Complex Analysis

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Abstract

Complex analysis investigates functions of complex variables. Holomorphic functions (also called analytic functions) are the heart of complex analysis. These functions behave very nicely — for instance, they are infinitely differentiable and are equal to their own Taylor series. These notes are taken during a spring 2022 undergrad course of MTH305: Complex Analysis course at IISER Mohali.

1 Complex numbers

1.1 Algebra of complex numbers

Using elementary algebra, one finds the square root of a complex number $\alpha + i\beta$ as

$$\sqrt{\alpha + i\beta} = \pm \left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i \frac{\beta}{|\beta|} \sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \right) \tag{1.1}$$

provided that $\beta \neq 0$. If $\beta = 0$ the square roots are $\pm\sqrt{\alpha}$ if $\alpha \geq 0$, $\pm i\sqrt{-\alpha}$ if $\alpha < 0$.

A complex number $\alpha + i\beta$ may be represented in various ways. It can be associated to the coordinate (α, β) in the two-dimensional plane or the matrix

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

One can verify that these representations are all isomorphic to one another. $\mathbb{C}$ can also be thought as the field $\mathbb{R}[x]/(x^2 + 1)$.

The absolute value (or modulus) has the following properties

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2 \text{ Re } z_1\bar{z}_2, \tag{1.2}$$
$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \text{ Re } z_1\bar{z}_2. \tag{1.3}$$

The triangle inequality is particularly important and shows that $\mathbb{C}$ is a metric space.

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.4}$$

Cauchy-Schwarz inequality states that

$$|\sum_{j=1}^{n} a_j b_j|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2. \tag{1.5}$$

for complex numbers a_j and b_j.

1.2 Geometry of complex numbers

de Moivre's formula. The following formula gives an easy way to express $\cos n\phi$ and $\sin n\phi$ in terms of $\cos\phi$ and $\sin\phi$.

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi. \tag{1.6}$$

It follows from the above formula that the nth roots of a complex number $z = r(\cos\phi + i\sin\phi)$ are given by

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + \sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right]. \tag{1.7}$$

where $k = 0, 1, \ldots, n-1$. These nth roots are the vertices of a regular n-gon. For a particularly interesting case, set

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}. \tag{1.8}$$

Then the nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

Analytic geometry. Often analytic geometry on the complex plane gives an easier or more elegant way to solve problems than using x and y in $\mathbb{R}^2$.

A directed line $z = a + bt$ determines a right half plane consisting of all points z with $\mathrm{Im}\ (z-a)/b < 0$ and a left half plane with $\mathrm{Im}\ (z-a)/b > 0$.

The Riemann sphere. Consider the unit sphere S in the three-dimensional space given by $x_1^2 + x_2^2 + x_3^3 = 1$. To each (x_1, x_2, x_3) on S except $(0,0,1)$ we associate a complex number

$$z = \frac{x_1 + ix_2}{1 - x_3}. \tag{1.9}$$

This correspondence is one-one. A little computation yields

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2},\ x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)},\ x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \tag{1.10}$$

The correspondence is completed by associating the point at ∞ to $(0,0,1)$. This way we regard the sphere as a representation of the extended complex plane. The hemisphere $x_3 < 0$ corresponds to the disk $|z| < 1$ and the hemisphere $x_3 > 0$ to its outside $|z| > 1$.

2 Holomorphic functions

2.1 Differentiation of complex functions

Suppose f is a complex-valued function of a single complex variable z. The derivative of f at a point z_0 in its domain is defined by the limit:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{2.1}$$

For the limit to exist, it must have the same value for any sequence of complex values for z that approach z_0 on the complex plane. Just like real differentiability, complex differentiability is linear and obeys the product rule, quotient rule and the chain rule.

The function f is said to be holomorphic on an open set U if f is complex differentiable at every point in U. If f is complex differentiable on some open neighbourhood of z_0, f is said to be holomorphic at z_0.

Example. The function $f(z) = |z|^2$ is complex differentiable at exactly one point ($z_0 = 0$). Thus it is not holomorphic at 0.

2.2 Cauchy-Riemann equations

Suppose $f(x + iy) = u + iv$ is a holomorphic function. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.2}$$

These are called the Cauchy-Riemann equations.

A holomorphic function also satisfies:

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{2.3}$$

which implies that f is functionally independent from $\bar{z}$.

If a function $f(z) = u+iv$ satisfies the Cauchy-Riemann equations and if the partial derivates are also continuous then f is holomorphic.

The derivative of f is given by:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{2.4}$$

Using the Cauchy-Riemann equations, $f'(z)$ can be written in four different ways. Also,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

That is, $|f'(z)|^2$ is the Jacobian of u and v with respect to x and y.

Laplace's equation. If $f(z) = u + iv$ is holomorphic then $u(x, y)$ and $v(x, y)$ satisfy the Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2.5}$$

Any function that satisfies the Laplace's equation is said to be harmonic. If two harmonic functions u and v satisfy the Cauchy-Riemann equations, then v is said to be the conjugate harmonic function of u and u is the conjugate harmonic function of $-v$.

The conjugate of a harmonic function can be found by integration. It is determined only upto an additive constant.

There is another way to compute the holomorphic function $f(z)$ whose real part is a given harmonic function $u(x, y)$ without use of integration. We treat $\bar{f}(\bar{z})$, the conjugate of $f(z)$, as a function of $\bar{z}$ only and write

$$u(x, y) = \frac{1}{2}\left[f(x + iy) + \bar{f}(x - iy)\right].$$

This is a formal identity and it holds even when x and y are complex. If we substitute $x = z/2, y = z/2i$ we obtain

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left[f(z) + \bar{f}(0)\right].$$

Since $f(z)$ is only determined up to a purely imaginary constant, we assume that $f(0)$ is real, which implies $\bar{f}(0) = u(0, 0)$. Then $f(z)$ can be computed by

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0).$$

A purely imaginary constant can be added at will.

2.3 Polynomials and rational functions

Polynomials. Consider the nth degree polynomial in complex coefficients:

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n. \tag{2.6}$$

where $a_n \neq 0$. It is trivial to see that $P(z)$ is holormorphic. We can write

$$P(z) = a_n(z - \alpha_1) \cdots (z - \alpha_n). \tag{2.7}$$

This factorization is unique except for the order of the factors. If α_i repeats k times ($k \leq n$) then we say the order of the zero α_i is k.

Suppose α is a zero of order k. Then $P(z) = (z - \alpha)^k P_k(z)$ where $P_k(\alpha) \neq 0$. Also, $P(\alpha) = P'(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0$ and $P^{(k)}(\alpha) \neq 0$ (by successive differentiation). A simple zero is a zero of order 1 and it satisfies $P(\alpha) = 0, P'(\alpha) \neq 0$.

Theorem 2.1 (Lucas's theorem)**.** *If all zeroes of a polynomial $P(z)$ lie in a half plane, then all zeroes of the derivative $P'(z)$ lie in the same half plane.*

Proof. From **??** we obtain

$$\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \cdots + \frac{1}{z - \alpha_n}. \tag{2.8}$$

[ADD MORE CONTENT] □

Rational functions. Consider the rational function

$$R(z) = \frac{P(z)}{Q(z)}.$$

where $P(z)$ and $Q(z)$ has no common factors. We put $R(z) = \infty$ when $Q(z) = 0$ such that $R(z)$ is continuous in the extended complex plane. The zeroes of $Q(z)$ are poles of $R(z)$ and the order of a pole is equal to the order of the corresponding zero of $Q(z)$.

The derivative

$$R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q(z)^2}. \tag{2.9}$$

exists when $Q(z) \neq 0$. As a rational function, $R'(z)$ has the same poles as that of $Q(z)$.

To define $R(\infty)$ we put $R(1/z) = R_1(z)$ and equate $R(\infty) = R_1(0)$. Suppose

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}.$$

Then

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}.$$

If $m > n$ $R(z)$ has a zero of order $m - n$ at ∞, if $m < n$ the point at ∞ is a pole of order $n - m$ and if $m = n$ $R(\infty) = a_n/b_n$.

The number of zeroes, including those at ∞, is equal to $\max(m, n)$. The number of poles is the same. This common number is called the order of $R(z)$.

$R(z) - a$ and $R(z)$ have the same order for a constant a. Thus a rational funcrtion $R(z)$ of order p has p zeroes and p poles, and every equation $R(z) = a$ has exactly p roots.

Rational functions of order 1 are called linear transformations. The linear transformation $z + a$ is called a parallel translation and $1/z$ is an inversion.

Partial fractions. Suppose $Q(z)$ is a polynomial with distinct roots $\alpha_1, \ldots, \alpha_n$ and if $P(z)$ is a polynomial of degree $< n$ then

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)}. \tag{2.10}$$

This is also called Heavyside's cover-up method of decomposing a rational function into partial fractions. We can also write

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{A_k}{(z - \alpha_k)}$$

where A_k are constants and use the method of undetermined coefficients to compute A_k. If α_s is a zero of order p and every other zero is of order 1 then we put

$$\frac{P(z)}{Q(z)} = \sum_{k=1, k \neq s}^{n} \frac{A_k}{(z - \alpha_k)} + \frac{A_{s1}}{(z - \alpha_s)} + \frac{A_{s2}}{(z - \alpha_s)^2} + \cdots + \frac{A_{sp}}{(z - \alpha_s)^p}.$$

We do the same for every zero of order > 1.

2.4 Power series

Sequences. Often it becomes difficult to prove convergence of sequence through explicitly determining its limit. An easier way is to prove that it is a Cauchy sequence.

Suppose $\{b_n\}$ and $\{a_n\}$ are two sequences such that $|b_m - b_n| \leq |a_m - a_n|$ for all pairs of subscripts then under Cauchy's condition, if $\{a_n\}$ is a Cauchy sequence, so is $\{b_n\}$.

Series. To an infinite series

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{2.11}$$

we associate a sequence of partial sums

$$s_n = a_1 + a_2 + \cdots + a_n. \tag{2.12}$$

The series converges iff the sequence of its partial sums converges and the limit of the sequence is the sum of the series.

The series **??** can be associated with another series

$$|a_1| + |a_2| + \cdots + |a_n| + \cdots . \tag{2.13}$$

Since $|a_n + a_{n+1} + \cdots + a_{n+p}| \leq |a_n| + |a_{n+1}| + \cdots + |a_{n+p}|$, the convergence of **??** implies that the original series **??** is convergent. A series with the property that the series formed by the absolute values of the terms converges is said to be absolutely convergent.

Uniform convergence vs. pointwise convergence. Consider the limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right) x = x.$$

This is true for all x. However, to have $|(1+1/n)x - x| = |x|/n < \epsilon$ for $n \geq n_0$ it is necessary that $n_0 > |x|/\epsilon$. Such an n_0 exists for every fixed x but the requirement cannot be met simultaneously for all x. This is the case of pointwise convergence.

A sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on the set E if to every $\epsilon > 0$ there exists an n_0 such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$ and all x in E. Unlike pointwise convergence, n_0 does not depend of x.

Cauchy's condition for uniform convergence would then be: the sequence $\{f_n(x)\}$ converges uniformly on E iff to every $\epsilon > 0$ there exists an n_0 such that $|f_m(x) - f_n(x)| < \epsilon$ for all $m, n \geq n_0$ and all x in E.

Also if $|f_m(x) - f_n(x)| \leq |a_m - a_n|$ on E and $\{a_n\}$ is convergent then $\{f_n(x)\}$ converges uniformly on E.

A series with variable terms

$$f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots$$

has the series with positive terms

$$a_1 + a_2 + \cdots + a_n + \cdots$$

for a majorant if $|f_n(x)| \leq M a_n$ for some constant M and for all sufficiently large n. The first series is a minorant of the second. Then

$$|f_n(x) + f_{n+1}(x) + \cdots + f_{n+p}(x)| \leq M(a_n + a_{n+1} + \cdots + a_{n+p}).$$

If the majorant converges, the minorant converges uniformly. This is called the Weierstrass M test.

Power series. For every power series

$$a_0 + a_1 z + \cdots + a_n z^n + \cdots, \tag{2.14}$$

where $a_i, z \in \mathbb{C}$, there exists R, $0 \leq R \leq \infty$, called the radius of covergence such that

1. The series converges absolutely for every z with $|z| < R$. If $0 \leq \rho < R$ the convergence is uniform for $|z| \leq \rho$.
2. If $|z| > R$ the series diverges.
3. In $|z| < R$ the sum of the series is analytic whose derivative can be obtained by termwise differentiation and the derived series has the same radius of convergence.

Hadamard's formula gives

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}. \tag{2.15}$$

It follows that a power series with positive R has derivatives of all orders.

Theorem 2.2 (Abel's limit theorem)**.** *If $\sum_{n=0}^{\infty} a_n$ converges, then $f(z) = \sum_{n=0}^{\infty} a_n z^n \to f(1)$ as $z \to 1$ in such a way that $|1 - z|/(1 - |z|)$ remains bounded.*

2.5 Euler's formula

The exponential. The solution to the differential equation $f'(z) = f(z)$ with the initial value $f(0) = 1$ can be found by representing $f(z)$ as a power series and differentiating it. We call this solution the exponential e^z. We find that

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \tag{2.16}$$

Since $\sqrt[n]{n!} \to \infty$ the exponential e^z converges in the whole complex plane.

A consequence of the differential equation is

$$e^{a+b} = e^a \cdot e^b. \tag{2.17}$$

Since $e^z \cdot e^{-z} = 1$ it is clear that e^z is never 0. Since the series has real coefficients $e^{\bar{z}}$ is the complex conjugate of e^z. Hence $|e^{iy}|^2 = e^{iy} \cdot e^{-iy} = 1$ and $|e^{x+iy}| = e^x$.

The trigonometric functions. The trigonometric functions are defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.18}$$

Using the expansion of e^z we obtain

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

Also we obtain Euler's formula

$$e^{iz} = \cos z + i \sin z. \tag{2.19}$$

All the trigonometric identities including the definitions of other trigonometric functions can be derived. All trigonometric functions are rational functions of e^{iz}.

The logarithm. By definition, $z = \log w$ is a root of the equation $e^z = w$. For $w \neq 0$ the equation $e^{x+iy} = w$ is equivalent to

$$e^x = |w|, \qquad e^{iy} = \frac{w}{|w|}. \tag{2.20}$$

Then $x = \log |w|$. Also there is only one y satisfying the above equation in the interval $0 \leq y < 2\pi$. Of course, it has period 2π. Thus

$$\log w = \log |w| + i \arg w.$$

Every complex number other than 0 has infinitely many logarithms which differ from each other by multiples of $2\pi i$. Also if $a \neq 0$ we write

$$a^b = e^{b \log a}.$$

See that if a is complex and b is a rational number in the reduced form p/q then a^b will have exactly q values.

The addition theorem of the exponential function yields

$$\log(z_1 z_2) = \log z_1 + \log z_2$$
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2,$$

in the sense that both sides represent the same infinite set of complex numbers. If we want to compare a value on the left with a value on the right, then we can merely assert that they differ by a multiple of $2\pi i$ (or 2π).

The inverse trigonometric functions. The inverse cosine can be obtained by solving the equation

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = w$$

which is a quadratic equation in e^{iz} with the roots $e^{iz} = w \pm \sqrt{w^2 - 1}$ and hence

$$z = \arccos w = -i \log(w \pm \sqrt{w^2 - 1}).$$

Since $w + \sqrt{w^2 - 1}$ and $w - \sqrt{w^2 - 1}$ are reciprocal numbers we may write

$$\arccos w = \pm i \log(w + \sqrt{w^2 - 1}).$$

The inverse sine can then be defined by

$$\arcsin w = \frac{\pi}{2} - \arccos w.$$

3 Mappings