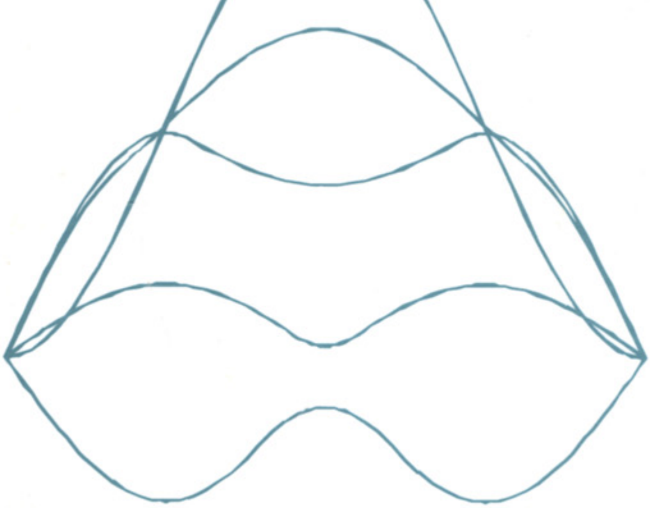

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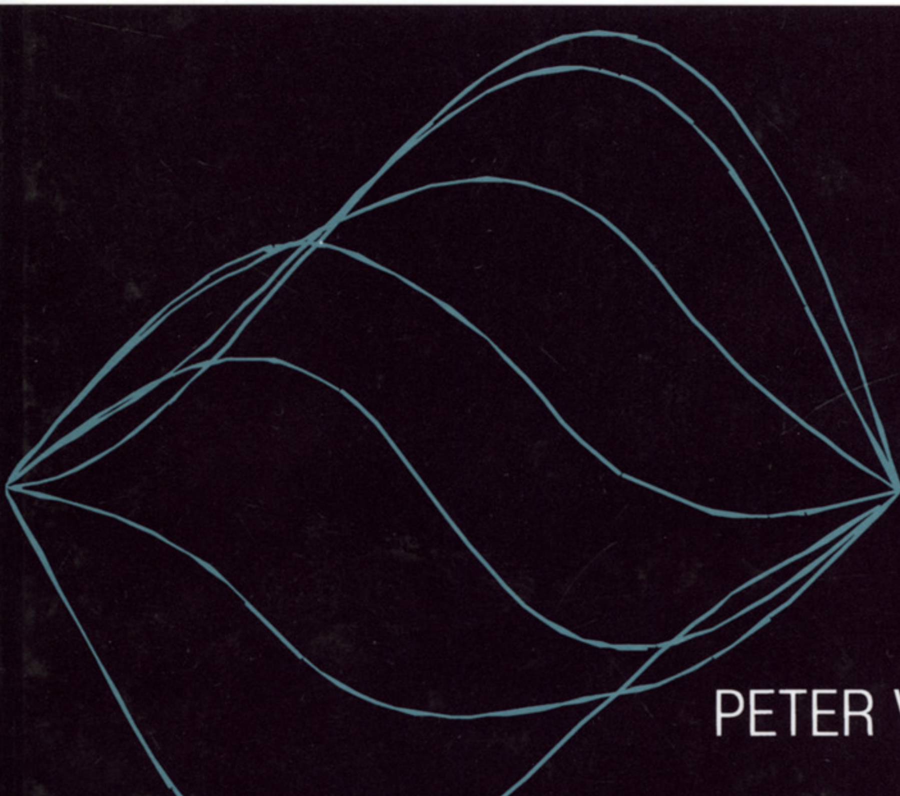
BEGINNING PARTIAL DIFFERENTIAL EQUATIONS

Second Edition



PETER V. O'NEIL

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Beginning Partial Differential Equations

PURE AND APPLIED MATHEMATICS

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Beginning Partial Differential Equations

Second Edition

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Preface

This book is a first course in partial differential equations. The first chapter covers first-order equations, solution of linear and quasi-linear equations, and the role of characteristics in the Cauchy problem. Chapter 2 is devoted to linear second-order equations, classification, the second order Cauchy problem, and the significance of characteristics in existence and uniqueness of solutions, and as carriers of discontinuities. Chapter 3 is a review of Fourier series, integrals, and transforms, and Chapters 4, 5 and 6 develop properties of solutions, and techniques for finding solutions in particular cases, for the wave equation, the heat equation, and Dirichlet and Neumann problems.

Chapters 7 and 8 are new to this edition and are independent of each other. Chapter 7 begins with a classical proof of an existence theorem for the Dirichlet problem. This existence question is then reformulated as a problem of representing a linear functional as an inner product in a Hilbert space, serving as an introduction to the use of function spaces in the study of partial differential equations. The chapter concludes with a brief introduction to distributions and the formulation of another existence theorem.

Chapter 8 is a collection of independent additional topics, including the solution of boundary value problems by eigenfunction expansions, numerical methods, and explicit solutions of Burger's equation, the telegraph equation, and Poisson's equation.

Particularly in working with solutions of wave and heat equations, it is often instructive to use computational software to carry out numerical approximations, to gauge the effects of parameters on solutions, to construct graphs, and to manipulate special functions such as Bessel functions. If such routines are not available, parts of some exercises can be omitted.

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Chapter 1

First-Order Equations

1.1 Notation and Terminology

A *partial differential equation* is an equation that contains at least one partial derivative. Examples are

$$\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xuy^2$$

and

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = f(x, y, z).$$

We often use subscripts to denote partial derivatives. In this notation, $u_x = \partial u/\partial x, u_{xx} = \partial^2 u/\partial x^2, u_{xy} = \partial^2 u/\partial y\,\partial x$, and so on. The partial differential equations listed above can be written, respectively,

$$u_x - xu_y = xuy^2$$

and

$$h_{xx} + h_{yy} + h_{zz} = f(x, y, z). \tag{1.1}$$

A *solution* of a partial differential equation is any function that satisfies the equation. We will often seek solutions satisfying certain conditions and perhaps having the independent variables confined to a specified set of values.

As an example of a solution, the equation

$$4u_x + 3u_y + u = 0 \tag{1.2}$$

has the solution

$$u(x, y) = e^{-x/4} f(3x - 4y),$$

in which f can be any differentiable function of a single variable. This can be verified by substituting $u(x, y)$ into the partial differential equation. Chain rule differentiations yield

$$\begin{aligned} u_x &= -\frac{1}{4}e^{-x/4}f(3x-4y) + e^{-x/4}\frac{d}{d(3x-4y)}[f(3x-4y)]\frac{d(3x-4y)}{dx} \\ &= -\frac{1}{4}e^{-x/4}f(3x-4y) + 3e^{-x/4}f'(3x-4y) \end{aligned}$$

and similarly,

$$u_y = -4e^{-x/4}f'(3x-4y).$$

Upon substitution into equation 1.2, we obtain

$$\begin{aligned} 4u_x + 3u_y + u = &-e^{-x/4}f(3x-4y) \\ &+ 12e^{-x/4}f'(3x-4y) - 12e^{-x/4}f'(3x-4y) \\ &+ e^{-x/4}f(3x-4y) = 0. \end{aligned}$$

Because of the freedom to choose f, equation 1.2 has infinitely many solutions.

The *order* of a partial differential equation is the order of the highest partial derivative occurring in the equation. Equation 1.2 is of order one and equation 1.1 is of order two.

A partial differential equation is *linear* if it is linear in the unknown function and its partial derivatives. An equation that is not linear is *nonlinear*. For example,

$$x^2 u_{xx} - yu_{xy} = u$$

is linear, whereas

$$x^2 u_{xx} - yu_{xy} = u^2$$

is nonlinear because of the u^2 term, and

$$(u_{xx})^{1/2} - 4u_{yy} = xu$$

is nonlinear because of the $(u_{xx})^{1/2}$ term.

A partial differential equation is *quasi-linear* if it is linear in its highest-order derivative term(s). The second-order equation

$$u_{xx} + 4yu_{yy} - (u_x)^3 + u_x u_y = \cos(u)$$

is quasi-linear because it is linear in its second derivative (highest-order) terms u_{xx} and u_{yy}. This equation is not linear because of the $\cos(u), u_x u_y$, and $(u_x)^3$ terms. Any linear equation is also quasi-linear.

We now have the vocabulary to begin studying partial differential equations, starting with first order.

Problems for Section 1.1

1. Show that

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is a solution of $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$.

2. Let c be a positive constant. Show that $u(x,t) = f(x+ct) + g(x-ct)$ is a solution of $u_{tt} = c^2 u_{xx}$ for any twice-differentiable functions f and g of one variable.

3. Show that

$$u(x,t) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

is a solution of $u_{tt} = c^2 u_{xx}$ for any φ that is twice differentiable and ψ that is differentiable for all real x. c is a positive constant. Show that this solution satisfies the conditions

$$u(x,0) = \varphi(x); u_t(x,0) = \psi(x)$$

for all real x.

4. Show that if p is a continuously differentiable function of one variable, the first-order partial differential equation

$$u_t = p(u)u_x$$

has a solution implicitly defined by

$$u(x,t) = \varphi(x + p(u)t),$$

in which φ can be any continuously differentiable function of one variable.

Use this idea to determine (perhaps implicitly) a solution of each of the following equations.

(a) $u_t = ku_x$, with k a nonzero constant

(b) $u_t = uu_x$

(c) $u_t = \cos(u)u_x$

(d) $u_t = e^u u_x$

(e) $u_t = u\sin(u)u_x$

5. Show that

$$u(x,y) = \ln((x-x_0)^2 + (y-y_0)^2)$$

satisfies $u_{xx} + u_{yy} = 0$ for all pairs $(x,\, y)$ of real numbers except (x_0, y_0).

6. Let v and w be solutions of

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + g(x,y)u = 0.$$

Show that $\alpha v + \beta w$ is also a solution for any numbers α and β.

7. In each of the following, classify the equation as linear, quasi-linear and not linear, or not quasi-linear.

   (a) $u^2u_{xx} + u_y = \cos(u)$

   (b) $x^2u_x + y^2u_y + u_{xy} = 2xy$

   (c) $(x - y)u_x^2 + u_{xy} = 1$

   (d) $(x - y)u_x^2 + 2u_y = 4y$

   (e) $x^2u_{yy} - yu_{xx} = \tan(u)$

   (f) $u_x + u_y^2 - u_{xx} = 4$

   (g) $u_x - u_xu_y - u_y = 0$

   (h) $uu_x + u_{xy} = u^2$

   (i) $u_{xy} - u_x^2 + u_y^2 - \sin(u_x) = 0$

   (j) $u_y/u_x = x^2$

8. Let k be a positive constant. Let

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4kt} f(\xi)\, d\xi,$$

   in which f is continuous on the real line. Show that $u_t = ku_{xx}$ for $-\infty < x < \infty, t > 0$. Also determine $u(x,\, t)$ when $f(x) = 1$ for all real x. Hint: Use a change of variables and the standard result that

$$\int_{-\infty}^{\infty} e^{-w^2}\, dw = \sqrt{\pi}.$$

1.2 The Linear First-Order Equation

We will solve the linear first-order partial differential equation in two independent variables:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y). \tag{1.3}$$

Assume that a, b, c, and f are continuous in some region of the plane, and that $a(x,\, y)$ and $b(x,\, y)$ are not both zero for the same $(x,\, y)$.

The key is to determine a change of variables

$$\xi = \varphi(x,y), \eta = \psi(x,y),$$

which transforms equation 1.3 to the simpler linear equation

$$w_\xi + h(\xi,\eta)w = F(\xi,\eta), \tag{1.4}$$

where $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$. We will want this transformation to be one-to-one, at least for all $(x,\, y)$ in some set $\mathcal{D}$ of points in the x,y - plane. In this event, on $\mathcal{D}$, we can, at least in theory, solve for x and y as functions of ξ and η. To ensure this, we will require that the Jacobian of the transformation does not vanish in $\mathcal{D}$:

$$J = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$$

for $(x,\, y)$ in $\mathcal{D}$.

Begin the search for a suitable transformation by computing the chain rule derivatives

$$u_x = w_\xi\xi_x + w_\eta\eta_x, u_y = w_\xi\xi_y + w_\eta\eta_y.$$

Substitute these into equation 1.3 to obtain

$$a(w_\xi\xi_x + w_\eta\eta_x) + b(w_\xi\xi_y + w_\eta\eta_y) + cw = f,$$

which we can write as

$$(a\xi_x + b\xi_y)w_\xi + (a\eta_x + b\eta_y)w_\eta + cw = f. \tag{1.5}$$

This is nearly in the form of equation 1.4 if we choose η so that

$$a\eta_x + b\eta_y = 0$$

for $(x,\, y)$ in $\mathcal{D}$. If $\eta_y \neq 0$, this requires that

$$\frac{\eta_x}{\eta_y} = -\frac{b}{a}.$$

Suppose for the moment that there is such an η. Putting $\eta(x,y) = c$, with c an arbitrary constant, then

$$d\eta = \eta_x\, dx + \eta_y\, dy = 0$$

implies that

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{b}{a}.$$

This means that $\eta = \psi(x,y)$ is an integral of the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{1.6}$$

Equation 1.6 is called the *characteristic equation* of the linear equation 1.3. The equation $\eta(x,y) = k =$ constant defines a family of curves in the plane called *characteristic curves*, or *characteristics*, of equation 1.3. We will say more about these in the next section.

Thus far we have found that we can make the coefficient of w_η in the transformed equation 1.5 vanish if we choose $\eta = \psi(x,y)$, with $\psi(x,y) = k$ an equation defining the general solution of the characteristic equation 1.6. With this step alone, equation 1.5 comes close to the transformed equation 1.4 we want to achieve. We can now choose ξ to suit our convenience and the condition that $J \neq 0$. One simple choice is

$$\xi = \varphi(x, y) = x$$

because then

$$J = \begin{vmatrix} 1 & 0 \\ \eta_x & \eta_y \end{vmatrix} = \eta_y$$

and this is nonzero in $\mathcal{D}$ by previous assumption.

Since $\xi_x = 1$ and $\xi_y = 0$, substitution of

$$\xi = x, \eta = \psi(x, y)$$

into equation 1.5 results in

$$a(x, y)w_\xi + c(x, y)w = f(x, y).$$

In this equation, replace each x by ξ and y by $y(\xi, \eta)$ to obtain an equation of the form

$$A(\xi, \eta)w_\xi + C(\xi, \eta)w = p(\xi, \eta).$$

Finally, restricting the variables to a set in which $A(\xi, \eta) \neq 0$, we have

$$w_\xi + \frac{C}{A}w = \frac{p}{A},$$

and this is in the form of equation 1.4 with

$$h(\xi, \eta) = \frac{C(\xi, \eta)}{A(\xi, \eta)} \text{ and } F(\xi, \eta) = \frac{p(\xi, \eta)}{A(\xi, \eta)}.$$

Example 1.1 *The linear equation*

$$x^2u_x + yu_y + xyu = 1$$

has the form of equation 1.3 with $a(x, y) = x^2, b(x, y) = y, c(x, y) = xy$, *and* $f(x, y) = 1$. *We will transform this equation to the simpler equation 1.4. The characteristic equation is*

$$\frac{dy}{dx} = \frac{b}{a} = \frac{y}{x^2}.$$

This is a separable first-order ordinary differential equation. Write

$$\frac{1}{y}\,dy = \frac{1}{x^2}\,dx.$$

Integrate and rearrange terms to obtain

$$\ln(y) + \frac{1}{x} = k$$

for $y > 0$ *and* $x \neq 0$ *and* k *an arbitrary constant. This is an integral of the characteristic equation and we choose*

$$\eta = \psi(x, y) = \ln(y) + \frac{1}{x}.$$

Graphs of $\ln(y) + 1/x = k$ *are the characteristics of this partial differential equation. Now, choosing* $\xi = x$, *we have the Jacobian*

$$J = \eta_y = \frac{1}{y} \neq 0,$$

as required. Since $\xi = x$,

$$\eta = \ln(y) + \frac{1}{\xi},$$

so

$$\ln(y) = \eta - \frac{1}{\xi}$$

and

$$y = e^{\eta - 1/\xi}.$$

Now apply the transformation

$$\xi = x, \eta = \ln(y) + \frac{1}{x},$$

to the partial differential equation, with

$$w(\xi, \eta) = u(x, y).$$

Compute

$$u_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi + w_\eta \left(-\frac{1}{x^2}\right) = w_\xi - \frac{1}{\xi^2} w_\eta$$

and

$$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\eta \frac{1}{y} = w_\eta \frac{1}{e^{\eta - 1/\xi}}.$$

The partial differential equation $x^2 u_x + y u_y + xyu = 1$ *transforms to*

$$\xi^2 \left(w_\xi - \frac{1}{\xi^2} w_\eta\right) + e^{\eta - 1/\xi} w_\eta \frac{1}{e^{\eta - 1/\xi}} + \xi e^{\eta - 1/\xi} w = 1$$

or

$$\xi^2 w_\xi + \xi e^{\eta - 1/\xi} w = 1.$$

Then

$$w_\xi + \frac{1}{\xi} e^{\eta - 1/\xi} w = \frac{1}{\xi^2},$$

and this has the form of equation 1.4 in any region of the $\xi, \eta -$ *plane with* $\xi \neq 0$. $\diamond$

The point to transforming equation 1.3 to the form of equation 1.4 is that we can solve this transformed equation. Think of

$$w_\xi + h(\xi,\eta)w = F(\xi,\eta)$$

as a linear first-order ordinary differential equation in ξ, with η carried along as a parameter. Following the method for ordinary differential equations, multiply this equation by

$$e^{\int h(\xi,\eta)d\xi}$$

to obtain

$$e^{\int h(\xi,\eta)d\xi}w_\xi + h(\xi,\eta)e^{\int h(\xi,\eta)d\xi}w = F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}.$$

Recognize this as

$$\frac{\partial}{\partial\xi}\left(e^{\int h(\xi,\eta)d\xi}w\right) = F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}.$$

Integrate with respect to ξ. Since η is being carried through this process as a parameter, the constant of integration may depend on η. We obtain

$$e^{\int h(\xi,\eta)d\xi}w = \int F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}\,d\xi + g(\eta),$$

in which g is any differentiable function of one variable. Then

$$w(\xi,\eta) = e^{-\int h(\xi,\eta)d\xi}\int F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}\,d\xi + g(\eta)e^{-\int h(\xi,\eta)d\xi}. \tag{1.7}$$

This is the *general solution* of the transformed equation (by general solution we mean one that contains an arbitrary function). Now obtain the general solution of the original equation 1.3 by substituting $\xi = \xi(x,y), \eta = \eta(x,y)$. This general solution will have the form

$$u(x,y) = e^{\alpha(x,y)}[M(x,y) + g(\psi(x,y))], \tag{1.8}$$

in which g is any differentiable function of one variable.

Example 1.2 *We will solve the constant coefficient equation*

$$au_x + bu_y + cu = 0,$$

in which a, b and c are numbers and $a \neq 0$. The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{a}$$

with general solution defined by the equation

$$bx - ay = k,$$

with k any number. Put

$$\xi = x \text{ and } \eta = bx - ay.$$

The characteristics of this differential equation are the straight-line graphs of $bx - ay = k$.

With this transformation, we find by a routine calculation that the partial differential equation transforms to

$$aw_\xi + cw = 0$$

or

$$w_\xi + \frac{c}{a}w = 0.$$

Multiply this equation by $e^{\int (c/a)d\xi}$, *which is* $e^{c\xi/a}$, *to get*

$$e^{c\xi/a}w_\xi + \frac{c}{a}we^{c\xi/a} = 0.$$

This is

$$\frac{\partial}{\partial \xi}(e^{c\xi/a}w) = 0.$$

Integrate with respect to ξ *to get*

$$e^{c\xi/a}w = g(\eta),$$

in which g can be any differentiable function of one variable. Then

$$w(\xi, \eta) = e^{-c\xi/a}g(\eta).$$

Finally, transform this solution back in terms of x and y:

$$u(x, y) = e^{-cx/a}g(bx - ay).$$

This solution is readily verified by substitution into the partial differential equation. ◇

Observe that the solution in this example has the form specified by equation 1.8.

Example 1.3 *The linear equation*

$$u_x + \cos(x)u_y + u = xy$$

has characteristic equation

$$\frac{dy}{dx} = \cos(x).$$

Integrate the characteristic equation to get

$$y - \sin(x) = k,$$

with k any number. This defines the transformation

$$\xi = x, \eta = y - \sin(x).$$

Graphs of $y - \sin(x) = k$ are the characteristics of this partial differential equation.

Now we have

$$y = \eta + \sin(x) = \eta + \sin(\xi)$$

and the partial differential equation transforms to

$$w_\xi + w = \xi[\eta + \sin(\xi)].$$

Multiply this equation by $e^{\int d\xi}$, which is e^ξ, to obtain

$$e^\xi w_\xi + we^\xi = \eta\xi e^\xi + \xi e^\xi \sin(\xi).$$

Write this equation as

$$\frac{\partial}{\partial \xi}(we^\xi) = \eta\xi e^\xi + \xi e^\xi \sin(\xi).$$

Integrate with respect to ξ to obtain

$$\begin{aligned} we^\xi &= \int \eta\xi e^\xi \, d\xi + \int \xi e^\xi \sin(\xi) \, d\xi \\ &= \eta e^\xi(\xi - 1) + \frac{1}{2}\xi e^\xi(\sin(\xi) - \cos(\xi)) + \frac{1}{2}e^\xi \cos(\xi) + g(\eta). \end{aligned}$$

Then

$$w(\xi, \eta) = \eta(\xi - 1) + \frac{1}{2}\xi(\sin(\xi) - \cos(\xi)) + \frac{1}{2}\cos(\xi) + e^{-\xi}g(\eta).$$

Finally,

$$\begin{aligned} u(x, y) = &(y - \sin(x))(x - 1) + \frac{1}{2}x(\sin(x) - \cos(x)) \\ &+ \frac{1}{2}\cos(x) + e^{-x}g(y - \sin(x)), \end{aligned}$$

in which g is any differentiable function of a single variable. ◇

Contrast the idea of the general solution for the linear first-order ordinary differential equation with that for the linear first-order partial differential equation. In the former case, the general solution of

$$y' + d(x)y = p(x)$$

contains an arbitrary constant. Graphs of the solutions obtained by making choices of the constant are curves in the $x, y-$ plane. If we require that $y(x_0) = y_0$, we pick out the unique solution whose graph passes through (x_0, y_0).

But if u is the general solution of the linear first-order partial differential equation 1.3, then $z = u(x, y)$ defines a family of surfaces in 3 - space, each surface corresponding to a choice of the arbitrary function g in equation 1.8. In the next section we investigate the kind of information that should be given in order to pick out one of these surfaces and determine a unique solution.

Problems for Section 1.2

For each of Problems 1 through 12, (a) solve the characteristic equation and sketch graphs of some of the characteristics, (b) define a transformation of the partial differential equation to the form of equation 1.4 and obtain the transformed equation, (c) find the general solution of the transformed equation, (d) find the general solution of the given partial differential equation, and (e) verify the solution by substituting it into the partial differential equation.

1. $3u_x + 5u_y - xyu = 0$
2. $u_x - u_y + yu = 0$
3. $u_x + 4u_y - xu = x$
4. $-2u_x + u_y - yu = 0$
5. $xu_x - yu_y + u = x$
6. $x^2u_x - 2u_y - xu = x^2$
7. $u_x - xu_y = 4$
8. $x^2u_x + xyu_y + xu = x - y$
9. $u_x + u_y - u = y$
10. $u_x - y^2u_y - yu = 0$
11. $u_x + yu_y + xu = 0$
12. $xu_x + yu_y + 2 = 0$
13. Find the general solution of

$$u_x + \alpha(y-1)u_y = \frac{1}{2}\beta f(x)(y-1)u,$$

in which α and β are real numbers and f is continuous on the real line. Use the general solution to find a solution satisfying

$$u(0, y) = y^n,$$

in which n is a nonnegative integer.

1.3 The Significance of Characteristics

In the preceding section we used the characteristic equation of $a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$ to define a transformation of this equation to a simpler form that we could solve. In this section we look more closely at the significance of characteristic curves, beginning with an example that will suggest the point we want to make. The characteristic equation of

$$2u_x + 3u_y + 8u = 0$$

is

$$\frac{dy}{dx} = \frac{3}{2}$$

and the characteristics are the straight-line graphs of $3x - 2y = k$. We find that this partial differential equation has the general solution

$$u(x,y) = e^{-4x}g(3x - 2y),$$

in which g can be any differentiable function defined over the real line.

Notice that simply specifying that the solution is to have a given value at a particular point does not determine g uniquely, hence does not determine a unique solution as occurs with ordinary differential equations.

Instead of specifying the solution value at a point, try specifying values of $u(x,\ y)$ along a curve Γ in the plane. To be specific for this discussion, let Γ be the x - axis and give values of $u(x,\ y)$ at points on Γ, say

$$u(x,0) = \sin(x) \text{ for } x \text{ real.}$$

We need

$$u(x,0) = e^{-4x}g(3x) = \sin(x),$$

so

$$g(3x) = e^{4x}\sin(x).$$

Putting $t = 3x$ yields

$$g(t) = e^{4t/3}\sin(t/3).$$

This determines g, and the solution satisfying the condition $u(x,0) = \sin(x)$ on Γ is

$$\begin{aligned} u(x,y) &= e^{-4x}g(3x - 2y) = e^{-4x}e^{4(3x-2y)/3}\sin\left(\frac{1}{3}(3x - 2y)\right) \\ &= e^{-8y/3}\sin\left(x - \frac{2}{3}y\right). \end{aligned}$$

In this example, specifying values of u along Γ uniquely determined the arbitrary function in the general solution, hence determined the unique solution of the partial differential equation having these given values.

Try another choice of Γ, say the line $y = x$. We will try to find a solution having given values along this curve, say

$$u(x,x) = x^4.$$

From the general solution, this requires that

$$u(x,x) = e^{-4x}g(x) = x^4,$$

so

$$g(x) = x^4e^{4x}.$$

Then

$$u(x,y) = e^{-4x}g(3x-2y) = e^{8(x-y)}(3x-2y)^4.$$

This is the unique solution satisfying $u(x,x) = x^4$ along the curve Γ given by $y = x$.

Can we find a unique solution having given values along any (reasonable) curve Γ in the plane? The answer is no, as we can see by taking Γ to be the line $3x - 2y = 1$, and prescribing values $u(x,\, y)$ is to have along Γ, say

$$u\left(x, \frac{1}{2}(3x-1)\right) = x^2.$$

This requires that we choose g so that

$$e^{-4x}g\left(3x - 2\frac{1}{2}(3x-1)\right) = x^2$$

or

$$g(1) = e^{4x}x^2.$$

This is impossible, because $e^{4x}x^2$ is not constant. There is no solution taking the value x^2 at points $(x,\, y)$ on this Γ.

Why did some choices of Γ give a solution, whereas another choice gave no solution? The difference was that the x - axis and the line $y = x$ are not characteristics of the partial differential equation, whereas the line $3x - 2y = 1$ is a characteristic. This suggests that characteristics have some significance in the context of existence and uniqueness of solutions. To understand this significance, go back to the general solution

$$u(x,y) = e^{\alpha(x,y)}[M(x,y) + g(\psi(x,y))]$$

of the linear first-order partial differential equation 1.3. Suppose that we prescribe $u(x,y) = q(x)$ along a characteristic. Now a characteristic is specified by $\psi(x,y) = k$. If $y = y(x)$ along this characteristic, then

$$q(x) = e^{\alpha(x,y(x))}[M(x,y(x)) + g(k)]$$

or

$$q(x) = e^{\alpha(x,y(x))}[M(x,y(x)) + C], \tag{1.9}$$

in which C is constant. The functions $M(x,\, y)$ and $\alpha(x, y)$ are determined by the partial differential equation and are not under our control, so equation 1.9 places a constraint on the given function $q(x)$. If $q(x)$ is not of this form for any constant C, there is no solution taking on these prescribed values on Γ. On the other hand, if $q(x)$ is of this form for some C, there are infinitely many such solutions, because we can choose for g any differentiable function such that $g(k) = C$.

Example 1.4 *To illustrate these remarks, begin by finding the general solution of*

$$xu_x + 2x^2u_y - u = x^2e^x. \tag{1.10}$$

The characteristic equation is

$$\frac{dy}{dx} = 2x$$

with general solution defined by $y - x^2 = k$. *The characteristics are parabolas. Let*

$$\xi = x \text{ and } \eta = y - x^2$$

to obtain

$$\xi w_\xi - w = \xi^2 e^\xi,$$

which we write as

$$w_\xi - \frac{1}{\xi}w = \xi e^\xi.$$

Multiply this equation by $e^{\int(-1/\xi)d\xi}$, *which is* $1/\xi$, *to obtain*

$$\frac{1}{\xi}w_\xi - \frac{1}{\xi^2}w = e^\xi$$

or

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}w\right) = e^\xi.$$

Integrate with respect to ξ *to get*

$$\frac{1}{\xi}w = e^\xi + g(\eta),$$

so that

$$w = \xi e^\xi + \xi g(\eta).$$

The general solution of equation 1.10 is

$$u(x, y) = xe^x + xg(y - x^2).$$

Now attempt to find solutions satisfying given conditions along various curves. Suppose that first we seek a solution such that $u(x, y) = \sin(x)$ *on the curve* $y = x^2 + 4$. *Notice that information is being specified along a characteristic. We will need*

$$u(x, x^2 + 4) = xe^x + xg(4) = \sin(x).$$

We must be able to find a constant C such that

$$xe^x + Cx = \sin(x)$$

for all x, and this is impossible. There is no solution satisfying $u(x,y) = \sin(x)$ on the given curve.

Next, suppose we want a solution such that $u(x,y) = xe^x - x$ on the parabola $y = x^2 + 4$. Now we need

$$u(x, x^2 + 4) = xe^x + xg(4) = xe^x - x.$$

This equation requires that $g(4) = -1$. This problem has infinitely many solutions because we can choose g to be any differentiable function of one variable such that $g(4) = -1$. Even though data is specified on a characteristic, the form of the data allows infinitely many solutions.

Finally, suppose we want a solution such that $u(x,y) = \cos(x)$ along the noncharacteristic parabola $y = x^2 + 4x$. Now we need

$$u(x, x^2 + 4x) = xe^x + xg(4x) = \cos(x).$$

This requires that

$$g(4x) = \frac{\cos(x) - xe^x}{x}.$$

Choose

$$g(t) = 4\frac{\cos(t/4) - (t/4)e^{t/4}}{t}$$

for, say, $t > 0$. The solution of the problem (for $x > 0$) is

$$\begin{aligned} u(x,y) &= xe^x + xg(y - x^2) \\ &= xe^x + 4x\left(\frac{\cos\left(\frac{y-x^2}{4}\right) - \frac{1}{4}(y - x^2)e^{(y-x^2)/4}}{y - x^2}\right). \diamond \end{aligned}$$

The problem of finding a solution of equation 1.3 taking on prescribed values on a given curve is called the *Cauchy problem* for this equation, and the information given on the curve is called *Cauchy data*. The examples suggest that we can expect a unique solution of a Cauchy problem if the curve along which data is specified is not characteristic, and either no solution or infinitely many solutions if the curve is characteristic.

Problems for Section 1.3

In each of Problems 1 through 6, determine the characteristic equation, solve it and sketch graphs of some of the characteristics, find the general solution of the partial differential equation, and attempt to find particular solutions satisfying the Cauchy data on the given curves.

1. $3yu_x - 2xu_y = 0$

   (a) $u(x,y) = x^2$ on the line $y = x$

(b) $u(x, y) = 1 - x^2$ on the line $y = -x$
(c) $u(x, y) = 2x$ on the ellipse $3y^2 + 2x^2 = 4$

2. $u_x - 6u_y = y$

(a) $u(x, y) = e^x$ on the line $y = -6x + 2$
(b) $u(x, y) = 1$ on the parabola $y = -x^2$
(c) $u(x, y) = -4x$ on the line $y = -6x$

3. $4u_x + 8u_y - u = 1$

(a) $u(x, y) = \cos(x)$ on the line $y = 3x$
(b) $u(x, y) = x$ on the line $y = 2x$
(c) $u(x, y) = 1 - x$ on the curve $y = x^2$

4. $-4yu_x + u_y - yu = 0$

(a) $u(x, y) = x^3$ on the line $x + 2y = 3$
(b) $u(x, y) = -y$ on $y^2 = x$
(c) $u(x, y) = 2$ on $x + 2y^2 = 1$

5. $yu_x + x^2u_y = xy$

(a) $u(x, y) = 4x$ on the curve $y = \frac{1}{3}x^{3/2}$
(b) $u(x, y) = x^3$ on the curve $3y^2 = 2x^3$
(c) $u(x, y) = \sin(x)$ on the line $y = 0$

6. $y^2u_x + x^2u_y = y^2$

(a) $u(x, y) = x$ on $y = 4x$
(b) $u(x, y) = -2y$ on $y^3 = x^3 - 2$
(c) $u(x, y) = y^2$ on $y = -x$

1.4 The Quasi-Linear Equation

For the general first-order linear equation 1.3, characteristics are certain curves in the x, y–plane, defined so that the Cauchy problem has a unique solution when the Cauchy data is specified along a noncharacteristic.

For the first-order quasi-linear partial differential equation

$$f(x, y, u)u_x + g(x, y, u)u_y = h(x, y, u), \tag{1.11}$$

characteristics are defined as curves in x, y, u - space determined as solutions of

$$\frac{dx}{dt} = f(x, y, u), \frac{dy}{dt} = g(x, y, u), \frac{du}{dt} = h(x, y, u). \tag{1.12}$$

We will show that a solution $u(x, y)$ of equation 1.11 may be interpreted as a surface made up of such characteristics. This idea can be used to obtain solutions containing given noncharacteristic curves, and hence provides a way of solving the Cauchy problem when the partial differential equation is quasi-linear. To understand this process, we need two facts.

Fact 1 Suppose that $u = \varphi(x, y)$ is a solution of equation 1.11 defining a surface $\sum$ and that $P_0 : (x_0, y_0, u_0)$ is a point on $\sum$, so $u_0 = \varphi(x_0, y_0)$. Then the characteristic passing through P_0 lies entirely on $\sum$.

To see why this is true, suppose that the characteristic has parametric equations

$$x = x(t), y = y(t), u = u(t).$$

Because P_0 is on this characteristic, for some t_0,

$$x(t_0) = x_0, y(t_0) = y_0, u(t_0) = u_0.$$

Because this curve is characteristic, we can use equations 1.12 to compute

$$\begin{aligned}\frac{d}{dt}\varphi(x(t), y(t)) &= \varphi_x \frac{dx}{dt} + \varphi_y \frac{dy}{dt} \\ &= \varphi_x f(x, y, u) + \varphi_y g(x, y, u) = h(x, y, u) = \frac{du}{dt}.\end{aligned}$$

Therefore,

$$u(t) = \varphi(x(t), y(t)) + k$$

for some constant k. But then $k = 0$, because we know that

$$u_0 = u(t_0) = \varphi(x(t_0), y(t_0)).$$

Therefore,

$$u(t) = \varphi(x(t), y(t))$$

and the characteristic lies on $\sum$.

Fact 2 If we begin with an arbitrary noncharacteristic curve Γ and construct the family of characteristics passing through points of Γ, the resulting surface $\sum$ is the graph of a solution of the partial differential equation.

To see why this is true, assume that $\sum$ is the graph of $u = \varphi(x, y)$. We want to show that φ is a solution of equation 1.11.

Suppose that Γ is parametrized by

$$x = x(s), y = y(s), z = z(s).$$

At any (x, y, u) on $\sum$,

$$\frac{dx}{ds} = f(x, y, u), \frac{dy}{ds} = g(x, y, u), \frac{du}{ds} = h(x, y, u)$$

because the surface is made up of characteristics. Then

$$\frac{du}{ds} = h(x, y, u) = \varphi_x \frac{dx}{ds} + \varphi_y \frac{dy}{ds} = f\varphi_x + g\varphi_y,$$

so φ is a solution.

These observations suggest the *method of characteristics* for solving the Cauchy problem when the partial differential equation is quasi-linear. Suppose that we want the solution of equation 1.11 assuming prescribed values on a given curve Γ that is not characteristic. Construct the characteristic through each point of Γ. This defines a surface in 3 - space, and this surface is the graph of the solution of this Cauchy problem.

This strategy also suggests why we do not want to specify data along a characteristic C. If we did so, the characteristic through each point of C would be just C itself, and this construction yields just the curve C, not a surface representing a solution of the partial differential equation.

Here are two illustrations of the method of characteristics.

Example 1.5 *We want the solution of*

$$yu_x - xu_y = e^u$$

that passes through the curve Γ *given by* $y = \sin(x), u = 0$. *This means that we want a solution* $u(x, y)$ *satisfying*

$$u(x, \sin(x)) = 0.$$

The characteristics of this partial differential equation are specified by

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -x, \frac{du}{dt} = e^u.$$

From the first two of these equations we can write

$$\frac{dy}{dx} = -\frac{x}{y}$$

or

$$y\,dy + x\,dx = 0,$$

with general solution (in terms of t)

$$x = a\cos(t) + b\sin(t) \text{ and } y = b\cos(t) - a\sin(t),$$

with a and b constant. From $du/dt = e^u$ *we obtain*

$$-e^{-u} = t + c.$$

The characteristics therefore have parametric representation

$$x = a\cos(t) + b\sin(t), y = b\cos(t) - a\sin(t), e^{-u} = c - t.$$

Parametrize Γ *as*

$$x = s, y = \sin(s), u = 0.$$

We use s as a parameter on Γ *to distinguish between points on* Γ *and points on characteristics. We want to construct a characteristic through each point of* Γ.