has in fact infinitely many. The adjective 'right' can be dropped since we're in a commutative ring by Corollary 1. See[Kaplansky].)

2. Fredholm Alternative

Now we can discuss something of interest, make a closer look of Corollary 2 and we know that it resembles the *Fredholm Alternative* since the integral operator is linear and bounded if the integration interval is finite, say $Int(f)_{(a,b)} := \int_a^b f dx, f \in L^2(\mathbb{R})$. But it's hard to checked some collection of operators form a ring. For example, the $Int(f)_{(a,b)}$with a and b fixed in $\mathbb{R}\backslash\{+\infty, -\infty\}$form a one-parameter(regard the functions $f, g \in L^2(\mathbb{R})$ as the parameter) additive group under the operation $Int(Af + Bg)_{(a,b)} =: AInf(f)_{(a,b)} + BInt(g)_{(a,b)}$, and the multiplication is the double-convolution $\int_{(a,b)\times(a,b)} f(y-x)g(x)dxdy =: Inf(f)_{(a,b)} * Int(g)_{(a,b)}$. This structure is rather good and we can usually not expect such a structure in the case of differential operators. And it's clear why the *Fredholm Alternative* remains true when the Hilbert space is replaced by some complex Banach algebra.

> **Corollary** 2(Spectral Mapping theorem)$P_2 := \mathbb{C}(\sigma) = \{c \in \mathbb{C} | c = a_0 + a_1\sigma + ... + a_n\sigma^n + ..., a_m \in \mathbb{C}, m \in \mathbb{N}, \sigma \in \sigma(T)\}$,then $P_1 \cong P_2$.

Moreover, if we restrict ourselves to $P_2^N := \mathbb{C}(\sigma) = \{B \in B(H) | B = a_0 + a_1\sigma + ... + a_n\sigma^n + ... + a_N\sigma^N, a_m \in \mathbb{C}, m \in \mathbb{N}, \sigma \in \sigma(T)\}$, then $P_2^N$is a semisimple ring because this ring can have finitely many prime ideals $(\sigma - \sigma_i)$due to the *Fundamental Algebra Theorem*. Since the ring$P_1^N := \{B \in B(H) | B = a_0 + a_1 T + ... + a_n T^n + ... + a_N T^N, a_m \in \mathbb{C}, m \in \mathbb{N}, \sigma \in \sigma(T)\}$is a semisimple ring due to the fact presented by Corollary 3 that , we use the *Wedderburn-Artin theorem* to assert the structure of this whole collection of *finite degree* polynomial rings generated by those non-nilpotent operators. And by the essential part of the proof of *Wedderburn-Artin theorem*, we also know the division rings are the epimorphism rings.

> **Theorem 3**(Wedderburn-Artin, [Lam]P35)Let R be any left semisimple ring. Then $R \cong Mat_{n_1}(D_1) \times ... \times Mat_{n_r}(D_r)$ for suitable division rings $D_1, ...D_r$ and positive integers $n_1, ...n_r$. The number r is uniquely determined, as are the pairs $(n_1, D_1), ...(n_r, D_r)$ (up to a permutation). There are exactly r mutually nonisomorphic left simple modules over R .

Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Furthermore, let $E_\lambda, \lambda \in \mathbb{R}$ be the projection of H onto the nullspace $Nul(T_\lambda^+) := Nul(T_\lambda^2)^{1/2}, T_\lambda = T - \lambda I$ of the positive part T_λ^+ofT_λ. Then $E_\lambda, \lambda \in \mathbb{R}$ is a spectral family on the interval $[m, M] \subseteq \mathbb{R}$ due to the spectral theorem of the self-adjoint linear operators. From the results of matrix analysis[Horn], we can have similar results considering the self-adjoint linear operators in finite-dimensional Hilbert space.

3. Linear Algebra and Statistics

> **Theorem 4**(Rayleigh-Ritz, [Horn]P176) Let $A \in Mat_n(\mathbb{C})$ be Hermitian, and let the eigenvalues of A be ordered as $\lambda_{min} = \lambda_1 \le ... \le \lambda_n = \lambda_{max}$. Then $\lambda_1 x^* x \le x^* Ax \le \lambda_n x^* x$ for all $x \in \mathbb{C}^n$. Furthermore, $\lambda_1 = min_{x \neq \theta} \frac{x^* Ax}{x^* x} = min_{x^* x = 1} x^* Ax$,and $\lambda_n = max_{x \neq \theta} \frac{x^* Ax}{x^* x} = max_{x^* x = 1} x^* Ax$.

Theorem 5 (Weyl,[Horn]P184). Let $A, B \in Mat_n(\mathbb{C})$, be Hermitian matrices, and let the eigenvalues of A, B and $A+B$ be arranged in increasing order $\lambda_{min} = \lambda_1 \leq ... \leq \lambda_n = \lambda_{max}$. Then for every pair of integers $j, k s.t. 1 \leq j, k \leq n$ and $j + k \geq n + 1$ we have $\lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda(B)$

Even the norm is parallel to the general definition. And the second theorem exactly explains why the spectral family have the property of $E_\lambda E_\mu = E_\lambda = E_\mu E_\lambda if \lambda \leq \mu$.Indeed, this property can also be explained as a consequence of the *Sylvester Inertia Theorem.*

This understanding leads to an important part of the multivariate statistics called the Fisher Projection method in the discriminant analysis. When we are confused with the classification of a stack of samples, the usual method to distinguish among these samples is to find the projection direction$a \in \mathbb{C}^n, s.t. \Delta(a) := \frac{a^*Ba}{a^*Aa} = \left(max_{x \neq \theta} \frac{x^*Bx}{x^*Ax}\right)$to maximize the difference between groups, where $A = \sum_{t=1}^{k} \sum_{j=1}^{n_t} (X_{(j)}^{(t)} - \bar{X^{(t)}})(X_{(j)}^{(t)} - \bar{X^{(t)}})', B = \sum_{t=1}^{k} n_t(\bar{X^{(t)}} - \bar{X})(\bar{X^{(t)}} - \bar{X})'$are the pooled in-group variance matrix and the between-group variance matrix. The $\left(X_{(j)}^{(t)}\right)$is the j-th component of the sample from t-th group. And this method can be extended to a more general situation called canonical correlation analysis:

Theorem 6([H.X.Gao]P347)Let $Z = (X\ Y)', X$and Y are correspondingly random variables of dimension p and q, say$p \leq q$.Given $E(Z) = 0_{p+q}, D(Z) = \begin{bmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{bmatrix} > 0, T = \sum_{11}^{-1/2} \sum_{12} \sum_{22}^{-1/2}$.Let the eigenvalues of TT' be ordered as $0 < \lambda_{min} = \lambda_p \leq ... \leq \lambda_1 = \lambda_{max}$ with the corresponding unitary orthogonal eigenvectors $l_1, ...l_n . a_k = \sum_{11}^{-1/2} l_k, b_k = \lambda_k^{-1} \sum_{22}^{-1} \sum_{21} a_k, k = 1, 2, ...p$,then$V_k = a_k' X, W_k = b_k' Y$ are the k-th pair canonical correlational variables of random variable X and Y, and λ_k is the k-th canonical correlational coefficient.

Proposition 3(c.f. the parallel result[Horn]P413)If $T \in B(H)$, then it may be written in the form $T = PU$ where P is a positive operator(square-root operator) and the U is a unitary operator.

From the above theorem we can see the notion of SVD(Singular Value Decomposition) can be use to extract the representative information from a stack of variables. And the Proposition 3 we proved above can be regarded as an explanation why we should introduce the positive part $T_\lambda^+$of$T_\lambda$into our proof of spectral theorem for self-adjoint linear operators. We must first extract the 'scalar part' of T_λ to see somehow its 'scalar strength' in the given Hilbert space$H$. And the rest part$U = T_\lambda^{-1} T$can be checked to be an unitary one. The spectral decomposition has its counterpart as finding a set of coordinate to substitute the current one in the Hilbert space H, and if the coordinate is substitute, we can see the $\sigma_p$are the axis directions while the $\sigma_c$are the continuous transformations of unit vectors. Moreover, such a representation can be realized in terms of algebra if the dimension is finite. And that's why we should construct such a weired thing in the proof of spectral theorem.

To sum up, the spectral theorem for self-adjoint linear operators is a result occuring in many branches of mathematics, henceforth it has corresponding many interpretations as well as applications.

4. Reference

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