The matrix exponential

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October 3, 2014

1 Definition and basic properties

These notes serve as a complement to Chapter 7 in Ahmad and Ambrosetti. In Chapter 7, the general solution of the equation

$$\bar{x}' = A\bar{x} \tag{1}$$

is computed using eigenvalues and eigenvectors. This method works when A has n distinct eigenvalues or, more generally, when there is a basis for $\mathbb{C}^n$ consisting of eigenvectors of A. If the latter case, we say that A is diagonalizable. In this case, the general solution is given by

$$\bar{x}(t) = c_1 e^{t\lambda_1}\bar{v}_1 + \cdots + c_n e^{t\lambda_n}\bar{v}_n,$$

where $\lambda_1, \ldots, \lambda_n$ are the, not necessarily distinct, eigenvalues of A and $\bar{v}_1, \ldots, \bar{v}_n$ the corresponding eigenvectors. Example 7.4.5 in the book illustrates what can happen if A is not diagonalizable, i.e. if there is no basis of eigenvectors of A. In these notes we will discuss this question in a more systematic way.

Our starting point is to write the general solution of (1) as

$$\bar{x}(t) = e^{tA}\bar{c},$$

where $\bar{c}$ is an arbitrary vector in $\mathbb{C}^n$, in analogy with the scalar case $n = 1$. We then have to begin by defining e^{tA} and show that the above formula actually gives a solution of the equation. Note that it is simpler to work in the complex vector space $\mathbb{C}^n$ than $\mathbb{R}^n$, since even if A is real it might have complex eigenvalues and eigenvectors. If A should happen to be real, we can restrict to real vectors $\bar{c} \in \mathbb{R}^n$ to obtain real solutions.

Recall from Taylor's formula that

$$e^t = \sum_{k=0}^{n} \frac{t^k}{k!} + r_n(t),$$

where

$$r_n(t) = \frac{e^s t^{n+1}}{(n+1)!},$$

for some s between 0 and t. Since $\lim_{n\to\infty} r_n(t) = 0$, we have

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}, \quad t \in \mathbb{R}.$$

This is an example of a power series.

More generally a power series is a series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k,$$

where the the coefficients a_k are complex numbers. The number

$$R := \sup\{r \geq 0 \colon \{|a_k| r^k\}_{k=0}^{\infty} \text{ is bounded}\}$$

is called the radius of convergence of the series. The significance of the radius of convergence is that the power series converges inside of it, and that the series can be manipulated as though it were a finite sum there (e.g. differentiated termwise). To prove this we first show a preliminary lemma.

Lemma 1. *Suppose that $\{f_n\}$ is a sequence of continuously differentiable functions on an interval I. Assume that $f_n' \to g$ uniformly on I and that $\{f_n(a)\}$ converges for some $a \in I$. Then $\{f_n\}$ converges pointwise on I to some function f. Moreover, f is continuously differentiable with $f'(x) = g(x)$.*

Proof. We have

$$f_n(x) = f_n(a) + \int_a^x f_n'(s)\, ds, \quad \forall n.$$

Since $f_n'(s) \to g(s)$ uniformly on the interval between a and x, we can take the limit under the integral and obtain that $f_n(x)$ converges to $f(x)$ defined by

$$f(x) = f(a) + \int_a^x g(s)\, ds, \quad x \in I,$$

where $f(a) = \lim_{n\to\infty} f_n(a)$. The fundamental theorem of calculus now shows that f is continuously differentiable with $f'(x) = g(x)$. $\square$

Theorem 2. *Assume that the power series*

$$\sum_{n=0}^{\infty} a_k (x - x_0)^k \tag{2}$$

has positive radius of convergence. The series converges uniformly and absolutely in the interval $[x_0 - r, x_0 + r]$ whenever $0 < r < R$ and diverges when $|x - x_0| > R$. The limit is infinitely differentiable and the series can be differentiated termwise.

Proof. If $|x - x_0| > R$ the sequence $a_k(x - x_0)^k$ is unbounded, so the series diverges. Consider an interval $|x - x_0| \leq r$, where $r \in (0, R)$. Choose $r < \tilde{r} < R$. Then

$$|a_k (x - x_0)^k| \leq |a_k| \tilde{r}^k \left(\frac{r}{\tilde{r}}\right)^k \leq C \left(\frac{r}{\tilde{r}}\right)^k,$$

when $|x - x_0| \le r$, where C is a constant such that $|a_k|\tilde{r}^k \le C \ \forall k$. Since $r/\tilde{r} < 1$ we find that

$$\sum_{k=0}^{\infty} \left(\frac{r}{\tilde{r}}\right)^k < \infty.$$

Weierstrass' M-test then shows that (2) converges uniformly when $|x - x_0| \le r$ to some function $S(x)$. S is at least continuous since it's the uniform limit of a sequence of continuous functions. If S is differentiable and can be differentiated termwise, then the derivative is

$$\sum_{k=0}^{\infty} (k+1)a_{k+1}(x - x_0)^k.$$

This is a again a power series with radius of convergence R (prove this!). Hence it converges uniformly on $[x_0 - r, x_0 + r]$, $0 < r < R$. It follows from the previous lemma that S is continuously differentiable on $(x_0 - R, x_0 + R)$ and that the power series can be differentiated termwise. An induction argument shows that S is infinitely many times differentiable. $\square$

The interval $(x_0 - R, x_0 + R)$ is called the interval of convergence. The power series also converges for complex x with $|x - x_0| < R$ (hence the name radius of convergence). All of the above properties still hold for complex x, but the derivative must now be understood as a complex derivative. This concept is studied in courses in complex analysis; we will not linger on it here.

Let us now return to matrices. The set of complex $n \times n$ matrices is denoted $\mathbb{C}^{n\times n}$.

Definition 3. If $A \in \mathbb{C}^{n\times n}$, we define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \quad t \in \mathbb{R}.$$

In particular,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

For this to make sense, we have to show that the series converges for all $t \in \mathbb{R}$. A matrix sequence $\{A_k\}_{k=1}^{\infty}$, $A_k \in \mathbb{C}^{n\times n}$, is said to converge if it converges element-wise, i.e. $[A_k]_{ij}$ converges as $k \to \infty$ for all i and j, where the notation $[A]_{ij}$ is used to denote the element on row i and column j of A (this will also be denoted a_{ij} in some places). As usual, a matrix series is said to converge if the corresponding (matrix) sequence of partial sums converges. Instead of checking if all the elements converge, it is sometimes useful to work with the matrix norm.

Definition 4. The matrix norm of A is defined by

$$\|A\| = \max_{\bar{x}\in\mathbb{C}^n, |\bar{x}|=1} |A\bar{x}|.$$

Equivalently, $\|A\|$ is the smallest $K \geq 0$ s.t.

$$|A\bar{x}| \leq K|\bar{x}|, \qquad \forall \bar{x} \in \mathbb{C}^n$$

since

$$|A\bar{x}| = \left| A\left(|\bar{x}| \frac{\bar{x}}{|\bar{x}|} \right) \right| = \left| A\left(\frac{\bar{x}}{|\bar{x}|} \right) \right| |\bar{x}| \quad \text{and} \quad \left| \frac{\bar{x}}{|\bar{x}|} \right| = 1.$$

The following proposition follows from the definition (prove this!).

Proposition 5. *The matrix norm satisfies*

(1) $\|A\| \geq 0$, *with equality iff* $A = 0$.

(2) $\|zA\| = |z|\|A\|$, $z \in \mathbb{C}$.

(3) $\|A + B\| \leq \|A\| + \|B\|$.

A more interesting property is the following.

Proposition 6.

$$\|AB\| \leq \|A\|\|B\|.$$

Proof. We have

$$|AB\bar{x}| \leq \|A\||B\bar{x}| \leq \|A\|\|B\||\bar{x}| = \|A\|\|B\| \quad \text{if} \quad |\bar{x}| = 1. \qquad \square$$

Proposition 7.

(1) $|a_{ij}| \leq \|A\|$.

(2) $\|A\| \leq \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$.

Proof. (1) Recall that $a_{ij} = [A]_{ij}$. Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$, be the standard basis. Then

$$A\bar{e}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \quad \Rightarrow \quad |a_{ij}| \leq \|A\bar{e}_j\| \leq \|A\||\bar{e}_j| \leq \|A\|.$$

(2) We have that

$$A\bar{x} = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} \bar{x} = \begin{pmatrix} R_1\bar{x} \\ \vdots \\ R_n\bar{x} \end{pmatrix},$$

where

$$R_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}$$

is row i of A. Hence

$$\|A\bar{x}\|^2 = \sum_{i=1}^{n} \|R_i\bar{x}\|^2 \underset{\text{Cauchy-Schwarz}}{\leq} \left(\sum_{i=1}^{n} |R_i|^2 \right) |\bar{x}|^2 = \left(\sum_{i,j} |a_{ij}|^2 \right) |\bar{x}|^2. \qquad \square$$

The following corollary is an immediate consequence of the above proposition.

Corollary 8. *Let $\{A_k\}_{k=1}^{\infty} \subset \mathbb{C}^{n\times n}$ and $A \in \mathbb{C}^{n\times n}$. Then*

$$\lim_{k\to\infty} A_k = A \Leftrightarrow \lim_{k\to\infty} \|A_k - A\| \to 0.$$

We can now show that our definition of the matrix exponential makes sense.

Proposition 9. *The series $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ defining e^{tA} converges uniformly on compact intervals. Moreover, the function $t \mapsto e^{tA}$ is differentiable with derivative Ae^{tA}.*

Proof. Each matrix element of $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ is a power series in t with coefficients $[A^k]_{ij}/k!$. The radius of convergence is infinite, $R = \infty$, since

$$\left|\frac{[A^k]_{ij}}{k!}\right| r^k \le \frac{\|A^k\| r^k}{k!} \le \frac{\|A\|^k r^k}{k!} \to 0$$

as $k \to \infty$ for all $r \ge 0$. The series thus converges uniformly on any compact interval and pointwise on $\mathbb{R}$. Differentiating termwise, we obtain

$$\frac{d}{dt}\sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=1}^{\infty} \frac{t^{k-1} A^k}{(k-1)!} \underset{j=k-1}{=} A \sum_{j=0}^{\infty} \frac{t^j A^j}{j!} = Ae^{tA}. \qquad \square$$

Since $e^{0\cdot A} = I$, we obtain the following result.

Theorem 10. *The general solution of* (1) *is given by*

$$\bar{x}(t) = e^{tA}\bar{c}, \quad \bar{c} \in \mathbb{C}^n$$

and the unique solution of the IVP

$$\bar{x}' = A\bar{x}, \quad \bar{x}(0) = \bar{x}_0$$

is given by

$$\bar{x}(t) = e^{tA}\bar{x}_0.$$

In certain cases it is possible to compute the matrix exponential directly from the definition.

Example 11. Suppose that

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is a diagonal matrix. Then

$$A^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix},$$

so

$$
\begin{aligned}
e^{tA} &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k \lambda_1^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k \lambda_2^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{t^k \lambda_n^k}{k!} \end{pmatrix} \\
&= \begin{pmatrix} e^{t\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{t\lambda_n} \end{pmatrix}.
\end{aligned}
$$

Example 12. Let's compute e^{tA} where

$$
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

We find that

$$
A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A^{4j+r} = A^r.
$$

Hence,

$$
e^{tA} = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.
$$

Let's now look at the case of a general diagonalizable matrix A. As already mentioned, A is diagonalizable if there is a basis for $\mathbb{C}^n$ consisting of eigenvectors $\bar{v}_1, \ldots, \bar{v}_n$ of A. The corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ don't have to be distinct. In this case we know from Chapter 7 in Ahmad and Ambrosetti that the general solution of (1) is given by

$$
\bar{x}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + \cdots + c_n e^{\lambda_n t} \bar{v}_n.
$$

In matrix notation, we can write this as

$$
\bar{x}(t) = \begin{pmatrix} | & & | \\ e^{\lambda_1 t} \bar{v}_1 & \cdots & e^{\lambda_n t} \bar{v}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},
$$

or

$$
\bar{x}(t) = \begin{pmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{pmatrix} \begin{pmatrix} e^{t\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{t\lambda_n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.
$$

Denote the matrix to the left by T. If we want to find the solution with $\bar{x}(0) = \bar{x}_0$, we have to solve the system of equations

$$
T\bar{c} = \bar{x}_0 \Leftrightarrow \bar{c} = T^{-1} \bar{x}_0.
$$

Thus the solution of the IVP is

$$\bar{x}(t) = Te^{tD}T^{-1}\bar{x}_0,$$

where

$$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

This can also be seen by computing the matrix exponential of e^{tA} using the following proposition.

Proposition 13.

$$e^{TBT^{-1}} = Te^{B}T^{-1}$$

Proof. This follows by noting that

$$\begin{aligned}(TBT^{-1})^k &= (TBT^{-1})(TBT^{-1})\cdots(TBT^{-1}) \\ &= TB(T^{-1}T)B(T^{-1}T)\cdots(T^{-1}T)BT^{-1} \\ &= TB^kT^{-1},\end{aligned}$$

whence

$$\sum_{k=0}^{\infty} \frac{(TBT^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{TB^kT^{-1}}{k!} = Te^{B}T^{-1}. \qquad \square$$

Returning to the discussion above, if A is diagonalizable, then it can be written

$$A = TDT^{-1} \tag{3}$$

with T and D as above. D is the matrix for the linear operator $\bar{x} \mapsto A\bar{x}$ in the basis $\bar{v}_1, \dots, \bar{v}_n$. The matrix for this linear operator in the standard basis is simply A. Equation (3) is the change of basis formula for matrices. For convenience we will also refer to D as the matrix for A in the basis $\bar{v}_1, \dots, \bar{v}_n$, although this is a bit sloppy. From (3) and Proposition 13 it follows that

$$e^{tA} = Te^{tD}T^{-1}.$$

The matrix for e^{tA} in the basis $\bar{v}_1, \dots, \bar{v}_n$ is thus given by e^{tD}. The solution of the IVP is given by

$$e^{tA}\bar{x}_0 = Te^{tD}T^{-1}\bar{x}_0,$$

confirming our previous result.

We finally record some properties of the matrix exponential which will be useful for matrices which can't be diagonalized.

Lemma 14. *$AB = BA \Rightarrow e^{A}B = Be^{A}$.*

Proof. $A^kB = A^{k-1}BA = \cdots = AB^k$. Thus

$$\lim_{N\to\infty} \left(\sum_{k=0}^{N} \frac{A^k}{k!}\right) B = \lim_{N\to\infty} B \left(\sum_{k=0}^{N} \frac{A^k}{k!}\right). \qquad \square$$

Proposition 15.

(1) $(e^A)^{-1} = e^{-A}$.

(2) $e^A e^B = e^{A+B} = e^B e^A$ *if* $AB = BA$.

(3) $e^{tA} e^{sA} = e^{(t+s)A}$.

Proof. (1) We have that

$$\frac{d}{dt} e^{tA} e^{-tA} = A e^{tA} e^{-tA} + e^{tA}(-A e^{-tA}) = (A - A) e^{tA} e^{-tA} = 0,$$

where we have used the previous lemma to interchange the order of e^{tA} and A. Hence

$$e^{tA} e^{-tA} \equiv C \text{ (constant matrix)}.$$

Setting $t = 0$, we find that $C = I$ (identity matrix). Setting $t = 1$, we find that $e^A e^{-A} = I$.

(2) We have that

$$\begin{aligned}\frac{d}{dt} e^{t(A+B)} e^{-tA} e^{-tB} &= (A+B) e^{t(A+B)} e^{-tA} e^{-tB} - e^{t(A+B)} A e^{-tA} e^{-tB} - e^{t(A+B)} e^{-tA} B e^{-tB} \\ &= (A + B - (A+B)) e^{tA} e^{tB} e^{-t(A+B)} \\ &= 0,\end{aligned}$$

where we have used the previous lemma in the second line. As in (1) we obtain $e^A e^B e^{-(A+B)} = I \Rightarrow e^A e^B = e^{A+B}$, where we have used (1).

(3) follows from (2) since tA and sA commute. ☐

Let's use this to compute the matrix exponential of a matrix which can't be diagonalized.

Example 16. Let

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$A = D + N = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The matrix A is not diagonalizable, since the only eigenvalue is 2 and $C\bar{x} = 2\bar{x}$ has the solution

$$\bar{x} = z \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad z \in \mathbb{C}.$$

Since D is diagonal, we have that

$$e^{tD} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

Moreover, $N^2 = 0$ (confirm this!), so

$$e^{tN} = I + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Since D and N commute, we find that

$$e^{tA} = e^{tD+tN} = e^{tD}e^{tN} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

Exercise 5 below shows that the hypothesis that A and B commute in Proposition 15b is essential.

2 Generalized eigenvectors

We know now how to compute the matrix exponential when A is diagonalizable. In the next section we will discuss how this can be done when A is not diagonalizable. In order to do that, we need to introduce some more advanced concepts from linear algebra. When A is diagonalizable there is a basis consisting of eigenvectors. The main idea when A is not diagonalizable is to replace this basis by a basis consisting of 'generalized eigenvectors'.

Definition 17. Let $A \in \mathbb{C}^{n\times n}$. A vector $\bar{v} \in \mathbb{C}^n$, $\bar{v} \neq 0$, is called a generalized eigenvector corresponding to the eigenvalue λ if

$$(A - \lambda I)^m \bar{v} = 0$$

for some integer $m \geq 1$.

Note that according to this definition, an eigenvector also qualifies as a generalized eigenvector. We also remark that λ in the above definition has to be an eigenvalue since if $m \geq 1$ is the smallest positive integer such that $(A-\lambda I)^m \bar{v} = 0$, then $\bar{w} = (A - \lambda I)^{m-1}\bar{v} \neq 0$ and

$$(A - \lambda I)\bar{w} = (A - \lambda I)(A - \lambda I)^{m-1}\bar{v} = (A - \lambda I)^m \bar{v} = 0,$$

so $\bar{w}$ is an eigenvector and λ is an eigenvalue.

The goal of this section is to prove the following theorem.

Theorem 18. *Let $A \in \mathbb{C}^{n\times n}$. Then there is a basis for $\mathbb{C}^n$ consisting of generalized eigenvectors of A.*

We will also discuss methods for constructing such a basis.

Before proving the main theorem, we discuss a number of preliminary results. Let V be a vector space and let $V_1, \dots, V_k$ be subspaces. We say that V is the direct sum of $V_1, \dots, V_k$ if each vector $\bar{x} \in V$ can be written in a unique way as

$$\bar{x} = \bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_k, \quad \text{where} \quad \bar{x}_i \in V_i, \quad i = 1, \dots, k.$$

If this is the case we use the notation

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$

We say that a subspace W of V is invariant under A if

$$x \in W \Rightarrow Ax \in W.$$

Example 19. Suppose that A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\bar{v}_1, \dots, \bar{v}_n$. It then follows that the vectors $\bar{v}_1, \dots, \bar{v}_n$ are linearly independent (see Theorem 7.1.2 in Ahmad and Ambrosetti) and thus form a basis for $\mathbb{C}^n$. Let

$$\ker(A - \lambda_i I) = \operatorname{span}\{\bar{v}_i\}, \quad i = 1, \dots, n,$$

be the corresponding eigenspaces. Here

$$\ker B = \{\bar{x} \in \mathbb{C}^n \colon B\bar{x} = 0\}$$

is the kernel (or null space) of an $n \times n$ matrix B. Then

$$\mathbb{C}^n = \ker(A - \lambda_1 I) \oplus \ker(A - \lambda_2 I) \oplus \cdots \oplus \ker(A - \lambda_n I)$$

by the definition of a basis. It is also clear that each eigenspace is invariant under A.

More generally, suppose that A is diagonalizable, i.e. that it has k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ and that the geometric multiplicity of each eigenvalue λ_i equals the algebraic multiplicity a_i. Let $\ker(A - \lambda_i I)$, $i = 1, \dots, k$, be the corresponding eigenspaces. We can then find a basis for each eigenspace consisting of a_i eigenvectors. The union of these bases consists of $a_1 + \cdots + a_k = n$ elements and is linearly independent, since eigenvectors belonging to different eigenvalues are linearly independent (this follows from an argument similar to Theorem 7.1.2 in Ahmad and Ambrosetti). We thus obtain a basis for $\mathbb{C}^n$ and it follows that

$$\mathbb{C}^n = \ker(A - \lambda_1 I) \oplus \ker(A - \lambda_2 I) \oplus \cdots \oplus \ker(A - \lambda_k I).$$

In this basis, A has the matrix

$$D = \begin{pmatrix} \lambda_1 I_1 & & \\ & \ddots & \\ & & \lambda_k I_k \end{pmatrix}$$

where each I_i is an $a_i \times a_i$ unit matrix. In other words, D is a diagonal matrix with the eigenvalues on the diagonal, each repeated a_i times. This explains why A is called diagonalizable.

Let $A \in \mathbb{C}^{n\times n}$ be a square matrix and $p(\lambda) = a_0\lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$ a polynomial. We define

$$p(A) = a_0 A^m + a_1 A^{n-1} + \cdots + a_{m-1}A + a_m I.$$

Theorem 20 (Cayley-Hamilton)**.** *Let $p_A(\lambda) = \det(A - \lambda I)$ be the characteristic polynomial of A. Then $p_A(A) = 0$.*

Proof. Cramer's rule from linear algebra says that

$$(A - \lambda I)^{-1} = \frac{1}{p_A(\lambda)} \operatorname{adj}(A - \lambda I),$$

where the adjugate matrix $\operatorname{adj} B$ is a matrix whose elements are the cofactors of B. More specifically, the element on row i and column j of the adjugate matrix for B is C_{ji}, where $C_{ji} = (-1)^{j+i} \det B_{ji}$, in which B_{ji} is the $(n-1) \times (n-1)$ matrix obtained by eliminating row j and column i from B (see sections 7.1.2–7.1.3 of Ahmad and Ambrosetti). Note that each element of $\operatorname{adj}(A-\lambda I)$ is a polynomial in λ of degree at most $n-1$ (since at least one element of the diagonal is eliminated). Thus,

$$p_A(\lambda)(A-\lambda I)^{-1} = \lambda^{n-1} B_{n-1} + \cdots + \lambda B_1 + B_0,$$

for some constant $n \times n$ matrices $B_0, \ldots, B_{n-1}$. Multiplying with $A - \lambda I$ gives

$$\begin{aligned} p_A(\lambda) I &= p_A(\lambda)(A-\lambda I)(A-\lambda I)^{-1} \\ &= -\lambda^n B_{n-1} + \lambda^{n-1}(AB_{n-1} - B_{n-2}) + \cdots + \lambda(AB_1 - B_0) + AB_0. \end{aligned}$$

Thus,

$$\begin{aligned} -B_{n-1} &= a_0 I, \\ AB_{n-1} - B_{n-2} &= a_1 I, \\ &\vdots \\ AB_1 - B_0 &= a_{n-1} I, \\ AB_0 &= a_n I, \end{aligned}$$

where

$$p_A(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

Multiplying the rows by A^n, A^{n-1}, …, A, I and adding them, we get

$$\begin{aligned} p_A(A) &= a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I \\ &= -A^n B_{n-1} + A^{n-1}(AB_{n-1} - B_{n-2}) + \cdots + A(AB_1 - B_0) + AB_0 \\ &= -A^n B_{n-1} + A^n B_{n-1} - A^{n-1} B_{n-2} + \cdots + A^2 B_1 - AB_0 + AB_0 \\ &= 0. \end{aligned}$$

□

Lemma 21. *Suppose that $p(\lambda) = p_1(\lambda)p_2(\lambda)$ where p_1 and p_2 are relatively prime. If $p(A) = 0$ we have that*

$$\mathbb{C}^n = \ker p_1(A) \oplus \ker p_2(A)$$

and each subspace $\ker p_i(A)$ is invariant under A.

Proof. The invariance follows from $p_i(A)Ax = Ap_i(A)x = 0$, $x \in \ker p_i(A)$. Since p_1 and p_2 are relatively prime, it follow by Euclid's algorithm that there exist polynomials q_1, q_2 such that

$$p_1(\lambda)q_1(\lambda) + p_2(\lambda)q_2(\lambda) = 1.$$

Thus

$$p_1(A)q_1(A) + p_2(A)q_2(A) = I.$$

Applying this identity to the vector $\bar{x} \in \mathbb{C}^n$, we obtain

$$\bar{x} = \underbrace{p_1(A)q_1(A)\bar{x}}_{\bar{x}_2} + \underbrace{p_2(A)q_2(A)\bar{x}}_{\bar{x}_1},$$

where

$$p_2(A)\bar{x}_2 = p_2(A)p_1(A)q_1(A)\bar{x} = p(A)q_1(A)\bar{x} = 0,$$

so that $\bar{x}_2 \in \ker p_2(A)$. Similarly $\bar{x}_1 \in \ker p_1(A)$. Thus $V = \ker p_1(A) + \ker p_2(A)$.

On the other hand, if

$$\bar{x}_1 + \bar{x}_2 = \bar{x}_1' + \bar{x}_2', \qquad \bar{x}_i, \bar{x}_i' \in \ker p_i(A), j = 1, 2,$$

we obtain that

$$\bar{y} = \bar{x}_1 - \bar{x}_1' = \bar{x}_2' - \bar{x}_2 \in \ker p_1(A) \cap \ker p_2(A),$$

so that

$$\bar{y} = p_1(A)q_1(A)\bar{y} + p_2(A)q_2(A)\bar{y} = q_1(A)p_1(A)\bar{y} + q_2(A)p_2(A)\bar{y} = 0.$$

It follows that the representation $\bar{x} = \bar{x}_1 + \bar{x}_2$ is unique and therefore

$$\mathbb{C}^n = \ker p_1(A) \oplus \ker p_2(A).$$

□

Recall that the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ can be factorized as

$$p_A(\lambda) = (-1)^n(\lambda - \lambda_1)^{a_1} \cdots (\lambda - \lambda_k)^{a_k}$$

where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A and $a_1, \ldots, a_k$ the corresponding algebraic multiplicities. Applying Theorem 20 and Lemma 21 to $p_A(\lambda)$ we obtain the following important result.

Theorem 22. *We have that*

$$\mathbb{C}^n = \ker(A - \lambda_1 I)^{a_1} \oplus \cdots \oplus \ker(A - \lambda_k I)^{a_k},$$

where each $\ker(A - \lambda_i I)^{a_i}$ *is invariant under* A.

Proof. We begin by noting that the polynomials $(\lambda - \lambda_i)^{a_i}$, $i = 1, \ldots, k$, are relatively prime. Repeated application of Lemma 21 therefore shows that

$$\mathbb{C}^n = \ker(A - \lambda_1 I)^{a_1} \oplus \cdots \oplus \ker(A - \lambda_k I)^{a_k},$$

with each $\ker(A - \lambda_i I)^{a_i}$ invariant. □

The space $\ker(A - \lambda_i I)^{a_i}$ is called the generalized eigenspace corresponding to λ_i. We leave it as an exercise to show that this is the space spanned by all generalized eigenvectors of A corresponding to λ_i.

We can now prove Theorem 18.

Proof of Theorem 18. If we select a basis $\{\bar{v}_{i,1}, \dots, \bar{v}_{i,n_i}\}$ for each subspace $\ker(A - \lambda_i I)^{a_i}$, then the union $\{\bar{v}_{1,1}, \dots, \bar{v}_{1,n_1}, \bar{v}_{2,1}, \dots, \bar{v}_{2,n_2}, \dots, \bar{v}_{k,1}, \dots, \bar{v}_{k,n_k}\}$ will be a basis for $\mathbb{C}^n$ consisting of generalized eigenvectors. The fact these vectors are linearly independent follows from Theorem 22. Indeed, suppose that

$$\sum_{i=1}^{k} \left(\sum_{j=1}^{n_i} \alpha_{i,j} \bar{v}_{i,j} \right) = 0.$$

Then by the definition of a direct sum, we have $\sum_{j=1}^{n_i} \alpha_{i,j} \bar{v}_{i,j}$ for each i. But then for each i, $\alpha_{i,j} = 0$, $j = 1, \dots, n_i$ since $\{\bar{v}_{i,1}, \dots, \bar{v}_{i,n_i}\}$ is linearly independent. $\square$

When A is diagonalizable it takes the form of a diagonal matrix in the eigenvector basis. We might therefore ask what a general matrix looks like in a basis of generalized eigenvalues. Since each generalized eigenspace is invariant under A, the matrix in the new basis will be block diagonal:

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

where each B_i is an $n_i \times n_i$ matrix, $n_i = \dim \ker(A - \lambda_i I)^{a_i}$. Moreover, B_i only has one eigenvalue λ_i. Indeed, B_i is the matrix for the restriction of A to $\ker(A - \lambda_i I)^{a_i}$ in the basis $\bar{v}_{i,1}, \dots, \bar{v}_{i,n_i}$ and if $A\bar{v} = \lambda \bar{v}$ for some non-zero $\bar{v} \in \ker(A - \lambda_i I)^{a_i}$, then

$$0 = (A - \lambda_i I)^{a_i} \bar{v} = (\lambda - \lambda_i)^{a_i} \bar{v} \Rightarrow \lambda = \lambda_i.$$

It follows that the dimension of $\ker(A - \lambda_i I)^{a_i}$ equals the algebraic multiplicity of the eigenvalue λ_i, that is, $n_i = a_i$. This follows since

$$\begin{aligned} (-1)^n (\lambda - \lambda_1)^{a_1} \cdots (\lambda - \lambda_k)^{a_k} &= \det(A - \lambda I) \\ &= \det(B - \lambda I) \\ &= \det(B_1 - \lambda I_1) \cdots \det(B_k - \lambda I_k) \\ &= (-1)^n (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}, \end{aligned}$$

where we have used the facts that the determinant of a matrix is independent of basis, and that the determinant of a block diagonal matrix is the product of the determinants of the blocks.

Set $N_i = B_i - \lambda_i I_i$, where I_i is the $n_i \times n_i$ unit matrix. Then $N_i^{a_i} = 0$ by the definition of the generalised eigenspaces. A linear operator N with the property that $N^m = 0$ for some m is called nilpotent.

We can summarize our findings as follows.

Theorem 23. *Let $A \in \mathbb{C}^{n \times n}$. There exists a basis for $\mathbb{C}^n$ in which A has the block diagonal form*

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

and $B_i = \lambda_i I_i + N_i$, where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A, I_i is the $a_i \times a_i$ unit matrix and N_i is nilpotent.

We remark that matrix B in the above theorem are not unique. Apart from the order of the blocks B_i, the blocks themselves depend on the particular bases chosen for the generalized eigenspaces. There is a particularly useful way of choosing these bases which gives rise to the Jordan normal form. Although the Jordan normal form will not be required to compute the matrix exponential we present it here for completeness and since it is mentioned in Ahmad and Ambrosetti.

Theorem 24. *Let $A \in \mathbb{C}^{n\times n}$. There exists an invertible $n \times n$ matrix T such that*

$$T^{-1}AT = J,$$

where J is a block diagonal matrix,

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}$$

and each block J_i is a square matrix of the form

$$J_i = \lambda I + N = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix},$$

where λ is an eigenvalue of A, I is a unit matrix and N has ones on the line directly above the diagonal and zeros everywhere else. In particular, N is nilpotent.

See `http://en.wikipedia.org/wiki/Jordan_normal_form` or any advanced textbook in linear algebra for more information. Note in particular that there is an alternative version of the Jordan normal form for real matrices with complex eigenvalues, which is briefly mentioned in Ahmad and Ambrosetti and discussed in more detail on the Wikipedia page.

3 Computing the matrix exponential

We will now use the results of the previous section in order to find an algorithm for computing e^{tA} when A is not diagonalizable. Note that our main interest is actually in solving the equation $\bar{x}' = A\bar{x}$, possibly with the initial condition $\bar{x}(0) = \bar{x}_0$. As we will see, this doesn't actually require computing e^{tA} explicitly.

As previously mentioned, when A is diagonalizable, the general solution of $\bar{x}' = A\bar{x}$ is given by

$$\bar{x}(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + \cdots + c_n e^{\lambda_n t}\bar{v}_n. \tag{4}$$

If we want to solve the IVP

$$\bar{x}' = A\bar{x}, \quad \bar{x}(0) = \bar{x}_0, \tag{5}$$

we simply choose $c_1, \dots, c_n$ so that

$$\bar{x}(0) = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n = \bar{x}_0.$$

In other words, the numbers c_i are the coordinates for the vector $\bar{x}_0$ in the basis $\bar{v}_1, \dots, \bar{v}_n$. Note that each term $e^{\lambda_i t}\bar{v}_i$ in the solution is actually $e^{tA}\bar{v}_i$. Since the jth column of the matrix e^{tA} is given by $e^{tA}\bar{e}_j$, where $\{\bar{e}_1, \dots, \bar{e}_n\}$ is the standard basis, we can compute the matrix exponential by repeating the above steps with initial data $\bar{x}_0 = \bar{e}_j$ for $j = 1, \dots, n$.

The same approach works when A is not diagonalizable, with the difference that the basis vectors $\bar{v}_i$ are now generalized eigenvectors instead of eigenvectors. Denote the basis vectors $\bar{v}_{i,j}$, $i = 1, \dots, k$, $j = 1, \dots, a_i$, as in the proof of Theorem 18 (recall that the dimension of each generalized eigenspace is the algebraic multiplicity of the corresponding eigenvalue). Then the general solution is

$$\bar{x}(t) = \sum_{i=1}^{k}\sum_{j=1}^{a_i} c_{i,j} e^{tA}\bar{v}_{i,j}$$

We thus need to compute

$$e^{tA}\bar{v}_{i,j},$$

where $\bar{v}_{i,j}$ is a generalized eigenvector corresponding to the eigenvalue λ_i. Since

$$(A - \lambda_i I)^{a_i}\bar{v}_{i,j} = 0,$$

we find that

$$\begin{aligned}
e^{tA}\bar{v}_{i,j} &= e^{t\lambda_i I + t(A-\lambda_i I)}\bar{v}_{i,j} \\
&= e^{t\lambda_i I} e^{t(A-\lambda_i I)}\bar{v}_{i,j} \\
&= e^{\lambda_i t} e^{t(A-\lambda_i I)}\bar{v}_{i,j} \\
&= e^{\lambda_i t}\left(I + t(A-\lambda_i I) + \frac{t^2}{2}(A-\lambda_i I)^2 + \cdots + \frac{t^{a-1}}{(a-1)!}(A-\lambda_i I)^{a_i-1}\right)\bar{v}_{i,j},
\end{aligned}$$

where we have also used the fact that I and $(A - \lambda I)$ commute and the definition of the matrix exponential. The general solution can therefore be written

$$\bar{x}(t) = \sum_{i=1}^{k}\sum_{j=1}^{a_i} c_{i,j} e^{\lambda_i t}\left(\sum_{\ell=0}^{a_i-1}\frac{t^\ell}{\ell!}(A-\lambda_i I)^\ell\right)\bar{v}_{i,j}. \tag{6}$$

In order to solve the IVP (5), we simply have to express $\bar{x}_0$ in the basis $\bar{v}_{i,j}$ to find the coefficients $c_{i,j}$. Finally, to compute e^{tA} we repeat the above steps for each standard basis vector $\bar{e}_j$.

Remark 25. Formula (6) shows that the general solution is a linear combination of exponential functions multiplied with polynomials. The polynomial factors can only appear for eigenvalues with (algebraic) multiplicity two or higher. This is precisely the same structure which we encountered for homogeneous higher order scalar linear differential equations with constant coefficients.

Remark 26. When A is diagonalizable we see from (4) that the general solution is just a linear combination of exponential functions, without polynomial factors. This seems to contradict the previous remark. A closer look reveals that in this case

$$(A - \lambda_i I)^2 \bar{v}_{i,j} = 0 \tag{7}$$

for each j. Indeed, we know that $\ker(A - \lambda_i I) \subseteq \ker(A - \lambda_i I)^{a_i}$. If A is diagonalizable then then the geometric multiplicity $\dim \ker(A - \lambda_i I)$ equals the algebraic multiplicity $a_i = \dim \ker(A - \lambda_i I)^{a_i}$, so the two subspaces coincide. This means that every generalized eigenvector is in fact an eigenvector, so that (7) holds. This implies that

$$(A - \lambda_i I)^{\ell} \bar{v}_{i,j} = 0, \quad \ell \geq 2$$

in (6) even if $a_i \geq 2$.

Remark 27. More generally, it might happen that $(A - \lambda_i I)^{\ell}$ vanishes on the generalized eigenspace $\ker(A - \lambda_i I)^{a_i}$ for some $\ell < a_i - 1$. One can show that there is a unique monic polynomial $p_{\min}(\lambda)$ such that $p_{\min}(A) = 0$. $p_{\min}(\lambda)$ is called the minimal polynomial. It divides the characteristic polynomial $p_A(\lambda)$ and can therefore be factorized as

$$p_{\min}(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k},$$

with $m_i \leq a_i$ for each i. Repeating the proof of Theorem 22, we obtain that

$$\mathbb{C}^n = \ker(A - \lambda_1 I)^{m_1} \oplus \cdots \oplus \ker(A - \lambda_k I)^{m_k},$$

so that $\ker(A - \lambda_i I)^{a_i} = \ker(A - \lambda_i I)^{m_i}$, i.e. $(A - \lambda_i I)^{m_i}$ vanishes on $\ker(A - \lambda_i I)^{a_i}$. In fact, one can show that $(A - \lambda_i I)^m$ vanishes on $\ker(A - \lambda_i I)^{a_i}$ if and only if $m \geq m_i$. In the diagonalizable case we have

$$p_{\min}(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_k),$$

i.e. $m_i = 1$ for all i.

We now consider some examples.

Example 28. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The characteristic polynomial is $(\lambda - 1)^2 - 4 = (\lambda + 1)(\lambda - 3)$, so the distinct eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. Since the eigenvalues are distinct, there is a basis consisting of eigenvectors. Solving the equations $A\bar{x} = -\bar{x}$ and $A\bar{x} = 3\bar{x}$, we find the eigenvectors $\bar{v}_1 = (1, -1)$ and $\bar{v}_2 = (1, 1)$. The general solution of the system $\bar{x}' = A\bar{x}$ is thus given by

$$\bar{x}(t) = c_1 e^{-t} \bar{v}_1 + c_2 e^{3t} \bar{v}_2 = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In order to compute the matrix exponential, we find the solutions with $\bar{x}(0) = \bar{e}_1$ and $\bar{x}(0) = \bar{e}_2$, respectively. In the first case, we obtain the equations

$$\begin{aligned} & c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Leftrightarrow & \begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2}. \end{cases} \end{aligned}$$

In the second case, we find that

$$\begin{aligned} & c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \Leftrightarrow & \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = \frac{1}{2}. \end{cases} \end{aligned}$$

Hence,

$$e^{At}\bar{e}_1 = \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t} + e^{3t} \\ -e^{-t} + e^{3t} \end{pmatrix}$$

and

$$e^{At}\bar{e}_2 = -\frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -e^{-t} + e^{3t} \\ e^{-t} + e^{3t} \end{pmatrix}.$$

Finally,

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^{-t} + e^{3t} & -e^{-t} + e^{3t} \\ -e^{-t} + e^{3t} & e^{-t} + e^{3t} \end{pmatrix}.$$

Example 29. Let

$$A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}.$$

The characteristic polynomial is $(\lambda + 1)^2$, so $\lambda_1 = -1$ is the only eigenvalue. This means that any vector belongs to the generalized eigenspace $\ker(A + I)^2$, so that

$$\begin{aligned} e^{At}\bar{v} &= e^{-t}e^{(A+I)t}\bar{v} \\ &= e^{-t}(I + t(A + I))\bar{v} \\ &= e^{-t}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\right)\bar{v} \\ &= e^{-t}\begin{pmatrix} 1 - 2t & 4t \\ -t & 1 + 2t \end{pmatrix}\bar{v}. \end{aligned}$$

In particular

$$e^{At} = e^{-t}\begin{pmatrix} 1 - 2t & 4t \\ -t & 1 + 2t \end{pmatrix}.$$

We could come to the same conclusion by using the standard basis vectors $\bar{e}_1$ and $\bar{e}_2$ as our basis $\bar{v}_{1,1}, \bar{v}_{1,2}$ in the solution formula (6). This would give the general solution

$$\bar{x}(t) = c_1 e^{tA}\bar{e}_1 + c_2 e^{tA}\bar{e}_2 = c_1 e^{-t}\begin{pmatrix} 1-2t \\ -t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 4t \\ 1+2t \end{pmatrix}.$$

There is one more possibility for 2×2 matrices. The matrix could be diagonalizable and still have a double eigenvalue. We leave it as an exercise to show that this can happen if and only if A is a scalar multiple of the identity matrix, i.e. $A = \lambda I$ for some number λ (which will be the only eigenvalue). In this case $e^{tA} = e^{\lambda t} I$.

For 3×3 matrices there are more possibilities.

Example 30. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

The characteristic polynomial of A is $p_A(\lambda) = -\lambda^2(\lambda - 2)$. Thus, A has the only eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ with algebraic multiplicities $a_1 = 2$ and $a_2 = 1$, respectively. We find that

$$A\bar{x} = 0 \iff \bar{x} = z(1, 0, -1),$$

$$A\bar{x} = 2\bar{x} \iff \bar{x} = z(0, 1, 0),$$

$z \in \mathbb{C}$. Thus $\bar{v}_1 = (1, 0, -1)$ and $\bar{v}_2 = (0, 1, 0)$ are eigenvectors corresponding to λ_1 and λ_2, respectively. We see that A is not diagonalizable.

The generalised eigenspace corresponding to λ_2 is simply the usual eigenspace $\ker(A - 2I)$, but the one corresponding to λ_1 is $\ker A^2$. Calculating

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we find e.g. the basis $\bar{v}_{1,1} = \bar{v}_1 = (1, 0, -1)$, $\bar{v}_{1,2} = (1, 0, 0)$ for $\ker A^2$ and we previously found the basis $\bar{v}_{2,1} = \bar{v}_2 = (0, 1, 0)$ for $\ker(A - 2I)$.

We have

$$e^{tA}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$e^{tA}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = e^{0\cdot t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Finally,

$$e^{tA}\begin{pmatrix}1\\0\\0\end{pmatrix} = (I+tA)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1+t & 0 & t\\ 0 & 1+2t & 0\\ -t & 0 & 1-t\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1+t\\0\\-t\end{pmatrix}.$$

We can thus already write the general solution as

$$\bar{x}(t) = c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1+t\\0\\-t\end{pmatrix} + c_3e^{2t}\begin{pmatrix}0\\1\\0\end{pmatrix}, \tag{8}$$

where we chose to use the simpler notation c_1, c_2, c_3 for the coefficients instead of $c_{1,1}, c_{1,2}, c_{2,1}$ from eq. (6).

In order to compute e^{tA}, we need to compute $e^{tA}\bar{e}_i$ for the standard basis vectors. Note that we have already computed

$$e^{tA}\bar{e}_1 = \begin{pmatrix}1+t\\0\\-t\end{pmatrix}$$

and

$$e^{tA}\bar{e}_2 = e^{2t}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

so it remains to compute $e^{tA}\bar{e}_3$. We thus need to solve the equation

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\1\\0\end{pmatrix},$$

and a simple calculation gives $c_1 = -1$, $c_2 = 1$ and $c_3 = 0$, so that

$$e^{tA}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}1+t\\0\\-t\end{pmatrix} = \begin{pmatrix}t\\0\\1-t\end{pmatrix}.$$

Thus,

$$e^{tA} = \begin{pmatrix}1+t & 0 & t\\ 0 & e^{2t} & 0\\ -t & 0 & 1-t\end{pmatrix}.$$

Example 31. Suppose that we in the previous example wanted to solve the IVP

$$\bar{x}' = A\bar{x}, \quad \bar{x}(0) = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Of course, once we have the formula for the matrix exponential we can find the solution by calculating

$$\bar{x}(t) = e^{tA}\bar{x}(0) = \begin{pmatrix}1+t & 0 & t\\ 0 & e^{2t} & 0\\ -t & 0 & 1-t\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}t\\e^{2t}\\1-t\end{pmatrix}.$$

We could however also solve the problem by using the general solution (8) and finding c_1, c_2, c_3 to match the initial data. We thus need to solve the system

$$\begin{pmatrix}0\\1\\1\end{pmatrix} = c_1 \begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2 \begin{pmatrix}1\\0\\0\end{pmatrix} + c_3 \begin{pmatrix}0\\1\\0\end{pmatrix},$$

giving $c_1 = -1$, $c_2 = 1$ and $c_3 = 1$. Thus,

$$\bar{x}(t) = -\begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}1+t\\0\\-t\end{pmatrix} + e^{2t}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}t\\e^{2t}\\1-t\end{pmatrix},$$

which coincides with the result of our previous computation.

Example 32. Let

$$A = \begin{pmatrix}3 & 1 & -1\\0 & 2 & 0\\1 & 1 & 1\end{pmatrix}.$$

The characteristic polynomial of A is $p_{\mathrm{A}}(\lambda) = -(\lambda-2)^3$. Thus, A has the only eigenvalue $\lambda_1 = 2$ with algebraic multiplicity 3. The corresponding generalized eigenspace is the whole of $\mathbb{C}^3$. Just like in Example 29 we can therefore compute the matrix exponential directly through the formula

$$e^{tA} = e^{2t}e^{t(A-2I)} = e^{2t}\left(I + t(A-2I) + \frac{t^2}{2}(A-2I)^2\right).$$

We find that

$$A - 2I = \begin{pmatrix}1 & 1 & -1\\0 & 0 & 0\\1 & 1 & -1\end{pmatrix} \quad \text{and} \quad (A-2I)^2 = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}.$$

Thus the term $\frac{t^2}{2}(A-2I)^2$ vanishes and we obtain

$$\begin{aligned}
e^{tA} &= e^{2t}(I + t(A-2I))\\
&= e^{2t}\left(\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} + t\begin{pmatrix}1 & 1 & -1\\0 & 0 & 0\\1 & 1 & -1\end{pmatrix}\right)\\
&= \begin{pmatrix}(1+t)e^{2t} & te^{2t} & -te^{2t}\\0 & e^{2t} & 0\\te^{2t} & te^{2t} & (1-t)e^{2t}\end{pmatrix}.
\end{aligned}$$

Example 33. Let

$$A = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 1\\-1 & 0 & 2\end{pmatrix}.$$

Again $p_A(\lambda) = -(\lambda - 2)^3$ and thus 2 is the only eigenvalue. This time

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (A - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that

$$\begin{aligned} e^{tA} &= e^{2t}\left(I + t(A - 2I) + \frac{t^2}{2}(A - 2I)^2\right) \\ &= e^{2t}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} e^{2t} & 0 & 0 \\ -\frac{t^2}{2}e^{2t} & e^{2t} & te^{2t} \\ -te^{2t} & 0 & e^{2t} \end{pmatrix}. \end{aligned}$$

The 4×4 case can be analyzed in a similar way. In general, the computations will get more involved the higher n is. Most computer algebra systems have routines for computing the matrix exponential. In Maple this can be done using the command `MatrixExponential` from the `LinearAlgebra` package.

We can also formulate the above algorithm using the block diagonal representation of A from Theorem 23. Let T be the matrix for the corresponding change of basis, i.e. T is the matrix whose columns are the coordinates for the basis vectors in which A takes the block diagonal form B. Then $A = TBT^{-1}$ and

$$e^{tA} = Te^{tB}T^{-1},$$

where

$$e^{tB} = \begin{pmatrix} e^{tB_1} & & \\ & \ddots & \\ & & e^{tB_k} \end{pmatrix},$$

and

$$e^{tB_i} = e^{t(\lambda_i I_i + N_i)} = e^{t\lambda_i I_i}e^{tN_i} = e^{\lambda_i t}\left(I_i + tN_i + \cdots + \frac{t^{a_i - 1}}{(a_i - 1)!}N_i^{m_i - 1}\right),$$

since $N_i^m = 0$ for $m \geq a_i$.

In the last two examples above, the matrices were already in block diagonal form, so that we could take $T = I$. Let us instead consider Example 30 from this perspective.

Example 34. For the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

in Example 30 we found that the generalized eigenspace $\ker A^2$ had the basis

$$\bar{v}_{1,1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \bar{v}_{1,2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and the eigenspace $\ker(A - 2I)$ the basis

$$\bar{v}_{2,1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus, we take

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

and find that

$$T^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since $A\bar{v}_{1,1} = 0$, $A\bar{v}_{1,2} = \bar{v}_{1,1}$ and $A\bar{v}_{2,1} = 2\bar{v}_{2,1}$, The corresponding block diagonal matrix is

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = 2.$$

We find that

$$e^{tB_1} = I + tB_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad e^{tB_2} = e^{2t}.$$

Thus,

$$e^{tB} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$$

and

$$\begin{aligned} e^{tA} = Te^{tB}T^{-1} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1+t & 0 & t \\ 0 & e^{2t} & 0 \\ -t & 0 & 1-t \end{pmatrix}. \end{aligned}$$

in agreement with our previous calculation.

Exercises

1. Compute e^A by summing the power series when

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **b)** $A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

2. Compute e^{tA} by diagonalising the matrix, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. Show that

$$\|e^A\| \le e^{\|A\|}.$$

4. **a)** Show that

$$(e^A)^* = e^{A^*}.$$

b) Show that e^S is unitary if S is skew symmetric, that is, $S^* = -S$.

5. Show that the following identities (for all $t \in \mathbb{R}$) imply $AB = BA$.

a) $Ae^{tB} = e^{tB}A$, **b)** $e^{tA}e^{tB} = e^{t(A+B)}$.

6. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 3 \\ 0 & -2 & 1 \end{pmatrix}.$$

Calculate the generalized eigenspaces and determine a basis consisting of generalized eigenvectors in each case.

7. Calculate e^{tA_j} for the matrices A_j in the previous exercise.

8. Solve the initial-value problem

$$\bar{x}' = A\bar{x}, \qquad \bar{x}(0) = \bar{x}_0,$$

where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 1 & -4 \\ 5 & 1 & -4 \end{pmatrix} \quad \text{and} \quad \bar{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

9. The matrix

$$A = \begin{pmatrix} 18 & 3 & 2 & -12 \\ 0 & 2 & 0 & 0 \\ -2 & 12 & 2 & 1 \\ 24 & 6 & 3 & -16 \end{pmatrix}$$

has the eigenvalues 1 and 2. Find the corresponding generalized eigenspaces and a determine a basis consisting of generalized eigenvectors.

10. Consider the initial value problem

$$\begin{cases} x_1' = x_1 + 3x_2, \\ x_2' = 3x_1 + x_2, \end{cases} \qquad \bar{x}(0) = \bar{x}_0.$$

For which initial data $\bar{x}_0$ does the solution converge to zero as $t \to \infty$?

11. Can you find a general condition on the eigenvalues of A which guarantees that all solutions of the IVP

$$\bar{x}' = A\bar{x}, \quad \bar{x}(0) = \bar{x}_0.$$

converge to zero as $t \to \infty$?

12. The matrices A_1 and A_2 in Exercise 6 have the same eigenvalues. If you've solved Exercise 7 correctly, you will notice that all solutions of the IVP corresponding to A_1 are bounded for $t \geq 0$ while there are unbounded solutions of the IVP corresponding to A_2. Explain the difference and try to formulate a general principle.