Standard Euclidean (Distance) Norm

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Introduction to Norms

- The Euclidean norm, also known as the l_2 norm or standard distance norm, measures the "length" of a vector.
- The Euclidean norm of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is given by:
  $$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$
  It represents the length or magnitude of the vector in Euclidean space.
- The Euclidean norm is the most common way of measuring distances in Euclidean space.

Definition of Euclidean Norm

Euclidean Norm

Given a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, the Euclidean norm (or 2-norm) is defined as:

$$\|\mathbf{v}\|_2 = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2}$$

- This norm represents the "length" or "magnitude" of the vector.
- It is commonly used in various fields such as geometry, physics, and machine learning.

Properties of the Euclidean Norm

The Euclidean norm satisfies the following properties:

- **Non-negativity:** $\|\mathbf{v}\|_2 \geq 0$ and $\|\mathbf{v}\|_2 = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- **Scalar multiplication:** For any scalar $a \in \mathbb{R}$ and vector $\mathbf{v}$,

$$\|a\mathbf{v}\|_2 = |a|\|\mathbf{v}\|_2$$

- **Triangle inequality:** For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$$

Example 1: 2D y 3D Vector

Example

Consider the vector $\mathbf{v} = (3, 4)$. The Euclidean norm is given by:

$$\|\mathbf{v}\|_2 = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

Therefore, the length of the vector is 5.

Explanation: This is a simple 2D vector where the norm is computed using the Pythagorean theorem. It corresponds to the length of the vector from the origin to the point $(3, 4)$ in 2D space.

Example

Consider the vector $\mathbf{v} = (1, 2, 2)$ in $\mathbb{R}^3$. The Euclidean norm is:

$$\|\mathbf{v}\|_2 = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$$

Hence, the magnitude of the vector is 3.

Example

Find the Euclidean norm of the vector $\mathbf{v} = (-2, 1, 3)$.
Solution:

$$\|\mathbf{v}\| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

The Euclidean norm of the vector $\mathbf{v}$ is $\sqrt{14}$

Explanation: In 3D space, we extend the idea of the 2D Euclidean norm by adding the square of the third component. The result is the length of the vector in 3D space.

Example 2: 3D Vector

Example 3: Higher Dimensional Vector

Find the Euclidean norm of the vector $\mathbf{v} = (1, -1, 2, -2)$.
Solution:

$$\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 2^2 + (-2)^2} = \sqrt{1 + 1 + 4 + 4} = \sqrt{10}$$

The Euclidean norm of the vector $\mathbf{v}$ is $\sqrt{10}$.

Explanation: In higher dimensions, the Euclidean norm follows the same principle as in 2D and 3D. Here, the vector lies in 4D space, and its norm is still calculated by summing the squares of its components and taking the square root.

Example 4: Zero Vector

Find the Euclidean norm of the zero vector $\mathbf{v} = (0, 0, 0)$.
Solution:

$$\|\mathbf{v}\| = \sqrt{0^2 + 0^2 + 0^2} = \sqrt{0} = 0$$

The Euclidean norm of the zero vector is 0.
Explanation: The zero vector has no length, as all its components are zero. Therefore, the Euclidean norm is 0, which makes sense geometrically since there is no distance from the origin.

Conclusion

The Euclidean norm provides a measure of the length or magnitude of a vector in any dimensional space. It is a fundamental tool in various applications, such as physics, engineering, and computer science, for measuring distances and understanding vector magnitudes.

- The Euclidean norm is a fundamental tool for measuring distances between points or vectors in space.
- It is used in various applications, including solving optimization problems, physics, and computer science.
- Understanding its properties and how to compute it is essential for working with vectors in any dimension.