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Partial Differential Equations

An Introduction

Nita H. Shah and Mrudul Y. Jani

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Contents

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Dr. Mrudul Y. Jani

Preface

Differential equations play a noticeable role in physics, engineering, economics, and other disciplines. Differential equations permit us to model changing forms in both mathematical and physical problems. These equations are precisely used when a deterministic relation containing some continuously varying quantities and their rates of change in space and/or time is recognized or postulated. So, the study of partial differential equations is much more important in comparison with ordinary differential equations. The important role of partial differential equations is precisely replicated by the fact that it is the motivation of all the books on the differential equation. The partial differential equation can be used to describe an extensive variety of occurrences, such as heat, fluid dynamics, diffusion, quantum mechanics, sound, elasticity, electrostatics, and electrodynamics. Therefore, we turn our attention to the area of partial differential equations.

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1 Introduction of Partial Differential Equations

Partial differential equations arise quite often in numerous engineering and physical problems when the functions contain two or more independent variables. Several problems in fluid mechanics and solid mechanics, electromagnetic theory, heat transfer, vibrations, and many other thrust areas of engineering lead to the study of partial differential equations.

While solving the problems of an ordinary differential equation, one should find the general solution first and then determine the arbitrary constants using the initial conditions and finally evaluate the particular solution. But, the same method cannot be applicable in the case of partial differential equations. Instead, in most of the problems of partial differential equations in a region, initial conditions are used to get the particular solution; and boundary conditions are used to evaluate the arbitrary constants or arbitrary functions at the boundary of the region.

In this chapter, we will study the definition of partial differential equations with some examples, Order of partial differential equations, the formation of partial differential equations, and the Direct Integration Method to solve some particular types of partial differential equations.

1.1 PARTIAL DIFFERENTIAL EQUATIONS

A differential equation that contains two or more independent variables is called a **partial differential equation**. A partial differential equation for the function $z(x_1, \dots x_n)$ is an equation of the form $f\left(x_1, \dots x_n; \frac{\partial z}{\partial x_1}, \dots \frac{\partial z}{\partial x_n}; \frac{\partial^2 z}{\partial x_1 \partial x_1}, \dots \frac{\partial^2 z}{\partial x_1 \partial x_n}; \dots\right) = 0$.

Some standard types of partial differential equations are

$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$	One-dimensional heat equation
$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$	One-dimensional wave equation
$u_{xx} + u_{yy} = 0$	Two-dimensional Laplace equation
$u_{xx} + u_{yy} + u_{zz} = 0$	Three-dimensional Laplace equation

The **order** of a partial differential equation is the order of the highest derivatives in the equation. The order of the above equations is 2.

Whereas, the order of $\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 5$ is one.

Usual Notations: If $z = f(x, y)$ be a function of two independent variables x and y then we use following usual notations for partial derivatives,

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y},\ t = \frac{\partial^2 z}{\partial y^2}.$$

Verification of Solution of Partial Differential Equations

Example 1.1: Verify that the equation $u = e^x \cos y$ is the solution to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or not.

Solution: Given equation is $u = e^x \cos y$.

Now, let $\frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \cos y$ and

let $\frac{\partial u}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$$\begin{aligned} \text{L. H. S.} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= e^x \cos y - e^x \cos y \\ &= 0 \\ &= \text{R. H. S.} \end{aligned}$$

Hence the proof.

1.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

The partial differential equations can be formed in two different ways:

I. Elimination of arbitrary constants that are present in the functional relationship between variables.
II. Elimination of arbitrary functions from the given relations.

I. **By Eliminating Arbitrary Constants**

Note: The number of arbitrary constants in the functional relation is equal to the number of times partial derivative one has to take to obtain the partial differential equation.

Consider, the function $f(x, y, z, a, b) = 0$. Where, a and b are independent arbitrary constants.

Step 1:

$$f(x, y, z, a, b) = 0 \tag{1.1}$$

Step 2:

$$\text{Find } \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0 \tag{1.2}$$

Step 3: Eliminating a and b from equations (1.1) and (1.2), the partial differential equation of form $F(x, y, z, p, q) = 0$ can be obtained.

II. **By Eliminating Arbitrary Functions**

Note: The number of arbitrary functions is equal to the order of partial differential equations.

Consider, the function either in the form $f(u, v) = 0$ or $v = f(u)$.

Step 1: Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Step 2: Find Jacobian $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Step 3: Equate $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$ and the partial differential equation of the form $F(x, y, z, p, q) = 0$ can be obtained.

Example 1.2: Eliminate the constants a and b from $z = (x + a)(y + b)$.

Solution: Taking partial derivatives w.r.t., x and y,

$$p = y + b, \quad q = x + a.$$

Substitute $p = y + b$ and $q = x + a$ in $z = (x + a)(y + b)$ and eliminate a and b; the partial differential equation is $z = pq$.

Example 1.3: Form a partial differential equation by eliminating a, b, and c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Taking partial derivatives w.r.t., x and y,

$$\frac{2x}{a^2} + \frac{2z}{c^2}p = 0,$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}q = 0 \tag{1.3}$$

Taking partial derivative of equation (1.3) w.r.t., y,

$$0 + \frac{2}{c^2}(zs + qp) = 0 \Rightarrow zs + qp = 0.$$

NOTE: In this problem, more than one partial differential equations are possible. These partial differential equations are $yzt + yq^2 - zq = 0$, $xzr + xp^2 - zp = 0$.

Example 1.4: Eliminate function f from the relation $f(xy + z^2, x + y + z) = 0$.

Solution: Let $u = xy + z^2$ and $v = x + y + z$

Taking partial derivatives of u *and* v w.r.t., x *and* y,

$$u_x = y + 2zp, \quad u_y = x + 2zq, \quad v_x = 1 + p, \quad v_y = 1 + q$$

$$\text{Let, } J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} y + 2zp & x + 2zq \\ 1 + p & 1 + q \end{vmatrix} = 0$$

$$\Rightarrow \frac{1+p}{1+q} = \frac{y+2zp}{x+2zq}$$

Example 1.5: By eliminating an arbitrary function ϕ form the partial differential equation from the relation $xyz = \phi(x + y + z)$.

Solution: Let $u = x + y + z$ and $v = xyz$

$$\text{Let, } J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 + p & 1 + q \\ yz + xyp & xz + xyq \end{vmatrix} = 0$$

$$\Rightarrow \frac{1+p}{1+q} = \frac{yz+xyp}{xz+xyq}$$

Example 1.6: Eliminate the arbitrary functions f and g from the relation $z = f(x + ay) + g(x - ay)$.

Solution

Hint: In the given problem, a number of functions are two so the order of partial differential equation is two.

Taking partial derivatives of $z = f(x + ay) + g(x - ay)$ w.r.t., x *and* y,

$$\frac{\partial z}{\partial x} = f'(x + ay) + g'(x - ay); \quad \frac{\partial z}{\partial y} = af'(x + ay) - ag'(x - ay)$$

Once again take partial derivatives w.r.t., x *and* y,

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ay) + g''(x - ay); \quad \frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 g''(x - ay)$$

From the above second-order partial derivatives, the resultant partial differential equation is

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Example 1.7: Eliminate the arbitrary functions f and ϕ from the relation $z = f(x) + e^y g(x)$.

Solution: Taking partial derivatives of $z = f(x) + e^y g(x)$ w.r.t., x and y,

$$\frac{\partial z}{\partial x} = f'(x) + e^y g'(x); \quad \frac{\partial z}{\partial y} = e^y g(x)$$

Now, substitute $e^y g(x) = \frac{\partial z}{\partial y}$ in $z = f(x) + e^y \phi(x)$

So, $z = f(x) + \frac{\partial z}{\partial y}$

Take once again partial derivative w.r.t., y.

The resultant partial differential equation is $\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$.

1.3 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

A relation between dependent variables and independent variables which satisfies the partial differential equation is called a **solution of a partial differential equation**. It is also called the **integral of a partial differential equation**.

Complete Solution or Complete Integral

A solution that contains an equal number of arbitrary constants and independent variables is called a complete solution or complete integral.

Particular Solution

In a complete solution by substituting the particular values of the arbitrary constants one can obtain a particular solution.

Singular Solution

If $f(x, y, z, a, b) = 0$ is the complete solution of the partial differential equation $F(x, y, z, p, q) = 0$ then eliminate a and b by taking $\frac{\partial f}{\partial a} = 0$, $\frac{\partial f}{\partial b} = 0$, if it exists, is called a singular solution.

General Solution

In the complete solution $f(x, y, z, a, b) = 0$, take assumption $b = \phi(a)$, so it can be written as

$$f(x, y, z, a, \phi(a)) = 0.$$

Take differentiation of $f(x, y, z, a, b) = 0$ w.r.t., a,

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\phi'(a) = 0.$$

Eliminating 'a' from $f(x, y, z, a, \phi(a)) = 0$ and $\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\phi'(a) = 0$, if it exists, is called a general solution of $F(x, y, z, p, q) = 0$.

1.3.1 Direct Integration Method to Solve Partial Differential Equations

The partial differential equations which contain only a single partial derivative term can be solved using this method.

Example 1.8: Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

Solution: Given a partial differential equation is $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 + y^2$.

Take integration on both the sides w.r.t., x and consider y as a constant.

$$\frac{\partial z}{\partial y} = \frac{x^3}{3} + xy^2 + f(y)$$

Now, integrating both the sides w.r.t., y and consider x as a constant.

where $F(y) = \int f(y)\, dy$

Example 1.9: Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$ and $z = 0$, when y is an odd multiple of $\frac{\pi}{2}$.

Solution: Given a partial differential equation is $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$.

Take integration on both the sides w.r.t., x and consider y as a constant.

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$$

Now given that when $x = 0 \Rightarrow \frac{\partial z}{\partial y} = -2\sin y$

$$\begin{aligned} \therefore\ -2\sin y &= -\cos 0 \sin y + f(y) \\ \Rightarrow\ -2\sin y &= -\sin y + f(y) \\ \Rightarrow f(y) &= -\sin y \end{aligned}$$

$$\therefore \frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Now, integrating both the sides w.r.t., y and consider x as a constant.

$$\therefore z = \cos x \cos y + \cos y + g(x)$$

Now, it is given that when y is an odd multiple of $\frac{\pi}{2}$ then $z = 0$.

That means if $y = (2k + 1)\frac{\pi}{2}$, $k = 0, \pm 1, \pm 2 \ldots$ then $z = 0$

$$\begin{aligned} \therefore\ 0 &= \cos x \cos(2k + 1)\frac{\pi}{2} + \cos(2k + 1)\frac{\pi}{2} + g(x) \\ \Rightarrow g(x) &= 0 \end{aligned}$$

$\therefore z = \cos x \cos y + \cos y$ is the required solution.

Example 1.10: Solve $\frac{\partial^2 z}{\partial x^2} = z$

Solution: Given partial differential equation is $\frac{\partial^2 z}{\partial x^2} - z = 0$.

Now, in the given problem, x is the only independent variable.

So, this partial differential equation can be considered as a second-order homogeneous ordinary differential equation with constant coefficients.

$$\text{i.e. } \frac{d^2 z}{dx^2} - z = 0$$

So, the operator form is $(D^2 - 1)z = 0$, where $D = \frac{d}{dx}$.

Now, the auxiliary equation is $m^2 - 1 = 0$.
$\therefore m = \pm 1$ are the roots which are real and distinct.

Therefore, the general solution is $z = f_1(y)e^x + f_2(y)e^{-x}$.

EXERCISES

Q1 Verify that the equation $u = \log(x^2 + y^2)$ is the solution to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or not.

Q2 Verify that the equation $u = \tan^{-1}\left(\frac{y}{x}\right)$ is the solution to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or not.

Q3 Verify that the equation $u = \sin 9t \sin\left(\frac{x}{4}\right)$ is the solution of a one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ for any suitable value of a.

Q4 Verify that the equation $u(x, t) = f(x + at) + g(x - at)$ is the solution of a one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ or not.

Q5 Verify that the equation $u = e^{-2t} \cos x$ is the solution of the heat equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ for any suitable value of a.

Q6 Form partial differential equation from $z = (x - 2)^2 (y - 3)^2$.

Q7 Form partial differential equation from $z = (x^2 + a)(y^2 + b)$.

Q8 By eliminating an arbitrary function f form the partial differential equation from the relation $z = xy + f(x^2 + y^2)$.

Q9 By eliminating an arbitrary function ϕ form the partial differential equation from the relation $z = f\left(\frac{x}{y}\right)$.

Q10 By eliminating an arbitrary function f form the partial differential equation from the relation $f(x^2 - y^2, xyz) = 0$.

Q11 By eliminating an arbitrary function ϕ form the partial differential equation from the relation $\phi(x + y + z, x^2 + y^2 + z^2) = 0$.

Q12 By eliminating the arbitrary functions f and g form the partial differential equation from the relation $z = xf(x + t) + g(x + t)$.

Q13 Solve $\frac{\partial^2 u}{\partial x \partial y} = e^{-y} \cos x$.

Q14 Solve $\frac{\partial^2 z}{\partial x^2} = -z$, given that when $x = 0$ then $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Q15 Solve $\frac{\partial^3 u}{\partial x^2 \partial y} = \cos(2x + 3y)$.

ANSWERS

1 Yes
2 Yes
3 For $a = 36$, u is the solution
4 Yes
5 For $a = \sqrt{2}$, u is the solution
6 $4z = p^2 + q^2$
7 $pq = 4xyz$

8 $\frac{p-y}{q-x} = \frac{x}{y}$

9 $\frac{p}{q} = \frac{-y}{x}$

10 $\frac{yz+xyp}{xz+xyq} = \frac{-x}{y}$

11 $\frac{1+p}{1+q} = \frac{x+zp}{y+zq}$

12 $2\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2}$

13 $u(x, y) = -e^{-y}\sin x + F(y) + g(x)$

14 $z = e^{y}\cos x + \sin x$

15 $u(x, y) = \frac{-\sin(2x+3y)}{12} + xF(y) + G(y) + h(x)$

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