Circular Coordinates under Different Cost Functions

Group 8 Project

ICERM August 2019

Applied Mathematical Modeling with Topological Techniques

Problem Setup

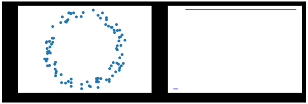
- We have a (simplicial) complex K from the dataset X, we can consider the homology and cohomology with a fixed coefficient field $\mathbb{K}$.
- Homology and boundary operator: $\partial : C_k(K) \to C_{k-1}(K)$. e.g. $\partial[a, b, c] = [a, b] + [b, c] + [c, a]$
- Cohomology and coboundary operator: $\delta : C^k(K) \to C^{k+1}$. e.g. $\delta \begin{bmatrix} a \mapsto 1 \\ b \mapsto 0 \\ c \mapsto 0 \end{bmatrix} = -[a, b]^* - [a, c]^*$ with $[a, b]^* = \begin{bmatrix} ab \mapsto 1 \\ bc \mapsto 0 \\ ca \mapsto 0 \end{bmatrix}$.
- **Intuition:** If you think of boundary operators as "derivation", then the coboundary operator is like "anti-derivation".

- **Theorem** (*Circular Coordinates*): Given a $[f] \in H^1(K)$, f can be made into a function $X \to S^1$. This **circular coordinate** can be found through the optimization problem $\min_{z \in C^0(X)} \|f - \delta z\|_{L^2}$.

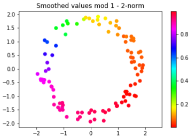
- **Problem** (New): How about we change the cost function $\|x\|_{L^2} := (\sum_i x_i^2)^{1/2}$ into:
  - $\|x\|_{L^1} := (\sum_i |x_i|)$ L1-norm
    It may introduce sparsity across coordinates instead of smoothness.
  - $(1-\lambda)\|x\|_{L^1} + \lambda\|x\|_{L^2}$ elastic net
    It may find a balance between L1 and L2 norms.
  - $\|x\|_{L^1} + \lambda\|x\|_{L^p}$
    It may produces some other kind of smoothness.
  - Localized penalty. Only take a penalty norm for some subvector of x.
  - In addition, we can penalized not only $x = f - \delta z$ but also
    - $x = \delta z$ (minimize edits?)
    - $x = z$ or $x = z \bmod 1$ (smaller values for functions as $X \to S^1$?)

Optimization Problem: Gradient Descent

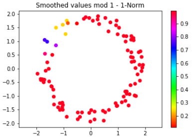
- Generic matrix optimization without Jacobian estimate: Slow and inefficient.
- Matrix optimization using Gradient Descent with Jacobian.
  - Example 1: Annulus.



    - L2 norm ($x = f - \delta z$ and $x = z \bmod 1$)

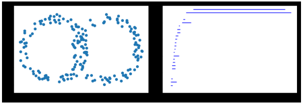


    - L1 norm ($x = f - \delta z$ and $x = z \bmod 1$)



Another Example

- Matrix optimization using Gradient Descent with Jacobian.
- Example 2: Double Annulus.



  - Mixed L2 norm $((1-\lambda)\|f-\delta z\|_{L^2}+\lambda\|f-\delta z\|_{L^1})$ with $\lambda \in [0,1]$.