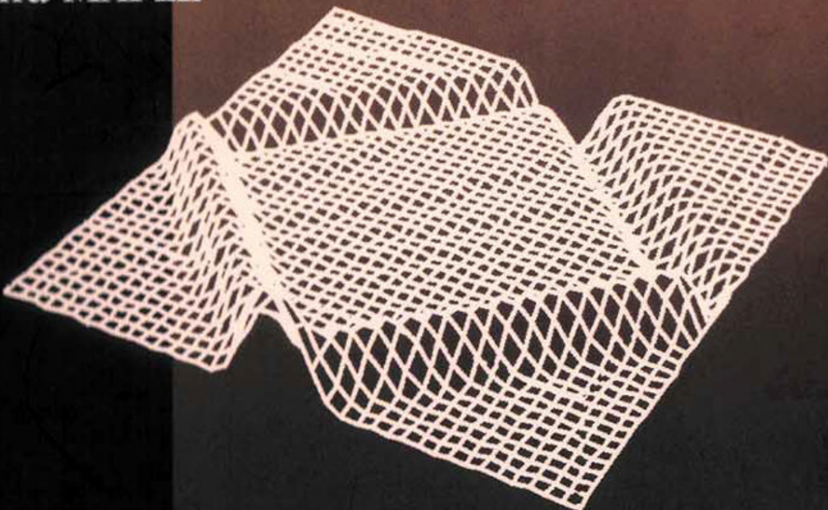
PARTIAL DIFFERENTIAL EQUATIONS (Second Edition)

An Introduction with Mathematica and MAPLE



Ioannis P Stavroulakis
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An Introduction with Mathematica and Maple (Second Edition)

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To our wives
Georgia and Mariam
and our children
Petros, Maria–Christina and Ioannis
and
Takuhi and Lusina

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Preface

In this second edition the section "Weak Derivatives and Weak Solutions" was removed to Chapter 5 to be together with advanced concepts such as discontinuous solutions of nonlinear conservation laws. The figures were rearranged, many points in the text were improved and the errors in the first edition were corrected.

Many thanks are due to G. Barbatis for his comments. Also many thanks to our graduate students over several semesters who worked through the text and the exercises making useful suggestions.

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September, 2003

Ioannis P. Stavroulakis,
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Preface to the First Edition

This textbook is a self-contained introduction to Partial Differential Equations (PDEs). It is designed for undergraduate and first year graduate students who are mathematics, physics, engineering or, in general, science majors.

The goal is to give an introduction to the basic equations of mathematical physics and the properties of their solutions, based on classical calculus and ordinary differential equations. Advanced concepts such as weak solutions and discontinuous solutions of nonlinear conservation laws are also considered.

Although much of the material contained in this book can be found in standard textbooks, the treatment here is reduced to the following features:

- To consider first and second order linear classical PDEs, as well as to present some ideas for nonlinear equations.
- To give explicit formulae and derive properties of solutions for problems with homogeneous and inhomogeneous equations; without boundaries and with boundaries. To consider the one dimensional spatial case before going on to two and three dimensional cases.
- To illustrate the effects for different problems with model examples: To use Mathematics software products as *Mathematica* and MAPLE in ScientifiCWorkPlacE in both graphical and computational aspects; To give a number of exercises completing the explanation to some advanced problems.

The book consists of eight Chapters, each one divided into several sections.

In Chapter I we present the theory of first-order PDEs, linear, quasilinear, nonlinear, the method of characteristics and the Cauchy problem. In Chapter II we give the classification of second-order PDEs in two variables based on the method of characteristics. A classification of almost-linear second-order PDEs in $n-$variables is also given. Chapter III is concerned with the one dimensional wave equation on the whole line, half-line and the mixed problem using the reflection method. The inhomogeneous equation as well as weak derivatives and weak solutions of the wave equation are also discussed. In Chapter IV the one dimensional diffusion equation is presented. The Maximum-minimum principle, the Poisson formula with applications and the reflection method are given. Chapter V contains an introduction to the theory of shock waves and conservation laws. Burgers' equation and Hopf–Cole transformation are discussed. The notion of weak solutions, Riemann problem, discontionuous solutions and Rankine–Hugoniot condition are considered. In Chapter VI the Laplace equation on the plane and space is considered. Maximum principles, the mean value property, Green's identities and the representation formulae are given. Green's functions for the half-space and sphere are discussed, as well as Harnack's inequalities and theorems. In Chapter VII some basic theorems on Fourier series and orthogonal systems are given. Fourier methods for the wave, diffusion and Laplace equations are also considered. Finally in Chapter VIII two and three dimensional wave and diffusion equations are considered. Kirchoff's formula and Huygens' principle as well as Fourier method are presented.

Model examples are given illustrated by software products as *Mathematica* and MAPLE in ScientifiCWorkPlacE. We also present the programs in *Mathematica* for those examples. For further details in *Mathematica* the reader is referred to Wolfram [49], Ross [34] and Vvedensky [47].

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June, 1999

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Contents

Chapter 1

First-order Partial Differential Equations

1.1 Introduction

Let $u = u(x_1, ..., x_n)$ be a function of n independent variables $x_1, ..., x_n$. A Partial Differential Equation (PDE for short) is an equation that contains the independent variables $x_1, ..., x_n$, the dependent variable or the unknown function u and its partial derivatives up to some order. It has the form

$$F(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_n}, u_{x_1 x_1}, ..., u_{x_i x_j}, ...) = 0, \tag{1.1}$$

where F is a given function and $u_{x_j} = \partial u/\partial x_j$, $u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$, $i, j = 1, ..., n$ are the partial derivatives of u. The *order* of a PDE is the order of the highest derivative which appears in the equation.

A set Ω in the $n-$dimensional Euclidean space $\mathbf{R}^n$ is called a *domain* if it is an open and connected set. A *region* is a set consisting of a domain plus, perhaps, some or all of its boundary points. We denote by $C(\Omega)$ the space of continuous functions in Ω and by $C^k(\Omega)$ the space of continuously differentiable functions up to the order k in Ω. Suppose (1.1) is a PDE of order m. By a *solution* of the equation (1.1) we mean a function $u \in C^m(\Omega)$ such that the substitution of u and its derivatives up to the order m in (1.1) makes it an identity in $(x_1, ..., x_n) \in \Omega$.

Some examples of PDEs (all of which occur in Physics) are:

1. $u_x + u_y = 0$ (transport equation)

2. $u_x + uu_y = 0$ (shock waves)

3. $u_x^2 + u_y^2 = 1$ (eikonal equation)

4. $u_{tt} - u_{xx} = 0$ (wave equation)

5. $u_t - u_{xx} = 0$ (heat or diffusion equation)

6. $u_{xx} + u_{yy} = 0$ (Laplace equation)

7. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ (biharmonic equation)

8. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)

9. $u_t + cuu_x = \varepsilon u_{xx}$ (Burgers' equation)

10. $u_t + cuu_x + u_{xxx} = 0$ (Korteweg–de Vries equation)

11. $\left(1 - u_t^2\right) u_{xx} + 2u_x u_t u_{xt} - \left(1 + u_x^2\right) u_{tt} = 0$ (Born–Infeld equation)

12. $u_{xy}^2 - u_{xx}u_{yy} = f\left(x, y\right)$ (Monge–Ampère equation) .

Each one of these equations has two independent variables denoted either by x, y or x, t. Equations 1, 2 and 3 are of first-order. Equations numbered as 4, 5, 6, 8 , 9, 11 and 12 are of second-order; 10 is of third-order; 7 is of fourth-order. Examples 2, 3, 8, 9, 10, 11 and 12 are distinguished from the others in that they are not "linear".

Linearity means the following. The correspondence

$$u(x_1, ..., x_n) \longmapsto Lu := F(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_i x_j}, ...)$$

defines an operator L. The operator L is said to be *linear* iff (if and only if)

$$L\left(c_1 u_1 + c_2 u_2\right) = c_1 L u_1 + c_2 L u_2 \tag{1.2}$$

for any functions u_1, u_2 and any constants $c_1, c_2 \in \mathbf{R}$.

The operator L is *nonlinear* if (1.2) is not satisfied. For instance, the equation 2 is nonlinear because $(u_1 + u_2)(u_1 + u_2)_y = u_1 u_{1y} + u_2 u_{2y}$ is not satisfied for any functions u_1 and u_2.

Nonlinearity may be of various types. An equation is said to be *almost-linear* if it is of the form $Lu + f\left(x, u\right) = 0$, where $f\left(x, u\right)$ is a nonlinear function with respect to u. An equation is said to be *quasi-linear* if it is linear with respect to highest order derivatives and *fully-nonlinear* if it is nonlinear with respect to highest order derivatives. For instance, the equation 8 is almost-linear, the equations 2, 9, 10 and 11 are quasi-linear, while the equations 3 and 12 are fully-nonlinear.

The general form of a first-order PDE for a function $u = u(x_1, ..., x_n)$ of n independent variables $(x_1, ..., x_n)$ is

$$F(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_n}) = 0,$$

where F is a given function and $u_{x_j} = \partial u/\partial x_j$, $j = 1, ..., n$ are the partial derivatives of the unknown function u. In the case of two independent variables x, y the above form is

$$F(x, y, u, u_x, u_y) = 0.$$

Equations of this type occur in the calculus of variations, geometrical optics, particle mechanics, etc. The philosophy of treatment of first-order PDEs is in many ways different from that of the more commonly encountered second-order PDEs appearing in physics and science. First-order PDEs may always be reduced to a system of Ordinary Differential Equations (ODEs for short).

If the operator L is linear then the equation

$$Lu = 0$$

is called a linear *homogeneous* equation, while

$$Lu = f,$$

where $f \neq 0$, is called a linear *inhomogeneous* equation. It is clear that Examples 1, 4, 5, 6 and 7 are linear homogeneous equations.

A partial differential equation subject to certain conditions in the form of initial or boundary conditions is known as an *initial value problem* (IVP for short) or *boundary value problem* (BVP for short). The initial conditions, also known as *Cauchy conditions,* are the values of the unknown function u and of an appropriate number of its derivatives at the initial point, while the boundary conditions are the values on the boundary ∂D of the domain D under consideration. The three most important kinds of boundary conditions are:

(i) *Dirichlet conditions* or *boundary conditions of the first kind* are the values of u prescribed at each point of the boundary ∂D.

(ii) *Neumann conditions* or *boundary conditions of the second kind* are the values of the normal derivative of u prescribed at each point of the boundary ∂D.

(iii) *Robin conditions* or *mixed boundary conditions* or *boundary conditions of the third kind* are the values of a linear combination of u and its normal derivative prescribed at each point of the boundary ∂D.

In this textbook we concentrate on problems for first-order PDEs (linear, quasi-linear and fully-nonlinear), the three classical linear second-order PDEs (wave, heat or diffusion and Laplace equations) as well as the Burgers' equation. We first consider one spatial dimension before going on two and three dimensions; problems without boundaries before problems with boundary conditions; homogeneous equations before inhomogeneous equations.

1.2 Linear First-order Equations

A linear first-order PDE in two independent variables x, y and the dependent variable u has the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d\,(x,y)\,, \tag{1.3}$$

where $a, b, c, d \in C^1(\Omega)\,, \Omega \subset \mathbf{R}^2$ and $a^2 + b^2 \neq 0$, that is, at least one of the coefficients a or b does not vanish on Ω. If we consider the differential operator

$$L := a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c,$$

then equation (1.3) is written as

$$Lu = d,$$

while the homogeneous equation corresponding to (1.3) is

$$Lu = 0. \tag{1.4}$$

By a *general solution* of (1.4) we mean a relation involving an arbitrary function such that for any choice of the arbitrary function we derive a solution of equation (1.4). If u_h denotes the general solution of the homogeneous equation and u_p a particular solution of the inhomogeneous equation (1.3), then the general solution of (1.3) is

$$u = u_h + u_p. \tag{1.5}$$

Indeed, (1.5) is a solution of equation (1.3), since by the linearity property of the operator L, we have

$$Lu = L(u_h + u_p) = Lu_h + Lu_p = 0 + d = d.$$

Conversely, if v is a solution of (1.3), then we will show that it is of the form (1.5). Take the function $v - u_p$. Then

$$L(v - u_p) = Lv - Lu_p = d - d = 0,$$

that is, $v - u_p$ is a solution of the homogeneous equation (1.4) and therefore $u_h = v - u_p$ for some choice of the arbitrary function which appears in u_h. Thus $v = u_h + u_p$.

Example 1.1. *Find the general solution of the equation*

$$\frac{\partial u}{\partial x} + u = e^{-x}. \tag{1.6}$$

Solution. The corresponding homogeneous equation is

$$\frac{\partial u}{\partial x} + u = 0. \tag{1.7}$$

Integrating with respect to x (holding y as a constant), we have

$$u(x, y) = e^{-x} f(y),$$

where f is an arbitrary continuously differentiable function. This is the general solution of (1.7). Observe that a particular solution of (1.6) is

$$u_p = xe^{-x}.$$

Thus the general solution of the inhomogeneous equation (1.6) is

$$u(x, y) = e^{-x} f(y) + xe^{-x},$$

where f is an arbitrary continuously differentiable function.

We could also work as in the case of ordinary differential equations. Thus, from (1.6), (considering y as a constant), we derive the solution

$$\begin{aligned}
u(x,y) &= e^{-\int dx}\left[f(y)+\int e^{-x}e^{\int dx}dx\right] \\
&= e^{-x}\left[f(y)+\int e^{-x}e^{x}dx\right] \\
&= e^{-x}f(y)+xe^{-x},
\end{aligned}$$

that is, the same result. Observe here that f is an arbitrary continuously differentiable function of y (instead of an arbitrary constant C that we have in the case of ODEs).

Next we will derive the form of the general solution of the linear first-order homogeneous equation

$$a(x,y)u_x+b(x,y)u_y+c(x,y)u=0, \tag{1.8}$$

where $a,b,c\in C^1(\Omega),\Omega\subset\mathbf{R}^2$. Consider the transformation

$$\begin{cases} \xi=\xi(x,y) \\ \eta=\eta(x,y) \end{cases}, \quad (x,y)\in\Omega,$$

with Jacobian

$$J=\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}\neq 0 \text{ on } \Omega.$$

Since

$$\begin{aligned}
u_x &= u_\xi\xi_x+u_\eta\eta_x, \\
u_y &= u_\xi\xi_y+u_\eta\eta_y,
\end{aligned}$$

the equation (1.8) is transformed into the following equation

$$(a\xi_x+b\xi_y)\,u_\xi+(a\eta_x+b\eta_y)\,u_\eta+cu=0, \tag{1.9}$$

where the coefficients are now expressed in terms of the new variables ξ,η. Our aim is to simplify equation (1.9), by choosing η such that

$$a\eta_x+b\eta_y=0. \tag{1.10}$$

This is accomplished as follows. Assume, without loss of generality, that $a(x,y)\neq 0$ and consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{1.11}$$

Let the general solution of equation (1.11) be

$$\eta(x,y) = K, \tag{1.12}$$

where $\eta_y \neq 0$ and K is an arbitrary constant. Then, for this function $\eta(x,y)$

$$\eta_x dx + \eta_y dy = 0$$

and, in view of (1.11), equation (1.10) is satisfied.

The one-parameter family of curves (1.12) defined by equation (1.11) are called *characteristic curves* of the differential equation (1.8).

Now choose

$$\xi = \xi(x,y) = x.$$

Then

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \eta_y \neq 0$$

and the transformation constructed in this manner, that is

$$\begin{aligned} \xi &= x \\ \eta &= \eta(x,y), \end{aligned}$$

where $\eta(x,y) = K$ is the general solution of the ODE (1.11), is invertible. The equation (1.9) reduces to the following simple form

$$a(\xi,\eta)\, u_\xi + c(\xi,\eta)\, u = 0, \tag{1.13}$$

called the *canonical form* for the linear equation (1.8), and it can be solved as an ODE (cf. Example 1.1).

In the case of the inhomogeneous equation (1.3) we derive the following form

$$a(\xi,\eta)\, u_\xi + c(\xi,\eta)\, u = d(\xi,\eta). \tag{1.14}$$

Example 1.2. *Find the general solution of the linear equation*

$$xu_x - yu_y + y^2u = y^2, \quad x, y \neq 0. \tag{1.15}$$

Solution. The coefficients are

$$a = x, \quad b = -y, \quad c = y^2, \quad d = y^2.$$

Consider the homogeneous equation

$$xu_x - yu_y + y^2u = 0. \tag{1.16}$$

The equation (1.11) is

$$\frac{dy}{dx} = -\frac{y}{x}$$

and its general solution (which gives the family of the characteristic curves) is

$$xy = K, \quad K \text{ a constant.}$$

An appropriate transformation is

$$\begin{aligned} \xi &= x \\ \eta &= xy, \end{aligned}$$

since the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = x \neq 0.$$

The coefficients a and c with respect to ξ and η become

$$a(\xi, \eta) = \xi, \quad c(\xi, \eta) = \frac{\eta^2}{\xi^2}$$

and therefore equation (1.13) yields

$$\xi u_\xi + \frac{\eta^2}{\xi^2}u = 0.$$

This is the canonical form of the homogeneous equation (1.16). The general solution of the last equation is

$$u = f(\eta)\, e^{-\int \frac{\eta^2}{\xi^3} d\xi} = f(\eta)\, e^{\frac{\eta^2}{2\xi^2}}$$

and therefore the general solution to equation (1.16) is

$$u(x, y) = f(xy)\, e^{\frac{y^2}{2}}.$$

Observe that the constant 1 is a particular solution of equation (1.15) and therefore the general solution of the inhomogeneous equation (1.15) is given by the function

$$u(x, y) = f(xy)\, e^{\frac{y^2}{2}} + 1, \tag{1.17}$$

where f is an arbitrary continuously differentiable function.

Note that we could consider from the beginning the inhomogeneous equation (1.15) and using the same transformation, equation (1.14) yields

$$\xi u_\xi + \frac{\eta^2}{\xi^2} u = \frac{\eta^2}{\xi^2},$$

with general solution

$$\begin{aligned}
u &= e^{-\int \frac{\eta^2}{\xi^3} d\xi} \left[f(\eta) + \int \frac{\eta^2}{\xi^3} e^{\int \frac{\eta^2}{\xi^3} d\xi} d\xi \right] \\
&= e^{\frac{\eta^2}{2\xi^2}} \left[f(\eta) + \int \frac{\eta^2}{\xi^3} e^{-\frac{\eta^2}{2\xi^2}} d\xi \right] \\
&= e^{\frac{\eta^2}{2\xi^2}} \left[f(\eta) + e^{-\frac{\eta^2}{2\xi^2}} \right] \\
&= f(\eta)\, e^{\frac{\eta^2}{2\xi^2}} + 1.
\end{aligned}$$

Thus the general solution of equation (1.15) is given by (1.17).

Exercises

1. Find the general solutions of the following equations:
(a) $xu_x + yu_y = nu$ (Euler's relation)
(b) $xu_x + yu_y = x^n$
(c) $au_x + bu_y + cu = d$, where a, b, c, d constants and $a^2 + b^2 \neq 0$.

2. (Extension of the linear equation in $n-$variables). Consider the equation

$$\sum_{j=1}^{n} a_j (x_1, ..., x_n) u_{x_j} = 0.$$

The characteristic system

$$\frac{dx_1}{a_1} = ... = \frac{dx_n}{a_n}$$

describes the family of the characteristic curves. If

$$u_1 (x_1, ..., x_n) = c_1, ..., u_{n-1} (x_1, ..., x_n) = c_{n-1}$$

are $n - 1$ functionally independent solutions of the characteristic system, then the general solution is given by

$$u = f (u_1, ..., u_{n-1}).$$

The functions $u_1 (x_1, ..., x_n), ..., u_{n-1} (x_1, ..., x_n)$ are functionally independent if

$$rank \begin{bmatrix} u_{1,x_1} & \cdots & u_{1,x_n} \\ \vdots & & \vdots \\ u_{n-1,x_1} & \cdots & u_{n-1,x_n} \end{bmatrix} = n - 1.$$

In the case of the linear equation

$$\sum_{j=1}^{n} a_j (x_1, ..., x_n) u_{x_j} + c (x_1, ..., x_n) u = 0,$$

the general solution is given by

$$u = vf (u_1, ..., u_{n-1}),$$

where v is a particular solution.

3. Find the general solutions of the equations
(a) $(y - z) u_x + (z - x) u_y + (x - y) u_z = 0.$
(b) $x (y - z) u_x + y (z - x) u_y + z (x - y) u_z = 0.$
(c) $a_1u_x + a_2u_y + a_3u_z + cu = 0$, where a_1, a_2, a_3, c are constants and $a_i \neq 0$ for some $i = 1, 2, 3$.

1.3 The Cauchy Problem for First-order Quasi-linear Equations

We consider the case of the quasi-linear equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{1.18}$$

Quasi-linearity means that the operator

$$u\,(x,y) \longmapsto Lu := a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u),$$

is nonlinear, but it is linear with respect to the derivatives (u_x, u_y). For instance, the equation 2 in the Introduction is nonlinear, but it is *quasi-linear* because

$$u(c_1v_1 + c_2v_2)_y = c_1uv_{1y} + c_2uv_{2y},$$

for any functions u, v_1, v_2 and any constants c_1, c_2.

A solution of (1.18) defines an integral surface $S : u = u\,(x,y)$ in the Euclidean (x,y,u) space. The normal to this surface at the point $P(x,y,u)$ is the vector $\overrightarrow{n}_P\,(u_x, u_y, -1)$ and let $\overrightarrow{v_P}$ be the vector $(a(x,y,u), b(x,y,u), c(x,y,u))$.

Then the equation (1.18) can be interpreted as the condition that at each point P of the integral surface S the vector $\overrightarrow{v_P}$ is tangent to the surface S.

Suppose that $P \in \Omega$, where Ω is a domain in the (x,y,u) space and consider the vector field $V = \{\overrightarrow{v_P} : P \in \Omega\}$. We define as *characteristic curves*

$$\Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ u = u(t) \end{cases} \qquad t \in [a,b]\,,$$

the integral curves in Ω of the *characteristic system*

$$\begin{cases} \dfrac{dx}{dt} = a(x,y,u) \\ \dfrac{dy}{dt} = b(x,y,u) \\ \dfrac{du}{dt} = c(x,y,u). \end{cases} \tag{1.19}$$

The last system can be rewritten shortly as

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)},$$

which is an *autonomous system* of ODEs. Assuming a, b and c to be of class $C^1(\Omega)$ by the existence and uniqueness theorem for ODEs it follows that through each point $P_0(x_0, y_0, u_0) \in \Omega$ passes exactly one characteristic curve Γ_0.

There is a 2-parameter family of characteristic curves in Ω of (1.19) and the curves do not change by translating the independent variable t. Note that if a surface $S : u = u(x, y)$ is a union of characteristic curves, then S is an integral surface and conversely every integral surface S is a union of characteristic curves.

Theorem 1.1. *Let the characteristic curve*

$$\Gamma_0 : \begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \end{cases} \qquad t \in [a, b],$$

intersect the integral surface S at the point $P_0(x_0, y_0, u_0) \in \Omega$. Then $\Gamma_0 \subset S$ which means

$$u_0(t) = u(x_0(t), y_0(t)), \quad a \le t \le b.$$

Proof. Let $U(t) = u_0(t) - u(x_0(t), y_0(t))$. As $P_0(x_0, y_0, u_0) \in S \cap \Gamma_0$, there exists $t_0 \in [a, b]$ such that

$$x_0 = x_0(t_0), y_0 = y_0(t_0), u_0 = u_0(t_0) \text{ and } U(t_0) = 0.$$

We have

$$\begin{aligned} \frac{dU}{dt} &= \frac{du_0}{dt} - \frac{dx_0}{dt}u_x - \frac{dy_0}{dt}u_y \\ &= c(x_0(t), y_0(t), u_0(t)) - a(x_0(t), y_0(t), u_0(t))u_x \\ &\quad -b(x_0(t), y_0(t), u_0(t))u_y \\ &= c(x_0(t), y_0(t), U(t) + u(t)) - a(x_0(t), y_0(t), U(t) + u(t))u_x \\ &\quad -b(x_0(t), y_0(t), U(t) + u(t))u_y, \end{aligned} \tag{1.20}$$

where $u(t) = u(x_0(t), y_0(t))$. The equation (1.20) is an ODE with initial condition $U(t_0) = 0$ and by the uniqueness theorem for the Cauchy[1] problem for ODEs it follows

$$U(t) = u_0(t) - u(x_0(t), y_0(t)) = 0, \quad a \le t \le b. \blacksquare$$

[1] Augustin Louis Cauchy, 21.08.1789–23.05.1857.

As a consequence of Theorem 1.1 we have that if two integral surfaces S_1 and S_2 have a common point P_0 , then they intersect along the characteristic curve Γ_0 through P_0.

The selection of an individual surface $S : u = u(x, y)$ among all integral surfaces, containing a prescribed curve Γ constitutes the Cauchy problem for (1.18). This is formulated as

Find a solution $u = u(x, y)$ of (1.18) *for which*

$$u_0(s) = u\left(x_0(s), y_0(s)\right), \quad a \le s \le b,$$

where

$$\Gamma : \begin{cases} x = x_0(s) \\ y = y_0(s) \\ u = u_0(s) \end{cases}$$

is an initial curve.

We shall consider the local solvability of the Cauchy problem, i.e. the existence of an integral surface in a neighborhood of the curve Γ. The main tool for solving the local problem is the well known Inverse Mapping Theorem (IMT), which also has a local character.

Theorem 1.2. **(IMT).** *Let* $D \subset \mathbf{R}^2_{s,t}$ *and* $D' \subset \mathbf{R}^2_{x,y}$ *be domains* $\Phi : D \to D'$ *be of class* $C^1(D), P_0(s_0, t_0) \in D, Q_0(x_0, y_0) \in D', \Phi(P_0) = Q_0,$

$$\Phi : \begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases}$$

and

$$J\Phi(P_0) = \frac{\partial(x,y)}{\partial(s,t)}(P_0) = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}(P_0) \neq 0.$$

Then there exist neighborhoods U *of* $P_0 \in D$ *and* U' *of* $Q_0 \in D'$ *and a mapping* $\Phi^{-1} \in C^1(U')$ *such that* $\Phi^{-1}(U') = U$ *and*

$$J\Phi^{-1}(Q_0) = \left(J\Phi(P_0)\right)^{-1}.$$

Now we prove a local existence theorem for the Cauchy problem

Theorem 1.3. *(Existence and Uniqueness Theorem) Consider the first-order quasi-linear PDE in the domain* $\Omega \subset \mathbf{R}^3$

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

where a, b and c are of class $C^1(\Omega)$

$$\Gamma : \begin{cases} x = x_0(s) \\ y = y_0(s) \\ u = u_0(s) \end{cases} \qquad 0 \le s \le 1,$$

is an initial smooth curve in Ω and

$$\frac{dx_0}{ds} b\left(x_0(s), y_0(s), u_0(s)\right) - \frac{dy_0}{ds} a\left(x_0(s), y_0(s), u_0(s)\right) \neq 0, \; 0 \le s \le 1. \quad (1.21)$$

Then there exists one and only one solution $u = u(x, y)$ defined in a neighborhood N of the initial curve Γ, which satisfies the equation (1.18) and the initial condition

$$u_0(s) = u\left(x_0(s), y_0(s)\right), \; 0 \le s \le 1.$$

Proof. Let us consider the Cauchy problem for the ODEs system

$$(C) : \begin{cases} \dfrac{dx}{dt} = a(x, y, u) \\ \dfrac{dy}{dt} = b(x, y, u) \\ \dfrac{du}{dt} = c(x, y, u) \end{cases}$$

with initial conditions

$$x(s, 0) = x_0(s), \quad y(s, 0) = y_0(s), \quad u(s, 0) = u_0(s).$$

¿From the existence and uniqueness theorem for ODEs the problem has a unique solution

$$x = x(s, t), \quad y = y(s, t), \quad u = u(s, t),$$

defined for $t : \alpha(s) \le t \le \beta(s)$ where $0 \in [\alpha(s), \beta(s)]$, $\alpha(s)$ and $\beta(s)$ are continuous functions and

$$D = \{(s, t) : 0 \le s \le 1, \alpha(s) \le t \le \beta(s)\} \subset \Omega' = \Pr_{oxy} \Omega.$$

According to (1.21) for the mapping

$$\Phi : \begin{cases} x = x(s, t) \\ y = y(s, t) \end{cases}$$

$$J\Phi\,|_{t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = \frac{dx_0}{ds} b - \frac{dy_0}{ds} a \neq 0.$$

By IMT there exists a unique inverse mapping $\Phi^{-1} : D' \to D$,

$$\Phi^{-1} : \begin{cases} s = s(x, y) \\ t = t(x, y) \end{cases}$$

defined in a neighborhood N' of $\Gamma' = \Pr_{oxy} \Gamma$. Consider now

$$u = u\left(s(x, y), t(x, y)\right) = \varphi\left(x, y\right).$$

We find that

$$\begin{aligned} a\varphi_x + b\varphi_y &= a\left(u_s s_x + u_t t_x\right) + b\left(u_s s_y + u_t t_y\right) \\ &= u_s\left(a s_x + b s_y\right) + u_t\left(a t_x + b t_y\right) \\ &= u_s\left(x_t s_x + y_t s_y\right) + u_t\left(x_t t_x + y_t t_y\right) \\ &= u_s.0 + u_t.1 \\ &= u_t = c \end{aligned}$$

and

$$\begin{aligned} \varphi\left(x_0\left(s\right), y_0\left(s\right)\right) &= u\left(s(x_0\left(s\right), y_0\left(s\right)), t(x_0\left(s\right), y_0\left(s\right))\right) \\ &= u\left(s, 0\right) = u_0\left(s\right). \end{aligned}$$

Moreover $\varphi\left(x, y\right)$ is a unique solution. Indeed let $\varphi_1\left(x, y\right)$ and $\varphi_2\left(x, y\right)$ be two solutions satisfying the initial condition and $S_j = \varphi_j\left(x, y\right), j = 1, 2$ be the corresponding integral surfaces. Considering the systems of ODEs

$$\begin{cases} \dfrac{dx}{dt} = a(x, y, \varphi_j\left(x, y\right)) \\ \dfrac{dy}{dt} = b(x, y, \varphi_j\left(x, y\right)) \end{cases}$$

with initial conditions

$$x\left(s, 0\right) = x_0\left(s\right), \quad y\left(s, 0\right) = y_0\left(s\right),$$

we find solutions $\left(x_j\left(s, t\right), y_j\left(s, t\right)\right)$. Then $\left(x_j\left(s, t\right), y_j\left(s, t\right), \varphi_j\left(s, t\right)\right)$ are solutions of system (C). Therefore by the uniqueness theorem for ODEs

$$\left(x_j\left(s, t\right), y_j\left(s, t\right), \varphi_j\left(s, t\right)\right), \quad j = 1, 2,$$

coincide in the common domain of definition . It follows that the characteristics Γ_1 and Γ_2 starting from the point $P\left(x_0\left(s\right), y_0\left(s\right), u_0\left(s\right)\right)$ also coincide. ■

Remark. Note that condition (1.21) implies that the vector (a, b, c) is not tangent to the initial curve Γ at the point (x_0, y_0, u_0) . For if it were

$$\left(\frac{dx_0}{ds}, \frac{dy_0}{ds}, \frac{du_0}{ds}\right) = k\,(a, b, c)$$

or

$$\frac{dx_0}{ds} = ka, \ \frac{dy_0}{ds} = kb, \ \frac{du_0}{ds} = kc,$$

for some constant k. Thus

$$\frac{dx_0}{ds}b - \frac{dy_0}{ds}a = kab - kba = 0,$$

which contradicts (1.21).

In the following example it is shown that when (1.21) is violated, i.e. if

$$\frac{dx_0}{ds}b\left(x_0\left(s\right), y_0\left(s\right), u_0(s)\right) - \frac{dy_0}{ds}a\left(x_0(s), y_0(s), u_0(s)\right) = 0, \ 0 \le s \le 1.$$

then for the Cauchy problem there may not exist a solution or there may exist infinitely many distinct solutions. In other words, either there is no existence of a solution or there is no uniqueness.

Example 1.3. *Consider the equation*

$$yu_x - xu_y = 0.$$

Show that there exist initial curves such that when (1.21) *holds with the equality sign, then the Cauchy problem has no solution or there exist infinitely many distinct solutions.*

Solution. It is easy to see that the characteristic curves are given by

$$x^2 + y^2 = k$$

and the general solution is

$$u = f\left(x^2 + y^2\right),$$

where f is an arbitrary function. Consider the following three cases:

(i) The initial curve is given by the parametric equations

$$\Gamma_1 : x = x_0(s) = s, \quad y = y_0(s) = 0, \quad u = u_0(s) = s^2.$$

This curve is the parabola

$$u = x^2, \quad y = 0,$$

which lies in the (x, u) plane. We have

$$\frac{dx_0}{ds} b(x_0, y_0, u_0) - \frac{dy_0}{ds} a(x_0, y_0, u_0) = -s \neq 0,$$

and by Theorem 1.3 there exists a unique solution. Indeed the integral surfaces $u = f(x^2 + y^2)$ are surfaces of revolution about the u axis. The condition that such a surface contains Γ_1 is

$$f(x_0^2 + y_0^2) = f(s^2) = s^2,$$

that is, $f(t) = t$, which leads to the unique solution

$$u = x^2 + y^2.$$

This surface is a circular paraboloid.

(ii) The initial curve is given by

$$\Gamma_2 : x = x_0(s) = \cos s, \; y = y_0(s) = \sin s, \; u = u_0(s) = \sin s,$$

that is, Γ_2 is the ellipse

$$x^2 + y^2 = 1, \quad u = y.$$

Here

$$\frac{dx_0}{ds} b(x_0, y_0, u_0) - \frac{dy_0}{ds} a(x_0, y_0, u_0) = (-\sin s)(-\cos s) - (\cos s)(\sin s) = 0.$$

If $u = f(x^2 + y^2)$ is a solution, then on the circle $x^2 + y^2 = 1$ one has $u = f(1)$ a constant. This is incompatible with the requirement $u = y$ and therefore no solution exists. Note that the given curve Γ_2 is such that its projection on the (x, y) plane coincides with the projection on the (x, y) plane of a characteristic curve, but Γ_2 itself is non-characteristic. Indeed the tangent vector $(-\sin s, \cos s, \cos s)$ to Γ_2 is nowhere parallel to the characteristic vector $(\sin s, -\cos s, 0)$ along Γ_2.

(iii) The initial curve is given by

$$\Gamma_3 : x = x_0(s) = \cos s,\ y = y_0(s) = \sin s,\ u = u_0(s) = 1,$$

that is, Γ_3 is the circle

$$x^2 + y^2 = 1, \quad u = 1.$$

Here again

$$\frac{dx_0}{ds} b(x_0, y_0, u_0) - \frac{dy_0}{ds} a(x_0, y_0, u_0) = 0.$$

In order for $u = f(x^2 + y^2)$ to be a solution it should satisfy $f(1) = 1$ which is possible for any function f such that $f(1) = 1$ (i.e. $f(w) = w^n$). For such a function $f, u = f(x^2 + y^2)$ is an integral surface which contains Γ_3. Clearly there are infinitely many solutions in this case. Observe that the initial curve Γ_3 is now a characteristic curve. Indeed the tangent vector $(-\sin s, \cos s, 0)$ to Γ_3 is parallel to the characteristic vector $(\sin s, -\cos s, 0)$ along Γ_3.

Example 1.4. *Solve the PDE* $uu_x + u_y = 1/2$, *with initial condition* $u(s, s) = s/4, \quad 0 \le s \le 1$.

Solution. The initial curve

$$\Gamma : \begin{cases} x = s \\ y = s \\ u = s/4 \end{cases}$$

where $0 \le s \le 1$ satisfies (1.21)

$$\frac{dx_0}{ds} b - \frac{dy_0}{ds} a = 1 - \frac{s}{4} \neq 0$$

for $s \neq 4$. The characteristic system

$$\begin{cases} \dfrac{dx}{dt} = u \\ \dfrac{dy}{dt} = 1 \\ \dfrac{du}{dt} = \dfrac{1}{2} \end{cases}$$

with initial conditions

$$x(s, 0) = s, \quad y(s, 0) = s, \quad u(s, 0) = s/4,$$

has a solution

$$\begin{cases} x = s + st/4 + t^2/4 \\ y = s + t \\ u = s/4 + t/2. \end{cases}$$

Solving with respect to s, t in terms of x, y we obtain

$$\begin{cases} s = \dfrac{4x - y^2}{4 - y} \\ t = \dfrac{4\,(y - x)}{4 - y} \end{cases}$$

and the unique solution of the problem is

$$u = \frac{8y - 4x - y^2}{4\,(4 - y)}$$

for $y = s \neq 4$.

The integral surface S through the initial curve Γ is plotted in the Figure 1.1 using the *Mathematica* program

```
f1=ParametricPlot3D[{s+(t^2+st)/4,t+s,(2t+s)/4},
{s,0,1},{t,-1,1},PlotPoints->10]
f2=ParametricPlot3D[{s,s,s/4},{s,-0.5,1.5}]
Show[f1,f2,Shading->False,
PlotLabel->"Integral surface through initial curve"]
```

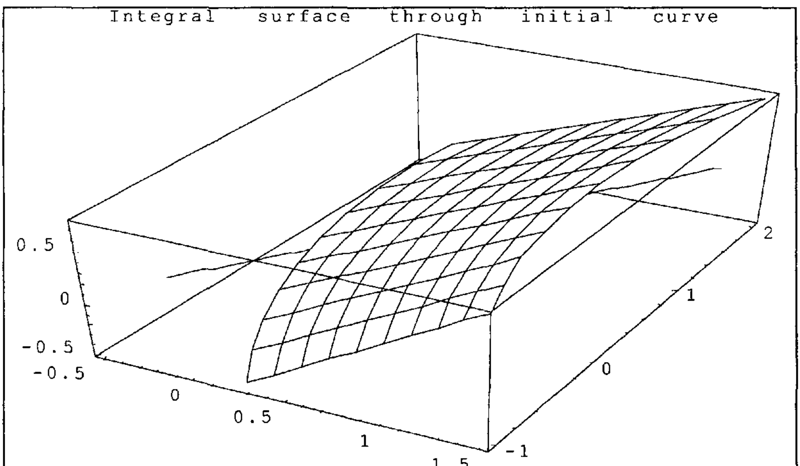

Figure 1.1. Graph of the function $u = \dfrac{8y - 4x - y^2}{4\,(4 - y)}$.

Example 1.5. *The solution of the equation* $u_y + uu_x = 0$ *can be interpreted as a vector field on the* $x-$*axis varying with the time* y. *Find the integral surface satisfying the initial condition* $u(s,0) = h(s)$, *where* h *is a given function.*

Solution. The characteristic system

$$\begin{cases} \dfrac{dx}{dt} = u \\ \dfrac{dy}{dt} = 1 \\ \dfrac{du}{dt} = 0 \end{cases}$$

with initial conditions

$$x(s,0) = s, \quad y(s,0) = 0, \quad u(s,0) = h(s)$$

has the solution

$$\begin{cases} x = s + h(s)t \\ y = t \\ u = h(s). \end{cases}$$

As before, (s,t) can be expressed in terms of (x,y) when

$$\begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = 1 + h'(s)t \neq 0,$$

i.e. $y = t \neq -\dfrac{1}{h'(s)}$. In this case for the solution

$$u = u(x,y) = u_0(s) = u_0(x,y)$$

we have

$$u_x = h'(s)s_x = \frac{h'(s)}{1 + h'(s)t}.$$

Hence for $h'(s) < 0$, u_x becomes infinite at the positive time

$$T = -\frac{1}{h'(s)}.$$

The smallest y for which this happens corresponds to the value $s = s_0$ at which $h'(s)$ has a minimum. At the time

$$T_0 = -\frac{1}{h'(s_0)}$$

the solution has a *gradient catastrophe* or *blow up*. There can not exist a smooth solution beyond the time T_0.

As an example, consider

$$\begin{aligned} u_0(s) &= s^3 - 3s^2 + 4, \quad 0 \le s \le 2. \\ h'(s) &= 3(s^2 - 2s) < 0, \quad 0 < s < 2 \\ h''(s) &= 3(2s - 2) = 0 \text{ for } s = 1, h'''(s) = 6. \end{aligned}$$

Then $h'(s)$ has a minimum at $s_0 = 1$ and $T_0 = 1/3$.

We plot the curves c_t

$$c_t : \begin{cases} x = s + t(s^3 - 3s^2 + 4), \\ u = s^3 - 3s^2 + 4, \end{cases}$$

in the Figure 1.2 for the instants $t = 0,\ 0.2,\ 0.3,\ 0.33,\ 0.333,\ 0.4$ to demonstrate the effect of blow up with the *Mathematica* program

```
u[s_]:=s^3-3s^2+4
x[s_,t_]:=s+tu[s]
h0=ParametricPlot[Evaluate[x[s,0],u[s]],{s,0,2},
PlotRange->{0,4},PlotLabel->"y=0"]
h1=ParametricPlot[Evaluate[x[s,0.2],u[s]],{s,0,2},
PlotRange->{0,4},PlotLabel->"y=0.2"]
h2=ParametricPlot[Evaluate[x[s,0.3],u[s]],{s,0,2},
PlotRange->{0,4},PlotLabel->"y=0.3"]
h3=ParametricPlot[Evaluate[x[s,0.33],u[s]],{s,0,2},
PlotRange->{0,4},PlotLabel->"y=0.33"]
h4=ParametricPlot[Evaluate[x[s,0.333],u[s]],{s,0,2},
PlotRange->{0,4},PlotLabel->"y=0.333"]
h5=ParametricPlot[Evaluate[x[s,0.4],u[s]],{s,0,2},
PlotRange->{0,4},PlotLabel->"y=0.4"]
Show[GraphicsArray[{{h0,h1},{h2,h3},{h4,h5}}],
Frame->True,FrameTicks->None]
```

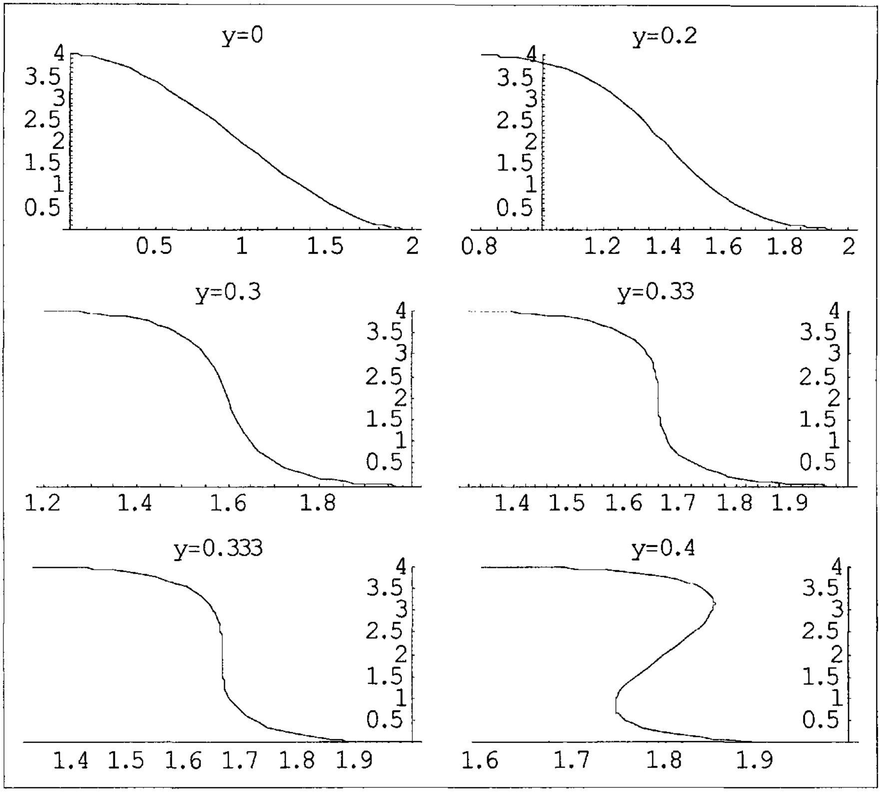

Figure 1.2. Curves c_t at the instants $t = 0, 0.2, 0.3, 0.33, 0.333, 0.4$.

Exercises

1. Prove that if two integral surfaces S_1 and S_2 of the equation (1.18) intersect transversally along a curve Γ, which means that at each point P of Γ the normal vectors $\overrightarrow{n_{P,1}}$ and $\overrightarrow{n_{P,2}}$ are linearly independent, then Γ is a characteristic curve.

2. Solve the following initial value problems:
(a) $u_x + yu_y = 2u, \quad u(1,s) = s.$
(b) $u_x + u_y = u^2, \quad u(s,0) = s^2.$
(c) $xu_x + (y + x^2)u_y = u, \quad u(2,s) = s - 4.$

3. Show that the solution of the quasi-linear PDE $u_y + a(u)u_x = 0$ with the initial condition $u(s,0) = h(s)$ is given implicitly by $u = h\left(x - a\left(u\right)y\right).$ Show that the solution becomes singular for some positive y unless $a(h(s))$ is a nondecreasing function.

1.4 General Solutions of Quasi-linear Equations

Suppose that for $P(x, y, z) \in \Omega$, $\vec{V}(a, b, c) \neq (0, 0, 0)$.

The characteristic curve

$$\Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ u = u(t) \end{cases}$$

can be represented as the intersection of two surfaces

$$\begin{aligned} \Gamma &= S_1 \cap S_2, \\ S_1 &: v(x, y, u) = c_1, \\ S_2 &: w(x, y, u) = c_2, \end{aligned} \tag{1.22}$$

for which the normal vectors $\vec{n}_1(v_x, v_y, v_u)$ and $\vec{n}_2(w_x, w_y, w_u)$ are linearly independent at each point P, which means that

$$rank \begin{bmatrix} v_x & v_y & v_u \\ w_x & w_y & w_u \end{bmatrix} = 2. \tag{1.23}$$

A continuously differentiable function $v(x, y, u)$ is said to be a *first integral* of (1.18) if it is a constant on characteristic curves.

Definition 1.1. *The first integrals $v(x, y, u)$ and $w(x, y, u)$ of* (1.18) *are functionally independent if (*1.23*) is fulfilled.*

Suppose $v(x, y, u)$ and $w(x, y, u)$ are functionally independent first integrals and (1.22) holds. From

$$\begin{aligned} v(x(t), y(t), u(t)) &= c_1, \\ w(x(t), y(t), u(t)) &= c_2, \end{aligned}$$

it follows

$$\begin{aligned} v_x\dot{x} + v_y\dot{y} + v_u\dot{u} &= 0, \\ w_x\dot{x} + w_y\dot{y} + w_u\dot{u} &= 0, \end{aligned}$$

where $\dot{x} = \dfrac{dx}{dt}$ and

$$\begin{aligned} v_x a + v_y b + v_u c &= 0, \\ w_x a + w_y b + w_u c &= 0. \end{aligned}$$

From (1.23) it follows that v and w are functionally independent first integrals iff

$$\frac{a}{\begin{vmatrix} v_y & v_u \\ w_y & w_u \end{vmatrix}} = \frac{b}{\begin{vmatrix} v_u & v_x \\ w_u & w_x \end{vmatrix}} = \frac{c}{\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}} \tag{1.24}$$

which geometricaly means that the vector $\vec{n}_1 \times \vec{n}_2$ is a tangent vector to Γ at P.

Theorem 1.4. *Let $v(x,y,u)$ and $w(x,y,u)$ be functionally independent first integrals of (1.18). Then the general solution of (1.18) is*

$$F(v(x,y,u), w(x,y,u)) = 0,$$

where F is an arbitrary continuously differentiable function of two variables.

Proof. Let $u = u(x,y)$ be a function for which

$$F(v(x,y,u(x,y)), w(x,y,u(x,y))) = 0. \tag{1.25}$$

Differentiating (1.25) with respect to x, y, we have

$$\begin{aligned} F_v(v_x + v_u u_x) + F_w(w_x + w_u u_x) &= 0, \\ F_v(v_y + v_u u_y) + F_w(w_y + w_u u_y) &= 0. \end{aligned}$$

Assuming $(F_v, F_w) \neq (0,0)$ it follows

$$\begin{vmatrix} v_x + v_u u_x & w_x + w_u u_x \\ v_y + v_u u_y & w_y + w_u u_y \end{vmatrix} = 0,$$

or

$$(v_u w_y - v_y w_u)u_x + (v_x w_u - v_u w_x)u_y = v_y w_x - v_x w_y. \tag{1.26}$$

¿From (1.26) and (1.24) it follows $au_x + bu_y = c$.

Conversely let $u = u(x,y)$ be a solution of (1.18), $v(x,y,u)$ and $w(x,y,u)$ be functionally independent first integrals of (1.18). Then , by (1.24) , it follows (1.26).

We have for the functions $V = v(x,y,u(x,y))$ and $W = w(x,y,u(x,y))$

$$\begin{aligned} \begin{vmatrix} V_x & W_x \\ V_y & W_y \end{vmatrix} &= \begin{vmatrix} v_x + v_u u_x & w_x + w_u u_x \\ v_y + v_u u_y & w_y + w_u u_y \end{vmatrix} \\ &= (v_u w_y - v_y w_u)u_x + (v_x w_u - v_u w_x)u_y - (v_y w_x - v_x w_y) \\ &= \lambda(au_x + bu_y - c) = 0. \end{aligned}$$

¿From the rank theorem of Calculus it follows that one of the functions V and W can be expressed as a function of the other, i.e. there exists a function f such that

$$v(x,y,u(x,y)) = f(w(x,y,u(x,y))). \blacksquare$$

Example 1.6. *Find the general solution of the equation*

$$(u-y)\,u_x + yu_y = x+y.$$

and solve the initial value problem $u(s,1) = 2+s$.

Solution. The characteristic system is

$$\underset{(i)}{\frac{dx}{u-y}} = \underset{(ii)}{\frac{dy}{y}} = \underset{(iii)}{\frac{du}{x+y}},$$

Using the proportion property $(i)+(iii) = (ii)$ we have

$$\frac{d(u+x)}{u+x} = \frac{dy}{y}.$$

Then

$$v = \frac{u+x}{y} = c_1$$

is a first integral. From $(i)+(ii) = (iii)$

$$\frac{d(x+y)}{u} = \frac{du}{x+y}$$

it follows that

$$w = (x+y)^2 - u^2 = c_2$$

is a second first integral. We have

$$\begin{bmatrix} v_x & v_y & v_u \\ w_x & w_y & w_u \end{bmatrix} = \begin{bmatrix} 1/y & -(x+u)/y^2 & 1/y \\ 2(x+y) & 2(x+y) & -2u \end{bmatrix}$$

and for $y \neq 0, x+y+u \neq 0$ we get the relation (1.24). The general solution is

$$F\left(\frac{u+x}{y}, (x+y)^2 - u^2\right) = 0$$

or

$$\frac{u+x}{y} = f((x+y)^2 - u^2). \tag{1.27}$$

We plot the surfaces $v = 0, 5, 10$ and $w = 0, 5, 10$ for $0 \le x \le 1, 0 \le y \le 1$ in Figure 1.3 using the *Mathematica* program

```
f0=Plot3D[-x,{x,0,1},{y,0,1},PlotPoints->10]
f1=Plot3D[5y-x,{x,0,1},{y,0,1},PlotPoints->10]
f2=Plot3D[10y-x,{x,0,1},{y,0,1},PlotPoints->10]
g1=Show[f0,f1,f2,Shading->False]
h0=Plot3D[x+y,{x,0,1},{y,0,1},PlotPoints->10]
h1=Plot3D[Sqrt[(x+y)^2+5],{x,0,1},{y,0,1},
PlotPoints->10]
h2=Plot3D[Sqrt[(x+y)^2+10],{x,0,1},{y,0,1},
PlotPoints->10]
g2=Show[h0,h1,h2,Shading->False]
Show[GraphicsArray[{g1,g2}]]
```

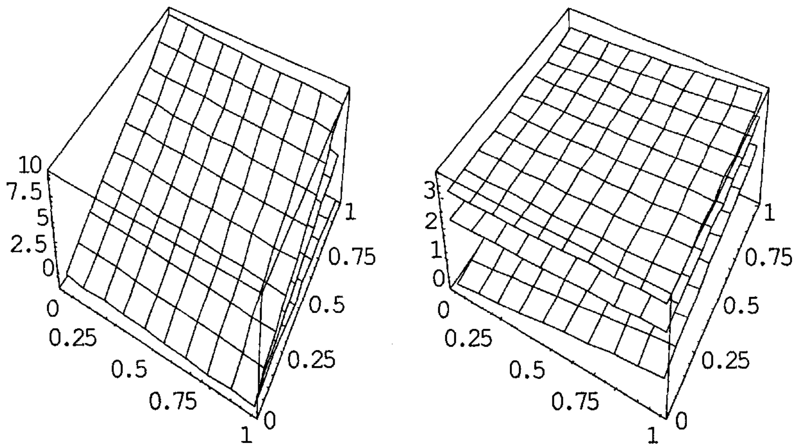

Figure 1.3. Two families of characteristics in Example 1.3.

To solve the initial value problem we substitute the initial conditions in (1.27)

$$\begin{aligned} 2 + s + s &= f((s+1)^2 - (s+2)^2), \\ 2(s+1) &= f(-2s-3), \end{aligned}$$

so $f(t) = -t - 1$ and the solution is

$$\frac{u+x}{y} + (x+y)^2 - u^2 + 1 = 0, \quad y \neq 0.$$

We now indicate (cf. Exercise 2, Section 1.2) how to proceed in the case of more than two variables. For the quasi-linear equation

$$\sum_{j=1}^{n} a_j\left(x_1, ..., x_n, u\right) u_{x_j} = c\left(x_1, ..., x_n, u\right)$$

the characteristic system is

$$\frac{dx_1}{a_1} = ... = \frac{dx_n}{a_n} = \frac{du}{c}.$$

If

$$w_1\left(x_1, ..., x_n, u\right) = c_1, ..., w_n\left(x_1, ..., x_n, u\right) = c_n$$

are n functionally independent first integrals of the characteristic system then the general solution is implicitly given by

$$F\left(w_1, ..., w_n\right) = 0,$$

where F is an arbitrary continuously differentiable function.

Exercises

1. Find the general solutions of the equations
(a) $(x-y)\, y^2 u_x - (x-y)\, x^2 u_y - \left(x^2+y^2\right) u = 0.$
(b) $(y-u)\, u_x + (u-x)\, u_y = x - y.$
(c) $x\,(y-u)\, u_x + y\,(u-x)\, u_y = (x-y)\, u.$
(d) $uu_x + (u^2 - x^2)u_y = -x.$
(e) $\left(1+\sqrt{u-x-y}\right) u_x + u_y = 2.$

2. Solve the initial value problems
(a)

$$\begin{cases} xu_x + yzu_z = 0, \\ u\,(x, y, 1) = x^y. \end{cases}$$

(b)

$$\begin{cases} (z-y)^2\, u_x + zu_y + yu_z = 0, \\ u\,(0, y, z) = 2y\,(y-z)\,. \end{cases}$$

3. The Euler PDE for a homogeneous function $u\,(x, y, z)$ is

$$xu_x + yu_y + zu_z = \alpha u.$$

Show that the initial value problem $u(x,y,1) = h(x,y)$ has a solution $u = z^{\alpha}h\left(\frac{x}{z},\frac{y}{z}\right), z \neq 0$ and $u(\lambda x, \lambda y, \lambda z) = \lambda^{\alpha}u(x,y,z)$.

4. Verify that:
(a) The general solution of the differential equation

$$u_y = \left(\frac{y}{x}u\right)_x \tag{1.28}$$

is $u = xf\left(x^2+y^2\right)$.
(b) The function

$$I(x,y) = \frac{x}{y}\int\limits_0^{\infty} e^{-y\sqrt{1+t^2}}\cos xt dt$$

satisfies the equation (1.28).
(c) The following identity is satisfied

$$\int\limits_0^{\infty} e^{-y\sqrt{1+t^2}}\cos xt dt = \frac{y}{\sqrt{x^2+y^2}}\int\limits_0^{\infty} e^{-\sqrt{(1+t^2)(x^2+y^2)}}dt, \quad y>0.$$

1.5 Fully-nonlinear First-order Equations

The general first-order equation for a function $u = u(x,y)$ has the form

$$F(x,y,u,p,q) = 0, \tag{1.29}$$

where $p = u_x, q = u_y$, F is a twice continuously differentiable function with respect to its arguments x,y,u,p,q and $F_p^2 + F_q^2 \neq 0$.

We assume now that the operator

$$u \mapsto F(x,y,u,p,q)$$

is nonlinear with respect to (p,q). In this case we say that (1.29) is a fully-nonlinear first-order equation. For instance the so called eikonal equation $u_x^2 + u_y^2 = 1$ arising in geometric optics is nonlinear because there exist u_1 and u_2 such that

$$(u_1+u_2)_x^2 + (u_1+u_2)_y^2 \neq (u_{1x}^2 + u_{1y}^2) + (u_{2x}^2 + u_{2y}^2).$$

The equation (1.29) can be viewed as a relation between the coordinates of the point $P(x,y,u)$ on an integral surface $S: u = u(x,y)$ and the direction of the normal vector $\vec{n}_P\,(p, q, -1)$ at P. The tangent plane $T_S\,(P_0)$ at the point $P_0\,(x_0, y_0, u_0) \in S$ is given by

$$u - u_0 = p\,(x - x_0) + q\,(y - y_0)\,,$$

where

$$F\,(x_0, y_0, u_0, p, q) = 0. \tag{1.30}$$

Given the values (x_0, y_0, u_0) in (1.30), different values of p therein will yield different values of q and hence a one-parameter family of tangent planes, parametrized by p. The envelope of these tangent planes is called the *Monge*[2] *cone* for (1.29) at P_0.

Recall that the envelope of a family of smooth surfaces $S_\lambda : u = G\,(x, y, \lambda)$, depending on a parameter $\lambda \in [a, b]$, is a surface Σ for which at each point $P \in \Sigma$ there exists $\lambda_0 \in [a, b]$ such that

$$T_{S_{\lambda_0}}\,(P) = T_\Sigma\,(P)\,.$$

The equation of Σ is implicitly given by the system

$$\Sigma : \begin{cases} u = G(x, y, \lambda) \\ G_\lambda(x, y, \lambda) = 0 \end{cases}. \tag{1.31}$$

In the case of the Monge cone, assuming $q = q\,(p)$ the system (1.31) is

$$M : \begin{cases} u - u_0 = p(x - x_0) + q\,(p)\,(y - y_0) \\ 0 = x - x_0 + q'\,(p)\,(y - y_0) \end{cases}. \tag{1.32}$$

Recall that a set $K \subset \mathbf{R}^3$ is said to be a *cone* with a vertex P_0 if for every $P \in K$ the point $\lambda P + (1 - \lambda)\,P_0 \in K$, for every $\lambda \in \mathbf{R}$. It is easy to see that the Monge cone is a cone with vertex P_0.

Example 1.7. *Find the equation of the Monge cone at $P_0(x_0, y_0, u_0)$ for the equation $u_x^2 + u_y^2 = 1$.*

Solution. By $p^2 + q^2 = 1$ we have $q = \pm\sqrt{1 - p^2}$ and the system (1.32) has the form

$$\begin{cases} u - u_0 = p(x - x_0) \pm \sqrt{1 - p^2}(y - y_0) \\ \\ 0 = (x - x_0) \mp \dfrac{p}{\sqrt{1 - p^2}}(y - y_0) \end{cases}$$

[2] Gaspard Monge, 10.05.1746–28.07.1818.

Taking squares of both equations and adding we get

$$(x-x_0)^2+(y-y_0)^2=(u-u_0)^2.$$

Assuming $x_0=y_0=u_0=0$ the cone, represented as an envelope of planes for which $p=0,\ 1,\ -\sqrt{2}/2$ is given in Figure 1.4, using the *Mathematica* program

```
f0=ParametricPlot3D[{uCos[v],uSin[v],u},
{u,0,1},{v,0,2Pi}, PlotPoints->20]
f1=Plot3D[y,{x,-1,1},{y,-1,1},PlotPoints->10]
f2=Plot3D[x,{x,-1,1},{y,-1,1},PlotPoints->10]
f3=Plot3D[(-x-y)/Sqrt[2],{x,-1,1},{y,-1,1},PlotPoints->10]
Show[f0,f1,f2,f3,Shading->False]
```

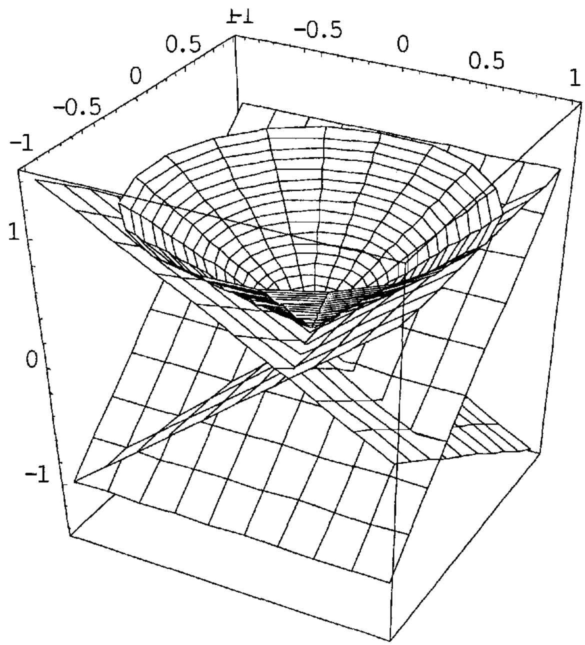

Figure 1.4. Monge cone

The equation (1.29) defines a cone field. Namely, let us consider the Monge cone $M(P)$ at each point $P(x,y,u) \in \mathbf{R}^3$. The completion of these cones $\{M(P) : P \in \mathbf{R}^3\}$ is the cone field.

A surface S in $\mathbf{R}^3$ solves the equation (1.29) iff it remains tangent to the cone $M(P)$ at each point $P \in S$.

Assuming $q = q(p)$, by (1.29), we have

$$\frac{dF}{dp} = F_p + F_q \frac{dq}{dp} = 0,$$

so that $\frac{dq}{dp}$ may be eliminated in (1.32) and the equations describing the Monge cone are

$$F(x_0, y_0, u_0, p, q) = 0,$$

$$p(x - x_0) + q(y - y_0) = u - u_0,$$

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q},$$

or

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q} = \frac{u - u_0}{pF_p + qF_q}.$$

The characteristic curves are determined as integral curves of the ODE system

$$\begin{cases} \dfrac{dx}{dt} = F_p \\ \dfrac{dy}{dt} = F_q \\ \dfrac{du}{dt} = pF_p + qF_q \end{cases} \tag{1.33}$$

or

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q}.$$

It is clear that the three equations (1.33) are not sufficient to determine the characteristic curves comprising the integral surface. The reason is that there are three equations only for the five unknown functions x, y, u, p, q. However for $p = p(x(t), y(t))$ and $q = q(x(t), y(t))$ we have

$$\begin{cases} \dfrac{dp}{dt} = p_x \dfrac{dx}{dt} + p_y \dfrac{dy}{dt} = p_x F_p + p_y F_q \\ \\ \dfrac{dq}{dt} = q_x \dfrac{dx}{dt} + q_y \dfrac{dy}{dt} = q_x F_p + q_y F_q \end{cases} \tag{1.34}$$

and from $F(x, y, u, p(x,y), q(x,y)) = 0$ it follows

$$\begin{cases} \dfrac{dF}{dx} = F_x + F_u p + F_p p_x + F_q q_x = 0 \\[2ex] \dfrac{dF}{dy} = F_y + F_u q + F_p p_y + F_q q_y = 0. \end{cases}$$

Since

$$p_y = u_{xy} = u_{yx} = q_x$$

equations (1.34) may be written as

$$\begin{cases} \dfrac{dp}{dt} = -F_x - pF_u, \\ \dfrac{dq}{dt} = -F_y - qF_u. \end{cases} \tag{1.35}$$

Equations (1.35) associated with (1.33) give a system of five ODEs for the five functions x, y, u, p, q depending on t. This system is called a *characteristic system* related to the equation (1.29). The equation (1.29) together with the characteristic system provides a system of six equations for the unknown functions $x(t), y(t), u(t), p(t), q(t)$

$$F(x, y, u, p, q) = 0, \tag{1.36}$$

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{-F_x - pF_u} = \frac{dq}{-F_y - qF_u}. \tag{1.37}$$

This system is overdetermined; however (1.36) follows from (1.37) for it is a first integral of (1.37). Indeed if $x(t), y(t), u(t), p(t), q(t)$ is a solution of (1.37) :

$$\begin{aligned} \frac{dF}{dt} &= \frac{d}{dt}F(x(t), y(t), u(t), p(t), q(t)) \\ &= F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_u\frac{du}{dt} + F_p\frac{dp}{dt} + F_q\frac{dq}{dt} \\ &= F_xF_p + F_yF_q + F_u(pF_p + qF_q) \\ &\quad + F_p(-F_x - pF_u) + F_q(-F_y - qF_u) \\ &= 0, \end{aligned}$$

which means that $F(x(t), y(t), u(t), p(t), q(t)) = const.$

If $F = 0$ is satisfied at an "initial point" $(x_0, ..., q_0)$ for $t = 0$, then the solution of (1.37) satisfies $F(x(t), y(t), u(t), p(t), q(t)) = 0$ for every t.

A solution of (1.37) can be interpreted as a *strip*. This means a space curve

$$\Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ u = u(t) \end{cases}$$

and along its point $P(x(t), y(t), u(t))$ the tangent plane $T(P)$ with the normal vector $n_P(p(t), q(t), -1)$.

Note that not any five functions define a strip. Namely, we require that the planes be tangent to the curve Γ which means that

$$\frac{du}{dt} = p(t)\frac{dx}{dt} + q(t)\frac{dy}{dt}, \tag{1.38}$$

called the *strip condition.* The strip condition is guaranteed by the system (1.37) because

$$\frac{du}{dt} = pF_p + qF_q = p(t)\frac{dx}{dt} + q(t)\frac{dy}{dt}.$$

We call the strips which are solutions of (1.37) *characteristic strips*, and their corresponding curves *characteristic curves.*

We consider the structure of integral surfaces and the initial value problem for (1.29). We formulate without proofs theorems which correspond to Theorems 1.1 and 1.3 of the quasilinear case.

Theorem 1.5. *If a characteristic strip has an element $(x_0, y_0, u_0, p_0, q_0)$ in common with an integral surface $u = u(x, y)$, then it lies completely on the surface, which means that if $(x(t), y(t), u(t), p(t), q(t))$ is a solution of (1.37) and there exists t_0 such that $x(t_0) = x_0, ..., q(t_0) = q_0$ then*

$$\begin{aligned} u(t) &= u(x(t), y(t)), \\ p(t) &= u_x(x(t), y(t)), \\ q(t) &= u_y(x(t), y(t)). \end{aligned}$$

Theorem 1.6. *Consider the PDE (1.29), where F has continuous second-order derivatives with respect to its variables $x, ..., q$ and suppose that*

$$\Gamma : \begin{cases} x = x_0(s) \\ y = y_0(s) \qquad 0 \le s \le 1 \\ u = u_0(s) \end{cases}$$

is an initial curve, $p_0(s)$, $q_0(s)$ *are two differentiable functions satisfying*

$$F(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s)) = 0,$$

$$\frac{du_0}{ds} = p_0(s)\frac{dx_0}{ds} + q_0(s)\frac{dy_0}{ds},$$

and

$$\frac{dx_0}{ds}F_q(x_0, y_0, u_0, p_0, q_0) - \frac{dy_0}{ds}F_p(x_0, y_0, u_0, p_0, q_0) \neq 0. \tag{1.39}$$

Then there exists a unique solution $u = u(x, y)$ *of* (1.29) *in a neighborhood* N' *of* $\Gamma' = \mathrm{Pr}_{Oxy}\Gamma$, *which contains the initial strip, i.e.*

$$\begin{aligned} u(x_0(s), y_0(s)) &= u_0(s), \\ u_x(x_0(s), y_0(s)) &= p_0(s), \\ u_y(x_0(s), y_0(s)) &= q_0(s). \end{aligned}$$

As before, the proofs are based on the existence and uniqueness theorem for ODEs and IMT.

Example 1.8. *Find the solution of the eikonal equation*

$$u_x^2 + u_y^2 = 1$$

through the initial curve

$$\Gamma : x = \cos s, \ y = \sin s, \ u = 1, \quad 0 \le s \le 2\pi.$$

Solution. Functions $p_0(s)$ and $q_0(s)$ such that

$$p_0^2(s) + q_0^2(s) = 1,$$

$$\frac{du_0}{ds} = 0 = p_0(s)(-\sin s) + q_0(s)(\cos s)$$

are $p_0(s) = \cos s$ and $q_0(s) = \sin s$.

For these functions the condition (1.39) is fulfilled

$$\frac{dx_0}{ds}F_q - \frac{dy_0}{ds}F_p = -2q_0 \sin s - 2p_0 \cos s = -2 \neq 0.$$

Integrating the system

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{u} = 2 \\ \dot{p} = 0 \\ \dot{q} = 0 \end{cases}$$

with initial conditions

$$\begin{aligned} x(s,0) &= \cos s, \\ y(s,0) &= \sin s, \\ u(s,0) &= 1, \\ p(s,0) &= \cos s, \\ q(s,0) &= \sin s, \end{aligned}$$

we get

$$\begin{cases} x = (2t+1)\cos s \\ y = (2t+1)\sin s \\ u = (2t+1). \end{cases}$$

Then $x^2 + y^2 = u^2$ is the integral surface for which

$$\begin{aligned} u(\cos s, \sin s) &= 1, \\ u_x(\cos s, \sin s) &= \cos s, \\ u_y(\cos s, \sin s) &= \sin s. \end{aligned}$$

The surface with Monge cones at the points $(1,0,1)$ and $(-1,0,1)$ is given in Figure 1.5 using the *Mathematica* program:

```
f0=ParametricPlot3D[{uCos[v], uSin[v],u},
{u,0,2},{v,0,2Pi}, PlotPoints->15,PlotRange->{0,2}]
f1=ParametricPlot3D[{1+uCos[v],uSin[v],1+u},
{u,0,1},{v,0,2Pi}, PlotPoints->15]
f2=ParametricPlot3D[{u Cos[v]-1,u Sin[v],1+u},
{u,0,1},{v,0,2Pi}, PlotPoints->15]
Show[f0,f1,f2,Shading->False]
```

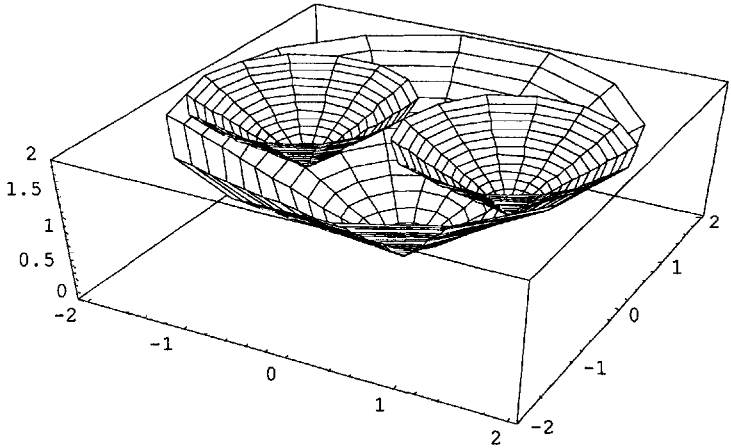

Figure 1.5. Integral surface with Monge cones.

Exercises

1. Solve the following initial value problems:

(a)

$$\begin{cases} u_x u_y = 2, \\ \Gamma : x = s, \ y = s, \ u = 3s, \quad 0 \le s \le 1. \end{cases}$$

(b)

$$\begin{cases} u_x u_y = u, \\ \Gamma : x = s, \ y = 1, \ u = s, \quad 0 \le s \le 1. \end{cases}$$

(c)

$$\begin{cases} u_x^2 + u_y^2 + 2\left(u_x - x\right)\left(u_y - y\right) - 2u = 0, \\ \Gamma : x = s, \ y = 0, \ u = 0, \quad 0 \le s \le 1. \end{cases}$$

(d)

$$\begin{cases} u_x^3 - u_y = 0, \\ \Gamma : x = s, \ y = 0, \ u = 2s\sqrt{s}, \quad 0 \le s \le 1. \end{cases}$$

(e)

$$\begin{cases} u_x + \frac{1}{2} u_y^2 = 1, \\ \Gamma : x = 0, \ y = s, \ u = s^2, \quad 0 \le s \le 1. \end{cases}$$

2. Consider the differential equation

$$u_x^2 + x u_y = 0. \tag{1.40}$$

Making the so called *Legendre*[3] *transformation*

$$v = px + qy - u,$$

where $p = u_x, q = u_y$, show that v satisfies the equation

$$p^2 + qv_p = 0.$$

Show that the solution of (1.40) can be expressed in parametric form as

$$\begin{cases} x = -\dfrac{p^2}{q} \\ y = \dfrac{p^3}{3q^2} + f'(q) \\ u = -\dfrac{p^3}{3q} + qf'(q) - f(q) \end{cases}$$

where f is an arbitrary continuously differentiable function.

[3] Adrien Marie Legendre, 18.09.1752–10.01.1833.

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Chapter 2

Second-order Partial Differential Equations

2.1 Linear Equations

The general form of a linear second-order equation in two independent variables x, y is

$$\begin{aligned} & a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d\left(x,y\right)u_x + e\left(x,y\right)u_y + f\left(x,y\right)u \\ = \; & g\left(x,y\right), \end{aligned} \tag{2.1}$$

where $a, b, c, d, e, f, g \in C^2\left(\Omega\right), \Omega \subseteq \mathbf{R}^2$ and $a^2+b^2+c^2 \neq 0$ in Ω. If we consider the partial differential operator

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) := a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f, \tag{2.2}$$

then the equation (2.1) is written as

$$Lu = g,$$

while the homogeneous equation corresponding to (2.1) is

$$Lu = 0. \tag{2.3}$$

The operator L is linear since the condition (1.2) is satisfied for every pair of functions $u_1, u_2 \in C^2(\Omega)$ and any constants $c_1, c_2 \in \mathbf{R}$. From the linearity of the operator it follows that if

$$u_1, ..., u_n,$$

are solutions of the homogeneous equation (2.3), then for every choice of constants $c_1, ..., c_n$ the function

$$c_1u_1 + ... + c_nu_n$$

is also a solution of (2.3). Furthermore, if u_p is a particular solution of Eq. (2.1), then

$$L(c_1u_1 + ... + c_nu_n + u_p) = L(c_1u_1 + ... + c_nu_n) + Lu_p = Lu_p = g.$$

Thus

$$u = c_1u_1 + ... + c_nu_n + u_p$$

is also a solution of Eq. (2.1) for every choice of constants $c_1, ..., c_n$.

We shall now consider the simplest case when the coefficients in Eq. (2.1) are real constants. Assume also that the given function g is a real-valued analytic function in Ω. Then in some cases we can obtain the *general solution* of Eq. (2.3), i.e. a relation involving two arbitrary $C^2(\Omega)$ functions such that for every choice of the arbitrary functions a solution of Eq. (2.3) results. If u_h denotes the general solution of the homogeneous equation (2.3) and u_p is any particular solution of the inhomogeneous equation (2.1), then

$$u = u_h + u_p$$

is termed the *general solution of the inhomogeneous equation.*

We classify linear differential operators $L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ into two types which we shall study separately. We say that:

(i) $L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is *reducible* or *factorable* if it can be written as a product of linear first-order factors of the form $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c$.

(ii) $L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is *irreducible* or *non-factorable* if it cannot be so written.

(i) Reducible Equations

In this case the general solution can be found with the aid of results of Section 1.2. Suppose L is such that

$$\begin{aligned} L &= L_1 L_2 \\ &= \left(a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1\right)\left(a_2 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + c_2\right). \end{aligned} \tag{2.4}$$

Since the coefficients are constants and $\dfrac{\partial^2}{\partial x \partial y} = \dfrac{\partial^2}{\partial y \partial x}$, the operators L_1, L_2 *commute*, i.e. $L_1 L_2 = L_2 L_1$. If u_1 is a solution of the linear first-order equation $L_1 u = 0$, then

$$L u_1 = (L_1 L_2)\, u_1 = (L_2 L_1)\, u_1 = L_2 (L_1 u_1) = L_2 (0) = 0,$$

that is, u_1 is a solution of (2.3). Similarly if u_2 is a solution of $L_2 u = 0$, then u_2 is a solution of (2.3). Since L is a linear operator, then $u = u_1 + u_2$ is also a solution. Accordingly, if $a = a_1 a_2 \neq 0$ and the factors L_1, L_2 are distinct, then the general solution of (2.3) is given by

$$u_h = e^{-\frac{c_1}{a_1} x} \varphi (b_1 x - a_1 y) + e^{-\frac{c_2}{a_2} x} \psi (b_2 x - a_2 y), \tag{2.5}$$

where φ and ψ are arbitrary twice continuously differentiable functions. If $L_1 = L_2$, that is

$$L = L_1 L_1 = \left(a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1\right)^2,$$

then the general solution is

$$u_h = e^{-\frac{c_1}{a_1} x} \left(x \varphi (b_1 x - a_1 y) + \psi (b_1 x - a_1 y)\right). \tag{2.6}$$

The operator L is always reducible when it is a *homogeneous operator*, that is, of the form

$$L = a \frac{\partial^2}{\partial x^2} + 2b \frac{\partial^2}{\partial x \partial y} + c \frac{\partial^2}{\partial y^2}.$$

If $a \neq 0$ and λ_1, λ_2 are the roots of the quadratic equation

$$a\lambda^2 + 2b\lambda + c = 0,$$

then

$$L = a\left(\frac{\partial}{\partial x} - \lambda_1 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \lambda_2 \frac{\partial}{\partial y}\right).$$

If $a = 0$, then

$$L = \frac{\partial}{\partial y}\left(2b\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}\right).$$

Note that the roots λ_1, λ_2 are real iff $b^2 - ac \geq 0$.

Example 2.1. *Find the general solution of the equation*

$$u_{xx} + u_x = u_{yy} + u_y.$$

Solution. This equation is written as $Lu = 0$, where L is the operator

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

The operator reduces to

$$L = L_1 L_2 = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 1\right)$$

and according to (2.5) the general solution is

$$u = \varphi(x + y) + e^{-x}\psi(x - y),$$

which may also be written in the form

$$\begin{aligned} u &= \varphi(x + y) + e^{-x}e^{x-y}h(x - y) \\ &= \varphi(x + y) + e^{-y}h(x - y), \end{aligned}$$

where φ, ψ and h are arbitrary functions.

A linear second-order equation in n independent variables $x_1, ..., x_n$ has the form

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = G. \tag{2.7}$$

If we consider the operator

$$L = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i} + C,$$

then Eq. (2.7) is written as

$$Lu = G$$

and the corresponding homogeneous equation is

$$Lu = 0. \tag{2.8}$$

Assume that the coefficients A_{ij}, B_i, C in L are real numbers and $A_{ij} = A_{ji}, i, j = 1, ..., n$. When L is reducible

$$\begin{aligned} L &= L_1 L_2 \\ &= \left(a_1 \frac{\partial}{\partial x_1} + ... + a_n \frac{\partial}{\partial x_n} + c\right)\left(b_1 \frac{\partial}{\partial x_1} + ... + b_n \frac{\partial}{\partial x_n} + d\right), \end{aligned}$$

then we can work as in the case of two independent variables. Accordingly, the general solution of Eq. (2.8) is

$$\begin{aligned} u_h &= e^{-\frac{c}{a_1} x_1} \varphi\left(a_2 x_1 - a_1 x_2, ..., a_n x_1 - a_1 x_n\right) \\ &\quad + e^{-\frac{d}{b_1} x_1} \psi\left(b_2 x_1 - b_1 x_2, ..., b_n x_1 - b_1 x_n\right), \end{aligned}$$

where φ, ψ are arbitrary functions. If either a_1 or b_1 is zero, the form of the general solution is modified appropriately. The general solution of the inhomogeneous equation (2.7) is

$$u = u_h + u_p,$$

where u_p is a particular solution.

(ii) Irreducible Equations

When the operator $L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is irreducible it is not always possible to find the general solution, but it is possible to construct solutions which contain as many arbitrary constants as we wish. This is achieved by attempting exponential type solutions of the form

$$u = e^{\alpha x + \beta y},$$

where α and β are constants to be determined. Since

$$\frac{\partial u}{\partial x} = \alpha u, \quad \frac{\partial u}{\partial y} = \beta u,$$

it is easy to see that

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) e^{\alpha x + \beta y} = L(\alpha, \beta)\, e^{\alpha x + \beta y}$$

and therefore $u = e^{\alpha x + \beta y}$ is a solution of the homogeneous equation (2.3), when

$$L(\alpha, \beta) = 0.$$

Suppose that the last relation is solved for β so as to obtain a functional relationship $\beta = h(\alpha)$. Then the function

$$u = e^{\alpha x + h(\alpha) y}$$

is a solution of (2.3). Also

$$u = \varphi(\alpha)\, e^{\alpha x + h(\alpha) y},$$

for arbitrary choice of the function φ is a solution. More generally the superpositions

$$u = \sum_{\alpha} \varphi(\alpha)\, e^{\alpha x + h(\alpha) y}, \quad u = \int \varphi(\alpha)\, e^{\alpha x + h(\alpha) y} d\alpha$$

are solutions whenever they define $C^2(\Omega)$ functions, and differentiation within the summation sign or within the integral sign is legitimate. The preceding ideas extend to Eq. (2.8) when the coefficients are constants.

As an example, let us consider the heat equation

$$u_{xx} - \frac{1}{k} u_t = 0, \quad k > 0 \text{ constant.} \tag{2.9}$$

The operator

$$L = \frac{\partial^2}{\partial x^2} - \frac{1}{k}\frac{\partial}{\partial t}$$

is irreducible. Looking for solutions of the form $u = e^{\alpha x+\beta t}$, we obtain

$$\alpha^2 - \frac{1}{k}\beta = 0.$$

Thus $\beta = k\alpha^2$, and for any value of α the function

$$u = e^{\alpha x + k\alpha^2 t}$$

is a solution. If we take $\alpha = in$, then the function

$$u = e^{inx - kn^2 t}$$

is a solution and also superpositions of the form

$$\begin{aligned} u &= \sum_{n=1}^{\infty} C_n e^{inx-kn^2 t} \\ &= \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx\right) e^{-kn^2 t}, \end{aligned}$$

are solutions of Eq. (2.9).

Exercises

1. Check whether the operators in the following equations are reducible and in the case they are find the general solution
(a) $u_{tt} - c^2 u_{xx} = 0$,
(b) $u_{xx} - \frac{1}{k}u_t = 0, \quad k > 0$,
(c) $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$,
(d) $u_{tt} = au_{xx} + 2bu_{xy} + cu_{yy}, \quad a, b, c$ positive constants and $b^2 - ac = 0$.

2. Find solutions of the exponential type $e^{\alpha x + \beta y}$ for the equations
(a) $u_{xx} - \frac{1}{c^2}u_{tt} = 0$,
(b) $u_{xx} - \frac{1}{k}u_t = 0$,
(c) $u_{xx} + u_{yy} = 0$.

3. Find the general solution of the equation

$$u_{xx} - u_{tt} = t + e^{2x}.$$

4. Show that

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left[e^{\alpha x+\beta y}\varphi(x,y)\right] = e^{\alpha x+\beta y} L\left(\frac{\partial}{\partial x} + \alpha, \frac{\partial}{\partial y} + \beta\right)\varphi(x,y).$$

Then find a particular solution of the equations
(a) $u_{xx} - u_t = e^{3x+2t}$,
(b) $u_{xx} - u_t = e^{2x+4t}$,
(c) $u_{xx} - u_t = A\cos(\alpha x + \beta t)$,
(d) $u_{xx} - u_t = Ax^2 + Bxt + C$.

5. (a) Using the change of independent variables

$$\xi = \ln x, \quad \eta = \ln y,$$

show that the equation

$$ax^2 u_{xx} + 2bxyu_{xy} + cy^2 u_{yy} + dxu_x + eyu_y + fu = 0,$$

where a, b, c, d, e, f are constants, is transformed into an equation with constant coefficients.

(b) Find the general solution of the equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

2.2 Classification and Canonical Forms of Equations in Two Independent Variables

Consider the *linear* equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{2.10}$$

and the *almost-linear* equation in two variables

$$au_{xx} + 2bu_{xy} + cu_{yy} + F(x, y, u, u_x, u_y) = 0, \tag{2.11}$$

where $a, ..., g$ are of class $C^2(\Omega)$, $\Omega \subseteq \mathbf{R}^2$ is a domain and $(a, b, c) \neq (0,0,0)$ in Ω.

The expression

$$au_{xx} + 2bu_{xy} + cu_{yy}$$

is called the *principal part* of each of these equations. Since the principal part mainly determines the properties of solutions we shall classify the more general form (2.11) instead of (2.10).

The function Δ defined by

$$\Delta(x,y) = b^2(x,y) - a(x,y)\,c(x,y)$$

is called the *discriminant* of Eq. (2.11).

The sign of the discriminant is invariant under invertible transformations of variables.

Theorem 2.1. *Let*

$$\Phi : \begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases}$$

be a smooth change of variables, for which

$$J\Phi(P) = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$$

and equation (2.11) *is transformed into*

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + \Psi(\xi,\eta,u,u_\xi,u_\eta) = 0. \tag{2.12}$$

Then the sign of the discriminant at $Q = \Phi(P)$ is the same as at P.

Proof. Making the change of variables we have:

$$\begin{aligned}
u_x &= u_\xi\xi_x + u_\eta\eta_x, \\
u_y &= u_\xi\xi_y + u_\eta\eta_y, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}, \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}.
\end{aligned}$$

Substituting in (2.11) we obtain the equation (2.12) where:

$$\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\
B &= a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y, \\
C &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2.
\end{aligned}$$

Applying the MAPLE procedure *Simplify* we have

$$\begin{aligned}\Delta' &= B^2 - AC = -\left(-\xi_x\eta_y + \eta_x\xi_y\right)^2\left(-b^2 + ca\right) \\ &= \left(\xi_x\eta_y - \eta_x\xi_y\right)^2\left(b^2 - ac\right) \\ &= J^2\Phi\left(P\right)\Delta.\end{aligned}$$

Since $J\Phi\left(P\right) \neq 0$ the proof is complete. ∎

¿From the above it is clear that we can classify Eq. (2.11) according to the sign of the discriminant.

Definition 2.1. *We say that the equation* (2.11) *at a point* $P\left(x,y\right) \in \Omega$ *is:*

(i) hyperbolic, if $\Delta\left(x,y\right) > 0$,
(ii) parabolic, if $\Delta\left(x,y\right) = 0$,
(iii) elliptic, if $\Delta\left(x,y\right) < 0$.

The equation is hyperbolic (parabolic, elliptic) in a subset $G \subset \Omega$ *if it is hyperbolic (parabolic, elliptic) at every point of* G.

Next we will show that we can find new coordinates ξ and η so that in terms of the new coordinates the form of Eq. (2.11) is such that its principal part is particularly simple. Then we say that the equation is in *canonical form.*

Theorem 2.2. *Assume that Eq.* (2.11) *is hyperbolic, parabolic or elliptic in a neighborhood of a point* $P_0\left(x_0,y_0\right)$. *Then there exists an invertible change of variables*

$$\Phi : \begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases},$$

defined in a neighborhood of the point $P_0\left(x_0,y_0\right)$ *such that the equation* (2.11) *can be reduced to one of the three forms, as follows:*
(i) if $P_0\left(x_0,y_0\right)$ *is a hyperbolic point*

$$u_{\xi\eta} + \Psi\left(\xi,\eta,u,u_\xi,u_\eta\right) = 0, \tag{2.13}$$

(first canonical form for hyperbolic equations);
(ii) if $P_0\left(x_0,y_0\right)$ *is a parabolic point*

$$u_{\eta\eta} + \Psi\left(\xi,\eta,u,u_\xi,u_\eta\right) = 0; \tag{2.14}$$

(iii) if $P_0\left(x_0,y_0\right)$ *is an elliptic point*

$$u_{\xi\xi} + u_{\eta\eta} + \Psi\left(\xi,\eta,u,u_\xi,u_\eta\right) = 0. \tag{2.15}$$

In the case of hyperbolic equations the transformation

$$\begin{aligned}\alpha &= \xi+\eta,\\ \beta &= \xi-\eta,\end{aligned}$$

reduces (2.13) to

$$u_{\alpha\alpha}-u_{\beta\beta}+\Theta\left(\alpha,\beta,u,u_\alpha,u_\beta\right)=0,$$

called the *second canonical form for hyperbolic equations.*

Proof of Theorem 2.2. (i) Let $P_0\left(x_0,y_0\right)$ be a hyperbolic point. We choose ξ and η in order to have

$$\begin{aligned}A &= a\xi_x^2+2b\xi_x\xi_y+c\xi_y^2=0,\\ C &= a\eta_x^2+2b\eta_x\eta_y+c\eta_y^2=0.\end{aligned}$$

So ξ and η are solutions of the first-order nonlinear equation

$$a\varphi_x^2+2b\varphi_x\varphi_y+c\varphi_y^2=0. \tag{2.16}$$

By the theory of Section 1.5

$$\frac{d\varphi}{dt}=pF_p+qF_q=2\left(ap^2+2bpq+cq^2\right)=0,$$

so along the characteristics of (2.16)

$$\varphi\left(x,y\right)=const. \tag{2.17}$$

If we suppose $\varphi_y\left(x_0,y_0\right)\neq 0$ we can determine $y=y\left(x\right)$ as an implicit function in a neighborhood of the point x_0 and

$$y'=\frac{dy}{dx}=-\frac{\varphi_x\left(x,y\right)}{\varphi_y\left(x,y\right)}.$$

By (2.16) the function $y\left(x\right)$ satisfies the ODE

$$ay'^2-2by'+c=0. \tag{2.18}$$

If we suppose $\varphi_x\left(x_0,y_0\right)\neq 0$ we can determine $x=x\left(y\right)$ as an implicit function in a neighborhood of the point y_0 and

$$x'=\frac{dx}{dy}=-\frac{\varphi_y\left(x,y\right)}{\varphi_x\left(x,y\right)}.$$

Then the function $x(y)$ satisfies the ODE

$$cx'^2 - 2bx' + a = 0. \tag{2.19}$$

Both equations (2.18) and (2.19) can be presented in the differential form

$$a\,(dy)^2 - 2bdxdy + c\,(dx)^2 = 0. \tag{2.20}$$

Without loss of generality we can suppose $a\,(x_0, y_0) \neq 0$ or $c\,(x_0, y_0) \neq 0$, because if $a\,(x_0, y_0) = c\,(x_0, y_0) = 0$, then $b\,(x_0, y_0) \neq 0$ and dividing (2.11) by $b\,(x_0, y_0)$ we obtain the form (2.13).

Let us suppose $a\,(x_0, y_0) \neq 0$ and $a\,(x, y) \neq 0$ in a neighborhood $\mathcal{N}$ of the point (x_0, y_0). The equation (2.18) reduces to two ODEs

$$y_1' = \frac{b + \sqrt{\Delta}}{a}, \quad y_2' = \frac{b - \sqrt{\Delta}}{a}, \quad \Delta = b^2 - ac. \tag{2.21}$$

Suppose $\xi(x, y) = C_1$ and $\eta(x, y) = C_2$ are respectively their general solutions defined in a domain $\mathcal{N}_1 \subset \mathcal{N}$. Then

$$\xi_y \neq 0 \text{ and } \eta_y \neq 0 \text{ for } (x, y) \in \mathcal{N}_1.$$

The change of variables

$$\Phi : \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases}$$

reduces (2.11) to the form (2.13). It is invertible, because by

$$\begin{aligned} y_1' &= -\frac{\xi_x}{\xi_y} = \frac{b + \sqrt{\Delta}}{a}, \\ y_2' &= -\frac{\eta_x}{\eta_y} = \frac{b - \sqrt{\Delta}}{a}, \end{aligned}$$

it follows

$$\xi_x \eta_y - \xi_y \eta_x = -\frac{2\sqrt{\Delta}}{a} \xi_y \eta_y \neq 0.$$

The case $c\,(x_0, y_0) \neq 0$ is treated similarly.

Next we describe the parabolic and elliptic cases.

(ii) Let $P_0\,(x_0, y_0)$ be a parabolic point. We should choose ξ and η such that $A = B = 0$. Since $b^2 - ac = 0$ it follows that one of the two coefficients a or c is not zero. Otherwise b should also be zero which contradicts $(a, b, c) \neq (0, 0, 0)$. If $a \neq 0$ equation (2.20) reduces to

$$y' = \frac{b}{a}.$$

Suppose its general solution is

$$\xi(x, y) = K.$$

Take $\eta = \eta(x, y)$ a simple function such that

$$J\Phi(P) = \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0.$$

Then $A = B = 0$ and $C \neq 0$. The change of variables

$$\Phi : \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases}$$

reduces (2.11) to the canonical form (2.14). In the case that $a = 0$, then $c \neq 0$ and we follow a similar procedure.

(iii) Let $P_0(x_0, y_0)$ be an elliptic point. We should choose ξ and η such that $A = C$ and $B = 0$. Since $b^2 - ac < 0$ it follows that $a \neq 0$ and (2.20) reduces to ODEs of the complex form

$$y_1' = \frac{b + i\sqrt{-\Delta}}{a}, \quad y_2' = \frac{b - i\sqrt{-\Delta}}{a}.$$

Let $\varphi(x, y) = \xi(x, y) + i\eta(x, y) = K$ be the general solution of the first equation. By (2.16) it follows that $A = C$ and $B = 0$. Then the change of variables

$$\Phi : \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases}$$

reduces (2.11) to the form (2.15). ■

The equation (2.20) is called the *characteristic equation* of (2.11), while its solutions are *characteristics*. In the hyperbolic domain the equation (2.11) admits two families of real characteristics, which intersect transversally. In the parabolic domain the equation (2.11) admits one family of real characteristics, while in the elliptic domain it has no real characteristics.

Example 2.2. *Determine the type of the equation*

$$x^2u_{xx} - y^2u_{yy} - 2yu_y = 0,$$

reduce it to the canonical form in the hyperbolic domain and find the general solution.

Solution . The discriminant is $b^2 - ac = x^2y^2$ and the equation is hyperbolic in $\mathbf{R}^2 \backslash \{(x, y) : x = 0, y = 0\}$. On the lines $x = 0$ and $y = 0$ the equation is parabolic.

Let us consider the hyperbolic domain. We apply MAPLE procedures of ScientifiCWorkPlacE to realize the algorithm of canonization. Namely

1. Solution of characteristic equation. Replacing y' by λ in (2.18) we derive the equation

$$a\lambda^2 - 2b\lambda + c = 0.$$

For

$$a = x^2, \quad b = 0, \quad c = -y^2$$

applying *Solve* to the last equation we get the solutions

$$\lambda_1 = \frac{y}{x}, \quad \lambda_2 = -\frac{y}{x}.$$

2. Applying *Solve ODE* we find solutions of equations

$$\frac{dy}{dx} = \frac{y}{x}, \quad \text{exact solution is } \frac{y}{x} = C_1$$

and

$$\frac{dy}{dx} = -\frac{y}{x}, \quad \text{exact solution is } xy = C_2.$$

3. The new variables are

$$\begin{cases} \xi = \dfrac{y}{x} \\ \eta = xy. \end{cases}$$

3.1. For ξ and η we apply *VectorCalculus+Jacobian,*

$$\left(\frac{y}{x}, xy\right), \quad \text{Jacobian is} \begin{bmatrix} -\dfrac{y}{x^2} & \dfrac{1}{x} \\ y & x \end{bmatrix}.$$

3.2. For ξ and η we apply *VectorCalculus+Hessian*

$$y/x, \quad \text{Hessian is} \begin{bmatrix} 2\dfrac{y}{x^3} & -\dfrac{1}{x^2} \\ -\dfrac{1}{x^2} & 0 \end{bmatrix},$$

$$xy, \quad \text{Hessian is} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3.3. Denote

$$\alpha = -\frac{1}{x^2}y, \quad \beta = \frac{1}{x}, \quad \gamma = y, \quad \delta = x,$$

$$\alpha_1 = \frac{2}{x^3}y, \quad \alpha_2 = -\frac{1}{x^2}, \quad \alpha_3 = 0,$$

$$\beta_1 = 0, \quad \beta_2 = 1, \quad \beta_3 = 0.$$

Applying *Simplify* we compute:

$$u_x = u_\xi \alpha + u_\eta \gamma = y\frac{-u_\xi + u_\eta x^2}{x^2},$$

$$u_y = u_\xi \beta + u_\eta \delta = \frac{u_\xi + u_\eta x^2}{x},$$

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\alpha^2 + 2u_{\xi\eta}\alpha\gamma + u_{\eta\eta}\gamma^2 + u_\xi \alpha_1 + u_\eta \beta_1 \\ &= y\frac{u_{\xi^2}y - 2u_{\xi\eta}yx^2 + u_{\eta^2}yx^4 + 2u_\xi x}{x^4}, \end{aligned}$$

$$\begin{aligned} u_{xy} &= u_{\xi\xi}\alpha\beta + u_{\xi\eta}(\alpha\delta + \beta\gamma) + u_{\eta\eta}\gamma\delta + u_\xi\alpha_2 + u_\eta\beta_2 \\ &= \frac{-u_{\xi^2}y + u_{\eta^2}yx^4 - u_\xi x + u_\eta x^3}{x^3}, \end{aligned}$$

$$\begin{aligned} u_{yy} &= u_{\xi\xi}\beta^2 + 2u_{\xi\eta}\beta\delta + u_{\eta\eta}\delta^2 + u_\xi\alpha_3 + u_\eta\beta_3 \\ &= \frac{u_{\xi^2} + 2u_{\xi\eta}x^2 + u_{\eta^2}x^4}{x^2}. \end{aligned}$$

Then

$$\begin{aligned} x^2u_{xx} - y^2u_{yy} - 2yu_y &= -2y\left(2u_{\xi\eta}y + u_\eta x\right) \\ &= -4u_{\xi\eta}y^2 - 2u_\eta xy \\ &= -4\xi\eta u_{\xi\eta} - 2\eta u_\eta \\ &= -4\xi\eta\left(u_{\xi\eta} + \frac{1}{2\xi}u_\eta\right) \end{aligned}$$

because from $\xi = \dfrac{y}{x}, \eta = xy$ it follows $y^2 = \xi\eta$.

The canonical form of the equation in the hyperbolic domain is

$$u_{\xi\eta} + \frac{1}{2\xi}u_\eta = 0.$$

The substitution $u_\eta = v$ reduces the last equation to the first-order equation

$$v_\xi + \frac{1}{2\xi}v = 0,$$

with general solution

$$v\left(\xi, \eta\right) = \xi^{-1/2}f\left(\eta\right).$$

Integrating with respect to η, we have

$$u\left(\xi, \eta\right) = \xi^{-1/2}\varphi\left(\eta\right) + \psi\left(\xi\right).$$

Therefore the general solution is

$$u\left(x, y\right) = \left(\frac{x}{y}\right)^{1/2}\varphi\left(xy\right) + \psi\left(\frac{y}{x}\right).$$

Example 2.3. *Determine the type of the equation*

$$y^2u_{xx} + 2xyu_{xy} + 2x^2u_{yy} + yu_y = 0$$

and reduce it to the canonical form in the elliptic domain.

Solution. As $\Delta = -x^2y^2$, the equation is elliptic if $x \neq 0, y \neq 0$. For

$$a = y^2, b = xy, c = 2x^2$$

the equation

$$a\lambda^2 - 2b\lambda + c = 0$$

has solutions

$$\lambda_1 = \frac{1}{2y^2}\left(2xy + 2ixy\right), \quad \lambda_2 = \frac{1}{2y^2}\left(2xy - 2ixy\right).$$

Now *Solve ODE*

$$\frac{dy}{dx} = \frac{1}{2y^2}\left(2xy + 2ixy\right)$$

yields the solution

$$\frac{1}{2}y^2(x) - \frac{1}{2}x^2 - \frac{1}{2}ix^2 = C_1.$$

New variables are

$$\begin{cases} \xi = y^2 - x^2, \\ \eta = -x^2. \end{cases}$$

For $\left(y^2 - x^2, -x^2\right)$, Jacobian is $\begin{bmatrix} -2x & 2y \\ -2x & 0 \end{bmatrix}$.

For $y^2 - x^2$, Hessian is $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$, for $-x^2$, Hessian is -2.

Denote:

$$\alpha = -2x, \quad \beta = 2y, \quad \gamma = -2x, \quad \delta = 0,$$

$$\alpha_1 = -2, \quad \alpha_2 = 0, \quad \alpha_3 = 2,$$

$$\beta_1 = -2, \quad \beta_2 = 0, \quad \beta_3 = 0,$$

and compute:

$$u_x = u_\xi\alpha + u_\eta\gamma = -2u_\xi x - 2u_\eta x,$$

$$u_y = u_\xi\beta + u_\eta\delta = 2u_\xi y,$$

$$\begin{aligned}
u_{xx} &= u_{\xi\xi}\alpha^2 + 2u_{\xi\eta}\alpha\gamma + u_{\eta\eta}\gamma^2 + u_\xi\alpha_1 + u_\eta\beta_1 \\
&= 4u_{\xi^2}x^2 + 8u_{\xi\eta}x^2 + 4u_{\eta^2}x^2 - 2u_\xi - 2u_\eta,
\end{aligned}$$

$$\begin{aligned}
u_{xy} &= u_{\xi\xi}\alpha\beta + u_{\xi\eta}(\alpha\delta + \beta\gamma) + u_{\eta\eta}\gamma\delta + u_\xi\alpha_2 + u_\eta\beta_2 \\
&= -4u_{\xi^2}xy - 4u_{\xi\eta}xy,
\end{aligned}$$

$$\begin{aligned}
u_{yy} &= u_{\xi\xi}\beta^2 + 2u_{\xi\eta}\beta\delta + u_{\eta\eta}\delta^2 + u_\xi\alpha_3 + u_\eta\beta_3 \\
&= 4u_{\xi^2}y^2 + 2u_\xi.
\end{aligned}$$

Then

$$\begin{aligned}
& y^2u_{xx} + 2xyu_{xy} + 2x^2u_{yy} + yu \\
= \; & 4y^2u_{\xi^2}x^2 + 4y^2u_{\eta^2}x^2 - 2y^2u_\eta + 4x^2u_\xi \\
= \; & 4y^2x^2\left(u_{\xi^2} + u_{\eta^2} - \frac{1}{2x^2}u_\eta + \frac{1}{y^2}u_\xi\right) \\
= \; & 4y^2x^2\left(u_{\xi^2} + u_{\eta^2} + \frac{1}{2\eta}u_\eta + \frac{1}{\xi-\eta}u_\xi\right).
\end{aligned}$$

The canonical form in the elliptic domain is

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{2\eta}u_\eta + \frac{1}{\xi-\eta}u_\xi = 0.$$

Exercises

1. Determine the type of the following equations and reduce them to the canonical form. Using *Mathematica* plot the two families of real characteristics in hyperbolic domains.

(a) $u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$.

(b) $u_{xx} - 6u_{xy} + 10u_{yy} + u_x - 3u_y = 0$.

(c) $u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$.

(d) $\left(1+x^2\right)u_{xx} + \left(1+y^2\right)u_{yy} + xu_x + yu_y = 0$.

(e) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} + (e^{2y} - e^{x+y})u_y = 0$.

2. Find the general solutions of the following equations in the domains of constant type.

(a) $u_{xx} - 2u_{xy} - 3u_{yy} = 0.$

(b) $3u_{xx} - 5u_{xy} - 2u_{yy} + 3u_x + u_y = 2.$

(c) $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$

(d) $x^2 u_{xx} - y^2 u_{yy} - 2y u_y = 4xy.$

3. Find PDEs, whose general solutions are of the form

(a) $u(x,y) = \varphi(x+y) + \psi(x-2y)$,

(b) $u(x,y) = x\varphi(x+y) + y\psi(x+y)$,

(c) $u(x,y) = \varphi(xy) + \psi(x/y)$,

(d) $u(x,y) = 1/x\,(\varphi(x-y) + \psi(x+y))$,where φ and ψ are arbitrary differentiable functions.

4. Consider the Tricomi[1] equation

$$yu_{xx} + u_{yy} = 0.$$

Show that this equation is:
(a) elliptic for $y > 0$ and with the change of variables

$$\xi = x, \quad \eta = \frac{2}{3} y^{\frac{3}{2}}$$

it reduces to

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta} u_\eta = 0.$$

(b) hyperbolic for $y < 0$ and with the change of variables

$$\xi = x - \frac{2}{3}(-y)^{\frac{3}{2}}, \quad \eta = x + \frac{2}{3}(-y)^{\frac{3}{2}}$$

[1] Francesco Jacopo Tricomi, 05.05.1897–21.11.1978.

it reduces to

$$u_{\xi\eta} - \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta) = 0.$$

Plot the picture of characteristics in the hyperbolic domain.

5. The Born–Infeld[2] equation is

$$(1 - \varphi_t^2)\varphi_{xx} + 2\varphi_x\varphi_t\varphi_{xt} - (1 + \varphi_x^2)\varphi_{tt} = 0. \tag{2.22}$$

Show that:

(a) Introducing new variables

$$\begin{aligned} \xi &= x - t, \\ \eta &= x + t, \\ u &= \varphi_\xi(\xi, \eta), \\ v &= \varphi_\eta(\xi, \eta) \end{aligned}$$

the equation (2.22) is equivalent to the system

$$\begin{cases} u_\eta - v_\xi = 0 \\ v^2 u_\xi - (1 + 2uv)\, u_\eta + u^2 v_\eta = 0. \end{cases} \tag{2.23}$$

(b) If $u_\eta = v_\xi$ and $\xi = \xi(u, v), \eta = \eta(u, v)$ is the inverse mapping then

$$\xi_v = \eta_u, \quad u_\xi \xi_u = v_\eta \eta_v.$$

(c) In the new variables (u, v) the system (2.23) is equivalent to the system

$$\begin{cases} \xi_v - \eta_u = 0 \\ v^2 \eta_v + (1 + 2uv)\, \xi_v + u^2 \xi_u = 0, \end{cases}$$

or to the equation

$$u^2 \xi_{uu} + (1 + 2uv)\, \xi_{uv} + v^2 \xi_{vv} + 2(u\xi_u + v\xi_v) = 0. \tag{2.24}$$

(d) Determine the hyperbolic domain of the equation (2.24) and show that the characteristics in (u, v) plane are the lines

$$u = C_1^2 v + C_1, v = C_2^2 u + C_2, u = 0, v = 0,$$

and their envelope is the hyperbola $1 + 4uv = 0$. Plot the picture of characteristics using *Mathematica*.

[2]Max Born, 11.12.1882–05.01.1970,
Leopold Infeld, 20.08.1898–15.01.1968.

2.3 Classification of Almost-linear Equations in $\mathbf{R}^n$

Let D be a domain in the $n-$dimensional Euclidean space $\mathbf{R}^n$. Denote by $x = (x_1, ..., x_n)$ a point of $\mathbf{R}^n$ and by $< .,. >$ the usual scalar product in $\mathbf{R}^n$. An *almost-linear second-order equation* in $\mathbf{R}^n$ has the form

$$\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + F(x, u, \nabla u) = 0, \tag{2.25}$$

where the coefficients $a_{ij}(x)$ are assumed to be continuously differentiable functions in x, $a_{ij}(x) = a_{ji}(x)$, $u(x)$ is an unknown function and $\nabla u = (u_{x_1}, ..., u_{x_n})$ is the gradient of u. Almost-linearity means that the equation (2.25) is linear with respect to second-order derivatives $u_{x_i x_j}$. The linear operator

$$L := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

is called the *principal part* of the operator appearing in equation (2.25). A function $u(x) \in C^2(D)$ is a solution of the equation (2.25) in D, if the substitution of u and its derivatives in (2.25) results an identity in $x \in D$.

A main requirement for a classification of the equation (2.25) is to be invariant under nonsingular changes of independent variables. As before, we make a classification locally, i.e. for a fixed point $x^0 \in D$.

Let

$$y = \phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}$$

and

$$x = \psi(y) = \begin{bmatrix} \psi_1(y) \\ \vdots \\ \psi_n(y) \end{bmatrix}$$

be nonsingular mappings defined in neighborhoods N and N' of x^0 and $y^0 = \phi(x^0)$ respectively, such that

$$y = \phi(\psi(y)), \quad y \in N' \text{ and } x = \psi(\phi(x)), \quad x \in N.$$

Let $u(x) \in C^2(D)$ be a solution of the equation (2.25) in D and

$$v(y) = u(\psi(y)), \quad y \in N'.$$

Then

$$u(x) = v(\phi(x)), \quad x \in N$$

and

$$u_{x_i} = \sum_{k=1}^{n} v_{y_k}(\phi(x)) \frac{\partial \phi_k}{\partial x_i},$$

$$u_{x_i x_j} = \sum_{k,l=1}^{n} v_{y_k y_l}(\phi(x)) \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_l}{\partial x_j} + \sum_{k=1}^{n} v_{y_k}(\phi(x)) \frac{\partial^2 \phi_k}{\partial x_i \partial x_j}.$$

A substitution in (2.25) leads to

$$\sum_{i,j=1}^{n} a_{ij}(x) \sum_{k,l=1}^{n} v_{y_k y_l}(\phi(x)) \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_l}{\partial x_j} + \Phi(y, v, \nabla v) = 0,$$

or

$$\sum_{k,l=1}^{n} b_{kl}(y) v_{y_k y_l}(y) + \Phi(y, v, \nabla v) = 0, \tag{2.26}$$

where

$$b_{kl}(y) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_l}{\partial x_j}. \tag{2.27}$$

The classification of the equation (2.25) at the point x^0 is based on the classification of the *characteristic form*

$$Q(x^0, \xi) = \sum_{i,j=1}^{n} a_{ij}(x^0) \xi_i \xi_j, \quad \xi \in \mathbf{R}^n. \tag{2.28}$$

Let A be the symmetric matrix

$$A = \begin{bmatrix} a_{11}(x^0) & \cdots & a_{1n}(x^0) \\ \vdots & & \vdots \\ a_{n1}(x^0) & \cdots & a_{nn}(x^0) \end{bmatrix}.$$

We have

$$Q\left(x^0,\xi\right) =< A\xi,\xi >$$

and if

$$\xi = \Lambda\eta, \quad \eta \in \mathbf{R}^n,$$

where Λ is a $n \times n$ matrix, then

$$Q\left(x^0,\xi\right) =< A\Lambda\eta, \Lambda\eta >=< \Lambda^T A\Lambda\eta, \eta > . \tag{2.29}$$

Here Λ^T denotes the *transpose matrix* of Λ. Let

$$\begin{aligned}
\lambda_{ki} &= \frac{\partial \phi_k}{\partial x_i}\left(x^0\right), \quad k,i = 1,...,n, \\
\Lambda &= \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{bmatrix}, \\
y &= \Lambda^T x.
\end{aligned} \tag{2.30}$$

Note that

$$y_k = \sum_{i=1}^{n} \lambda_{ki} x_i, \quad \frac{\partial y_k}{\partial x_i} = \lambda_{ki}.$$

If we make the linear change of variables (2.30) we obtain the transformed coefficients as

$$b_{kl} = \sum_{i,j=1}^{n} a_{ij}\left(x^0\right) \lambda_{ki}\lambda_{kj},$$

which coincide by (2.27). Denote by B the matrix with elements b_{kl}, i.e.

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}.$$

It is symmetric and

$$B = \Lambda^T A \Lambda.$$

By (2.29) the characteristic forms of (2.25) and the transformed equation at x^0 and $y^* = \Lambda^T x^0$ are equal:

$$Q\left(x^0, \xi\right) = Q\left(y^*, \eta\right), \quad \xi = \Lambda\eta.$$

By the basic theorem for quadratic forms there exists a nonsingular matrix Λ such that $Q\left(x^0, \xi\right)$ reduces to the *canonical form*

$$Q\left(y^*, \eta\right) = \eta_1^2 + ... + \eta_p^2 - \eta_{p+1}^2 - ... - \eta_{p+q}^2 \tag{2.31}$$

where $p \geq 0,\ q \geq 0,\ p+q \leq n$. The number p of positive terms in (2.31) is called the *positive index*, the number q of negative terms the *negative index*, $r = p+q$ - *rank* and $\nu = n - r$ - *nullity* of the characteristic form (2.28). The important statement is that these numbers are invariant with respect to nonsingular linear transformations of the variables ξ and x . Therefore the classification of (2.25) is made regardless of the canonical form of the characteristic form (2.28).

Let

$$\xi = \Lambda\eta,$$

be the nonsingular linear transformation reducing the characteristic form (2.28) to its canonical form (2.31). Then the transformation

$$y = \Lambda^T x,$$

reduces the equation (2.31) at the point x^0 to the form

$$\sum_{i=1}^{p} v_{y_i y_i} - \sum_{i=1}^{q} v_{y_{p+i} y_{p+i}} + \Phi\left(y, v, \nabla v\right) = 0,$$

where

$$v\left(y\right) = u\left(\left(\Lambda^T\right)^{-1} y\right).$$

The equation (2.25) at the point x^0 is said to be:

(1) *elliptic*, if

$$\nu = n - p - q = 0, \text{and either } p = 0 \text{ or } q = 0,$$

(2) *hyperbolic*, if

$$\nu = 0, \text{ and either } p = n - 1 \text{ and } q = 1, \text{ or } p = 1 \text{ and } q = n - 1,$$

(3) *ultrahyperbolic,* if

$$\nu = 0 \text{ and } 1 < p < n - 1,$$

(4) *parabolic*, if $\nu > 0$.

The equation (2.25) is said to be elliptic (hyperbolic, ultrahyperbolic, parabolic) in D, if it is elliptic (hyperbolic, ultrahyperbolic, parabolic) at every point of D .

The classification can also be made with respect to the eigenvalues of the coefficient matrix A, i.e. the roots of the equation

$$\begin{vmatrix} a_{11}\left(x^0\right) - \lambda & \cdots & a_{1n}\left(x^0\right) \\ \vdots & & \vdots \\ a_{n1}\left(x^0\right) & \cdots & a_{nn}\left(x^0\right) - \lambda \end{vmatrix} = 0.$$

¿From linear algebra it is known that since the matrix A is symmetric its eigenvalues are all real. Moreover the number of positive, zero and negative eigenvalues of the matrix A remains invariant under nonsingular changes of independent variables. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the matrix A of the principal part of Eq. (2.25).

The equation (2.25) at the point x^0 is said to be:

(1) *elliptic*, if $\lambda_1, ..., \lambda_n$ are nonzero and have the same sign,

(2) *hyperbolic*, if $\lambda_1, ..., \lambda_n$ are nonzero and all except one have the same sign,

(3) *ultrahyperbolic,* if $\lambda_1, ..., \lambda_n$ are nonzero and at least two of them are positive and two negative,

(4) *parabolic*, if any one of $\lambda_1, ..., \lambda_n$ is zero.

For instance, the Laplace equation

$$\Delta u := u_{x_1 x_1} + ... + u_{x_n x_n} = 0,$$

is elliptic in $\mathbf{R}^n$, the wave equation

$$u_{tt} - c^2 \Delta u = 0,$$

where c is a constant, is hyperbolic in $\mathbf{R}^{n+1}$, while the heat or diffusion equation

$$u_t - a^2 \Delta u = 0,$$

where a is a constant, is parabolic in $\mathbf{R}^{n+1}$. The equation

$$u_{x_1x_1} + u_{x_2x_2} - u_{x_3x_3} - u_{x_4x_4} - u_{x_5x_5} = 0,$$

is ultrahyperbolic in $\mathbf{R}^5$.

Example 2.4. Reduce the equation

$$2u_{x_1x_1} + 3u_{x_2x_2} - \frac{1}{2}u_{x_3x_3} + 6u_{x_1x_2} - 2u_{x_2x_3} = 0$$

to the canonical form. Determine the type and change of variables.

Solution. The characteristic form of the equation is

$$\begin{aligned} & 2\xi_1^2 + 3\xi_2^2 - \frac{1}{2}\xi_3^2 + 6\xi_1\xi_2 - 2\xi_2\xi_3 \\ = & \left(\sqrt{2}\xi_1 + \sqrt{2}\xi_2 - \frac{1}{\sqrt{2}}\xi_3\right)^2 + (\xi_1+\xi_2)^2 - (\xi_1-\xi_3)^2 \\ = & \eta_1^2 + \eta_2^2 - \eta_3^2. \end{aligned}$$

The change of variables

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

has an inverse

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 & 1 \\ \sqrt{2} & -1 & -1 \\ -\sqrt{2} & 2 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}.$$

Then the linear transformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 2 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

reduces the original equation to the canonical form

$$v_{y_1 y_1} + v_{y_2 y_2} - v_{y_3 y_3} = 0.$$

So the equation is hyperbolic on the whole space. Note that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 1 & 1 \\ \sqrt{2} & 1 & 0 \\ -\frac{1}{2}\sqrt{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

and

$$v(y_1, y_2, y_3) = u\left(\sqrt{2}y_1 + y_2 + y_3, \sqrt{2}y_1 + y_2, -\frac{1}{\sqrt{2}}y_1 - y_3\right).$$

Exercises.

1. Reduce the following equations to the canonical form.

(a) $4u_{x_1x_1} + 3u_{x_2x_2} + \frac{3}{2}u_{x_3x_3} + 2u_{x_1x_2} + 4u_{x_1x_3} + 2u_{x_2x_3} = 0$.

(b) $3u_{x_1x_1} + 4u_{x_2x_2} - \frac{2}{3}u_{x_3x_3} + 4u_{x_1x_2} + 2u_{x_2x_3} = 0$.

(c) $3u_{x_1x_1} + 2u_{x_2x_2} - 2u_{x_3x_3} - \frac{1}{2}u_{x_4x_4} + 6u_{x_1x_2} - 2u_{x_1x_4}$
$+2u_{x_2x_3} - 2u_{x_2x_4} + 2u_{x_3x_4} = 0$.

(d) $4u_{x_1x_1} + u_{x_2x_2} + \frac{1}{3}u_{x_3x_3} - 2u_{x_1x_2} + 2u_{x_1x_3} = 0$.

2. Suppose that $a_i, b_i, i = 1, ..., n$ and c are constants and $a_i \neq 0$. Find a function w such that the change of the dependent variable $u = wv$ reduces the equation

$$\sum_{i=1}^{n} a_i u_{x_i x_i} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f(x)$$

to the form

$$\sum_{i=1}^{n} a_i v_{x_i x_i} + Cu = F(x).$$

3. Classify the equation (2.11) with respect to the eigenvalues λ_1, λ_2 of the coefficent matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Compare with Definition 2.1.

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Chapter 3

One Dimensional Wave Equation

3.1 The Wave Equation on the Whole Line. D'Alembert Formula

The simplest hyperbolic second-order equation is the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{3.1}$$

where x signifies the spatial variable or "position", t the "time" variable, $u = u(x,t)$ the unknown function and c is a given positive constant. The wave equation describes vibrations of a string. Physically $u(x,t)$ represents the "value" of the normal displacement of a particle at position x and time t.

Using the theory of Section 2.2 the characteristic equation of (3.1) is

$$(dx)^2 - c^2 (dt)^2 = 0$$

and

$$\begin{cases} x + ct = c_1 \\ x - ct = c_2 \end{cases}$$

are two families of real characteristics. Introducing the new variables

$$\Phi : \begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}, \qquad \Phi^{-1} : \begin{cases} x = (\xi + \eta)/2 \\ t = (\xi - \eta)/2c \end{cases}$$

and the function

$$U(\xi,\eta) = u\left(\left(\xi+\eta\right)/2, \left(\xi-\eta\right)/2c\right),$$

the equation (3.1) reduces to

$$U_{\xi\eta}(\xi,\eta) = 0. \tag{3.2}$$

Therefore

$$U_{\xi}(\xi,\eta) = F(\xi),$$

$$U(\xi,\eta) = \int F(\xi)\,d\xi + g(\eta) = f(\xi) + g(\eta),$$

and in the original variables $u(x,t)$ is of the form

$$u(x,t) = f(x+ct) + g(x-ct), \tag{3.3}$$

known as the general solution of (3.1). It is the sum of the function $g(x-ct)$ which presents a shape traveling without change to the right with speed c and the function $f(x+ct)$ - another shape, traveling to the left with speed c.

Consider the Cauchy (initial value) problem for (3.1)

$$(CW): \begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbf{R}, \quad t > 0, \\ u(x,0) = \varphi(x) & x \in \mathbf{R}, \\ u_t(x,0) = \psi(x) & x \in \mathbf{R}, \end{cases}$$

where φ and ψ are arbitrary functions of x. Further we denote $\mathbf{R}^+ = \{t : t \geq 0\}$.

Theorem 3.3. *(D'Alembert[1] formula). If $\varphi \in C^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R})$ the problem (CW) has a unique solution $u \in C^2(\mathbf{R} \times \mathbf{R}^+)$ given by the formula*

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \tag{3.4}$$

Proof. We are looking for a solution of the problem in the form (3.3) satisfying the initial conditions at $t = 0$

[1] Jean Le Rond D'Alembert, 16.11.1717–29.10.1783.

$$f(x) + g(x) = \varphi(x), \tag{3.5}$$

$$cf'(x) - cg'(x) = \psi(x).$$

Differentiating (3.5) with respect to x and solving the linear system for f' and g', we obtain

$$f'(x) = \frac{1}{2}\varphi'(x) + \frac{1}{2c}\psi(x), \tag{3.6}$$

$$g'(x) = \frac{1}{2}\varphi'(x) - \frac{1}{2c}\psi(x). \tag{3.7}$$

Integrating (3.6) and (3.7) from 0 to x we get

$$f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds + \left(f(0) - \frac{1}{2}\varphi(0)\right),$$

$$g(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds + \left(g(0) - \frac{1}{2}\varphi(0)\right).$$

By (3.5) $f(0) + g(0) = \varphi(0)$. Therefore

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) \\ &\quad + \frac{1}{2c}\left(\int_0^{x+ct} \psi(s)\,ds - \int_0^{x-ct} \psi(s)\,ds\right) \\ &= \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \end{aligned}$$

Conversely it is easy to see that for $\varphi \in C^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R})$ this formula gives the solution $u \in C^2(\mathbf{R} \times \mathbf{R}^+)$ of (CW). Note that if $\varphi = \psi = 0$, then it follows $u = 0$. ■

Some corollaries from D'Alembert formula are as follows:

1. Domain of dependence. The value of u at (x_0, t_0) is determined by the restriction of initial functions φ and ψ in the interval $[x_0 - ct_0, x_0 + ct_0]$ on the $x-$axis, whose end-points are cut out by the characteristics:

$$x - x_0 = \pm c\,(t - t_0)\,,$$

through the point (x_0, t_0).

The characteristic triangle $\Delta\,(x_0, t_0)$ is defined as the triangle in $\mathbf{R} \times \mathbf{R}^+$ with vertices

$$A_0\,(x_0 - ct_0, 0)\,, \quad B_0\,(x_0 + ct_0, 0)\,, \quad P_0\,(x_0, t_0)\,.$$

For every $(x_1, t_1) \in \Delta\,(x_0, t_0)$

$$[x_1 - ct_1, x_1 + ct_1] \subset [x_0 - ct_0, x_0 + ct_0],$$

$$\Delta\,(x_1, t_1) \subset \Delta\,(x_0, t_0)$$

and $u\,(x_1, t_1)$ is determined by the values of φ and ψ on $[x_1 - ct_1, x_1 + ct_1]$.

2. Domain of influence. The point $(x_0, 0)$ on the $x-$axis influences the value of u at (x, t) in the wedge-shaped region

$$I\,(x_0) = \{(x, t) : x_0 - ct \leq x \leq x_0 + ct, \quad t \geq 0\}.$$

For any

$$P_1\,(x_1, t_1) \in I\,(x_0)\,, \quad \Delta\,(x_1, t_1) \cap I\,(x_0) \neq \emptyset,$$

$$P_1\,(x_1, t_1) \notin I\,(x_0)\,, \quad \Delta\,(x_1, t_1) \cap I\,(x_0) = \emptyset.$$

3. Well-posedness.

The problem (CW) is *well-posed* in the sense of Hadamard[2] if the following three requirements are satisfied:

(i) There exists a solution;

(ii) The solution is unique;

(iii) The solution is stable.

Statement (iii) means that *small variations* of the initial data yield small variations on the corresponding solutions. This is also referred to as continuous

[2] Jacques Hadamard, 18.12.1865–17.10.1963.

dependence upon the initial data. The meaning of small variation is made precise in terms of the topology suggested by the problem. A problem that does not satisfy any one of these conditions is called *ill-posed.*

For $v(x) \in C(\mathbf{R})$ and $w(x,t) \in C(\mathbf{R} \times [0,\infty))$ introduce uniform norms

$$||v||_\infty = \sup_{x\in\mathbf{R}} |v(x)|$$

and

$$||w||_{\infty,T} = \sup_{x\in\mathbf{R},0\leq t\leq T} |w(x,t)|.$$

For a given $T > 0$ by (3.4) it follows

$$\begin{aligned} ||u||_{\infty,T} &\leq \frac{1}{2}(||\varphi||_\infty + ||\varphi||_\infty) + \frac{1}{2c}||\psi||_\infty \int_{x-ct}^{x+ct} ds \\ &\leq ||\varphi||_\infty + T||\psi||_\infty. \end{aligned}$$

Then for any $\varepsilon > 0$ there exists $\delta \in \left(0, \dfrac{\varepsilon}{1+T}\right)$ such that if $||\varphi||_\infty < \delta$ and $||\psi||_\infty < \delta$ it follows $||u||_{\infty,T} < \varepsilon$, which proves the continuous dependence.

Example 3.3. *Solve the problem* (CW) *with* $c = 1$, $\psi = 0$ *and*

$$\varphi(x) = \begin{cases} \cos^3 x & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0 & \text{if } x \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \end{cases}$$

Solution. The solution of the problem is $u(x,t) = \frac{1}{2}(\varphi(x+t) + \varphi(x-t))$. Using *Mathematica* the profile of $u(x,t)$ is presented in Figure 3.1 at successive instants $t = 0,\ 1,\ \frac{\pi}{2},\ 3,\ 4,\ 5$. Note that at $t = 0$ the amplitude is 1. After the instant $t = \frac{\pi}{2}$ the profile breaks up into two traveling waves moving in opposite directions with speed 1 and amplitude $\frac{1}{2}$. The surface $u = u(x,t)$ is presented in Figure 3.2. We use the *Mathematica* program

```
f[x_]:=Which[-Pi/2<=x<=Pi/2,Cos[x]^3,True,0]
u[x_,t_]:=(f[x+t]+f[x-t])/2
h0=Plot[Evaluate[u[x,0]],{x,-8,8},
PlotRange->{0,1},PlotLabel->"Wave at t=0"]
h1=Plot[Evaluate[u[x,1]],{x,-8,8},
PlotRange->{0,1},PlotLabel->"Wave at t=1"]
h2=Plot[Evaluate[u[x,2]],{x,-8,8},
```

```
PlotRange->{0,1},PlotLabel->"Wave at t=Pi/2"]
h3=Plot[Evaluate[u[x,3]],{x,-8,8},
PlotRange->{0,1},PlotLabel->"Wave at t=3"]
h4=Plot[Evaluate[u[x,4]],{x,-8,8},
PlotRange->{0,1},PlotLabel->"Wave at t=4"]
h5=Plot[Evaluate[u[x,5]],{x,-8,8},
PlotRange->{0,1},PlotLabel->"Wave at t=5"]
Show[GraphicsArray[{{h0,h1},{h2,h3},{h4,h5}}],
Frame->True,FrameTicks->None]
Plot3D[u[x,t],{x,-8,8},{t,0,5},PlotPoints->40,
AxesLabel->"Position","Time","Value",
PlotRange->{0,1},Shading->False]
```

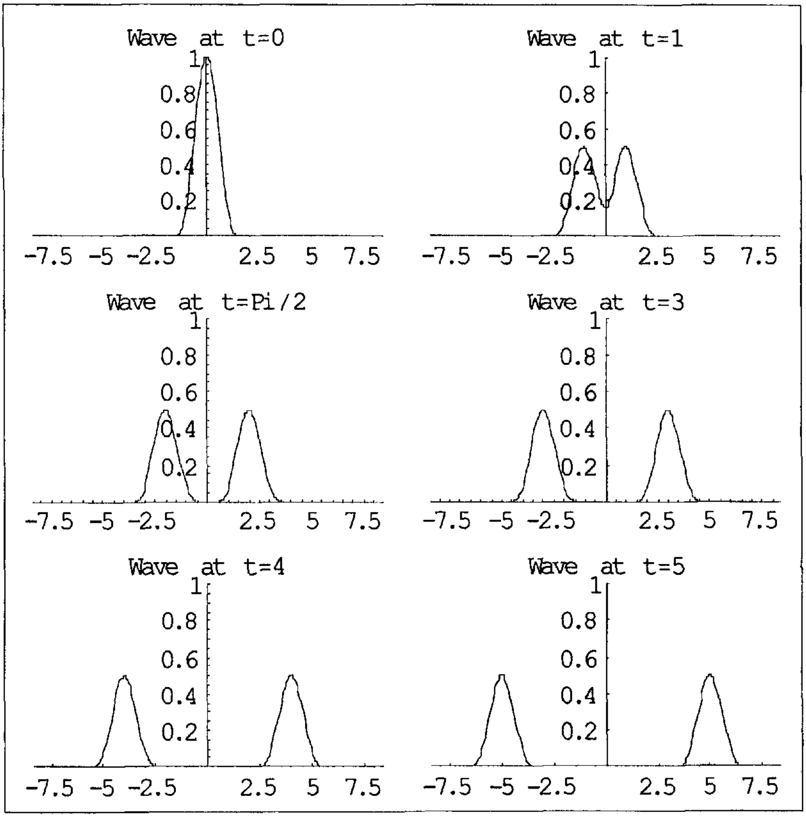

Figure 3.1. The wave at instants $t = 0,\ 1,\ \frac{\pi}{2},\ 3,\ 4,\ 5$.

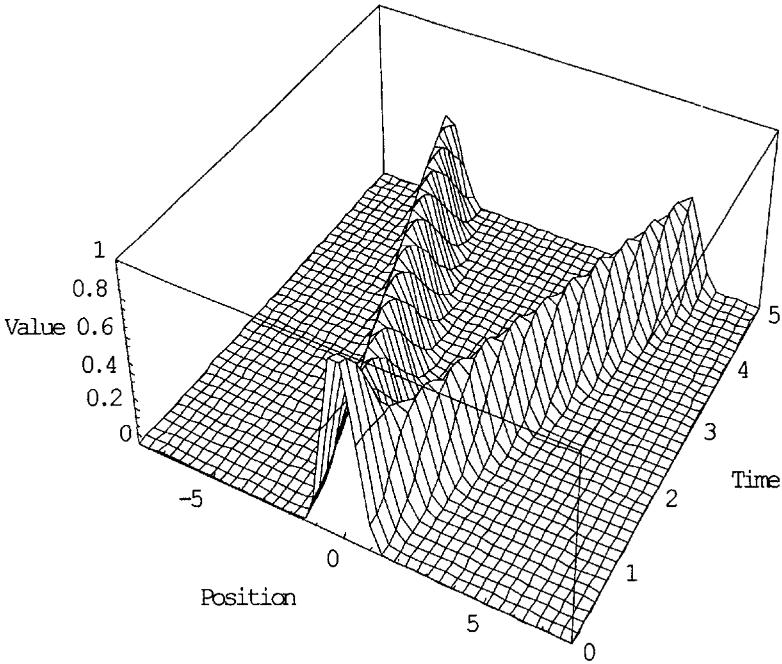

Figure 3.2. Graph of the function $u = u(x,t)$ in Example 3.3.

Example 3.4. *Solve the problem (CW) with* $c = \sqrt{\pi}, \varphi = 0$ *and* $\psi(x) = e^{-x^2}$.

Solution. Let $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-s^2} ds$ be the *error function* used in statistics. The solution can be expressed in terms of erf as

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi}} \int\limits_{x-\sqrt{\pi}t}^{x+\sqrt{\pi}t} e^{-s^2} ds \\ &= \frac{1}{4}\left(\operatorname{erf}\left(x+\sqrt{\pi}t\right) - \operatorname{erf}\left(x-\sqrt{\pi}t\right)\right). \end{aligned}$$

Using *Mathematica* the profile of $u(x,t)$ is presented in Figure 3.3 at the successive instants $t = 0,\ 1,\ 2,\ 3$. Note that at $t = 1$ the amplitude is $1/2$ and it remains the same for all next instants.The surface $u = u(x,t)$ is plotted in Figure 3.4. We use the following program

```
u[x_,t_]:=(Erf[x+Sqrt[Pi]t]-Erf[x-Sqrt[Pi]t])/4
h0=Plot[Evaluate[u[x,0]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=0"]
h1=Plot[Evaluate[u[x,1]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=1"]
h2=Plot[Evaluate[u[x,2]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=2"]
h3=Plot[Evaluate[u[x,3]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=3"]
Show[GraphicsArray[{{h0,h1},{h2,h3}}],
Frame->True,FrameTicks->None]
Plot3D[u[x,t],{x,-8,8},{t,0,4},
AxesLabel->"Position","Time","Value",PlotPoints->20,
PlotRange->{0,0.5},Shading->False]
```

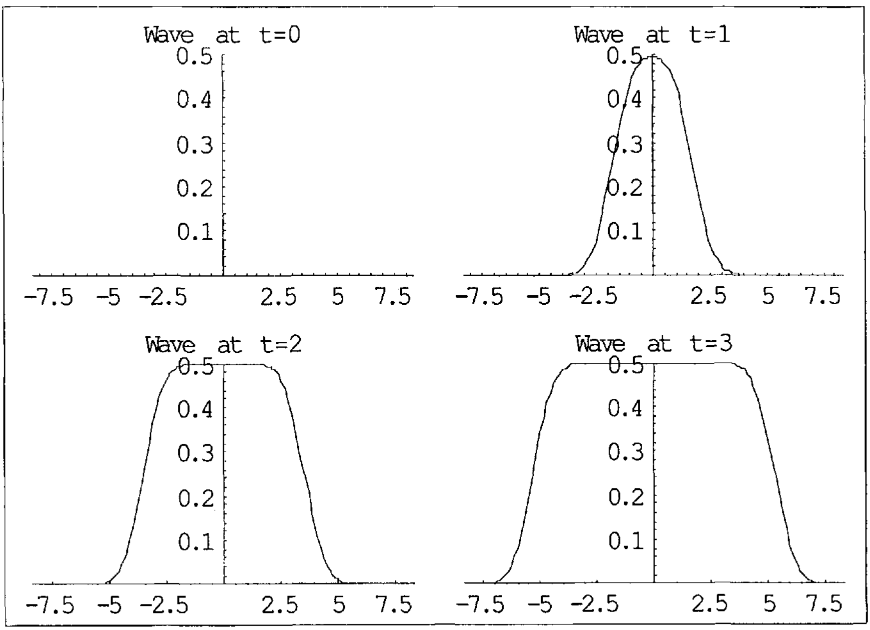

Figure 3.3. Wave at instants $t = 0,\ 1,\ 2,\ 3$.

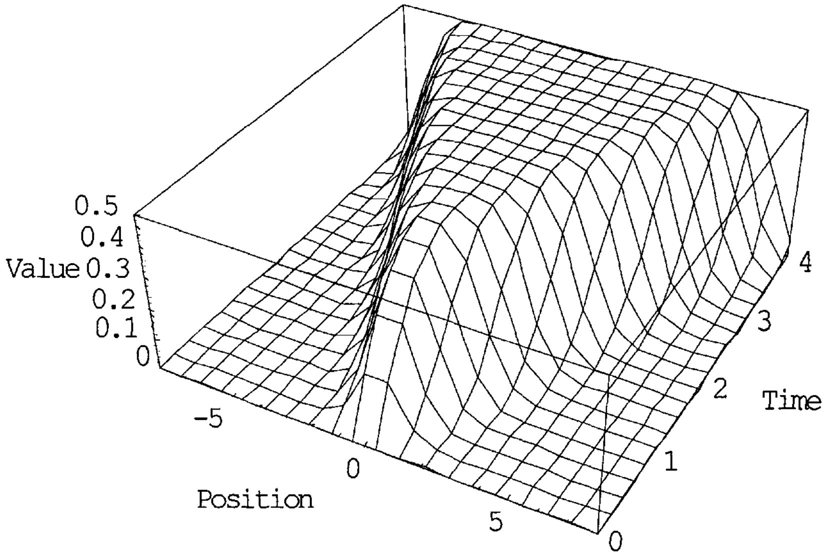

Figure 3.4. Graph of the function $u = \frac{1}{4}\left(\operatorname{erf}\left(x + \sqrt{\pi}t\right) - \operatorname{erf}\left(x - \sqrt{\pi}t\right)\right)$

Exercises

1. Prove the formula for the general solution of the wave equation (3.1) reducing it to the system of first order equations:

$$\begin{cases} v_t - cv_x = 0 \\ u_t + cu_x = v. \end{cases}$$

2. Suppose

$$\begin{aligned} &A(x,t), \quad B(x+cs, t+s), \\ &C(x + c(s-\tau), t+s+\tau), \quad D(x - c\tau, t+\tau) \end{aligned}$$

are vertices of a characteristic parallelogram, where s, τ are positive parameters. Prove that if $u \in C^2\left(\mathbf{R}^2\right)$ is a solution of the wave equation (3.1) then

$$u(A) + u(C) = u(B) + u(D). \tag{3.8}$$

Conversely, prove that if u is of class $C^2\left(\mathbf{R}^2\right)$ and satisfies (3.8) for every $(s,t) \in \mathbf{R}^2$, then u is a solution of the equation (3.1).

3. (a) Prove that if $u(x,y,z) = u(\rho)\,, \rho = \sqrt{x^2+y^2+z^2}$, then

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = u_{\rho\rho} + \frac{2}{\rho} u_\rho.$$

(b) Making the change of variables $v(\rho,t) = \rho u(\rho,t)$ show that the general solution of the three-dimensional wave equation

$$u_{tt} - c^2 (u_{\rho\rho} + \frac{2}{\rho} u_\rho) = 0$$

is

$$u(\rho,t) = \frac{1}{\rho} \left(f(\rho+ct) + g(\rho-ct) \right).$$

(c) Prove that the initial problem for the *spherical wave equation* with conditions

$$u(\rho,0) = \varphi(\rho)\,, \quad u_t(\rho,0) = \psi(\rho)$$

has a solution

$$u(\rho,t) = \frac{1}{2\rho}((\rho+ct)\varphi(\rho+ct) + (\rho-ct)\varphi(\rho-ct)) + \frac{1}{2c\rho} \int_{\rho-ct}^{\rho+ct} s\psi(s)\,ds.$$

Note that this solution exists provided $\rho \geq ct$.

4. Show that for $\psi \in C^1(\mathbf{R})$ the function $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\,ds$ verifies the problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \psi(x)\,. \end{cases}$$

Check it also using *Mathematica*.

5. (a) Prove that if $\varphi(s)$ is a continuous function, then $\varphi(x \pm ct)$ are "weak" solutions of the equation $u_{tt} - c^2 u_{xx} = 0$ in the sense

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \varphi(x \pm ct)\left(v_{tt} - c^2 v_{xx}\right) dxdt = 0,$$

for every test function $v(x,t)$ of the space

$$C_0^\infty\left(\mathbf{R}^2\right) = \left\{f \in C^\infty\left(\mathbf{R}^2\right) : \operatorname{supp} f \text{ is compact}\right\},$$

where

$$\operatorname{supp} f = \overline{\left\{(x,t) \in \mathbf{R}^2 : f(x,t) \neq 0\right\}}.$$

(b) Prove that if $\varphi(s)$ is a continuous function, then the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = 0 \end{cases}$$

has a weak solution $u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right)$ in the sense

$$\int_{\mathbf{R}}\int_{\mathbf{R}} u(x,t)\left(v_{tt} - c^2 v_{xx}\right) dxdt = 0,$$

for every test function $v(x,t) \in C_0^\infty\left(\mathbf{R}^2\right)$.

(c) Using *Mathematica* draw the profile of the solution of the problem (3.1) with

$$c = 1, \quad \varphi(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

at each of the instants $t = 0,\ 0.2,\ 0.6,\ 0.8,\ 1.2.$

3.2 The Wave Equation on the Half-line. Reflection Method

Let us consider the problem (CW) on the half-line $(0,\infty)$ with Dirichlet[3] boundary condition at the endpoint $x=0$.

This is the problem:

$$(CDW):\begin{cases} u_{tt}-c^2u_{xx}=0 & 0<x<\infty, t>0,\\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & 0<x<\infty,\\ u(0,t)=0 & t\geq 0.\end{cases}$$

It can be interpreted as vibrations of a very long string with a clamped one end.

We are looking for a solution of (CDW) given by an explicit formula. In fact we shall reduce the problem (CDW) to a problem (CW) by the odd *reflection method.* It consists in considering the odd extensions of the initial functions $\varphi_o(x)$ and $\psi_o(x)$ where

$$\varphi_o(x):=\begin{cases} \varphi(x) & \text{if } x>0,\\ -\varphi(-x) & \text{if } x<0,\\ 0 & \text{if } x=0.\end{cases}$$

The problem

$$(CW_o):\begin{cases} v_{tt}-c^2v_{xx}=0 & x\in\mathbf{R}, t>0,\\ v(x,0)=\varphi_o(x) & x\in\mathbf{R},\\ v_t(x,0)=\psi_o(x) & x\in\mathbf{R}.\end{cases}$$

has the solution

$$v(x,t)=\frac{1}{2}\left(\varphi_o(x+ct)+\varphi_o(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_o(s)\,ds.$$

Its restriction

$$u(x,t)=\left.v(x,t)\right|_{x\geq 0}$$

is the unique solution of the problem (CDW).

If $0<x<ct$, then

[3]Lejeune Peter Gustav Dirichlet, 13.02.1805–05.05.1859.

$$\begin{aligned}\int\limits_{x-ct}^{x+ct} \psi_o(s)\,ds &= \int\limits_{0}^{x+ct} \psi(s)\,ds - \int\limits_{x-ct}^{0} \psi(-s)\,ds \\ &= \int\limits_{0}^{x+ct} \psi(s)\,ds + \int\limits_{ct-x}^{0} \psi(s)\,ds = \int\limits_{ct-x}^{x+ct} \psi(s)\,ds.\end{aligned}$$

Therefore we have

$$u(x,t) = \begin{cases} \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \dfrac{1}{2c}\displaystyle\int\limits_{x-ct}^{x+ct} \psi(s)\,ds & \text{if } x > ct, \\ \frac{1}{2}\left(\varphi(x+ct) - \varphi(ct-x)\right) + \dfrac{1}{2c}\displaystyle\int\limits_{ct-x}^{ct+x} \psi(s)ds & \text{if } 0 < x < ct. \end{cases} \tag{3.9}$$

Note that $u(x,t)$ is a continuous function if the *compatibility condition* $\varphi(0) = 0$ is satisfied. Otherwise $u(x,t)$ is a discontinuous solution and the jump of $u(x,t)$ on the characteristic $x = ct$ is

$$u(ct+0,t) - u(ct-0,t) = \varphi(0).$$

We have

Theorem 3.4. *Let $\varphi(x) \in C^2(\mathbf{R}^+)$, $\psi(x) \in C^1(\mathbf{R}^+)$ and the following compatibility conditions be satisfied:*

$$\varphi(0) = \varphi''(0) = \psi(0) = 0. \tag{3.10}$$

Then the function $u(x,t)$ defined by (3.9) is the unique solution of the problem (CDW) of class $C^2(\mathbf{R}^+ \times \mathbf{R}^+)$.

Proof. The function $u(x,t)$ is of class C^2 in domains $\{(x,t) : x > ct > 0\}$ and $\{(x,t) : 0 < x < ct\}$. We shall prove that the derivatives of $u(x,t)$ up to order two are continuous along the line $x = ct$. We have

$$u_x(x,t) = \begin{cases} \frac{1}{2}\left(\varphi'(x+ct) + \varphi'(x-ct)\right) + \dfrac{1}{2c}\left(\psi(x+ct) - \psi(x-ct)\right), & x > ct, \\ \frac{1}{2}\left(\varphi'(x+ct) + \varphi'(ct-x)\right) + \dfrac{1}{2c}\left(\psi(x+ct) + \psi(ct-x)\right), & 0 < x < ct. \end{cases}$$

Therefore by (3.10)

$$u_x(ct+0,t) - u_x(ct-0,t) = -\frac{1}{c}\psi(0) = 0.$$

By the same way

$$u_{xx}(ct+0,t) - u_{xx}(ct-0,t) = \varphi''(0) = 0,$$

$$u_t(ct+0,t) - u_t(ct-0,t) = \psi(0) = 0,$$

$$u_{tx}(ct+0,t) - u_{tx}(ct-0,t) = -c\varphi''(0) = 0,$$

$$u_{tt}(ct+0,t) - u_{tt}(ct-0,t) = c^2\varphi''(0) = 0.$$

Moreover the function $u(x,t)$ satisfies the equation, boundary and initial conditions of the problem (CDW). ■

We can do the same for the problem with the Neumann[4] boundary condition, considering even extensions of initial data.

Let us consider the problem

$$(CNW): \left\{ \begin{array}{cc} u_{tt} - c^2u_{xx} = 0 & 0 < x < \infty, t > 0, \\ u(x,0) = \varphi(x), \; u_t(x,0) = \psi(x) & 0 < x < \infty, \\ u_x(0,t) = 0 & t \geq 0. \end{array} \right.$$

In this case we reduce the problem (CNW) to (CW) with initial functions $\varphi_e(x)$ and $\psi_e(x)$, where

$$\varphi_e(x) := \left\{ \begin{array}{ll} \varphi(x) & \text{if } x \geq 0, \\ \varphi(-x) & \text{if } x \leq 0. \end{array} \right.$$

As before we can show that the problem (CNW) has a unique solution

$$u(x,t) = \left\{ \begin{array}{ll} \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}(\Psi(x+ct) - \Psi(x-ct)), & x > ct, \\ \\ \frac{1}{2}(\varphi(x+ct) + \varphi(ct-x)) + \frac{1}{2c}(\Psi(x+ct) + \Psi(ct-x)), & 0 < x < c,. \end{array} \right.$$

[4]Karl Gottfried Neumann, 07.05.1832–27.03.1925

where $\Psi(t) = \int\limits_0^t \psi(s)\,ds$.

Example 3.5. *Solve the problem* (CDW) *with* $c = 1, \psi = 0$ and

$$\varphi(x) = \begin{cases} \cos^3 x & x \in (3\pi/2, 5\pi/2), \\ 0 & x \in \mathbf{R}^+ \backslash (3\pi/2, 5\pi/2). \end{cases}$$

Solution. The solution of the odd extended problem

$$\begin{cases} v_{tt} - v_{xx} = 0 & x \in \mathbf{R}, t > 0, \\ v(x,0) = \varphi_o(x) & x \in \mathbf{R}, \\ v_t(x,0) = 0 & x \in \mathbf{R}, \end{cases}$$

is $v(x,t) = \frac{1}{2}(\varphi_o(x+t) + \varphi_o(x-t))$.

The original problem has the solution

$$u(x,t) = \begin{cases} \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) & x > t, \\ \frac{1}{2}(\varphi(x+t) - \varphi(t-x)) & 0 < x < t. \end{cases}$$

The profile of $u(x,t)$ is presented in Figure 3.5 at successive instants $t = 0$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, 2π, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$ using the *Mathematica* program

```
f[x_]:=Which[3Pi/2<=x<=5Pi/2,Cos[x]^3,True,0]
g[x_]:=Which[x<0,-f[-x],True,f[x]]
u[x_,t_]:=(f[x+t]+g[x-t])/2
h0=Plot[Evaluate[u[x,0]],{x,0,8Pi},
PlotRange->{-1,1},PlotLabel->"Wave at t=0"]
h1=Plot[Evaluate[u[x,Pi/2]],{x,0,8Pi},
PlotRange->{-1,1},PlotLabel->"Wave at t=Pi/2"]
h2=Plot[Evaluate[u[x,3Pi/2]],{x,0,8Pi},
PlotRange->{-1,1},PlotLabel->"Wave at t=3Pi/2"]
h3=Plot[Evaluate[u[x,2Pi]],{x,0,8Pi},
PlotRange->{-1,1},PlotLabel->"Wave at t=2Pi"]
h4=Plot[Evaluate[u[x,5Pi/2]],{x,0,8Pi},
PlotRange->{-1,1},PlotLabel->"Wave at t=5Pi/2"]
h5=Plot[Evaluate[u[x,7Pi/2]],{x,0,8Pi},
PlotRange->{-1,1},PlotLabel->"Wave at t=7Pi/2"]
Show[GraphicsArray[{{h0,h1},{h2,h3},{h4,h5}}]]
```

```
Frame->True,FrameTicks->None]
Plot3D[u[x,t],{x,0,8Pi},{t,0,4Pi},
AxesLabel->"Position","Time","Value",PlotPoints->40,
PlotRange->{-1,1},Shading->False]
```

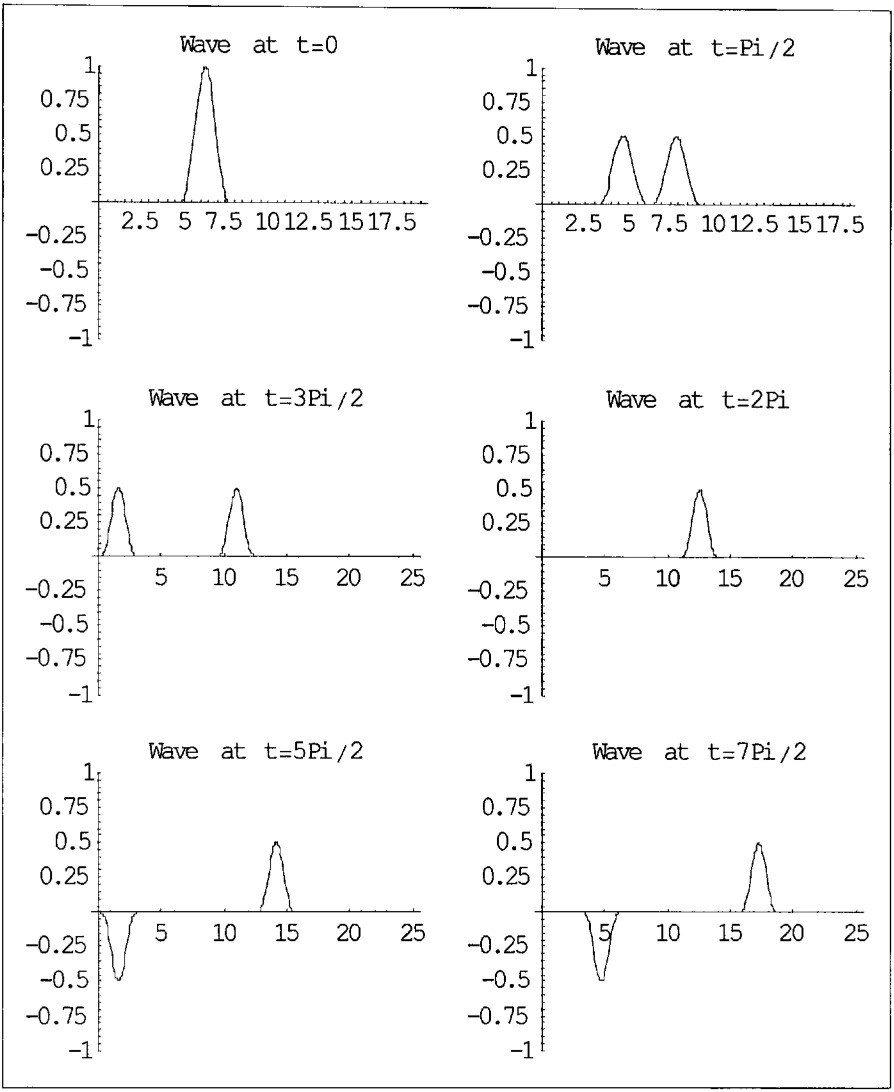

Figure 3.5. Wave at the instants $t = 0$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, 2π , $\frac{5\pi}{2}$, $\frac{7\pi}{2}$.

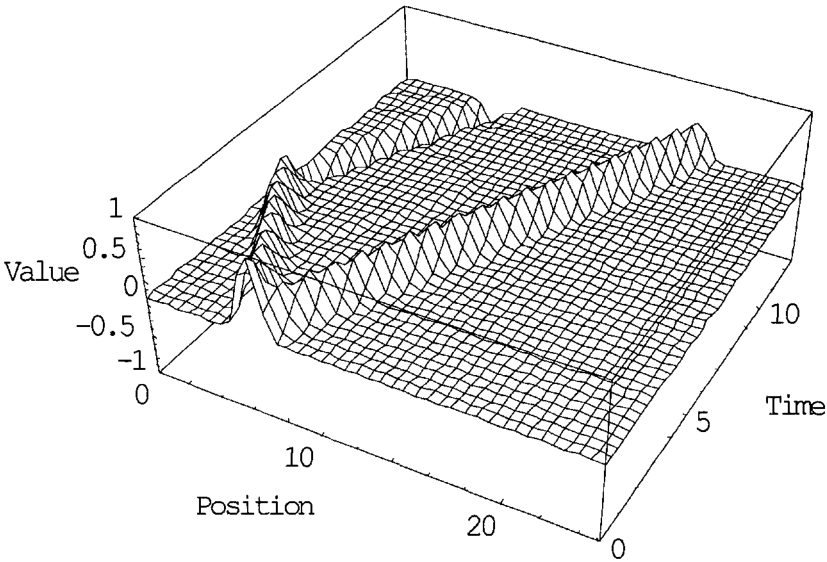

Figure 3.6. Graph of the function $u = u(x,t)$ in Example 3.5

Note that the initial profile splits into two profiles with amplitude $\frac{1}{2}$ up to the instant 2π, when the left one turns to zero and after this instant it changes its direction. This fact is known as a "*swimmer effect*". The graph of the function $u(x,t)$ on the rectangle $R_1 = \{(x,t) : 0 \le x \le 8\pi, 0 \le t \le 4\pi\}$ is plotted in Figure 3.6.

Exercises

1. Prove that for a function $f(x) \in C^2(\mathbf{R}^+)$ its odd extension $f_o(x) \in C^2(\mathbf{R})$ if and only if $f(0) = f''(0) = 0$.

2. Solve the problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \infty, t > 0, \\ u(x,0) = \sin^3 x, \quad u_t(x,0) = 0 & 0 < x < \infty, \\ u(0,t) = 0 & t \ge 0. \end{cases}$$

Prove that the solution $u(x,t) \in C^2((0,\infty) \times \mathbf{R})$.

3.3 Mixed Problem for the Wave Equation

Let us consider the problem (CW) on a finite interval $[0,l]$ with Dirichlet boundary conditions at the end-points $x=0$ and $x=l$. This is the problem

$$(MDW):\begin{cases} u_{tt}-c^2u_{xx}=0, & 0<x<l, t>0,\\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x), & 0\le x\le l,\\ u(0,t)=u(l,t)=0, & t\ge 0.\end{cases}$$

It can be interpreted as vibrations of a string with clamped ends, for instance vibrations of a guitar string.

We can get the solution of the problem (MDW) again using the method of reflection in this case through both ends. We extend the initial data $\varphi(x)$ and $\psi(x)$ given on the interval $(0,l)$ to the whole line using "odd"extensions $\varphi_{eo}(x)$ and $\psi_{eo}(x)$ with respect to both sides $x=0$ and $x=l$, where

$$\varphi_{eo}(x):=\begin{cases} \varphi(x) & 0<x<l,\\ -\varphi(-x) & -l<x<0,\\ \text{extended to be of period } 2l. \end{cases}$$

Consider the problem (CW_{eo}) :

$$(CW_{eo}):\begin{cases} v_{tt}-c^2v_{xx}=0 & x\in\mathbf{R}, t>0,\\ v(x,0)=\varphi_{eo}(x) & x\in\mathbf{R},\\ v_t(x,0)=\psi_{eo}(x) & x\in\mathbf{R}.\end{cases}$$

By Section .3.1 it has a solution

$$v(x,t)=\frac{1}{2}\left(\varphi_{eo}(x+ct)+\varphi_{eo}(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{eo}(s)\,ds.$$

Its restriction

$$u(x,t)=\left.v(x,t)\right|_{0\le x\le l}$$

gives the unique solution of the problem (MDW). Note that the solution formula is characterized by a number of reflections at each end $x=0$ and $x=l$ along characteristics through reflecting points. They divide the domain $R=\{(x,t):0<x<l, t>0\}$ into diamond-shaped domains with sides parallel to characteristics and within each diamond the solution $u(x,t)$ is given by a different formula.

On the data φ and ψ we impose the compatibility condition

$$\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0. \tag{3.11}$$

In this case the solution $u(x,t)$ is a continuous function on R. Note that $u(x,t) \in C^2(R)$ if

$$\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \psi(0) = \psi(l) = 0. \tag{3.12}$$

We can do the same for the problem with the Neumann boundary condition, considering even extensions of initial data. Namely, let us consider the problem

$$(MNW): \begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l, t > 0, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x) & 0 < x < l, \\ u_x(0,t) = u_x(l,t) = 0 & t \geq 0. \end{cases}$$

In this case we reduce the problem (MNW) to (CW_{ee}) with initial functions $\varphi_{ee}(x)$ and $\psi_{ee}(x)$,where

$$\varphi_{ee}(x) := \begin{cases} \varphi(x), & 0 < x < l, \\ \varphi(-x), & -l < x < 0, \\ \text{extended to be of period } 2l. \end{cases}$$

As before the problem (MNW) admits the unique solution

$$u(x,t) = w(x,t)|_{0 \leq x \leq l},$$

where $w(x,t)$ is the solution of the problem

$$(CW_{ee}): \begin{cases} w_{tt} - c^2 w_{xx} = 0 & x \in \mathbf{R}, t > 0, \\ w(x,0) = \varphi_{ee}(x) & x \in \mathbf{R}, \\ w_t(x,0) = \psi_{ee}(x) & x \in \mathbf{R}. \end{cases}$$

Example 3.6. *Solve the problem* (MDW) *with* $c = 1, \psi = 0$ and

$$\varphi(x) = \begin{cases} \cos^3 x & x \in [(3\pi/2, 5\pi/2)], \\ 0 & x \in [0, 3\pi/2) \cup (5\pi/2, 4\pi]. \end{cases}$$

Solution. The solution of the odd extended problem

$$\begin{cases} v_{tt} - v_{xx} = 0 & x \in \mathbf{R}, t > 0, \\ v(x,0) = \varphi_{eo}(x) & x \in \mathbf{R}, \\ v_t(x,0) = 0 & x \in \mathbf{R}, \end{cases}$$

is $v(x,t) = \frac{1}{2}\left(\varphi_{eo}(x+t) + \varphi_{eo}(x-t)\right)$.

The original problem has a solution

$$u(x,t) = v(x,t)|_{0 \leq x \leq 4\pi}.$$

The graph of the function $u(x,t)$ on the rectangle

$$R_2 = \{(x,t) : 0 \leq x \leq 4\pi, 0 \leq t \leq 4\pi\}$$

is presented in Figure 3.7 using the *Mathematica* program

```
f[x_]:=Which[3Pi/2<=x<=5Pi/2,Cos[x]^3,True,0]
g0[x_]:=Which[x<0,-f[-x],True,f[x]]
g1[x_]:=Which[x>4Pi,-f[x-4Pi],True,f[x]]
u[x_,t_]:=(g1[x+t]+g0[x-t])/2
Plot3D[u[x,t],{x,0,4Pi},{t,0,4Pi},
AxesLabel->"Position","Time","Value", PlotPoints->40,
PlotRange->{-1,1},Shading->False]
```

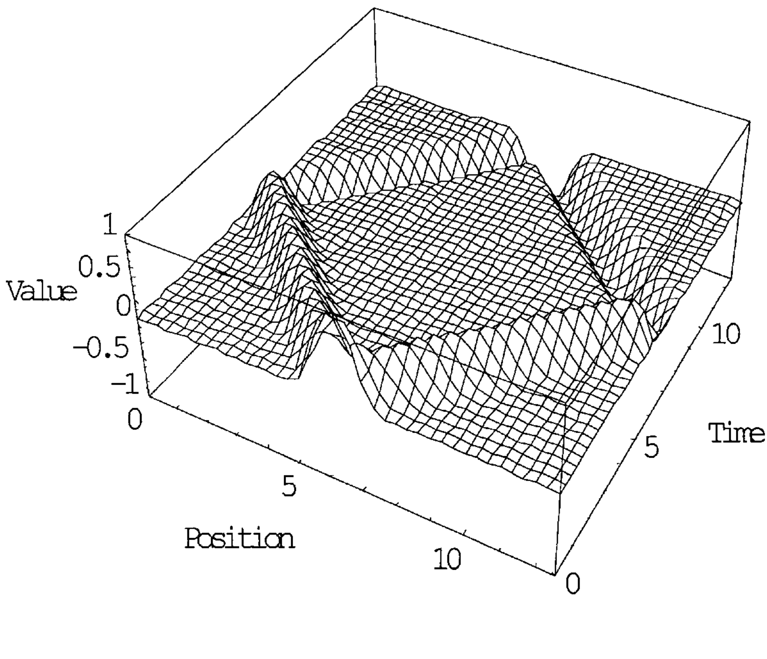

Figure 3.7. Reflection of a wave.

Note that the initial profile splits into two profiles with amplitude $\frac{1}{2}$ up to the instant 2π, when both turn to zero and after this instant they change their direction up to the instant 4π.

Exercises

1. Find the values $u\left(\frac{1}{2},1\right)$, $u\left(\frac{3}{4},\frac{1}{2}\right)$ where $u(x,t)$ is the solution of the problem

$$\begin{cases} u_{tt}-u_{xx}=0 & 0<x<1, t>0, \\ u(x,0)=x^2(1-x),\ u_t(x,0)=0 & 0<x<1, \\ u_x(0,t)=u_x(1,t)=0 & t\geq 0. \end{cases}$$

2. Solve the problem

$$\begin{array}{ll} u_{tt}-u_{xx}=0, & 0<x<4\pi, t>0, \\ u(x,0)=\begin{cases} \cos^3 x, & 3\pi/2\leq x\leq 5\pi/2, \\ 0, & x\in[0,4\pi]\setminus(3\pi/2,5\pi/2), \end{cases} & \\ u_t(x,0)=0, & 0<x<4\pi, \\ u_x(0,t)=u_x(4\pi,t)=0, & t\geq 0. \end{array}$$

Plot the graph of the function $u(x,t)$ on the rectangle $R_3=\{(x,t):0<x<4\pi, 0<t<4\pi\}$ using *Mathematica*.

3.4 Inhomogeneous Wave Equation

Let $f\in C^1(\mathbf{R}^2)$ and consider the inhomogeneous Cauchy problem

$$(ICW):\begin{cases} u_{tt}-c^2u_{xx}=f & x\in\mathbf{R}, t>0, \\ u(x,0)=\varphi(x) & x\in\mathbf{R}, \\ u_t(x,0)=\psi(x) & x\in\mathbf{R}. \end{cases}$$

It can be split into two problems - one homogeneous with nonzero initial data (CW), which we solve, and one inhomogeneous with zero initial data

$$\begin{array}{ll} u_{tt}-c^2u_{xx}=f & x\in\mathbf{R}, t>0, \\ u(x,0)=0 & x\in\mathbf{R}, \\ u_t(x,0)=0 & x\in\mathbf{R}. \end{array} \tag{3.13}$$

If $u_1(x,t)$ and $u_2(x,t)$ are solutions of (CW) and (3.13) respectively, then $u(x,t) = u_1(x,t) + u_2(x,t)$ is a solution of (ICW).

Let us consider (3.13). Making change of variables

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct, \end{aligned}$$

we transform (3.13) into

$$U_{\xi\eta} = -\frac{1}{4c^2} F(\xi, \eta), \tag{3.14}$$

$$U(\xi, \xi) = U_\xi(\xi, \xi) = U_\eta(\xi, \xi) = 0, \tag{3.15}$$

where

$$\begin{aligned} U(\xi, \eta) &= u(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}), \\ F(\xi, \eta) &= f(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}). \end{aligned}$$

Integrating (3.14) with respect to η we have

$$U_\xi(\xi, \xi) - U_\xi(\xi, \eta) = -\frac{1}{4c^2} \int_\eta^\xi F(\xi, s) ds,$$

which, in view of (3.15), yields

$$U_\xi(\xi, \eta) = \frac{1}{4c^2} \int_\eta^\xi F(\xi, s) ds.$$

Integrating the last equation with respect to ξ

$$U(\xi, \eta) - U(\eta, \eta) = \frac{1}{4c^2} \int_\eta^\xi \int_\eta^z F(z, s) ds dz,$$

$$U(\xi, \eta) = \frac{1}{4c^2} \int_\eta^\xi \int_\eta^z F(z, s) ds dz. \tag{3.16}$$

Let us make change of variables

$$\begin{aligned} s &= \sigma - c\tau, \\ z &= \sigma + c\tau, \end{aligned}$$

which has Jacobian $J = \dfrac{\partial(s,z)}{\partial(\sigma,\tau)} = 2c$. The last change transforms the domain of integration

$$D = \{(s,z) : \eta \le s \le z, \quad \eta \le z \le \xi\}$$

into

$$D' = \{(\sigma,\tau) : x - c(t-\tau) \le \sigma \le x + c(t-\tau), \quad 0 \le \tau \le t\}.$$

Indeed by, $\eta \le s \le z \le \xi$, we have

$$x - ct \le \sigma - c\tau \le \sigma + c\tau \le x + ct.$$

Then it follows

$$0 \le 2c\tau \le 2ct \Leftrightarrow 0 \le \tau \le t,$$

and

$$x - c(t-\tau) \le \sigma \le x + c(t-\tau).$$

The solution of (3.13) is

$$u_2(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)\,d\sigma d\tau = \frac{1}{2c}\iint_{\Delta(x,t)} f(\sigma,\tau)\,d\sigma d\tau \tag{3.17}$$

where $\Delta(x,t)$ denotes the characteristic triangle.

We prove that problem (ICW) has a solution given by the exact formula

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) \\ &+ \frac{1}{2c}\left(\int_{x-ct}^{x+ct} \psi(s)\,ds + \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)\,d\sigma d\tau\right). \end{aligned} \tag{3.18}$$

Note that from (3.18) it follows the well-posedness of (ICW).

Indeed, as in Section 3.1, for $0 \le t \le T$, we have

$$|u(x,t)| \le \|\varphi\|_\infty + \frac{1}{2c}\|\psi\|_\infty 2cT + \frac{1}{2c}\|f\|_{\infty,T}\iint_{\Delta(x,t)} d\sigma d\tau$$

$$\le \|\varphi\|_\infty + T\|\psi\|_\infty + \frac{T^2}{2}\|f\|_{\infty,T},$$

because

$$\iint_{\Delta} d\sigma d\tau = S(\Delta) \leq \frac{2cT.T}{2}.$$

Then for $\varepsilon > 0$, there exists $\delta \in \left(0, \frac{\varepsilon}{1+T+T^2/2}\right)$ such that if $\|\varphi\|_\infty < \delta$, $\|\psi\|_\infty < \delta$ and $\|f\|_{\infty,T} < \delta$, it follows $\|u\|_{\infty,T} < \varepsilon$.

Example 3.7. *Solve the problem*

$$\begin{cases} u_{tt} - u_{xx} = xt & x \in \mathbf{R}, t > 0, \\ u(x,0) = 0 & x \in \mathbf{R}, \\ u_t(x,0) = 0 & x \in \mathbf{R}. \end{cases}$$

Solution. The solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \sigma\tau d\sigma d\tau \\ &= \frac{1}{4}\int_0^t \tau((x+(t-\tau))^2 - (x-(t-\tau))^2)d\tau \\ &= x\int_0^t \tau(t-\tau)d\tau = x\left(\frac{t^3}{2} - \frac{t^3}{3}\right) \\ &= \frac{1}{6}xt^3. \end{aligned}$$

Exercises

1. Solve the problems

(a)

$$\begin{cases} u_{tt} - u_{xx} = e^{x-t} & x \in \mathbf{R}, t > 0, \\ u(x,0) = 0 & x \in \mathbf{R}, \\ u_t(x,0) = 0 & x \in \mathbf{R}. \end{cases}$$

(b)

$$\begin{cases} u_{tt} - u_{xx} = \sin x & x \in \mathbf{R}, t > 0, \\ u(x,0) = \cos x & x \in \mathbf{R}, \\ u_t(x,0) = x & x \in \mathbf{R}. \end{cases}$$

(c)

$$\begin{cases} u_{tt} - u_{xx} = x^2 & x \in \mathbf{R}, t > 0, \\ u(x,0) = \cos x & x \in \mathbf{R}, \\ u_t(x,0) = 0 & x \in \mathbf{R}. \end{cases}$$

2. (a) Prove the formula

$$\frac{d}{dt}\int_0^{\alpha(t)} f(x,t)dx = \int_0^{\alpha(t)} \frac{\partial f}{\partial t}(x,t)dx + f(\alpha(t),t)\alpha'(t),$$

where $f(x,t)$ and $\alpha(t)$ are differentiable functions.

(b) Verify that the function

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(\sigma,\tau)d\sigma d\tau$$

satisfies the problem

$$u_{tt} - u_{xx} = f(x,t); \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

where $f(x,t) \in C^1(\mathbf{R}^2)$.

Check the problem using a *Mathematica* program. One program is

```
g[w_]:=Integrate[f[v,w],v,x-t+w,x+t-w]
u[x_,t_]:=Integrate[g[w],w,0,t]/2
utt=D[u[x,t],t,2]
uxx=D[u[x,t],x,2]
Simplify[utt-uxx]
```

Give another program.

3. Prove the formula (3.17) applying Green's[5] identity

$$\int_{\partial\Delta} c^2 v dt + u dx = \iint_{\Delta} \left(u_t - c^2 v_x\right) dx dt,$$

to the equation $u_{tt} - c^2 u_{xx} = f(x,t)$,where $\partial\Delta$ is the oriented boundary of the characteristic triangle $\Delta = \Delta(x,t)$.

[5] George Green, 14.07.1793–31.03.1841.

3.5 Conservation of the Energy

Let

$$R = \{(x,t) : 0 < x < l,\ 0 < t < \infty\},$$

and $u \in C^2(R)$ be a solution of the problem

$$(MDW) : \begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l, t > 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & 0 < x < l, \\ u(0,t) = u(l,t) = 0, & t \geq 0. \end{cases}$$

The quantity

$$KE(t) = \frac{1}{2} \int_0^l u_t^2(x,t)\, dx$$

is known as the *kinetic energy,* the quantity

$$PE(t) = \frac{1}{2} \int_0^l c^2 u_x^2(x,t)\, dx$$

is the *potential energy.* The sum of the kinetic and potential energy

$$E(t) = KE(t) + PE(t) = \frac{1}{2} \int_0^l \left(u_t^2(x,t) + c^2 u_x^2(x,t)\right) dx$$

is the *total energy* of the system at the instant t. The conservation of energy is one of the most basic facts about the wave equation. For the above mentioned problem (MDW) we show that the total energy $E(t)$ is a constant independent of t. This is the law of *conservation of energy.*

Theorem 3.5. *If $u \in C^2(R)$ is a solution of the problem (MDW), then the energy $E(t)$ is a constant $E(t) = E(0)$.*

Proof. Multiplying the equation by u_t , using the identities

$$u_t u_{tt} = \frac{1}{2} \frac{\partial}{\partial t} \left(u_t^2\right),$$

$$u_t u_{xx} = \frac{\partial}{\partial x} \left(u_x u_t\right) - \frac{1}{2} \frac{\partial}{\partial t} \left(u_x^2\right),$$

and integrating by parts we get

$$
\begin{aligned}
0 &= \int_0^l \left(u_{tt} - c^2 u_{xx}\right) u_t dx \\
&= \int_0^l \left(\frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2\right) - c^2\left(\frac{\partial}{\partial x}\left(u_x u_t\right) - \frac{1}{2}\frac{\partial}{\partial t}\left(u_x^2\right)\right)\right)dx \\
&= \frac{1}{2}\frac{d}{dt}\int_0^l \left(u_t^2\left(x,t\right) + c^2 u_x^2\left(x,t\right)\right)dx \\
&\quad -c^2 u_x\left(l,t\right)u_t\left(l,t\right) + c^2 u_x\left(0,t\right)u_t\left(0,t\right) \\
&= \frac{dE}{dt}.
\end{aligned}
$$

Therefore for $t > 0$

$$
\begin{aligned}
E\left(t\right) &= E\left(0\right) = \frac{1}{2}\int_0^l \left(u_t^2\left(x,0\right) + c^2 u_x^2\left(x,0\right)\right)dx \qquad (3.19) \\
&= \frac{1}{2}\int_0^l \left(\psi^2(x) + c^2\varphi'^2(x)\right)dx,
\end{aligned}
$$

so the energy is conserved.

¿From (3.19) it follows that if $\varphi = \psi = 0$ then $u = 0$ on R. ■

Exercises

1. Consider the problem

$$
\begin{cases}
u_{tt} - u_{xx} + u_t = 0 & 0 < x < l, \quad t > 0, \\
u\left(x,0\right) = \varphi\left(x\right), \quad u_t\left(x,0\right) = \psi\left(x\right) & 0 < x < l, \\
u\left(0,t\right) = u\left(l,t\right) = 0 & t \geq 0.
\end{cases}
$$

Prove that the energy

$$
E\left(t\right) = \frac{1}{2}\int_0^l \left(u_t^2\left(x,t\right) + u_x^2\left(x,t\right)\right)dx
$$

is a decreasing function.

2. Let $u \in C^2\left(\mathbf{R}^2\right)$ be a solution of the wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

and

$$D^T = \{(x,t) : a - ct \leq x \leq b + ct, 0 \leq t \leq T\},$$

$$D_T = \{(x,t) : a + ct \leq x \leq b - ct, 0 \leq t \leq T\}.$$

(a) Using

$$u_t\left(u_{tt} - c^2 u_{xx}\right) = \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right)_t - \left(c^2 u_x u_t\right)_x$$

and Green's identity in D^T prove that

$$\int_{a-cT}^{b+cT} \left(u_t^2 + c^2 u_x^2\right)(x,T)\,dx - \int_a^b \left(u_t^2 + c^2 u_x^2\right)(x,0)\,dx$$

$$= \int_{a-cT}^{a} \left(u_t - cu_x\right)^2\left(x, \frac{a-x}{c}\right)dx + \int_b^{b+cT} \left(u_t + cu_x\right)^2\left(x, \frac{x-b}{c}\right)dx.$$

(b) From the last identity it follows

$$\int_{a-cT}^{b+cT} \left(u_t^2 + c^2 u_x^2\right)(x,T)\,dx \geq \int_a^b \left(u_t^2 + c^2 u_x^2\right)(x,0)\,dx.$$

(c) Applying Green's identity in D_T prove that

$$\int_{a+cT}^{b-cT} \left(u_t^2 + c^2 u_x^2\right)(x,T)\,dx \leq \int_a^b \left(u_t^2 + c^2 u_x^2\right)(x,0)\,dx.$$

3. Consider the problem

$$\begin{cases} u_{tt} - u_{xx} = f & x \in \mathbf{R}, t > 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & x \in \mathbf{R}, \end{cases}$$

and suppose that $f \in C^1(\mathbf{R} \times \mathbf{R}^+) \cap L^2(\mathbf{R} \times \mathbf{R}^+), \varphi \in C^2(\mathbf{R}), \varphi' \in L^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Let

$$E(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \left(u_t^2(x,t) + u_x^2(x,t)\right) dx.$$

Prove that

$$\sqrt{E(t)} \leq \frac{1}{\sqrt{2}} \int_0^t \left(\int_{-\infty}^{+\infty} f^2(x,s)\, dx \right)^{\frac{1}{2}} ds + \sqrt{E(0)}.$$

Chapter 4

One Dimensional Diffusion Equation

4.1 Maximum-minimum Principle for the Diffusion Equation

In this section we consider the homogeneous *one-dimensional diffusion (heat) equation*

$$u_t - \alpha^2 u_{xx} = 0, \tag{4.1}$$

which appears in the study of heat conduction and other diffusion processes. As a model for equation (4.1), we consider a thin metal bar of length l whose sides are insulated. Denote by $u(x, t)$ the temperature of the bar at the point x at the time t. The constant $k = \alpha^2$ is known as the *thermal conductivity.* The parameter k depends only on the material from which the bar is made. The units of k are $(\text{length})^2/\text{time}$. Some values of k are as follows: Silver 1.71, Copper 1.14, Aluminium 0.86, Water 0.0014. In order to determine the temperature in the bar at any time t we need to know:

(1) initial temperature distribution

$$u(x, 0) = \varphi(x), \qquad 0 \leq x \leq l,$$

where $\varphi(x)$ is a given function.

(2) boundary conditions at the ends of the bar.

For instance, we assume that the temperatures at the ends are fixed

$$u(0,t) = T_1, \quad u(l,t) = T_2, \quad t > 0.$$

However it turns out that it suffices to consider the case $T_1 = T_2 = 0$ only. We can also assume that the ends of the bar are insulated, so that no heat can pass through them, which implies

$$u_x(0,t) = u_x(l,t) = 0, \quad t > 0.$$

A "well posed" problem for a diffusion process is

$$u_t - ku_{xx} = 0, \quad 0 < x < l, \; t > 0, \tag{4.2}$$

where $u\,(x,t)$ satisfies the initial condition

$$u(x,0) = \varphi(x), \quad 0 < x < l \tag{4.3}$$

and the boundary conditions

$$u(0,t) = u(l,t) = 0, \; t > 0 \tag{4.4}$$

or

$$u_x(0,t) = u_x(l,t) = 0, \quad t > 0. \tag{4.5}$$

The problem (4.2), (4.3), (4.4) is known as the Dirichlet problem for the diffusion equation, while (4.2), (4.3), (4.5) as the Neumann problem.

At first we discuss a property of the diffusion equation, known as the maximum-minimum principle.

Let $R = \{(x,t) : 0 \leq x \leq l, 0 \leq t \leq T\}$ be a closed rectangle and

$$\Gamma = \{(x,t) \in R : t = 0 \text{ or } x = 0 \text{ or } x = l\}.$$

Theorem 4.1. *(Maximum-minimum principle). Let $u(x,t)$ be a continuous function in R which satisfies equation (4.2) in $R \backslash \Gamma$. Then*

$$\max_R u(x,t) = \max_\Gamma u(x,t), \tag{4.6}$$

$$\min_R u(x,t) = \min_\Gamma u(x,t). \tag{4.7}$$

By Theorem 4.1 the maximum (minimum) of $u(x,t)$ cannot be assumed anywhere inside the rectangle but only on the bottom or lateral sides (unless u is a constant).

Proof of Theorem 4.1. Denote

$$M = \max_{\Gamma} u(x,t).$$

We shall show that $\max\limits_{R} u(x,t) \le M$ which implies (4.6).

Consider the function $v(x,t) = u(x,t) + \varepsilon x^2$, where ε is a positive constant. We have for $(x,t) \in R \setminus \Gamma$

$$Lv(x,t) = Lu(x,t) - 2k\varepsilon = -2\varepsilon k < 0 \tag{4.8}$$

and

$$v(x,t) \le M + \varepsilon l^2, \text{ if } (x,t) \in \Gamma.$$

If $v(x,t)$ attains its maximum at an interior point (x_1,t_1) it follows that $Lv(x_1,t_1) \ge 0$, which contradicts (4.8). Therefore $v(x,t)$ attains its maximum at a point of $\partial R = \Gamma \cup \gamma, \gamma = \{(x,t) \in R : t = T\}$. Suppose $v(x,t)$ has a maximum at a point $(\bar{x},T) \in \gamma$, $0 < \bar{x} < l$. Then $v_x(\bar{x},T) = 0$, $v_{xx}(\bar{x},T) \le 0$. As

$$v(\bar{x},T) \ge v(\bar{x},T-\delta), \quad 0 < \delta < T,$$

we have

$$v_t(\bar{x},T) = \lim_{\delta \to 0} \frac{v(\bar{x},T-\delta) - v(\bar{x},T)}{-\delta} \ge 0.$$

Therefore $Lv(\bar{x},T) \ge 0$, which contradicts (4.8). Hence

$$M_1 = \max_{R} v(x,t) = \max_{\Gamma} v(x,t) \le M + \varepsilon l^2,$$

which implies $u(x,t) \le M + \varepsilon(l^2 - x^2)$, on R for every $\varepsilon > 0$.

Letting $\varepsilon \to 0$, we obtain $u(x,t) \le M$ on R which means that

$$\max_{R} u(x,t) = \max_{\Gamma} u(x,t).$$

Considering the function $w(x,t) = -u(x,t)$ we get (4.7). ■

By the maximum-minimum principle it follows the uniqueness of the solution of the Dirichlet problem for the diffusion equation

$$(IDD):\begin{cases} u_t - ku_{xx} = f(x,t) & 0 < x < l, 0 < t \le T, \\ u(x,0) = \varphi(x) & 0 \le x \le l, \\ u(0,t) = g(t), \quad u(l,t) = h(t) & 0 \le t \le T. \end{cases}$$

Suppose

$$\begin{aligned} f(x,t) &\in C(R), \quad \varphi(x) \in C[0,l], \\ g(t) &\in C[0,T], \quad h(t) \in C[0,T], \\ \varphi(0) &= g(0), \quad \varphi(l) = h(0). \end{aligned}$$

By a solution we mean a function $u \in C(R)$ which is differentiable inside R and satisfies the equation along with the initial and the boundary conditions of (IDD).

Theorem 4.2. *The problem (IDD) has no more than one solution.*

Proof. Suppose $u_1(x,t)$ and $u_2(x,t)$ are two solutions of (IDD). Let $w(x,t) = u_1(x,t) - u_2(x,t)$. Then

$$\begin{cases} w_t - kw_{xx} = 0 & 0 < x < l, 0 < t \le T, \\ w(x,0) = 0 & 0 \le x \le l, \\ w(0,t) = w(l,t) = 0 & 0 \le t \le T. \end{cases}$$

By Theorem 4.1 it follows

$$\max_R w(x,t) = \min_R w(x,t) = 0.$$

Therefore $w(x,t) \equiv 0$, so that $u_1(x,t) \equiv u_2(x,t)$ for every $(x,t) \in R$. ■

Consider the problem (IDD) with $f = g = h = 0$,that is

$$(HDD):\begin{cases} u_t - ku_{xx} = 0 & 0 < x < l, 0 < t \le T, \\ u(x,0) = \varphi(x) & 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & 0 \le t \le T. \end{cases}$$

As a Corollary of Theorem 4.1 the continuous dependence of solutions of (HDD) with respect to initial data follows.

Corollary 4.1. *Let $u_j(x,t)$ be a solution of (HDD) with initial data $\varphi_j(x)$, $j = 1,2$. Then*

$$\max_{0 \le x \le l} |u_1(x,t) - u_2(x,t)| \le \max_{0 \le x \le l} |\varphi_1(x) - \varphi_2(x)|, \tag{4.9}$$

for every $t \in [0,T]$.

Proof. Consider the function $w(x,t) = u_1(x,t) - u_2(x,t)$, which satisfies

$$\begin{cases} w_t - kw_{xx} = 0 & 0 < x < l, 0 < t \le T, \\ w(x,0) = \varphi_1(x) - \varphi_2(x) & 0 \le x \le l, \\ w(0,t) = w(l,t) = 0 & 0 \le t \le T. \end{cases}$$

By Theorem 4.1 it follows that

$$\begin{aligned} u_1(x,t) - u_2(x,t) &\le \max\{\max_{0\le x\le l} (\varphi_1(x) - \varphi_2(x)), 0\} \\ &\le \max_{0\le x\le l} |\varphi_1(x) - \varphi_2(x)|, \end{aligned}$$

and

$$\begin{aligned} u_1(x,t) - u_2(x,t) &\ge \min\{\min_{0\le x\le l} (\varphi_1(x) - \varphi_2(x)), 0\} \\ &\ge -\max\{\max_{0\le x\le l} (\varphi_1(x) - \varphi_2(x)), 0\} \\ &\ge -\max_{0\le x\le l} |\varphi_1(x) - \varphi_2(x)|, \end{aligned}$$

which imply (4.9). ■

The uniqueness and stability of solutions to (HDD) can be derived by another approach, known as the energy method. We have already used this method in Section 3.5 for the wave equation.

Let u be a solution of the problem (HDD). The quantity

$$H(t) = \int_0^l u^2(x,t)dx$$

is referred to as the *thermal energy* at the instant t. In contrast to the wave equation where the energy is a constant, we shall show that $H(t)$ is a decreasing function.

Theorem 4.3. *(a) Let $u(x,t)$ be a solution of (HDD). Then*

$$H(t_1) \ge H(t_2), \text{ if } 0 \le t_1 \le t_2 \le T.$$

(b) Let $u_j(x,t)$ be a solution of (HDD) corresponding to the initial data $\varphi_j(x)$, $j = 1,2$. Then

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 dx \le \int_0^l (\varphi_1(x) - \varphi_2(x))^2 dx.$$

Proof. (a) Multiplying the equation by u, using

$$uu_t = \frac{1}{2}\frac{\partial}{\partial t}(u^2), \qquad uu_{xx} = \frac{\partial}{\partial x}(uu_x) - u_x^2$$

and integrating, we obtain

$$\begin{aligned}
0 &= \int_0^l (u_t - ku_{xx})u dx \\
&= \int_0^l \left(\frac{1}{2}\frac{\partial}{\partial t}(u^2) - k\frac{\partial}{\partial x}(uu_x) + ku_x^2\right) dx \\
&= \frac{1}{2}\frac{d}{dt}\left(\int_0^l u^2(x,t)dx\right) - k((uu_x)(l,t) - (uu_x)(0,t)) \\
&\quad + k\int_0^l u_x^2 dx \\
&\geq \frac{1}{2}\frac{dH}{dt}(t).
\end{aligned}$$

Therefore $H(t)$ is a decreasing function, so if $0 \leq t_1 \leq t_2 \leq T$, then $H(t_1) \geq H(t_2)$.

(b) The function $w(x,t) = u_1(x,t) - u_2(x,t)$ satisfies (HDD) with $\varphi(x) = \varphi_1(x) - \varphi_2(x)$. Therefore for $t \geq 0$ by (a)

$$\begin{aligned}
\int_0^l (u_1(x,t) - u_2(x,t))^2 dx &\leq \int_0^l (u_1(x,0) - u_2(x,0))^2 dx \\
&= \int_0^l (\varphi_1(x) - \varphi_2(x))^2 dx. \blacksquare
\end{aligned}$$

Exercises

1. Consider the mixed problem for the diffusion equation

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 2,\ 0 < t, \\ u(x,0) = x(2-x) & 0 \leq x \leq 2, \\ u(0,t) = u(2,t) = 0 & 0 \leq t. \end{cases}$$

Show that:

(a) $0 < u(x,t) < 1$ for every $t > 0$ and $0 < x < 2$,

(b) $u(x,t) = u(2-x,t)$ for every $t \geq 0$ and $0 \leq x \leq 2$,

(c) $\int_0^2 u^2(x,t)dx \leq \frac{16}{15}$ for every $t \geq 0$.

2. The maximum principle is not valid for parabolic equations with variable coefficients. Verify that the equation $u_t - xu_{xx} = 0$ in the rectangle $R = \{(x,t) : -2 \le x \le 2,\ 0 \le t \le 1\}$ has a solution $u(x,t) = -2xt - x^2$ and $\max_R u(x,t) = u(-1,1) = 1$.

3. Consider the thermal energy $H(t)$ of the problem (HDD).
(a) Show that

$$H'(t) = -2k \int_0^l u_x^2(x,t)dx$$

and

$$H''(t) = 4 \int_0^l u_t^2(x,t)dx.$$

(b) Using the Cauchy–Schwarz inequality derive that

$$H'^2(t) \le H(t)H''(t).$$

(c) Show that for every $0 \le t_1 < t < t_2 \le T$ the inequality

$$H(t) \le H(t_1)^{\frac{t_2-t}{t_2-t_1}} H(t_2)^{\frac{t-t_1}{t_2-t_1}},$$

holds, known as logarithmic convexity of $H(t)$.

4.2 The Diffusion Equation on the Whole Line

In this section we give an explicit formula for the solution of the *Cauchy problem for the diffusion equation* on the whole line

$$(CD) : \begin{cases} u_t - ku_{xx} = 0 & x \in \mathbf{R},\ 0 < t < T, \\ u(x,0) = \varphi(x) & x \in \mathbf{R}. \end{cases}$$

We shall prove that the solution of (CD) is given by the Poisson[1] formula

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-\dfrac{(x-\xi)^2}{4kt}} \varphi(\xi)d\xi, \tag{4.10}$$

assuming that $\varphi(x)$ is continuous and bounded on $\mathbf{R}$.

Notice from (4.10) that the value of $u(x,t)$ depends on the values of the initial data $\varphi(\xi)$ for all $\xi \in \mathbf{R}$. Conversely, the value of φ at a point x_0 has

[1]Simeon Denis Poisson, 21.06.1781–25.04.1840.

an immediate effect everywhere for $t > 0$. This effect is known as infinite speed of propagation which is in contrast to the wave equation. Moreover the solution given by (4.10) is infinitely differentiable for $t > 0$. It is known that the diffusion is a smoothing process going forward in time. Going backward (antidiffusion) the process becomes chaotic. Therefore, we would not expect well-posedness of the backward-in-time problem for the diffusion equation.

A natural way to derive (4.10) is the Fourier transform, but we do not consider it in our text. In order to prove that (4.10) satisfies the problem (CD) we need some preliminaries on *improper integrals*. Recall some definitions and properties.

Let $f(x,y)$ be a continuous function in $(x,y) \in \mathbf{R} \times [a,b]$. Suppose the integral

$$I(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{4.11}$$

is convergent for every $y \in [a,b]$.

Definition 4.1. *We say that the integral* (4.11) *is uniformly convergent for* $y \in [a,b]$, *if for every* $\varepsilon > 0$ *there exists* $A_0 = A_0(\varepsilon)$, *such that if* $A > A_0$, *then*

$$\left| \int_{-\infty}^{-A} f(x,y)dx \right| + \left| \int_{A}^{+\infty} f(x,y)dx \right| < \varepsilon,$$

for every $y \in [a,b]$.

Theorem 4.4. *If the integral* (4.11) *is uniformly convergent for* $y \in [a,b]$, *then the function* $I(y)$ *is continuous in* $[a,b]$.

Theorem 4.5. *Suppose* $f(x,y)$ *and* $\frac{\partial f}{\partial y}(x,y)$ *are continuous functions in* $\mathbf{R} \times [a,b]$, $I(y)$ *is convergent for every* $y \in [a,b]$ *and*

$$J(y) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)dx$$

is uniformly convergent for $y \in [a,b]$. *Then* $I(y)$ *is a differentiable function in* (a,b) *and*

$$I'(y) = J(y).$$

A criterion on uniform convergence of integrals is the following.

Theorem 4.6. *(Weierstrass[2] criterion) Suppose there exists a function $g(x)$ such that $|f(x,y)| \leq g(x)$ for every $y \in [a,b]$ and the integral*

$$\int_{-\infty}^{\infty} g(x)dx$$

is convergent. Then the integral (4.11) is uniformly convergent for $y \in [a,b]$.

Definition 4.2. *A differentiable function $u(x,t)$ is a solution of the problem (CD) if it satisfies the equation $u_t - ku_{xx} = 0$ in $\mathbf{R} \times (0,T)$ and*

$$\lim_{t \downarrow 0} u(x,t) = \varphi(x) \tag{4.12}$$

Theorem 4.7. *Let $\varphi(x) \in C(\mathbf{R})$ and $|\varphi(x)| \leq M$. Then Poisson formula (4.10) defines an infinitely differentiable function $u(x,t)$ which is a solution of the problem (CD) and $|u(x,t)| \leq M$.*

Proof. From (4.10), making the change of variables $\xi = x - p\sqrt{kt}$, we have

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} \varphi(x - p\sqrt{kt})dp. \tag{4.13}$$

By Poisson identity

$$\int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} dp = 2\sqrt{\pi}$$

we obtain

$$|u(x,t)| \leq \frac{M}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} dp = M.$$

Let us show that (4.12) is fulfilled. Note that the formula (4.10) has a meaning for $t > 0$ and the initial condition is satisfied in the limit sense. We have

$$u(x,t) - \varphi(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} (\varphi(x - p\sqrt{kt}) - \varphi(x))dp. \tag{4.14}$$

Let $\varepsilon > 0$ be fixed. As $\varphi(x)$ is continuous, there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2}, \text{ if } |x - y| < \delta,$$

or

$$\left|\varphi(x - p\sqrt{kt}) - \varphi(x)\right| < \frac{\varepsilon}{2}, \text{ if } |p| < \frac{\delta}{\sqrt{kt}}. \tag{4.15}$$

[2]Karl Theodor Wilhelm Weierstrass, 31.10.1815–19.02.1897.

As the integral $\int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} dp$ is convergent there exists a sufficiently small t_0 such that

$$\frac{1}{2\sqrt{\pi}} \int_{|p|>\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}} dp < \frac{1}{2\sqrt{\pi}} \int_{|p|>\frac{\delta}{\sqrt{kt_0}}} e^{-\frac{p^2}{4}} dp < \frac{\varepsilon}{4M}, \tag{4.16}$$

if $0 < t < t_0$. Then, by (4.14), (4.15) and (4.16) for $t \in (0, t_0)$ we have

$$\begin{aligned} |u(x,t) - \varphi(x)| &\le \frac{1}{2\sqrt{\pi}} \int_{|p|<\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}} \left|\varphi(x - p\sqrt{kt}) - \varphi(x)\right| dp \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_{|p|>\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}} \left(\left|\varphi(x - p\sqrt{kt}) - \varphi(x)\right|\right) dp \\ &< \frac{\varepsilon}{4\sqrt{\pi}} \int_{|p|<\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}} dp + 2M \frac{\varepsilon}{4M} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which means that (4.12) is fulfilled.

It remains to show that $u(x,t)$ satisfies the equation $u_t - ku_{xx} = 0$ in $\mathbf{R} \times (0,T)$. Let $Lu \equiv u_t - ku_{xx}$. As

$$L\left(\frac{1}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{4kt}}\right) = 0$$

it sufficies to show that

$$Lu = \frac{1}{2\sqrt{\pi k}} \int_{-\infty}^{\infty} L\left(\frac{1}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{4kt}}\right) \varphi(\xi) d\xi.$$

Suppose that

$$(x,t) \in S := [-A, A] \times [\delta, T],$$

where $A > 0$ and $0 < \delta < T$ are fixed.

By Theorem 4.5, in order to show that

$$u_t(x,t) = \frac{1}{2\sqrt{\pi k}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \left(\frac{(x-\xi)^2}{4k\sqrt{t^5}} - \frac{1}{2\sqrt{t^3}}\right) \varphi(\xi) d\xi \tag{4.17}$$

we need to prove that the last integral is uniformly convergent for $(x,t) \in S$. We have

$$\left|\frac{(x-\xi)^2}{4k\sqrt{t^5}} - \frac{1}{2\sqrt{t^3}}\right| \le \frac{(x-\xi)^2}{4k\sqrt{\delta^5}} + \frac{1}{2\sqrt{\delta^3}} \tag{4.18}$$

$$\begin{aligned} &\leq \frac{x^2+\xi^2}{2k\sqrt{\delta^5}}+\frac{1}{2\sqrt{\delta^3}} \\ &\leq \frac{A^2+\xi^2}{2k\sqrt{\delta^5}}+\frac{1}{2\sqrt{\delta^3}}. \end{aligned}$$

¿From the elementary inequality

$$(x-\xi)^2 \geq \frac{\xi^2-2x^2}{2}$$

it follows

$$\begin{aligned} \frac{(x-\xi)^2}{4kt} &\geq \frac{\xi^2-2x^2}{8kt} \\ &\geq \frac{\xi^2-2A^2}{8kT} \end{aligned} \tag{4.19}$$

Then, by (4.18), (4.19) and Theorem 4.6, we have

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}}\left(\frac{(x-\xi)^2}{4k\sqrt{t^5}}-\frac{1}{2\sqrt{t^3}}\right)\varphi(\xi)d\xi\right| \\ \leq\ & M\int_{-\infty}^{\infty} e^{\frac{2A^2-\xi^2}{8kT}}\left(\frac{A^2+\xi^2}{2k\sqrt{\delta^5}}+\frac{1}{2\sqrt{\delta^3}}\right)d\xi \\ \leq\ & C\int_{-\infty}^{\infty} e^{-\xi^2}\left(\xi^2+1\right)d\xi, \end{aligned}$$

where $C=C\left(M,A,k,T,\delta\right)$ is a constant, so it follows that the integral (4.13) is uniformly convergent. Note that the integral $\int_{-\infty}^{\infty}\xi^2e^{-\xi^2}d\xi$ is convergent and

$$\begin{aligned} \int_{-\infty}^{\infty}\xi^2e^{-\xi^2}d\xi &= -\frac{1}{2}\int_{-\infty}^{\infty}\xi de^{-\xi^2} \\ &= -\frac{1}{2}\left(\xi e^{-\xi^2}\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}e^{-\xi^2}d\xi\right) \\ &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$

Following the same way we show that

$$u_x(x,t)=\frac{1}{2\sqrt{\pi k}}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2}{4kt}}\left(\frac{\xi-x}{2k\sqrt{t^3}}\right)\varphi(\xi)d\xi,$$

$$u_{xx}(x,t) = \frac{1}{2\sqrt{\pi k}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \left(\frac{(x-\xi)^2}{4k^2\sqrt{t^5}} - \frac{1}{2k\sqrt{t^3}} \right) \varphi(\xi) d\xi$$

and the last integrals are uniformly convergent.

Then

$$u_t - ku_{xx} = 0, \text{ if } (x,t) \in [-A, A] \times [\delta, T].$$

As $A > 0$ and $0 < \delta < T$ are arbitrary

$$u_t - ku_{xx} = 0, \text{ if } (x,t) \in \mathbf{R} \times (0, T],$$

which completes the proof. ■

The initial data $\varphi(x)$ in (CD) is a continuous function in Theorem 4.4. It can be supposed $\varphi(x)$ to have a *jump discontinuity.*

A function $\varphi(x)$ is said to have a jump at x_0, if both left and right limits of $\varphi(x)$ exist

$$\varphi(x_0 - 0) = \lim_{\substack{x \to x_0 \\ x < x_0}} \varphi(x), \quad \varphi(x_0 + 0) = \lim_{\substack{x \to x_0 \\ x > x_0}} \varphi(x)$$

and

$$\varphi(x_0 - 0) \neq \varphi(x_0 + 0).$$

The function $\varphi(x)$ is said to be *piecewise continuous* if in each finite interval it has a finite number of jumps and is continuous at all other points.

Theorem 4.8. *Let $\varphi(x)$ be a bounded piecewise continuous function. Then formula* (4.10) *defines an infinitely differentiable function $u(x,t)$, which is a solution of the equation*

$$u_t - ku_{xx} = 0, \quad (x,t) \in \mathbf{R} \times (0, T)$$

and

$$\lim_{t \downarrow 0} u(x,t) = \frac{1}{2}(\varphi(x+0) + \varphi(x-0))$$

for all $x \in \mathbf{R}$.

Proof. Let x_0 be a point of jump discontinuity of $\varphi(x)$. As in the proof of Theorem 4.7 we show that

$$\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-\frac{p^2}{4}}\varphi(x_0 - p\sqrt{kt})dp \to \frac{1}{2}\varphi(x_0 - 0), \tag{4.20}$$

$$\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{0} e^{-\frac{p^2}{4}}\varphi(x_0 - p\sqrt{kt})dp \to \frac{1}{2}\varphi(x_0 + 0), \tag{4.21}$$

as $t \to 0,\ t > 0$. Let us prove (4.20). Suppose $\varepsilon > 0$ and $\delta > 0$ are such that

$$\left|\varphi(x_0 - p\sqrt{kt}) - \varphi(x_0 - 0)\right| < \varepsilon, \text{ if } 0 < p < \frac{\delta}{\sqrt{kt}}.$$

As $\int_0^{+\infty} e^{-\frac{p^2}{4}}dp = \sqrt{\pi}$ there exists $t_0 > 0$, such that

$$\int_{p>\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}}dp < \frac{\varepsilon\sqrt{\pi}}{2M}, \quad \text{if } 0 < t < t_0,$$

where, as before, $|\varphi(x)| \le M$ for every x. Then

$$\begin{aligned}
&\left|\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-\frac{p^2}{4}}\varphi(x_0 - p\sqrt{kt})dp - \frac{1}{2}\varphi(x_0 - 0)\right| \\
= &\left|\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-\frac{p^2}{4}}\Big(\varphi(x_0 - p\sqrt{kt}) - \varphi(x_0 - 0)\Big)dp\right| \\
\le &\frac{1}{2\sqrt{\pi}}\int_{0<p<\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}}\left|\varphi(x_0 - p\sqrt{kt}) - \varphi(x_0 - 0)\right|dp \\
&+\frac{1}{2\sqrt{\pi}}\int_{p>\frac{\delta}{\sqrt{kt}}} e^{-\frac{p^2}{4}}\Big(\left|\varphi(x_0 - p\sqrt{kt})\right| + |\varphi(x_0 - 0)|\Big)dp \\
< &\frac{1}{2\sqrt{\pi}}\left(\varepsilon\sqrt{\pi} + 2M\frac{\varepsilon\sqrt{\pi}}{2M}\right) = \varepsilon,
\end{aligned}$$

which proves (4.20). Analogously it can be proved (4.21), which completes the proof. ■

Example 4.1. *Solve the problem*

$$\begin{cases} u_t - u_{xx} = 0, & x \in \mathbf{R}, t > 0 \\ u(x,0) = e^{-x}, & x \in \mathbf{R}. \end{cases}$$

Solution. By the Poisson formula we have

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t} - \xi} d\xi.$$

Using

$$\begin{aligned} \frac{(x-\xi)^2}{4t} + \xi &= \frac{1}{4t}(x^2 - 2\xi x + \xi^2 + 4\xi t) \\ &= \frac{1}{4t}(x^2 + \xi^2 + 4t^2 - 2\xi x + 4\xi t - 4xt + 4xt - 4t^2) \\ &= \frac{(\xi + 2t - x)^2}{4t} + x - t \end{aligned}$$

and making change of variable $\dfrac{\xi + 2t - x}{2\sqrt{t}} = p$, we have

$$\begin{aligned} u(x,t) &= \frac{e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(\xi+2t-x)^2}{4t}} d\xi \\ &= \frac{e^{t-x} 2\sqrt{t}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-p^2} dp \\ &= e^{t-x}. \end{aligned}$$

Observe that $\lim_{t \downarrow 0} u(x,t) = e^{-x}$ for every $x \in \mathbf{R}$.

Example 4.2. *Solve the problem*

$$\begin{cases} u_t - u_{xx} = 0, & x \in \mathbf{R}, t > 0 \\ u(x,0) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| > 1. \end{cases} \end{cases}$$

Solution. We express the solution in terms of the error function of statistics

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp,$$

already used in Section 3.1, Example 3.4. Note that

$$\begin{aligned} \operatorname{erf}(0) &= 0\,, \quad \lim_{x \to +\infty} \operatorname{erf}(x) = 1, \\ \operatorname{erf}(-x) &= -\operatorname{erf}(x)\,. \end{aligned} \tag{4.22}$$

By the Poisson formula and the change of variable $\xi = x - 2\sqrt{t}p$

$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-1}^{1} e^{-\frac{(x-\xi)^2}{4t}} d\xi \\
&= \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-p^2} dp \\
&= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{t}}\right)\right).
\end{aligned}$$

By Theorem 4.8 and (4.22) we have

$$\begin{aligned}
\lim_{t\downarrow 0} u(1,t) &= \lim_{t\downarrow 0} \frac{1}{2}\left(\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right) - 0\right) \\
&= \frac{1}{2} = \frac{1}{2}(u(1+0,0) + u(1-0,0)).
\end{aligned}$$

Example 4.3. *Solve the Cauchy problem for the diffusion equation $u_t - u_{xx} = 0$ with initial data*

$$\varphi(x) = \begin{cases} 1-x, & 0 \le x \le 1, \\ 1+x, & -1 \le x \le 0, \\ 0, & |x| \ge 1. \end{cases}$$

Show that $u(x,t) \to 0$ as $t \to +\infty$ for every x.

Solution. Making the change of variable $\xi = x - 2\sqrt{t}p$ we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}}\left(\int_{-1}^{0} (1+\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi + \int_{0}^{1} (1-\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi\right) \\
&= \frac{1}{\sqrt{\pi}}\left((1+x)\int_{\frac{x}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-p^2} dp + (1-x)\int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x}{2\sqrt{t}}} e^{-p^2} dp \right. \\
&\quad \left. - 2\sqrt{t}\int_{\frac{x}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} p e^{-p^2} dp + 2\sqrt{t}\int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x}{2\sqrt{t}}} p e^{-p^2} dp\right) \\
&= \frac{1}{2}(1+x)\left(\operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)
\end{aligned}$$

$$+\frac{1}{2}(1-x)\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)-\operatorname{erf}\left(\frac{x-1}{2\sqrt{t}}\right)\right)$$
$$+\sqrt{\frac{t}{\pi}}\left(e^{-\frac{(x+1)^2}{4t}}-2e^{-\frac{x^2}{4t}}+e^{-\frac{(x-1)^2}{4t}}\right).$$

The graph of this function is given in Figure 4.1 using MAPLE in ScientifiCWorkPlacE.

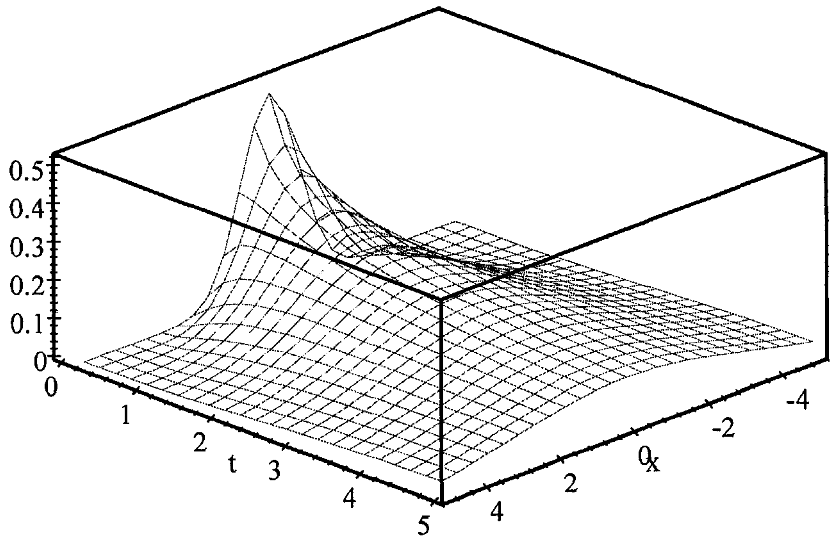

Figure 4.1. Graph of the function $u = u(x,t)$ in Example 4.3.

Note that for every $x \in \mathbf{R}$

$$\left|\sqrt{t}\int_{\frac{x}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} pe^{-p^2}\,dp\right| \le \sqrt{t}\int_{\frac{x}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} |p|\,dp = A(t) \to 0$$

as $t \to +\infty$. Indeed for $x \geq 0$

$$A(t) = \sqrt{t}\int_{\frac{x}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} pdp = \frac{1}{2}\left(\frac{(x+1)^2}{4\sqrt{t}} - \frac{x^2}{4\sqrt{t}}\right) \to 0,$$

as $t \to +\infty$.

If $-1 < x < 0$

$$\begin{aligned} A(t) &= \sqrt{t}\left(-\int_{\frac{x}{2\sqrt{t}}}^{0} pdp + \int_{0}^{\frac{x+1}{2\sqrt{t}}} pdp\right) \\ &= \frac{1}{2}\left(\frac{x^2}{4\sqrt{t}} + \frac{(x+1)^2}{4\sqrt{t}}\right) \to 0, \end{aligned}$$

as $t \to +\infty$.

Finally for $x \leq -1$

$$\begin{aligned} A(t) &= -\sqrt{t}\int_{\frac{x}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} pdp \\ &\leq \frac{1}{2}\left(-\frac{(x+1)^2}{4\sqrt{t}} + \frac{x^2}{4\sqrt{t}}\right) \to 0, \end{aligned}$$

as $t \to +\infty$.

The same way

$$\sqrt{t}\int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x}{2\sqrt{t}}} pe^{-p^2}dp \to 0 \text{ as } t \to +\infty.$$

Then by $\mathrm{erf}(0) = 0$ it follows that $\lim_{t\to+\infty} u(x,t) = 0$ for every $x \in \mathbf{R}$. Note that

$$u(0,t) = \mathrm{erf}\left(\frac{1}{2\sqrt{t}}\right) + 2\sqrt{\frac{t}{\pi}}\left(e^{-\frac{1}{4t}} - 1\right).$$

The graph of this function is given in Figure 4.2.

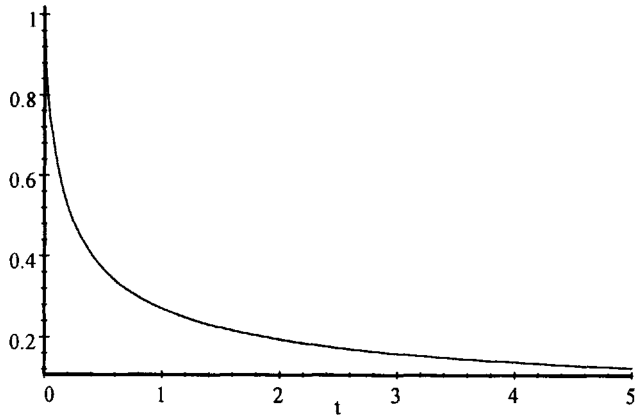

Figure 4.2. Graph of the function $u = \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right) + 2\sqrt{\frac{t}{\pi}}\left(e^{-\frac{1}{4t}} - 1\right)$

Exercises.

1. Solve the Cauchy problem for the diffusion equation $u_t - u_{xx} = 0$, $x \in \mathbf{R}$, $t > 0$ with initial data

(a) $u(x,0) = e^{-x^2}$,

(b) $u(x,0) = e^{-|x|}$,

(c) $u(x,0) = \begin{cases} 2, & \text{if } x > 0, \\ 4, & \text{if } x < 0. \end{cases}$

Compute $u(0,t)$ in the cases (a), (b) and show that $\lim\limits_{t\to\infty} u(0,t) = 0$. Compute $\lim\limits_{t\downarrow 0} u(0,t)$.

2. Consider the Cauchy problem for the diffusion equation with the initial condition $u(x,0) = \varphi(x)$. Show that if $\varphi(x)$ is an odd (even) function, then the solution $u(x,t)$ is also an odd (even) function of x.

3. Solve the Cauchy problem for the diffusion equation with constant dissipation

$$\begin{cases} u_t - ku_{xx} + bu = 0, & (x,t) \in \mathbf{R} \times (0,\infty), \\ u(x,0) = \varphi(x), & x \in \mathbf{R}. \end{cases}$$

4. Solve the Cauchy problem for the diffusion equation with convection

$$\begin{cases} u_t - ku_{xx} + vu_x = 0, & (x,t) \in \mathbf{R} \times (0,\infty), \\ u(x,0) = \varphi(x), & x \in \mathbf{R}. \end{cases}$$

4.3 Diffusion on the Half-line

Let us consider the diffusion equation on the half-line $(0, \infty)$ and take the Dirichlet boundary condition at the end-point $x = 0$.

Using the reflection method considered in Section 3.2 for the wave equation we shall treat the problem

$$\begin{cases} u_t - ku_{xx} = 0 & x \in (0, +\infty),\ t > 0, \\ u(x,0) = \varphi(x) & x \in (0, +\infty), \\ u(0,t) = 0 & 0 \le t. \end{cases} \tag{4.23}$$

We are looking for a solution formula for (4.23) analogous to the Poisson formula.

Let us consider the problem (CD) with initial data φ_o, which is the odd extension of $\varphi(x)$ on the whole line

$$\begin{cases} u_t - ku_{xx} = 0 & x \in \mathbf{R},\ t > 0, \\ u(x,0) = \varphi_o(x) & x \in \mathbf{R}, \end{cases} \tag{4.24}$$

where

$$\varphi_o(x) = \begin{cases} \varphi(x) & x > 0, \\ -\varphi(-x) & x < 0, \\ 0 & x = 0. \end{cases}$$

Let $u_o(x,t)$ be the unique solution of (4.24) which, by the Poisson formula, is

$$\begin{aligned} u_o(x,t) &= \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \varphi_o(\xi) d\xi \\ &= \frac{1}{2\sqrt{\pi kt}} \left(\int_0^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \varphi(\xi) d\xi - \int_{-\infty}^{0} \varphi(-\xi) e^{-\frac{(x-\xi)^2}{4kt}} d\xi \right) \\ &= \frac{1}{2\sqrt{\pi kt}} \left(\int_0^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \varphi(\xi) d\xi - \int_0^{\infty} \varphi(\xi) e^{-\frac{(x+\xi)^2}{4kt}} d\xi \right) \\ &= \frac{1}{2\sqrt{\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4kt}} - e^{-\frac{(x+\xi)^2}{4kt}} \right) \varphi(\xi) d\xi. \end{aligned} \tag{4.25}$$

The restriction

$$u(x,t) = u_o(x,t)|_{x \ge 0}$$

is the unique solution of the problem (4.23). Note that $u_o(x,t)$ satisfies the diffusion equation and is an odd function $u_o(-x,t) = -u_o(x,t)$, which easily follows from (4.25). Then $u(0,t) = u_o(0,t) = 0$ and $u(x,t)$ satisfies the diffusion equation. Moveover, $u(x,t)$ satisfies the initial condition for $x > 0$.

Let us consider now the Neumann boundary condition at the end point $x = 0$ for the diffusion equation on the half-line. Namely, let us consider the problem

$$\begin{cases} u_t - ku_{xx} = 0 & x \in (0,+\infty),\ t > 0, \\ u(x,0) = \varphi(x) & x \in (0,+\infty), \\ u_x(0,t) = 0 & t > 0. \end{cases} \tag{4.26}$$

In this case we use the even reflection of $\varphi(x)$

$$\varphi_e(x) = \begin{cases} \varphi(x) & x \geq 0, \\ \varphi(-x) & x \leq 0. \end{cases}$$

Let $u_e(x,t)$ be the solution of the problem

$$\begin{cases} u_t - ku_{xx} = 0, & x \in \mathbf{R},\ t > 0 \\ u(x,0) = \varphi_e(x), & x \in \mathbf{R}. \end{cases}$$

As before, we have

$$\begin{aligned} u_e(x,t) &= \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \varphi_e(\xi) d\xi \\ &= \frac{1}{2\sqrt{\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4kt}} + e^{-\frac{(x+\xi)^2}{4kt}} \right) \varphi(\xi) d\xi. \end{aligned}$$

The restriction $u(x,t) = u_e(x,t)|_{x\geq 0}$ is the solution of (4.26). Note that

$$\frac{\partial u_e}{\partial x}(x,t) = \frac{-1}{\sqrt{\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-\xi)^2}{4kt}} \left(\frac{x-\xi}{4kt} \right) + e^{-\frac{(x+\xi)^2}{4kt}} \left(\frac{x+\xi}{4kt} \right) \right) \varphi(\xi) d\xi$$

and

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \frac{\partial u_e}{\partial x}(0,t) \\ &= \frac{-1}{\sqrt{\pi kt}} \int_0^{\infty} \left(e^{-\frac{\xi^2}{4kt}} \left(\frac{-\xi}{4kt} \right) + e^{-\frac{\xi^2}{4kt}} \left(\frac{\xi}{4kt} \right) \right) \varphi(\xi) d\xi = 0. \end{aligned}$$

As before, $u(x,t)$ satisfies the diffusion equation and the initial condition.

Example 4.4. *Solve (4.23) with $\varphi(x) = e^{-x}$ and $k = 1$.*

Solution. By the solution formula for (4.23)

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_0^\infty \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) e^{-\xi} d\xi \\
&= \frac{1}{2\sqrt{\pi t}} \int_0^\infty \left(e^{-\left(\frac{(x-\xi)^2}{4t} + \xi\right)} - e^{-\left(\frac{(x+\xi)^2}{4t} + \xi\right)} \right) d\xi.
\end{aligned}
$$

Using

$$
\begin{aligned}
\frac{(x-\xi)^2}{4t} + \xi &= \frac{(\xi + 2t - x)^2}{4t} + x - t \\
\frac{(x+\xi)^2}{4t} + \xi &= \frac{(\xi + 2t + x)^2}{4t} - (x + t)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}} \left(e^{t-x} \int_0^\infty e^{-\frac{(\xi+2t-x)^2}{4t}} d\xi - e^{t+x} \int_0^\infty e^{-\frac{(\xi+2t+x)^2}{4t}} d\xi \right) \\
&= \frac{e^{t-x}}{\sqrt{\pi}} \int_{\frac{2t-x}{2\sqrt{t}}}^\infty e^{-p^2} dp - \frac{e^{t+x}}{\sqrt{\pi}} \int_{\frac{2t+x}{2\sqrt{t}}}^\infty e^{-p^2} dp \\
&= \frac{1}{2}\left(e^{t-x} - e^{t+x}\right) + \frac{1}{2}\left(e^{t+x} \operatorname{erf}\left(\frac{x+2t}{2\sqrt{t}}\right) + e^{t-x} \operatorname{erf}\left(\frac{x-2t}{2\sqrt{t}}\right)\right).
\end{aligned}
$$

Note that $u(0,t) = \frac{1}{2}e^t \left(\operatorname{erf}\left(\sqrt{t}\right) + \operatorname{erf}\left(-\sqrt{t}\right)\right) = 0$.

Example 4.5. *Solve (4.26) with $\varphi(x) = e^{-x}$ and $k = 1$.*

Solution. By the solution formula for (4.26) and previous calculations we obtain that the solution for $x \geq 0$ is

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_0^\infty \left(e^{-\left(\frac{(x-\xi)^2}{4t} + \xi\right)} + e^{-\left(\frac{(x+\xi)^2}{4t} + \xi\right)} \right) d\xi \\
&= \frac{1}{2}\left(e^{t-x} + e^{t+x}\right) + \frac{1}{2}\left(e^{t-x} \operatorname{erf}\left(\frac{x-2t}{2\sqrt{t}}\right) - e^{t+x} \operatorname{erf}\left(\frac{x+2t}{2\sqrt{t}}\right)\right)
\end{aligned}
$$

As $\text{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ we have

$$\begin{aligned} u_x(x,t) &= \frac{1}{2}\left(e^{t+x} - e^{t-x}\right) + \frac{1}{\sqrt{2}}[-e^{t-x}\,\text{erf}\left(\frac{x-2t}{2\sqrt{t}}\right) \\ &\quad + e^{t-x}\frac{1}{\sqrt{\pi t}}e^{-\frac{(x-2t)^2}{4t}} - e^{t+x}\,\text{erf}\left(\frac{x+2t}{2\sqrt{t}}\right) - e^{t+x}\frac{1}{\sqrt{\pi t}}e^{-\frac{(x+2t)^2}{4t}}], \\ u_x(0,t) &= 0. \end{aligned}$$

Exercises.

1. Prove the following maximum principle for the problem (4.26). If $\varphi(x)$ is a bounded continuous function, then the solution $u(x,t)$ of (4.26) satisfies

$$|u(x,t)| \leq \sup_{x\geq 0} |\varphi(x)|.$$

2. Let $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{p^2}{2}}dp$ be the density function of the standard normal distribution. Show that $\Phi(+\infty) = 1$ and $\Phi(-x) = 1 - \Phi(x)$.

(a) Derive that the problem (4.26) with the initial condition $\varphi(x) = e^{-x}$ has a solution

$$u(x,t) = e^{kt-x}\left(1 - \Phi\left(\frac{2kt-x}{\sqrt{2kt}}\right)\right) + e^{kt+x}\left(1 - \Phi\left(\frac{2kt+x}{\sqrt{2kt}}\right)\right).$$

(b) Using the maximum principle for (4.26) show that

$$\begin{aligned} e^{-ab}\Phi(a-b) + e^{ab}\Phi(-(a+b)) &\leq e^{-\frac{b^2}{2}}, \quad a \in \mathbf{R},\ b > 0, \\ 1 + \frac{1}{2}e^{-\frac{x^2}{2}} - e^{-\frac{3x^2}{8}} &\leq \Phi(x) \leq 1, \quad x \geq 0. \end{aligned}$$

4.4 Inhomogeneous Diffusion Equation on the Whole Line

Consider the problem of finding a function $u(x,t)$ such that

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x \in \mathbf{R},\ t > 0, \\ u(x,0) = \varphi(x) & x \in \mathbf{R}, \end{cases} \tag{4.27}$$

known as Cauchy problem for the inhomogeneous diffusion equation. By linear properties of the operator $Lu \equiv u_t - ku_{xx}$, the solution of (4.27) $u(x,t)$ is the sum of the solutions $v(x,t)$ and $w(x,t)$ of the problems

$$\begin{cases} v_t - kv_{xx} = 0 & x \in \mathbf{R},\ t > 0, \\ v(x,0) = \varphi(x) & x \in \mathbf{R}, \end{cases} \tag{4.28}$$

and

$$\begin{cases} w_t - kw_{xx} = f(x,t) & x \in \mathbf{R},\ t > 0, \\ w(x,0) = 0 & x \in \mathbf{R}, \end{cases} \tag{4.29}$$

respectively.

We have that

$$v(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \varphi(\xi) d\xi \tag{4.30}$$

and will show that

$$w(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi k(t-\tau)}} e^{-\frac{(x-\xi)^2}{4k(t-\tau)}} f(\xi,\tau) d\xi d\tau. \tag{4.31}$$

Denote

$$G(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}},$$

known as Green's function or *fundamental solution* of the diffusion operator L. It is clear that

$$L(G(x,t)) = 0, \qquad (x,t) \in \mathbf{R} \times (0,\infty), \tag{4.32}$$

$$L(G(x-\xi,t-\tau)) = 0, \qquad (x,t) \in \mathbf{R} \times (0,\infty),\ t \neq \tau.$$

¿From (4.30) and (4.31) it follows that the solution of (4.27) is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} G(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau. \tag{4.33}$$

Assuming that $f(x,t)$ is bounded and continuous on $\mathbf{R} \times (0,\infty)$, we prove that the function $w(x,t)$ given by (4.31) satisfies the problem (4.29).

By the maximum principle for the diffusion equation on the whole line it follows that the function $u(x,t)$ given by (4.33) is the unique solution of (4.27).

Theorem 4.9. *Let $f(x,t) \in C(\mathbf{R} \times (0,\infty))$ be a bounded function and*

$$v(x,t,\tau) = \int_{-\infty}^{\infty} G(x-\xi,t-\tau)f(\xi,\tau)d\xi.$$

Then the function

$$w(x,t) = \int_0^t v(x,t,\tau)d\tau$$

is a solution of the problem (4.29).

Proof. By virtue of the estimates of Section 4.2 and the assumption of the theorem the integrals

$$\int_0^t v(x,t,\tau)d\tau, \quad \int_0^t v_t(x,t,\tau)d\tau, \quad \int_0^t v_{xx}(x,t,\tau)d\tau$$

are uniformly convergent over bounded and closed intervals of $\mathbf{R}$.

By the formula for differentiation of integrals depending on parameters

$$\frac{\partial}{\partial t}\int_0^t f(x,t)dx = \int_0^t f_t(x,t)dx + f(t,t)$$

and we have

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial}{\partial t}\int_0^t v(x,t,\tau)d\tau = \frac{\partial}{\partial t}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} v(x,t,\tau)d\tau \\
&= \lim_{\varepsilon\to 0}\frac{\partial}{\partial t}\int_0^{t-\varepsilon} v(x,t,\tau)d\tau \\
&= \int_0^t v_t(x,t,\tau)d\tau + \lim_{\varepsilon\to 0} v(x,t,t-\varepsilon) \\
&= \int_0^t kv_{xx}(x,t,\tau)d\tau + \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} G(x-\xi,\varepsilon)f(\xi,t-\varepsilon)d\xi \\
&= k\frac{\partial^2}{\partial x^2}\left(\int_0^t v(x,t,\tau)d\tau\right) + \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2} f(x-2p\sqrt{k\varepsilon},t-\varepsilon)dp \\
&= k\frac{\partial^2 w}{\partial x^2} + f(x,t).
\end{aligned}$$

To show that the initial condition is satisfied observe that

$$\begin{aligned}
|w(x,t)| &\le \int_0^t |v(x,t,\tau)|\,d\tau \\
&\le \int_0^t \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\left|f(x-2p\sqrt{k(t-\tau)},\tau)\right| dp d\tau \\
&\le t\sup|f(x,t)|
\end{aligned}$$

and therefore $\lim_{t\downarrow 0} w(x,t) = 0$. ■

Consider now the inhomogeneous problem on the half-line with the Dirichlet boundary condition

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x>0, \quad t>0, \\ u(0,t) = h(t) & t>0, \\ u(x,0) = \varphi(x) & x>0. \end{cases} \tag{4.34}$$

We reduce (4.34) to a simpler problem letting $v(x,t) = u(x,t) - h(t)$. Then $v(x,t)$ satisfies the problem

$$\begin{cases} v_t - kv_{xx} = f(x,t) - h'(t) & x > 0, \quad t > 0, \\ v(0,t) = 0 & t > 0, \\ v(x,0) = \varphi(x) - h(0) & x > 0. \end{cases} \tag{4.35}$$

The solution $v(x,t)$ of (4.35) is the sum $v(x,t) = v_1(x,t) + v_2(x,t)$, where $v_1(x,t)$ and $v_2(x,t)$ are solutions of the problems

$$\begin{cases} v_{1t} - kv_{1xx} = 0 & x > 0, \quad t > 0, \\ v_1(0,t) = 0 & t > 0, \\ v_1(x,0) = \varphi(x) - h(0) & x > 0, \end{cases} \tag{4.36}$$

and

$$\begin{cases} v_{2t} - kv_{2xx} = f(x,t) - h'(t) & x > 0, \quad t > 0, \\ v_2(0,t) = 0 & t > 0, \\ v_2(x,0) = 0 & x > 0. \end{cases} \tag{4.37}$$

The solution of (4.36) is found by the reflection method of the previous section. Note that (4.37) can be solved again by the reflection method using the odd extension of the source function $F(x,t) = f(x,t) - h'(t)$. Namely, let $F_o(x,t)$ be the odd extension of $F(x,t)$ with respect to x and $w(x,t)$ be the solution of the problem

$$\begin{cases} w_t - kw_{xx} = F_o(x,t), & x \in \mathbf{R}, \quad t > 0 \\ w(x,0) = 0, \quad x \in \mathbf{R}, \end{cases}$$

given by

$$w(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-\tau) F_o(\xi,\tau) d\xi d\tau.$$

Then

$$v_2(x,t) = w(x,t)|_{x \geq 0}$$

is the solution of (4.37).

Let us show that $v_2(0,t) = 0$ for $t > 0$. We have

$$\begin{aligned} v_2(x,t) &= -\int_0^t \int_{-\infty}^{0} G(x-\xi, t-\tau) F(-\xi,\tau) d\xi d\tau \\ &\quad + \int_0^t \int_0^{\infty} G(x-\xi, t-\tau) F(\xi,\tau) d\xi d\tau \\ &= \int_0^t \int_0^{\infty} (G(x-\xi, t-\tau) - G(x+\xi, t-\tau)) F(\xi,\tau) d\xi d\tau, \\ v_2(0,t) &= \int_0^t \int_0^{\infty} (G(-\xi, t-\tau) - G(\xi, t-\tau)) F(\xi,\tau) d\xi d\tau = 0, \end{aligned}$$

because $G(x,t)$ is an even function with respect to x. ■

Exercise

1. Solve the inhomogeneous Neumann problem on the half-line

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x > 0, \quad t > 0, \\ u_x(0,t) = h(t) & t > 0, \\ u(x,0) = \varphi(x) & x > 0. \end{cases}$$

Chapter 5

Weak Solutions, Shock Waves and Conservation Laws

5.1 Weak Derivatives and Weak Solutions

Consider the Cauchy problem (CW). It was noted in Section 3.1 that to have a solution $u \in C^2(\mathbf{R} \times \mathbf{R}^+)$ of (CW) we require $\varphi \in C^2(\mathbf{R})$ and $\psi \in C^1(\mathbf{R})$. If the last assumptions are not satisfied then the solution given by D'Alembert formula is not a classical solution. How to justify the meaning of a solution in this case? There exist two main approaches. One is to introduce the so called weak derivatives so that the wave equation is satisfied in a form of integral identity. The other is the sequential approach. Consider approximating problems with smooth data $(\varphi_k, \psi_k) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$. It is possible to define a *weak (generalized) solution* of the problem by passing to the limit in L^2 spaces of corresponding solutions u_k. We prove that in some sense these two approaches are equivalent.

Let $L^2(\Omega)$ be the usual Lebesgue space of square integrable functions $u : \Omega \to \mathbf{R}$, where Ω is a measurable domain in $\mathbf{R}^n$. $L^2(\Omega)$ is a Banach space with a norm

$$||u||^2_{L^2(\Omega)} = \int\limits_{\Omega} u^2(x)\, dx.$$

Denote by $C_0^\infty(\Omega)$ the space of test functions, i.e. all functions $\rho(x) \in$

$C^\infty(\Omega)$ with compact support

$$\mathrm{supp}\rho = \overline{\{x \in \mathbf{R}^n : \rho(x) \neq 0\}}.$$

Next $L^2_{loc}(\Omega)$ is the space of functions $u : \Omega \to \mathbf{R}$, such that for every compact subset $K \subset \Omega$ it holds $u|_K \in L^2(K)$.

Definition 5.1. *A function $v \in L^2_{loc}(\Omega)$ is said to be the weak $\dfrac{\partial}{\partial x_j}$ derivative of a given function $u \in L^2_{loc}(\Omega)$ iff*

$$\int_\Omega v(x)\,\rho(x)\,dx = -\int_\Omega u(x)\,\frac{\partial \rho}{\partial x_j}(x)\,dx,$$

for every test function $\rho(x) \in C_0^\infty(\Omega)$.

If $u \in C^1(\Omega)$ it is easy to see that the weak $\frac{\partial}{\partial x_j}$ derivative of $u(x)$ is equal to $\dfrac{\partial u}{\partial x_j}$.

The *Sobolev*[1] *space* $W^{2,1}(\Omega)$ is introduced as the space of $L^2(\Omega)$ functions $u : \Omega \to \mathbf{R}$ for which there exist weak derivatives $\dfrac{\partial u}{\partial x_j} \in L^2(\Omega)$, $\quad j = 1, ..., n.$

$W^{2,1}(\Omega)$ is a Banach space with the norm

$$||u||^2_{W^{2,1}(\Omega)} = \int_\Omega \left(u^2 + \sum_{j=1}^n (\frac{\partial u}{\partial x_j})^2 \right) dx.$$

By $W^{2,1}_{loc}(\Omega)$ we denote the space of functions $u : \Omega \to \mathbf{R}$ such that for every compact subset $K \subset \Omega$ it holds $u|_K \in W^{2,1}(K)$. Every function of $W^{2,1}(\Omega)$ can be approximated by a sequence of smooth functions with respect to $W^{2,1}$ norm on compact subdomains of Ω.

Consider the so-called *Dirac*[2] *kernels* or *mollifying kernels* or the *Friedrichs*[3] *mollifiers.* Let $\varepsilon > 0$ and $\rho_0(x) \in C_0^\infty(\Omega)$

$$\rho_0(x) = \begin{cases} c_0 \exp\left(\frac{-1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

[1] Sergej Lvovich Sobolev, 08.10.1908–03.01.1989.
[2] Paul Adrien Maurice Dirac, 1902–1982.

[3] Kurt Otto Friedrichs, 1901-1982.

where the constant c_0 is such that $\int\limits_{\mathbf{R}^n} \rho_0(x)\,dx = 1$. Define the sequence of mollifiers

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho_0\left(\frac{x}{\varepsilon}\right)$$

for which $\rho_\varepsilon(x) \in C_0^\infty(\Omega)$, $\operatorname{supp}\rho_\varepsilon(x) = B_\varepsilon(0)$ and $\int\limits_{\mathbf{R}^n} \rho_\varepsilon(x)\,dx = 1$.

Mollifications or *Regularizations* $J_\varepsilon u$ of a function $u \in L^1_{loc}(\Omega)$ are defined as

$$J_\varepsilon u(x) = \int\limits_\Omega \rho_\varepsilon(x-y)\,u(y)\,dy.$$

If $u \in L^1_{loc}(\Omega)$, then $J_\varepsilon u \in C^\infty(\mathbf{R}^n)$ and

$$D_x^k J_\varepsilon u(x) = \int\limits_{\mathbf{R}^n} D_x^k \rho_\varepsilon(x-y)\,u(y)\,dy = (-1)^{|k|}\int\limits_{\mathbf{R}^n} D_y^k \rho_\varepsilon(x-y)\,u(y)\,dy.$$

Here $D_x^k = \dfrac{\partial^{|k|}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}$, $|k| = k_1 + \ldots + k_n$, is a partial derivative of order $|k|$.

Theorem 5.1. *For a given function $u \in W^{2,1}_{loc}(\Omega)$ the regularizations $J_\varepsilon u$ tend to u in $W^{2,1}(K)$ for every compact $K \subset \Omega$, i.e.*

$$||J_\varepsilon u - u||_{W^{2,1}(K)} \to 0 \text{ as } \varepsilon \to 0.$$

Proof. Let $\varepsilon < dist(K, \partial\Omega)$. By the change of variables $y = x - \varepsilon z$ and the Cauchy–Schwarz[4] inequality we have

$$\begin{aligned} J_\varepsilon u(x) &= \int\limits_\Omega \rho_\varepsilon(x-y)\,u(y)\,dy \\ &= \frac{1}{\varepsilon^n}\int\limits_{|y-x|\le\varepsilon} \rho_0\left(\frac{x-y}{\varepsilon}\right)u(y)\,dy \\ &= \int\limits_{|z|\le 1} \rho_0(z)\,u(x-\varepsilon z)\,dz, \end{aligned}$$

[4]Hermann Amandus Schwarz, 25.01.1843- 30.11.1921

$$\begin{aligned}
|J_\varepsilon u(x)| &\le \int\limits_{|z|\le 1} \rho_0^{1/2}(z)\,\rho_0^{1/2}(z)\,|u(x-\varepsilon z)|\,dz \\
&\le \left(\int\limits_{|z|\le 1} \rho_0(z)\,dz\right)^{1/2} \left(\int\limits_{|z|\le 1} \rho_0(z)\,u^2(x-\varepsilon z)\,dz\right)^{1/2} \\
&= \left(\int\limits_{|z|\le 1} \rho_0(z)\,u^2(x-\varepsilon z)\,dz\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}
\int\limits_K |J_\varepsilon u(x)|^2\,dx &\le \int\limits_{|z|\le 1} \rho_0(z)\left(\int\limits_K u^2(x-\varepsilon z)\,dx\right)dz \\
&\le \int\limits_{|z|\le 1} \rho_0(z)\,dz \int\limits_{K_\varepsilon} u^2(x)\,dx = \int\limits_{K_\varepsilon} u^2(x)\,dx
\end{aligned}$$

or

$$\|J_\varepsilon u(x)\|_{L^2(K)} \le \|u\|_{L^2(K_\varepsilon)},$$

where $K_\varepsilon = \{x \in \Omega : dist(x, K) \le \varepsilon\}$.

Let $\delta < \frac{\varepsilon}{3}$. There exist $v \in C(K_\varepsilon)$ such that

$$\|u - v\|_{L^2(K_\varepsilon)} < \delta.$$

By Exercise 2, (a) it follows

$$\|J_{\varepsilon_1} v - v\|_{C(K)} \to 0,$$

as $\varepsilon_1 \to 0$. Then for sufficiently small $\varepsilon_1 < \varepsilon$

$$\|J_{\varepsilon_1} v - v\|_{L^2(K)} < \delta.$$

Moreover

$$\|J_{\varepsilon_1}(u - v)\|_{L^2(K)} \le \|u - v\|_{L^2(K_\varepsilon)} < \delta,$$

and then

$$\begin{aligned}
||J_{\varepsilon_1}u - u||_{L^2(K)} &\leq ||J_{\varepsilon_1}(u-v)||_{L^2(K)} + ||J_{\varepsilon_1}v - v||_{L^2(K)} \\
&\quad + ||u-v||_{L^2(K)} \\
&\leq 2\delta + ||u-v||_{L^2(K_\varepsilon)} \\
&\leq 3\delta < \varepsilon.
\end{aligned}$$

A similar procedure is used to prove that $||D_j(J_{\varepsilon_1}u - u)||_{L^2(K)} \to 0$ as $\varepsilon_1 \to 0$ and

$$D_j(J_\varepsilon u)(x) = J_\varepsilon(D_j u)(x),$$

for $u \in W^{1,2}_{loc}(\Omega)$ –Exercise 4. ■

Definition 5.2. *A function $u \in W^{2,1}_{loc}(\mathbf{R}^2)$ is said to be a weak solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ iff*

$$\int_{\mathbf{R}^2} u\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt = 0$$

for every test function $\rho \in C_0^\infty(\mathbf{R}^2)$.

Definition 5.3. *A function $u \in W^{2,1}_{loc}(\mathbf{R}^2)$ is said to be a weak solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ iff there exists a sequence of smooth solutions $u_k(x,t) \in C^2(\mathbf{R}^2)$ of the wave equation such that for every compact set $K \subset \mathbf{R}^2$, $||u_k - u||_{W^{2,1}(K)} \to 0$ as $k \to \infty$.*

Theorem 5.2. *Definition 5.2 is equivalent to Definition 5.3.*

Proof. (a) Definition 5.3 $\Rightarrow$ Definition 5.2.

Let $\rho \in C_0^\infty(\mathbf{R}^2)$, supp$\rho \subset K$ and (u_k) be a sequence of C^2 smooth solutions of the wave equation such that $||u_k - u||_{W^{2,1}(K)} \to 0$ as $k \to \infty$. Integrating by parts, we have

$$\begin{aligned}
0 &= \int_{\mathbf{R}^2} \left(u_{ktt} - c^2u_{kxx}\right)\rho dxdt = \int_{\mathbf{R}^2} u_k\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt \\
&= \int_K u_k\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt.
\end{aligned}$$

By Definition 5.3 and the Cauchy–Schwarz inequality it follows

$$\begin{aligned}
\left|\int_K u\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt\right| &= \left|\int_K (u - u_k)\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt\right| \\
&\leq \int_K |u - u_k|\left|\rho_{tt} - c^2\rho_{xx}\right| dxdt \\
&\leq ||u - u_k||_{L^2(K)}||\rho_{tt} - c^2\rho_{xx}||_{L^2(K)} \\
&\leq ||u - u_k||_{W^{2,1}(K)}||\rho_{tt} - c^2\rho_{xx}||_{L^2(K)} \to 0
\end{aligned}$$

as $k \to \infty$.

Therefore

$$\int_K u\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt = \int_{\mathbf{R}^2} u\left(\rho_{tt} - c^2\rho_{xx}\right) dxdt = 0.$$

(b) Definition 5.2 $\Rightarrow$ Definition 5.3.

We use regularizations to construct approximating sequences of solutions. Let $u \in W^{2,1}_{loc}\left(\mathbf{R}^2\right)$ be a weak solution of the wave equation in the sense of Definition 5.2 and $u_\varepsilon = J_\varepsilon u$. For every compact set $K \subset \mathbf{R}^2$, $||u_\varepsilon - u||_{W^{2,1}(K)} \to 0$ as $\varepsilon \to 0$. It remains to prove that u_ε is a smooth solution of the wave equation. Denote for simplicity

$$\begin{aligned}
X &= (x,t), \quad dX = dxdt, \\
Y &= (y,\tau), \quad dY = dyd\tau,
\end{aligned}$$

$$L_X u = u_{tt} - c^2 u_{xx}, \quad L_Y u = u_{\tau\tau} - c^2 u_{yy}.$$

By integration by parts and Fubini theorem we have

$$\begin{aligned}
\int\limits_{\mathbf{R}^2} L_X u_\varepsilon(X)\,\rho(X)\,dX &= \int\limits_{\mathbf{R}^2} u_\varepsilon(X)\,L_X\rho(X)\,dX \\
&= \int\limits_{\mathbf{R}^2_X}\int\limits_{\mathbf{R}^2_Y} \rho_\varepsilon(X-Y)\,u(Y)\,L_X\rho(X)\,dY dX \\
&= \int\limits_{\mathbf{R}^2_Y} u(Y) \int\limits_{\mathbf{R}^2_X} \rho_\varepsilon(X-Y)\,L_X\rho(X)\,dX dY \\
&= \int\limits_{\mathbf{R}^2_Y} u(Y) \int\limits_{\mathbf{R}^2_X} L_X\rho_\varepsilon(X-Y)\,\rho(X)\,dX dY \\
&= \int\limits_{\mathbf{R}^2_Y} u(Y) \int\limits_{\mathbf{R}^2_X} L_Y\rho_\varepsilon(X-Y)\,\rho(X)\,dX dY \\
&= \int\limits_{\mathbf{R}^2_X} \rho(X) \int\limits_{\mathbf{R}^2_Y} u(Y)\,L_Y\rho_\varepsilon(X-Y)\,dY dX = 0.
\end{aligned}$$

As $\rho(X)$ is arbitrary it follows that $L_X u_\varepsilon(X) = 0$ which completes the proof. ■

Exercises

1. Show that the function $\dfrac{1}{(1-|x|)^m} \in L^2_{loc}(|x|<1)$ for every m, but $\dfrac{1}{(1-|x|)^m} \in L^2(|x|<1)$ for $m < \frac{1}{2}$.

2. (a) Let $u \in C(\bar{\Omega})$ and $K \subset \Omega$ be a compact set. Prove that

$$||J_\varepsilon u - u||_{C(K)} = \max_{x\in K} |\,(J_\varepsilon u - u)(x)\,| \to 0,$$

as $\varepsilon \to 0$.

(b) Let $u \in C^m(\bar{\Omega})$ and $K \subset \Omega$ be a compact set. Using

$$\begin{aligned}
D^k_x J_\varepsilon u(x) &= (-1)^{|k|} \int\limits_{\Omega} D^k_y \rho_\varepsilon(x-y)\,u(y)\,dy \\
&= \int\limits_{\Omega} \rho_\varepsilon(x-y)\,D^k_y u(y)\,dy,
\end{aligned}$$

for $|k| \leq m$, prove that

$$||J_\varepsilon u - u||_{C^m(K)} = \sum_{|k| \leq m} ||D^k (J_\varepsilon u - u)||_{C(K)} \to 0,$$

as $\varepsilon \to 0$.

3. (a) Let $u(x,t) = |x|$, $B = \{(x,t) : x^2 + t^2 < 1\}$. Verify that u has weak derivatives

$$\frac{\partial u}{\partial x} = sgn x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}, \quad \frac{\partial u}{\partial t} = 0$$

in B.

(b) Let $u(x,t) = sgn x$, $B = \{(x,t) : x^2 + t^2 < 1\}$. Show that u has not a weak derivative $\dfrac{\partial u}{\partial x}$ in B.

4. Prove that if $u \in W^{1,2}_{loc}(\Omega)$, then

$$D_j (J_\varepsilon u)(x) = J_\varepsilon (D_j u)(x).$$

5.2 Conservation Laws

We consider solutions of hyperbolic systems of conservation laws. These are systems of PDEs of the form

$$u_t(x,t) + (f(u(x,t)))_x = 0, \tag{5.1}$$

where $u : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^m$ is a vector function

$$u(x,t) = \begin{bmatrix} u_1(x,t) \\ \vdots \\ u_m(x,t) \end{bmatrix},$$

and $f : \mathbf{R}^m \to \mathbf{R}^m$ is a mapping

$$f(u(x,t)) = \begin{bmatrix} f_1(u_1(x,t), \ldots, u_m(x,t)) \\ \vdots \\ f_m(u_1(x,t), \ldots, u_m(x,t)) \end{bmatrix}.$$

The function u describes physical quantities as mass, momentum, energy in fluid dynamical problems. The mapping $f(u)$ is called a *flux function*. The system (5.1) is *hyperbolic* iff the Jacobian matrix

$$A = Jf(u) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{bmatrix},$$

has only real eigenvalues and is diagonalizable, i.e. there exists a complete set of m linearly independent eigenvectors.

The Euler[5] system in gas dynamics is a system of conservation laws. In one space dimension these equations are

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix} = 0, \tag{5.2}$$

where $\rho = \rho(x,t)$ is the density, v is the velocity, ρv is the momentum, p is the pressure and E is the energy. The equations (5.2) are known as

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, && \text{conservation of mass,} \\ (\rho v)_t + \left(\rho v^2 + p\right)_x &= 0, && \text{conservation of momentum,} \\ E_t + (v(E+p))_x &= 0, && \text{conservation of energy.} \end{aligned}$$

Introducing new variables

$$u(x,t) = \begin{bmatrix} \rho(x,t) \\ \rho(x,t)\, v(x,t) \\ E(x,t) \end{bmatrix} = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{bmatrix},$$

the system can be written in the form (5.1) with

$$f(u) = \begin{bmatrix} u_2 \\ \frac{u_2^2}{u_1} + p(u) \\ \frac{u_2}{u_1}(u_3 + p(u)) \end{bmatrix}.$$

[5]Leonard Euler, 15.04.1707–18.09.1783.

Example 5.1. *Assume $p(u) = u_1$. Then the system (5.2) is hyperbolic in* $\mathbf{R}^3 \setminus \{u : u_1 = 0\}$.

Solution. The Jacobian of

$$f(u) = \begin{bmatrix} u_2 \\ \frac{u_2^2}{u_1} + u_1 \\ \frac{u_2}{u_1}(u_3 + u_1) \end{bmatrix},$$

is the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{u_2^2}{u_1^2} + 1 & 2\frac{u_2}{u_1} & 0 \\ -\frac{u_2}{u_1}\left(\frac{u_3 + u_1}{u_1} - 1\right) & \frac{u_3 + u_1}{u_1} & \frac{u_2}{u_1} \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = \frac{u_2}{u_1}, \quad \lambda_2 = \frac{u_2}{u_1} + 1, \quad \lambda_3 = \frac{u_2}{u_1} - 1.$$

Corresponding eigenvectors are

$$v^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 1 \\ \frac{u_2}{u_1} + 1 \\ \frac{u_3 + u_2}{u_1} + 1 \end{bmatrix},$$

$$v^3 = \begin{bmatrix} 1 \\ \frac{u_2}{u_1} - 1 \\ \frac{u_3 - u_2}{u_1} + 1 \end{bmatrix},$$

which are linearly independent, because the determinant

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & \frac{u_2}{u_1} + 1 & \frac{u_2}{u_1} - 1 \\ 1 & \frac{u_3 + u_2}{u_1} + 1 & \frac{u_3 - u_2}{u_1} + 1 \end{vmatrix} = -2.$$

The simple initial value problem for the system (5.1) is the Cauchy problem in which (5.1) holds for $x \in \mathbf{R}$, $t > 0$ and

$$u(x,0) = u_0(x), \tag{5.3}$$

where $u_0(x)$ is a prescribed function.

Let us consider the Cauchy problem for the simplest equation of the form (5.1) in one dimension

$$\begin{aligned} u_t + au_x &= 0, \quad x \in \mathbf{R}, t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned} \tag{5.4}$$

where a is a constant, known as the linear advection equation or one-side wave equation. The problem (5.4) has the unique solution

$$u(x,t) = u_0(x - at), \tag{5.5}$$

if the initial function $u_0(x) \in C^1(\mathbf{R})$. It can be found by the method of characteristics of Chapter I. If $u_0 \in C^k(\mathbf{R})$ then $u(x,t) \in C^k(\mathbf{R} \times [0,\infty))$. The solution presents a right moving profile (graph) of the function $u_0(x)$ with speed a.

Example 5.2. *Solve the problem*

$$\begin{aligned} u_t + u_x &= 0, \\ u_0(x) &= \begin{cases} \cos^3 x & \text{if } x \in [3\pi/2, 5\pi/2], \\ 0 & \text{if } x \notin [3\pi/2, 5\pi/2]. \end{cases} \end{aligned}$$

The solution is

$$u(x,t) = u_0(x - t).$$

Note that $u_0 \in C^1(\mathbf{R})$, because

$$\left(\cos^3 x\right)' |_{x=3\pi/2, 5\pi/2} = 0.$$

Then $u(x,t) = u_0(x-t) \in C^1(\mathbf{R} \times [0,\infty))$. The graphs of $u(x,t)$ at the instants $t = 0$, 2, 4, 6 are plotted in Figure 5.1 using the *Mathematica* program

```
f[x_]:=Which[3Pi/2<=x<=5Pi/2, Cos[x]^3,True,0]
u[x_,t_]:=f[x-t]
h0=Plot[Evaluate[u[x,0],{x,5Pi/4,5Pi},
PlotRange->{0,2},PlotLabel->"Wave at t=0"]
h1=Plot[Evaluate[u[x,2],{x,5Pi/4,5Pi},
```

```
PlotRange->{0,2},PlotLabel->"Wave at t=2"]
h2=Plot[Evaluate[u[x,4],{x,5Pi/4,5Pi},
PlotRange->{0,2},PlotLabel->"Wave at t=4"]
h3=Plot[Evaluate[u[x,6],{x,5Pi/4,5Pi},
PlotRange->{0,2},PlotLabel->"Wave at t=6"]
Show[GraphicsArray[{{ho,h1},{h2,h3}}],
Frame->True, FrameTics->None]
```

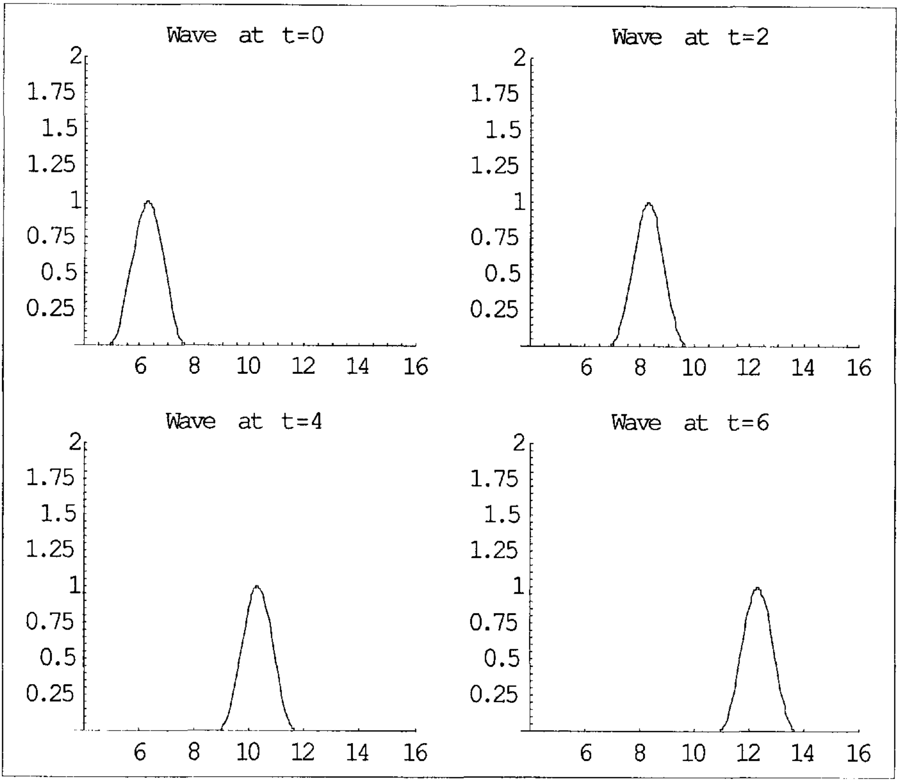

Figure 5.1. The wave $u(x,t)$ at the instants $t = 0$, 2, 4, 6.

Suppose now that $u_0(x)$ is not a smooth function. Then the function (5.5) is not smooth and does not satisfy (5.4) in the usual sense. It satisfies (5.4) in a weak (generalized) notion. An approach to generalize the notion of solution is to satisfy an integral identity.

Denote by $C_0^1(\mathbf{R}\times[0,\infty))$ the space of C^1 functions ρ vanishing outside of a compact set in $t\geq 0$, i.e. there exists $T>0$ such that $supp\rho\subseteq[-T,T]\times[0,T]$, so that $\rho=0$ outside of $[-T,T]\times[0,T]$ and on the lines $t=T,\ x=-T$ and $x=T$.

Definition 5.4. *Assume that* $u_0(x)\in L^1_{loc}(\mathbf{R})$. *A function* $u(x,t)\in L^1_{loc}(\mathbf{R}\times[0,\infty))$ *is a weak solution of* (5.4) *iff*

$$\int_0^\infty\int_{-\infty}^\infty u(\rho_t+a\rho_x)\,dxdt+\int_{-\infty}^\infty u_0(x)\,\rho(x,0)\,dx=0,$$

for every test function $\rho\in C_0^1(\mathbf{R}\times[0,\infty))$.

Proposition 5.1. *Let* $u(x,t)$ *be a smooth solution of the problem* (5.4). *Then* $u(x,t)$ *is a weak solution of the problem.*

Proof. Obviously $u(x,t)\in L^1_{loc}(\mathbf{R}\times[0,\infty))$. Let $\rho(x,t)\in C_0^1(\mathbf{R}\times[0,\infty))$ and supp$\rho\subseteq[-T,T]\times[0,T]$. Multiplying (5.4) by ρ, integrating in $[-T,T]\times[0,T]$ and using $\rho(\pm T,t)=\rho(x,T)=0$, we obtain

$$\begin{aligned}
0 &= \int_0^T\int_{-T}^T(u_t+au_x)\,\rho dxdt\\
&= \int_0^T\int_{-T}^T\left((u\rho)_t+a\,(u\rho)_x-u\,(\rho_t+a\rho_x)\right)dxdt\\
&= \int_{-T}^T\int_0^T(u\rho)_t\,dtdx-\int_0^T\int_{-T}^T u\,(\rho_t+a\rho_x)\,dxdt\\
&= -\left(\int_{-T}^T u_0(x)\,\rho(x,0)\,dx+\int_0^T\int_{-T}^T u\,(\rho_t+a\rho_x)\,dxdt\right). \blacksquare
\end{aligned}$$

Another approach to generalize the notion of solution of (5.4) is to approximate the nonsmooth initial function $u_0(x)$ with a sequence of smooth functions $u_{n0}(x)$. The function $u_n(x,t)=u_{n0}(x-at)$ is the solution of the problem (5.4) with initial data $u_{n0}(x)$. Then a generalized solution of (5.4) is defined as a L^1- limit of the sequence $u_n(x,t)$.

Definition 5.5. *Assume that* $u_0(x) \in L^1(\mathbf{R})$. *The function* $u = u(x,t) \in L^1(\mathbf{R} \times [0,\infty))$ *is a strong solution of (5.4) iff*

$$\lim_{n\to\infty} \int_0^\infty \int_{-\infty}^\infty |u(x,t) - u_{n0}(x-at)|\, dxdt = 0,$$

for any sequence (u_{n0}) *of smooth functions such that*

$$\lim_{n\to\infty} \int_{-\infty}^\infty |u_{n0}(x) - u_0(x)|\, dx = 0.$$

It can be proved (Exercise 2, b) that a strong solution is a weak solution. Unfortunately the sequential approach is not appropriate for nonlinear differential equations.

As an extension of one-side wave equations we consider linear *strictly hyperbolic systems*

$$\begin{aligned} u_t + Au_x &= 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{5.6}$$

Here $u : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^m$, $A \in \mathbf{R}^{m\times m}$ is a constant matrix. The system (5.6) is strictly hyperbolic iff the matrix A is diagonalizable and has m distinct real eigenvalues. Let

$$A = R\Lambda R^{-1}, \tag{5.7}$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_m), \quad R = [r_1 \mid r_2 \mid \ldots \mid r_m]$$

is the matrix of right eigenvectors

$$Ar_k = \lambda_k r_k, \quad k = 1, 2, \ldots, m.$$

Changing the variables

$$v = R^{-1}u,$$

by $R^{-1}u_t + \Lambda R^{-1}u_x = 0$, we obtain

$$v_t + \Lambda v_x = 0, \tag{5.8}$$

or componentwise

$$(v_k)_t + \lambda_k (v_k)_x = 0, \quad k = 1, 2, \ldots, m.$$

The initial conditions to (5.8) are

$$\begin{aligned} v(x,0) &= v_0(x) = R^{-1} u_0(x), \\ v_k(x,0) &= v_{k0}(x) = \left(R^{-1} u_0(x)\right)_k \end{aligned} \tag{5.9}$$

Then

$$v_k(x,t) = v_{k0}(x - \lambda_k t) \tag{5.10}$$

is the solution of (5.8), (5.9).

The solution of (5.6) is

$$u(x,t) = \sum_{k=1}^{m} v_{k0}(x - \lambda_k t) r_k.$$

Example 5.3. *Solve the problem*

$$\begin{aligned} u_t + A u_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned}$$

where $u : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^3$,

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix}$$

$$u_0(x) = \begin{bmatrix} f_1(x) - \frac{7}{2} f_2(x) + f_3(x) \\ f_1(x) + f_2(x) \\ f_1(x) - \frac{13}{2} f_2(x) + f_3(x) \end{bmatrix},$$

$$\begin{aligned} f_1(x) &= \begin{cases} -\sin^3 x & x \in [-3\pi, -2\pi] \\ 0 & x \notin [-3\pi, -2\pi] \end{cases}, \\ f_2(x) &= \begin{cases} -\sin^3 x & x \in [-\pi, 0] \\ 0 & x \notin [-\pi, 0] \end{cases}, \\ f_3(x) &= \begin{cases} -\sin^3 x & x \in [\pi, 2\pi] \\ 0, & x \notin [\pi, 2\pi] \end{cases}. \end{aligned}$$

Solution. The matrix

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{bmatrix},$$

has eigenvalues

$$\lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = 1$$

and corresponding eigenvectors

$$r_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, r_2 = \begin{bmatrix} -\frac{7}{2} \\ 1 \\ -\frac{13}{2} \end{bmatrix}, r_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$R = [r_1 \mid r_2 \mid r_3] = \begin{bmatrix} 1 & -\frac{7}{2} & 1 \\ 1 & 1 & 0 \\ 1 & -\frac{13}{2} & 1 \end{bmatrix},$$

$$R^{-1} = \begin{bmatrix} -\frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{5}{2} & -1 & -\frac{3}{2} \end{bmatrix}, \Lambda = R^{-1}AR = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$v = \begin{bmatrix} f_1(x-2t) \\ f_2(x+t) \\ f_3(x-t) \end{bmatrix},$$

$$u = Rv = \begin{bmatrix} f_1(x-2t) - \frac{7}{2} f_2(x+t) + f_3(x-t) \\ f_1(x-2t) + f_2(x+t) \\ f_1(x-2t) - \frac{13}{2} f_2(x+t) + f_3(x-t) \end{bmatrix}.$$

The graph of the function $u_1(x,t)$ is given in Figure 5.2 plotted by the *Mathematica* program

```
Clear[f,g,h,u]
f[x_]:=Which[-3Pi <=x<= -2Pi, -Sin[x]^3,True,0]
g[x_]:=Which[-Pi <=x<= 0, -Sin[x]^3,True,0]
h[x_]:=Which[Pi <=x<= 2Pi, -Sin[x]^3,True,0]
u[x_,t_]:=f[x-2t]-7g[x+t]/2+h[x-t]
Plot3D[u[x,t],{x,-4Pi,8Pi},{t,0,6Pi}
AxesLabel->"Position","Time","Value u1",
PlotPoints->40, PlotRange->{-4,1}, Shading->False]
```

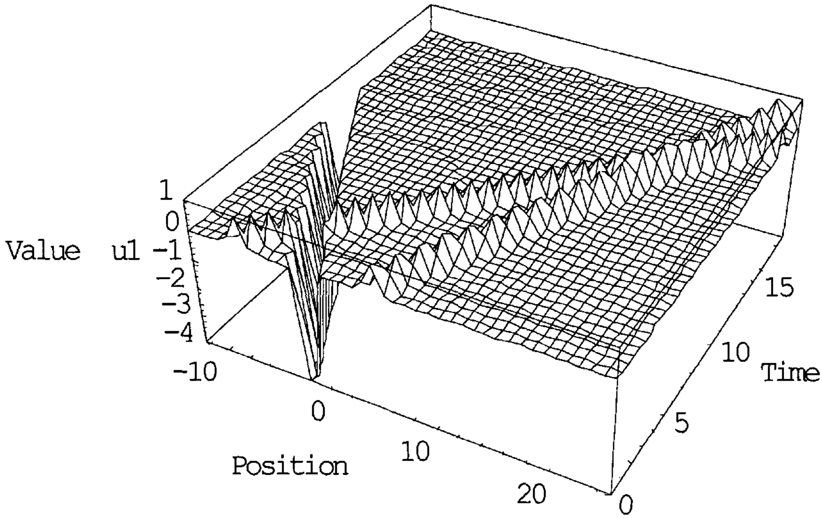

Figure 5.2. Graph of the function $u = u_1(x,t)$.

All calculations in Examples 5.1 and 5.3 are made by MAPLE in ScientifiCWorkPlacE.

Exercises.

1. Consider the problem

$$\begin{aligned} u_t + Au_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned}$$

with

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Show that the system is hyperbolic, but not strictly hyperbolic. Diagonalize it and solve the Cauchy problem with initial conditions

$$u_0(x) = \begin{bmatrix} 2f_1(x) + f_2(x) \\ f_1(x) - 2f_2(x) - 2f_3(x) \\ 2f_1(x) + f_3(x) \end{bmatrix},$$

where $f_1(x), f_2(x), f_3(x)$ are given in Example 3. Plot the graph of the function $u_2(x,t)$.

2. (a) Let $u_0(x) \in L^1_{loc}(\mathbf{R})$ be any locally integrable function. Prove that the function $u(x,t) = u_0(x-t)$ is a weak solution of the Cauchy problem

$$\begin{aligned} u_t + u_x &= 0, \quad x \in \mathbf{R}, t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned}$$

in the sense of Definition 5.4, i.e. $u(x,t)$ satisfies the identity

$$\int_0^\infty \int_{-\infty}^\infty u_0(x-t)(v_t + v_x)\, dxdt + \int_{-\infty}^\infty u_0(x) v(x,0)\, dx = 0,$$

for any $v \in C_0^1(\mathbf{R} \times [0,\infty))$.

(b) Prove that if the function $u = u(x,t) \in L^1(\mathbf{R} \times [0,\infty))$ is a strong solution of (5.4) then it is a weak solution of (5.4).

5.3 Burgers' Equation

The simplest equation combining both nonlinear propagation and diffusion effects is the Burgers'[6] equation

$$u_t + uu_x = \varepsilon u_{xx}. \tag{5.11}$$

The equation (5.11) was studied at first in a physical context by Bateman (1915). Subsequently, Burgers (1948) rederived it as a model equation in the theory of turbulence. Around 1950, Hopf[7] and independently Cole[8], showed that the exact solution of (5.11) could be found by using the transformation

[6] J.M. Burgers. A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech., 45 (1948), 171–199.

[7] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. 3(1950), 201–230.

[8] J.D. Cole. On a quasilinear parabolic equation occuring in aerodynamics. Q. Appl. Math 9 (1951), 225–236.

$$u = \psi_x, \quad \psi = -2\varepsilon \ln \varphi. \tag{5.12}$$

This, known now as *Hopf–Cole transformation,* reduces (5.11) to the diffusion equation

$$\varphi_t = \varepsilon \varphi_{xx}. \tag{5.13}$$

The motivation of (5.12) is as follows. Let us rewrite (5.11) as a conservation law

$$u_t - \left(\varepsilon u_x - \frac{1}{2} u^2 \right)_x = 0$$

and try to find $\psi \in C^2$ such that

$$\begin{cases} \psi_x = u, \\ \psi_t = \varepsilon u_x - \frac{1}{2} u^2. \end{cases} \tag{5.14}$$

Then $\psi_{xt} = \psi_{tx}$ implies (5.11). From (5.14) it follows

$$\psi_t = \varepsilon \psi_{xx} - \frac{1}{2} \psi_x^2. \tag{5.15}$$

Now introducing

$$\varphi(x,t) = e^{-\frac{1}{2\varepsilon} \psi(x,t)},$$

it is easy to show that (5.15) is equivalent to the diffusion equation (5.13).

Let us consider the Cauchy problem for equation (5.11) with initial condition

$$u(x,0) = u_0(x).$$

Under the transformation (5.12) the initial condition reduces to

$$\varphi(x,0) = e^{-\frac{1}{2\varepsilon} \int\limits_0^x u_0(s)\, ds}. \tag{5.16}$$

By the Poisson formula the problem (5.13), (5.16) has the unique solution

$$\varphi(x,t) = \frac{1}{2\sqrt{\varepsilon\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\varepsilon}\int_0^{\xi} u_0(s)\,ds - \frac{1}{4\varepsilon t}(x-\xi)^2} d\xi$$

$$= \frac{1}{2\sqrt{\varepsilon\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\varepsilon} g(x,t,\xi)} d\xi,$$

where

$$g(x,t,\xi) = \frac{(x-\xi)^2}{2t} + \int_0^{\xi} u_0(s)\,ds.$$

The exact solution of (5.11) is

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} e^{-\frac{1}{2\varepsilon} g(x,t,\xi)} d\xi}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\varepsilon} g(x,t,\xi)} d\xi}. \tag{5.17}$$

We consider the behavior of the solution (5.17) as $\varepsilon \to 0$, while $(x, t, u_0(x))$ is fixed.

Let us recall an asymptotic formula derived by the so called *steepest descent* method

$$\int_{-\infty}^{\infty} f(x)\, e^{-\frac{1}{2\varepsilon} g(x)} dx \sim \sqrt{\frac{4\varepsilon\pi}{g''(s)}} f(s)\, e^{-\frac{1}{2\varepsilon} g(s)}, \tag{5.18}$$

where s is a strong local minimum point of $g(x)$:

$$g'(s) = 0, \quad g''(s) > 0.$$

The estimate is motivated by Taylor's[9] formula and the Poisson integral

$$\int_{-\infty}^{\infty} f(x)\, e^{-\frac{1}{2\varepsilon} g(x)} dx \quad \sim \quad \int_{-\infty}^{\infty} f(s)\, e^{-\frac{1}{2\varepsilon}\left(g(s) + \frac{1}{2} g''(s)(x-s)^2\right)} dx$$

[9] Brook Taylor, 18.09.1685–29.12.1731.

$$= f(s)\, e^{-\frac{1}{2\varepsilon} g(s)} \int_{-\infty}^{\infty} e^{-\frac{1}{4\varepsilon} g''(s)(x-s)^2} dx$$

$$= \sqrt{\frac{4\varepsilon\pi}{g''(s)}} f(s)\, e^{-\frac{1}{2\varepsilon} g(s)}.$$

In order to apply the asymptotic formula (5.18) to (5.17) we need to study critical points of the function $g(x, t, \xi)$ with respect to ξ.

Suppose that there is only one strong local minimum $\xi(x, t)$ which satisfies

$$\frac{\partial g}{\partial \xi} = u_0(\xi) - \frac{x - \xi}{t} = 0. \tag{5.19}$$

¿From (5.17), in view of (5.18), it follows

$$u(x, t) \sim \frac{x - \xi}{t} = u_0(\xi).$$

The asymptotic solution may be rewritten as

$$\begin{cases} u = u_0(\xi), \\ x = \xi + t u_0(\xi), \end{cases}$$

or in implicit form

$$u = u_0(x - ut). \tag{5.20}$$

The last function is exactly the solution of the problem

$$\begin{aligned} u_t + u u_x &= 0, \quad x \in \mathbf{R}, t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned} \tag{5.21}$$

found by the method of characteristics - Chapter I. The solution (5.20) is smooth for small t if $u_0(x)$ is a smooth function. Differentiating (5.20) with respect to x , we have

$$u_x = u_0'(\xi)(1 - t u_x)$$

or

$$u_x = \frac{u_0'(\xi)}{1 + u_0'(\xi)\, t},$$

if

$$1 + u_0'(\xi)\, t \neq 0.$$

Suppose $u_0'(x) < 0$ for every x. Then $u_x = \infty$ if $t = -\dfrac{1}{u_0'(\xi)}$. The first instant T_0 when $u_x = \infty$, known as *gradient catastrophe,* corresponds to a s_0 where $u_0'(x)$ has a minimum

$$T_0 = -\frac{1}{u_0'(s_0)}, \quad u_0''(s_0) = 0, \; u_0'''(s_0) > 0.$$

Example 5.4. *Find the instant of gradient catastrophe for the problem*

$$\begin{aligned} u_t + uu_x &= 0, \quad x \in \mathbf{R}, t > 0, \\ u(x,0) &= -\tanh\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbf{R}. \end{aligned}$$

Solution. The solution of the problem in implicit form is

$$u(x,t) = -\tanh\left(\frac{1}{\varepsilon}\left(x - u(x,t)\, t\right)\right).$$

For the function $u_0(x) = -\tanh\left(\frac{x}{\varepsilon}\right)$

$$u_0'(x) = -\frac{1}{\varepsilon \cosh^2 \frac{x}{\varepsilon}}$$

which has a minimum at $x = 0$

$$\begin{aligned} \min u_0'(x) &= -\max \frac{1}{\varepsilon \cosh^2 \frac{x}{\varepsilon}} \\ &= -\frac{1}{\varepsilon \min \cosh^2 \frac{x}{\varepsilon}} \\ &= -\frac{1}{\varepsilon}, \end{aligned}$$

because

$$\cosh^2 \frac{x}{\varepsilon} = \left(\frac{e^{x/\varepsilon} + e^{-x/\varepsilon}}{2}\right)^2 \geq 1.$$

Then $T_0 = \varepsilon \to 0$ as $\varepsilon \to 0$. Note that

$$\lim_{\varepsilon \to 0} \left(-\tanh\left(\frac{x}{\varepsilon}\right)\right) = 1 - 2H(x),$$

where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

The graphs of the functions $-\tanh\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ are given in Figure 5.3.

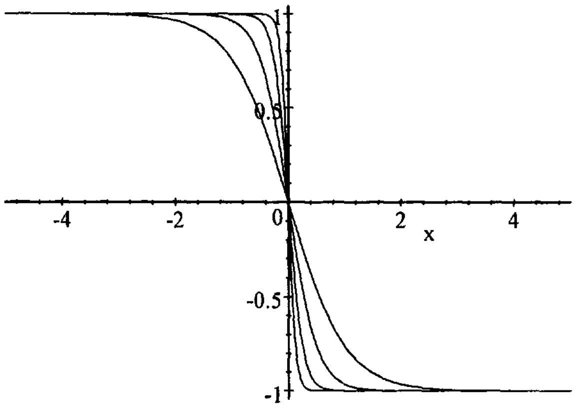

Figure 5.3. Graphs of the functions $-\tanh\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

The gradient catastrophe is demonstrated in Figure 5.4 by the *Mathematica* program

```
u[s_]:=-Tanh[s]
x[s_,t_]:=s+tu[s]
h0=ParametricPlot[Evaluate[Evaluate[x[s,0],u[s]],{s,-5,5},
PlotRange->{-1.01,1.01},PlotLabel->"t=0"]
h1=ParametricPlot[Evaluate[Evaluate[x[s,1],u[s]],{s,-5,5},
PlotRange->{-1.01,1.01},PlotLabel->"t=1"]
h2=ParametricPlot[Evaluate[Evaluate[x[s,2],u[s]],{s,-5,5},
PlotRange->{-1.01,1.01},PlotLabel->"t=2"]
h3=ParametricPlot[Evaluate[Evaluate[x[s,3],u[s]],{s,-5,5},
```

```
PlotRange->{-1.01,1.01},PlotLabel->"t=3"]
Show[GraphicsArray[{{h0,h1},{h2,h3}}],
Frame->True, FrameTics->None]
```

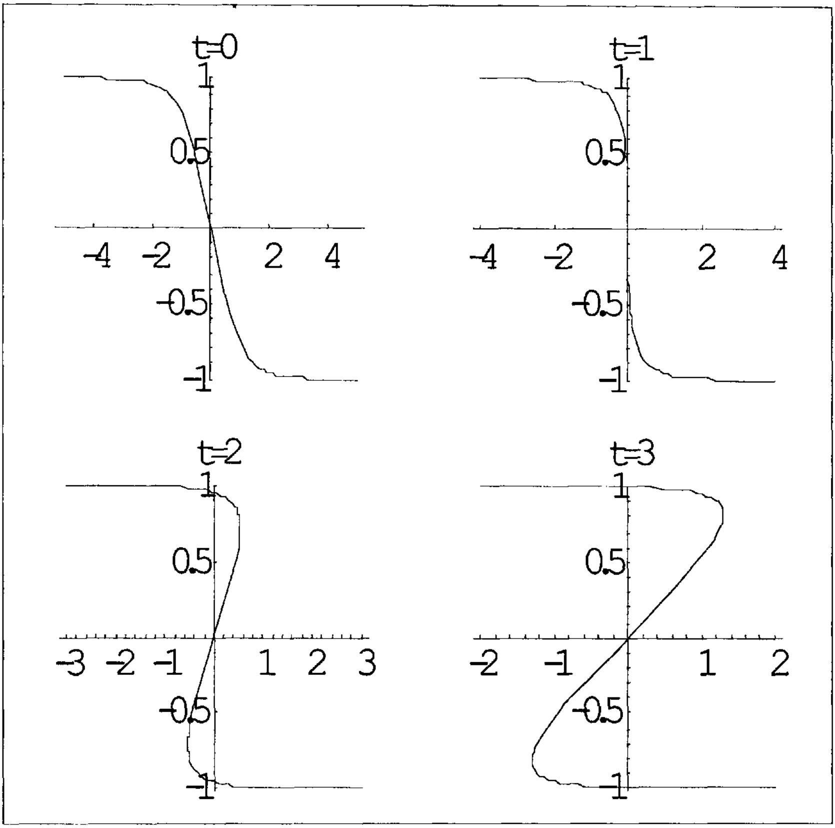

Figure 5.4. Gradient catastrophe.

In Figure 5.5 it is given the surface in $\mathbf{R}^3$

$$S : \begin{cases} x = s - t\tanh s, \\ u = -\tanh s, \\ t = t. \end{cases}$$

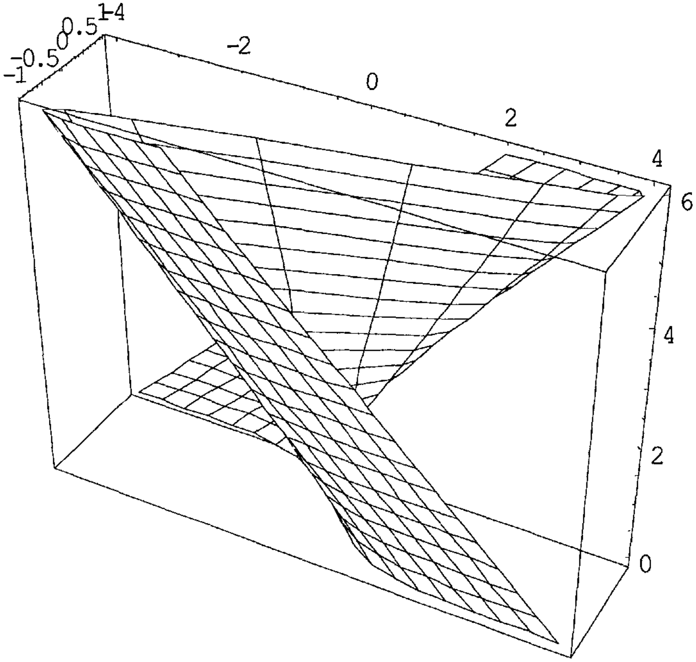

Figure 5.5. The surface $S(x(s,t), u(s,t), t)$.

Let us return to the asymptotic behavior of solution (5.17) as $\varepsilon \to 0$. For a given $u_0(x)$ and fixed (x,t) it is possible to have three solutions of equation (5.19) $\xi_1 < \xi_0 < \xi_2$, such that ξ_1 and ξ_2 are points of local minima of $u_0(x)$, while ξ_0 is a local maximum. It is possible to have

$$g(x,t,\xi_1) < g(x,t,\xi_2) \tag{5.22}$$

or

$$g(x,t,\xi_1) > g(x,t,\xi_2) \tag{5.23}$$

For the function $u_0(x) = e^{-x^2}$ the situation (5.22) is demonstrated in Figure 5.6.a and Figure 5.6.b while (5.23) in Figure 5.7.a and Figure 5.7.b.

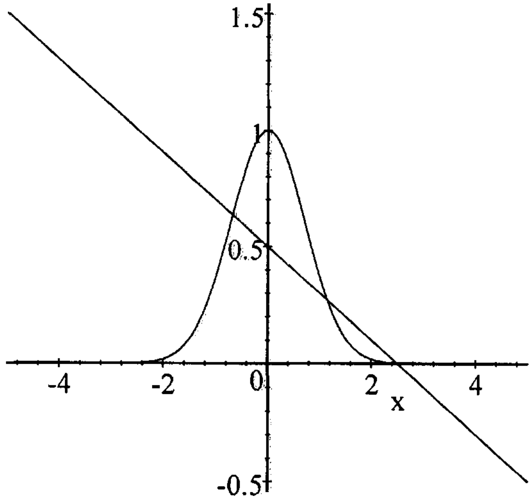

Figure 5.6 a. Graphs of functions $y = e^{-x^2}$ and $y = \frac{2.5-x}{5}$.

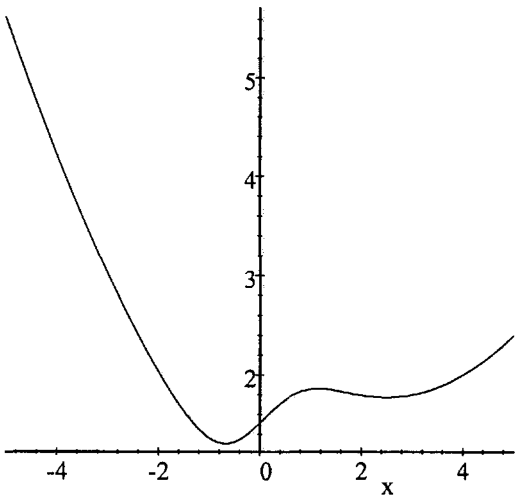

Figure 5.6.b. Graph of $g(x,t,\xi) = \frac{(2.5-x)^2}{10} + \int_{-\infty}^{x} e^{-t^2} dt$,
(the case $g(x,t,\xi_1) < g(x,t,\xi_2)$)

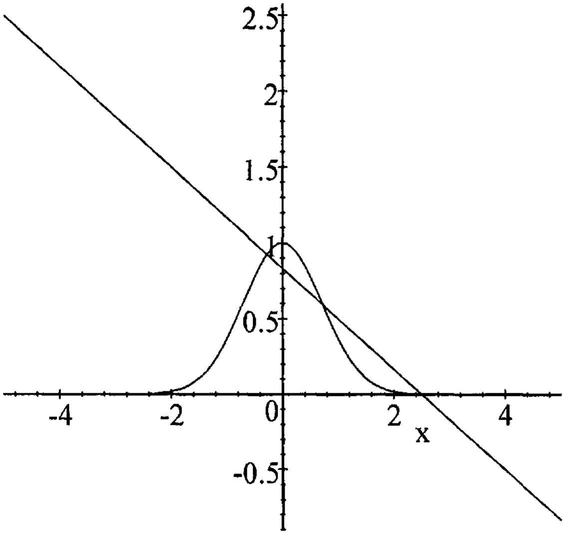

Figure 5.7 a. Graphs of functions $y = e^{-x^2}$ and $y = \frac{2.5-x}{3}$.

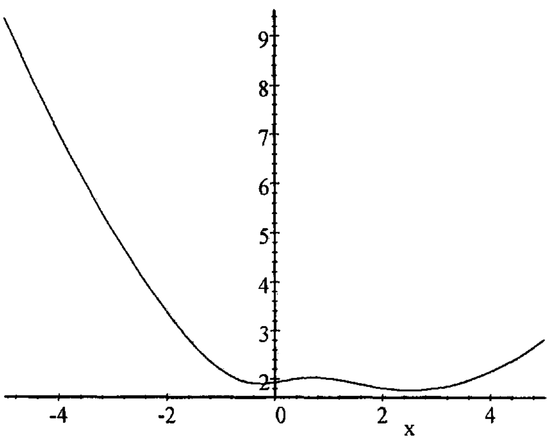

Figure 5.7.b. Graph of $g(x,t,\xi) = \frac{(2.5-x)^2}{6} + \int_{-\infty}^{x} e^{-t^2} dt$,
(the case $g(x,t,\xi_1) > g(x,t,\xi_2)$)

In the case (5.22), by virtue of (5.18), we have

$$\begin{aligned}
u(x,t) &\sim \frac{\frac{x-\xi_1}{t}\frac{1}{\sqrt{g''(\xi_1)}}e^{-\frac{g(\xi_1)}{2\varepsilon}}+\frac{x-\xi_2}{t}\frac{1}{\sqrt{g''(\xi_2)}}e^{-\frac{g(\xi_2)}{2\varepsilon}}}{\frac{1}{\sqrt{g''(\xi_1)}}e^{-\frac{g(\xi_1)}{2\varepsilon}}+\frac{1}{\sqrt{g''(\xi_2)}}e^{-\frac{g(\xi_2)}{2\varepsilon}}} \\
&= \frac{\frac{x-\xi_1}{t}+\frac{x-\xi_2}{t}\sqrt{\frac{g''(\xi_1)}{g''(\xi_2)}}e^{-\frac{g(\xi_2)-g(\xi_1)}{2\varepsilon}}}{1+\sqrt{\frac{g''(\xi_1)}{g''(\xi_2)}}e^{-\frac{g(\xi_2)-g(\xi_1)}{2\varepsilon}}} \\
&\to \frac{x-\xi_1}{t}=u_0(\xi_1),
\end{aligned}$$

as $\varepsilon \to 0$. Similarly in the case (5.23)

$$u(x,t) \sim \frac{x-\xi_2}{t}=u_0(\xi_2),$$

as $\varepsilon \to 0$. Both ξ_1 and ξ_2 depend on (x,t).

The inequality $g(x,t,\xi_1) < g(x,t,\xi_2)$ or its opposite determines the behavior of $u(x,t)$ as $\varepsilon \to 0$ at a given (x,t). For a fixed x the changeover from ξ_1 to ξ_2 occurs at an instant τ such that $g(x,\tau,\xi_1(x,\tau)) = g(x,\tau,\xi_2(x,\tau))$ which implies

$$\frac{1}{2\tau}\left((x-\xi_1)^2-(x-\xi_2)^2\right)=\int_{\xi_1}^{\xi_2}u_0(s)\,ds. \tag{5.24}$$

The last equation means that the regions in (ξ,u) plane between the graphs of $u_0(\xi)$ and $\frac{x-\xi}{\tau}$ for $\xi \in [\xi_1,\xi_0]$ and $\xi \in [\xi_0,\xi_2]$ have equal areas. From (5.24) it follows

$$\frac{u_0(\xi_1)+u_0(\xi_2)}{2}=\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}u_0(s)\,ds. \tag{5.25}$$

We summarize these observations in

Theorem 5.3. *Let $u_\varepsilon(x,t)$ be a solution of the problem*

$$\begin{aligned}
u_t+uu_x &= \varepsilon u_{xx}, \quad x\in\mathbf{R}, t>0, \\
u(x,0) &= u_0(x), \quad x\in\mathbf{R}.
\end{aligned}$$

Suppose that for a given x, there exist τ and $\xi_1 < \xi_2$ such that

$$u_0\left(\xi_j\right) = \frac{x - \xi_j}{\tau}, \quad j = 1, 2$$

and (5.25) *is satisfied. Then*

$$\begin{aligned} u_\varepsilon\left(x,t\right) &\sim u_0\left(\xi_1\right), \quad \text{if } t > \tau, \\ u_\varepsilon\left(x,t\right) &\sim u_0\left(\xi_2\right), \quad \text{if } t < \tau, \end{aligned}$$

as $\varepsilon \to 0$.

Exercises

1. Show that any solution of the problem

$$\begin{aligned} u_t + uu_x &= u^n, \quad x \in \mathbf{R}, t > 0, \\ u\left(x,0\right) &= u_0\left(x\right), \quad x \in \mathbf{R}, \end{aligned}$$

satisfies the functional relation

$$u = e^t u_0\left(x - u + ue^{-t}\right), \quad n = 1,$$

$$u = \frac{u_0\left(x - \ln\left(1 - tu\right)\right)}{1 - tu_0\left(x - \ln\left(1 - tu\right)\right)}, \quad n = 2,$$

$$\begin{aligned} u^{1-n} &= (1-n)\,t + u_0^{1-n}\left(x + \frac{1}{n-2}\left(u^{2-n} - \left(u^{1-n} + (n-1)\,t\right)^{\frac{n-2}{n-1}}\right)\right), \\ n &\neq 1, 2. \end{aligned}$$

2. Consider *Liouville's*[10] *equation*

$$\frac{\partial^2 u}{\partial x \partial y} = e^u \tag{5.26}$$

Show that:
(a) Making the change of variables $(x, y, u) \longmapsto (x', y', u')$ such that

[10] Josef Liouville, 24.03.1809–08.09.1882.

$$\begin{cases} \dfrac{\partial u'}{\partial x'} = \dfrac{\partial u}{\partial x} + \beta e^{\frac{1}{2}(u+u')} \\ \dfrac{\partial u'}{\partial y'} = -\dfrac{\partial u}{\partial y} - \dfrac{2}{\beta} e^{\frac{1}{2}(u-u')} \\ x' = x \\ y' = y \end{cases},$$

where β is a constant, the equation (5.26) reduces to

$$\frac{\partial^2 u'}{\partial x' \partial y'} = 0.$$

(b) The solution of the problem

$$\begin{cases} \dfrac{du}{dx} + p(x)\, e^{\frac{1}{2}u} = q(x) \\ u(x_0) = u_0 \end{cases}$$

is

$$u(x) = 2 \ln \frac{e^{\frac{1}{2}\int\limits_{x_0}^{x} q(s)ds}}{e^{-\frac{u_0}{2}} + \frac{1}{2}\int\limits_{x_0}^{x} p(t)\, e^{\frac{1}{2}\int\limits_{x_0}^{t} q(s)ds} dt}.$$

(c) The general solution of the equation (5.26) is

$$u(x,y) = 2 \ln \frac{e^{\frac{1}{2}(f(x)-g(y))}}{\frac{\beta}{2}\int\limits_{x_0}^{x} e^{f(t)}dt + \frac{1}{\beta}\int\limits_{y_0}^{y} e^{-g(t)}dt},$$

where f and g are functions, x_0 and y_0 are constants.

3. (a) Consider the Cauchy problem for the advection-diffusion equation

$$\begin{aligned} u_t + a u_x &= \varepsilon u_{xx}, \quad x \in \mathbf{R}, t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned} \tag{5.27}$$

with $u_0(x) \in L^1(\mathbf{R})$. Making a change of variables $v(x,t) = u(x+at,t)$ it reduces to Cauchy problem for the diffusion equation

$$\begin{aligned} v_t &= \varepsilon v_{xx}, \quad x \in \mathbf{R}, t > 0, \\ v(x,0) &= u_0(x), \quad x \in \mathbf{R}. \end{aligned}$$

(b) Show that the solution $u_\varepsilon(x,t) \in C^\infty(\mathbf{R} \times (0,\infty))$ of (5.27) is

$$u_\varepsilon(x,t) = \frac{1}{2\sqrt{\varepsilon\pi t}} \int_{-\infty}^{\infty} u_0(\xi)\, e^{-\frac{1}{4\varepsilon t}(x - at - \xi)^2} d\xi$$

and $u_\varepsilon(x,t) \sim u_0(x - at)$ as $\varepsilon \to 0$.

5.4 Weak Solutions. Riemann Problem

Consider the Cauchy problem for the quasilinear equation

$$\begin{aligned} u_t + uu_x &= 0, \quad x \in \mathbf{R}, t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned} \tag{5.28}$$

which is a limit case of Burgers' equation as $\varepsilon \to 0$. If $u_0(x)$ is nonsmooth we introduce, as in Section 5.2, a notion of weak solution.

Definition 5.6. *Assume $u_0(x) \in L^1_{loc}(\mathbf{R})$. A function $u(x,t) \in L^2_{loc}(\mathbf{R} \times [0,\infty))$ is a weak solution of* (5.28) *iff*

$$\int_0^\infty \int_{-\infty}^\infty \left(u\rho_t + \frac{u^2}{2}\rho_x \right) dxdt + \int_{-\infty}^\infty u_0(x)\,\rho(x,0)\,dx = 0, \tag{5.29}$$

for every test function $\rho \in C_0^1(\mathbf{R} \times [0,\infty))$.

We have

Proposition 5.2. *Let $u \in C^1(\mathbf{R} \times [0,\infty))$ be a smooth solution of the equation $u_t + uu_x = 0$ and a weak solution of the problem* (5.28)*. If $u_0(x)$ is continuous at a point x_0, then $u(x_0,0) = u_0(x_0)$.*

Proof. Let $\rho(x,t) \in C_0^1(\mathbf{R} \times [0,\infty))$. As in Proposition 5.1 we are led to

$$\int_{-\infty}^{\infty} (u(x,0) - u_0(x))\, \rho(x,0)\, dx = 0.$$

Suppose $u(x_0,0) > u_0(x_0)$. By continuity there exists a neighborhood U such that

$$u(x,0) > u_0(x), \quad x \in U.$$

Take $\rho(x,t) \in C_0^1(\mathbf{R} \times [0,\infty))$ such that

$$\begin{aligned} \operatorname{supp}\rho(x,0) &= [a,b] \subset U, \quad \rho(x) > 0, \quad x \in (a,b), \\ \rho(a) &= \rho(b) = 0. \end{aligned}$$

Then

$$\begin{aligned} &\int_{-\infty}^{\infty} (u(x,0) - u_0(x))\, \rho(x,0)\, dx \\ = &\int_{a}^{b} (u(x,0) - u_0(x))\, \rho(x,0)\, dx > 0, \end{aligned}$$

which is a contradiction. Similarly $u(x_0,0) < u_0(x_0)$ is impossible. Then $u(x_0,0) = u_0(x_0)$. ■

The problem (5.28) with discontinuous initial data is known as a *Riemann*[11] *problem.* Let us consider the initial data

$$u_0(x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0, \end{cases} \tag{5.30}$$

where u_l and u_r are constants.

The two cases $u_l > u_r$ and $u_l < u_r$ are quite different with respect to the solvability of problem (5.28). It can be proved that if $u_l > u_r$, then the weak solution is unique, while if $u_l < u_r$, then there exist infinitely many solutions.

Case I. $u_l > u_r$

Consider the problem

[11] Georg Friedrich Bernhard Riemann, 17.09.1826–20.07.1866.

$$\begin{aligned} u_t + uu_x &= \varepsilon u_{xx}, \quad x \in \mathbf{R}, t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbf{R}. \end{aligned}$$

If $u_l > u_r$ we are in a situation to apply Theorem 5.1.

Let $x > 0$ be fixed and

$$s = \frac{u_l + u_r}{2}.$$

The instant τ of Theorem 5.3 is determined by the slope k of the straight line through the points $(x, 0)$ and $(0, s)$

$$k = -\frac{1}{\tau} = -\frac{s}{x}.$$

Then

$$\tau = \frac{x}{s},$$

and by Theorem 5.3

$$u(x,t) \sim \begin{cases} u_l & x < st, \\ u_r & x > st, \end{cases} \qquad \text{as } \varepsilon \to 0.$$

The unique solution of (5.28) is known as a *shock wave*, while $s = (u_l + u_r)/2$ is a *shock speed*, the speed at which the discontinuity of the solution travels.

Proposition 5.3. *The function*

$$u(x,t) = \begin{cases} u_l & x < st, \\ u_r & x > st, \end{cases} \tag{5.31}$$

is a weak solution of the problem (5.28) *with initial data* (5.30), *where*

$$u_l > u_r, \quad s = \frac{u_l + u_r}{2}.$$

Proof. Let $\rho(x,t) \in C_0^1(\mathbf{R} \times [0, \infty))$. Denote for simplicity

$$A := \int_0^\infty \int_{-\infty}^\infty \left(u\rho_t + \frac{u^2}{2}\rho_x \right) dxdt,$$

$$B := -\int_{-\infty}^{\infty} u(x,0)\,\rho(x,0)\,dx.$$

We have

$$A = \int_0^{\infty}\int_{-\infty}^{st}\left(\rho_t u_l + \rho_x \frac{u_l^2}{2}\right)dxdt + \int_0^{\infty}\int_{st}^{\infty}\left(\rho_t u_r + \rho_x \frac{u_r^2}{2}\right)dxdt,$$

$$\begin{aligned} A_1 \quad : \quad &= \int_0^{\infty}\int_{-\infty}^{st}\left(\rho_t u_l + \rho_x \frac{u_l^2}{2}\right)dxdt \\ &= u_l \int_0^{\infty}\left(\int_{-\infty}^{st}\rho_t dx\right)dt + \frac{u_l^2}{2}\int_0^{\infty}\left(\int_{-\infty}^{st}\rho_x dx\right)dt. \end{aligned}$$

By

$$\int_{-\infty}^{st}\rho_t(x,t)\,dx = \frac{d}{dt}\int_{-\infty}^{st}\rho(x,t)\,dx - \rho(st,t)\,s$$

and

$$\int_{-\infty}^{st}\rho_x(x,t)\,dx = \rho(st,t) - \rho(-\infty,t) = \rho(st,t),$$

it follows

$$\int_0^{\infty}\int_{-\infty}^{st}\rho_t(x,t)\,dxdt = \int_0^{\infty}\left(\frac{d}{dt}\left(\int_{-\infty}^{st}\rho(x,t)\,dx\right) - \rho(st,t)\,s\right)dt,$$

$$\begin{aligned} \int_0^{\infty}\left(\frac{d}{dt}\int_{-\infty}^{st}\rho(x,t)\,dx\right)dt \quad &= \quad \int_{-\infty}^{\infty}\rho(x,\infty)\,dx - \int_{-\infty}^{0}\rho(x,0)\,dx \\ &= \quad -\int_{-\infty}^{0}\rho(x,0)\,dx. \end{aligned}$$

Then

$$\int_0^\infty \int_{-\infty}^{st} \rho_t(x,t)\,dxdt = -\int_{-\infty}^0 \rho(x,0)\,dx - s\int_0^\infty \rho(st,t)\,dt,$$

$$A_1 = -u_l\left(\int_{-\infty}^0 \rho(x,0)\,dx + s\int_0^\infty \rho(st,t)\,dt\right) + \frac{u_l^2}{2}\int_0^\infty \rho(st,t)\,dt.$$

Similarly

$$\begin{aligned}
A_2 \quad &: = \int_0^\infty \int_{st}^\infty \left(\rho_t u_r + \rho_x \frac{u_r^2}{2}\right) dxdt \\
&= u_r\left(\int_0^\infty \left(\int_{st}^\infty \rho_t dx\right) dt\right) + \frac{u_r^2}{2}\left(\int_0^\infty \int_{st}^\infty \rho_x dxdt\right) \\
&= -u_r\left(\int_0^\infty \rho(x,0)\,dx - s\int_0^\infty \rho(st,t)\,dt\right) - \frac{u_r^2}{2}\int_0^\infty \rho(st,t)\,dt
\end{aligned}$$

because

$$\int_{st}^\infty \rho_t(x,t)\,dx = \frac{d}{dt}\int_{st}^\infty \rho(x,t)\,dx + s\rho(st,t),$$

$$\int_{st}^\infty \rho_x(x,t)\,dx = -\rho(st,t).$$

Then

$$\begin{aligned}
A \quad &= \quad A_1 + A_2 \\
&= -u_l\left(\int_{-\infty}^0 \rho(x,0)\,dx + s\int_0^\infty \rho(st,t)\,dt\right) + \frac{u_l^2}{2}\int_0^\infty \rho(st,t)\,dt \\
&\quad -u_r\left(\int_0^\infty \rho(x,0)\,dx - s\int_0^\infty \rho(st,t)\,dt\right) - \frac{u_r^2}{2}\int_0^\infty \rho(st,t)\,dt.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
B &= -\int_{-\infty}^{\infty} u(x,0)\,\rho(x,0)\,dx \\
&= -\int_{-\infty}^{0} u(x,0)\,\rho(x,0)\,dx - \int_{0}^{\infty} u(x,0)\,\rho(x,0)\,dx \\
&= -u_l \int_{-\infty}^{0} \rho(x,0)\,dx - u_r \int_{0}^{\infty} \rho(x,0)\,dx.
\end{aligned}
$$

Finally

$$A = B + \left(s(u_r - u_l) + \frac{u_l^2 - u_r^2}{2}\right) \int_0^{\infty} \rho(st,t)\,dt,$$

and since

$$s(u_r - u_l) + \frac{u_l^2 - u_r^2}{2} = \frac{u_r + u_l}{2}(u_r - u_l) + \frac{u_l^2 - u_r^2}{2} = 0,$$

we obtain $A = B$. ■

Case II. $u_l < u_r$

In this case there exist more than one weak solutions. One is (5.31). We show

Proposition 5.4. *The function*

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t \le x \le u_r t \\ u_r & x > u_r t \end{cases}.$$

is a weak solution of the problem (5.28) *with initial data* (5.30).

Proof. Let $\rho(x,t) \in C_0^1(\mathbf{R} \times [0,\infty))$. For simplicity we take $u_l = -1$ and $u_r = 1$ and denote

$$C := \int_0^{\infty} \int_{-\infty}^{\infty} \left(u\rho_t + \frac{u^2}{2}\rho_x\right) dxdt,$$

$$D := -\int_{-\infty}^{\infty} u(x,0)\,\rho(x,0)\,dx = \int_{-\infty}^{0} \rho(x,0)\,dx - \int_{0}^{\infty} \rho(x,0)\,dx.$$

The function x/t for $t \neq 0$ satisfies the equation $u_t + uu_x = 0$. We have

$$C = \int_0^\infty \int_{-\infty}^{-t} \left(-\rho_t + \frac{1}{2}\rho_x\right) dxdt + \int_0^\infty \int_{-t}^{t} \left(\frac{x}{t}\rho_t + \frac{1}{2}\left(\frac{x}{t}\right)^2 \rho_x\right) dxdt$$
$$+ \int_0^\infty \int_t^\infty \left(\rho_t + \frac{1}{2}\rho_x\right) dxdt = C_1 + C_2 + C_3,$$

where

$$C_1 := \int_0^\infty \int_{-\infty}^{-t} \left(-\rho_t + \frac{1}{2}\rho_x\right) dxdt = \int_{-\infty}^{0} \rho(x,0)\, dx - \frac{1}{2}\int_0^\infty \rho(-t,t)\, dt,$$

$$C_2 := \int_0^\infty \int_{-t}^{t} \left(\frac{x}{t}\rho_t + \frac{1}{2}\left(\frac{x}{t}\right)^2 \rho_x\right) dxdt,$$

and

$$C_3 := \int_0^\infty \int_t^\infty \left(\rho_t + \frac{1}{2}\rho_x\right) dxdt = -\int_0^\infty \rho(x,0)\, dx + \frac{1}{2}\int_0^\infty \rho(t,t)\, dt.$$

Because C_2 has a singularity at 0

$$C_2 = \lim_{\varepsilon \to 0} C_{2,\varepsilon}$$

where

$$C_{2,\varepsilon} := \int_\varepsilon^\infty \int_{-t}^{t} \left(\frac{x}{t}\rho_t + \frac{1}{2}\left(\frac{x}{t}\right)^2 \rho_x\right) dxdt.$$

We have

$$\left(\frac{x}{t}\rho\right)_t = -\frac{x}{t^2}\rho + \frac{x}{t}\rho_t,$$

$$\left(\frac{1}{2}\left(\frac{x}{t}\right)^2 \rho\right)_x = \frac{x}{t^2}\rho + \frac{1}{2}\left(\frac{x}{t}\right)^2 \rho_x,$$

$$\frac{x}{t}\rho_t + \frac{1}{2}\left(\frac{x}{t}\right)^2 \rho_x = \left(\frac{x}{t}\rho\right)_t + \left(\frac{1}{2}\left(\frac{x}{t}\right)^2 \rho\right)_x,$$

$$\int_{-t}^{t} \left(\frac{1}{2}\left(\frac{x}{t}\right)^2 \rho\right)_x dx = \frac{1}{2}\rho(t,t) - \frac{1}{2}\rho(-t,t),$$

$$\int_{-t}^{t} \left(\frac{x}{t}\rho\right)_t dx = \frac{d}{dt}\int_{-t}^{t} \left(\frac{x}{t}\rho\right) dx - (\rho(t,t) - \rho(-t,t)).$$

Then

$$\begin{aligned} C_{2,\varepsilon} &= \int_{\varepsilon}^{\infty} \frac{d}{dt}\left(\int_{-t}^{t} \left(\frac{x}{t}\rho\right) dx\right) dt - \frac{1}{2}\int_{\varepsilon}^{\infty} (\rho(t,t) - \rho(-t,t))\, dt \\ &= -\int_{-\varepsilon}^{\varepsilon} \frac{x}{\varepsilon}\rho(x,\varepsilon)\, dx - \frac{1}{2}\int_{\varepsilon}^{\infty} (\rho(t,t) - \rho(-t,t))\, dt. \end{aligned}$$

By the mean value theorem

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} \frac{x}{\varepsilon}\rho(x,\varepsilon)\, dx &= \frac{x_\varepsilon}{\varepsilon}\rho(x_\varepsilon,\varepsilon)\, 2\varepsilon \\ &= 2x_\varepsilon \rho(x_\varepsilon,\varepsilon), \end{aligned}$$

where $x_\varepsilon \in (-\varepsilon, \varepsilon)$. Because the function ρ is bounded and $x_\varepsilon \to 0$ as $\varepsilon \to 0$ it follows

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \frac{x}{\varepsilon}\rho(x,\varepsilon)\, dx = 0.$$

Then

$$C_2 = -\frac{1}{2}\int_{0}^{\infty} (\rho(t,t) - \rho(-t,t))\, dt.$$

Finally

$$
\begin{aligned}
C &= C_1 + C_2 + C_3 \\
&= \int_{-\infty}^{0} \rho(x,0)\,dx - \frac{1}{2}\int_{0}^{\infty} \rho(-t,t)\,dt - \frac{1}{2}\int_{0}^{\infty} (\rho(t,t) - \rho(-t,t))\,dt \\
&\quad - \int_{0}^{\infty} \rho(x,0)\,dx + \frac{1}{2}\int_{0}^{\infty} \rho(t,t)\,dt \\
&= \int_{-\infty}^{0} \rho(x,0)\,dx - \int_{0}^{\infty} \rho(x,0)\,dx = D
\end{aligned}
$$

which completes the proof. ■

Exercises

1. Show that the equation $u_t + uu_x = \varepsilon u_{xx}$ has a traveling wave solution of the form $u_\varepsilon(x,t) = w(x - at)$, where w satisfies the equation

$$\varepsilon w'(y) + aw(y) = \frac{1}{2}w^2(y) + C. \tag{5.32}$$

Verify that the function

$$w(y) = a - \sqrt{a^2 - 2C}\tanh\frac{1}{2\varepsilon}\sqrt{a^2 - 2C}y, \quad C \le \frac{a^2}{2},$$

satisfies the equation (5.32). Determine the behavior of this solution as $\varepsilon \to 0$.

2. There exist infinitely many weak solutions of the problem (5.28) with initial data (5.30) in the case $u_l < u_r$. Show that every function

$$u(x,t) = \begin{cases} u_l & x < st, \\ u_m & st \le x \le u_m t, \\ \dfrac{x}{t} & u_m t \le x \le u_r t, \\ u_r & u_r t < x, \end{cases}$$

where $u_m \in [u_l, u_r]$ and $s = \dfrac{u_l + u_m}{2}$ is a weak solution of the problem.

5.5 Discontinuous Solutions of Conservation Laws. Rankine–Hugoniot Condition.

Let us consider the Cauchy problem for a general conservation law

$$\begin{cases} u_t + (f(u))_x = 0, & x \in \mathbf{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{5.33}$$

For the sake of simplicity we assume that in (5.1) $m = 1$. Suppose f is a C^1 function and let

$$g(u) = \frac{df(u)}{du}. \tag{5.34}$$

If u is a classical solution of (5.33), then

$$u_t + g(u)\,u_x = 0, \quad x \in \mathbf{R},\ t > 0, \tag{5.35}$$

which is a quasilinear first order equation. The characteristics of (5.35) in (x,t) plane are the curves

$$c : \begin{cases} x = x(t) \\ t = t \end{cases},$$

such that

$$\frac{dx(t)}{dt} = g(u(x(t),t)). \tag{5.36}$$

Along the characteristics u is a constant because

$$\begin{aligned} \frac{du(x(t),t)}{dt} &= u_x(x(t),t)\,\frac{dx(t)}{dt} + u_t(x(t),t) \\ &= u_x(x(t),t)\,g(u(x(t),t)) + u_t(x(t),t) = 0. \end{aligned}$$

By (5.36) it follows that a characterictic through the point $(x_0, 0)$ is the straight line

$$x = x_0 + g(u_0(x_0))\,t,$$

with slope

$$k_0 = \frac{1}{g(u_0(x_0))}, \quad g(u_0(x_0)) \neq 0.$$

Assume that there exist two points $x_1 < x_2$ such that

$$0 < k_1 = \frac{1}{g(u_0(x_1))} < k_2 = \frac{1}{g(u_0(x_2))}.$$

Then the characteristics c_1 and c_2 through $(x_1, 0)$ and $(x_2, 0)$ intersect at some point P. At this point $u(P) = u_0(x_1) = u_0(x_2)$, which is impossible. Hence the solution can not be continuous at P. So, the existence of a classical solution depends on the intersection of the characteristics of equation (5.35) and is independent of smoothness of the functions $u_0(x)$ and $f(u)$. If the function $g(u_0(x))$ is monotone increasing then the classical solution exists for $t > 0$; otherwise it can't be defined for all $t > 0$. Assume that

$$\frac{dg(u_0(x))}{dx} < 0,$$

for some x. It can be shown that the solution u is smooth up to the instant

$$T_0 = -\frac{1}{\min_{x\in\mathbf{R}} \frac{dg(u_0(x))}{dx}}.$$

The above considerations lead us to introduce a weak solution of the Cauchy problem (5.33).

Definition 5.7. Assume that $u_0(x) \in L^1_{loc}(\mathbf{R})$. *A function is a weak solution of* (5.33) *iff* $u \in L^1_{loc}(\mathbf{R} \times [0, \infty))$, $f(u) \in L^1_{loc}(\mathbf{R} \times [0, \infty))$ and

$$\int_0^\infty \int_{-\infty}^\infty (u\rho_t + f(u)\rho_x)\, dxdt + \int_{-\infty}^\infty u_0(x)\rho(x, 0)\, dx = 0, \tag{5.37}$$

for every test function $\rho \in C^1_0(\mathbf{R} \times [0, \infty))$.

We consider now weak solutions of (5.33) which are piecewise smooth only. We show that not every discontinuity is admissible.

We say that u is piecewise smooth in $\mathbf{R} \times [0, \infty)$ if there exist a finite number of smooth curves $\Gamma_j \subset \mathbf{R} \times [0, \infty)$, $j = 1, ..., k$ outside of which u is a C^1 function and across Γ_j it has a jump discontinuity. Let Γ be the curve of discontinuities

$$\Gamma : \begin{cases} x = \gamma(t) \\ t = t \end{cases}.$$

Assume that Γ is a smooth curve, the tangent and normal vectors to Γ at (x, t) are $\vec{\tau}(\dot{\gamma}(t), 1)$ and $\vec{\nu}(1, -\dot{\gamma}(t))$, where $\dot{\gamma}(t) = \frac{d\gamma}{dt}$. Denote

$$u_\pm(\gamma(t), t) = \lim_{\varepsilon\to 0} u(\gamma(t) \pm \varepsilon, t \mp \varepsilon\dot{\gamma}(t)),$$

the limits of u on each side of Γ.

Theorem 5.4. *Let $u : \mathbf{R} \times [0, \infty) \to \mathbf{R}$ be a piecewise C^1 function. Then u is a weak solution of* (5.33) *iff*

(1) u is a classical solution in domains where u is a C^1 function,

(2) u satisfies the jump condition

$$\frac{d\gamma(t)}{dt} = \frac{f(u_+(\gamma(t), t)) - f(u_-(\gamma(t), t))}{u_+(\gamma(t), t) - u_-(\gamma(t), t)},$$

along every discontinuity curve $\Gamma : x = \gamma(t)$.

The jump condition is known as *Rankine-Hugoniot*[12] *condition.* For the case of Burgers' equation $f(u) = \frac{u^2}{2}$, it reduces to

$$\frac{d\gamma(t)}{dt} = \frac{u_+(\gamma(t), t) + u_-(\gamma(t), t)}{2}.$$

If $\Gamma : x = x_0 + kt$ is a straight line the last equation means

$$k = \frac{u_+(\gamma(t), t) + u_-(\gamma(t), t)}{2}.$$

Proof of Theorem 5.4. Suppose u is a piecewise C^1 function, which is a weak solution of (5.33).

As in the proof of Proposition 5.2 u is a classical solution in domains where u is a C^1 function. Assume that $\Gamma : x = \gamma(t)$ is a discontinuity curve, $P \in \Gamma$ and $B \subset \mathbf{R} \times (0, \infty)$ is a small ball centered at P, which does not intersect other curves of discontinuity. Let $\rho \in C_0^1(B)$. As u is a weak solution and $supp \rho \subset B$, we have

$$\begin{aligned} 0 &= \int_0^{\infty} \int_{-\infty}^{\infty} (u\rho_t + f(u)\rho_x)\, dxdt = \iint_B (u\rho_t + f(u)\rho_x)\, dxdt \\ &= \iint_{B^+} (u\rho_t + f(u)\rho_x)\, dxdt + \iint_{B^-} (u\rho_t + f(u)\rho_x)\, dxdt, \end{aligned}$$

where $B^{\pm}$ are the two open components of B on each side of Γ. By Green's identity

[12] William John Macqorn Rankine 1820–1872. W.J.M.Rankine. On the thermodynamic theory of waves of finite longitudial disturbance. Phil. Trans. 160(1870), 277–288.

Pierre Henri Hugoniot, 1851–1887. H. Hugoniot. Sur la propagation du mouvement dans les corps et specialement dans les gaz parfaits. J. l'Ecole Polytech. 58(1889), 1–125.

$$\begin{aligned}
0 &= \iint_{B^+} (u\rho_t + f(u)\rho_x)\,dxdt + \iint_{B^-} (u\rho_t + f(u)\rho_x)\,dxdt \\
&= -\iint_{B^+} (u_t + (f(u))_x)\,\rho dxdt - \int\limits_{B\cap\Gamma} (-\dot{\gamma}(t)\,u_+ + f(u_+))\,\rho ds \\
&\quad -\iint_{B^-} (u_t + (f(u))_x)\,\rho dxdt + \int\limits_{B\cap\Gamma} (-\dot{\gamma}(t)\,u_- + f(u_-))\,\rho ds \\
&= \int\limits_{B\cap\Gamma} (\dot{\gamma}(t)(u_+ - u_-) - (f(u_+) - f(u_-))\,\rho ds.
\end{aligned}$$

Since $\rho \in C_0^1(B)$ is arbitrary we obtain the jump relation

$$\dot{\gamma}(t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Conversely, it is easy to check that if u is a piecewise C^1 function which satisfies (1) and (2), then it is a weak solution of (5.33). ■

Example 5.6. *Consider the Cauchy problem*

$$u_t + uu_x = 0,$$

$$u(x,0) = \begin{cases} 1 & x \le 0, \\ 1-x & 0 \le x \le 1, \\ 0 & x > 1. \end{cases}$$

Determine the time of existence of a continuous weak solution and find a discontinuous weak solution.

Solution. The characteristic through the point $(x_0, 0)$ is $c_0 : x = x_0 + tu_0(x_0)$, so that

$$c_0 : x = \begin{cases} x_0 + t & x_0 \le 0, \\ x_0 + t(1-x_0) & 0 \le x_0 \le 1, \\ x_0 & x_0 > 1. \end{cases}$$

The picture of characteristics is given in Figure 5.8.

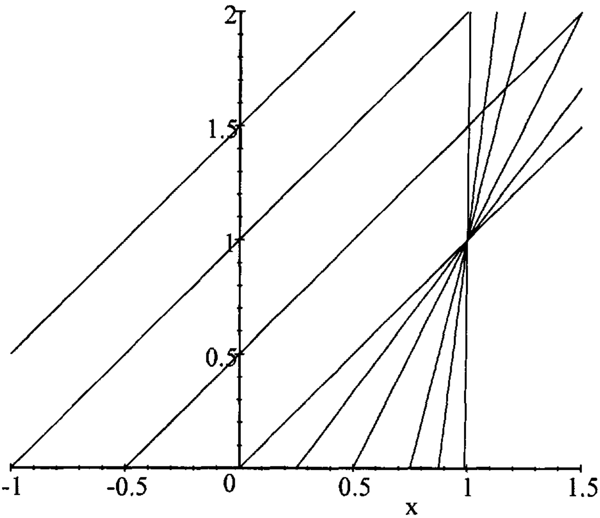

Figure 5.8. Characteristics in Example 5.6.

No pair of characteristics intersect for $t < 1$. A continuous weak solution for $t < 1$ is

$$u(x,t) = \begin{cases} 1 & x \le t, \\ \dfrac{1-x}{1-t} & t \le x \le 1, \\ 0 & x \ge 1. \end{cases}$$

Characteristics intersect for $t \ge 1$. In this case we are looking for a weak solution of the form

$$u(x,t) = \begin{cases} 1, & x < 1 + k(t-1), \\ 0, & x > 1 + k(t-1). \end{cases}$$

By Theorem 5.2 and the Rankine–Hugoniot condition we should have

$$k = \frac{u_+ + u_-}{2} = \frac{1}{2}.$$

So, for $t \ge 1$ the function

$$u(x,t) = \begin{cases} 1, & x < 1/2\,(t+1), \\ 0, & x > 1/2\,(t+1), \end{cases}$$

is a weak solution of the problem.

Exercises

1. Consider the Cauchy problem

$$u_t + uu_x = 0,$$

$$u(x,0) = \begin{cases} \dfrac{k-1}{k+1} & x \leq 0, \\ \dfrac{k^2+1}{k^2-1} - \sqrt{\left(\dfrac{k^2+1}{k^2-1}\right)^2 - (1-x)^2} & 0 \leq x \leq 1, \\ 0 & x \geq 1, \end{cases}$$

where $k \geq 2$ is an integer.

(a) Find the characteristics of the problem and show that their envelope is the curve

$$x = 1 + \frac{k^2+1}{k^2-1}\left(t - \sqrt{1+t^2}\right),$$

in (x,t) plane.

(b) Verify that the continuous solution exists for

$$0 \leq t < \frac{2k}{k^2-1},$$

and does not exist for $t \geq \dfrac{2k}{k^2-1}$.

(c) Plot the picture of characteristics and their envelope with *Mathematica* in the case $k = 2$.

2. (a) Find the characteristics and the solution of the problem

$$u_t + uu_x = 0,$$

$$u(x,0) = \begin{cases} \alpha a + \beta & x \leq a, \\ \alpha x + \beta & a \leq x \leq b, \\ \alpha b + \beta & x \geq b. \end{cases}$$

(b) Show that:
if $\alpha \geq 0$ the solution is differentiable for $t \geq 0$,
if $\alpha < 0$ the solution is continuous for $0 \leq t < -\dfrac{1}{\alpha}$.

3. Consider the Cauchy problem

$$u_t + uu_x = 0,$$

$$u(x,0) = \begin{cases} 1 & x \leq 0, \\ \cos^2 x & 0 \leq x \leq \pi/2, \\ 0 & x \geq \pi/2. \end{cases}$$

(a) Determine the characteristics and show that they have an envelope of two branches.

(b) Plot the picture of chracteristics and their envelope with *Mathematica.*

(c) Find a weak solution.

4. Consider the problem

$$\begin{aligned} u_t + uu_x + au &= 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

Show that the characteristics of the problem are

$$x = x_0 + \frac{1 - e^{-at}}{a} u_0(x_0).$$

Discuss the question of breaking of solutions.

Chapter 6

The Laplace Equation

6.1 Harmonic Functions. Maximum-minimum Principle.

The Laplace[1] equation or potential equation is

$$\Delta u = 0, \tag{6.1}$$

where Δu is the Laplacian of the function u

$$\begin{aligned} \Delta u &= \nabla^2 u = u_{xx} + u_{yy} \quad \text{in two dimensions,} \\ \Delta u &= \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \quad \text{in three dimensions.} \end{aligned}$$

A function $u \in C^2(\Omega)$ which satisfies the Laplace equation is called a *harmonic function.* The inhomogeneous Laplace equation

$$\Delta u = f,$$

where f is a given function is known as the Poisson equation.

The Laplace equation is very important in applications. It appears in physical phenomena such as

1. Steady-state heat conduction in a homogeneous body with constant heat capacity and constant conductivity.
2. Steady-state incompressible fluid flow.
3. Electrical potential of a stationary electrical field in a region without charge.

[1]Pierre Simon Laplace, 23.03.1749–05.03.1827.

The basic mathematical problem is to solve the Laplace or Poisson equation in a given domain possibly with a condition on its boundary $\partial\Omega = \bar{\Omega}\backslash\Omega$.

Let φ and ψ be continuous functions on $\partial\Omega$. The problem of finding a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$(DL): \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

is called the Dirichlet or first boundary value problem (BVP) for the Laplace equation. Historically, the name boundary value problem was attributed to only problems for which the PDE was of the elliptic type. Today we use this term in a much wider sense.

The Neumann or second BVP is

$$(NL): \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \psi & \text{on } \partial\Omega, \end{cases}$$

where $\vec{n}$ denotes the outward unit normal to $\partial\Omega$ and $\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$ is the normal derivative.

The Robin or third BVP is

$$(RL): \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \sigma u + \dfrac{\partial u}{\partial n} = \psi & \text{on } \partial\Omega, \end{cases}$$

where σ is a continuous function on $\partial\Omega$.

A boundary value problem for the Laplace equation is well posed in the sense of Hadamard with respect to a class of boundary data if

1. A solution of the problem exists;
2. The solution is unique;
3. Small variations of the boundary data yield small variations on the corresponding solutions.

The Cauchy problem for Laplace equation is ill-posed. A modification of Hadamard's example follows.

Example 6.1. *Consider the problem*

$$(CL_n): \begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \mathbf{R} \times (0, \infty), \\ u(x,0) = 0, \ u_y(x,0) = \dfrac{\cos nx}{n^2}. \end{cases}$$

Show that $u_n(x,y) = \dfrac{1}{n^3} \sinh ny \cos nx$ *is a solution of the problem* (CL_n) *but*

$$\lim_{n\to\infty} \|u_n(x,y)\|_{C(\mathbf{R}\times[0,\infty))} = 0 \tag{6.2}$$

is not fulfilled.

Solution. It can be easily seen that $u_n(x,y) = \frac{1}{n^3}\sinh ny \cos nx$ is a solution of (CL_n). Let $\lambda \in (0,1)$. There exist x_0 and $n_k \to +\infty$, such that $\cos n_k x_0 \to \lambda$ as $k \to \infty$. This follows from the fact that if x is an irrational multiple of π, then the set of points $\{(\cos nx, \sin nx) : n \in \mathbf{N}\}$ is a dense set in the unit circle $\mathbf{S} = \{\,(x,y) : x^2 + y^2 = 1\}$. For every $y > 0$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{\sinh ny}{n^3} &= \frac{1}{2}\lim_{n\to\infty} \frac{e^{ny} - e^{-ny}}{n^3} \\ &= \frac{1}{2}\lim_{n\to\infty} \frac{e^{ny}}{n^3} = +\infty.\end{aligned}$$

Then

$$\lim_{k\to\infty} \frac{1}{n_k^3} \sinh n_k y \cos n_k x_0 = +\infty,$$

for $y > 0$, which implies that (6.2) is not true.

In contrast to the Cauchy problem for Laplace equation, the Dirichlet problem is well posed. This follows by the maximum-minimum principle for harmonic functions.

Let $\Omega \subset \mathbf{R}^N$, $N = 2$ or 3 be a bounded domain and $\Gamma = \partial\Omega$ be its boundary.

Denote by P a point of Ω,

$$P = (x,y),\ |P| = \sqrt{x^2 + y^2} \quad \text{if } N = 2$$

or

$$P = (x,y,z),\ |P| = \sqrt{x^2 + y^2 + z^2} \quad \text{if } N = 3.$$

Theorem 6.1. *(Maximum-minimum principle). Suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function in a bounded domain Ω. Then*

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u, \tag{6.3}$$

$$\min_{\bar{\Omega}} u = \min_{\Gamma} u. \tag{6.4}$$

Proof. Consider the case $N = 2$. Let $\varepsilon > 0$ and consider the modified function $v(P) = u(P) + \varepsilon|P|^2$. Then

$$\Delta v = \Delta u + \varepsilon\Delta(x^2 + y^2) = 4\varepsilon > 0,$$

while $\Delta v = v_{xx} + v_{yy} \leq 0$ at an interior maximum point by the second derivative test in Calculus. Since $v(P)$ has no interior maximum in Ω, being a continuous function, it should attain its maximum on $\partial\Omega$ with

$$v(P_1) = \max_{\Gamma} v(P) = \max_{\bar{\Omega}} v(P).$$

Then for $P \in \bar{\Omega}$

$$u(P) < v(P) \leq v(P_1) = u(P_1) + \varepsilon|P_1|^2 \leq \max_{\Gamma} u + \varepsilon R^2, \tag{6.5}$$

where R is such that $\Omega \subset B_R(0)$. Since ε is an arbitrary by (6.5) it follows

$$u(P) \leq \max_{\Gamma} u \leq \max_{\bar{\Omega}} u.$$

As $P \in \bar{\Omega}$ is arbitrary by the last inequality (6.3) follows. Because $-u$ is also a harmonic function and $\min\limits_{\bar{\Omega}} u = -\max\limits_{\bar{\Omega}} (-u)$ (6.4) also follows. ■

Corollary 6.1. *Let Ω be a bounded domain, $f \in C(\bar{\Omega})$ and $\varphi \in C(\Gamma)$. Then the Dirichlet problem*

$$\begin{cases} \Delta u = f(P), & P \in \Omega, \\ u(P) = \varphi(P), & P \in \Gamma, \end{cases} \tag{6.6}$$

has no more than one solution.

Proof. Suppose $u_j(P)$, $j = 1, 2$ are two solutions of (6.6) and $u = u_1 - u_2$. Then $u \in C(\bar{\Omega})$ is a harmonic function and $u = 0$ on Γ. By the Maximum-minimum principle it follows that $u \equiv 0$ on Ω. ■

Exercises.

1. (a) Show that in polar coordinates

$$\begin{cases} x = \rho \cos\theta, \\ y = \rho \sin\theta, \end{cases}$$

the two dimensional Laplacian is

$$\Delta u(x, y) = u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta}.$$

(b) A harmonic function $u(x, y)$ is rotationaly invariant if $u(\rho, \theta)$ depends only on ρ. Prove that $u(\rho) = c_1 \ln\rho + c_2$ if u is rotationaly invariant.

(c) In spherical coordinates

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\varphi, \end{cases}$$

the three dimensional Laplacian is expressed as

$$\Delta u(x,y,z) = u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2\sin\varphi}(\sin\varphi u_\varphi)_\varphi + \frac{1}{\rho^2\sin^2\varphi}u_{\theta\theta}$$

(d) A harmonic function $u(x,y,z)$ is spherically symmetric if $u(\rho,\theta,\varphi)$ depends only on ρ. Show that $u(\rho) = C_1\frac{1}{\rho} + C_2$ if u is spherically symmetric.

2. Prove that the function $u(x,y) = \dfrac{1-x^2-y^2}{x^2+(y-1)^2}$ is harmonic in $\mathbf{R}^2\backslash(0,1)$. Find the maximum M and minimum m of $u(x,y)$ in the disk $B_\rho(0,0), \rho < 1$ and show that $Mm = 1$. Plot the graphic of $u(x,y)$, where $(x,y) \in B_{0.9}(0,0)$ using polar coordinates.

6.2 Green's Identities

Let $u, v \in C^2(\bar{\Omega})$, Ω be a domain with smooth boundary $\partial\Omega$, $\vec{n}$ be the outward unit normal vector to $\partial\Omega$. Recall the following notations of field theory

$$\begin{aligned} \text{grad}u &= \nabla u = (u_x, u_y, u_z), \\ \text{div}\vec{F} &= \nabla\cdot\vec{F} = f_x + g_y + h_z, \\ \text{rot}\vec{F} &= \nabla\times\vec{F} = (h_y - g_z, f_z - h_x, g_x - f_y), \\ \Delta u &= \text{div}(\nabla u) = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}, \end{aligned}$$

where $\vec{F}(f,g,h)$ is a vector field. Denote $dV = dxdydz$, dS_P a surface element and ds_P an arc length element at P on $\partial\Omega$.

We have the divergence theorem or the Gauss–Ostrogradskii[2] formula

$$\iiint\limits_{\Omega} \text{div}\vec{F}dV = \iint\limits_{\partial\Omega} \vec{F}\cdot\vec{n}dS_P. \tag{6.7}$$

[2]Karl Friedrich Gauss, 30.04.1777– 23.02.1855,
Michail Vasilievich Ostrogradskii, 12.09.1801– 20.12.1861

If $F = \nabla u$, we have

$$\iiint\limits_{\Omega} \Delta u dV = \iint\limits_{\partial\Omega} \frac{\partial u}{\partial n} dS_P, \tag{6.8}$$

known as *Gauss formula.*

By the product rule

$$(vu_x)_x = v_x u_x + vu_{xx}$$

it follows

$$\operatorname{div}(v\nabla u) = \nabla v \cdot \nabla u + v\Delta u$$

and by (6.7).

$$\iiint\limits_{\Omega} v\Delta u dV + \iiint\limits_{\Omega} \nabla v \cdot \nabla u dV = \iint\limits_{\partial\Omega} v\frac{\partial u}{\partial n} dS_P, \tag{6.9}$$

known as *Green's first identity.*

Changing the role of v and u we have

$$\iiint\limits_{\Omega} u\Delta v dV + \iiint\limits_{\Omega} \nabla u \cdot \nabla v dV = \iint\limits_{\partial\Omega} u\frac{\partial v}{\partial n} dS_P. \tag{6.10}$$

Subtracting (6.9) from (6.10) we obtain

$$\iiint\limits_{\Omega} (u\Delta v - v\Delta u) dV = \iint\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS_P, \tag{6.11}$$

known as *Green's second identity.*

Consider the two dimensional case.

Let $D \subset \mathbf{R}^2$ be a bounded domain with smooth closed oriented boundary Σ, $u, v \in C^2(D) \cap C(\bar{D})$. Include $\mathbf{R}^2 \subset \mathbf{R}^3$ by $(x, y) \to (x, y, 0)$ and consider the cylinder $K \subset \mathbf{R}^3$ with base D and altitude 1. As u and v do not depend on z

$$\iiint\limits_{K} (u\Delta v - v\Delta u) dx dy dz = \iint\limits_{D} (u\Delta v - v\Delta u)\, dx dy,$$

$$\iint\limits_{\partial K} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS_P = \oint_{\Sigma} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds_P.$$

Then, by (6.11), we obtain Green's second identity in $\mathbf{R}^2$

$$\iint\limits_D (u\Delta v - v\Delta u)\, dxdy = \oint_\Sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds_P. \tag{6.12}$$

Consider some applications of Green's identities.

1. Mean value property.

Theorem 6.2. *The average value of any harmonic function over any sphere is equal to its value at the center.*

Proof. Let $u(P)$ be a harmonic function on B, where

$$\begin{aligned} B &= B_a(P_0) = \left\{P \in \mathbf{R}^3 : |P - P_0| \le a\right\}, \\ S &= S_a(P_0) = \left\{P \in \mathbf{R}^3 : |P - P_0| = a\right\}. \end{aligned}$$

By (6.9) it follows

$$0 = \iiint\limits_B \Delta u dV = \iint\limits_S \frac{\partial u}{\partial n} dS_P. \tag{6.13}$$

For the sphere S the unit normal vector at $P \in S$ is

$$\vec{n} = \frac{P - P_0}{a} = \left(\frac{x - x_0}{a}, \frac{y - y_0}{a}, \frac{z - z_0}{a} \right).$$

Let us make the change of variables

$$\begin{cases} x = x_0 + \rho\cos\theta\sin\varphi, \\ y = y_0 + \rho\sin\theta\sin\varphi, \\ z = z_0 + \rho\cos\varphi. \end{cases}$$

Then for

$$u(\rho, \theta, \varphi) = u(x_0 + \rho\cos\theta\sin\varphi,\ y_0 + \rho\sin\theta\sin\varphi,\ z_0 + \rho\cos\varphi)$$

we have

$$\begin{aligned} \left.\frac{\partial u}{\partial n}\right|_S &= \frac{x - x_0}{a}u_x + \frac{y - y_0}{a}u_y + \frac{z - z_0}{a}u_z \\ &= \cos\theta\sin\varphi u_x + \sin\theta\sin\varphi u_y + \cos\varphi u_z \\ &= \left.\frac{\partial}{\partial\rho}u(\rho, \theta, \varphi)\right|_{\rho = a}. \end{aligned}$$

Therefore (6.13) becomes

$$\begin{aligned} 0 &= \iint_S \frac{\partial}{\partial \rho} u(\rho,\theta,\varphi)\big|_{\rho=a} dS_p \\ &= \int_0^{2\pi}\int_0^{\pi} u_\rho \left(\rho,\theta,\varphi\right)\big|_{\rho=a} a^2 \sin\varphi d\varphi d\theta \end{aligned}$$

and as $a > 0$

$$0 = \int_0^{2\pi}\int_0^{\pi} u_\rho \left(\rho,\theta,\varphi\right)\big|_{\rho=a} \sin\varphi d\varphi d\theta.$$

The last identity is valid for every $a > 0$, so that we can consider a as a variable r and we have

$$\frac{d}{dr}\left(\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi)\sin\varphi d\varphi d\theta\right) = 0.$$

Then

$$I(r) = \int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi)\sin\varphi d\varphi d\theta$$

is independent of r. Letting $r \to 0$, we get

$$\begin{aligned} \lim_{r\to 0} I(r) &= \int_0^{2\pi}\int_0^{\pi} u(0,\theta,\varphi)\sin\varphi d\varphi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} u(P_0)\sin\varphi d\varphi d\theta \\ &= 4\pi u(P_0). \end{aligned}$$

Then it follows

$$\begin{aligned} 4\pi u(P_0) &= \int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)\sin\varphi d\varphi d\theta, \\ 4\pi a^2 u(P_0) &= \int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)a^2\sin\varphi d\varphi d\theta \\ &= \iint_S u(P) dS_P \end{aligned}$$

or

$$u(P_0) = \frac{1}{4\pi a^2}\iint\limits_S u(P)dS_P. \blacksquare$$

Note that the mean value property is also valid in the two dimensional case. Namely, if $u(x,y)$ is a harmonic function in $\mathbf{R}^2$, $P_0(x_0,y_0)\in\mathbf{R}^2$ and

$$K_a=\left\{P\in\mathbf{R}^2:|P-P_0|\le a\right\}$$

is a disk, $C_a=\partial K_a$ then

$$u(P_0)=\frac{1}{2\pi a}\oint_{C_a}u(P)ds_P,$$

which is the mean value formula for the two dimensional equation.

By the mean value property it follows the maximum-minimum principle as well as uniqueness for solutions of Dirichlet problem in domains of $\mathbf{R}^3$.

Theorem 6.3. *Let $u(P)$ be a harmonic function in the domain Ω and u be bounded from above. Then u attains* $\sup u$ *in Ω, iff u is a constant.*

Proof. As Ω is a connected set it can not be represented as a union of two nonempty open subsets O_1 and O_2 whose intersection is empty.

Let $M=\sup\limits_{\Omega} u=u(P_0)$, $P_0\in\Omega$ and $O_1=\{P\in\Omega:u(P)=M\}$. As u is a continuous function O_1 is relatively closed and $O_2=\Omega\backslash O_1$ is open. We shall prove that O_1 is an open set. Then, as Ω is connected, we have $O_2=\emptyset$ because $O_1\neq\emptyset$ and $\Omega=O_1$ which means that u is a constant in Ω.

Let $P_1\in O_1$ and $B_r(P_1)\subset\Omega$, where $B_r(P_1)=\{P:|P-P_1|<r\}$. We shall prove that $B_r(P_1)\subset O_1$, which means that O_1 is an open set. As $M=\sup\limits_{\Omega} u$, we have $u\le M$ on the boundary $S_r(P_1)=\{P:|P-P_1|=r\}$. Suppose there is a point $P_2\in S_r(P_1)$ such that $u(P_2)<M$. By the continuity of u there is a neighborhood N of P_2 such that $u(P)<M$ if $P\in N$. Let $\sigma=N\cap S_r(P_1)$. By the mean value property

$$\begin{aligned}
u(P_1) &= \frac{1}{|S_r(P_1)|}\iint_{S_r(P_1)}u(P)dS_P\\
&= \frac{1}{|S_r(P_1)|}\left(\iint_{\sigma}u(P)dS_P+\iint_{S_r(P_1)\backslash\sigma}u(P)dS_P\right)\\
&< \frac{1}{|S_r(P_1)|}\left(M|\sigma|+M|S_r(P_1)\backslash\sigma|\right)=M,
\end{aligned}$$

which is a contradiction. Therefore $u(P)=M$ if $P\in S_r(P_1)$. By the same way $u(P)=M$ if $P\in S_\rho(P_1)$ for every $\rho\in(0,r)$. Finally $u(P)=M$ in $B_r(P_1)$ and this means that O_1 is an open set, which completes the proof. ■

As a direct consequence we have

Corollary 6.2. *Let Ω be a bounded domain with smooth boundary $\Gamma = \partial\Omega$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be an harmonic function. Then*

$$\begin{aligned}\max_{\bar{\Omega}} u &= \max_{\Gamma} u,\\ \min_{\bar{\Omega}} u &= \min_{\Gamma} u\,.\end{aligned}$$

Corollary 6.3. *Let Ω be a bounded domain, $\varphi(P) \in C(\Gamma)$, $f(P) \in C(\Omega)$. Then the Dirichlet problem*

$$\begin{cases} \Delta u = f(P), \ P \in \Omega, \\ u(P) = \varphi(P), \ P \in \Gamma, \end{cases}$$

has no more than one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

2. Dirichlet principle

Theorem 6.4. *Let $\Omega \subset \mathbf{R}^3$ be a domain with boundary $\Gamma = \partial\Omega$. Among all functions $v(P) \in C^2(\Omega)\cap C(\bar{\Omega})$ that satisfy the Dirichlet boundary condition*

$$v(P) = \varphi(P) \text{ on } \Gamma, \tag{6.14}$$

where $\varphi(P) \in C(\Gamma)$, the lowest energy

$$E(v) = \frac{1}{2} \iiint\limits_{\Omega} |\nabla v|^2 dV,$$

*is attained by a harmonic function satisfying (*6.14).

Proof. We prove that if u is the unique harmonic function, such that $u(P) = \varphi(P)$ on Γ, then for every $v \in C^2(\Omega) \cap C(\bar{\Omega})$ with $v(P) = \varphi(P)$ on Γ,we have

$$E(v) \geq E(u).$$

We can represent $v = u - w$, where $w(P) = 0$ on Γ.

By the Green's first identity

$$\begin{aligned} E(v) &= E(u-w) = \frac{1}{2}\iiint_{\Omega} \left(|\nabla u|^2 - 2\nabla u \nabla w + |\nabla w|^2\right) dV \\ &= E(u) + E(w) + \iiint_{\Omega} w \Delta u dV - \iint_{\Gamma} w \frac{\partial u}{\partial n} dS_P \\ &= E(u) + E(w) \geq E(u), \end{aligned}$$

which completes the proof. ■

3. Representation formula

Theorem 6.5. *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be such that $\Delta u \in L^1(\Omega)$. Then for every $P \in \Omega$, if $N = 3$,*

$$\begin{aligned} u(P) &= \frac{1}{4\pi} \iint_{\partial\Omega} \left(\frac{1}{|Q-P|} \frac{\partial u(Q)}{\partial n} - u(Q) \frac{\partial}{\partial n} \left(\frac{1}{|Q-P|} \right) \right) dS_Q \\ &\quad - \frac{1}{4\pi} \iiint_{\Omega} \frac{\Delta u(Q)}{|Q-P|} dV_Q, \end{aligned}$$

and if $N = 2$

$$\begin{aligned} u(P) &= \frac{1}{2\pi} \oint_{\partial\Omega} \left(u(Q) \frac{\partial}{\partial n} \ln|Q-P| - \ln|Q-P| \frac{\partial u(Q)}{\partial n} \right) ds_Q \\ &\quad + \frac{1}{2\pi} \iint_{\Omega} \ln|Q-P| \Delta u(Q) dx dy. \end{aligned}$$

Proof. Consider the three dimensional case. Fix $P \in \Omega$ and let ε be sufficiently small such that $B_\varepsilon(P) \subset \Omega$. Let us apply the Green's second identity in $\Omega \backslash B_\varepsilon(P)$ for the functions $u(Q)$ and $v(Q) = \dfrac{1}{|Q-P|}$, which is harmonic for $Q \neq P$. Denote for simplicity

$$\Omega_\varepsilon = \Omega \backslash B_\varepsilon(P),\ B_\varepsilon = B_\varepsilon(P),\ S_\varepsilon = \partial B_\varepsilon(P).$$

On S_ε we have

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}, \quad r = |Q-P| \tag{6.15}$$

It follows by the Green's second identity that

$$\begin{aligned} & - \iiint_{\Omega_\varepsilon} \frac{\Delta u(Q)}{|Q-P|} dV_Q \\ = & \iint_{\partial\Omega} \left(u \frac{\partial}{\partial n} \left(\frac{1}{|Q-P|} \right) - \frac{1}{|Q-P|} \frac{\partial u}{\partial n} \right) dS_Q \\ & + \iint_{S_\varepsilon} \left(u \frac{\partial}{\partial n} \left(\frac{1}{|Q-P|} \right) - \frac{1}{|Q-P|} \frac{\partial u}{\partial n} \right) dS_Q. \end{aligned} \tag{6.16}$$

By (6.15)

$$A_\varepsilon \ : = \iint_{S_\varepsilon} \left(u \frac{\partial}{\partial n} \left(\frac{1}{|Q-P|} \right) - \frac{1}{|Q-P|} \frac{\partial u}{\partial n} \right) dS_Q \tag{6.17}$$

$$\begin{aligned}
&= \iint_{S_\varepsilon}\left(u\left(-\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)_{r=\varepsilon}+\frac{1}{\varepsilon}\frac{\partial u}{\partial r}\right)dS \\
&= \frac{1}{\varepsilon^2}\iint_{S_\varepsilon} u dS+\frac{1}{\varepsilon}\iint_{S_\varepsilon}\frac{\partial u}{\partial r}dS \\
&= 4\pi\frac{1}{|S_\varepsilon|}\iint_{S_\varepsilon} u dS+4\pi\varepsilon\frac{1}{|S_\varepsilon|}\iint_{S_\varepsilon}\frac{\partial u}{\partial r}dS \\
&= 4\pi M_\varepsilon(u)+4\pi\varepsilon M_\varepsilon\left(\frac{\partial u}{\partial r}\right),
\end{aligned}$$

where $M_\varepsilon(u)$ denotes the mean value of u over S_ε and $|S_\varepsilon|$ the area of S_ε. As $u \in C^1(\bar{\Omega})$, letting $\varepsilon \to 0$, by $\int$, we have

$$\lim_{\varepsilon\to 0} A_\varepsilon = 4\pi u(P).$$

Then, by (6.16), we obtain as $\varepsilon \to 0$

$$\begin{aligned}
-\iiint_\Omega \frac{\Delta u(Q)}{|Q-P|}dV_Q &= \iint_{\partial\Omega}\left(u\frac{\partial}{\partial n}\left(\frac{1}{|Q-P|}\right)-\frac{1}{|Q-P|}\frac{\partial u}{\partial n}\right)dS_Q \\
&\quad +4\pi u(P),
\end{aligned}$$

or

$$u(P)=\frac{1}{4\pi}\iint\limits_{\partial\Omega}\left(\frac{1}{|Q-P|}\frac{\partial u}{\partial n}-u\frac{\partial}{\partial n}\left(\frac{1}{|Q-P|}\right)\right)dS_Q-\frac{1}{4\pi}\iiint\limits_{\Omega}\frac{\Delta u(Q)dV_Q}{|Q-P|}. \blacksquare$$

Motivated by the representation formula we set

$$F(Q,P)=\begin{cases} \dfrac{1}{4\pi|Q-P|} & \text{if } N=3, \\ -\frac{1}{2\pi}\ln|Q-P| & \text{if } N=2. \end{cases}$$

The function $F(Q,P)$ is called a *fundamental solution* of the Laplacian with pole at P.

In the case of an harmonic function u we get the following conclusions.

Corollary 6.4. *Let $u \in C^2(\Omega)\cap C^1(\bar{\Omega})$ be a harmonic function in Ω. Then*

$$u(P)=\iint\limits_{\partial\Omega}\left((F(Q,P)\frac{\partial u(Q)}{\partial n}-u(Q)\frac{\partial F(Q,P)}{\partial n}\right)dS_Q, \qquad (6.18)$$

for every $P\in\Omega$.

Corollary 6.5. *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then $u \in C^\infty(\Omega)$ and every partial derivative of u is a harmonic function in Ω.*

Proof. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ the conclusion follows by (6.18). If $u \in C^2(\Omega)$ we can apply (6.18) to any subdomain $\Omega' \subset \Omega$ with smooth boundary. ■

Exercises

1. A function $u \in C(\Omega)$ is called *subharmonic* if for every $P \in \Omega$, there exists a ball $B_r(P) \subset \Omega$ such that for every $\rho < r$

$$u(P) \le \frac{1}{4\pi\rho^2} \iint\limits_{S_\rho(P)} u(Q)dS_Q.$$

Prove that, if u is subharmonic and bounded from above, then u attains $\sup u$ in Ω, iff u is a constant.

2. Prove the vector form of the Green's second identity

$$\iiint\limits_{\Omega} (\vec{u} \cdot rot\, rot\, \vec{v} - \vec{v} \cdot rot\, rot\, \vec{u})dV = \iint\limits_{\partial\Omega} (\vec{u} \times rot\, \vec{v} - \vec{v} \times rot\, \vec{u}) \cdot \vec{n}\, dS,$$

where $\vec{u}(P)$ and $\vec{v}(P)$ are smooth vector-valued functions, Ω is a domain with smooth boundary Γ, $\vec{n}$ is the outward normal vector to Γ ($\vec{u} \times \vec{v}$ means the vector product of vectors $\vec{u}$ and $\vec{v}$.)

3. (a) Prove the Green's first identity for the biharmonic operator Δ^2

$$\iiint\limits_{\Omega} v\Delta^2 u dV = \iiint\limits_{\Omega} \Delta u \Delta v dV - \iint\limits_{\partial\Omega} \Delta u \frac{\partial v}{\partial n} dS + \iint\limits_{\partial\Omega} v \frac{\partial}{\partial n}(\Delta u) dS,$$

where $u,\ v \in C^4(\Omega) \cap C^3(\bar{\Omega})$.

(b) Prove Dirichlet principle for biharmonic functions. Among all functions $v \in C^4(\Omega) \cap C^3(\bar{\Omega})$ satisfying the boundary conditions

$$v(P) = \varphi(P), \quad \frac{\partial v}{\partial n}(P) = \psi(P), P \in \Gamma, \tag{6.19}$$

where $\varphi(P)$ and $\psi(P) \in C(\partial\Omega)$, the lowest energy

$$E(v) = \frac{1}{2} \iiint\limits_{\Omega} |\Delta v|^2 dV,$$

is attained by a biharmonic function u, i.e. a function satisfying $\Delta^2 u = 0$ and (6.19).

4. (a) Show that if u is a solution of the Neumann problem

$$\begin{aligned} \Delta u &= f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= \psi \text{ on } \partial\Omega, \end{aligned}$$

then

$$\iiint\limits_{\Omega} f(P)dV = \iint\limits_{\partial\Omega} \psi(P)dS.$$

(b) Prove Dirichlet principle for the Neumann boundary condition. Among all functions $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying the boundary condition

$$\frac{\partial}{\partial n} v(P) = \psi(P), \quad P \in \partial\Omega, \tag{6.20}$$

where $\psi(P) \in C(\partial\Omega)$ and $\iint_{\partial\Omega} \psi(P)dS = 0$, the lowest energy

$$E(v) = \frac{1}{2} \iiint\limits_{\Omega} |\Delta v|^2 dV - \iint\limits_{\partial\Omega} \psi(P)v(P)dS,$$

is attained by a harmonic function u, which satisfies (6.20).

5. (a) Prove that if $u \in C^1(\bar{\Omega})$ and $P_0 \in \Omega$ is an interior point, then the mean value $M_\varepsilon(P_0) = \dfrac{1}{4\pi\varepsilon^2} \displaystyle\iint_{S_\varepsilon(P_0)} u(Q)dS \to u(P_0)$ as $\varepsilon \to 0$.

(b) Show that the last statement is not true if u is a discontinuous function.

6.3 Green's Functions

Now we use Green's indentities to study the Dirichlet problem. Consider the problem of finding a function $\Phi(Q,P) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\begin{cases} \Delta_Q \Phi(Q,P) = 0, & Q \in \Omega, \\ \Phi(Q,P) = F(Q,P), & Q \in \partial\Omega, \end{cases} \tag{6.21}$$

where $P \in \Omega$ is fixed and

$$F(Q,P) = \begin{cases} \dfrac{1}{4\pi|Q-P|} & \text{if } N = 3, \\ -\frac{1}{2\pi} \ln|Q-P| & \text{if } N = 2. \end{cases}$$

Suppose that (6.21) has a solution and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a harmonic function. By the Green's second identity we have

$$\iint\limits_{\partial\Omega} \left(\Phi(Q,P)\frac{\partial u(Q)}{\partial n_Q} - u(Q)\frac{\partial \Phi}{\partial n_Q}(Q,P) \right) dS_Q = 0. \tag{6.22}$$

Consider the Dirichlet problem of finding a function $u(P) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$\begin{cases} \Delta u(P) = 0, \ P \in \Omega, \\ u(P) = \varphi(P), \ P \in \partial\Omega, \end{cases} \tag{6.23}$$

where $\varphi(P) \in C(\partial\Omega)$. By the representation formula

$$u(P) = \iint\limits_{\partial\Omega} \left(F(Q,P)\frac{\partial u(Q)}{\partial n_Q} - u(Q)\frac{\partial F(Q,P)}{\partial n_Q} \right) dS_Q \tag{6.24}$$

Substracting (6.24) from (6.22) and using (6.21), we obtain

$$u(P) = -\iint\limits_{\partial\Omega} \varphi(Q)\frac{\partial}{\partial n_Q} G(Q,P) dS_Q, \tag{6.25}$$

where $G(Q,P) = F(Q,P) - \Phi(Q,P)$ is known as *Green's function* for the Laplacian in Ω. Formula (6.25) is an integral representation of any solution of the Dirichlet problem (6.23).

A main property of the Green's function is its *symmetry.*

Lemma 6.1. *The Green's function for the Laplacian in Ω is symmetric, i.e. for every P_1 and $P_2 \in \Omega$*

$$G(P_1, P_2) = G(P_2, P_1). \tag{6.26}$$

Proof. Let $\varepsilon > 0$ be small enough such that $B_\varepsilon(P_i) \subset \Omega$, $i = 1, 2$ and $B_\varepsilon(P_1) \cap B_\varepsilon(P_2) = \varnothing$. The functions $u(P) = G(P, P_1)$ and $v(P) = G(P, P_2)$ are harmonic in $\Omega_\varepsilon = \Omega \backslash (B_\varepsilon(P_1) \cup B_\varepsilon(P_2))$. Applying the Green's second identity to $u(P)$ and $v(P)$ in Ω_ε, we have

$$\begin{aligned} 0 &= \iiint_{\Omega_\varepsilon} (u\Delta v - v\Delta u) dV \qquad (6.27) \\ &= \iint_{S_\varepsilon(P_1)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS + \iint_{S_\varepsilon(P_2)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS, \end{aligned}$$

because $u(P) = v(P) = 0$ on $\partial\Omega$.

Letting $\varepsilon \to 0$ we observe that

$$\lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_1)} u(P)\frac{\partial v(P)}{\partial n}dS = 0, \quad \lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_2)} v(P)\frac{\partial u(P)}{\partial n}dS = 0.$$

Therefore by (6.27)

$$\lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_1)} v\frac{\partial u}{\partial n}dS = \lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_2)} u\frac{\partial v}{\partial n}dS.$$

As $\Phi(P, P_i) \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $i = 1, 2$ by

$$\begin{aligned}
\frac{\partial u(P)}{\partial n} &= \frac{\partial F(P, P_1)}{\partial n} - \frac{\partial \Phi(P, P_1)}{\partial n}, \\
\frac{\partial v(P)}{\partial n} &= \frac{\partial F(P, P_2)}{\partial n} - \frac{\partial \Phi(P, P_2)}{\partial n},
\end{aligned}$$

and, as before, letting $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_1)} v(P)\frac{\partial \Phi(P, P_1)}{\partial n}dS = \lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_2)} u(P)\frac{\partial \Phi(P, P_2)}{\partial n}dS = 0.$$

Then

$$\lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_1)} v(P)\frac{\partial F(P, P_1)}{\partial n}dS = \lim_{\varepsilon\to 0} \iint\limits_{S_\varepsilon(P_2)} u(P)\frac{\partial F(P, P_2)}{\partial n}dS.$$

Calculating the limits in the last expression, as in (6.17), we obtain

$$v(P_1) = u(P_2),$$

which means that $G(P_1, P_2) = G(P_2, P_1)$. ■

Corollary 6.6. *The functions* $P \longmapsto G(Q, P)$ *and* $P \longmapsto \dfrac{\partial}{\partial n_Q}G(Q, P)$ *are harmonic in* Ω *for every* $Q \in \partial\Omega$.

Proof. As $G(Q, P) = G(P, Q)$ and $F(P, Q) = F(Q, P)$ it follows that $\Phi(Q, P) = \Phi(P, Q)$. As $\Delta_P \Phi(P, Q) = 0$ by the definition it follows that

$$\Delta_P \Phi(Q, P) = \Delta_P \Phi(P, Q) = 0,$$

which implies that $\Delta_P G(Q, P) = 0$ for $Q \in \partial\Omega,\ P \in \Omega$. Moreover we have

$$\Delta_P \frac{\partial}{\partial n_Q} G(Q, P) = \frac{\partial}{\partial n_Q} \Delta_P G(Q, P) = 0. \blacksquare$$

The solution of the Dirichlet problem (6.23) is given by formula (6.25). By Corollary 6.6 the function $u(P)$ is harmonic since

$$\Delta_P u(P) = -\iint\limits_{\partial\Omega} \varphi(Q)\Delta_P \frac{\partial}{\partial n_Q} G(Q,P) dS_Q = 0.$$

It remains to show the boundary condition in the sense

$$\lim_{P\to Q} u(P) = \varphi(Q), \quad Q \in \partial\Omega.$$

This can be shown if the problem (6.21) has a solution, i.e. if the Green's function for the Laplacian in Ω is determined.

The Green's function also allows us to solve Dirichlet problem for Poisson equation. Namely, the solution of the problem

$$\begin{cases} \Delta u(P) = f(P) \text{ in } \Omega, \\ u(P) = \varphi(P) \text{ on } \partial\Omega, \end{cases} \tag{6.28}$$

is given by

$$u(P) = -\iiint\limits_{\Omega} f(Q)G(Q,P)dV_Q - \iint\limits_{\partial\Omega} \varphi(Q)\frac{\partial G(Q,P)}{\partial n_Q} dS_Q. \tag{6.29}$$

Solving the Dirichlet problem (6.28) reduces to solving the Dirichlet problem (6.21). We solve the problem (6.21) for some regions Ω with simple geometry.

6.4 Green's Functions for a Half-space and Sphere

6.4.1 Half-space

Let $D \subset \mathbf{R}^3$ be the half-space of points $P(x,y,z)$, $z > 0$. Each point $P(x,y,z) \in D$ has a reflected point $P^*(x,y,-z) \notin D$. Suppose $Q(\xi,\eta,0) \in \partial D = \{z=0\}$. By symmetry of P and P^* with respect to ∂D

$$|Q-P| = |Q-P^*|. \tag{6.30}$$

As D is an infinite region, all the properties on Green's function are still valid if we impose the so-called boundary condition at infinity, that is the function and its derivatives are tending to 0 as $|P| \to \infty$.

We assert that the Green's function for D is

$$G(Q,P) = \frac{1}{4\pi|Q-P|} - \frac{1}{4\pi|Q-P^*|}.$$

Indeed we have to show that $\Phi(Q,P) = \dfrac{1}{4\pi|Q-P^*|}$ is a solution of the problem

$$\begin{cases} \Delta_Q \Phi(Q,P) = 0 \text{ in } D, \\ \Phi(Q,P) = \dfrac{1}{4\pi|Q-P|} \text{ on } \partial D. \end{cases}$$

As the function $\dfrac{1}{4\pi|Q-P^*|}$ is harmonic in D, because $P^* \notin D$ it remains to prove that the boundary conditions are satisfied. As for $Q \in \partial D$, $|Q-P| = |Q-P^*|$ it follows that

$$\Phi(Q,P) = \frac{1}{4\pi|Q-P^*|} = \frac{1}{4\pi|Q-P|}.$$

We find an explicit formula for the Dirichlet problem for the Laplace equation in the half-space

$$\begin{cases} \Delta u(x,y,z) = 0 \text{ in } z > 0, \\ u(x,y,0) = \varphi(x,y), \quad (x,y) \in \mathbf{R}^2. \end{cases} \tag{6.31}$$

Denote $P(x,y,z) \in D$, $\quad Q = (\xi,\eta,\zeta)$ and observe that

$$\begin{aligned} \frac{\partial G(Q,P)}{\partial n_Q} &= -\left.\frac{\partial G(Q,P)}{\partial \zeta}\right|_{\zeta=0} \\ &= \left.\frac{1}{4\pi}\left(\frac{\zeta - z}{|Q-P|^3} - \frac{\zeta + z}{|Q-P^*|^3}\right)\right|_{\zeta=0} \\ &= -\left.\frac{1}{2\pi}\frac{z}{|Q-P|^3}\right|_{\zeta=0} \\ &= -\frac{1}{2\pi}\frac{z}{\left((\xi-x)^2 + (\eta-y)^2 + z^2\right)^{3/2}}. \end{aligned}$$

Then, in view of (6.25), the solution of the problem (6.31) is

$$u(x,y,z) = \frac{z}{2\pi}\iint\limits_{\mathbf{R}^2} \frac{\varphi(\xi,\eta)d\xi d\eta}{\left((x-\xi)^2 + (y-\eta)^2 + z^2\right)^{3/2}} \tag{6.32}$$

Lemma 6.2. *For all $z > 0$ and for all $(x,y) \in \mathbf{R}^2$*

$$\frac{z}{2\pi}\iint\limits_{\mathbf{R}^2} \frac{d\xi d\eta}{\left((x-\xi)^2 + (y-\eta)^2 + z^2\right)^{3/2}} = 1.$$

Proof. Making the change of variables $\xi = x + \rho\cos\theta$, $\eta = y + \rho\sin\theta$ we have for $z > 0$

$$
\begin{aligned}
& \frac{z}{2\pi}\iint_{\mathbf{R}^2} \frac{d\xi d\eta}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}} \\
= & \frac{z}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\rho\, d\rho\, d\theta}{(\rho^2+z^2)^{3/2}} \\
= & \frac{z}{2}\left.\frac{\left(\rho^2+z^2\right)^{-\frac{1}{2}}}{-\frac{1}{2}}\right|_0^{\infty} \\
= & 1. \blacksquare
\end{aligned}
$$

By Lemma 6.2 there follows a maximum principle for harmonic functions in a half-space.

Corollary 6.7. *If $\varphi(Q) \in C(\mathbf{R}^2)\cap L^\infty(\mathbf{R}^2)$, $|\varphi(Q)| \le M$ then the function $u(x, y, z)$ given by (6.32) is harmonic and $|u(P)| \le M$.*

6.4.2 Sphere

Let B_R be the open ball of radius R centered at the origin O, $S_R = \partial B_R$. Let $P \in B_R$ and P^* be the inverse point of P with respect to S_R defined as

$$\overrightarrow{OP^*} = \frac{R^2}{|P|^2}\overrightarrow{OP},$$

which implies

$$|P|\,|P^*| = R^2. \tag{6.33}$$

If $Q \in S_R$ by (6.33) it follows that the two triangles QOP^* and POQ are similar, because they have a common angle $\sphericalangle QOP = \sphericalangle QOP^*$, $|Q| = R$ and

$$\frac{|P|}{|Q|} = \frac{|Q|}{|P^*|}.$$

Then it follows

$$\frac{|Q-P|}{|Q-P^*|} = \frac{|P|}{R}$$

or

$$\frac{1}{|Q-P|} = \frac{R}{|P||Q-P^*|} = \frac{|P^*|}{R|Q-P^*|}, \tag{6.34}$$

if $Q \in S_R$, $P \in B_R$.

For each $P \in B_R \backslash \{O\}$ the solution of (6.21) is given by

$$\Phi(Q,P) = \begin{cases} \dfrac{R}{4\pi|P||Q-P*|} & \text{if } P \neq O, \\[2ex] \dfrac{1}{4\pi R} & \text{if } P = O. \end{cases}$$

The function $\Phi(Q,P)$ is harmonic with respect to P in $B_R \backslash \{P\}$ because $\dfrac{1}{|Q-P|}$ and $\dfrac{1}{|Q-P*|}$ are harmonic. By (6.34) it follows that the boundary condition $\Phi(Q,P) = \dfrac{1}{4\pi|Q-P|}$, $Q \in S_R$ is satisfied.

In the two dimensional case we have

$$\Phi(Q,P) = \begin{cases} -\dfrac{1}{2\pi} \ln |Q - P^*| \dfrac{|P|}{R} & \text{if } P \neq O, \\[2ex] -\dfrac{1}{2\pi} \ln R & \text{if } P = O. \end{cases}$$

Thus the Green's function for the sphere S_R in $\mathbf{R}^3$ is

$$G(Q,P) = \frac{1}{4\pi} \left(\frac{1}{|Q-P|} - \frac{R}{|P||Q-P*|} \right).$$

In order to solve the Dirichlet problem according to the formula (6.25) let us compute the normal derivative $\frac{\partial}{\partial n_Q} G(Q,P)$, which for the sphere S_R is

$$\left. \frac{\partial}{\partial |Q|} G(Q,P) \right|_{Q \in S_R}.$$

Applying the cosine theorem to the triangles OQP and OQP^* we have

$$|Q-P|^2 = |P|^2 + |Q|^2 - 2|P||Q| \cos \gamma,$$

$$|Q-P^*|^2 = \frac{R^4}{|P*|^2} + |Q|^2 - 2\frac{R^2|Q|}{|P*|} \cos \gamma,$$

where $\gamma = \sphericalangle QOP$. Therefore for $Q \in S_R$, γ fixed

$$\begin{aligned} \frac{\partial |Q-P|}{\partial |Q|} &= \frac{|Q| - |P| \cos \gamma}{|Q-P|}, \\ \frac{\partial |Q-P*|}{\partial |Q|} &= \frac{|Q| - \frac{R^2}{|P*|} \cos \gamma}{|Q-P*|}, \end{aligned}$$

$$
\begin{aligned}
-\frac{\partial}{\partial n_Q}G(Q,P)\bigg|_{|Q|=R} &= -\frac{\partial}{\partial |Q|}G(Q,P)\bigg|_{|Q|=R} \\
&= \frac{1}{4\pi}\left(\frac{|Q|-|P|\cos\gamma}{|Q-P|^3} - \frac{R\left(|Q|-\dfrac{R^2}{|P|}\cos\gamma\right)}{|P||Q-P|^3}\right) \\
&= \frac{1}{4\pi}\left(\frac{|Q|-|P|\cos\gamma}{|Q-P|^3} - \frac{|P|\left(|Q|\,|P|-R^2\cos\gamma\right)}{R^2|Q-P|^3}\right) \\
&= \frac{1}{4\pi}\frac{R^3-R|P|^2}{R^2|Q-P|^3} \\
&= \frac{1}{4\pi}\frac{R^2-|P|^2}{R|Q-P|^3}.
\end{aligned}
\tag{6.35}
$$

Substituting (6.35) in (6.25) we obtain

Theorem 6.6. *Let $u \in C^2(B_R) \cap C^1(\overline{B_R})$ be a solution of the Dirichlet problem*

$$
\begin{cases} \Delta u = 0 \text{ in } B_R, \\ u(P) = \varphi(P) \text{ on } S_R. \end{cases} \tag{6.36}
$$

Then

$$
u(P) = \frac{R^2-|P|^2}{4\pi R}\iint\limits_{S_R}\frac{\varphi(Q)}{|Q-P|^3}dS_Q. \tag{6.37}
$$

Setting $u(P) = 1$ in $\overline{B_R}$ we obtain

Corollary 6.8. *For every $P \in B_R$*

$$
\iint\limits_{S_R}\frac{1}{|Q-P|^3}dS_Q = \frac{4\pi R}{R^2-|P|^2}. \tag{6.38}
$$

Note that (6.37) and (6.38) are known as Poisson formulae. In the two dimensional case we have respectively

$$
u(P) = \frac{R^2-|P|^2}{2\pi R}\oint_{C_R}\frac{\varphi(Q)}{|Q-P|^2}ds_Q, \tag{6.39}
$$

$$
\oint_{C_R}\frac{1}{|Q-P|^2}ds_Q = \frac{2\pi R}{R^2-|P|^2}. \tag{6.40}
$$

Introducing polar coordinates for

$$P(\rho\cos\theta, \rho\sin\theta),\ Q(R\cos\tau, R\sin\tau),$$

$$u(\rho,\theta) = u(\rho\cos\theta, \rho\sin\theta),\ \varphi(\tau) = \varphi(R\cos\tau, R\sin\tau)$$

by (6.39), (6.40) we obtain

$$u(\rho,\theta) = \frac{R^2-\rho^2}{2\pi}\int_0^{2\pi}\frac{\varphi(\tau)d\tau}{R^2-2R\rho\cos(\theta-\tau)+\rho^2},$$

$$\int_0^{2\pi}\frac{d\tau}{R^2-2R\rho\cos(\theta-\tau)+\rho^2} = \frac{2\pi}{R^2-\rho^2},$$

known also as Poisson formulae.

The formula (6.37) gives the unique solution of the Dirichlet problem for the sphere, which is the statement of the following

Theorem 6.7. *For every $\varphi(P) \in C(S_R)$, the function*

$$u(P) = \frac{R^2-|P|^2}{4\pi R}\iint\limits_{S_R}\frac{\varphi(Q)dS_Q}{|Q-P|^3}, \tag{6.41}$$

is the unique solution of the Dirichlet problem.

Proof. By Corollary 6.6, u given by (6.41) is a harmonic function in B_R. To prove the boundary condition in the sense

$$\lim_{P\to Q} u(P) = \varphi(Q), \quad Q \in S_R,$$

let us fix a $Q_0 \in S_R$ and $\varepsilon > 0$. As $\varphi \in C(S_R)$ there exists $\delta > 0$ such that

$$|\varphi(Q)-\varphi(Q_0)| < \frac{\varepsilon}{2} \text{ if } |Q-Q_0| < \delta,\ Q \in S_R.$$

Moreover let $M > 0$ be such that $|\varphi(P)| \le M,\ P \in S_R$. By (6.37) and (6.38) we have

$$\begin{aligned}
|u(P)-\varphi(Q_0)| &\le \frac{R^2-|P|^2}{4\pi R}\iint_{S_R}\frac{|\varphi(Q)-\varphi(Q_0)|}{|Q-P|^3}dS_Q \\
&\le \frac{R^2-|P|^2}{4\pi R}\left(\frac{\varepsilon}{2}\iint_{S_1}\frac{dS_Q}{|Q-P|^3} + \iint_{S_2}\frac{|\varphi(Q)-\varphi(Q_0)|}{|Q-P|^3}dS_Q\right) \\
&= \frac{\varepsilon}{2} + \frac{R^2-|P|^2}{4\pi R}\iint_{S_2}\frac{|\varphi(Q)-\varphi(Q_0)|}{|Q-P|^3}dS_Q,
\end{aligned} \tag{6.42}$$

where

$$S_1 = S_R \cap \{Q : |Q - Q_0| < \delta\}, \quad S_2 = S_R \backslash S_1.$$

Let $\delta_1 \in (0, \delta)$ and $P \in B_R$, $|P - Q_0| < \delta_1$.

Observe that for $Q \in S_2$

$$\delta \leq |Q - Q_0| \leq |P - Q_0| + |P - Q| < \delta_1 + |P - Q|,$$

$$\delta - \delta_1 \leq |P - Q|,$$

$$R = |Q_0| \leq |P - Q_0| + |P| < \delta_1 + |P|,$$

$$R - \delta_1 \leq |P|,$$

and

$$\begin{aligned} J(P) \quad : \quad &= \frac{R^2 - |P|^2}{4\pi R} \iint_{S_2} \frac{|\varphi(Q) - \varphi(Q_0)| dS_Q}{|Q - P|^3} \\ &\leq \frac{R^2 - (R - \delta_1)^2}{4\pi R} \frac{2M}{(\delta - \delta_1)^3} |S_2| \\ &\leq \frac{(2R\delta_1 - \delta_1^2)\, M\, |S_R|}{2\pi R(\delta - \delta_1)^3}. \end{aligned}$$

The last estimate implies

$$\lim_{P \to Q_0} J(P) = 0.$$

Taking $\delta_1 \in (0, \delta)$ such that $J(P) < \frac{\varepsilon}{2}$, by (6.42) we obtain that for $|P - Q_0| < \delta_1$, $P \in B_R$ it follows

$$|u(P) - \varphi(Q_0)| < \varepsilon.$$

It means $\lim\limits_{P \to Q_0} u(P) = \varphi(Q_0)$.

By the maximum-minimum principle it follows that $u(P)$ is the unique solution of the Dirichlet problem. ■

Exercises

1. (a) Find the Green's function for the half-plane $G = \{(x, y) : y > 0\} \subset \mathbf{R}^2$.

(b) If $u \in C^2(G) \cap C^1(\bar{G})$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta u(x, y) = 0, \ x \in \mathbf{R}, \ y > 0, \\ u(x, 0) = \varphi(x), \ x \in \mathbf{R}, \end{cases}$$

then

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\xi)d\xi}{y^2+(\xi-x)^2}. \tag{6.43}$$

(c) Prove that

$$\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\xi}{y^2+(\xi-x)^2} = 1$$

and if $\varphi(x) \in C(R) \cap L^\infty(R)$ then the function (6.43) satisfies $\lim\limits_{(x,y)\to(x_0,0)} u(x,y) = \varphi(x_0)$.

2. (a) Find the Green's function for the exterior sphere $B_R^e = \{P \in \mathbf{R}^3 : |P| > R\}$

(b) If $u \in C^2(B_R^e) \cap C^1(\overline{B_R^e})$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta u(P) = 0, \ P \in B_R^e, \\ u(P) = \varphi(P), \ P \in S_R, \end{cases}$$

then

$$u(P) = \frac{|P|^2 - R^2}{4\pi R}\iint\limits_{S_R}\frac{\varphi(Q)dS_Q}{|Q-P|^3}, \quad P \in B_R^e. \tag{6.44}$$

(c) Prove that if $\varphi(P) \in C(S_R)$ the function (6.44) is harmonic,

$$\lim_{P\to Q} u(P) = \varphi(Q), \text{ if } Q \in S_R$$

and

$$\lim_{|P|\to\infty} u(P) = 0.$$

3. Let Ω be a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $P \in \Omega$. The function

$$R(Q,P) = F(Q,P) - \psi(Q,P)$$

where

$$\begin{cases} \Delta_Q\psi(Q,P) = 0, \quad Q \in \Omega, \\ \frac{\partial}{\partial n_Q}\psi(Q,P) + a\psi(Q,P) = \frac{\partial}{\partial n_Q}F(Q,P) + aF(Q,P), \ Q \in \Gamma, \end{cases}$$

$a \neq 0$ is a constant, is called Robin function. Prove that:

(a) $R(P_1,P_2) = R(P_2,P_1)$ if $P_1, P_2 \in \Omega$, $P_1 \neq P_2$.

(b) if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of the Robin problem

$$\begin{cases} \Delta u = f(P), \ P \in \Omega, \\ \frac{\partial u}{\partial n} + au = \varphi(P), \ P \in \Gamma, \end{cases}$$

then

$$u(P) = \iint\limits_{\Gamma}\varphi(Q)R(Q,P)dS_Q - \iiint\limits_{\Omega} f(Q)R(Q,P)dV_Q.$$

6.5 Harnack's Inequalities and Theorems

As an application of Poisson formula (6.37) we derive, the so called [3] *Harnack's inequalities* for harmonic functions.

Theorem 6.8. *Let u be a nonnegative harmonic function in Ω. Then for every $P_0 \in \Omega$, $B_R(P_0) \subset \Omega$ and every $\rho \in (0, R)$, $P \in S_\rho(P_0)$*

$$\frac{R(R-\rho)}{(R+\rho)^2}u(P_0) \le u(P) \le \frac{R(R+\rho)}{(R-\rho)^2}u(P_0). \tag{6.45}$$

Proof. Using a translation of the argument if it is necessary we may assume $P_0 = O$. We have by Poisson formula and the mean value property

$$\begin{aligned}
u(P) &= \frac{R^2-|P|^2}{4\pi R}\iint_{S_R}\frac{u(Q)dS_Q}{|Q-P|^3}\\
&\le \frac{R^2-|P|^2}{4\pi R}\iint_{S_R}\frac{u(Q)dS_Q}{(|Q|-|P|)^3}\\
&= \frac{R^2-|P|^2}{4\pi R(R-|P|)^3}\iint_{S_R}u(Q)dS_Q\\
&= \frac{R(R^2-\rho^2)}{(R-\rho)^3}u(O)\\
&= \frac{R(R+\rho)}{(R-\rho)^2}u(O).
\end{aligned}$$

By the same way

$$\begin{aligned}
u(P) &\ge \frac{R^2-\rho^2}{4\pi R}\iint_{S_R}\frac{u(Q)dS_Q}{(|Q|+|P|)^3}\\
&= \frac{R^2-\rho^2}{4\pi R(R+\rho)^3}\iint_{S_R}u(Q)dS_Q\\
&= \frac{R(R^2-\rho^2)}{(R+\rho)^3}u(O)\\
&= \frac{R(R-\rho)}{(R+\rho)^2}u(O). \blacksquare
\end{aligned}$$

[3] Alex Harnack, 1851–1888. A. Harnack. Grundlagen des logaritmischen Potentiales. Leipzig, 1887.

Corollary 6.9. *(Liouville's theorem). A nonnegative harmonic function in* $\mathbf{R}^3$ *is a constant.*

Proof. Fix $P_0 \in \mathbf{R}^3$ and $\rho > 0$. By (6.45) letting $R \to +\infty$ we obtain $u(P_0) = u(P)$ for every P, $|P - P_0| = \rho$. As P_0 and ρ are arbitrary, u should be a constant in $\mathbf{R}^3$. ■

As a consequence of Liouville's theorem we derive the unique continuation property for harmonic functions on a half-space.

Theorem 6.9. *Every* $\varphi(x, y) \in C(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ *has a unique bounded harmonic extension in* $\mathbf{R}^2 \times \{z > 0\}$, *given by*

$$u(x, y, z) = \frac{z}{2\pi} \iint_{\mathbf{R}^2} \frac{\varphi(\xi, \eta) d\xi d\eta}{((x - \xi)^2 + (y - \eta)^2 + z^2)^{3/2}}. \tag{6.46}$$

Proof. It is easy to see that the function $u(x, y, z)$, given by (6.46), is harmonic for $z > 0$. By Corollary 6.7 it is a bounded function.

Suppose v_1 and v_2 are two bounded harmonic functions on $\mathbf{R}^2 \times \{z > 0\}$ such that $v_1|_{z=0} = v_2|_{z=0} = \varphi$. Then $w = v_1 - v_2$ is harmonic on $\mathbf{R}^2 \times \{z > 0\}$ and $w|_{z=0} = 0$. The function

$$\tilde{w}(x, y, z) = \begin{cases} w(x, y, z), & z > 0 \\ -w(x, y, -z), & z \leq 0 \end{cases}$$

is bounded and harmonic on the whole $\mathbf{R}^3$. By Liouville's theorem it must be a constant. As $w|_{z=0} = 0$, then $w \equiv 0$ on $\mathbf{R}^3$.

The existence part shows that (6.46) is the unique harmonic extension of φ to $\mathbf{R}^2 \times \{z > 0\}$. ■

Denote for simplicity by $\Pi(Q, P)$ the Poisson kernel in $\mathbf{R}^3$,

$$\Pi(Q, P) := \frac{R^2 - |P|^2}{4\pi R |Q - P|^3},$$

which is a harmonic function

$$\Delta_P \Pi(Q, P) = 0, \quad P \neq Q.$$

By Theorem 6.7

$$u(P) = \iint_{S_R} \varphi(Q) \Pi(Q, P) dS_Q$$

is a harmonic function in B_R.

Theorem 6.10. *(Harnack's first theorem) Let $\{u_n(P)\}$ be a sequence of harmonic functions in a domain Ω, uniformly convergent on every compact $K \subset \Omega$. Then the limit function $u(P)$ is harmonic in Ω.*

Proof. It is clear that the limit function $u(P)$ is a continuous function. Let B be an open ball, $S = \partial B$ and $P \in B$. Passing to the limit in

$$u_n(P) = \iint_S u_n(Q)\Pi(Q, P)dS_Q$$

we obtain

$$u(P) = \iint_S u(Q)\Pi(Q, P)dS_Q.$$

As $\Pi(Q, P)$ is a harmonic function in B, by

$$\Delta_P u(P) = \iint_S u(Q)\Delta_P \Pi(Q, P)dS_Q = 0$$

it follows that $u(P)$ is harmonic in B. As B is arbitrary, $u(P)$ is harmonic in Ω. ■

Theorem 6.11. *(Harnack's second theorem). Let $u_1(P) \leq u_2(P) \leq ... \leq u_n(P) \leq ...$be a monotone increasing sequence of harmonic functions in Ω, which is convergent in a point $Q \in \Omega$. Then $\{u_n\}$ is uniformly convergent on every compact subset $K \subset \Omega$ and the limit function u is a harmonic in Ω.*

Proof. Let O_1 be the set of points P of Ω where the sequence $\{u_n(P)\}$ is convergent. We shall show that O_1 is an open set. Let $Q \in O_1$ and $2R = dist(Q, \partial\Omega)$. We show that the sequence $\{u_n(P)\}$ is convergent in the ball $B_{R/3}(Q)$. By the monotone property and Harnack's inequality we have

$$v_{n,m}(P) = u_{n+m}(P) - u_n(P) \geq 0, \quad m > 0$$

and

$$v_{n,m}(P) \leq \frac{R(R+\rho)}{(R-\rho)^2} v_{n,m}(Q) \tag{6.47}$$

for every P, $|P - Q| = \rho \leq \frac{R}{3}$. Let $\varepsilon > 0$ be arbitrary and $N(\varepsilon)$ be such that for $n > N(\varepsilon)$, $m > 0$

$$u_{n+m}(Q) - u_n(Q) < \frac{\varepsilon}{3}.$$

¿From (6.47) we have

$$0 \le v_{n,m}(P) < \frac{R\left(R + \frac{R}{3}\right)}{\left(R - \frac{R}{3}\right)^2} \cdot \frac{\varepsilon}{3} = \varepsilon, \tag{6.48}$$

which means that $\{u_n(P)\}$ is convergent for every P, $|P - Q| < \frac{R}{3}$.

Thus O_1 is an open set. Let now $O_2 = \Omega \backslash O_1$. We prove that O_2 is also open. As Ω is a connected set and $O_1 \neq \varnothing$ it implies that $O_2 = \varnothing$ and $\Omega = O_1$.

Let $Q \in O_2$ be arbitrary and $4r = dist(Q, \partial\Omega)$. If there exists a point $Q_0 \in O_1$, $|Q - Q_0| < \frac{r}{3}$ it follows by previous observation that $Q \in O_1$. Therefore $B_{\frac{r}{3}}(Q) \subset O_2$, which shows that O_2 is open. As we have noted, this implies that $\Omega = O_1$, so $\{u_n(P)\}$ is convergent at every point of Ω. By (6.48) it follows that $\{u_n\}$ is uniformly convergent on closed balls. If $K \subset \Omega$ is a compact subset covering K by a finite number of balls with appropriate radius, we obtain that $\{u_n\}$ is uniformly convergent on K. By Harnack's first theorem the limit function is a harmonic function. ■

Exercises

1 (a) Prove the Harnack's inequalities in the two dimensional case. For every $P_0 \in \Omega$, $K_R(P_0) \subset \Omega$ and every $\rho \in (O, R)$, $P \in C_\rho(P_0)$

$$\frac{R-\rho}{R+\rho} u(P_0) \le u(P) \le \frac{R+\rho}{R-\rho} u(P_0),$$

where $u(P)$ is a nonnegative harmonic function on $\Omega \subset \mathbf{R}^2$.

(b) Prove Liouville's theorem in $\mathbf{R}^2$.

2. Prove that Liouville's theorem holds for harmonic functions in $\mathbf{R}^3$, bounded from above (below).

3 (a) Prove that if $u(P)$ is a harmonic function in $\mathbf{R}^3$, $R > 0$, then

$$\begin{aligned} \frac{\partial u}{\partial x}(P) &= \frac{3}{4\pi R^3} \iint_{S_R(P)} u(Q) n_x dS, \\ \left| \frac{\partial u(P)}{\partial x} \right| &\le \frac{3}{R} \max_{S_R(P)} |u|. \end{aligned}$$

(b) Prove that, if $u(P)$ is a harmonic function in $\mathbf{R}^3$ and for every $Q \in \mathbf{R}^3$

$$\lim_{|Q-P| \to \infty} \left(\frac{u(P)}{|Q - P|} \right) = 0,$$

then u is a constant.

4 (a) Let B be the unit ball centered at the origin and u be the unique solution of

$$\begin{cases} \Delta u = 0 \text{ in } B, \\ u|_{\partial B} = \varphi. \end{cases}$$

Prove that if $\varphi \in C(\partial B)$ and $\varphi(x,y,z) = -\varphi(x,y,-z)$ then $u(x,y,z) = -u(x,y,-z)$.

(b) Let u be a harmonic function in $B^+ = B \cap \{z > 0\}$ vanishing for $z = 0$. Extend u to a harmonic function on B.

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Chapter 7

Fourier Series and Fourier Method for PDEs

7.1 Fourier Series

7.1.1 Fourier coefficients. Convergence of Fourier series

In this chapter we consider Fourier series and the Fourier method in order to solve boundary value problems for linear PDEs in terms of series. This approach was used by Joseph Fourier[1] , who had developed his ideas on trigonometric series studying heat conduction.

Let us begin with the *Fourier sine series.*

Let $f(x)$ be a piecewise continuous function, $x \in [0, l]$ and $f(x)$ be expressed as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}. \tag{7.1}$$

The problem is, how to find the coefficients b_n if $f(x)$ is a given function? Observe that

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0 \quad \text{if } m \neq n, \tag{7.2}$$

$$\int_0^l \sin^2 \frac{n\pi x}{l} dx = \frac{l}{2} \qquad\qquad n \in \mathbf{N}, \tag{7.3}$$

[1]Jean Baptiste Joseph Fourier, 21.03.1768–16.05.1830.
His results on the representation of functions by trigonometric series, presented to the Academy of Sciences in Paris in 1807 and 1811, were criticized (most strongly by Lagrange) for a lack of rigor and were not published until 1822.

known as orthogonality property of the trigonometric system

$$\left\{\sin\frac{n\pi x}{l} \,:\, n \in \mathbf{N}\right\}.$$

Suppose that we can integrate (7.1) term by term.

Multiplying (7.1) by $\sin\frac{m\pi x}{l}$, integrating from 0 to l and using (7.2) and (7.3), we obtain

$$b_m = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx, \ m \in \mathbf{N}. \tag{7.4}$$

Similarly suppose that $f(x)$ is expanded in *Fourier cosine series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \quad x \in [0, l].$$

Using orthogonality of the trigonometric system $\left\{\cos\frac{n\pi x}{l} \,:\, n \in \mathbf{N}\cup\{0\}\right\}$

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0, \quad m \neq n,$$

$$\int_0^l \cos^2\frac{n\pi x}{l}dx = \frac{l}{2}, \quad n \in \mathbf{N},$$

we obtain that the coefficients a_m are expressed as

$$\begin{aligned} a_0 &= \frac{2}{l}\int_0^l f(x)dx, \\ a_m &= \frac{2}{l}\int_0^l f(x)\cos\frac{m\pi x}{l}dx, \quad m \in \mathbf{N}. \end{aligned}$$

A full Fourier series, or simply *Fourier series,* of a function $f(x)$, where $x \in (-l, l)$ is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right). \tag{7.5}$$

Observe that

$$\int_{-l}^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0, \quad n \neq m, \tag{7.6}$$

$$\int_{-l}^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0, \quad n \neq m,$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0, \quad \forall n, \forall m \in \mathbf{N}\cup\{0\}$$

and

$$\int_{-l}^{l} \cos^2\frac{n\pi x}{l}dx = \int_{-l}^{l} \sin^2\frac{n\pi x}{l}dx = l, \tag{7.7}$$

$$\int_{-l}^{l} dx = 2l,$$

known as orthogonality of the trigonometric system

$$\left\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, ..., \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}, ...\right\}.$$

If we can integrate (7.5) term by term, then using orthogonality we obtain the coefficients a_m and b_m as

$$a_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}dx, \quad m \in \mathbf{N}\cup\{0\} \tag{7.8}$$

and

$$b_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{m\pi x}{l}dx, \quad m \in \mathbf{N}. \tag{7.9}$$

Recall some facts for series of functions.

Let $I = [a, b]$ be a closed and bounded interval, $u_n(x) : I \to R$, $n \in \mathbf{N}$, be a function. The series

$$\sum_{n=1}^{\infty} u_n(x) \tag{7.10}$$

is *pointwise convergent* to a function $u(x)$ on I iff for every $x_0 \in I$ the series

$$\sum_{n=1}^{\infty} u_n(x_0)$$

converges to $u(x_0)$.

This means that for every $x_0 \in I$ and every $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon, x_0)$, such that if $N > N_0$

$$\left|u(x_0) - \sum_{n=1}^{N} u_n(x_0)\right| < \varepsilon.$$

The series (7.10) *converges uniformly* to $u(x)$ on I iff for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that if $N > N_0$

$$\left|u(x) - \sum_{n=1}^{N} u_n(x)\right| < \varepsilon,$$

for every $x \in I$.

The series (7.10) *converges in the mean-square* (or in L^2) sense to $u(x)$ on I iff for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that if $N > N_0$

$$\int_a^b \left| u(x) - \sum_{n=1}^{N} u_n(x) \right|^2 dx < \varepsilon.$$

Note that uniform convergence is stronger than both pointwise and mean-square convergence.

Remark 7.1. If $v(x)$ is a bounded function on I,

$$|v(x)| \leq M$$

for every $x \in I$ and $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on I, then the series $\sum_{n=1}^{\infty} u_n(x)v(x)$ is uniformly convergent.

The series (7.10) is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |u_n(x)|$ is convergent. A criterion for uniform convergence is

Theorem 7.1. *(Weierstrass criterion) Let there exist constants c_n, $n \in \mathbf{N}$ such that*

$$|u_n(x)| \leq c_n, \quad \forall n, \quad \forall x \in I$$

and the series

$$\sum_{n=1}^{\infty} c_n$$

be convergent. Then the series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on I.

For instance, if the coefficients a_n and b_n of the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{7.11}$$

are such that the series

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|)$$

is convergent, then (7.11) is uniformly convergent on $\mathbf{R}$.

Some basic statements for uniformly convergent series with respect to continuity, differentiability and integrability of the sum are as follows:

Theorem 7.2. *Let $u_n(x)$, $x \in I$ be a continuous function and $\sum_{n=1}^{\infty} u_n(x)$ be uniformly convergent on I. Then the sum $u(x) = \sum_{n=1}^{\infty} u_n(x)$ is a continuous function on I.*

Theorem 7.3. *Let $u_n(x)$, $x \in I$ be an integrable function and $\sum_{n=1}^{\infty} u_n(x)$ be uniformly convergent on I. Then*

$$\int_a^b \sum_{n=1}^{\infty} u_n(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx.$$

Theorem 7.4. *Let $u_n(x)$, $x \in I$ be a continuously differentiable function and the series $\sum_{n=1}^{\infty} u_n'(x)$ be uniformly convergent on I. If $\sum_{n=1}^{\infty} u_n(x)$ is pointwise convergent and $u(x)$ is its sum, then $u(x)$ is differentiable on I and*

$$u'(x) = \sum_{n=1}^{\infty} u_n'(x).$$

Denote $I_l := [-l, l]$ and suppose the series (7.5) is uniformly convergent on I_l. By Remark 7.1, Theorem 7.3, the orthogonality properties (7.6) and (7.7) we obtain the coefficients formulae (7.8) and (7.9), known as *Fourier coefficients* of the function $f(x)$.

Suppose $f(x) : \mathbf{R} \to \mathbf{R}$ is a *periodic function* with period $2l$, i.e.

$$f(x) = f(x + 2l), \quad \forall x \in \mathbf{R},$$

and $f(x)$ is *absolutely integrable* on I_l

$$\int_{-l}^{l} |f(x)|dx < \infty.$$

By

$$\left| \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \right| \le \int_{-l}^{l} |f(x)|dx, \tag{7.12}$$

$$\left| \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \right| \le \int_{-l}^{l} |f(x)|dx, \tag{7.13}$$

the Fourier coefficients (7.8), (7.9) are well-defined.

Let us associate to the function f its Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right).$$

The natural question is: Does $f(x)$ coincide with the sum of its Fourier series and what kind of convergence appears? There are answers to this question for some classes of functions $f(x)$.

The function $f(x)$ has a *jump discontinuity* at a point $x_0 \in I$ if the one side limits

$$f(x_0 + 0) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x), \quad f(x_0 - 0) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x),$$

exist but are not equal. The value of the jump discontinuity is the number

$$Jf(x_0) := f(x_0 + 0) - f(x_0 - 0).$$

The function $f(x)$ is said to be *piecewise continuous* on I, if there exist a finite number of points x_j, $j = 1, ..., n$, $a \leq x_1 < x_2 < ... < x_n \leq b$, such that $f(x)$ is continuous on each open interval (x_j, x_{j+1}), $j = 1, ..., n-1$ and $f(x)$ has a jump discontinuity at each point x_j.

Theorem 7.5. *Suppose $f(x)$ and $f'(x)$ are piecewise continuous functions on I_l and*

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),$$

is the Fourier series of f, where the coefficients a_n and b_n are given by (7.8) and (7.9). Then the sum of the Fourier series $s(x)$ is equal to $f(x)$ at each point $x \in (-l, l)$ where f is continuous and is equal to

$$\frac{1}{2}(f(x_0 + 0) + f(x_0 - 0)),$$

if f has a jump discontinuity at x_0. At $x = \pm l$, the series converges to

$$\frac{1}{2}(f(l - 0) + f(-l + 0)).$$

Theorem 7.6. *Let $f(x)$ be a continuous function on I_l, $f(-l) = f(l)$ and $f'(x)$ be piecewise continuous on I_l. Then the Fourier series of f converges uniformly to $f(x)$ on I_l.*

Theorem 7.6 is based on a result on the mean-square convergence of the Fourier series.

Theorem 7.7. *Let f be a $2l$-periodic piecewise continuous function on I_l. Then the Fourier series for f converges to $f(x)$ in the mean-square sense and*

$$\frac{1}{l}\int_{-l}^{l}|f(x)|^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right). \tag{7.14}$$

The equation (7.14) is known as *Parseval's*[2] *equality.*

Proof of Theorem 7.6. As $f'(x)$ is piecewise continuous on I_l, it is absolutely integrable and square integrable. Denote by a_n' and b_n' the Fourier coefficients of f'. By Theorem 7.7

$$\frac{1}{l}\int_{-l}^{l}|f'(x)|^2dx=\frac{a_0'^2}{2}+\sum_{n=1}^{\infty}\left(a_n'^2+b_n'^2\right)<\infty \tag{7.15}$$

Integrating (7.8) and (7.9) by parts and using $f(l)=f(-l)$, we obtain

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx \\ &= -\frac{1}{n\pi}\int_{-l}^{l}f'(x)\sin\frac{n\pi x}{l}dx \\ &= -\frac{l}{n\pi}b_n', \end{aligned} \tag{7.16}$$

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx \\ &= \frac{1}{n\pi}\int_{-l}^{l}f'(x)\cos\frac{n\pi x}{l}dx \\ &= \frac{l}{n\pi}a_n'. \end{aligned} \tag{7.17}$$

Using the elementary inequality

$$\frac{1}{n}(|a|+|b|)\le a^2+b^2+\frac{1}{2n^2},$$

[2] *Mark-Antoine Parseval des Chêmes, 1755–1833.*

by (7.15) we obtain that the series

$$\sum_{n=1}^{\infty}\left(\frac{|a'_n|}{n}+\frac{|b'_n|}{n}\right)$$

is convergent. Then, by (7.16) and (7.17), it follows that the series

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)$$

is convergent. By the Weierstrass criterion the Fourier series for f is uniformly convergent and the sum is a continuous function which coincides with $f(x)$ on I_l. ■

Using integration by parts we have the following estimates for Fourier coefficients.

Lemma 7.1. *Let $f(x)$ be a $2l$-periodic function, absolutely integrable on I_l, a_n and b_n be the Fourier coefficients. Then*

$$|a_n| \leq M, \quad |b_n| \leq M, \quad n \in \mathbf{N}, \tag{7.18}$$

where

$$M = \frac{1}{l}\int_{-l}^{l}|f(x)|dx.$$

Moreover, suppose that $f(x)$ is differentiable and $f'(x)$ is absolutely integrable on I_l. Then

$$|a_n| \leq \frac{M_1}{n}, \quad |b_n| \leq \frac{M_1}{n}, \quad n \in \mathbf{N}, \tag{7.19}$$

where

$$M_1 = \frac{1}{\pi}\int_{-l}^{l}|f'(x)|dx.$$

If $f'(x)$ is continuous and $f''(x)$ is absolutely integrable on I_l, then

$$|a_n| \leq \frac{M_2}{n^2}, \quad |b_n| \leq \frac{M_2}{n^2}, \quad n \in \mathbf{N}, \tag{7.20}$$

where

$$M_2 = \frac{l}{\pi^2}\int_{-l}^{l}|f''(x)|dx.$$

Proof. It is easy to see that (7.18) follows from (7.12) and (7.13). In order to obtain (7.19) we integrate by parts

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \\ &= \frac{1}{n\pi}f(x)\sin\frac{n\pi x}{l}\Big|_{-l}^{l} - \frac{1}{n\pi}\int_{-l}^{l} f'(x)\sin\frac{n\pi x}{l}dx \\ &= -\frac{1}{n\pi}\int_{-l}^{l} f'(x)\sin\frac{n\pi x}{l}dx. \end{aligned} \tag{7.21}$$

Then

$$|a_n| \leq \frac{1}{n\pi}\int_{-l}^{l}|f'(x)|dx = \frac{M_1}{n}.$$

Similarly, in view of $f(l) = f(-l)$, which follows by continuity, we have

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{n\pi}\int_{-l}^{l} f'(x)\cos\frac{n\pi x}{l}dx \tag{7.22}$$

and

$$|b_n| \leq \frac{M_1}{n}.$$

In order to obtain (7.20) we integrate by parts in (7.21) and (7.22). By $f'(l) = f'(-l)$ we have

$$\begin{aligned} a_n &= -\frac{l}{(n\pi)^2}\int_{-l}^{l} f''(x)\cos\frac{n\pi x}{l}dx, \\ b_n &= -\frac{l}{(n\pi)^2}\int_{-l}^{l} f''(x)\sin\frac{n\pi x}{l}dx. \end{aligned}$$

Then it follows

$$|a_n| \leq \frac{M_2}{n^2}, \quad |b_n| \leq \frac{M_2}{n^2}. \blacksquare$$

7.1.2 Even and odd functions. The complex form of the full Fourier series

A function f defined on $\mathbf{R}$ or on an interval I_l is said to be *even* if for every x

$$f(x) = f(-x).$$

The function f is called *odd* if for every x

$$f(-x) = -f(x).$$

The graph of an even function is symmetric with respect to axis Oy. If f is an integrable even function on I_l then

$$\int_{-l}^{l} f(x)dx = 2\int_0^l f(x)dx.$$

If f is an odd function then $f(0) = 0$. The graph of an odd function is symmetric with respect to origin O. If f is an integrable odd function on I_l

$$\int_{-l}^{l} f(x)dx = 0.$$

It is easy to see that:

(1) The sum of two even (odd) functions is an even (odd) function.

(2) The product of an even and an odd function is an odd function, while the product of two odd (even) functions is an even function.

Let $f(x)$ be a function defined on the interval $(0, l)$. It can be extended to $(-l, l)$ as an even function by

$$f_e(x) = \begin{cases} f(x) & 0 < x < l, \\ f(-x) & -l < x < 0. \end{cases}$$

The even extension is not necessarily defined at 0.

The function $f(x)$ can be extended to $(-l, l)$ as an odd function by

$$f_o(x) = \begin{cases} f(x) & 0 < x < l, \\ -f(-x) & -l < x < 0, \\ 0 & x = 0. \end{cases}$$

Let us return to Fourier series. Suppose $f(x)$ is an even function, $2l$-periodic and absolutely integrable on $(-l, l)$. Then for the Fourier coefficients we have

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}\cup\{0\}, \\ b_n &= 0, \quad n \in \mathbf{N}. \end{aligned} \tag{7.23}$$

If $f(x)$ is an odd function, $2l$-periodic and absolutely integrable on $(-l, l)$, then

$$\begin{aligned} a_n &= 0, \quad \in \mathbf{N}\cup\{0\} \\ b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}. \end{aligned} \tag{7.24}$$

Finally, given a function $f(x)$ on $(0,l)$ it can be expanded both in Fourier cosine or Fourier sine series. Namely, let us consider the $2l$-periodic extension of the even extension $f_e(x)$ and calculate the Fourier series with coefficients (7.23).

Restricting to the interval $(0,l)$ we obtain the Fourier cosine series of $f(x)$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}\cup\{0\}.$$

Similarly, taking the $2l$-periodic extension of the odd extension $f_o(x)$ we obtain a Fourier series with coefficients (7.24). Restricting to the interval $(0,l)$ we get the Fourier sine series of $f(x)$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}.$$

Let us consider some examples of Fourier series.

Example 7.1. *Expand the 2π-periodic function $f : \mathbf{R} \to \mathbf{R}$,*

$$f(x) = x, \quad x \in (-\pi, \pi)$$

in Fourier series.

Solution. As the given function is odd in $(-\pi,\pi)$, we have

$$\begin{aligned}
a_n &= 0, \quad n = 0, 1, 2, \ldots \\
b_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx dx = -\frac{2}{n\pi}\int_0^{\pi} x d\cos nx \\
&= -\frac{2}{n\pi}\left(x\cos nx|_0^{\pi} - \int_0^{\pi}\cos nx dx\right) \\
&= \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

The Fourier series is

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

By Theorem 7.5 we have

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad -\pi < x < \pi.$$

If $x = \pi$, then

$$\begin{aligned} \frac{1}{2}\left(f(\pi-0)+f(-\pi+0)\right) &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi \\ &\iff \\ \frac{1}{2}(\pi-\pi) &= 0. \end{aligned}$$

Example 7.2. *Expand the 2π-periodic function $f : \mathbf{R} \to \mathbf{R}$,*

$$f(x) = x^2, \quad x \in (-\pi, \pi)$$

in Fourier series.

Solution. As the function is even, we have

$$\begin{aligned} b_n &= 0, \quad n \in \mathbf{N}, \\ a_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx, \quad n \in \mathbf{N}, \\ a_0 &= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}, \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi n}\int_0^{\pi} x^2 d\sin nx \\ &= \frac{2}{\pi n}\left(x^2 \sin nx\Big|_0^{\pi} - 2\int_0^{\pi} x \sin nx dx\right) \\ &= \frac{4}{\pi n^2}\int_0^{\pi} x d\cos nx \\ &= \frac{4}{\pi n^2}\left(x \cos nx\Big|_0^{\pi} - \int_0^{\pi} \cos nx dx\right) \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

The Fourier series of the function is

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

As the function is continuous in $[-\pi, \pi]$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad x \in [-\pi, \pi].$$

Taking $x = \pi$ and $x = 0$ we obtain the identities

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Example 7.3. *Expand the* 2π*-periodic function* $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} \dfrac{\pi - x}{2} & 0 \le x < \pi, \\ -\dfrac{\pi + x}{2} & -\pi < x < 0. \end{cases}$$

in Fourier series.

Solution. As the function is odd in $(-\pi, \pi)\backslash\{0\}$, we have

$$\begin{aligned} a_n &= 0, \quad n \in \mathbf{N} \cup \{0\}, \\ b_n &= \frac{2}{\pi}\int_0^{\pi} \frac{\pi - x}{2} \sin nx dx \\ &= -\frac{2}{\pi n}\int_0^{\pi} \frac{\pi - x}{2} d\cos nx \\ &= -\frac{2}{\pi n}\left(\frac{\pi - x}{2} \cos nx\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos nx dx\right) \\ &= \frac{1}{n}, \quad n = 1, 2, \ldots \end{aligned}$$

The Fourier series is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

The coefficients and the partial sums can be calculated by the *Mathematica* program as follows

```
Clear[a, x, k, n, f, fs]
f[x_] := Which[x <= 0, -(x + Pi)/2, True, (Pi - x)/2]
f1[x_] := (Pi - x)/2
a[k_] := (2/Pi)Integrate[f1[x] Sin[k x], {x, 0, Pi}]
fs[x_, 12] := Sum[a[k] Sin[k x], {k, 1, 12}]
     g1 = Plot[Evaluate[f[x]], {x, -Pi, Pi}]
g11 = Plot[Evaluate[f[x - 2Pi]], {x, Pi, 3Pi}]
g2 = Plot[Evaluate[fs[x, 12]], {x, -Pi, Pi}]
g22 = Plot[Evaluate[fs[x - 2Pi, 12]], {x, Pi, 3Pi}]
Show[g1, g11, g2, g22]
```

The graphs of $f(x)$ and the partial sum $S_{12}(x)$ are plotted in Figure 7.1.

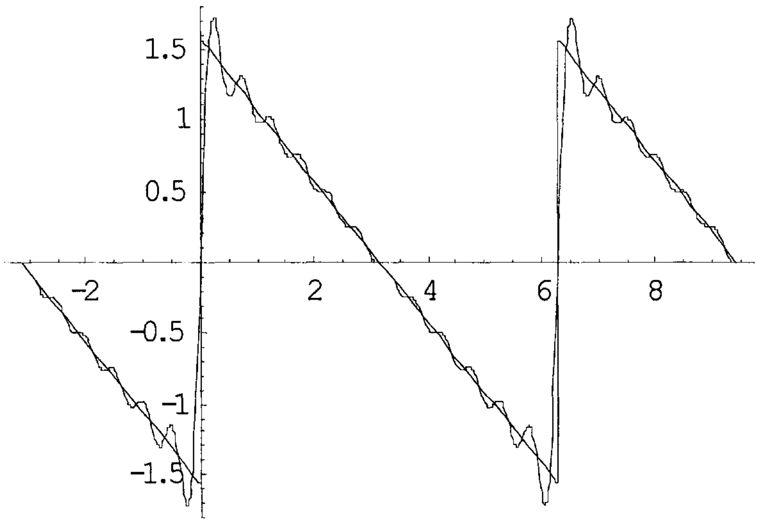

Figure 7.1. Graphs of $S_{12}(x)$ and $f(x)$ in Example 7.3.

Consider the partial sum

$$\begin{aligned} S_N(x) &= \sum_{n=1}^{N} \frac{\sin nx}{n} = \sum_{n=1}^{N} \int_0^x \cos nt dt \\ &= \int_0^x \sum_{n=1}^{N} \cos nt dt. \end{aligned}$$

We have the trigonometric identity

$$\frac{1}{2} + \sum_{n=1}^{N} \cos nt = \frac{\sin\left(N + \frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

where

$$D_N(t) := \frac{\sin\left(N + \frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

is known as the *Dirichlet kernel.*

Then

$$S_N(x) = \int_0^x \left(D_N(t) - \frac{1}{2}\right) dt = \int_0^x D_N(t)dt - \frac{x}{2}. \tag{7.25}$$

Let us consider now the so called *Gibbs phenomenon*[3] describing the difference between the partial sums of the Fourier series and the value of the function near a jump point. Gibbs showed that the limit deviation of $S_N(x)$ in a neighborhood of a jump point x_0 is greater than the jump $j(x_0)$ at x_0 by an amount of 18%.

Let us illustrate the Gibbs phenomenon by Example 7.3. We have

$$S_N(x) = \int_0^x \frac{\sin\left(N + \frac{1}{2}\right)t}{t} dt + \int_0^x \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) \sin\left(N + \frac{1}{2}\right) t dt - \frac{x}{2} \tag{7.26}$$

It is easy to see that

$$\lim_{t\to 0} \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) = 0,$$

so that the function

$$g_N(t) = \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) \sin\left(N + \frac{1}{2}\right) t$$

is bounded in a neighborhood of 0. Therefore

$$\int_0^x g_N(t)dt \to 0 \text{ as } x \to 0$$

[3]Josiah Willard Gibbs, 11.02.1839–28.04.1903.
In a letter to Nature 59 (1899) he described the Gibbs phenomenon. For this and other contributions Gibbs has been honored by a prize with his name by the American Mathematical Society.

uniformly in N. Making the change of variable $s = \left(N+\frac{1}{2}\right)t$ in the first integral of (7.26) we obtain

$$\lim_{x\to 0} S_N(x) = \lim_{x\to 0} \int_0^{\left(N+\frac{1}{2}\right)x} \frac{\sin s}{s} ds. \tag{7.27}$$

Let us consider the so called *Sine integral*

$$\mathrm{Si}(t) = \int_0^t \frac{\sin s}{s} ds.$$

It is easy to see that $\mathrm{Si}(t)$ is an odd and bounded function.
The maximum of $\mathrm{Si}(t)$ is attained at π, $\mathrm{Si}(\pi) \approx 1.8519$.
The graph of the function Si is plotted in Figure 7.2.

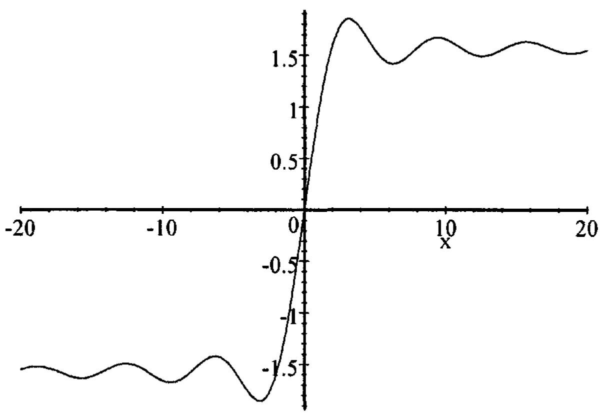

Figure 7.2. Graph of the function $\mathrm{Si}\,(x)$.

As the function $\mathrm{Si}(t)$ is monotone increasing on $[-\pi,\pi]$, it is invertible and for every $p_0 \in [-\mathrm{Si}(\pi), \mathrm{Si}(\pi)]$ there exists a unique $t_0 = \mathrm{Si}^{-1}(p_0) \in [-\pi,\pi]$ such that

$$p_0 = \int_0^{t_0} \frac{\sin s}{s} ds.$$

Taking $x_N = \dfrac{t_0}{N+1} \to 0$ as $N \to \infty$, by (7.27) we get

$$\lim_{N\to\infty} S_N(x_N)$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \int_0^{\frac{N+1/2}{N+1}t_0} \frac{\sin s}{s} ds \\
&= \int_0^{t_0} \frac{\sin s}{s} ds \\
&= p_0 \in [-\operatorname{Si}(\pi), \operatorname{Si}(\pi)].
\end{aligned}
$$

If $p_0 = \operatorname{Si}(\pi)$ then $t_0 = \pi$ and we get

$$\lim_{N\to\infty} S_N\left(\frac{\pi}{N+1}\right) = \operatorname{Si}(\pi) \approx 1.8519,$$

while

$$\lim_{N\to\infty} f\left(\frac{\pi}{N+1}\right) = f(0) = \frac{\pi}{2} \approx 1.5707.$$

Note that $Jf(0) = f(0+0) - f(0-0) = \pi$ and the Gibbs amount is

$$
\begin{aligned}
\frac{\lim_{N\to\infty} \left(S_N\left(\frac{\pi}{N+1}\right) - S_N\left(-\frac{\pi}{N+1}\right)\right) - Jf(0)}{Jf(0)} &\approx \frac{1.8519 - 1.5707}{1.5707} \\
&= 0.17902 \approx 18\%.
\end{aligned}
$$

Finally in this Section let us note that there exists a complex form of Fourier series based on Euler's formulae

$$e^{ix} = \cos x + i\sin x, \quad e^{-ix} = \cos x - i\sin x \tag{7.28}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{7.29}$$

Let us consider a Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}), \tag{7.30}$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}\cup\{0\}, \\
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}.
\end{aligned}
$$

Substituting (7.29) in (7.30), we obtain

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n)e^{i\frac{n\pi x}{l}} + \frac{1}{2}(a_n + ib_n)e^{-i\frac{n\pi x}{l}}$$

or more briefly

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{l}}, \tag{7.31}$$

where

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx,$$

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2l}\int_{-l}^{l} \left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right) f(x)dx \\ &= \frac{1}{2l}\int_{-l}^{l} e^{-i\frac{n\pi x}{l}} f(x)dx,\ n \in \mathbf{N}, \end{aligned}$$

and

$$c_{-n} = \overline{c_n} = \frac{1}{2}(a_n + ib_n),\ n \in \mathbf{N},$$

where $\overline{c_n}$ is the complex conjugate of c_n. Simply we have

$$c_n = \frac{1}{2l}\int_{-l}^{l} e^{-i\frac{n\pi x}{l}} f(x)dx,\ n \in \mathbf{Z}, \tag{7.32}$$

$$\overline{c_n} = c_{-n}. \tag{7.33}$$

The series (7.31) with coefficients (7.32) is known as the *complex form of the Fourier series.*

Let us note that the system of complex exponentials

$$\left\{1, e^{i\frac{\pi x}{l}}, e^{i\frac{2\pi x}{l}}, \dots, e^{-i\frac{\pi x}{l}}, e^{-i\frac{2\pi x}{l}}, \dots\right\}$$

has the orthogonality property, that is

$$\int_{-l}^{l} e^{i\frac{n\pi x}{l}}\overline{e^{i\frac{m\pi x}{l}}}dx = \int_{-l}^{l} e^{i\frac{(n-m)\pi x}{l}}dx$$

$$= \frac{l}{i\pi(n-m)}\left(e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right) = 0,$$

if $n \neq m$ and

$$\int_{-l}^{l} e^{i\frac{n\pi x}{l}}.\overline{e^{i\frac{n\pi x}{l}}}dx = \int_{-l}^{l} 1dx = 2l.$$

7.2 Orthonormal Systems. General Fourier Series

The system of complex-valued functions $\{\varphi_n(x)\}$ defined on the interval $I = [a, b]$ is said to be *orthonormal* iff

$$\begin{aligned} \int_a^b \varphi_n(x)\bar{\varphi}_m(x)dx &= 0, \quad m \neq n, \\ \int_a^b |\varphi_n(x)|^2 dx &= 1, \end{aligned}$$

for every $m, n \in \mathbf{N}$.

For example, the system $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}$ is orthonormal on $[-\pi, \pi]$.

Let $\{\varphi_n(x)\}$ be an orthonormal system and $f(x)$ be an absolutely integrable function on $[a, b]$. Then the numbers

$$c_n = \int_a^b f(x)\overline{\varphi_n(x)}dx, \quad n \in \mathbf{N},$$

are well-defined and are called the *Fourier coefficients of f with respect to* $\{\varphi_n\}$. We write as before

$$f(x) \sim \sum_{n=1}^{\infty} c_n\varphi_n(x),$$

where the series is called the *Fourier series of f with respect to* $\{\varphi_n\}$.

The partial sums of the Fourier series have a minimal property in L^2-sense.

Theorem 7.8. *Let $f \in L^2[a,b]$ and $\{\varphi_n\}$ be an orthonormal system in $L^2[a,b]$. Then the function*

$$\Phi(\gamma_1, ..., \gamma_n) = \int_a^b \left| f(x) - \sum_{k=1}^{n} \gamma_k \varphi_k(x) \right|^2 dx$$

attains its minimum at the point $(c_1, ..., c_n)$, where c_k is the k-th Fourier coefficient of f.

Proof. By the orthonormality of the system $\{\varphi_n\}$ we have

$$\begin{aligned}
\Phi(\gamma_1, ..., \gamma_n) &= \int_a^b |f|^2 dx - \sum_{k=1}^{n} \left(\bar{\gamma}_k \int_a^b f \bar{\varphi}_k dx + \gamma_k \int_a^b \bar{f} \varphi_k dx \right) \\
&\quad + \sum_{k=1}^{n} |\gamma_k|^2 \\
&= \int_a^b |f|^2 dx - \sum_{k=1}^{n} (\bar{\gamma}_k c_k + \gamma_k \bar{c}_k) + \sum_{k=1}^{n} |\gamma_k|^2 \\
&= \int_a^b |f|^2 dx + \sum_{k=1}^{n} |c_k - \gamma_k|^2 - \sum_{k=1}^{n} |c_k|^2 \\
&\geq \int_a^b |f|^2 dx - \sum_{k=1}^{n} |c_k|^2. \qquad (7.34)
\end{aligned}$$

By (7.34) it follows that $\Phi(\gamma_1, ..., \gamma_n)$ is minimal at $(c_1, ..., c_n)$ and

$$\min \Phi(\gamma_1, ..., \gamma_n) = \int_a^b |f|^2 dx - \sum_{k=1}^{n} |c_k|^2 \geq 0. \qquad (7.35)$$

Corollary 7.1. *Let $f \in L^2[a,b]$ and $\{\varphi_n\}$ be an orthonormal system in $L^2[a,b]$. Then*

$$\int_a^b |f(x)|^2 dx \geq \sum_{n=1}^{\infty} |c_n|^2. \qquad (7.36)$$

Proof. The result follows from (7.35). As the series $\sum_{n=1}^{\infty} |c_n|^2$ is convergent it follows that $|c_n| \to 0$ or $c_n \to 0$ as $n \to \infty$. ■

The inequality (7.36) is known as *Bessel inequality*[4].

[4]Friedrich Wilhelm Bessel, 22.07.1784–17.03.1846.

Let us consider some special examples of orthonormal systems generated by boundary value problems for linear second-order differential equations.

7.2.1 The Bessel functions

The *Bessel equation of order p* is

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \tag{7.37}$$

or

$$y'' + \frac{1}{x}y' + \left(1 - \frac{p^2}{x^2}\right)y = 0, \quad x \neq 0,$$

where p is a nonnegative constant. As the equation (7.37) is a linear second-order differential equation its general solution is of the form

$$y = c_1y_1 + c_2y_2,$$

where y_1 and y_2 are two linearly independent solutions of (7.37), and c_1 and c_2 are arbitrary constants.

The *Bessel function of the first kind of order p* is defined as

$$J_p(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(x/2)^{2n+p}}{n!\Gamma(p+n+1)}, \tag{7.38}$$

where

$$\Gamma(p) = \int_0^{\infty} t^{p-1}e^{-t}dt$$

is known as the *Gamma function.*

If p is not an integer, then $J_{-p}(x)$ is a second linearly independent solution and the general solution is

$$y = c_1J_p(x) + c_2J_{-p}(x).$$

If p is an integer, then

$$J_{-p}(x) = (-1)^pJ_p(x),$$

and so $J_{-p}(x)$ is not a second linearly independent solution. In this case the function

$$Y_p(x) = \lim_{q\to p} \frac{J_q(x)\cos q\pi - J_{-q}(x)}{\sin q\pi},$$

known, as the *Bessel function of the second kind of order p*, is a second linearly independent solution.

Let us recall some properties of the Gamma function $\Gamma(p)$.

1. $\Gamma(1) = 1$.

2. $\Gamma(p+1) = p\Gamma(p)$.

3. $\Gamma(n+1) = n!$ if n is a positive integer.

4. $|\Gamma(-n)| = \infty$, $n = 0, 1, 2, ...$

5. $\Gamma(p) = \dfrac{\Gamma(p+k)}{p\,(p+1)\,...(p+k-1)}$ if $p \neq 0, -1, ..., -k+1$.

The solution (7.38) is derived by the power series method for solving second-order differential equations. For simplicity let us consider it for the case $p = 0$.

Suppose the solution of the equation

$$xy'' + y' + xy = 0 \tag{7.39}$$

is presented in the form of power series

$$y = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ... \tag{7.40}$$

Differentiating (7.40) twice and substituting in (7.39) we obtain

$$\begin{aligned}&(2c_2 x + 3.2c_3 x^2 + 4.3c_4 x^3 + ...)\\ &+(c_1 + 2c_2 x + 3c_3 x^2 + ...) + (c_0 x + c_1 x^2 + c_2 x^3 + ...) = 0.\end{aligned} \tag{7.41}$$

Equating the coefficients of the powers x^k, $k = 0, 1, 2, ...$ in (7.41) to zero we obtain

$$c_1 = 0,$$

$$(n+2)^2 c_{n+2} + c_n = 0, \quad n \in \mathbf{N}.$$

As $c_1 = 0$ it follows

$$c_3 = c_5 = ... = c_{2k+1} = ... = 0,$$

and

$$c_{2k} = (-1)^k \frac{c_0}{2^2.4^2...(2k)^2} = (-1)^k \frac{c_0}{2^{2k}(k!)^2}.$$

Then a solution of (7.39) has a form

$$y = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}. \tag{7.42}$$

Note that, by D'Alembert criterion, (7.42) is absolutely convergent for every x . If $c_0 = 1$ we get the Bessel function of the zeroth-order

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}\,(n!)^2}.$$

It is a solution of (7.39) with initial data

$$y(0) = 1, \quad y'(0) = 0.$$

Similarly, the Bessel function of the first-order

$$J_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1} n!(n+1)!}$$

is the solution of the equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

with initial data

$$y(0) = 0, \quad y'(0) = \frac{1}{2}.$$

Let us note that the Gamma function and the Bessel functions of the first and the second kind of order p are denoted in *Mathematica* by Gamma[p], Bessel$J[p, x]$, Bessel$Y[p, x]$ respectively.

The graphs of the functions $J_0(x), J_1(x)$ and $J_2(x)$ are plotted in Figure 7.3. Note that the functions $J_k(x)$, $k = 0, 1, 2,,$ have a countable number of positive zeros. The first three of them are computed by the *Mathematica* program

```
a[1, 1] = FindRoot[BesselJ[0, x] == 0, {x, 1}]
a[1, 2] = FindRoot[BesselJ[0, x] == 0, {x, 5}]
a[1, 3] = FindRoot[BesselJ[0, x] == 0, {x, 10}]
a[2, 1] = FindRoot[BesselJ[1, x] == 0, {x, 1}]
a[2, 2] = FindRoot[BesselJ[1, x] == 0, {x, 5}]
a[2, 3] = FindRoot[BesselJ[1, x] == 0, {x, 10}]
a[3, 1] = FindRoot[BesselJ[2, x] == 0, {x, 1}]
```

```
a[3, 2] = FindRoot[BesselJ[2, x] == 0, {x, 5}]
a[3, 3] = FindRoot[BesselJ[2, x] == 0, {x, 7}]
Table[a[i, j], {i, 1, 3}, {j, 1, 3}]
Do[g[n] = Plot[BesselJ[n, x], {x, 0, 20}, PlotStyle -> {GrayLevel[0.3], {}}],
{n, 0, 2}]
Show[g[0], g[1], g[2], PlotLabel -> "Bessel's functions J[n,x], n=0,1,2"]
```

The first three zeros are as follows:

$J_0(x)$: x -> 2.40483, x -> 5.52008, x -> 8.65373,
$J_1(x)$: x -> 1.32349 10^{-20} , x -> 3.83171, x -> 10.1735,
$J_2(x)$: x -> 0.00086538, x -> 5.13562, x -> 11.6198.

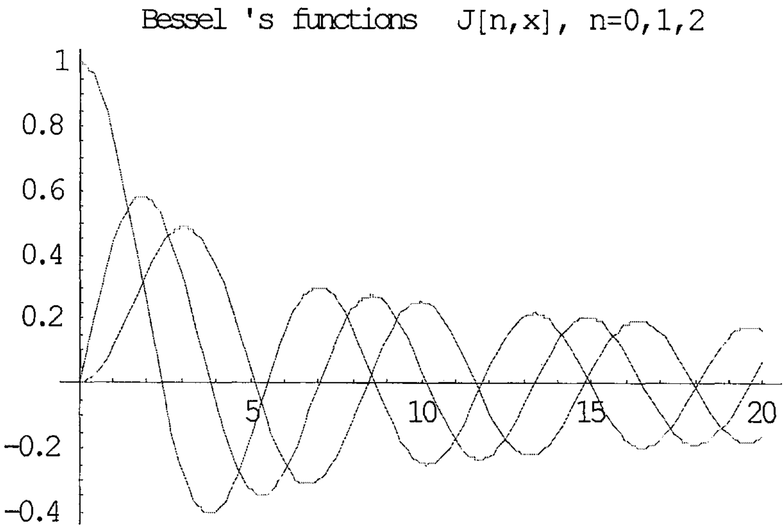

Figure 7.3. Graphs of $J_0(x), J_1(x)$ and $J_2(x)$.

Let us consider the orthogonality of the functions $\{\sqrt{x}J_0(\mu_k x)\}$, where μ_k are distinct positive zeros of $J_0(x)$. Note that the function $J_0(\lambda x)$ satisfies the differential equation

$$y'' + \frac{1}{x}y' + \lambda^2 y = 0.$$

Lemma 7.2. *The system of the functions* $\{\sqrt{x}J_0(\mu_k x)\}$ *is orthogonal on* $[0, 1]$

$$\int_0^1 xJ_0(\mu_k x)J_0(\mu_n x)dx = 0, \text{ if } n \neq k,$$

and

$$\int_0^1 xJ_0^2(\mu_k x)dx = \frac{1}{2}J_0'^2(\mu_k).$$

Proof. Let $y_1(x) = J_0(\lambda_1 x)$ and $y_2(x) = J_0(\lambda_2 x)$ for $\lambda_1 \neq \lambda_2$. Multiplying by xy_2 the equation

$$y_1'' + \frac{1}{x}y_1' + \lambda_1^2 y_1 = 0,$$

by xy_1 the equation

$$y_2'' + \frac{1}{x}y_2' + \lambda_2^2 y_2 = 0$$

and substracting we have

$$x(y_1''y_2 - y_2''y_1) + (y_1'y_2 - y_2'y_1) + (\lambda_1^2 - \lambda_2^2)xy_1y_2 = 0$$

or

$$(x(y_1'y_2 - y_2'y_1))' + (\lambda_1^2 - \lambda_2^2)xy_1y_2 = 0.$$

Then

$$\begin{aligned}(\lambda_1^2 - \lambda_2^2)\int_0^1 xy_1y_2dx &= -y_1'(1)y_2(1) + y_2'(1)y_1(1) \qquad (7.43)\\ &= -\lambda_1 J_0'(\lambda_1)J_0(\lambda_2) + \lambda_2 J_0'(\lambda_2)J_0(\lambda_1).\end{aligned}$$

If $\lambda_1 = \mu_k$ and $\lambda_2 = \mu_n$ are distinct positive zeros of $J_0(x)$ it follows

$$\int_0^1 xJ_0(\mu_k x)J_0(\mu_n x)dx = 0.$$

By (7.43)

$$\int_0^1 xJ_0(\mu_k x)J_0(\lambda x)dx = \frac{\mu_k J_0'(\mu_k)J_0(\lambda)}{\lambda^2 - \mu_k^2}.$$

Letting $\lambda \to \mu_k$, we have

$$\begin{aligned}\int_0^1 xJ_0^2(\mu_k x)dx &= \lim_{\lambda\to\mu_k}\frac{\mu_k J_0'(\mu_k)J_0(\lambda)}{\lambda^2 - \mu_k^2}\\ &= \lim_{\lambda\to\mu_k}\frac{\mu_k J_0'(\mu_k)J_0'(\lambda)}{2\lambda}\\ &= \frac{J_0'^2(\mu_k)}{2}. \blacksquare\end{aligned}$$

7.2.2 Legendre polynomials

The *Legendre differential equation*[5] is

$$((1-x^2)y'(x))' + n(n+1)y(x) = 0, \tag{7.44}$$

where $n \geq 0$ is an integer. The points $x = \pm 1$ are singular points and not every solution of (7.44) is bounded on the interval $[-1, 1]$. For $n \geq 0$ an integer, bounded solutions of (7.44) are polynomials, known as *Legendre polynomials*, defined as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}((x^2-1)^n).$$

Using *Mathematica* we find that the first six Legendre polynomials are as follows

$$P_0(x) = 1,$$

$$P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}\left(-1 + 3x^2\right),$$

$$P_3(x) = \frac{1}{2}\left(-3x + 5x^3\right),$$

$$P_4(x) = \frac{1}{8}\left(3 - 30x^2 + 35x^4\right),$$

$$P_5(x) = \frac{1}{8}\left(15x - 70x^3 + 63x^5\right).$$

They are plotted in Figure 7.4 by the program

```
Table[LegendreP[n, x], {n, 0, 5}]

Do[g[n] = Plot[LegendreP[n, x], {x, -1, 1}, PlotStyle ->
{GrayLevel[0.3], {}}], {n, 0, 5}]

Show[g[0], g[1], g[2], g[3], g[4], g[5]]
```

[5] Adrien Marie Legendre, 18.09.1752–10.01.1833.

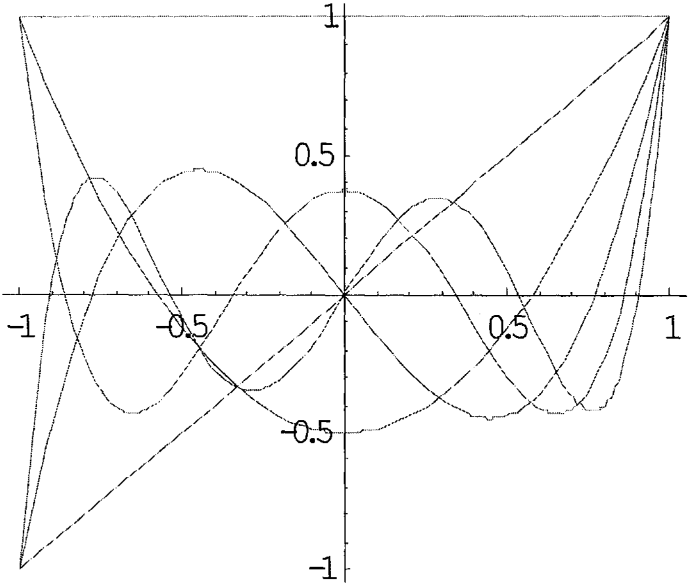

Figure 7.4. Graphs of the first six Legendre polynomials

We list some properties of $P_n(x)$ as follows:

1°. $P_n(1) = 1$, $P_n(-1) = (-1)^n$ for every $n = 0, 1, 2, \ldots$

2°. $P_n(x)$ is an even function if n is an even integer, and $P_n(x)$ is an odd function if n is an odd integer.

3°. $|P_n(x)| \leq 1$ if $|x| \leq 1$.

4°. $P_n(x)$ has n real simple zeros on the interval $(-1, 1)$ for $n \geq 1$.

5°. $P_n(x)$ are orthogonal on $(-1, 1)$.

6°. $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$.

7°. $\int_{-1}^{1} P_n^2(x)dx = \dfrac{2}{2n+1}$.

Let us prove 5° using the procedure considered in Lemma 7.2. Multiplying by $P_m(x)$ the equation

$$((1-x^2)P_n')' + n(n+1)P_n = 0,$$

by $P_n(x)$ the equation

$$((1-x^2)P_m')' + m(m+1)P_m = 0$$

and subtracting we have

$$P_m((1-x^2)P_n')' - P_n((1-x^2)P_m')' = (m(m+1) - n(n+1))P_nP_m$$

or

$$((1-x^2)(P_mP_n' - P_nP_m'))' = (m(m+1) - n(n+1))P_nP_m.$$

Integrating over $[-1,1]$ we obtain

$$(m(m+1) - n(n+1))\int_{-1}^{1} P_n(x)P_m(x)dx = 0.$$

Therefore for $m \neq n$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0.$$

Let us prove 7° using 6°. Multiplying the identity 6° by $P_{n-1}(x)$ and integrating in $[-1,1]$ by 5° we have

$$(2n+1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = n\int_{-1}^{1} P_{n-1}^2(x)dx. \tag{7.45}$$

Replacing n by $n-1$ in 6°, multiplying by $P_n(x)$ and integrating over $[-1,1]$, we get

$$(2n-1)\int_{-1}^{1} xP_{n-1}(x)P_n(x)dx = n\int_{-1}^{1} P_n^2(x)dx.$$

Therefore by (7.45) we obtain the identity

$$(2n+1)\int_{-1}^{1} P_n^2(x)dx = (2n-1)\int_{-1}^{1} P_{n-1}^2(x)dx,$$

valid for $n = 1, 2, \dots$. Hence

$$\begin{aligned}
\int_{-1}^{1} P_n^2(x)dx &= \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)dx \\
&= \frac{2n-1}{2n+1}\cdot\frac{2n-3}{2n-1}\int_{-1}^{1} P_{n-2}^2(x)dx \\
&= \dots \\
&= \frac{2n-1}{2n+1}\cdot\frac{2n-3}{2n-1}\cdot\ \dots\ \cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)dx \\
&= \frac{2}{2n+1}. \quad \blacksquare
\end{aligned}$$

Exersices

1. Expand the following functions in Fourier series:

(a) $e^{ax}, \quad -\pi < x < \pi$, where $a = const \neq 0$.

(b) $\sin^3 x, \quad -\pi < x < \pi$.

(c) $f(x) = \begin{cases} 0 & -\pi < x < 0, \\ e^x & 0 \le x \le \pi. \end{cases}$

(d) $g(x) = \begin{cases} \sin x & 0 \le x < \pi/2, \\ 0 & \pi/2 < x \le \pi. \end{cases}$

2. (a) Expand in Fourier series the function

$$y = \operatorname{Si}(x), \quad -\pi < x < \pi.$$

(b) Show that:

$$\pi \operatorname{Si}\left(\frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(\operatorname{Si} 2(k+1)\pi - \operatorname{Si} 2k\pi + 2 \operatorname{Si} \pi\right).$$

3. (a) Using the Euler formula and the expansion

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots, \quad |z| < 1,$$

prove that if $-\pi < x < \pi$, then

$$\begin{aligned} \ln\left(2\cos\frac{x}{2}\right) &= \cos x - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \ldots, \\ \frac{x}{2} &= \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots . \end{aligned}$$

(b) Find the sums of the series

$$\begin{aligned} &\frac{\cos 2x}{3} - \frac{\cos 3x}{8} + \ldots + (-1)^n \frac{\cos nx}{n^2-1} + \ldots, \\ &\frac{\sin 2x}{3} - \frac{\sin 3x}{8} + \ldots + (-1)^n \frac{\sin nx}{n^2-1} + \ldots, \end{aligned}$$

if $-\pi < x < \pi$.

4. (a) The *Bernoulli numbers* B_n[6] are the coefficients of the Taylor series of the function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n x^n.$$

Prove that:

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \dots + \binom{n}{n-1} B_{n-1} = 0,$$

$$B_{2n-1} = 0 \quad \text{if } n \geq 2.$$

Compute the first seven Bernoulli numbers.

(b) The *Bernoulli polynomials* $B_n(t)$ are defined as the coefficients of x^n in the expansion

$$\frac{xe^{xt}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(t)\, x^n.$$

Prove that:

$$B_n(t) = \binom{n}{0} B_0 t^n + \binom{n}{1} B_1 t^{n-1} + \dots + \binom{n}{n-1} B_{n-1} t + B_n,$$

$$B_n(t+1) - B_n(t) = nt^{n-1},$$

$$B_1(t) = t - \frac{1}{2} = -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin 2n\pi t}{n}.$$

Compute first seven Bernoulli polynomials and plot their graphs with *Mathematica.*

(c) Prove the identity

$$\sum_{k=0}^{n} \binom{n}{k} B_k(s)\, B_{n-k}(t) = -(n-1)\, B_n(s+t) + n(s+t-1)\, B_{n-1}(s+t).$$

5. Find the general solution of the differential equation

$$y'' + \frac{5}{x} y' + y = 0,$$

using the change of variables $u = x^2 y$.

[6] Daniel I Bernoulli, 29.01.1700–17.03.1782.

6. Prove that

$$\int_0^1 xJ_p^2(\lambda x)\,dx = \frac{1}{2}J_{p+1}^2(\lambda) \quad \text{if} \quad J_p(\lambda)=0, \quad p>-\frac{1}{2},$$

$$\int_0^1 xJ_p^2(\lambda x)\,dx = \frac{1}{2}\left(1-\frac{p^2}{\lambda^2}\right)J_p^2(\lambda) \quad \text{if} \quad J_p'(\lambda)=0, \quad p\geq 0.$$

7.3 Fourier Method for the Diffusion Equation

7.3.1 Homogeneous equation and boundary conditions

Consider the boundary-value problem

$$(MDH):\begin{cases} u_t-\alpha^2u_{xx}=0 & 0<x<l,\ t>0,\\ u(x,0)=\varphi(x) & 0\leq x\leq l,\\ u(0,t)=u(l,t)=0 & t\geq 0.\end{cases}$$

Our goal is to find the solution of (MDH) using the method of *separation of variables* or Fourier method. A separable solution is a solution of the form

$$u(x,t)=X(x)T(t)$$

to the problem

$$(SD):\begin{cases} u_t-\alpha^2u_{xx}=0 & 0<x<l,\ t>0,\\ u(0,t)=u(l,t)=0 & t\geq 0.\end{cases}$$

Plugging this into the diffusion equation, we get

$$X(x)T'(t)-\alpha^2X''(x)T(t)=0$$

or

$$\frac{T'(t)}{\alpha^2T(t)}=\frac{X''(x)}{X(x)}.$$

In order for the last relation to be an equality each side must be identically equal to a constant:

$$\frac{T'(t)}{\alpha^2T(t)}=\frac{X''(x)}{X(x)}=\lambda.$$

By the boundary conditions

$$u(0,t) = X(0)T(t) = 0, \; u(l,t) = X(l)T(t) = 0$$

it follows

$$X(0) = X(l) = 0$$

so that $X(x)$ satisfies the following *eigenvalue (Sturm–Liouville) problem*

$$\begin{cases} X''(x) = \lambda X(x) & 0 < x < l, \\ X(0) = X(l) = 0, \end{cases} \tag{7.46}$$

while $T(t)$ satisfies the equation

$$T'(t) - \lambda \alpha^2 T(t) = 0. \tag{7.47}$$

We are looking for the values of λ which lead to nontrivial solutions. Consider the following three cases:

(i) Let $\lambda = \beta^2 > 0, \; \beta > 0$. Then the equation (7.46) has the general solution

$$X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x}.$$

By the boundary conditions it follows

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\beta l} + c_2 e^{-\beta l} = 0 \end{cases}$$

and $c_1 = c_2 = 0$ because

$$\Delta = \begin{vmatrix} 1 & 1 \\ e^{\beta l} & e^{-\beta l} \end{vmatrix} = e^{-\beta l} - e^{\beta l} \neq 0.$$

(ii) If $\lambda = 0$, $X(x)$ has the form

$$X(x) = c_1 x + c_2.$$

It follows again $c_1 = c_2 = 0$.

So in the first two cases the problem (7.46) admits the trivial solution only.

(iii) If $\lambda = -\beta^2 < 0, \; \beta > 0$, then the equation (7.46) has the general solution

$$X(x) = c_1 \cos \beta x + c_2 \sin \beta x.$$

By the boundary conditions it follows that for a nontivial solution

$$c_1 = 0 \text{ and } \sin\beta l = 0.$$

Then

$$\beta l = n\pi, \; n \in \mathbf{Z}.$$

So the only nontrivial solution of (7.46) appears when

$$\lambda = \lambda_n = -\left(\frac{n\pi}{l}\right)^2, \; n \in \mathbf{N}$$

and has the form

$$X_n(x) = a_n \sin\frac{n\pi x}{l}, \; n \in \mathbf{N}.$$

The above values λ_n are called *eigenvalues* and the functions $X_n(x)$ *eigenfunctions.*

Solving (7.47) with $\lambda = \lambda_n$, we obtain

$$T_n(t) = b_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}.$$

Therefore functions of the form

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right), \quad n \in \mathbf{N}, \tag{7.48}$$

are solutions of the problem (SD).

In order to find a solution of (MDH) we take a *superposition* of $u_n(x,t)$. Namely, the function

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\frac{n\pi x}{l} \tag{7.49}$$

is the solution of (MDH) provided that

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}, \quad 0 \le x \le l.$$

The last identity means that A_n are the Fourier sine coefficients of $\varphi(x)$, i.e.

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx, \quad n = 1, 2, \dots . \tag{7.50}$$

So the formal solution of (MDH) is the function (7.49) with coefficients A_n determined by (7.50).

In order to justify the method we prove

Theorem 7.9. *Let $\varphi \in C[0,l]$, $\varphi'(x)$ be a piecewise continuous function and $\varphi(0) = \varphi(l) = 0$. Then the problem (MDH) has a unique solution given by* (7.49) *and* (7.50) .

Proof. Let us note that the function $\varphi(x)$ satisfies the assumptions of Theorem 7.6. Taking the Fourier expansions of the odd extension of $\varphi(x)$ on $[-l,l]$, we have

$$\varphi(x) \sim \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}, \quad 0 \le x \le l,$$

and

$$\varphi'(x) \sim \sum_{n=1}^{\infty} A_n' \cos\frac{n\pi x}{l}, \quad 0 \le x \le l.$$

As in Theorem 7.6, we have

$$A_n = \frac{l}{n\pi} A_n'$$

and the series $\sum_{n=1}^{\infty} |A_n|$ is convergent because

$$|A_n| \le \frac{1}{2}\left(\left(\frac{l}{n\pi}\right)^2 + A_n'^2\right),$$

and by Bessel inequality

$$\sum_{n=1}^{\infty} A_n'^2 \le \frac{2}{l}\int_0^l \varphi'^2(x)dx.$$

Since

$$\left|A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\frac{n\pi x}{l}\right| \le |A_n|$$

by the Weierstrass criterion the series (7.49) is uniformly convergent and $u(x,t)$ is a continuous function.

By uniform convergence it follows

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \varphi(x), \quad 0 \le x \le l.$$

Since the boundary conditions are satisfied let us prove that $u(x,t)$ satisfies the heat equation.

Let us fix $0 < \delta < T$. Suppose $\delta \le t \le T$ and $0 < x < l$. Formal differentiation yields

$$u_t(x,t) = -\sum_{n=1}^{\infty} A_n \left(\frac{n\pi\alpha}{l}\right)^2 e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\frac{n\pi x}{l}, \tag{7.51}$$

$$u_x(x,t) = \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l}\right) e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \cos\frac{n\pi x}{l}, \tag{7.52}$$

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l}\right)^2 e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\frac{n\pi x}{l}. \tag{7.53}$$

In order to prove convergence of (7.51) observe that there exists n_0 such that for $n \ge n_0$

$$\left(\frac{n\pi\alpha}{l}\right)^2 e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \le \left(\frac{n\pi\alpha}{l}\right)^2 e^{-\left(\frac{n\pi\alpha}{l}\right)^2 \delta} \le 1,$$

because

$$\lim_{x\to\infty} xe^{-\delta x} = 0.$$

Then

$$\left| A_n \left(\frac{n\pi\alpha}{l}\right)^2 e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\frac{n\pi x}{l} \right| \le |A_n|$$

for $n \ge n_0$.

As the series $\sum_{n=1}^{\infty} |A_n|$ is convergent, by the Weierstrass criterion and Theorem 7.4, we obtain (7.51). Similarly, we obtain (7.52) and (7.53). Combining (7.51) and (7.53) we get $u_t = \alpha^2 u_{xx}$ in $(0,l) \times [\delta, T]$. As δ and T are arbitrary we have $u_t = \alpha^2 u_{xx}$ in $(0,l) \times (0,\infty)$.

By the maximum-minimum principle for the diffusion equation it follows that the solution obtained is unique. ■

7.3.2 Inhomogeneous equation and boundary conditions

Let us consider now the boundary-value problem for the inhomogeneous diffusion equation

$$(MDI) \quad \begin{cases} u_t - \alpha^2 u_{xx} = f(x,t) & 0 < x < l,\ t > 0, \\ u(x,0) = \varphi(x) & 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & t > 0. \end{cases}$$

To solve (MDI) we apply the method of *variation of constants (parameters)* looking for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\frac{n\pi x}{l}.$$

Substituting formally $u(x,t)$ into the equation

$$u_t - \alpha^2 u_{xx} = f(x,t),$$

we obtain

$$\sum_{n=1}^{\infty} A_n'(t) e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi x}{l}, \tag{7.54}$$

where

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi x}{l},$$

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}dx.$$

Then equating coefficients in (7.54), we obtain

$$A_n'(t) = e^{\left(\frac{n\pi\alpha}{l}\right)^2 t} f_n(t)$$

or

$$A_n(t) = A_n(0) + \int_0^t e^{\left(\frac{n\pi\alpha}{l}\right)^2 s} f_n(s)ds.$$

In order to calculate $A_n(0)$ observe that

$$u(x,0) = \sum_{n=1}^{\infty} A_n(0)\sin\frac{n\pi x}{l} = \varphi(x)$$

and therefore

$$A_n(0) = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx.$$

Then the solution of (MDI) is

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} + \int_0^t e^{\left(\frac{n\pi\alpha}{l}\right)^2 (s-t)} f_n(s)ds\right)\sin\frac{n\pi x}{l},$$

where

$$f_n(s) = \frac{2}{l}\int_0^l f(x,s)\sin\frac{n\pi x}{l}dx,$$

$$a_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx.$$

In order to justify the solution we consider the case when $f(x,t) = f(x)$.

Theorem 7.10. *Suppose $\varphi(x) \in C[0,l]$ and $f(x) \in C[0,l]$, $\varphi'(x)$ and $f'(x)$ are piecewise continuous functions and*

$$\varphi(0) = \varphi(l) = f(0) = f(l) = 0.$$

Then the problem (MDI) has the unique solution

$$u(x,t) = \sum_{n=1}^{\infty}\left(\left(a_n - \left(\frac{l}{n\pi a}\right)^2 f_n\right)e^{-\left(\frac{n\pi a}{l}\right)^2 t} + \left(\frac{l}{n\pi a}\right)^2 f_n\right)\sin\frac{n\pi x}{l},$$

where

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx, \\ f_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}. \end{aligned}$$

Since the proof is similar to the proof of Theorem 7.9 it is left to the reader as an exercise.

Finally in this section we consider the case of inhomogeneous boundary conditions and the *method of shifting the data.* Consider the diffusion equation with sources at both endpoints

$$\begin{cases} u_t - a^2 u_{xx} = 0 & 0 < x < l,\ t > 0, \\ u(x,0) = 0 & 0 \le x \le l, \\ u(0,t) = p(t), \quad u(l,t) = q(t) & t \ge 0. \end{cases}$$

The last problem can be reduced to a problem (MDI) subtracting from u any known function satisfying the boundary conditions (BCs)

$$u(0,t) = p(t),\ u(l,t) = q(t).$$

The linear combination

$$s(x,t) = \left(1 - \frac{x}{l}\right) p(t) + \frac{x}{l} q(t), \quad 0 \leq x \leq l$$

satisfies the BCs. Consider now

$$v(x,t) = u(x,t) - s(x,t).$$

Then, as

$$\begin{aligned} s_t &= \left(1 - \frac{x}{l}\right) p'(t) + \frac{x}{l} q'(t), \\ s_{xx} &= 0, \end{aligned}$$

the function $v(x,t)$ satisfies the problem (MDI)

$$\begin{cases} v_t - \alpha^2 v_{xx} = -\left(1 - \dfrac{x}{l}\right) p'(t) - \dfrac{x}{l} q'(t) & 0 < x < l,\ t > 0, \\ v(x,0) = -\left(1 - \dfrac{x}{l}\right) p(0) - \dfrac{x}{l} q(0) & 0 \leq x \leq l, \\ v(0,t) = 0, \quad v(l,t) = 0 & t \geq 0. \end{cases}$$

which has been considered before.

If we have an inhomogeneous equation and inhomogeneous BCs

$$(P) \begin{cases} u_t - \alpha^2 u_{xx} = f(x,t) & 0 < x < l,\ t > 0, \\ u(x,0) = \varphi(x) & 0 \leq x \leq l, \\ u(0,t) = p(t), \quad u(l,t) = q(t) & t \geq 0, \end{cases}$$

we can split it into two problems

$$(P_1) \begin{cases} v_t - \alpha^2 v_{xx} = f(x,t) & 0 < x < l,\ t > 0, \\ v(x,0) = \varphi(x) & 0 \leq x \leq l, \\ v(0,t) = v(l,t) = 0 & t \geq 0, \end{cases}$$

and

$$(P_2) \begin{cases} w_t - \alpha^2 w_{xx} = 0 & 0 < x < l,\ t > 0, \\ w(x,0) = 0 & 0 \leq x \leq l, \\ w(0,t) = p(t), \quad w(l,t) = q(t) & t \geq 0. \end{cases}$$

Solving (P_1) and (P_2) by previous procedures we obtain that $u(x,t) = v(x,t) + w(x,t)$ is a solution of (P).

Example 7.4. *Solve the heat conduction problem for the copper rod of length one*

$$\begin{cases} u_t - 1.14u_{xx} = 0 & 0 < x < 1,\ t > 0, \\ u(x,0) = \sin 2\pi x + 4x & 0 \le x \le 1, \\ u(0,t) = 2, \quad u(1,t) = 6 & t \ge 0. \end{cases}$$

Solution.

The function $s(x) = 2(1-x) + 6x = 2 + 4x$, known as a *steady state solution* satisfies

$$\begin{cases} s_t - 1.14s_{xx} = 0, \\ s(0) = 2, \quad s(1) = 6. \end{cases}$$

Then for $v(x,t) = u(x,t) - s(x)$ we have

$$\begin{cases} v_t - 1.14v_{xx} = 0 & \text{if } 0 < x < 1,\ t > 0, \\ v(x,0) = \sin 2\pi x - 2 & 0 \le x \le 1, \\ v(0,t) = v(1,t) = 0 & t \ge 0. \end{cases}$$

The last problem has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin n\pi x\, e^{-1.14n^2\pi^2 t},$$

where

$$\begin{aligned} c_n &= 2\int_0^1 (\sin 2\pi x - 2) \sin n\pi x\, dx \\ &= \begin{cases} 1 & n = 2, \\ \frac{4}{n\pi}\left((-1)^n - 1\right) & n \ne 2. \end{cases} \end{aligned}$$

The solution of the original problem is

$$\begin{aligned} u(x,t) &= 2 + 4x + \sin 2\pi x\, e^{-4.56\pi^2 t} \\ &\quad + \frac{4}{\pi} \sum_{n \ge 3} \frac{(-1)^n - 1}{n} \sin n\pi x\, e^{-1.14n^2\pi^2 t}. \end{aligned}$$

7.4 Fourier Method for the Wave Equation

7.4.1 Homogeneous equation and boundary conditions

Let us consider the Dirichlet boundary value problem (BVP) for the homogeneous wave equation

$$(MWH): \begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l,\ t > 0, \\ u(x,0) = \varphi(x) & 0 \le x \le l, \\ u_t(x,0) = \psi(x) & 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & t > 0, \end{cases}$$

which describes the motion of the vibrating string.

Our goal is to find the solution of (MWH) using the Fourier method.

A separable solution is a solution of the form

$$u(x,t) = X(x)T(t)$$

to the problem

$$(SW): \begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) = 0 & t \ge 0. \end{cases}$$

Plugging the form into the wave equation, we get

$$X(x)T''(t) - c^2 X''(x)T(t) = 0$$

or

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{7.55}$$

By the boundary conditions

$$u(0,t) = X(0)T(t) = 0,\ u(l,t) = X(l)T(t) = 0$$

it follows

$$X(0) = X(l) = 0. \tag{7.56}$$

So $X(x)$ satisfies the problem

$$(P): \begin{cases} X''(x) - \lambda X(x) = 0, & 0 < x < l, \\ X(0) = X(l) = 0. \end{cases}$$

As before the problem (P) has nontrivial solutions

$$X_n(x) = a_n \sin\frac{n\pi x}{l}, \quad n = 1, 2, ...$$

corresponding to

$$\lambda = \lambda_n = -\left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \dots \tag{7.57}$$

Plugging (7.57) into (7.55), we obtain the ODE

$$T''(t) + \left(\frac{n\pi c}{l}\right)^2 T(t) = 0$$

with general solution

$$T_n(t) = b_n \cos\frac{n\pi c}{l}t + c_n \sin\frac{n\pi c}{l}t, \quad n \in \mathbf{N}.$$

Therefore functions of the form

$$u_n(x,t) = \left(A_n \cos\frac{n\pi c}{l}t + B_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}, \quad n \in \mathbf{N},$$

known as *normal modes* of vibration, are solutions of the problem (SW). In order to find a solution of (MWH) we take a superposition of $u_n(x,t)$. Namely, we are looking for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi c}{l}t + B_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}. \tag{7.58}$$

Formally the last function satisfies the initial conditions if

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l},$$

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}.$$

Using the Fourier-sine series for $\varphi(x)$ and $\psi(x)$ we obtain that

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}, \tag{7.59}$$

$$B_n = \frac{2}{n\pi c}\int_0^l \psi(x)\sin\frac{n\pi x}{l}dx, \quad n \in \mathbf{N}. \tag{7.60}$$

In order to justify the formal solution we prove

Theorem 7.11. *Suppose* $\varphi \in C^2[0,l]$, $\varphi'''(x)$ *is piecewise continuous* $\psi \in C^1[0,l], \psi''(x)$ *is piecewise continuous, and*

$$\begin{aligned}\varphi(0) &= \varphi''(0) = \varphi(l) = \varphi''(l) = 0, \\ \psi(0) &= \psi(l) = 0.\end{aligned} \tag{7.61}$$

Then the function (7.58), where the coefficients A_n and B_n are determined by (7.59) and (7.60), is the unique solution of the problem (MWH).

Proof. As in Theorem 7.9, the main tool to justify differentiation of the series (7.58) is Theorem 7.6. We show that (7.58) and the series

$$\begin{aligned}u_{xx}(x,t) &= -\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^2\left(A_n\cos\frac{n\pi c}{l}t + B_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}, \\ u_{tt}(x,t) &= -\sum_{n=1}^{\infty}\left(\frac{n\pi c}{l}\right)^2\left(A_n\cos\frac{n\pi c}{l}t + B_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}\end{aligned} \tag{7.62}$$

are uniformly convergent in $(0,l)\times(0,\infty)$.

Since

$$\left|\left(\frac{n\pi}{l}\right)^2\left(A_n\cos\frac{n\pi c}{l}t + B_n\sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}\right| \le \left(\frac{n\pi}{l}\right)^2(|A_n|+|B_n|)$$

it suffices to show that the series

$$\sum_{n=1}^{\infty} n^2(|A_n|+|B_n|) \tag{7.63}$$

is convergent.

Let us take the Fourier sine series of $\varphi(x)$ and the Fourier cosine series of $\varphi'''(x)$ for $x \in [0,l]$

$$\begin{aligned}\varphi(x) &\sim \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{l}, \\ \varphi'''(x) &\sim \sum_{n=1}^{\infty} A_n'''\cos\frac{n\pi x}{l}.\end{aligned} \tag{7.64}$$

Similarly for the Fourier sine series of $\psi(x)$ and $\psi''(x)$

$$\begin{aligned}\psi(x) &\sim \sum_{n=1}^{\infty} B_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l}, \\ \psi''(x) &\sim \sum_{n=1}^{\infty} B_n''\sin\frac{n\pi x}{l}.\end{aligned} \tag{7.65}$$

Integrating by parts (7.59) and (7.60) and using conditions (7.61), we have

$$A_n = -\left(\frac{l}{n\pi}\right)^3 A_n''' \text{ and } B_n = \frac{1}{c}\left(\frac{l}{n\pi}\right)^3 B_n''. \tag{7.66}$$

By the Bessel inequality, we obtain

$$\sum_{n=1}^{\infty} A_n'''^2 \leq \frac{2}{l}\int_0^l \varphi'''^2(x)dx$$

and

$$\sum_{n=1}^{\infty} B_n''^2 \leq \frac{2}{l}\int_0^l \psi''^2(x)dx.$$

Then the series (7.63) is convergent, because by (7.66)

$$\begin{aligned} n^2\left(|A_n|+|B_n|\right) &= \frac{l^3}{\pi^3}\cdot\frac{1}{n}\left(|A_n'''|+\frac{1}{c}|B_n''|\right) \\ &\leq \frac{l^3}{2\pi^3}\left(\frac{1}{n^2}+\left(|A_n'''|+\frac{1}{c}|B_n''|\right)^2\right) \\ &\leq \frac{l^3}{2\pi^3}\left(\frac{1}{n^2}+2\left(A_n'''^2+\frac{1}{c^2}B_n''^2\right)\right). \end{aligned}$$

By the convergence of the series (7.63) it follows also that the series (7.64) and (7.65) are uniformly convergent in $[0, l]$ and the initial conditions

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}, \quad x \in [0, l],$$

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}, \quad x \in [0, l].$$

are satisfied. The uniqueness of the solution follows by the energy method - Section 3.2. ■

7.4.2 Inhomogeneous equation and boundary conditions

Consider now the mixed BVP for the inhomogeneous wave equation

$$(MWI): \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < l,\ t > 0, \\ u(x,0) = \varphi(x) & 0 \le x \le l, \\ u_t(x,0) = \psi(x) & 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & t > 0. \end{cases}$$

The solution of (MWI) can be constructed by superposing the unique solution of (MWH) with the unique solution of the problem

$$(WI): \begin{cases} v_{tt} - c^2 v_{xx} = f(x,t) & 0 < x < l,\ t > 0, \\ v(x,0) = 0 & 0 \le x \le l \\ v_t(x,0) = 0 & 0 \le x \le l, \\ v(0,t) = v(l,t) = 0 & t > 0. \end{cases}$$

The last problem can be solved by reducing it to the Cauchy problem for the inhomogeneous wave equation, by odd reflection of $f(x,t)$ with respect to $x = 0$ and $x = l$.

Another approach is to expand $f(x,t)$ in the Fourier sine series

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}, \quad 0 < x < l,$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi x}{l} dx, \quad n \in \mathbf{N}. \tag{7.67}$$

Let us try to find a solution $v(x,t)$ of the form

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi x}{l}, \tag{7.68}$$

where

$$v_n(0) = v_n'(0) = 0. \tag{7.69}$$

Formally, substituting (7.68) into the wave equation, we get

$$v_n''(t) + \left(\frac{n\pi c}{l}\right)^2 v_n(t) = f_n(t).$$

The last linear second-order ODE with initial conditions (7.69) has the unique solution

$$v_n(t) = \frac{l}{n\pi c} \int_0^t f_n(\tau) \sin\left(\frac{n\pi c}{l}(t-\tau)\right) d\tau. \tag{7.70}$$

The solution of (WI) is (7.68), where $f_n(t)$ and $v_n(t)$ are determined by (7.67) and (7.70).

Let us consider finally the case of inhomogeneous boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < l,\ t > 0, \\ u(x,0) = \varphi(x) & 0 \le x \le l, \\ u_t(x,0) = \psi(x) & 0 \le x \le l, \\ u(0,t) = g(t),\ u(l,t) = h(t) & t > 0. \end{cases}$$

The solution of the last problem can be found by superposing the solution of the problem (MWI) with the solution W of the problem with zero initial data and source

$$(WB) : \begin{cases} w_{tt} - c^2 w_{xx} = 0 & 0 < x < l,\ t > 0, \\ w(x,0) = 0 & 0 \le x \le l, \\ w_t(x,0) = 0 & 0 \le x \le l, \\ w(0,t) = g(t),\ w(l,t) = h(t) & t > 0. \end{cases}$$

In order to solve the last problem, we use the method of shifting the data. Namely, considering

$$W(x,t) = lw(x,t) - ((l-x)g(t) + xh(t))$$

we reduce (WB) again to a problem of the type (MWI) :

$$\begin{cases} W_{tt} - c^2 W_{xx} = -((l-x)g''(t) + xh''(t)) & 0 < x < l,\ t > 0, \\ W(x,0) = -((l-x)g(0) + xh(0)) & 0 \le x \le l, \\ W_t(x,0) = -((l-x)g'(0) + xh'(0)) & 0 \le x \le l, \\ W(0,t) = W(l,t) = 0 & t > 0. \end{cases}$$

7.5 Fourier Method for the Laplace Equation

7.5.1 BVPs for the Laplace equation in a rectangle

We consider now the Laplace equation

$$u_{xx} + u_{yy} = 0 \text{ in } D, \tag{7.71}$$

where $D = \{(x,y) : 0 < x < a, 0 < y < b\}$ is a rectangle in a plane. On each side of D we assume that either Dirichlet or Neumann boundary conditions are prescribed. These problems can be solved by the method of separation of variables.

Example 7.5. *Solve (7.71) with the boundary conditions*

$$\begin{aligned} u(x,0) &= 0, \quad u(x,b) = 0, \quad 0 \le x \le a, \\ u(0,y) &= g(y), \quad u_x(a,y) = h(y), \quad 0 \le y \le b. \end{aligned}$$

Solution. The solution of the problem has a form $u = u_1 + u_2$, where u_1 and u_2 satisfy (7.71) respectively with the boundary conditions

$$(BC_1) : \begin{cases} u_1(x,0) = u_1(x,b) = 0 & 0 \le x \le a, \\ u_1(0,y) = g(y), \quad u_{1x}(a,y) = 0 & 0 \le y \le b, \end{cases}$$

and

$$(BC_2) : \begin{cases} u_2(x,0) = u_2(x,b) = 0 & 0 \le x \le a, \\ u_2(0,y) = 0, \quad u_{2x}(a,y) = h(y) & 0 \le y \le b. \end{cases}$$

We find each one of u_1 and u_2 by the Fourier method. Separating variables for $u_1(x,y) = X(x)Y(y)$ we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \text{ in } D.$$

This implies that

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < a, \tag{7.72}$$

$$Y''(y) - \lambda Y(y) = 0, \quad 0 < y < b, \tag{7.73}$$

for a constant λ. Since the function u_1 satisfies (BC_1) we should have

$$Y(0) = Y(b) = 0 \tag{7.74}$$

$$X'(a) = 0. \tag{7.75}$$

Nontrivial solutions of (7.73), (7.74) are

$$Y_n(y) = \sin\frac{n\pi y}{b}$$

corresponding to

$$\lambda = \lambda_n = -\left(\frac{n\pi}{b}\right)^2, \quad n \in \mathbf{N}.$$

The differential equation for $X(x)$

$$X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0$$

implies that

$$X(x) = C_1 \cosh\frac{n\pi x}{b} + C_2 \sinh\frac{n\pi x}{b}.$$

The condition (7.75) is satisfied if

$$\frac{C_2}{C_1} = -\tanh\frac{n\pi a}{b}.$$

Then $X(x)$ has the form

$$X_n(x) = a_n\left(\cosh\frac{n\pi x}{b} - \tanh\frac{n\pi a}{b}\sinh\frac{n\pi x}{b}\right).$$

We are looking for a solution u_1 of the form

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n\left(\cosh\frac{n\pi x}{b} - \tanh\frac{n\pi a}{b}\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}. \tag{7.76}$$

It satisfies the boundary condition

$$u_1(0,y) = g(y), \quad 0 \le y \le b,$$

when

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b} = g(y), \quad 0 \le y \le b,$$

which implies that

$$a_n = \frac{2}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy. \tag{7.77}$$

Suppose now $u_2(x,y) = X(x)Y(y)$ satisfies (7.71) and (BC_2). As before, we have the equations (7.72) and (7.73) for $X(x)$ and $Y(y)$ with the boundary conditions

$$Y(0) = Y(b) = 0$$

and

$$X(0) = 0.$$

Then

$$Y_n(y) = \sin \frac{n\pi y}{b}$$

corresponding to

$$\lambda = \lambda_n = -\left(\frac{n\pi}{b}\right)^2, \quad n \in \mathbf{N}.$$

For $X(x)$

$$\begin{aligned} X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) &= 0, \\ X(0) &= 0, \end{aligned}$$

which implies

$$X_n(x) = b_n \sinh \frac{n\pi x}{b}, \quad n \in \mathbf{N}.$$

Looking for $u_2(x,y)$ in the form

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

the condition

$$u_{2x}(a,y) = h(y),$$

should be satisfied, which yields

$$b_n = \frac{2}{n\pi} \frac{1}{\cosh \frac{n\pi a}{b}} \int_0^b h(y) \sin \frac{n\pi y}{b} dy. \tag{7.78}$$

Finally the solution is

$$u(x,y) = u_1(x,y) + u_2(x,y),$$

where a_n and b_n are determined by (7.77) and (7.78).

7.5.2 Dirichlet problem for Laplace equation in a disk

Let us consider the problem

$$(DL) : \begin{cases} u_{xx} + u_{yy} = 0 & \text{in } x^2 + y^2 < a^2, \\ u(x,y) = g(x,y) & \text{on } x^2 + y^2 = a^2. \end{cases}$$

In polar coordinates the problem reduces to

$$u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} = 0 \text{ if } 0 \le \rho < a, \tag{7.79}$$

with boundary condition

$$u(a,\theta) = h(\theta), \quad 0 \le \theta \le 2\pi, \tag{7.80}$$

where $u(\rho,\theta) = u(\rho\cos\theta, \rho\sin\theta)$ and $h(\theta) = g(a\cos\theta, a\sin\theta)$. The separation of variables means seeking a solution of the form $u(\rho,\theta) = R(\rho)\Theta(\theta)$. Plugging into equation (7.79) we find that

$$R''\Theta + \frac{1}{\rho}R'\Theta + \frac{1}{\rho^2}R\Theta'' = 0.$$

Dividing by $R\Theta \neq 0$ and multiplying by ρ^2 , we obtain

$$\Theta'' + \lambda\Theta = 0 \tag{7.81}$$

and

$$\rho^2 R'' + \rho R' - \lambda R = 0 \tag{7.82}$$

for some constant λ. For the function $\Theta(\theta)$ it is natural to require the periodic boundary condition

$$\Theta(\theta) = \Theta(\theta + 2\pi). \tag{7.83}$$

Then (7.81) and (7.83) imply that $\lambda = n^2$ and

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

is the corresponding solution. Note that if $\lambda = 0$,

$$\Theta_0(\theta) = a_0 \neq 0$$

is a nontrivial solution of (7.81).

If $\lambda = n^2, \quad n \in \mathbf{N}$, then the equation (7.82) is an Euler equation. Making change of variables $\rho = e^t$ it reduces to

$$\ddot{R} - n^2 R = 0$$

with general solution

$$R_n(t) = c_n e^{nt} + d_n e^{-nt}.$$

Then

$$R_n(\rho) = c_n \rho^n + d_n \frac{1}{\rho^n}, \quad n \in \mathbf{N}$$

and we have a separable solution of the form

$$u_n(\rho,\theta) = (a_n \cos n\theta + b_n \sin n\theta)\left(c_n\rho^n + d_n\frac{1}{\rho^n}\right).$$

If $\lambda = 0$ the equation (7.82) reduces to

$$\rho R'' + R' = 0$$

with general solution

$$R_0(\rho) = c_0 + d_0 \ln \rho.$$

So we have a separable solution

$$u_0 = a_0 + b_0 \ln \rho$$

for some constants a_0 and b_0.

The functions u_n and u_0 are harmonic in D. At $\rho = 0$ some of these solutions are infinite if $d_n \neq 0$ and $b_0 \neq 0$. We reject these terms in order to have bounded solutions. We are looking for a solution of the form

$$u(\rho,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \rho^n (A_n \cos n\theta + B_n \sin n\theta), \tag{7.84}$$

which satisfies the boundary condition if

$$h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta).$$

By the Fourier expansion of $h(\theta)$ we observe

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\tau) \cos n\tau d\tau, \quad n \in \mathbf{N} \cup \{0\}, \tag{7.85}$$

and

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\tau) \sin n\tau d\tau, \quad n \in \mathbf{N}. \tag{7.86}$$

The solution of the problem (7.79), (7.80) is the series (7.84) with coefficients (7.85) and (7.86). It is surprising that this series is summable explicitly and the result coinsides with the Poisson integral formula for the Dirichlet problem in a disk.

Proposition 7.1. *Let $r \in [0,1)$. Then*

$$S \ : \ = \sum_{n=1}^{\infty} r^n \sin nt = \frac{r \sin t}{1 - 2r \cos t + r^2},$$

$$C \ : \ = \sum_{n=1}^{\infty} r^n \cos nt = \frac{r \cos t - r^2}{1 - 2r \cos t + r^2}.$$

Proof. Consider the partial sums $C_n = \sum_{k=1}^{n-1} r^k \cos kt$ and $S_n = \sum_{k=1}^{n-1} r^k \sin kt$. By the Euler formula we have

$$\begin{aligned}
C_n + iS_n &= \sum_{k=1}^{n-1} r^k (\cos kt + i \sin kt) \\
&= \sum_{k=1}^{n-1} r^k e^{ikt} \\
&= \frac{1 - r^n e^{int}}{1 - re^{it}} - 1 \\
&= \frac{\left(1 - r^n e^{int}\right)\left(1 - re^{-it}\right)}{(1 - r\cos t)^2 + (r \sin t)^2} - 1 \\
&= \frac{1}{1 - 2r\cos t + r^2} (A_n + iB_n) - 1,
\end{aligned}$$

where

$$\begin{aligned}
A_n &= 1 - r\cos t - r^n \cos nt + r^{n+1} \cos(n-1)t, \\
B_n &= r \sin t - r^n \sin nt + r^{n+1} \sin(n-1)t.
\end{aligned}$$

Then

$$\begin{aligned}
C_n &= \frac{A_n}{1 - 2r\cos t + r^2} - 1, \\
S_n &= \frac{B_n}{1 - 2r\cos t + r^2}.
\end{aligned}$$

We have $r^n \to 0$ as $n \to \infty$ because $0 < r < 1$ and

$$\begin{aligned}
C &= \lim_{n\to\infty} C_n = \frac{1 - r\cos t}{1 - 2r\cos t + r^2} - 1 \\
&= \frac{r\cos t - r^2}{1 - 2r\cos t + r^2}, \\
S &= \lim_{n\to\infty} S_n = \frac{r \sin t}{1 - 2r\cos t + r^2}. \quad \blacksquare
\end{aligned}$$

Theorem 7.12. *Suppose that the function $h(\theta)$ is continuous and 2π-periodic. Then the sum of the series*

$$u(\rho, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \rho^n (A_n \cos n\theta + B_n \sin n\theta)$$

with coefficients (7.85) and (7.86) *is*

$$u(\rho,\theta) = \frac{a^2-\rho^2}{2\pi}\int_0^{2\pi} \frac{h(\tau)d\tau}{a^2 - 2a\rho\cos(\theta-\tau)+\rho^2}. \tag{7.87}$$

Proof. We have

$$\begin{aligned} u(\rho,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\tau)d\tau \\ &\quad + \sum_{n=1}^{\infty} \frac{\rho^n}{\pi a^n}\int_0^{2\pi} h(\tau)(\cos n\tau\cos n\theta + \sin n\tau \sin n\theta)d\tau \\ &= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^n \cos n(\theta-\tau)\right) h(\tau)d\tau. \end{aligned}$$

By Proposition 7.1 as $\frac{\rho}{a} < 1$ we have

$$\begin{aligned} &\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^n \cos n(\theta-\tau) \\ &= \frac{1}{2} + \frac{\left(\frac{\rho}{a}\right)\cos(\theta-\tau) - \frac{\rho^2}{a^2}}{1 - 2\left(\frac{\rho}{a}\right)\cos(\theta-\tau) + \frac{\rho^2}{a^2}} \\ &= \frac{1}{2} + \frac{a\rho\cos(\theta-\tau) - \rho^2}{a^2 - 2a\rho\cos(\theta-\tau)+\rho^2} \\ &= \frac{a^2-\rho^2}{2\left(a^2 - 2a\rho\cos(\theta-\tau)+\rho^2\right)}. \blacksquare \end{aligned}$$

Exercises

1. Solve the heat conduction problems for an aluminium bar of length 2

$$u_t - 0.86u_{xx} = 0, \quad 0 < x < 2, t > 0$$

with initial and boundary conditions as follows:

(a)

$$\begin{aligned} u(x,0) &= \sin\frac{\pi x}{2}, \\ u(0,t) &= u(2,t) = 0. \end{aligned}$$

(b)

$$\begin{aligned} u(x,0) &= 10\cos x, \\ u(0,t) &= u(2,t) = 0. \end{aligned}$$

(c)

$$u(x,0) = \begin{cases} 0 & 0 \le x < 1, \\ 40 & 1 \le x \le 2, \end{cases}$$
$$u(0,t) = u(2,t) = 0.$$

(d)

$$u(x,0) = \begin{cases} 20x & 0 \le x \le 1, \\ 20(2-x) & 1 \le x \le 2, \end{cases}$$
$$u_x(0,t) = u_x(2,t) = 0.$$

2. Solve the BVPs for the diffusion equation:

(a)

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < 1, t > 0, \\ u(x,0) = T_0 & 0 < x < 1, \\ u(0,t) = T_1, \quad u(1,t) = T_2 & t > 0. \end{cases}$$

(b)

$$\begin{cases} u_t - u_{xx} + 2u = 0 & 0 < x < 1, t > 0, \\ u(x,0) = \cos x & 0 < x < 1, \\ u(0,t) = u(1,t) = 0 & t > 0. \end{cases}$$

3. Solve the following mixed problems for the wave equation

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 3, t > 0:$$

(a)

$$\begin{array}{cc} u(x,0) = 1 - \cos\dfrac{\pi x}{3} & 0 \le x \le 3, \\ u_t(x,0) = 0 & 0 \le x \le 3, \\ u(0,t) = u(3,t) = 0 & t > 0. \end{array}$$

(b)

$$\begin{array}{cc} u(x,0) = 0 & 0 \le x \le 3, \\ u_t(x,0) = 1 & 0 \le x \le 3, \\ u_x(0,t) = u_x(3,t) = 0 & t > 0. \end{array}$$

(c)

$$u(x,0) = \begin{cases} 2x & 0 \le x \le 1, \\ 2 & 1 \le x \le 2, \\ 6 - 2x & 2 \le x \le 3, \end{cases}$$
$$\begin{array}{cc} u_t(x,0) = 0 & 0 \le x \le 3, \\ u(0,t) = u(3,t) = 0 & t > 0. \end{array}$$

(d)

$$\begin{aligned} u(x,0) &= x\cos\pi x & 0<x<1,\\ u_t(x,0) &= 1 & 0\le x\le 1,\\ u(0,t) = u(1,t) &= 0 & t>0. \end{aligned}$$

4. Prove Theorem 7.10.

5. Solve the following mixed problems for the inhomogeneous wave equations:

(a)

$$\begin{cases} u_{tt}-4u_{xx}=\sin 3x & 0<x<\pi,\ t>0,\\ u(x,0)=0 & 0\le x\le\pi,\\ u_t(x,0)=\cos x & 0\le x\le\pi,\\ u(0,t)=u(\pi,t)=0 & t>0, \end{cases}$$

(b)

$$\begin{cases} u_{tt}-4u_{xx}=2\sin x & 0<x<1,\ t>0,\\ u(x,0)=0 & 0\le x\le 1,\\ u_t(x,0)=0 & 0\le x\le 1,\\ u(0,t)=1, \quad u(1,t)=1, & t>0. \end{cases}$$

6. Solve the mixed problem

$$\begin{cases} u_{tt}-c^2u_{xx}+au=0 & 0<x<l,\ t>0,\\ u(x,0)=\varphi(x) & 0\le x\le l,\\ u_t(x,0)=0 & 0\le x\le l,\\ u(0,t)=u(l,t)=0 & t>0, \end{cases}$$

where a is a constant.

7. Solve the mixed problem for the wave equation

$$u_{tt}-c^2u_{xx}=A\sin\omega t, \quad 0<x<l, t>0,$$

with zero initial and boundary conditions. For which ω does the resonance (growth in time) occur ?

8. Solve the boundary value problems for the Laplace equation in the square $K=\{(x,y):0<x<\pi,0<y<\pi\}$:

(a) $u_y(x,0)=u_y(x,\pi)=u_x(0,y)=0, \quad u_x(\pi,y)=\cos 3y,$

(b) $u(0,y)=u_y(x,0)+u(x,0)=u_x(\pi,y)=0, \quad u(x,\pi)=\sin\frac{3x}{2}.$

9. (a) Using Euler's formula prove the identities:

$$
\begin{aligned}
\sin^{2m} t &= \frac{1}{4^m}\left(2\sum_{k=0}^{m-1}(-1)^{m+k}\binom{2m}{k}\cos\left(2\left(m-k\right)t\right)+\binom{2m}{m}\right),\\
\sin^{2m+1} t &= \frac{1}{4^m}\left(\sum_{k=0}^{m-1}(-1)^{m+k}\binom{2m+1}{k}\sin\left(\left(2\left(m-k\right)+1\right)t\right)\right),\\
\cos^{2m} t &= \frac{1}{4^m}\left(2\sum_{k=0}^{m-1}\binom{2m}{k}\cos\left(2\left(m-k\right)t\right)+\binom{2m}{m}\right),\\
\cos^{2m+1} t &= \frac{1}{4^m}\left(\sum_{k=0}^{m-1}\binom{2m+1}{k}\cos\left(\left(2\left(m-k\right)+1\right)t\right)\right).
\end{aligned}
$$

(b) Solve the Dirichlet problem

$$
\begin{cases}
u_{xx}+u_{yy}=0 & \text{in } x^2+y^2<1,\\
u\left(x,y\right)=x^4-y^3 & \text{on } x^2+y^2=1.
\end{cases}
$$

10. Let $U_n(x,y)$ be the solution of the problem

$$
\begin{cases}
u_{xx}+u_{yy}=0 & \text{in } x^2+y^2<1,\\
u\left(x,y\right)=y^n & \text{on } x^2+y^2=1.
\end{cases}
$$

Prove that:
(a)

$$
\begin{aligned}
& U_{2m}(\rho,\theta)\\
=\ & \frac{1}{2^{2m}}\left(\left(2\sum_{k=0}^{m-1}(-1)^{m+k}\binom{2m}{k}\rho^{2(m-k)}\cos\left(2\left(m-k\right)\theta\right)\right)+\binom{2m}{m}\right),
\end{aligned}
$$

$$
\begin{aligned}
& U_{2m+1}(\rho,\theta)\\
=\ & \frac{1}{2^{2m}}\left(\sum_{k=0}^{m-1}(-1)^{m+k}\binom{2m+1}{k}\rho^{2(m-k)+1}\sin\left(\left(2\left(m-k\right)+1\right)\theta\right)\right).
\end{aligned}
$$

(b)

$$
\begin{aligned}
1 &\geq U_2(x,y) \geq U_4(x,y) \geq \dots \geq U_{2m}(x,y) \geq 0, \\
1 &\geq U_1(x,y) \geq U_3(x,y) \geq \dots \geq U_{2m+1}(x,y) \geq 0 \text{ if } y \geq 0, \\
-1 &\leq U_1(x,y) \leq U_3(x,y) \leq \dots \leq U_{2m+1}(x,y) \leq 0 \text{ if } y \leq 0.
\end{aligned}
$$

Plot the graphs of the functions $U_1(x,y), U_3(x,y)$ and $U_5(x,y)$ using *Mathematica.*

(c) The series $\sum\limits_{m=1}^{\infty} \frac{1}{4^m}\binom{2m}{m}$ is divergent.

(d) The series $\sum\limits_{m=1}^{\infty} U_{2m}(x,y)$ is divergent for $x^2 + y^2 < 1$,while the series $\sum\limits_{m=1}^{\infty} q^m U_{2m+1}(x,y)$ is convergent for $0 \leq q < 1, x^2 + y^2 < 1$.

Chapter 8

Diffusion and Wave Equations in Higher Dimensions

8.1 Diffusion Equation in Three Dimensional Space

Let us consider the Cauchy problem for the diffusion equation in $\mathbf{R}^3$

$$(CD_3) : \begin{cases} u_t = k\Delta u = k(u_{xx} + u_{yy} + u_{zz}), & t > 0 \\ u(P, 0) = \phi(P), \end{cases}$$

where $P = (x, y, z) \in \mathbf{R}^3$ and $\phi(P)$ is a given function.

At first observe

Proposition 8.1. *Suppose $u_1(x,t)$, $u_2(y,t)$ and $u_3(z,t)$ are solutions of the one-dimensional diffusion equation $u_t - ku_{ss} = 0$, where $s \in \{x, y, z\}$. Then $u(x, y, z, t) = u_1(x,t)u_2(y,t)u_3(z,t)$ is a solution of $u_t - k\Delta u = 0$ in $\mathbf{R}^3$.*

Proof. We have

$$\begin{aligned} u_t &= u_{1t}u_2u_3 + u_1u_{2t}u_3 + u_1u_2u_{3t} \\ &= k(u_{1xx}u_2u_3 + u_1u_{2yy}u_3 + u_1u_2u_{3zz}) \\ &= k\Delta(u_1u_2u_3) \\ &= k\Delta u. \blacksquare \end{aligned}$$

The function

$$G(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}}$$

is a *fundamental solution* of the diffusion equation $u_t - ku_{xx} = 0$.

By Proposition 8.1 the function

$$\begin{aligned} G_3(P,t) &= G(x,t)G(y,t)G(z,t) \\ &= \frac{1}{8\sqrt{(\pi kt)^3}} e^{-\frac{1}{4kt}(x^2+y^2+z^2)} \\ &= \frac{1}{8\sqrt{(\pi kt)^3}} e^{-\frac{1}{4kt}|P|^2} \end{aligned}$$

is a solution of

$$u_t - k\Delta u = 0, \quad P \in \mathbf{R}^3,\ t > 0. \tag{8.1}$$

$G_3(P,t)$ is again called the Green's function or fundamental solution of (8.1).

Observe that

$$\begin{aligned} &\int_{\mathbf{R}^3} G_3(P,t)dP \qquad (8.2) \\ = &\int_{-\infty}^{\infty} G(x,t)dx \int_{-\infty}^{\infty} G(y,t)dy \int_{-\infty}^{\infty} G(z,t)dz \\ = &\ 1. \end{aligned}$$

We consider the case when the initial data $\phi(P)$ is a function with separable variables

$$\phi(P) = \varphi(x)\psi(y)\sigma(z). \tag{8.3}$$

Proposition 8.2. Suppose that $\phi(P)$ is a function with separable variables (8.3), where φ, ψ and σ are bounded and continuous functions.

Then

$$u(P,t) = \int_{\mathbf{R}^3} G_3(P-Q,t)\phi(Q)dQ, \tag{8.4}$$

with $Q = (\xi, \eta, \zeta) \in \mathbf{R}^3$ is a solution of (CD_3).

Proof. Separating integration we have

$$\begin{aligned} u(P,t) &= \int_{\mathbf{R}^3} G_3(P-Q,t)\phi(Q)dQ \\ &= \int_{-\infty}^{\infty} G(x-\xi,t)\varphi(\xi)d\xi \int_{-\infty}^{\infty} G(y-\eta,t)\psi(\eta)d\eta \int_{-\infty}^{\infty} G(z-\zeta,t)\sigma(\zeta)d\zeta \\ &= u_1(x,t)u_2(y,t)u_3(z,t). \end{aligned}$$

By Theorem 4.7 and Proposition 8.1 it follows that

$$u_t - k\Delta u = 0 \text{ for } (P,t) \in \mathbf{R}^3 \times (0,\infty)$$

and

$$\begin{aligned} \lim_{t\downarrow 0} u(P,t) &= \lim_{t\downarrow 0} u_1(x,t) \lim_{t\downarrow 0} u_2(y,t) \lim_{t\downarrow 0} u_3(z,t) \\ &= \varphi(x)\psi(y)\sigma(z) \\ &= \phi(P). \blacksquare \end{aligned}$$

By linearity Proposition 8.2 can be extended for any initial data which is a finite linear combination of functions with separable variables of the form

$$\phi_n(P) = \sum_{k=1}^{n} c_k \varphi_k(x)\psi_k(y)\sigma_k(z) \tag{8.5}$$

Let us show that any continuous and bounded function on $\mathbf{R}^3$ can be uniformly approximated by functions of type (8.5) on bounded domains. Recall *Bernstein's*[1] *polynomial* for a bounded function on the interval $[0,1]$, given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}.$$

Theorem 8.1. *(Bernstein). Let $f(x) \in C[0,1]$. Then $B_n(x) \to f(x)$ uniformly for $x \in [0,1]$ as $n \to +\infty$.*

[1] Sergej Natanovich Bernstein, 06.03.1880–26.10.1968.

Proposition 8.3. *Let $\phi(P) \in C\left([0,1]^3\right)$, $|\phi(P)| \le M$ and $\varepsilon > 0$. There exists a function with separable variables $\phi_n(P) \in C\left([0,1]^3\right)$, such that $|\phi_n(P)| \le M$ and*

$$|\phi(P) - \phi_n(P)| < \varepsilon \text{ if } P \in [0,1]^3.$$

Proof. Let $\varepsilon > 0$. By Theorem 8.1 there exists n such that

$$\left|\phi(x,y,z) - \sum_{k=0}^{n} \binom{n}{k} \phi\left(\frac{k}{n}, y, z\right) x^k (1-x)^{n-k}\right| < \frac{\varepsilon}{2},$$

for $(x,y,z) \in [0,1]^3$.

By the same way there exists n_k such that

$$\left|\phi\left(\frac{k}{n}, y, z\right) - \sum_{m=0}^{n_k} \binom{n_k}{m} \phi\left(\frac{k}{n}, \frac{m}{n_k}, z\right) y^m (1-y)^{n_k-m}\right| < \frac{\varepsilon}{2}.$$

for $(y,z) \in [0,1]^2$.

Let

$$\phi_n(x,y,z) = \sum_{k=0}^{n} \sum_{m=0}^{n_k} \binom{n}{k} \binom{n_k}{m} \phi\left(\frac{k}{n}, \frac{m}{n_k}, z\right) x^k (1-x)^{n-k} y^m (1-y)^{n_k-m}.$$

We have that $\phi_n(x,y,z) \in C\left([0,1]^3\right)$ is a function with separable variables and

$$\begin{aligned}
&|\phi(x,y,z) - \phi_n(x,y,z)| \\
\le\ & \left|\phi(x,y,z) - \sum_{k=0}^{n} \binom{n}{k} \phi\left(\frac{k}{n}, y, z\right) x^k (1-x)^{n-k}\right| \\
&+ \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left|\phi\left(\frac{k}{n}, y, z\right) - \sum_{m=0}^{n_k} \binom{n_k}{m} \phi\left(\frac{k}{n}, \frac{m}{n_k}, z\right) y^m (1-y)^{n_k-m}\right| \\
<\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \\
=\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

By the construction of $\phi_n(x,y,z)$ we have $|\phi_n(x,y,z)| \le M$. ■

Let $R > 0$. By rescaling the variables $(x, y, z) \longmapsto \left(\frac{x+R}{2R}, \frac{y+R}{2R}, \frac{z+R}{2R}\right)$ we can prove that every function $\phi(x, y, z) \in C\left([-R, R]^3\right)$ can be uniformly approximated by a function $\phi_n(x, y, z) \in C\left([-R, R]^3\right)$ with separable variables.

Theorem 8.2. *Suppose $\phi(P) \in C(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$.*
Then the function

$$u(P, t) = \int_{\mathbf{R}^3} G_3(P - Q, t)\phi(Q)dQ$$

is a solution of the diffusion equation (8.1) *on $\mathbf{R}^3$ and*

$$\lim_{t \downarrow 0} u(P, t) = \phi(P) \tag{8.6}$$

uniformly on bounded sets of $\mathbf{R}^3$.

Proof. By Proposition 8.2 it follows that $u(P, t)$ satisfies (8.1). Let us show that (8.6) holds. Suppose $\varepsilon > 0$ and $B \subset \mathbf{R}^3$ is a bounded set. Making the change of variables $Q = P - 2\sqrt{kt}P'$ we have

$$u(P, t) = \frac{1}{\sqrt{\pi^3}} \int_{\mathbf{R}^3} e^{-|P'|^2} \phi(P - 2\sqrt{kt}P')dP', \tag{8.7}$$

where $P' = (p, q, r) \in \mathbf{R}^3$. Let $|\phi(P)| \leq M,\ P \in \mathbf{R}^3$ and denote

$$K_R = [-R, R]^3,\ \tilde{K}_R = \mathbf{R}^3 \backslash K_R.$$

There exist $R > 0$ and $\phi_n(P) \in C\left([-R, R]^3\right)$ with separable variables such that:

$$\frac{1}{\sqrt{\pi^3}} \int_{\tilde{K}_R} e^{-|P'|^2} dP' < \frac{\varepsilon}{8M} \text{ and } B \subset [-R, R]^3, \tag{8.8}$$

$$|\phi(P) - \phi_n(P)| < \frac{\varepsilon}{4} \text{ if } P \in [-R, R]^3. \tag{8.9}$$

By continuity of $\phi(P)$ there exists $\delta > 0$ such that if $t \in (0, \delta)$, then

$$\max_{P' \in K_R} \left|\phi(P - 2\sqrt{kt}P') - \phi(P)\right| < \frac{\varepsilon}{4} \text{ for } P \in K_R. \tag{8.10}$$

Finally for $P \in B \subset K_R$ and $t \in (0, \delta)$, by (8.8), (8.9) and (8.10), we have

$$
\begin{aligned}
&|u(P,t) - \phi(P)| \\
\leq\ & \int_{\mathbf{R}^3} G_3(P-Q,t)|\phi(Q) - \phi_n(P)|dQ + |\phi_n(P) - \phi(P)| \\
=\ & \frac{1}{\sqrt{\pi^3}} \int_{\mathbf{R}^3} e^{-|P'|^2} |\phi(P - 2\sqrt{kt}P') - \phi_n(P)|dP' + |\phi_n(P) - \phi(P)| \\
\leq\ & \frac{2M}{\sqrt{\pi^3}} \int_{\tilde{K}_R} e^{-|P'|^2} dP' + \frac{1}{\sqrt{\pi^3}} \int_{K_R} e^{-|P'|^2} |\phi(P - 2\sqrt{kt}P') - \phi(P)|dP' \\
& + \frac{1}{\sqrt{\pi^3}} \int_{K_R} e^{-|P'|^2} |\phi(P) - \phi_n(P)|dP' + |\phi_n(P) - \phi(P)| \\
<\ & 2M.\frac{\varepsilon}{8M} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,
\end{aligned}
$$

which completes the proof. ■

The same arguments hold as well for the diffusion equation in higher space dimensions. The fundamental solution for the diffusion equation in $\mathbf{R}^n$ is given by

$$
G_n(P,t) = \frac{1}{2^n\sqrt{(\pi kt)^n}} e^{-\frac{|P|^2}{4kt}},
$$

where $P = (x_1, ..., x_n)$ and $|P| = \sqrt{x_1^2 + ... + x_n^2}$. Following previous steps one can prove that the solution of the Cauchy problem

$$
\begin{cases} u_t - k\Delta u = 0, & (P,t) \in \mathbf{R}^n \times (0,\infty), \\ u(P,0) = \phi(P) & P \in \mathbf{R}^n \end{cases}
$$

is given by

$$
u(P,t) = \int_{\mathbf{R}^n} G_n(P-Q,t)\phi(Q)dQ.
$$

The solution of the inhomogeneous problem

$$
\begin{cases} u_t - k\Delta u = f(P,t) & (P,t) \in \mathbf{R}^n \times (0,\infty), \\ u(P,0) = 0 & P \in \mathbf{R}^n, \end{cases}
$$

is given by

$$
u(P,t) = \int_0^t \int_{\mathbf{R}^n} G_n(P-Q,t-\tau)f(Q,\tau)dQd\tau.
$$

The maximum principle holds in higher dimensions as well. Let $\Omega \subset \mathbf{R}^n$ be a domain

$$
\Omega_T = \Omega \times (0,T), \ \Pi = \partial\Omega_T \backslash \{(P,t) : P \in \Omega, t = T\}.
$$

Suppose that $u \in C^2(\Omega_T) \cap C(\bar{\Omega}_T)$ is a solution of the diffusion equation $u_t = k\Delta u$ in Ω_T. Then

$$\begin{aligned} \max_{\bar{\Omega}_T} u &= \max_{\Pi} u, \\ \min_{\bar{\Omega}_T} u &= \min_{\Pi} u. \end{aligned}$$

Exercises

1. Find solutions of the problems

(a)

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbf{R}^3 \times (0, \infty), \\ u(x, y, z, 0) = x^2 yz \end{cases}$$

(b)

$$\begin{cases} u_t - \Delta u = 0 \text{ in } \mathbf{R}^3 \times (0, \infty), \\ u(x, y, z, 0) = x^2 yz - xyz^2. \end{cases}$$

2. The function $T_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ is called a trigonometric polynomial of n-th order if $a_n^2 + b_n^2 \neq 0$. $T_n(x)$ is even if $b_1 = b_2 = ... = b_n = 0$. Prove that:

(a) The function $\cos^k x$ can be represented as an even trigonometric polynomial of k-th order.

(b) If $f(x) \in C[0, \pi]$ and $\varepsilon > 0$, then there exists an even trigonometric polynomial $T_n(x)$ such that for every $x \in [0, \pi]$

$$|f(x) - T_n(x)| < \varepsilon.$$

3. Using the reflection method find a formula for the solution of the BVP for the diffusion equation in half-plane.

$$\begin{cases} u_t - k\Delta u = 0 \text{ in } \{(x, y, t) : x > 0, y \in \mathbf{R}, t > 0\}. \\ u(0, y, t) = 0 \quad (y, t) \in \mathbf{R} \times \mathbf{R}^+, \\ u(x, y, 0) = \phi(x, y) \quad (x, y) \in (0, \infty) \times \mathbf{R}. \end{cases}$$

4. Find a formula for the solution of the BVP for the diffusion equation in half-space

$$\begin{cases} u_t - k\Delta u = 0 \text{ in } \{(x, y, z, t) : (x, y) \in \mathbf{R}^2, z > 0, t > 0\}, \\ \frac{\partial u}{\partial z}(x, y, 0, t) = 0 \quad (x, y) \in \mathbf{R}^2, \\ u(x, y, z, 0) = \phi(x, y, z) \quad (x, y, z) \in \mathbf{R}^2 \times (0, \infty). \end{cases}$$

8.2 Fourier Method for the Diffusion Equation in Higher Dimensions

In this Section we shall apply the Fourier method to the diffusion equation

$$u_t = k\Delta u$$

in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with standard initial and boundary conditions on $\partial\Omega$. Such BVPs are as follows:

$$u_t = k\Delta u \text{ in } \Omega \times (0, \infty),$$

$$u(x, y, 0) = \phi(x, y) \quad (x, y) \in \Omega, \tag{8.11}$$

$$u(x, y, t) = 0 \text{ on } \partial\Omega \times [0, \infty)$$

or

$$\frac{\partial u}{\partial n}(x, y, t) = 0 \text{ on } \partial\Omega \times [0, \infty)$$

or

$$\frac{\partial u}{\partial n}(x, y, t) + \sigma u(x, y, t) = 0 \text{ on } \partial\Omega \times [0, \infty).$$

Separating variables

$$u(x, y, t) = \Phi(x, y)T(t)$$

and substituting into (8.11) we see that Φ and T must satisfy

$$\frac{T'(t)}{kT(t)} = \frac{\Delta\Phi(x, y)}{\Phi(x, y)} = -\lambda,$$

where λ is constant. This leads to the eigenvalue problem for the Laplacian

$$-\Delta\Phi = \lambda\Phi \quad \text{in } \Omega$$

with boundary condition

$$\Phi = 0 \quad \text{on } \partial\Omega \tag{8.12}$$

or

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{8.13}$$

or

$$\frac{\partial \Phi}{\partial n} + \sigma\Phi = 0 \quad \text{on } \partial\Omega. \tag{8.14}$$

It can be shown that for each one of the boundary conditions (8.12)–(8.14) there is an infinite sequence of eigenvalues

$$\lambda_n \to \infty \quad \text{as } n \to \infty$$

and an infinite set of orthogonal eigenfunctions which is complete. Denote by Φ_n the eigenfunction corresponding to λ_n with the understanding that not all of λ_n are distinct. Solving the *ODE* for $T(t)$

$$T'(t) + k\lambda_n T(t) = 0$$

we find

$$T(t) = a_n e^{-k\lambda_n t}.$$

We are looking for a solution of the form

$$u(x,y,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t}\Phi_n(x,y), \tag{8.15}$$

which satisfies the initial condition if

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \Phi_n(x,y).$$

By the orthogonality of (Φ_n) it follows that

$$A_n = \frac{\iint_\Omega \phi(x,y)\Phi_n(x,y)dxdy}{\iint_\Omega \Phi_n^2(x,y)dxdy}. \tag{8.16}$$

If we suppose $\phi(x,y) \in L^2(\Omega)$ it can be shown that the series (8.15) is convergent for $t > 0$ and $u(x,y,t) \to \phi(x,y)$ as $t \to 0$ in the mean-square sense in Ω.

As an example, consider the heat transfer problem on a circular plate

$$\begin{cases} u_t = a^2\Delta u & x^2+y^2 < R^2,\ t > 0, \\ u(x,y,0) = \phi(\sqrt{x^2+y^2}) & x^2+y^2 < R^2, \\ u(x,y,t) = 0 & x^2+y^2 = R^2,\ t \geq 0. \end{cases}$$

Using polar coordinates for $u(P,t) = u(\sqrt{x^2+y^2}, t) = u(\rho, t)$, we have

$$\begin{cases} u_t = a^2(u_{\rho\rho} + \frac{1}{\rho}u_\rho) & 0 \le \rho < R,\ t > 0, \\ u(\rho, 0) = \phi(\rho) & 0 \le \rho < R, \\ u(\rho, t) = 0 & \rho = R,\ t \ge 0. \end{cases}$$

Separating variables

$$u(\rho, t) = U(\rho)T(t),$$

we have

$$\frac{T'(t)}{a^2 T(t)} = \frac{U''(\rho) + \frac{1}{\rho}U'(\rho)}{U(\rho)} = -\lambda^2.$$

As before

$$T(t) = c_k e^{-(a\lambda)^2 t}$$

while $U(\rho)$ satisfies the Bessel equation

$$U''(\rho) + \frac{1}{\rho}U'(\rho) + \lambda^2 U(\rho) = 0. \tag{8.17}$$

The first solution of (8.17) is the Bessel function of zeroth-order

$$U(\rho) = J_0(\lambda\rho) \tag{8.18}$$

while a second linearly independent solution of (8.17) is $Y_0(\lambda\rho)$, which we do not take into account because it is infinite at zero.

The boundary condition $U|_{\rho=R} = 0$ is satisfied if

$$J_0(\lambda R) = 0. \tag{8.19}$$

Then the eigenvalues of the problem (8.17), (8.19) are

$$\lambda_k = \frac{\mu_k}{R},$$

where $\mu_k \to +\infty$ as $k \to \infty$ are the zeros of the Bessel function $J_0(x)$.

We are looking for a solution of the form

$$u(\rho, t) = \sum_{k=1}^{\infty} c_k e^{-(a\lambda_k)^2 t} J_0(\lambda_k \rho), \tag{8.20}$$

which satisfies the initial condition if

$$\phi(\rho) = \sum_{k=1}^{\infty} c_k J_0(\lambda_k \rho).$$

For $r = \frac{\rho}{R} \in [0,1]$ we have

$$\phi(Rr) = \sum_{k=1}^{\infty} c_k J_0(\mu_k r).$$

By the properties of the Bessel function, Subsection 7.2.1, it follows

$$c_k = \frac{2}{J_0'^2(\mu_k)} \int_0^1 r J_0(\mu_k r)\phi(Rr)dr. \tag{8.21}$$

The solution of the problem is (8.20) with coefficients given by (8.21).

Example 8.1. *Solve the problem*

$$\begin{cases} u_t = \Delta u, \quad x^2+y^2<1,\ t>0, \\ u(x,y,0) = J_0(\mu_1\sqrt{x^2+y^2}) + J_0(\mu_2\sqrt{x^2+y^2}), \quad x^2+y^2<1, \\ u(x,y,t) = 0 \quad x^2+y^2 = 1,\ t \geq 0. \end{cases}$$

Solution. In polar coordinates the problem is

$$\begin{cases} u_t = u_{\rho\rho} + \frac{1}{\rho}u_\rho, \quad 0 \leq \rho < 1,\ t>0, \\ u(\rho,0) = J_0(\mu_1\rho) + J_0(\mu_2\rho), \quad 0 \leq \rho < 1, \\ u(1,t) = 0, \quad t \geq 0. \end{cases}$$

The solution is

$$u(\rho,t) = \sum_{k=1}^{\infty} c_k e^{-{\mu_k}^2 t} J_0(\mu_k \rho),$$

where

$$c_k = \frac{2}{J_0'^2(\mu_k)} \int_0^1 r J_0(\mu_k r)\left(J_0(\mu_1 r) + J_0(\mu_2 r)\right)dr.$$

By the orthogonality of $\{\sqrt{r}J_0(\mu_k r)\}$ it follows

$$c_k = \begin{cases} 1 & \text{if } k = 1,\ 2, \\ 0 & \text{if } k \geq 3. \end{cases}$$

The solution is

$$u(\rho, t) = e^{-\mu_1^2 t} J_0(\mu_1 \rho) + e^{-\mu_2^2 t} J_0(\mu_2 \rho),$$

where $\mu_1 = 2.4$ and $\mu_2 = 5.52$. The surface

$$(\rho \cos\theta, \rho \sin\theta, u(\rho, t)), (\rho, \theta) \in [0, 1] \times [0, 2\pi]$$

is plotted in Figure 8.1 at the instants $t = 0,\ 0.1,\ 0.4$ using the *Mathematica* program

```
Clear[a,b,x,y,u]
a=2.4
b=5.52
x[r_,v_]:=rCos[v]
y[[r_,v_]:=r Sin[v]
u[r_,v_,t_]:=Exp[-a^2 t] BesselJ[0,a r]+Exp[-b^2 t] BesselJ[0,b r]
h0=ParametricPlot3D[Evaluate[x[r,v],y[r,v],u[r,v,0]],
{r,0,1},{v,0,2Pi}, Shading->False,PlotRange->{-1,2}]
h1=ParametricPlot3D[Evaluate[x[r,v],y[r,v],u[r,v,0.1]],
{r,0,1},{v,0,2Pi}, Shading->False,PlotRange->{-1,1}]
h2=ParametricPlot3D[Evaluate[x[r,v],y[r,v],u[r,v,0.4]],
{r,0,1},{v,0,2Pi}, Shading->False,PlotRange->{-1,1}]
Show[GraphicsArray[{h0,h1,h2}],
Frame->True,FrameTicks->None]
```

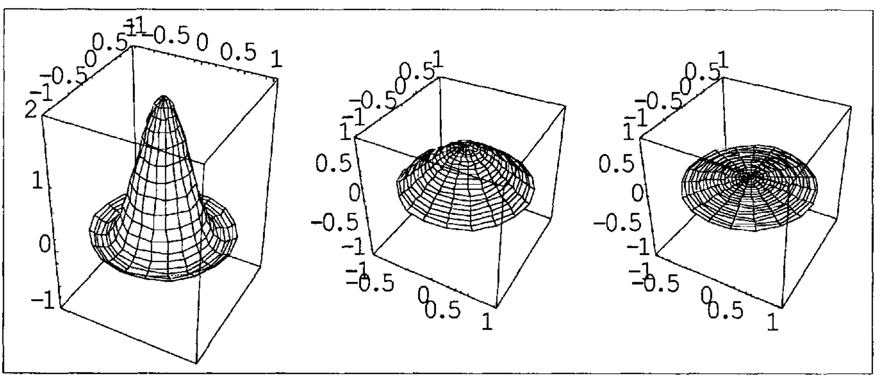

Figure 8.1. Temperature $u(\rho, t)$ at the instants $t = 0,\ 0.1,\ 0.4$

Finally in this Section we consider the problem with the Neumann condition

$$\begin{cases} u_t = a^2 \Delta u & x^2+y^2<R^2,\ t>0, \\ u(x,y,0) = \phi(\sqrt{x^2+y^2}) & x^2+y^2<R^2, \\ \frac{\partial u}{\partial n}(x,y,t) = 0 & x^2+y^2=R^2,\ t\geq 0. \end{cases}$$

Using polar coordinates for $u(P,t) = u(\rho,t)$, we have

$$\begin{cases} u_t = a^2(u_{\rho\rho} + \frac{1}{\rho}u_\rho) & 0\leq\rho<R,\ t>0, \\ u(\rho,0) = \phi(\rho) & 0\leq\rho<R, \\ \frac{\partial}{\partial\rho}u(\rho,t)|_{\rho=R} = 0 & \rho=R,\ t\geq 0. \end{cases}$$

Separating variables

$$u(\rho,t) = U(\rho)T(t)$$

for $U(\rho)$ we find

$$U(\rho) = J_0(\lambda\rho) \text{ and } U'(R) = 0.$$

Then

$$J_0'(\lambda R) = -J_1(\lambda R) = 0$$

and the eigenvalues are

$$\lambda_k = \frac{\nu_k}{R},$$

where ν_k are the zeros of the Bessel function J_1. Looking for a solution of the form

$$u(\rho,t) = \sum_{k=1}^{\infty} c_k e^{-(a\lambda_k)^2 t} J_0\left(\frac{\nu_k\rho}{R}\right), \tag{8.22}$$

by

$$u(\rho,0) = \phi(\rho),$$

we find

$$c_k = \frac{1}{\gamma_k}\int_0^1 rJ_0(\nu_k r)\phi(Rr)dr,$$

where

$$\gamma_k = \int_0^1 rJ_0^2(\nu_k r)dr.$$

Working as in Lemma 7.2, we have

$$\begin{aligned}\int_0^1 rJ_0^2(\nu_k r)dr &= \lim_{\lambda\to\nu_k} \frac{\lambda J_0(\nu_k)J_0'(\lambda)}{\nu_k^2-\lambda^2} \\ &= -\frac{1}{2}J_0(\nu_k)J_0''(\nu_k) \\ &= \frac{1}{2}J_0^2(\nu_k).\end{aligned}$$

Therefore the solution of the problem is (8.22), where

$$c_k = \frac{2}{J_0^2(\nu_k)}\int_0^1 rJ_0(\nu_k r)\phi(Rr)dr.$$

Exercises

1. Solve the problem

$$\begin{cases} u_t = u_{\rho\rho} + \frac{1}{\rho}u_\rho & 0 \le \rho < 1,\ t > 0, \\ u(\rho,0) = 1-\rho^2 & 0 \le \rho < 1, \\ u(1,t) = 0 & t \ge 0. \end{cases}$$

The solution is

$$u(\rho,t) = 8\sum_{k=1}^{\infty} e^{-\mu_k^2 t}\frac{J_0(\mu_k\rho)}{\mu_k^3 J_1(\mu_k)}.$$

Show that

$$u(0,t) \approx \frac{16}{\mu_1^4}e^{-\mu_1^2 t}.$$

2. Solve the problem

$$\begin{cases} u_t = a^2(u_{\rho\rho} + \frac{1}{\rho}u_\rho) & 0 \le \rho < R,\ t > 0, \\ u(\rho,0) = T & 0 \le \rho < R, \\ \frac{\partial}{\partial\rho}u(\rho,t)|_{\rho=R} = q & t \ge 0. \end{cases}$$

8.3 Kirchoff's Formula for the Wave Equation. Huygens' Principle

8.3.1 Kirchoff's formula. Spherical means.

The linear wave equation in $\mathbf{R}^3$ is

$$u_{tt} - c^2 \Delta u = 0, \tag{8.23}$$

where $u = u(P,t)$ and $P = (x,y,z) \in \mathbf{R}^3$. We are looking for the solution of (8.23) with initial conditions

$$\begin{aligned} u(P,0) &= \phi(P), \\ u_t(P,0) &= \psi(P), \end{aligned} \tag{8.24}$$

as in the D'Alembert formula. Assume that $\phi(P) \in C^3(\mathbf{R}^3)$ and $\psi(P) \in C^2(\mathbf{R}^3)$. Then there exists a unique solution to the problem (8.23), (8.24) given by the formula

$$u(P,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{|Q-P|=ct} \phi(Q)dS_Q\right) + \frac{1}{4\pi c^2 t}\iint\limits_{|Q-P|=ct} \psi(Q)dS_Q, \tag{8.25}$$

which is due to Poisson but known as *Kirchoff's formula.*[2] To derive it we shall use the so called *spherical means* introduced by Poisson. Let us denote

$$\bar{u}(P,r,t) = \frac{1}{4\pi r^2}\iint\limits_{|Q-P|=r} u(Q,t)dS_Q$$

to be the mean value of $u(P,t)$ over the sphere $S_r(P) = \{Q : |Q-P| = r\}$ with center P and radius r. Some properties of $\bar{u}(P,r,t)$ are as follows:

1^0.

$$\bar{u}(P,r,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi,t)\sin\varphi d\varphi d\theta, \tag{8.26}$$

where

$$u(r,\theta,\varphi,t) = u(x + r\cos\theta\sin\varphi, y + r\sin\theta\sin\varphi, z + r\cos\varphi, t).$$

2^0. If $u(P,t)$ is a continuous function, then

$$\lim_{r\to 0}\bar{u}(P,r,t) = \bar{u}(P,0,t) = u(P,t).$$

[2] Gustav Robert Kirchoff, 12.03.1824–17.10.1887.

3^0. If $u(P,t)$ is differentiable in t , then

$$\frac{\partial}{\partial t}\bar{u}(P,r,t) = \overline{u_t}(P,r,t). \tag{8.27}$$

4^0. If $u(P,t)$ is twice differentiable in (x,y,z) , then

$$\overline{\Delta u}(P,r,t) = \bar{u}_{rr}(P,r,t) + \frac{2}{r}\bar{u}_r(P,r,t). \tag{8.28}$$

5^0. If $u(P,t) \in C^m\left(\mathbf{R}^4\right)$, $m \in \mathbf{N}$, then $P \longmapsto \bar{u}(P,r,t) \in C^m\left(\mathbf{R}^4\right)$.

Proof of $4°$. Let us change to spherical coordinates for $Q(\xi,\eta,\zeta)$

$$\begin{aligned}
\xi &= x + r\cos\theta\sin\varphi \\
\eta &= y + r\sin\theta\sin\varphi \\
\zeta &= z + r\cos\varphi,
\end{aligned}$$

where $\theta \in [0,2\pi]$ and $\varphi \in [0,\pi)$. From

$$\Delta u(P,t) = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi u_\varphi\right) + \frac{1}{r^2}\frac{1}{\sin^2\varphi}u_{\theta\theta},$$

(8.26), the Fubini theorem and the periodicity of $u(r,\theta,\varphi,t)$ with respect to θ, we have

$$\begin{aligned}
\overline{\Delta u}(P,r,t) &= \frac{1}{4\pi r^2}\iint_{|Q-P|=r}\Delta u(Q,t)dS_Q \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(u_{rr} + \frac{2}{r}u_r\right)\sin\varphi d\varphi d\theta \\
&\quad + \frac{1}{4\pi r^2}\left(\int_0^{2\pi}\int_0^{\pi}\frac{\partial}{\partial\varphi}\left(\sin\varphi u_\varphi\right)d\varphi d\theta + \int_0^{\pi}\frac{1}{\sin\varphi}\int_0^{2\pi}u_{\theta\theta}d\varphi d\theta\right) \\
&= \frac{\partial^2}{\partial r^2}\left(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u\sin\varphi d\varphi d\theta\right) + \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u\sin\varphi d\varphi d\theta\right) \\
&\quad + \frac{1}{4\pi r^2}\left(\int_0^{2\pi}\sin\varphi u_\varphi\,|_0^{\pi}\,d\theta + \int_0^{\pi}\frac{1}{\sin\varphi}u_\theta\,|_0^{2\pi}\,d\varphi\right) \\
&= \bar{u}_{rr}(P,r,t) + \frac{2}{r}\bar{u}_r(P,r,t). \blacksquare
\end{aligned}$$

Proposition 8.4. *Let* $\phi(r) \in C^3(0,\infty)$ *and* $\omega(r) \in C^2(0,\infty)$. *The* *solution $v(r,t)$ of the problem*

$$\begin{cases} v_{tt} - c^2\left(v_{rr} + \frac{2}{r}v_r\right) = 0, \quad (r,t) \in (0,\infty)\times(0,\infty), \\ v(r,0) = \phi(r), \quad r \in (0,\infty), \\ v_t(r,0) = \omega(r), \quad r \in (0,\infty) \end{cases}$$

satisfies

$$v(0,t) = \lim_{r\to 0} v(r,t) = \frac{d}{dt}\left(t\phi(ct)\right) + t\omega(ct).$$

Proof. Introducing $w(r,t) = rv(r,t)$ for $r \geq 0$ we easily find that

$$\begin{cases} w_{tt} - c^2 w_{rr} = 0, \quad (r,t) \in (0,\infty)\times(0,\infty), \\ w(r,0) = r\phi(r), \quad r \in (0,\infty), \\ w_t(r,0) = r\omega(r), \quad r \in (0,\infty) \\ w(0,t) = 0, \quad t \in (0,\infty). \end{cases}$$

This problem for the wave equation on the half-line has a solution

$$w(r,t) = \begin{cases} \frac{1}{2c}\left(\frac{\partial}{\partial t}\int_{r-ct}^{r+ct} s\phi(s)ds + \int_{r-ct}^{r+ct} s\omega(s)ds\right), & \text{if } r > ct, \\ \frac{1}{2c}\left(\frac{\partial}{\partial t}\int_{ct-r}^{r+ct} s\phi(s)ds + \int_{ct-r}^{r+ct} s\omega(s)ds\right), & \text{if } 0 \leq r \leq ct. \end{cases}$$

In order to find $v(0,t)$ observe that

$$\begin{aligned} v(0,t) &= \lim_{r\to 0} v(r,t) = \lim_{r\to 0}\frac{w(r,t)}{r} \\ &= \lim_{r\to 0}\frac{w(r,t)-w(0,t)}{r} = \frac{\partial}{\partial r}w(r,t)\,|_{r=0}\,, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial r}\int_{ct-r}^{r+ct} s\phi(s)ds\bigg|_{r=0} &= \lim_{r\to 0}\frac{1}{r}\int_{ct-r}^{r+ct} s\phi(s)ds \\ &= \lim_{r\to 0}\frac{2r}{r}s_r\phi(s_r) = 2ct\phi(ct), \end{aligned}$$

where $s_r \in [ct-r, ct+r]$, $s_r \to ct$ as $r \to 0$.

Then

$$\begin{aligned} v(0,t) &= \frac{\partial}{\partial r}w(r,t)\,|_{r=0} \\ &= \frac{2c}{2c}\frac{d}{dt}\left(t\phi(ct)\right) + \frac{2ct}{2c}\omega(ct) \\ &= \frac{d}{dt}\left(t\phi(ct)\right) + t\omega(ct). \ \blacksquare \end{aligned}$$

Derivation of Kirchoff's formula

Applying the mean-sphere operator to (8.23), (8.24) by properties $1° - 5°$ for $v(r,t) = \overline{u}(P,r,t)$, we have

$$\begin{cases} v_{tt} - c^2\left(v_{rr} + \frac{2}{r}v_r\right) = 0, & \text{if } (r,t) \in (0,\infty) \times (0,\infty), \\ v(r,0) = \overline{\phi}(P,r) \\ v_t(r,0) = \overline{\psi}(P,r). \end{cases}$$

Then by Proposition 8.4. and $2°$ we obtain

$$\begin{aligned} u(P,t) &= \overline{u}(P,0,t) = v(0,t) \\ &= \frac{\partial}{\partial t}\left(t\overline{\phi}(P,ct)\right) + t\overline{\psi}(P,ct) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{4\pi c^2 t^2}\iint_{|Q-P|=ct}\phi(Q)dS_Q\right) + \frac{t}{4\pi c^2 t^2}\iint_{|Q-P|=ct}\psi(Q)dS_Q \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{|Q-P|=ct}\phi(Q)dS_Q\right) + \frac{1}{4\pi c^2 t}\iint_{|Q-P|=ct}\psi(Q)dS_Q, \end{aligned}$$

which is Kirchoff's formula.

In the case of the one-dimensional wave equation the solution given by D'Alembert formula is as regular as the initial data. However, in the three-dimensinal case, because of the t -derivative in the Kirchoff's formula, the solution is less regular than the data ϕ and ψ. In general if

$$\phi \in C^{m+1}\left(\mathbf{R}^3\right) \text{ and } \psi \in C^m\left(\mathbf{R}^3\right), \quad m \geq 2,$$

then $u \in C^m\left(\mathbf{R}^3 \times \mathbf{R}^+\right)$. If ϕ and ψ are of class $C^2\left(\mathbf{R}^3\right)$ then the second derivatives of u might blow up at some point even though the second derivatives of ϕ and ψ are bounded. This is known as the *focusing effect.*

8.3.2 Wave equation on $\mathbf{R}^2$. Method of descent. Huygens' principle.

We can derive the solution formula for the wave equation on $\mathbf{R}^2 \times \mathbf{R}^+$

$$(CW_2): \begin{cases} u_{tt} - c^2\Delta u = 0, & P \in \mathbf{R}^2, \quad t > 0, \\ u(P,0) = \phi(P), & P \in \mathbf{R}^2, \\ u_t(P,0) = \psi(P), & P \in \mathbf{R}^2. \end{cases}$$

Assume that $\phi \in C^3(\mathbf{R}^2)$ and $\psi \in C^2(\mathbf{R}^2)$ .Let us transform the integral

$$I = \frac{1}{4\pi c^2 t} \iint_{|\overline{Q}-\overline{P}|=ct} g(\xi,\eta) dS_{\overline{Q}},$$

where $\overline{P} = (P,0) = (x,y,0) \in \mathbf{R}^2 \times \{0\}$ and $\overline{Q} = (Q,\zeta) = (\xi,\eta,\zeta) \in \mathbf{R}^3$. We have

$$I = \frac{1}{4\pi c^2 t}\left(\iint_{S+} g(\xi,\eta) dS_{\overline{Q}} + \iint_{S-} g(\xi,\eta) dS_{\overline{Q}}\right),$$

where

$$\begin{aligned} S^+ &= \left\{(\xi,\eta,\zeta) : \zeta = \sqrt{(ct)^2-(x-\xi)^2-(y-\eta)^2}\right\}, \\ S^- &= \left\{(\xi,\eta,\zeta) : \zeta = -\sqrt{(ct)^2-(x-\xi)^2-(y-\eta)^2}\right\} \end{aligned}$$

are the upper and lower hemisphere. For both hemispheres

$$dS_{\bar{Q}} = \sqrt{1+\zeta_\xi^2+\zeta_\eta^2}\, d\xi d\eta = \frac{ct\, d\xi d\eta}{\sqrt{(ct)^2-(x-\xi)^2-(y-\eta)^2}}$$

and

$$\begin{aligned} I &= \frac{2}{4\pi c^2 t}\iint_{(x-\xi)^2+(y-\eta)^2\le(ct)^2} \frac{ct\, g(\xi,\eta)d\xi d\eta}{\sqrt{(ct)^2-(x-\xi)^2-(y-\eta)^2}} \\ &= \frac{1}{2\pi c}\iint_{(x-\xi)^2+(y-\eta)^2\le(ct)^2} \frac{g(\xi,\eta)d\xi d\eta}{\sqrt{(ct)^2-(x-\xi)^2-(y-\eta)^2}} \\ &= \frac{1}{2\pi c}\iint_{|P-Q|\le ct} \frac{g(Q)dQ}{\sqrt{(ct)^2-|P-Q|^2}}. \end{aligned}$$

In order to solve (CW_2) we can consider it as a problem in $\mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$. By Kirchoff's formula

$$u(P,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{|\bar{Q}-\bar{P}|=ct} \phi(\xi,\eta) dS_{\bar{Q}}\right) + \frac{1}{4\pi c^2 t}\iint_{|\bar{Q}-\bar{P}|=ct} \psi(\xi,\eta) dS_{\bar{Q}},$$

where $\overline{P} = (x,y,0)$ and $\overline{Q} = (Q,\zeta) = (\xi,\eta,\zeta)$. By previous calculation the solution of (CW_2) is

$$u(P,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\iint_{|P-Q|\le ct} \frac{\phi(Q)dQ}{\sqrt{(ct)^2-|P-Q|^2}}\right) + \frac{1}{2\pi c}\iint_{|P-Q|\le ct} \frac{\psi(Q)dQ}{\sqrt{(ct)^2-|P-Q|^2}}$$

which is Kirchoff's formula for the wave equation on $\mathbf{R}^2$.

As an application of Kirchoff's formula we consider the so called *Huygens' principle*[3]. According to it, in the three dimensional space , the values of ϕ and ψ at the point $P_0 \in \mathbf{R}^3$ influence the solution on the sphere $|P - P_0| = ct$ only.

Suppose for simplicity $\phi = 0, \psi$ has a compact support $K = \overline{\{P : \psi(P) \neq 0\}}$ and $\psi > 0$ in the interior of K. Denote for $P \notin K$

$$\begin{aligned} \bar{d}(P, K) &= \min\{|P - Q| : Q \in K\}, \\ \tilde{d}(P, K) &= \max\{|P - Q| : Q \in K\}, \end{aligned}$$

which exist because the function $g(Q) = |P - Q|$ is continuous on K, a compact set. Kirchoff's formula

$$u(P, t) = \frac{1}{4\pi c^2 t} \iint\limits_{|Q-P|=ct} \psi(Q) dS_Q$$

implies

$$u(P, t) = 0 \quad \text{if} \quad ct < \bar{d}(P, K) \quad \text{or} \quad ct > \tilde{d}(P, K),$$

because the sphere $|P - Q| = ct$ does not intersect K for these values of t. If $u(P, t)$ is a sound produced on K, with $\psi(P, t)$ as an initial speed, it is heard at the point $P \notin K$ from the instant $\dfrac{\bar{d}(P, K)}{c}$ until $\dfrac{\tilde{d}(P, K)}{c}$. This means that the sound passes through P for $\dfrac{\tilde{d}(P, K) - \bar{d}(P, K)}{c}$ time and moves with speed c.

Let us introduce the forward and backward wave fronts at the instant t_0 as

$$\bar{W}(t_0) = \left\{ P \in \mathbf{R}^3 : \begin{array}{c} u(P, t) = 0 \quad \text{if } t < t_0 \quad \text{and } \exists \delta > 0 \\ u(P, t) \neq 0 \quad \text{if } t \in (t_0, t_0 + \delta). \end{array} \right\}$$

and

$$\tilde{W}(t_0) = \left\{ P \in \mathbf{R}^3 : \begin{array}{c} u(P, t) = 0 \quad \text{if } t > t_0 \quad \text{and } \exists \delta > 0 \\ u(P, t) \neq 0 \quad \text{if } t \in (t_0 - \delta, t_0). \end{array} \right\}$$

In the case under consideration $\phi = 0$ and $\psi\,|_K \geq 0$ implies

$$\{P : \bar{d}(P, K) = ct_0\} \subset \bar{W}(t_0)$$

and

$$\left\{P : \tilde{d}(P, K) = ct_0\right\} \subset \tilde{W}(t_0),$$

[3] Christian Huygens, 14.04.1629 – 08.07.1695.

so $\bar{W}(t_0)$ and $\tilde{W}(t_0)$ are non empty sets. Namely, this is the *Huygens' principle.* It does not hold in $\mathbf{R}^2$ where $\tilde{W}(t_0) = \emptyset$. This means that there exists $\{t_j\}, \quad t_j \to \infty$ such that $u(t_j) \neq 0$.

For instance, in the case $\phi = 0$ and $\psi|_K \geq 0$ in $\mathbf{R}^2$ by Kirchoff's formula

$$u(P,t) = \frac{1}{2\pi c} \iint_{|Q-P| \leq ct} \frac{\psi(Q)}{\sqrt{(ct)^2 - |Q-P|^2}} dQ$$

it follows that $u(P,t) > 0$ if $t > \dfrac{\bar{d}(P,K)}{c}$ because the disk $|Q-P| \leq ct$ intersects K for $t \in \left(\dfrac{\bar{d}(P,K)}{c}, \dfrac{\tilde{d}(P,K)}{c} \right)$ and contains K for $t > \dfrac{\tilde{d}(P,K)}{c}$. The sound produced on K will be heard infinitely. This means that Huygens' principle does not occur in $\mathbf{R}^2$.

We are lucky to live in a three dimensional space because we are able to hear every sound for a finite interval of time. This phenomenon does not occur in the two dimensional world (Flatland) where sounds are heard forever.

Exercises

1. Verify that Kirchoff's formula gives the solution of the problem (8.23), (8.24) in the case $\phi = 0$. Namely, for $\psi \in C^2(\mathbf{R}^3)$ show that

$$u(P,t) = \frac{1}{4\pi c^2 t} \iint_{|Q-P|=ct} \psi(Q) dS_Q$$

satisfies:

(a) $u(P,0) = 0, \quad u_t(P,0) = \psi(P)$

(b) $u_t = \frac{u}{t} + \frac{1}{4\pi ct} \iiint_{|Q-P| \leq ct} \Delta\psi(Q) dQ$

(c) $\frac{\partial}{\partial t} \left(\frac{1}{4\pi ct} \iiint_{|Q-P| \leq ct} \Delta\psi(Q) dQ \right) = c \iint_{|Q-P|=ct} \Delta\psi(Q) dS_Q$

(d)

$$\begin{aligned} u_{tt} &= \frac{1}{4\pi ct} \frac{\partial}{\partial t} \iiint_{|Q-P| \leq ct} \Delta\psi(Q) dQ \\ &= c^2 \Delta \left(\frac{1}{4\pi c^2 t} \iint_{|Q-P|=ct} \psi(Q) dS_Q \right) \\ &= c^2 \Delta u. \end{aligned}$$

8.4 Fourier Method for the Wave Equation on the Plane. Nodal Sets

In this section we shall apply the Fourier method to the wave equation

$$u_{tt} = c^2 \Delta u \tag{8.29}$$

in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with standard initial and boundary conditions on $\partial\Omega$ as follows:

$$\begin{aligned} u(x,y,0) &= \phi(x,y), \quad (x,y) \in \Omega, \\ u_t(x,y,0) &= \psi(x,y), \quad (x,y) \in \Omega, \end{aligned} \tag{8.30}$$

$$u(x,y,t) = 0 \text{ on } \partial\Omega \times [0, \infty) \tag{8.31}$$

or

$$\frac{\partial u}{\partial n}(x,y,t) = 0 \text{ on } \partial\Omega \times [0, \infty) \tag{8.32}$$

or

$$\frac{\partial u}{\partial n}(x,y,t) + \sigma u(x,y,t) = 0 \text{ on } \partial\Omega \times [0, \infty). \tag{8.33}$$

Separating variables

$$u(x,y,t) = \Phi(x,y)T(t)$$

and substituting into (8.29) we see that Φ and T must satisfy

$$\frac{T''(t)}{c^2 T(t)} = \frac{\Delta\Phi(x,y)}{\Phi(x,y)} = -\lambda$$

where λ is constant. This leads to the eigenvalue problem for the Laplacian

$$-\Delta\Phi = \lambda\Phi \quad \text{in } \Omega \tag{8.34}$$

with boundary condition

$$\Phi = 0 \quad \text{on } \partial\Omega \tag{8.35}$$

or

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{8.36}$$

or

$$\frac{\partial \Phi}{\partial n} + \sigma \Phi = 0 \quad \text{on } \partial\Omega. \tag{8.37}$$

Denote again by Φ_n the eigenfunction corresponding to λ_n with the understanding that not all of λ_n are distinct. Solving the *ODE* for $T(t)$

$$T''(t) + c^2 \lambda_n T(t) = 0$$

we find

$$T(t) = A_n \cos c\sqrt{\lambda_n} t + B_n \sin c\sqrt{\lambda_n} t.$$

We are looking for a solution of the form

$$u(x,y,t) = \sum_{n=1}^{\infty} \left(A_n \cos c\sqrt{\lambda_n} t + B_n \sin c\sqrt{\lambda_n} t \right) \Phi_n(x,y), \tag{8.38}$$

which satisfies the initial conditions if

$$\begin{aligned} \phi(x,y) &= \sum_{n=1}^{\infty} A_n \Phi_n(x,y), \\ \psi(x,y) &= \sum_{n=1}^{\infty} c\sqrt{\lambda_n} B_n \Phi_n(x,y). \end{aligned}$$

By the orthogonality of (Φ_n) it follows

$$\begin{aligned} A_n &= \frac{\iint_\Omega \phi(x,y)\Phi_n(x,y)dxdy}{\iint_\Omega \Phi_n^2(x,y)dxdy}, \\ B_n &= \frac{\iint_\Omega \psi(x,y)\Phi_n(x,y)dxdy}{c\sqrt{\lambda_n}\iint_\Omega \Phi_n^2(x,y)dxdy}. \end{aligned} \tag{8.39}$$

Example 8.2. *Determine the radial vibrations* $u\left(\sqrt{x^2+y^2}, t\right) = u(\rho, t)$ *of the circular drum* $D = \{(x,y) : x^2 + y^2 \le 1\}$ *satisfying the problem*

$$\begin{cases} u_{tt} = \Delta u, \quad x^2 + y^2 < 1, \ t > 0, \\ u(x, y, 0) = J_0(\mu_1\sqrt{x^2 + y^2}) + J_0(\mu_2\sqrt{x^2 + y^2}), \quad x^2 + y^2 < 1, \\ u_t(x, y, 0) = 0, \quad x^2 + y^2 < 1, \\ u(x, y, t) = 0, \quad x^2 + y^2 = 1, \ t \geq 0. \end{cases}$$

Solution. In polar coordinates the problem is

$$\begin{cases} u_{tt} = u_{\rho\rho} + \frac{1}{\rho}u, \quad 0 \leq \rho < 1, \ t > 0, \\ u(\rho, 0) = J_0(\mu_1\rho) + J_0(\mu_2\rho), \quad 0 \leq \rho < 1, \\ u_t(\rho, 0) = 0, \quad 0 \leq \rho < 1, \\ u(1, t) = 0, \quad t \geq 0. \end{cases}$$

Separating variables as in Section 8.2, we find the solution

$$u(\rho, t) = \sum_{k=1}^{\infty} (a_k \cos \mu_k t + b_k \sin \mu_k t)\, J_0(\mu_k \rho),$$

where

$$\begin{aligned} a_k &= \frac{2}{J_0'^2(\mu_k)} \int_0^1 r J_0(\mu_k r) u(r, 0) dr, \\ b_k &= \frac{2}{\mu_k J_0'^2(\mu_k)} \int_0^1 r J_0(\mu_k r) u_t(r, 0) dr. \end{aligned}$$

In our case

$$\begin{aligned} a_k &= \frac{2}{J_0'^2(\mu_k)} \int_0^1 r J_0(\mu_k r)\,(J_0(\mu_1 r) + J_0(\mu_2 r))\, dr, \\ b_k &= 0. \end{aligned}$$

By the orthogonality of $\{\sqrt{r} J_0(\mu_k r)\}$ it follows

$$a_k = \begin{cases} 1 & \text{if } k = 1, 2, \\ 0 & \text{if } k \geq 3. \end{cases}$$

The solution is

$$u(\rho, t) = J_0(\mu_1 \rho) \cos \mu_1 t + J_0(\mu_2 \rho) \cos \mu_2 t,$$

where $\mu_1 = 2.4$ and $\mu_2 = 5.52$. The surface

$$D_t : (\rho\cos\theta, \rho\sin\theta, u(\rho,t)), (\rho,\theta) \in [0,1]\times[0,2\pi]$$

is plotted in Figure 8.2 at the instants $t = 0,\ \pi/\mu_2,\ \pi/\mu_1$ using the *Mathematica* program

```
Clear[a,b,x,y,u]
a=2.4
b=5.52
x[r_,v_]:=rCos[v]
y[[r_,v_]:=r Sin[v]
u[r_,v_,t_]:=Cos[a t] BesselJ[0,a r]+Cos[b t] BesselJ[0,b r]
h0=ParametricPlot3D[Evaluate[x[r,v],y[r,v],u[r,v,0]],
{r,0,1},{v,0,2Pi}, Shading->False,PlotRange->{-1,2}]
h1=ParametricPlot3D[Evaluate[x[r,v],y[r,v],u[r,v,Pi/b]],
{r,0,1},{v,0,2Pi}, Shading->False,PlotRange->{-1,1}]
h2=ParametricPlot3D[Evaluate[x[r,v],y[r,v],u[r,v,Pi/a]],
{r,0,1},{v,0,2Pi}, Shading->False,PlotRange->{-1,1}]
Show[GraphicsArray[{h0,h1,h2}],
Frame->True,FrameTicks->None]
```

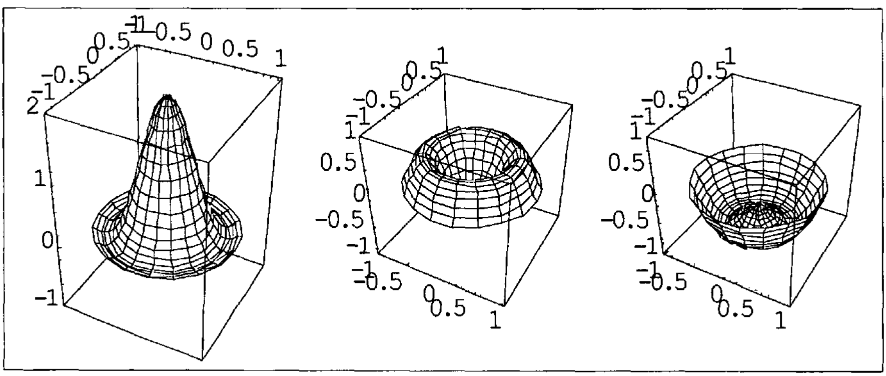

Figure 8.2. The surface D_t at the instants $t = 0,\ \pi/\mu_2,\ \pi/\mu_1$

Next we consider the wave equation on rectangular domains in $\mathbf{R}^2$. Let $\Pi = (0,a) \times (0,b)$ be a rectangle and consider the wave equation

$$u_{tt} = c^2 \left(u_{xx} + u_{yy}\right), \quad (x,y) \in \Pi, t > 0,$$

with the usual initial condition and the Dirichlet boundary condition

$$u|_{x=0} = u|_{x=a} = u|_{y=0} = u|_{y=b} = 0.$$

The eigenfunctions of the problem

$$-v_{xx} - v_{yy} = \lambda v, \quad (x,y) \in \Pi, \tag{8.40}$$

$$v|_{x=0} = v|_{x=a} = v|_{y=0} = v|_{y=b} = 0 \tag{8.41}$$

are

$$v_{mn}(x,y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad (m,n) \in \mathbf{N}^2$$

corresponding to the eigenvalues

$$\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

We shall discuss the *nodal set*

$$\mathcal{N}_v = \{(x,y) : v(x,y) = 0\}$$

of an eigenfunction v of the problem (8.39), (8.41). Note that boundary points of Π do not belong to $\mathcal{N}_v$.

The nodal set has a physical meaning, because it presents the set where the rectangular membrane Π does not move at all. The nodal sets consist of points, curves and surfaces in one, two and three dimensions respectively. There is a physical experiment due to Chladny[4] which allows one to visualize the nodal set in two dimensions. Covering the membrane Π with fine sand and vibrating it with a given frequency the sand particles take the place of the nodal set. These sets are known also as *Chladny's figures.*

Let us consider nodal sets of some eigenfunctions of (8.39) in the square $K = (0,\pi) \times (0,\pi)$. Both eigenfunctions v_{mn} and v_{nm} have the eigenvalue

---

[4]Ernst Florens Friedrich Chladny, 30.11.1756–03.04.1827.

$$\lambda = \lambda_{mn} = \lambda_{nm} = m^2 + n^2$$

and vibrate with angular frequency

$$\omega = c\sqrt{\lambda_{mn}}.$$

For each $\alpha \in [0,1]$

$$v_\alpha = \alpha v_{mn}(x,y) + (1-\alpha)v_{nm}(x,y)$$

is a mode of vibration with frequency ω. The nodal set N_{v_α} is the curve

$$c_\alpha : \alpha v_{mn}(x,y) + (1-\alpha)v_{nm}(x,y) = 0.$$

It varies from the nodal set $N_{v_{mn}}$ to the nodal set $N_{v_{nm}}$ and divides the square K into several different regions which vibrate independently.

We consider now in detail the nodal sets

$$\mathcal{N}_n = \{(x,y) \in K : V_n(x,y) = 0\}$$

of the function

$$V_n(x,y) = \sin 2nx \sin y + \sin x \sin 2ny.$$

It is clear that if $(x,y) \in \mathcal{N}_n$, then the points

$$\begin{array}{lll} (y,x), & (\pi - x, y), & (\pi - y, x), \\ (y, \pi - x), & (x, \pi - y), & (\pi - x, \pi - y) \end{array}$$

also belong to $\mathcal{N}_n$.

Using mathematical induction one can prove:

Claim 1^0.

$$V_n(x,y) = V_{n-1}(x,y) + 2\sin x \sin y\,(\cos(2n-1)x + \cos(2n-1)y)$$

Claim 2^0.

$$\begin{aligned} V_n(x,y) &= \sin x \sin y (\cos x + \cos y) \\ & \left(2^{2n-1}\left(\cos^{2(n-1)} x + \cos^{2(n-1)} y\right) + \ldots + (-1)^{n-1} 2n\right) \end{aligned}$$

for $n \geq 2$.

The first four functions $V_n(x,y)$ are expanded as follows:

$$V_1(x,y) = 2\sin x \sin y (\cos x + \cos y),$$

$$\begin{aligned} V_2(x,y) &= \sin x \sin y (\cos x + \cos y) \\ & \left(8\cos^2 x - 2\cos x \cos y - 4 + 8\cos^2 y\right), \end{aligned}$$

$$V_3(x,y) = \sin x \sin y (\cos x + \cos y) f(x,y),$$

where

$$\begin{gathered} f(x,y) = 32(\cos^4 x + \cos^4 y - \cos^3 x \cos y - \cos x \cos^3 y \\ + \cos^2 x \cos^2 y + \cos x \cos y - \cos^2 x - \cos^2 y) + 6 \end{gathered}$$

and

$$V_4(x,y) = \sin x \sin y (\cos x + \cos y) g(x,y),$$

where

$$\begin{aligned} g(x,y) &= 8(16\left(\cos^6 x + \cos^6 y\right) - 16\left(\cos^5 x \cos y - \cos y \cos^5 x\right) \\ & +16\left(\cos^4 y \cos^2 x + 6\cos^2 y \cos^4 x\right) - 16\cos^3 y \cos^3 x \\ & -24\left(\cos^4 y - \cos^4 x\right) + 24\left(\cos^3 y \cos x + \cos y \cos^3 x\right) \\ & -24\cos^2 y \cos^2 x + 10\left(\cos^2 y + \cos^2 x\right) - 10\cos y \cos x - 1). \end{aligned} \tag{8.42}$$

Computations are made by MAPLE in Scientific WorkPlacE using *Expand+Factor*. The nodal sets $\mathcal{N}_n$ or $n = 1, 2, 4$ are presented using *Plot2D+Implicit*.

For the function

$$V_1(x,y,a) = \sin 2x \sin y + a \sin x \sin 2y = 2\sin x \sin y (\cos x + a\cos y)$$

the curves $c_a : \cos x + a\cos y = 0$ for $a = 0.5,\ 1,\ 1.5$ are presented in Figure 8.3.

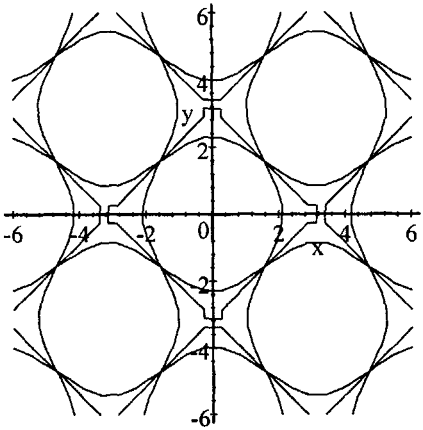

Figure 8.3. Curves $c_a : \cos x + a \cos y = 0$ for $a = 0.5,\ 1,\ 1.5$

Using the expansion of $V_2(x, y) = \sin 4x \sin y + \sin x \sin 4y$ the nodal set $\mathcal{N}_2$ is presented in Figure 8.4. It divides K into 4 regions.

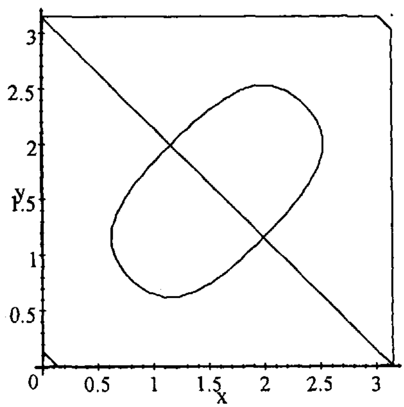

Figure 8.4. Nodal set $\mathcal{N}_2$

The nodal set $\mathcal{N}_g$, where g is given by (8.42), is presented in Figure 8.5. The nodal set $\mathcal{N}_g$ divides K into 8 regions.

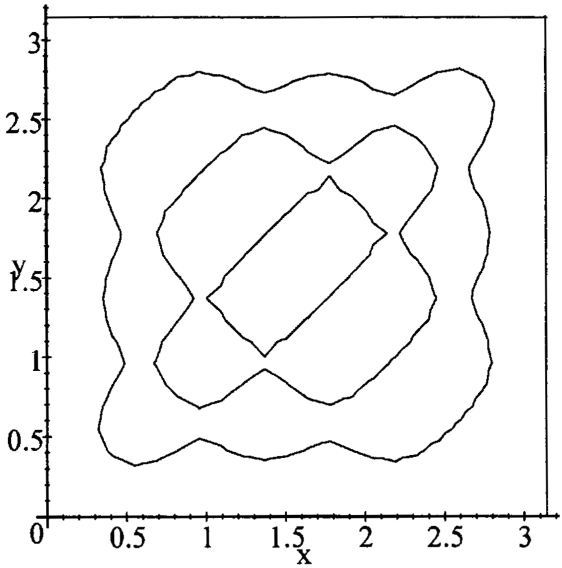

Figure 8.5. Nodal set $\mathcal{N}_g$

Claim 3^0. The nodal set $\mathcal{N}_n$ divides K into $2n$ regions.

Exercises

1. Solve the problem

$$(P_1): \begin{cases} u_{tt} - c^2\left(u_{xx} + u_{yy}\right) = f(x,y)\sin\omega t & (x,y) \in \Pi, t > 0, \\ u\left(x,y,t\right) = 0 & (x,y) \in \partial\Pi, t > 0, \\ u\left(x,y,0\right) = 0 & (x,y) \in \Pi, \\ u_t\left(x,y,0\right) = 0 & (x,y) \in \Pi, \end{cases}$$

using the method of separation of variables. Consider separately the "nonresonance" case $\omega \neq \omega_{mn} = \pi c\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}$ and the "resonance" $\omega = \omega_{m_0n_0}$ for some (m_0, n_0) .

2. Consider the axis symmetric bounded solutions $u = u(\rho, \varphi)$ of the Laplace equation in a ball B_R :

$$(P_2) : \begin{cases} \sin\varphi \frac{\partial}{\partial \rho}\left(\rho^2 u_\rho\right) + \frac{\partial}{\partial \varphi}\left(\sin\varphi u_\varphi\right) = 0 & 0 \le \rho < R, 0 \le \varphi \le \pi, \\ u(R, \varphi) = h(\varphi) & 0 \le \varphi \le \pi. \end{cases}$$

(a) Separating variables

$$u(\rho, \varphi) = R(\rho)\,\Phi(\varphi)$$

show that $R(\rho)$ satisfies the Euler equation

$$\rho^2 R''(\rho) + 2\rho R'(\rho) - \nu(\nu+1)\,R(\rho) = 0,$$

while $\Phi(\varphi) = \bar{\Phi}(x)$ satisfies the Legendre equation

$$\frac{d}{dx}\left(\left(1 - x^2\right)\frac{d\bar{\Phi}}{dx}\right) + \nu(\nu+1)\,\bar{\Phi} = 0,$$

where $\Phi(\varphi) = \Phi(\arccos x) = \bar{\Phi}(x)$.

(b) The solution of the problem (P_2) is

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} c_n \rho^n P_n(\cos\varphi), \quad 0 \le \varphi \le \pi$$

where P_n is the n–th Legendre polynomial and

$$c_n = \frac{2n+1}{2R^n}\int_0^{\pi} h(\varphi)\,P_n(\cos\varphi)\sin\varphi d\varphi.$$

(c) Solve the problem (P_2) with $R = 1$ and $h(\varphi) = \cos 2\varphi$. Plot the surface

$$S_\rho : \begin{cases} \rho\cos\theta\sin\varphi \\ \rho\sin\theta\sin\varphi \\ u(\rho, \varphi)\cos\varphi \end{cases},$$

with *Mathematica* for $(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$ and $\rho = 0.25,\ 0.5,\ 1$.

3. Prove Claims 1^0–3^0.

4. Plot the nodal set $\mathcal{N}_3$ using the expansion

$$V_3(x,y) = \sin y \sin x (\cos x + \cos y) f(x,y)$$

where

$$\begin{aligned} f(x,y) = 32(&\cos^4 x + \cos^4 y - \cos^3 x \cos y - \cos x \cos^3 y \\ &+ \cos^2 x \cos^2 y + \cos x \cos y - \cos^2 x - \cos^2 y) + 6. \end{aligned}$$

Show that $\mathcal{N}_f$ intersects the diagonal $x = y$ of K at the points

$$(\pi/6, \pi/6), \ (\pi/3, \pi/3), \ (2\pi/3, 2\pi/3), \ (5\pi/6, 5\pi/6).$$

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Answers and Hints to Exercises

Chapter 1

Section 1.2

1. (a) $u = x^n f\left(\frac{y}{x}\right)$, (b) $u = f\left(\frac{y}{x}\right) + \frac{x^n}{n}$,(c) $u = e^{-cx/a} f(ay - bx) + \frac{d}{c}$.

3. (a) It is a linear equation in three variables and the characteristic system is

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}.$$

The general solution is $u = f\left(x + y + z, x^2 + y^2 + z^2\right)$.

(b) $u = f(xyz, x + y + z)$, (c) $u = u = e^{-cx/a_1} f(a_2 x - a_1 y, a_3 x - a_1 z)$, where $a_1 \neq 0$.

Section 1.3

2. (a) $u = ye^{x-1}$, (b) $u = \dfrac{(x-y)^2}{1 - y(x-y)^2}$ (c) $u = y - x^2$.

Section 1.4

1. (a) $F\left(x^3 + y^3, \dfrac{x-y}{u}\right) = 0$, (b) $F\left(x + y + u, x^2 + y^2 + u^2\right) = 0$,

(c) $F(x + y + u, xyu) = 0$, (d) $F\left(x^2 + u^2, (x+u)^2 - 2y\right) = 0$,

(e) $F\left(u - 2y, y + 2\sqrt{u - x - y}\right) = 0$.

2. (a) $u = \dfrac{x^y}{z}$, (b) $u = 2x + (y-z)^2 + y^2 - z^2$.

Section 1.5

1. (a) $u = y + 2x$, (b) $u = xy$, (c) $u = 2xy - \dfrac{3}{2}y^2$,

(d) $u = \dfrac{2x\sqrt{x}}{\sqrt{1-28y}}$, (e) $u = x + \dfrac{y^2}{1+2x}$.

Chapter 2

Section 2.1

1. (a) $u = f(x+ct) + g(x-ct)$, (b) Irreducible,

(c) $u = f(3x-y) + g\left(\frac{1}{3}x - y\right)$,

(d) $u = f\left(x+\sqrt{a}t, y+\sqrt{c}t\right) + g\left(x-\sqrt{a}t, y-\sqrt{c}t\right)$.

2. (a) $u = e^{\alpha x + \alpha ct}, u = e^{\alpha x - \alpha ct}$, (b) $u = e^{\alpha x + k\alpha^2 t}$, (c) $u = e^{\alpha(x+iy)}$, $i = \sqrt{-1}$.

3. $u = f(x+ct) + g(x-ct) + \frac{t^3}{6} - \frac{e^{2x}}{4}$.

4. (a) $u_p = \frac{1}{7}e^{3x+2t}$, (b) $u_p = \frac{1}{4}xe^{2x+4t}$,

(c) $u_p = -\frac{A}{\alpha^4+\beta^2}\left[\alpha^2\cos(\alpha x+\beta t) + \beta\sin(\alpha x+\beta t)\right]$,

(d) $u_p = \frac{A}{12}x^4 - \frac{B}{2}xt^2 - \frac{C}{3}t^3$.

5. (a) $au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} + (d-a)u_\xi + (e-c)u_\eta + fu = 0$.

(b) $u = f\left(\frac{y}{x}\right) + xg\left(\frac{y}{x}\right)$.

Section 2.2

1. (a) $u_{\xi\eta} - \frac{1}{16}(u_\xi - u_\eta) = 0, \quad \xi = x - y,\ \eta = 3x + y$.

(b) $u_{\xi\xi} + u_{\eta\eta} + u_\xi = 0, \quad \xi = x,\ \eta = 3x + y$.

(c) $u_{\xi\eta}=0 \quad \xi=x+2\sqrt{-y},\ \eta=x-2\sqrt{-y}, \quad \text{if } y<0;$

$u_{\xi\xi}+u_{\eta\eta}=0 \quad \xi=x,\ \eta=2\sqrt{y}, \quad \text{if } y>0.$

(d) $u_{\xi\xi}+u_{\eta\eta}=0 \quad \xi=\ln\left(x+\sqrt{1+x^2}\right),\ \eta=\ln\left(y+\sqrt{1+y^2}\right).$

(e) $u_{\eta\eta}=0, \quad \xi=e^{-x}-e^{-y},\ \eta=x.$

2. (a) $u(x,y)=f(y-x)+g(y+3x).$

(b) $u(x,y)=f(2x+y)\,e^{\frac{1}{8}(3y-x)}+\frac{2}{8}(2x+y)+g(3y-x).$

(c) $u(x,y)=f(y-\cos x+x)+g(y-\cos x-x).$

(d) $u(x,y)=-2xy+f(xy)\sqrt{-\frac{x}{y}}+g\left(\frac{y}{x}\right)$, if $xy<0$.

3. (a) $2u_{xx}+u_{xy}-u_{yy}=0$, (b) $u_{xx}-2u_{xy}+u_{yy}=0$,

(c) $x^2u_{xx}-y^2u_{yy}=0$, (d) $x(u_{xx}-u_{yy})+2u_x=0.$

Section 2.3

1. (a) $v_{y_1y_1}+v_{y_2y_2}+v_{y_3y_3}=0,$
$y_1=\frac{1}{3}\sqrt{2}x_1+\frac{1}{3}\sqrt{2}x_2-\frac{1}{3}\sqrt{2}x_3,$
$y_2=\frac{2}{3}x_1-\frac{1}{3}x_2-\frac{2}{3}x_3,\ y_3=-\frac{1}{3}x_1-\frac{1}{3}x_2+\frac{4}{3}x_3.$

(b) $v_{y_1y_1}+v_{y_2y_2}-v_{y_3y_3}=0,\ y_1=\frac{1}{5}\sqrt{3}x_1+\frac{1}{5}\sqrt{3}x_2-\frac{1}{5}\sqrt{3}x_3,$
$y_2=\frac{3}{5}x_1-\frac{2}{5}x_2-\frac{3}{5}x_3,\ y_3=-\frac{1}{5}x_1-\frac{1}{5}x_2+\frac{6}{5}x_3.$

(c) $v_{y_1y_1}+v_{y_2y_2}-v_{y_3y_3}-v_{y_4y_4}=0,$
$y_1=\sqrt{2}x_1-\sqrt{2}x_2-\sqrt{2}x_3-\sqrt{2}x_4,$
$y_2=-x_1+2x_2+2x_3+2x_4,$
$y_3=-x_1+x_2,\ y_4=-x_1+x_2+x_3.$

(d) $v_{y_1y_1}+v_{y_2y_2}=0,\ y_1=\frac{1}{3}\sqrt{3}x_1+\frac{1}{3}\sqrt{3}x_2,\ y_2=-x_2,$
$y_3=-\frac{1}{3}x_1-\frac{1}{3}x_2+x_3.$

2. Substitute $u=wv$ in the equation and choose w such that $2a_iw_{x_i}+b_iw=0.$

Chapter 3

Section 3.2

2. $u(x,y) = \frac{1}{2}\left(\sin^3(x+t) + \sin^3(x-t)\right)$.

Section 3.3

1. $u\left(\frac{1}{2},1\right) = \frac{1}{8}, \quad u\left(\frac{3}{4},\frac{1}{2}\right) = \frac{3}{32}$.

Section 3.4

1. (a) $u(x,y) = \frac{1}{4}\left(e^{x+t} - e^{x-t}(1+2t)\right)$.

(b) $u(x,y) = xt + \cos x \cos t + \sin x\,(1-\cos t)$.

(c) $u(x,y) = \frac{1}{6}\left(3x^2t^2 + \frac{1}{2}t^4\right) + \cos x \cos t$.

Chapter 4

Section 4.1

3. Hint. (c) The result follows from (b) if $H(t) > 0$ for $t \in [t_1, t_2]$ because

$$\frac{d^2}{dt^2}\ln H(t) = \frac{H''H - H'^2}{H^2} \geq 0.$$

If $H(t) \geq 0$, replacing $H(t)$ by $H_\varepsilon(t) = H(t) + \varepsilon$ then for some $\varepsilon \in (0,1)$, one can prove the inequality for H_ε.
Then let $\varepsilon \to 0$.

Section 4.2

1. (a) $u(x,t) = \frac{1}{\sqrt{1+4t}}e^{-\frac{x^2}{1+4t}}$,

(b) $u(x,t) = \frac{e^t}{2}\left(e^{-x}\left(1+\operatorname{erf}\left(\frac{x}{2\sqrt{t}} - \sqrt{t}\right)\right) + e^x\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right)\right)\right)$,

(c) $u(x,t) = \frac{1}{2}\left(6 - 2\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)$.

3. Hint. Make the change of dependent variable $u(x,t) = e^{-bt}v(x,t)$.

4. Hint. Make the change of independent variable $y = x - vt$.

Section 4.4

1. Hint. Consider the function $v(x,t) = u(x,t) - xh(t)$ and use the even extension of the source function.

Chapter 5

Section 5.2

1. The eigenvalues of A are $\lambda_1 = 8$ and $\lambda_{2,3} = -1$. The solution is

$$u(x,t) = \begin{bmatrix} 2f_1(x-8t) + f_2(x+t) \\ f_1(x-8t) - 2f_2(x+t) - 2f_3(x+t) \\ 2f_1(x-8t) + f_3(x+t) \end{bmatrix}$$

Section 5.4

2. Use the characteristic method.

Section 5.5

2. (a) The charactersistics are

$$c_{x_0} : \begin{cases} x_0 + t\,(\alpha a + \beta) & x_0 \le a, \\ x_0 + t\,(\alpha x_0 + \beta) & a \le x_0 \le b, \\ x_0 + t\,(\alpha b + \beta) & x_0 \ge b. \end{cases}$$

The solution is

$$u\,(x,t) = \begin{cases} \alpha a + \beta & x \le a + t\,(\alpha a + \beta)\,, \\ \dfrac{\alpha x + \beta}{1 + \alpha t} & a + t\,(\alpha a + \beta) \le x \le b + t\,(\alpha b + \beta)\,, \\ \alpha b + \beta & x \ge b + t\,(\alpha b + \beta)\,, \end{cases}$$

3. (a) The envelope is

$$x = \begin{cases} \frac{1}{2}\arcsin\frac{1}{t} + \frac{1}{2}\left(1 + \sqrt{1 - \frac{1}{t^2}}\right) \\ \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{1}{t} + \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{t^2}}\right) \end{cases}, \ t \ge 1.$$

Chapter 6

Section 6.2

2. Hint. Use that $div(\vec{u} \times \vec{v}) = \vec{v} \cdot rot\vec{u} - \vec{u} \cdot rot\vec{v}$.

Section 6.4

1. (a) $G(Q,P) = \frac{1}{2\pi}\left(\ln\frac{1}{|Q-P|} - \ln\frac{1}{|Q-P*|}\right)$, where $Q(\xi,\eta)$, $P(x,y)$, $P^*(x,-y)$.

(b) Show that

$$\frac{\partial G(Q,P)}{\partial \eta} = -\frac{\partial G}{\partial \eta}\bigg|_{\eta=0} = \frac{2y}{(x-\xi|^2+y^2}.$$

2. (a) $G(Q,P) = \frac{1}{4\pi}\left(\frac{1}{|Q-P|} - \frac{R}{|P||Q-P*|}\right)$, $|P|.|P^*| = R^2$.

(b) Use that

$$-\frac{\partial}{\partial n_Q}G(Q,P)\bigg|_{|Q|=R} = \frac{\partial}{\partial |Q|}G(Q,P) = \frac{|P|^2-R^2}{4\pi R|Q-P|^3}.$$

(c) To show the boundary condition apply the Green's second identity to $G(Q,P)$ and $\frac{R}{|Q|}$ in the region $\{P : R \le |Q| \le T,\ |Q-P| \ge \varepsilon\}$.
Letting $T \to \infty$ and $\varepsilon \to 0$ obtain that

$$\frac{R}{|P|} = \frac{|P|^2-R^2}{4\pi R}\iint\limits_{S_R}\frac{dS_Q}{|Q-P|^3} \text{ if } |P| > R.$$

Then

$$\lim_{P\to Q}\frac{|P|^2-R^2}{4\pi R}\iint\limits_{S_R}\frac{dS_Q}{|Q-P|^3} = 1 \text{ if } |Q| = R$$

and $\lim\limits_{P\to Q} u(P) = \varphi(Q)$, follows as in Theorem 6.8.

Section 6.5

3. (a) Use the mean value property of harmonic functions

(b) Show that the first partial derivatives are equal to zero.

Chapter 7

Section 7.2

1. (a)
$e^{ax} = \frac{e^{a\pi}-e^{-a\pi}}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2}(a\cos nx - n\sin nx)\right), \quad -\pi < x < \pi.$

(b) $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x, \quad -\pi \le x \le \pi.$

(c)

$$f(x) = \frac{e^{\pi}-1}{2\pi} + \frac{e^{\pi}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{1+n^2}(\cos nx - n\sin nx).$$

(d)

$$g(x) = \frac{1}{2}\sin x - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2-1}\sin 2nx, \quad 0 < x \le \pi.$$

2. (a) $\mathrm{Si}\,(x) = \frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n}\left(\mathrm{Si}\,(n+1)\,\pi - \mathrm{Si}\,(n-1)\,\pi + 2\,(-1)^{n-1}\,\mathrm{Si}\,\pi\right),$
$-\pi < x < \pi.$

3. Hint. (a) Use that

$$\ln\left(1+e^{ix}\right) = \ln 2\cos\frac{x}{2} + i\frac{x}{2},$$

(b) Use that $\dfrac{1}{n^2-1} = \dfrac{1}{2}\left(\dfrac{1}{n-1} - \dfrac{1}{n+1}\right)$
and results of (a).

4. (a) Equate the coefficients of x^n in both sides of the identity

$$x = \left(\sum_{n=0}^{\infty}\frac{1}{n!}B_n x^n\right)\left(\sum_{n=1}^{\infty}\frac{1}{n!}x^n\right).$$

$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}.$

(c) Evaluate the coefficient to t^n in the product
$\left(\sum\limits_{k=0}^{\infty}\frac{1}{k!}B_k(s)x^k\right)\left(\sum\limits_{l=0}^{\infty}\frac{1}{l!}B_l(t)x^l\right)$ and use that

$$\frac{x^2 e^{(s+t)x}}{(e^x-1)^2} = -x^2 \frac{d}{dx}\left(\frac{e^{(s+t)x}}{e^x-1}\right) + (s+t-1)\,\frac{x^2 e^{(s+t)x}}{e^x-1}.$$

5. $y(x) = \frac{1}{x^2}(C_1 J_2(x) + C_2 Y_2(x))$.

6. Hint. Use the indentity $xJ_p'(x) - pJ_p(x) = -xJ_{p+1}(x)$.

Section 7.5

1. (a) $u(x,t) = e^{-0.215\pi^2 t}\sin\frac{\pi x}{2}$

(b) $u(x,t) = 20\pi\sum_{n=1}^{\infty}\frac{n}{n^2\pi^2-4}(1-(-1)^n\cos 2)e^{-0.215n^2\pi^2 t}\sin\frac{n\pi x}{2}.$

(c)
$u(x,t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\left(\cos\frac{n\pi}{2} - (-1)^n\right)\frac{1}{n}e^{-0.215n^2\pi^2 t}\sin\frac{n\pi x}{2}.$

(d)
$u(x,t) = 10 + \frac{80}{\pi^2}\sum_{n=1}^{\infty}\left(2\cos\frac{n\pi}{2} - 1 - (-1)^n\right)\frac{1}{n^2}e^{-0.215n^2\pi^2 t}\cos\frac{n\pi x}{2}.$

2. (a)
$u(x,t) = T_1 + (T_2 - T_1)x$
$+\frac{2}{\pi}\sum_{n=1}^{\infty}(T_0 - T_1 + (-1)^n(T_2 - T_0))\,\frac{e^{-kn^2\pi^2 t}}{n}\sin n\pi x$

(b)
$u(x,t) = 2\pi\sum_{n=1}^{\infty}\frac{(1-(-1)^n)n}{n^2\pi^2-1}e^{-(2+n^2\pi^2)t}\sin n\pi x.$

3. (a)
$u(x,t) = \frac{4}{\pi}\cos\frac{\pi t}{3}\sin\frac{\pi x}{3} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n-1}{n^2-1}\cos\frac{n\pi t}{3}\sin\frac{n\pi x}{3}$

(b) $u(x,t) = t$

(c) $u(x,t) = \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin 2n\pi/3}{n^2}\sin\frac{n\pi x}{3}\cos\frac{n\pi t}{3}$

(d) $u(x,t) = \left(-\frac{1}{2\pi}\cos\pi t + \frac{4}{\pi^2}\sin\pi t\right)\sin\pi x$
$+\frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n n}{n^2-1}\cos n\pi t + \frac{1-(-1)^n}{\pi n^2}\sin n\pi t\right)\sin n\pi x.$

5. (a)
$u(x,t) = \frac{1}{36}(1 - \cos 6t)\sin 3x + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} \sin 2nt \sin nx$

(b)
$u(x,t) = 1 + \frac{1}{2\pi^2}(1 - \cos 2\pi t)\sin \pi x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n} \cos 2n\pi t \sin n\pi x.$

8. (a) $u(x,t) = \dfrac{\cosh 3x}{3\sinh 3\pi}\cos 3y$

(b) $u(x,t) = \dfrac{3\cosh\frac{3y}{2} - 2\sinh\frac{3y}{2}}{3\cosh\frac{3\pi}{2} - 2\sinh\frac{3\pi}{2}} \sin\dfrac{3x}{2}$

9. (b)
$u(\rho,\theta) = \frac{3}{8} - \frac{3}{4}\rho\sin\theta + \frac{\rho^2}{2}\cos 2\theta + \frac{\rho^3}{4}\sin 3\theta + \frac{\rho^4}{8}\cos 4\theta.$

Chapter 8

Section 8.1

1. (a) $u(x,y,z,t) = x^2yz + 2tyz,$

(b) $u(x,y,z,t) = y(xz - 2t)(x - z).$

3. $u(x,y,t)$
$= \int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} (G_2(x-\xi, y-\eta, t) - G_2(x+\xi, y-\eta, t))\,\phi(\xi,\eta)\,d\xi d\eta.$

4. $u(x,y,z,t)$
$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} (G_3(x-\xi, y-\eta, z-\zeta, t)$
$+ G_3(x+\xi, y-\eta, z-\zeta, t))\,\phi(\xi,\eta,\zeta)\,d\xi d\eta d\zeta.$

Section 8.2

2. $u(\rho,t) = T + qR\left(2\dfrac{a^2t}{R^2} - \dfrac{1}{4}\left(1 - 2\dfrac{\rho^2}{R^2}\right)\right)$

$-\sum\limits_{n=1}^{\infty} \dfrac{2e^{-(a\mu_n/R)^2 t}}{\mu_n^2 J_0(\mu_n)} J_0\left(\dfrac{\mu_n\rho}{R}\right)$,where μ_n are positive zeros of $J_1(\mu) = 0$.

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