NOTES ON STUDENTIZED RANGED DISTRIBUTION

H.R.LAW

Notation

We are basically focus on the one-way analysis of variance model(ANOVA). Assume that there are n observations in all separated into ν groups with sizes r_i. Then we should make sure we know[1]:

$$\begin{gathered}
\bar{x_i} = \frac{1}{r_i}\sum_{j=1}^{r_i} x_{ij} \\
\bar{x} = \frac{1}{n}\sum_{i=1}^{\nu}\sum_{j=1}^{r_i} x_{ij} \\
MSE = \frac{1}{n-\nu}\sum_{i=1}^{\nu}\sum_{j=1}^{r_i}(x_{ij}-\bar{x_i})^2, dfe = n-\nu \\
MStot = \frac{1}{n-1}\sum_{i=1}^{\nu}\sum_{j=1}^{r_i}(x_{ij}-\bar{x})^2, dftot = n-1 \\
(\nu-1)MST = (n-1)\cdot MStot - (n-\nu)\cdot MSE, dft = \nu - 1
\end{gathered}$$

1. What is Studentized Ranged Distribution?

1.1. **Motivation and History of Studentized Ranged Distribution.** Loosely speaking, the studentized ranged distribution emerges from the studentized 'gaps' inbetween different means of groups in a normal population. Or just differences between means of different normal populations. [2]The simplest studentized ranged distribution emerges from the distribution of difference between the means of two normal populations, which is, the t-distribution.[3] In the normal theory, we are very familiar with the t-distribution which we used to test the equal-mean hypothesis. In case of unequal variance, we have[4]:

$$\frac{\bar{x_1}-\bar{x_2}}{\sqrt{\frac{s_1^2}{r_1}+\frac{s_2^2}{r_2}}} \sim t(Welch)$$

and for the equal variance case we have:

$$\frac{\bar{x_1}-\bar{x_2}}{\sqrt{\frac{(r_1-1)s_1^2+(r_2-1)s_2^2}{r_1+r_2-2}}} \sim t(r_1+r_2-2)$$

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[1] [Dean&Voss]

[2] In [Hartley2], Hartley mentioned that:'Although theoretically such an estimate is not efficient and certainly not sufficient, it is nevertheless of considerable importance in many fields of application because of its simplicity.'

[3] In fact, we have $q_\alpha = \sqrt{2}t_\alpha$ when $n=2$. See [Newman](8)(10).

[4] $s_i^2 = \frac{1}{r_i-1}\sum_{j=1}^{r_i}(x_{ij}-\bar{x_i})^2$

This test against equal mean hypothesis is also known as the Student's t-test[Student]. Student is the alias of W.Gosset, who firstly discovered that in the small size sample, the distribution of 'studentized' mean is actually different from the normal distribution.[5]

In [Student], Gosset derived the t-distribution by using moment-equalizing method in terms of the moments of normal distribution. Note that in his paper, Gosset only proved that s^2 and $\bar{x}$ are not correlated instead of being independent. They are indeed independent by Cochran's theorem, but that is too strong a condition for the t-distribution.

While studentization is later formalized as the procedure of dividing by the sample standard deviation, we used the term 'standardization' more often emphasizing its nature of scaling. Although the studentized ranged distribution is always attributed to J.W.Tukey, Tukey himself only provided a few unpublished papers on this topic besides [Tukey].

A major contributor is H.O.Hartley, who presented in his paper [Hartley1, Hartley2] how his methods can be used to obtained a generalized version of t-distribution, which is now called studentized ranged distribution.

1.2. **Tukey's Contribution (HSD).** However, Tukey won his fame for solid reasons[6]. Tukey pointed out in [Tukey] a full procedure of how to use the gaps to compare means of groups. What Tukey summarized is[7]:

(1)(Pairwise significant, excessive gaps)There is unduly wide gap between adjacent variety means

(2)(MST significant, stragglers)One variety mean struggles too much from the grand mean.(Nair test[Nair])

(3)($Var(\bar{x_i})$ significant, excessive variability)The variety means taken together are too variable.(F-test)

Tukey used this distribution in multi-comparison in order to construct a pairwise confident interval, which is known as 'Tukey's Honest Significant Differences'(HSD). HSD improved the earlier version of 'Fisher's Least Significant Differences'(LSD). Suppose that in a one-way ANOVA model we have n observations with r treatment groups. Each group has r_i observations with mean $\bar{x}$.

$$LSD = t_\alpha(n-\nu)\sqrt{MSE\left(\frac{1}{r_1}+\frac{1}{r_2}\right)} \overset{r_i \equiv r_0}{=} t_\alpha(n-r)\sqrt{MSE\left(\frac{2}{r_0}\right)}$$

A difference between the means of two groups is regarded significant if $|\bar{x}_1 - \bar{x}_2| > LSD$.

$$HSD = q_\alpha(n,\nu)\sqrt{\frac{1}{2}MSE\left(\frac{1}{r_1}+\frac{1}{r_2}\right)} \overset{r_i \equiv r_0}{=} q_\alpha(n,r)\sqrt{MSE\left(\frac{1}{r_0}\right)}$$

A difference between the means of two groups is regarded significant if $|\bar{x}_1 - \bar{x}_2| > HSD$.

[5][Student]...But, as we decrease the number of experiments, the values of the standard deviation found from the sample pf experiments becomes itself subject to an increasing error until judgments reached in this way may become altogether misleading.

[6][Scheffé]pp.67, footnote. "Tukey was the first to devise a method of simultaneously estimating all contrasts..."

[7][Tukey]

An improvement which can be achieved using studentized ranged distribution on LSD is [Hayter], which is a bit more powerful than Tukey's HSD:

$$MLSD = q_\alpha(n,\nu-1)\sqrt{MSE\left(\frac{1}{r_1}+\frac{1}{r_2}\right)} \overset{r_i \equiv r_0}{=} q_\alpha(n,r-1)\sqrt{MSE\left(\frac{2}{r_0}\right)}$$

An obvious corollary is that the lengths are $MLSD \geq LSD \geq HSD$, the powers are $MLSD \leq LSD \leq HSD$ if we know that when $n = 2$ the studentized distribution degenerates into a t-distribution.

2. Derivation of Studentized Ranged Distribution

The following derivation is a streamlined version of [Hartley1, Hartley2]. The importance of studentization is mentioned in [Hartley1] as :'...By this we mean that a statistic whose sampling distribution involves the unknown standard deviation of the population is modified so that its distribution involves only quantities calculated from the sample.[8,9]

2.1. Derivation of General Studentized Statistic Distribution.

The idea of studentization is proposed by Gosset, and his student Hartley developed a method of obtaining the studentized distribution from the original statistics' distribution. Let us consider such a class of statistics as described by [Hartley1]: Let W be a *positive* statistic calculated from a sample drawn from a normal population $N\left(\mu,\sigma^2\right)$. If we denote by s^2 the *unbiased* estimate of σ^2 independent of the sample from which W is calculated, then the distribution of $\frac{W}{s} = \frac{W}{\sigma}/\frac{s}{\sigma}$[10] will not include an expression of σ. So without loss of generality we can assume that $W \sim N(\mu,1)$ and $f_W(w)$ be the pdf of it. In the following arguments, we used Hartley's 'method of quadratures' to obtain a numerical approximation of the distribution of $\frac{W}{\rho_{dfe}}$.

Lemma 1. *Define $\rho^2_{dfe} := dfe \cdot s^2$, dfe is the degree of freedom in the sample variance[11], then the joint pdf of (ρ_{dfe}, W) is $f_{(\rho_{dfe},W)}(\rho,w) = 2\cdot\frac{1}{\Gamma(\frac{dfe}{2})(\sqrt{2})^{dfe}}\rho^{dfe-1}e^{-\frac{1}{2}\rho^2}f_W(w)$.*

Proof. By Cochran's theorem and our assumption that $\sigma^2 = 1$, $\rho^2 \sim \chi^2(\nu)$. And also by Cochran, we know that W is independent of ρ. Then this is simply a joint distribution of independent pairs. Note that the multiplier 2 is just to compensate the fact that we have two square roots for one positive numbers. □

Proposition 2. *The pdf of the random variable $Q = \frac{W}{\rho_{dfe}}$ is*

$$\begin{aligned} f_Q(q = \tfrac{w}{\rho}) &= 2\cdot\frac{1}{\Gamma(\frac{dfe}{2})(\sqrt{2})^{dfe}}\int_0^\infty \rho^{dfe-1}e^{-\frac{1}{2}\rho^2}\left[\int_0^{q\rho} f_W(w)dw\right]d\rho \\ &= 2\cdot\frac{1}{\Gamma(\frac{dfe}{2})(\sqrt{2})^{dfe}}\int_0^\infty \rho^{dfe-1}e^{-\frac{1}{2}\rho^2}F_W(q\rho)d\rho \end{aligned}$$

[8] In this sense studentization is simply interpreted as replacing the variance of the true population distribution with the sample variance.

[9] [Hartley1]...Although the importance of studentizing problems is now well understood, there is a relatively small number of distribution functions for which the studentized form is known, and frequently large-sample results have to be consulted...It is therefore the purpose of this paper to estimate the accuracy with which a studentized distribution function may be approximated to by the corresponding large-sample distribution.

[10] Bear in mind that $X \sim N(0,1); W = \sqrt{n}\bar{x} \sim N(\frac{\mu}{n},1)$ in the following arguments.

[11] In one-way ANOVA, it is taken as $dfe = n - \nu$, and in [Hartley1] it is always denoted as n.

Proof. Use Lemma 1 and the change of variable formula and notice that $W > 0$. In the next steps, we want to use the cdf of W to approximate the cdf of $Q_{dfe} = \frac{W}{\sqrt{dfe}}$. □

The following result is the major result obtained by Hartley in [Hartley1], which is actually used for numerical calculation of the pdf of the distribution of studentized statistic Q_{dfe}. This method is of advantages since Q_{dfe} is a statistic which has been already studentized, while W is a statistic whose distribution is much more easily obtained.The proof is to expand F_W using Taylor series first, and then control each Taylor coefficient using Stirling's formula. We do not want to calculate the cdf of Q_{dfe} directly probably because that is a double integral involving a gamma function which loses much more precision when approximated.

Theorem 3. *(Hartley's inequality)For the studentized statistic $Q_{dfe} = \frac{W}{1\cdot\sqrt{dfe}}, q_{dfe} = \frac{w}{\sqrt{dfe}}$, we can approximate its pdf $f_{Q_{dfe}}$ using the cdf F_W of W.*

$$|f_{Q_{dfe}}(q_{dfe}) - F_W(w)| \leq \left\{ |\frac{dF_W}{dW}(w)| \cdot w + max_w |\frac{d^2F_W}{dW^2}(w)| \cdot w^2 \right\} \cdot \frac{1}{4 \cdot dfe} \cdot \frac{1}{90 \cdot dfe^2}$$

Using Hartley's inequality, we can obtain the studentized-W distribution as long as we have a full knowledge of W distribution.

2.2. **Application to the Studentized Ranged Distribution.** [Newman] has a fuller explanation on this direction of numerical calculation method, Hartley later published a paper [Hartley2] explaining how this studentized ranged distribution emerge from the study of range of a sample population$\{x_1 \leq x_2 \leq \cdots \leq x_N\}$. We firstly derive the distribution of the range statistic $W_q = R := x_{max} - x_{min}$, and then studentize it using the method described in the previous section. [12]

The first result proved below is about the discrete case.

Proposition 4. *The range statistic $R := x_{max} - x_{min}$ of a random sample of size n whose population pdf is f_X can be written as:*

$$f_R(n, h, m-1, \xi) = \sum_{i=-\infty}^{+\infty} \left\{ \left(\int_{\xi+ih}^{\xi+(i+m)h} f_X(x)dx \right)^n - \left(\int_{\xi+(i+1)h}^{\xi+(i+m)h} f_X(x)dx \right)^n \right\}$$

where h is the length of a partition on $(-\infty, \infty)$, $\xi_i = \xi + (i + \frac{1}{2})h$ are midpoints of each partition component, the value $f_R(n, h, m-1, \xi)$ is the probability that the range $x_{max} - x_{min} \geq (m-1)h \iff x_{max} - x_{min} > mh$.

Proof. Notice that the probability of a sample's n points falling exactly in the interval $(\xi_i, \xi_i + (m-1)h]$ is $\left(\int_{\xi+ih}^{\xi+(i+m)h} f_X(x)dx \right)^n - \left(\int_{\xi+(i+1)h}^{\xi+(i+m)h} f_X(x)dx \right)^n$. Take $h \to 0$ with $(m-1)h$ fixed, we deduced the following Theorem 6. □

Corollary 5. *The range of mean*

$$\Xi = h \cdot lim_{m\to\infty} \{(m+1)f_R(n, h, m, \xi) - f_R(0) - f_R(1) - \cdots - f_R(m)\}$$

with sample size n fixed, the midpoints ξ fixed and the group size h fixed.[Hartley2]

[12]Since that if we assume that $r_i \equiv r_0$ in the following discussion for convenience, then $q := \frac{max_{1\leq i\leq\nu}\{\bar{x_i}\} - min_{1\leq i\leq\nu}\{\bar{x_i}\}}{\sqrt{MSE/r_0}} = \frac{W_q}{\rho_{dfe_q}}$ is of the form we discuss in the previous section.

The second result proved below is about the continuous case where the population is normally distributed. See also [Newman], where there is a classic example of potato yields and the Table III in pp.25 in this article is exactly part of what we used today in [Dean&Voss]. A comparison between these two tables ([Newman] Table III, [Dean&Voss] Table A.8 in Appendix A) shows that the interpolation method has been improved over decades.

Theorem 6. *If $X \sim f_X$, then $f_{R,n}(W := mh) = n\int_{-\infty}^{\infty} f_X(x)\left(\int_x^{x+W} f_X(y)dy\right)dx$. The statistics W can be regarded as the fixed lower bound $x_{max} - x_{min} > W$.*[Hartley2]

Corollary 7. *If $X \sim f_X = N(0,1)$, then $f_{R,n}(W := mh) = \left(\int_{-\frac{1}{2}W}^{\frac{1}{2}W} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}dx\right)^n + 2n\int_{\frac{1}{2}W}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\left(\int_{y-W}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}dz\right)^{n-1} dy$. The statistics W can be regarded as the fixed lower bound $x_{max} - x_{min} > W$.*[Hartley2]

How this studentized ranged distribution can be used in the ANOVA?[13] We can compare multiple groups at the same time by using range-test. That is to say, instead of comparing a pair of treatment at one time and make Bonferroni bound, we can simply make the studentized range

$$q := \frac{max_{1\leq i\leq \nu}\{\bar{x}_i\} - min_{1\leq i\leq \nu}\{\bar{x}_i\}}{\sqrt{MSE/r_0}} \overset{W=[max_{1\leq i\leq \nu}\{\bar{x}_i\}-min_{1\leq i\leq \nu}\{\bar{x}_i\}],s=\sqrt{\frac{MSE}{r_0}}}{\longleftarrow} \frac{W}{s}$$

Definition 8. (Studentized Ranged Distribution)And we just give the distribution of q a name after we use the quadrature method to obtain the exact values of this distribution, the studentized ranged distribution. More generally if we have $Z_i \overset{i.i.d}{\sim} N(\mu, \sigma^2), i = 1, \cdots, n$,then $q := \frac{max_{1\leq i\leq n}\{Z_i\} - min_{1\leq i\leq n}\{Z_i\}}{Est(\sigma^2)} = \frac{max_{1\leq i\leq j\leq n}|Z_i - Z_j|}{Est(\sigma^2)} \sim q_{n-\nu,\nu}$where ν is the degree of freedom used to estimate σ^2, if we use the unbiased one it is $q_{1,n-1}$. Therefore we have acquainted how many normal independent random variables ' studentized range will distribute, the studentized range will be distributed as a studentized range distribution(Or Tukey's distribution).

as the testing statistic. And after calculating this statistic from the observed sample, we compare it with the α-quantile of the studentized ranged distribution $q_\alpha(n, \nu)$. The reason why the degree of freedom of the sample variance matters in this studentized distribution is that we have to use the W's cdf to approximate Q_{dfe}'s pdf via Theorem 3, which is relatively easy since F_W can be obtained via integration of $f_{R,n}$ not involving dfe. Or more precisely, the studentization itself intrinsically requires the degree of freedom of sample variance to involve.

The third result proved below is about the continuous case where the population is uniformly distributed[14]. This is omitted here.

Summing up the cases above, we have full recognition of the studentized ranged distribution.

Proposition 9. *When the sample size n goes up[15], the studentized ranged distribution is more heavy-tailed; When the degree of freedom ν goes up, the studentized ranged distribution is less heavy-tailed.*

[13] This application to ANOVA is first presented by Newman in his paper[Newman].

[14] [Hartley2]

[15] Or $dfe = (n - v)$ goes up with ν fixed, some software use this as a parameter.

3. Some Applications

3.1. **An Interesting Digression.** When I first encountered the problem of figuring out the distribution of $\frac{Range}{Sample\,Variance}$, the intuitive way is not the method of quadrature but the more popular way of figuring $\left(R, S^2\right)$'s joint distribution first and then perform the transformation $\begin{cases} X &= \frac{R}{S^2} \\ Y &= S^2 \end{cases}$ to get the distribution (X, Y). After that we can get a marginal distribution of $X = \frac{R}{S^2}$. This method is intuitive since we already know that the sample variance from a normal population $S^2 \sim \frac{\sigma^2}{n-1}\chi^2(n-1)$. And the range distribution can be deduced using binomial techniques such as that used in [Casella&Berger] pp.231 Example 5.4.7 as $f_{(X_{(1)},X_{(n)})}(u,w) = \frac{n!}{(n-2)!} f_X(u) f_X(w) \left[F_X(w) - F_X(u)\right]^{n-2}, u < w$. Using the transformation $\begin{cases} R = & X_{(n)} - X_{(1)} \\ V = & X_{(n)} + X_{(1)} \end{cases}, \begin{cases} X_{(1)} = & \frac{V-R}{2} \\ X_{(n)} = & \frac{V+R}{2} \end{cases}$ with $Jacobian = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$. The transformed joint distribution is $f_{(R,V)}(r,v) = \frac{n(n-1)}{2} \cdot f_X(\frac{v-r}{2}) f_X(\frac{v+r}{2}) \left[F_X(\frac{v+r}{2}) - F_X(\frac{v-r}{2})\right]^{n-2}$, and we have got

$$f_R(r) = \int_{-\infty}^{\infty} \frac{n(n-1)}{2} \cdot f_X(\frac{v-r}{2}) f_X(\frac{v+r}{2}) \left[F_X(\frac{v+r}{2}) - F_X(\frac{v-r}{2})\right]^{n-2} dv$$

and in case of two normal independent random variables it has the closed form:

$$f_R(r) = \int_{-\infty}^{\infty} \frac{n(n-1)}{2} \cdot \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} \cdot \left(2 \cdot (\frac{v}{2} - \mu)^2 + \frac{v^2}{2}\right)} \left[\int_{\frac{v-r}{2}}^{\frac{v+r}{2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt\right]^{n-2} dv$$

Is this arguement valid? Yes. Can we proceed as we planned before? No. This is because R, S^2 is actually not independent, we cannot find their joint transformation. However, we can of course compare the numerical value gained from this method and the Hartley's quadrature method. In that way we can see the correlation is actually not very significant.

3.2. **In Nonparametric Test(Steel-Dwass-Critchlow-Fligner Test[16]).** To carry out a nonparametric test for the one-way layout:

The observation is assumed to satisfying following nonparametric result:[17]

(A1)The random variables $x_{ij}, i = 1, \cdots \nu, j = 1, \cdots r_i$ are mutually independent.

(A2)In each treatment group these $r_i, i = 1, \cdots, \nu$ random variables are from the same population distribution F_i

(A3)The underlying distributions only differ in locations $F_i(t) = F(t - \tau_i), t \in \mathbb{R}$[18]

$H_0 : \tau_u = \tau_v, 1 \leq u < v \leq \nu$ v.s. $H_1 : \tau_{u_0} = \tau_{v_0}, for\ some\ (u_0, v_0)$

The Steel-Dwass-Critchlow-Fligner Multiple comparison is an extension of Tukey's HSD into this nonparametric ν-sample location problem. SDCF test considered the standardized Wilcoxon rank sum statistics for two groups.

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[16][Hollander&Wolfe&Chicken]Sec 6.5

[17]This section is basically a re-paraphrase of Sec 6.5 in [Hollander&Wolfe&Chicken], I refer the reader to this book. However, it is natural to extend Tukey's HSD into such a nonparametric test after we know HSD controls the maximum Type I error for all simultaneous cofident intervals.

[18]Note that this is normality assumption if we are still in the one-way ANOVA model.

Definition 10. (Standardized Wilcoxon Rank-sum Statistics) Let W_{uv} be the Wilcoxon Rank-sum statistics of two groups u, v in the one-way layout specified by (A1)-(A3). The standardized Wilcoxon Rank-sum statistics $W_{uv}^{*}=\sqrt{2}\left[\frac{W_{uv}-E_0(W_{uv})}{Var_0(W_{uv})}\right]=\sqrt{2}\frac{W_{uv}-\frac{r_v(r_u+r_v+1)}{2}}{\sqrt{\frac{r_ur_v(r_u+r_v+1)}{24}}}, \forall 1\leq i<j\leq\nu$.

Proposition 11. *([Hollander&Wolfe&Chicken]pp.257,pp.263[19]) When H_0 is true and $r_1=\cdots=r_\nu=r\rightarrow\infty$, then the random vector is asymptotically distributed as*

$$\left(W_{11}^{*},W_{12}^{*},\cdots,W_{1r_1}^{*},\cdots,W_{\nu 1}^{*},W_{\nu 2}^{*},\cdots,W_{\nu r_\nu}^{*}\right)\dot{\sim}N\left(0_{1\times\sum_{k=1}^{\nu}r_k},\Sigma\right),$$
$$max_{1\leq u<v\leq\nu}\left|W_{uv}^{*}\right|\dot{\sim}max_{1\leq k<l\leq\nu}\left|Z_k-Z_l\right|$$
$$\Sigma=Corr\left(Z_1-Z_2,Z_1-Z_3,\cdots,Z_1-Z_{r_1},\cdots,Z_{\nu-1}-Z_\nu\right)_{1\times\left(\begin{array}{c}\nu\\2\end{array}\right)},$$

where $Z_k\sim N(0,1), k=1,\cdots,\nu$.

Corollary 12. *([Hollander&Wolfe&Chicken]pp.263)$max_{1\leq u<v\leq\nu}\left|W_{uv}^{*}\right|\dot{\sim}max_{1\leq k<l\leq\nu}\left|Z_k-Z_l\right|$where $Z_k\sim N(0,1), k=1,\cdots,\nu$.*

And now we can see that this is exactly the definition of studentized ranged distribution.

3.3. **Adjusted P-values and R-commands.** Actually, the adjusted p-values in the studentized range test are simply the p-values obtained using the testing statistics. The adjective 'adjusted' is used to emphasized that this is a simultaneous p-value. For example, the Bonferroni might give a p-value of $1-n\alpha^{*}$ instead of $1-\alpha^{*}$ where α^{*} is the assigned p-value you want; in Tukey's HSD, the p-value is directly calculated using the studentized ranged distribution.

The adjusted p-value in R is obtained using the studentized ranged statistic $q_{n-\nu,\nu}:=\frac{max_{1\leq i\leq\nu}\{\bar{x}_i\}-min_{1\leq i\leq\nu}\{\bar{x}_i\}}{\sqrt{MSE/r_0}}$ in 4.4.4 of [Dean&Voss].

```
nematode =
    matrix(c(0,0,0,0,1000,1000,1000,1000,5000,5000,5000,5000,10000,10000,10000,10000,
10.8,9.1,13.5,9.2,11.1,11.1,8.2,11.3,5.4,4.6,7.4,5.0,5.8,5.3,3.2,7.5),
byrow=FALSE,ncol=2,nrow=16)
nematode=data.frame(nematode)
names(nematode) = c("Num.Nematodes","SeedlingGrowth")
nematode$Num.Nematodes = as.factor(nematode$Num.Nematodes)
nematode.aov = aov(SeedlingGrowth ~ Num.Nematodes, data = nematode)

trmt.0 = c(10.8,9.1,13.5,9.2)
trmt.1000 = c(11.1,11.1,8.2,11.3)
trmt.5000 = c(5.4,4.6,7.4,5.0)
trmt.10000 = c(5.8,5.3,3.2,7.5)
trmt = c(trmt.0,trmt.1000,trmt.5000,trmt.10000)
sse =
    sum(var(trmt.0)*(length(trmt.0)-1)+var(trmt.1000)*(length(trmt.1000)-1)+
var(trmt.5000)*(length(trmt.5000)-1)+var(trmt.10000)*(length(trmt.10000)-1))
ssto = var(trmt)*(length(trmt)-1)
```

[19]Here the author referred to [Miller] pp.155-156, which is another invaluable source

```
sst = ssto - sse
dfe <- length(trmt.0)-1 + length(trmt.1000)-1 +       length(trmt.5000)-1
+  length(trmt.10000)-1 dfto <- length(trmt)-1 dft <- dfto - dfe
mst <- sst / dft mse <- sse / dfe

TukeyHSD(nematode.aov)
Tukeyq=(abs(mean(trmt.0)-mean(trmt.1000)))/sqrt(mse/4)
Tukeyq
ptukey(Tukeyq,4,dfe,nranges=1,lower.tail = FALSE)
```

Since the R provides an obscure description of its 'qtukey' function, I rewrite a function providing the Tukey's table [20]. The typical grammar should be

```
qtukey(0.05,5,30,lower.tail=FALSE)
```

calculates the upper quantile $q_{0.05}(30,5)(Or \quad q_{5,30,0.05})$.

```
QTable<-function(dfrange=10,nurange=20,alpha=0.05,digs=3){
   ROWS<-dfrange
   COLS<-nurange
   tabl<-matrix(nrow=ROWS,ncol=COLS)
   for(a in 2:COLS){
      tabl[1,a]=a
   }
   for(b in 2:ROWS){
      tabl[b,1]=b
   }
   for(i in 2:ROWS){
      for(j in 2:COLS){
         tabl[i,j]<-qtukey(alpha,j,i,nranges=1,lower.tail=FALSE)
         #R has a wrong description for the parameter 'nmeans',
         #From the description of R:
         #i=n-nu;j=nu->n=i+j;nu=j->number of groups nu=j;
         #Each group has n/nu elements, which is (i+j)/j
         #The above interpretation is wrong, the parameter 'nmeans'
         #should be regarded as number of groups when 'nranges'=1
         #To get a correct result, always set 'nranges'=1.
         tabl[i,j]<-round(tabl[i,j],digs)
      }
   }
   tmp<-as.data.frame(tabl)
   colnames(tmp)<-as.character(tmp[1,])
   tmp<-tmp[-1,]
   rownames(tmp)<-as.character(tmp[,1])
   tmp<-tmp[,-1]
   message("significant level=",alpha)
   message("nu=1:",nurange,", df=1:",dfrange)
   message("Rows are the value sequence of df;Columns are the nu sequence.")
   print(tmp) }
   QTable(dfrange=20,nurange=8,alpha=0.01)
```

[20] Table A.8 in Appendix of [Dean&Voss]

The output turns out to be[21]:

```
significant level=0.01 nu=1:8, df=1:20
Rows are the value sequence of df;Columns are the nu sequence.
   2      3      4      5      6      7      8
2  13.902 19.015 22.564 25.372 27.757 29.856 31.730
3   8.260 10.620 12.170 13.322 14.239 14.998 15.646
4   6.511  8.120  9.173  9.958 10.583 11.101 11.542
5   5.702  6.976  7.804  8.421  8.913  9.321  9.669
6   5.243  6.331  7.033  7.556  7.972  8.318  8.612
7   4.949  5.919  6.542  7.005  7.373  7.678  7.939
8   4.745  5.635  6.204  6.625  6.959  7.237  7.474
9   4.596  5.428  5.957  6.347  6.657  6.915  7.134
10  4.482  5.270  5.769  6.136  6.428  6.669  6.875
11  4.392  5.146  5.621  5.970  6.247  6.476  6.671
12  4.320  5.046  5.502  5.836  6.101  6.320  6.507
13  4.260  4.964  5.404  5.726  5.981  6.192  6.372
14  4.210  4.895  5.322  5.634  5.881  6.085  6.258
15  4.167  4.836  5.252  5.556  5.796  5.994  6.162
16  4.131  4.786  5.192  5.489  5.722  5.915  6.079
17  4.099  4.742  5.140  5.430  5.659  5.847  6.007
18  4.071  4.703  5.094  5.379  5.603  5.787  5.944
19  4.046  4.669  5.054  5.334  5.553  5.735  5.889
20  4.024  4.639  5.018  5.293  5.510  5.688  5.839
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