Circular Coordinates under Different Cost Functions

Group 8 Project

ICERM August 2019

Applied Mathematical Modeling with Topological Techniques

Problem Setup

- Given a simplicial complex K from the dataset X, we can consider the homology and cohomology with a fixed coefficient field $\mathbb{k}$.
- Homology and boundary operator: $\partial : C_k(K) \to C_{k-1}(K)$. e.g. $\partial[a, b, c] = [a, b] + [b, c] + [c, a]$
- Cohomology and coboundary operator:

$$\delta : C^k(K) \to C^{k+1}(K). \text{ e.g. } \delta \begin{bmatrix} a \mapsto 1 \\ b \mapsto 0 \\ c \mapsto 0 \end{bmatrix} = -[a, b]^* + [c, a]^*$$

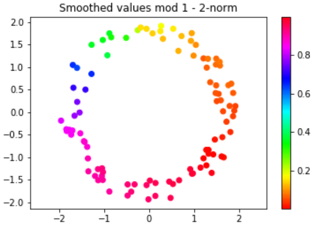
with $[a, b]^* = \begin{bmatrix} ab \mapsto 1 \\ bc \mapsto 0 \\ ca \mapsto 0 \end{bmatrix}$.

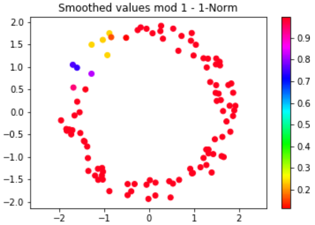
- **Intuition:** If you think of boundary operators as "derivation", then the coboundary operator is like "anti-derivation".

- **Theorem** (*Circular Coordinates*): Given a $[f] \in H^1_{\mathbb{Z}}(X)$, f can be made into a function $X \to S^1$. This **circular coordinate** can be found through the optimization problem $\min_{z \in C^0(X)} \|f - \delta z\|_{L^2}$.
- **Problem** (New): How about adding a regularization term $\mathcal{R}(x)$ on $x = f - \delta z$?
  - $\|x\|_{L^2} := (\sum_i |x_i|)$ L2-norm
  - $\|x\|_{L^1} := (\sum_i |x_i|)$ L1-norm
    It may introduce sparsity across coordinates instead of smoothness.
  - $(1 - \lambda)\|x\|_{L^1} + \lambda\|x\|_{L^2}$ elastic net
    It may find a balance between L1 and L2 norms.
- Generic matrix optimization without Jacobian estimate: Slow and inefficient.
- Matrix optimization using Gradient Descent with Jacobian.

Optimization Problem: Gradient Descent

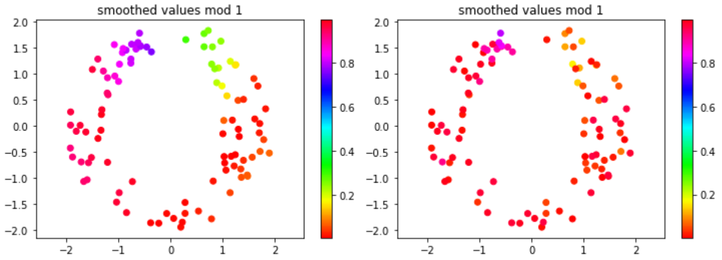
- Example 1: Annulus, L2 norm ($\|f - \delta z\|_2$) and L1 norm ($\|f - \delta z\|_1$)





Another Example

- Example 2: Annulus with thin and thick parts, L2 norm ($\|f - \delta z\|_2$) and L1 norm ($\|f - \delta z\|_1$)



Another Example

- Example 3: Double Annulus, mixed L2 and L1 norm $((1-\lambda)\|f-\delta z\|_{L^2}+\lambda\|f-\delta z\|_{L^1})$ with $\lambda \in [0,1]$.