Lecture 3

Econ 2001

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Lecture 3 Outline

1. Metric and Metric Spaces
2. Norm and Normed Spaces
3. Sequences and Subsequences
4. Convergence
5. Monotone and Bounded Sequences

Announcements:
- Friday's exam will be at 3pm, in WWPH 4716; recitation will be at 1pm. The exam will last an hour.

Vector Space Over the Reals

Definition

A vector space V over $\mathbf{R}$ is a 4-tuple $(V, \mathbf{R}, +, \cdot)$ where V is a set, $+ : V \times V \to V$ is vector addition, $\cdot : \mathbf{R} \times V \to V$ is scalar multiplication, satisfying:

1. Closure under addition: $\forall \mathbf{x}, \mathbf{y} \in V$, $(\mathbf{x} + \mathbf{y}) \in V$
2. Associativity of $+$: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
3. Commutativity of $+$: $\forall \mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
4. Existence of vector additive identity: $\exists! \mathbf{0} \in V$ s.t. $\forall \mathbf{x} \in V$, $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$
5. Existence of vector additive inverse: $\forall \mathbf{x} \in V$ $\exists! \mathbf{y} \in V$ s.t. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = \mathbf{0}$. Denote this unique $\mathbf{y}$ as $(-\mathbf{x})$, and define $\mathbf{x} - \mathbf{y}$ as $\mathbf{x} + (-\mathbf{y})$.
6. Distributivity of scalar multiplication over vector addition: $\forall \alpha \in \mathbf{R}$ $\forall \mathbf{x}, \mathbf{y} \in V$, $\alpha \cdot (\mathbf{x} + \mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y}$
7. Distributivity of scalar multiplication over scalar addition: $\forall \alpha, \beta \in \mathbf{R}$ $\forall \mathbf{x} \in V$, $(\alpha + \beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$
8. Associativity of $\cdot$: $\forall \alpha, \beta \in \mathbf{R}, \forall \mathbf{x} \in V$, $(\alpha \cdot \beta) \cdot \mathbf{x} = \alpha \cdot (\beta \cdot \mathbf{x})$
9. Multiplicative identity: $\forall \mathbf{x} \in V$, $1 \cdot \mathbf{x} = \mathbf{x}$

Fun With Vector Spaces

Supose we have a vector space V over $\mathbf{R}$ and $\mathbf{x} \in V$.

- What is $0 \cdot \mathbf{x} =$**?**

  $$0 \cdot \mathbf{x} = (0+0) \cdot \mathbf{x} \overbrace{=}^{\text{distributivity of scalar muliplication}} 0 \cdot \mathbf{x} + 0 \cdot \mathbf{x} \overbrace{=}^{\text{existence of vector additive identity}} \mathbf{0}$$

  where the first zero is a scalar while the last is a vector.
- Using this result:

  $$\mathbf{0} = 0 \cdot \mathbf{x} = (1-1) \cdot \mathbf{x} = 1 \cdot \mathbf{x} + (-1) \cdot \mathbf{x} = \mathbf{x} + (-1) \cdot \mathbf{x}$$

- So we conclude

  $$(-1) \cdot \mathbf{x} = (-\mathbf{x}).$$

  which is not something implied by the nine properties of a vector space.

Distance

- We work with abstract sets (typically vector spaces) and our main objective is to be able to talk about their properties as well as properties of functions defined over them.
- For example, we would like to define notions like limits and convergence, continuity of a function from one of these sets to another, and so on.
- Continuity of a function $f : \mathbf{R} \to \mathbf{R}$ is defined as

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ such that } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

- How can we construct a similar definition when the domain is some abstract set X?
  - ε, δ, and $f(\cdot)$ are scalars, but we do not know how to measure the distance between two points.
- So, next we define function that measures "distance" between any two points in a set.
  - using this function, one can express the idea that two points are "close" (say within δ of each other).
  - Notice that the even more general case in which f is a function from X to Y does not complicate things.
- First, we define distance for abstract settings; next, specialize to vector spaces.

Metric Spaces and Metrics

Definition

A metric d on a set X is a function $d : X \times X \to \mathbf{R}_+$ satisfying

1. $d(\mathbf{x}, \mathbf{y}) \geq 0$, with $d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y} \qquad \forall \mathbf{x}, \mathbf{y} \in X$.
2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \qquad \forall \mathbf{x}, \mathbf{y} \in X$ (symmetry).
3. triangle inequality:
$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \qquad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$$

A metric space (X, d) is a set X with a metric d defined on it.

- The value $d(\mathbf{x}, \mathbf{y})$ is interpreted as the distance between $\mathbf{x}$ and $\mathbf{y}$.
- This definition generalizes the concept of distance to abstract spaces.
- Why "triangle inequality"?

$$\begin{array}{ccccc} & & y & & \\ & \nearrow & & \searrow & \\ x & & \rightarrow & & z \end{array}$$

- Notice that a metric space need not be a vector space.

Metrics: Examples

Example

The discrete metric on any given set X is defined as

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

It trivially satisfies 1. and 2., and as for triangle inequality: $1 \leq 2$.

Example

The l^1 metric (taxicab) on $\mathbf{R}^2$ is defined as $d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}_+$:

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

Example

The l^∞ metric on $\mathbf{R}^2$ is defined as $d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}_+$:

$$d(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|\}$$

Metrics and Applications

Definition

In a metric space (X, d), a subset $S \subseteq X$ is bounded if

$$\exists \mathbf{x} \in X \text{ and } \beta \in \mathbf{R} \text{ such that } \forall \mathbf{s} \in S,\ d(\mathbf{s}, \mathbf{x}) \leq \beta$$

- Given a metric space, boundedness is a propety of subsets (it is relative!).

Definition

In a metric space (X, d), we define the diameter of a subset $S \subseteq X$ by

$$diam\ (S) = \sup\{d(\mathbf{s}, \mathbf{s}') : \mathbf{s}, \mathbf{s}' \in S\}$$

- A metric can be used to define a "ball of a given radius around a point ".

Definition

In a metric space (X, d), we define an open ball of center $\mathbf{x}$ and radius ε as

$$B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in X : d(\mathbf{y}, \mathbf{x}) < \varepsilon\}$$

and a closed ball of and center x and radius ε as

$$B_\varepsilon[\mathbf{x}] = \{\mathbf{y} \in X : d(\mathbf{y}, \mathbf{x}) \leq \varepsilon\}$$

Distance Between Sets

We can also define the distance from a point to a set:

$$d(A, \mathbf{x}) = \inf_{\mathbf{a} \in A} d(\mathbf{a}, x)$$

and the distance between sets:

$$\begin{aligned} d(A, B) &= \inf_{\mathbf{a} \in A} d(B, \mathbf{a}) \\ &= \inf\{d(\mathbf{a}, \mathbf{b}) : \mathbf{a} \in A, \mathbf{b} \in B\} \end{aligned}$$

- Note that $d(A, B)$ is **not** a metric (why?).

Norms and Normed Spaces

- The following defines a vector's length (its size).

Definition

Let V be a vector space. A norm on V is a function $\|\cdot\| : V \to \mathbf{R}_+$ satisfying

1. $\|\mathbf{x}\| \geq 0 \ \forall \mathbf{x} \in V$.
2. $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0} \ \forall \mathbf{x} \in V$.
3. triangle inequality: $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \ \forall \mathbf{x}, \mathbf{y} \in V$
4. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\| \ \forall \alpha \in \mathbf{R}, \forall \mathbf{x} \in V$

- Why is this also called "triangle" inequality?

$$\begin{array}{ccccc} & & \mathbf{x} & & \\ \mathbf{x} & \nearrow & & \searrow & \mathbf{y} \\ 0 & & \rightarrow & & \mathbf{x}+\mathbf{y} \\ \mathbf{y} & \searrow & & \nearrow & \mathbf{x} \\ & & \mathbf{y} & & \end{array}$$

Definition

A normed vector space is a vector space over $\mathbf{R}$ equipped with a norm.

- There are many different norms: if $\|\cdot\|$ is a norm on V, so are $2\|\cdot\|$ and $3\|\cdot\|$ and $k\|\cdot\|$ for any $k > 0$.

Normed Spaces

Can one relate norm and distance on a given normed vector space?

In $\mathbf{R}^n$, the standard norm is defined as

$$\|\mathbf{z}\| = \left(\sum_{i=1}^{n} z_i^2 \right)^{\frac{1}{2}}$$

while the Euclidean distance between two vectors $\mathbf{x}$ and $\mathbf{y}$

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Norm and Euclidean distance

Here, the distance between $\mathbf{x}$ and $\mathbf{y}$ is the length of the vector given by their difference:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

- This idea generalizes.

Norm and Distance

- In any normed vector space, the norm can be used to define a distance.

Theorem

Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : V \times V \Rightarrow \mathbf{R}_+$ be defined by

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

Then (V, d) is a metric space.

- In a normed vector space, we define a metric using the norm.
- Remember that $\|\mathbf{x}\| : V \to \mathbf{R}_+$ is the lenght of a vector $\mathbf{x} \in V$.
- Here $\mathbf{x} = \mathbf{v} - \mathbf{w}$.
- Hence, the distance between two vectors is the length of the difference vector.

Proof that (V, d) is a Metric Space I

We must verify that the d defined satisfies the three properties of a metric.

- First property: $d(\mathbf{x}, \mathbf{y}) \geq 0$, with $d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}\ \forall \mathbf{x}, \mathbf{y} \in X$

Proof.

Let $\mathbf{v}, \mathbf{w} \in V$. By definition, $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| \geq 0$. Moreover

$$
\begin{aligned}
d(\mathbf{v}, \mathbf{w}) = 0 \quad &\Leftrightarrow \quad \|\mathbf{v} - \mathbf{w}\| = 0 \\
&\Leftrightarrow \quad \mathbf{v} - \mathbf{w} = \mathbf{0} && \text{by def of } \|\cdot\| \\
&\Leftrightarrow \quad (\mathbf{v} + (-\mathbf{w})) + \mathbf{w} = \mathbf{w} && \text{by vector additive inverse} \\
&\Leftrightarrow \quad \mathbf{v} + ((-\mathbf{w}) + \mathbf{w}) = \mathbf{w} && \text{by associative property} \\
&\Leftrightarrow \quad \mathbf{v} + \mathbf{0} = \mathbf{w} && \text{by vector additive inverse} \\
&\Leftrightarrow \quad \mathbf{v} = \mathbf{w}
\end{aligned}
$$

$\square$

Proof that (V, d) is a Metric Space II

- For symmetry, we need to show that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})\ \forall \mathbf{x}, \mathbf{y} \in X$

Proof.

Let $\mathbf{v}, \mathbf{w} \in V$.

$$\begin{array}{rcll} d(\mathbf{v}, \mathbf{w}) & = & \|\mathbf{v} - \mathbf{w}\| & \\ & = & |-1|\|\mathbf{v} - \mathbf{w}\| & \\ & = & \|(-1)(\mathbf{v} + (-\mathbf{w}))\| & \text{by def of } \|\cdot\| \\ & = & \|(-1)\mathbf{v} + (-1)(-\mathbf{w})\| & \\ & = & \|-\mathbf{v} + \mathbf{w}\| & \text{by the first slide} \\ & = & \|\mathbf{w} + (-\mathbf{v})\| & \\ & = & \|\mathbf{w} - \mathbf{v}\| & \\ & = & d(\mathbf{w}, \mathbf{v}) & \end{array}$$

□

Proof that (V, d) is a Metric Space III

- The triangle inequality is $d(\mathbf{x}, z) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})\ \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$.

Proof.

Let $\mathbf{u}, \mathbf{w}, \mathbf{v} \in V$.

$$
\begin{aligned}
d(\mathbf{u}, \mathbf{w}) &= \|\mathbf{u} - \mathbf{w}\| \\
&= \|\mathbf{u} + (-\mathbf{v} + \mathbf{v}) - \mathbf{w}\| \\
&= \|\mathbf{u} - \mathbf{v} + \mathbf{v} - \mathbf{w}\| \\
&\leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| && \text{by triangle for } \|\cdot\| \\
&= d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})
\end{aligned}
$$

$\square$

Summary

Since we have shown that $d(\cdot) = \|\cdot\|$ satisfies the three properites of a metric on V, we conclude that (V, d) is a metric space.

Normed Vector Spaces: Examples

R^n admits many norms

The n-dimensional Euclidean space $\mathbf{R}^n$ supports the following norms

- the standard (Euclidean) norm:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i)^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

  where $\cdot$ (the "dot product" of vectors) is defined as:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

- the L^1 norm:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

- the L^p norm: $$\|\mathbf{x}\|_\infty = \left(|x_1|^p + \ldots + |x_n|^p\right)^{\frac{1}{p}}$$
- the maximum norm, or sup norm, or L^∞ norm:

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

Normed Vector Spaces: Results

Theorem (Cauchy-Schwarz Inequality)

If $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$*, then*

$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 \leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$$

- This can also be written as

  $$\|\mathbf{v} \cdot \mathbf{w}\|^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$$

  or

  $$\|\mathbf{v} \cdot \mathbf{w}\| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

  where $\| \cdot \|$ is the standard norm.
- Prove this as an exercise.

Different Norms

Different norms on a given vector space yield different geometric properties.

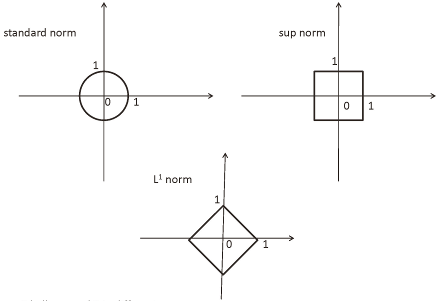
- In $\mathbf{R}^n$, a closed ball around the origin of radius ε is:

$$B_\varepsilon[\mathbf{0}] = \{\mathbf{x} \in \mathbf{R}^n : d(\mathbf{x}, \mathbf{0}) \leq \varepsilon\} = \{\mathbf{0} \in X : \|\mathbf{x}\| \leq \varepsilon\}$$

- Draw a ball in $\mathbf{R}^2$ ($\{\mathbf{x} \in \mathbf{R}^2 : \|\mathbf{x}\| = 1\}$) in 3 different cases:

standard norm ($\|\cdot\|_2$) L^1 norm ($\|\cdot\|_1$) sup-norm ($\|\cdot\|_\infty$)

- $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2} = 1, \quad \|\mathbf{x}\|_1 = |x_1| + |x_2| = 1, \quad \|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|\} = 1$



unit balls around 0 in different norms

Equivalent Norms

- Do different norm imply different results when applied to ideas like convergence, continuity and so on? Hopefully not.

Definition

Two norms $\|\cdot\|$ and $\|\cdot\|^*$ on the same vector space V are said to be Lipschitz-equivalent (or equivalent) if $\exists m, M > 0$ such that $\forall \mathbf{x} \in V$,

$$m\|\mathbf{x}\| \leq \|\mathbf{x}\|^* \leq M\|\mathbf{x}\|$$

Equivalently, $\exists m, M > 0$ such that $\forall \mathbf{x} \in V, \mathbf{x} \neq \mathbf{0}$,

$$m \leq \frac{\|\mathbf{x}\|^*}{\|\mathbf{x}\|} \leq M$$

Equivalent Norms

- Equivalent norms define the same notions of convergence and continuity.
  - For topological purposes, equivalent norms are indistinguishable.

Compare Different Norms

- Let $\|\cdot\|$ and $\|\cdot\|^*$ be equivalent norms on the vector space V, and fix $\mathbf{x} \in V$.
- Let

  $B_\varepsilon(\mathbf{x}, \|\cdot\|) = \{\mathbf{y} \in V : \|\mathbf{x}-\mathbf{y}\| < \varepsilon\}$ and $B_\varepsilon(\mathbf{x}, \|\cdot\|^*) = \{\mathbf{y} \in V : \|\mathbf{x}-\mathbf{y}\|^* < \varepsilon\}$
- Then, for any $\varepsilon > 0$,

$$B_{\frac{\varepsilon}{M}}(\mathbf{x}, \|\cdot\|) \subseteq B_\varepsilon(\mathbf{x}, \|\cdot\|^*) \subseteq B_{\frac{\varepsilon}{m}}(\mathbf{x}, \|\cdot\|)$$

  ε balls are 'close' to each other.

Result

- In $\mathbf{R}^n$ (or any finite-dimensional normed vector space), all norms are equivalent.
  - Roughly, up to a difference in scaling, for topological purposes there is a unique norm in $\mathbf{R}^n$.

Sequences of Real Numbers

Definition

A sequence of real numbers is a function $f : \mathbb{N} \longrightarrow \mathbb{R}$.

Notation

- If $f(n) = a_n$, for $n \in \mathbb{N}$, we denote the sequence by the symbol $\{a_n\}_{n=1}^{\infty}$, or sometimes by $\{a_1, a_2, a_3, \dots\}$.
- The values of f, that is, the elements a_n, are called the terms of the sequence.

Examples

- Let $a_n = n$ then $\{a_n\}_{n=1}^{\infty} = \{1, 2, 3, \dots\}$
- Let $a_1 = 1$, $a_{n+1} = a_n + 2, \forall\, n > 1$ then $\{a_n\}_{n=1}^{\infty} = \{1, 3, 5, \dots\}$
- Let $a_n = (-1)^n$, $n \geq 1$ then $\{a_n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \dots\}$

Limit of a Sequence of Real Numbers

Definition

A sequence of real numbers $\{a_n\}$ is said to converge to a point $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \text{ there exists } N(\varepsilon) \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |a_n - a| < \varepsilon$$

- The sequence converges to a limit point if for every strictly positive epsilon, there exists a natural number (possibly dependent on epsilon) such that all the points in the sequence on and after that number are at most epsilon away from the limit point.

Notation

When $\{a_n\}$ converges to a we usually write

$$a_n \longrightarrow a$$

or

$$\lim_{n \to \infty} a_n = a$$

- Generalize to arbitrary sequences next.

Sequences

- The objects along the sequence are now elements of some set.

Definition

Let X be a set. A sequence is a function $f : \mathbb{N} \longrightarrow X$.

Notation

We denote the sequence by the symbols $\{\mathbf{x}_n\}_{n=1}^{\infty}$, or sometimes by $\{\mathbf{x}_n\}$.

- When $X = \mathbf{R}^n$ each term in the sequence is a vector in $\mathbf{R}^n$.

Convergence

- To define convergence, X must be a metric space so that one talk about points being close to each other using the metric to measure distance.

Definition

Let (X, d) be a metric space. A sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{x} \in X$ if

$$\forall \varepsilon > 0 \; \exists N(\varepsilon) \in \mathbf{N} \text{ such that } n > N(\varepsilon) \Rightarrow d(\mathbf{x}_n, \mathbf{x}) < \varepsilon$$

Notation

When $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$ we write

$$\mathbf{x}_n \to \mathbf{x} \qquad \text{or} \qquad \lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$$

- For any strictly positive epsilon, there exists a number such that all points in the sequence from then on are at most epsilon away from the limit point.
- Similar to real numbers: we replaced the standard distance with a metric.

Remark

The limit point must be an element of X.

Uniqueness of Limits for Real Numbers

Theorem

Let $\{a_n\}$ be a sequence of real numbers that converges to two real numbers b and b'; then, $b = b'$.

Proof.

By contradiction. Assume (wlog) that $b > b'$ and let

$$\varepsilon = \frac{1}{2}(b - b') > 0.$$

- By definition, there are $N(\varepsilon)$ and $N'(\varepsilon)$ such that

  $$\forall\, n \geq N(\varepsilon) \quad |b - a_n| < \varepsilon \qquad \text{and} \qquad \forall n \geq N'(\varepsilon) \quad |b' - a_n| < \varepsilon$$

- But this is impossible since it implies that

  $$2\varepsilon = b - b' \leq |b - a_n| + |b' - a_n| < 2\varepsilon$$

  for $n > \max\{N(\varepsilon), N'(\varepsilon)\}$. $\square$

Uniqueness of Limits

Theorem (Uniqueness of Limits)

In a metric space (X, d), *if* $\mathbf{x}_n \to \mathbf{x}$ *and* $\mathbf{x}_n \to \mathbf{x}'$, *then* $\mathbf{x} = \mathbf{x}'$.

Proof.

By contradiction. Suppose $\{\mathbf{x}_n\}$ is a sequence in X, $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{x}_n \to \mathbf{x}'$, and $\mathbf{x} \neq \mathbf{x}'$. Since $\mathbf{x} \neq \mathbf{x}'$, $d(\mathbf{x}, \mathbf{x}') > 0$. Let $\varepsilon = \frac{d(\mathbf{x},\mathbf{x}')}{2}$

Then, there exist $N(\varepsilon)$ and $N'(\varepsilon)$ such that $\begin{array}{c} n > N(\varepsilon) \Rightarrow d(\mathbf{x}_n, \mathbf{x}) < \varepsilon \\ n > N'(\varepsilon) \Rightarrow d(\mathbf{x}_n, \mathbf{x}') < \varepsilon \end{array}$.

Choose $n > \max\{N(\varepsilon), N'(\varepsilon)\}$ Then

$$
\begin{array}{rcl}
d(\mathbf{x}, \mathbf{x}') & \leq & d(\mathbf{x}, \mathbf{x}_n) + d(\mathbf{x}_n, \mathbf{x}') \\
& < & \varepsilon + \varepsilon \\
& = & 2\varepsilon \\
& = & d(\mathbf{x}, \mathbf{x}') \\
d(\mathbf{x}, \mathbf{x}') & < & d(\mathbf{x}, \mathbf{x}')
\end{array}
$$

a contradiction. $\square$

Subsequences

If $\{\mathbf{x}_n\}$ is a sequence and $n_1 < n_2 < n_3 < \cdots$ then $\{\mathbf{x}_{n_k}\}$ is called a subsequence.

Definition

A subsequence of the sequence described by f from $\mathbf{N}$ to X is the sequence given by a composite function

$$f \circ g : \mathbf{N} \longrightarrow X$$

where

$$g : \mathbf{N} \longrightarrow \mathbf{N}$$

and $g(m) > g(n)$ whenever $m > n$.

- A subsequence is formed by taking some of the elements of the parent sequence in the same order.

Example

$x_n = \frac{1}{n}$, so $\{x_n\} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right)$. If $n_k = 2k$, then $\{x_{n_k}\} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$.

Monotone Sequences of Reals

Definitions

A sequence of real numbers $\{x_n\}$ is increasing if $x_{n+1} \geq x_n$ for all n.
A sequence of real numbers $\{x_n\}$ is decreasing if $x_{n+1} \leq x_n$ for all n.

- We say strictly increasing or decreasing if the above inequalities are strict.

Definition

A sequence of real numbers is monotone if it is either increasing or decreasing.

Definition

If $\{x_n\}$ is a sequence of real numbers, $\{x_n\}$ tends to infinity (written $x_n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$) if

$$\forall K \in \mathbf{R}\ \exists N(K) \text{ such that } n > N(K) \Rightarrow x_n > K$$

Similarly define $x_n \to -\infty$ or $\lim x_n = -\infty$.

Facts about Convergence of Sequences of Reals

Convergence and Subsequences

$\{a_n\}$ converges to b if and only if every subsequence of $\{a_n\}$ converges to b.

Convergence Operations

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$
2. for $c \in \mathbb{R}$, $\lim_{n\to\infty} ca_n = ca$,
3. $\lim_{n\to\infty} a_n b_n = ab$
4. $\lim_{n\to\infty}(a_n)^k = a^k$
5. $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ provided $a_n \neq 0$ $(n = 1, 2, 3, \dots)$, and $a \neq 0$
6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b_n \neq 0$ $(n = 1, 2, 3, \dots)$, and $b \neq 0$

Boundedness for Sequences of Real Numbers

Definition

The sequence of real numbers $\{a_n\}$ is bounded above if

$$\exists \overline{m} \in \mathbb{R} \qquad \text{such that} \qquad a_n \leq \overline{m} \; \forall \, n \in \mathbb{N}$$

Definition

The sequence of real numbers $\{a_n\}$ is bounded below if

$$\exists \underline{m} \in \mathbb{R} \qquad \text{such that} \qquad a_n \geq \underline{m} \; \forall \, n \in \mathbb{N}$$

Boundedness and Convergence

Theorem

If a sequence of real numbers $\{a_n\}$ converges then it is bounded.

Proof.

Suppose $a_n \longrightarrow a$.

- Then, there is an integer N such that $n > N$ implies $|a_n - a| < 1$.
- Define r as follows:

$$r = max\{1, |a_1 - a|, \ldots, |a_N - a|\}$$

- Then $|a_n - a| \leq r$ for $n = 1, 2, 3, \ldots$.

$\square$

Boundedness and Convergence

Theorem

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that

$$a_n \leq b_n \leq c_n, \qquad \forall\, n \in \mathbb{N}$$

and we have that both

$$a_n \longrightarrow a, \qquad c_n \longrightarrow a$$

Then

$$b_n \longrightarrow a$$

Proof in the Problem Set.

Monotonicity and Convergence

Theorem

A monotone sequence of real numbers is convergent if and only if it is bounded.

Proof.

We have already shown that a convergent sequence is bounded (monotone or not), so we need only prove that a bounded and monotone sequence converges.
Suppose $\{a_n\}$ is bounded and increasing. Let $\overline{a} = \sup\{a_n : n \in \mathbb{N}\}$.
We want $\overline{a}$ to be the limit of $\{a_n\}$; so, we need to show that

$$\forall \varepsilon > 0, \ \exists\, N \in \mathbb{N} \quad \text{such that} \quad |\overline{a} - a_n| < \varepsilon, \ \forall\, n \geq N$$

For any $\varepsilon > 0$, $\overline{a} - \varepsilon$ is not an upper bound of $\{a_n\}$; thus $\exists\, N$ such that $a_N > \overline{a} - \varepsilon$.
Since $\{a_n\}$ is increasing, for every $n \geq N$ we have $a_N \leq a_n$
Putting these observations together

$$\overline{a} - \varepsilon < a_N \leq a_n \leq \overline{a}$$

where the last inequality follows from $a_n \leq \overline{a}$, $\forall\, n$.
Thus,

$$|a_n - \overline{a}| < \varepsilon, \qquad \forall\, n \geq N$$

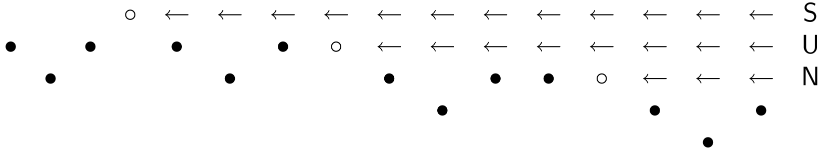
Prove the case in which $\{a_n\}$ is bounded and decreasing as an exercise. □

A Monotone Subsequence Always Exist

Theorem (Rising Sun Lemma)

Every sequence of real numbers has a monotone subsequence.

- Why the name?



Proof.

Exercise ☐

Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass)

Every sequence of real numbers that is bounded has a convergent subsequence.

Proof.

This is done by using the Rising Sun Lemma and the previous theorem:

$\boxed{\text{every real sequence has a monotone subsequence}}$ and

$\boxed{\text{a monotone sequence is convergent if and only if it is bounded}}$ □

Tomorrow

- We start defining open and closed sets, then we move to functions and define limits and continuity.

1. Open and Closed Set
2. Limits of functions
3. Continuity