Combinatorics: Cheatsheet

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Contents

These notes closely follow the book of *Principles and Techniques of Combinatorics*.

1 Inclusion-Exclusion

Inclusion-exclusion principle. Suppose $A_1, A_2, \ldots, A_n$ are finite sets then

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i<j}^{n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|.$$

Generalized inclusion-exclusion principle. Let S is an n-element set and $\{P_1, P_2, \ldots, P_q\}$ be a set of q properties which may be satisfied by elements of S. We define

$$\omega(P_{i1}P_{i2}\cdots P_{im}) = |\cap_{j=1}^{m} A_{ij}|$$

where $A_{ij} \in S$ is the set of elements having property P_{ij}. We define

$$\omega(m) = \sum(\omega(P_{i1}P_{i2}\cdots P_{im}).$$

That is, $\omega(m)$ is the number of elements of S having at least m properties.

We define

$$E(m) = \text{number of elements of } S \text{ having exactly } m \text{ properties.}$$

Then

$$\begin{aligned}E(m) =& \omega(m) - \binom{m+1}{m}\omega(m+1) + \binom{m+2}{m}\omega(m+2) \\ & \cdots + (-1)^{q-m}\binom{q}{m}\omega(q) \\ =& \sum_{k=m}^{q}(-1)^{k-m}\binom{k}{m}\omega(k).\end{aligned}$$

Proof. Will be updated. □

We define $\omega(0) = |S|$.

As a special case, we have

$$\begin{aligned}E(0) &= \omega(0) - \omega(1) + \cdots + (-1)^q\omega(q) \\ &= \sum_{k=0}^{q}(-1)^k\omega(k).\end{aligned}$$

Let $A_1, A_2, \ldots, A_q$ be any q subsets of a finite set S. Then

$$\begin{aligned}E(0) =& |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_q| \\ =& |S| - \sum_{i=1}^{q}|A_i| + \sum_{i<j}^{q}|A_i \cap A_j| - \sum_{i<j<k}^{q}|A_i \cap A_j \cap A_k| + \cdots \\ & (-1)^q|A_1 \cap A_2 \cap \cdots \cap A_q|\end{aligned}$$

which leads to the familiar exclusion-inclusion principle.

Stirling numbers of the second kind. The number of onto functions from A to B where $|A| = n$ and $|B| = m$ is

$$F(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n. \tag{1.1}$$

Proof. Will be updated. $\square$

$F(n,m)$ is the number of ways of distributing n distinct objects into m distinct boxes so that no box is empty. Then it follows that

$$F(n,m) = m! S(n,m) \tag{1.2}$$

which gives a formula to compute $S(n,m)$.

Permutations with fixed positions. Let $0 \le k \le r \le n$. We define $D(n,r,k)$ to be number of r-permutations of $\{1,2,\ldots,n\}$ that have exactly k fixed positions. From the generalized inclusion-exclusion principle, we can obtain

$$D(n,r,k) = \frac{\binom{r}{k}}{(n-r)!} \sum_{i=0}^{r-k} (-1)^i \binom{r-k}{i} (n-k-i)!. \tag{1.3}$$

Proof. Will be updated. $\square$

Euler φ-function. We define $\varphi(n)$ to be the number of integers between 1 and n which are co-prime to n. Let $n \in \mathbb{N}$ and let

$$n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_k^{m_k}$$

be its prime factorization. Using the generalized inclusion-exclusion principle, we can obtain

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right). \tag{1.4}$$

Proof. Will be updated. □