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M.Sc. [Mathematics]

II - Semester

311 24

PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

In mathematics, a Partial Differential Equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. Differential equations allow us to model changing patterns in both physical and mathematical problems. Principally, a differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations, thus, play a prominent role in engineering, physics, economics and other disciplines. The term differential equation was coined by Leibniz in 1676 for a relationship between the two differentials dx and dy for the two variables x and y . These equations are specifically used whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated.

Partial differential equations are specifically used to formulate problems involving functions of several variables. A special case is Ordinary Differential Equations (ODEs), which deals with functions of a single variable and their derivatives. Fundamentally, the partial differential equations can be used to describe a wide variety of phenomena, such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. The ordinary differential equations often model one-dimensional dynamical systems, while the partial differential equations often model multidimensional systems. Both ordinary and partial differential equations are broadly classified as linear and nonlinear.

This book, *Partial Differential Equations*, is divided into four blocks that are further divided into fourteen units which will help you understand how to solve the ordinary differential equations in more than two variables, surfaces and curves, simultaneous differential equations of the first order and the first degree in three variables, methods of solutions to PDEs, methods of solution of $dx/P=dy/Q=dz/R$, Pfaffian differential forms and equations, partial differential equations of the first order, origins of first order partial differential equations, Cauchy's problem for first order equations, linear equations of the first order, surfaces orthogonal to a given system of surfaces, Cauchy's method of characteristics, compatible systems of first order equations, Charpit's method, Jacobi's method, partial differential equations of the second order, origin of second order equations, linear partial differential equations with constant coefficients, equations with variable coefficients, method of integral transforms, Laplace's equations, elementary solutions of Laplace's equation, boundary value problems, the wave equation, elementary solutions of the one dimensional wave equations, the diffusion equation and elementary solutions of the diffusion equation.

NOTES

Introduction

NOTES

The book follows the self-instruction mode or the SIM format wherein each unit begins with an ‘Introduction’ to the topic followed by an outline of the ‘Objectives’. The content is presented in a simple, organized and comprehensive form interspersed with ‘Check Your Progress’ questions and answers for better understanding of the topics covered. A list of ‘Key Words’ along with a ‘Summary’ and a set of ‘Self Assessment Questions and Exercises’ is provided at the end of the each unit for effective recapitulation.

BLOCK I
ODE IN MORE THAN TWO VARIABLES AND
PFAFFIAN DIFFERENTIAL EQUATIONS

NOTES

**UNIT 1 ORDINARY DIFFERENTIAL
EQUATIONS IN MORE
THAN TWO VARIABLES**

Structure

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1.0 INTRODUCTION

In mathematics, an Ordinary Differential Equation (ODE) is a differential equation containing one or more functions of one independent variable and the derivatives of those functions. The term ordinary is used in contrast with the term partial differential equation which may be with respect to more than one independent variable. Differential equations are essential for a mathematical description of nature—they lie at the core of many physical theories. For example, let us just mention Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schrodinger's equation for quantum mechanics, and Einstein's equation for the general theory of gravitation.

A solution of a differential equation is a function that satisfies the equation. The solutions of a homogeneous linear differential equation form a vector space. In the ordinary case, this vector space has a finite dimension, equal to the order of the equation. All solutions of a linear differential equation are found by adding to a particular solution any solution of the associated homogeneous equation.

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In this unit, you will be able to illustrates the concept of ordinary differential equations in more than two variables with the help of examples and problems. Surfaces and curves in three dimensions are also focused on in the end of this unit.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Know about the general form of a linear differential equation of n th order
- Solve problems based on ordinary differential equations in more than two variables
- Understand homogeneous and non-homogenous equations with constant coefficients
- Discuss surfaces and curves in three dimensions

1.2 ORDINARY DIFFERENTIAL EQUATIONS IN MORE THAN TWO VARIABLES

The general form of a linear differential equation of n th order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where P_1, P_2, \dots, P_n and Q are functions of x alone or constants.

The linear differential equation with constant coefficients are of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = Q \quad (1.1)$$

where P_1, P_2, \dots, P_n are constants and Q is a function of x .

The equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \quad (1.2)$$

is then called the *Reduced Equation* (R.E.) of the Equation (1.1)

If $y = y_1(x), y = y_2(x), \dots, y = y_n(x)$ are n -solutions of this reduced equation, then $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is also a solution of the reduced equation where c_1, c_2, \dots, c_n are arbitrary constants.

The solution $y = y_1(x), y = y_2(x), y = y_3(x), \dots, y = y_n(x)$ are said to be linearly independent if the *Wronskian* of the functions is not zero where the Wronskian of the functions y_1, y_2, \dots, y_n , denoted by $W(y_1, y_2, \dots, y_n)$, is defined by

NOTES

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & y_3 \dots y_n \\ y'_1 & y'_2 & y'_3 \dots y'_n \\ y''_1 & y''_2 & y''_3 \dots y''_n \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} \dots y_n^{(n-1)} \end{vmatrix}$$

Since the general solution of a differential equation of n th order contains n arbitrary constants, $u = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is its complete solution.

Let v be any solution of the differential Equation (3.31), then

$$\frac{d^n v}{dx^n} + P_1 \frac{d^{n-1} v}{dx^{n-1}} + P_2 \frac{d^{n-2} v}{dx^{n-2}} + \dots + P_{n-1} \frac{dv}{dx} + P_n v = Q \quad (1.3)$$

Since u is a solution of Equation (1.2), we get

$$\frac{d^n u}{dx^n} + P_1 \frac{d^{n-1} u}{dx^{n-1}} + P_2 \frac{d^{n-2} u}{dx^{n-2}} + \dots + P_{n-1} \frac{du}{dx} + P_n u = 0 \quad (1.4)$$

Now adding Equation (1.3) and (1.4), we get

$$\frac{d^n(u+v)}{dx^n} + P_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + P_2 \frac{d^{n-2}(u+v)}{dx^{n-2}} + \dots + P_{n-1} \frac{d(u+v)}{dx} + P_n(u+v) = Q$$

This shows that $y = u + v$ is the complete solution of the Equation (1.1).

Introducing the operators D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$ etc. The Equation (1.1) can be written in the form

$$\begin{aligned} D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \dots + P_{n-1} D y + P_n y &= Q \\ \text{or } (D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_{n-1} D + P_n) y &= Q \\ \text{or } F(D) y &= Q \text{ where } F(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_{n-1} D + P_n \end{aligned}$$

From the above discussions it is clear that the general solution of $F(D)y = Q$ consists of two parts:

- (i) The Complementary Function (C.F.) which is the complete primitive of the Reduced Equation (R.E.) and is of the form

$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ containing n arbitrary constants.

- (ii) The Particular Integral (P.I.) which is a solution of $F(D)y = Q$ containing no arbitrary constant.

Rules for Finding The Complementary Function

Let us consider the 2nd order linear differential equation

$$\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad (1.5)$$

Let $y = A e^{mx}$ be a trial solution of the Equation (1.5); then the Auxiliary Equation (A.E.) of Equation (1.5) is given by

$$m^2 + P_1 m + P_2 = 0 \quad (1.6)$$

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The Equation (1.6) has two roots $m = m_1, m = m_2$. We discuss the following cases:

(i) When $m_1 \neq m_2$, then the complementary function will be

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

(ii) When $m_1 = m_2$, then the complementary function will be

$$y = (c_1 + c_2 x) e^{m_1 x} \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

(iii) When the auxiliary Equation (3.36) has complex roots of the form $\alpha + i\beta$ and $\alpha - i\beta$, then the complementary function will be

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

Let us consider the equation of order n

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \quad (1.7)$$

Let $y = A e^{mx}$ be a trial solution of Equation (1.7), then the auxiliary equation is

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots + P_{n-1} m + P_n = 0 \quad (1.8)$$

Rule (1): If $m_1, m_2, m_3, \dots, m_n$ be n distinct real roots of (1.8), then the general solution will be

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

where $c_1, c_2, c_3, \dots, c_n$ are arbitrary constants.

Rule (2): If the two roots m_1 and m_2 of the auxiliary equation are equal and each equal to m , the corresponding part of the general solution will be $(c_1 + c_2 x) e^{mx}$ and if the three roots m_3, m_4, m_5 are equal to α the corresponding part of the solution is $(c_3 + c_4 x + c_5 x^2) e^{\alpha x}$ and others are distinct, the general solution will be

$$y = (c_1 + c_2 x) e^{mx} + (c_3 + c_4 x + c_5 x^2) e^{\alpha x} + c_6 e^{m_6 x} + \dots + c_n e^{m_n x}$$

Rule (3): If a pair of imaginary roots $\alpha \pm i\beta$ occur twice, the corresponding part of the general solution will be

$$e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$$

and the general solution will be

$$y = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$

where c_1, c_2, \dots, c_n are arbitrary constants and m_5, m_6, \dots, m_n are distinct real roots of Equation (1.8).

Rule (4): If the two roots (real) be m and $-m$, the corresponding part of the general solution will be $c_1 e^{mx} + c_2 e^{-mx}$

$$= c_1 (\cosh mx + \sinh mx) + c_2 (\cosh mx - \sinh mx)$$

$$= c'_1 \cosh mx + c'_2 \sinh mx \text{ where } c'_1 = c_1 + c_2, c'_2 = c_1 - c_2$$

and general solution will be

$$y = c'_1 \cosh mx + c'_2 \sinh mx + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

where $c'_1, c'_2, c'_3, \dots, c'_n$ are arbitrary constants and $m_3, m_4 \dots m_n$ are distinct real roots of Equation (1.8).

Rules for Finding Particular Integrals

Any particular solution of $F(D)y = f(x)$ is known as its Particular Integral (P.I.).
The P.I. of $F(D)y = f(x)$ is symbolically written as

$$\text{P.I.} = \frac{1}{F(D)} \{f(x)\} \text{ where } F(D) \text{ is the operator.}$$

The operator $\frac{1}{F(D)}$ is defined as that operator which, when operated on $f(x)$ gives a function $\phi(x)$ such that $F(D)\phi(x) = f(x)$

$$\text{i.e., } \frac{1}{F(D)} \{f(x)\} = \phi(x) (\text{= P.I.})$$

$$\therefore F(D) \left\{ \frac{1}{F(D)} f(x) \right\} = f(x) \quad \left[\because \frac{1}{F(D)} f(x) = \phi(x) \right]$$

Obviously $F(D)$ and $1/F(D)$ are inverse operators.

Case I: Let $F(D) = D$, then $\frac{1}{D} f(x) = \int f(x) dx$.

Proof: Let $y = \frac{1}{D} \{f(x)\}$, operating by D , we get $Dy = D \cdot \frac{1}{D} \{f(x)\}$ or $Dy = f(x)$ or

$$\frac{dy}{dx} = f(x) \text{ or } dy = f(x) dx$$

Integrating both sides with respect to x , we get

$$y = \int f(x) dx, \text{ since particular integrating does not contain any arbitrary constant.}$$

Case II: Let $F(D) = D - m$ where m is a constant, then

$$\frac{1}{D-m} \{f(x)\} = e^{mx} \int e^{-mx} f(x) dx.$$

Proof: Let $\frac{1}{D-m} \{f(x)\} = y$, then operating by $D - m$, we get

$$(D - m) \cdot \frac{1}{D-m} \{f(x)\} = (D - m) y$$

$$\text{or } f(x) = \frac{dy}{dx} - my$$

or $\frac{dy}{dx} - my = f(x)$ which is a first order linear differential equation and I.F. = $e^{\int -mdx} = e^{-mx}$.

Then multiplying above equation by e^{-mx} and integrating with respect to x , we get

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$y e^{-mx} = \int f(x)e^{-mx} dx$, since particular integral does not contain any arbitrary constant

or

$$y = e^{mx} \int f(x)e^{-mx} dx.$$

Note: If $\frac{1}{F(D)} = \frac{a_1}{D-m_1} + \frac{a_2}{D-m_2} + \dots + \frac{a_n}{D-m_n}$ where a_i and m_i ($i = 1, 2, \dots, n$) are constants, then

$$\begin{aligned} \frac{1}{F(D)} \{f(x)\} &= a_1 e^{m_1 x} \int f(x)e^{-m_1 x} dx + a_2 e^{m_2 x} \int f(x)e^{-m_2 x} dx + \\ &\dots + a_n e^{m_n x} \int f(x)e^{-m_n x} dx \\ &= \sum_{i=1}^n a_i e^{m_i x} \int f(x)e^{-m_i x} dx \end{aligned}$$

We now discuss methods of finding particular integrals for certain specific types of right hand functions

Type I: $f(D) y = e^{mx}$ where m is a constant.

$$\text{Then P.I.} = \frac{1}{F(D)} \{e^{mx}\} = \frac{e^{mx}}{F(m)} \text{ if } F(m) \neq 0$$

If $F(m) = 0$, then we replace D by $D + m$ in $F(D)$,

$$\text{P.I.} = \frac{1}{F(D)} \{e^{mx}\} = e^{mx} \cdot \frac{1}{F(D+m)} \{1\}$$

Example 1.1: $(D^3 - 2D^2 - 5D + 6)y = (e^{2x} + 3)^2 + e^{3x} \cosh x$.

Solution: The reduced equation is

$$(D^3 - 2D^2 - 5D + 6)y = 0 \quad \dots(1)$$

Let $y = Ae^{mx}$ be a trial solution of (1). Then the auxiliary equation is

$$m^3 - 2m^2 - 5m + 6 = 0 \text{ or } m^3 - m^2 - m^2 + m - 6m + 6 = 0$$

$$\text{or } m^2(m-1) - m(m-1) - 6(m-1) = 0$$

$$\text{or } (m-1)(m^2 - m - 6) = 0 \text{ or } (m-1)(m^2 - 3m + 2m - 6) = 0$$

$$\text{or } (m-1)(m-3)(m+2) = 0 \text{ or } m = 1, 3, -2$$

\therefore The complementary function is

$$y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x} \text{ where } c_1, c_2, c_3 \text{ are arbitrary constants.}$$

$$\text{Again } (e^{2x} + 3)^2 + e^{3x} \cosh x = e^{4x} + 6e^{2x} + 9 + e^{3x} \left(\frac{e^x + e^{-x}}{2} \right).$$

$$= e^{4x} + 6e^{2x} + 9e^{0.x} + \frac{e^{4x}}{2} + \frac{e^{2x}}{2}$$

$$= \frac{3}{2}e^{4x} + \frac{13}{2}e^{2x} + 9e^{0.x}$$

\therefore The particular integral is

$$\begin{aligned}
 y &= \frac{1}{D^3 - 2D^2 - 5D + 6} \left\{ \frac{3}{2} e^{4x} + \frac{13}{2} e^{2x} + 9e^{0.x} \right\} \\
 &= \frac{1}{(D-1)(D-3)(D+2)} \left\{ \frac{3}{2} e^{4x} + \frac{13}{2} e^{2x} + 9e^{0.x} \right\} \\
 &= \frac{3}{2} \frac{1}{(D-1)(D-3)(D+2)} e^{4x} + \frac{13}{2} \frac{1}{(D-1)(D+2)(D-3)} \{e^{2x}\} \\
 &\quad + 9 \frac{1}{(D-1)(D-3)(D+2)} e^{0.x} \\
 &= \frac{3}{2} \frac{e^{4x}}{(4-1)(4-3)(4+2)} + \frac{13}{2} \frac{e^{2x}}{(2-1)(2+2)(2-3)} \\
 &\quad + 9 \frac{e^{0.x}}{(0-1)(0-3)(0+2)} \\
 &= \frac{3}{2} \frac{e^{4x}}{3.1.6} + \frac{13}{2} \frac{e^{2x}}{1.4.(-1)} + 9 \frac{e^{0.x}}{(-1)(-3).2} \\
 &= \frac{e^{4x}}{12} - \frac{13}{8} e^{2x} + \frac{3}{2}.
 \end{aligned}$$

Hence the general solution is

$$\begin{aligned}
 y &= \text{C.F.} + \text{P.I.} \\
 &= c_1 e^x + c_2 e^{3x} + c_3 e^{-2x} + \frac{e^{4x}}{12} - \frac{13}{8} e^{2x} + \frac{3}{2}.
 \end{aligned}$$

Notes: 1. When $F(m) = 0$ and $F'(m) \neq 0$, P.I. = $\frac{1}{F(D)} \{e^{mx}\} = x \frac{1}{F'(D)} \{e^{mx}\}$

$$= \frac{x e^{mx}}{F'(m)}$$

2. When $F(m) = 0$, $F'(m) = 0$ and $F''(m) \neq 0$, then P.I. = $\frac{1}{F(D)} \{e^{mx}\}$
 $= x^2 \frac{1}{F''(D)} \{e^{mx}\} = \frac{x^2 e^{mx}}{F''(m)}$

and so on.

Type II: $f(x) = e^{mx} V$ where V is any function of x .

Here the particular integral (P.I.) of $F(D)y = f(x)$ is

$$\text{P.I.} = \frac{1}{F(D)} \{e^{mx} V\} = e^{mx} \frac{1}{F(D+m)} \{V\}.$$

Example 1.2: Solve $(D^2 - 5D + 6)y = x^2 e^{3x}$

Solution: The reduced equation is

$$(D^2 - 5D + 6)y = 0$$

(1)

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Let $y = Ae^{mx}$ be a trial solution of Equation (1) and then auxiliary equation is

$$m^2 - 5m + 6 = 0 \text{ or } m^2 - 3m - 2m + 6 = 0$$

$$\text{or } m(m-3) - 2(m-3) = 0 \text{ or } (m-3)(m-2) = 0$$

$$\therefore m = 2, 3$$

\therefore The complementary function is

$$y = c_1 e^{2x} + c_2 e^{3x} \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

The particular integral is

$$\begin{aligned} y &= \frac{1}{D^2 - 5D + 6} \{x^2 e^{3x}\} = \frac{e^{3x}}{(D+3)^2 - 5(D+3) + 6} \{x^2\} \\ &= e^{3x} \frac{1}{D^2 + 6D + 9 - 5D - 15 + 6} \{x^2\} = e^{3x} \frac{1}{D^2 + D} \{x^2\} \\ &= e^{3x} \frac{1}{D(1+D)} \{x^2\} = e^{3x} \frac{1}{D} (1+D)^{-1} \{x^2\} \\ &= \frac{e^{3x}}{D} (1 - D + D^2 - D^3 + D^4 - \dots) \{x^2\} \\ &= \frac{e^{3x}}{D} \{x^2 - 2x + 2\} = e^{3x} \left(\frac{x^3}{3} - x^2 + 2x \right) \end{aligned}$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{2x} + c_2 e^{3x} + e^{3x} \left(\frac{x^3}{3} - x^2 + 2x \right).$$

Recall: (i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

Type III: (a) $F(D)y = \sin ax$ or $\cos ax$ where $F(D) = \phi(D^2)$.

$$\text{Here P.I.} = \frac{1}{F(D)} \{\sin ax\} = \frac{1}{\phi(-a^2)} \sin ax \text{ (if } \phi(-a^2) \neq 0\text{)}$$

$$\text{or P.I.} = \frac{1}{F(D)} \{\cos ax\} = \frac{1}{\phi(-a^2)} \cos ax \text{ (if } \phi(-a^2) \neq 0\text{)}$$

[Note D^2 has been replaced by $-a^2$ but D has not been replaced by $-a$.]

(b) $F(D)y = \sin ax$ or $\cos ax$ and $F(D) = \phi(D^2, D)$

$$\begin{aligned} \text{Here P.I.} &= \frac{1}{F(D)} \{\sin ax\} = \frac{1}{\phi(D^2, D)} \{\sin ax\} = \frac{1}{\phi(-a^2, D)} \{\sin ax\} \\ &\quad \text{if } \phi(-a^2, D) \neq 0 \end{aligned}$$

$$\text{or } y = \frac{1}{F(D)} \{\cos ax\} = \frac{1}{\phi(D^2, D)} \{\cos ax\} = \frac{1}{\phi(-a^2, D)} \{\cos ax\}$$

if $\phi(-a^2, D) \neq 0$

$$(c) F(D)y = \sin ax \text{ or } \cos ax \text{ and } F(D) = \frac{\psi(D)}{\phi(D^2)}$$

Here P.I. = $\frac{1}{F(D)}\{\sin ax\} = \frac{\psi(D)}{\phi(D^2)}\{\sin ax\} = \frac{\psi(D)}{\phi(-a^2)}\{\sin ax\}$ if $\phi(-a^2) \neq 0$

$$\text{or } y = \frac{1}{F(D)}\{\cos ax\} = \frac{\psi(D)}{\phi(D^2)}\{\cos ax\}$$

$$= \frac{\psi(D)}{\phi(-a^2)}\{\cos ax\} \text{ if } \phi(-a^2) \neq 0$$

(d) $F(D)y = \sin ax \text{ or } \cos ax, F(D) = \phi(D^2)$ but $\phi(-a^2) = 0$.

Here P.I. = $\frac{1}{F(D)}\{\sin ax \text{ or } \cos ax\} = x \frac{1}{F'(D)}\{\sin ax \text{ or } \cos ax\}$

Alternatively, $\sin ax$ and $\cos ax$ can be written in the form $\sin ax = \frac{e^{ixa} - e^{-ixa}}{2i}$

and $\cos ax = \frac{e^{ixa} + e^{-ixa}}{2}$, then find P.I. by the method of Type I.

Example 1.3: Solve $(D^4 + 2D^2 + 1)y = \cos x$.

Solution: The reduced equation is $(D^4 + 2D^2 + 1)y = 0$

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equation is

$$m^4 + 2m^2 + 1 = 0 \text{ or } [(m^2 + 1)]^2 = 0 \text{ or } m = \pm i, \pm i$$

\therefore C.F. = $(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x$ where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\therefore \text{P.I.} = \frac{1}{D^4 + 2D^2 + 1}\{\cos x\}$$

$$= x \frac{1}{4D^3 + 4D}\{\cos x\}$$

$$[\because \phi(D^2) = D^4 + 2D^2 + 1]$$

$$\phi(-1^2) = 1 - 2 + 1 = 0, \text{ then } \frac{1}{F(D)}\{f(x)\} = x \frac{1}{F'(D)}\{f(x)\}$$

$$= x \frac{1}{4D^3 + D}\{\cos x\} = \frac{x}{4} \cdot \frac{x}{3D^2 + 1}\{\cos x\}$$

$$= \frac{x^2}{4} \frac{1}{3D^2 + 1}\{\cos x\} = \frac{x^2}{4} \cdot \frac{\cos x}{-3 + 1} = -\frac{x^2}{8} \cos x$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

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$$= (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x - \frac{x^2}{8} \cos x.$$

Example 1.4: Solve $(D^2 - 4)y = \sin 2x$.

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Solution: The reduced equation is

$$(D^2 - 4)y = 0$$

Let $y = Ae^{mx}$ be a trial solution and then auxiliary equation is

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

The complementary function is

$$y = c_1 e^{2x} + c_2 e^{-2x} \text{ where } c_1, c_2 \text{ are arbitrary constants.}$$

The particular integral is

$$\begin{aligned} y &= \frac{1}{D^2 - 4} \{\sin 2x\} = \frac{1}{-2^2 - 4} \sin 2x \quad [\text{Replace } D^2 \text{ by } -2^2] \\ &= -\frac{1}{8} \sin 2x \end{aligned}$$

The general solution is $y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} \sin 2x$.

Example 1.5: Solve $(3D^2 + 2D - 8)y = 5 \cos x$.

Solution: The reduced equation is

$$(3D^2 + 2D - 8)y = 0$$

Let $y = Ae^{mx}$ be a trial solution and then the auxiliary equation is

$$3m^2 + 2m - 8 = 0 \text{ or } 3m^2 + 6m - 4m - 8 = 0$$

$$\text{or } 3m(m+2) - 4(m+2) = 0 \text{ or } (m+2)(3m-4) = 0$$

$$\text{or } m = -2, m = \frac{4}{3}$$

\therefore The complementary function is

$$y = c_1 e^{-2x} + c_2 e^{\frac{4}{3}x} \text{ when } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

The particular integral is

$$\begin{aligned} y &= \frac{1}{3D^2 + 2D - 8} \{5 \cos x\} = 5 \frac{1}{(3D-4)(D+2)} \{\cos x\} \\ &= 5 \frac{(3D+4)(D-2)}{(9D^2-16)(D^2-4)} \{\cos x\} = 5 \frac{(3D+4)(D-2)}{[9(-1^2)-16][-1^2-4]} \{\cos x\} \\ &\quad [D^2 \text{ is replaced by } -1^2 \text{ in the denominator}] \left[\frac{\Psi(D)}{\phi(D^2)} \text{ form} \right] \\ &= \frac{5}{(-25)(-5)} [3D^2 - 6D + 4D - 8] \{\cos x\} = \frac{1}{25} [3D^2 - 2D - 8] \cos x \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{25} \left(3 \frac{d^2}{dx^2}(\cos x) - 2 \frac{d}{dx}(\cos x) - 8 \cos x \right) \\
 &= \frac{1}{25} [-3 \cos x + 2 \sin x - 8 \cos x] = \frac{1}{25} (2 \sin x - 11 \cos x)
 \end{aligned}$$

The general solution is

$$\begin{aligned}
 y &= \text{C.F.} + \text{P.I.} \\
 &= c_1 e^{-2x} + c_2 e^{4/3x} + \frac{1}{25} (2 \sin x - 11 \cos x).
 \end{aligned}$$

Type IV: $F(D)y = x^n$, n is a positive integer.

$$\text{Here } \text{P.I.} = \frac{1}{F(D)} \{x^n\} = [F(D)]^{-1} \{x^n\}$$

In this case, $[F(D)]^{-1}$ is expanded in a binomial series in ascending powers of D upto D^n and then operate on x^n with each term of the expansion. The terms in the expansion beyond D^n need not be considered, since the result of their operation on x^n will be zero.

Example 1.6: Solve $D^2(D^2 + D + 1)y = x^2$.

Solution: The reduced equation is

$$D^2(D^2 + D + 1)y = 0 \quad (1)$$

Let $y = Ae^{mx}$ be a trial solution of Equation (2) and then the auxiliary equation is

$$m^2(m^2 + m + 1) = 0$$

$$\therefore m = 0, 0 \text{ and } m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

\therefore The complementary function is

$$\begin{aligned}
 y &= (c_1 + c_2 x) e^{0 \cdot x} + e^{-\frac{1}{2}x} \left(c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x \right) \\
 &= c_1 + c_2 x + e^{-\frac{1}{2}x} \left(c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x \right)
 \end{aligned}$$

where c_1, c_2, c_3, c_4 are the arbitrary constant.

The particular integral is

$$\begin{aligned}
 y &= \frac{1}{D^2(D^2 + D + 1)} \{x^2\} = \frac{1}{D^2} (1 + D + D^2)^{-1} \{x^2\} \\
 &= \frac{1}{D^2} \{1 - (D + D^2) + (D + D^2)^2 - (D + D^2)^3 + \dots\} \{x^2\} \\
 &= \frac{1}{D^2} \{1 - (D + D^2) + (D^2 + 2D^3 + D^4) - (D + D^2)^3 + \dots\} \{x^2\}
 \end{aligned}$$

NOTES

$$\begin{aligned}
 &= \frac{1}{D^2} \{x^2 - (2x + 2) + (2) + 0\} \\
 &= \frac{1}{D^2} \{x^2 - 2x\} = \frac{1}{D} \left\{ \frac{x^3}{3} - x^2 \right\} = \frac{x^4}{12} - \frac{x^3}{3}
 \end{aligned}$$

The general solution is $y = C.F. + P.I.$

$$= c_1 + c_2 x + e^{-x/2} \left(c_3 \cos \frac{\sqrt{3}}{2} x + c_4 \sin \frac{\sqrt{3}}{2} x \right) + \frac{x^4}{12} - \frac{x^3}{3}.$$

Example 1.7: Solve $(D^2 + 4)y = x \sin^2 x$.

Solution: The reduced equation is

$$(D^2 + 4)y = 0$$

The trial solution $y = A e^{mx}$ gives the auxiliary equation as

$$m^2 + 4 = 0, m = \pm 2i$$

The complementary function is $y = c_1 \cos 2x + c_2 \sin 2x$

The particular integral is $y = \frac{1}{D^2 + 4} \{x \sin^2 x\}$

$$\begin{aligned}
 &= \frac{1}{D^2 + 4} \left\{ \frac{x}{2} (1 - \cos 2x) \right\} = \frac{1}{D^2 + 4} \left\{ \frac{x}{2} - \frac{x}{2} \cos 2x \right\} \\
 &= \frac{1}{D^2 + 4} \left\{ \frac{x}{2} \right\} - \frac{1}{D^2 + 4} \left\{ \frac{x}{2} \frac{(e^{2ix} + e^{-2ix})}{2} \right\} \\
 &= \frac{1}{4} \left(1 + \frac{D^2}{4} \right)^{-1} \left\{ \frac{x}{2} \right\} - \frac{1}{4} \frac{e^{2ix}}{(D + 2i)^2 + 4} \{x\} - \frac{e^{-2ix}}{4(D - 2i)^2 + 4} \{x\} \\
 &= \frac{1}{4} \frac{x}{2} - \frac{e^{2ix}}{4} \frac{1}{D^2 + 4Di - 4 + 4} \{x\} - \frac{e^{-2ix}}{4} \frac{1}{D^2 - 4Di - 4 + 4} \{x\} \\
 &= \frac{x}{8} - \frac{e^{2ix}}{4} \frac{1}{4Di \left(1 + \frac{D}{4i} \right)} \{x\} - \frac{e^{-2ix}}{4 \cdot (-4Di) \left(1 - \frac{D}{4i} \right)} \{x\} \\
 &= \frac{x}{8} - \frac{e^{2ix}}{4} \cdot \frac{1}{4Di} \left(1 + \frac{D}{4i} \right)^{-1} \{x\} - \frac{e^{-2xi}}{4(-4Di)} \left(1 - \frac{D}{4i} \right)^{-1} \{x\} \\
 &= \frac{x}{8} - \frac{e^{2ix}}{4} \cdot \frac{1}{4Di} \left(1 - \frac{D}{4i} + \frac{D^2}{-16} \dots \right) \{x\} - \frac{e^{-2xi}}{4(-4Di)} \left(1 + \frac{D}{4i} + \dots \right) \{x\} \\
 &= \frac{x}{8} - \frac{e^{2ix}}{4} \cdot \frac{1}{4Di} \left(x - \frac{1}{4i} \right) + \frac{e^{-2xi}}{4 \cdot 4Di} \left(x + \frac{1}{4i} \right) \\
 &= \frac{x}{8} - \frac{e^{2ix}}{2 \cdot 8i} \left(\frac{x^2}{2} - \frac{x}{4i} \right) + \frac{e^{-2xi}}{2 \cdot 8i} \left(\frac{x^2}{2} + \frac{x}{4i} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{x}{8} - \frac{x^2}{2.8} \left(\frac{e^{2ix} - e^{-2xi}}{2i} \right) + \frac{x}{2.16.i^2} \left(\frac{e^{2ix} + e^{-2xi}}{2} \right) \\
 &= \frac{x}{8} - \frac{x^2}{2.8} \sin 2x - \frac{x}{2.16} \cos 2x \\
 &= \frac{x}{8} - \frac{x^2}{16} \sin 2x - \frac{x}{32} \cos 2x
 \end{aligned}$$

Hence the general solution is $y = C.F. + P.I.$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{8} - \frac{x^2}{16} \sin 2x - \frac{x}{32} \cos 2x.$$

Example 1.8: Solve $(D^4 + D^3 - 3D^2 - 5D - 2)y = 3xe^{-x}$.

Solution: The reduced equation is

$$(D^4 + D^3 - 3D^2 - 5D - 2)y = 0 \quad (1)$$

The trial solution $y = Ae^{mx}$ gives the auxiliary equation as

$$\begin{aligned}
 &m^4 + m^3 - 3m^2 - 5m - 2 = 0 \\
 \text{or } &m^4 + m^3 - 3m^2 - 3m - 2m - 2 = 0 \\
 \text{or } &m^3(m+1) - 3m(m+1) - 2(m+1) \\
 \text{or } &(m+1)(m^3 - 3m - 2) = 0 \text{ or } (m+1)\{m^3 + m^2 - m^2 - m - 2m - 2\} = 0 \\
 \text{or } &(m+1)\{m^2(m+1) - m(m+1) - 2(m+1)\} = 0 \\
 \text{or } &(m+1)(m+1)(m^2 - m - 2) = 0 \\
 \text{or } &(m+1)^2(m^2 - 2m + m - 2) = 0 \\
 \text{or } &(m+1)^2(m+1)(m-2) = 0 \\
 \therefore &m = -1, -1, -1, 2
 \end{aligned}$$

The complementary function is $y = (c_1 + c_2x + c_3x^2)e^{-x} + c_4e^{2x}$.

The particular integral is

$$\begin{aligned}
 y &= \frac{1}{(D+1)^3(D-2)} \{3e^{-x}x\} \\
 &= 3e^{-x} \frac{1}{(D-1+1)^3(D-3)} \{x\} = 3e^{-x} \frac{1}{D^3(-3)(1-D/3)} \{x\} \\
 &= -e^{-x} \frac{1}{D^3} \left(1 - \frac{D}{3} \right)^{-1} \{x\} = -e^{-x} \frac{1}{D^3} \left(1 + \frac{D}{3} + \frac{D^2}{9} + \dots \right) \{x\} \\
 &= -e^{-x} \frac{1}{D^3} \left(x + \frac{1}{3} \right) = -e^{-x} \frac{1}{D^2} \left(\frac{x^2}{2} + \frac{x}{3} \right) = -e^{-x} \frac{1}{D} \left(\frac{x^3}{6} + \frac{x^2}{6} \right) \\
 &= -e^{-x} \left(\frac{x^4}{24} + \frac{x^3}{18} \right)
 \end{aligned}$$

The general solution is $y = C.F. + P.I.$

NOTES

$$= (c_1 + c_2 x + c_3 x^2) + c_4 e^{2x} - e^{-x} \left(\frac{x^4}{24} + \frac{x^3}{18} \right).$$

Type V: (a) $F(D)y = xV$ where V is a function of x .

NOTES

$$\text{Here P.I.} = \frac{1}{F(D)} \{xV\} = \left\{ x - \frac{1}{F(D)} F'(D) \right\} \frac{1}{F(D)} \{V\}.$$

Example 1.9: Solve $(D^2 + 9)y = x \sin x$.

Solution: The reduced equation is $(D^2 + 9)y = 0$ (1)

The trial solution $y = Ae^{mx}$ gives the auxiliary equation as

$$m^2 + 9 = 0 \text{ or } m = \pm 3i$$

\therefore C.F. = $c_1 \cos 3x + c_2 \sin 3x$ where c_1 and c_2 are arbitrary constants.

and P.I. = $\frac{1}{F(D)} \{x \sin x\}$ where $F(D) = D^2 + 9$

$$\begin{aligned} &= \left\{ x - \frac{1}{F(D)} F'(D) \right\} \frac{1}{F(D)} \{\sin x\} \\ &= \left\{ x - \frac{2D}{D^2 + 9} \right\} \frac{1}{D^2 + 9} \{\sin x\} \\ &= \left\{ x - \frac{2D}{D^2 + 9} \right\} \frac{\sin x}{-1 + 9} = \left\{ x - \frac{2D}{D^2 + 9} \right\} \left\{ \frac{\sin x}{8} \right\} \\ &= \frac{x \sin x}{8} - \frac{1}{4} \frac{1}{-1 + 9} D\{\sin x\} = \frac{x \sin x}{8} - \frac{1}{32} \cos x \end{aligned}$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 3x + c_2 \sin 3x + \frac{x \sin x}{8} - \frac{1}{32} \cos x$$

(b) $F(D)y = x^n V$ where V is any function of x .

$$\text{Here P.I.} = \frac{1}{F(D)} \{f(x)\} = \frac{1}{F(D)} \{x^n V\} = \left\{ x - \frac{F'(D)}{F(D)} \right\}^n \frac{1}{F(D)} \{V\}$$

Example 1.10: Solve $(D^2 - 1)y = x^2 \sin x$

Solution: The reduced equation is $(D^2 - 1)y = 0$ (2)

Let $y = Ae^{mx}$ be a trial solution. Then the auxiliary equation is

$$m^2 - 1 = 0 \text{ or } m = \pm 1$$

\therefore C.F. = $c_1 e^x + c_2 e^{-x}$ where c_1 and c_2 are arbitrary constants.

\therefore P.I. = $\frac{1}{F(D)} \{x^2 \sin x\}$ where $F(D) = D^2 - 1$

$$= \left\{ x - \frac{F'(D)}{F(D)} \right\}^2 \frac{1}{F(D)} \{\sin x\} = \left\{ x - \frac{1}{D^2 - 1} 2D \right\}^2 \frac{1}{D^2 - 1} \{\sin x\}$$

NOTES

$$\begin{aligned}
 &= \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \left\{ \frac{1}{-1^2 - 1} \sin x \right\} \\
 &= \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \{-1/2 \sin x\} \\
 &= \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \left\{ -\frac{x}{2} \sin x + \frac{1}{D^2 - 1} \right\} \{\cos x\} \\
 &= \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \left\{ -\frac{x}{2} \sin x - \frac{1}{2} \cos x \right\} \\
 &= -\frac{x^2}{2} \sin x - \frac{x}{2} \cos x + \frac{1}{D^2 - 1} \{D(x \sin x + \cos x)\} \\
 &= -\frac{x^2}{2} \sin x - \frac{x}{2} \cos x + \frac{1}{D^2 - 1} \{\sin x + x \cos x - \sin x\} \\
 &= -\frac{x^2}{2} \sin x - \frac{x}{2} \cos x + \frac{1}{D^2 - 1} \{x \cos x\}
 \end{aligned}$$

$$\begin{aligned}
 \text{Again } \frac{1}{D^2 - 1} \{x \cos x\} &= \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \frac{1}{D^2 - 1} \{\cos x\} \\
 &= \left\{ x - \frac{1}{D^2 - 1} 2D \right\} \left\{ \frac{1}{-1^2 - 1} \cos x \right\} \\
 &= -\frac{1}{2} x \cos x + \frac{1}{D^2 - 1} \{-\sin x\} \\
 &= -\frac{1}{2} x \cos x - \frac{\sin x}{-1^2 - 1} = -\frac{1}{2} x \cos x + \frac{1}{2} \sin x
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{P.I.} &= -\frac{x^2}{2} \sin x - \frac{x}{2} \cos x - \frac{x}{2} \cos x + \frac{1}{2} \sin x \\
 &= -\frac{1}{2} x^2 \sin x - x \cos x + \frac{1}{2} \sin x
 \end{aligned}$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{-x} - \frac{1}{2} x^2 \sin x - x \cos x + \frac{1}{2} \sin x.$$

1.2.1 Classification of Linear Partial Differential Equations of Second Order

Consider the following linear partial differential equation of the second order in two independent variables,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

Where A, B, C, D, E, F and G are functions of x and y .

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This equation when converted to quasi-linear partial differential equation takes the form,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

These equations are said to be of:

1. Elliptic type if $B^2 - 4AC < 0$
2. Parabolic type if $B^2 - 4AC = 0$
3. Hyperbolic type if $B^2 - 4AC > 0$

Let us consider some examples to understand this:

- $\frac{\partial^2 u}{\partial x^2} - 2x \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial u}{\partial y} = 0$

$$\Rightarrow u_{xx} - 2xu_{xy} + x^2u_{yy} - 2u_y = 0$$

Comparing it with the general equation we find that,

$$A = 1, B = -2x, C = x^2$$

Therefore

$$B^2 - 4AC = (-2x)^2 - 4x^2 = 0, \forall x \text{ and } y \neq 0$$

So the equation is parabolic at all points.

- $y^2u_{xx} + x^2u_{yy} = 0$

Comparing it with the general equation we get,

$$A = y^2, B = 0, C = x^2$$

Therefore

$$B^2 - 4AC = 0 - 4x^2y^2 < 0, \forall x \text{ and } y \neq 0$$

So the equation is elliptic at all points.

- $x^2u_{xx} - y^2u_{yy} = 0$

Comparing it with the general equation we find that,

$$A = x^2, B = 0, C = -y^2$$

Therefore

$$B^2 - 4AC = 0 - 4x^2y^2 > 0, \forall x \text{ and } y \neq 0$$

So the equation is hyperbolic at all points.

Following three are the most commonly used partial differential equations of the second order:

1. Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is equation is of elliptic type.

2. One-dimensional heat flow equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This equation is of parabolic type.

3. One-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This is a hyperbolic equation.

1.2.2 Homogeneous and Non-homogeneous Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients

$$\text{Let } f(D, D')z = V(x, y) \quad (1.9)$$

Then if

$$f(D, D') = A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'_2 + \cdots + A_n D'^n \quad (1.10)$$

where A_1, A_2, \dots, A_n are constants.

Then Equation (1.9) is known as Homogeneous equation and takes the form

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n)z = V(x, y) \quad (1.11)$$

Complementary Function

Consider the equation,

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n)z = 0 \quad (1.12)$$

Let

$$z = \phi(y + mx) \quad (1.13)$$

be a solution of Equation (1.12)

Now $D^r z = m^r \phi^r(y + mx)$

$$D'^s z = \phi^{(e)}(y + mx)$$

$$\text{and } D^r D'^s z = m^r \phi^{(r+s)}(y + mx)$$

Therefore, on substituting (1.13) in Equation (1.12), we get

$$(A_0 m^n + A_1 m^{n-s} + A_2 m^{n-2} + \cdots + A_n) \phi^{(n)}(y + mx) = 0$$

which will be satisfied if

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$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0 \quad \dots(1.14)$$

Equation (1.14) is known as the Auxiliary Equation.

Let m_1, m_2, \dots, m_n be the roots of the Equation (1.14),

Then the following three cases arise:

Case I: Roots m_1, m_2, \dots, m_n are Distinct

Part of C.F. corresponding to $m = m_1$ is

$$z = \phi_1(y + m_1 x)$$

where ' ϕ_1 ' is an arbitrary function.

Part of C.F. corresponding to $m = m_2$ is

$$z = \phi_2(y + m_2 x)$$

where ϕ_2 is any arbitrary function.

Now since our equation is linear, so the sum of solutions is also a solution.

Therefore, our complimentary function becomes,

$$\text{C.F.} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$$

Case II: Roots are Imaginary

Let the pair of complex roots of the Equation (1.14) be

$$u \pm iv$$

then the corresponding part of complimentary function is

$$z = \phi_1(y + ux + ivx) + \phi_2(y + ux - ivx) \quad \dots(1.15)$$

Let $y + ux = P$ and $vix = Q$

$$\text{Then } z = \phi_1(P + iQ) + \phi_2(P - iQ)$$

$$\text{Or } z = (\phi_1 + \phi_2)P + (\phi_1 - \phi_2)iQ$$

$$\text{If } \phi_1 + \phi_2 = \xi_1$$

$$\text{And } \phi_1 - \phi_2 = \xi_2$$

Then

$$\varphi_1 = \frac{1}{2}(\xi_1 + i\xi_2)$$

and

$$\varphi_2 = \frac{1}{2}(\xi_1 - i\xi_2)$$

Substituting these values in Equation (1.15), we get

$$z = \frac{1}{2} \xi_1(P + iQ) + \frac{1}{2} i\xi_2(P + iQ) + \frac{1}{2} \xi_1(P - iQ) - \frac{1}{2} i\xi_2(P - iQ)$$

or

$$z = \frac{1}{2} \{\xi_1(P + iQ) + \xi_1(P - iQ)\} + \frac{1}{2} i\{\xi_2(P + iQ) - \xi_2(P - iQ)\}$$

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Case III: Roots are Repeated

Let m be the repeated root of Equation (1.14).

Then we have,

$$(D - mD')(D - mD')z = 0$$

Putting $(D - mD')z = U$, we get (1.16)

$$(D - mD')U = 0 \quad (1.17)$$

Since the equation is linear, it has the following subsidiary equations,

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dU}{0} \quad (1.18)$$

Two independent integrals of Equation (1.18) are

$$y + mx = \text{constant}$$

and $U = \text{constant}$

$$\therefore U = \phi(y + mx)$$

is a solution of Equation (1.17) where ϕ is an arbitrary function.

Substituting in Equation (1.16)

$$\frac{\partial z}{\partial x} - m \frac{\partial z}{\partial y} = \phi(y + mx) \quad (1.19)$$

which has the following subsidiary equations,

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y + mx)}$$

Two independent integrals of Equation (1.16) are

$$y + mx = \text{constant}$$

and $z = x\phi(y + mx) + \text{constant}$

$$\text{Therefore } z = x\phi(y + mx) + \psi(y + mx) \quad (1.20)$$

is a solution of Equation (1.19) where ψ is an arbitrary function.

Equation (1.20) is the part of C.F. corresponding to two times repeated root.

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In general, if the root m is repeated r times, the corresponding part of C.F. is

$$z = x^{r-1}\phi_1(y + mx) + x^{r-2}\phi_2(y + mx) + \dots + \phi_r(y + mx)$$

where $\phi_1, \phi_2, \dots, \phi_r$ are arbitrary functions.

Example 1.11: Solve the equation, $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.

Solution: The A.E. of the given equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$\text{or } (m-1)^3 = 0$$

$$\Rightarrow m = 1, 1, 1$$

$$\therefore \text{C.F.} = x^2\phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+x).$$

Non-Homogeneous Linear Equations with Constant Coefficients

If all the terms on left hand side of Equation (1.19) are not of same degree then Equation (1.19) is said to be **Non-Homogeneous equation**. Equation is said to be **reducible** if the symbolic function $f(D, D')$ can be resolved into factors each of which is of first degree in D and D' and irreducible otherwise.

For example, the equation

$$f(D, D')z = (D^2 - D'^2 + 2D + 1)z = (D + D' + 1)(D - D' + 1)z = x^2 + xy$$

is reducible while the equation

$$f(D, D')z = (DD' + D'^3)z = D'(D + D'^2)z = \cos(x + 2y)$$

is irreducible.

Reducible Non Homogeneous Equations

In the equation,

$$f(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2) \dots (a_nD + b_nD' + c_n) \quad \dots (1.21)$$

where a 's, b 's and c 's are constants.

The complementary function takes the form

$$(a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2) \dots (a_nD + b_nD' + c_n)z = 0 \quad (1.22)$$

Any solution of the equation given by

$$(a_iD + b_iD' + c_i)z = 0 \quad (1.23)$$

is a solution of the Equation (1.22)

Forming the Lagrange's subsidiary equations of Equation (1.23),

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{-c_i z} \quad (1.24)$$

The two independent integrals of Equation (1.24) are

$$b_i x - a_i y = \text{constant}$$

and $z = \text{constant } e^{\frac{-c_i x}{a_i}}$, if $a_i \neq 0$
or

$$z = \text{constant } e^{\frac{-c_i y}{b_i}}, \text{ if } b_i \neq 0$$

Therefore

$$z = e^{\frac{-c_i x}{a_i}} \phi_i(b_i x - a_i y), \text{ if } a_i \neq 0$$

or

$$z = e^{\frac{-c_i y}{b_i}} \psi_i(b_i x - a_i y) \quad \text{if } b_i \neq 0$$

is the general solution of Equation (1.23). Here ϕ_i and ψ_i are arbitrary functions.

Example 1.12: Solve the differential equations

$$(D^2 - D'^2 - 3D + 3D')z = 0.$$

Solution: The equation can also be written as

$$(D - D')(D + D' - 3)z = 0$$

$$\therefore \text{C.F.} = \phi_1(y + x) + e^{3x} \phi_2(x - y)$$

Or

$$\psi_1(y + x) + e^{3y} \psi_2(x - y)$$

When the Factors are Repeated

Let the factor is repeated two times and is given by,

$$(aD + bD' + c)$$

Consider the equation

$$(aD + bD' + c)(aD + bD' + c)z = 0 \quad (1.25)$$

$$\text{Put } (aD + bD' + c)z = U \quad (1.26)$$

Then the Equation (1.21) reduces to

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$$(aD + bD' + c)U = 0 \quad (1.27)$$

General solution of Equation (1.27) is

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$$U = e^{-\frac{c}{a}x} \phi(bx - ay) \text{ if } a \neq 0 \quad (1.28)$$

Or

$$U = e^{-\frac{c}{b}y} \psi(bx - ay) \text{ if } b \neq 0 \quad (1.29)$$

Substituting Equation (1.28) in Equation (1.26), we obtain

$$(aD + bD' + c)z = e^{-\frac{c}{a}x} \phi(bx - ay) \quad (1.30)$$

The subsidiary equations are,

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{e^{-\frac{c}{a}x} \phi(bx - ay) - cz} \quad (1.31)$$

The two independent integrals of Equations (1.31) are given by

$$bx - ay = \text{constant} = \lambda \quad (1.32)$$

$$\text{and } \frac{dz}{dx} + \frac{c}{a}z = \frac{1}{a}e^{-\frac{c}{a}x} \phi(bx - ay) = \frac{1}{a}e^{-\frac{c}{a}x} \phi(\lambda) \quad (1.33)$$

The Equation (1.33) being an ordinary linear equation has the following solution:

$$ze^{\frac{c}{a}x} = \frac{1}{a}x\phi(\lambda) + \text{constant}$$

$$\text{or } ze^{\frac{c}{a}x} = \frac{1}{a}x\phi(bx - ay) + \text{constant}$$

Therefore, general solution of Equation (1.30) is

$$\begin{aligned} z &= \frac{x}{a}e^{-\frac{c}{a}x} \phi(bx - ay) + \phi_1(bx - ay)e^{-\frac{c}{a}x} \\ &= e^{-\frac{c}{a}x} \{x\phi_2(bx - ay) + \phi_1(bx - ay)\} \end{aligned} \quad \dots(1.34)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Similarly from Equations (1.29) and (1.26), we get

$$z = e^{-\frac{c}{b}y} \{y\psi_2(bx - ay) + \psi_1(bx - ay)\}$$

where ψ_1 and ψ_2 are arbitrary functions.

In general, for r times repeated factor, $(aD + bD' + c)$

$$z = e^{-\frac{c}{a}x} \sum_{i=1}^r x^{i-1} \phi_i(bx - ay) \quad \text{if } a \neq 0$$

Or

$$z = e^{-\frac{c}{b}y} \sum_{i=1}^r y^{i-1} \psi_i(bx - ay) \quad \text{if } b \neq 0$$

where $\phi_1, \phi_2, \dots, \phi_r$ and $\psi_1, \psi_2, \dots, \psi_r$ are arbitrary functions.

Example 1.13: Solve the differential equation,

$$(2D - D' + 4)(D + 2D' + 1^2 z = 0)$$

Solution: C.F. corresponding to the factor $(2D - D' + 4)$ is

$$e^{4y} \phi(x + 2y)$$

C.F. corresponding to the factor $(D = 2D' + 1)^2$ is

$$e^{-x} \{x\phi_2(2x - y) + \phi_1(2x - y)\}$$

$$\text{Hence C.f.} = e^{4y} \phi(x + 2y) + e^{-x} \{x\phi_2(2x - y) + \phi_1(2x - y)\}$$

Irreducible Non-Homogeneous Equations

For solving the equation

$$f(D, D')z = 0 \tag{1.35}$$

$$\text{Substitute } z = ce^{ax+by} \text{ where } a, b \text{ and } c \text{ are constants} \tag{1.36}$$

$$\text{Now } D^r z = ca^r e^{ax+by}$$

$$D^r D'^a z = ca^r b^s e^{ax+by}$$

$$\text{and } D'^s z = cb^s e^{ax+by}$$

Substituting Equation (1.36) in Equation (1.35), we get,

$$cf(a, b)e^{ax+by} = 0$$

which will hold if

$$f(a, b) = 0 \tag{1.37}$$

For any selected value of a (or b) Equation (1.37) gives one or more values of b (or a). Thus there exists infinitely many pairs of numbers (a_i, b_i) satisfying Equation (1.37).

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Thus

$$z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$$

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(1.38)

where $f(a_i, b_i) = 0 \quad \forall i$, is a solution of the Equation (1.35),

If

$$f(D, D') = (D + hD' + k)g(D, D') \quad (1.39)$$

then any pair (a, b) such that

$$a + hb + k = 0 \quad (1.40)$$

satisfies Equation (1.37). There are infinite number of such solutions.

From Equation (1.40)

$$a = -(hb + k)$$

Thus

$$\begin{aligned} z &= \sum_{i=1}^{\infty} c_i e^{-(hb_i + k)x + b_i y} \\ &= e^{-kx} \sum_{i=1}^{\infty} c_i e^{b_i(y - hx)} \end{aligned} \quad (1.41)$$

is a part of C.F. corresponding to a linear factor $(D + hD' + k)$ given in Equation (1.39).

Equation (1.41) is equivalent to

$$e^{-kx} \phi(y - hx)$$

where ' ϕ ' is an arbitrary function.

Equation (1.38) is the general solution if $f(D, D')$ has no linear factor otherwise general solution will be composed of both arbitrary functions and partly arbitrary constants.

Example 1.14: Solve the differential equation $(2D^4 + 3D^2D' + D'^2)z = 0$.

Solution: The given equation is equivalent to

$$(2D^2 + D')(D^2 + D')z = 0$$

C.F. corresponding to the first factor

$$= \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$$

where a_i and b_i are related by

$$2a_i^2 + b_i = 0$$

$$\text{or } b_i = -2a_i^2$$

Therefore, part of C.F. corresponding to the first factor

$$\sum_{i=1}^{\infty} d_i e^{e_i(x - e_i y)}$$

where e_i and d_i are arbitrary constants.

$$\therefore \text{C.F.} = \sum_{i=1}^{\infty} c_i e^{a_i(x - 2a_i y)} + \sum_{i=1}^{\infty} d_i e^{e_i(x - e_i y)}$$

Particular Integral

In the equation,

$$f(D, D')z = V(x, y) \quad \dots(1.42)$$

$f(D, D')$ is a non homogeneous function of D and D' .

$$\text{P.I.} = \frac{1}{f(D, D')} V(x, y) \quad \dots(1.43)$$

Here if $V(x, y)$ is of the form e^{ax+by} where ' a ' and ' b ' are constants then we use the following theorem to evaluate the particular integral:

Theorem 1.1: If $f(a, b) \neq 0$, then

$$\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$$

Proof: By differentiation

$$D^r D^s e^{ax+by} = a^r b^s e^{ax+by}$$

$$D^r e^{ax+by} = a^r e^{ax+by}$$

$$D^s e^{ax+by} = b^s e^{ax+by}$$

$$\therefore f(D, D') e^{ax+by} = f(a, b) e^{ax+by}$$

$$e^{ax+by} = f(a, b) \frac{1}{f(D, D')} e^{ax+by}$$

Dividing the above equation by $f(a, b)$

$$\frac{1}{f(a, b)} e^{ax+by} = \frac{1}{f(D, D')} e^{ax+by}$$

NOTES

$$\text{or } \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$$

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Example 1.15: Solve the equation $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$

Solution: The given equation is equivalent to

$$(D - D')(D + D' - 3)z = e^{x-2y}$$

$$\text{C.F.} = \phi_1(y+x) + e^{3x}\phi_2(y-x)$$

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)} e^{x-2y}$$

$$= -\frac{1}{12} e^{x-2y}$$

$$\text{Therefore, } z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{12} e^{x-2y}$$

But in case $V(x, y)$ is of the form $e^{ax+by}\phi(x, y)$ where 'a' and 'b' are constants then following theorem is used to evaluate the particular integral:

Theorem 1.2: If $\phi(x, y)$ is any function, then

$$\frac{1}{f(D, D')} e^{ax+by} \phi(x, y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x, y)$$

Proof: From Leibnitz's theorem for successive differentiation, we have

$$\begin{aligned} D^r \{e^{ax+by} \phi(x, y)\} &= e^{ax+by} \{D^r \phi(x, y) + {}^r c_1 a D^{r-1} \phi(x, y)\} \\ &\quad + {}^r c_2 a^2 D^{r-2} \phi(x, y) + \dots + {}^r c_r a^r \phi(x, y) \\ &= e^{ax+by} \{D^r + {}^r c_1 D^{r-1} + {}^r c_2 a^2 D^{r-2} + \dots + {}^r c_r a^r\} \phi(x, y) \\ &= e^{ax+by} (D + a)^r \phi(x, y). \end{aligned}$$

Similarly

$$D'^s \{e^{ax+by} \phi(x, y)\} = e^{ax+by} (D'+b)^s \phi(x, y)$$

$$\begin{aligned} \text{and } D^r D'^s \{e^{ax+by} \phi(x, y)\} &= D^r [e^{ax+by} (D'+b) \phi(x, y)] \\ &= e^{ax+by} (D + a)^r (D' + b)^s \phi(x, y) \end{aligned}$$

$$\text{So } f(D, D') \{e^{ax+by} \phi(x, y)\} = e^{ax+by} f(D + a, D' + b) \phi(x, y) \quad (1.44)$$

$$\text{Put } f(D + a, D' + b) \phi(x, y) = \psi(x, y)$$

$$\therefore \phi(x, y) = \frac{1}{f(D+a, D'+b)} \psi(x, y)$$

Substituting in Equation (1.44), we get

$$f(D, D') \left\{ e^{ax+by} \frac{1}{f(D+a, D'+b)} \psi(x, y) \right\} = e^{ax+by} \psi(x, y)$$

Operating on the equation by $\frac{1}{f(D, D')}$

$$e^{ax+by} \frac{1}{f(D+a, D'+b)} \psi(x, y) = \frac{1}{f(D, D')} \left\{ e^{ax+by} \psi(x, y) \right\}$$

Replacing $\psi(x, y)$ by $\phi(x, y)$, we have

$$\frac{1}{f(D, D')} \left(e^{ax+by} \phi(x, y) \right) = e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x, y)$$

Example 1.16: Solve $(D^2 - D'^2 - 3D + 3D')y = xy + e^{x+av}$.

Solution: The given equation is equivalent to,

$$(D - D')(D + D' - 3)y = xy + e^{x+2y}$$

$$\text{C.F.} = \phi_1(y+x) + e^{3x} \phi_2(x-y)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - D')(D + D' - 3)} xy + \frac{1}{(D - D')(D + D' - 3)} e^{x+2y} \\ &= -\frac{1}{3D} \left\{ 1 - \frac{D'}{D} \right\}^{-1} \left\{ 1 - \frac{D + D'}{3} \right\}^{-1} xy \\ &\quad + e^{x+2y} \frac{1}{(D + 1 - D' - 2)(D + 1 + D' + 2 - 3)} 1. \\ &= -\frac{1}{3D} \left\{ 1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \dots \right\} \left\{ 1 + \frac{D + D'}{3} + \frac{2}{9} DD' + \dots \right\} xy + e^{x+2y} \\ &\quad \frac{1}{(D - D' - 1)(D + D')} 1. \\ &= -\frac{1}{3D} \left\{ 1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \dots \right\} \left\{ xy + \frac{x+y}{3} + \frac{2}{9} \cdot \right\} + e^{x+2y} \frac{1}{(-1)(D + D')} 1. \end{aligned}$$

NOTES

$$= -\frac{1}{3D} \left\{ xy + \frac{2}{3}x + \frac{x^2}{2} + \frac{1}{3}y + \frac{2}{9} \right\} - xe^{x+2y}$$

NOTES

$$= -\frac{1}{3} \left\{ \frac{x^2y}{2} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{1}{3}xy + \frac{2}{9}x \right\} - xe^{x+2y}.$$

$$\therefore z = \phi_1(y+x) + e^{3x} \phi_2(x-y) - xe^{x+2y} - \frac{1}{6}x^2y - \frac{1}{9}x^2 - \frac{x^3}{18} - \frac{1}{9}xy - \frac{2}{27}x$$

Example 1.17: Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y + xy + 1$.

Solution: Equation is equivalent to

$$(D - 1)(D - D' + 1)z = \cos(x + 2y) + e^y + xy + 1$$

Complementary Function = $e^x \varphi_1(y) + e^y \varphi_2(x+y)$.

Particular integral corresponding to $\cos(x + 2y)$ is

$$\frac{1}{D^2 - DD' + D' - 1} \cos(x + 2y)$$

$$= \frac{1}{(-1) - (-2) + D' - 1} \cos(x + 2y)$$

$$= \frac{1}{D'} \cos(x + 2y)$$

$$= \frac{1}{2} \sin(x + 2y)$$

Corresponding to e^y , the particular integral is

$$= \frac{1}{D^2 - DD' + D' - 1} e^y$$

$$= \frac{1}{D' - 1} e^y$$

$$= e^y \cdot \frac{1}{D'} \cdot 1$$

$$= ye^y.$$

Particular Integral corresponding to the part $(xy + 1)$ is

$$= \frac{1}{(D-1)(D-D'+1)} (xy + 1)$$

$$- \{1 - D\}^{-1} \{1 + (D - D')\}^{-1} (xy + 1)$$

NOTES

$$\begin{aligned}
 &= -\{1 + D + D^2 + \dots\} \{1 - (D - D') + (D - D')^2 - \dots\} (xy + 1) \\
 &= -\{1 + D + D^2 + \dots\} \{(xy + 1) - (y - x) - 2\} \\
 &= -\{1 + D + D^2 + \dots\} \{(xy - y + x - 1)\} \\
 &= -\{(xy - y + x - 1) + (y + 1)\} \\
 &= -(xy + x) \\
 &= -x(y + 1)
 \end{aligned}$$

$$\therefore z = e^x \varphi_1(y) + e^y \varphi_2(x + y) + \frac{1}{2} \sin(x + 2y) + ye^y - x(y + 1)$$

1.2.3 Partial Differential Equations Reducible to Equations with Constant Coefficients

The equation,

$$f(xD, yD')z = V(x, y)$$

$$\text{where } f(xD, yD') = \sum_{r,s} c_{rs} x^r y^s D^r D'^s, c_{rs} = \text{constant.} \quad (1.45)$$

is reduced to linear partial differential equation with constant coefficients by the following substitution:

$$u = \log x, v = \log y \quad (1.46)$$

By substitution of Equation (1.46)

$$\begin{aligned}
 xD &= x \frac{\partial}{\partial x} \\
 &= x \frac{\partial}{\partial u} \frac{\partial u}{\partial x} \\
 &= \frac{\partial}{\partial u} = d \text{(say)}
 \end{aligned}$$

And

$$\begin{aligned}
 x^2 D^2 &= x^2 D \left(\frac{1}{x} \frac{\partial}{\partial u} \right) \\
 &= x^2 \left(-\frac{1}{x^2} \frac{\partial}{\partial u} + \frac{1}{x^2} \frac{\partial^2}{\partial u^2} \right) \\
 &= \frac{\partial^2}{\partial u^2} - \frac{\partial}{\partial u} \\
 &= d(d-1)
 \end{aligned}$$

NOTES

Therefore,

$$x^r D^r = d(d-1)(d-2)\dots(d-r-1)$$

$$\text{and } y^s D'^s = d'(d'-1)(d'-2)\dots(d'-s-1)$$

$$\begin{aligned} \text{Hence } f(xD, yD') &= \sum c_{rs} d(d-1)\dots(d-r-1) d'(d'-1)\dots(d'-s-1) \\ &= g(d, d') \end{aligned}$$

Here the coefficients in $g(d, d')$ are constants.

Thus by substitution Equation (1.45) is reduced to

$$g(d, d')z = V(e^u, e^v)$$

$$\text{Or } g(d, d')z = U(u, v) \quad (1.47)$$

Equation (1.47) can be solved by methods that have been described for solving partial differential equations with constant coefficients.

Example 1.18: Solve the differential equation,

$$(x^2 D^2 - 4xy DD' + 4y^2 D'^2 + 6y D')z = x^3 y^4$$

Solution: Put $u = \log x$

$$v = \log y$$

The given equation can be reduced to

$$\{d(d-1) - 4dd' + 4d'(d'-1) + 6d'\}z = e^{3u+4v}$$

$$\text{or } \{(d-2d')^2 - (d-2d')\}z = e^{3u+4v}$$

$$\text{or } (d-2d')(d-2d'-1)z = e^{3u+4v}$$

The complementary function is $\phi_1(2u+v) + e^u \phi_2(2u+v)$

$$\begin{aligned} &= \phi_1(\log x^2 y) + x\phi_2(\log x^2 y) \\ &= \psi_1(x^2 y) + x\psi_2(x^2 y) \end{aligned}$$

And the particular integral is $\frac{1}{(d-2d')(d-2d'-1)} e^{3u+2v}$

$$= \frac{1}{30} e^{3u+4v}$$

$$= \frac{1}{30} x^3 y^4$$

∴

$$z = \psi_1(x^2y) + x\psi_2(x^2y) + \frac{1}{30}x^3y^4.$$

Example 1.19: Find the solution of $(x^2D^2 - y^2D'^2 - yD' + xD)z = 0$

Solution: Put

$$u = \log x$$

$$v = \log y$$

The given differential can be reduced to

$$\begin{aligned} & \{d(d-1) - d'(d'-1) - d' + d\}z = 0 \\ \Rightarrow & (d^2 - d'^2)z = 0 \\ & \text{A.E. is} \\ & m^2 - 1 = 0 \\ \Rightarrow & m = 1, -1 \\ \Rightarrow & z = \phi_1(v+u) + \phi_2(v-u) \\ & = \phi_1(\log xy) + \phi_2\left(\log \frac{y}{x}\right) \\ & = \Psi_1(xy) + \Psi_2\left(\frac{y}{x}\right). \end{aligned}$$

NOTES

Example 1.20: Determine the solution of the following equation:
 $(x^2D^2 + 2xyDD' + y^2D'^2)z + nz = n(xD + yD')z + x^2 + y^2 + x^3$

Solution: Put

$$u = \log x$$

$$v = \log y$$

The Equation reduces to

$$\{d(d-1) + 2dd' + d'(d'-1)\}z - n(d+d')z + nz = e^{2u} + e^{2v} + e^{3u}$$

or

$$\{[d+d']^2 - (d+d')\}z - n(d+d')z + nz = e^{2u} + e^{2v} + e^{3u}$$

or

$$\{(d+d')(d+d'-1) - n(d+d') + n\}z = e^{2u} + e^{2v} + e^{3u}$$

or

$$\{(d+d')^2 - (n+1)(d+d') + n\}z = e^{2u} + e^{2v} + e^{3u}$$

or

$$(d+d'-n)(d+d'-1)z = e^{2u} + e^{2v} + e^{3u}$$

$$C.F. = e^{nu}\phi_1(u-v) + e^{u}\phi_2(u-v)$$

NOTES

$$= x^n \psi_1\left(\frac{x}{y}\right) + x \psi_2\left(\frac{x}{y}\right)$$

$$P.I. = \frac{1}{(d+d'-n)(d+d'-1)} \{e^{2u} + e^{2v} + e^{3u}\}$$

$$= \frac{1}{2-n} e^{2u} + \frac{1}{2-n} e^{2v} + \frac{1}{(3-n)^2} e^{3u}$$

$$= -\frac{x^2 + y^2}{n-2} - \frac{1}{2} \cdot \frac{1}{n-3} x^3$$

$$\therefore z = x^n \psi_1\left(\frac{x}{y}\right) + x \psi_2\left(\frac{x}{y}\right) - \frac{x^2 + y^2}{n-2} - \frac{1}{2} \frac{x^3}{n-3}$$

Example 1.21: Solve $(x^2 D^2 - xyDD' - 2y^2 D'^2 + xD - 2yD')z = \log \frac{y}{x} - \frac{1}{2}$

Solution: Put $u = \log x$

$$v = \log y$$

Our equation reduces to

$$\{d(d-1) - dd' - 2d'(d'-1) + d - 2d'\}z = v - u - \frac{1}{2}$$

$$(d^2 - dd' - 2d'^2)z = v - u - \frac{1}{2}$$

$$\text{or } (d - 2d')(d + d')z = v - u - \frac{1}{2}$$

$$C.F. = \phi_1(2u + v) + \phi_2(u - v)$$

$$= \psi_1(x^2 y) + \psi_2\left(\frac{x}{y}\right)$$

$$P.I. = \frac{1}{(d-2d')(d+d')} \left(v - u - \frac{1}{2} \right)$$

NOTES

$$\begin{aligned}
 &= \frac{1}{d-2d'} \cdot \frac{1}{d} \left\{ 1 - \frac{d'}{d} \dots \right\} \left(v - u - \frac{1}{2} \right) \\
 &= \frac{1}{d-2d'} \cdot \frac{1}{d} \left\{ v - u - \frac{1}{2} - u \right\} \\
 &= \frac{1}{d-2d'} \left(uv - u^2 - \frac{1}{2}u \right) \\
 &= \frac{1}{d} \left\{ 1 + \frac{2d'}{d} + \frac{4d'^2}{d^2} + \dots \right\} \left(uv - u^2 - \frac{1}{2}u \right) \\
 &= \frac{1}{d} \left\{ uv - u^2 - \frac{1}{2}u + u^2 \right\} \\
 &= \frac{u^2 v}{2} - \frac{u^2}{4} \\
 &= \frac{1}{2} (\log x)^2 \log y - \frac{1}{4} (\log x)^2
 \end{aligned}$$

$\therefore z = \psi_1(x^2 y) + \psi_2\left(\frac{x}{y}\right) + \frac{1}{2} (\log x)^2 \log y - \frac{1}{4} (\log x)^2.$

Example 1.22: Solve the differential equation,

$$(x^2 D^2 + 2xyDD' + y^2 D'^2)z = (x^2 + y^2)^{\frac{n}{2}}$$

Solution: Put

$$u = \log x$$

$$v = \log y$$

The equation is reduced to $\{d(d-1) + 2dd' + d'(d'-1)\}z = (e^{2u} + e^{2v})^{\frac{n}{2}}$

or

$$\{(d+d')^2 - (d+d')\}z = (e^{2u} + e^{2v})^{\frac{n}{2}}$$

or

$$(d+d')(d+d'-1)z = (e^{2u} + e^{2v})^{\frac{n}{2}}$$

C.F.

$$= \phi_1(u-v) + e^u \phi_2(u-v)$$

$$= \phi_1\left(\log \frac{x}{y}\right) + x \phi_2\left(\log \frac{x}{y}\right)$$

$$= \Psi_1\left(\frac{x}{y}\right) + x \Psi_2\left(\frac{x}{y}\right)$$

Particular Integral is $\frac{1}{(d+d')(d+d'-1)}(e^{2u} + e^{2v})^{\frac{n}{2}}$

NOTES

Substituting $Z = \frac{1}{d+d'-1}(e^{2u} + e^{2v})^{\frac{n}{2}}$

$$\text{or } \frac{\partial Z}{\partial u} + \frac{\partial Z}{\partial v} = Z + (e^{2u} + e^{2v})^{\frac{n}{2}}$$

$$\text{The subsidiary equations are } \frac{du}{1} = \frac{dv}{1} = \frac{dZ}{Z + (e^{2u} + e^{2v})^{\frac{n}{2}}}$$

Two independent integrals of Equation are given by

$$u - v = \text{constant} = a \text{ (say)}$$

and

$$\frac{dZ}{dv} - Z = (e^{2u} + e^{2v})^{\frac{n}{2}}$$

$$= e^{nv} (e^{2a} + 1)^{\frac{n}{2}}$$

Since this equation is linear, therefore

$$Ze^{-v} = \frac{e^{(n-1)v}}{(n-1)} (e^{2a} + 1)^{\frac{n}{2}}$$

$$\therefore Z = \frac{e^{nv}}{n-1} (e^{2a} + 1)^{\frac{n}{2}}$$

$$= \frac{(e^{2u} + e^{2v})^{\frac{n}{2}}}{(n-1)}$$

$$\therefore P.I. = \frac{1}{d+d'} \left\{ \frac{(e^{2u} + e^{2v})^{\frac{n}{2}}}{n-1} \right\}$$

$$= \frac{1}{(n-1)} \int_{a=v-u} \{e^{2u} + e^{2a+2u}\}^{\frac{n}{2}} du$$

$$= \frac{1}{n-1} \left\{ \int (e^{2a} + 1)^{\frac{n}{2}} \int e^{nu} du \right\}_{a=v-u}$$

$$\begin{aligned}
 &= \frac{1}{n(n-1)} \left\{ e^{nu} (e^{2a} + 1)^{\frac{n}{2}} \right\}_{a=v-u} \\
 &= \frac{1}{n(n-1)} (e^{2u} + e^{2v})^{\frac{n}{2}} \\
 &= \frac{1}{n(n-1)} (x^2 + y^2)^{\frac{n}{2}} \\
 \therefore z &= \psi_1 \left(\frac{x}{y} \right) + x \psi_2 \left(\frac{x}{y} \right) + \frac{1}{n(n-1)} (x^2 + y^2)^{\frac{n}{2}}.
 \end{aligned}$$

Example 1.23: Solve $(x^2 D^2 - 2xyDD' + y^2 D'^2 - xD + 3yD')z = \frac{8y}{x}$

Solution: Put $u = \log x$

$$v = \log y$$

Our Equation reduces to

$$\{d(d-1) - 2dd' + d'(d'-1) - d + 3d'\}z = 8e^{v-u}$$

$$\text{or } \{(d-d')^2 - 2(d-d')\}z = 8e^{v-u}$$

$$\text{or } (d-d')(d-d'-2)z = 8e^{v-u}$$

$$\text{C.F. } = \phi_1(u+v) + e^{2u} \phi_2(u+v)$$

$$= \psi_1(xy) + x^2 \psi_2(xy)$$

$$\begin{aligned}
 \text{P.I. } &= 8 \cdot \frac{1}{(d-d')(d-d'-2)} e^{v-u} \\
 &= e^{v-u}
 \end{aligned}$$

$$= \frac{y}{x}$$

$$\therefore z = \psi(xy) + x^2 \psi_2(xy) + \frac{y}{x}.$$

Example 1.24: Solve $(x^2 D^2 + 2xyDD' + y^2 D'^2)z = x^m y^n$

Solution: Put $u = \log x$

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$$v = \log y$$

The equation reduces to

$$\{d(d-1) + 2dd' + d'(d'-1)\}z = e^{mu+nv}$$

$$\text{or } \{(d+d')^2 - (d+d')\}z = e^{mu+nv}$$

$$\text{or } (d+d')(d+d'-1)z = e^{mu+nv}$$

$$\text{C.F. } = \phi_1(u-v) + e^u \phi_2(u-v)$$

$$= \psi_1\left(\frac{x}{y}\right) + x\psi_2\left(\frac{x}{y}\right)$$

$$\text{P.I. } = \frac{1}{(d+d')(d+d'-1)} e^{mu+nv}$$

$$= \frac{1}{(m+n)(m+n-1)} e^{mu+nv}$$

$$= \frac{1}{(m+n)(m+n-1)} x^m y^n$$

$$\therefore z = \psi_1\left(\frac{x}{y}\right) + x\psi_2\left(\frac{x}{y}\right) + \frac{1}{(m+n)(m+n-1)} x^m y^n.$$

Check Your Progress

1. What is P.I.?
2. What is the condition for a quasi-linear partial differential equation to be of elliptic type?
3. What is a non-homogenous equation?

1.3 SURFACES AND CURVES IN THREE DIMENSIONS

Principal Normal

The angular change of an arcual length between a pair of points is known as curvature denoted as κ (kappa) and is characterized by its centre, radius and circularity.

The angle of inclination of the curve varies at a definite rate per unit arc length and the rate of change of angle of a circle is called curvature of the curve.

For a circle, the radius of curvature at any point is the radius of the circle. The values of curvature range from ∞ for a straight line to 0 for an arcual length of infinity.

Let AP form an arcual length of ds and an angle of $d\psi$ at the centre. Then curvature is conveniently denoted by $\frac{d\psi}{ds}$ (Refer Figure 1.1). The value of curvature becomes equal to the radius in the case of any two points in a circular system (Refer Figure 1.2).

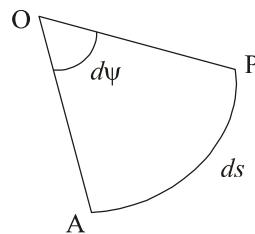


Fig. 1.1 Curvature of Curve

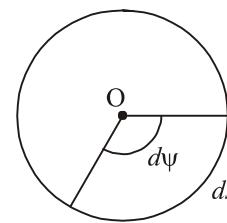


Fig. 1.2 Curvature of Circle

The reciprocal of the curvature of the curve is the radius of curvature. It is denoted by P .

Radius of Curvature

Radius of curvaline of a curve can be calculated using any one of the following formulae.

$$\text{Cartesian form: } \rho = \frac{(1+y'^2)^{3/2}}{y''} \quad x = f(y)$$

$$\text{where } y' = \frac{dy}{dx} \text{ and } y'' = \frac{d^2y}{dx^2}$$

Parametric form:

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

where $x = x(t)$ and $y = y(t)$

$$\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}$$

$$\ddot{x} = \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}$$

$$\text{Polar form: } \rho = \frac{(r^2 + \dot{r}^2)^{3/2}}{r^2 + 2\dot{r}^2 - r\ddot{r}}$$

NOTES

where $r = f(\theta)$

$$\dot{r} = \frac{dr}{d\theta},$$

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$$\ddot{r} = \frac{d^2 r}{d\theta^2}$$

Note: If $\frac{dy}{dx}$ becomes infinity at any point on the curve then the radius of curvature at that point is given by

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)} = \frac{1}{\left(\frac{d^2x}{dy^2}\right)}$$

Table 1.1 List of Formulae

$y = f(x)$	$\frac{dy}{dx} = f'(x)$	$y = f(x)$	$\frac{dy}{dx} = f'(x)$
1. x^n	nx^{n-1}	2. $\log x$	$\frac{1}{x}$
3. e^{ax}	$a e^{ax}$	4. $\sin ax$	$a \cos ax$
5. $\cos ax$	$-a \sin ax$	6. $\tan ax$	$a \sec^2 ax$
7. $\operatorname{cosec} ax$	$-a \operatorname{cosec} ax \cot ax$	8. $\cot ax$	$-a \operatorname{cosec}^2 ax$
9. $\sec ax$	$a \sec ax \tan ax$	10. a^x	$a^x \log a$
11. x^x	$x^x [\log x + 1]$	12. $\cos h x$	$\sin h x$
13. $\sin h x$	$\cos h x$	14. $\tanh x$	$\sec h^2 x$
15. $\cot h x$	$-\operatorname{cosec} h^2 x$	16. $\sec h x$	$-\sec h x \tan h x$
17. $\operatorname{cosec} h x$	$-\operatorname{cosec} h x \cot h x$	18. $\cos h^{-1} x$	$\pm \frac{1}{\sqrt{x^2 - 1}}$
19. $\sin h^{-1} x$	$\frac{1}{\sqrt{x^2 + 1}}$	20. $\tan h^{-1} x$	$\frac{1}{1 - x^2}; -1 < x < 1$
21. $\cot h^{-1} x$	$\frac{1}{1 - x^2}; x > +1 \text{ & } x < -1$		
22. $\sec h^{-1} x$	$\frac{-1}{x\sqrt{1 - x^2}}$	23. $\operatorname{cosec} h^{-1} x$	$\frac{-1}{x\sqrt{x^2 + 1}}$

24. $\log(\sec x + \tan x)$ $\sec x$

25. $\sin^{-1} x$ $\frac{1}{\sqrt{1-x^2}}$

26. $\log(\cosec x - \cot x)$ $\cosec x$

27. $\cos^{-1} x$ $\frac{-1}{\sqrt{1-x^2}}$

28. $\tan^{-1} x$ $\frac{1}{1+x^2}$

29. $\cot^{-1} x$ $-\frac{1}{1+x^2}$

30. $\sec^{-1} x$ $\frac{1}{x\sqrt{x^2-1}}$

31. $\cosec^{-1} x$ $-\frac{1}{x\sqrt{x^2-1}}$

Example 1.25: Find the radius of curvature at $(a, a/2)$ on the curve $4ay^2 = (2a - x)^3$.

Solution: Since $4ay^2 = (2a - x)^3$,

Differentiating the equation with respect to x , we get

$$8ay \frac{dy}{dx} = 3(2a - x)^2(-1)$$

$$\frac{dy}{dx} = -\frac{3}{8} \frac{(2a - x)^2}{ay},$$

$$\left. \frac{dy}{dx} \right|_{(a, a/2)} = -\frac{3}{8} \frac{a^2}{a \cdot a/2} = -\frac{3}{4}$$

Differentiating again with respect to x , we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{3}{8} \left[\frac{ay \cdot 2(2a - x)(-1) - (2a - x)^2 \cdot a \frac{dy}{dx}}{a^2 y^2} \right] \\ \left. \frac{d^2y}{dx^2} \right|_{(a, a/2)} &= -\frac{3}{8} \left[\frac{a(a/2)2(a)(-1) - a^2 \cdot a(-3/4)}{a^2 \times a^2/4} \right] \\ &= -\frac{3}{2} \left[\frac{-a^3 + (3a^3/4)}{a^4} \right] \\ &= -\frac{3}{2a} \left(-1 + \frac{3}{4} \right) = -\frac{3}{2a} \left(-\frac{1}{4} \right) = \frac{3}{8a} \end{aligned}$$

The radius of curvature,

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{\left(1+\frac{9}{16}\right)^{3/2}}{3/8a} = \frac{5^3}{4^3} \times \frac{8a}{3} = \frac{125a}{24}$$

NOTES

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If $r(s)$ is an arc length parametrized curve, then $r'(s)$ is a unit vector and hence $\mathbf{r}' \cdot \mathbf{r}' = 1$. Differentiating this relation, we obtain

$$\mathbf{r}' \cdot \mathbf{r}'' = 0, \quad (1.48)$$

which states that \mathbf{r}'' is orthogonal to the tangent vector, provided it is not a null vector. This fact can also be interpreted from the definition of the second derivative $r''(s)$.

$$\mathbf{r}''(s) = \lim_{\Delta s \rightarrow 0} \frac{\mathbf{r}'(s + \Delta s) - \mathbf{r}'(s)}{\Delta s}. \quad (1.49)$$

The direction of $\mathbf{r}'(s + \Delta s) - \mathbf{r}'(s)$ becomes perpendicular to the tangent vector as $\Delta s \rightarrow 0$. The unit vector

$$\mathbf{n} = \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|} = \frac{\mathbf{t}'(s)}{|\mathbf{t}'(s)|}, \quad (1.50)$$

which has the direction and sense of $t'(s)$ is called the *unit principal normal vector* at s . The plane determined by the unit tangent and normal vectors $t(s)$ and $n(s)$ is called the *osculating plane* at s . It is also well known that the plane through three consecutive points of the curve approaching a single point defines the osculating plane at that point

When $\mathbf{r}'(s + \Delta s)$ is moved from Q to P , then $\mathbf{r}'(s)$, $\mathbf{r}'(s + \Delta s)$ and $\mathbf{r}'(s + \Delta s) - \mathbf{r}'(s)$ form an isosceles triangle, since $\mathbf{r}'(s + \Delta s)$ and $\mathbf{r}'(s)$ are unit tangent vectors. Thus we have

$$|\mathbf{r}'(s + \Delta s) - \mathbf{r}'(s)| = \Delta\theta \cdot 1 = \Delta\theta = |\mathbf{r}''(s)\Delta s| \text{ as } \Delta s \rightarrow 0 \text{ and hence}$$

$$|\mathbf{r}''(s)| = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\Delta s} = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\varrho \Delta\theta} = \frac{1}{\varrho} \equiv \kappa. \quad (1.51)$$

κ is called the *curvature*, and its reciprocal ϱ is called the *radius of curvature* at s . It follows that

$$\mathbf{r}'' = \mathbf{t}' = \kappa \mathbf{n}. \quad (1.52)$$

The vector $\mathbf{k} = \mathbf{r}'' = \mathbf{t}'$ is called the *curvature vector*, and measures the rate of change of the tangent along the curve. By definition κ is nonnegative, thus the sense of the normal vector is the same as that of $\mathbf{r}''(s)$.

The curvature for arbitrary speed curve can be obtained as follows. First we evaluate $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ by the chain rule

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} = \mathbf{t}v, \quad (1.53)$$

$$\ddot{\mathbf{r}} = \frac{d}{dt}[\mathbf{t}v] = \frac{d\mathbf{t}}{ds}v^2 + \mathbf{t}\frac{dv}{dt} = \kappa\mathbf{n}v^2 + \mathbf{t}\frac{dv}{dt}, \quad (1.54)$$

where $v = \frac{ds}{dt}$ is the parametric speed. Taking the cross product of $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ we obtain

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \kappa v^3 \mathbf{t} \times \mathbf{n}. \quad (1.55)$$

For the planar curve, we can give the curvature κ a sign by defining the normal vector such that $(\mathbf{t}, \mathbf{n}, \mathbf{e}_z)$ form a right-handed screw, where $\mathbf{e}_z = (0, 0, 1)^T$. The point where the curvature changes sign is called an *inflection point*.

According to this definition the unit normal vector of the plane curve is given by

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(-\dot{y}, \dot{x})^T}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad (1.56)$$

and hence from Equation (4.8) we have

$$\kappa = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \mathbf{e}_z}{v^3} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \quad (1.57)$$

Also

$$|\dot{\mathbf{r}}| = \frac{ds}{dt}, \quad (1.58)$$

For a space curve, by taking the norm of Equation (1.55) and using Equation (1.58), we obtain

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}. \quad (1.59)$$

The normal vector for the arbitrary speed curve can be obtained from $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, where \mathbf{b} is the unit *binormal vector*.

The unit tangent vector is

$$\mathbf{t} = \pm \frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}}. \quad (1.60)$$

The unit principal normal vector and curvature for implicit curves can be obtained as follows. For the planar curve the normal vector can be deduced by combining Equations (1.56) and (1.60) yielding

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$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(f_x, f_y)^T}{\sqrt{f_x^2 + f_y^2}} = \frac{\nabla f}{|\nabla f|}, \quad (1.61)$$

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where only the $+$ sign of \mathbf{t} was used (although it is not necessary).

We will introduce a derivative operator with respect to arc length so that the derivation becomes simple. If we rewrite the plane implicit curve as $f(x(s), y(s)) = 0$ where s is arc length along the implicit curve, the total derivative with respect to the arc length becomes

$$\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}. \quad (1.62)$$

Now if we replace $\frac{dx}{ds}$ and $\frac{dy}{ds}$, we obtain the derivative operator with respect to arc length

$$\frac{d}{ds} = \frac{1}{|\nabla f|} \left(f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y} \right). \quad (1.63)$$

By applying the operator from Equation (1.63) to Equation (1.60) and equating with $\kappa \mathbf{n}$ (using Equations (1.62) and (1.61)), we obtain

$$\kappa = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_x^2f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}}. \quad (1.64)$$

For a 3-D implicit curve, we can deduce a derivative operator similar to Equation (1.63),

$$\frac{d}{ds} = \frac{1}{|\boldsymbol{\alpha}|} \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right), \quad (1.65)$$

where $\boldsymbol{\alpha}$ is the tangent vector of the 3-D implicit curve given by

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = \nabla f \times \nabla g, \quad (1.66)$$

and

$$\alpha_1 = \frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial g}{\partial y} \frac{\partial f}{\partial z}, \quad (1.67)$$

$$\alpha_2 = \frac{\partial g}{\partial x} \frac{\partial f}{\partial z} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial z}, \quad (1.68)$$

$$\alpha_3 = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}. \quad (1.69)$$

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By applying the derivative operator from Equation (4.18) to $|\alpha|t = \alpha$ we obtain

$$\frac{d|\alpha|t}{ds} = \frac{1}{|\alpha|} \left(\alpha_1 \frac{\partial \alpha}{\partial x} + \alpha_2 \frac{\partial \alpha}{\partial y} + \alpha_3 \frac{\partial \alpha}{\partial z} \right), \quad (1.70)$$

which gives

$$|\alpha|^2 \kappa n + |\alpha| |\alpha|' t = \left(\alpha_1 \frac{\partial \alpha}{\partial x} + \alpha_2 \frac{\partial \alpha}{\partial y} + \alpha_3 \frac{\partial \alpha}{\partial z} \right). \quad (1.71)$$

Taking the cross product of $|\alpha|t = \alpha$ and Equation (1.71) yields

$$|\alpha|^3 \kappa b = \alpha \times \left(\alpha_1 \frac{\partial \alpha}{\partial x} + \alpha_2 \frac{\partial \alpha}{\partial y} + \alpha_3 \frac{\partial \alpha}{\partial z} \right). \quad (1.72)$$

Thus,

$$\kappa = \frac{|\alpha \times (\alpha_1 \frac{\partial \alpha}{\partial x} + \alpha_2 \frac{\partial \alpha}{\partial y} + \alpha_3 \frac{\partial \alpha}{\partial z})|}{|\alpha|^3}. \quad (1.73)$$

Curves With Torsion and Binormal

The following section defines curves with the help of torsion and binormal.

Torsion

While at each point of a curve C its curvature measures the deviation of the curve from the tangent, its torsion $\tau(s)$ measures the deviation of C from the osculating plane. The tangent t and normal n vectors are orthogonal to each other and lie in the osculating plane. Define a unit binormal vector b such that (t, n, b) form a right-handed screw such that

$$b = t \times n, \quad t = n \times b, \quad n = b \times t, \quad (1.74)$$

The plane defined by normal and binormal vectors is called the *normal plane* and the plane defined by binormal and tangent vectors is called the *rectifying plane*. The plane defined by tangent and normal vectors is called the *osculating plane*. The binormal vector for the arbitrary speed curve with nonzero curvature can be obtained by using Equation (1.55) and the first equation of (1.74) as follows:

$$b = \frac{\dot{r} \times \ddot{r}}{|\dot{r} \times \ddot{r}|}. \quad (1.75)$$

The binormal vector is perpendicular to the osculating plane and its rate of change is expressed by the vector

$$b' = \frac{d}{ds}(t \times n) = \frac{dt}{ds} \times n + t \times \frac{dn}{ds} = t \times n',$$

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where we used the fact that $\frac{dt}{ds} = \mathbf{r}'' = \kappa \mathbf{n}$.

Since n is a unit vector $n \cdot n = 1$, we have $n \cdot n' = 0$. Therefore n' is parallel to the rectifying plane (b, t) , and hence n' can be expressed as a linear combination of b and t :

$$\mathbf{n}' = \mu \mathbf{t} + \tau \mathbf{b}. \quad (1.77)$$

Thus, using Equations (1.76) and (1.77), we obtain

$$\mathbf{b}' = \mathbf{t} \times (\mu \mathbf{t} + \tau \mathbf{b}) = \tau \mathbf{t} \times \mathbf{b} = -\tau \mathbf{b} \times \mathbf{t} = -\tau \mathbf{n}. \quad (1.78)$$

The coefficient τ is called the *torsion* and measures how much the curve deviates from the osculating plane. By taking the dot product with $-n$, we obtain the torsion of the curve at a nonzero curvature point

$$\tau = -\mathbf{n} \cdot \mathbf{b}' = -\frac{\mathbf{r}''}{\kappa} \cdot \left(\mathbf{r}' \times \frac{\mathbf{r}'''}{\kappa} \right)' = -\frac{\mathbf{r}''}{\kappa} \cdot \left(\mathbf{r}' \times \frac{\mathbf{r}'''}{\kappa} \right) = \frac{(\mathbf{r}' \mathbf{r}'' \mathbf{r}''')}{\mathbf{r}'' \cdot \mathbf{r}''}, \quad (1.79)$$

where Equation (1.52) is used and $(\mathbf{r}' \mathbf{r}'' \mathbf{r}''')$ is a *triple scalar product*.

The torsion for an arbitrary speed curve is given by

$$\tau = \frac{(\ddot{\mathbf{r}} \ddot{\mathbf{r}} \ddot{\mathbf{r}})}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})}. \quad (1.80)$$

While the curvature is determined only in magnitude, except for plane curves, torsion is determined both in magnitude and sign. Torsion is positive when the rotation of the osculating plane is in the direction of a right-handed screw moving in the direction of \mathbf{t} as s increases. If the torsion is zero at all points, the curve is planar.

The binormal vector of a 3-D implicit curve can be obtained from Equation (1.72) as follows:

$$\mathbf{b} = \frac{\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z} \right)}{|\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z} \right)|}. \quad (1.81)$$

The torsion for a 3-D implicit curve can be derived by applying the derivative operator (1.65) to (1.72), which gives

$$\begin{aligned} & \frac{d}{ds}(|\boldsymbol{\alpha}|^3 \kappa \mathbf{b}) \\ &= \frac{1}{|\boldsymbol{\alpha}|} \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right) \left(\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z} \right) \right) \end{aligned} \quad (1.82)$$

and therefore

$$\begin{aligned} |\boldsymbol{\alpha}|(|\boldsymbol{\alpha}|^3\kappa)' \mathbf{b} - |\boldsymbol{\alpha}|^4\kappa\tau \mathbf{n} &= \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right) \\ &\quad \left(\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z} \right) \right) \end{aligned} \quad (1.83)$$

Taking the dot product with Equation (1.71) we obtain

$$\begin{aligned} -|\boldsymbol{\alpha}|^6\kappa^2\tau &= \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z} \right) \cdot \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right) \\ &\quad \left(\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z} \right) \right) \end{aligned} \quad (1.84)$$

from which we calculate τ .

Example 1.26: A circular helix in parametric representation is given by $\mathbf{r}(t) = (a \cos t, a \sin t, bt)^T$.

Find the curvature and torsion of a circular helix with $a = 2$, $b = 3$, for $0 \leq t \leq 6\pi$.

Solution: The parametric speed is easily computed as $|\dot{\mathbf{r}}(t)| = \sqrt{a^2 + b^2} \equiv c$, which is a constant. Therefore the curve is regular and its arc length is

$$s(t) = \int_0^t |\dot{\mathbf{r}}| dt = \int_0^t \sqrt{a^2 + b^2} dt = ct.$$

We can easily re-parameterize the curve with arc length by replacing t by $\frac{s}{c}$ yielding $\mathbf{r} = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c})^T$. The first three derivatives are evaluated as

$$\mathbf{r}'(s) = \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c} \right)^T,$$

$$\mathbf{r}''(s) = \left(-\frac{a}{c^2} \cos \frac{s}{c}, -\frac{a}{c^2} \sin \frac{s}{c}, 0 \right)^T,$$

$$\mathbf{r}'''(s) = \left(\frac{a}{c^3} \sin \frac{s}{c}, -\frac{a}{c^3} \cos \frac{s}{c}, 0 \right)^T.$$

The curvature and torsion are evaluated as follows:

$$\kappa^2 = \mathbf{r}'' \cdot \mathbf{r}''' = \frac{a^2}{c^4} \left(\cos^2 \frac{s}{c} + \sin^2 \frac{s}{c} \right) = \frac{a^2}{c^4} = \text{constant},$$

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$$\tau = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}'' \cdot \mathbf{r}'''} = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\kappa^2} = \frac{c^4}{a^2} \begin{vmatrix} -\frac{a}{c} \sin \frac{s}{c} & \frac{a}{c} \cos \frac{s}{c} & \frac{b}{c} \\ -\frac{a}{c^2} \cos \frac{s}{c} & -\frac{a}{c^2} \sin \frac{s}{c} & 0 \\ \frac{a}{c^3} \sin \frac{s}{c} & -\frac{a}{c^3} \cos \frac{s}{c} & 0 \end{vmatrix}$$

$$= \frac{c^4}{a^2} \frac{b}{c} \frac{a^2}{c^5} \left(\cos^2 \frac{s}{c} + \sin^2 \frac{s}{c} \right) = \frac{b}{c^2} = \text{constant}.$$

1.3.1 Surfaces

A surface is a locus of points whose coordinates are functions of two independent parameters u and v . thus the parametric equations for a surface defined in a Cartesian system are

$$x = f_1(u, v) \quad y = f_2(u, v) \quad z = f_3(u, v)$$

and the surface is defined by some function $F(u, v)=0$. Consider any curve drawn on the surface. Let s be the arc length of the curve measured from some fixed point on the curve to the current point $\{x, y, z\}$. then the tangent to the curve is the vector $\{x', y', z'\}$ where the "''' refers to differentiation with respect to s . Now the straight line generated by the tangent to a point $\{x, y, z\}$ is normal to the vector $\{F_x, F_y, F_z\}$, where F_x is the partial derivative of F with respect to x . F remains constant along any curve as s varies.

Thus

$$F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_z \frac{dz}{ds} = 0$$

Thus $\{x', y', z'\}$ and $\{F_x, F_y, F_z\}$ are perpendicular. So all tangent lines at a point are perpendicular to this vector and thus lie in a plane through $\{x, y, z\}$ perpendicular to this vector. This is the tangent plane. The normal to the plane at the point of contact is the normal to the surface at that point. In particular the parametric curves are those for which

$$u = \text{constant} \text{ or } v = \text{constant}$$

Then if we denote

$$r_1 = \frac{\partial r}{\partial u} \quad r_2 = \frac{\partial r}{\partial v}$$

We have r_1 as a vector tangent to the curve $v = \text{constant}$, at the point r . Consider two neighbouring points on the surface with position vectors r and $r + dr$ corresponding to the parameter values (u, v) and $(u+du, v+dv)$ respectively.

$$\text{Then } dr = r_1 du + r_2 dv$$

Since the two points are arbitrarily closely spaced points on a curve passing through them, the length ds of the element of arc joining them is equal to the actual distance $|dr|$ between them. Thus

$$ds^2 = r_1^2 du^2 + 2r_1 r_2 dudv + r_2^2 dv^2$$

We define $E = r_1^2$

$$F = r_1 r_2$$

$$G = r_2^2$$

These quantities are called the *fundamental magnitudes of the first order*.

Also we define a quantity H given as,

$$H^2 = EG - F^2$$

By definition the normal to the surface at any point is perpendicular to every tangent line through that point. Hence it is perpendicular to both r_1 and r_2 . Thus the unit normal is given by

$$n = r_1 \times r_2 / H$$

In a similar manner the second derivatives of r are denoted by

$$r_{11} = \frac{\partial^2 r}{\partial u^2} \quad r_{12} = \frac{\partial^2 r}{\partial u \partial v} \quad r_{22} = \frac{\partial^2 r}{\partial v^2}$$

The second order magnitudes of the surface are then defined as

$$L = n \cdot r_{11}$$

$$M = n \cdot r_{12}$$

$$N = n \cdot r_{22}$$

$$T^2 = LN - M^2$$

The curvature of a curve at a point can hence be defined. Consider any plane which intersects the surface at a particular point P and which contains the normal to surface at that point. The result of such a normal section is a curve and we may evaluate the curvature of that curve at the point P . However there are infinitely many planes through P which contain the normal to the surface at P .

Given a point P on the surface any curve on the surface passing through the point will have a tangent vector defined at the point. The plane containing all the tangent vectors for any curve passing through the point is called the tangent plane for that point. Suppose we intersect the tangent plane with the surface and examine the rate at which the surface deviates from the plane along any particular direction. We will find that there are two directions on the surface at right angles to each other such that in one direction the surface deviates the quickest from the plane and in the other direction the surface deviates the slowest. Both of these directions have the property that the normal at a consecutive point separated by an infinitesimal

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distance in either direction meets the normal at P . This means that the curve for a section along one of these directions has no torsion and is subject to curvature in only one direction. These are the *principal directions*.

The values for the curvature of the curves obtained by taking normal sections along these principal directions are extrema. In other words as we change the direction of the section, the curvature for a normal section achieves a maximum at one of the principal directions. It achieves a minimum at the other principal direction. Let the curvatures of these special sections be κ_a and κ_b respectively. In the case of a plane, the curvature of each normal section is identical. In this case any pair of perpendicular directions may be taken as the principal directions. A curve on the surface such that the normals at consecutive points intersect is called a line of curvature. The point of intersection of consecutive normals along a line of curvature at P is the centre of curvature at P , its distance from P is a principal radius of curvature and the reciprocal is a principal curvature.

Thus at each point there are two principal curvatures κ_a and κ_b . These are the normal curvatures of the surface in the directions of the lines of curvature. Given the principal curvatures, the curvature of the surface can be described in a number of ways. The first curvature of a surface is defined by

$$J = \kappa_a + \kappa_b$$

The second curvature of a surface also called the Gaussian curvature is defined by

$$K = \kappa_a \kappa_b$$

These are related to the fundamental magnitudes by

$$J = \frac{EN - 2FM + GL}{H^2}$$

$$K = \frac{LN - M^2}{H^2}$$

The first curvature is analogous to the curvature of a curve while the second curvature is analogous to the torsion of a curve.

Check Your Progress

4. What is curvature?
5. What is the range of values of curvature?
6. If $f(x) = e^{ax}$. What is $f'(x)$?

1.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The Particular Integral (P.I.) which is a solution of $F(D)y = Q$ containing no arbitrary constant.
2. $B^2 - 4AC < 0$.
3. If all the terms on left hand side of an equation are not of same degree then the equation is said to be non-homogeneous equation.
4. The angular change of an arcual length between a pair of points is known as curvature.
5. The values of curvature range from ∞ for a straight line to 0 for an arcual length of infinity.
6. $a e^{ax}$.

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1.5 SUMMARY

- The general form of a linear differential equation of n th order is
$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$
- $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = 0$
is then called the *Reduced Equation* (R.E.)
- Any particular solution of $F(D)y = f(x)$ is known as its Particular Integral (P.I.). The P.I. of $F(D)y = f(x)$ is symbolically written as

$$\text{P.I.} = \frac{1}{F(D)} \{f(x)\}$$
 where $F(D)$ is the operator.

- $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$

Where A, B, C, D, E, F and G are functions of x and y .

This equation when converted to quasi-linear partial differential equation takes the form,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

These equations are said to be of:

1. Elliptic type if $B^2 - 4AC < 0$
2. Parabolic type if $B^2 - 4AC = 0$
3. Hyperbolic type if $B^2 - 4AC > 0$

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- Let $f(D, D')z = V(x, y)$ (1.9)

Then if

$$f(D, D') = A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'_2 + \cdots + A_n D'^n$$

where A_1, A_2, \dots, A_n are constants.

Then Equation (1.9) is known as Homogeneous equation and takes the form

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n)z = V(x, y)$$

- The angular change of an arcual length between a pair of points is known as curvature denoted as κ (kappa) and is characterized by its centre, radius and circularity.
- The angle of inclination of the curve varies at a definite rate per unit arc length and the rate of change of angle of a circle is called curvature of the curve.

- Cartesian form: $\rho = \frac{(1 + y'^2)^{3/2}}{y''}$

- Parametric form:

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

- Polar form: $\rho = \frac{(r^2 + \dot{r}^2)^{3/2}}{r^2 + 2\dot{r}^2 - r\ddot{r}}$

- A surface is a locus of points whose coordinates are functions of two independent parameters u and v . thus the parametric equations for a surface defined in a Cartesian system are

$$x = f_1(u, v) \quad y = f_2(u, v) \quad z = f_3(u, v)$$

and the surface is defined by some function $F(u, v) = 0$.

1.6 KEY WORDS

- Ordinary differential equation:** An ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and the derivatives of those functions.
- Curvature of curve:** The angle of inclination of the curve varies at a definite rate per unit arc length and the rate of change of angle of a circle is called curvature of the curve.
- Surface:** A surface is a locus of points whose coordinates are functions of two independent parameters.

1.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Find the reduced equation of the general form of a linear differential equation of n th order.
2. What are complimentary functions? Discuss.
3. Give an example equation of the following:
 - (a) Elliptic type
 - (b) Parabolic type
 - (c) Hyperbolic type
4. Give $f'(x)$ for the following $f(x)$:
 - (a) x^n
 - (b) $\log(\sec x + \tan x)$
 - (c) $\sin^{-1} x$
5. What is curvature of a curve and curvature of a circle? Discuss.
6. Define curves with the help of torsion and binormal.
7. Write a short note on surfaces.

Long Answer Questions

1. Find the general solution of $(D^3 - 2D^2)y = (e^2x + 3) + ex \cosh x$.
2. Solve $(D^2 + 2D - 1)y = x e^3x$
3. Solve $(D^2 + 1)y = \sin 2x$.
4. Solve $(D^2 - 2D + 8)y = 3 \cos x$.
5. Solve $(D^2 - 7)y = 2x \sin^2 x$.
6. Solve the differential equation $(D^2 + 2D'^2 - 1)z = 0$.
7. Solve the differential equation $(D^4 + 2D^2D' - 2D'^2)z = 0$.
8. Solve the equation $(D^2 + D'^2 - 2D + 3D')z = e^{x-y}$.
9. Solve the differential equation $(x^2D^2 + xyDD' + 3y^2D')z = (x^2 - y)^{n/2}$.
Find the radius of curvature at $(a/2, a/2)$ on the curve $2ay^2 = (2a - x)3$.

NOTES

1.8 FURTHER READINGS

Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.

Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.

*Ordinary Differential
Equations in More Than
Two Variables*

NOTES

- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 2 SIMULTANEOUS DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

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2.0 INTRODUCTION

The term differential equation, sometimes called ordinary differential equation to distinguish it from partial differential equations and other variants, is an equation involving two variables, an independent variable x and a dependent variable y , as well as the derivatives (first and possibly higher) of y with respect to x . The term first-order differential equation is used for any differential equation whose order is 1. A first-degree differential equation is a differential equation that is linear in its highest-order derivative.

In this unit, you will learn how differential equations are identified and also about their formation and definition. Ordinary differential equations and partial differential equations of the order occur with the highest order differential coefficient.

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You will also learn how to solve differential equations of first order and first degree. The relation between the dependent and independent variables are formed to satisfy the equation. Homogeneous and non-homogeneous equations with constant coefficients can be recognized and solved with the help of a trial solution. You will also learn how equations are reduced to their linear forms. The Cauchy and Legendre linear equations can be derived from Euler's equation. You will be able to deduce and find simultaneous linear equation using constant coefficients. This unit also explains the methodology for finding the integrating factors when an equation is an exact differential equation.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Describe differential equations of first order and first degree
- Solve differential equations of first order and first degree
- Discuss homogeneous and non-homogeneous equations
- Analyze integrating factors
- Discuss simultaneous differential equations of first order and first degree

2.2 DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND THE FIRST DEGREE

In mathematics, a differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.

Basic Definitions

Equations in which an unknown function, and its derivatives or differentials occur are called *differential equations*.

For example,

$$(i) \quad x + y \frac{dy}{dx} = 3y \qquad (ii) \quad \frac{dy}{dx} + \frac{x-y}{x+y} = 0$$

$$(iii) \quad \frac{d^2y}{dx^2} + y = \sin x, \text{ and} \qquad (iv) \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

are all differential equations.

If, in a differential equation, the unknown function is a function of one independent variable, then it is known as an *ordinary differential equation*. If the unknown function is a function of two or more independent variables and the equation involves partial derivatives of the unknown function, then it is known as a

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partial differential equation. In the above examples, equations from (i) to (iii) are ordinary differential equations while the fourth is a partial differential equation.

The order of a differential equation is the order of the highest differential coefficient which occurs in it. In the examples given above, (i) and (ii) are of first order while (iii) and (iv) are of second order.

The degree of a differential equation is the degree of the highest order differential coefficient which occurs in it, after the equation has been cleared of radicals and fractions. The above listed equations are all of degree 1. To decide, for example, the degree of the differential equation,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2}, \text{ we rewrite it as}$$

$$\rho^2 \left(\frac{d^2y}{dx^2} \right)^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 \text{ and observe that it is of degree 2.}$$

If in a differential equation, the derivatives and the dependent variables appear in first power and there are no products of these, and also the coefficients of the various terms are either constants or functions of the independent variable, the equation is said to be a *linear differential equation*.

2.2.1 Solution of Differential Equations

A relation between the dependent and independent variables, which, when substituted in the equation, satisfies it, is known as a *solution* or a *primitive* of the equation. Note that, in the solution, the derivatives of the dependent variable should not be present.

The solution, in which the number of arbitrary constants occurring is equal to the order of the equation, is known as the *general solution* or the *complete integral*. By giving particular values to the arbitrary constants appearing in the general solution we obtain *particular solutions* of the equation.

For example, $y = Ae^{2x} + Be^{-2x}$, $y = 3e^{2x} + 2e^{-2x}$ are respectively the general solution and the particular solution of the equation $\frac{d^2y}{dx^2} - 4y = 0$.

Solutions of equations which do not contain any arbitrary constants and which are not derivable from the general solution by giving particular values to one or more of the arbitrary constants, are called *singular solutions*.

2.2.2 Formation of Differential Equations

Consider the relation $y = Ae^x + Be^{-x}$... (2.1)

Where A and B are constants.

$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

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$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$$\frac{d^2y}{dx^2} = y \quad \dots(3.2)$$

Equation (2.2), which is obtained from Equation (2.1) by eliminating the arbitrary constants A and B , is the differential equation whose primitive is Equation (2.1). Further, Equation (2.1) is the general solution of (2.2).

The order of the ordinary differential equation and the number of arbitrary constants appearing in the general solution will be always equal. Also, if there are n arbitrary constants in the relation between the dependent and the independent variables, then by eliminating them, we will arrive at a differential equation of order n .

Example 2.1: Obtain the differential equation associated with the primitive $y = Cx + C - C^3$, where C is an arbitrary constant.

Solution:

$$y = Cx + C - C^3 \quad \dots(1)$$

$$\frac{dy}{dx} = C \quad \dots(2)$$

Using equations (2) and (1), we get the differential equation,

$$\left(\frac{dy}{dx}\right)^3 - (x+1)\frac{dy}{dx} + y = 0$$

Example 2.2: Form the differential equation satisfied by all circles having their centres on the straight line $y = 10$ and touching the x -axis.

Solution: As the centre lies on the line $y = 10$ and the circle touches the x -axis, the centre of the circle is $(a, 10)$ and its radius will be 10.

Equation of the circle is,

$$(x-a)^2 + (y-10)^2 = 100 \quad \dots(1)$$

Where a is an arbitrary constant. Differentiating equation (1) with respect to x ,

$$2(x-a) + 2(y-10)\frac{dy}{dx} = 0$$

$$(x-a) = -(y-10)\frac{dy}{dx} \quad \dots(2)$$

Substituting equation (2) in (1) we get,

$$(y-10)^2 \left(\frac{dy}{dx}\right)^2 + (y-10)^2 = 100$$

$$\text{i.e., } (y - 10)^2 \left[1 + \left(\frac{dy}{dx} \right)^2 \right] = 100$$

This is the equation for which equation (1) is the primitive.

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Example 2.3: Eliminate A and B from $y = Ae^x + Be^{2x}$

Solution: $y = Ae^x + Be^{2x}$... (1)

$$\frac{dy}{dx} = Ae^x + 2Be^{2x}$$
 ... (2)

$$\frac{d^2y}{dx^2} = Ae^x + 4Be^{2x}$$
 ... (3)

$$\frac{dy}{dx} - 2y = -Ae^x$$
 ... (4)

$$\frac{d^2y}{dx^2} - 4y = -3Ae^x$$
 ... (5)

Eliminating A between equations (4) and (5), we get the following differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

2.2.3 Solution of Differential Equation of First Order and First Degree

An ordinary differential equation of first order and first degree can be written as

$$\frac{dy}{dx} = f(x, y) \quad \dots (2.3)$$

The differential equation can be classified as

- (1) Exact equations.
- (2) Equations solvable by separation of variables.
- (3) Homogeneous equations.
- (4) Linear equation of first order.

Exact Equations

An equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

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is called exact if there exists a function $u(x, y)$ such that

$$du = Mdx + Ndy$$

A necessary and sufficient condition for the exactness of the differential equation $Mdx + Ndy = 0$ is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the differential equation $Mdx + Ndy = 0$ is exact, then the general solution of this differential equation can be obtained by the following rules:

(1) Integrate Mdx taking y as constant.

(2) Integrate Ndy taking x as constant.

(3) Then add the two integrates ignoring the repeated terms if there be any but add a constant of integration to get the general solution.

Example 2.4: Solve $(x^2 - y)dx + (y^2 - x)dy = 0$.

Solution: Here $M = x^2 - y$ and $N = y^2 - x$

Now $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence the differential equation is exact.

Now integrating M w.r.t. x keeping y as constant, we get

$$\int M dx = \int (x^2 - y) dx = \frac{x^3}{3} - xy$$

Integrating N w.r.t. y keeping x as constant, we get

$$\int N dy = \int (y^2 - x) dy = \frac{y^3}{3} - xy$$

Then the general solution (or complete primitive) is $\frac{x^3}{3} - xy + \frac{y^3}{3} = c$, where

c is an arbitrary constant.

Definition: A function $f(x, y)$ is said to be an *Integrating Factor* (I.F.) of the equation $Mdx + Ndy = 0$ if we can find a function $u(x, y)$ such that $f(x, y)(Mdx + Ndy) = du$. In other words, an I.F. is a multiplying factor by which the equation can be made exact.

For solving differential equations, the following results will be of much help:

Sl. No.	Given Expression	Integrating Factor	Exact Differential
1.	$x \, dy - y \, dx$	$\frac{1}{x^2}$	$\frac{x \, dy - y \, dx}{x^2} = d\left(\frac{y}{x}\right)$
2.	$x \, dy - y \, dz$	$\frac{1}{x^2 + y^2}$	$\frac{x \, dy - y \, dx}{x^2 + y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$
3.	$x \, dx + y \, dy$	$\frac{1}{x^2 + y^2}$	$\begin{aligned} \frac{x \, dx + y \, dy}{x^2 + y^2} &= \frac{\frac{1}{2}d(x^2 + y^2)}{x^2 + y^2} \\ &= \frac{1}{2}d[\log(x^2 + y^2)] \end{aligned}$
4.	$x \, dy - y \, dx$	$\frac{1}{xy}$	$\begin{aligned} \frac{x \, dy - y \, dx}{xy} &= \frac{dy}{y} - \frac{dx}{x} \\ &= d\left\{\log\left(\frac{y}{x}\right)\right\} \end{aligned}$
5.	$y \, dx - x \, dy$	$\frac{1}{y^2}$	$\frac{y \, dx - x \, dy}{y^2} = d\left(\frac{x}{y}\right)$

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Example 2.5: Solve $x \, dx + y \, dy + \frac{x \, dy - y \, dx}{x^2 + y^2} = 0$.

Solution: Here $x \, dx + y \, dy + \frac{x \, dy - y \, dx}{x^2 + y^2} = 0$

$$\text{or } \frac{1}{2}d(x^2 + y^2) + \frac{\frac{x \, dy - y \, dx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = 0$$

$$\text{or } \frac{1}{2}d(x^2 + y^2) + d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = 0$$

By integrating, we get

$$\frac{1}{2}(x^2 + y^2) + \tan^{-1} \frac{y}{x} = c, \text{ where } c \text{ is an arbitrary constant.}$$

Example 2.6: Solve $x \, dx + y \, dy = m(x \, dy - y \, dx)$.

Solution: Here $x \, dx + y \, dy = m(x \, dy - y \, dx)$

$$\text{or } \frac{x \, dx + y \, dy}{x^2 + y^2} = m\left(\frac{x \, dy - y \, dx}{x^2 + y^2}\right) \quad \left(\text{Multiplying both sides by } \frac{1}{x^2 + y^2}\right)$$

$$\text{or} \quad \frac{\frac{1}{2}d(x^2 + y^2)}{x^2 + y^2} = m \frac{x dy - y dx}{x^2}$$

$$1 + \frac{y^2}{x^2}$$

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$$\text{or} \quad \frac{1}{2}d\{\log(x^2 + y^2)\} = m d\left\{\tan^{-1}\frac{y}{x}\right\}$$

By integrating, we get $\frac{1}{2}\log(x^2 + y^2) = m \tan^{-1}\frac{y}{x} + c$, where c is an arbitrary constant.

2.2.4 Variable Separable

Differential equations of the form $f(x) dx = g(y) dy$ are called equations with variables separated.

Integrating the equation, we get,

$$\int f(x) dx = \int g(y) dy + C$$

Example 2.7: Solve $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$

Solution: Rearranging we get,

$$\frac{3e^x}{1 - e^x} dx = -\frac{\sec^2 y}{\tan y} dy$$

$$-3 \int \frac{e^x}{1 - e^x} dx = \int \frac{\sec^2 y}{\tan y} dy$$

$$3 \log(1 - e^x) = \log \tan y + \log C$$

$$(1 - e^x)^3 = C \tan y$$

Example 2.8: Solve $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$

Solution:
$$\frac{dy}{dx} = -\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}$$

$$\int \frac{dy}{\sqrt{1 - y^2}} = - \int \frac{dx}{\sqrt{1 - x^2}}$$

$$\sin^{-1} y = -\sin^{-1} (x) + C$$

2.2.5 Homogeneous Equations

These equations are of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of x and y of the same degree.

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To solve these equations, put $y = vx$. Then,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting this in the given equation it reduces to the type in which the variables are separable.

Example 2.9: Solve $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$

Solution: Put $y = vx$, then,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Using these in the given equation we get,

$$\begin{aligned} v + x \frac{dv}{dx} &= \frac{vx^3}{x^3 + v^3 x^3} \\ \therefore x \frac{dv}{dx} &= \frac{v^4}{1 + v^3} \\ \frac{1 + v^3}{v^4} dv &= -\frac{dx}{x} \\ \int \frac{1}{v^4} dv + \int \frac{1}{v} dv &= -\int \frac{dx}{x} \\ -\frac{1}{3v^3} + \log v &= C - \log x \\ \log vx &= C + \frac{1}{3v^3} \\ \therefore \log y &= C + \frac{x^3}{3y^3} \text{ or } y = A e^{\frac{x^3}{3y^3}} \end{aligned}$$

2.2.6 Homogeneous Equations with Constant Coefficients

An equation of the form,

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \quad \dots(2.4)$$

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Where a_1, a_2, \dots, a_n are constants and the equation is called a homogeneous linear partial equation of n^{th} order with constant coefficients.

$$\text{Let, } D^n = \frac{\partial z}{\partial x^n}; D'^n = \frac{\partial z}{\partial y^n}$$

Substituting above equation in (3.4) we get,

$$(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n)z = F(x, y)$$

$$f(D, D')z = F(x, y)$$

The solution will contain two parts:

1. Complementary Function (CF)
2. Particular Integral (PI)

To find complementary function, consider $f(D, D')z = 0$.

Auxiliary equation is,

$$f(m, 1) = 0 \text{ when } (D = m, D' = 1)$$

Now let us find the roots of the equation.

Roots	CF
1. m_1, m_2, m_3, \dots	$\phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x) + \dots$
2. m_1, m_1, m_1, \dots	$\phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) + \dots$

For example,

1. If the roots are $-1, 2, 6$

$$\text{CF} = \phi_1(y - x) + \phi_2(y + 2x) + \phi_3(y + 6x)$$

2. If the roots are $-2, -2, -2$

$$\text{CF} = \phi_1(y - 2x) + x\phi_2(y - 2x) + x^2\phi_3(y - 2x)$$

3. If the roots are $1, 1, 2$

$$\text{CF} = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x)$$

Rules to Find PI

Let, $u(x, y)$ be the PI

Then, $f(D, D') u(x, y) = F(x, y)$

$$\therefore u(x, y) = \frac{1}{f(D, D')} F(x, y)$$

1. $\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}, f(a, b) \neq 0$

2. (a) $\frac{1}{f(D, D')} \sin(ax+by)$ (b) $\frac{1}{f(D, D')} \cos(ax+by)$

Replace D^2 by $-a^2$, D'^2 by $-b^2$, DD' by $-ab$

3. $\frac{1}{f(D, D')} x^r y^s$

Expand $\frac{1}{f(D, D')}$ in ascending powers of $\frac{D'}{D}$ up to $\left(\frac{D'}{D}\right)^r$ and operate on $x^r y^s$.

or,

Expand $\frac{1}{f(D, D')}$ in ascending powers of $\frac{D}{D'}$ up to $\left(\frac{D}{D'}\right)^s$ and operate on $x^r y^s$.

4. $\frac{1}{f(D, D')} e^{ax+by} g(x, y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} g(x, y)$

5. $\frac{1}{D - mD'} f(x, y)$

Change y to $y - mx$ in $f(x, y)$ and integrate with respect to x treating y as constant. In the resulting integral change y to $y + mx$.

Notes: 1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \infty$; 2. $(1-x)^{-1} = 1 + x + x^2 + \dots \infty$.

Example 2.10: Solve $\frac{1}{D^2 + DD' - 6D'^2} y \cos x$

Solution: $\frac{1}{(D-2D')(D+3D')} y \cos x$

Replace y by $y + 3x$. Keep y as a constant.

$$= \frac{1}{(D-2D')} \int (y+3x) \cos x \, dx$$

$$= \frac{1}{D-2D'} [(y+3x) \sin x - 3(-\cos x)]$$

After integration replace y by $y - 3x$

$$= \frac{1}{D-2D'} y \sin x + 3 \cos x$$

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Replace y by $y - 2x$. While integrating keep y as a constant.

$$\begin{aligned} &= \int [(y - 2x)\sin x + 3\cos x] dx \\ &= (y - 2x)(-\cos x) - (-2)(-\sin x) + 3\sin x \\ \text{After integration replace } y \text{ by } y + 2x \\ &= (y + 2x - 2x)(-\cos x) - 2\sin x + 3\sin x \\ &= -y\cos x + \sin x \end{aligned}$$

Example 2.11: Solve $\frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$

$$\begin{aligned} \text{Solution:} \quad &= \frac{1}{(D - 2D')^2} e^{2x+y} \text{ when, } D = 2, D' = 1, Dr = 0 \\ &= \frac{1}{(D - 2D')(D - 2D')} e^{2x+y} \\ &= \frac{1}{D - 2D'} \int e^{2x+y-2x} dx \\ &= \frac{1}{D - 2D'} e^y \cdot x \\ &= \frac{1}{D - 2D'} e^{y+2x} \cdot x \\ &= \int e^{y-2x+2x} \cdot x dx = \int e^y \cdot x dx \\ &= e^y \frac{x^2}{2} \\ &= e^{y+2x} \cdot \frac{x^2}{2} \end{aligned}$$

Example 2.12: Solve $(2D^2 + 5DD' + 2D'^2)z = 0$

Solution: Auxiliary equation is,

$$\begin{aligned} 2m^2 + 5m + 2 &= 0 \\ 2m^2 + 4m + m + 2 &= 0 \\ 2m(m+2) + 1(m+2) &= 0 \\ (2m+1)(m+2) &= 0 \\ m &= \frac{-1}{2}, -2 \\ \therefore z &= \phi_1 \left(y - \frac{x}{2} \right) + \phi_2(y - 2x) \end{aligned}$$

Example 2.13: Solve $(D^3 - 4D^2D' + 4DD'^2)z = 0$

Solution:

Auxiliary equation is,

$$\begin{aligned}m^3 - 4m^2 + 4m &= 0 \\m(m^2 - 4m + 4) &= 0 \\m(m-2)^2 &= 0 \\m &= 0, 2, 2\end{aligned}$$

$$\therefore z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$$

2.2.7 Non-Homogeneous Equations

The general form of the equation of this type is,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \dots(2.5)$$

Where atleast one c_1 and c_2 is non-zero.

The cases of this type are considered below:

Case (1): When $b_1 = -a_2$

Then the equation becomes,

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{-b_1x + b_2y + c_2}$$

Cross-multiplying we get,

$$\begin{aligned}(-b_1x + b_2y + c_2)dy &= (a_1x + b_1y + c_1)dx \\b_2ydy + c_2dy - a_1xdx - c_1dx &= b_1(ydx + xdy) \\i.e., \quad (b_2y + c_2)dy - (a_1x + c_1)dx &= b_1d(xy)\end{aligned}$$

Integrating we get,

$$b_2 \frac{y^2}{2} + c_2y - a_1 \frac{x^2}{2} - c_1x = b_1xy + K$$

Example 2.14: Solve $\frac{dy}{dx} = \frac{2x-y+3}{x-3y-1}$

Solution: Cross-multiplying and rearranging the terms, we get,

$$(xdy + ydx) = (2x+3)dx + (3y+1)dy$$

Integrating we get,

$$\int d(xy) = \int (2x+3)dx + \int (3y+1)dy$$

$$xy = x^2 + 3x + \frac{3}{2}y^2 + y + K \text{ is the solution.}$$

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Case (2): When $a_1 = b_2$ and $b_1 = a_2$
Then the equation becomes,

$$\begin{aligned}\frac{dy}{dx} &= \frac{a_1x + b_1y + c_1}{b_1x + a_1y + c_2} \\ \frac{dy}{a_1x + b_1y + c_1} &= \frac{dx}{b_1x + a_1y + c_2} \\ &= \frac{d(x+y)}{(a_1 + b_1)(x+y) + c_1 + c_2} \\ &= \frac{d(x-y)}{(b_1 - a_1)(x-y) + c_2 - c_1} \\ \int \frac{d(x+y)}{(a_1 + b_1)(x+y) + c_1 + c_2} &= \int \frac{d(x-y)}{(b_1 - a_1)(x-y) + c_2 - c_1} \\ \frac{1}{(a_1 + b_1)} \log[(a_1 + b_1)(x+y) + (c_1 + c_2)] &\\ &= \frac{1}{(b_1 - a_1)} \log[(b_1 - a_1)(x-y) + (c_2 - c_1)] + K\end{aligned}$$

Example 2.15: Solve $\frac{dy}{dx} = \frac{3x+4y-1}{4x+3y+2}$

Solution:

$$\begin{aligned}\frac{dy}{3x+4y-1} &= \frac{dx}{4x+3y+2} = \frac{d(x+y)}{7(x+y)+1} = \frac{d(x-y)}{(x-y)+3} \\ \int \frac{d(x+y)}{7(x+y)+1} &= \int \frac{d(x-y)}{(x-y)+3} \\ \frac{1}{7} \log[7(x+y)+1] &= \log[(x-y)+3] + \log c \\ \frac{1}{7} \log[7(x+y)+1] &= \log c[(x-y)+3] \\ \log [7(x+y)+1] &= \log c[(x-y)+3]^7 \\ 7x+7y+1 &= K(x-y+3)^7\end{aligned}$$

Non-homogeneous Equations with Constant Coefficients

Consider the equation of the form,

$$f(D, D')z = F(x, y) \quad \dots(2.6)$$

If $f(D, D')$ is not homogeneous then Equation (2.6) is called a non-homogeneous linear equation.

PI can be found as in homogeneous linear equations.

$$\text{To find CF, consider } f(D, D')z = 0 \quad \dots(2.7)$$

Assume, $z = Ce^{hx+ky}$ as a trial solution.

Substituting in Equation (2.7) we get, $f(h, k) = 0$

Find k in terms of h [or h in terms of k]

Let the r values of k be,

$$f_1(h), f_2(h), \dots, f_r(h)$$

Then, $z = C_n e^{hx+f_n(h)y}$, $n = 1, 2, \dots, r$

will be the separate solution of Equation (2.7).

The general solution of Equation (2.7) is of the form,

$$z = \sum C_1 e^{hx+f_1(h)y} + \sum C_2 e^{hx+f_2(h)y} + \dots + \sum C_r e^{hx+f_r(h)y}$$

Example 2.16: Solve $(D^2 - DD' + D' - 1)z = e^x$

Solution:

To find CF, consider,

$$(D^2 - DD' + D' - 1)z = 0$$

Let, $z = Ce^{hx+ky}$ as a trial solution.

$$f(h, k) = 0$$

$$h^2 - hk + k - 1 = 0$$

$$\begin{aligned} h &= \frac{k \pm \sqrt{k^2 - 4(k-1)}}{2} \\ &= \frac{k \pm \sqrt{(k-2)^2}}{2} \\ &= \frac{k \pm (k-2)}{2} = k-1, 1 \end{aligned}$$

$$\begin{aligned} \therefore \text{CF} &= \sum C_1 e^{(k-1)x+ky} + \sum C_2 e^{x+ky} \\ &= e^{-x} \sum C_1 e^{k(x+y)} + e^x \sum C_2 e^{ky} \\ &= e^{-x} \phi_1(y+x) + e^x \phi_2(y) \end{aligned}$$

$$\text{PI} = \frac{1}{D^2 - DD' + D' - 1} e^x$$

Put, $D = 1, D' = 0, Dr = 0$

$$\begin{aligned} &= \frac{1}{(D-D'+1)(D-1)} e^x \\ &= \frac{1}{D-1} e^x \end{aligned}$$

NOTES

$$= e^x \frac{1}{D+1-1} (1) \\ = xe^x$$

NOTES

∴

$$z = e^{-x}\phi_1(y+x) + e^x\phi_2(y) + xe^x$$

Example 2.17: Solve $(D^2 + DD' + D' - 1)z = \cos(x-y)$

Solution:

To find CF, consider

$$(D^2 + DD' + D' - 1)z = 0$$

Assume, $z = Ce^{hx+ky}$ as a trial solution.

$f(h, k) = 0$ becomes, $h^2 + hk + k - 1 = 0$

$$h = \frac{-k \pm \sqrt{k^2 - 4(k-1)}}{2} \\ = \frac{-k \pm \sqrt{(k-2)^2}}{2} \\ = \frac{-k \pm (k-2)}{2} \\ = -1, -k+1$$

$$\text{CF} = \sum C_1 e^{-x+ky} + \sum C_2 e^{(-k+1)x+ky} \\ = e^{-x} \sum C_1 e^{ky} + e^x \sum C_2 e^{k(y-x)} \\ = e^{-x}\phi_1(y) + e^x\phi_2(y-x)$$

$$\text{PI} = \frac{1}{D^2 + DD' + D' - 1} \cos(x-y)$$

Put,

$$D^2 = -a^2 = -1$$

$$DD' = -ab = 1$$

$$D'^2 = -b^2 = -1$$

$$= \frac{1}{-1+1+D'-1} \cos(x-y)$$

$$= \frac{1}{D'-1} \cos(x-y)$$

$$= \frac{(D'+1)}{D'^2-1} \cos(x-y)$$

$$= -\frac{1}{2}(\sin(x-y) + \cos(x-y))$$

$$\therefore z = e^{-x} f_1(y) + e^x f_2(y-x) - \frac{1}{2}[\sin(x-y) + \cos(x-y)]$$

NOTES

Example 2.18: Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{x-y} + x^2y$

Solution: To find CF, consider

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = 0$$

$f(h, k) = 0$ becomes,

$$h^2 + 2hk + k^2 - 2h - 2k = 0$$

$$h^2 + 2h(k-1) + k^2 - 2k = 0$$

$$h = \frac{-2(k-1) \pm \sqrt{4(k-1)^2 - 4(k^2 - 2k)}}{2}$$

$$= \frac{-2(k-1) \pm \sqrt{4}}{2}$$

$$= 1 - k \pm 1$$

$$= 2 - k, -k$$

$$\begin{aligned} \text{CF} &= \sum C_1 e^{(2-k)x+ky} + \sum C_2 e^{-kx+ky} \\ &= e^{2x} \sum C_1 e^{k(y-2x)} + \sum C_2 e^{k(y-x)} \\ &= e^{2x} \varphi_1(y-2x) + \varphi_2(y-x) \end{aligned}$$

$$\text{PI}_1 = \frac{1}{(D+D'-2)(D+D')} e^{x-y}$$

$$= \frac{1}{(1-1-2)(D+D')} e^{x-y}$$

$$= -\frac{1}{2} \int e^{x-(y+x)} dx$$

$$= -\frac{1}{2} e^{-y} x$$

$$= -\frac{1}{2} x e^{x-y}$$

$$\text{PI}_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} x^2 y$$

$$= \frac{1}{-2D \left[1 + \frac{D'}{D} - \frac{D'^2}{2D} - D' - \frac{D}{2} \right]} x^2 y$$

NOTES

$$\begin{aligned}
 &= \frac{1}{-2D \left[1 - \left(\frac{D}{2} + D' - \frac{D'}{D} \right) \right]} x^2 y \quad (\because D'^2(x^2 y) = 0) \\
 &= \frac{1}{-2D \left[1 - \left(\frac{D}{2} + D' - \frac{D'}{D} \right) \right]^{-1}} x^2 y \\
 &= \\
 &- \frac{1}{2D} \left[1 + \frac{D}{2} + D' - \frac{D'}{D} + \frac{D^2}{4} + DD' - D' - \frac{3}{4} DD' + \frac{3}{4} D^2 D' \right] x^2 y \\
 &= -\frac{1}{2D} \left[1 + \frac{D}{2} - \frac{D'}{D} + \frac{1}{4} DD' + \frac{D^2}{4} + \frac{3}{4} D^2 D' \right] x^2 y \\
 &= -\frac{1}{2} \left[\frac{1}{D} + \frac{1}{2} - \frac{D'}{D^2} + \frac{D'}{4} + \frac{D}{4} + \frac{3}{4} DD' \right] x^2 y \\
 &= -\frac{1}{2} \left[\frac{x^3}{3} y + \frac{1}{2} x^2 y - \frac{x^4}{12} + \frac{x^2}{4} + \frac{xy}{2} + \frac{3x}{2} \right] \\
 \therefore z &= e^{2x} \varphi_1(y - 2x) + \varphi_2(y - x) - \frac{x}{2} e^{x-y} - \frac{1}{2} \left[\frac{x^3 y}{3} + \frac{x^2 y}{2} - \frac{x^4}{12} + \frac{x^2}{4} + \frac{xy}{2} + \frac{3x}{2} \right]
 \end{aligned}$$

2.2.8 Exact Differential Equations and Integrating Factors

In mathematics, an integrating factor is a function that is used to solve the given equation with the help of differential equations.

Exact Differential Equation

A differential equation is said to be exact if it can be derived directly from its primitive without any further operation of elimination or reduction. Thus the differential equation,

$$M(x, y)dx + N(x, y)dy = 0 \quad \dots(2.8)$$

is exact if it can be derived by equating the differential of some function $U(x, y)$ to zero.

Let, $U(x, y) = C$ be the solution of the Equation (2.8).

Differentiating this we get,

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = 0 \quad \dots(2.9)$$

Equations (2.8) and (2.9) are identical,

$$M = \frac{\partial U}{\partial x}, \quad N = \frac{\partial U}{\partial y}$$

If we eliminate U between these by means of the equivalence of the relation, then we get,

$$\frac{\partial}{\partial x} \left(\frac{\partial U}{\partial y} \right) = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial U}{\partial x} \right)$$

And, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Thus, the condition for $M dx + N dy = 0$, to be an exact equation is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Rules for Solving $Mdx + Ndy = 0$, when it is Exact

- (i) First integrate M with respect to x regarding y as a constant.
- (ii) Then integrate with respect to y those terms in N which do not contain x .
- (iii) The sum of the expressions obtained in (i) and (ii), when equated to an arbitrary constant, will be the solution.

Example 2.19: Solve $(\sin x \cos y + e^{2x})dx + (\cos x \sin y + \tan y)dy = 0$

Solution:

Here, $M = \sin x \cos y + e^{2x}$
 $N = \cos x \sin y + \tan y$
 $\frac{\partial M}{\partial y} = -\sin x \sin y; \quad \frac{\partial N}{\partial x} = -\sin x \sin y$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Integrating M with respect to x

regarding y as a constant, we get, $\left[-\cos x \cos y + \frac{1}{2}e^{2x} \right]$

In N , the term not involving x namely $\tan y$ is integrated with respect to y giving $\log \sec y$.

\therefore The solution is,

$$-\cos x \cos y + \frac{e^{2x}}{2} + \log \sec y = C$$

NOTES

Example 2.20: Solve $(ye^{xy} - 2y^3)dx + (xe^{xy} - 6xy^2 - 2y)dy = 0$

Solution: $M = ye^{xy} - 2y^3, \frac{\partial M}{\partial y} = e^{xy} + xye^{xy} - 6y^2$

NOTES

$$N = xe^{xy} - 6xy^2 - 2y, \frac{\partial N}{\partial x} = e^{xy} + xye^{xy} - 6y^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$\begin{aligned}\therefore \int M dx &= \int (ye^{xy} - 2y^3)dx \\ &= y \frac{e^{xy}}{y} - 2xy^3 = e^{xy} - 2xy^3\end{aligned}$$

Integrating those terms in N which do not contain x with respect to y , we get,

$$\int N dy = \int -2y dy = -y^2, \text{ omitting terms involving } x \text{ in } N.$$

\therefore The solution is, $e^{xy} - 2xy^3 - y^2 = C$

Note: Sometimes when the equation is not apparently exact, by suitably regrouping the terms we can find an integrating factor, which when multiplied by the equation will make it an exact equation.

Example 2.21: Solve $y(2x^2y + e^x)dx - (e^x + y^3)dy = 0$

Solution:

$$\begin{aligned}M &= 2x^2y^2 + ye^x, \quad \frac{\partial M}{\partial y} = 4x^2y + e^x \\ N &= -(e^x + y^3), \quad \frac{\partial N}{\partial x} = -e^x\end{aligned}$$

As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact. However, we can rearrange the equation as,

$$ye^x dx - e^x dy + (2x^2 dx - y dy)y^2 = 0$$

Now dividing by y^2 , we have

$$\frac{ye^x dx - e^x dy}{y^2} + 2x^2 dx - y dy = 0$$

i.e., $d\left(\frac{e^x}{y}\right) + 2x^2 dx - y dy = 0$

Integrating, we find the solution as, $\frac{e^x}{y} + \frac{2x^3}{3} - \frac{y^2}{2} = C$

NOTES

Note that, in this example, we used the integrating factor $\frac{1}{y^2}$.

Rules for Finding Integrating Factors

Rule I When $Mx + Ny = 0$, and the equation is a homogeneous one, then $\frac{1}{Mx + Ny}$ is an integrating factor.

Example 2.22: Solve $x^2 y dx - (x^3 + y^3) dy = 0$

Solution: The equation is not exact and $Mx + Ny = -y^4 \neq 0$. Hence, $-\frac{1}{y^4}$ can be used as an integrating factor. Then,

$$-\frac{x^2}{y^3} dx + \left(\frac{x^3 + y^3}{y^4} \right) dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$$

$$\frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$$

Hence, the equation has become exact,

$$\int M dx = -\int \frac{x^2}{y^3} dx = -\frac{x^3}{3y^3}$$

In N , integrating the term not containing x , namely $\frac{1}{y}$ with respect to y we get $\log y$.

$$\therefore \text{The solution is } -\frac{x^3}{3y^3} + \log y = C$$

Rule II If the equation is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor.

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Example 2.23: Solve $y(x^2y^2 + xy + 1)dx + x(x^2y^2 - xy + 1)dy = 0$

Solution: The equation is not exact, since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$Mx - Ny = x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy = 2x^2y^2 \neq 0$$

Using, $\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$ as an integrating factor we get,

$$\left(\frac{x^2y^2 + xy + 1}{2x^2y} \right) dx + \left(\frac{x^2y^2 - xy + 1}{2xy^2} \right) dy = 0$$

$$\left(y + \frac{1}{x} + \frac{1}{x^2y} \right) dx + \left(x - \frac{1}{y} + \frac{1}{xy^2} \right) dy = 0$$

Now, $\frac{\partial M}{\partial y} = 1 - \frac{1}{x^2y^2}$ and $\frac{\partial N}{\partial x} = 1 - \frac{1}{x^2y^2}$

\therefore The equation is exact and the solution is,

$$\int \left(y + \frac{1}{x} + \frac{1}{x^2y} \right) dx + \int -\frac{1}{y} dy = C$$

$$xy + \log x - \frac{1}{xy} - \log y = C$$

Rule III

(i) If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor.

(ii) If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is a function of y alone, say $g(y)$, then $e^{\int g(y)dy}$ is an integrating factor.

Example 2.24: Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

Solution: The equation is not exact and,

$$\frac{\partial M}{\partial y} = 3xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{y} = g(y)$$

$e^{\int g(y)dy} = e^{\log y} = y$ is an integrating factor.

Multiplying by y we get the equation,

$$(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$$

Now, the equation is exact and the solution is,

$$\int (xy^4 + y^2)dx + 2 \int y^5 dy = C$$

$$\text{i.e., } 3y^4x^2 + 6xy^2 + 2y^6 = C$$

Rule IV If the equation $Mdx + Ndy = 0$, is of the form $x^a y^b (my dx + nx dy) + x^r y^s (py dx + qxdy) = 0$, where a, b, m, n, r, s, p, q are constants, then $x^h y^k$, is an integrating factor, where h and k are determined using the condition that after multiplication by $x^h y^k$, the equation becomes exact.

Example 2.25: Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$

Solution: The equation is not an exact one and it can be rewritten as,

$$y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0$$

$$\text{or, } y^2(ydx + 2xdy) + x^2(-2ydx - xdy) = 0$$

So that, this is of the form mentioned in Rule IV above.

Multiplying the equation by $x^h y^k$ we get,

$$(x^h y^{3+k} - 2x^{h+2} y^{k+1})dx + (2x^{h+1} y^{k+2} - x^{h+3} y^k)dy = 0 \quad \dots(1)$$

$$\text{Now, } \frac{\partial M}{\partial y} = (3+k)x^h y^{k+2} - 2(k+1)x^{h+2} y^k \quad \text{and,}$$

$$\frac{\partial N}{\partial x} = 2(h+1)x^h y^{k+2} - (h+3)x^{h+2} y^k$$

NOTES

Using the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and equating the coefficients of like powered

terms on both sides, we get

NOTES

$$\begin{aligned} 3 + k &= 2(h + 1) && \text{or, } k - 2h \\ = -1 & & & \\ 2k + 2 &= h + 3 && \text{or, } 2k - h = 1 \end{aligned}$$

Solving them we get, $k = 1$, $h = 1$, so that the integrating factor is xy .

The equation (1) for these values of h and k becomes,

$$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0$$

As this equation is exact, the solution is,

$$\int (xy^4 - 2x^3y^2)dx = C$$

$$\frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$$

$$x^2y^4 - x^4y^2 = K$$

Check Your Progress

1. What is an ordinary differential equation?
2. What is a partial differential equation?
3. What is complete integral?
4. Write the form of a differential equation with variables separated.

2.3 SIMULTANEOUS DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND THE FIRST DEGREE IN THREE VARIABLES

A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x and is said to be linear equation in y .

Multiplying both sides by $e^{\int P dx}$ we get,

$$e^{\int P dx} \left(\frac{dy}{dx} + Py \right) = Q e^{\int P dx}$$

$$\frac{d}{dx} \left(y e^{\int P dx} \right) = Q e^{\int P dx}$$

Integrating, we get the solution as $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + C$

Note: $e^{\int Pdx}$ is called the integrating factor.

NOTES

Example 2.26: Solve $\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$, given that $y = 0$, when $x = \pi/2$.

Solution: Comparing with $\frac{dy}{dx} + Py = Q$ we find that,

$$P = \cot x, \quad Q = 4x \operatorname{cosec} x$$

$$\int Pdx = \int \cot x dx = \log \sin x$$

$$e^{\int Pdx} = e^{\log \sin x} = \sin x$$

$$\text{Solution is, } y \sin x = \int 4x \operatorname{cosec} x \sin x dx$$

$$= \int 4x dx = 2x^2 + C$$

$$y = 0, \text{ when } x = \frac{\pi}{2} \text{ gives, } C = -\frac{\pi^2}{2}. \therefore \text{The solution is, } y \sin x = 2x^2 - \frac{\pi^2}{2}$$

2.3.1 Equations Reducible to Linear Equations

Consider the equation $\frac{dy}{dx} + Py = Qy^n$

Where, P and Q are functions of x .

Dividing by y^n we get,

$$y^{-n} \frac{dy}{dx} + y^{1-n} P = Q \quad \dots(2.10)$$

$$\text{Putting, } v = y^{1-n}, \quad \frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$

Using this, Equation (2.10) becomes,

$$\frac{dv}{dx} + (1-n)v P = (1-n)Q$$

Which is a linear equation in v and hence, can be solved by the previous method.

NOTES

Example 2.27: Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Solution: Dividing by $\cos^2 y$ we get,

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \quad \dots(1)$$

Let $v = \tan y$. Then, $\frac{dv}{dx} = \sec^2 y \frac{dy}{dx}$

\therefore Equation (1) becomes, $\frac{dv}{dx} + 2vx = x^3$

$$P = 2x; \quad Q = x^3$$

$$\int P dx = \int 2x dx = x^2 \text{ and,}$$

$$e^{\int P dx} = e^{x^2}$$

$$\text{Solution is, } ve^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$\therefore ve^{x^2} = \int x^3 e^{x^2} dx + C = \int x x^2 e^{x^2} dx + C$$

$$\text{Put, } t = x^2; \quad dt = 2x dx$$

$$\therefore ve^{x^2} = \frac{1}{2} \int t e^t dt$$

$$ve^{x^2} = \frac{1}{2}(te^t - e^t) + C = \frac{1}{2}(x^2 e^{x^2} - e^{x^2}) + C$$

$$v = \frac{1}{2}(x^2 - 1) + Ce^{-x^2}$$

The solution is,

$$\tan y = \frac{1}{2}(x^2 - 1) + Ce^{-x^2}$$

2.3.2 Cauchy's and Legendre's Linear Equations

The equation of the form $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = X$ $\dots(2.11)$

Where a_0, a_1, \dots, a_n are constants and X is a function of x is called *Cauchy's linear equation derived from Euler's linear equation*.

Its more general form is,

$$a_0(bx+c)^n \frac{d^n y}{dx^n} + a_1(bx+c)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(bx+c) \frac{dy}{dx} + a_n y = X$$

Which includes Equation (2.11) for $b=1, c=0$. This is known as *Legendre's linear equation*.

The Equation (2.12) with the coefficient of $x^n \frac{d^n y}{dx^n}$ as unity namely,

$$x^n \frac{d^n y}{dx^n} + b_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + b_{n-1} x \frac{dy}{dx} + b_n y = X \quad \dots(2.12)$$

is called a *Homogeneous linear equation* of the n^{th} order.

Equation (2.11), which has variable coefficients x^n, x^{n-1}, \dots can be reduced to an equation with constant coefficients by putting $x = e^z$ or $z = \log x$.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz}$$

$$x \frac{dy}{dx} = \theta y, \text{ where } \theta \equiv \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right)$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dz} \right)$$

$$= -\frac{1}{x^2} \theta y + \frac{1}{x} \frac{d}{dz} \left(\frac{dy}{dz} \right) \frac{dz}{dx} = -\frac{1}{x^2} \theta y + \frac{1}{x^2} \theta^2 y$$

$$\therefore x^2 \frac{d^2 y}{dx^2} = \theta(\theta-1)y$$

Proceeding like this we may show that,

$$x^n \frac{d^n y}{dx^n} = \theta(\theta-1)(\theta-n+1)y$$

Example 2.28: Solve, $x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} + 13y = \log x$

Solution: $x^2 D^2 + 8xD + 13y = \log x$

Substituting $x = e^z$ transforms equation to,

NOTES

$$[\theta(\theta-1) + 80 + 13]y = z, \text{ where } \theta = \frac{d}{dz}$$

NOTES

AE is, $m^2 + 7m + 13 = 0$

$$m = \frac{-7 \pm \sqrt{49 - 52}}{2} = -\frac{7}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\text{CF} = e^{\frac{-7z}{2}} \left(A \cos \frac{\sqrt{3}z}{2} + B \sin \frac{\sqrt{3}z}{2} \right)$$

$$\text{PI} = \frac{1}{\theta^2 + 7\theta + 13} z$$

$$= \frac{1}{13 \left(1 + \frac{\theta^2 + 7\theta}{13} \right)} z = \frac{1}{13} \left[1 + \frac{\theta^2 + 7\theta}{13} \right]^{-1} z$$

$$= \frac{1}{13} \left[1 - \left(\frac{\theta^2 + 7\theta}{13} \right) + \text{Higher powers of } \theta \right] z$$

$$= \frac{1}{13} \left[1 - \frac{7}{13} \theta \right] z = \frac{1}{13} \left[z - \frac{7}{13} \right]$$

∴ The solution is,

$$y = e^{-7z/2} \left(A \cos \frac{\sqrt{3}}{2} z + B \sin \frac{\sqrt{3}}{2} z \right) + \frac{1}{13} \left(z - \frac{7}{13} \right)$$

$$= x^{\frac{-7}{2}} \left(A \cos \frac{\sqrt{3}}{2} \log x + B \sin \frac{\sqrt{3}}{2} \log x \right) + \frac{1}{13} \left(\log x - \frac{7}{13} \right)$$

Example 2.29: Solve, $(x^2 D^2 - 3xD + 4)y = x^2$ given that, $y(1) = 1$ and $y'(1) = 0$

Solution: Substituting x with e^z the equation is transformed as,

$$[\theta(\theta-1) - 3\theta + 4]y = e^2, \text{ where } \theta = \frac{d}{dz}$$

$$\text{i.e., } (\theta^2 - 4\theta + 4)y = e^{2z}$$

$$\text{AE is, } m^2 - 4m + 4 = 0; \therefore m = 2, 2$$

$$\text{CF is, } e^{2z}(A + Bz)$$

$$\text{PI is, } \frac{1}{D^2 - 2D + 4} e^{2z} = \frac{z}{2D - 2} e^{2z} = \frac{z^2}{2} e^{2z}$$

∴ The solution is,

$$\begin{aligned}y &= e^{2z}(A + Bz) + \frac{z^2}{2}e^{2z} \\&= x^2(A + B \log x) + \frac{x^2(\log x)^2}{2}\end{aligned}$$

$y(1) = 1$ gives, $A = 1$

$$\begin{aligned}\frac{dy}{dx} &= x^2\left(\frac{B}{x}\right) + (A + B \log x)2x + \frac{1}{2}\left[\frac{x^2 \cdot 2 \log x}{x} + (\log x)^2\right] \\ \frac{dy}{dx} &= 0, \text{ at } x = 1 \text{ gives, } 2A + B = 0 \quad \therefore B = -2\end{aligned}$$

∴ The solution is, $y = x^2(1 + 2 \log x) + \frac{x^2(\log x)^2}{2}$

NOTES

2.3.3 Simultaneous Linear Equations with Constant Coefficients

A pair of equations of the form,

$$\begin{aligned}f_1(D)x + f_2(D)y &= \phi_1(t) \\f_3(D)x + f_4(D)y &= \phi_2(t)\end{aligned}$$

Where f_1, f_2, f_3 and f_4 are polynomials in the operator $D\left(=\frac{d}{dt}\right)$ is called a

system of simultaneous linear equations. The variables x and y are the dependent variables and they are functions of the independent variable t . One of the dependent variables, say y can be eliminated by an algebraic manipulation of the operator D , and the resultant equation can be solved to obtain the expression for the dependent variable x . The number of arbitrary constants in the solution will be equal to the order of the resultant equation. The expression for the dependent variable y in terms of the independent variable t can be obtained by substituting the expression for x already obtained, in one of the given equations.

The number of arbitrary constants appearing in the general solution of the system of equations is equal to the index of the highest power of D in the expansion of

$$\begin{vmatrix} f_1(D) & f_2(D) \\ f_3(D) & f_4(D) \end{vmatrix}$$

If more arbitrary constants are introduced in the expression for x and y , then the extra constants are to be expressed in terms of the other constants.

NOTES

Example 2.30: Solve, $\frac{dx}{dt} - \frac{dy}{dt} - y = -e^t$, $x + \frac{dy}{dt} - y = e^{2t}$

$$\text{Solution: } Dx - (D + 1)y = -e^t \quad \dots(1)$$

$$x + (D - 1)y = e^{2t} \quad \dots(2)$$

Operating equation (2) by D we get,

$$Dx + (D^2 - D)y = D(e^{2t}) = 2e^{2t} \quad \dots(3)$$

Subtracting equation (1) from (3) gives,

$$[-(D + 1) - (D^2 - D)]y = -2e^{2t} - e^t$$

$$\text{or, } (D^2 + 1)y = 2e^{2t} + e^t$$

AE is, $m^2 + 1 = 0$ or $m = \pm i$

CF = $A \cos t + B \sin t$

$$\text{PI is, } \frac{1}{D^2 + 1}(2e^{2t} + e^t) = \frac{2}{5}e^{2t} + \frac{1}{2}e^t$$

$$\therefore y = A \cos t + B \sin t + \frac{2}{5}e^{2t} + \frac{1}{2}e^t$$

$$\text{From equation (2) we get, } x = e^{2t} - \frac{dy}{dt} + y$$

$$= (A + B)\sin t + (A - B)\cos t + \frac{3}{5}e^{2t}$$

$$\therefore \text{The solution is, } x = (A + B)\sin t + (A - B)\cos t + \frac{3}{5}e^{2t}$$

$$y = A \cos t + B \sin t + \frac{2}{5}e^{2t} + \frac{1}{2}e^t$$

Example 2.31: Solve, $\frac{dx}{dt} + 5x + \frac{dy}{dt} + 7y = 2e^t$, $2\frac{dx}{dt} + x + 3\frac{dy}{dt} + y = e^t$

Solution:

$$(D + 5)x + (D + 7)y = 2e^t \quad \dots(1)$$

$$(2D + 1)x + (3D + 1)y = e^t \quad \dots(2)$$

Multiplying equation (1) by $(2D + 1)$ and (2) by $(D + 5)$ we get,

$$(2D + 1)(D + 5)x + (2D + 1)(D + 7)y = 2(2D + 1)e^t \quad \dots(3)$$

$$(D + 5)(2D + 1)x + (D + 5)(3D + 1)y = (D + 5)e^t \quad \dots(4)$$

Subtracting equation (3) from (4) gives,

$$[(2D + 1)(D + 7) - (D + 5)(3D + 1)]y = 0$$

$$(-D^2 - D + 2)y = 0; (D^2 + D - 2)y = 0$$

AE is, $m^2 + m - 2 = 0$

$$(m + 2)(m - 1) = 0; m = -2, 1$$

Solution is, $y = Ae^{-2t} + Be^t$

If we substitute the values of y and $\frac{dy}{dt}$ in equation (1) or (2), we get a first order equation in x solving which we find that another constant of integration appears in the expression for x . Substituting the expressions for x and y in equations (1) or (2), we find that this third constant is expressible in terms of the other two constants. To avoid this computation we eliminate the term $\frac{dx}{dt}$ from equations (1) and (2) as below and substitute for y and $\frac{dy}{dt}$.

Equation (1) multiplied by (2) gives, $(2D + 10)x + (2D + 14)y = 4e^t$... (5)

Subtracting equation (5) from (2) gives, $9x - (D - 13)y = 3e^t$

$$9x = 3e^t + \frac{dy}{dt} - 13y; x = \frac{1}{3}(e^t - 5Ae^{-2t} - 4Be^t)$$

\therefore The solution is, $x = \frac{1}{3}(e^t - 5Ae^{-2t} - 4Be^t); y = Ae^{-2t} + Be^t$

Check Your Progress

5. What is Cauchy's linear equation derived from Euler's linear equation?
6. Write Legendre's linear equation.

2.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. If, in a differential equation, the unknown function is a function of one independent variable, then it is known as an ordinary differential equation.
2. If the unknown function is a function of two or more independent variables and the equation involves partial derivatives of the unknown function, then it is known as a partial differential equation.

NOTES

3. The solution, in which the number of arbitrary constants occurring is equal to the order of the equation, is known as the general solution or the complete integral.

4. $f(x) dx = g(y) dy.$

5. The equation of the form $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = X$

Where a_0, a_1, \dots, a_n are constants and X is a function of x is called *Cauchy's linear equation derived from Euler's linear equation*.

6. $a_0 (bx + c)^n \frac{d^n y}{dx^n} + a_1 (bx + c)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} (bx + c) \frac{dy}{dx} + a_n y = X$

2.5 SUMMARY

- Equations in which an unknown function, and its derivatives or differentials occur are called *differential equations*.
- If in a differential equation, the derivatives and the dependent variables appear in first power and there are no products of these, and also the coefficients of the various terms are either constants or functions of the independent variable, the equation is said to be a *linear differential equation*.
- A relation between the dependent and independent variables, which, when substituted in the equation, satisfies it, is known as a *solution* or a *primitive* of the equation.
- The solution, in which the number of arbitrary constants occurring is equal to the order of the equation, is known as the *general solution* or the *complete integral*.
- Solutions of equations which do not contain any arbitrary constants and which are not derivable from the general solution by giving particular values to one or more of the arbitrary constants, are called *singular solutions*.
- The order of the ordinary differential equation and the number of arbitrary constants appearing in the general solution will be always equal. Also, if there are n arbitrary constants in the relation between the dependent and the independent variables, then by eliminating them, we will arrive at a differential equation of order n .
- An ordinary differential equation of first order and first degree can be written as

$$\frac{dy}{dx} = f(x, y)$$

- A function $f(x, y)$ is said to be an *Integrating Factor* (I.F.) of the equation $Mdx + Ndy = 0$ if we can find a function $u(x, y)$ such that

$f(x, y) (M dx + N dy) = du$. In other words, an I.F. is a multiplying factor by which the equation can be made exact.

- Differential equations of the form $f(x) dx = g(y) dy$ are called equations with variables separated.
- In mathematics, an integrating factor is a function that is used to solve the given equation with the help of differential equations.

- The equation $x^n \frac{d^n y}{dx^n} + b_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + b_{n-1} x \frac{dy}{dx} + b_n y = X$

is called a *Homogeneous linear equation* of the n th order.

- A pair of equations of the form,

$$f_1(D)x + f_2(D)y = \phi_1(t)$$

$$f_3(D)x + f_4(D)y = \phi_2(t)$$

Where f_1, f_2, f_3 and f_4 are polynomials in the operator $D\left(=\frac{d}{dt}\right)$ is called a system of simultaneous linear equations.

NOTES

2.6 KEY WORDS

- **Differential equation:** Equations in which an unknown function, and its derivatives or differentials occur are called differential equations.
- **Order:** The order of a differential equation is the order of the highest differential coefficient which occurs in it.
- **Degree:** The degree of a differential equation is the degree of the highest order differential coefficient which occurs in it, after the equation has been cleared of radicals and fractions.

2.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Obtain the differential equation associated with the primitive $y = Cx - 2C + 3C^3$, where C is an arbitrary constant.
2. Form the differential equation satisfied by all circles having their centres on the straight line $x = 10$ and touching the y -axis.
3. Eliminate A and B from $y = Ae^{2x} + Be^{2x}$.
4. What are exact equations? Discuss.
5. Solve $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$.

6. Discuss non-homogenous equations with constant coefficients.
7. What are simultaneous linear equations with constant coefficients?

Long Answer Questions

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1. Solve the following equations:
 - (a) $(x^2 + 2y)dx + (y^2 + 2x)dy = 0$
 - (b) $x \, dx - y \, dy = m(x \, dy + y \, dx)$

2. Solve $\frac{dy}{dx} = \frac{xy^2}{x^2 + y^2}$

3. Solve $\frac{1}{D^2 - DD' + D'^2} y \sin x$

4. Solve $\frac{dy}{dx} = \frac{x+y+3}{2x-3y-1}$

5. Solve $(D^2 + 2DD' - 5D' - 5)z = 2e^x$

6. Solve $(ye^{xy} + y^3)dx + (xe^{xy} + xy^2 + y)dy = 0$

7. Solve $x^2y^2dx + (x^3 - y^3)dy = 0$

8. Solve $(x^2D^2 + xD + 2)y = x^2$ given that, $y(1) = 1$ and $y'(1) = 0$

9. Solve $\frac{dx}{dt} + \frac{dy}{dt} - y = e^t, x - \frac{dy}{dt} + y = e^{2t}$

2.8 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 3 METHODS OF SOLUTIONS TO PDEs

NOTES

Structure

- 3.0 Introduction
- 3.1 Objectives
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 - 3.2.1 Methods of Solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
- 3.3 Differential Equations of First Order but not of First Degree
 - 3.3.1 Singular Solution
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3.0 INTRODUCTION

A simultaneous differential equation is one of the mathematical equations for an indefinite function of one or more than one variables that relate the values of the function. Differentiation of an equation in various orders. Differential equations play an important function in engineering, physics, economics, and other disciplines.

In this unit, you will study the formation of simultaneous differential equations and the methods of finding solutions of these equations and, finally, understand a few interesting applications of simultaneous differential equations.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss that the solution set of simultaneous differential equations is a two-parameter family of space curves
- Use the methods of solution of simultaneous differential equations
- Find orthogonal trajectories of a system of curves on a given surface

3.2 METHODS OF SOLUTION OF $dx/P=dy/Q=dz/R$

Let us consider two families of surfaces

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$$u(x,y,z) = c_1, v(x,y,z) = c_2, \dots (3.1)$$

c_1 , and c_2 being the parameters.

You know that these surfaces intersect in a two-parameter family of space curves. Also, along any curve of the family, $du = 0$ and $dv = 0$.

Now,

$$du = 0 \quad u_x dx + u_y dy + u_z dz = 0 \quad \dots (3.2)$$

$$dv = 0 \quad v_x dx + v_y dy + v_z dz = 0 \quad \dots (3.3)$$

with (dx, dy, dz) being the projections of the tangent vector to the curve.

Solving Equations (3.2) and (3.3) for dx, dy and dz , we obtain

$$\frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}$$

$$\text{or} \quad \frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} \quad \dots (3.4)$$

where P, Q, R are known functions of x, y, z .

Equation (3.4) are the simultaneous differential equations of the two-parameter family of space curves in which two families of surfaces $u(x,y,z) = c_1$, $v(x,y,z) = c_2$, intersect.

Example 3.1: Find the differential equations of the space curves in which the two families of surfaces

$$u = x^2 + y^2 + z^2 = c_1, v = x + z = c_2$$

If (dx, dy, dz) are the projections of the tangent vector to the space curve in which the given surfaces intersect, then along any curve of the family, we have

$$du = 0 \quad 2x dx + 2y dy + 2z dz = 0 \quad \dots (1)$$

$$\text{and} \quad dv = 0 \quad dx + dz = 0 \quad \dots (2)$$

Solving Equations (1) and (2), we get

$$\frac{dx}{2y - 0} = \frac{dy}{2z - 2x} = \frac{dz}{0 - 2y}$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{z-x} = \frac{dz}{-y},$$

which are the required differential equations of the space curves.

3.2.1 Methods of Solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

You have seen that the curves of intersection of family of surfaces, given by Equation (3.1) namely, $u(x,y,z) = c_1$, $v(x,y,z) = c_2$, are defined by the system of simultaneous differential Equation (3.4), i.e.,

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

where

Thus, if we can derive from Equation (3.4), two relations of the form (3.1) involving two arbitrary constants c_1 and c_2 then, by varying these constants, we obtain a two-parameter family of curves satisfying Equation (3.4).

In this section, we shall describe the methods of finding the surfaces of the type (3.1) starting with Equation (3.4) for which the functions P, Q, R are known.

Method of Multipliers

You may recall that any tangential direction through a point (x, y, z) to the surface $u(x, y, z) = c_1$ satisfies the relation

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \dots (3.5)$$

If $u(x, y, z) = c_1$ is a suitable one-parameter system of surfaces for the system of Equations (3.4), viz.,,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

then the tangential direction to the integral curve through the point (x, y, z) is also a tangential direction to this surface and hence, from Equations (3.4) and (3.5), we get

$$P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} + R \frac{\partial u}{\partial z} = 0$$

Similarly, for surface $v(x, y, z) = c_2$, we have

$$P \frac{\partial v}{\partial x} + Q \frac{\partial v}{\partial y} + R \frac{\partial v}{\partial z} = 0$$

To find u and v we try to determine functions (P_1, Q_1, R_1) and (P_2, Q_2, R_2) with the properties

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$$P_1 = \frac{\partial u}{\partial x}, Q_1 = \frac{\partial u}{\partial y}, R_1 = \frac{\partial u}{\partial z}$$

$$P_2 = \frac{\partial v}{\partial x}, Q_2 = \frac{\partial v}{\partial y}, R_2 = \frac{\partial v}{\partial z}$$

such that

$$PP_1 + QQ_1 + RR_1 = 0 \quad \dots (3.6)$$

$$PP_2 + QQ_2 + RR_2 = 0 \quad \dots (3.7)$$

From componendo-dividendo in algebra, we know that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1} = \frac{P_2 dx + Q_2 dy + R_2 dz}{PP_2 + QQ_2 + RR_2} \quad \dots (3.8)$$

Thus, in view of Equations (3.6) and (3.7), we get from Equation (3.8),

$$P_1 dx + Q_1 dy + R_1 dz = 0 \quad \dots (3.9)$$

$$P_2 dx + Q_2 dy + R_2 dz = 0 \quad \dots (3.10)$$

Now, if Equations (3.9) and (3.10) are exact, then

$$du = P_1 dx + Q_1 dy + R_1 dz = 0$$

$$dv = P_2 dx + Q_2 dy + R_2 dz = 0$$

On integrating these equations, we get the surfaces

$$u(x, y, z) = c_1$$

and

The curves of intersection of these surfaces are the integral curves of the system of Equation (3.4).

Example 3.2: Solve the equations

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Solution: Here $P = mz-ny$, $Q = nx-lz$, $R = ly-mx$

If we take $P_1 = 1$, $Q_1 = m$, $R_1 = n$

and $P_2 = x$, $Q_2 = y$, $R_2 = z$

then

$$\begin{aligned} PP_1 + QQ_1 + RR_1 &= 1(mz-ny) + m(nx-lz) + n(ly-mx) \\ &= lmz-lyn+mnx-mlz+lny-mnx = 0 \end{aligned}$$

and

$$PP_2 + QQ_2 + RR_2 = x(mz - ny) + y(nx - lz) + z(ly - mx)$$

$$= mxz - nxy + nxy - lyz + lyz - mxz = 0$$

For the given system, we have

$$\frac{dx}{mx-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{l dx + m dy + n dz}{0} = \frac{x dx + y dy + z dz}{0}$$

Also,

$$l dx + m dy + n dz = d(lx + my + nz) = du \text{ (say)}$$

and,

$$xdx + ydy + zdz = d\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right) = dv \text{ (say)}$$

Therefore the integral curves are given by the intersection of family of surfaces

$$lx + my + nz = c_1$$

$$\text{and } x^2 + y^2 + z^2 = c_2.$$

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3.3 DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

The differential equation of first order and n th degree is generally represented by

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \dots + A_{n-1} p + A_n = 0$$

where p denotes $\frac{dy}{dx}$ and $A_1, A_2, A_3, \dots, A_{n-1}, A_n$ are functions of x and y .

The above equation can also be written as $f(x, y, p) = 0 \dots (3.11)$

The differential Equation (3.11) can be classified into three categories:
Equations solvable for p , i.e., of the form $p = f(x, y)$, Equations solvable for y , i.e., of the form $y = f(x, p)$ and Equations solvable for x , i.e., of the form $x = f(y, p)$.

Equation Solvable for p

Let the equation $p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \dots + A_{n-1} p + A_n = 0 \dots (3.12)$

be solvable for ' p ' and can be written in form

$$[p - F_1(x, y)][p - F_2(x, y)][p - F_3(x, y)] \dots [p - F_n(x, y)] = 0 \dots (3.13)$$

$$\Rightarrow [p - F_1(x, y)] = 0, [p - F_2(x, y)] = 0$$

$$[p - F_3(x, y)] = 0, \dots, [p - F_n(x, y)] = 0$$

$$\Rightarrow \frac{dy}{dx} = F_1(x, y), \frac{dy}{dx} = F_2(x, y), \frac{dy}{dx} = F_3(x, y), \dots, \frac{dy}{dx} = F_n(x, y)$$

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which are the equations of first order and first degree.

Let the solutions of these equations be

$$f_1(x, y, c_1) = 0, f_2(x, y, c_2) = 0, f_3(x, y, c_3) = 0, \dots, f_n(x, y, c_n) = 0$$

where, $c_1, c_2, c_3, \dots, c_n$ are arbitrary constants.

Then, the most general solution of the differential equation (3.12) is

$$[f_1(x, y, c_1)][f_2(x, y, c_2)][f_3(x, y, c_3)] \dots [f_n(x, y, c_n)] = 0$$

As the Equation (3.12) is of first order thus, its general solution can contain only one arbitrary constant. Hence, without the loss of generality we can take $c_1 = c_2 = c_3 = \dots = c_n = c$ (say)

Thus, the general solution of the Equation (3.12) can be written as

$$[f_1(x, y, c)][f_2(x, y, c)][f_3(x, y, c)] \dots [f_n(x, y, c)] = 0$$

Example 3.3: Solve the differential equation $x^2 \left(\frac{dy}{dx} \right)^2 - 2xy \frac{dy}{dx} + 2y^2 - x^2 = 0$

Solution: The given equation is $x^2 \left(\frac{dy}{dx} \right)^2 - 2xy \frac{dy}{dx} + 2y^2 - x^2 = 0 \quad \dots (1)$

By putting $\frac{dy}{dx} = p$ in Equation (1), we get $x^2 p^2 - 2xyp + 2y^2 - x^2 = 0 \quad \dots (2)$

Solving Equation (2), we get

$$\begin{aligned} p &= \frac{2xy \pm \sqrt{4x^2 y^2 - 4x^2 (2y^2 - x^2)}}{2x^2} \\ &= \frac{2xy \pm \sqrt{4x^2 y^2 - 8x^2 y^2 + 4x^4}}{2x^2} \\ &= \frac{2xy \pm \sqrt{4x^4 - 4x^2 y^2}}{2x^2} \\ &= \frac{2xy \pm \sqrt{4x^2 (x^2 - y^2)}}{2x^2} = \frac{2xy \pm 2x\sqrt{x^2 - y^2}}{2x^2} = \frac{y \pm \sqrt{x^2 - y^2}}{x} \\ \Rightarrow \frac{dy}{dx} &= \frac{y \pm \sqrt{x^2 - y^2}}{x} \end{aligned} \quad \dots (3)$$

which is a homogeneous equation in x and y .

Let $y = vx$

Differentiating both sides with respect to x , we get

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

By putting the values of y and $\frac{dy}{dx}$ in equation (iii), we get

$$v + x \frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2 x^2}}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = v \pm \sqrt{1 - v^2} \Rightarrow x \frac{dv}{dx} = \pm \sqrt{1 - v^2}$$

Separating the variables, we get

$$\frac{dv}{\sqrt{1 - v^2}} = \pm \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 - v^2}} = \pm \int \frac{dx}{x} + c$$

$$\Rightarrow \sin^{-1} v = \pm \log x \pm \log c \Rightarrow \sin^{-1} v = \pm \log xc \Rightarrow \sin^{-1} \frac{y}{x} = \pm \log cx,$$

which is the required solution

Equation Solvable for y

A differential equation is said to be solvable for y , if it can be expressed as

$$y = f(x, p) \quad \dots (3.14)$$

Differentiating both sides with respect to x , we get

$$\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right) \quad \dots (3.15)$$

The differential Equation (3.15) is in two variables p and x .

Let the solution of the Equation (3.15) be

$$\phi(x, p, c) = 0 \quad \dots (6.15)$$

So, by eliminating p from Equation (3.14) and Equation (3.16), we get the solution of differential Equation (3.14). If it is not easier to eliminate p , then put (3.14) and (3.16) in the form

$$x = \phi_1(p, c), y = \phi_2(p, c)$$

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In this case these two equations together constitute the solution of Equation (3.14).

Note: When the Equation (3.15) can be written as $f_1(x, p) \cdot f_2\left(x, p, \frac{dp}{dx}\right) = 0$, neglect the factor which does not contain $\frac{dp}{dx}$, i.e., $f_1(x, p)$ since if $f_1(x, p) = 0$, then elimination of p from this equation and Equation (3.14) does not give a general solution.

Example 3.4: Solve the differential equation $y = 3x + \log p$.

Solution: The given equation is $y = 3x + \log p$... (1)

Differentiating both sides with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= 3 + \frac{1}{p} \frac{dp}{dx} \Rightarrow p = 3 + \frac{1}{p} \frac{dp}{dx} \\ \left[\because \frac{dy}{dx} = p \right] \\ \Rightarrow \quad dx &= \frac{dp}{p(p-3)} \Rightarrow dx = \frac{1}{3} \left[\frac{1}{p-3} - \frac{1}{p} \right] dp \end{aligned}$$

(By partial fraction)

Integrating both sides, we get

$$\begin{aligned} x &= \frac{1}{3} [\log(p-3) - \log p] + \log c_1 \\ \Rightarrow \quad 3x &= \log \frac{p-3}{p} + \log c_2 \Rightarrow 3x = \log \left(\frac{p-3}{p} \right) \cdot c_2 \end{aligned}$$

Taking exponent of both sides, we get

$$\begin{aligned} e^{3x} &= \frac{p-3}{p} \cdot c_2 \\ \Rightarrow \quad \frac{p-3}{p} &= ce^{3x} \Rightarrow 1 - \frac{3}{p} = ce^{3x} \Rightarrow p = \frac{3}{1-ce^{3x}} \end{aligned}$$

Substituting this value of p in (i), we get $y = 3x + \log \frac{3}{1-ce^{3x}}$

which is the required solution.

Equation Solvable for x

A differential equation is said to be solvable for x , if x can be expressed as

$$x = f(y, p) \quad \dots (3.17)$$

Differentiating both sides with respect to y , we get $\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \dots (3.18)$

The differential Equation (3.18) is in two variables y and p .

Let the solution of the Equation (3.18) be $\phi(y, p, c) = 0 \dots (3.19)$

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Then, by eliminating p from Equations (3.17) and (3.19), we get the solution of differential Equation (3.17). If it is not easier to eliminate p , then express (3.17) and (3.19) in the form

$$x = \phi_1(p, c), y = \phi_2(p, c)$$

In such case, the above two equations taken together constitute the solution of Equation (3.17).

Note: When the Equation (3.18) can be written as $f_1(y, p) \cdot f_2\left(y, p, \frac{dp}{dy}\right) = 0$,

neglect the factor which does not contain $\frac{dp}{dy}$, i.e., $f_1(y, p)$ since if $f_1(y, p) = 0$, then elimination of p from this equation and equation (i) doe not give a general solution.

Example 3.5: Solve the differential equation $x = y + p^2$

Solution: The given equation is $x = y + p^2 \dots (1)$

Differentiating both sides with respect to y , we get

$$\begin{aligned} \frac{dx}{dy} &= 1 + 2p \frac{dp}{dy} \\ \Rightarrow \quad \frac{1}{p} - 1 &= 2p \frac{dp}{dy} \Rightarrow 1 - p = 2p^2 \frac{dp}{dy} \Rightarrow dy = \left(\frac{2p^2}{1-p}\right) dp \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \int dy &= \int \left(\frac{2p^2}{1-p}\right) dp + c \\ \Rightarrow \quad y &= -2 \int \left[p + 1 + \frac{1}{p-1}\right] dp + c \\ \Rightarrow \quad y &= c - 2 \left[\frac{p^2}{2} + p + \log(p-1)\right] \Rightarrow y = c - [p^2 + 2p + 2\log(p-1)] \end{aligned}$$

Substituting the value of y in Equation (1), we get

$$x = c - [2p + 2\log(p-1)] \text{ which is the required solution.}$$

NOTES**3.3.1 Singular Solution**

A singular solution is a solution of differential equation which cannot be obtained from its general solution by assigning any particular values to the arbitrary constants.

Discriminant

The discriminant of a polynomial is an expression which gives information about the nature of the polynomial's roots. We know that the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$. Here, if $b^2 - 4ac > 0$, the equation has two real roots, if $b^2 - 4ac = 0$, the equation has equal roots which are real, and if $b^2 - 4ac < 0$, the equation has two imaginary roots. But if equation has higher degree than two, then the condition that the equation $f(x) = 0$ has two equal roots is obtained by eliminating x between $f(x) = 0$ and $f'(x) = 0$.

p -Discriminant and c -Discriminant $\left(p = \frac{dy}{dx} \right)$

$$\text{Let } f(x, y, p) = 0 \quad \dots (3.21)$$

be the differential equation having

$$\phi(x, y, c) = 0 \quad \dots (3.22)$$

as the its solution, where, c is an arbitrary constant.

Differentiating both sides of Equation (3.22) partially with respect to c , we get

$$\frac{\partial}{\partial c}(\phi(x, y, c)) = 0 \quad \dots (3.23)$$

Differentiating both sides of Equation (3.21) partially with respect to p , we get

$$\frac{\partial}{\partial p}(f(x, y, p)) = 0 \quad \dots (3.24)$$

By eliminating c between the Equations (3.22) and (3.23), we get c -discriminant. Thus, c -discriminant represents the locus of each point which has equal values of c in $\phi(x, y, c) = 0$.

By eliminating p between the Equations (3.21) and (3.24), we get p -discriminant. Thus, p -discriminant represents the locus of each point which has equal values of p in

$$f(x, y, p) = 0.$$

- Note:** (i) The envelope which is present in both c -discriminant and p -discriminant is known as the singular solution.
- (ii) If the singular solution of a differential equation $f(x, y, p) = 0$ whose primitive is $\phi(x, y, c) = 0$ is given by $F(x, y) = 0$, then $F(x, y)$ will be a

factor of both the discriminates.

- (iii) In the Clairaut's form of the differential equation both c -discriminant and p -discriminant are same.

Method For Determining Singular Solution

Use the following steps find the singular solutions of a differential equation

$$f(x, y, p) = 0 \quad \dots (3.25)$$

Step 1: Find its complete solution $\phi(x, y, c) = 0$... (3.26)

where, c is arbitrary constant.

Step 2: Find p -discriminant of Equation (3.25) and c -discriminant of equation (3.26). Both contain the singular solution.

Now, when p -discriminant is equated to zero, it may include the following as a factor:

- Envelope, i.e., singular solution once (E)
- Cusp locus once (C)
- Tac locus, twice (T^2)

So p -discriminant $\equiv ET^2C$, which does not contain nodal locus

and when c -discriminant is equated to zero it may include the following as a factor:

- Envelope, i.e, singular solution once (E)
- Cusp locus thrice (C^3)
- Nodal locus twice (N^2)

So c -discriminant $\equiv EN^2C^3$

Example 3.6: Obtain the primitive and the singular solution of

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + 4x = 0$$

Solution: The given differential equation is $x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} + 4x = 0$... (1)

By putting $\frac{dy}{dx} = p$ in Equation (1), we get

$$\Rightarrow xp^2 - 2yp + 4x = 0 \quad \dots (2)$$

$$\Rightarrow y = \frac{xp}{2} + \frac{2x}{p}$$

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Differentiating both sides, with respect to x , we get

$$\frac{dy}{dx} = \frac{x}{2} \frac{dp}{dx} + \frac{p}{2} + \frac{2}{p} - \frac{2x}{p^2} \cdot \frac{dp}{dx}$$

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$$\Rightarrow p = \frac{x}{2} \cdot \frac{dp}{dx} + \frac{p}{2} + \frac{2}{p} - \frac{2x}{p^2} \cdot \frac{dp}{dx}$$

$$\Rightarrow \frac{p}{2} - \frac{2}{p} = \left(\frac{x}{2} - \frac{2x}{p^2} \right) \frac{dp}{dx}$$

$$\Rightarrow \frac{p^2 - 4}{2p} = \frac{x}{2p^2} (p^2 - 4) \frac{dp}{dx}$$

$$\Rightarrow (p^2 - 4) \left(1 - \frac{x}{p} \frac{dp}{dx} \right) = 0$$

$$\Rightarrow p^2 - 4 = 0 \text{ or } 1 - \frac{x}{p} \cdot \frac{dp}{dx} = 0 \Rightarrow p = 2 \text{ or } \frac{x}{p} \cdot \frac{dp}{dx} = 1$$

Now, for $\frac{x}{p} \cdot \frac{dp}{dx} = 1$

$$\Rightarrow \frac{dp}{p} = \frac{dx}{x}$$

[Separating the variables]

Integrating both sides, we get

$$\log p = \log x + \log c$$

where c is any arbitrary constant

$$\Rightarrow \log p = \log cx \Rightarrow p = cx \quad \dots (3)$$

Substituting the value of p from Equation (3) in Equation (2), we get

$$c^2 x^3 - 2y c x + 4x = 0 \Rightarrow c^2 x^2 - 2y c + 4 = 0 \quad \dots (4)$$

which is the complete primitive of (1),

From Equation (4), we have p -discriminant (ET²C) is $4y^2 - 16x^2 = 0$

$$\Rightarrow y^2 - 4x^2 = 0$$

Also from Equation (2), c -discriminant (EN²C³) is $4y^2 - 16x^2 = 0$

Since, $y^2 - 4x^2$ is non-repeated common factor in p and c -discriminants and it satisfies the differential Equation (2).

Hence the singular solution of (1) is

$$y^2 - 4x^2 = 0 \Rightarrow y = \pm 2x$$

Check Your Progress

1. Write the general form an equation solvable for y.
2. Define singular solution.

NOTES**3.4 ORTHOGONAL TRAJECTORIES OF A SYSTEM OF CURVES ON A SURFACE**

A curve which cuts every member of a given family of curves according to a definite law is called a trajectory of the given family.

Orthogonal trajectory

A curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the given family.

Orthogonal trajectories of a given family of curves, themselves form a family of curves.

We know that the angle between the two curves is equal to the angle between the tangents at their common point of intersection. The slope of these tangents is

given by $\frac{dy}{dx}$.

\therefore When the curves intersect orthogonally,

Product of their slopes = - 1

Let the common point of intersection of two curves c_1 and c_2 be P and the angle between these curves at P is right angle, then

$$\left(\frac{dy}{dx} \right)_{c_1} \times \left(\frac{dy}{dx} \right)_{c_2} = -1 \Rightarrow \left(\frac{dy}{dx} \right)_{c_1} = - \left(\frac{dx}{dy} \right)_{c_2}$$

Oblique trajectory

A curve which cuts every member of a given family of curves at certain angle other than right angle is called an oblique trajectory of the given family.

3.4.1 Differential Equation of Orthogonal Trajectories**Cartesian coordinates**

Step 1: Let the equation of the given family of curves be

$$f(x, y, c) = 0 \quad \dots (3.27)$$

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where c is the parameter of the family.

Step 2: Differentiating (3.27) with respect to x and eliminating c between (3.27) and the resulting equation, we get the differential equation of the given family as

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \quad \dots (3.28)$$

Step 3: Putting $\frac{dy}{dx} = -\frac{dx}{dy}$ in (3.28), we get

$$F\left(x, y, -\frac{dx}{dy}\right) = 0 \quad \dots (3.29)$$

which is the differential equation of orthogonal trajectory.

Step 4: Integrate (3.29) to get the required equation of orthogonal trajectories.

Example 3.7: Find the orthogonal trajectories of the family of co-axial circles $x^2 + y^2 + 2gx + c = 0$, where g is a parameter and c is a constant.

Solution: The given equation is $x^2 + y^2 + 2gx + c = 0$... (1)

Differentiating (1) with respect to x , we get

$$2x + 2y \frac{dy}{dx} + 2g = 0 \Rightarrow x + y \frac{dy}{dx} + g = 0 \quad \dots (2)$$

Eliminating g between (1) and (2), we get the differential equation of the given family of circles as $x^2 + y^2 - 2x\left(x + y \frac{dy}{dx}\right) + c = 0$

$$\Rightarrow y^2 - x^2 - 2xy \frac{dy}{dx} + c = 0 \quad \dots (3)$$

By replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$, we get

$$\begin{aligned} & y^2 - x^2 + 2xy \frac{dx}{dy} + c = 0 \\ \Rightarrow & 2xy \frac{dx}{dy} - x^2 = -c - y^2 \end{aligned} \quad \dots (4)$$

Let $x^2 = v$. Then $2x \frac{dx}{dy} = \frac{dv}{dy}$

\therefore The Equation (4) becomes

$$y \frac{dv}{dy} - v = -c - y^2 \Rightarrow \frac{dv}{dy} - \frac{1}{y}v = -\frac{c}{y} - y$$

which is a linear differential equation in v .

$$\therefore \text{I.F.} = e^{-\int 1/y dy} = e^{-\log y} = e^{\log y^{-1}} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

Thus, the required solution is

$$\begin{aligned} v \frac{1}{y} &= \int \left(-\frac{c}{y} - y \right) \frac{1}{y} dy \\ \Rightarrow v \frac{1}{y} &= \frac{c}{y} - y - k, \text{ where } k \text{ is an arbitrary constant} \\ \Rightarrow v &= c - y^2 - ky \Rightarrow x^2 + y^2 + ky - c = 0 \end{aligned}$$

which is the orthogonal trajectories of the family of co-axial circles $x^2 + y^2 + 2gx + c = 0$.

Polar Co-ordinates

Step 1: Let the equation of the given family of polar curves be

$$f(r, \theta, c) = 0 \quad \dots (3.30)$$

where c is the parameter of the family.

Step 2: Differentiating (3.30) with respect to θ and eliminating c between (3.30) and the resulting equation, we get the differential equation of the given family as

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \dots (3.31)$$

Step 3: From differential calculus, we know that $r \frac{d\theta}{dr}$ is the tangent of the angle between the tangent to a curve and the radius vector at any point (r, θ) . Since, the curves intersect orthogonally,

$$\left(r \frac{d\theta}{dr}\right)_{c_1} \times \left(r \frac{d\theta}{dr}\right)_{c_2} = -1, \text{ where } c_1 \text{ and } c_2 \text{ are two curves.}$$

$$r \frac{d\theta}{dr} = -\frac{1}{r} \frac{dr}{d\theta} \Rightarrow \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\text{Replace } \frac{dr}{d\theta} \text{ by } -r^2 \frac{d\theta}{dr} \text{ in (ii), we get } F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \dots (3.32)$$

which is the differential equation of orthogonal trajectory.

Step 4: Integrate (3.32) to get the required equation of the orthogonal trajectories.

Example 3.8: Find the orthogonal trajectories of the curves $r^n \sin n\theta = a^n$.

Solution: The given equation of the system of curve is $r^n \sin n\theta = a^n \quad \dots (1)$

Taking log on both sides, we get

$$n \log r + \log \sin n\theta = n \log a$$

Differentiating with respect to θ , we get

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$$n \frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{1}{\sin n\theta} \cdot n \cos n\theta = 0$$

$$\frac{1}{r} \frac{dr}{d\theta} + \cot n\theta = 0 \Rightarrow r \frac{d\theta}{dr} = -\tan n\theta$$

... (2)

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (2) we get

$$-\frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta \Rightarrow \frac{dr}{r} = \tan n\theta d\theta$$

Integrating, we get $\int \frac{dr}{r} = \int \tan n\theta d\theta + c_1$, where c_1 is any arbitrary constant.

$$\log r = \frac{1}{n} \log \sec n\theta + \log c$$

[Putting $c_1 = \log c$]

$$\Rightarrow \log r = \log (\sec n\theta)^{1/n} + \log c$$

$$\Rightarrow \log r = \log c (\sec n\theta)^{1/n}$$

$$\Rightarrow r = c (\sec n\theta)^{1/n} \Rightarrow r^n = c^n \sec n\theta \Rightarrow r^n \cos n\theta = c^n$$

which is required equation of the orthogonal trajectories.

Check Your Progress

3. When two curves intersect orthogonally, what is the product of their slopes?
4. Define oblique trajectory.

3.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. $y = f(x, p)$.
2. A singular solution is a solution of differential equation which cannot be obtained from its general solution by assigning any particular values to the arbitrary constants.
3. -1 .
4. A curve which cuts every member of a given family of curves at certain angle other than right angle is called an oblique trajectory of the given family.

3.6 SUMMARY

- $\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$ are the simultaneous differential equations of the two-parameter family of space curves in which two families of surfaces $u(x,y,z) = c_1, v(x,y,z) = c_2$, intersect.
- The differential equation of first order and n th degree is generally represented by

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \dots + A_{n-1} p + A_n = 0$$

where p denotes $\frac{dy}{dx}$ and $A_1, A_2, A_3, \dots, A_{n-1}, A_n$ are functions of x and y .

- A differential equation is said to be solvable y , if it can be expressed as

$$y = f(x, p)$$

- A differential equation is said to be solvable for x , if x can be expressed as

$$x = f(y, p)$$

- A singular solution is a solution of differential equation which cannot be obtained from its general solution by assigning any particular values to the arbitrary constants.
- A curve which cuts every member of a given family of curves according to a definite law is called a trajectory of the given family.
- A curve which cuts every member of a given family of curves at certain angle other than right angle is called an oblique trajectory of the given family.

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3.7 KEY WORDS

- **Simultaneous differential equation:** A simultaneous differential equation is one of the mathematical equations for an indefinite function of one or more than one variables that relate the values of the function.
- **Trajectory:** A curve which cuts every member of a given family of curves according to a definite law is called a trajectory of the given family.

3.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. How do you form simultaneous differential equations?
2. Solve the differential equation $y = 2x + \log p$.

3. Solve the differential equation $x = y + p$.
4. Find the orthogonal trajectories of the curves $r^n \cos \theta = a^n$.

Long Answer Questions

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1. Find the differential equations of the space curves in which the two families of surfaces $u = c_1$ and $v = c_2$ intersect.
 - (a) $u = 3x + 4y + z = c_1$, $v = x + z = c_2$
 - (b) $u = x^2 + y^2 = c_2$, $v = 3x + 4z = c_2$
2. Find the integral curves of the equations

$$\frac{dx}{y(x+y)+az} = \frac{dy}{x(x+y)-az} = \frac{dz}{z(x+y)}$$

3. Find the integral curves of the equations

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)}$$

4. Find the system of orthogonal trajectories on plane $z=0$, to the system of straight lines

3.9 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 4 PFAFFIAN DIFFERENTIAL FORMS AND EQUATIONS

NOTES

Structure

- 4.0 Introduction
- 4.1 Objectives
- 4.2 Pfaffian Differential Forms, Functions and Equations
- 4.3 Solution of Pfaffian Differential Equations in Three Variables
- 4.4 Answers to Check Your Progress Questions
- 4.5 Summary
- 4.6 Key Words
- 4.7 Self Assessment Questions and Exercises
- 4.8 Further Readings

4.0 INTRODUCTION

In mathematics, Pfaffian functions are defined as a certain distinct form or arrangement of functions whose derivative can be written in terms of the original function. They were originally introduced by Askold Georgevich Khovanskii in the 1970s, but are named after German mathematician Johann Pfaff. In many real-world problems, the given mathematical relations between the forms dx, dy, du or dz , etc., typically termed as Pfaffian equations are used for finding functions which relate the variables x, y, u or z , etc. The theory of Pfaffian equations is reasonably a challenging subject. The Pfaffian Differential Equations are also sometimes abbreviated as PfDE.

In this unit, you will study about the Pfaffian differential forms and equations, and will know the methods to find solutions of Pfaffian differential equations in three variables.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the significance of Pfaffian differential forms of equations
- Define what Pfaffian differential equations are
- Solve Pfaffian differential equations of three variables

4.2 PFAFFIAN DIFFERENTIAL FORMS, FUNCTIONS AND EQUATIONS

In mathematics, Pfaffian functions are defined as a certain distinct form or arrangement of functions whose derivative can be written in terms of the original

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function. They were originally introduced by Askold Georgevich Khovanskii in the 1970s, but are named after German mathematician Johann Pfaff. In many real-world problems, the given mathematical relations between the forms dx, dy, dz or du , etc., typically termed as Pfaffian equations are used for finding functions which relate the variables x, y, z or u , etc. The theory of Pfaffian equations is reasonably a challenging subject. The Pfaffian Differential Equations are sometimes abbreviated as PfDE.

Pfaffian Functions

On differentiation, the results obtained from some functions can be written in terms of the original function, for example the exponential function, $f(x) = e^x$. If we differentiate this function then we get e^x again, that is,

$$f'(x) = f(x)$$

Another similar example of this function is the reciprocal function represented as, $g(x) = 1/x$. On differentiation this function has the form,

$$g'(x) = -g(x)^2$$

Some of the other additional functions may not have these properties, however, their derivatives can be written similarly in terms of functions as defined above. For example, consider the function $h(x) = e^x \log(x)$ which can be written as,

$$h'(x) = e^x \log x + x^{-1} e^x = h(x) + f(x)g(x). \quad \dots(4.1)$$

Such functions form the links in the Pfaffian chain. Such a Pfaffian chain is considered as a sequence of functions, say in the region of f_1, f_2, f_3 , etc., having the property that when any of the functions in this chain is differentiated then the result can be written in terms of the function itself and all the functions preceding it in the chain, specifically as a polynomial in those functions and the variables involved. Consequently, for the functions given in Equation (4.1) we can state that f, g, h is a Pfaffian chain.

A **Pfaffian function** is then just a **polynomial** in the **functions** appearing in a **Pfaffian chain** and the **function argument**.

Consequently, using the Pfaffian chain the following function will be the Pfaffian function,

$$F(x) = x^3 f(x)^2 - 2g(x)h(x)$$

Significant Definition: Let U be an open domain in \mathbf{R}^n . Then a Pfaffian chain of order $r \geq 0$ and degree $\alpha \geq 1$ in U is a sequence of real analytic functions f_1, \dots, f_r in U satisfying differential equations of the form,

$$\frac{\partial f_i}{\partial x_j} = P_{i,j}(\mathbf{x}, f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))$$

for $i = 1, \dots, r$, where $P_{ij} \in \mathbf{R}[x_1, \dots, x_n, y_1, \dots, y_r]$ are the polynomials of degree $\leq \alpha$. As a consequence, a function f on U is termed as a Pfaffian function of order r and degree (α, β) if,

$$f(x) = P(x, f_1(x), \dots, f_r(x))$$

where $P \in \mathbf{R}[x_1, \dots, x_n, y_1, \dots, y_r]$ is a polynomial of degree at most $\beta \geq 1$. The numbers r, α and β are collectively termed as the format of the Pfaffian function, and give a useful measure of its complexity.

Pfaffian Equations

As per the ‘Encyclopedia of Mathematics’, an equation of the form is termed as the Pfaffian equation if,

$$w \equiv a_1(x) dx_1 + \dots + a_n(x) dx_n = 0, n \geq 3 \quad \dots(4.2)$$

Where $x \in D \subset \mathbf{R}^n$, w is a differential 1-Form and the functions $a_j(x), j=1, \dots, n$, are real-valued. Let $a_j(x) \in C^1(D)$ and suppose that the vector field $a(x) = (a_1(x), \dots, a_n(x))$ does not have critical points in the domain D .

A manifold $M^k \subset \mathbf{R}^n$ of dimension $k \geq 1$ and of class C^1 is termed as an integral manifold of the Pfaffian Equation (4.2) if $w \equiv 0$ on M^k . The Pfaffian equation is said to be completely integrable if there is one and only one integral manifold of maximum possible dimension $n-1$ through each point of the domain D .

Frobenius Theorem 4.1: A necessary and sufficient condition for the Pfaffian Equation (4.2) to be completely integrable is,

$$d w \wedge w \equiv 0 \quad \dots(4.3)$$

Here $d w$ is the differential form of degree 2 obtained from w by exterior differentiation, and \wedge is the exterior product. In this state, the integration of the Pfaffian equation reduces to the integration of a system of ordinary differential equations.

In a three-dimensional Euclidean space, a Pfaffian equation has the form,

$$Pdx + Qdy + Rdz = 0, \quad \dots(4.4)$$

Where P, Q and R are functions of x, y and z , and condition given in Equation (4.3) for complete integrability has the form,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad \dots(4.5)$$

Or, $(\text{curl } F, F) = 0$, where $F = (P, Q, R)$

In this condition, there exist smooth functions $\mu, U (\mu \neq 0)$ such that,

$$Pdx + Qdy + Rdz \equiv \mu dU,$$

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And the integral surfaces of the Pfaffian Equation (4.4) are given by the equations,

$$U(x, y, z) = \text{Constant}$$

If F is a certain force field, then the field $\mu^{-1} F$ has U as a potential function. If the Pfaffian Equation (4.4) is not completely integrable, then it does not have integral surfaces but can have integral curves.

If the arbitrary functions $x = x(t)$, $y = y(t)$ are given, then Equation (4.4) will be an ordinary differential equation in 'z' and the curve $x = x(t)$, $y = y(t)$, $z = z(t)$ will be an integral curve.

Then, J. Pfaff provided the solution to the problem for Equation (4.2) for arbitrary $n \geq 3$ and then reducing the differential 1-Form w to a canonical form. The condition given in Equation (4.5) was first obtained by L. Euler in 1755.

Through an even change of variables any Pfaffian equation can locally be brought to the form,

$$dy_0 - \sum_{j=1}^p z_j dy_j = 0. \quad \dots(4.6)$$

Where $y_0, \dots, y_p, z_1, \dots, z_p$ are the new independent variables ($2p + 1 \leq n$, $p \geq 0$). The number $2p + 1$ is called the class of the Pfaffian equation; here p is the largest number such that the differential form $w \wedge dw \wedge \dots \wedge dw$ of degree $2p + 1$ is not identically zero. When $p = 0$, then the Pfaffian equation is completely integrable. The functions $y_0(x), \dots, y_p(x)$, are termed as the first integrals of the Pfaffian Equation (4.6) and the integral manifolds of maximum possible dimension $n - p - 1$ are given by the equations,

$$y_0(x) = c_0, \dots, y_p(x) = c_p$$

A Pfaffian system is a system of equations of the form,

$$w_1 = 0, \dots, w_k = 0, k < n \quad \dots(4.7)$$

Where $x \in D \subset \mathbf{R}^n$ and w_i are differential 1-Forms expressed as,

$$\omega_j = \sum_{q=1}^n \omega_{jq}(x) dx_q, \quad j = 1, \dots, k.$$

The rank r of the matrix $\|w_{jk}(x)\|$ is the rank of the Pfaffian system at the point x . A Pfaffian system is said to be completely integrable if there is one and only one integral manifold of maximum possible dimension $n - r$ through each point $x \in U$.

Frobenius Theorem 4.2: A necessary and sufficient condition for a Pfaffian system given in Equation (4.7) of rank k to be completely integrable is,

$$dw_j \wedge w_1 \wedge \dots \wedge w_k = 0, j = 1, \dots, k$$

The problem of integrating any finite non-linear system of partial differential equations is equivalent to the problem of integrating a certain Pfaffian system. A number of results has been obtained on the analytic theory of Pfaffian systems.

A completely-integrable Pfaffian system has the form,

$$dy = x^{-p} f dx + z^{-q} g dz$$

Now of m equations which has been considered for formulation, and where p and q are the positive integers and the vector functions $f(x, y, z)$, $g(x, y, z)$ are holomorphic at the point $x=0, y=0, z=0$; then the sufficient conditions must be established or specified for the existence of a holomorphic solution at the origin.

Theorem 4.3: Let $\Lambda^1(M)$ be the space of external differential type of 1-Forms on M . Then a Pfaffian equation can be defined as a module $E \subset \Lambda^1(M)$ over the ring of smooth functions generated by a single 1-Form $w \in \Lambda^1(M)$. This specific form represents the Pfaffian equation. Thus, the Pfaffian equation which is represented or symbolized by means of 1-Forms are equal when they are multiplied by a specific function that is precisely non-vanishing on M .

Let, the notation used for a Pfaffian equation E is $\{w=0\}$, where ' w ' is considered as one of the forms that represents E .

Now, if $E : \{w=0\}$ is a Pfaffian equation, where w is non-vanishing on M , then E can be considered as a distribution of co-dimension 1 on M , considering that M is a field of hyperplanes, since in the tangent space $T_{\alpha} M$ ($\alpha \in M$) the hyperplane that consists or comprises of tangent vectors which are typically annihilated by the form $w|_{\alpha}$ is defined.

Furthermore, a contact structure can be defined as a Pfaffian equation for the field of kernels of a contact 1-Form on an odd-dimensional manifold. Different types of Pfaffian equations can be obtained or established by the restriction of a contact structure either on an odd-dimensional manifold or on an even-dimensional manifold. The following example defines the concept of contact structure.

The contact structure $(dz + x dy + u dv = 0)$ that is in \mathbf{R}^5 domain is restricted to the three-dimensional manifold $(u = v = 0)$ and coincides with the Pfaffian equation of the form $(dz + x dy = 0)$, i.e., a contract structure in \mathbf{R}^3 domain.

Example 4.1. Find the solution of the following Pfaffian differential equation:

$$(y + z) dx + dy + dz = 0$$

Solution: The Pfaffian differential equation can also be written as follows:

$$dx + \frac{dy + dz}{y + z} = 0$$

This equation has the solution,

$$x + \ln(y + z) - \text{Constant}$$

Or,

$$z = -y + ae^{-x}$$

Where ‘ a ’ is an arbitrary function.

Example 4.2: Solve the Pfaffian equation $xdy - ydx = 0$.

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Solution: This Pfaffian equation can be solved using the quotient rule. The equation will take the form,

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2} \equiv x^{-2}(xdy - ydx)$$

Now the equation $xdy - ydx = 0$ is multiplied by x^{-2} , which is termed as an integrating factor of the equation. The equation now becomes,

$$x^{-2}(xdy - ydx) = 0$$

The left-hand side of the above equation is termed as an ‘exact differential’, namely, $d(y/x)$. Considering the following new ‘Rule of Thumb’, we can state that ‘ $y/x = \text{Constant}$ ’ gives a family of solutions to $xdy - ydx = 0$.

General Form of Pfaffian Equations in Two Variables

The general form of Pfaffian equations in two variables x and y is written as,

$$P dx + Q dy = 0$$

Where $P = P(x, y)$ and $Q = Q(x, y)$ are functions of x and y , respectively.

This equation can be simply written as $w = 0$, where $w = P dx + Q dy$. If the functions $f = f(x, y)$ and $g = g(x, y)$ can be obtained such that $w = gdf$, then $w = 0$ can be reduced to $df = 0$ with solutions $f(x, y) = c$, where c is any constant. The general solution $f(x, y) = c$ represents a family of curves with ‘ c ’ as a parameter.

4.3 SOLUTION OF PFAFFIAN DIFFERENTIAL EQUATIONS IN THREE VARIABLES

In the case of three variables, the Pfaffian Differential Equations or PfDE takes the form,

$$P dx + Q dy + R dz = 0 \quad \dots(4.8)$$

Where P, Q and R being the functions of x, y and z .

The system of a simultaneous differential equations can be associated with the Equation (4.8), namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots(4.9)$$

The system of Equation (4.9) defines a two-parameter family of space curves.

Let us now find the relationship between the system of Equations (4.8) and (4.9), through the Pfaffian equation of the form,

$$dx + dy + dz = 0 \quad \dots(4.10)$$

On integrating the Equation (4.10), we obtain,

$$x + y + z = c, -\infty < c < \infty \quad \dots(4.11)$$

Where 'c' is an arbitrary constant.

This is a one-parameter family of parallel planes in the xyz - space.

Now the system of simultaneous differential equations associated with Equation (4.10) is,

$$dx/1 = dy/1 = dz/1 \quad \dots(4.12)$$

The solution set of Equation (4.12) is a two-parameter family of straight lines of the form,

$$x - y = c_1 \text{ and } y - z = c_2 \quad \dots(4.13)$$

Where c_1 and c_2 are two arbitrary constants.

As per the standard rule, a family of straight lines intersect a family of planes orthogonally if the direction cosines of the family of straight lines are the same as the direction cosines of normal to the family of planes.

Write the Equation (4.13) in the form,

$$\frac{x-c_1}{1} = \frac{y}{1} = \frac{z+c_2}{1}$$

Observe that the direction cosines of the family of lines become,

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Furthermore, the direction ratios of the normal to the family of planes are $(1, 1, 1)$ so that the direction cosines of normal to planes are,

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Thus, the family of straight lines (as per Equation (4.13)) intersect the family of planes orthogonally.

Example 4.3: For the given PfDE or Pfaffian differential equation find the family of curves.

$$z dx + dy + dz = 0 \quad \dots(1)$$

Solution: The above Equation (1) cannot directly be integrated to obtain a family of surfaces.

We have to first find the system of simultaneous differential equations that are associated with Equation (1) as follows,

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$$\frac{dx}{z} = \frac{dy}{1} = \frac{dz}{1} \quad \dots(2)$$

Taking the second and third ratios of Equation (2), we obtain,

$$y - z = c_1$$

Similarly, from first and third ratios of Equation (2), we obtain,

$$x - \frac{1}{2} z^2 = c_2$$

Where c_1 and c_2 are two arbitrary constants.

Thus the solution set of Equation (2) is as follows;

$$y - z = c_1 \quad \text{and} \quad x - \frac{1}{2} z^2 = c_2 \quad \dots(3)$$

Which is defined as a two-parameter family of curves obtained as intersection of two families of surfaces.

In this condition, the integral surfaces of Equation (1) do not exist. However, in the plane $x = k_1$, a constant, we have $dx = 0$ and then we can integrate Equation (1) and obtain the following form of equation,

$$y + z = c_1$$

Thus, we can state that the equations, $x = k_1$ and $y + z = c_1$ define a one-parameter family of straight lines, all of which lie on the plane $x = k_1$. These lines are considered orthogonal to the curves specified by Equation (3).

Similarly, we can state that on the plane,

$$y = k_2, dy = 0$$

Thus, on integrating Equation (1), we obtain,

$$x + \ln |z| = c_2$$

Therefore, the given equations define a one-parameter family of curves all of which lie on the plane $y = k_2$ and they are orthogonal to the curves specified by Equation (3).

In consequence, to determine, we can state that the Pfaffian Differential Equations or PfDE of the form given in Equation (4.8) is not necessarily integrable. The following two conditions hold:

Condition 1: If the Pfaffian Differential Equations or PfDE of the form given in Equation (4.8) is integrable, then its integral curve is considered as a one-parameter family of surfaces which are orthogonal to the two-parameter family of space curves specifically defined by the associated system of simultaneous Equation (4.9).

Condition 2: If the Pfaffian Differential Equations or PfDE of the form given in Equation (4.8) is not integrable, then it can still have solutions in the sense that it can be determined on the basis of a given surface, say,

$$z = F(x, y)$$

This is referred to as a one-parameter family of curves. These curves are again considered as orthogonal to the two-parameter family of space curves specifically defined by the associated system of simultaneous Equation (4.9).

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Check Your Progress

1. What is Pfaffian function?
2. What is Pfaffian equation?
3. When is Pfaffian equation said to be completely integrable?
4. Define the form of Pfaffian equation in three-dimensional Euclidean space.
5. Give the general form of Pfaffian equations in two variables.
6. State the general form of Pfaffian equations in three variables.

4.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A Pfaffian function is just a polynomial in the functions appearing in a Pfaffian chain and the function argument.
2. As per the ‘Encyclopedia of Mathematics’, an equation of the form is termed as the Pfaffian equation if,

$$w \equiv a_1(x) dx_1 + \dots + a_n(x) dx_n = 0, n \geq 3$$

Where $x \in D \subset \mathbf{R}^n$, w is a differential 1-Form and the functions $a_j(x), j = 1, \dots, n$, are real-valued.

3. Consider the Pfaffian equation of the form,

$$w \equiv a_1(x) dx_1 + \dots + a_n(x) dx_n = 0, n \geq 3 \quad \dots(1)$$

Where $x \in D \subset \mathbf{R}^n$, w is a differential 1-Form and the functions $a_j(x), j = 1, \dots, n$, are real-valued. Let $a_j(x) \in C^1(D)$ and suppose that the vector field $a(x) = (a_1(x), \dots, a_n(x))$ does not have critical points in the domain D . Then a manifold $M^k \subset \mathbf{R}^n$ of dimension $k \geq 1$ and of class C^1 is termed as an integral manifold of the Pfaffian Equation (1) if $w \equiv 0$ on M^k . The Pfaffian equation is said to be completely integrable if there is one and only one integral manifold of maximum possible dimension $n-1$ through each point of the domain D .

4. Pfaffian equation in three-dimensional Euclidean space has the form,

$$Pdx + Qdy + Rdz = 0$$

Where P, Q and R are functions of x, y and z .

5. The general form of Pfaffian equations in two variables x and y is written as,

$$P dx + Q dy = 0$$

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- Where $P = P(x, y)$ and $Q = Q(x, y)$ are functions of x and y , respectively.
6. In the case of three variables, the Pfaffian Differential Equations or PfDE takes the form,

$$P \, dx + Q \, dy + R \, dz = 0$$

Where P, Q and R being the functions of x, y and z .

4.5 SUMMARY

- In mathematics, Pfaffian functions are defined as a certain distinct form or arrangement of functions whose derivative can be written in terms of the original function. They were originally introduced by Askold Georgievich Khovanskiî in the 1970s, but are named after German mathematician Johann Pfaff.
- In many real-world problems, the given mathematical relations between the forms dx, dy, dz or du , etc., typically termed as Pfaffian equations are used for finding functions which relate the variables x, y, z or u , etc.
- The Pfaffian Differential Equations are sometimes abbreviated as PfDE.
- Some functions form the links in the Pfaffian chain. Such a Pfaffian chain is considered as a sequence of functions, say in the region of f_1, f_2, f_3 , etc., having the property that when any of the functions in this chain is differentiated then the result can be written in terms of the function itself and all the functions preceding it in the chain, specifically as a polynomial in those functions and the variables involved.
- A Pfaffian function is just a polynomial in the functions appearing in a Pfaffian chain and the function argument.
- As per the ‘Encyclopedia of Mathematics’, an equation of the form is termed as the Pfaffian equation if,

$$w \equiv a_1(x) \, dx_1 + \dots + a_n(x) \, dx_n = 0, \quad n \geq 3$$

Where $x \in D \subset \mathbf{R}^n$, w is a differential 1-Form and the functions $a_j(x), j = 1, \dots, n$, are real-valued.

- Pfaffian equation in three-dimensional Euclidean space has the form,

$$P \, dx + Q \, dy + R \, dz = 0$$

Where P, Q and R are functions of x, y and z .

- The rank r of the matrix $\| w_{jk}(x) \|$ is the rank of the Pfaffian system at the point x . A Pfaffian system is said to be completely integrable if there is one and only one integral manifold of maximum possible dimension $n-r$ through each point $x \in U$.
- A completely-integrable Pfaffian system has the form,

$$dy = x^{-p} f \, dx + z^{-q} g \, dz$$

- The general form of Pfaffian equations in two variables x and y is written as,

$$P \, dx + Q \, dy = 0$$

Where $P = P(x, y)$ and $Q = Q(x, y)$ are functions of x and y , respectively.

- In the case of three variables, the Pfaffian Differential Equations or PfDE takes the form,

$$P \, dx + Q \, dy + R \, dz = 0$$

Where P, Q and R being the functions of x, y and z .

- As per the standard rule, a family of straight lines intersect a family of planes orthogonally if the direction cosines of the family of straight lines are the same as the direction cosines of normal to the family of planes.

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4.6 KEY WORDS

- Pfaffian function:** A Pfaffian function is just a polynomial in the functions appearing in a Pfaffian chain and the function argument.
- Rank r of Pfaffian system:** The rank r of the matrix $\|w_{jk}(x)\|$ is the rank of the Pfaffian system at the point x , and a Pfaffian system is said to be completely integrable if there is one and only one integral manifold of maximum possible dimension $n-r$ through each point $x \in U$.
- Pfaffian equations in two variables:** The general form of Pfaffian equations in two variables x and y is, $P \, dx + Q \, dy = 0$, where $P = P(x, y)$ and $Q = Q(x, y)$ are functions of x and y , respectively.
- Pfaffian equations in three variables:** In the case of three variables, the Pfaffian Differential Equations or PfDE takes the form, $P \, dx + Q \, dy + R \, dz = 0$, where P, Q and R being the functions of x, y and z .

4.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

- What is PfDE?
- Who gave the theory of Pfaffian equations?
- Define the terms Pfaffian function and Pfaffian equation.
- Give the definitions and notations for Pfaffian function and Pfaffian equation.
- Differentiate the function, $f'(x) = f(x)$.
- What is a differential 1-Form?
- Give the Pfaffian equations for two variables and three variables.

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8. Prove that the completely-integrable Pfaffian system has the form
 $dy = x^{-p} f dx + z^{-q} g dz,$

Long Answer Questions

1. Explain the significance of Pfaffian equation in estimating differential equations.
2. ‘In mathematics, Pfaffian functions are defined as a certain distinct form or arrangement of functions whose derivative can be written in terms of the original function’. Justify the statement giving appropriate examples.
3. Briefly discuss the concept of Pfaffian chain with the help of appropriate examples.
4. ‘A Pfaffian function is just a polynomial in the functions appearing in a Pfaffian chain and the function argument’. Justify the statement giving appropriate examples.
5. Briefly discuss using the concept of Pfaffian chain that the following function will be the Pfaffian function,

$$F(x) = x^3 f(x)^2 - 2g(x)h(x)$$

6. Discuss the Pfaffian equation in three variables and then explain the three-dimensional Euclidean space on the basis of a Pfaffian equation.
7. Explain the necessary and sufficient conditions for a Pfaffian system on the basis of Frobenius Theorems.
8. Solve the Pfaffian equation, $x dy - y dx = 0$.
9. Solve the Pfaffian system of the equation,

$$x dy - y dx + dz = 0 \text{ and } y dx + dz = 0$$

4.8 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

BLOCK II

LINEAR AND NON LINEAR PDE

*Partial Differential
Equations of the First Order*

UNIT 5 PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

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Structure

- 5.0 Introduction
- 5.1 Objectives
- 5.2 Partial Differential Equations
- 5.3 Origins of First Order Partial Differential Equations
- 5.4 Answers to Check Your Progress Questions
- 5.5 Summary
- 5.6 Key Words
- 5.7 Self Assessment Questions and Exercises
- 5.8 Further Readings

5.0 INTRODUCTION

In mathematics, a Partial Differential Equation (PDE) is a differential equation that contains beforehand unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a computer model. A special case is Ordinary Differential Equations (ODEs), which deal with functions of a single variable and their derivatives.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

In this unit, you will study about Partial Differential Equation (PDE), origin of first order partial differential equations.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what Partial Differential Equation (PDE) is

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- Explain the origin of first order partial differential equations
- Solve various types of partial differential equations
- Form partial differential equations by eliminating arbitrary constants and Function
- Explain Laplace equations, Poisson equations and wave equations

5.2 PARTIAL DIFFERENTIAL EQUATIONS

Let $z = f(x, y)$ be a function of two independent variables x and y . Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

are the first order partial derivatives; $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.

Any equation which contains one or more partial derivatives is called a partial differential equation. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$; $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for Partial Differential Equation (PDE) of first order and second order respectively.

We use the following notations for partial derivatives,

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Partial differential equation may be formed by eliminating (i) arbitrary constants (ii) arbitrary functions.

Example 5.1: Form the parital differential equation by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$.

Solution: Given, $z = ax + by + a^2 + b^2$... (5.1)

Here we have two arbitrary constants a and b . Therefore, we need two more equations to eliminate a and b . Differentiating equation (1) partially with respect to x and y respectively we get,

$$\frac{\partial z}{\partial x} = p = a \quad \dots(5.2)$$

$$\frac{\partial z}{\partial y} = q = b \quad \dots(5.3)$$

From Equations (5.2) and (5.3), we get,

$$a = p, b = q$$

Substituting values of a and b in (5.1) we get,

$$z = px + qy + p^2 + q^2$$

This is the required partial differential equation.

Example 5.2: Eliminate a and b from $z = (x + a)(y + b)$.

Solution: Differentiating partially with respect to x and y ,

$$p = y + b, q = x + a$$

Eliminating a and b , we get $z = pq$.

Example 5.3: Form the partial differential equation by eliminating the arbitrary constants in $z = (x - a)^2 + (y - b)^2$.

Solution: Given, $z = (x - a)^2 + (y - b)^2$... (5.4)

Here we have two arbitrary constants a and b . To eliminate these two arbitrary constants we need two more equations connecting a and b . Therefore, differentiating Equation (5.4) partially with respect to x and y , we get,

$$\frac{\partial z}{\partial x} = p = 2(x - a) \quad \dots(5.5)$$

$$\frac{\partial z}{\partial y} = q = 2(y - b) \quad \dots(5.6)$$

From Equation (5.5), we get,

$$x - a = \frac{p}{2} \quad \dots(5.7)$$

From Equation (5.6), we get,

$$y - b = \frac{q}{2} \quad \dots(5.8)$$

Substituting Equations (5.7) and (5.8) in (5.4) we get,

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

Simplifying we get, $4z = p^2 + q^2$

This gives the partial differential equation after elimination of a and b .

Example 5.4: Form the partial differential equation by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$.

Solution: Given, $z = (x^2 + a)(y^2 + b)$... (5.9)

Here we have two arbitrary constants a and b .

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Differentiating Equation (5.9) partially with respect to x and y we get,

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \quad \dots(5.10)$$

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$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \quad \dots(5.11)$$

$$\text{From Equation (5.10) we get, } \frac{p}{2x} = y^2 + b \quad \dots(5.12)$$

$$\text{From Equation (5.11) we get, } \frac{q}{2y} = x^2 + a \quad \dots(5.13)$$

Substituting Equations (5.12) and (5.13) in (5.4), we get,

$$z = \frac{p}{2x} \cdot \frac{q}{2y}$$

$$pq = 4xyz$$

This gives the required partial differential equation.

Example 5.5: Form the partial differential equation by eliminating a , b , c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution: Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$...(5.14)

Differentiate partially with respect to x and y we get,

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot p = 0 \quad \dots(5.15)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot q = 0 \quad \dots(5.16)$$

Differentiating Equation (5.15) partially with respect to y ,

$$0 + \frac{2}{c^2} (zs + qp) = 0$$

$$zs + qp = 0$$

Note: More than one partial differential equation is possible in this problem. These partial differential equations are,

$$xsr + xp^2 - zp = 0, yzt + yq^2 - zq = 0$$

Formation of Partial Differential Equation by Eliminating Arbitrary Functions

The partial differential equations can be formed by eliminating arbitrary junctions. The following examples will make the concept clear.

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Example 5.6: Eliminate arbitrary function from,

$$z = f(x^2 + y^2) \quad \dots(5.17)$$

Solution: Differentiating partially with respect to x and y , we get,

$$p = f'(x^2 + y^2).2x \quad \dots(5.18)$$

$$q = f'(x^2 + y^2).2y \quad \dots(5.19)$$

Eliminating $f'(x^2 + y^2)$ from Equation (5.18) and (5.19), we get, $py = qx$

Example 5.7: Form the partial differential equation by eliminating the arbitrary function ϕ from $xyz = \phi(x^2 + y^2 - z^2)$.

Solution: Given, $xyz = \phi(x^2 + y^2 - z^2) \quad \dots(5.20)$

This equation contains only one arbitrary function ϕ and we have to eliminate it.

Differentiating Equation (5.20) partially with respect to x and y we get,

$$yz + xyp = \phi'(x^2 + y^2 - z^2)(2x - 2zp) \quad \dots(5.21)$$

$$xz + xyq = \phi'(x^2 + y^2 - z^2)(2y - 2zq) \quad \dots(5.22)$$

From Equation (5.21), we get,

$$\phi'(x^2 + y^2 - z^2) = \frac{yz + xyp}{2x - 2zp} \quad \dots(5.23)$$

From Equation (5.22), we get,

$$\phi'(x^2 + y^2 - z^2) = \frac{xz + xyq}{2y - 2zq} \quad \dots(5.24)$$

Since, LHS of Equations (5.23) and (5.24) are equal, we have,

$$\frac{yz + xyp}{2x - 2zp} = \frac{xz + xyq}{2y - 2zq}$$

$$(yz + xyp)(y - zq) = (xz + xyq)(x - zp)$$

$$\text{i.e., } y(z + xp)(y - zq) = x(z + yq)(x - zp) \quad \dots(5.25)$$

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On simplifying Equation (5.25) we get,

$$px(y^2 + z^2) - qy(z^2 + x^2) = z(x^2 - y^2)$$

Which gives the required partial differential equation.

Example 5.8: Eliminate the arbitrary function from $z = (x + y)f(x^2 - y^2)$

Solution: Given, $z = (x + y)f(x^2 - y^2)$... (5.26)

Differentiating partially with respect to x and y we get,

$$p = (x + y)f'(x^2 - y^2)2x + f(x^2 - y^2) \cdot 1 \quad \dots(5.27)$$

$$q = (x + y)f'(x^2 - y^2)(-2y) + f(x^2 - y^2) \cdot 1 \quad \dots(5.28)$$

Eliminating $f'(x^2 - y^2)$ from Equations (5.27) and (5.28) we get,

$$\frac{2x(x + y)}{-2y(x + y)} = \frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)}$$

$$2x[q - f(x^2 - y^2)] = -2y[p - f(x^2 - y^2)]$$

$$xq - xf(x^2 - y^2) = -yp + yf(x^2 - y^2)$$

$$xq + yp = (x + y)f(x^2 - y^2)$$

$$= (x + y) \frac{z}{(x + y)}$$

$$\therefore z = xq + yp$$

This is a required equation.

Example 5.9: Eliminate the arbitrary function from $z = xy + f(x^2 + y^2)$

Solution: Given, $z = xy + f(x^2 + y^2)$... (5.29)

Differentiating partially Equation (5.29) with respect to x and y we get,

$$p = y + f'(x^2 + y^2) \cdot 2x \quad \dots(6.30)$$

$$q = x + f'(x^2 + y^2) \cdot 2y \quad \dots(5.31)$$

Eliminating $f'(x^2 + y^2)$ from Equations (5.30) and (5.31) we get,

$$(p - y)y = (q - x)x$$

$$py - y^2 = qx - x^2$$

$$py - qx = y^2 - x^2$$

Which is a required equation.

Example 5.10: Eliminate the arbitrary functions f and ϕ from the relation $z = f(x + ay) + \phi(x - ay)$

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Solution: Differentiating partially with respect to x and y we get,

$$p = f'(x + ay) + \phi'(x - ay) \quad \dots(5.32)$$

$$q = af'(x + ay) - a\phi'(x - ay) \quad \dots(5.33)$$

Differentiating these again, with respect to x and y we get,

$$\frac{\partial^2 z}{\partial x^2} = r = f''(x + ay) + \phi''(x - ay) \quad \dots(5.34)$$

$$\frac{\partial^2 z}{\partial y^2} = t = a^2 f''(x + ay) + a^2 \phi''(x - ay) \quad \dots(5.35)$$

From Equations (5.34) and (5.35) we get,

$$t = a^2 r$$

Equations Solvable by Direct Integration

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.

In complete integral, if we give particular values to the arbitrary constants, we get particular integral. If $\phi(x, y, z, a, b) = 0$, is the complete integral of a partial

differential equation, then the eliminant of a and b from the equations $\frac{\partial \phi}{\partial a} = 0$, $\frac{\partial \phi}{\partial b} = 0$, is called singular integral.

Let us consider four standard types of nonlinear partial differential equations and the procedure for obtaining their complete solution.

Type I Equations of the form $F(p, q) = 0$. In this type of equations we have only p and q and there is no x, y and z . To solve this type of problems, let us assume that $z = ax + by + c$ be the solution and then proceed as in the following examples.

Example 5.11: Solve $p^2 + q^2 = 4$

Solution: Given, $p^2 + q^2 = 4 \quad \dots(5.36)$

Let us assume that $z = ax + by + c$ be a solution of Equation (5.36). $\dots(5.37)$

Partially differentiating Equation (5.36) with respect to x and y , we get,

$$\frac{\partial z}{\partial x} = p = a \text{ and } \frac{\partial z}{\partial y} = q = b \quad \dots(5.38)$$

Substituting Equation (5.38) in (5.36) we get,

$$a^2 + b^2 = 4 \quad \dots(5.39)$$

To get the complete integral we have to eliminate any one of the arbitrary constants from Equation (5.37).

From Equation (5.39) we get,

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$$b = \pm\sqrt{4-a^2} \quad \dots(5.40)$$

Substituting Equation (5.40) in (5.37) we get,

$$z = ax \pm y\sqrt{4-a^2} + C \quad \dots(5.41)$$

Which contains only two constants (equal to number of independent variables). Therefore, it gives the complete integral.

To check for Singular Integral:

Differentiating Equation (5.38) partially with respect to a and c and equating to zero, we get,

$$\frac{\partial z}{\partial a} = x \pm \frac{1}{2\sqrt{4-a^2}}(-2a) = 0 \quad \dots(5.42)$$

and, $\frac{\partial z}{\partial c} = 1 = 0$

Here, $1 = 0$ is not possible.

Hence, there is no singular integral.

Example 5.12: Solve $p^2 + q^2 = npq$

Solution. The solution is, $z = ax + by + c$, where $a^2 + b^2 = nab$

$$\text{Solving, } b = \frac{a(n \pm \sqrt{n^2 - 4})}{2}$$

The compete integral is,

$$z = ax + \frac{ay}{2}(n \pm \sqrt{n^2 - 4}) + c$$

Differentiating partially with respect to c , we see that there is no singular integral, as we get an absurd result.

Example 5.13: Solve $p + q = pq$

Solution: This equation is of the type, $F(p, q) = 0$.

\therefore The complete solution is of the form, $z = ax + by + c$ $\dots(5.43)$

Differentiating Equation (5.43) partially with respect to x and y we get,

$$p = a, q = b$$

Therefore, the given equation becomes,

$$a + b = ab$$

$$a = b(a - 1); \quad b = \frac{a}{a-1}$$

Therefore, the complete solution is,

$$z = ax + \left(\frac{a}{a-1} \right) y + c$$

This type of equation has no singular solution.

Let, $c = \varphi(a)$

$$z = ax + \left(\frac{a}{a-1} \right) y + \phi(a) \quad \dots(5.44)$$

Differentiating partially with respect to a ,

$$\begin{aligned} 0 &= x + \left[\frac{(a-1)1-a}{(a-1)^2} \right] y + \phi'(a) \\ 0 &= -\frac{1}{(a-1)^2} y + \phi'(a) \end{aligned} \quad \dots(5.45)$$

The elimination of a between Equations (5.44) and (5.45) gives the general solution.

Type II Equation of the form $z = px + qy + F(p, q)$ (Clairaut's form). In this type of problems assume that, $z = ax + by + F(a, b)$ be the solution.

Example 5.14: Solve $z = px + qy + ab$

Solution: This equation is of Clairaut's type. Therefore, the complete solution is obtained by replacing p by a and q by b , where a and b are arbitrary constants.

i.e., the complete solution is, $z = ax + by + ab$ $\dots(5.46)$

Differentiating Equation (5.46) partially with respect to a and b , and equating these to zero we get,

$$0 = x + b \quad \dots(5.47)$$

$$0 = y + a \quad \dots(5.48)$$

Eliminating a and b from Equations (5.46), (5.47) and (5.48) we get,

$$z = -xy - xy + xy$$

i.e., $z + xy = 0$

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This gives the singular solution of the given partial differential equation and to get the general solution.

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Put, $b = \phi(a)$ in Equation (5.46)

$$\therefore z = ax + \phi(a)y + a\phi(a) \quad \dots(5.49)$$

Differentiating partially with respect to a we get,

$$0 = x + \phi'(a)y + a\phi'(a) + \phi(a) \quad \dots(5.50)$$

Eliminating a from Equations (5.49) and (5.50) we get the general solution.

Example 5.15: Obtain the complete solution and singular solution of,

$$z = px + qy + p^2 + pq + q^2$$

Solution: This equation is of Clairaut's form. Therefore, the complete solution is,

$$z = ax + by + a^2 + ab + b^2 \quad \dots(5.51)$$

Where, a and b are arbitrary constants.

Differentiating Equation (5.51) partially with respect to a and b we get,

$$0 = x + 2a + b \quad (5.52)$$

$$0 = y + 2b + a \quad (5.53)$$

$$2x - y = 3a, \text{ and } 2y - x = 3b$$

$$a = \frac{2x - y}{3}, b = \frac{2y - x}{3}$$

Substituting this in Equation (5.51) we get,

$$z = \left(\frac{2x - y}{3}\right)x + \left(\frac{2y - x}{3}\right)y + \left(\frac{2x - y}{3}\right)^2$$

$$+ \frac{(2x - y)(2y - x)}{9} + \left(\frac{2y - x}{3}\right)^2$$

Simplifying we get, $3z = xy - x^2 - y^2$. This is the singular solution.

To find singular integral:

Differentiating Equation (5.52) partially with respect to a and b , and then equating to zero, we get,

$$\frac{\partial z}{\partial a} = x + \frac{a}{\sqrt{1+a^2+b^2}} = 0 \quad \dots(5.54)$$

$$\frac{\partial z}{\partial b} = y + \frac{b}{\sqrt{1+a^2+b^2}} = 0 \quad \dots(5.55)$$

From Equation (5.54), we get,

$$x^2 = \frac{a^2}{1+a^2+b^2} \quad \dots(5.56)$$

From Equation (5.55), we get,

$$y^2 = \frac{b^2}{1+a^2+b^2} \quad \dots(5.57)$$

From Equations (5.56) and (5.57) we get,

$$\begin{aligned} x^2 + y^2 &= \frac{a^2 + b^2}{1+a^2+b^2} \\ \therefore 1 - (x^2 + y^2) &= 1 - \frac{a^2 + b^2}{1+a^2+b^2} \\ &= \frac{1}{1+a^2+b^2} \\ \text{i.e., } 1 - x^2 - y^2 &= \frac{1}{1+a^2+b^2} \\ \therefore \sqrt{1+a^2+b^2} &= \frac{1}{\sqrt{1-x^2-y^2}} \end{aligned} \quad \dots(5.58)$$

Substituting Equation (5.58) in (5.54) and (5.55) we get,

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}, \quad b = \frac{-y}{\sqrt{1-x^2-y^2}} \quad \dots(5.59)$$

Substituting Equations (5.58) and (5.59) in (5.52) we get,

$$\begin{aligned} z &= \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}} \\ &= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} \\ \therefore z &= \sqrt{1-x^2-y^2} \text{ or, } z^2 = 1-x^2-y^2 \\ \therefore x^2 + y^2 + z^2 &= 1 \end{aligned}$$

This is the singular integral.

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Type III Equation of the form, $F(z, p, q) = 0$

Example 5.16: Solve $z = p^2 + q^2$

Solution: Given, $z = p^2 + q^2$... (5.60)

Assume that, $z = f(x + ay)$ is a solution of Equation (5.60). ... (5.61)

Put, $x + ay = u$ in Equation (5.61)

Then, $z = f(u)$... (5.62)

Partially differentiating Equation (5.62) with respect to x and y we get,

$$p = \frac{dz}{du}, q = a \frac{dz}{du} \quad \dots(5.63)$$

$$\left(\because \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} \right)$$

Substituting Equation (5.63) in (5.60) we get,

$$z = \left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2$$

$$\text{i.e.,} \quad \left(\frac{dz}{du} \right)^2 (1 + a^2) = z$$

$$\text{i.e.,} \quad \frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$\text{i.e.,} \quad \frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}} \quad \dots(5.64)$$

Integrating Equation (5.64) we get,

$$\int \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}} \int du$$

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + b$$

$$\text{i.e.,} \quad 2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + b$$

This gives the complete integral.

Example 5.17: Solve $ap + bq + cz = 0$

Solution: Given, $ap + bq + cz = 0$... (5.65)

Let us assume that, $z = f(x + ky)$... (5.66)

By the solution of Equation (5.66).

Put $x + ky = u$ in Equation (5.66)

$$\therefore z = f(u) \quad \dots(5.67)$$

$$p = \frac{dz}{du}; q = k \frac{dz}{du} \quad \dots(5.68)$$

Substituting Equation (5.67) in (5.68) we get,

$$a \cdot \frac{dz}{du} + b \cdot k \frac{dz}{du} + c \cdot z = 0$$

$$\text{i.e., } \frac{dz}{du}(a + bk) = -cz$$

$$\therefore \frac{dz}{du} = -\frac{cz}{a + bk}$$

$$\text{i.e., } \frac{dz}{z} = -\frac{c}{a + bk} du \quad \dots(5.69)$$

Integrating Equation (5.69) we get,

$$\int \frac{dz}{z} = -\frac{c}{a + bk} \int du$$

$$\log z = -\frac{c}{a + bk} (u) + \log b$$

$$\text{i.e., } \log z = A[x + ky] + \log b, \text{ where } A = -\frac{c}{a + bk}$$

$$\text{i.e., } \log z - \log b = A(x + ky)$$

$$\log \left(\frac{z}{b} \right) = A(x + ky)$$

$$\frac{z}{b} = e^{A(x+ky)}$$

$$\therefore z = b e^{A(x+ky)}$$

This gives the complete integral.

Type IV Equation of the form, $F_1(x, p) = F_2(y, q)$

Example 5.18: Solve the equation, $p + q = x + y$

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Solution: We can write the equation in the form, $p - x = y - q$

Let, $p - x = a$, then $y - q = a$

Hence, $p = x + a$, $q = y - a$

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy \\ &= (x + a)dx + (y - a)dy \end{aligned}$$

On Integrating,

$$z = \frac{(x + a)^2}{2} + \frac{(y - a)^2}{2} + b$$

There is no singular integral and the general integral is found as usual.

Example 5.19: Solve $p^2 + q^2 = x + y$

Solution: Given, $p^2 + q^2 = x + y$

$$p^2 - x = y - q^2 = k$$

$$\therefore p^2 - x = k; y - q^2 = k$$

$$p = \pm\sqrt{x+k}, q = \pm\sqrt{y-k}$$

$$dz = pdx + qdy$$

$$= \pm(\sqrt{x+k})dx \pm(\sqrt{y-k})dy$$

Integrating we get the complete solution.

$$\begin{aligned} z &= \pm\frac{2}{3}(x+k)^{3/2} \pm\frac{2}{3}(y-k)^{3/2} + C \\ &= \pm\frac{2}{3}\left[(x+k)^{3/2} + (y-k)^{3/2}\right] + C \end{aligned}$$

Example 5.20: Solve $p + q = \sin x + \sin y$

Solution:

$$p - \sin x = \sin y - q = k$$

$$\therefore p = k + \sin x; q = \sin y - k$$

$$dz = pdx + qdy$$

$$= (k + \sin x)dx + (\sin y - k)dy$$

On integrating, we get,

$$\begin{aligned} z &= (kx - \cos x) - (ky + \cos y) + C \\ z &= k(x - y) - (\cos x + \cos y) + C \end{aligned}$$

This is the complete solution.

NOTES

Laplace Equation

In mathematics, Laplace equation is a second order partial differential equation. It is named after Pierre-Simon Laplace and is written as,

$$\nabla^2 \varphi = 0$$

Here ∇^2 is the Laplace operator and φ is a scalar function of 3 variables. Laplace equation and Poisson equation are examples of elliptic partial differential equations. The universal theory of solutions to Laplace equation is termed as potential theory. The solutions of Laplace equation are harmonic functions and have great importance in many fields of science.

Twice differentiable real-valued functions f of real variables x, y and z are found using the following notations.

In Cartesian Coordinates:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

In Cylindrical Coordinates:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

In Spherical Coordinates:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial f}{\partial \varphi} \right) + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 f}{\partial \theta^2} = 0$$

The Laplace equation $\nabla^2 \varphi = 0$ can also be written as $\nabla \cdot \nabla \varphi = 0$.

It is also sometimes written using the notation $\nabla \varphi = 0$, where Δ is also the Laplace operator.

Solutions of Laplace equation are harmonic functions. If the right-hand side is specified as a given function, $f(x, y, z)$ then the whole equation can be written as,

$$\nabla \varphi = f$$

This is the Poisson equation. The Laplace equation is also considered as a special type of the Helmholtz equation.

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Laplace Equation in Two Dimensions

The Laplace equation in two independent variables has the form,

$$\varphi_{xx} + \varphi_{yy} = 0$$

Analytic Functions: Both the real and imaginary parts of a complex analytic function satisfy the Laplace equation. If $z = x + iy$ and also if $f(z) = u(x,y) + iv(x,y)$ then the necessary condition that $f(z)$ be analytic is that it must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$, where u_x is the first partial derivative of u with respect to x . It follows the notation,

$$u_{yy} = (-v_x)_y = -\left(v_y\right)_x = -(u_x)_x$$

Thus u satisfies the Laplace equation. Similarly it can be proved that v also satisfies the Laplace equation. Conversely, for a harmonic function it is the real part of an analytic function $f(z)$.

For a trial form, $f(z) = \varphi(x, y) + i\psi(x, y)$, the Cauchy-Riemann equations is satisfied if,

$$\Psi_x = -\varphi_y, \quad \Psi_y = \varphi_x$$

This relation does not determine ψ , but only its increments as $d\psi = \varphi_y dx + \varphi_x dy$.

The Laplace equation for φ implies that the integrability condition for ψ satisfied as ψ_{xy} is ψ_{yx} ; and thus ψ can be defined using a line integral. Both the integrability condition and Stokes' theorem implies that the value of the line integral connecting two points is independent of the path. The resulting pair of solutions of the Laplace equation is termed as **conjugate harmonic functions**. This construction is valid only locally or provided that the path does not loop around a singularity. There is a close connection between the Laplace equation and analytic functions which implies that any solution of the Laplace equation has derivatives of all orders and can be expanded in a power series within a circle that does not enclose a singularity. Also there is a close association between power series and Fourier series. If a function f is expanded in a power series inside a circle of radius R then it means that,

$$f(z) \sum_{n=0}^{\infty} C_n Z^n$$

These are correctly defined coefficients whose real and imaginary parts are given as,

$$c_n = a_n + ib_n.$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} [a_n r^n \cos n\theta - b_n r^n \sin n\theta] + i \sum_{n=1}^{\infty} [a_n r^n \sin n\theta + b_n r^n \cos n\theta],$$

This is a Fourier series for f .

Laplace Equation in Three Dimensions

Partial Differential
Equations of the First Order

A fundamental solution of Laplace's equation satisfies the equation,

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = -\delta(x - x', y - y', z - z'),$$

Here the Dirac delta function δ denotes a unit source concentrated at the points (x', y', z') . No other function has this specific property. It can be taken as a limit of functions whose integrals over space are unity and which can shrink to a point in the region where the function is non-zero. Basically, a different sign convention is taken for this equation while defining fundamental solutions. This sign is very helpful because $-\Delta$ is a positive operator. Thus, the definition of the fundamental solution implies that if the Laplacian of u is integrated over any volume that encloses the source point, then it is denoted as,

$$\iiint_V \nabla \cdot \nabla u \, dV = -1$$

The Laplace equation remains unchanged during rotation of coordinates and hence a fundamental solution can be obtained that only depends upon the distance r from the source point. For example, if we consider the volume of a ball of radius a around the source point, then Gauss' divergence theorem implies that,

$$-1 = \iiint_V \nabla \cdot \nabla u \, dV = \iint_S u_r \, dS = 4\pi a^2 u_r(a)$$

It can be denoted as,

$$u_r(r) = -\frac{1}{4\pi r^2},$$

It is on a sphere of radius r which is centered around the source point.

Thus,

$$u = \frac{1}{4\pi r}$$

Poisson Equation

In mathematics, Poisson equation is a partial differential equation. It is named after the French mathematician, geometer and physicist Siméon-Denis Poisson. The Poisson equation is,

$$\Delta \varphi = f$$

Here Δ is the Laplace operator and f and φ are real or complex-valued functions on a manifold. If the manifold is Euclidean space, then the Laplace operator is denoted as Δ^2 and hence Poisson equation can be written as,

$$\nabla^2 \varphi = f$$

In three dimensional Cartesian coordinates, it takes the form:

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$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi(x, y, z) = f(x, y, z).$$

NOTES

For disappearing f , this equation becomes Laplace equation and is denoted as,

$$\Delta \varphi = 0.$$

The Poisson equation may be solved using a Green's function; a general exposition of the Green's function for the Poisson equation is given in the article on the screened Poisson equation. There are various methods for numerical solution. The relaxation method, an iterative algorithm, is one example.

A second order partial differential equation is of the form, $\nabla^2 \Psi = -4\pi\rho$. If $\rho = 0$, then it reduces to Laplace equation. It can also be considered as Helmholtz differential equation of the form,

$$\nabla^2 \Psi + k^2 \Psi = 0$$

Wave Equation

The wave equation is an important second-order linear partial differential equation of waves. It is analysed on the basis of sound waves, light waves and water waves. The wave equation is considered as a hyperbolic partial differential equation. In its simplest form, the wave equation refers to a scalar function $u = (x_1, x_2, \dots, x_n, t)$ that satisfies,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Here ∇^2 is the spatial Laplacian and c is a fixed constant equal to the propagation speed of the wave and is also known as the non-dispersive wave equation. For a sound wave in air at 20°C this constant is about 343 m/s (speed of sound). For a spiral spring, it can be as slow as a meter per second. The differential equations for waves are based on the speed of wave propagation that varies with the frequency of the wave. This specific phenomenon is known as dispersion. In such a case, c must be replaced by the phase velocity as shown below:

$$v_p = \frac{\omega}{k}$$

The speed can also depend on the amplitude of the wave which will lead to a nonlinear wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = c(u)^2 \nabla^2 u$$

A wave can be superimposed onto another movement. In that case the scalar u will contain a Mach factor which is positive for the wave moving along the flow and negative for the reflected wave.

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The elastic wave equation in three dimensions describes the propagation of waves in an isotropic homogeneous elastic medium. Most of the solid materials are elastic, hence this equation is used to analyse the phenomena such as seismic waves in the Earth and ultrasonic waves which detect flaws in materials. In its linear form, this equation has a more complex form compared to the equations discussed above because it accounts for both longitudinal and transverse motion using the notation:

$$\rho \ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u})$$

Where:

- λ and μ are termed as Lamé parameters which describe the elastic properties of the medium.
- ρ is the density.
- \mathbf{f} is the source function or driving force.
- $\ddot{\mathbf{u}}$ is the displacement vector.

In this equation, both the force and the displacement are vector quantities. Hence, this equation is also termed as the vector wave equation.

General Solution of One Dimensional Wave Equation

The one dimensional wave equation for a partial differential equation has a general solution of the form that defines new variables as,

$$\xi = x - ct \quad ; \quad \eta = x + ct$$

It changes the wave equation into,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

This leads to the general solution of the form,

$$u(\xi, \eta) = F(\xi) + G(\eta) \Rightarrow u(x, t) = F(x - ct) + G(x + ct)$$

Basically, solutions of the one dimensional wave equation are sums of a right traveling function F and a left traveling function G . Here the term ‘Traveling’ refers the shape of the individual arbitrary functions with respect to x which stays constant, though the functions are transformed left and right with time at the speed c .

As per the Helmholtz equation, named for Hermann von Helmholtz, is the elliptic partial differential equation of the form $\nabla^2 A + k^2 A = 0$, where ∇^2 is the Laplace operator, k is the wavenumber and A is the amplitude.

NOTES

Check Your Progress

1. Define complete solution.
2. What is singular integral?
3. How is a partial differential equation formed?
4. Explain the terms 'complete integral' and 'particular integral'.
5. Explain Laplace equation.
6. What is the other way to write a Laplace equation?

5.3 ORIGINS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

The classical theory of first order Partial Differential Equation or PDE started in about 1760 with Euler and D'Alembert and ended in about 1890 with the work of Lie. In the prevailing period the great mathematicians Lagrange, Charpit, Monge, Pfaff, Cauchy, Jacobi and Hamilton have made significant contributions to the study of partial differential equations. Demidov quoted, "Lie developed the connection between 'groups of infinitely small transformations' and finite continuous groups of transformations in three theorems which make the foundation of the theory of Lie algebras. Lie also discovered the connections while studying linear homogeneous PDEs of first order. Thus these equations were evolved in the field of differential equations on which the theory of Lie groups originally rooted itself.

The discovery of first order PDE is considered as one of the most remarkable mathematical achievements of 19th century. Unfortunately, these mathematical theories are no longer considered essential for studying the basic of PDEs.

The theory of first order PDE as in Goursat (1917), Courant and Hilbert (1937, 1962), Sneddon (1957) and Garabedian (1964), states that the first order partial differential equations can be evaluated by defining an integral surface in the space of independent and dependent variables. This essentially involves theories and notations to develop Monge curves and Monge strips which will give a system of ordinary differential equations, called Charpit equations and a complicated geometrical proofs for existence and uniqueness of the solution of a Cauchy problem.

Characteristically, the Monge curves and Monge strips (in $(x_1, x_2, \dots, x_m, u)$ -space of independent variables (x_1, x_2, \dots, x_m) and dependent variable (u)) have been called characteristic curves and characteristic strips by all other mathematicians. However, the word 'characteristics' is specifically associated with the projections of Monge curves on the space of independent variables and is consistent with the use of a higher order equation or a systems of equations.

Additionally, in the method of characteristics of a first order PDE the Charpit equations (1784) are used for derivation. Though the credit of these equations are not given to Charpit by Courant and Hilbert, Garabedian and even by Goursat, who called these equations simply as characteristic equations.

This happened because Charpit died before he could follow up his manuscript describing these theories and notations which he sent to Paris Academy of Sciences for authentication and validation. Later Lacroix published his results in 1814 (A.R. Forsyth, Treatise DE, 1885-1928) and finally Charpit's manuscript was found in the beginning of the 20th century. Charpit found these characteristic equations while trying to discover the complete integrals.

A Partial Differential Equation or PDE is a mathematical equation that comprises of two or more independent variables, an unknown function which is dependent on those variables, and partial derivatives of the unknown function with respect to the independent variables. Characteristically, the equations that involve the partial derivatives of one dependent variable are termed as the 'partial differential equations'. The order of a partial differential equation is the order of the highest derivative involved.

A first order partial differential equation is a partial differential equation that involves only first derivatives of the unknown function of n variables. The equation takes the form,

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = 0.$$

Such equations originate in the formation of characteristic surfaces for hyperbolic partial differential equations, in the calculus of variations, in some geometrical problems, and in simple models for gas dynamics whose solution involves the method of characteristics.

A solution or a particular solution to a partial differential equation is a function that solves the equation or, in other words, turns it into an identity when substituted into the equation. A solution is called general if it contains all particular solutions of the equation concerned.

Check Your Progress

7. Explain analytical function.
8. How can a Poisson equation be correlated to a second order partial differential equation and a Helmholtz equation?
9. What is first order partial differential equation ?
10. What turns into an identity when substituted into the equation?

NOTES

5.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.
2. In complete integral, if we give particular values to the arbitrary constants, we get particular integral. If $j(x, y, z, a, b) = 0$, is the complete integral of a partial differential equation, then the eliminant of a and b from the equations

$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$, is called singular integral.

3. Partial differential equation may be formed by eliminating (i) arbitrary constants (ii) arbitrary functions.
4. A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.

In complete integral, if we give particular values to the arbitrary constants, we get particular integral. If $\varphi(x, y, z, a, b) = 0$, is the complete integral of a partial differential equation, then the eliminant of a and b from the equations

$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$, is called singular integral.

5. In mathematics, Laplace equation is a second order partial differential equation. It is named after Pierre-Simon Laplace and is written as,

$$\nabla^2 \varphi = 0$$

Here ∇^2 is the Laplace operator and φ is a scalar function of 3 variables. Laplace equation and Poisson equation are examples of elliptic partial differential equations. The universal theory of solutions to Laplace equation is termed as potential theory. The solutions of Laplace equation are harmonic functions and have great important in many fields of science.

6. The Laplace equation $\nabla^2 \varphi = 0$ can also be written as $\nabla \cdot \nabla \varphi = 0$.
7. Both the real and imaginary parts of a complex analytic function satisfy the Laplace equation. If $z = x + iy$ and also if $f(z) = u(x, y) + iv(x, y)$ then the necessary condition that $f(z)$ be analytic is that it must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$, where u_x is the first partial derivative of u with respect to x . It follows the notation,

$$u_{yy} = (-v_x)_y = -(-v_y)_x = -(u_x)_x.$$

8. A second order partial differential equation is of the form, $\nabla^2 \Psi = -4\pi\rho$. If $\rho = 0$, then it reduces to Laplace equation. It can also be considered as Helmholtz differential equation of the form,

$$\nabla^2 \Psi + k^2 \Psi = 0$$

9. A first order partial differential equation is a partial differential equation that involves only first derivatives of the unknown function of n variables. The equation takes the form, $F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = 0$.
10. A solution or a particular solution to a partial differential equation is a function that solves the equation or, in other words, turns it into an identity when substituted into the equation.

NOTES

5.5 SUMMARY

- Let $z = f(x, y)$ be a function of two independent variables x and y . Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives; $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.
- Any equation which contains one or more partial derivatives is called a partial differential equation. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$; $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for Partial Differential Equation (PDE) of first order and second order respectively.
- Partial differential equation may be formed by eliminating (i) arbitrary constants (ii) arbitrary functions.
- The partial differential equations can be formed by eliminating arbitrary junctions.
- Given, $p^2 + q^2 = 4$

Let us assume that $z = ax + by + c$ be a solution.

Partially differentiating Equation with respect to x and y , we get, $\frac{\partial z}{\partial x} = p = a$

and $\frac{\partial z}{\partial y} = q = b$

- In mathematics, Laplace equation is a second order partial differential equation. It is named after Pierre-Simon Laplace and is written as, $\nabla^2 \varphi = 0$.
- Here ∇^2 is the Laplace operator and φ is a scalar function of 3 variables. Laplace equation and Poisson equation are examples of elliptic partial differential equations.
- The universal theory of solutions to Laplace equation is termed as potential theory. The solutions of Laplace equation are harmonic functions and have great important in many fields of science.

NOTES

- Twice differentiable real-valued functions f of real variables x, y and z are found using the following notations.
- The Laplace equation $\nabla^2 \varphi = 0$ can also be written as $\nabla \cdot \nabla \varphi = 0$. It is also sometimes written using the notation $\nabla \varphi = 0$, where Δ is also the Laplace operator.
- Solutions of Laplace equation are harmonic functions. If the right-hand side is specified as a given function, $f(x, y, z)$ then the whole equation can be written as, $\nabla \varphi = f$.

This is the Poisson equation. The Laplace equation is also considered as a special type of the Helmholtz equation.

- The Laplace equation in two independent variables has the form,
$$\varphi_{xx} + \varphi_{yy} = 0.$$
- Both the real and imaginary parts of a complex analytic function satisfy the Laplace equation. If $z = x + iy$ and also if $f(z) = u(x, y) + iv(x, y)$ then the necessary condition that $f(z)$ be analytic is that it must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$, where u_x is the first partial derivative of u with respect to x . It follows the notation,

$$u_{yy} = (-v_x)_y = -\left(v_y\right)_x = -(u_x)_x.$$

- For a trial form, $f(z) = \varphi(x, y) + i\psi(x, y)$, the Cauchy-Riemann equations is satisfied if, $\Psi_x = -\varphi_y$, $\Psi_y = \varphi_x$. This relation does not determine ψ , but only its increments as $d\psi = \varphi_y dx + \varphi_x dy$.
- The Laplace equation for φ implies that the integrability condition for ψ satisfied as $\psi_{xy} = \psi_{yx}$; and thus ψ can be defined using a line integral. Both the integrability condition and Stokes' theorem implies that the value of the line integral connecting two points is independent of the path. The resulting pair of solutions of the Laplace equation is termed as conjugate harmonic functions.

- A fundamental solution of Laplace's equation satisfies the equation,

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = -\delta(x - x', y - y', z - z'),$$

- Basically, a different sign convention is taken for this equation while defining fundamental solutions. This sign is very helpful because $-\Delta$ is a positive operator. Thus, the definition of the fundamental solution implies that if the Laplacian of u is integrated over any volume that encloses the source point,

then it is denoted as, $\iiint_V \nabla \cdot \nabla u \, dV = -1$.

NOTES

- The Laplace equation remains unchanged during rotation of coordinates and hence a fundamental solution can be obtained that only depends upon the distance r from the source point. For example, if we consider the volume of a ball of radius a around the source point, then Gauss' divergence theorem

implies that, $-1 = \iiint_V \nabla \cdot \nabla u \, dV = \iint_S u_r \, dS = 4\pi a^2 u_r(a)$.

- It can be denoted as, $u_r(r) = -\frac{1}{4\pi r^2}$, It is on a sphere of radius r which is centered around the source point. Thus, $u = \frac{1}{4\pi r}$
- Poisson equation is a partial differential equation. It is named after the French mathematician, geometer and physicist Siméon-Denis Poisson. The Poisson equation is, $\Delta\varphi = f$.
- The wave equation is an important second-order linear partial differential equation of waves. It is analysed on the basis of sound waves, light waves and water waves.
- The wave equation is considered as a hyperbolic partial differential equation. In its simplest form, the wave equation refers to a scalar function $u = (x_1, x_2, \dots, x_n, t)$ that satisfies, $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$.

- A wave can be superimposed onto another movement. In that case the scalar u will contain a Mach factor which is positive for the wave moving along the flow and negative for the reflected wave.
- The one dimensional wave equation for a partial differential equation has a general solution of the form that defines new variables as,

$$\xi = x - ct \quad ; \quad \eta = x + ct$$

It changes the wave equation into, $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$. This leads to the general solution of the form,

$$u(\xi, \eta) = F(\xi) + G(\eta) \Rightarrow u(x, t) = F(x - ct) + G(x + ct)$$

5.6 KEY WORDS

- Partial differential equation:** A Partial Differential Equation or PDE is a mathematical equation that comprises of two or more independent variables, an unknown function which is dependent on those variables, and partial derivatives of the unknown function with respect to the independent variables.

NOTES

- **First order partial differential equation:** A first order partial differential equation is a partial differential equation that involves only first derivatives of the unknown function of n variables. The equation takes the form,

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}) = 0.$$

- **First order partial derivatives:** $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives.
- **Second order partial derivatives:** $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.
- **Complete integral:** A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution.
- **Partial differential equation:** Any equation which contains one or more partial derivatives is called a partial differential equation. $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z;$ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ are examples for partial differential equation of first order and second order respectively.
- **Laplace equation:** In mathematics, Laplace equation is a second order partial differential equation. It is named after Pierre-Simon Laplace and is written as, $\nabla^2 \varphi = 0$.

Here ∇^2 is the Laplace operator and φ is a scalar function of 3 variables.

- **Poisson equation:** In mathematics, Poisson equation is a partial differential equation. It is named after the French mathematician, geometer and physicist Siméon-Denis Poisson. The Poisson equation is, $\Delta \varphi = f$.

Here Δ is the Laplace operator and f are j real or complex-valued functions on a manifold. If the manifold is Euclidean space, then the Laplace operator is denoted as ∇^2 and hence Poisson equation can be written as,

$$\nabla^2 \varphi = f$$

- **Wave equation:** In its simplest form, the wave equation refers to a scalar function $u = (x_1, x_2, \dots, x_n, t)$ that satisfies, $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

Here ∇^2 is the spatial Laplacian and c is a fixed constant equal to the propagation speed of the wave and is also known as the non-dispersive wave equation.

5.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is partial differential equation?
2. Which equations are called as singular integral.
3. How will you identify the order of a partial differential equation? Give an example.
4. Derive the equation to find singular integral.
5. What is Laplace equation?
6. Name two methods of converting ordinary equations to partial differential equations.
7. Define complete integral.
8. Give the Laplace equations for the following:
 - Cartesian coordinates
 - Cylindrical coordinates
 - Spherical coordinates
9. What is analytic function?

Long Answer Questions

1. Explain formation of partial differential equation by eliminating arbitrary functions.
2. Derive the Laplace equation in two dimensions with the help of examples.
3. Explain and derive Laplace equation in three dimensions.
4. Elaborate a note on Poisson equation.
5. Explain wave equation with the help of examples.
6. Give the general solution of one dimensional wave equation.
7. Obtain a partial differential equation by eliminating the arbitrary constants of the following:

$$(i) z = ax + by + \sqrt{a^2 + b^2}$$

$$(ii) \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

$$(iii) z = xy + y\sqrt{x^2 - a^2} + b$$

$$(iv) z = ax^3 + by^3$$

$$(v) (x - a)^2 + (y - b)^2 + z^2 = a^2 + b^2$$

$$(vi) 2z = (ax + y)^2 + b$$

8. Eliminate the arbitrary function from the following:

$$(i) z = e^y f(x + y)$$

$$(ii) z = f(my - lx)$$

NOTES

- (iii) $z = f(x^2 + y^2 + z^2)$ (iv) $z = x + y + f(xy)$
 (v) $z = f(x) + e^y g(x)$ (vi) $z = f(x + 4y) + g(x - 4y)$
 (vii) $z = f(2x + 3y) + y g(2x + 3y)$ (viii) $z = f(x + y) \cdot \phi(x - y)$

NOTES

9. Solve the following differential equations:

- (i) $(3z - 4y)p + (4x - 2z)q = 2y - 3x$ (ii) $y^2zp + x^2zq = y^2x$
 (iii) $x^2p - y^2q = (x - y)z$ (iv) $xp + yq = 2z$
 (v) $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

10. Eliminate the arbitrary function(s) from the following and form the partial differential equations:

- (i) $xy + yz + zx = f\left(\frac{z}{x+y}\right)$ (ii) $z = f(x^2 + y^2 + z^2)$
 (iii) $u = e^y f(x - y)$ (iv) $z = f(\sin x + \cos y)$
 (v) $\phi(x + y + z, x^2 + y^2 - z^2) = 0$ (vi) $z = f(2x + 3y) + \phi(y + 2x)$
 (vii) $u = f(x^2 + y) + g(x^2 - y)$ (viii) $u = x f(ax + by) + g(ax + by)$

11. Find the complete solution of the following partial differential equations:

- (i) $pq + p + q = 0$ (ii) $p^3 = q^3$
 (iii) $p = e^q$ (iv) $z = px + qy + p^2 + pq + q^2$
 (v) $z = px + qy + \log pq$ (vi) $z = px + qy + p^2 - q^2$
 (vii) $z^2 = 1 + p^2 + q^2$

5.8 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 6 CAUCHY'S PROBLEM FOR FIRST ORDER EQUATIONS

NOTES

Structure

- 6.0 Introduction
- 6.1 Objectives
- 6.2 First Order Linear Equations
- 6.3 Cauchy's Problem for First Order Linear Equations
 - 6.3.1 Quasi-linear PDE
 - 6.3.2 Cauchy Problem for Quasi-linear PDE
- 6.4 Integral Surfaces Passing Through a Given Curve
 - 6.4.1 Integral Surface Passing Through a Given a Curve
- 6.5 Answers to Check Your Progress Questions
- 6.6 Summary
- 6.7 Key Words
- 6.8 Self Assessment Questions and Exercises
- 6.9 Further Readings

6.0 INTRODUCTION

A first order differential equation is an equation of the form $F(t, y, \dot{y}) = 0$. A solution of a first order differential equation is a function $f(t)$ that makes $F(t, f(t), f'(t))=0$ for every value of t . Here, F is a function of three variables which we label t , y , and \dot{y} . It is understood that \dot{y} will explicitly appear in the equation although t and y need not. The term “first order” means that the first derivative of y appears, but no higher order derivatives do.

A Cauchy problem in mathematics asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain. A Cauchy problem can be an initial value problem or a boundary value problem or it can be either of them. It is named after Augustin Louis Cauchy.

In this unit, you will study about first order linear equations and Cauchy's problems based on first order linear equations.

6.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss first order linear equations
- State Cauchy's problems for first order linear equations
- Find the integral surface passing through a given surface

6.2 FIRST ORDER LINEAR EQUATIONS

NOTES

Linear Equation of First Order

A linear differential equation of the first order and first degree can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \dots(6.1)$$

where P and Q are either functions of x or constants (including zero).

The solution of such a differential equation can be obtained in the following way:

1. $e^{\int P dx}$ is an Integrating Factor (I.F.) of Equation (6.1).
2. Multiplying both sides of Equation (6.1) by I.F. = $e^{\int P dx}$. The equation (6.1) becomes

$$\frac{dy}{dx} e^{\int P dx} + P e^{\int P dx} y = Q e^{\int P dx}$$

or $\frac{d}{dx} \left\{ y e^{\int P dx} \right\} = Q e^{\int P dx}$

Integrating, we get $y e^{\int P dx} = \int [Q e^{\int P dx}] dx + c$ where c is an arbitrary constant.

\therefore The general solution is $y = e^{-\int P dx} \left[c + \int \{Q e^{\int P dx}\} dx \right]$.

Example 6.1: Solve $x \cos x \frac{dy}{dx} + (x \sin x + \cos x)y = 1$.

Solution: The given equation is

$$\frac{dy}{dx} + \frac{x \sin x + \cos x}{x \cos x} y = \frac{1}{x \cos x} \quad \dots(1)$$

which is a first order linear differential equation.

$$\begin{aligned} \therefore \text{I.F.} &= e^{\int \frac{x \sin x + \cos x}{x \cos x} dx} = e^{\int \frac{\sin x}{\cos x} dx + \int \frac{dx}{x}} \\ &= e^{-\log \cos x + \log x} = e^{\log \left(\frac{x}{\cos x} \right)} = \frac{x}{\cos x} \end{aligned}$$

Multiplying equation (1) by I.F. and integrating w.r.t. x , we get

$$\begin{aligned} y \cdot \frac{x}{\cos x} &= \int \frac{1}{x \cos x} \times \frac{x}{\cos x} dx + c \quad (\text{where } c \text{ is an arbitrary constant}) \\ &= \int \sec^2 x dx + c \\ &= \tan x + c \\ \therefore xy &= \tan x \cos x + c \cos x = \sin x + c \cos x. \end{aligned}$$

Equations Reducible to Linear Form

We begin with a definition.

Definition: An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \quad \dots(6.2)$$

where P and Q are functions of x alone or constant, is known as the *Bernoulli's equation*.

Such an equation can be reduced to the linear form by a substitution and then solved as follows:

The Equation (6.2) can be written in the form

$$y^{-n} \frac{dy}{dx} + y^{-n+1}P = Q \quad \dots(6.3)$$

Putting $y^{1-n} = z$, we get $(1-n)y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$

Then Equation (6.3) becomes

$$\frac{1}{1-n} \frac{dz}{dx} + zP = Q$$

$$\text{or } \frac{dz}{dx} + P(1-n)z = (1-n)Q \quad \dots(6.4)$$

which is a linear differential equation in z . Its I.F. = $e^{\int P dx}$

Multiplying Equation (6.4) by I.F., we now have

$$e^{(1-n)\int P dx} \frac{dz}{dx} + P(1-n)e^{(1-n)\int P dx} = Q e^{(1-n)\int P dx}$$

$$\text{or } \frac{d}{dx} \left[z \cdot e^{(1-n)\int P dx} \right] = Q e^{\int (1-n) P dx}$$

Integrating, $z \cdot e^{(1-n)\int P dx} = \int [Q e^{\int (1-n) P dx}] dx + c$ where c is an arbitrary constant.

∴ The general solution is

$$y^{1-n} = e^{-(1-n)\int P dx} \left[c + \int [Q e^{(1-n)\int P dx}] dx \right]$$

Example 6.2: Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$.

Solution: The given equation is $-\frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \quad \dots(1)$

Let $\frac{1}{y^2} = z$. Then $-\frac{2}{y^3} \frac{dy}{dx} = \frac{dz}{dx}$. The Equation (1) now becomes

$$\frac{dz}{dx} + 2xz = 2e^{-x^2} \quad \dots(2)$$

which is a first order linear equation and its I.F. = $e^{\int 2x dx} = e^{x^2}$.

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Multiplying Equation (2) by I.F. and integrating w.r.t. x , we get

$$z \cdot e^{x^2} = 2 \int e^{-x^2} \cdot e^{x^2} dx + c = 2x + c$$

NOTES

or $\frac{e^{x^2}}{y^2} = 2x + c.$

Example 6.3: Solve $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}$... (1)

Solution: The given equation is a first order linear differential equation and its I.F.

$$= e^{\int \frac{4x}{x^2+1} dx} = e^{2 \log(1+x^2)} = e^{\log(1+x^2)2} = (1+x^2)^2.$$

Multiplying Equation (1) by I.F. $= (1+x^2)^2$ and integrating w.r.t. x , we get

$$\begin{aligned} y \cdot (1+x^2)^2 &= \int \frac{1}{(1+x^2)^3} (1+x^2)^2 dx + c \text{ where } c \text{ is an arbitrary constant.} \\ &= \int \frac{1}{1+x^2} dx + c = \tan^{-1} x + c \end{aligned}$$

\therefore The general solution is $y \cdot (1+x^2)^2 = \tan^{-1} x + c.$

Example 6.4: Solve $(x+y+1)dy = dx.$

Solution: The given equation is $\frac{dx}{dy} = x + y + 1$

$$\frac{dx}{dy} - x = 1 + y \quad \dots (1)$$

which is the first order linear differential equation and I.F. $= e^{-\int dy} = e^{-y}.$

Multiplying Equation (1) by I.F. $= e^{-y}$ and integrating w.r.t. y , we get

$$x e^{-y} = \int (1+y) e^{-y} dy + c \text{ where } c \text{ is an arbitrary constant.}$$

$$\begin{aligned} &= \int e^{-y} dy + \int y e^{-y} dy + c = \int e^{-y} dy - y e^{-y} + \int e^{-y} dy + c \\ &= -2 e^{-y} - y e^{-y} + c \end{aligned}$$

$$\text{or } x = -2 - y + c e^y \text{ or } x + y + 2 = c e^y.$$

Hence the general solution is

$$x + y + 2 = c e^y.$$

Example 6.5: Solve $x \frac{dy}{dx} + y = x y^2.$

Solution: The equation can be written as $\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = 1$... (1)

Let $\frac{1}{y} = z$. Then $-\frac{1}{y^2} \frac{dy}{dx} = \frac{dz}{dx}$. The Equation (1) now becomes

$$-\frac{dz}{dx} + \frac{z}{x} = 1 \text{ or } \frac{dz}{dx} - \frac{1}{x} z = -1 \quad \dots (2)$$

which is first order linear differential equation and

Cauchy's Problem for First Order Equations

$$\text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}.$$

Multiplying Equation (2) by I.F. $= \frac{1}{x}$ and integrating w.r.t. x , we get

$$z \cdot \frac{1}{x} = \int (-1) \frac{1}{x} dx + c = -\log x + c \text{ where } c \text{ is an arbitrary constants.}$$

$$\text{or } \frac{1}{y} \cdot \frac{1}{x} = -\log x + c \text{ or } xy(c - \log x) = 1.$$

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6.3 CAUCHY'S PROBLEM FOR FIRST ORDER LINEAR EQUATIONS

An equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_{n-1} x \frac{dy}{dx} + p_n y = f(x) \quad \dots(6.5)$$

where p_1, p_2, \dots, p_n are constants, is called a homogeneous linear equation of degree n . This equation is also known as *Euler-Cauchy equation*.

To solve such an equation, we transform this equation into a linear equation with constant coefficients by the transformation $\log x = z$ or $x = e^z$ and solve that by the methods discussed above.

Let us denote $\frac{d}{dz} = Q$.

$$\text{Now } \log x = z \quad \therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$\text{or } \frac{1}{x} \frac{dy}{dz} = \frac{dy}{dx} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz}$$

$$\text{or} \quad x \frac{dy}{dx} = Qy$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx} = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2}$$

$$\text{or} \quad x^2 \frac{d^2 y}{dx^2} = \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) = (Q^2 - Q)y = Q(Q-1)y$$

Similarly $x^3 \frac{d^3 y}{dx^3} = Q(Q-1)(Q-2)y$ and so on.

$$\therefore x^n \frac{d^n y}{dx^n} = Q(Q-1)(Q-2) \dots (Q-n+1)y$$

NOTES

Thus the transformation $\log x = z$ reduce the Equation (6.5) to the form

$$[Q(Q-1)(Q-2)\dots(Q-n+1) + p_1 Q(Q-1)(Q-2)\dots(Q-n+2) + \dots + p_{n-1} Q + p_n] y = f(e^z)$$

which is clearly a linear equation with constant coefficients.

The following examples illustrates the method.

Example 6.6: Solve $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$.

Solution: Let $\log x = z$.

$$\text{Then } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz} = Qy \quad \text{where} \quad Q = \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx} = -\frac{1}{x^2} \frac{dy}{dx} + \frac{1}{x^2} \frac{d^2 y}{dz^2}$$

$$\text{or} \quad x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = (Q^2 - Q)y = Q(Q-1)y$$

$$\frac{d^3 y}{dx^3} = \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = \frac{d}{dx} \left(\frac{1}{x^2} \frac{d^2 y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz} \right)$$

$$= -\frac{2}{x^3} \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) + \frac{1}{x^2} \left(\frac{d^3 y}{dz^3} - \frac{d^2 y}{dz^2} \right) \frac{dz}{dx}$$

$$= -\frac{2}{x^3} \left(\frac{d^2 y}{dx^2} - \frac{dy}{dz} \right) + \frac{1}{x^3} \left(\frac{d^3 y}{dz^3} - \frac{d^2 y}{dz^2} \right)$$

$$\text{or} \quad x^3 \frac{d^3 y}{dx^3} = \frac{d^3 y}{dz^3} - 3 \frac{d^2 y}{dz^2} + \frac{2dy}{dz} = Q(Q^2 - 3Q + 2)y = Q(Q-1)(Q-2)y$$

Then the given equation becomes

$$(Q^3 - 3Q^2 + 2Q)y + 3(Q^2 - Q)y + Qy + 8y = 65 \cos z$$

$$\text{or} \quad (Q^3 + 8)y = 65 \cos z \quad \dots(1)$$

$$\text{The reduced equation of Equation (1) is } (Q^3 + 8)y = 0 \quad \dots(2)$$

Let $y = Ae^{mz}$ be a trial solution of Equation (2) and then the auxiliary equation is

$$m^3 + 8 = 0 \quad \text{or} \quad (m+2)(m^2 - 2m + 4) = 0$$

$$\therefore m = -2, 1 \pm \sqrt{3}i$$

The complementary function is $c_1 e^{-2z} + e^z (c_2 \cos \sqrt{3}z + c_3 \sin \sqrt{3}z)$ where c_1, c_2 and c_3 are arbitrary constants.

$$\begin{aligned} \text{P.I.} &= \frac{1}{Q^3 + 8} 65 \cos z = 65 \frac{1}{Q^2(Q) + 8} \cos z \\ &= 65 \frac{1}{8-Q} \cos z \quad (\because Q^2 \text{ is replace by } -1^2) \end{aligned}$$

$$= 65 \frac{8+Q}{64-Q^2} \cos z = \frac{65}{64-(-1)} (8+Q) \cos z = 8 \cos z - \sin z$$

∴ The general solution is

$$\begin{aligned} y &= c_1 e^{-2z} + e^z (c_2 \cos \sqrt{3} z + c_3 \sin \sqrt{3} z) + 8 \cos z - \sin z \\ &= c_1 x^{-2} + x [c_2 \cos (\sqrt{3} \log x) + c_3 \sin (\sqrt{3} \log x)] + 8 \cos (\log x) \\ &\quad - \sin (\log x). \end{aligned}$$

NOTES

Example 6.7: Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2) \frac{dy}{dx} - 3y = x^2 + x + 1$.

Solution: Let $(3x+2) = e^z$ or $z = \log(3x+2)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{3}{3x+2} \text{ or } (3x+2) \frac{dy}{dx} = 3 \frac{dy}{dz}$$

$$\begin{aligned} \text{and } \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{3}{3x+2} \cdot \frac{dy}{dz} \right) = -\frac{9}{(3x+2)^2} \frac{dy}{dz} + \frac{3}{3x+2} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} \\ &= -\frac{9}{(3x+2)^2} \frac{dy}{dz} + \frac{3}{3x+2} \cdot \frac{d^2y}{dz^2} \cdot \frac{3}{3x+2} \end{aligned}$$

$$\text{or } (3x+2)^2 \frac{d^2y}{dx^2} = 9 \left(\frac{d^2y}{dz^2} - \frac{dy}{dz} \right)$$

Then the given equation becomes

$$\begin{aligned} 9 \left(\frac{d^2y}{dz^2} - \frac{dy}{dz} \right) + 5 \cdot 3 \frac{dy}{dz} - 3y &= \left(\frac{e^z - 2}{3} \right)^2 + \frac{e^z - 2}{3} + 1 \\ \text{or } 9 \frac{d^2y}{dz^2} + 6 \frac{dy}{dz} - 3y &= \frac{e^{2z} - 4e^z + 4}{9} + \frac{e^z - 2}{3} + 1 \\ &= \frac{1}{9} (e^{2z} - e^z + 7) \end{aligned} \quad \dots(1)$$

Let $y = Ae^{mz}$ be a trial solution of the reduced Equation (1) and then the auxiliary equation is $9m^2 + 6m - 3 = 0$ or $9m^2 + 9m - 3m - 3 = 0$

$$\text{or } 9m(m+1) - 3(m+1) = 0 \text{ or } 3(m+1)(3m-1) = 0$$

$$\therefore m = 1/3, -1$$

∴ C.F. = $c_1 e^{z/3} + c_2 e^{-z}$ where c_1 and c_2 are arbitrary constants.

$$\begin{aligned} \text{P.I.} &= \frac{1}{9Q^2 + 6Q - 3} \{e^{2z} - e^z + 7\} \frac{1}{9} \text{ where } \frac{d}{dz} = Q \\ &= \frac{1}{9} \left[\frac{e^{2z}}{9 \cdot 4 + 6 \cdot 2 - 3} - \frac{e^z}{9 \cdot 1 + 6 \cdot 1 - 3} + \left(-\frac{1}{3} \right) (1 - 2Q - 3Q^2)^{-1} \{7\} \right] \\ &= \frac{1}{9} \left[\frac{e^{2z}}{45} - \frac{e^z}{12} - \frac{7}{3} \right] \end{aligned}$$

Hence the general solution is

$$y = c_1 e^{z/3} + c_2 e^{-z} + \frac{1}{9} \left[\frac{e^{2z}}{45} - \frac{e^z}{12} - \frac{7}{3} \right]$$

NOTES

$$= c_1 (3x+2)^{1/3} + c_2 (3x+2)^{-1} + \frac{1}{9} \left[\frac{(3x+2)^2}{45} - \frac{(3x+2)}{12} - \frac{7}{3} \right].$$

6.3.1 Quasi-linear PDE

Consider the quasi-linear PDE given by

$$a(x, y, u) ux + b(x, y, u) uy = c(x, y, u), \quad (6.6)$$

where a, b, c are continuously differentiable functions on a domain $\Omega \subseteq \mathbb{R}^3$. Let Ω_0 be the projection of Ω to the XY -plane.

Definition (Integral Surface): Let $\subseteq \Omega_0$ and $u : D \rightarrow \mathbb{R}$ be a solution of the Equation (6.6). The surface S represented by $z = u(x, y)$ is called an Integral Surface.

6.3.2 Cauchy Problem for Quasi-linear PDE

Cauchy Problem

To find an integral surface $z = u(x, y)$ of the quasi-linear PDE (6.6), containing a given space curve Γ whose parametric equations are

$$x = f(s), y = g(s), z = h(s), s \in I, \quad \dots(6.7)$$

where f, g, h are assumed to be continuously differentiable on the interval I and $h(s) = u(f(s), g(s))$ for $s \in I$.

Initial Value Problem

Initial value problem for the quasi-linear PDE (6.6) is a special Cauchy problem for (6.6), wherein the initial curve Γ lies in the ZX -plane and the y variable has an interpretation of the time-variable. That is, Γ has the following parametric form:

$$x = f(s), y = 0, z = h(s), s \in I, \quad \dots(6.8)$$

We consider three Cauchy problems for linear PDEs where the PDEs can be solved explicitly. These three examples illustrate that all three possibilities concerning a mathematical problem can occur, namely,

- (i) Cauchy problem has a unique solution. (Example 6.8)
- (ii) Cauchy problem has an infinite number of solutions. (Example 6.9)
- (iii) Cauchy problem has no solution. (Example 6.10)

Consider the following equation

$$ux = cu + d(x, y), \quad \dots(6.9)$$

where $c \in \mathbb{R}$ and d is a continuously differentiable function. The Equation (6.9) can be thought of as an ODE where y appears as a parameter. Its explicit solution is given by

$$u(x, y) = e^{cx} \left(\int_0^x e^{-c\xi} d(\xi, y) d\xi + u(0, y) \right) \quad \dots(6.10)$$

Example 6.8: (Cauchy Problem 1: Existence of a Unique solution) The Cauchy data is prescribed on the Y -axis:

$$u_x = cu + d(x, y), \quad u(0, y) = y.$$

The unique solution is given by

$$u(x, y) = e^{cx} \left(\int_0^x e^{-c\xi} d(\xi, y) d\xi + y \right) \quad \dots(6.11)$$

Example 6.9: (Cauchy Problem 2: Non-uniqueness of solutions) Cauchy data is prescribed on the X -axis:

$$u_x = cu, \quad u(x, 0) = e^{cx}. \quad \dots(6.12)$$

This Cauchy problem has infinitely many solutions:

$$u(x, y) = e^{cx} T(y),$$

T is any function of a single variable such that $T(0) = 2$.

Example 6.10: (Cauchy Problem 3: Non-Existence of solutions) Cauchy data is prescribed on the X -axis:

$$u_x = cu, \quad u(x, 0) = \sin x. \quad \dots(6.13)$$

This Cauchy problem has no solution. For, if it has a solution then, in view of the formula (6.10), the solution satisfies

$$\sin x = u(x, 0) = e^{cx} u(0, 0), \text{ for all } x \in \mathbb{R}.$$

The above equation cannot hold and hence the Cauchy problem has no solution.

6.4 INTEGRAL SURFACES PASSING THROUGH A GIVEN CURVE

The following section illustrates the basics of curve.

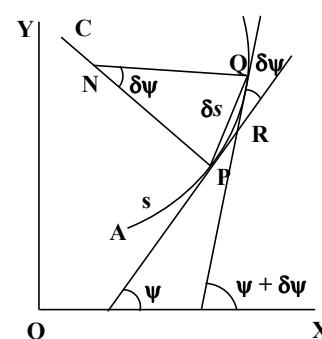


Fig. 6.1 Curve

NOTES

In Figure 6.1, Let P, Q be two neighbouring points on a curve AB. Let arc AP = s and arc AQ = $s + ds$ so that the length of the arc PQ = ds , A being the fixed point on the curve, from where arc is measured. Let the tangents at P and Q make angles ψ and $\psi + d\psi$, respectively with a fixed line say x-axis. Then, angle $d\psi$ through which the tangent turns as its point of contact travels along the arc PQ is called the total curvature of arc PQ.

The ratio $\frac{d\psi}{ds}$ represents the average rate of change in the angle ψ per unit of arc length along the curve. It is called the average curvature of arc PQ.

The limiting value of the average curvature when $Q \rightarrow P$ is called the curvature of the curve at the point P.

In general, the ratio $\frac{d\psi}{ds}$ approaches a limit $\frac{d\psi}{ds}$ as $ds \rightarrow 0$.

Thus, the curvature at a point P = $\lim_{Q \rightarrow P} \frac{d\psi}{ds} = \lim_{ds \rightarrow 0} \frac{d\psi}{ds} = \frac{d\psi}{ds}$.

The curvature of a curve C at a point (x, y) on C is usually denoted by the

Greek letter κ (kappa). It is given by the equation $\kappa = \left| \frac{d\psi}{ds} \right|$

where s is the arc length measured along the curve and ψ is the angle made by tangent line to C at (x, y) with positive x-axis.

The reciprocal of the curvature of the curve at P, is called the radius of curvature of the curve at P and is usually denoted by ρ . Thus, $\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$

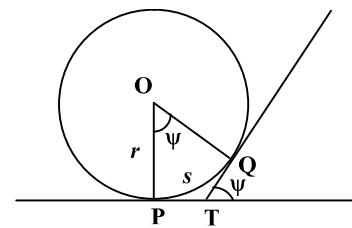
If PC is normal at P and PC = ρ , then C is called the centre of curvature of the curve at P.

The circle with centre C and radius PC = ρ is called the circle of curvature of the curve at P.

The length of the chord drawn through P, intercepted by the circle of curvature at P, is called a chord of curvature.

Note: A straight line does not bend at all (because $\frac{d\psi}{ds}$ is zero as ψ is constant).

Therefore, the curvature of a straight line is zero (Refer Figure 6.2).



NOTES

Fig. 6.2 Chord of Curvature

Curvature of a Circle

Let O and r be the centre and radius of a circle, respectively. Let P and Q be two points on the circle.

So that, arc $PQ = s$ and the tangent at Q make an angle ψ with the tangent at P. Then, $\angle POQ = \psi$. Therefore, $s (= \text{arc } PQ) = r\psi$.

Differentiating with respect to ψ , we get $\frac{ds}{d\psi} = r$

$$\therefore \text{Curvature} = \frac{d\psi}{ds} = \frac{1}{r} \text{ (constant)}$$

Thus, the curvature at every point of the circle is equal to the reciprocal of its radius and therefore, it is constant.

Note: Circle is the only curve of constant curvature.

Radius of Curvature for Intrinsic Curves

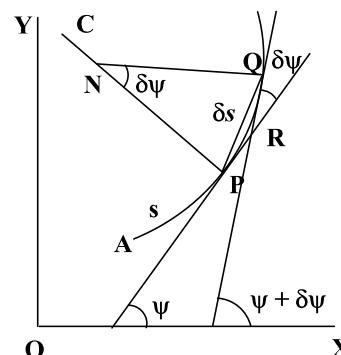


Fig. 6.3 Radius of Curvature for Intrinsic Curves

Let P, Q be two neighbouring points on a curve AB. Let the lengths of arc AP = s and arc AQ = $s + \delta s$.

Therefore, the length of the arc $PQ = \delta s$.

Let angles made by the tangents at P and Q with x-axis be ψ be $\psi + \delta\psi$, respectively. Also let the normals at P and Q intersect at N. Join P and Q.

NOTES

$$\therefore \angle PNQ = \delta\psi$$

From the triangle PQN , by sine-rule, we get

$$\begin{aligned} \frac{PN}{\sin PQN} &= \frac{\text{chord } PQ}{\sin \delta\psi} \\ \Rightarrow PN &= \frac{\text{chord } PQ}{\sin \delta\psi} \cdot \sin PQN \\ &= \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin PQN \end{aligned}$$

If ρ be the radius of curvature, then we have

$$\begin{aligned} \rho &= \lim_{Q \rightarrow P} PN \\ &= \lim_{\delta s \rightarrow 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin PQN \\ [\because \delta s \rightarrow 0, \delta\psi \rightarrow 0, \lim_{\delta s \rightarrow 0} \frac{\text{chord } PQ}{\delta s} &= 1, \angle PQN \rightarrow \frac{\pi}{2} \text{ and } \frac{\delta\psi}{\sin \delta\psi} \rightarrow 1] \end{aligned}$$

$$\therefore \rho = \frac{ds}{d\psi}$$

The angle between the tangents to the curve at P and Q i.e., $\delta\psi$ is called the angle of contingence of the arc PQ. The relation between s and ψ for a curve is called its intrinsic equation.

Thus, we can say that at a point P

$$\rho = \lim_{Q \rightarrow P} \left(\frac{\text{arc } PQ}{\text{the angle of contingence of arc } PQ} \right).$$

6.4.1 Integral Surface Passing Through a Given a Curve

Consider the first order linear PDE $Pp + Qq = R$

We know that the auxiliary system associated with the given PDE is given by

$$\frac{dx}{Q} = \frac{dy}{P} = \frac{dz}{R}$$

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ represent the integral surface of the above system.

Suppose that $[x(t), y(t), z(t)]$ be the parametric form of the curve passing through the above integral surface.

$$\text{i.e., } u[x(t), y(t), z(t)] = 0$$

and $v[x(t), y(t), z(t)] = 0$

Cauchy's Problem for First Order Equations

The General integral of the given PDE is $f(u, v) = 0$, subject to the condition that $f(c_1, c_2) = 0$.

Exercise 6.11: Find the equation of the integral surface of the PDE

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

Solution: The auxiliary system is given by

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \quad \dots(1)$$

Taking the 1st and 3rd fraction of Equation (1), we get

$$\begin{aligned} \frac{dx}{2y(z-3)} &= \frac{dz}{y(2x-3)} \\ \Rightarrow \quad \frac{dx}{2(z-3)} &= \frac{dz}{2x-3} \\ \Rightarrow \quad (2x-3)dx &= (2z-6)dz \end{aligned}$$

Integrating we get

$$\begin{aligned} x^2 - 3x &= z^2 - 6z + c_1 \\ \text{or} \quad x^2 - 3x - z^2 + 6z &= c_1 \end{aligned} \quad \dots(2)$$

Using $(0, y, -1)$ as multipliers each fraction of Equation (1) is equal to

$$\frac{ydy - dz}{2xy - yz - 2yx + 3y} = \frac{ydy - dz}{-yz + 3y} = \frac{ydy - dz}{y(3-z)}$$

Equating this expression with 1st fraction of Equation (1) we get

$$\begin{aligned} \frac{dx}{2y(z-3)} &= \frac{ydy - dz}{y(3-z)} \\ \Rightarrow \quad \frac{dx}{2y(z-3)} &= \frac{ydy - dz}{y(3-z)} \\ \Rightarrow \quad \frac{dx}{2} &= \frac{ydy - dz}{-1} \\ \Rightarrow \quad dx + 2ydy - 2dz &= 0 \end{aligned}$$

Integrating we get

$$x + y^2 - 2z = c_2 \quad \dots(3)$$

Now the given curve is $x^2 + y^2 = 2x$, $z = 0$,

This equation can also be written as $(x-1)^2 + (y-0)^2 = 1$ which is circle with centre $(1, 0)$ and radius 1. The corresponding parametric equation is

$$x - 1 = \cos \theta, \quad y - 0 = \sin \theta, \quad z = 0.$$

$$\text{or} \quad x = 1 + \cos \theta, \quad y = \sin \theta, \quad z = 0.$$

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By the given condition, the integral surface passes through the above circle. Therefore from Equation (3) we get

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$$\begin{aligned} & 1 + \cos \theta + \sin^2 \theta - 2(0) = c_2 \\ \Rightarrow & 1 + \cos \theta + 1 - \cos^2 \theta = c_2 \\ \Rightarrow & 2 + \cos \theta - \cos^2 \theta = c_2 \end{aligned} \quad \dots(4)$$

Also from Equation (2), we get

$$\begin{aligned} & (1 + \cos \theta)^2 - 3(1 + \cos \theta) - 0^2 + 6(0) = c_1 \\ \Rightarrow & 1 + 2 \cos \theta + \cos^2 \theta - 3 - 3 \cos \theta = c_1 \\ \Rightarrow & \cos^2 \theta - 2 - \cos \theta = c_1 \\ \Rightarrow & -(2 + \cos \theta - \cos^2 \theta) = c_1 \end{aligned} \quad \dots(5)$$

From Equation (4) and (5), we get

$$\begin{aligned} & c_1 = -c_2 \\ \text{or } & c_1 + c_2 = 0 \\ \text{or } & x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0 \\ \text{or } & x^2 + y^2 - z^2 - 2x + 4z = 0 \end{aligned}$$

which is required surface.

Integral Surface Orthogonal to Given Surface

Consider the linear partial differential equation

$$Pp + Qq = R \quad \dots(6.14)$$

$$\text{Let } f(x, y, z) = c \quad \dots(6.15)$$

be the integral surface of Equation (6.14), also for any surface

$$z = g(x, y) \quad \dots(6.16)$$

Let $p(x, y, z)$ be any point on the line such that Equations (6.15) and (6.16) are orthogonal at p . We have the direction ratios respectively for (6.15) and (6.16) as

$$\langle f_x, f_y, f_z \rangle \quad \text{and} \quad \langle p, q, -1 \rangle$$

Now the condition for the orthogonality suggests that

$$f_x p + f_y q + f_z (-1) = 0$$

$$\text{or } f_x p + f_y q = f_z$$

which is of the form $Pq + Qq = R$

$$\text{where } P = f_x, Q = f_y, R = f_z,$$

Example 6.12: Find the surface which is orthogonal to one parameter system $z = cxy(x^2 + y^2)$ and passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$.

Solution: The given one parameter system is $\frac{xy(x^2+y^2)}{z} = \frac{1}{c}$

NOTES

$$\text{Let } f(x, y, z) = \frac{xy(x^2+y^2)}{z}$$

$$\text{Now, } P = f_x = \frac{y(x^2+y^2)+2x^2y}{z}$$

$$Q = f_y = \frac{x(x^2+y^2)+2xy^2}{z}$$

$$\text{and } R = f_z = \frac{-xy(x^2+y^2)}{z^2}$$

now auxiliary system of equations are

$$\frac{dx}{y(x^2+y^2)+2x^2y} = \frac{dy}{x(x^2+y^2)+2xy^2} = \frac{dz}{-xy(x^2+y^2)}$$

$$\text{or } \frac{dx}{y(x^2+y^2)+2x^2y} = \frac{dy}{x(x^2+y^2)+2xy^2} = \frac{zdz}{-xy(x^2+y^2)} \quad \dots(1)$$

Using multipliers $(x, y, 1)$, each ratio of Equation (1) is equal to

$$\frac{xdx+ydy+zdz}{3x^3y+xy^3+x^3y+3xy^3-x^3y-xy^3} = \frac{xdx+ydy+zdz}{3xy(x^2+y^2)}$$

Equating this with 3rd term of Equation (1), we get

$$\frac{xdx+ydy+zdz}{3xy(x^2+y^2)} = \frac{zdz}{-xy(x^2+y^2)}$$

$$\Rightarrow xdx + ydy + zdz = -3zdz$$

$$\Rightarrow xdx + ydy + 4zdz = 0$$

Integrating

$$\begin{aligned} \frac{x^2}{2} + \frac{y^2}{2} + \frac{4z^2}{2} &= \frac{c_1}{2} \\ \Rightarrow x^2 + y^2 + 4z^2 &= c_1 \end{aligned} \quad \dots(2)$$

Using multipliers $(x, y, 0)$ and $(x, -y, 0)$ and equating the two fractions we get

$$\begin{aligned} \frac{xdx+ydy}{3x^3y+xy^3+x^3y+3xy^3} &= \frac{xdx-ydy}{3x^3y+xy^3-x^3y-3xy^3} \\ \Rightarrow \frac{xdx+ydy}{4x^3y+4xy^3} &= \frac{xdx-ydy}{2x^3y-2xy^3} \\ \Rightarrow \frac{xdx+ydy}{x^2+y^2} &= \frac{2(xdx-ydy)}{x^2-y^2} \end{aligned}$$

NOTES

Integrating, we get

$$\log(x^2 + y^2) = 2 \log(x^2 - y^2) + \log c_2$$

$$\Rightarrow \frac{x^2 + y^2}{(x^2 - y^2)^2} = c_2 \quad \dots(3)$$

Now parametric systems o hyperbola is

$$x = a \sec \theta, \quad x = a \tan \theta, \quad z = 0$$

\therefore from Equation (3),

$$c_1 = a^2 \sec^2 \theta + a^2 \tan^2 \theta$$

$$\Rightarrow c_1 = a^2 (\sec^2 \theta + \tan^2 \theta)$$

And from Equation (3)

$$c_2 = \frac{a^2 \sec^2 \theta + a^2 \tan^2 \theta}{(a^2 \sec^2 \theta - a^2 \tan^2 \theta)^2}$$

$$\Rightarrow c_2 = \frac{\sec^2 \theta + \tan^2 \theta}{a^2(\sec^2 \theta - \tan^2 \theta)^2}$$

$$\Rightarrow c_2 = \frac{\sec^2 \theta + \tan^2 \theta}{a^2(1)^2}$$

$$\Rightarrow c_2 = \frac{\frac{c_1}{a^2}}{a^2(1)^2}$$

$$\Rightarrow c_2 = \frac{c_1}{a^4}$$

$$\Rightarrow a^4 c_2 = c_1$$

$$\Rightarrow c_1 = a^4 c_2$$

\therefore The required surace orthogonal to the given system is

$$\frac{(x^2 - y^2)^2 (x^2 + y^2 + 4z^2)}{x^2 + y^2} = a^4$$

Some Elementary Curves

Some curves which are useful in drawing the approximate shape of the curves near the origin when either x -axis or y -axis is a tangent at the origin as given in Figure 6.4(a), (b), (c), (d), (e), (f), (g) and (h).

These figure display the cubic parabola.

Figure 6.5(a), (b), (c) and (d) show semi-cubic parabola.

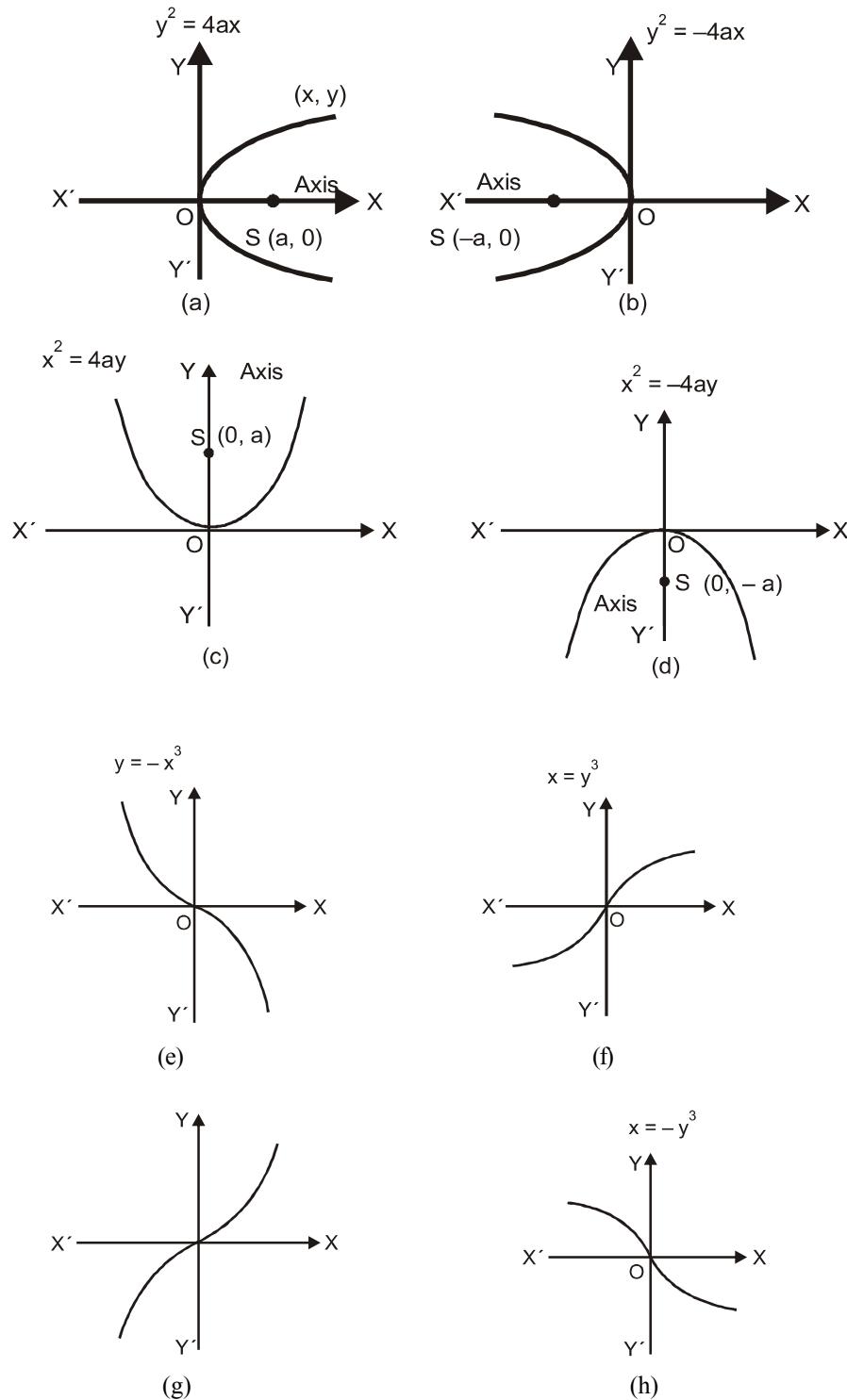


Fig. 6.4 Cubical Parabola

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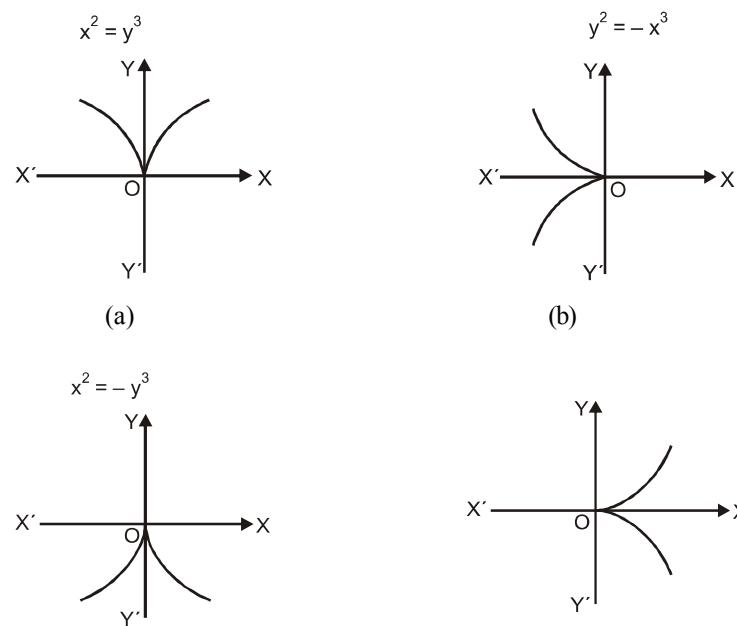


Fig. 6.5 Semi-Cubical Parabola

Check Your Progress

1. What is the integrating factor of $\frac{dy}{dx} + P(x)y = Q(x)$?
2. Write Bernoulli's equation.

6.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. $e^{\int P dx}$
2. $\frac{dy}{dx} + Py = Qy^n$

6.6 SUMMARY

- A linear differential equation of the first order and first degree can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- An equation of the form

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are functions of x alone or constant, is known as the *Bernoulli's equation*.

Cauchy's Problem for First Order Equations

- An equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + p_{n-1} x \frac{dy}{dx} + p_n y = f(x)$$

where p_1, p_2, \dots, p_n are constants, is called a homogeneous linear equation of degree n . This equation is also known as *Euler-Cauchy equation*.

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- Let $\subseteq \Omega$ and $u : D \rightarrow \mathbb{R}$ be a solution of the Equation (6.6). The surface S represented by $z = u(x, y)$ is called an Integral Surface.
- The curvature at every point of the circle is equal to the reciprocal of its radius and therefore, it is constant.

6.7 KEY WORDS

- **First order differential equation:** A first order differential equation is an equation of the form $F(t, y, \dot{y})=0$.
- **Cauchy's problem:** A Cauchy's problem asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain.

6.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is Euler-Cauchy equation?
2. What is Bernoulli's equation? Explain.
3. Discuss linear equations of first order with the help of examples.

Long Answer Questions

1. Solve $2x^2 \frac{dy}{dx} = xy + y^2$
2. Solve $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$
3. Find the equation of the integral surface of the PDE.
4. Explain Cauchy's problems with examples.

NOTES

6.9 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 7 ORTHOGONAL SURFACES AND NONLINEAR PDEs OF THE FIRST ORDER

NOTES

Structure

- 7.0 Introduction
- 7.1 Objectives
- 7.2 Surfaces Orthogonal to a Given System of Surfaces
- 7.3 Nonlinear Partial Differential Equations of the First Order
- 7.4 Cauchy's Method of Characteristics Equations
- 7.5 Answers to Check Your Progress Questions
- 7.6 Summary
- 7.7 Key Words
- 7.8 Self Assessment Questions and Exercises
- 7.9 Further Readings

7.0 INTRODUCTION

Families of surfaces which are mutually orthogonal. Up to three families of surfaces may be orthogonal in three dimensions. The simplest example of three orthogonal surfaces in three dimensions are orthogonal planes, but three confocal conic surfaces are also mutually orthogonal.

In mathematics and physics, a nonlinear partial differential equation is a partial differential equation with nonlinear terms. They describe many different physical systems, ranging from gravitation to fluid dynamics, and have been used in mathematics to solve problems such as the Poincaré conjecture and the Calabi conjecture.

The method of characteristics is a technique for solving partial differential equations. Typically, it applies to first-order equations, although more generally the method of characteristics is valid for any hyperbolic partial differential equation. The method is to reduce a partial differential equation to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hypersurface.

This unit gives you an overview of a surface orthogonal to a given system of surfaces and nonlinear partial differential equation.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Find a surface orthogonal to a given system of surfaces

- Discuss nonlinear partial differential equation of the first order
- Understand Cauchy's method of Characteristics Equations

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7.2 SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

One of the most significant applications of the first order partial differential equation is to find the system of surfaces that are orthogonal to a given or specified system of surfaces.

Consider the following equation which represents the one-parameter family of surfaces,

$$F(x, y, z) = C \quad \dots(7.1)$$

Now, we have to determine the system of surfaces which characteristically intersect each of the given surfaces orthogonally. By the Equation (7.1), we can state that (x, y, z) be a point on the surface where the normal to the surface can be defined by the direction ratios $(\partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$, which can also be denoted using the notations P, Q and R .

$$\text{Let, } Z = \phi(x, y) \quad \dots(7.2)$$

Figure 7.1 illustrates the surfaces which intersect or cut each of the given system of surfaces orthogonally.

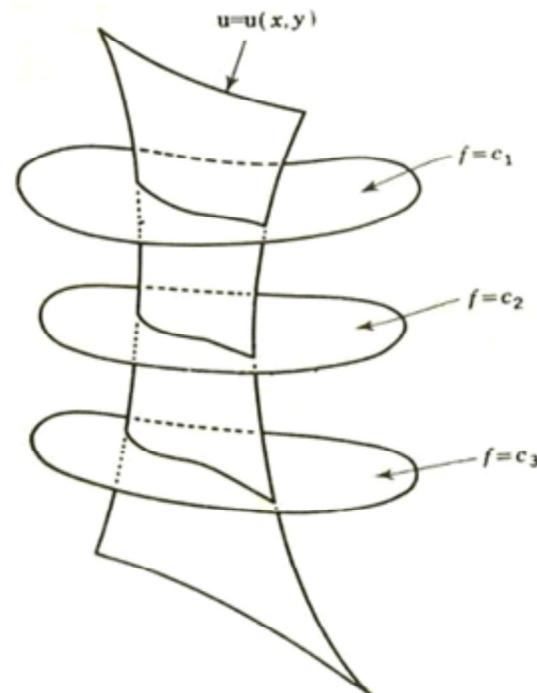


Fig. 7.1 Orthogonal Surfaces

Definition: Angle between two surfaces at a point of intersection is the angle between their tangent planes.

Suppose a one-parameter family of surfaces is characterized by the equation,

$$f(x, y, u) = c \quad \dots(7.3)$$

We have to find a collection of surfaces which cut each of these given surfaces at right angles.

Consider that the surface is represented by the equation,

$$u(x, y) - u = 0 = F(x, y, u)$$

Figure 7.1 illustrates the orthogonal surfaces where each surface cuts the given system of surfaces orthogonally.

At a point of intersection (x, y, u) , we can state that,

$$\nabla f = \langle f_x, f_y, f_u \rangle \quad \dots(7.3)$$

This is the normal to the surface for Equation (7.1). Similarly at (x, y, u) ,

$$\nabla F = \langle u_x, u_y, -1 \rangle \quad \dots(7.4)$$

This is the normal to the surface for Equation (7.2).

Since both the surfaces intersect orthogonally, hence we can state that at point of intersection (x, y, u) their respective normals are perpendicular.

Consequently we have a quasi-linear partial differential equation of the form,

$$\nabla f \cdot \nabla F = f_x u_x + f_y u_y - f_u = 0 \quad \dots(7.5)$$

Therefore integral surface of quasi-linear partial differential equation as depicted in Equation (7.5) is orthogonal to the given surface $f(x, y, u) = c$. In addition, due to the Lagrange's theorem, any point (x, y, u) on the integral surface satisfies the following equations,

$$\frac{dx}{\partial f / \partial x} = \frac{dy}{\partial f / \partial y} = \frac{du}{\partial f / \partial u}$$

Example 7.1: Find the surface which intersects the surfaces of the system $u(x + y) = c$ ($3u + 1$) orthogonally and which passes through the circle $x^2 + y^2 = 1$, $u = 1$. Here, c is a real parameter.

Solution: Here,

$$f = \frac{u(x + y)}{3u + 1}$$

And hence we have,

$$\frac{\partial f}{\partial x} = \frac{u}{3u + 1}, \quad \frac{\partial f}{\partial y} = \frac{u}{3u + 1}, \quad \frac{\partial f}{\partial u} = \frac{x + y}{(3u + 1)^2}$$

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The integral curves are thus given by,

$$\frac{dx}{u/(3u+1)} = \frac{dy}{u/(3u+1)} = \frac{du}{(x+y)/(3u+1)^2}$$

$$\Rightarrow \frac{dx}{u(3u+1)} = \frac{dy}{u(3u+1)} = \frac{du}{x+y}.$$

Taking the first two, we obtain,

$$x - y = c_1$$

And then on further solving we obtain,

$$\begin{aligned} \frac{(x+y)dx + (x+y)dy}{2} &= u(3u+1)du \\ \Rightarrow (x+y)^2 - 4u^3 - 2u^2 &= c_2. \end{aligned}$$

The equation for the given curve can be written in parametric form as,

$$\{(x(0) = x_0(s), y(0) = y_0(s), u(0) = u_0(s)) : s \in I\}$$

You will have the equation of the form,

$$\begin{aligned} 2x_0(s)y_0(s) &= 1 - c_1^2, \\ x_0(s)^2 + y_0(s)^2 + 2x_0(s)y_0(s) - 4u_0(s)^3 - 2u_0^2(s) &= c_2 \\ \Rightarrow 1 + 1 - c_1^2 - 6 &= c_2 \end{aligned}$$

This gives a relation between c_1 and c_2 as,

$$c_2 + c_1^2 + 4 = 0.$$

Thus the required surface is,

$$x^2 + y^2 = 2u^3 + u^2 - 2.$$

7.3 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Qualitative theory of differential equations studies the properties of solutions of ordinary differential equations without finding the solutions themselves.

The foundations of the qualitative theory of differential equations were laid at the end of the 19th century by H. Poincare and A.M. Lyapunov. Poincare made extensive use of geometric methods, regarding the solutions of systems of differential equations as curves in an appropriate space. On this basis he created a general theory of the behaviour of solutions of second-order differential equations and solved a number of fundamental problems on the dependence of solutions on parameters. Lyapunov studied the behaviour of solutions in a neighbourhood of

an equilibrium position and founded the modern theory of stability of motion. The geometric approach of Poincare was developed in the 1920s by George Birkhoff, who discovered many important facts in the qualitative theory of higher-dimensional systems of differential equations.

Non Linear Systems

General systems of non linear differential equations are considered in the normal form:

$$\frac{dy}{dx} = Y(y, x), \quad y \in \mathbf{R}^n. \quad \dots(7.6)$$

Autonomous systems are given by the equation,

$$\frac{dy}{dx} = Y(y). \quad \dots(7.7)$$

The space of vectors y for the system (7.7) is called phase space. The system (7.6) can be reduced to the autonomous form (7.7) by increasing the order by one. An autonomous system of the form (7.7) defines a dynamical system if all its solutions can be extended to the whole axis $-\infty < x < +\infty$.

Let $y = y(x, y_0)$ be the solution to (7.7) with initial data $x = 0, y = y_0$. The curve $y = y(x, y_0), -\infty < x < +\infty$, in the phase space is called a trajectory, while the parts corresponding to $x \geq 0, x \geq 0$ are called semi trajectories. A special role is played by trajectories which degenerate to a point $y(x, y_0) \equiv y_0$ when $Y(y_0) = 0$. Such points are called equilibrium positions. Another important type of trajectory is that of a periodic solution, representing a closed curve in the phase space. A closed trajectory is called a limit cycle if at least one other trajectory converges to it.

An important problem in the qualitative theory of non linear systems is the study of the asymptotic behaviour of all solutions as $x \rightarrow \pm\infty$. For autonomous systems of the form (7.7), this problem reduces to the study of the structure of the limit sets of all the semi trajectories and the ways the trajectories approach these sets. The limit set of each semi trajectory is closed and invariant. A subset of the phase space is called invariant if it consists of complete trajectories. If a semi trajectory is bounded, then its limit set is connected.

If $n = 2$, i.e., when the phase space is a plane, Poincare and I. Bendixson have given an exhaustive description of the possible arrangements of the trajectories. Under the hypothesis that the equation $Y(y) = 0$ has only a finite number of solutions in any bounded part of the plane, they proved that the limit set of any bounded semi trajectory can only be one of the following three types:

- (i) A single equilibrium state;
- (ii) A single closed trajectory; or
- (iii) A finite number of equilibrium states and trajectories converging to these equilibrium states as $x \rightarrow \pm\infty$.

Poincare and A. Denjoy considered the case of a first-order equation of the type (7.6) whose right-hand side is periodic in both arguments y and x . The

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structure of the solutions in this case depends essentially on the rotation number, defined by the formula

$$\mu = \lim_{x \rightarrow \infty} \frac{y(x, y_0)}{x}.$$

If μ is rational, then there exists a periodic solution and if μ is irrational, then all solutions are quasi-periodic functions with two frequencies.

For $n > 2$ it is not possible to give such a clear description of the behaviour of the trajectories. There is, however, a lot of information about the limiting behaviour of higher-dimensional autonomous systems. Let a closed bounded invariant set of the phase space be called minimal if it contains no proper subset with the same properties. Then each minimal set is the closure of a recurrent trajectory. Thus, the limit set of each bounded semi-trajectory contains a recurrent trajectory.

In the important particular case when the system has an invariant measure, the study of general regularity of the behaviour of the solutions has been carried out in great detail.

Of special interest for applications are structurally-stable systems, i.e., systems which are stable under a perturbation of the right-hand sides which is small in the sense of C^1 . For $n = 2$, in any bounded part of the plane there are only a finite number of periodic solutions. For $n > 2$ the behaviour of a structurally-stable system is considerably more complicated. S. Smale has given an example of a structurally-stable system having an infinite number of periodic solutions in a bounded part of the phase space.

Numerous investigations have been devoted to the study of global properties of concrete systems of differential equations. In connection with investigations in the theory of automatic control, a new branch of the qualitative theory of differential equations evolved in the 1950s, namely the theory of stability of motion in the large. An important role in the theory of oscillations is played by dissipative systems of the form (7.6) for which all solutions fall into some bounded domain as time increases. The properties of dissipative systems have been studied in great detail. Relatively reliable methods have been constructed enabling one to establish the dissipativeness of concrete systems.

One of the problems in the qualitative theory of differential equations is that of the existence of periodic solutions. For the proof of the existence of such solutions use is often made of topological devices, in particular the various criteria for the existence of a fixed point.

A complete qualitative study of non-linear systems of differential equations has only been achieved in very special cases. For example, it has been proved that the Lienard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ has, under very natural hypotheses, a unique periodic solution, while all its other solutions converge to this periodic one.

With regard to the Van der Pol equation with perturbation,

$$\ddot{x} + k(x^2 - 1)\dot{x} + x = kb\lambda \sin \lambda t,$$

the following interesting facts have been established for large values of the parameter k . For a special choice of the parameter b the equation has two asymptotically-stable solutions with periods $(2n+1)2\pi/\lambda$ and $(2n-1)2\pi/\lambda$, where n is a sufficiently large integer, and the majority of remaining solutions converge to these two. In addition, there is a countable set of unstable periodic solutions and a continuum of recurrent non-periodic ones.

Lyapunov's Method to Determine Stability for Nonlinear Systems

An equilibrium point is Lyapunov stable if all solutions of the dynamical system that start out near an equilibrium point \mathbf{x}_e stay near \mathbf{x}_e forever. More strongly, if \mathbf{x}_e is Lyapunov stable and all solutions that start out near \mathbf{x}_e converge to \mathbf{x}_e , then \mathbf{x}_e is asymptotically stable.

The general study of the stability of solutions of differential equations is known as stability theory. Lyapunov stability theorems give only sufficient condition.

Lyapunov, in his original 1892 work proposed two methods for demonstrating stability. The first method developed the solution in a series which was then proved convergent within limits. The second method, which is almost universally used nowadays, makes use of a *Lyapunov function* $V(x)$ which has an analogy to the potential function of classical dynamics. It is introduced as follows. Consider a function $V(x) : R^n \rightarrow R$ such that

- $V(x) \geq 0$ with equality if and only if $x = 0$ (positive definite.)
- $\dot{V}(x) = \frac{d}{dt}V(x) \leq 0$ with equality if and only if (negative definite).

Then $V(x)$ is called a Lyapunov function candidate and the system is asymptotically stable in the sense of Lyapunov. $V(0) = 0$ is required otherwise $V(x) = 1/(1 + |x|)$ would prove that $\dot{x}(t) = x$ is locally stable. An additional condition called properness or radial unboundedness is required in order to conclude global asymptotic stability.

It is easier to visualize this method of analysis by thinking of a physical system (for example, vibrating spring and mass) and considering the energy of such a system. If the system loses energy over time and the energy is never restored then eventually the system must grind to a stop and reach some final resting state. This final state is called the attractor. However, finding a function that gives the precise energy of a physical system can be difficult, and for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable.

Lyapunov's realization was that stability can be proven without requiring knowledge of the true physical energy, providing a Lyapunov function can be found to satisfy the constraints.

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7.4 CAUCHY'S METHOD OF CHARACTERISTICS EQUATIONS

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In mathematics, the method of characteristics is a technique for solving partial differential equations. Typically, it is applied for solving the first order equations, even though the method of characteristics is specifically used for any hyperbolic partial differential equation. For a first order PDE (Partial Differential Equation), the method of characteristics discovers curves, termed as the characteristic curves or just characteristics, along which the PDE becomes an Ordinary Differential Equation (ODE). Once the ODE is found, it can be solved along the characteristic curves and transformed into a solution for the original PDE.

The Cauchy method is named after the prolific 19th century French mathematical analyst Augustin Louis Cauchy. Principally, the Cauchy problem tests for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain. A Cauchy problem can be an initial value problem or a boundary value problem or it can be either of them.

For a partial differential equation defined on \mathbf{R}^{n+1} and a smooth manifold $S \subset \mathbf{R}^{n+1}$ of dimension n , where S is called the Cauchy surface, the Cauchy problem consists of finding the unknown functions u_1, \dots, u_N of the differential equation with respect to the independent variables t, x_1, \dots, x_n that satisfies,

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = F_i \left(t, x_1, \dots, x_n, u_1, \dots, u_N, \dots, \frac{\partial^k u_j}{\partial t^{k_0} \partial x_i^{k_1} \dots \partial x_n^{k_n}}, \dots \right)$$

For $i, j = 1, 2, \dots, N; k_0 + k_1 + \dots + k_n = k \leq n_j; k_0 < n_j$

Subject to the condition, for some value of $t = t_0$.

$$\frac{\partial^k u_i}{\partial t^k} = \phi_i^{(k)}(x_1, \dots, x_n) \quad \text{for } k = 0, 1, 2, \dots, n_i - 1$$

Where, $\phi_i^{(k)}(x_1, \dots, x_n)$ are given functions defined on the surface S , collectively acknowledged as the Cauchy data of the problem. The derivative of order zero means that the function itself is specified.

A Cauchy boundary condition augments an ordinary differential equation or a partial differential equation with conditions that the solution must satisfy on the boundary, preferably to ensure that a unique solution exists. A Cauchy boundary condition specifies both the function value and normal derivative on the boundary of the domain.

The Cauchy problem for partial differential equations of order exceeding '1' is solved on the basis of the analyticity assumption for the equation or for the Cauchy data specified in the Cauchy-Kovalevskaya theorem.

Cauchy–Kowalevski Theorem

The Cauchy–Kowalevski theorem states that, “If all the functions F_i are analytic in some neighborhood of the point $(t^0, x_1^0, x_2^0, \dots, \phi_{j,k_0,k_1,\dots,k_n}^0, \dots)$, and if all the functions $\phi_j^{(k)}$ are analytic in some neighborhood of the point $(x_1^0, x_2^0, \dots, x_n^0)$, then the Cauchy problem has a unique analytic solution in some neighborhood of the point $(t^0, x_1^0, x_2^0, \dots, x_n^0)$.

Following questions are associated with Cauchy problems:

1. Does there exist (even though only locally) a solution?
2. If the solution exists, then what is its domain of existence?
3. Is the solution unique?
4. If the solution is unique, then is the solution in some sense a continuous function of the initial data?

The Cauchy problem is typically used to find a solution (an integral) of a differential equation satisfying the initial conditions (initial data). The initial data are specified for $t = 0$ and the solution is required for $t \geq 0$.

The simplest Cauchy problem is to find a function $u(x)$ defined on the half-line $x \geq x_0$, satisfying a first order ordinary differential equation of the form,

$$\frac{du}{dx} = f(x, u) \quad \dots (7.8)$$

For f being a given function and taking a specified value u_0 at $x = x_0$:

$$u(x_0) = u_0 \quad \dots (7.9)$$

In geometrical terms this means that, considering the family of integral curves of Equation (7.8) in the (x, u) -plane, one wishes to find the curve passing through the point (x_0, u_0) .

The first proposition concerning the existence of such a function, on the assumption that f is continuous for all x and continuously differentiable with respect to u , was first proved by A.L. Cauchy (1820–1830) and then generalized by E. Picard (1891–1896), who replaced differentiability by a Lipschitz condition with respect to u . Therefore, under those specific conditions the Cauchy problem has a unique solution which, moreover, depends continuously on the initial data.

The following Hadamard’s example illustrates the Cauchy problem for the Laplace equation of the form,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

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With initial conditions,

$$u(x, y, 0) = \phi_0(x, y), \quad \frac{\partial u}{\partial z}(x, y, 0) = 0$$

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This has no solution if $\phi_0(x, y)$ is not an analytic function.

Consider the following first order partial differential equation for the dependent variables $u(x, y)$,

$$a(x, y, u) \frac{\partial u}{\partial x} + b(x, y, u) \frac{\partial u}{\partial y} = c(x, y, u) \quad \dots(7.10)$$

This is a quasi-linear partial differential equation, because it is linear in the derivatives of $u(x, y)$.

The equation is to be integrated subject to Cauchy data: $u(x, y)$ is given on a curve Γ . In parametric form, this corresponds to,

$$u = u_0(\xi) \quad \text{on} \quad x = x_0(\xi), \quad y = y_0(\xi) \quad \dots(7.11)$$

Where the parameter $\xi_1 < \xi < \xi_2$. Here u_0, x_0 and y_0 are smooth functions of ξ and there is no value of ξ for which $dx_0/d\xi = dy_0/d\xi = 0$.

On the curve Γ , $\frac{du_0}{d\xi} = \frac{\partial u}{\partial x} \frac{dx_0}{d\xi} + \frac{\partial u}{\partial y} \frac{dy_0}{d\xi}$

But the partial derivatives are also determined from Equation (7.10) and so,

$$\begin{pmatrix} a \\ dx_0/d\xi & dy_0/d\xi \end{pmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} c \\ du_0/d\xi \end{pmatrix}$$

Consequently for a unique solutions, we require that,

$$\left| \begin{array}{cc} a & b \\ dx_0/d\xi & dy_0/d\xi \end{array} \right| \equiv a \frac{dy_0}{d\xi} - b \frac{dx_0}{d\xi} \neq 0, \infty$$

Characteristics

We write $\frac{dx}{ds} = a$ and $\frac{dy}{ds} = b$ for some parameter s and then the partial differential equation as defined in Equation (7.10) and Cauchy data in Equation (7.11) are given by,

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b \quad \text{and} \quad \frac{du}{ds} = c.$$

This is subject to $x = x_0(\xi)$, $y = y_0(\xi)$, $u = u_0(\xi)$ for $\xi_1 < \xi < \xi_2$ on $s = 0$. The projection of the solution $u(x, y)$ onto the (x, y) -plane is termed the characteristic projection and the curves $\frac{dx}{ds} = a$ and $\frac{dy}{ds} = b$ are the **characteristics**.

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Check Your Progress

1. Write the equation which represents the one-parameter family of surfaces.
2. What is the general systems of nonlinear differential equations?
3. The Cauchy method is named after which mathematical analyst?

7.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. $F(x, y, z) = C$.
2. $\frac{dy}{dx} = Y(y, x)$, $y \in R^n$.
3. The Cauchy method is named after the prolific 19th century French mathematical analyst Augustin Louis Cauchy.

7.6 SUMMARY

- One of the most significant applications of the first order partial differential equation is to find the system of surfaces that are orthogonal to a given or specified system of surfaces.
- $F(x, y, z) = C$ represents the one-parameter family of surfaces.
- Angle between two surfaces at a point of intersection is the angle between their tangent planes.
- In mathematics, the method of characteristics is a technique for solving partial differential equations. Typically, it is applied for solving the first order equations, even though the method of characteristics is specifically used for any hyperbolic partial differential equation.
- A nonlinear partial differential equation is a partial differential equation with nonlinear terms.
- A Cauchy boundary condition augments an ordinary differential equation or a partial differential equation with conditions that the solution must satisfy on the boundary, preferably to ensure that a unique solution exists. A Cauchy boundary condition specifies both the function value and normal derivative on the boundary of the domain.

NOTES

- **Orthogonal surfaces:** Families of surfaces which are mutually orthogonal.
- **Cauchy problem:** The Cauchy problem is typically used to find a solution (an integral) of a differential equation satisfying the initial conditions (initial data).

7.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Draw a diagram to illustrate the surfaces which intersect or cut each of the given system of surfaces orthogonally.
2. Discuss nonlinear system of partial differential equation of first order.
3. State Cauchy–Kowalevski Theorem.
4. Discuss method of characteristics.

Long-Answer Questions

1. Find the surface which intersects the surfaces of the system $u(x - 2y) = c(3u - 7)$ orthogonally and which passes through the circle $2x^2 - y^2 = 1$, $u = 1$. Here, c is a real parameter.
2. Find the surface which intersects the surfaces of the system $u(x + y) = c(3u + 1)$ orthogonally and which passes through the circle $x^2 - 2y^2 = 1$, $u = 1$. Here, c is a real parameter.
3. Solve the PDE $u_x u_y - u = 0$ subject to the condition $u(x, -x) = 1$

7.9 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 8 COMPATIBLE SYSTEMS OF FIRST ORDER EQUATIONS

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8.0 INTRODUCTION

The method of finding the complete integral of non-linear PDEs of the first order is first given by the Italian mathematician Lagrange (1736-1813). But it was the French mathematician Charpit who perfected it. The method considers the compatible system of first order equations. Charpit's method cannot be generalized directly to any number of independent variables. However, there is another method of solving non-linear PDEs which is due to the German mathematician Jacobi (1804-1851), which can be easily generalized for any number of variables. We have also considered this method in this unit. The methods due to Charpit and Jacobi, as you shall see later in this unit, aim at constructing the integral containing a number of arbitrary constants. The results obtained by these methods do not indicate any particular suggestion of Cauchy's theorem and do not help in finding a solution to initial-data problem.

This unit will discuss the general method of finding the complete integral of non-linear PDEs of the first order. It also defines the compatible system of equations.

8.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss compatible systems of first order equations
- Analyze the condition for two systems of first order non-linear PDEs to be compatible
- Apply Charpit's method for finding the complete integral of a non- linear PDE of first order

8.2 COMPATIBLE SYSTEMS OF FIRST ORDER EQUATIONS

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Definition: A system of two first order Partial Differential Equations (PDEs) of the form,

$$f(x, y, u, p, q) = 0 \quad \dots(8.1)$$

and,

$$g(x, y, u, p, q) = 0 \quad \dots(8.2)$$

are said to be **compatible** if they have a **common solution**.

Theorem 8.1. The above given Equations (8.1) and (8.2) are compatible on a domain D if,

$$1. \quad J = \frac{\partial(f, g)}{\partial(p, q)} = \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0 \text{ on } D$$

2. p and q can be explicitly solved from the given Equations (8.1) and (8.2) as $p = \mathcal{A}(x, y, u)$ and $q = \mathcal{B}(x, y, u)$.

Further we can say that the following equation is integrable,

$$du = \mathcal{A}(x, y, u) dx + \mathcal{B}(x, y, u) dy$$

Theorem 8.2. A necessary and sufficient condition for the integrability of the equation $du = \mathcal{A}(x, y, u) dx + \mathcal{B}(x, y, u) dy$ is,

$$\boxed{[f, g] \equiv \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p \frac{\partial(f, g)}{\partial(u, p)} + q \frac{\partial(f, g)}{\partial(u, q)} = 0.} \quad \dots(8.3)$$

Alternatively, the Equations (8.1) and (8.2) are compatible iff Equation (8.3) holds.

For the compatibility of the equations $f(x, y, u, p, q) = 0$ and $g(x, y, u, p, q) = 0$, it is not necessary that every solution of the equation $f(x, y, u, p, q) = 0$ be a solution of the equation $g(x, y, u, p, q) = 0$ or vice-versa. For example, the following system of equations,

$$f \text{ aH } xp \text{ H } yq \text{ H } x = 0 \quad \dots(8.4)$$

$$g \text{ aH } x^2p + q \text{ H } xu = 0 \quad \dots(8.5)$$

are compatible because they have common solutions $u = x + c(1 + xy)$, where c is an arbitrary constant. Remember that $u = x(y + 1)$ is a solution of Equation (8.4) but not of Equation (8.5).

Solution of Equations (8.1) and (8.2)

The Equations (8.1) and (8.2) can be solved in order to obtain explicit expressions for p and q in the form,

$$p = f(x, y, z), q = g(x, y, z) \quad \dots(8.6)$$

If,

$$J = \frac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{vmatrix} \neq 0 \quad \dots(8.7)$$

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For the condition that the pair of Equations (8.1) and (8.2) should be compatible, the equations reduces to the following condition such that the equation becomes integrable,

$$dz = p \, dx + q \, dy = f \, dx + y \, dy \quad \dots(8.8)$$

Additionally, the Equation (8.8) is a Pfaffian differential equation and the necessary and sufficient condition for its integrability is as follows,

$$\mathbf{X} \cdot \text{curl } \mathbf{X} = 0$$

Where $\mathbf{X} = (f, y, -1)$

$$\begin{aligned} \text{Now, } \mathbf{X} \cdot \text{curl } \mathbf{X} &= (f \mathbf{i} + y \mathbf{j} - \mathbf{k}) \cdot [\mathbf{i}(-y_z) + \mathbf{j}(f_z) + \mathbf{k}(y_x - f_y)] \\ &= f(-y_z) + y(f_z) - (y_x - f_y) \end{aligned}$$

Therefore, $\mathbf{X} \cdot \text{curl } \mathbf{X} = 0$

$$\Rightarrow y_x + f y_z = f_y + y - f_z \quad \dots(8.9)$$

Thus this proves that Equations (8.1) and (8.2) are compatible iff and only iff the Equation (8.9) is satisfied where f and y are given by the Equation (8.6). In order to obtain the compatibility condition in terms of ‘ f ’ and ‘ g ’, we substitute in Equations (8.1) and (8.2) the values of ‘ p ’ and ‘ q ’ from the Equation (8.6) and then differentiate the resulting Equations (8.1) and (8.2) with respect to x and y , and obtain,

$$f_x + f_z \phi + f_p (\phi_x + \phi_z \phi) + f_q (\psi_x + \psi_z \phi) = 0 \quad \dots(8.10)$$

$$g_x + g_z \phi + g_p (\phi_x + \phi_z \phi) + g_q (\psi_x + \psi_z \phi) = 0 \quad \dots(8.11)$$

$$f_y + f_z \psi + f_p (\phi_y + \phi_z \psi) + f_q (\psi_y + \psi_z \psi) = 0 \quad \dots(8.12)$$

$$g_y + g_z \psi + g_p (\phi_y + \phi_z \psi) + g_q (\psi_y + \psi_z \psi) = 0 \quad \dots(8.13)$$

Multiplying Equations (8.10) and (8.11) by g_p and f_p , respectively, and subtracting the resulting equations, we obtain,

$$\begin{aligned} (f_x g_p - g_x f_p) + \phi (f_z g_p - g_z f_p) + (\phi_x + \phi_z \phi) (f_p g_p - g_p f_p) + (\psi_x + \psi_z \phi) (f_q g_p - g_q f_p) &= 0 \\ \Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p \frac{\partial(f,g)}{\partial(z,p)} + (\psi_x + \psi_z \phi) \frac{\partial(f,g)}{\partial(q,p)} &= 0 \end{aligned} \quad \dots(8.14)$$

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In this, the relation given in Equation (8.6) is used and replaced ‘f’ by ‘p’ in the second term.

Similarly, on multiplying the Equations (8.12) and (8.13) by g_q and f_q , respectively, and then subtracting the resulting equations, we obtain,

$$\Rightarrow \frac{\partial(f,g)}{\partial(y,p)} + q \frac{\partial(f,g)}{\partial(z,q)} - (\phi_y + \phi_z \psi) \frac{\partial(f,g)}{\partial(q,p)} = 0 \quad \dots(8.15)$$

Adding Equations (8.14) and (8.15) and using the relation given in Equation (8.9), we obtain the following compatibility condition for Equations (8.1) and (8.2) as,

$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p \frac{\partial(f,g)}{\partial(z,p)} + q \frac{\partial(f,g)}{\partial(z,q)} = 0 \quad \dots(8.16)$$

The expression on the left hand side of the Equation (8.16) is denoted by $[f, g]$. Thus, we have,

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p \frac{\partial(f,g)}{\partial(z,p)} + q \frac{\partial(f,g)}{\partial(z,q)} = 0$$

Or,

$$f_p \frac{\partial g}{\partial x} + f_q \frac{\partial g}{\partial y} + (p f_p + q f_q) \frac{\partial g}{\partial z} - (f_x + p f_z) \frac{\partial g}{\partial p} - (f_y + q f_z) \frac{\partial g}{\partial q} = 0 \quad \dots(8.17)$$

Additionally, the Equation (8.17) is used, which is a first order PDE, for finding the equation ‘ $g=0$ ’ that is compatible with the given equation ‘ $f=0$ ’. Once ‘ g ’ is identified we can find ‘ p ’ and ‘ q ’ and then integrate the Equation (8.8) to obtain one parameter family of solutions in the form,

$$h(x, y, z) = 0 \quad \dots(8.18)$$

Here ‘ b ’ is an arbitrary constant. The solution given by relation Equation (8.18) shall satisfy both the Equations (8.1) and (8.2). Thus, compatible equations have a one-parameter family of common solutions.

Example 8.1: Show that the following partial differential equation,

$$p^2x + q^2y - z = 0$$

is compatible with,

$$p^2x - q^2y = 0$$

and then find their one-parameter family of common solutions.

Solution: Let,

$$f = p^2x + q^2y - z = 0 \quad \dots(1)$$

$$\text{And, } g = p^2x - q^2y = 0 \quad \dots(2)$$

Using condition (8.17) we obtain,

$$\begin{aligned} [f,g] &= 2xp^3 - 2yq^3 - (p^2 - q) 2px + (q^2 - p) 2qy \\ &= 2(q^2y - p^2x) = 0 \end{aligned}$$

Hence the given system of PDEs are compatible.

On solving Equations (1) and (2) for ‘p’ and ‘q’, we have,

$$p = \pm \left(\frac{z}{2x} \right)^{\frac{1}{2}}, q = \pm \left(\frac{z}{2y} \right)^{\frac{1}{2}} \quad \dots(3)$$

Taking only the positive sign and substituting in ‘ $dz = p dx + q dy$ ’, we obtain,

$$\frac{\sqrt{2}}{\sqrt{z}} dz = \frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$$

Or,

$$\sqrt{2z} = \sqrt{x} + \sqrt{y} + b$$

Which is the one-parameter family of common solutions.

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8.3 CHARPIT'S METHOD

Charpit's method is used to find the solution of most general partial differential equation of order one, given by

$$F(x, y, z, p, q) = 0 \quad (8.19)$$

The primary idea in this method is the introduction of a second partial differential equation of order one,

$$f(x, y, z, p, q, a) = 0 \quad (8.20)$$

containing an arbitrary constant ‘a’ and satisfying the following conditions :

1. Equations (8.19) and (8.20) can be solved to give

$$p = p(x, y, z, a) \text{ and } q = q(x, y, z, a)$$

2. The equation

$$dz = p(x, y, z, a)dx + q(x, y, z, a)dy \quad (8.21)$$

is integrable.

When a function ‘f’ satisfying the conditions 1 and 2 has been found, the solution of Equation (8.21) containing two arbitrary constants (including ‘a’) will be a solution of Equation (8.19). The condition 1 will hold if

$$J = \frac{\partial(F, f)}{\partial(p, q)} = \begin{vmatrix} \frac{\partial F}{\partial p} & \frac{\partial f}{\partial p} \\ \frac{\partial F}{\partial q} & \frac{\partial f}{\partial q} \end{vmatrix} \neq 0 \quad (8.22)$$

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Condition 2 will hold when

$$p\left(\frac{\partial p}{\partial z}\right) + q\left(-\frac{\partial p}{\partial z}\right) - \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0$$

$$\Rightarrow p \frac{\partial q}{\partial z} + \frac{\partial q}{\partial x} = q \frac{\partial p}{\partial z} + \frac{\partial p}{\partial y} \quad (8.23)$$

Substituting the values of p and q as functions of x, y and z in Equations (8.19) and (8.20) and differentiating with respect to x

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \frac{\partial q}{\partial x} = 0$$

$$\text{and } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x} = 0$$

Therefore,

$$\left(\frac{\partial F}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial F}{\partial q} \frac{\partial f}{\partial p} \right) \frac{\partial q}{\partial x} = \frac{\partial F}{\partial x} \frac{\partial f}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial f}{\partial x}$$

$$\text{or } \frac{\partial q}{\partial x} = \frac{1}{J} \left\{ \frac{\partial F}{\partial x} \frac{\partial f}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial f}{\partial x} \right\}$$

$$\text{Similarly } \frac{\partial p}{\partial y} = \frac{1}{J} \left\{ -\frac{\partial F}{\partial y} \frac{\partial f}{\partial q} + \frac{\partial F}{\partial q} \frac{\partial f}{\partial y} \right\}$$

$$\frac{\partial p}{\partial z} = \frac{1}{J} \left\{ -\frac{\partial F}{\partial z} \frac{\partial f}{\partial q} + \frac{\partial F}{\partial q} \frac{\partial f}{\partial z} \right\}$$

$$\text{and } \frac{\partial q}{\partial z} = \frac{1}{J} \left\{ \frac{\partial F}{\partial z} \frac{\partial f}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial f}{\partial z} \right\} \quad (8.24)$$

Substituting the values from Equation (8.24) in Equation (8.23)

$$\frac{1}{J} \left[p \left(\frac{\partial F}{\partial z} \frac{\partial f}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial f}{\partial z} \right) + \left(\frac{\partial F}{\partial x} \frac{\partial f}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial f}{\partial x} \right) \right]$$

$$= \frac{1}{J} \left[q \left(-\frac{\partial F}{\partial z} \frac{\partial f}{\partial q} + \frac{\partial F}{\partial p} \frac{\partial f}{\partial z} \right) + \left(-\frac{\partial F}{\partial y} \frac{\partial f}{\partial q} + \frac{\partial F}{\partial q} \frac{\partial f}{\partial y} \right) \right]$$

$$\text{or } \left(-\frac{\partial F}{\partial p} \right) \frac{\partial f}{\partial x} + \left(-\frac{\partial F}{\partial q} \right) \frac{\partial f}{\partial y} + \left(-p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q} \right) \frac{\partial f}{\partial z} \\ + \left(p \frac{\partial F}{\partial z} + \frac{\partial F}{\partial x} \right) \frac{\partial f}{\partial p} + \left(q \frac{\partial F}{\partial z} + \frac{\partial F}{\partial y} \right) \frac{\partial f}{\partial q} = 0 \quad (8.25)$$

The Equation (8.25) being linear in variable x, v, z, p, q and f has the following subsidiary equations:

$$-\frac{dx}{\partial p} = -\frac{dy}{\partial q} = -\frac{dz}{p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z}} \quad (8.26)$$

If any of the integrals of Equations (8.26) involve p or q then it is of the form of Equation (8.20).

Then we solve Equations (8.19) and (8.20) for p and q and integrate Equation (8.21).

Example 8.2: Get complete integral of the equation,

$$p^2 + q^2 - 2px - 2qy + 2xy = 0 \quad (1)$$

Solution: The subsidiary equations are

$$\frac{dp}{2(y-p)} = \frac{dq}{2(x-q)} = \frac{dx}{-2(p-x)} = \frac{dy}{-2(q-y)} \quad (2)$$

$$\therefore \frac{dp + dq}{2y + 2x - 2p - 2q} = \frac{dx + dy}{2x + 2y - 2p - 2q}$$

$$\therefore dp + dq = dx + dy$$

Integrating, we get

$$p + q = x + y + a$$

where a is constant

$$\therefore (p - x) + (q - y) = a \quad (3)$$

Equation (1) can also be written as

$$(p - x)^2 + (q - y)^2 = (x - y)^2$$

$$\text{Now } \{(p - x) - (q - y)\}^2 + \{(p - x) + (q - y)\}^2$$

$$= 2 \{(p - x)^2 + (q - y)^2\}$$

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$$\therefore (p - x) - (q - y) = \sqrt{2(x - y)^2 - a^2} \quad (4)$$

Adding Equations (3) and (4),

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$$(p - x) = \frac{1}{2}a + \frac{1}{2}\sqrt{2(x - y)^2 - a^2}$$

$$\text{or } p = \frac{a}{2} + x + \frac{1}{2}\sqrt{2(x - y)^2 - a^2}$$

Similarly subtracting Equation (4) from Equation (3)

$$q = y + \frac{a}{2} - \frac{1}{2}\sqrt{2(x - y)^2 - a^2}$$

$$\therefore dz = pdx + qdy$$

or

$$\begin{aligned} dz &= \left\{ \frac{a}{2} + x + \frac{1}{2}\sqrt{2(x - y)^2 - a^2} \right\} dx + \left\{ y + \frac{a}{2} - \frac{1}{2}\sqrt{2(x - y)^2 - a^2} \right\} dy \\ &= \frac{1}{2}d(x^2 + y^2) + \frac{a}{2}d(x + y) + \frac{1}{2}\sqrt{2(x - y)^2 - a^2}d(x - y) \end{aligned}$$

On integrating

$$z + b = \frac{x^2 + y^2}{2} + \frac{a}{2}(x + y) + \frac{1}{2} \int (2U^2 - a^2)^{\frac{1}{2}} dU$$

where $U = x - y$ and b is an arbitrary constant

$$\begin{aligned} z + b &= \frac{x^2 + y^2}{2} + \frac{a}{2} \left(x + y + \frac{1}{\sqrt{2}} \left\{ \frac{U\sqrt{U^2 - \frac{a^2}{2}}}{2} - \frac{a^2}{4} \log \left(U + \sqrt{U^2 - \frac{a^2}{2}} \right) \right\} \right) \\ &= \frac{x^2 + y^2}{2} + \frac{a}{2}(x + y) + \frac{(x - y)\sqrt{2(x - y)^2 - a^2}}{4} \\ &\quad - \frac{a^2}{4\sqrt{2}} \log \left((x - y) + \sqrt{(x - y)^2 - \frac{a^2}{2}} \right) \end{aligned}$$

Example 8.3: Determine the complete integral of the equation

$$p^2 + q^2 - 2px - 2qy + 1 = 0 \quad (1)$$

Solution: The subsidiary equations are

$$\frac{dx}{-(2-2xp)} = \frac{dy}{-(2q-2y)} = \frac{dp}{-2p} = \frac{dq}{-2q} \quad (2)$$

With

$$\frac{dp}{p} = \frac{dq}{q}$$

On integrating, we get

$$p = aq \quad (3)$$

where 'a' is an arbitrary constant.

Substituting the value of p from Equation (3) in Equation (1)

$$q^2(1+a^2) - 2q(ax+y) + 1 = 0$$

$$\therefore q = (ax+y) + \sqrt{(ax+y)^2 - (1+a^2)}$$

$$\therefore dz = pdx + qdy$$

which gives

$$\begin{aligned} dz &= q(adx + dy) \\ &= d(ax+y)\left\{ (ax+y) + \sqrt{(ax+y)^2 - (1+a^2)} \right\} \end{aligned}$$

Integrating

$$\begin{aligned} z + b &= \frac{1}{2}(ax+y)^2 + \frac{(ax+y)\sqrt{(ax+y)^2 - (1+a^2)}}{2} \\ &\quad - \frac{(a^2+1)}{2} \log \left\{ (ax+y) + \sqrt{(ax+y)^2 - (1+a^2)} \right\} \end{aligned}$$

where b is an arbitrary constant.

Example 8.4: Find complete integral of the following equation

$$2(pq + py + qx) + x^2 + y^2 = 0 \quad (1)$$

Solution: The subsidiary equations of Equation (1) are

$$\frac{dx}{-(2q+2y)} = \frac{dy}{-(2p+2x)} = \frac{dp}{(2q+2x)} = \frac{dq}{(2p+2y)} \quad (2)$$

$$\therefore dp + dq + dx + dy = 0$$

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Integrating

$$p + q + x + y = \text{constant} = a \text{ (say)}$$

$$\text{or } (p + x) + (q + y) = a \quad (3)$$

Equation (1) can be written as

$$2(p + x)(q + y)(x - y)^2 = 0$$

$$\text{or } (p + x)(q + y) = -\frac{1}{2}(x - y)^2$$

$$\begin{aligned}\therefore (p + x) - (q + y) &= \sqrt{(p + x) + (q + y))^2 - 4(p + x)(q + y)} \\ &= \sqrt{a^2 + 2(x - y)^2}\end{aligned}$$

Adding Equation (3) and (4),

$$2(p + x) = a + \sqrt{a^2 + 2(x - y)^2}$$

$$\text{or } p = -x + \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + 2(x - y)^2}$$

Subtracting Equation (4) from Equation (3)

$$q = -y + \frac{a}{2} - \frac{1}{2}\sqrt{a^2 + 2(x - y)^2}$$

$$\therefore dz = pdx + qdy$$

giving

$$\begin{aligned}dz &= -(xdx + ydy) + \frac{a}{2}(dx + dy) + \frac{1}{2}\sqrt{a^2 + 2(x - y)^2}d(x - y) \\ &= -\frac{1}{2}d(x^3 + y^3) + \frac{a}{2}d(x + y) + \frac{1}{2}\sqrt{a^2 + 2(x - y)^2}d(x - y)\end{aligned}$$

Integrating the above equation, we get

$$\begin{aligned}2z + b &= -(x^2 + y^2) + a(x + y) + \sqrt{2} \int \sqrt{\frac{a^2}{2}(x - y)^2} d(x - y) \\ &= -(x^2 + y^2) + a(x + y) + \frac{\sqrt{2}(x - y)\sqrt{\frac{a^2}{2} + (x - y)^2}}{2}\end{aligned}$$

$$\begin{aligned}
 & + \sqrt{2} \frac{a^2}{4} \log \left\{ (x-y) + \sqrt{\frac{a^2}{2} + (x-y)^2} \right\} \\
 & = -(x^2 + y^2) + a(x+y) + \frac{(x-y)\sqrt{a^2 + 2(x-y)^2}}{2} \\
 & \quad + \frac{a^2}{2\sqrt{2}} \log \left\{ (x-y) + \sqrt{\frac{a^2}{2} + (x-y)^2} \right\}.
 \end{aligned}$$

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Example 8.5: Find complete integral of the equation,

$$p^2 + q^2 - 2pq \tanh 2y = \sec h^2 2y$$

Solution: The subsidiary equations are,

$$\begin{aligned}
 \frac{dx}{-(2p-2q \tanh 2y)} &= \frac{dy}{(-2q-2p \tanh 2y)} = \frac{dp}{0} \\
 &= \frac{dq}{-4pq \sec h^2 2y + 4 \sec h^2 2y \tanh 2y}
 \end{aligned}$$

$$\therefore dp = 0$$

$$\text{or } p = \text{constant} = a \text{ (say)}$$

Therefore

$$q^2 - 2a \tanh 2y \cdot q + a^2 - \sec h^2 2y = 0$$

$$\begin{aligned}
 \therefore q &= a \tanh 2y + \sqrt{a^2 \tanh^2 2y - a^2 + \sec h^2 2y} \\
 &= a \tanh 2y + \sqrt{1-a^2} \sec h 2y
 \end{aligned}$$

$$\therefore dz = pdz + qdy$$

gives

$$\begin{aligned}
 dz &= adx + \left(a \tanh 2y + \sqrt{1-a^2} \sec h 2y \right) dy \\
 &= d \left(ax + \frac{a}{2} \log \cosh 2y \right) + \sqrt{1-a^2} \sec h 2y dy
 \end{aligned}$$

Integrating

$$z + b = ax + \frac{a}{2} \log \cosh 2y + \sqrt{1-a^2} \int \frac{2dy}{e^{2y} + e^{-2y}}$$

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$$= ax + \frac{a}{2} \log \cosh 2y + \sqrt{1-a^2} \int \frac{2e^{2y} dy}{1+e^{4y}}$$

$$= ax + \frac{a}{2} \log \cosh 2y + \sqrt{1-a^2} (\tan^{-1} e^{2y}).$$

Example 8.6: Find complete integral

$$xy + 3yq = 2(z - x^2 q^2) \quad (1)$$

Solution: The subsidiary equations are

$$\frac{dx}{-x} = \frac{dy}{-3y - 4x^3 q} = \frac{dp}{p - 2p + 4xq^2} = \frac{dq}{3q - 2q}$$

$$\therefore \frac{dq}{q} = \frac{dx}{-x}$$

$$\Rightarrow qx = \text{constant} = a$$

$$\Rightarrow q = \frac{a}{x}$$

Substituting in Equation (1) we get

$$p = \frac{2(z - a^3)}{x} - \frac{3ya}{x^2}$$

$$\therefore dz = pdx + qdy$$

$$\text{gives } dz = \left\{ \frac{2(z - a^3)}{x} - \frac{3ya}{x^2} \right\} dx + \frac{a}{x} dy$$

Multiplying by x^2

$$x^3 dz = 2x(z - a^3) dx - 3yadx + axdy$$

$$\text{i.e., } x^4 d\left(\frac{z - a^3}{x^2}\right) = -3aydx + axdy$$

$$\text{i.e., } d\left(\frac{z - a^3}{x^2}\right) = \frac{a}{x^3} dy - \frac{3ay}{x^4} dx = d\left(\frac{ay}{x^2}\right)$$

$$\text{On integrating, we get } \frac{z - a^3}{x^2} = \frac{ay}{x^3} + b$$

$$\text{or } z = a\left(a + \frac{y}{x}\right) + bx^2 \text{ where, } a \text{ and } b \text{ are arbitrary constants.}$$

8.4 SPECIAL TYPES OF FIRST ORDER EQUATIONS

The most general first order differential equation can be written as,

$$\frac{dy}{dt} = f(y, t) \quad \dots(8.27)$$

Fundamentally, there is no general formula for the solution to the Equation (8.27). The following methods are used for solving the given equations.

Type I: Equations Involving only ‘p’ and ‘q’

The non-linear PDEs of first order which do not contain the variables x, y, z explicitly and involve only p and q are of the form,

$$f(p, q) = 0 \quad \dots(8.28)$$

This solution gives us the compatible equation, say,

$$p = \text{constant} = a$$

Once the value of p is known to us, we can obtain the corresponding value of q from the Equation (8.28) in the form

$$q = Q(a), \text{ a constant.}$$

Therefore, the equation,

$$dz = p dx + q dy$$

reduces to the form,

$$dz = a dx + Q(a) dy$$

Integrating, we get complete integral of Equation (8.28) as,

$$z = ax + Q(a) y + b$$

Where b is an arbitrary constant.

Example 8.7: Find the complete integral of the equation,

$$pq = 1$$

Solution: For the given equation, we consider only the following derivation of the Charpit's equations,

$$\frac{dx}{p} = \frac{dp}{0} = \frac{dq}{0}$$

On solving, from 1st and 2nd fractions, we obtain

$$dp = 0$$

$$\Rightarrow p = \text{constant} = a, \text{ say.}$$

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Subsequently, from the above equation and the given equation, we obtain,

$$aq = 1 \Rightarrow q = \frac{1}{a}$$

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Substituting for p and q in the equation,

$$dz = p dx + q dy$$

We obtain the following equation,

$$dz = a dx + \frac{1}{a} dy$$

Integrating, we get the complete integral of the given equation as,

$$z = ax + \frac{1}{a} y + b$$

Type II: Equations not Involving the Independent Variables

Equations of this type are of the form,

$$f(z, p, q) = 0 \quad \dots(8.29)$$

Consider the following form of Charpit's Equation,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

The last two fractions of the above equation will yield,

$$\frac{dp}{p} = \frac{dq}{q}$$

On integrating the above equation, the compatible system is obtained as,

$$p = aq \quad \dots(8.30)$$

Where 'a' is a constant.

From the relations given in the Equations (8.29) and (8.30), we obtain

$$\begin{aligned} f(z, aq, q) &= 0 \\ \Rightarrow q &= Q(a, z) \end{aligned} \quad \dots(8.31)$$

Consider the equation,

$$dz = p dx + q dy$$

On further solving the above equation and substituting the values of Equation (8.30) in it we obtain,

$$dz = aq dx + q dy$$

$$= (adx + dy)q$$

$$= (adx + dy) Q(a,z)$$

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Thus, the complete integral of the Equation (8.29) is given by,

$$\int \frac{dz}{Q(a,z)} = ax + y + b,$$

Where 'b' is an arbitrary constant.

Wave Equation

For deriving the equation governing small transverse vibrations of an elastic string, we position the string along the x -axis, extend it to its length L and fix it at its ends $x = 0$ and $x = L$. Distort the string and at some instant, say $t = 0$, release it to vibrate. Now the problem is to find the deflection $u(x, t)$ of the string at point x and at any time $t > 0$.

To obtain $u(x, t)$ as the result of a partial differential equation we have to make simplifying assumptions as follows:

1. The string is homogeneous. The mass of the string per unit length is constant.
The string is perfectly elastic and hence does not offer any resistance to bending.
2. The tension in the string is constant throughout.
3. The vibrations in the string are small so the slope at each point remains small.

For modeling the differential equation, consider the forces working on a small portion of the string. Let the tension be T_1 and T_2 at the endpoints P and Q of the chosen portion. The horizontal components of the tension are constant because the points on the string move vertically according to our assumption. Hence we have,

$$T_1 \cos \alpha = T_2 \cos \beta = T = \text{const} \quad (8.32)$$

The two forces in the vertical direction are $-T_1 \sin \alpha$ and $T_2 \sin \beta$ of T_1 and T_2 . The negative sign shows that the component is directed downward. If ρ is the mass of the undeflected string per unit length and Δx is the length of that portion of the string that is undeflected then by Newton's second law the resultant of these two forces is equal to the mass $\rho \Delta x$ of the portion times the acceleration $\partial^2 u / \partial t^2$

$$T_2 \sin \beta - T_1 \sin \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}.$$

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On dividing the above equation by $T_2 \cos \beta = T_1 \cos \alpha = T$, we get

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \tan \beta - \tan \alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \quad (8.33)$$

Since $\tan \alpha$ and $\tan \beta$ are the slopes of the string at x and $x + \Delta x$, therefore

$$\tan \alpha = \left(\frac{\partial u}{\partial x} \right)_{x} \quad \text{and} \quad \tan \beta = \left(\frac{\partial u}{\partial x} \right)_{x+\Delta x}.$$

By dividing Equation (8.33) by Δx and substituting the values of $\tan \alpha$ and $\tan \beta$, we have

$$\frac{1}{\Delta x} \left[\left(\frac{\partial u}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

As Δx approaches zero, the equation becomes the linear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho} \quad (8.34)$$

which is the one-dimensional wave equation governing the vibrations of an elastic string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (8.35)$$

To determine the solution we use the boundary conditions, $x = 0$ and $x = L$,

$$u(0, t) = 0, \quad u(L, t) = 0 \quad \text{for all } t \quad (8.36)$$

The initial velocity and initial deflection of the string determine the form of motion. If $f(x)$ is the original deflection and $g(x)$ is the initial velocity, then our initial conditions are,

$$u(x, 0) = f(x) \quad (8.37)$$

and

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x). \quad (8.38)$$

I. Now the problem is to get the solution of Equation (3.5) satisfying the conditions (8.36)-(8.38).

By using the method of separation of variables, verify solutions of the wave equation (8.35) of the form

$$u(x, t) = F(x)G(t) \quad (8.39)$$

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8.5 SOLUTIONS SATISFYING GIVEN CONDITIONS

The Laplace transformation is an important operational method for solving linear differential equations. It is particularly useful in solving *initial value problems* connected with linear differential equations (ordinary and partial). The advantage of Laplace transformation in solving initial value problems lies in the fact that initial conditions are taken care of at the outset and the specific particular solution required is obtained without first obtaining the general solution of the linear differential equation.

Laplace Transforms of Elementary Functions

Using the definitions, we find the Laplace transformation of some simple functions.

(i) Transform of $f(t) = 1$, $t \geq 0$

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} dt = \left[-\frac{e^{-st}}{s} \right]_0^{\infty} \\ e^{-st} &\rightarrow 0 \text{ as } t \rightarrow \infty, \text{ if } s > 0 \\ \therefore L(1) &= \frac{1}{s}, s > 0 \end{aligned} \quad \dots(8.40)$$

(ii) Transform of e^{-at} , where a is a constant.

$$L[e^{-at}] = \int_0^{\infty} e^{-st} e^{-at} dt = \int_0^{\infty} e^{-(s+a)t} dt = \left[-\frac{e^{-(s+a)t}}{s+a} \right]_0^{\infty}$$

$e^{-(s+a)t}$ tends to zero as, $t \rightarrow \infty$, if $(s + a > 0)$

$$\therefore L[e^{-at}] = \frac{1}{s+a}, s > -a \quad \dots(8.41)$$

(iii) Transform of e^{at} , where a is a constant.

$$L[e^{at}] = \frac{1}{s-a}, s > a \quad \dots(8.42)$$

The result follows from (ii) by changing a to $-a$.

(iv) Transform of $\sin at$, where a is a constant.

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$$\begin{aligned} L[\sin at] &= \int_0^\infty e^{-st} \sin at dt = \left[-\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty \\ &= \frac{-s}{s^2 + a^2} \underset{t \rightarrow \infty}{\text{Lt}} (e^{-st} \sin at) - \frac{a}{s^2 + a^2} \underset{t \rightarrow \infty}{\text{Lt}} (e^{-st} \cos at) = \frac{a}{s^2 + a^2} \end{aligned}$$

(when $s > 0$, both $(e^{-st} \sin at)$ and $(e^{-st} \cos at)$ tend to zero as $t \rightarrow \infty$).

$$\therefore L[\sin at] = \frac{a}{s^2 + a^2}, s > 0 \quad \dots(8.43)$$

(v) Transform of $\cos at$, where a is a constant.

$$\begin{aligned} L[\cos at] &= \int_0^\infty e^{-st} \cos at dt = \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^\infty \\ &= \frac{-s}{s^2 + a^2} \underset{t \rightarrow \infty}{\text{Lt}} (e^{-st} \cos at) + \frac{a}{s^2 + a^2} \underset{t \rightarrow \infty}{\text{Lt}} (e^{-st} \sin at) \\ &\quad + \frac{s}{s^2 + a^2}, s > 0, \text{ by the results stated in equation} \quad \dots(8.44) \end{aligned}$$

$$\therefore L[\cos at] = \frac{s}{s^2 + a^2}, s > 0$$

(vi) Transform of t^n , where n is a positive integer.

$$L[t^n] = \int_0^\infty e^{-st} t^n dt = \left[-\frac{t^n e^{-st}}{s} \right]_0^\infty - \int_0^\infty \left(-\frac{1}{s} \right) e^{-st} \cdot n t^{n-1} dt$$

On integration by parts,

$$= -\frac{1}{s} \cdot \underset{t \rightarrow 0}{\text{Lt}} \frac{t^n}{e^{st}} + \frac{n}{s} \int_0^\infty e^{-st} \cdot n t^{n-1} dt$$

If $s > 0$, by applying L'Hospital's rule successively, it can be shown that as $t \rightarrow \infty$,

$$L[t^n] = \frac{n}{s} L[t^{n-1}], s > 0 \quad \dots(8.45)$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} L[t^{n-2}], s > 0$$

By repeated application of equation (6.16)

$$L[t^n] = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} L[t^0]$$

But,

$$L[t^0] = L(1) = \frac{1}{s}$$

$$\therefore L[t^n] = \frac{n.(n-1)\dots2.1}{s^n} \cdot \frac{1}{s}$$

or $L[t^n] = \frac{n!}{s^n + 1}, s > 0$

Note: $L[t] = [1/s^2]$, $L[t^2] = (2/s^3)$

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8.6 JACOBI'S METHOD

Let q, p be canonical variables. Their time evolution satisfies the Hamilton's equations,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$$

Let the initial data be $q(0) = Q$ and $p(0) = P$. Hamilton's equation induce a time-dependent mapping,

$$(Q, P) \mapsto (q(t), p(t))$$

such that,

$$q = q(Q, P, t) \text{ and } p = p(Q, P, t)$$

The Hamiltonian dynamics define a flow in phase space. We can also refer to the inverse mapping,

$$Q = Q(q, p, t) \text{ and } P = P(q, p, t)$$

This mapping is canonical. We will find this mapping, i.e., we will express the coordinates and momenta at time t given their values at time zero. The aim of Hamilton-Jacobi theory is to find this mapping. Since this mapping is canonical, so it has a certain structure that its Jacobian must satisfy. Besides, the transformed variables satisfy Hamiltonian dynamics with a transformed Hamiltonian. In the present case, we want the new variables (Q, P) to be stationary in time, since we want

$$q(t) = q(Q(t), P(t), t) = q(Q(0), P(0), t)$$

A way to impose it is to require the transformed Hamiltonian K to be, say, zero (could be any function of time only). Suppose furthermore that we try to generate this transformation using a generating function of the form $F_2(q, P, t)$ (here again, the choice of generating function is somewhat arbitrary). Then,

$$K = H(q, p, t) + \frac{\partial F_2}{\partial t}(q, P, t) = 0$$

And

$$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial p_i}$$

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Using the first of the canonical relations and substituting it into the Hamiltonian transformation, we obtain

$$\frac{\partial F_2}{\partial t}(q, P, t) + H\left(q, \frac{\partial F_2}{\partial q}(q, P, t), t\right) = 0 \quad (8.46)$$

This is the Hamilton-Jacobi equation. Fixing P , the initial data for the momentum, and defining

$$S(q, t) = F_2(q, P, t)$$

Equation (15) takes the form

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

It is a first order partial differential equation in $(n + 1)$ variables. Its solution is a generating function with P as a parameter which we can use in order to find the mapping between the coordinates at time 0 and time t .

Note: A partial differential equation in $n + 1$ variables is by no means simpler than a system of $2n$ ordinary differential equations. The transition from first-order partial differential equations and system of ordinary differential equations is standard in the analysis of hyperbolic partial differential equations.

Now, suppose we wish to solve the equation for S . Since it is a first-order equation in $n + 1$ variables, the solution involves $n + 1$ integration constant. Note however that S is only defined up to an additive constant, which will not affect the generating function anyways. Thus, without loss of generality, S depends on n integration constants α ,

$$S = S(q, \alpha, t)$$

Since these constants are arbitrary, we are free to identify them with the momenta P , i.e., set

$$F_2(q, P, t) = S(q, P, t)$$

Note that the way of writing a solution with n integration constants is not unique. Now we can proceed,

$$p_i = \frac{\partial S}{\partial q_i}(q, P, t) \text{ and } Q_i = \frac{\partial S}{\partial P_i}(q, P, t)$$

Inverting these equations provides the required solution.

Let us take the example of a harmonic oscillator,

$$H(q, p) = \frac{1}{2m}(p^2 + m^2\omega^2q^2)$$

The Hamilton-Jacobi equation for $S(q, t)$ is,

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial q} \right)^2 + m^2\omega^2q^2 \right] = 0$$

We have to find a solution of the form,

$$S(q, t) = A(q) - \alpha t$$

Substituting we get,

$$\alpha = \frac{1}{2m}[(A'(q))^2 + m^2\omega^2q^2]$$

i.e.,

$$S(q, t) = \int_0^q \sqrt{2m\alpha - m^2\omega^2r^2} dr - \alpha t$$

We can now identify the constant of integration α with the initial momentum P . The transformation equations are,

$$p = \frac{\partial S}{\partial q} = \sqrt{2mP - m^2\omega^2q^2}$$

And

$$Q = \frac{\partial S}{\partial P} = -t + m \int_0^q \frac{1}{\sqrt{2mP - m^2\omega^2r^2}} dr$$

The last integral can be easily solved,

$$Q = -t + \sqrt{\frac{m}{2P}} \int_0^q \frac{1}{\sqrt{1 - \frac{m\omega^2}{2P}r^2}} dr = -t + \frac{1}{\omega} \sin^{-1} \left(\sqrt{\frac{m\omega^2}{2P}} q \right)$$

$$\Rightarrow q = \sqrt{\frac{2P}{m\omega^2}} \sin[\omega(t + Q)]$$

or,

$$p = \sqrt{2mP} \cos[\omega(t + Q)]$$

Q, P can be any pair of variables referring to time 0.

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Jacobi's Theorem

Let $S(q_1, q_2, \dots, q_n, \alpha_1, \alpha_2, \dots, \alpha_n, t)$, i.e., $S(q_i, \alpha_i, t), i = 1, 2, \dots, n$, be any integral of the equation,

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_i}, q_i, t\right) = 0$$

$$\text{Then } \beta_i = \frac{\partial S}{\partial \alpha_i} \text{ and } p_i = \frac{\partial S}{\partial q_i}.$$

These $2n$ equations link p_i, q_i to α_i 's and β_i 's, and hence they provide the general solution to the original canonical equations. The crux of the first form of Jacobi's theorem is that α_i 's and β_i 's are constants of motion which we have to prove.

Proof: Fix $\alpha_1 = P_1, \alpha_2 = P_2, \dots$ and consider the function, $F_2(q_1, \dots, q_n; P_1, \dots, P_n, t) \equiv S(q_1, \dots, q_n, P_1, \dots, P_n, t)$.

Such function generates a contact transformation. Hence,

$$Q_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial S}{\partial P_i}$$

$$P_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial S}{\partial q_i} \quad \dots(8.47)$$

And

$$K = H + \frac{\partial S}{\partial t} = 0$$

Since K is 0, the equations of motion are

$$P_i = -\frac{\partial K}{\partial Q_i} = 0 \text{ and } Q_i = -\frac{\partial K}{\partial P_i} = 0$$

Solving these equations, we get

$$\begin{aligned} P_i &= \text{constant} = \alpha_i, \quad i = 1, 2, \dots, n \\ Q_i &= \text{constant} = \beta_i, \quad i = 1, 2, \dots, n \end{aligned} \quad \dots(8.48)$$

Hence we have,

$$Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i}$$

So it is established that α_i and β_i are constants of motion and S is a solution of the equation, $K = H + \frac{\partial F_2}{\partial t}$ such that $K \equiv 0$

Here, S is known as Hamilton's principle or special function.

Method of Separation of Variables in HJ Equation

The Hamilton-Jacobi Equation (HJE) is of most use when it can be solved via additive separation of variables, which directly identifies constants of motion. For example, the time t can be separated if the Hamiltonian does not depend on time explicitly. In that case, the time derivative $\partial S / \partial t$ in the HJE must be a constant, usually denoted by $-E$, providing the separated solution,

$$S = W(q_1, q_2, \dots, q_n) - Et$$

where the time-independent function $W(\mathbf{q})$ is sometimes called **Hamilton's characteristic function**. The reduced Hamilton-Jacobi equation can now be expressed as,

$$H\left(q, \frac{\partial S}{\partial q}\right) = E$$

To illustrate separability for other variables, we assume that a certain generalized coordinate q_k and its derivative $\partial S / \partial q_k$ appear together as a single function,

$$\psi\left(q_k, \frac{\partial S}{\partial q_k}\right)$$

in the Hamiltonian,

$$H = H(q_1, q_2, \dots, q_{k-1}, q_{k+1}, \dots, q_N; p_1, p_2, \dots, p_{k-1}, p_{k+1}, \dots, p_N; \psi; t)$$

In that case, the function S can be partitioned into two functions, one that depends only on q_k and another that depends only on the remaining generalized coordinates

$$S = S_k(q_k) + S_{rem}(q_1, \dots, q_{k-1}, q_{k+1}, \dots, q_N, t)$$

Substitution of these formulae into the Hamilton-Jacobi equation shows that the function ψ must be a constant (denoted here as $x\Gamma_k$), yielding a first-order ordinary differential equation for $S_k(q_k)$.

$$\psi\left(q_k, \frac{dS_k}{dq_k}\right) = \Gamma_k$$

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In many cases, the function S can be separated completely into N functions $S_m(q_m)$,

$$S = S_1(q_1) + S_2(q_2) + \cdots + S_N(q_N) - Et$$

In such a case, the problem devolves to N ordinary differential equations.

Check Your Progress

1. What is the condition for a system of two first order Partial Differential Equations (PDEs) to be compatible?
2. What is the primary idea in Charpit's method?
3. Write the wave equation.
4. What is the advantage of Laplace transformation?
5. Write the reduced form of Hamilton-Jacobi equation.

8.7 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A system of two first order Partial Differential Equations (PDEs) of the form, $f(x, y, u, p, q) = 0$ and, $g(x, y, u, p, q) = 0$ are said to be compatible if they have a common solution.
2. The primary idea in Charpit's method is the introduction of a second order partial differential equation of order one.
3. $u(x, t) = F(x)G(t)$.
4. The advantage of Laplace transformation in solving initial value problems lies in the fact that initial conditions are taken care of at the outset and the specific particular solution required is obtained without first obtaining the general solution of the linear differential equation.

$$5. H\left(q, \frac{\partial S}{\partial q}\right) = E$$

8.8 SUMMARY

- The method of finding the complete integral of non-linear PDEs of the first order is first given by the Italian mathematician Lagrange (1736-1813).
- A system of two first order Partial Differential Equations (PDEs) of the form, $f(x, y, u, p, q) = 0$ and, $g(x, y, u, p, q) = 0$ are said to be compatible if they have a common solution.
- Charpit's method is used to find the solution of most general partial differential equation of order one.

NOTES

- The non-linear PDEs of first order which do not contain the variables x, y, z explicitly and involve only p and q are of the form, $f(p, q) = 0$.
- The Laplace transformation is an important operational method for solving linear differential equations. It is particularly useful in solving initial value problems connected with linear differential equations (ordinary and partial).

8.9 KEY WORDS

- **Compatible system:** A system of two first order Partial Differential Equations (PDEs) of the form, $f(x, y, u, p, q) = 0$ and, $g(x, y, u, p, q) = 0$ are said to be compatible if they have a common solution.
- **Charpit's method:** A method for finding a complete integral of the general first-order partial differential equation in two independent variables.

8.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is compatible system of partial differential equations?
2. Discuss special types of first order equations.
3. Find the complete integral of $p(q - 1) = 1$.
4. Find the complete integral of $p^2 + q^2 = 1$.
5. Discuss Jacobi's method.

Long Answer Questions

1. Show that $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$.
2. Show that the equations $f(x, y, p) = 0$ and $g(x, y, q) = 0$ are compatible if $f_p g_x - f_y g_q = 0$.
3. Show that the equations $(y - z)p + (z - x)q = x - y$ and $z - px - qy = 0$ are compatible.
4. Find the complete integral of $z^2 = pqxy$ by Charpit's method.
5. Using Charpit's method find the complete integral of the equation $(p^2 + q^2)y = qz$.

8.11 FURTHER READINGS

Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.

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- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

BLOCK III
PDE WITH CONSTANT COEFFICIENTS AND
INTEGRAL TRANSFORMS

NOTES

**UNIT 9 PARTIAL DIFFERENTIAL
EQUATIONS OF THE
SECOND ORDER**

Structure

- 9.0 Introduction
- 9.1 Objectives
- 9.2 Origin of Second Order Equations
- 9.3 Partial Differential Equations of the Second Order
- 9.4 Answers to Check Your Progress Questions
- 9.5 Summary
- 9.6 Key Words
- 9.7 Self Assessment Questions and Exercises
- 9.8 Further Readings

9.0 INTRODUCTION

In mathematics, a Partial Differential Equation (PDE) is a differential equation that contains beforehand unknown multi-variable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a computer model. A special case is Ordinary Differential Equations (ODEs), which deal with functions of a single variable and their derivatives.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

In this unit, you will study about origin of second order equations, partial differential equations of second order in detail.

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9.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand origin of second order equations
- Explain partial differential equations of second order

9.2 ORIGIN OF SECOND ORDER EQUATIONS

In mathematics, the history of differential equations is traced from the period of development of 'differential equations' from calculus, which was independently invented by the English physicist Isaac Newton and German mathematician Gottfried Leibniz. Figure 9.1 illustrates the overview of the historical origin of differential equations, a unique mathematical tool that was invented independently by Isaac Newton (1676) and Gottfried Leibniz (1693).

Year	Mathematician	Description
1676	 Isaac Newton (1643-1727) (IQ=190-200) English physicist and mathematician	Solved his first differential equation, by the use of infinite series, eleven years after his discovery of calculus in 1665.
1693	 Gottfried Leibniz (1646-1716) (IQ=205) German mathematician	Solved his first differential equation, the year in which Newton first published his results.

Fig. 9.1 Origin of Differential Equations by Isaac Newton and Gottfried Leibniz

The exact chronological origin and history to the subject of differential equations is a bit difficult subject because of a number of reasons. First being the secretiveness, second being the private publication issues as the private works published only decades later, and in the words of English polymath Thomas Young the third and most significant being the 'nature of the battle of mathematical and scientific discovery'.

The history of the origin and the studies of differential equations, thus can be credited to Newton, Leibniz, the Bernoulli brothers, and other mathematicians and physicists.

Newton-Leibniz Years

In circa 1671, the English physicist Isaac Newton wrote his then-unpublished book, 'The Method of Fluxions and Infinite Series', which was actually published in the year 1736. In his book he classified the first order differential equations into three classes as shown in Table 9.1, and named it as fluxional equations. Table 9.1 illustrates the classification of Newton that follows the modern notation.

Table 9.1 Classification of Newton for First Order Differential Equations

Partial Differential
Equations of the
Second Order

Ordinary Differential Equations		Partial Differential Equations
Class 1	Class 2	Class 3
$\frac{dy}{dx} = f(y)$	$\frac{dy}{dx} = f(x, y)$	$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$

As per the Newton classification, the first two classes contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, and are today termed as 'ordinary differential equations', while the third class involves the partial derivatives of one dependent variable and are termed as 'partial differential equations'.

Differential equations, thus, first came into existence with the invention of calculus by Newton and Leibniz. In Chapter 2 of his 1671 work 'Methodus fluxionum et Serierum Infinitarum', Sir Isaac Newton listed the following three types of differential equations:

$$\begin{aligned}\frac{dy}{dx} &= f(x) \\ \frac{dy}{dx} &= f(x, y) \\ x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} &= y\end{aligned}$$

He solved these examples and other related problems using infinite series and discussed the non-uniqueness of solutions.

Jacob Bernoulli proposed the Bernoulli differential equation in 1695. This is an ordinary differential equation of the form, $y' + P(x)y = Q(x)y^n$ for which in the subsequent year Leibniz obtained solutions by simplifying it.

Historically, the problem of a vibrating string, such as of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. In 1746, d'Alembert discovered the one-dimensional wave equation and within next ten years Euler discovered the three-dimensional wave equation.

According to British mathematician Prof. Edward Lindsay Ince, the study of 'differential equations' started in 1675, when German mathematician Gottfried Leibniz wrote the following equation:

$$\int x dx = \frac{1}{2}x^2$$

In 1676, Newton solved his first differential equation. In the same year, Leibniz introduced the term 'differential equations' which is derived from the Latin

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word 'aequatio differentialis' to denote a relationship between the differentials dx and dy of two variables x and y .

In 1693, Leibniz solved his first differential equation and that same year Newton published the results of previous differential equation solution methods, this year is said to mark the inception for the differential equations as a distinct field in mathematics.

Bernoulli Years

Swiss mathematicians and brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748), in Basel, Switzerland, were among the first interpreters of Leibniz' version of differential calculus.

The first book on the subject of differential equations, theoretically, was written between 1701 and 1704 and published (1707) in Latin was by the Italian mathematician Gabriele Manfredi's named as 'On the Construction of First-degree Differential Equations'. The book was principally based on the interpretations of the Leibniz and the Bernoulli brothers. Most of the publications on differential equations and partial differential equations published in the 18th century seemed to expand on the version developed by Leibniz, a methodology, employed by those as Leonhard Euler, Daniel Bernoulli, Joseph Lagrange, and Pierre Laplace.

Integrating Factor

In 1739, Swiss mathematician Leonhard Euler began using the integrating factor as a methodology to derive differential equations that were integrable in finite form.

Differential equations are described by their order, which is typically determined by the term with the highest derivatives. An equation containing only first derivatives is termed as a first order differential equation, while an equation containing the second derivative is termed as a second order differential equation, and so on. Differential equations that describe natural phenomena essentially always have only first and second order derivatives in them, but there are some exceptions, such as the thin film equation, which is a fourth order partial differential equation. Differential equations differ from ordinary equations of mathematics in that in addition to variables and constants they also contain derivatives of one or more of the variables involved.

9.3 PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

These are the equations containing one or more partial derivatives and are concerned with at least two independent variables. The order of a partial differential equation is the order of its highest derivative appearing in the equation:

[A] The Partial Differential Equation of the First Order: Let a relation

$$\phi(x, y, z, a, b) = 0 \quad \dots (9.1)$$

be derived from the partial differential equation

$$F(x, y, z, p, q) = 0 \quad \dots (9.2)$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

The solution (9.1) consisting of as many arbitrary constants as the number of independent variables is called the *Complete Integral* of (9.2). If we give particular values to a and b in (9.1), then it becomes *Particular Integral*.

Since the envelope of all the surfaces given by (9.1) is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope with p and q belonging to the envelope at that point must satisfy (9.2).

Hence the relation found by eliminating a and b between $\phi(x, y, z, a, b) = 0$, $\frac{d\phi}{da}$

$= 0$, $\frac{d\phi}{db} = 0$ is called the *Singular Integral*.

If $b = f(a)$ then (9.1) becomes $\phi[x, y, z, a, f(a)] = 0$.

The elimination of $f(a)$ between this equation and $\frac{d\phi}{da} = 0$ gives the *General Integral*.

Methods of Solution

(i) Lagrange's Method

Lagrange's equation is of the form

$$Pp + Qq = R \quad \dots (9.3)$$

P, Q, R being functions of x, y, z .

If $u = f(x, y, z) = a$ satisfies (9.3), then we get on differentiating (9.3) partially w.r.t. x and y ,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q = 0$$

giving $p = -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial z}$, $q = -\frac{\partial u}{\partial y} / \frac{\partial u}{\partial z}$

so that (9.3) yields,

$$P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} + R \frac{\partial u}{\partial z} = 0 \quad \dots (9.4)$$

Obviously $u = a$ satisfies (9.4) and hence comparing (9.3) and (9.4) we get equations known as *Lagrange's Subsidiary equations*, i.e.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots (9.5)$$

which are also satisfied by $u = a$.

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If $v = b$ is another integral of (9.5), then $\phi(u, v) = 0$ or $u = \phi(v)$, ϕ , being arbitrary function, is an integral of (9.3).

Example 9.1: Solve $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$

Lagrange's subsidiary equations are

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)} \quad \dots(9.6)$$

$$\text{First two fractions give } \frac{dy}{dx} = \frac{2y^4 - x^3y}{y^3x - 2x^4}$$

which being a homogeneous equation may be solved by putting $y = vx$ whence may we get

$$\frac{dx}{x} = \frac{v^3 - 2}{v(v+1)(v^2 - v+1)} dv = \left(-\frac{2}{v} + \frac{1}{v+1} + \frac{2v-1}{v^2-v+1} \right) dv$$

Integrating $\log x + \log A = -2 \log v + \log(v+1) + \log(v^2-v+1)$

$$\text{i.e., } Ax = \frac{(v+1)(v^2-v+1)}{v^2} = \left(\frac{y}{x} + 1 \right) \left(\frac{y^2}{x^2} - \frac{y}{x} + 1 \right) / \frac{y^2}{x^2}$$

$$\text{or } Ax^2y^2 = x^3 + y^3, \text{i.e., } \frac{x^2y^2}{x^3 + y^3} = A$$

Again from (9.6) we have

$$\frac{dx}{y^3 - 2x^3} = \frac{dy}{2y^3 - x^3} = \frac{dz}{9z(x^3 - y^3)} = \frac{\frac{dx}{x} + \frac{dy}{y}}{-3(x^3 - y^3)}$$

$$\text{The last two fractions give } \frac{dx}{x} + \frac{dy}{y} = \frac{dz}{-3z}$$

$$\text{Integrating } x^3y^3 = \frac{1}{Bz} \text{ or } z = \frac{1}{x^3y^3} \phi \left(\frac{x^3 + y^3}{x^2y^2} \right) \text{ by taking } \frac{1}{B} = \phi \left(\frac{1}{A} \right)$$

$$\text{or } z = \frac{1}{x^3y^3} \phi \left(\frac{x}{y^2} + \frac{y}{x^2} \right)$$

which is the required solution.

(ii) Standard Methods

Standard I. Equations involving p and q only as

$$F(p, q) = 0 \quad \dots(9.7)$$

have their complete integrals

$$z = ax + by + c \quad \dots(9.8)$$

where a and b are connected by the relation $F(a, b) = 0$.

Example 9.2: Solve $3p^2 - 2q^2 = 4pq$.

Its solution is $z = ax + by + c$ provided $3a^2 - 2b^2 = 4ab$

$$\text{i.e., } b = \frac{-4a \pm \sqrt{16a^2 + 24a^2}}{4} = a \left(-1 \pm \frac{\sqrt{10}}{2} \right)$$

Hence the complete integral is

$$z = a \left[x + y \left(-1 \pm \frac{\sqrt{10}}{2} \right) \right] + c$$

Standard II. The equation analogous to Clairaut's form

$\left(i.e., y = x \frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \text{ which has its solution as } y = xc + f(c) \right)$, such that

$$z = px + qy + f(p, q)$$

has for its complete integral, $z = ax + by + f(a, b)$.

Example 9.3: Solve $z = px + qy - 2\sqrt{pq}$

Its complete integral is $z = ax + by - 2\sqrt{ab}$

Standard III. The equations of the form $F(z, p, q) = 0$ are solved by

putting $q = ap$, (a being an arbitrary constant) and changing p into $\frac{dz}{dX}$ where $X = x + ay$ and then solving the resulting ordinary differential equations between z and X .

Example 9.4: Solve $q(p^2z + q^2) = 4$.

Putting $q = ap$ where $p = \frac{dz}{dX}$ and $X = x + ay$, we have

$$p = \frac{dz}{dX} = \pm \frac{1}{3\sqrt{z+a^2}}, i.e., dX = \frac{3}{2} (z+a^2)^{\frac{1}{2}} dz, \text{ taking +ve sign}$$

Integrating $X + b = (z + a^2)^{3/2}$ or $(x + ay + b)^2 = (z + a^2)^3$

Standard IV. Equations of the form $f_1(x, p) = f_1(y, q)$ are solved by putting $f_1(x, p) = f_2(y, q) = a$ (an arbitrary constant).

These equations give p and q which when substituted in $dz = p dx + q dy$ give the complete integral.

Example 9.5: Solve $q = 2yp^2$.

We have $p^2 = \frac{q}{2y} = a^2$ (say).

When $p^2 = a^2$, $p = \frac{dz}{dx} = a$ we have $z = ax + \text{constant}$

and when $q = \frac{dz}{dy} = 2a^2y$ we have $z = a^2y^2 + \text{constant}$.

The complete integral is $z = ax + a^2y^2 + b$.

(iii) Charpit's Method

Let the partial differential equation be

$$F(x, y, z, p, q) = 0 \quad \dots (9.9)$$

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Since z depends upon x and y both, therefore

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p dx + q dy \quad \dots (9.10)$$

Forming Charpit's auxiliary equations (proof is not required)

$$\frac{dp}{\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z}} = \frac{dz}{-p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

We may find a relation

$$f(p, q) = 0 \quad \dots (9.11)$$

between p and q . (9.9) and (9.11) will yield p and q which when substituted in (9.10) give the required solution.

Example 9.6: Solve $2zx - px^2 - 2qxy + pq = 0$

Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

Whence $dq = 0$ gives $q = a$ (constant)

Putting $q = a$ in the given equation we get $p = \frac{2x(z - ay)}{x^2 - a}$

Now substituting values of p and q in $dz = p dx + q dy$ we have

$$dz = \frac{2x(z - ay)}{x^2 - a} dx + a dy$$

or $\frac{dx - a dy}{z - ay} = \frac{2x}{x^2 - a} dx$

which gives on integration, $z - ay = c(x^2 - a)$ i.e., $z = ay + c(x^2 - a)$

[B] Partial Differential Equations of the Second and Higher Orders

Such an equation of second order is of the form

$$Rr + Ss + Tt = V$$

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$ and R, S, T, V are functions of x, y, z, p, q ,

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

The complete solution of such equation will contain two arbitrary functions as

$$z = f(x + ay) + \phi(x - ay).$$

Methods of Solution:

(i) **By inspection:** Method is clear from the following Problem.

Example 9.7: Solve $ar = xy$, i.e., $a \frac{\partial^2 z}{\partial x^2} = xy$.

Integrating with regard to x , $a \frac{\partial z}{\partial x} = \frac{x^2}{2} y + \phi(y)$, constant of integration with regard to x being possibly a function of y .

Integrating again with regard to x ,

$$\begin{aligned} az &= \int \frac{x^2}{2} y \, dy + \int \phi(y) \, dx + \text{const.} \\ &= \frac{x^3 y}{6} + x \phi(y) + \psi(y). \end{aligned}$$

(ii) Monge's Method

The equation is $Rr + Ss + Tt = V$... (9.12)

Total differential of p and q being

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r \, dx + s \, dy$$

$$\text{and } dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s \, dx + t \, dy$$

$$\text{We have } r = \frac{dp - s \, dy}{dx}, \quad t = \frac{dq - s \, dx}{dy}.$$

Substituting these values of r and t in (9.12) we may get

$$(R \, dp \, dy + T \, dp \, dx - V \, dx \, dy) - s (R \, dy^2 - S \, dx \, dy + T \, dx^2) = 0 \quad \dots (9.13)$$

If any relation between x , y , z , p , q makes each of the bracketed expressions in (9.13) vanish this relation will also satisfy (9.13). Hence from (9.13) we have so-called *Monge's subsidiary equations* as

$$R \, dy^2 - S \, dx \, dy + T \, dx^2 = 0 \quad \dots (9.14)$$

$$R \, dp \, dy + T \, dq \, dx - V \, dx \, dy = 0 \quad \dots (9.15)$$

Let (9.14) resolve into two linear equations

$$dy - m_1 dx = 0 \quad \dots (9.16)$$

$$dy - m_2 dx = 0 \quad \dots (9.17)$$

Combining (9.16) with (9.15) and with $dz = p \, dx + q \, dy$ if necessary, we may get two integrals $u_1 = a$, $v_1 = b$ giving an *intermediary integral* $u_1 = f_1(v_1)$, f_1 being an arbitrary function.

Similarly, from (9.17) and (9.15) along with $dz = p \, dx + q \, dy$, we may find another intermediary integral $u_2 = f_2(v_2)$, f_2 being arbitrary.

These two intermediary integrals can yield p and q which when substituted in $dz = p \, dx + q \, dy$ will yield the complete integral on integration.

In case $m_1 = m_2$ either of the intermediary integrals may be integrated to give the complete integral.

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Example 9.8: Solve $r + (a + b)s + abt = xy$.

Putting $r = \frac{dp - s dy}{dx}$ and $t = \frac{dq - s dx}{dy}$ in the given equation we get

Monge's subsidiary equations as $dy^2 - (a + b) + dx dy + ab dx^2 = 0$

... (9.18)

$dp dy + ab dq dx - xy dx dy = 0$... (9.19)

(9.18) yields, $dy - a dx = 0$ and $dy - b dx = 0$

Integrating them, $y - ax = c_1$ and $y - bx = c_2$

Comparing these with (9.19) we get a $dp + ab dq - ax(c_1 + ax) dx = 0$

$$b dp + a ab dq - bx(c_2 + ax) dx = 0$$

$$\text{Their integration yields } p + bq - c_1 \frac{x^2}{2} - \frac{ax^3}{3} = k_1$$

$$p + aq - c_2 \frac{x^2}{2} - \frac{bx^3}{3} = k_2$$

$$\text{or } p + bq - (y - ax) \frac{x^2}{2} - \frac{ax^3}{3} = \phi_1(c_1) = \phi_1(y - ax) \quad \dots (9.20)$$

$$p + aq - (y - bx) \frac{x^2}{2} - \frac{bx^3}{3} = \phi_2(c_2) = \phi_2(y - bx) \quad \dots (9.21)$$

Solving (9.20) and (9.21),

$$p = \frac{1}{a-b} \left[\frac{yx^2}{2}(a-b) - (a^2 - b^2) \frac{x^3}{6} + a\phi_1(y - ax) - b\phi_2(y - bx) \right]$$

$$\text{and } q = \frac{1}{a-b} \left[\frac{x^3}{6}(a-b) - \phi_1(y - ax) + \phi_2(y - bx) \right]$$

Substituting these values of p and q in $dz = p dx + q dy$, we find

$$\begin{aligned} dz &= \left[\frac{yx^2}{2} - (a+b) \frac{x^3}{6} + \frac{a\phi_1(y - ax)}{a-b} - \frac{b\phi_2(y - bx)}{a-b} \right] dx \\ &\quad + \left[\frac{x^3}{6} - \frac{\phi_1(y - ax)}{a-b} + \frac{\phi_2(y - bx)}{a-b} \right] dy \\ &= -(a+b) \frac{x^3}{6} dx + \frac{3x^2 y dx + x^3 dy}{6} - \frac{1}{a-b} [\phi_1(y - ax)(dy - adx)] \\ &\quad + \frac{1}{a-b} [\phi_2(y - bx)(dy - bdx)] \end{aligned}$$

$$\text{Integrating, } z = -(a+b) \frac{x^4}{24} + \frac{yx^3}{6} + \psi_1(y - ax) + \psi_2(y - bx).$$

(iii) Monge's Method of Integrating

$Rr + Ss + Tt + U(rt - s^2) = V$, R, S, T, U, V , being functions of x, y, z, p, q .

Putting $r = \frac{dp - s dy}{dx}$ and $t = \frac{dq - s dx}{dy}$ in the given equation we get

$$(R dp dy + T dq dx + U dp dq - V dx dy) - s (R dy^2 - S dx dy + T dx^2 + U dp + dx + U dq dy) = 0$$

Say $N - sM = 0$

Consider,

$$\begin{aligned} M + \lambda N &= R dy^2 + T dx^2 - (S + \lambda V) dx dy + U dp dx + U dq dy \\ &\quad + R dp dy + \lambda R dp dy + \lambda T dq dx + \lambda U dp dq \\ &\equiv (A dy + B dx + C dp) (E dy + F dx + G dq) \text{ (say)} \end{aligned}$$

Then equating the coefficients of $dy^2, dx^2, dpdq$ we get

$$AE = R, BF = T, GC = \lambda U$$

Also taking $A = R, E = 1, B = kT, F = \frac{1}{k}, C = mU, G = \frac{\lambda}{m}$ and equating the coefficients of the other five terms, we may find

$$kT + \frac{R}{k} = -(S + \lambda V) \quad \dots(9.22)$$

$$\frac{\lambda R}{m} = U \quad \dots(9.23)$$

$$\frac{kT\lambda}{m} = \lambda T, \quad \dots(9.24)$$

$$mU = \lambda R \quad \dots(9.25)$$

$$\frac{mU}{k} = U \quad \dots(9.26)$$

From (9.26), $m = k$ which satisfies (9.24).

From (9.23) or (9.25) $m = \frac{\lambda R}{U} = k$ and hence from (9.22),

$$\lambda^2 (RT + UV) + \lambda US + U^2 = 0 \quad \dots(9.27)$$

So if λ is a root of (9.27), the required factors of $M + \lambda N$ are

$$\left(R dy + \lambda \frac{RT}{U} dx + \lambda R dp \right) \left(dy + \frac{U}{\lambda R} dx + \frac{U}{R} dq \right)$$

i.e., $\frac{R}{U} (U dy + \lambda T dx + \lambda U dp) \frac{1}{\lambda R} (\lambda R dy + U dx + \lambda U dq)$

We thus obtain integrals from the linear equations

$$U dy + \lambda T dx + \lambda U dp = 0 \quad \dots(9.28)$$

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$$\lambda R \, dy + U \, dx + \lambda U \, dq = 0 \quad \dots(9.29)$$

If $u_1 = f_1(v_1)$ and $u_2 = f_2(v_2)$ be the two intermediary integrals so obtained for finding p and q and substituting in $dz = p \, dx + q \, dy$ we get the required solution after integration.

Example 9.9: Solve $z(I + q^2)r - 2pqzs + z(I + p^2)t - z^2(s^2 - rt) + I + p^2 + q^2 = 0$

Here $R = z(1 + q^2)$, $S = -2pqz$, $T = (1 + p^2)z$, $U = z^2V$, $V = -(1 + p^2 + q^2)$
 \therefore quadratic in λ is $(RT + UV)\lambda^2 + \lambda US + U^2 = 0$, i.e., $(pd\lambda - z)^2 = 0$

giving $\lambda = \frac{z}{pq}$.

Now the system of intermediary integrals is

$$U \, dy + \lambda T \, dx + \lambda U \, dp = 0, \quad U \, dx + \lambda R \, dy + \lambda U \, dq = 0$$

$$i.e., pq \, dy + (1 + p^2) \, dx + z \, dp = 0, \quad \dots(9.30)$$

$$pq \, dx + (1 + q^2) \, dy + z \, dq = 0 \quad \dots(9.31)$$

$$Also \quad dz = p \, dx + q \, dy \quad \dots(9.32)$$

So (9.30) can be written as $dx + p(p \, dx + q \, dy) + z \, dp = 0$

or $dx + p \, dz + z \, dp = 0$ giving $x + pz = A$

Similarly from (9.31), $y + qz = B$

So that $p = \frac{A-x}{z}$ and $q = \frac{B-y}{z}$

Putting in $dz = p \, dx + q \, dy$ and integrating we get the required solution.

$$dz = \frac{A-x}{z} \, dx + \frac{B-y}{z} \, dy \quad \text{or} \quad -z \, dz = (A-x)(-dx) + (B-y)(-dy)$$

$$-\frac{z^2}{2} = \frac{(A-x)^2}{z} + \frac{(B-y)^2}{z} + \text{const.}, \text{i.e., } z^2 + (x-A)^2 + (y-B)^2 = C^2$$

[C] General Linear Partial Differential Equations of an Order Higher than the First

Such equations are of the form

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \cdots + M \frac{\partial z}{\partial x} + N \frac{\partial z}{\partial y} + Pz = f(x, y)$$

$$\text{or } [A_0 D^n + A_1 D^{n-1} D' + \cdots + A_n D'^n + B_0 D^{n-1} + \cdots + MD + ND' + P]z = f(x, y)$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$

i.e., $F(D, D')z = f(x, y)$.

Its complete solution consists of two parts: (i) *Complementary Function* (C.F.) (ii) *Particular Integral* (P.I.).

The complementary function is obtained from $F(D, D') = 0$.

Methods of Solution

(i) Complementary Function of a Homogeneous Partial Differential Equation with Constant Coefficients

Such an equation is of the form

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n) z = f(x, y)$$

Taking the trial solution as $z = \phi(y + mx)$, so that

$$D^n(z) = m^n \phi^{(n)}(y + mx), D'^n(z) = \phi^{(n)}(y + mx)$$

and in general $D^r D'^s = m^r \phi^{(r+s)}(y + mx)$

The *auxiliary equation* of the given equation becomes

$$A_0 m^n + A_1 m^{n-1} + \cdots + A_n = 0$$

If the n roots given by it be m_1, m_2, \dots, m_n , then the required complementary function is $z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + mx)$

Example 9.10: Solve $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$, i.e., $(D^2 - a^2 D'^2) z = 0$.

Putting $z = \phi(y + mx)$, the auxiliary equation is $m^2 - a^2 = 0$ giving $m = \pm a$

Hence the solution is $z = \phi(y + ax) + \psi(y - ax)$

Note. If an equation has repeated roots such as $(D - mD')^2 z = 0$, the solution is $z = x\phi(y + mx) + \psi(y + mx)$

If a root is repeated thrice then $z = x^2\phi(y + mx) + x\psi(y + mx) + \chi(y + mx)$ and so on.

(ii) Particular Integral of Homogeneous Equations

Let the equation be $F(D, D')z = \phi(x, y)$. Then

$$P.I. = \frac{1}{F(D, D')} \phi(x, y).$$

Case I. If $\phi(x, y) = e^{ax+by}$ then $\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$

provided $F(a, b) \neq 0$.

Case II. If $\phi(x, y) = \sin ax$ or $\cos ax$ then express $F(D, D')$ as $F(D^2, DD', D'^2)$ and write

$$\frac{\sin ax \text{ or } \cos ax}{F(D^2, DD', D'^2)} = \frac{\sin ax \text{ or } \cos ax}{F(-a^2, -ab, -b^2)} \text{ provided } F(-a^2, -ab, -b^2) \neq 0$$

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Case III. If $\phi(x, y) = x^m y^n$, then $\frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n$,

expanding binomially and operating $x^m y^n$ on every term.

Case IV. If $\phi(x, y) = e^{ax+by}$ V which may also arise in case of failures of cases I and II.

$\frac{1}{F(D, D')} e^{ax+by} V = e^{ax+by} \frac{1}{F(D+a, D'+b)} V$ which reduces to any of the above cases.

Example 9.11: Find the Particular Integral of $(D^2 - 2DD' + D'^2)z = 12xy$.

$$\begin{aligned} \text{We have, } P.I. &= \frac{1}{F(D^2 - 2DD' + D'^2)} 12xy \\ &= 12 \frac{1}{(D - D')^2} xy = \frac{12}{D^2} \left(1 - \frac{D'}{D}\right)^{-2} xy \\ &= \frac{12}{D^2} \left(1 + \frac{2D'}{D} + \dots\right) xy = 12 \cdot \frac{1}{D^2} \left(xy + \frac{2}{D} x\right) \\ &= 12 \left[\frac{1}{D^2} xy + \frac{2}{D^3} x \right] = 12 \left[\frac{x^3 y}{6} + \frac{x^4}{12} \right] = 2x^3 y + x^4. \end{aligned}$$

Complete Integral is $z = x\phi(y+x) + \psi(y+x) + 2x^3 y + x^4$.

COROLLARY 1. In case $f(x, y) = f(ax + by)$ and $F(D, D')$ is a homogeneous function of D, D' of degree n , then

$$D^n f(ax + by) = a^n f^n(ax + by)$$

$$D'^n f(ax + by) = b^n f^n(ax + by)$$

So that $F(D, D')(ax + by) = F(a, b)f^n(ax + by)$

$$\therefore \frac{f^n(ax + by)}{F(D, D')} = \frac{f(ax + by)}{F(a, b)} \text{ when } F(a, b) \neq 0$$

Hence $\frac{f(ax + by)}{F(D, D')}$ may be evaluated by integrating $f(ax + by)$, n times

with regard to $(ax + by)$ and then dividing by $F(a, b)$.

Example 9.12: Solve $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 12(x + z)$, i.e., $(D^2 + D'^2)V = 12(x + y)$.

Auxiliary equation is $m^2 + 1 = 0$, i.e., $m = \pm i$ so that C.F. is $\phi(y + ix) + x(y - ix)$

$$\text{and } P.I. = \frac{12(x+y)}{D^2 + D'^2} = \frac{12 \cdot \frac{(x+y)^2}{1^2 + 1^2}}{D^2 + D'^2} = (x+y)^2$$

Hence the solution is $V = (x + y)^3 + \phi(y + ix) + \psi(y - ix)$.

COROLLARY 2. In case, method of Cor. 1 fails, i.e., $F(a, b) = 0$, then consider

$$(D - mD')z = p - mq = x^r \psi(y + mx) \dots (1)$$

$$\text{Lagrange's subsidiary equations are } \frac{dx}{1} = \frac{dy}{-m} = \frac{dx}{x^r \psi(y + mx)}$$

of which first two fractions, give $dy + mdx = 0$, i.e., $y + mx = \text{const.} = c$ (say). From first and third fractions, we have

$$dz = \frac{dz}{x^r \phi(y + mx)} = \frac{dz}{x^r \psi(c)} \text{ or } dz = x^r \psi(c) dx$$

$$\text{Integrating } z = \frac{x^{r+1}}{r+1} \psi(c) = \frac{x^{r+1}}{r+1} \psi(y + mx)$$

$$\text{Thus 1 yields, } (D - mD') \frac{x^{r+1}}{r+1} \psi(y + mx) = x^r \psi(y + mx).$$

$$\text{i.e., } \frac{1}{D - mD'} x^r \psi(y + mx) = \frac{x^{r+1}}{r+1} \psi(y + mx)$$

$$\text{Hence } \frac{1}{(D - mD')^n} \psi(y + mx) = \frac{1}{(D - mD')^{n-1}} \frac{x}{1} \psi(y + mx)$$

$$= \frac{1}{(D - mD')^{n-2}} \frac{x^2}{1.2} (y + mx)$$

.....

$$= \frac{x^n}{[n]} \psi(y + mx)$$

Example 9.13: Solve $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$, i.e., $(D - 3D')^2 z = 6x + 2y$.

C.F. is clearly, $x\phi(y + 3x) + \psi(y + 3x)$

$$\begin{aligned} P.I. &= \frac{1}{(D - 3D')^2} \cdot (6x + 2y) = \frac{2}{(D - 3D')^2} \cdot (y + 3x) \\ &= 2 \cdot \frac{x^2}{[2]} (y + 3x) = x^2 (y + 3x). \end{aligned}$$

Solution is $z = x^2(3x + y) + x\phi(y + 3x) + \psi(y + 3x)$.

COROLLARY 3. General Rule

Consider $(D - mD')z = p - mq = f(x, y)$

$$\text{Lagrange's subsidiary equations are } \frac{dx}{1} = \frac{dy}{-m} = \frac{dx}{f(x, y)}$$

First two fractions give $y + mx = c$

and then first and third fractions yield $dz = f(x, y) dx = f(x, c - mx) dx$

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$$\therefore z = \int f(x, c - mx) dx + \text{const.}$$

Hence, $\frac{f(x, y)}{D - mD'} = \int f(x, c - mx) dx$, where c is replaced by $y + mx$

after integration.

Example 9.14: Solve $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$, i.e., $(2D - 3D')(D + D')z = 5e^{x-y}$.

Clearly, C.F. is $\phi(y - x) + \psi(2y + 3x)$

$$\begin{aligned} P.I. &= \frac{1}{(D + D')(2D - 3D')} 5e^{x-y} = \frac{1}{D + D'} \frac{5e^{x+y}}{2 - 3(-1)} \text{ by Cor. 1} \\ &= \frac{1}{(D + D')} e^{x-y} = \int e^{x-(c+x)} dx = \int e^{-c} dx = xe^{-c} = xe^{x-y} \end{aligned}$$

Solution is $z = xe^{x-y} + \phi(y - x) + \psi(2y + 3x)$.

(iii) **Non-homogeneous Linear Equations** (complementary function).

Consider $(D - mD' - n)z = 0$, i.e., $p - mq = nz$

Lagrange's subsidiary equations are $\frac{dx}{1} = \frac{dx}{-m} = \frac{dx}{nz}$

First two fractions give $y + mx = \text{const.}$

First and third fractions give $n dx = \frac{dz}{z}$, i.e., $nx = \log z - \log k$ or $z = k e^{nx}$.

Hence the integral of the given equation is $z = e^{nx} \phi(y + mx)$.

Note 1. If factors are repeated say $(D - mD' - n)^2 z = 0$, then

$$z = xe^{nx} \phi(y + mx) + e^{nx} \psi(y + mx).$$

Note 2. The Particular Integral is obtained by the methods already discussed.

Example 9.15: Solve

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - 2 \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial y} + 2 \frac{\partial z}{\partial x} = e^{2x+3y} + \sin(2x+y) + xy.$$

Given equation is $(D^2 - DD' + 2D'^2 + 2D' + 2D)z = e^{2x+3y} + \sin(2x+y) + xy$

or $(D + D')(D - 2D' + 2)z = e^{2x+3y} + \sin(2x+y) + xy$.

C.F. is $\phi(y - x) + e^{2x} \psi(y + 2x)$ and by usual methods

$$P.I. = -\frac{1}{10} e^{2x+3y} - \frac{1}{6} \cos(2x+y) + \frac{x}{24} (6xy + 9x - 2x^2 - 6y - 12)$$

Hence the solution.

(iv) **Equations Reducible to Linear Form**

Consider an equation all of whose terms are of the form

$$Ax^m y^n = \frac{\partial^{m+n} z}{\partial x^m \partial y^n}$$

Put $x = e^u$, i.e., $u = \log x$ and $y = e^v$, i.e., $v = \log y$, then it is easy to verify that if

$D \equiv \frac{\partial}{\partial u} = x \frac{\partial}{\partial x}$; $D' \equiv \frac{\partial}{\partial v} = y \frac{\partial}{\partial y}$, then $x^2 \frac{\partial^2}{\partial x^2} = D(D - 1)$, $x^3 \frac{\partial^3}{\partial x^3} = D(D - 1)(D - 2)$ etc.

$$y^2 \frac{\partial^2}{\partial y^2} = D'(D' - 1), y^3 \frac{\partial^3}{\partial y^3} = D'(D' - 1)(D' - 2) \text{ etc.}$$

$$\begin{aligned} \text{or in general } x^m y^n \frac{\partial^{m+n}}{\partial x^m \partial y^n} &= x^m \frac{\partial^m}{\partial x^m} \cdot y^n \frac{\partial^n}{\partial y^n} \\ &= D(D - 1) \dots (D - m + 1) \cdot D'(D' - 1) \dots (D' - n + 1). \end{aligned}$$

So that the given equation will reduce to $F(D, D')z = V$ which can be integrated by usual methods.

Example 9.16: Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0$.

Putting $u = \log x$, $v = \log y$ with $D \equiv \frac{\partial}{\partial u}$, $D' \equiv \frac{\partial}{\partial v}$, this becomes

$$\{D(D - 1) - D'(D' - 1) - D' + D\} z = 0, \text{ i.e., } (D^2 - D'^2) z = 0$$

$$\text{or } (D - D')(D + D')z = 0$$

$$\begin{aligned} \text{So that solution is } z &= f(v + u) + F(v - u) = f(\log xy) + F\left(\log \frac{y}{x}\right) \\ &= \phi(xy) + \psi\left(\frac{y}{x}\right). \end{aligned}$$

(v) Equations in which Linear Factors of $F(D, D') = 0$ are not Possible

Consider $(D^2 - D')z = 0$.

Assume $z = Ae^{\alpha x + \beta y}$, giving $D'z = A\beta e^{\alpha x + \beta y}$ and $D^2z = A\alpha^2 e^{\alpha x + \beta y}$

So that the given equation, yields $\alpha^2 - \beta = 0$, i.e., $\alpha^2 = \beta$

Hence the Complementary function is $z = Ae^{\alpha x + \alpha^2 y} = Ae^{\alpha(x + \alpha y)}$ or in general, $z = \sum Ae^{\alpha(x + \alpha y)}$.

Check Your Progress

1. Define complete integral.
2. What is particular integral?
3. What is Monge's method?
4. Give Charpit's auxillary equation.

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9.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

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1. The solution containing of as many arbitrary constants as the number of independent variables is called as complete integral.
2. A solution of a differential equation formed by assigning values to the arbitrary constants in the complete primitive. Non-singular solution of a differential equation containing no arbitrary constants.
3. A general integral of a first-order partial differential equation is a relation between the variables in the equation involving one arbitrary function such that the equation is satisfied when the relation is substituted in it, for every choice of the arbitrary function.
4. In the mathematical theory of partial differential equations, a Monge equation, named after Gaspard Monge, is a first-order partial differential equation for an unknown function u in the independent variables x_1, \dots, x_n

$$F\left(u, x_1, x_2, \dots, x_n, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$$

that is a polynomial in the partial derivatives of u . Any Monge equation has a Monge cone.

Classically, putting $u = x_\theta$, a Monge equation of degree k is written in the form

$$\sum_{i_0+\dots+i_n=k} P_{i_0\dots i_n}(x_0, x_1, \dots, x_k) dx_0^{i_0} dx_1^{i_1} \cdots dx_n^{i_n} = 0$$

and expresses a relation between the differentials dx_k . The Monge cone at a given point (x_0, \dots, x_n) is the zero locus of the equation in the tangent space at the point.

The Monge equation is unrelated to the (second-order) Monge–Ampère equation.

4. Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

Whence $dq = 0$ gives $q = a$ (constant)

Putting $q = a$ in the given equation we get $p = \frac{2x(z - ay)}{x^2 - a}$

Now substituting values of p and q in $dz = p dx + q dy$ we have

$$dz = \frac{2x(z - ay)}{x^2 - a} dx + a dy$$

or $\frac{dx - a dy}{z - ay} = \frac{2x}{x^2 - a} dx$
which gives on integration, $z - ay = c(x^2 - a)$ i.e., $z = ay + c(x^2 - a)$

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9.5 SUMMARY

- The order of a partial differential equation is the order of its highest derivative appearing in the equation.
- The partial differential equation of the first order: Let a relation

$$\phi(x, y, z, a, b) = 0$$

be derived from the partial differential equation

$$F(x, y, z, p, q) = 0$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

- The solution consisting of as many arbitrary constants as the number of independent variables is called the *Complete Integral* of. If we give particular values to a and b , then it becomes *Particular Integral*.
- Since the envelope of all the surfaces given is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope with p and q belonging to the envelope at that point must satisfy. Hence the relation

found by eliminating a and b between $\phi(x, y, z, a, b) = 0$, $\frac{d\phi}{da} = 0$, $\frac{d\phi}{db} = 0$ is called the *Singular Integral*.

If $b = f(a)$ then it becomes $\phi[x, y, z, a, f(a)] = 0$.

The elimination of $f(a)$ between this equation and $\frac{d\phi}{da} = 0$ gives the *General Integral*.

- Lagrange's equation is of the form

$$Pp + Qq = R$$

P, Q, R being functions of x, y, z .

If $u = f(x, y, z) = a$ satisfies, then we get on differentiating partially w.r.t. x and y ,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q = 0$$

$$\text{giving } p = -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial z}, q = -\frac{\partial u}{\partial y} / \frac{\partial u}{\partial z}$$

so that it yields,

$$P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} + R \frac{\partial u}{\partial z} = 0$$

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- Lagrange's subsidiary equations are

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$$

First two fractions give $\frac{dy}{dx} = \frac{2y^4 - x^3y}{y^3x - 2x^4}$

which being a homogeneous equation may be solved by putting $y = vx$
whence may we get

$$\frac{dx}{x} = \frac{v^3 - 2}{v(v+1)(v^2 - v + 1)} dv = \left(-\frac{2}{v} + \frac{1}{v+1} + \frac{2v-1}{v^2 - v + 1} \right) dv$$

- Equations involving p and q only as

$$F(p, q) = 0$$

have their complete integrals

$$z = ax + by + c$$

where a and b are connected by the relation $F(a, b) = 0$.

- The equations of the form $F(z, p, q) = 0$ are solved by putting $q = ap$, (a being an arbitrary constant) and changing p into $\frac{dz}{dX}$ where $X = x + ay$ and then solving the resulting ordinary differential equations between z and X .

- $Rr + Ss + Tt = V$

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$ and R, S, T, V are functions of x, y, z, p, q ,

$$\text{where } p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}.$$

- Integrating with regard to x , a $\frac{\partial z}{\partial x} = \frac{x^2}{2} y + \phi(y)$, constant of integration with regard to x being possibly a function of y .
Integrating again with regard to x ,

$$\begin{aligned} az &= \int \frac{x^2}{2} y dy + \int \phi(y) dx + \text{const.} \\ &= \frac{x^3 y}{6} + x \phi(y) + \psi(y). \end{aligned}$$

- Monge's Method of Integrating

$Rr + Ss + Tt + U(rt - s^2) = V$, R, S, T, U, V , being functions of x, y, z, p, q .

Putting $r = \frac{dp - s dy}{dx}$ and $t = \frac{dq - s dx}{dy}$ in the given equation we get

$$(R dp dy + T dq dx + U dp dq - V dx dy) - s (R dy^2 - S dx dy + T dx^2 + U dp + dx + U dq dy) = 0$$

- Lagrange's subsidiary equations are $\frac{dx}{1} = \frac{dx}{-m} = \frac{dx}{nz}$

First two fractions give $y + mx = \text{const.}$

First and third fractions give $n dx = \frac{dz}{z}$, i.e., $nx = \log z - \log k$ or $z = k e^{nx}$.

Hence the integral of the given equation is $z = e^{nx} \phi(y + mx)$.

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9.6 KEY WORDS

- **Complete integral:** The solution containing of as many arbitrary constants as the number of independent variables is called as complete integral.
- **Particular integral:** A solution of a differential equation formed by assigning values to the arbitrary constants in the complete primitive. Non-singular solution of a differential equation containing no arbitrary constants.
- **General integral:** A general integral of a first-order partial differential equation is a relation between the variables in the equation involving one arbitrary function such that the equation is satisfied when the relation is substituted in it, for every choice of the arbitrary function.

9.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Give the partial differential equation of the first order.
2. What is Lagrange's method?
3. What is standard method?
4. Give the equation of Charpit's method.
5. Give the equation for partial differential equation of the second and higher order.
6. Brief about Monge's method of integration.

Long Answer Questions

1. Explain the general linear partial differential equations of an order higher than the first.
2. Provide the complementary functions of homogeneous partial differential equation with constant coefficients.

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3. Explain and derive the equations for non-homogeneous linear equations.
4. Give the equations reducible to linear form.

5. Solve $z = px + qy - 2\sqrt{pq}$

Its complete integral is $z = ax + by - 2\sqrt{ab}$

6. Solve $r + (a + b)s + abt = xy$.
7. Find the Particular Integral of $(D^2 - 2DD' + D^2) z = 12xy$.
8. Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0$.

9.8 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 10 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Structure

- 10.0 Introduction
- 10.1 Objectives
- 10.2 Linear Partial Differential Equations with Constant Coefficients
- 10.3 Answers to Check Your Progress Questions
- 10.4 Summary
- 10.5 Key Words
- 10.6 Self Assessment Questions and Exercises
- 10.7 Further Readings

10.0 INTRODUCTION

In mathematics, a Partial Differential Equation (PDE) is a differential equation that contains beforehand unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a computer model. A special case is Ordinary Differential Equations (ODEs), which deal with functions of a single variable and their derivatives.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

In this unit, you will study about linear partial differential equations with constant coefficients in detail.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what partial differential equations are
- Explain partial differential equations with constant coefficients

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10.2 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Generally, an ordinary equation of second order is of the form

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X$$

where P, Q, X are functions of x only.

I. Linear Equations with Constant Coefficients

The general solution is found by usual methods of finding the complementary function and particular integral of the equation. Although the student is presumed to have a sound knowledge of the methods to be employed for finding the complementary function and particular integral, but still then, we summarize them as below:

Let there be a differential equation of the type

$$\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = X$$

where p_1 and p_2 are constants and X is a function of x .

In terms of the D operator, it may be written

$$(D^2 + p_1 D + p_2) y = X \text{ where } D \text{ stands for } \frac{d}{dx}, \text{ i.e., } D \equiv \frac{d}{dx}$$

or we may write thus, $f(D) y = X$.

To Find the Complementary Function (C.F.). The X is removed and replaced by zero. Then an auxiliary equation is written either by putting $y = e^{mx}$ whence, we get

$$m^2 + p_1 m + p_2 = 0,$$

or simply writing $D^2 + p_1 D + p_2 = 0$, i.e., $f(D) = 0$.

In either case, we get the roots of the quadratic.

Case I: If the roots are of the type m_1 and m_2 (real and distinct) C.F. is

$$C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case II: If $m_1 = m_2$ i.e., both the roots are real and equal, C.F. is

$$(C_1 + C_2 x) e^{m_1 x}.$$

Case III: If the roots are imaginary i.e., of the type $\alpha \pm i\beta$, C.F. is

$$e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x] \quad \text{or} \quad C_1 e^{\alpha x} \cos (\beta x + C_2).$$

Case IV: If the roots are of the type $\alpha \pm \sqrt{\beta}$, C.F. is

$$C_1 e^{\alpha x} \cosh (\sqrt{\beta} x + C_2).$$

Note: The number of the arbitrary constants will be the same as the order of the equation.

To find the Particular Integral (P.I.) We have

$$\text{P.I.} = \frac{X}{f(D)} \text{ which for } f(D) = D - \alpha \Rightarrow \frac{X}{D - \alpha} = e^{\alpha x} \int X e^{-\alpha x} dx.$$

Case I: If $X = e^{\alpha x}$ where α is any constant.

$$\text{P.I.} = \frac{e^{\alpha x}}{f(D)} = \frac{e^{\alpha x}}{f(a)} \text{ if } f(a) \neq 0.$$

Case II: If $X = x^m$, where m is a positive integer

$$\text{P.I.} = \frac{x^m}{f(D)} = [f(D)]^{-1} x^m.$$

Expand $[f(D)]^{-1}$ binomially upto m th power of D and then operate x^m on every term.

Case III: If $X = \sin ax$ or $\cos ax$.

$$\text{P.I.} = \frac{\sin ax \text{ or } \cos ax}{f(D^2)} = \frac{\sin ax \text{ or } \cos ax}{f(-a^2)} \text{ provided } f(-a^2) \neq 0.$$

In case $f(-a^2) = 0$, $\frac{\sin ax}{f(D^2)} = \text{Imaginary part of } \frac{e^{iax}}{f(D^2)}$

and $\frac{\cos ax}{f(D^2)} = \text{Real part of } \frac{e^{iax}}{f(D^2)}$, which is Case I.

Case IV: If $X = e^{\alpha x} V$, where V is any function of x , then

$$\text{P.I.} = \frac{e^{\alpha x} V}{f(D)} = e^{\alpha x} \cdot \frac{1}{f(D+a)} \cdot V.$$

Case V: If $X = x \cdot V$, where V is any function of x , then

$$\begin{aligned} \text{P.I.} &= \frac{xV}{f(D)} = x \frac{1}{f(D)} \cdot V - \frac{1}{f(D)} f'(D) \frac{1}{f(D)} V \\ &= \left\{ x - \frac{1}{f(D)} f'(D) \right\} \frac{1}{f(D)} V. \end{aligned}$$

Hence, general solution = C.F. + P.I.

Example 10.1: Solve $\frac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x$

or $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$

Now its complementary function and particular integral may be found thus:

For complementary function, the auxiliary equation is

$$D^2 - 1 = 0 \quad \text{or} \quad D = \pm 1.$$

\therefore C.F. is $C_1 e^x + C_2 e^{-x}$.

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Particular integral

$$= \frac{x \sin x}{D^2 - 1} + \frac{(1+x^2)e^x}{D^2 - 1}$$

$$= \text{imaginary part in } \frac{xe^{ix}}{D^2 - 1} + e^x \frac{1}{(D+1)^2 - 1} \cdot (1+x^2)$$

$$= \text{imaginary part in } e^{ix} \cdot \frac{1}{(D+i)^2 - 1} \cdot x + e^x \cdot \frac{1}{D^2 + 2D} (1+x^2)$$

$$= \text{imaginary part in } e^{ix} \frac{1}{D^2 + 2iD - 2} x + \frac{e^x \cdot 1}{2D \left(1 + \frac{D}{2}\right)} (1+x^2)$$

$$= \text{imaginary part in } \frac{e^{ix}}{-2} \left[1 - \frac{2iD + D^2}{2} \right]^{-1} x + \frac{e^x}{2} \cdot D^{-1} \left\{ 1 + \frac{D}{2} \right\}^{-1} \cdot (1+x^2)$$

= imaginary part in

$$\frac{e^{ix}}{-2} \left[1 + \frac{2iD + D^2}{2} + \dots \right] \cdot x + \frac{e^x}{2} D^{-1} \left[1 - \frac{D}{2} + \frac{D^2}{4} - \dots \right] (1+x^2)$$

$$= \text{imaginary part in } \frac{e^{ix}}{-2} [x + i] + \frac{e^x}{2} D^{-1} \left[1 + x^2 - x + \frac{1}{2} \right]$$

$$= \text{imaginary part in } \left(\frac{\cos x + i \sin x}{-2} \right) (x + i) + \frac{e^x}{2} \int \left(x^2 - x + \frac{3}{2} \right) dx$$

$$= -\frac{1}{2} (x \sin x + \cos x) + \frac{e^x}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2} x \right]$$

Hence, the general solution is

$$y = C_1 e^x + C_2 e^{-x} - \frac{1}{2} (x \sin x + \cos x) + \frac{xe^x}{12} (2x^2 - 3x + 9).$$

II. Linear Equations with Variable Coefficients (Homogeneous Linear Equations)

Consider, $p_0 x^2 \frac{d^2 y}{dx^2} + p_1 x \frac{dy}{dx} + p_2 y = X$.

Put $x = e^z$, i.e., $z = \log x$; $\therefore \frac{dz}{dx} = \frac{1}{x}$.

Then $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$,

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{1}{x} \frac{d}{dz} \left(\frac{1}{x} \frac{dy}{dz} \right) = \frac{1}{x^2} \frac{d^2 y}{dz^2} - \frac{1}{x^3} \frac{dx}{dz} \frac{dy}{dz} \\ &= \frac{1}{x^2} \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) \end{aligned}$$

If we put $\frac{d}{dz} \equiv D$, we have

$$x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y, \dots, x^n \frac{d^n y}{dx^n} = D(D-1)(D-2)\dots(D-n+1)$$

Now substituting these values in (10.1), we get

$$p_0 D(D-1)y + p_1 Dy + p_2 y = X,$$

which may be solved by the method employed in I.

Example 10.2: Solve $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 2 \log x$.

Put $x = e^z$ and denote $\frac{d}{dz}$ by D , we have

$$D(D-1)y - Dy + y = 2z \quad \text{or} \quad (D^2 - 2D + 1)y = 2z.$$

Auxiliary equation is

$$D^2 - 2D + 1 = 0$$

or $(D-1)^2 = 0; \therefore D = 1$ (repeated twice).

$$\text{C.F.} = (c_1 + c_2 z)e^z.$$

$$\begin{aligned} \text{P.I.} &= \frac{2z}{D^2 - 2D + 1} = 2 [1 - (2D - D^2)]^{-1} \cdot z \\ &= 2(1 + 2D\dots) z = 2(z + 2) = 2z + 4. \end{aligned}$$

Therefore, general solution is $y = (c_1 + c_2 z)e^z + 2z + 4$

$$= (c_1 + c_2 \log x)x + 2 \log x + 4.$$

Note. Any equation of the type

$$(a + bx)^n \frac{d^n y}{dx^n} + P_1(a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_{n-1}(a + bx) \frac{dy}{dx} + P_n y = F(x)$$

can be reduced to the homogeneous linear form by putting $z = ax + b$ or this can be solved by putting $ax + b = e^z$ as above.

III. Exact Differential Equations and Equations of other Special Types

The equations for differential equations of special types is of the type

$$P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$$

where P_0, P_1 and P_2 are the functions of x , is said to be exact if

$$P_2 - P_1' + P_0'' = 0$$

or in general an equation of order n (say),

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = Q(x)$$

is exact if $P_n - P_{n-1}' + \dots + (-1)^n P_0^{(n)} = 0$,

where $P', P'', \dots, P^{(n)}$ are the successive derivatives of P .

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In case the equation is exact, its first integral is

$$P_0 y_{n-1} + (P_1 - P_0') y_{n-2} + (P_2 - P_1' + P_0'') y_{n-3} + \dots$$

$$\dots + \{P_{n-1} - P_{n-2}' + \dots + (-1)^n P_0^{n-1}\} y = \int Q(x) + C,$$

where y_n stands for $\frac{d^n y}{dx^n}$ etc.

Example 10.3: Solve $\sin^2 x \frac{d^2 y}{dx^2} = 2y$

$$\text{or } \frac{d^2 y}{dx^2} - 2y \operatorname{cosec}^2 x = 0$$

$$\text{or } \cot x \frac{d^2 y}{dx^2} - 2y \cot x \operatorname{cosec}^2 x = 0,$$

which is exact and hence its first integral is

$$\cot x \frac{dy}{dx} + \operatorname{cosec}^2 x = c_1$$

$$\text{or } \frac{dy}{dx} + \frac{1}{\sin x \cos x} y = c_1 \tan x$$

which is a linear differential equation of the first order.

$$\begin{aligned} \text{Integrating factor} &= e^{\int \frac{dx}{\sin x \cos x}} = e^{\int 2 \operatorname{cosec} 2x dx} \\ &= e^{\log \tan x} = \tan x. \end{aligned}$$

$$\therefore \text{The solution is } y \tan x = \int c_1 \tan^2 x dx + c_2$$

$$\begin{aligned} &= c_1 \int (\sec^2 x - 1) dx + c_2 \\ &= c_1 (\tan x - x) + c_2. \end{aligned}$$

Note 1: Sometimes the equation becomes exact by multiplying an integrating factor x^m , where m can be found by applying the condition of exactness.

Note 2: Equations of the form $\frac{d^2 y}{dx^2} = f(y)$ can be integrated on multiplying by

$2 \frac{dy}{dx}$, whence we get

$$\begin{aligned} \left(\frac{dy}{dx} \right)^2 &= 2 \int f(y) dy + c_1 \\ \text{or } \frac{dy}{dx} &= \sqrt{c_1 + 2 \int f(y) dy} \end{aligned}$$

which may further be integrated by any of the standard methods.

Note 3: Equations not containing x directly can be integrated by putting $\frac{dy}{dx} = p$.

Example 10.4: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0$.

$$\text{Put } \frac{dy}{dx} = p; \therefore \frac{d^2y}{dx^2} = p \frac{dp}{dy}.$$

$$\text{So we get } p \frac{dp}{dy} + p + p^3 = 0$$

$$\text{or } \frac{dp}{1+p^2} = -dy.$$

Integrating $\tan^{-1} p = c - y$, i.e., $\frac{dy}{dx} = p = \tan(c - y)$, which can further be integrated by standard methods.

Similarly equations not containing y can be integrated by putting

$$\frac{dy}{dx} = p \left(\because \frac{d^2y}{dx^2} = \frac{dp}{dx} = p \frac{dp}{dy} \text{ etc.} \right).$$

Note 4. Equations in which y appears in only two derivatives whose orders differ by unity can be integrated as follows:

Example 10.5: $a \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2}$.

$$\text{Put } \frac{dy}{dx} = q; \quad \therefore \quad \frac{d^2y}{dx^2} = \frac{dq}{dx}. \text{ We have}$$

$$a \frac{dq}{dx} = (1 + q^2)^{1/2}$$

$$\text{or } \frac{dq}{\sqrt{1+q^2}} = \frac{dx}{a}.$$

$$\text{Integrating } \sinh^{-1} q = \frac{x}{a} + c_1.$$

$$\therefore q = \frac{dy}{dx} = \sinh \left(\frac{x}{a} + c_1 \right).$$

$$\text{Integrating again, } y = a \cosh \left(\frac{x}{a} + c_1 \right) + c_2.$$

IV. The Complete Solutions in Terms of a Known Integral

Let $y = y_1$ be a known integral in the complementary function of

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X. \quad \dots(10.2)$$

NOTES

Put $y = vy_1$; so that $\frac{dy}{dx} = y_1 \frac{dv}{dx} + v \frac{dy_1}{dx}$

and $\frac{d^2y}{dx^2} = y_1 \frac{d^2y}{dx^2} + 2 \frac{dv}{dx} \cdot \frac{dy_1}{dx} + v \frac{d^2y_1}{dx^2}$.

With these substitutions, (10.2) gives

$$y_1 \left(\frac{d^2v}{dx^2} \right) + \frac{dv}{dx} \left(2 \frac{dy_1}{dx} + Py_1 \right) + v \left(\frac{d^2y_1}{dx^2} + P \frac{dy_1}{dx} + Qy_1 \right) = X$$

$$\text{or } \frac{d^2v}{dx^2} + \left(P + \frac{2}{y_1} \frac{dy_1}{dx} \right) \frac{dv}{dx} = \frac{X}{y_1}, \quad \dots(10.3)$$

since $\frac{d^2y_1}{dx^2} + P \frac{dy_1}{dx} + Qy_1 = 0$ by hypothesis.

Putting $\frac{dv}{dx} = p$, (10.3) becomes

$$\frac{dp}{dx} + \left(P + \frac{2}{y_1} \frac{dy_1}{dx} \right) p = \frac{X}{y_1}, \quad \dots(10.4)$$

which is a linear equation of first order and hence its integrating factor

$$= e^{\int \left(P + \frac{2}{y_1} \frac{dy_1}{dx} \right) dx} = e^{\int P dx + 2 \log y_1} = y_1^2 e^{\int P dx}$$

Hence solution of (10.3) in p is

$$p \cdot y_1^2 e^{\int P dx} = \int \left[y_1^2 \cdot \frac{X}{y_1} e^{\int P dx} \right] dx + c_1$$

$$\text{or } P = \frac{dv}{dx} = \frac{c_1 e^{-\int P dx}}{y_1^2} + \frac{e^{-\int P dx}}{y_1^2} \int \left(y_1 \cdot X \cdot e^{\int P dx} dx \right)$$

Integrating,

$$v = c_2 + c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + \int \frac{e^{-\int P dx}}{y_1^2} \int \left(y_1 \cdot X \cdot e^{\int P dx} (dx)^2 \right)$$

Hence the required solution of (10.2) is

$$y = vy_1 = c_2 y_1 + c_1 y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + y_1 \int \frac{e^{-\int P dx}}{y_1^2} \times \int \left(y_1 \cdot X \cdot e^{\int P dx} \right) (dx)^2.$$

Example 10.6: Solve $x^2 \frac{d^2y}{dx^2} - (x^2 + 2x) \frac{dy}{dx} + (x + 2)y = x^2 \cdot e^x$.

Obviously $y = x$ is a part of the complementary.

Putting $y = vx$; the given equation gives

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x^2 + 2x}{x^2} \right) \frac{dv}{dx} = \frac{x \cdot e^x}{x}.$$

Now put $\frac{dv}{dx} = p$; then we get

$$\frac{dp}{dx} - p = e^x$$

Being a linear equation of first order in p ,

$$\text{Integrating factor} = e^{-\int dx} = e^{-x}.$$

$$\begin{aligned} \text{Thus } p \cdot e^{-x} &= \int e^x \cdot e^{-x} dx + C_1 \\ &= x + C_1 \end{aligned}$$

$$\text{or } p = \frac{dv}{dx} = (x + C_1) e^x.$$

\therefore Required solution is $y = vx = (x^2 - x + C_1 x) e^x + C_2 x$.

Note: It will be helpful to find that

$y = x$ is a part of the complementary, if $P + Qx = 0$,

$y = x^2$ is a part of the complementary, if $2 + 2Px + Qx^2 = 0$,

$y = e^x$ is a part of the complementary, if $P + Q + 1 = 0$,

$y = e^{-x}$ is a part of the complementary, if $1 - P + Q = 0$,

$y = e^{ax}$ is a part of the complementary, $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$ etc.

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V. Transformation of Equation

(i) By Changing the Dependent Variable

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X \quad \dots (10.5)$$

Putting $y = vy_1$ this becomes

$$\frac{d^2y}{dx^2} + \left(P + \frac{2}{y_1} \frac{dy_1}{dy} \right) \frac{dy}{dx} + \frac{v}{y_1} \left(\frac{d^2y_1}{dx^2} + P \frac{dy_1}{dx} + Qy_1 \right) = \frac{X}{y_1} \quad \dots (10.6)$$

Choosing y_1 such that $\frac{d^2y_1}{dx^2} + P \frac{dy_1}{dx} + Qy_1 = 0$

$$(10.6) \text{ reduces to } \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} = \frac{X}{y_1}, \quad \dots (10.7)$$

$$\text{where } P_1 = P + \frac{2}{y_1} \frac{dy_1}{dx}, \text{ giving } y_1 = e^{-\frac{1}{2} \int P dx}$$

Now equation (10.6) may be solved as in the previous method.

Example 10.7: $\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} = n^2 y.$

Here $P_1 = \frac{2}{x}.$

NOTES

Take $y_1 = e^{-\frac{1}{2} \int P dx} = e^{-\frac{1}{2}} e^{-\frac{1}{2} \int \frac{2}{x} dx} = e^{-\log x} = 1/x.$

Now put $y = vy_1$ whence the given equation becomes

$$\frac{d^2v}{dx^2} - n^2 v = 0$$

Its solution is $v = C_1 e^{nx} + C_2 e^{-nx}.$

The required solution is $y = vy_1 = \frac{1}{x} (C_1 e^{nx} + C_2 e^{-nx})$

(ii) By Changing the Independent Variable

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X. \quad \dots (10.7)$$

We know that

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \text{ and } \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} \left(\frac{dz}{dx} \right)^2 + \frac{dy}{dz} \cdot \frac{d^2z}{dx^2}$$

Substituting these values in (10.7), we get

$$\frac{d^2y}{dz^2} \left(\frac{dz}{dx} \right)^2 + \frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + P \frac{dy}{dz} \cdot \frac{dz}{dx} + Qy = X$$

$$\text{or } \frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + P \frac{dz}{dx}}{\left(\frac{dz}{dx} \right)^2} \cdot \frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx} \right)^2} y = \frac{X}{\left(\frac{dz}{dx} \right)^2}$$

$$\text{or } \frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = X_1, \quad \dots (10.8)$$

$$\text{where } P_1 = \frac{\frac{d^2z}{dx^2} + P \frac{dz}{dx}}{\left(\frac{dz}{dx} \right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx} \right)^2} \text{ and } X_1 = \frac{X}{\left(\frac{dz}{dx} \right)^2}$$

If z be chosen such that $\frac{d^2z}{dx^2} + P \frac{dz}{dx} = 0,$

$$\text{i.e., } z = \int e^{-\int P dx} dx,$$

then the given equation changes into

$$\frac{d^2y}{dz^2} + Q_1 y = X_1,$$

which can be solved if Q_1 is a constant or a constant multiplied by $\frac{1}{z^2}$.

Again if z be chosen such that Q_1 be a^2 (a constant), then

$$a^2 \left(\frac{dz}{dx} \right)^2 = Q$$

$$\text{i.e., } az = \int \sqrt{Q} dx.$$

With this substitution (10.7) reduces to

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + a^2y = X,$$

which can be solved if P_1 is a constant.

Example 10.8: Solve $\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + \frac{a^4}{x^4}y = 0$.

We can choose z such that

$$\left(\frac{dz}{dx} \right)^2 = Q = \frac{a^2}{x^4},$$

$$\text{giving } z = \pm \frac{a}{x}.$$

Now changing the independent variable from x to z when $z = \frac{a}{x}$, we have

$$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P \frac{dz}{dx} \right)}{\left(\frac{dz}{dx} \right)^2} = 0.$$

The given equation reduces to

$$\frac{d^2y}{dz^2} + y = 0.$$

Its solution is $y = C_1 \cos z + C_2 \sin z$.

Required solution is $y = C_1 \cos \frac{a}{x} + C_2 \sin \frac{a}{x}$.

Check Your Progress

1. Expand PDE and define it.
2. What are linear equations?
3. Define the term complementary function.

NOTES

10.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. PDE stand for Partial Differential Equation (PDE), it is a differential equation that contains beforehand unknown multivariable functions and their partial derivatives
2. Linear equation can be defined as equation that may be put in the form where are the variables, and are the coefficients, which are often real numbers.
3. The complementary function is the general solution of the differential equation and a particular integral is any solution (i.e., the function of x) that satisfies the differential equation.

10.4 SUMMARY

- Generally, an ordinary equation of second order is of the form

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = X$$

where P, Q, X are functions of x only.

- Let there be a differential equation of the type

$$\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = X$$

where p_1 and p_2 are constants and X is a function of x .

- In terms of the D operator, it may be written

$$(D^2 + p_1 D + p_2) y = X \text{ where } D \text{ stands for } \frac{d}{dx}, \text{ i.e., } D \equiv \frac{d}{dx}$$

or we may write thus, $f(D) y = X$.

- To Find the Complementary Function (C.F.) is the X is removed and replaced by zero. Then an auxiliary equation is written either by putting $y = e^{mx}$ whence, we get

$$m^2 + p_1 m + p_2 = 0,$$

or simply writing $D^2 + p_1 D + p_2 = 0$, i.e., $f(D) = 0$.

- If the roots are of the type m_1 and m_2 (real and distinct) C.F. is

$$C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

- If $m_1 = m_2$ i.e., both the roots are real and equal, C.F. is

$$(C_1 + C_2 x) e^{m_1 x}.$$

- If the roots are imaginary i.e., of the type $a \pm ib$, C.F. is

$$e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x] \quad \text{or} \quad C_1 e^{\alpha x} \cos (\beta x + C_2).$$

- If the roots are of the type $a \pm \sqrt{\beta}$, C.F. is

$$C_1 e^{ax} \cosh(\sqrt{\beta}x + C_2).$$

- If $X = e^{ax}$ where a is any constant.

$$\text{P.I.} = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)} \text{ if } f(a) \neq 0.$$

- If $X = x^m$, where m is a positive integer

$$\text{P.I.} = \frac{x^m}{f(D)} = [f(D)]^{-1} x^m.$$

- If $X = e^{ax} V$, where V is any function of x , then

$$\text{P.I.} = \frac{e^{ax}V}{f(D)} = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V.$$

- The equations for differential equations of special types is of the type

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$$

- Sometimes the equation becomes exact by multiplying an integrating factor x^m , where m can be found by applying the condition of exactness.
- Sometimes the equation becomes exact by multiplying an integrating factor x^m , where m can be found by applying the condition of exactness.
- Equations of the form $\frac{d^2y}{dx^2} = f(y)$ can be integrated on multiplying by $2 \frac{dy}{dx}$, whence we get

$$\left(\frac{dy}{dx} \right)^2 = 2 \int f(y) dy + c_1$$

$$\text{or} \quad \frac{dy}{dx} = \sqrt{c_1 + 2 \int f(y) dy}$$

which may further be integrated by any of the standard methods.

NOTES

10.5 KEY WORDS

- Partial differential equation:** Partial Differential Equation (PDE) is a differential equation that contains beforehand unknown multivariable functions and their partial derivatives.
- Linear equation:** A linear equation is an equation that may be put in the form where are the variables, and are the coefficients, which are often real numbers.
- Complementary function:** The complementary function is the general solution of the differential equation and a particular integral is any solution (i.e., the function of x) that satisfies the differential equation.

10.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short Answer Questions

1. What is partial differential equation?
2. Write a note on linear equations with constant coefficient.
3. Write the exact differential equation of other special type.
4. Brief a note on transformation of equations by changing the independent variable.

Long Answer Questions

1. Prove that $\frac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x$
or $(D^2 - I)y = x \sin x + (1 + x^2)e^x$
2. Explain linear equations with variable coefficients (Homogeneous Linear Equations)
3. Prove that $\sin^2 x \frac{d^2y}{dx^2} = 2y$
4. Explain complete solutions in terms of a known integral.
5. Explain transformation of equation.
6. Prove that $\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + \frac{a^4}{x^4}y = 0$.

10.7 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 11 METHODS OF INTEGRAL TRANSFORMS

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Structure

- 11.0 Introduction
- 11.1 Objectives
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 - 11.2.1 Deduction of the Definition of the Laplace Transform from that of the Integral Transform
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 - 11.2.3 Some Methods for Finding Laplace Transforms
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 - 11.2.6 The Fourier Transforms
 - 11.2.7 Definition of Infinite Hankel Transform
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 - 11.2.9 Hankel Transforms of $\frac{d^2f}{dx^2}$, $\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx}$ and $\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{n^2}{x^2} f$ under Certain Conditions
- 11.3 Answers to Check Your Progress Questions
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- 11.7 Further Readings

11.0 INTRODUCTION

In mathematics, an integral transform maps an equation from its original domain into another domain where it might be manipulated and solved much more easily than in the original domain. The solution is then mapped back to the original domain using the inverse of the integral transform.

As an example of an application of integral transforms, consider the Laplace transform. This is a technique that maps differential or integro-differential equations in the time domain into polynomial equations in what is termed the complex frequency domain. (Complex frequency is similar to actual, physical frequency but rather more general. Specifically, the imaginary component ω of the complex frequency $s = -\sigma + i\omega$ corresponds to the usual concept of frequency, viz., the rate at which a sinusoid cycles, whereas the real component σ of the complex frequency corresponds to the degree of 'damping', i.e., an exponential decrease of the amplitude.) The equation cast in terms of complex frequency is readily solved in the complex frequency domain (roots of the polynomial equations in the complex frequency domain correspond to eigenvalues in the time domain), leading to a 'solution' formulated in the frequency domain. Employing the inverse transform, i.e., the inverse procedure of the original Laplace transform, one obtains

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a time- domain solution. In this example, polynomials in the complex frequency domain (typically occurring in the denominator) correspond to power series in the time domain, while axial shifts in the complex frequency domain correspond to damping by decaying exponentials in the time domain.

The Laplace transform finds wide application in physics and particularly in electrical engineering, where the characteristic equations that describe the behavior of an electric circuit in the complex frequency domain correspond to linear combinations of exponentially damped, scaled, and time-shifted sinusoids in the time domain. Other integral transforms find special applicability within other scientific and mathematical disciplines.

In this unit, you will study about integral transformation, methods of integral transformation and various examples based on integral transformation and its methods in detail.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what integral transformation is
- Explain methods of integral transformation
- Discuss various examples based on integral transformation and its methods

11.2 METHODS OF INTEGRAL TRANSFORMATIONS

All transforms as Laplace transform, Fourier-transform and Hankel transform are included in the term Integral transform and we define it as follows:

If there is a known function $K(\alpha, x)$ of two variables α and x such that the integral

$$\int_0^\infty K(\alpha, x) F(x) dx \quad \dots (11.1)$$

is convergent, then the integral (11.1) is termed as the Integral transform of the function $F(x)$ and is denoted by $\tilde{F}(x)$ or $T\{F(x)\}$, i.e.,

$$\tilde{F}(x) = T\{F(x)\} = \int_0^\infty K(\alpha, x) F(x) dx \quad \dots (11.2)$$

The function $K(\alpha, x)$ introduced here is sometimes known as the *Kernel* of the transformation and α is a parameter (real or complex) independent of x .

11.2.1 Deduction of the Definition of the Laplace Transform from that of the Integral Transform

The Integral transform of $F(x)$ can be defined as

$$T\{F(x)\} = \int_0^\infty K(\alpha, x) F(x) dx \quad \dots (11.3)$$

where $K(\alpha, x)$ is the Kernel of the transformation.

If we take the Kernel,

$$\left. \begin{aligned} K(\alpha, x) &= K(s, t) = 0 \text{ for } t < 0 \\ &= e^{-st} \text{ for } t \geq 0 \end{aligned} \right\} \quad \dots (11.4)$$

then the transform

$$T\{F(x)\} = \int_0^\infty e^{-st} F(t) dt \text{ for } t \geq 0 \quad \dots (11.5)$$

is known as the *Laplace transform*.

11.2.2 Definition of the Laplace Transform

If $F(t)$ be a function of t defined for all positive values of t (i.e., $t \geq 0$), then the Laplace transform of $F(t)$ denoted by $L\{F(t)\}$ or $\tilde{F}(s)$ or $f(s)$ is defined by the expression

$$L\{F(t)\} = \tilde{F}(t) = f(s) = \int_0^\infty e^{-st} F(t) dt \quad \dots (11.6)$$

where s is a parameter (real or complex).

If the integral $\int_0^\infty e^{-st} F(t) dt$ converges for some value of s , then the Laplace transform of $F(t)$ is said to exist, otherwise it does not exist.

Example 11.1: Find the Laplace transform of the following functions:

$$(i) F(t) = 1$$

$$(ii) F(t) = t$$

$$(iii) F(t) = t^n, n = 0, 1, 2, 3, \dots$$

By definition of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt \quad \dots (11.7)$$

(i) when $F(t) = 1$, (11.7) becomes

$$L\{1\} = \int_0^\infty e^{-st} 1 dt = \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{s}, s > 0$$

(ii) when $F(t) = t$, (1) gives

$$\begin{aligned} L\{t\} &= \int_0^\infty e^{-st} \cdot t dt \\ &= \left[\frac{e^{-st}}{-s} \cdot t \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt, (\text{integrating by parts}). \\ &= 0 + \frac{1}{s} \int_0^\infty e^{-st} dt = \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{s^2}, s > 0 \end{aligned}$$

(iii) when $F(t) = t^n$, the transform (11.7) reduces to

$$\begin{aligned} L\{t^n\} &= \int_0^\infty e^{-st} t^n dt \\ &= \left[\frac{e^{-st}}{-s} \cdot t^n \right]_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} \cdot t^{n-1} dt, (\text{integrating by parts}). \end{aligned}$$

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$$= 0 + \frac{n}{s} \left[\frac{e^{-st}}{-s} t^{n-1} \right]_0^\infty + \frac{n(n-1)}{s^2} \int_0^\infty e^{-st} \cdot t^{n-2} dt, \text{ (integrating by parts)}$$

$$= \frac{n(n-1)(n-2)}{s^3} \int_0^\infty e^{-st} \cdot t^{n-3} dt$$

(repeating the process of integration by parts,

whence first integral vanishes for both the limits)

$$= \dots$$

$$= \frac{n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1}{s^n} \int_0^\infty e^{-st} \cdot t^0 dt \text{ where } t^0 = 1$$

$$= \frac{|n|}{s^n} \cdot \frac{1}{s} = \frac{|n|}{s^{n+1}}, \quad s > 0. \text{ or } \frac{\sqrt{(n+1)}}{s^{n+1}}, \quad s > 0$$

Example 11.2: Find the Laplace transform of e^{at} .

Here $F(t) = e^{at}$

Hence by definition of Laplace transform, we have

$$\begin{aligned} L\{e^{at}\} &= \int_0^\infty e^{-st} \cdot e^{at} dt \\ &= \int_0^\infty e^{-(s-a)t} dt \\ &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty \\ &= \frac{1}{s-a}, \quad s > a \text{ for which the integral converges,} \\ &\qquad\qquad\qquad \text{otherwise it diverges for } s \leq a. \end{aligned}$$

Example 11.3: Find the Laplace transform of $\sin at$ and $\cos at$.

We have, by definition of Laplace transform

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

when $F(t) = \sin at$, then

$$\begin{aligned} L\{\sin at\} &= \int_0^\infty e^{-st} \sin at dt \\ &= \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty \\ &\quad \left[\text{Or } \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right] \\ &= \frac{a}{s^2 + a^2}, \quad s > 0 \end{aligned}$$

Again when $F(t) = \cos at$, then

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at dt$$

$$\begin{aligned}
 &= \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^\infty \\
 &\quad \left[: \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right] \\
 &= \frac{s}{s^2 + a^2}, \quad s > 0
 \end{aligned}$$

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Example 11.4: Find the Laplace transform of $\sinh at$ and $\cosh at$.

We have, by definition of Laplace transform

$$\begin{aligned}
 L\{\sinh at\} &= \int_0^\infty e^{-st} \sinh at dt \\
 &= \frac{1}{2} \int_0^\infty e^{-st} (e^{at} - e^{-at}) dt \\
 &= \frac{1}{2} \left[\int_0^\infty e^{-st} \cdot e^{at} dt - \int_0^\infty e^{-st} \cdot e^{-at} dt \right] \\
 &= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] \text{ by Example 11.2} \\
 &= \frac{a}{s^2 - a^2}, \quad s > |a|
 \end{aligned}$$

and $L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at dt$

$$\begin{aligned}
 &= \frac{1}{2} \int e^{-st} (e^{at} + e^{-at}) at \\
 &= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] \text{ by Example 11.2} \\
 &= \frac{s}{s^2 - a^2}, \quad s > |a|
 \end{aligned}$$

Example 11.5: Find the Laplace transform of the following functions:

(i) $F(t) = t \sin at$

(ii) $F(t) = t \cos at$

By definition, the Laplace transform of a function $F(t)$ is given by

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

(i) when $F(t) = t \sin at$, we have

$$\begin{aligned}
 L\{t \sin at\} &= \int_0^\infty e^{-st} \cdot t \sin at dt \\
 &= \left[t \cdot \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty \\
 &\quad - \int_0^\infty \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) dt
 \end{aligned}$$

[on integrating by parts, treating t as first function]

and

$e^{-st} \sin at$ as the second function and using the result

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx),$$

while the first integral vanishes for both the limits]

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$$\begin{aligned} &= \frac{s}{s^2 + a^2} \int_0^\infty e^{-st} at \sin dt + \frac{a}{s^2 + a^2} \int_0^\infty e^{-st} \cos at dt \\ &= \frac{s}{s^2 + a^2} \cdot \frac{a}{s^2 + a^2} + \frac{a}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2} \text{ by Example 11.3 for } s > 0 \\ &= \frac{2as}{(s^2 + a^2)^2}, \quad s > 0. \end{aligned}$$

(ii) when $F(t) = t \cos at$, we have

$$L\{t \cos at\} = \int_0^\infty e^{-st} \cdot t \cos at dt$$

On R.H.S., integrating by parts treating t as first function and $e^{-st} \cos at$ as second function and using the result $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$, while the first integral vanishes for both the limits, we are left with

$$\begin{aligned} L\{t \cos at\} &= \frac{s}{s^2 + a^2} \int_0^\infty e^{-st} \cos at dt - \frac{a}{s^2 + a^2} \int_0^\infty e^{-st} \sin at dt \text{ (as in (i))} \\ &= \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2} - \frac{a}{s^2 + a^2} \cdot \frac{a}{s^2 + a^2} \text{ by Problem 3 for } s > 0 \\ &= \frac{s^2 - a^2}{(s^2 + a^2)^2}, \quad s > 0. \end{aligned}$$

Example 11.6: Find the Laplace transform of t^a , where a is positive but not necessarily an integer.

Hint: Proceed just like in Example 11.1 (iii) and get the result $\frac{\Gamma(a+1)}{s^{n+1}}$, $s > 0$ since if a is not an integer, then $\lfloor a \rfloor$ is not defined.

Example 11.7: Find the Laplace transform of e^{-at}

Ans. $\frac{1}{s+a}$ (replace a by $-a$ in Example 11.2)

Example 11.8: Find the Laplace transform of the following functions.

$$(i) F(t) = e^{at} \sin bt \quad \text{Ans. } b/\{(s-a)^2 + b^2\}$$

$$(ii) F(t) = e^{at} \cos bt$$

Ans. $(s-a)/\{(s-a)^2 + b^2\}$

(Meerut 1981, 85, 86)

$$\begin{aligned} \text{Hint. } L\{e^{at} \sin bt\} &= \int_0^\infty e^{-st} \cdot e^{at} \sin bt dt \\ &= \int_0^\infty e^{-(s-a)t} \sin bt dt \\ &= \left[\frac{e^{-(s-a)t}}{(s-a)^2 + b^2} \{-(s-a) \sin bt - b \cos bt\} \right]_0^\infty \end{aligned}$$

Note. The results so far derived alongwith a few more can be tabulated as following:

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$F(t)$	$L\{F(t)\} = f(s) = \tilde{F}(s)$
1	$1/s, s > 0$
t	$1/s^2, s > 0$
$t^n, n = 0, 1, 2\dots$	\underline{n}/s^{n+1} or $\Gamma(n+1)/s^{n+1}, s > 0$
$t^a, a > 0$ but not necessarily an integer	$\Gamma(a+1)/s^{a+1}, a > 0$
$e^{at}; e^{-at}$	$\frac{1}{s-a}; \frac{1}{s+a}, s > a$
$\sin at$	$a/(s^2 + a^2), s > 0$
$\cos at$	$s/(s^2 + a^2), s > 0$
$\sinh at$	$a/(s^2 - a^2), s > 0$
$\cosh at$	$s/(s^2 - a^2), s > 0$
$t \sin at$	$2as/(s^2 + a^2)^2, s > 0$
$t \cos at$	$(s^2 - a^2)/(s^2 + a^2)^2$
$e^{at} \sin bt$	$b/\{(s-a)^2 + b^2\}$
$e^{at} \cos bt$	$(s-a)/\{(s-a)^2 + b^2\}$
$t^{n-1} e^{at}, n > 0$	$\underline{n}/(s-a)^n$
$J_0(at)$ and $t J_0(at)$	$1/(s^2 + a^2)$ and $s/(s^2 + a^2)^{3/2}$

11.2.3 Some Methods for Finding Laplace Transforms

[1] Direct Method

This is based on the definition of Laplace transforms, for example

$$\begin{aligned}
 L\{(t^2 + 1)^2\} &= L\{t^4 + 2t^2 + 1\} = L\{t^4\} + 2L\{t^2\} + L\{1\} \\
 &= \int_0^\infty e^{-st} t^4 \cdot dt + 2 \int_0^\infty e^{-st} \cdot t^2 dt + \int_0^\infty e^{-st} \cdot 1 \cdot dt \\
 &= \frac{|4|}{s^5} + 2 \cdot \frac{|2|}{s^3} + \frac{1}{s} = \frac{24 + 4s^2 + s^4}{s^5}
 \end{aligned}$$

[2] Series-Expansion Method

If the function $F(t)$ is expressible as a Power series, for example

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$$F(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

then the Laplace transform of $F(t)$ is obtained by taking Laplace transforms of each term in the series, for example

$$\begin{aligned} L\{\sin \sqrt{t}\} &= L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3} + \frac{(\sqrt{t})^5}{5} - \frac{(\sqrt{t})^7}{7} + \dots\right\} \\ &= L\{t^{1/2}\} - \frac{1}{3} L\{t^{3/2}\} + \frac{1}{5} L\{t^{5/2}\} - \frac{1}{7} L\{t^{7/2}\} + \dots \\ &= \frac{\sqrt{2}}{s^{3/2}} - \frac{\sqrt{5/2}}{3 s^{5/2}} + \frac{\sqrt{7/2}}{5 s^{7/2}} - \frac{\sqrt{9/2}}{7 s^{9/2}} + \dots \\ &= \frac{\sqrt{\pi}}{2s^{3/2}} \left[1 - \frac{1}{2^2 s} + \frac{1}{2} \left(\frac{1}{2^2 s} \right)^2 - \frac{1}{3} \left(\frac{1}{2^2 s} \right)^3 + \dots \right] \\ &= \frac{\sqrt{\pi}}{2s^{3/2}} \left[e^{-\frac{1}{2^2 s}} \right] = \frac{e^{-\frac{1}{4s}} \sqrt{\pi}}{2s^{3/2}} \end{aligned} \quad \dots (11.8)$$

[3] Method of Differential Equations

If a differential equation satisfied by the function $F(t)$ can be determined, then its Laplace transform may be evaluated by using the properties of Laplace transforms, for example

$$\begin{aligned} \text{if } F(t) = \sin \sqrt{t} \text{ then } F'(t) &= \frac{1}{2\sqrt{t}} \cos \sqrt{t} \\ \text{and } F''(t) &= \frac{1}{2} \left[-\frac{1}{2t^{3/2}} \cos \sqrt{t} - \frac{1}{2t} \sin \sqrt{t} \right] = \frac{1}{2t} \left[-F'(t) - \frac{1}{2} F(t) \right] \\ \text{i.e.,} \quad 4t F''(t) + 2F'(t) + F(t) &= 0 \quad \dots (11.9) \\ \text{which is clearly satisfied by } F(t) = \sin \sqrt{t} \end{aligned}$$

Assuming that $L\{F(t)\} = L\{\sin \sqrt{t}\} = f(s)$, then as per the standard notation,

$$\begin{aligned} L\{4t F''(t)\} &= 4L\{tF''(t)\} \\ &= -4 \frac{d}{ds} \left[s^2 f(s) - s F(0) - F(0) \right] \\ &= -8s f(s) - 4s^2 f'(s) - 4F(0) \\ L\{2F'(t)\} &= 2L\{f'(t)\} = 2[s f(s) - F(0)] \\ &= 2 s f(s) - 2F(0) \end{aligned}$$

and $L\{F(t)\} = f(s)$

\therefore Taking Laplace transform of (11.9), we get

$$L\{4t F''(t)\} + L\{2F'(t)\} + L\{F(t)\} = 0$$

$$\text{i.e., } -8s f(s) - 4s^2 f'(s) - 4F(0) + 2s f(s) - 2F(0) + f(s) = 0$$

$$\text{or } 4s^2 f'(s) + (6s - 1)f(s) = 0 \text{ since } F(0) = \sin \sqrt{0} = 0$$

$$\text{or } \frac{f'(s)}{f(s)} = \frac{1-6s}{4s^2} = \frac{1}{4s^2} - \frac{3}{2s}$$

Integrating with regard to 's', we find

$$\log f(s) = \log C - \frac{1}{4s} - \frac{3}{2} \log s, \text{ } C \text{ being constant of integration}$$

$$= \log \frac{C}{s^{3/2}} e^{-\frac{1}{4s}}$$

i.e., $f(s) = \frac{C e^{-\frac{1}{4s}}}{s^{3/2}}$... (11.10)

Now to determine C , apply the limits of initial-value theorem, i.e., when $t \rightarrow 0, s \rightarrow \infty$.

For t small, $\sin \sqrt{t} = \sqrt{t}$ so that $L\{\sqrt{t}\} = \frac{[3/2]}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

and for s large, $f(s) = \frac{C}{s^{3/2}}$ as $\lim_{s \rightarrow \infty} e^{-\frac{1}{4s}} = 1$

$\therefore L\{F(t) = f(s)\}$ gives for $t \rightarrow 0, s \rightarrow \infty$,

$$\frac{\sqrt{\pi}}{2s^{3/2}} = \frac{C}{s^{3/2}}, \text{ i.e., } C = \frac{\sqrt{\pi}}{2}.$$

$$\text{Hence } f(s) = \frac{\sqrt{\pi} e^{-\frac{1}{4s}}}{2s^{3/2}}. \quad \dots (11.11)$$

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[4] Method of Differentiation with Respect to a Parameter

This method is based on differentiation of Laplace transform of a known function w.r.t. a parameter, for example, if

$F(t) = t \sin at$ then we have to find $L\{t \sin at\}$

$$\text{Consider } L\{\cos at\} = \int_0^\infty e^{-st} \cos at dt = \frac{s}{s^2 + a^2}$$

Differentiating w.r.t. a , we get

$$\frac{d}{da} \int_0^\infty e^{-st} \cos at dt = \frac{d}{da} \left(\frac{s}{s^2 + a^2} \right)$$

$$\text{i.e., } \int_0^\infty e^{-st} (-t \sin at) dt = \frac{-2as}{(s^2 + a^2)^2}$$

or

$$+ L\{t \sin at\} = \frac{+2as}{(s^2 + a^2)^2} \quad \dots (11.12)$$

11.2.4 Fourier's Integral

If $f(x)$ satisfies the Dirichlet's condition in $-\pi \leq x \leq \pi$, and $\int_{-\infty}^{\infty} f(x) dx$ converges, i.e., is integrable in $-\infty < x < \infty$, then we have the Fourier series expansion for $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \quad \dots (11.13)$$

$$\begin{aligned} \text{where } a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \\ \text{and } b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \end{aligned} \quad \left. \right\} \quad \dots (11.14)$$

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This function of x may be developed into a trigonometric series for all values of x between $x = -c$ and $x = c$ by putting

$$z = \frac{\pi}{c}x, \text{ where } z = -\pi \text{ when } x = -c$$

and $z = \pi \text{ when } x = c,$

$$\text{i.e., } f(x) = f\left(\frac{c}{\pi}z\right)$$

Then the series (11.13) may be developed in terms of z as

$$f\left(\frac{c}{\pi}z\right) = a_0 + a_1 \cos z + a_2 \cos 2z + a_3 \cos 3z + \dots + b_1 \sin z + b_2 \sin 2z + b_3 \sin 3z + \dots \quad (11.15)$$

$$\text{where } \left. \begin{array}{l} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right) dz, \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right) \cos nz dz, \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right) \sin nz dz. \end{array} \right\} \dots (11.16)$$

$$\text{and}$$

Now if we replace z by $\frac{\pi}{c}x$, then (11.15) becomes

$$f(x) = a_0 + a_1 \cos \frac{\pi x}{c} + a_2 \cos \frac{2\pi x}{c} + a_3 \cos \frac{3\pi x}{c} + \dots + b_1 \sin \frac{\pi x}{c} + b_2 \sin \frac{2\pi x}{c} + b_3 \sin \frac{3\pi x}{c} + \dots \quad (11.17)$$

Its coefficients being the same as those of (11.15) is therefore valid from $x = -c$ to $x = c$, where

$$\left. \begin{array}{l} a_0 = \frac{1}{2\pi} \int_{-c}^c f(x) \cdot \frac{\pi}{c} dx = \frac{1}{2c} \int_{-c}^c f(x) dx \quad \text{if when } z = \frac{\pi}{c}x, dz = \frac{\pi}{c}dx \\ = \frac{1}{2c} \int_{-c}^c f(t) dt \text{ (say)} \\ a_n = \frac{1}{\pi} \int_{-c}^c f(x) \cos \frac{n\pi x}{c} \cdot \frac{\pi}{c} dx \\ = \frac{1}{c} \int_{-c}^c f(t) \cos \frac{n\pi t}{c} dt \\ b_n = \frac{1}{\pi} \int_{-c}^c f(x) \sin \frac{n\pi x}{c} \cdot \frac{\pi}{c} dx \\ = \frac{1}{2c} \int_{-c}^c f(t) \sin \frac{n\pi t}{c} dt. \end{array} \right\} \dots (11.18)$$

Now if we substitute the values of the coefficients $a_0, a_1, a_2, a_3, \dots, b_1, b_2, b_3, \dots$ given by (11.17) in (11.18), then we get

$$\begin{aligned} f(x) &= \frac{1}{2c} \int_{-c}^c f(t) dt + \frac{1}{c} \int_{-c}^c f(t) \cos \frac{\pi t}{c} dt + \frac{1}{c} \int_{-c}^c f(t) \cos \frac{2\pi t}{c} \cos \frac{2\pi x}{c} dt + \dots \\ &\quad + \frac{1}{c} \int_{-c}^c f(t) \sin \frac{\pi t}{c} \sin \frac{\pi x}{c} dt + \frac{1}{c} \int_{-c}^c f(t) \sin \frac{2\pi t}{c} \sin \frac{2\pi x}{c} dt + \dots \\ &= \frac{1}{c} \int_{-c}^c f(t) \left[\frac{1}{2} + \cos \frac{\pi t}{c} \cos \frac{\pi x}{c} + \cos \frac{2\pi t}{c} \cos \frac{2\pi x}{c} + \dots \right] dt \end{aligned}$$

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$$\begin{aligned}
 & + \sin \frac{\pi t}{c} \sin \frac{\pi x}{c} + \sin \frac{2\pi t}{c} \sin \frac{2\pi x}{c} + \dots \Big] dt \\
 & = \frac{1}{c} \int_{-c}^c f(t) \left[\frac{1}{2} + \left\{ \cos \frac{\pi t}{c} \cos \frac{\pi x}{c} + \sin \frac{\pi t}{c} \sin \frac{\pi x}{c} \right\} \right. \\
 & \quad \left. + \left\{ \cos \frac{2\pi t}{c} \cos \frac{2\pi x}{c} + \sin \frac{2\pi t}{c} \sin \frac{2\pi x}{c} \right\} + \dots \right] dt \\
 & = \frac{1}{c} \int_{-c}^c f(t) \left[\frac{1}{2} + \cos \frac{\pi}{c} (x-t) + \cos \frac{2\pi}{c} (x-t) + \dots \right] dt \\
 & = \frac{1}{2c} \int_{-c}^c f(t) \left[1 + 2 \cos \frac{\pi}{c} (x-t) + 2 \cos \frac{2\pi}{c} (x-t) + \dots \right] dt \\
 & = \frac{1}{2\pi} \int_{-c}^c f(t) \left[\frac{\pi}{c} \cos \frac{0\pi}{c} (x-t) + \frac{\pi}{c} \cos \frac{2\pi}{c} (x-t) \right. \\
 & \quad \left. + \frac{\pi}{c} \cos \left(\frac{-\pi}{c} \right) (x-t) + \frac{\pi}{c} \cos \frac{2\pi}{c} (x-t) + \frac{\pi}{c} \cos \left(\frac{-2\pi}{c} \right) (x-t) + \dots \right] dt \\
 & \quad \text{since } 2 \cos \phi = \cos \phi + \cos(-\phi) \\
 & = \frac{1}{2\pi} \int_{-c}^c f(t) \left[\dots \frac{\pi}{c} \cos \left(-\frac{n\pi}{c} \right) (x-t) + \dots + \frac{\pi}{c} \cos \left(-\frac{2\pi}{c} \right) (x-t) + \frac{\pi}{c} \cos \left(-\frac{\pi}{c} \right) (x-t) \right. \\
 & \quad \left. + \frac{\pi}{c} \cos \frac{0\pi}{c} (x-t) + \frac{\pi}{c} \cos \frac{\pi}{c} (x-t) + \frac{\pi}{c} \cos \frac{2\pi}{c} (x-t) + \dots \right. \\
 & \quad \left. + \frac{\pi}{c} \cos \frac{n\pi}{c} (x-t) + \dots \right] dt \\
 & = \frac{1}{2\pi} \int_{-c}^c f(t) \left[\lim_{n \rightarrow \infty} \sum_{r=-n}^{r=n} \frac{\pi}{c} \cos \frac{r\pi}{c} (x-t) \right] dt \\
 & = \frac{1}{2\pi} \int_{-c}^c f(t) \left[\lim_{n \rightarrow \infty} \sum_{r=-n}^{r=n} \frac{1}{c/\pi} \cos \frac{r}{c/\pi} (x-t) \right] dt.
 \end{aligned}$$

If c becomes indefinitely large, i.e., as $c \rightarrow \infty$, $\frac{c}{\pi} \rightarrow \infty$, we have

$$\lim_{c \rightarrow \infty} \sum_{r=-\infty}^{\infty} \frac{1}{c/\pi} \cos \frac{r}{c/\pi} (x-t) = \int_{-\infty}^{\infty} \cos u (x-t) du$$

(by the definition of integral as the limit of a sum).

$$\text{Hence } f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) dt \int_{-\infty}^{\infty} \cos u (x-t) du. \quad \dots (11.19)$$

This double integral is known as *Fourier's Integral* and holds if x is a point of continuity of $f(x)$.

$$\text{Aliter. We have } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \dots (11.20)$$

$$\text{where } a_n = \frac{1}{l} \int_{-l}^l f(u) \cos \frac{n\pi u}{l} du \text{ and } b_n = \frac{1}{l} \int_{-l}^l f(u) \sin \frac{n\pi u}{l} du \quad \dots (11.21)$$

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so that $a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} = \frac{1}{\pi} \int_{-l}^l f(u) \cos \frac{n\pi}{l}(u-x) du$

and $\frac{a_0}{2} = \frac{1}{2l} \int_{-l}^l f(u) du$

∴ (11.20) gives

$$f(x) = \frac{1}{2l} \int_{-l}^l f(u) du + \frac{1}{l} \sum_{n=1}^{\infty} f(u) \cos \frac{n\pi}{l}(u-x) du \quad \dots (11.22)$$

Assuming that $\int_{-\infty}^{\infty} |f(u)| du$ converges, the first term on R.H.S. of (11.22) approaches zero as $l \rightarrow \infty$ and hence (11.22) yields

$$f(x) = \lim_{l \rightarrow \infty} \frac{1}{l} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(u) \cos \frac{n\pi}{l}(u-x) du \quad \dots (11.23)$$

Putting $\frac{\pi}{l} = \Delta t$, (11.23) can be written as

$$f(x) = \lim_{\Delta t \rightarrow 0} \sum_{n=1}^{\infty} \Delta t F(n\Delta t) \quad \dots (11.24)$$

$$\text{where } F(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos t(u-x) du \quad \dots (11.25)$$

Thus (11.24) gives

$$f(x) = \int_0^{\infty} F(t) dt = \frac{1}{\pi} \int_0^{\infty} dt \int_{-\infty}^{\infty} f(u) \cos t(u-x) du \quad \dots (11.26)$$

which is Fourier's Integral formula.

Note: The complex form of Fourier's integral is

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} dt \int_{-\infty}^{\infty} f(u) e^{itu} du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{it(x-u)} du dt \end{aligned} \quad \dots (11.27)$$

11.2.5 Different Forms of Fourier's Integrals

Fourier's Integral is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) dt \int_{-\infty}^{\infty} \cos u(x-t) du \quad \dots (11.28)$$

$$\text{Here } \int_{-\infty}^{\infty} \cos u(x-t) du = \int_{-\infty}^0 \cos u(x-t) dt + \int_0^{\infty} \cos u(x-t) dt$$

$$\text{Say } I = I_1 + I_2$$

Replacing u by $-u$ in I_1 , we get

$$I_1 = - \int_{\infty}^0 \cos u(x-t) du = \int_0^{\infty} \cos u(x-t) du = I_2.$$

$$\text{Here } I = \int_{-\infty}^{\infty} \cos u(x-t) du = 2 \int_0^{\infty} \cos u(x-t) du.$$

Therefore (11.28) becomes

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) dt \int_0^{\infty} \cos u (x-t) du. \quad \dots (11.29)$$

Now as the limits of integration in (11.29) do not involve the variables u or t , the order of integration may be changed, i.e.,

$$f(x) = \frac{1}{\pi} \int_0^{\infty} du \int_{-\infty}^{\infty} f(t) \cos u (x-t) dt. \quad \dots (11.30)$$

If $f(x)$ be an odd function of x , i.e., $f(-x) = -f(x)$, then

$$\int_{-\infty}^{\infty} f(t) \cos u (x-t) dt = \int_{-\infty}^0 f(t) \cos u (x-t) dt + \int_0^{\infty} f(t) \cos u (x-t) dt.$$

Replacing t by $-t$, we have

$$\begin{aligned} \int_{-\infty}^0 f(t) \cos u (x-t) dt &= - \int_{\infty}^0 f(-t) \cos u (x+t) dt \\ &= - \int_0^{\infty} f(t) \cos u (x+t) dt \end{aligned}$$

$\therefore f(-t) = -f(t)$.

Thus $\int_{-\infty}^{\infty} f(t) \cos u (x-t) dt$

$$\begin{aligned} &= - \int_0^{\infty} f(t) \cos u (x+t) dt + \int_0^{\infty} f(t) \cos u (x-t) dt \\ &= \int_0^{\infty} f(t) [\cos u (x-t) - \cos u (x+t)] dt \\ &= \int_0^{\infty} f(t) \sin ux \sin ut dt \end{aligned}$$

Substituting it in (11.30), we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} du \int_0^{\infty} f(t) \sin ut \sin ux dt. \quad \dots (11.31)$$

Changing the order of integration this may be written as

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f(t) dt \int_0^{\infty} \sin ut \sin ux du. \quad \dots (11.32)$$

Again if $f(x)$ be an even function of (x) , i.e., $f(-x) = f(x)$, we have

$$\int_{-\infty}^{\infty} f(t) \cos u (x-t) dt = \int_{-\infty}^0 f(t) \cos u (x-t) dt + \int_0^{\infty} f(t) \cos u (x-t) dt.$$

Replacing t by $-t$ in the first integral on the right, we get

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) \cos u (x-t) dt &= - \int_{\infty}^0 f(-t) \cos u (x+t) dt + \int_0^{\infty} f(t) \cos u (x-t) dt \\ &= \int_0^{\infty} f(t) \cos u (x+t) dt + \int_0^{\infty} f(t) \cos u (x-t) dt \end{aligned}$$

$\therefore f(-t) = f(t)$

$$\begin{aligned} &= \int_0^{\infty} f(t) [\cos u (x+t) + \cos u (x-t)] dt \\ &= 2 \int_0^{\infty} f(t) \cos ut \cos ux dt \end{aligned}$$

Its substitution in (11.30) yields

$$f(x) = \frac{2}{\pi} \int_0^{\infty} du \int_0^{\infty} f(t) \cos ut \cos ux dt \quad \dots (11.33)$$

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$$= \frac{2}{\pi} \int_0^\infty f(t) dt \int_0^\infty \cos ut \cos ux du \quad (\text{On changing the order of integration})$$

Example 11.9: Show that the sum function of the Integral formula is $\frac{1}{2} [f(x+0) + f(x-0)]$ corresponding to the function $f(x)$ in the interval $0 < x < l$.

By Weierstrass test, since $\left| \int_{-\infty}^\infty f(u) \cos t(x-u) du \right| \leq \int_{-\infty}^\infty |f(u)| du$ (which converges), the integral $\int_{-\infty}^\infty f(u) \cos t(x-u) du$ converges absolutely and uniformly for all values of t . Thus by reversing the order of integration, we have,

$$\begin{aligned} \frac{1}{\pi} \int_{t=0}^l dt \int_{u=-\infty}^\infty f(u) \cos t(x-u) du &= \frac{1}{\pi} \int_{u=-\infty}^\infty f(u) du \int_{t=0}^l \cos t(x-u) dt \\ &= \frac{1}{\pi} \int_{u=-\infty}^\infty f(u) \frac{\sin l(u-x)}{u-x} dx \\ &= \frac{1}{\pi} \int_{p=-\infty}^\infty f(x+p) \frac{\sin lp}{p} dp \quad \text{when } u = x+p \\ &= \frac{1}{\pi} \int_{-\infty}^0 f(x+p) \frac{\sin lp}{p} dp + \frac{1}{\pi} \int_0^\infty f(x+p) \frac{\sin lp}{p} dp \\ &= \frac{1}{2} [f(x+0) f(x-0)] \text{ when } l \rightarrow \infty \end{aligned}$$

Example 11.10: Show that $\int_0^\infty \frac{\cos ux}{u^2 + 1} = \frac{\pi}{2} e^{-x}$, $x \geq 0$.

Putting $f(x) = e^{-x}$ in the Fourier's Integral form (11.33), we have

$$\begin{aligned} f(x) = e^{-x} &= \frac{2}{\pi} \int_0^\infty du \int_0^\infty e^{-t} \cos ut \cos ux dt = \frac{2}{\pi} \int_0^\infty \cos ux du \int_0^\infty e^{-t} \cos ut dt \\ &= \frac{2}{\pi} \int_0^\infty \cos ux du \left[\frac{e^{-t}}{1+u^2} (-\cos ut + u \sin ut) \right]_0^\infty \\ &= \frac{2}{\pi} \int_0^\infty \cos \frac{\cos ux}{1+u^2} du \\ \therefore \int_0^\infty \frac{\cos ux}{1+u^2} du &= \frac{\pi}{2} e^{-x} \end{aligned}$$

Note. If $x = 0$, this result reduces to $\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2}$.

11.2.6 The Fourier Transforms

[A] Fourier Sine Transforms: They can be subdivided in two, namely, the infinite Fourier sine transform and the Finite Fourier sine transforms.

[a₁] *The Infinite Fourier sine Transform* of a function $F(x)$ of x such that $0 < x < \infty$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^\infty F(x) \sin nx dx \quad \dots (11.34)$$

Here $F(x)$ is called as the *Inverse Fourier sine transform of $f_s(n)$* and defined as

$$F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx dx \quad \dots (11.35)$$

Thus if $f_s(n) = f_s[F(x)]$, then $F(x) = f_s^{-1}[f_s(n)]$... (11.36)
where f is the symbol for Fourier transform and f^{-1} for its inverse.

Example 11.11: Find the sine transform of e^{-x} .

We have

$$f_s(n) = \int_0^\infty e^{-x} \sin nx dx = \left[\frac{e^{-x}}{1+n^2} (-\sin nx - n \cos nx) \right]_0^\infty = \frac{n}{1+n^2}.$$

Example 11.12: Find the inverse sine transform of $e^{-\lambda n}$.

We have

$$\begin{aligned} f_s^{-1}[e^{-\lambda n}] &= \frac{2}{\pi} \int_0^\infty e^{-\lambda n} \sin nx dx = \frac{2}{\pi} \left[\frac{e^{-\lambda n}}{\lambda^2 + n^2} (-\lambda \sin nx - n \cos nx) \right]_0^\infty \\ &= \frac{2}{\pi} \cdot \frac{x}{\lambda^2 + n^2}. \end{aligned}$$

[a₂] The Finite Fourier sine transform of a function $F(x)$ of x such that $0 < x < l$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx \quad \dots (11.37)$$

In case $l = \pi$, this becomes

$$f_s(n) = \int_0^\pi F(x) \sin nx dx \quad \dots (11.38)$$

and the inversion formula is

$$F(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx \quad \dots (11.39)$$

whence a_n is the coefficient of $\sin nx$ in the expansion of $F(x)$ in a sine series and is given by

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi F(x) \sin nx dx \\ &= \frac{2}{\pi} f_s(n) \text{ by (5)} \end{aligned} \quad \dots (11.40)$$

Example 11.13: Find the Fourier sine transform of $F(x) = x$ such that $0 < x < 2$.

We have $f_s(n) = \int_0^2 F(x) \sin \frac{n\pi x}{2} dx$
as $l = 2$ in the existing case.

$$\begin{aligned} &= \int_0^2 x \sin \frac{n\pi x}{2} dx \\ &= \left[x \cdot \frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2 + \int_0^2 \frac{2}{n\pi} \cos \frac{n\pi x}{2} dx \end{aligned}$$

(on integrating by parts)

$$= \left[\frac{-2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right]_0^2 = -\frac{4}{n\pi} \cos n\pi.$$

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[B] Fourier Cosine Transforms.

They can also be subdivided into two, namely, Infinite and Finite cosine transforms.

[b₁] The Infinite Fourier Cosine Transform of $F(x)$ for $0 < x < \infty$, is defined as

$$f_c(n) = \int_0^\infty F(x) \cos nx dx, n \text{ being a positive integer.} \quad \dots (11.41)$$

Here the function $F(x)$ is called as the *Inverse cosine transform* of $f_c(n)$ and is defined as

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx dx \quad \dots (11.42)$$

$$\text{Thus if } f_c(n) = f_c[F(x)], \text{ then } F(x) = f_c^{-1}[f_c(n)] \quad \dots (11.43)$$

Example 11.14: Find the cosine transform of $x^n e^{-ax}$.

$$\text{We have } \int_0^\infty e^{-ax} \cos nx dx = \frac{a}{a^2 + n^2} \text{ and } f_c(n) = \int_0^\infty x^n e^{-ax} \cos nx dx$$

Differentiating the first relation n times w.r.t. 'a' we find

$$\begin{aligned} \int_0^\infty x^n e^{-ax} \cos nx dx &= (-1)^n \frac{d^n}{da^n} \left(\frac{a}{a^2 + n^2} \right) \\ &= \frac{\underline{n} \cos \left\{ (n+1) \tan^{-1} \frac{n}{a} \right\}}{(a^2 + n^2)^{(n+1)/2}} \text{ by usual method.} \end{aligned}$$

$$\text{Hence } f_c(n) = \frac{\underline{n} \cos \left\{ (n+1) \tan^{-1} \frac{n}{a} \right\}}{(a^2 + n^2)^{(n+1)/2}}$$

Example 11.15: Find $f_c^{-1}\{e^{-\lambda n}\}$

$$\begin{aligned} \text{We have } f_c^{-1}\{e^{-\lambda n}\} &= \frac{2}{\pi} \int_0^\infty e^{-\lambda n} \cos nx dx \\ &= \frac{2}{\pi} \left[\frac{e^{-\lambda n}}{\lambda^2 + n^2} (-\lambda \cos nx + x \sin nx) \right]_0^\infty = \frac{2}{\lambda^2 + n^2} \end{aligned}$$

[b₂] The Finite Fourier cosine transform of $F(x)$ for $0 < x < l$ is defined as

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx \quad \dots (11.44)$$

When $l = \pi$, this becomes

$$f_c(n) = \int_0^\pi F(x) \cos nx dx \quad \dots (11.45)$$

and the inversion formula is

$$F(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{n=1}^{\infty} f_c(n) \cos nx \quad \dots (11.46)$$

$$\text{when } f_c(0) = \int_0^\pi F(x) dx \quad \dots (11.47)$$

Also b_n the coefficient of $\cos nx$ in the expansion of $F(x)$ in a cosine series is given by

$$b_n = \frac{2}{\pi} \int_0^\pi F(x) \cos nx dx = \frac{2}{\pi} f_c(n) \text{ by (11.45)} \quad \dots (11.48)$$

Example 11.16: Find the finite Fourier cosine transform of x .

We have $f_c(n) = \int_0^\pi x \cos nx dx$

$$\begin{aligned} &= \left[\frac{x \sin nx}{n} \right]_0^\pi - \frac{1}{n} \int_0^\pi \sin nx dx \text{ (on integrating by parts)} \\ &= 0 - \frac{1}{n} \left[\frac{\cos nx}{-n} \right]_0^\pi \\ &= \frac{1}{n^2} \{(-1)^n - 1\}, \end{aligned}$$

$n = 1, 2, 3, \dots$

$$\text{But if } n = 0, \quad f_c(0) = \int_0^\pi x dx = \left[\frac{x^2}{2} \right]_0^\pi = \frac{\pi^2}{2}.$$

Note. On the next page are tabulated some useful Fourier sine and cosine transforms in a concise form.

[C] The Complex Fourier Transforms.

The Complex Fourier Transform of a function $F(x)$ for $-\infty < x < \infty$, is defined as

$$f(n) = \int_{-\infty}^{\infty} F(x) e^{inx} dx \quad \dots (11.49)$$

where e^{inx} is said to be the *Kernel* of the transform.

$$\text{The inversion formula is } F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(n) e^{-inx} dn \quad \dots (11.50)$$

$$\begin{aligned} \text{We have } f(n) &= \int_{-1}^1 (1-x^2) e^{inx} dx = \left[(1-x^2) \frac{e^{inx}}{in} + \frac{2}{in} \int_{-1}^1 x e^{inx} dx \right] \\ &\quad \text{(on integrating by parts)} \\ &= 0 + \frac{2}{in} \left[\frac{x e^{inx}}{in} \right]_{-1}^1 - \frac{2}{(in)^2} \int_{-1}^1 e^{inx} dx \\ &= \frac{2}{-n^2} \left[e^{in} + e^{-in} \right] + \frac{2}{n^2} \left[\frac{e^{inx}}{in} \right]_{-1}^1 = -\frac{2}{n^2} (e^{in} + e^{-in}) + \frac{2}{in^3} (e^{in} - e^{-in}) \\ &= -\frac{4}{n^2} \cos n + \frac{4}{n^3} \sin n = -\frac{4}{n^3} (n \cos n - \sin n). \end{aligned}$$

Example 11.18: Find the Complex Fourier transform of $e^{-|x|}$ and then invert it.

$$\text{We have } f(n) = \int_{-\infty}^{\infty} e^{-|x|} e^{inx} dx = \int_{-\infty}^0 e^{(1+in)x} dx + \int_0^{\infty} e^{-(1-in)x} dx$$

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<i>Infinite Fourier Sine Transforms</i>		<i>Finite Fourier Sine Transforms</i>	
$f_s(n) = \int_0^\infty F(x) \sin nx dx$, $F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx dn$	$f_s(n) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx$	$F(x)$	$f_c(n)$
e^{-x}	$n(1 + n^2)$	1	$\frac{1}{n} [1 + (-1)^{n+1}]$
$x e^{-x^2/2}$	$\sqrt{\pi/2} n e^{-x^2/2}$	x	$(-1)^{n+1} \cdot \pi/n$
$\frac{\sin x}{x}$	$\frac{1}{2} \log \left \frac{1+n}{1-n} \right $	$\begin{cases} x, & 0 < x \leq \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$	$\frac{2}{n^2} \sin \frac{n\pi}{2}$
$\frac{x}{x(1-x_0^2)} 2, 0 < x < 1, \nu > -1$	$\frac{-\pi 2^{v-1/2} \Gamma(v+1)}{n^{-1/2} J_{v+3/2}(m)}$	$1 - \frac{x}{\pi}$	$\frac{1}{n}$
$x^{m-1}, 0 < \mu < 1$	$\sqrt{\mu} \sin \left(\frac{\mu \pi}{2} \right) n^{-\mu}$	x^2, \dots	$\frac{\pi^2}{n} (-1)^{n-1} - \frac{2}{n^3} [1 - (-1)^n]$
$x^{\nu'} e^{-\mu_x}$	$\frac{2^v \mu^v (\Gamma v) n}{(\mu^2 + n^2)^{v+1}}$	$x(\pi - x)$	$\frac{2}{n^3} [1 - (-1)^n]$
$\{x^2 - a^2\}_{1/2}, x > a$	$J_0(an)$	x^3, \dots	$(-1)^n \pi [6/n^3 - \pi^2/n]$
		$x(\pi^2 - x^2)$	$(-1)^{n+1} 6\pi/n^3$
		$e^{\lambda x}, \dots$	$\frac{n}{n^2 + \lambda^2} [1 + (-1)^n e^{\lambda \pi}]$
		$\cos \lambda x$	$\frac{n}{n^2 - \lambda^2} [1 - (-1)^n \cos \lambda \pi]$
$\cos mx, m = 1, 2, 3, \dots$			$\begin{cases} \frac{n}{n^2 - m^2} [1 - (-1)^{n+m}], & n \neq m \\ 0, & n = m \end{cases}$
$\sin mx, m = 1, 2, 3, \dots$			$\begin{cases} 0, & n \neq m \\ \frac{n}{\pi/2 - \lambda^2}, & \lambda \neq 0, 1, 2, \dots \end{cases}$
			$\frac{n}{\lambda^2 + n^2}$

Infinite Fourier Cosine Transforms		Finite Fourier Cosine Transforms	
$f_c(n) = \int_0^\infty F(x) \cos nx dx$, $F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx dn$	$f_c(n) = \int_0^\pi F(x) \cos nx dx$, $F(x) = \frac{1}{\pi} f_c(\theta) + \sum_{n=1}^{\infty} f_c(n) \cos nx$	$F(x)$	$f_c(n)$
$\begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases} \dots$	$\frac{\sin na}{n}$	1	$\begin{cases} \pi, & n = 0 \\ 0, & n = 1, 2, 3, \dots \end{cases}$
$x^{\mu-1}, \quad 0 < \mu < 1$	$\sqrt{\mu} n^{-\mu} \sin \frac{\mu\pi}{2}$	$\begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases} \dots$	$\begin{cases} 0, & n = 0 \\ 2/\pi \sin n\pi/2, & n = 1, 2, 3, \dots \end{cases}$
$e^{-x} \dots$	$\frac{1}{1+n^2}$	x	$\begin{cases} \pi^2/2, & n = 0 \\ 1/n^2 [(-1)^n - 1], & n = 1, 2, 3, \dots \end{cases}$
$\operatorname{Sech} x\pi$	$\sqrt{\frac{\pi}{2}} \frac{1}{1+n^4}$	$x^2 \dots$	$\begin{cases} \pi^3/3, & n = 0 \\ 2\pi/n^2 (-1)^n, & n = 1, 2, 3, \dots \end{cases}$
e^{-x^2}	$\sqrt{\frac{\pi}{2}} e^{-n^2/2}$	x^3	$\begin{cases} \pi^4/4, & n = 0 \\ 3\pi^2/n^2 (-1)^n + 6/n^4 [(-1)^n - 1], & n = 1, 2, 3, \dots \end{cases}$
$\cos x, \quad 0 < x < a$ $0, x > a$	$\frac{1}{2} \left[\sin \frac{a(1-n)}{1-n} + \sin \frac{a(1+n)}{1+n} \right]$	$\left(1 - \frac{x}{\pi} \right)^2 \dots$	$\begin{cases} \pi/3, & n = 0 \\ 2/\pi n^2, & n = 1, 2, 3, \dots \end{cases}$
$\sin \frac{x^2}{2} \dots$	$\sqrt{\frac{\pi}{2}} \left[\cos \frac{n^2}{2} - \sin \frac{n^2}{2} \right]$	$e^{\lambda x}$	$\frac{\lambda}{\lambda^2 + n^2} [(-1)^n e^{\lambda x} - 1]$
$\cos \frac{x^2}{2}$	$\sqrt{\frac{\pi}{2}} \left[\cos \frac{n^2}{2} + \sin \frac{n^2}{2} \right]$	$\sin \lambda x \dots$	$\frac{\lambda}{n^2 - \lambda^2} [(-1)^n \cos \lambda \pi - 1], \quad n \neq \lambda$
$(1-x^2)^\nu, \quad 0 < x < 1$ $0, x > 1 \text{ and } \nu > -3/2$	$\sqrt{\pi} 2^{\nu-1/2} \Gamma(\nu+1) \cdot n^{-\nu-1/2} J_{\nu+1/2}(n)$	$\sin mx, \quad m = 1, 2, 3, \dots$	$\begin{cases} 0, & n = m \\ m/(n^2 - m^2) [(-1)^{n+m} - 1], & n \neq m \end{cases}$
		$\frac{\cos \lambda(\pi-x)}{\sinh \pi \lambda}$	$\frac{\lambda}{\lambda^2 + n^2}$

NOTES

Example 11.19: Find the Fourier Complex Transform of

$$F(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$= \frac{1}{1+in} + \frac{1}{1-in} = \frac{2}{1+n^2}$$

so that the inversion formula gives,

$F(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+n^2} e^{inx} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-inx}}{1+n^2} dx$ which may be integrated by contour Integration.

Note. Several other **Complex Fourier Transforms** have been tabulated on the next page.

[D] Parseval's Identity for Fourier Integrals

It is stated as $\int_{-\infty}^{\infty} |F(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(n)|^2 dn$... (11.51)

where $f(n)$ is the Fourier transform of $F(x)$.

Example 11.19. Find the Fourier transform of

$$F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

(Agra, 1982; Kanpur, 1970)

Hence or otherwise evaluate $\int_0^{\infty} \frac{\sin^2 nx}{n^2} dx$.

We have $f(n) = \int_{-\infty}^{\infty} F(x) e^{-inx} dx = \int_{-a}^a 1 \cdot e^{-inx} dx$

$$= \left[\frac{e^{-inx}}{-in} \right]_{-a}^a = \frac{e^{+ina} - e^{-ina}}{in} = \frac{2 \sin na}{n} \text{ for } n \neq 0$$

$$\text{For } n = 0, f(n) = 2 \lim_{n \rightarrow 0} \frac{\sin na}{n} = 2 \lim_{n \rightarrow 0} \frac{1}{n} \left[na - \frac{1}{3} n^3 a^3 + \dots \right] = 2a$$

Now using Parseval's identity, we find

$$\int_{-a}^a 1^2 \cdot dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4 \sin^2 na}{n^2} dn \text{ when } n \neq 0$$

$$\text{i.e., } \frac{1}{2\pi} \cdot 2 \int_0^{\infty} \frac{4 \sin^2 na}{n^2} dn = [x]_{-a}^a = 2a$$

$$\therefore \int_0^{\infty} \frac{\sin^2 nx}{n^2} dx = \frac{\pi x}{2}.$$

[E] Relation between the Fourier Transform of the Derivatives of a Function

If $f(n)$ be the Fourier transform of $F(x)$, then we have to express the Fourier transform of the function $\frac{d^m F}{dx^m}$ in terms of $f(n)$.

We have by the definition of Fourier-transform,

$$f \left[\frac{d^m F}{dx^m} \right] = \int_{-\infty}^{\infty} \frac{d^m F}{dx^m} e^{inx} dx = f^m(n) \text{ (say)} \quad \dots (11.52)$$

so that $f^m(n) = \int_{-\infty}^{\infty} \frac{d^m F}{dx^m} e^{inx} dx = \left[\frac{d^{m-1} F}{dx^{m-1}} e^{inx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (in) e^{inx} \frac{d^{m-1} F}{dx^{m-1}} dx$

NOTES

(on integrating by parts)

$$= -in \int_{-\infty}^{\infty} \frac{d^{m-1} F}{dx^{m-1}} d^{inx} dx,$$

under the assumption $\frac{d^{m-1} F}{dx^{m-1}} \rightarrow 0$ as $|x| \rightarrow \infty$

$$= -in f^{n-1}(n) \text{ by Equation (11.52)} \quad \dots (11.53)$$

Repeating the same process under the assumption

$$\frac{d^r F}{dx^r} \rightarrow 0 \text{ as } |x| \rightarrow \infty, r = 1, 2, 3, \dots (m-1)$$

we get after $(m-1)$ operations,

$$f^m(n) = (-in)^m f(n) \quad \dots (11.54)$$

which follows that the Fourier transform of $\frac{d^m F}{dx^m}$ is $(-in)^m$ times the

Fourier transform of $F(x)$ subject to the condition that $\frac{d^r F}{dx^r} \rightarrow 0$ when $|x| \rightarrow \infty$, for $r = 1, 2, 3, \dots (m-1)$.

By similar procedure we can find a relation between the sine and cosine Fourier transforms of the derivatives of a function, such as

$$f_c^m(n) = \int_0^{\infty} \frac{d^m F}{dx^m} \cos nx dx = \left[\frac{d^{m-1} F}{dx^{m-1}} \cos nx \right]_0^{\infty} + n \int_0^{\infty} \frac{d^{m-1} F}{dx^{m-1}} \sin nx dn$$

integrating by parts

$$= -\alpha_{n-1} + f_s^{m-1}(n) \quad \dots (11.55)$$

Under the assumptions,

$$\frac{d^{m-1} F}{dx^{m-1}} \rightarrow 0 \text{ as } x \rightarrow \infty \text{ and } \frac{d^{m-1} F}{dx^{m-1}} \rightarrow \alpha_{n-1} \text{ as } x \rightarrow 0.$$

$$\text{Similarly, integrating, } f_s^m(n) = \int_0^{\infty} \frac{d^m F}{dx^m} \sin nx dx \\ = -n f_c^{m-1}(n) \quad \dots (11.56)$$

Equation (11.54) and (11.55) yield,

$$f_c^m(n) = -\alpha_{n-1} - n^2 f_c^{m-2}(n) \quad \dots (11.57)$$

Repeating the procedure $f_c^m(n)$ may be expressed as the sum of α' s and either $f_c(n)$ or $f'_c(n)$ or $f_c'(n)$ or $f_c''(n)$. $f_c(n)$ will occur when x is odd and in that case we can write $\alpha_0 + n f_s(n)$ place of $f'_c(n)$. We thus have

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$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-inx} dx$	$f(n) = \int_{-\infty}^x F(x) e^{inx} dx$	$F(x)$	$f(n)$
$\frac{1}{ x }$	$\frac{1}{ n }$	$\frac{\sinh ax}{\sin \pi x}, -\pi < a < \pi$	$\frac{\sinh a}{\cosh n + \cos a}$
$e^{i\omega t}, a < x < b$ $0, x < a, x > b$	$i \frac{e^{i\omega t} - e^{ib(\omega+n)}}{n}$	$\frac{\cosh ax}{\cosh \pi x}, -\pi < a < \pi$	$2 \frac{\cos a / 2 \cosh a / 2}{\cosh n + \cos a}$
$e^{-\lambda x + i\omega x}, x > 0$ $0, x < 0$	$\frac{i}{\omega + n + i\lambda}$	$\frac{\sin \{b(a^2 + x^2)^{1/2}\}}{(a^2 + x^2)^{1/2}} \dots$	$\begin{cases} 0, n > b \\ \pi J_0 \left[a \sqrt{b^2 - n^2} \right], n < b \end{cases}$
$e^{-\lambda x^2}, R(\lambda) > 0$	$\frac{[(\lambda^2 + n^2)^{1/2} + \lambda]^{1/2}}{(\lambda^2 + n^2)^{1/2}}$	$\frac{\cosh \{b(a^2 - x^2)^{1/2}\}}{(a^2 - x^2)^{1/2}}, x < a$ $0, x > a$	$\pi J_0 \left[a \sqrt{n^2 + b^2} \right]$
$\sin \lambda x^2 \dots$	$\sqrt{\frac{\pi}{\lambda}} e^{-\pi^2/4\lambda}$	$\frac{\cos \{b(a^2 - x^2)^{1/2}\}}{\sqrt{a^2 - x^2}}, x < a$ $0, x > a$	$\pi J_0 \left[a \sqrt{n^2 - b^2} \right]$
$\cos \lambda x^2$	$\sqrt{\frac{\pi}{\lambda}} \sin \left(\frac{n^2}{4\lambda} + \frac{\pi}{4} \right)$	$P_n(x), x < 1$ $0, x > 1$	$i^v \sqrt{2} J_{v+1/2}(n)$
$0, x > a$	$\pi J_0(am)$		

$$f_c^{2m}(n) = - \sum_{r=0}^{m-1} (-1)^r \alpha_{2m-2r-1} n^{2r} + (-1)^m n^{2m} f_c(n) \quad \dots (11.58)$$

$$\text{and } f_c^{2m+1}(n) = - \sum_{r=0}^m (-1)^r \alpha_{2m-2r} n^{2r} + (-1)^m n^{2m+1} f_s(n) \quad \dots (11.59)$$

Similar procedure with the help of (11.54) and (11.55), will yield

$$f_s^m(n) = n \alpha_{n-2} - n^2 f_s^{m-2}(n) \quad \dots (11.60)$$

$$f_s^{2m}(n) = - \sum_{r=1}^m (-1)^r n^{2r-1} \alpha_{2n-2r} + (-1)^{m+1} n^{2m} f_s(n) \quad \dots (11.61)$$

$$\text{and } f_s^{2m+1}(n) = - \sum_{r=1}^m (-1)^r n^{2r-1} \alpha_{2n-2r+1} + (-1)^{m+1} n^{2m+1} f_s(n) \quad \dots (11.62)$$

Note 1: The following results are easily deducible

$$\left. \begin{array}{l} (i) \int_0^\infty \frac{d^2 F}{dx^2} \cos nx dx = -n^2 f_c(n) \\ (ii) \int_0^\infty \frac{d^4 F}{dx^4} \cos nx dx = -n^4 f_c(n) \end{array} \right\} \text{when } x=0, \frac{dF}{dx} = \frac{d^3 F}{dx^3} = 0 \quad \dots (11.63)$$

$$\left. \begin{array}{l} (iii) \int_0^\infty \frac{d^2 F}{dx^2} \sin nx dx = n^2 f_s(n) \\ (iv) \int_0^\infty \frac{d^4 F}{dx^4} \sin nx dx = n^4 f_c(n) \end{array} \right\} \text{when } x=0, \frac{dF}{dx} = \frac{d^2 F}{dx^2} = 0 \quad \dots (11.64)$$

$$(v) \int_0^\infty \frac{\partial F}{\partial t} \sin nx dx = \frac{\partial}{\partial t} \int_0^\infty F \sin nx dx = \frac{\partial f}{\partial t} \quad \dots (11.65)$$

Note 2. In case the transforms are finite, then consider

$$\int_0^\pi \frac{\partial F}{\partial x} \sin nx dx = [F(x) \sin nx]_0^\pi - n \int_0^\pi F(x) \cos nx dx, \text{ integrating by parts}$$

$$= -n f_c(n) \quad \dots (11.66)$$

under the assumption that $F(0)$ and $F(\pi)$ both are finite.

$$\begin{aligned} \text{Similarly, } \int_0^\pi \frac{\partial F}{\partial x} \cos nx dx &= [F(x) \cos nx]_0^\pi + n \int_0^\pi F(x) \sin nx dx \\ &= (-1)^n F(\pi) - F(0) + n f_s(n) \dots (11.67) \end{aligned}$$

Assuming that $F(x) \rightarrow 0$ at $x = \pi$ and at $x = 0$, (34) reduces to

$$\int_0^\pi \frac{\partial F}{\partial x} \cos nx dx = n f_s(n) \quad \dots (11.68)$$

and (11.66) reduces to

$$\begin{aligned} \int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx dx &= -n \int_0^\pi \frac{\partial F}{\partial x} \cos nx dx \\ &= n [(-1)^{n+1} F(\pi) + F(0)] - n^2 f_s(n) \text{ by (11.67)} \quad \dots (11.69) \end{aligned}$$

If $F(0) = F(\pi) = 0$, then (36) yields,

$$\int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx dx = -n^2 f_s(n) \quad \dots (11.70)$$

Similarly (11.67) yields

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$$\int_0^\pi \frac{\partial^2 F}{\partial x^2} \cos nx dx = (-1)^n F'(\pi) - F'(0) - n^2 f_c(n) \quad \dots (11.71)$$

In case $\frac{\partial^2 F}{\partial x^2}$ vanishes at $x = 0$ and at $x = \pi$, it is easy to see that (11.64) gives

$$\int_0^\pi \frac{\partial^4 F}{\partial x^4} \sin nx dx = -n^2 \int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx dx = n^4 f_s(n) \quad \dots (11.72)$$

and when $\frac{\partial F}{\partial x}, \frac{\partial^3 F}{\partial x^3}$ vanish at $x = 0$ and at $x = \pi$, (11.71) gives

$$\int_0^\pi \frac{\partial^2 F}{\partial x^2} \cos nx dx = -n^2 f_c(n) \quad \dots (11.73)$$

$$\text{So that } \int_0^\pi \frac{\partial^4 F}{\partial x^4} \cos nx dx = -n^4 f_c(n) \quad \dots (11.74)$$

Example 11.20: Determine the function F such that

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0, \quad 0 < x < \pi,$$

with the boundary condition $F = 0$ when $x = 0$ and $x = \pi$
 $= 0$ when $y = 0$
 $= F_0 (\text{const.})$ when $y = \pi$.

F being given to be zero when $x = 0$ and $x = \pi$, we have to use the finite sine transform, i.e., $f(n) = \int_0^\pi F(x) \sin nx dx$

Applying it to the given differential equation we have

$$\int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx dx + \int_0^\pi \frac{\partial^2 F}{\partial y^2} \sin nx dx = 0$$

with the condition, $f = 0$ when $y = 0$ and $f = \int_0^\pi F_0 \sin nx dx$ when $y = \pi$

By (11.70) we have, $\int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx dx = -n^2 f$

$$\therefore -n^2 f + \frac{\partial^2 F}{\partial y^2} = 0 \text{ where } \frac{\partial^2 f}{\partial y^2} = \int_0^\pi \frac{\partial^2 F}{\partial y^2} \sin nx dx$$

$$\text{or } \frac{\partial^2 f}{\partial y^2} - n^2 f = 0.$$

Its general solution is $f = A \sinh ny$

$$\begin{aligned} \text{But } f &= F_0 \int_0^\pi \sin nx dx \text{ when } y = \pi \\ &= F_0 \left[-\frac{\cos nx}{n} \right]_0^\pi = 0 \text{ when } n \text{ is even} \\ &\quad = -2 F_0 / n \text{ when } n \text{ is odd.} \end{aligned}$$

So that considering the two solutions for f we conclude

$$f = 0 \text{ when } n \text{ is even and } f = \frac{2F_0}{n} \operatorname{cosech} n\pi \sinh ny \text{ when } n \text{ is odd.}$$

Hence the inversion formula will give on replacing n by $2m + 1$,

$$F = \frac{4F_0}{\pi} \sum_{m=0}^{\infty} (2m+1)^{-1} \cosech(2m+1)\pi \sinh(2m+1)y \sin(2m+1)$$

x

[F] Multiple Fourier Transforms

If $F(x, y)$ be a function of two variables x and y , then assuming it to be the function of x only, its Fourier transform $\phi(n, y)$ is given by

$$\phi(n, y) = \int_{-\infty}^{\infty} F(x, y) e^{inx} dx \quad \dots (11.75)$$

Now if $f(n, l)$ be the Fourier complex transform of $f(n, y)$ which is regarded as function of y only then

$$f(n, l) = \int_{-\infty}^{\infty} \phi(n, y) e^{ily} dy \quad \dots (11.76)$$

These two results when combined, give

$$f(n, l) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i(nx+ly)} dx dy \quad \dots (11.77)$$

and the inversion formula is

$$f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(n, l) e^{-i(nx+ly)} dn dl \quad \dots (11.78)$$

Similarly in case of three variables x, y, z , we have

$$f(n, l, m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{i(nx+ly+mz)} dx dy dz \quad \dots (11.77A)$$

$$\text{and } f(x, y, z) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(n, l, m) e^{-i(nx+ly+mz)} dn dl dm \quad \dots (11.78A)$$

Note 1: The result may be generalized for any number of variables.

Note 2: In case the Fourier transforms are finite such that $F(x, y)$ is a function of two independent variables x, y where $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$, then the sine transform of $F(x, y)$ is given by

$$f_s(n, l) = \int_0^\pi \int_0^\pi F(x, y) \sin nx \sin ly dx dy \quad \dots (11.79)$$

and the inversion formula is

$$F(x, y) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} f_s(n, l) \sin nx \sin ly \quad \dots (11.80)$$

[G] Convolution or Faltung Theorem for Fourier Transforms

If $F(x)$ and $G(x)$ are two functions such that $-\infty < x < \infty$ then their Faltung or Convolution F^*G is defined as

$$H(x) = F^*G = \int_{-\infty}^{\infty} F(n) G(x-n) dn \quad \dots (11.81)$$

It is worth noting that the Fourier Transform of the Convolution of $F(x)$ and $G(x)$ is the product of their Fourier transforms, i.e.,

$$f[F^*G] = f[F] f[G] \quad \dots (11.82)$$

Since $f[F^*G] = \int_{-\infty}^{\infty} H(x) e^{-inx} dx$ by definition

$$\begin{aligned} &= \int_{-\infty}^{\infty} F(x) e^{-inx} dx \int_{-\infty}^{\infty} G(x) e^{-inx} dx \\ &= f[F] \cdot f[G]. \end{aligned}$$

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[H] Evaluation of Integrals with the help of Fourier Inversion Theorem

Let $I_1 = \int_0^\infty e^{-ax} \cos nx dx$ and $I_2 = \int_0^\infty e^{-ax} \sin nx dx$.

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Integrating by parts, we have

$$I_1 = \left[-\frac{1}{a} e^{-ax} \cos nx \right]_0^\infty - \frac{n}{a} \int_0^\infty e^{-ax} \sin nx dx = \frac{1}{a} - \frac{n}{a} I_2.$$

$$\text{Similarly } I_2 = \frac{n}{a} I_1$$

These give on solving $I_1 = \frac{a}{a^2 + n^2}$ and $I_2 = \frac{n}{a^2 + n^2}$

Thus taking $F(x) = e^{-ax}$, its sine and cosine Fourier transforms are $\frac{a}{a^2 + n^2}$

and $\frac{n}{a^2 + n^2}$ respectively, so that the inversion formula gives

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a}{a^2 + n^2} \cos nx dn \quad \dots (11.83)$$

$$\text{and } e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a}{a^2 + n^2} \sin nx dn \quad \dots (11.84)$$

$$\text{i.e., } \int_0^\infty \frac{\cos nx}{a^2 + n^2} dn = \frac{\pi}{2a} e^{-ax} \text{ and } \int_0^\infty \frac{n \sin nx}{a^2 + n^2} dn = \frac{\pi}{2} e^{-ax} \quad \dots (11.85)$$

11.2.7 Definition of Infinite Hankel Transform

If $J_n(px)$ be the Bessel function of the first kind of order n , then the Hankel transform of a function $f(x)$, ($0 < x < \infty$) denoted by $\tilde{f}(p)$ is defined as

$$\tilde{f}(p) = \int_0^\infty f(x) \cdot x J_n(px) dx \quad \dots (11.86)$$

Here $x J_n(px)$ is the *Kernel* of the transformation

We sometimes write (11.86) as

$$H\{f(x)\} = \tilde{f}(p) = \int_0^\infty f(x) \cdot x J_n(px) dx \quad \dots (11.87)$$

Example 11.21: Taking $x J_0(px)$ as the Kernel of the transformation, find the Hankel transform of the following functions:

$$(i) f(x) = \frac{e^{-ax}}{x}, (ii) f(x) = e^{-ax}, (iii) f(x) = \begin{cases} 1, & 0 < x < a, \\ 0, & x > a, \end{cases} \quad n \neq 0$$

$$(iv) f(x) = \begin{cases} a^2 - x^2, & 0 < x < a, \\ 0, & x > a, \end{cases} \quad n \neq 0$$

$$(i) \text{ We have } H\{f(x)\} = \tilde{f}(p) = \int_0^\infty \frac{e^{-ax}}{x} \cdot x J_0(px) dx \\ = \int_0^\infty e^{-ax} J_0(px) dx = (a^2 + p^2)^{-1/2}$$

$$(ii) \tilde{f}(p) = \int_0^\infty e^{-ax} x J_0(px) dx = \frac{a}{(a^2 + p^2)^{-3/2}}$$

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$$\begin{aligned}
 \textbf{Aliter. } \tilde{f}(p) &= \int_0^\infty e^{-ax} \cdot x J_0(px) dx = \left[-\frac{1}{a} e^{-x} \cdot x J_0(px) \right]_0^\infty \\
 &\quad + \frac{1}{a} \int_0^\infty e^{-ax} \frac{d}{dx} \{x J_0(px)\} dx \text{ (on integrating by parts)} \\
 &= 0 + \frac{1}{a} \int_0^\infty e^{-ax} \{J_0(px) + x J'_0(px)\} dx \\
 &= \frac{1}{a} \int_0^\infty e^{-ax} J_0(px) dx + \frac{1}{a} \int_0^\infty e^{-ax} x J'_0(px) dx
 \end{aligned}$$

But we have $x J'_n(x) = n J_n(x) - x J_{n+1}(x)$

Writing $n = 0$ and replacing x by px , we get $J'_0(xp) = -J_1(xp)$,

$$\begin{aligned}
 \therefore \tilde{f}(p) &= \frac{1}{a} \int_0^\infty e^{-ax} \{J_0(px)\} dx - \frac{1}{a} \int_0^\infty e^{-ax} x J_1(px) p dx \\
 &= \frac{1}{a(a^2 + p^2)^{3/2}} - \frac{1}{a} \cdot \frac{p^2}{a(a^2 + p^2)^{3/2}} \\
 &= \frac{a}{(a^2 + p^2)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 (iii) H\{f(x)\} &= \tilde{f}(p) = \int_0^\infty f(x) \cdot x J_0(px) dx \\
 &= \int_0^a 1 \cdot x J_0(px) dx + \int_a^\infty 0 \cdot x J_0(px) dx \\
 &= \frac{a}{p} J_1(ap)
 \end{aligned}$$

$$\begin{aligned}
 (iv) H\{f(x)\} &= \tilde{f}(p) = \int_0^\infty f(x) x \cdot J_0(px) dx \\
 &= \int_0^a (a^2 - x^2) x J_0(px) dx + \int_a^\infty 0 \cdot x J_0(px) dx \\
 &= \frac{4a}{p^2} J_1(pa) - \frac{2a^2}{p^2} J_0(pa)
 \end{aligned}$$

Note. On the next page we have tabulated some useful Hankel transforms.

11.2.8 Hankel Transform of the Derivatives of a Function

If $\tilde{f}_n(p)$ be the Hankel transform of order n of the function $f(x)$, i.e.,

$\tilde{f}_n(p) = \int_0^\infty x f(x) J_n(px)$, then the Hankel transform of $\frac{df}{dx}$ is

$$\tilde{f}'_n(p) = \int_0^\infty x \frac{df}{dx} J_n(px) dx \quad \dots (11.88)$$

$$\begin{aligned}
 &= [x f(x) J_n(px)]_0^\infty - \int_0^\infty f(x) \frac{d}{dx} \{x J_n(px)\} dx \quad \dots (11.89) \\
 &\quad \text{(on integrating by parts)}
 \end{aligned}$$

$$= - \int_0^\infty f(x) \{J_n(px) + x J'_n(px)\} dx \quad \dots (11.90)$$

under the assumption that $x f(x) \rightarrow 0$ and $x \rightarrow 0$ or $x \rightarrow \infty$

$$= - \int_0^\infty f(x) \{(1-n) J_n(px)\} dx - \int_0^\infty p x f(x) J_{n-1}(px) dx$$

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$$\begin{aligned}\therefore xJ_n(px) &= px J_n(px) - nJ_n(px) \\ &= (n-1) \int_0^\infty f(x) J_n(px) dx - p\tilde{f}_{n-1}(p) \\ &= (n-1) I - pf_{n-1}(p) \text{ (say) where } I \equiv \int_0^\infty f(x) J_n(px) dx \dots (11.91)\end{aligned}$$

The recurrence relation,

$$J_{n-1}(x) - \frac{2n}{x} J_n(x) + J_{n+1}(x) = 0 \text{ gives}$$

$$J_{n-1}(px) - \frac{2n}{px} J_n(px) + J_{n+1}(px) = 0 \text{ on replacing } x \text{ by } px$$

$$\text{i.e., } 2n J_n(x) = px J_{n-1}(x) + px J_{n+1}(x)$$

$$\begin{aligned}\text{so that } 2nI &= 2n \int_0^\infty f(x) J_n(px) dx \\ &= p \left[\int_0^\infty x f(x) J_{n-1}(px) dx + \int_0^\infty x f(x) J_{n+1}(px) dx \right] \\ &= p\tilde{f}_{n-1}(p) + p\tilde{f}_{n+1}(p) \dots (11.92)\end{aligned}$$

Hence (11.92) reduces to

$$\begin{aligned}\tilde{f}_n(p) &= \frac{n-1}{2n} p\tilde{f}_{n-1}(p) + \frac{n-1}{2n} p\tilde{f}_{n+1}(p) - p\tilde{f}_{n-1}(p) \\ &= -p \left[\frac{n+1}{2n} \tilde{f}_{n-1}(p) - \frac{n-1}{2n} \tilde{f}_{n+1}(p) \right] \dots (11.93)\end{aligned}$$

which gives the required Hankel transform of $\frac{df}{dx}$.

Replacing n by $(n-1)$ and $(n+1)$ in succession (11.93) yields,

$$\tilde{f}'_{n-1}(p) = -p \left[\frac{n}{2(n-1)} \tilde{f}_{n-2}(p) - \frac{n-2}{2(n-1)} \tilde{f}_n(p) \right]$$

$$\text{and } \tilde{f}'_{n+1}(p) = -p \left[\frac{n+2}{2(n+1)} \tilde{f}_n(p) - \frac{n}{2(n+1)} \tilde{f}_{n+2}(p) \right]$$

Using these results and replacing f by f' in (11.93) we find

$$\begin{aligned}\tilde{f}_{n+1}''(p) &= -p \left[\frac{n+1}{2n} \tilde{f}'_{n-1}(p) - \frac{n-1}{2n} \tilde{f}_{n+2}(p) \right] \\ &= \frac{p^2}{4} \left[\frac{n+1}{n-1} \tilde{f}_{n-2}(p) - 2 \frac{n^2-3}{n^2-1} \tilde{f}_n(p) + \frac{n-1}{n+1} \tilde{f}_{n+2}(p) \right] \dots (11.94)\end{aligned}$$

COROLLARY. Putting $n = 1, 2, 3$, successively in (11.93) we have

$$\tilde{f}_1(p) = -p \tilde{f}_0(p) \dots (11.95)$$

$$\tilde{f}_2(p) = -p \left[\frac{3}{4} \tilde{f}_1(p) - \frac{1}{4} \tilde{f}_3(p) \right] \dots (11.96)$$

$$\tilde{f}_3'(p) = -p \left[\frac{2}{3} \tilde{f}_2(p) - \frac{1}{3} \tilde{f}_4(p) \right] \dots (11.97)$$

11.2.9 Hankel Transforms of $\frac{d^2f}{dx^2}$, $\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx}$ and $\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{n^2}{x^2} f$ under Certain Conditions

We have $H\left\{\frac{d^2f}{dx^2}\right\} = \int_0^\infty \frac{d^2f}{dx^2} \cdot x J_n(px) dx = \left[\frac{df}{dx} x J_n(px) \right]_0^\infty - \int_0^\infty \frac{df}{dx} \cdot \frac{d}{dx} \{x J_n(px)\} dx.$

Assuming that $xf(x) \rightarrow 0$ as $x \rightarrow 0, x \rightarrow \infty$, we have

$$\begin{aligned} H\left\{\frac{d^2f}{dx^2}\right\} &= - \int_0^\infty \frac{df}{dx} [J_n(px) + px J'_n(px)] dx \\ &= \int_0^\infty x \frac{d^2f}{dx^2} \cdot J_n(px) dx \\ \text{or } \int_0^\infty x \left(\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} \right) J_n(px) dx &= -p \int_0^\infty \frac{df}{dx} \cdot x J'_n(px) dx \\ &= -p [f(x) x J'_n(px)]_0^\infty - \int_0^\infty f(x) \frac{d}{dx} \{x J'_n(px)\} dx \\ &= p \int_0^\infty f(x) \frac{d}{dx} \{x J'_n(px)\} dx \quad \dots (11.95) \\ &\quad (\because xf(x) \rightarrow 0 \text{ as } x \rightarrow 0 \text{ or } \infty) \end{aligned}$$

But $J_n(px)$ satisfies Bessel's differential equation, i.e.,

$$\begin{aligned} \frac{d}{dx} \left(x \frac{dy}{dx} \right) + \left(1 - \frac{n^2}{x^2} \right) xy &= 0, \\ \therefore \frac{d}{dx} \left(x \frac{dy}{dx} J_n(x) \right) + \left(1 - \frac{n^2}{x^2} \right) x \cdot J_n(x) &= 0 \\ \text{or } \frac{1}{p} \frac{d}{dx} [px \cdot J'_n(px)] &= - \left(p^2 - \frac{n^2}{x^2} \right) \frac{x}{p^2} J_n(px) \end{aligned}$$

on replacing x by px

$$\text{or } \frac{d}{dx} [x J'_n(px)] = - \left[p^2 - \frac{n^2}{x^2} \right] \frac{x}{p} J_n(px)$$

As such (11.95) reduces to

$$\begin{aligned} \int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} \right) \cdot x J_n(px) dx &= -p \int_0^\infty f(x) \left(p^2 - \frac{n^2}{x^2} \right) \frac{x}{p} J_n(px) dx \\ \text{or } \int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{n^2}{x^2} f \right) \cdot x J_n(px) dx &= -p^2 \int_0^\infty f(x) \cdot x J_n(px) dx \\ &= -p^2 \tilde{f}_n(p) \quad \dots (11.96) \end{aligned}$$

where $\tilde{f}_n(p)$ is the Hankel transform of order n of $f(x)$.

COROLLARY 1. If we put $n = 0$ in (11.96), we get

$$\int_0^\infty x \left(\frac{d^2f}{dx^2} + \frac{1}{x} \frac{df}{dx} \right) J_0(px) dx = -p^2 \tilde{f}_0(p) \quad \dots (11.97)$$

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where $\tilde{f}_0(p)$ is the Hankel Transform of Zeroeth order.

COROLLARY 2. If we put $n = 1$ in (11.96), we find

$$\int_0^\infty \left(\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{f}{x^2} \right) x J_1(px) dx = -p^2 \tilde{f}_1(p) \quad \dots (11.98)$$

$$\text{or} \quad \int_0^\infty x \frac{df}{dx} J_1(px) dx = -p \tilde{f}(p) \quad \dots (11.99)$$

where $\tilde{f}(p) = \int_0^\infty x f(x) J_0(px) dx$

Example 11.22: Find $H \left\{ \frac{\partial f}{\partial x} \right\}$ when $f = \frac{e^{-ax}}{x}$ and $n = 1$.

We have

$$H \left\{ \frac{\partial f}{\partial x} \right\} = \int_0^\infty x \frac{df}{dx} J_1(px) dx = -p \tilde{f}_0(p) \quad \text{by (8) of §11.6.}$$

$$= -p \int_0^\infty x f(x) J_0(px) dx = -p \int_0^\infty e^{-ax} J_0(px) dx = \frac{-ap}{(a^2 + p^2)^{3/2}}$$

Example 11.23: Find $H \left\{ \frac{d^2 f}{dt^2} \right\}$ when $f = f(x, t)$.

$$\begin{aligned} \text{We have } H \left\{ \frac{d^2 f}{dt^2} \right\} &= \int_0^\infty x \frac{\partial^2 f}{\partial t^2} J_n(px) dx = \frac{\partial^2}{\partial t^2} \int_0^\infty x f(x, t) J_n(px) dx \\ &= \frac{d^2}{dt^2} \tilde{f}(p, t). \end{aligned}$$

Example 11.24: Evaluate $\int_0^\infty \left(\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} \right) \cdot x J_0(px) dx$, when $f(x) = e^{-ax}$.

We have by (11.97)

$$\begin{aligned} \int_0^\infty x \left(\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} \right) J_0(px) dx &= -p^2 \tilde{f}_0(p) \\ &= -p^2 \int_0^\infty e^{-ax} J_0(px) dx = \frac{-ap^2}{(a^2 + p^2)^{3/2}}. \end{aligned}$$

Check Your Progress

1. What is function and how is it denoted?
2. What is Fourier integral?
3. In how many parts Fourier sine transform divided?
4. What is finite Fourier sine transform?
5. What is infinite Fourier cosine transform of $F(x)$?

11.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. If there is a known function $K(a, x)$ of two variables a and x such that the integral

$$\int_0^\infty K(a, x) F(x) dx$$

2. Fourier's Integral is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) dt \int_{-\infty}^{\infty} \cos u(x-t) du$$

3. Fourier sine transforms can be subdivided in two, namely, the infinite Fourier sine transform and the Finite Fourier sine transforms.

4. The Finite Fourier sine transform of a function $F(x)$ of x such that $0 < x < l$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx$$

5. The Infinite Fourier Cosine Transform of $F(x)$ for $0 < x < \infty$, is defined as $f_c(n) = \int_0^\infty F(x) \cos nx dx$, n being a positive integer.

Here the function $F(x)$ is called as the *Inverse cosine transform* of $f_c(n)$ and is defined as

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx dx$$

Thus if $f_c(n) = f_c[F(x)]$, then $F(x) = f_c^{-1}[f_c(n)]$

6. Find the Complex Fourier transform of $e^{-|x|}$ and then invert it.

7. Hankel Transforms of $\frac{d^2 f}{dx^2}$, $\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx}$ and $\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{n^2}{x^2} f$ under Certain Conditions

11.4 SUMMARY

- If there is a known function $K(a, x)$ of two variables a and x such that the integral

$$\int_0^\infty K(a, x) F(x) dx$$

is convergent, then the integral (1) is termed as the Integral transform of the function $F(x)$ and is denoted by $\tilde{F}(x)$ or $T\{F(x)\}$, i.e.,

$$\tilde{F}(x) = T\{F(x)\} = \int_0^\infty K(a, x) F(x) dx$$

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- The function $K(a, x)$ introduced here is sometimes known as the *Kernel* of the transformation and a is a parameter (real or complex) independent of x .
- If $F(t)$ be a function of t defined for all positive values of t (i.e., $t \geq 0$), then the Laplace transform of $F(t)$ denoted by $L\{F(t)\}$ or $\tilde{F}(s)$ or $f(s)$ is defined by the expression

$$L\{F(t)\} = \tilde{F}(t) = f(s) = \int_0^\infty e^{-st} F(t) dt$$

where s is a parameter (real or complex).

- If the integral $\int_0^\infty e^{-st} F(t) dt$ converges for some value of s , then the Laplace transform of $F(t)$ is said to exist, otherwise it does not exist.
- By definition, the Laplace transform of a function $F(t)$ is given by

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

- If the function $F(t)$ is expressible as a Power series, for example

$$F(t) = a_0 + a_1 t + a_2 t^2 + \dots = \sum_{n=0}^{\infty} a_n t^n$$

- If $f(x)$ satisfies the Dirichlet's condition in $-p \leq x \leq p$, and $\int_{-\infty}^{\infty} f(x) dx$ converges, i.e., is integrable in $-\infty < x < \infty$, then we have the Fourier series expansion for $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

- Fourier sine transforms can be subdivided in two, namely, the infinite Fourier sine transform and the Finite Fourier sine transforms.
- [a₂] *The Finite Fourier sine transform* of a function $F(x)$ of x such that $0 < x < l$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx$$

- The Complex Fourier Transform of a function $F(x)$ for $-\infty < x < \infty$, is defined as

$$f(n) = \int_{-\infty}^{\infty} F(x) e^{inx} dx$$

where e^{inx} is said to be the *Kernel* of the transform.

- If $F(x, y)$ be a function of two variables x and y , then assuming it to be the function of x only, its fourier transform $f(n, y)$ is given by

$$\phi(n, y) = \int_{-\infty}^{\infty} F(x, y) e^{inx} dx$$

- If $F(x)$ and $G(x)$ are two functions such that $-\infty < x < \infty$ then their Faltung or Convolution F^*G is defined as

$$H(x) = F^*G = \int_{-\infty}^{\infty} F(n) G(x - n) dn$$

- If $J_n(px)$ be the Bessel function of the first kind of order n , then the Hankel transform of a function $f(x)$, $(0 < x < \infty)$ denoted by $\tilde{f}(p)$ is defined as

$$\tilde{f}(p) = \int_0^\infty f(x) \cdot x J_n(px) dx$$

- If $\tilde{f}_n(p)$ be the Hankel transform of order n of the function $f(x)$, i.e.,

$\tilde{f}_n(p) = \int_0^\infty x f(x) J_n(px) dx$, then the Hankel transform of $\frac{df}{dx}$ is

$$\tilde{f}'_n(p) = \int_0^\infty x \frac{df}{dx} J_n(px) dx$$

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11.5 KEY WORDS

- **Integral transform:** An integral transform maps an equation from its original domain into another domain where it might be manipulated and solved much more easily than in the original domain. The solution is then mapped back to the original domain using the inverse of the integral transform.
- **Laplace transform:** The Laplace transform is an integral transform named after its inventor Pierre-Simon Laplace.
- **Series expansion:** A series expansion is a method for calculating a function that cannot be expressed by just elementary operators (addition, subtraction, multiplication and division).
- **Fourier integral:** Fourier integrals are generalizations of Fourier series. The series representation of a function is a periodic form obtained by generating the coefficients from the function's definition on the least period.

11.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is definition of the Laplace transform?
2. Find the Laplace transform of e^{at} .
3. Find the Laplace transform of t^a where a is positive but not necessarily an integer.
4. What is method of differential equation?
5. Find method of differentiation with respect to a parameter.
6. What is Fourier transform?

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Long Answer Questions

1. Find the Laplace transform of $\sin at$ and $\cos at$.
2. Find the Laplace transform of the following functions:
 - (i) $F(t) = t \sin at$
 - (ii) $F(t) = t \cos at$
3. Explain different forms of Fourier's integrals.
4. Show that the sum function of the Integral formula is $\frac{1}{2} [f(x+0) + f(x-0)]$ corresponding to the function $f(x)$ in the interval $0 < x < 1$.
5. Find the Fourier sine transform of $F(x) = x$ such that $0 < x < 2$.
7. Explain Hankel transform of the derivatives of a function.

11.7 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

BLOCK IV
LAPLACE, WAVE AND DIFFUSION EQUATIONS

UNIT 12 LAPLACE'S EQUATIONS

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Structure

- 12.0 Introduction
- 12.1 Objectives
- 12.2 Laplace's Equations: Elementary Solutions of Laplace's Equations
- 12.3 Answers to Check Your Progress Questions
- 12.4 Summary
- 12.5 Key Words
- 12.6 Self Assessment Questions and Exercises
- 12.7 Further Readings

12.0 INTRODUCTION

In mathematics, Laplace's equation is a second-order partial differential equation named after Pierre-Simon Laplace who first studied its properties. This is often written as

$$\nabla^2 f = 0 \quad \text{or} \quad \Delta f = 0,$$

Laplace's equation, second-order partial differential equation widely useful in physics because its solutions R (known as harmonic functions) occur in problems of electrical, magnetic, and gravitational potentials, of steady-state temperatures, and of hydrodynamics. The equation was discovered by the French mathematician and astronomer Pierre-Simon Laplace (1749–1827).

Laplace's equation states that the sum of the second-order partial derivatives of R , the unknown function, with respect to the Cartesian coordinates, equals zero:

$$\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} = 0.$$

The sum on the left often is represented by the expression $\nabla^2 R$, in which the symbol ∇^2 is called the Laplacian, or the Laplace operator.

Many physical systems are more conveniently described by the use of spherical or cylindrical coordinate systems. Laplace's equation can be recast in these coordinates; for example, in cylindrical coordinates, Laplace's equation is

$$\nabla^2 R = \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \frac{1}{r^2} \frac{\partial^2 R}{\partial \theta^2} + \frac{\partial^2 R}{\partial z^2} = 0.$$

In this unit, you will study about Laplace's equation and its elementary solutions of Laplace's equation.

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12.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what Laplace's equation is
- Discuss about Laplace's equation and its elementary solutions of Laplace's equation

12.2 LAPLACE'S EQUATIONS: ELEMENTARY SOLUTIONS OF LAPLACE'S EQUATIONS

Laplace's Equation

In mathematics, Laplace's equation is a second order partial differential equation named after Pierre-Simon Laplace who first studied its properties. A solution to Laplace's equation has the property that the average value over a spherical surface is equal to the value at the center of the sphere (as per Gauss's harmonic function theorem). Because Laplace's equation is linear, hence the superposition of any two solutions is also a solution.

Typically, the solution to Laplace's equation is uniquely determined if the value of the function is specified on all boundaries or the normal derivative of the function is specified on all boundaries.

Since the Laplace operator appears in the heat equation, hence the one physical interpretation can be given as, 'fix the temperature on the boundary of the domain according to the given specification of the boundary condition'. Allow heat to flow until a stationary state is reached in which the temperature at each point on the domain does not change anymore.

The Cartesian form of three-dimensional Laplace's equation as a particular case of steady heat flow in the form.

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots (12.1)$$

In cylindrical coordinates (r, θ, z) , can be expressed as,

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots (12.2)$$

and in Polar spherical coordinates (r, θ, ϕ) it can be shown as,

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \dots (12.3)$$

Two-dimensional Cartesian form of Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (12.4)$$

Taking u as independent of z , the two-dimensional Laplace's equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots (12.5)$$

and in Polar coordinates (r, θ) it resumes the same form as (12.5).

One-dimensional Laplace's equation is $\frac{\partial^2 u}{\partial x^2} = 0$.

Its solution being easy and straight has no points of worth consideration and hence we shall consider only two and three-dimensional Laplace equations.

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12.7 Two-Dimensional Laplace's Equation (Steady Flow of Heat)

[A] Solution of Two-Dimensional Laplace's Equation in Cartesian Coordinates

We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \dots (12.6)$$

(taking temperature as independent of time).

This can be solved either by the method of separation of variables or by the application of integral transforms as is evident from the following problems:

Example 12.1. Determine the steady state temperature distribution in a thin plate bounded by the lines $x = 0$, $x = l$, $y = 0$ and $y = \infty$, assuming that heat cannot escape from either surface of the plate, the edges $x = 0$ and $x = l$ being kept at a temperature zero and also the lower edge $y = 0$ is kept at temperature $F(x)$ and the edge $y = \infty$ at temperature zero.

The boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (12.7)$$

With conditions

- (i) $u(0, y) = 0$,
- (ii) $u(l, y) = 0$
- (iii) $u(x, 0) = F(x)$, and
- (iv) $u(x, \infty) = 0$.

In order to apply the method of separation of variables, assume that

$$u(x, y) = X(x) Y(y) \quad \dots (12.8)$$

So that $\frac{\partial^2 u}{\partial x^2} = Y \frac{d^2 X}{dx^2}$ and $\frac{\partial^2 u}{\partial y^2} = X \frac{d^2 Y}{dy^2}$

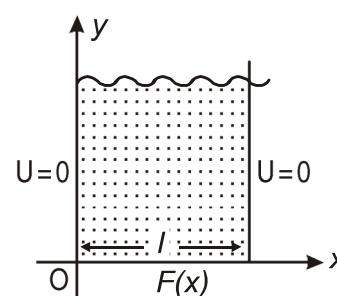


Fig. 12.1

\therefore Equation (12.7) gives $\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = -\lambda^2$ (say) as variables are separated.

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Here $\frac{1}{X} \frac{d^2 X}{dx^2} = -\lambda^2$ i.e. $\frac{d^2 X}{dx^2} + \lambda^2 X = 0$ gives

$$X = A \cos \lambda x + B \sin \lambda x \quad \dots (12.9)$$

and $\frac{1}{Y} \frac{d^2 Y}{dy^2} = -\lambda^2$ i.e. $\frac{d^2 Y}{dy^2} - \lambda^2 Y = 0$ gives $Y = C e^{\lambda y} + D e^{-\lambda y}$... (12.10)

As such a solution of Equation (12.7) is

$$u(x, y) = XY = (A \cos \lambda x + B \sin \lambda x)(C e^{\lambda y} + D e^{-\lambda y}) \quad \dots (12.11)$$

Applying condition (iv) we have $C = 0$ and applying (i) $A = 0$, so that (12.11) takes the form

$$u(x, y) = B \sin \lambda x \cdot e^{-\lambda y} \quad \dots (12.12)$$

But condition (ii) yields, $\sin \lambda l = 0$

i.e. $\lambda = \frac{n\pi}{l}$, n being an integer.

Hence for all distinct n , the general solution of Equation (12.7) is

$$u(x, y) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{l} e^{-n\pi y/l} \quad \dots (12.13)$$

which gives the required temperature in the thin plate, where

$$B_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx \quad \text{and} \quad F(x) = u(x, \infty) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi x}{l}$$

Example 12.2. (Temperature Distribution in a Finite Plate). Find the steady state temperature distribution of a thin rectangular plate bounded by the lines $x = 0$, $x = l$, $y = 0$, $y = b$ assuming that the edges $x = 0$, $x = l$ and $y = 0$ are maintained at temperature zero while the edge $y = b$ is maintained at temperature $F(x)$.

The boundary value problem is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$... (12.14)

with the conditions (i) $u(0, y) = 0$, (ii) $u(l, y) = 0$, (iii) $u(x, 0) = 0$ and (iv) $u(x, b) = F(x)$.

Proceeding just like in Example 12.1, we get the general solution of (12.14) as

$$u(x, y) = \sum_{n=1}^{\infty} [B_n e^{n\pi y/l} + C_n e^{-n\pi y/l}] \sin \frac{n\pi x}{l} \quad \dots (12.15)$$

In view of condition (i), $C_n = -B_n$ so that (12.15) reduces to

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} B_n [e^{n\pi y/l} - e^{-n\pi y/l}] \sin \frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} D_n \sinh \frac{n\pi y}{l} \sin \frac{n\pi x}{l} \quad \text{on setting } D_n = 2B_n \end{aligned}$$

By condition (iv), $F(x) = \sum_{n=1}^{\infty} D_n \sinh \frac{n\pi b}{l} \sin \frac{n\pi y}{l}$ so that

$$D_n \sinh \frac{n\pi b}{l} = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx.$$

Hence the solution is

$$u(x, y) = \frac{2}{l} \sum_{n=1}^{\infty} \sinh \frac{n\pi y}{l} \operatorname{cosech} \frac{n\pi b}{l} \sin \frac{n\pi x}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx.$$

Example 12.3. (Insulated at One Side): Determine the steady state temperature in a rectangular plate of length a and width b with sides maintained at temperature zero while the lower end is at temperature $F(x)$ and upper one insulated.

$$\text{The boundary value problem is } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (12.16)$$

with conditions (i) $u(0, y) = 0$, (ii) $u(a, y) = 0$, (iii) $u(x, 0) = F(x)$ and (iv) $u(x, b) = 0$

Proceeding just like in Example 12.1, we have

$$u(x, y) = \sum_{n=1}^{\infty} \left[C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a} \right] \sin \frac{n\pi x}{a} \quad \dots (12.17)$$

In view of (iii), we have

$$F(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \text{ so that } C_n = \frac{2}{a} \int_0^a F(x) \sin \frac{n\pi x}{a} dx$$

$$\text{and in view of (iv), } 0 = u_y(x, b) = \sum_{n=1}^{\infty} \frac{n\pi}{a} \left[C_n \cosh \frac{n\pi b}{a} + D_n \sinh \frac{n\pi b}{a} \right] \sin \frac{n\pi x}{a},$$

$$\text{giving } D_n = -C_n \tanh \frac{n\pi b}{a}$$

$$\text{Hence } u(x, y) = \frac{2}{a} \sum_{n=1}^{\infty} \left[\cosh \frac{n\pi y}{a} - \tanh \frac{n\pi y}{a} \sin \frac{n\pi y}{a} \right] \sin \frac{n\pi x}{a}$$

$$\int_0^a F(x) \sin \frac{n\pi x}{a} dx.$$

Example 12.4. Heat flows in a semi-infinite rectangular plate, the end $x = 0$ being kept at temperature $T^\circ C$ and the edges $y = a$ at temperature zero, then show that the temperature at any point (x, y) is given by

$$u(x, y) = \frac{4T}{\pi} \sum_{r=1}^{\infty} \frac{1}{2r+1} \sin \frac{(2r+1)\pi y}{a} e^{-(2r+1)\pi x/a}.$$

$$\text{The boundary value problem is } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (12.18)$$

with conditions (i) $u = 0$ when $y = 0$, (ii) $u = 0$ when $y = a$, (iii) $u = T$ when $x = 0$.

The solution by usual method is

$$u(x, y) = (A \cos ny + B \sin ny) e^{-nx}$$

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In view of (i), $0 = Ae^{-nx}$ i.e. $A = 0$.

In view of (ii), $0 = B \sin na e^{-nx}$ giving $\sin na = 0$ i.e., $na = (2r + 1)\pi$.

Hence the general solution is

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$$u(x, y) = \sum_{r=1}^{\infty} B_r \sin(2r+1) \frac{\pi y}{a} e^{-(2r+1)\pi x/a} \quad \dots (12.19)$$

In view of (iii), $T = \sum_{r=1}^{\infty} B_r \sin(2r+1) \frac{\pi y}{a}$ so that

$$B_r = \frac{2}{a} \int_0^a T \sin(2r+1) \frac{\pi y}{a} dy = \frac{4T}{(2r+1)\pi}$$

$$\text{Hence } u(x, y) = \frac{4T}{\pi} \sum_{r=1}^{\infty} \frac{1}{2r+1} \sin \frac{(2r+1)\pi y}{a} e^{-(2r+1)\pi x/a}.$$

Example 12.5. A square plate has its faces and its edges $x = 0$ and $x = \pi$ ($0 < y < \pi$) insulated. Its edges $y = 0$ and $y = \pi$ are kept at temperatures zero and $f(x)$ respectively. Show that the formula for its steady temperature is

$$u(x, y) = \frac{1}{2\pi} a_0 y + \sum_{r=1}^{\infty} a_r \frac{\sinh ny}{\sinh n\pi} \cos nx,$$

$$\text{where } a_r = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx, n = 0, 1, 2, \dots$$

Hint. The boundary value problem is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with conditions

(i) $u_x(0, y) = 0$, (ii) $u_x(\pi, y) = 0$, (iii) $u(x, 0) = 0$, (iv) $u(x, \pi) = f(x)$.

Applying method of separation of variables.

Example 12.6. If $u(x, y)$ denotes the electrostatic potential in a region bounded by the planes $x = 0$, $x = \pi$ and $y = 0$ in which there is a uniform distribution of space charge of density $\frac{h}{4\pi}$. If the planes $x = 0$ and $y = 0$, are kept at potential zero, the plane $x = \pi$ at an other fixed potential $u = 1$ and u is finite as $y \rightarrow \infty$, then find u .

The function $u(x, y)$ satisfies Poisson equation $\nabla^2 u = -4\pi\rho$ in two

dimensions, where $\rho = \frac{h}{4\pi}$ and hence the boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -h \quad (0 < x < \pi, y > 0) \quad \dots (12.20)$$

with conditions $u = 0$ when $x = 0$, $u = 1$ when $x = \pi$ $\dots (12.21)$

and $u = 0$ when $y = 0$, $(0 < x < \pi)$ and $u < M$ ($0 \leq x \leq \pi$, $y > 0$) $\dots (12.22)$

where M is some constant.

Using finite Fourier transform (12.20) gives

$$\int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin \lambda x \, dx + \int_0^\pi \frac{\partial^2 u}{\partial y^2} \sin \lambda x \, dx = -h \int_0^\pi \sin \lambda x \, dx$$

or $\frac{d^2 \tilde{u}}{dy^2} - p^2 \tilde{u} = p(-1)^p - h F_s \{1\}, \quad \dots (12.23)$

where $\tilde{u} = \int_0^\pi u \sin \lambda x \, dx$ (under the conditions $u = 0$ at $x = 0$ and $u = 1$ at $x = \pi$).

Finite Fourier transform of (12.22) gives $\tilde{u} = 0$ when $y = 0$ and $|\tilde{u}| < M\pi$ $\dots (12.24)$

\therefore Solution of (4) is $\tilde{u} = Ae^{-py} + Be^{py} + \frac{p(-1)^p - h F_s(1)}{-p^2}$

Since \tilde{u} is finite when $y \rightarrow \infty$ $\therefore B = 0$.

Also $y = 0$, $\tilde{u} = 0$ gives $A = \frac{p(-1)^p - h F_s(1)}{p^2}$

Hence $\tilde{u} = \frac{h F_s(1) - p(-1)^p}{p^2} (1 - e^{py})$

Applying the inversion formula for finite Fourier sine transform, we find

$$u(x, y) = \frac{2}{\pi} \sum \tilde{u} \sin nx.$$

Example 12.7. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$ under the boundary conditions.

$$u(x, 0) = x^2, u(x, \pi) = 0, u_x(0, y) = \frac{\partial}{\partial x} u_x(0, y) = 0 = u_x(\pi, y),$$

$$\text{Ans. } u(x, y) = \frac{\pi}{3}(\pi - y) + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\sinh n(\pi - y) \cos nx}{n^2 \sinh nx}$$

[B] Solution of Two-Dimensional Laplace's Equation in Cylindrical (or Polar) Coordinates

The Laplace's equation in this case is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots (12.25)$

Assume $u(r, \theta) = R(r) \Theta(\theta)$

So that $\frac{\partial^2 u}{\partial r^2} = \Theta \frac{\partial^2 R}{\partial r^2}$, $\frac{\partial u}{\partial r} = \Theta \frac{dR}{dr}$ and $\frac{\partial^2 u}{\partial \theta^2} = R \frac{d^2 \Theta}{d \theta^2}$.

$$\therefore (12.25) \text{ gives } \Theta \frac{d^2 R}{dr^2} + \frac{1}{r} \Theta \frac{dR}{dr} + \frac{1}{r^2} \frac{R d^2 \Theta}{d \theta^2} = 0 \quad \dots (12.26)$$

$$\text{or } \frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{\Theta} \frac{d^2 \Theta}{d \theta^2} = n^2 \text{ (say)} \quad \dots (12.27)$$

as variables are separated.

$$\text{Here } -\frac{1}{\Theta} \frac{d^2 \Theta}{d \theta^2} = n^2 \text{ i.e., } \frac{d^2 \Theta}{d \theta^2} + n^2 \Theta = 0 \text{ gives } \Theta = A \cos n\theta \text{ or } \Theta = B \sin n\theta \quad \dots (12.28)$$

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$$\text{and } \frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = n^2 \text{ i.e., } r^2 \frac{d^2 R}{dr^2} + \frac{dR}{dr} - n^2 R = 0$$

being homogeneous, takes the form $\frac{d^2 R}{ds^2} - n^2 R = 0$ on putting $r = e^s$ and then its solution is

$$R = C e^{ns} + D e^{-ns} \text{ i.e., } R = C r^n + D r^{-n} \quad \dots (12.29)$$

Taking $n = 0$ we have from (12.27)

$$\frac{d^2 \Theta}{d\theta^2} = 0 \text{ giving } \Theta = A_0 \theta + B_0 \quad \dots (12.30)$$

$$\text{and } r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} = 0 \text{ or } \frac{d^2 R}{ds^2} = 0 \text{ (when } r = e^s \text{) giving}$$

$$R = C_0 s + D_0 = C_0 \log r + D_0. \quad \dots (12.31)$$

The solution of Laplace's equation in cylindrical coordinates when u is independent of z is known as *Circular Harmonics* and n is the degree of the harmonic. Hence the Circular Harmonics of degree zero are given by

$$u_0 = (A\theta + B)(C \log r + D) \quad [\text{by (12.30) and (12.31)}]$$

and those of degree n are given by

$$u_n = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \quad [\text{by (12.28) and (12.29)}]$$

The general single-valued solution of (12.25) for all possible n may be written as

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) + C_0 \quad \dots (12.32)$$

where A_0, A_n, B_n, C_n, D_n and C_0 all are arbitrary constants.

Example 12.8. For a semi-circular plate of radius a with boundary diameter at 0°C and surface at 100°C , show that the temperature distribution is given by

$$u = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} \frac{\sin(2n-1)\theta}{a^{2n-1}}.$$

$$\text{The boundary value problem, } \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots (12.33)$$

Its solution by Equation (12.32) can be given as,

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) + C_0 \quad \dots (12.34)$$

But Temperature Being Finite at $r = 0$, (2) Should not Contain Terms of Log r and r^{-n} and This will be so if $A_0 = 0 = D_n$.

Moreover at $r = 0$, u being zero, we should have $C_0 = 0$

$$\begin{aligned} \text{Hence (12.34) reduces to } u &= \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) C_n r^n \\ &= \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n \text{ taking } a_n = A_n C_n \text{ etc.} \end{aligned}$$

Now a being the radius of the sphere and assuming $u = u(a)$ at $r = a$, we have

$$u(a) = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) a^n \quad \dots (12.35)$$

where $a_n = \frac{2}{\pi} \int_0^\pi \frac{u(a)}{a^n} \cos n\theta d\theta = \frac{200}{\pi a^n} \int_0^\pi \cos n\theta d\theta = 0$ and

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi \frac{u(a)}{a^n} \sin n\theta d\theta \\ \text{i.e., } b_n &= \frac{200}{\pi a^n} \int_0^\pi \sin n\theta d\theta = 0 \text{ when } n \text{ is even and } \frac{400}{n\pi a^n} \text{ when } n \text{ is odd.} \end{aligned}$$

Hence (12.35) reduces to

$$u = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} \frac{\sin(2n-1)\theta}{a^{2n-1}} \text{ for } n \text{ odd.}$$

Consider the following example. Determine the steady state temperature at any point of a semi-circular metal plate of radius a whose circumference is maintained to a given temperature of $T^0 C$ whereas the base is kept at zero temperature.

Given $u = u(r; \theta)$, the boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots (12.36)$$

with conditions (i) $u = 0$ when $\theta = 0$ for $0 \leq r < a$

(ii) u is finite when $r \rightarrow 0$

(iii) $u = T$ when $r = a$ for $0 < \theta < \pi$

Solution of (12.36) by (12.37) then becomes,

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) (C_n r^n + D_n R^{-n}) + C_0 \quad \dots (12.37)$$

In view of condition (ii), u being finite, (12.37) must not contain terms of $\log r$ and r^{-n} and this will be so when $A_0 = 0 = D_n$. Thus (12.37) reduces to

$$u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n + C_0 \text{ when } a_n = A_n C_n, b_n = B_n C_n \quad \dots (12.38)$$

In view of condition (i) $C_0 = 0$ and hence (12.38) yields

$$u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n \quad \dots (12.39)$$

By condition (iii), this gives, $T = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) a^n$ from which we find

$$a_n = \frac{2}{\pi} \int_0^\pi \frac{T}{a^n} \cos n\theta d\theta = \frac{2T}{\pi a^n} \int_0^\pi \cos n\theta d\theta = 0 \text{ and}$$

$$b_n = \frac{2}{\pi} \int_0^\pi \frac{T}{a^n} \sin n\theta d\theta = \frac{2T}{\pi a^n} \int_0^\pi \sin n\theta d\theta = \frac{2T}{n\pi a^n} (1 - \cos n\pi)$$

Hence (12.39) gives the required solution as

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$$u = \frac{2T}{n\pi a^n} - \sum_{n=1}^{\infty} (1 - \cos n\pi) r^n \sin n\theta.$$

Example 12.9. A long cylinder is made of two halves, the upper half is at the temperature T_1 and the lower half at the temperature T_2 . Find the distribution of temperature inside the cylinder.

Taking the axis of cylinder along z -axis, there is symmetry along z -axis and hence z -axis has no effect on the distribution of temperature. At the centre where $r = 0$, we have $u = \text{finite}$. The boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots (12.40)$$

Its solution is

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) (C_n r^n + D_n r^{-n}) + C_0 \quad \dots (12.41)$$

u being finite, we have to eliminate the terms of $\log r$ and r^{-n} so that $A_0 = 0 = D_n$

$$\therefore (12.41) \text{ becomes } u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) + C_0 \quad \dots (12.42)$$

where $a_n = A_n C_n$ etc.

Suppose that $u = F(\theta)$ at $r = R$ (say), then (12.42) gives

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) R^n \quad \dots (12.43)$$

$$\text{This gives } C_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$a_n = \frac{1}{R^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \quad \text{and} \quad b_n = \frac{1}{R^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta$$

But we are given that

$$f(\theta) = T_1 \text{ for } \pi > \theta > 0 \text{ (upper half)}$$

$$\text{and } f(\theta) = T_2 \text{ for } 2\pi > \theta > \pi \text{ (lower half)}$$

$$\therefore C_0 = \frac{1}{2\pi} \left[\int_0^\pi T_1 d\theta + \int_\pi^{2\pi} T_2 d\theta \right] = \frac{T_1 + T_2}{2}$$

$$a_n = \frac{1}{\pi R^n} \left[\int_0^\pi T_1 \cos n\theta d\theta + \int_\pi^{2\pi} T_2 \cos n\theta d\theta \right] = 0$$

$$\text{and } b_n = \frac{1}{\pi R^n} \left[\int_0^\pi T_1 \sin n\theta d\theta + \int_\pi^{2\pi} T_2 \sin n\theta d\theta \right]$$

$$= \frac{1}{\pi R^n} \left[\int_0^\pi \frac{T_1 + T_2}{n} (1 - \cos n\pi) d\theta \right] = \begin{cases} 2/n\pi R^n (T_1 - T_2) & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even.} \end{cases}$$

Hence the solution (12.42) reduces to

$$u = \frac{T_1 + T_2}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi R^n} (T_1 - T_2) \sin n\theta \cdot r^n \quad \text{where } n = 1, 3, 5, \dots$$

$$= \frac{T_1 + T_2}{2} + \frac{2}{\pi R} (T_1 - T_2) \sin \theta - r + \frac{2}{3\pi R^3} (T_1 - T_2) \sin 3\theta \times r^2 + \dots$$

which gives the required distribution of temperature.

Example 12.10. If u is a function of r and θ satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots (12.44)$$

within the region of the plane bounded by $r = a$, $r = 0$, $\theta = 0$, $\theta = \frac{\pi}{2}$ and also satisfying the boundary conditions $u = 0$ when $\theta = 0$, $u = 0$ when $\theta = \frac{\pi}{2}$, $u = 0$ when $r = b$ and $u = \theta \left(\frac{\pi}{2} - \theta \right)$ where $r = 0$ then show that

$$u = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(r/b)^{4n-2} (b/r)^{4n-2} \sin(4n-2)}{(a/b)^{4n-2} - (b/a)^{4n-2} (4n-2)}.$$

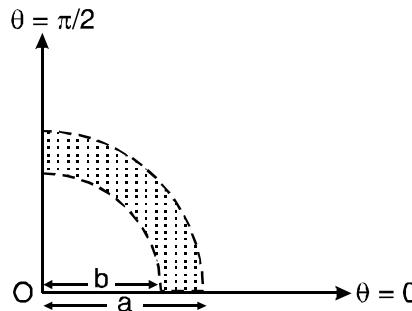


Fig. 12.2

The solution of Equation (12.44) is

$$u(r, \theta) = (A \cos m\theta + B \sin m\theta) \times (Cr^m + Dr^{-m}) \quad \dots (12.45)$$

where we have taken m^2 as constant of separation. Applying the boundary condition $u = 0$, when $\theta = 0$, (2) gives $0 = A(Cr^m + Dr^{-m})$ i.e. $A = 0$

∴ Equation (12.45) becomes $u(r, \theta) = (C' r^m + D' r^{-m}) \sin m\theta \quad \dots (12.46)$
where $BC = C'$ and $BD = D'$.

The condition $u = 0$ when $\theta = 0$, gives $0 = (C'r^m + D'r^{-m})$

$$\sin \frac{m\pi}{2} \text{ i.e. } \sin \frac{m\pi}{2} = 0$$

$$\text{or } \frac{m\pi}{2} = (2n-1)\pi \text{ giving } m = 4n-2.$$

Also the condition $u = 0$ when $r = b$ gives $0 = (C'r^m + D'r^{-m}) \sin m\theta$ i.e. $C'b^m + D'b^{-m} = 0$ which yields with $m = 4n-2$, $D' = -C'b^{2m} = -C'b^{8n-4}$.

As such (12.46) reduces to $u = C' [r^{4n-2} - b^{8n-4} r^{-(4n-2)}] \sin(4n-2)\theta$

Considering all possible n , the general solution becomes

$$u = \sum_{n=1}^{\infty} C'_n [r^{4n-2} - b^{8n-4} r^{-(4n-2)}] \sin(4n-2)\theta \quad \dots (12.47)$$

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Applying the condition $u = \theta \left(\frac{\pi}{2} - \theta \right)$ when $r = a$, (12.47) yields

$$\theta \left(\frac{\pi}{2} - \theta \right) = \sum_{n=1}^{\infty} C'_n \left[a^{4n-2} - b^{8n-4} a^{-(4\pi-2)} \right] \sin (4n-2)\theta$$

$$\begin{aligned} \text{So that } C'_n \left(a^{4n-2} - \frac{b^{8n-4}}{a^{4n-2}} \right) &= \frac{4}{\pi} \int_0^{\pi/2} \theta \left(\frac{\pi}{2} - \theta \right) \sin (4n-2)\theta d\theta \\ &= \frac{4}{\pi} \cdot \frac{4}{(4n-2)^3}. \end{aligned}$$

$$\text{Giving } C'_n = \frac{16}{\pi(4n-2)^3} \cdot \frac{a^{4n-2}}{a^{8n-4} - b^{8n-4}}.$$

Hence (12.47) reduces to

$$\begin{aligned} u &= \sum_{n=1}^{\infty} \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{\left\{ r^{4n-2} - b^{8n-4} r^{-(4n-2)} \right\} a^{4n-2}}{a^{8n-4} - b^{8n-4}} \frac{\sin (4n-2)\theta}{(4n-2)^3} \\ &= \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{r}{b} \right)^{4n-2} - \left(\frac{b}{r} \right)^{4n-2}}{\left(\frac{a}{b} \right)^{4n-2} - \left(\frac{b}{a} \right)^{4n-2}} \cdot \frac{\sin (4n-2)\theta}{(4n-2)^3}. \end{aligned}$$

Three-Dimensional Laplace's Equation

[A] Solution of Three-Dimensional Laplace-Equation in Cartesian Coordinates.

$$\text{The equation is } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots (12.48)$$

$$\text{Suppose } u = X(x) Y(y) Z(z) \quad \dots (12.49)$$

$$\text{Then (12.48) yields, } \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad \dots (12.50)$$

This relation being of form $F_1(X) + F_2(Y) + F_3(Z) = 0$ will be true only if F_1, F_2, F_3 are constant functions since x, y, z and so X, Y, Z are independent functions. We therefore take constants $-n^2, -m^2, +p^2$ such that $p^2 = m^2 + n^2$ and

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -n^2 \text{ i.e. } \frac{d^2 X}{dx^2} + n^2 X = 0 \text{ giving } X = A \cos nx + B \sin nx \quad \dots (12.51)$$

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -m^2 \text{ i.e. } \frac{d^2 Y}{dy^2} + m^2 Y = 0 \text{ giving } Y = C \cos my + D \sin my \quad \dots (12.52)$$

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = p^2 \text{ i.e. } \frac{d^2 Z}{dz^2} - p^2 Z = 0 \text{ giving } Z = E e^{pz} + F e^{-pz} \quad \dots (12.53)$$

As such the combined solution of (12.48) is

$$u = (A \cos nx + B \sin nx) (C \cos my + D \sin my) (E e^{pz} + F e^{-pz}) \quad \dots (12.54)$$

where $p^2 = m^2 + n^2$.

As an alternative this may be taken as

$$u = (Ae^{nx} + Be^{-nx}) (Ce^{my} + De^{-my}) [E \cos \nabla z + F \sin \nabla z] \quad \dots (12.55)$$

Note. We can easily verify that Laplace's equation $\nabla^2 u = 0$ is satisfied by

$$u = \frac{l}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} \quad \dots (12.56)$$

where l is a constant and (a, b, c) are coordinates of a fixed point.

[B] Solution of Three-Dimensional Laplace-Equation in Cylindrical Coordinates

$$\text{We have } \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots (12.57)$$

$$\text{Suppose that } u(r, \theta, z) = R(r) \Theta(\theta) Z(z). \quad \dots (12.58)$$

$$\text{Then (12.57) yields } \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{r}{R} \frac{dR}{dr} + \frac{1}{r^2 \Theta} = \frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad \dots (12.59)$$

As variables are separated, we may take

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda^2 \text{ and } \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -\mu^2 \text{ i.e. } \frac{d^2 Z}{dz^2} - \lambda^2 Z = 0$$

$$\text{and } \frac{d^2 \Theta}{d\theta^2} + \mu^2 \Theta = 0 \quad \dots (12.60)$$

$$\text{These yield } Z = e^{\pm \lambda z} \text{ and } \Theta = e^{\pm i\mu\theta} \quad \dots (12.61)$$

or in other words the solutions of (12.60) are

$$Z = A_1 e^{\lambda z} + B_1 e^{-\lambda z}, \Theta = A_2 \cos \mu\theta + B_2 \sin \mu\theta \quad \dots (12.62)$$

Also then (12.60) reduces to

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2} \right) R = 0 \quad \dots (12.63)$$

which is Bessel's equation and takes the form

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\mu^2}{x^2} \right) R = 0, \text{ on putting } \lambda r = x \quad \dots (12.64)$$

Its general solutions are

$$R = A_3 J_\mu(\lambda r) + B_3 J_{-\mu}(\lambda r), \text{ for fractional } \mu \quad \dots (12.65)$$

$$\text{and } R = A_3 J_\mu(\lambda r) + B_3 Y_\mu(\lambda r), \text{ for integral } \mu \quad \dots (12.66)$$

As such the solutions for (12.57), with the help of (12.62), (12.65) and (12.66) are

$$u(r, \theta, z) = (A_1 e^{\lambda z} + B_1 e^{-\lambda z}) (A_2 \cos \mu\theta + B_2 \sin \mu\theta) [A_3 J_\mu(\lambda r) + B_3 J_{-\mu}(\lambda r)] \quad \dots (12.67)$$

$$\text{and } u(r, \theta, z) = (A_1 e^{\lambda z} + B_1 e^{-\lambda z}) (A_2 \cos \mu\theta + B_2 \sin \mu\theta) [A_3 J_\mu(\lambda r) + B_3 Y_\mu(\lambda r)] \quad \dots (12.68)$$

Note 1. The general solution of (12.63) may be written as

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$$R = A_{\lambda\mu} J_\mu(\lambda r) + B_{\lambda\mu} Y_\mu(\lambda r) \quad \dots (12.69)$$

where $A_{\lambda\mu}$ and $B_{\lambda\mu}$ are constants.

Since $Y_\mu(\lambda r) \rightarrow \infty$ as $r \rightarrow 0$, therefore in a physical problem if μ is finite along the line $r = 0$, then we must have $B_{\lambda\mu} = 0$ and hence the solution of (12.58) may be written as

$$u = \sum_{\lambda} \sum_{\mu} A_{\lambda\mu} J_\mu(\lambda r) e^{\pm \lambda z \pm i\mu\theta} \quad \dots (12.70)$$

Note 2. Trying the superposition, the general solution of (12.57) may be written as

$$u(r, \theta, z) = \sum_{\mu=0}^{\infty} J_\mu(\lambda r) [e^{\lambda z} (A_\mu \cos \mu \theta + B_\mu \sin \mu \theta) + e^{-\lambda z} (C_\mu \cos \mu \theta + D_\mu \sin \mu \theta)] \quad \dots (12.71)$$

Note 3. In a problem if there is symmetry about z -axis, then we may take $\mu = 0$ and the solution will be

$$u(r, \theta, z) = \sum_{\lambda} A_{\lambda} J_0(\lambda r) e^{\pm \lambda z} \quad \dots (12.72)$$

Note 4. If in a problem of symmetry about z -axis, $u \rightarrow 0$ as $r \rightarrow 0$ and $z \rightarrow \infty$, then the solution is of the form $u(r, \theta, z) = \sum_{\lambda} A_{\lambda} J_0(\lambda r) e^{-\lambda z}$... (12.73)

[C] Solution of Three-Dimensional Laplace-Equation in Spherical Polar Coordinates

$$\text{We have } \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \dots (12.74)$$

$$\text{or equivalently, } r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} + \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \dots (12.75)$$

$$\text{Suppose } u(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \quad \dots (12.76)$$

Then (12.74) and (12.75) yield on dividing by $R \Theta \Phi$

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 \Theta} \frac{d^2 \Theta}{d\theta^2} + \frac{\cos \theta}{r^2 \Theta} \frac{d\Theta}{d\theta} + \frac{1}{r^2 \sin^2 \theta} \cdot \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \quad \dots (12.77)$$

$$\text{and } \left[\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{dR} \frac{dR}{dr} + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) \right] \sin^2 \theta = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = \lambda^2 \text{ (say)} \quad \dots (12.78)$$

$$\text{Considering (12.78), } -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = \lambda^2, \text{ i.e., } \frac{d^2 \Phi}{d\phi^2} + \lambda^2 \Phi = 0 \text{ gives } \Phi = C e^{\pm i\lambda\phi} \quad \dots (12.79)$$

$$\text{and } \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = -\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{\lambda^2}{\sin^2 \theta} = n(n+1) \text{ (say).}$$

$$\text{giving } \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - n(n+1)R = 0 \quad \dots (12.80)$$

$$\text{and } \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left[n(n+1) - \frac{\lambda^2}{\sin^2 \theta} \right] \Theta = 0 \quad \dots (12.81)$$

Considering (12.77), if we write $\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = -\lambda^2$ and $\frac{r^2}{R} \frac{d^2R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} = n(n+1)$, then we have

$\Phi = Ce^{\pm i\lambda\phi}$ which is (12.79); $r^2 \frac{dR}{dr^2} + \frac{2rdR}{dr} - n(n+1)R = 0$ which is (12.80) and

$$\frac{d^2\Theta}{d\theta^2} + \cot \theta \frac{d\Theta}{d\theta} + \left\{ n(n+1) - \frac{\lambda^2}{\sin^2 \theta} \right\} \Theta = 0 \text{ which is (12.81).}$$

Now the Equation (12.80) being homogeneous if we put $r = e^s$, then it reduces to $\{D(D-1) + 2D - n(n+1)\}R = 0$ where $D \equiv \frac{d}{ds}$ or $(D-n)(D+n+1)R = 0$ giving $R = A e^{ns} + B e^{-(n+1)s} = Ar^n + Br^{-n-1}$

$$\dots (12.82)$$

Again if we put $\cos \theta = \mu$ in (12.50) then since

$$\frac{d\Theta}{d\theta} = -\frac{d\Theta}{d\mu} \frac{d\mu}{d\theta} = -\sin \theta \frac{d\Theta}{d\theta} \text{ i.e. } \frac{1}{\sin \theta} \frac{d}{d\theta} \equiv -\frac{d}{d\mu},$$

$$\text{we have } \frac{d}{d\mu} \left\{ (1-\mu^2) \frac{d\Theta}{d\mu} \right\} + \left\{ n(n+1) - \frac{\lambda^2}{1-\mu^2} \right\} \Theta = 0$$

$$\text{i.e. } (1-\mu^2) \frac{d^2\Theta}{d\mu^2} - 2\mu \frac{d\Theta}{d\mu} + \left\{ n(n+1) - \frac{\lambda^2}{1-\mu^2} \right\} \Theta = 0 \quad \dots (12.83)$$

which is Legendre's associated equation and hence has the solution

$$\Theta = AP_n^\lambda(\mu) + BQ_n^\lambda(\mu) = AP_n^\lambda(\cos \theta) + BQ_n^\lambda(\cos \theta) \quad \dots (12.84)$$

In other words if we take $\Theta = \Theta(\cos \theta)$ from associated Legendre equation, then the solution of (12.74) is of the form $(Ar^n + Br^{-n-1}) \Theta(\cos \theta) e^{\pm i\lambda\phi}$

So that summing over for all n and trying superposition, the general solution of (12.74) may be written as

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) \Theta(\cos \theta) e^{\pm i\lambda\phi} \quad \dots (12.85)$$

Note 1. If $\lambda = 0$, then (12.83) reduces to $(1-\mu^2) \frac{d^2\Theta}{d\mu^2} - 2\mu \frac{d\Theta}{d\mu} + n(n+1)\Theta = 0$ which is Legendre's equation and hence we have for integral n

$$\Theta = P_n(\mu) = P_n(\cos \theta) = \frac{1}{2^n |n|} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n$$

$$\text{and also } \Theta = Q_n(\mu) = Q_n(\cos \theta) = \frac{1}{2} P_n(\mu) \log \frac{\mu+1}{\mu-1} - \sum_{j=0}^p \frac{2n-4j-1}{(2j+1)(n-j)} P_{n-2j-1}(\mu)$$

where $p = \frac{1}{2}(n-1)$ or $\frac{1}{2}n-1$ according as n is odd or even.

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$$\text{Thus } \Theta = C_n P_n(\mu) + D_n Q_n(\mu) \text{ so that } u = \sum_{n=0}^p \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) \\ \{C_n P_n(\cos \theta) + D_n Q_n(\cos \theta)\} e^{-i\lambda\phi} \quad \dots (12.86)$$

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In case $D_n = 0$ under specified boundary conditions, then

$$\Theta = C_n P_n(\cos \theta)$$

$$\text{Hence the solution is } u = \sum_{n=0}^p \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) \cdot e^{-i\lambda\phi} \quad \dots (12.87)$$

Note 2. If there is axial symmetry about z -axis, then u depends only on r and θ and so (12.74) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0 \quad \dots (12.88)$$

Its solution by putting $\phi = 0$ in (12.87), is

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) \{C_n P_n(\cos \theta) + D_n Q_n(\cos \theta)\} \quad \dots (12.89)$$

In case $D_n = 0$ under specified conditions, then

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) \quad \dots (12.90)$$

Example 12.11. If the surface S of a sphere of radius a is kept at a fixed distribution of electric potential $u = F(\theta)$, then find the potential u at all points in space which is assumed to be free of further charge.

In this case $\frac{\partial^2 \mu}{\partial \phi^2}$ being zero, we have the constant of separation i.e. $\lambda = 0$ and hence equation (12.83), reduces to

$$(1 - \mu^2) \frac{d^2 \Theta}{d\mu^2} - 2\mu \frac{d\Theta}{d\mu} + n(n+1) \Theta = 0 \quad \dots (12.91)$$

Being Legendre's equation the Legendre's polynomial $P_n(\mu) = P_n(\cos \theta)$ is the solution of (12.91) i.e. we have

$$u = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) \quad \dots (12.92)$$

Now to determine the potential u , we consider the problem in two cases:

Case I. Outside the sphere. Since the potential at infinity vanishes i.e.

$\lim_{r \rightarrow \infty} u = 0$, the boundary condition requires that any positive power of r should not be present in the solution (12.92), thereby giving $A_n = 0$ so that (12.92) reduces to

$$u = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \theta) \quad \dots (12.93)$$

Assuming that $u = F(\theta)$ when $r = a$, (12.93) gives

$$F(\theta) = f(\cos \theta) \text{ (say)} = \sum_{n=0}^{\infty} \frac{B_n}{a^{n+1}} P_n(\cos \theta) \quad \dots (12.94)$$

If we replace $\cos \theta$ by u in (12.94), we get $a^{n+1} f(u) = \sum_{n=0}^{\infty} B_n P_n(u)$

$$\text{so that } a^{n+1} \int_{-1}^1 f(u) P_n(u) du = \int_{-1}^1 \sum_n B_n P_n(u) \cdot P_n(u) du = B_n \int_{-1}^1 P_n^2(u) du \\ = \frac{2B_n}{2n+1} \text{ giving } B_n = \frac{2n+1}{2} a^{n+1} \int_0^\pi F(\theta) P_n(\cos \theta) \sin \theta d\theta \quad \dots (12.95)$$

on setting $u = \cos \theta$.

Putting this value of B_n in (12.93), we get the required potential outside the sphere *i.e.*

$$u = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left(\frac{a}{r} \right)^{n+1} P_n(\cos \theta) \int_0^\pi F(\theta) P_n(\cos \theta) \sin \theta d\theta \quad \dots (12.96)$$

Case II. Inside the sphere. Since the potential inside the sphere can not be infinite, therefore the general solution must not contain any negative power of r , thereby giving $B_n = 0$, so that (12.92) reduces to

$$u = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta), r < a \quad \dots (12.97)$$

when $r = a$, $u = F(\theta)$, therefore (12.97) gives $F(\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos \theta)$

from which we have as in case I,

$$A_n = \frac{2n+1}{a^n} \int_0^\pi F(\theta) P_n(\cos \theta) \sin \theta d\theta \quad \dots (12.98)$$

Substituting (12.98) in (12.97) we get the required potential inside the sphere.

Example 12.12. Find a solution of the equation

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0 \quad \dots (12.99)$$

in the form $\phi = f(r) \cos \theta$, given that (i) $-\frac{\partial \phi}{\partial r} = u \cos \theta$ when $r = a$ and

(ii) $-\frac{\partial \phi}{\partial r} = 0$ when $r = \infty$.

We have $\phi = f(r) \cos \theta$

$$\therefore \frac{\partial \phi}{\partial r} = f'(r) \cos \theta, \frac{\partial \phi}{\partial \theta} = -f(r) \sin \theta.$$

Their substitution in (12.99) gives

$$\frac{\partial}{\partial r} \{r^2 f'(r) \cos \theta\} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \{-\sin \theta f(r) \sin \theta\} = 0$$

$$\text{i.e., } r^2 f'(r) + 2r f'(r) - 2f(r) = 0 \quad \dots (12.100)$$

This equation being homogeneous, can be reduced to the form

$$\{D(D-1) + 2D - 2\} f(r) = 0 \text{ by putting } r = e^s \text{ and } D \equiv \frac{d}{ds}$$

$$\text{or } (D^2 + D - 2) f(r) = 0 \quad \text{or} \quad (D-1)(D+2) f(r) = 0$$

$$\therefore f(r) = A e^s + B e^{-2s} = Ar + \frac{B}{r^2} \text{ so that } \phi = Ar \cos \theta + \frac{B}{r^2} \cos \theta \quad \dots (12.101)$$

(12.101) gives $\frac{\partial \phi}{\partial r} = A \cos \theta - \frac{2B}{r^3} \cos \theta$.

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Applying the condition (i) $-u \cos \theta = A \cos \theta - \frac{2B}{a^3} \cos \theta$ (12.102)

and applying (ii) $0 = A \cos \theta$ i.e., $A = 0$ and then from (12.102) $B = \frac{a^3 u}{2}$.

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Hence (12.101) yields $\phi = \frac{1}{2} \frac{a^3 u}{r^3} \cos \theta$.

Example 12.13. Find the permanent temperature within a solid sphere of radius unit when one half of the surface of the sphere is kept at constant temperature 0°C and the other half of the surface at 1°C .

The distribution being symmetrical about z -axis,
we have

$$\begin{aligned} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) &= 0 \\ \text{i.e., } r^2 \frac{\partial^2 \phi}{\partial r^2} + 2r \frac{\partial \phi}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) &= 0 \end{aligned} \quad \dots (12.103)$$

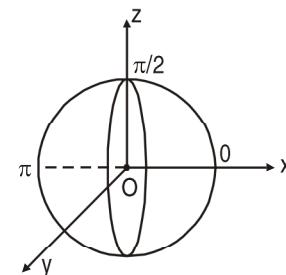


Fig. 12.3

With boundary conditions

$$(i) \phi = 1 \text{ for } 0 < \theta < \frac{\pi}{2} \text{ and } (ii) \phi = 0 \text{ for } \frac{\pi}{2} < \theta < \pi$$

under the consideration of distribution for upper half of the sphere i.e. for $0 < \theta < \pi$.

Assuming that $\phi = R(r) \Theta(\theta)$, (12.103) yields on dividing throughout by $R \Theta$,

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} = -\frac{1}{\Theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = 0 \quad \dots (12.104)$$

or separating the variables and taking λ^2 as constant of separation,

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} = \frac{1}{\Theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = \lambda^2 \text{ (say)}$$

so that $r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \lambda^2 R = 0$

and $\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \lambda^2 \Theta = 0$

Taking $\lambda^2 = n(n + 1)$, (12.105) yields $r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - n(n + 1) R = 0$ which being a homogeneous equation can be solved by substitution $r = e^s$, to give the solution $R = Ar^n + \frac{B}{r^{n+1}}$.

Also taking $\cos \theta = \mu$, (12.106) yields, $\frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) \frac{d\Theta}{d\mu} \right\} + n(n + 1) \Theta = 0$ which is Legendre's equation and hence $P_n(\mu)$ is a solution of it i.e., $\Theta = P_n(\mu) = P_n(\cos \theta)$

Combining the two solutions we have for all n

$$\phi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) \quad \dots (12.107)$$

Now the temperature at the centre being finite it is required that $B_n = 0$

$$\therefore (12.107) \text{ reduces to } \phi = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\mu) \quad \dots (12.108)$$

But by orthogonal properties of Legendre's polynomials, we have

$$\int_{-1}^1 P_n^2(\mu) d\mu = \frac{2}{2n + 1}$$

$$\text{Also } r = 1 \text{ gives } \phi = \sum A_n P_n(\mu) \quad \dots (12.109)$$

Multiplying (12.109) by $P_n(\mu)$ and integrating with regard to μ from -1 to 1 , we have

$$\begin{aligned} A_n &= \frac{2n+1}{2} \int_{-1}^1 \phi P_n(\mu) d\mu = \frac{2n+1}{2} \int_0^\pi \phi P_n(\cos \theta) \sin \theta d\theta \text{ when } \mu = \cos \theta \\ &= \frac{2n+1}{2} \left[\int_0^{\pi/2} \phi P_n(\cos \theta) \sin \theta d\theta + \int_{\pi/2}^\pi \phi P_n(\cos \theta) \sin \theta d\theta \right] \\ &= \frac{2n+1}{2} \int_0^{\pi/2} P_n(\cos \theta) \sin \theta d\theta \\ &\quad [\because \phi = 1 \text{ for } 0 < \theta < \pi/2 \text{ and } \phi = 0 \text{ for } \pi/2 < \theta < \pi] \end{aligned}$$

$$= \frac{2n+1}{2} \int_0^1 P_n(\mu) d\mu \text{ giving } A_0 = \frac{1}{2} \int_0^1 1 \cdot d\mu = \frac{1}{2}$$

and $A_1 = \frac{3}{2} \int_0^1 P_1(\mu) d\mu = \frac{3}{2} \int_0^1 \mu d\mu = \frac{3}{4}$

$$A_2 = \frac{5}{2} \int_0^1 P_2(\mu) d\mu = \frac{5}{2} \int_0^1 \frac{3\mu^2 - 1}{2} d\mu = \frac{5}{4} [\mu^2 - \mu]_0^1 = 0$$

$$A_3 = \frac{7}{2} \int_0^1 P_3(\mu) d\mu = \frac{7}{4} \int_0^1 (5\mu^2 - \mu) d\mu = \frac{7}{4} \left[\frac{5}{4}\mu^2 - \frac{3}{2}\mu^2 \right]_0^1 = -\frac{7}{16} \text{ etc.}$$

Hence $\phi = \frac{1}{2} + \frac{3}{4} \cdot r P_1(\cos \theta) - \frac{7}{16} r^3 P_3(\cos \theta) + \dots$

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General Properties of Harmonic Functions

We know that functions satisfying Laplace's differential equation are said to be the *Harmonic functions*. Now to discuss general properties of such functions, let us consider a vector point function \mathbf{A} and a scalar point function u satisfying Laplace's equation *i.e.*,

$$\nabla^2 u = 0 \quad \dots (12.110)$$

$$\text{such that} \quad \mathbf{A} = \nabla u \quad \dots (12.111)$$

$$\therefore \nabla \cdot \mathbf{A} = \nabla \cdot (\nabla u) = \nabla^2 u = 0 \quad \dots (12.112)$$

$$\text{But Gauss' divergence theorem gives } \iint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V (\nabla \cdot \mathbf{A}) du \quad \dots (12.113)$$

$$\text{which with the help of (12.112) yields } \iint_S \mathbf{A} \cdot d\mathbf{s} = \iint_S (\nabla u) \cdot d\mathbf{s} = 0 \quad \dots (12.114)$$

$$\text{If we take curl of both sides of (12.113), we get } \nabla \times \mathbf{A} = \nabla \times \nabla u = 0 \quad \dots (12.115)$$

But Stoke's theorem for a vector field \mathbf{A} is

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = 0 \quad \dots (12.116)$$

where integral being taken over the closed curve C bounding the open surface S .

$$(12.117) \text{ with the help of (12.113) reduces to } \oint_C (\nabla u) \cdot d\mathbf{r} = 0 \quad \dots (12.117)$$

From (12.114) and (12.117) certain important properties of harmonic functions can be deduced.

Applying Green's theorem *i.e.*

$$\iiint_V (v\nabla^2 w - w\nabla^2 v) du = \iint_S (v\nabla w - w\nabla v) \cdot d\mathbf{s} \quad \dots (12.118)$$

we can easily exhibit that if $\nabla^2 u = 0$ in a region bounded by a sphere of radius r then the value of u say u_0 at the centre of the sphere is given by

$$u_0 = \frac{1}{4\pi r^2} \iint_S u \, ds \quad \dots (12.119)$$

where integral is taken over the surface of the sphere.

These results may be categorically stated as:

- (i) From (12.119), the average value of a harmonic function on the surface of a sphere in which it has no singularity *i.e.*, the points where the function becomes infinite, is equal to its value at the centre of the sphere.
- (ii) From (12.114), it follows that a harmonic function having no singularity in a given region cannot have a maximum or minimum value in the region.
- (iii) From (ii) we conclude that a harmonic function without singularity within a region and being constant everywhere on the bounding surface of the region, has the same constant value everywhere inside the region.
- (iv) Two harmonic functions having identical values on a closed contour and having no singularity within the contour, are identical throughout the region bounded by the contour.

- (v) From Green's theorem it follows that if the normal derivative of a harmonic function is zero on a closed surface within which there is no singularity, the function is constant.
- (vi) It follows from (v) that if two harmonic functions have the same normal derivative on a closed surface within which there are no singularities, they differ at most by an additive constant.

NOTES**Check Your Progress**

1. Give the equation for two-dimensional Cartesian form of Laplace's equation.
2. What is the equation for one-dimensional Laplace's equation?
3. What is circular harmonics?
4. What is harmonic function?

12.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Two-dimensional Cartesian form of Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2. One-dimensional Laplace's equation is $\frac{\partial^2 u}{\partial x^2} = 0$.
3. The solution of Laplace's equation in cylindrical coordinates when u is independent of z is known as Circular Harmonics and n is the degree of the harmonic. Hence the Circular Harmonics of degree zero are given by

$$u_0 = (A\theta + B)(C \log r + D)$$

4. Functions satisfying Laplace's differential equation are said to be the Harmonic functions.

12.4 SUMMARY

- The Cartesian form of three-dimensional Laplace's equation as a particular case of steady heat flow in the form.

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- Two-dimensional Cartesian form of Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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- One-dimensional Laplace's equation is $\frac{\partial^2 u}{\partial x^2} = 0$.

- The function $u(x, y)$ satisfies Poisson equation $\nabla^2 u = -4\pi\rho$ in two dimensions, where $\rho = \frac{h}{4\pi}$ and hence the boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -h \quad (0 < x < \pi, y > 0)$$

- The solution of Laplace's equation in cylindrical coordinates when u is independent of z is known as Circular Harmonics and n is the degree of the harmonic. Hence the Circular Harmonics of degree zero are given by

$$u_0 = (A\theta + B)(C \log r + D)$$

- The general single-valued solution of (1) for all possible n may be written as

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) (C_n r^n + D_n r^{-n}) + C_0$$

where A_0, A_n, B_n, C_n and D_n all are arbitrary constants.

- The condition $u = 0$ when $\theta = 0$, gives $0 = (C'r^m + D'r^{-m})$

$$\sin \frac{m\pi}{2} \text{ i.e. } \sin \frac{m\pi}{2} = 0 \text{ or } \frac{m\pi}{2} = (2n-1)\pi \text{ giving } m = 4n-2.$$

- In a problem if there is symmetry about z -axis, then we may take $\mu = 0$ and the solution will be

$$u(r, \theta, z) = \sum_{\lambda} A_{\lambda} J_0(\lambda r) e^{\pm \lambda z}$$

- If in a problem of symmetry about z -axis, $u \rightarrow 0$ as $r \rightarrow 0$ and $z \rightarrow \infty$, then the solution is of the form $u(r, \theta, z) = \sum_{\lambda} A_{\lambda} J_0(\lambda r) e^{-\lambda z}$

- Functions satisfying Laplace's differential equation are said to be the Harmonic functions.

- The average value of a harmonic function on the surface of a sphere in which it has no singularity, i.e., the points where the function becomes infinite, is equal to its value at the centre of the sphere.

- Two harmonic functions having identical values on a closed contour and having no singularity within the contour, are identical throughout the region bounded by the contour.

- From Green's theorem it follows that if the normal derivative of a harmonic function is zero on a closed surface within which there is no singularity, the function is constant.

- That if two harmonic functions have the same normal derivative on a closed surface within which there are no singularities, they differ at most by an additive constant.

12.5 KEY WORDS

- **Harmonic functions:** Functions satisfying Laplace's differential equation are said to be harmonic functions.
- **Circular harmonics:** The solution of Laplace's equation in cylindrical coordinates when u is independent of z is known as circular harmonics and n is the degree of the harmonic.

NOTES

12.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Prove that a square plate has its faces and its edges $x = 0$ and $x = p$ ($0 < y < p$) insulated. Its edges $y = 0$ and $y = p$ are kept at temperatures zero and $f(x)$ respectively.

2. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < p$, $0 < p$, $0 < y < p$ under the boundary conditions.

3. Prove that for a semi-circular plate of radius a with boundary diameter at 0°C and surface at 100°C , show that the temperature distribution is given by

$$u = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} \frac{\sin(2n-1)\theta}{a^{2n-1}}.$$

4. A long cylinder is made of two halves, the upper half is at the temperature T_1 and the lower half at the temperature T_2 . Find the distribution of temperature inside the cylinder.

5. Find a solution of the equation

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0$$

in the form $\phi = f(r) \cos \theta$, given that (i) $-\frac{\partial \phi}{\partial r} = u \cos \theta$ when $r = a$ and

$$(ii) \quad -\frac{\partial \phi}{\partial r} = 0 \text{ when } r = \infty.$$

Long Answer Questions

1. Find the steady state temperature distribution of a thin rectangular plate bounded by the lines $x = 0$, $x = l$, $y = 0$, $y = b$ assuming that the edges $x = 0$, $x = l$ and $y = 0$ are maintained at temperature zero while the edge $y = b$ is maintained at temperature $F(x)$.

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2. Determine the steady state temperature in a rectangular plate of length a and width b with sides maintained at temperature zero while the lower end is at temperature $F(x)$ and upper one insulated.
3. Prove that heat flows in a semi-infinite rectangular plate, the end $x = 0$ being kept at temperature $T^\circ C$ and the edges $y = a$ at temperature zero, then show that the temperature at any point (x, y) is given by

$$u(x, y) = \frac{4T}{\pi} \sum_{r=1}^{\infty} \frac{1}{2r+1} \sin \frac{(2r+1)\pi y}{a} e^{-(2r+1)\pi x/a}.$$

4. If $u(x, y)$ denotes the electrostatic potential in a region bounded by the planes $x = 0$, $x = \pi$ and $y = 0$ in which there is a uniform distribution of space charge of density $\frac{h}{4\pi}$. If the planes $x = 0$ and $y = 0$, are kept at potential zero, the plane $x = \pi$ at an other fixed potential $u = 1$ and u is finite as $y \rightarrow \infty$, then find u .
5. Determine the steady state temperature at any point of a semi-circular metal plate of radius a whose circumference is maintained to a given temperature of $T^\circ C$ whereas the base is kept at zero temperature.

\mathbb{E} $u = u(r, \theta)$, the boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

with conditions (i) $u = 0$ when $\theta = 0$ for $0 \leq r < a$

(ii) u is finite when $r \rightarrow 0$

(iii) $u = T$ when $r = a$ for $0 < \theta < \pi$

6. If the surface S of a sphere of radius a is kept at a fixed distribution of electric potential $u = F(\theta)$, then find the potential u at all points in space which is assumed to be free of further charge.

12.7 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 13 THE WAVE EQUATIONS

Structure

- 13.0 Introduction
- 13.1 Objectives
- 13.2 The Wave Equations: Elementary Solutions to Wave Equations
- 13.3 Answers to Check Your Progress Questions
- 13.4 Summary
- 13.5 Key Words
- 13.6 Self Assessment Questions and Exercises
- 13.7 Further Readings

NOTES

13.0 INTRODUCTION

A wave is produced when a vibrating source periodically disturbs the first particle of a medium. This creates a wave pattern that begins to travel along the medium from particle to particle. The frequency at which each individual particle vibrates is equal to the frequency at which the source vibrates. Similarly, the period of vibration of each individual particle in the medium is equal to the period of vibration of the source. In one period, the source is able to displace the first particle upwards from rest, back to rest, downwards from rest, and finally back to rest. This complete back-and-forth movement constitutes one complete wave cycle.

The wave equation is an important second-order linear partial differential equation for the description of waves as they occur in classical physics such as mechanical waves (for example, water waves, sound waves and seismic waves) or light waves. It arises in fields like acoustics, electromagnetism, and fluid dynamics. Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. In 1746, d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation.

The wave equation in one space dimension can be derived in a variety of different physical settings. Most famously, it can be derived for the case of a string that is vibrating in a two-dimensional plane, with each of its elements being pulled in opposite directions by the force of tension. Another physical setting for derivation of the wave equation in one space dimension utilizes Hooke's Law. In the theory of elasticity, Hooke's Law is an approximation for certain materials, stating that the amount by which a material body is deformed (the strain) is linearly related to the force causing the deformation (the stress).

In this unit, you will study about the wave equations and its elementary solutions to wave equation in detail.

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13.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss about wave equations
 - Explain wave equations and its elementary solutions to wave equation
-

13.2 THE WAVE EQUATIONS: ELEMENTARY SOLUTIONS TO WAVE EQUATIONS

[A] Derivation of One-Dimensional Wave Equation

Consider a flexible string of length l tightly stretched between two points $x = 0$ and $x = l$ on x -axis, with its ends at these ends. If the string is set into small transverse vibration, the displacement say $u(x, t)$ from the x -axis of any point x of the string at any time t is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$, T

being tension and ρ the linear density.

$$\text{The equation } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots (13.1)$$

is known as one-dimensional wave equation.

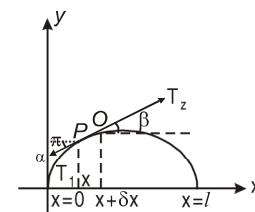


Fig. 13.1

Let the string (assumed to be perfectly flexible) of length l tightly stretched between the points $x = 0$ and $x = l$ on x -axis be distorted and then at a certain instant of time say $t = 0$, it is released and allowed to vibrate. To determine its deflection (displacement from x -axis) at any point x at any time t , let us take the following assumptions:

- (i) The string is uniform, i.e. its mass m per unit length is constant.
- (ii) The string is perfectly elastic and so offers no resistance to any bending.
- (iii) The tension T is so large that the action of gravitational force on the string is negligible.
- (iv) The motion of the string is a small transverse vibration in a vertical plane, i.e., each particle of the string moves strictly in the vertical plane so that the deflection and slope (gradient) at any point of the string are very small in absolute value.

Cosider the motion of an element PQ of length δs of the string. The string being perfectly elastic the tension T_1 at P and T_2 at Q are tangential to the curve of the string. Let T_1 and T_2 make angle α and β respectively with the horizontal.

There being no motion in the horizontal direction, we have

$$T_1 \cos \alpha = T_2 \cos \beta = T \text{ (say)} = \text{constant} \quad \dots (13.2)$$

Mass of the element PQ is $\rho \delta s$. By Newton's second law of motion we therefore have

$$T_2 \sin \beta - T_1 \sin \alpha = (\rho \delta s) \cdot \frac{\partial^2 u}{\partial t^2}, \quad \dots (13.3)$$

$\frac{\partial^2 u}{\partial t^2}$ being upward acceleration of PQ

Using (13.1), (13.2) yields $\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \frac{\rho \delta s}{T} \cdot \frac{\partial^2 u}{\partial t^2}$

$$\text{i.e.} \quad \tan \beta - \tan \alpha = \frac{\rho \delta s}{T} \frac{\partial^2 u}{\partial t^2} \quad \dots (13.4)$$

Replacing δs by δx since the gradient of the curve is very small, (13.4) gives

$$\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x = \frac{\rho \delta s}{T} \frac{\partial^2 u}{\partial t^2} \quad \dots (13.5)$$

since $\tan \alpha$ and $\tan \beta$ are slopes at x and $x + \delta x$ respectively.

$$\text{or} \quad \frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\text{i.e.} \quad \frac{u_x(x + \delta x, t) - u_x(x, t)}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

Proceeding to the limit as $\delta x \rightarrow 0$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \text{ where } \frac{1}{c^2} = \frac{\rho}{T}.$$

Note 1. $c^2 = \frac{T}{\rho}$ reveals that the constant $\frac{T}{\rho}$ is positive.

Note 2. Since u is dependent of x and t both, therefore we have used the partial derivative $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$.

Note 3. If a force $F(x, t)$ per unit of mass acts in the u -direction along the string, in addition to the tension of the string, then

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F.$$

[B] Derivative of Two-Dimensional Wave Equation

In case of a rectangular membrane, the two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots (13.6)$$

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Consider the motion of a stretched membrane supposed to be stretched and fixed along its entire boundary in the x - y plane. Let us take the following assumptions:

(i) The membrane is homogeneous, i.e., mass (say) ρ per unit area is constant.

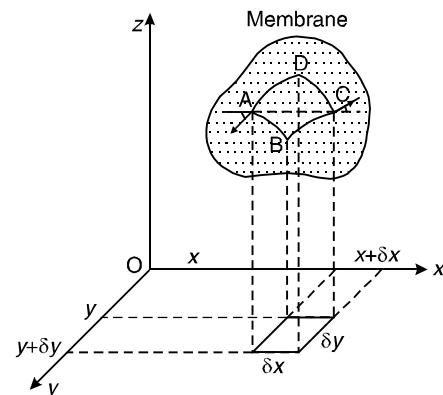


Fig. 13.2

(ii) The membrane is perfectly flexible and so thin that it offers no resistance to any bending.

(iii) The tension T per unit length caused by the stretching of the membrane is invariant during the motion and retains the same value at each of its points and in all the directions.

(vi) The deflection $u(x, y, t)$ of the membrane during the motion is negligible as compared to the size of the membrane. Also all the angles of inclination are small.

Consider the motion of an element $ABCD$ of the membrane. Let its area be $\delta x \delta y$. T being the tension per unit length, the force acting on the edges are $T \delta x$ and $T \delta y$ approximately. Also the membrane being perfectly flexible, the tensions $T \delta x$ and $T \delta y$ are tangential to the membrane. Let α, β be the inclinations of these tensions with the horizontal. Then the horizontal components of the forces at one pair of opposite edges are $T \delta y \cos \alpha$ and $T \delta y \cos \beta$. When α and β are very small $\cos \alpha \rightarrow 1$ and $\cos \beta \rightarrow 1$ so that $T \delta y \cos \alpha \rightarrow T \delta y$ and $T \delta y \cos \beta \rightarrow T \delta y$, i.e., the horizontal components of the forces at opposite edges are nearly equal and hence the motion of the particles of the membrane in horizontal direction is negligibly small. As such we assume that every particle of the membrane moves vertically.

The resultant vertical force

$$\begin{aligned} &= T \delta y \sin \beta - T \delta y \sin \alpha \\ &= T \delta y (\tan \beta - \tan \alpha) \\ &\quad (\text{as } \alpha, \beta \text{ being small } \sin \alpha = \alpha = \tan \alpha \\ &\quad \text{and } \sin \beta = \beta = \tan \beta) \end{aligned}$$

$$= T \delta y [u_x(x + \delta x, y_1) - u_x(x, y_2)] \quad \dots (13.7)$$

where u_x denotes the partial derivative of u w.r.t. x and y_1, y_2 are the values of y between y and $y + \delta y$.

Similarly, the resultant vertical force acting on the other two edges

$$= T \delta x [u_y(x_1, y + \delta y) - u_y(x_2, y)] \quad \dots (13.8)$$

where u_y denotes the partial derivative of u w.r.t. y and x_1, x_2 are the values of x between x and $x + \delta x$.

By Newton's second law of motion, we have

$$\text{Total vertical force on the element} = \rho \delta x \delta y \frac{\partial^2 u}{\partial t^2}$$

$$\text{i.e., } T \delta y [u_x(x + \delta x, y_1) - u_x(x, y_2)] + T \delta x [u_y(x_1, y + \delta y) - u_y(x_2, y)]$$

$$= \rho \delta x \delta y \frac{\partial^2 u}{\partial t^2}$$

where $\frac{\partial^2 u}{\partial t^2}$ is the acceleration of the element.

$$\text{Thus } \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \left[\frac{u_x(x + \delta x, y_1) - u_x(x, y_2)}{\delta x} \right] + \frac{T}{\rho} \left[\frac{u_y(x_1, y + \delta y) - u_y(x_2, y)}{\delta y} \right]$$

Proceeding to the limit as $\delta x \rightarrow 0$ and $\delta y \rightarrow 0$, we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} [u_{xx} + u_{yy}] = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = c^2 \nabla^2 \text{ where } \nabla^2 \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \dots (13.9)$$

$$\text{Note 1: If } u = v(x, y) e^{i\omega t}, \text{ (13.9) yields } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + k^2 v = 0 \text{ where } k^2 = \left(\frac{\omega}{c} \right)^2 \dots (13.10)$$

Note 2: The three-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \nabla^2 u \quad \dots (13.11)$$

$$\text{where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Green's Functions for the Wave Equation

The wave equation is

$$\nabla^2 \psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \quad \dots (13.12)$$

Also written as,

$$\nabla^2 \psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi = 0 \quad \dots (13.13)$$

If its solution be of the form

$$\psi(x, y, z, t) = \Psi(x, y, z) e^{\pm i\omega t} \quad \dots (13.14)$$

$$\text{then (13.12) gives, } \nabla^2 \Psi + \lambda^2 \Psi = 0 \quad \dots (13.15)$$

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which is known as *Space form of the wave equation* or *Helmholtz's equation*.

Taking $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ as the position vector of a point (x, y, z) and $\mathbf{r}' = x' \mathbf{i} + y' \mathbf{j} + z' \mathbf{k}$ as the position vector of an isolated point (x', y', z') , the *Green's function* $G(\mathbf{r}, \mathbf{r}')$ is defined as

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + \frac{1}{|\mathbf{r}' - \mathbf{r}|} \quad \dots (13.16)$$

$$\text{where } H(\mathbf{r}, \mathbf{r}') \text{ satisfies } \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} \right) H(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (13.17)$$

$$\begin{aligned} \text{Using Green's formula i.e. } \Psi(\mathbf{r}) &= \frac{1}{4\pi} \int_S \left\{ \frac{1}{|\mathbf{r}' - \mathbf{r}|} \frac{\partial \Psi(\mathbf{r}')}{\partial n} \right. \\ &\quad \left. - \Psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r}' - \mathbf{r}|} \right\} dS' \end{aligned} \quad \dots (13.18)$$

it may be shown that

$$\Psi(\mathbf{r}) = \frac{1}{4\pi} \int_S \left\{ G(\mathbf{r} - \mathbf{r}') \frac{\partial \Psi(\mathbf{r}')}{\partial n} - \Psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS' \quad \dots (13.19)$$

where \mathbf{n} is the unit outward drawn normal to the surface S .

Now we claim that the solution of space form of the wave equation under certain boundary conditions can be made to depend on the determination of the appropriate Green's function. Let us assume that $G(\mathbf{r}, \mathbf{r}')$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} \right) G(\mathbf{r}, \mathbf{r}') + \lambda^2 G(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (13.20)$$

under the assumption that $G(\mathbf{r}, \mathbf{r}')$ is finite and continuous w.r.t. either their variables x, y, z or x', y', z' for the points \mathbf{r}, \mathbf{r}' belonging to a region V bounded by a closed surface S except in \mathbf{r} -neighbourhood where there is a singularity of the type

$$\frac{e^{-i\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \text{ as } |\mathbf{r}'| \rightarrow |\mathbf{r}| \quad \dots (13.21)$$

Now $\Psi(\mathbf{r})$ being the solution of (13.15) and its partial derivatives of the first and second orders being continuous within the volume V on the closed surface S we have

$$\begin{aligned} \frac{1}{4\pi} \int_S \frac{e^{-i\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial \Psi(\mathbf{r}')}{\partial n} - \Psi(\mathbf{r}') \frac{\partial}{\partial n} e^{i\lambda|\mathbf{r}-\mathbf{r}'|} \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|} \int dS' \\ = \begin{cases} \Psi(\mathbf{r}), & \text{if } \mathbf{r} \text{ lies inside } V \\ 0, & \text{if } \mathbf{r}' \text{ does not lie inside } V \end{cases} \end{aligned} \quad \dots (13.22)$$

Using (13.21), we therefore have

$$\Psi(\mathbf{r}) = \frac{1}{4\pi} \int_S \left\{ G(\mathbf{r}, \mathbf{r}') \frac{\partial \Psi(\mathbf{r}')}{\partial n} - \Psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS' \quad \dots (13.23)$$

Taking $G(\mathbf{r}, \mathbf{r}')$ such that it satisfies the boundary condition

$$G_1(\mathbf{r}, \mathbf{r}') = 0 \quad \dots (13.24)$$

whereas the point \mathbf{r}' lies on the surface S , then (13.23) reduces to

$$\Psi(\mathbf{r}) = \frac{1}{4\pi} \int_S \Psi(\mathbf{r}') \frac{\partial G_1(\mathbf{r}, \mathbf{r}')}{\partial n} dS' \quad \dots (13.25)$$

which gives Ψ at any point \mathbf{r} within S .

Again if $G_2(\mathbf{r}, \mathbf{r}')$ is such a function satisfying $\frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} = 0$ for \mathbf{r}' lying inside S we have

$$\Psi(\mathbf{r}) = \frac{1}{4\pi} \int_S \frac{\partial \Psi(\mathbf{r}')}{\partial n} G_2(\mathbf{r}, \mathbf{r}') dS' \quad \dots (13.27)$$

which gives Ψ at any point within S provided $\frac{\partial \Psi}{\partial n}$ is known at every point of S

COROLLARY. *Green's function for Diffusion equation:*

The diffusion equation is $\frac{\partial u}{\partial t} = h^2 \nabla^2 u$... (13.28)

Let $u(\mathbf{r}, t)$ be a solution of it. Then for a volume V enclosed by a surface S , the boundary condition is

$$u(\mathbf{r}, t) = \phi(\mathbf{r}, t) \quad \dots (13.29)$$

when \mathbf{r} lies inside S .

The Initial condition is

$$u(\mathbf{r}, 0) = f(\mathbf{r}) \quad \dots (13.30)$$

when \mathbf{r} lies inside V .

If we define Green's function $G(\mathbf{r}, \mathbf{r}', t - t')$, $t > t'$

such that $\frac{\partial G}{\partial t} = h^2 \nabla^2 G$... (13.31)

With boundary condition

$$G(\mathbf{r}, \mathbf{r}', t - t') = 0 \quad \dots (13.32)$$

when \mathbf{r}' lies inside S and initial condition

$$\lim_{t \rightarrow t'} G \rightarrow 0 \quad \dots (13.33)$$

at the points of V except at the point \mathbf{r} where G takes the form

$$\frac{e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{4h^2(t-t')}}}{8[\pi h^2(t-t')]^{3/2}} \quad \dots (13.34)$$

Now G being a function of t and hence of $(t - t')$ only, (13.31) is equivalent to

$$\frac{\partial G}{\partial t'} + h^2 \nabla^2 G = 0 \quad \dots (13.35)$$

Physically interpreted $G(\mathbf{r}, \mathbf{r}', t - t')$ is the temperature at any point \mathbf{r}' at time t due to an instantaneous point source of unit strength generated at time t' of the point \mathbf{r} . Initially, the temperature of the solid is zero and the surface is kept at zero temperature.

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Equations (13.28) and (13.29) being valid for $t' < t$, can be rewritten as

$$\frac{\partial u}{\partial t'} = h^2 \nabla^2 u, t' < t \quad \dots (13.36)$$

and $u(\mathbf{r}', t) = \phi(\mathbf{r}', t)$ when \mathbf{r}' lies inside S $\dots (13.37)$

Equations (13.35) and (13.36) yield,

$$\frac{\partial}{\partial t'} (uG) = u \frac{\partial G}{\partial t'} + G \frac{\partial u}{\partial t'} = h^2 [G \nabla^2 u - u \nabla^2 G]$$

so that for an arbitrary small $\varepsilon > 0$, we find

$$\int_0^{t-\varepsilon} \left\{ \int_V \frac{\partial}{\partial t'} (uG) dv' \right\} dt' = h^2 \int_0^{t-\varepsilon} \left\{ \int_V [G \nabla^2 u - u \nabla^2 G] dv' \right\} dt' \quad \dots (13.38)$$

or, changing the order of integration,

$$\begin{aligned} & \int_V (uG)_{t'=t-\varepsilon} dv' - \int_V (uG)_{t'=0} dv' \\ &= u(\mathbf{r}, t) \int_V [G(\mathbf{r}, \mathbf{r}'), t - t']_{t'=t-\varepsilon} dv' - \int_V G(\mathbf{r}, \mathbf{r}', t) f(\mathbf{r}') dv' \end{aligned}$$

By Eqation (13.34), for $G(\mathbf{r}, \mathbf{r}', t - t')$ we have $\int_V G(\mathbf{r}, \mathbf{r}', t - t')_{t'=t-0} dv' = 1$
so that when $\varepsilon \rightarrow 0$, L.H.S. of (13.38)

$$= u(\mathbf{r}', t) - \int_V f(\mathbf{r}') G(\mathbf{r}, \mathbf{r}', t) dv'$$

Hence applying Green's theorem to the R.H.S. of (13.38) and using (13.29) and (13.32) we may find

$-h^2 \int_0^t dt' \int_S \phi(\mathbf{r}', t) \frac{\partial G}{\partial n} dS'$ in limit when $\varepsilon \rightarrow 0$ and $\frac{\partial G}{\partial n}$ denoting the derivative of G along outward drawn normal to the surface S .

We shall ultimately find,

$$u(\mathbf{r}, t) = \int_V f(\mathbf{r}') G(\mathbf{r}, \mathbf{r}', t) dv' - h^2 \int_0^t dt' \int_S \phi(\mathbf{r}', t) \frac{\partial G}{\partial n} dS' \quad \dots (13.39)$$

which gives the solution of (13.28) with boundary conditions (13.29) and (13.30).

Homogeneous and Inhomogeneous Wave Equations

In the next chapter we shall discuss Maxwell's electromagnetic field equations in the form $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\dots (13.40)$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \dots (13.41)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \dots (13.42)$$

and $\nabla \cdot \mathbf{D} = \rho \quad \dots (13.43)$

In addition to these equations, we have few more relations in a homogeneous isotropic medium.

$$\mathbf{D} = k\mathbf{E} \quad \dots (13.44)$$

$$\mathbf{B} = \mu \mathbf{H} \quad \dots (13.45)$$

and $\mathbf{J} = \sigma \mathbf{E} \quad \dots (13.46)$

The method of integration to be used here for electrodynamical equations actually leads us to homogeneous wave equation as shown below. For the purpose of their integration, we introduce a vector \mathbf{A} known as *magnetic vector potential* such that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \dots (13.47)$$

$$(13.40) \text{ and } (13.47) \text{ yield } \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \quad \dots (13.48)$$

(on changing the order of time and space derivatives).

$$\text{We can write (13.48) as } \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad \dots (13.49)$$

which follows that $\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right)$ is an irrotational vector and hence it is expressible as the gradient of a scalar point function such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi, \phi \text{ being a scalar potential} \quad \dots (13.50)$$

$$\text{or } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \quad \dots (13.51)$$

Multiply (13.41) by μ and using (13.45), we have

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t} \quad \dots (13.52)$$

$$\text{But } \nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad \dots (13.53)$$

$$\therefore \text{Equation (13.52) gives } \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t} \quad \dots (13.54)$$

Differentiation of (13.51) w.r.t. 't' yields

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \frac{\partial \phi}{\partial t} \quad \dots (13.55)$$

Elimination of $\frac{\partial \mathbf{E}}{\partial t}$ from (13.54) and (13.55) with the help of (13.44) gives

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} + \mu \kappa \left(\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \frac{\partial \phi}{\partial t} \right) \quad \dots (13.56)$$

$$\text{or } -\nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu \kappa \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} \mu \kappa \frac{\partial \phi}{\partial t} \right) \quad \dots (13.57)$$

It follows from (13.57) that curl of \mathbf{A} is specified by its divergence but div \mathbf{A} is not specified. But to find \mathbf{A} uniquely, curl \mathbf{A} and div \mathbf{A} both should be specified and hence let us assume that

$$\nabla \cdot \mathbf{A} = -\mu \kappa \frac{\partial \phi}{\partial t} \quad \dots (13.58)$$

$$\text{So that (13.57) yields } \nabla^2 \mathbf{A} - \mu \kappa \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \quad \dots (13.59)$$

$$\text{Also (13.43) with the help of (13.44) gives } \nabla \cdot \mathbf{E} = \frac{\rho}{\kappa} \quad \dots (13.60)$$

$$\text{which with the help of (13.42) becomes } \nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) = \frac{\rho}{\kappa} \quad \dots (13.61)$$

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$$\text{or} \quad -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) - \nabla^2 \phi = \frac{\rho}{\kappa} \quad \dots (13.62)$$

Elimination of $\nabla \mathbf{A}$ from (13.58) and (13.62), yields

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$$\nabla^2 \phi - \mu\kappa \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\kappa} \quad \dots (13.63)$$

If we put $c = \frac{1}{\sqrt{\mu\kappa}}$, (13.59) and (13.63) reduce to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \quad \dots (13.64)$$

$$\text{and} \quad \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\kappa} \quad \dots (13.65)$$

which have got the same form and known as *Inhomogeneous wave equations* or *Lorentz's equations* and they lead to the conclusion that magnetic vector potential \mathbf{A} and scalar potential ϕ are propagated in accordance with the equation of the form

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = -f(x, y, z, t) \quad \dots (13.66)$$

which is claimed to solve with initial conditions

$$u = 0 \text{ and } \frac{\partial u}{\partial t} = 0 \text{ at } t = 0 \quad \dots (13.67)$$

In order to use the method of Laplace transform, assume that

$$L\{u(x, y, z, t)\} = U(x, y, z, s) \text{ and } L\{f(x, y, z, t)\} = F(x, y, z, s) \quad \dots (13.68)$$

Taking Laplace transform of (13.66), we get

$$\nabla^2 U - \frac{s^2}{c^2} = -F \quad \dots (13.69)$$

$$\text{If we put } \kappa^2 = -\frac{s^2}{c^2}, \kappa = \frac{s}{c} i, i = \sqrt{-1}, \text{ then it becomes} \\ \nabla^2 U + \kappa^2 U + F = 0 \quad \dots (13.70)$$

which is *Helmholtz's equation*.

In particular case (13.70) can be taken as

$$\nabla^2 U_0 + \kappa^2 U_0 = 0 \quad \dots (13.71)$$

which is the standard form of Helmholtz's equation and its particular solution is

$$U_0 = \frac{e^{\pm i \kappa r}}{r} \quad \dots (13.72)$$

where r is the distance from a point and U_0 is determined at another point.

Using this particular solution, we can find the general solution of (13.70) as

$$U(x, y, z, s) = \frac{1}{4\pi} \iiint F(x_1, y_1, z_1) e^{\pm i \kappa r} \cdot \frac{1}{r} dv \quad \dots (13.73)$$

$$\text{where } r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \text{ and } dv = dx dy dz \quad \dots (13.73)$$

It may be verified that (13.73) satisfies (13.70).

Now substituting $\kappa = \frac{s}{c} i$ (13.74) becomes

$$U(x, y, z, s) = \frac{1}{4\pi} \iiint \frac{F(x_1, y_1, z_1)}{r} e^{\pm s/cr} dv \quad \dots (13.75)$$

Taking inverse Laplace transform of (13.75) we find the solution of inhomogeneous wave equation (13.66) as

$$u(x, y, z, s) = \frac{1}{4\pi} \iiint \frac{f\left(x, y, z, t - \frac{r}{c}\right)}{r} dv \quad \dots (13.76)$$

[Since we define the Laplace transform of $F(t)$ as $f(s) = L\{F(t)\}$

$= \int_0^\infty e^{-st} F(t) dt$. Under the condition that definite integral of $F(t)$ exists and $F(t) = 0$ for $t < 0$. Also we define the inverse transform $L^{-1}\{f(s)\} = F(t)$ and $L\left\{\frac{d^2F}{dt^2}\right\} = s^2 f - sF'(0) - s^2 F(0)$ where

$F'(0)$ is $\frac{dF}{dt}$ evaluated at $t = 0$ and

$$L^{-1}\left\{e^{as} f(s)\right\} = \begin{cases} 0, & t < a \\ F(t-a), & t > a. \end{cases} \text{ Also } L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 U$$

The equation (13.76) shows that the effects in variation of $F(x_1, y_1, z_1, t)$ do not approach the point (x, y, z) unless the time t is retarded by r/c .

As such we can write the solutions of (13.64) and (13.65) as

$$A = \frac{\mu}{4\pi} \iiint \frac{J(x_1, y_1, z_1, t) - \frac{r}{c}}{r} dv \quad \dots (13.77)$$

$$\text{and } \phi = \frac{1}{4\pi\kappa} \iiint \frac{\rho(x_1, y_1, z_1, t - \frac{r}{c})}{r} dv \quad \dots (13.78)$$

These give *retarded potentials* of electro-dynamics.

Theory of Wave Guides

Here we have to discuss the propagation of electromagnetic waves travelling in the longitudinal direction in a homogeneous isotropic medium filling the interior of a metal tube of infinite length, under the assumptions

- (i) The tube has a uniform cross-section.
- (ii) The tube is placed straight along x -axis.
- (iii) The conductivity of the tube is infinite.
- (iv) The medium is devoid of free charges.
- (v) x - y plane is the plane of cross-section of the tube.
- (vi) x -axis is along the wave guide.

Taking E_0 , H_0 , σ , κ and μ as electric intensity, magnetic intensity, conductivity, electric inductive capacity and magnetic inductive capacity respectively, we can write the fundamental Maxwell's equations in the forms

$$\nabla \times \mathbf{E}_0 = -\mu \frac{\partial \mathbf{H}_0}{\partial t}, \quad \dots (13.79)$$

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$$\nabla \times \mathbf{H}_0 = \sigma \mathbf{E}_0 + \kappa \frac{\partial \mathbf{E}_0}{\partial t} \quad \dots (13.80)$$

$$\nabla \cdot \mathbf{E}_0 = 0 \quad \dots (13.81)$$

and $\nabla \times \mathbf{H}_0 = 0 \quad \dots (13.82)$

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In order to discuss the possible oscillations propagating inside the wave guide, we can take \mathbf{E}_0 and \mathbf{H}_0 of the form $e^{i\omega t}$ such that

$$\mathbf{E}_0 = \mathbf{E} e^{i(\omega t - ax)} \quad \dots (13.83)$$

and $\mathbf{H}_0 = \mathbf{H} e^{i(\omega t - ax)} \quad \dots (13.84)$

The frequency of oscillation being given by

$$= \frac{\omega}{2\pi} \cdot a \text{ is known as the propagation constant.}$$

Now we have $\mathbf{E} = \mathbf{i} E_x + \mathbf{j} E_y = \mathbf{k} E_z \quad \dots (13.85)$

and $\mathbf{H} = \mathbf{i} H_x + \mathbf{j} H_y + \mathbf{k} H_z \quad \dots (13.86)$

If we substitute for \mathbf{E}_0 and \mathbf{H}_0 from (13.83) and (13.84) into (13.79) and (13.80) we get the Cartesian components as

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -i\omega\mu H_x; \quad \frac{\partial E_x}{\partial z} + aE_x = -i\omega\mu H_y; \\ aE_y + \frac{\partial E_x}{\partial y} &= -i\omega\mu H_z \end{aligned} \quad \dots (13.87)$$

and $\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= (\sigma + i\omega\kappa) E_x; \quad \frac{\partial H_x}{\partial z} + aH_z = (\sigma + i\omega\kappa) E_y; \\ -aH_y - \frac{\partial H_z}{\partial y} &= (\sigma + i\omega\kappa) E_x \end{aligned} \quad \dots (13.88)$

It is observed that there are two types of waves namely (i) TE (Transverse electric) or H waves, and (ii) TM (transverse magnetic) of E waves, which exist independently and satisfy equations (13.87) and (13.88).

Case I. TE or H waves are characterized by

$$E_x = 0 \text{ and } H_x \neq 0 \quad \dots (13.89)$$

which follows that in the direction of propagation, the *electric field* has no component while the *magnetic field* has a component.

If we put $E_x = 0$, (13.87) and (13.88) yield

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega\mu H_x; \quad aE_z = -i\omega\mu H_y; \quad aE_y = i\omega\mu H_z \quad \dots (13.90)$$

and $\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= 0; \quad \frac{\partial H_x}{\partial z} + aH_z = (\sigma + i\omega\kappa) E_y; \\ -aH_y - \frac{\partial H_z}{\partial y} &= (\sigma + i\omega\kappa) E_x \end{aligned} \quad \dots (13.91)$

Elimination of H_y , H_z , E_y and E_z yields

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} = -[a^2 - (\sigma + i\omega\kappa)i\omega\mu] H_x \quad \dots (13.92)$$

But in an electric region inside the wave guide $\sigma \ll \omega\kappa$ so that (13.92) reduces to

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} = -(a^2 + \omega^2 \mu\kappa) H_x = -K^2 H_x \quad \dots (13.93)$$

where $K^2 = a^2 + \omega^2 \mu\kappa$

Hence magnetic intensity H_x can be determined under given boundary conditions.

Now from (13.90) and (13.91) we can derive

$$\begin{aligned} H_y &= -\frac{a}{\kappa^2} \frac{\partial H_x}{\partial y}; \quad H_z = -\frac{a}{\kappa^2} \frac{\partial H_x}{\partial z}; \quad E_y = -\frac{i\omega\mu}{a} H_z; \\ E_z &= -\frac{i\omega\mu}{a} H_y \end{aligned} \quad \dots (13.94)$$

Thus \mathbf{E} and \mathbf{H} can be determined if H_x is known.

In case the surface of the metallic wave guide is a perfect conductor, then the tangential component of \mathbf{E} vanishes and for a *rectangular wave guide* with its sides parallel to y and z axes, $E_x = 0 = E_y$ at the surface of the wave guide. As such it follows from (13.94) that

$$\frac{\partial H_x}{\partial y} = 0 = \frac{\partial H_x}{\partial z}.$$

If n be the normal to the surface then at the surface of a wave guide of any cross-section, we have

$$\frac{\partial H_x}{\partial n} = 0.$$

Taking general coordinate system, we can write (13.93) as

$$\nabla_{y,z}^2 H_x + \kappa^2 H_x = 0 \quad \dots (13.95)$$

where $\nabla_{y,z}^2 \equiv \frac{\partial^2}{\partial y^2} \frac{\partial^2}{\partial z^2}$ subject to $E_z = -\frac{i\omega\mu}{a} H_y$.

Its solution therefore gives the possible value of K and hence the value of 'a' the constant of propagation such as

$$a = \sqrt{K^2 - \omega^2 \mu\kappa} \quad \dots (13.96)$$

These are imaginary values of 'a' which lead to possible wave propagation along wave guide while for real 'a', the wave is rapidly attenuated as it proceeds along x -axis of the wave guide.

Case II. TM or E waves are characterised by $H_x = 0$ and $E_x \neq 0$...(13.97)

Hence (13.87) and (13.88) for $H_x = 0$ yield

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0; \quad \frac{\partial E_x}{\partial z} + aE_z = -i\omega\mu H_y; \quad aE_y + \frac{\partial E_x}{\partial y} = i\omega\mu H_z \quad \dots (13.98)$$

and $\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + i\omega\kappa) E_x; \quad aH_z = (\sigma + i\omega\kappa) E_y; \quad -aH_y = (\sigma + i\omega\kappa) E_z$...(13.99)

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Elimination of E_z , E_y , H_y and H_z yields

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -(a^2 + \omega^2 \mu \kappa) E_x = -\kappa^2 E_x \quad \dots (13.100)$$

For general coordinate system, this can be written as

$$\nabla_{y,z}^2 E_x + \kappa^2 E_x = 0$$

where $\nabla_{y,z}^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$... (13.101)

In case of the surface of a perfectly conducting wave guide **E** has no tangential component at the surface, thereby giving $E_x = 0$ at the surface of the wave guide.

The determination of the possible values of K leads to the possible value of 'a' the propagation constant.

From (13.97) and (13.98) E_z , E_y , H_y and H_z can be determined.

Solution of One-Dimensional Wave Equation

The equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$... (12.102)

Its solution by D'Alembert's method has already been discussed. Here we solve it by the method of separation of variables.

Assume $u(x, t) = X(x) T(t)$... (13.103)

where X is a function of x alone and T that of t alone.

$$\therefore \frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2} \text{ and } \frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2}$$

which when substituted in (13.102) give

$$X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2}, \text{ i.e., } \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2}$$

on dividing throughout by XTc^2 .

As the variables are separated, taking λ as the constant of separation, we have

$$\begin{aligned} \frac{1}{X} \frac{d^2 X}{dx^2} &= \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \lambda \quad \text{giving} \\ \frac{d^2 X}{dx^2} &= \lambda X \text{ and } \frac{d^2 T}{dt^2} = c^2 \lambda T \end{aligned} \quad \dots (13.104)$$

There arise three possibilities:

$$(i) \lambda = 0, \text{ so that by (13.104)} \quad \frac{d^2 X}{dx^2} = 0, \frac{d^2 T}{dt^2} = 0 \quad \text{giving} \\ X = Ax + B, T = Ct + D \quad \dots (13.105)$$

$$(ii) \lambda = \mu^2, \text{ so that by (13.104)} \quad \frac{d^2 X}{dx^2} - \mu^2 X = 0, \frac{d^2 T}{dt^2} - \mu^2 c^2 T = 0 \quad \text{giving} \\ X = Ae^{\mu x} + Be^{-\mu x}, T = Ce^{\mu ct} + De^{-\mu ct} \quad \dots (13.106)$$

$$(iii) \lambda = -\mu^2, \text{ so that by (13.104)} \quad \frac{d^2 X}{dx^2} + \mu^2 X = 0, \frac{d^2 T}{dt^2} + \mu^2 c^2 T = 0 \quad \text{giving}$$

$$\begin{aligned} X &= A \cos \mu x + B \sin \mu x; \\ T &= C \cos \mu c t + D \sin \mu c t \end{aligned} \quad \dots (13.107)$$

If we impose the boundary conditions

$$u(o, t) = 0, u(l, t) = 0 \text{ for all } t \quad \dots (13.108)$$

and the initial condition

$$u(x, o) = F(x); \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x) \quad \dots (13.109)$$

then (13.108) asserts that

$$u(o, t) = X(o) T(t) = 0 \text{ and } u(l, t) = X(l) T(t) = 0$$

which imply that either $T(t) = 0$ or $X(o) = 0$ and $X(l) = 0$

Thus from (13.105) when $x = 0$, we have $B = 0$ and $X(l) = 0 = Al + B$ then giving $A = 0$.

Also from (13.106) when $x = 0$, we have

$$X(o) = 0 = A + B \text{ and } X(l) = 0 = Ae^{\mu l} + Be^{-\mu l} \text{ giving } A = B = 0.$$

In either case $A = B = 0$ give $X(x) = 0$, so that the solutions (13.105) and (13.106) fail to give that solution of (13.102) and it is the solution (13.107) which is periodic in time and is capable of giving a solution of (13.102).

Combining the two solutions of (13.107) we have a general solution of (13.102) as

$$u(x, t) = (A \cos \mu x + B \sin \mu x)(C \cos \mu c t + D \sin \mu c t) \quad \dots (13.110)$$

Now to determine the constants A, B and μ , we adjust them so as (13.110) satisfies (13.108), i.e.,

$$u(o, t) = 0 = A \{(\cos \mu c t + D \sin \mu c t)\} = 0 \text{ giving } A = 0$$

and $u(l, t) = 0 = (o + B \sin \mu l)(C \cos \mu c t + D \sin \mu c t)$ holds for

$$\mu l = n\pi \text{ i.e., } \mu = \frac{n\pi}{l}, n \text{ being a positive integer.}$$

Hence the solution of (13.102) satisfying the boundary conditions (13.108), may be written as

$$u_n(x, t) = \left(C_n \cos \frac{n\pi c t}{l} + D_n \sin \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l} \quad \dots (13.111)$$

Now applying the initial condition (13.109), (13.111) yields

$$u(x, o) = C_n \sin \frac{n\pi x}{l} = F(x)$$

$$\begin{aligned} \text{and } \left(\frac{\partial u}{\partial t} \right)_{t=0} &= \left[-\frac{n\pi c}{l} C_n \sin \frac{n\pi c t}{l} + \frac{n\pi c}{l} D_n \cos \frac{n\pi c t}{l} \right]_{t=0} \sin \frac{n\pi x}{l} \\ &= \frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x) \end{aligned}$$

It is notable that a mere single term as solution will not satisfy $u(x, o)$ and $\left(\frac{\partial u}{\partial t} \right)_{t=0}$

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In fact, the solution (13.103) is linear and homogeneous and hence it indicates that the sum of any number of distinct solutions of (13.102) is also a solution of (13.102). As such the required solution of (13.102) in place of (13.111) may be taken as

$$u(x, t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi c t}{l} + D_n \sin \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l} \quad \dots (13.112(a))$$

where $C_n \sin \frac{n\pi x}{l} = F(x)$ and $\frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x)$.

Of course, the solution (13.112(a)) satisfies (13.108) and hence together with (13.109) it provides

$$\begin{aligned} u(x, 0) &= \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = F(x) \text{ and } \left(\frac{\partial u}{\partial t} \right)_{t=0} \\ &= \sum_{n=1}^{\infty} \frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x) \end{aligned} \quad \dots (13.112(b))$$

The R.H.S.'s of (13.112(b)) being Fourier expansion, we have

$$C_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx \text{ and } \frac{n\pi c}{l} D_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} dx \quad \dots (13.113)$$

Hence (13.111) gives the required solution of (13.108), for all values of C_n and D_n given by (13.113) satisfying (13.108) and (13.109).

COROLLARY 1. In (13.109) if we assume $g(x) = 0$, the initial velocity, then (13.113) yields $D_n = 0$ and hence (13.112(b)) reduces to

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} C_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l} = \frac{1}{2} \sum_{n=1}^{\infty} C_n \left[\sin \frac{n\pi}{l} (x - ct) + \sin \frac{n\pi}{l} (x + ct) \right] \\ &= \frac{1}{2} \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} (x - ct) + \frac{1}{2} \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} (x + ct) \end{aligned} \quad \dots (13.114)$$

Thus replacing x by $x - ct$ and $x + ct$ successively in (13.111) we find two series

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} (x - ct) \text{ and } \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} (x + ct)$$

We may therefore conclude that

$$u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] \quad \dots (13.115)$$

which is the solution of wave equation (13.102), where f is the odd periodic extension of F with period $2l$.

COROLLARY 2. If we put $\lambda_n = \frac{n\pi c}{l}$ then the functions given by (13.110) are termed as the *Eigen functions* or *characteristic functions* and the values $\lambda_n = \frac{n\pi c}{l}$ are known as *Eigen Values* or *Characteristic Values* of the vibrating string and the set

$\mathbf{A} = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$ is known as the *Spectrum*.

We also observe that u_n represents a harmonic motion with frequency

$\frac{\lambda_n}{2\pi} = \frac{nc}{2l}$ cycles per unit time. We call this motion as the *nth normal mode* of the

string. In case $n = 1$, the normal mode is called as the *fundamental mode* while the normal modes for $n = 2, 3, 4, \dots$ are called as *Overtone*.

While discussing the D'Alembert's method for solving one-dimensional wave equation of the type

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots (13.116)$$

We have found a solution of it in the form

$$u(x, t) = \phi(x + ct) + \psi(x - ct) \quad \dots (13.117)$$

We require the verification of the boundary condition

$$u(0, t) = 0, u(l, t) = 0 \quad \dots (13.118)$$

and the initial conditions $u(x, 0) = f(x)$ $\dots (13.119)$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \dots (13.120)$

Obviously $u'(x, t) = \frac{\partial u}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct) \quad \dots (13.121)$

Applying (13.119) and (13.120) to (13.117) and (13.121) we get

$$u(x, 0) = \phi(x) + \psi(x) = F(x) \quad \dots (13.122)$$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = c \{ \phi'(x) - \psi'(x) \} = 0 \quad \dots (13.123)$

Assuming $g(x) = 0$ in particular (13.123) yields

$$\phi'(x) = \psi'(x) \text{ giving on integration } \phi(x) = \psi(x) + \lambda \quad \dots (13.124)$$

so that (13.122) and (13.124) render

$$\phi(x) = \frac{1}{2}[F(x) - \lambda] \quad \dots (13.125)$$

and $\psi(x) = \frac{1}{2}[F(x) + \lambda] \quad \dots (13.126)$

whence with the help of (13.125) and (13.126), (13.118) yields

$$u(x, t) = \frac{1}{2}[F(x + ct) + F(x - ct)] \quad \dots (13.127)$$

which reduces to

$$u(0, t) = \frac{1}{2}[F(ct) + F(-ct)] = 0$$

and $u(l, t) = \frac{1}{2}[F(l + ct) + F(l - ct)] = 0 \quad \dots (13.128)$

by the use of (13.119) and (13.120).

It follows from (13.128) that the function F is odd and periodic with period $(2l)$ and hence (13.127) is the solution of (13.116). Physically interpreted (13.117) represents two plane waves travelling in opposite directions with the same period.

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Example 13.1: A string is stretched between two fixed points $(0, 0)$ and $(1, 0)$ and released at rest from the positions $u = \lambda \sin \pi x$. Show that the formula for its subsequent displacement $u(x, t)$ is given by $u(x, t) = \lambda \cos(c\pi t) \sin(\pi x)$, c^2 being diffusivity.

The boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots (13.129)$$

with boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(1, t) = 0 \quad \dots (13.130)$$

and initial conditions

$$u(x, 0) = \lambda \sin \pi x = 0 \quad \text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \quad \dots (13.131)$$

By (13.129), we therefore have

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos(n\pi ct) \sin(n\pi x) \text{ where } C_n = 2 \int_0^1 \lambda \sin \pi x \sin n\pi x dx$$

$$\text{It is obvious that } C_n = 0 \text{ for } n = 2, 3, \dots \text{ but } C_1 = \lambda \int_0^1 \sin^2 \pi x dx = \lambda$$

$$\text{Hence } u(x, t) = \lambda \cos(\pi ct) \sin(\pi x).$$

Example 13.2: Show that the deflection of vibrating string of length π (its ends being fixed and $c^2 = 1$), corresponding to zero initial velocity and initial deflection $F(x) = \lambda (\sin x - \sin 2x)$ is given by $u(x, t) = \lambda (\cos t \sin x - \cos 2t \sin 2x)$.

The boundary value problem is $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ (as $c^2 = 1$) $\dots (13.132)$

with conditions $g(x) = 0$ and $F(x) = \lambda (\sin x - \sin 2x)$.

Hence by (13.126), we have (as $D_n = 0$)

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} C_n \cos nt \sin nx \quad \text{as } c = 1 \text{ and } \pi = l$$

$$\begin{aligned} \text{where } C_n &= \frac{2}{\pi} \int_0^\pi F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^\pi \lambda (\sin x - \sin 2x) \sin nx dx \\ &= \frac{2\lambda}{\pi} \int_0^\pi \sin x \sin nx dx - \frac{2\lambda}{\pi} \int_0^\pi \sin 2x \sin nx dx \end{aligned}$$

$$\text{Clearly } C_n = 0 \text{ for } n = 3, 4, 5, \dots \text{ and } C_1 = \lambda, C_2 = \lambda.$$

Hence the required deflection of the vibrating string is given by

$$u(x, t) = C_1 \cos t \sin x + C_2 \cos 2t \sin 2x = \lambda (\sin x - \sin 2x)$$

which verifies the assertion.

Example 13.3: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ if the string of length

$2a$ is originally plucked at the middle point by giving it an initial displacement d from the mean position.

The boundary value problem is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with the initial conditions:

$$F(x) = u(x, o) = \begin{cases} \frac{d}{a}, & 0 \leq x < a \\ \frac{d}{a}(2a - x), & a \leq x \leq 2a. \end{cases}$$

Also initial velocity being zero, i.e., $g(x) = 0$ we have $D_n = 0$

$$\therefore \text{By (13.126), } u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi ct}{2a} \sin \frac{n\pi x}{2a}.$$

$$\begin{aligned} \text{where } C_n &= \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{2a} \int_0^{2a} F(u) \sin \frac{n\pi u}{l} du \\ &= \frac{1}{a} \int_0^a \frac{d}{a} u \sin \frac{n\pi u}{l} du + \frac{1}{a} \int_0^{2a} (2a - u) \sin \frac{n\pi u}{2a} du = \frac{8d}{\pi^2 n^2} \sin \frac{n\pi}{2} \end{aligned}$$

which vanishes for $n = 2, 4, 6, \dots$

$$\begin{aligned} \text{Hence } u(x, t) &= \frac{8d}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{2a} \sin \frac{n\pi x}{2a} \\ &= \frac{8d}{\pi^2} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(2r-1)^2} \cos \frac{(2r-1)\pi ct}{2a} \sin \frac{(2r-1)\pi x}{2a}. \end{aligned}$$

Example 13.4: A string is stretched between two fixed points (o, o) and (l, o) and released at rest from the deflection given by

$$F(x) = \begin{cases} \frac{2\lambda}{l} x, & 0 < x < \frac{l}{2} \\ \frac{2\lambda}{l} (l-x), & \frac{l}{2} < x < l \end{cases}$$

Show that the deflection of the string at any time t is given by

$$u(x, t) = \frac{8\lambda}{\pi^2} \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

Put $a = \frac{l}{2}$ in the previous problem.

Example 13.5: The points of trisection of a string are pulled aside through a distance d on opposite sides of the equilibrium-position and the string is released from rest. Show that the displacement of the string at any subsequent time is given by

$$u(x, t) = \frac{9d}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \sin \frac{2r\pi}{3} \sin \frac{2r\pi}{3a} x \cos \frac{2r\pi ct}{3a}.$$

Also show that the mid-point of the string always remains at rest.

Consider OB as equilibrium-position of the string of length $3a$ (say), and C, D are points of trisection, which are pulled aside through a distance d on opposite sides and released.

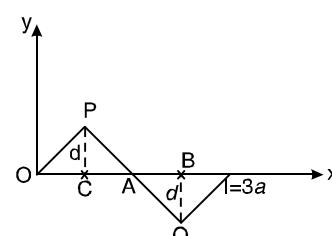


Fig. 13.3

NOTES

The boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots (13.133)$$

NOTES

Using the conceptions of coordinate geometry, the equation of line OP is

$$y = \frac{d-0}{a-0}(x-0), \text{ i.e., } y = \frac{d}{a}x$$

the equation of PQ is $y - d = \frac{-d-d}{2a-a}(x-a)$, i.e., $y = \frac{d(3a-2x)}{a}$ and the equation

of QB is $y - (-d) = \frac{0-(-d)}{3a-2a}(x-2a)$, i.e., $y = \frac{d(x-3a)}{a}$.

Hence the initial deflection is stated as

$$F(x) = \begin{cases} \frac{d}{a}x, & 0 \leq x \leq a \\ \frac{d}{a}(3a-2x), & a \leq x \leq 2a \text{ and the initial velocity } g(x) = 0 \\ & \text{so that } D_n = 0 \text{ in (13.126)} \\ \frac{d}{a}(x-3a), & 2a \leq x \leq 3a \end{cases}$$

Therefore, we have from (B)

$$\begin{aligned} C_n &= \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{3a} \frac{d}{a} \left[\int_0^a x \sin \frac{n\pi x}{3a} dx \right. \\ &\quad \left. + \int_a^{2a} (3a-2x) \sin \frac{n\pi x}{3a} dx + \int_{2a}^{3a} (x-3a) \sin \frac{n\pi x}{3a} dx \right] \\ &= \frac{18d}{n^2\pi^2} \{1 + (-1)^n\} \sin \frac{n\pi}{3} \text{ on integrating by parts and simiplifying.} \end{aligned}$$

Obviously $C_n = 0$ for $x = 1, 3, 5, 7, \dots$, i.e. being odd

and for even n , $C_n = \frac{18d}{n^2\pi^2} \cdot 2 \sin \frac{n\pi}{3} = \frac{36d}{n^2\pi^2} \sin \frac{n\pi}{3}$.

Hence by (13.126), the solution is

$$\begin{aligned} u(x, t) &= \frac{36}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{(2r)^2} \sin \frac{2r\pi}{3} \sin \frac{2r\pi}{3a} x \cos \frac{2r\pi ct}{3a} \\ &= \frac{9d}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \sin \frac{2r\pi}{3} \sin \frac{2r\pi}{3a} x \cos \frac{2r\pi ct}{3a} \end{aligned}$$

If we set $x = \frac{3a}{2}$, this result reduces to

$$u(x, t) = 0$$

since $\sin \frac{2r\pi}{3} \cdot \frac{3a}{2} = \sin r\pi = 0$ for each r .

This follows that the mid-point of the string always remains at rest.

Check Your Progress

1. What is magnetic potential?
2. What are electromagnetic waves?
3. What is d'Alembert's formula?
4. Give a three-dimensional wave equation.
5. How is TM wave characterised?

NOTES

13.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The term magnetic potential can be used for either of two quantities in classical electromagnetism: the magnetic vector potential, or simply vector potential, \mathbf{A} ; and the magnetic scalar potential ϕ . Both quantities can be used in certain circumstances to calculate the magnetic field \mathbf{B} .
2. Electromagnetic radiation (EM radiation or EMR) refers to the waves (or their quanta, photons) of the electromagnetic field, propagating (radiating) through space, carrying electromagnetic radiant energy. It includes radio waves, microwaves, infrared, (visible) light, ultraviolet, X-rays, and gamma rays.
3. In Partial Differential Equations (PDEs), d'Alembert's formula is the general solution to the one-dimensional wave equation:

$u_{tt}(x, t) = c^2 u_{xx}(x, t)$ (where subscript indices indicate partial differentiation, using the d'Alembert operator, the PDE becomes $u = 0$).

4. The three-dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \nabla^2 u$

$$\text{where } \Delta^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

5. *TM or E waves* are characterised by $H_x = 0$ and $E_x = 0$

13.4 SUMMARY

- Consider a flexible string of length l tightly stretched between two points $x=0$ and $x=l$ on x -axis, with its ends at these ends. If the string is set into small transverse vibration, the displacement say $u(x, t)$ from the x -axis of

any point x of the string at any time t is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$, T

being tension and ρ the linear density.

NOTES

The equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is known as one-dimensional wave equation.

- The tension T is so large that the action of gravitational force on the string is negligible.
- The motion of the string is a small transverse vibration in a vertical plane i.e. each particle of the string moves strictly in the vertical plane so that the deflection and slope (gradient) at any point of the string are very small in absolute value.
- $c^2 = \frac{T}{\rho}$ reveals that the constant $\frac{T}{\rho}$ is positive.
- If a force $F(x, t)$ per unit of mass acts in the u -direction along the string, in addition to the tension of the string, then

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F.$$

- In case of a rectangular membrane, the two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- The tension T per unit length caused by the stretching of the membrane is invariant during the motion and retains the same value at each of its points and in all the directions.
- The deflection $u(x, y, t)$ of the membrane during the motion is negligible as compared to the size of the membrane. Also all the angles of inclination are small.
- By Newton's second law of motion, we have total vertical force on the

$$\text{element} = \rho \delta x \delta y \frac{\partial^2 u}{\partial t^2}$$

$$\text{i.e., } T \delta y [u_x(x + \delta x, y_1) - u_x(x, y_2)] + T \delta x [u_y(x_1, y + \delta y) - u_y(x_2, y)]$$

$$= \rho \delta x \delta y \frac{\partial^2 u}{\partial t^2}$$

where $\frac{\partial^2 u}{\partial t^2}$ is the acceleration of the element.

- The three-dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \nabla^2 u$

$$\text{where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

- The wave equation is $\nabla^2\psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$

Also written as, $\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi = 0$

- Using Green's formula i.e. $\psi(\mathbf{r}) = \frac{1}{4\pi} \int_S \left\{ \frac{1}{|\mathbf{r}' - \mathbf{r}|} \frac{\partial \psi(\mathbf{r}')}{\partial n} \right.$

$$\left. - \psi(\mathbf{r}') \frac{\partial}{\partial n} \frac{1}{|\mathbf{r}' - \mathbf{r}|} \right\} dS' \text{ it may be shown that}$$

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \int_S \left\{ G(\mathbf{r} - \mathbf{r}') \frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \right\} dS' \text{ where } \mathbf{n} \text{ is the unit outward drawn normal to the surface S.}$$

- In order to discuss the possible oscillations propagating inside the wave guide, we can take \mathbf{E}_0 and \mathbf{H}_0 of the form $e^{i\omega t}$ such that

$$\mathbf{E}_0 = \mathbf{E} e^{i(\omega t - ax)} \text{ and } \mathbf{H}_0 = \mathbf{H} e^{i(\omega t - ax)}$$

- In case the surface of the metallic wave guide is a perfect conductor, then the tangential component of \mathbf{E} vanishes and for a rectangular wave guide with its sides parallel to y and z axes, $E_x = 0 = E_y$ at the surface of the wave guide.

$$\frac{\partial H_x}{\partial y} = 0 = \frac{\partial H_x}{\partial z}.$$

- If n be the normal to the surface then at the surface of a wave guide of any cross-section, we have

$$\frac{\partial H_x}{\partial n} = 0.$$

- In case of the surface of a perfectly conducting wave guide \mathbf{E} has no tangential component at the surface, thereby giving $E_x = 0$ at the surface of the wave guide.
- The determination of the possible values of K leads to the possible value of ' a ' the propagation constant.

NOTES

13.5 KEY WORDS

- Wave equation:** The wave equation is an important second-order linear partial differential equation for the description of waves—as they occur in classical physics such as mechanical waves (for example, water waves, sound waves and seismic waves) or light waves.
- Green's formula:** In mathematics, Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C .

NOTES

- **Diffusion equation:** The diffusion equation is a partial differential equation. In physics, it describes the behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle.

13.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is a wave equation? Derive it.
2. Give the derivation for one-dimensional wave equation.
3. Give the derivation for two-dimensional wave equation.
4. Give the Green's function for wave equation.
5. What are the theories of wave equations based on?

Long Answer Questions

1. A string is stretched between two fixed points $(0, 0)$ and $(1, 0)$ and released at rest from the positions $u = \lambda \sin \pi x$. Show that the formula for its subsequent displacement $u(x, t)$ is given by $u(x, t) = \lambda \cos(c\pi t) \sin(\pi x)$, c^2 being diffusivity.
2. Show that the deflection of vibrating string of length π (its ends being fixed and $c^2 = 1$), corresponding to zero initial velocity and initial deflection $F(x) = \lambda (\sin x - \sin 2x)$ is given by $u(x, t) = \lambda (\cos t \sin x - \cos 2t \sin 2x)$.
3. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ if the string of length $2a$ is originally plucked at the middle point by giving it an initial displacement d from the mean position.
4. A string is stretched between two fixed points $(0, 0)$ and $(l, 0)$ and released at rest from the deflection given by

$$F(x) = \begin{cases} \frac{2\lambda}{l} x, & 0 < x < \frac{l}{2} \\ \frac{2\lambda}{l} (l - x), & \frac{l}{2} < x < l \end{cases}$$

Show that the deflection of the string at any time t is given by

$$u(x, t) = \frac{8\lambda}{\pi^2} \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

5. The points of trisection of a string are pulled aside through a distance d on opposite sides of the equilibrium-position and the string is released from

rest. Show that the displacement of the string at any subsequent time is given by

$$u(x, t) = \frac{9d}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \sin \frac{2r\pi}{3} \sin \frac{2r\pi}{3a} x \cos \frac{2r\pi ct}{3a}.$$

The Wave Equations

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13.7 FURTHER READINGS

- Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.
- Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.
- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.

UNIT 14 THE DIFFUSION EQUATIONS

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Structure

- 14.0 Introduction
 - 14.1 Objectives
 - 14.2 The Diffusion Equations: Elementary Solutions
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-

14.0 INTRODUCTION

The diffusion equation is a partial differential equation. It describes the behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle. In mathematics, it is applicable in common to a subject relevant to the Markov process as well as in various other fields, such as the materials sciences, information science, life science, social science, and so on. These subjects described by the diffusion equation are generally called Brown problems.

The diffusion equation is continuous in both space and time. One may discretize space, time, or both space and time, which arise in application. Discretizing time alone just corresponds to taking time slices of the continuous system, and no new phenomena arise. In discretizing space alone, the Green's function becomes the discrete Gaussian kernel, rather than the continuous Gaussian kernel. In discretizing both time and space, one obtains the random walk.

In this unit, you will study about diffusion equations and its elementary solutions.

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the diffusion equation
- Explain diffusion equations and its elementary solutions

14.2 THE DIFFUSION EQUATIONS: ELEMENTARY SOLUTIONS

Assuming that the temperature at any point (x, y, z) of a solid at time t is $u(x, y, z, t)$, the thermal conductivity of the solid is K , the density of the solid is ρ and specific heat is σ , the heat equation

$$\frac{\partial u}{\partial t} = h^2 \nabla^2 u \quad \dots (14.1)$$

where $h^2 = \frac{K}{\rho\sigma} = k$ (say), k being known as diffusivity, is said to be the equation of diffusion or the Fourier equation of heat flow.

We know that heat flows from points at higher temperature to the points at lower temperature and the rate of decrease of temperature at any point varies with the direction. In other words the amount of heat say ΔH crossing an element of surface ΔS in Δt seconds is proportional to the greatest rate of decrease of the temperature u , i.e.,

$$\Delta H = K \Delta S \Delta t \left| \frac{du}{dt} \right| \quad \dots (14.2)$$

and v the velocity of heat flow is given by

$$v = -K \operatorname{grad} u = -K \cdot \nabla u \quad \dots (14.3)$$

Here $u(x, y, z, t)$ is the temperature of the solid at (x, y, z) at an instant of time t and K the thermal conductivity of the solid is a positive constant in cal/cm-sec °C units.

Let S be the surface of an arbitrary volume V of the solid. Then the total flux of heat flow across S per unit time is given by

$$H = \iint_S (-K \nabla u) \cdot \hat{n} dS \quad \dots (14.4)$$

where \hat{n} is the positive outward drawn normal vector to the element dS and the negative sign shows the increase of temperature with the increase of x so that

$\frac{\partial u}{\partial x}$ is positive and heat flows towards negative x from points of higher

temperature to those of lower temperature, thereby rendering the flux to be negative.

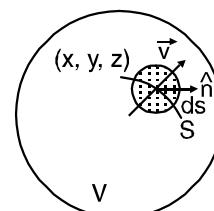


Fig. 14.1

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Now applying Gauss's divergence theorem according to which if V be the volume bounded by a closed surface S and A be a vector function of position with continuous derivative, we have the quantity of heat entering S per unit time as

$$\iint_S (K \nabla u) \cdot \hat{n} dS = \iiint_V \nabla \cdot (K \nabla u) dV \quad \dots (14.5)$$

$$i.e., \quad \iiint_V \nabla \cdot A dV = \iint_S A \cdot \hat{n} dS = \oint_S A \cdot dS. \quad \dots (14.6)$$

Taking volume element $dV = dx dy dz$, the heat contained in

$$V = \iiint_V \sigma \rho u dV.$$

\therefore The time rate of increase of heat is given by

$$\frac{\partial}{\partial t} \iiint_V \sigma \rho u dV = \iiint_V \sigma \rho \frac{\partial u}{\partial t} dV \quad \dots (14.7)$$

Equating R.H.S.'s of (14.5) and (14.7), we find

$$\iiint_V \left[\sigma \rho \frac{\partial u}{\partial t} - \nabla \cdot (K \nabla u) \right] dV = 0. \quad \dots (14.8)$$

But V being arbitrary and the integrand being assumed to be continuous the relation (14.8) will be identically zero for every point if

$$\sigma \rho \frac{\partial u}{\partial t} = \nabla \cdot (K \nabla u)$$

$$\text{or } \frac{\partial u}{\partial t} = \frac{K}{\sigma \rho} \nabla \cdot \nabla u = h^2 \nabla^2 u = k \nabla^2 u \text{ where } h^2 = k = \frac{K}{\sigma \rho}$$

$$\text{or } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k} \frac{\partial u}{\partial t} = \frac{1}{h^2} \frac{\partial u}{\partial t} \quad \dots (14.9)$$

This is three-dimensional diffusion equation.

COROLLARY 1. If the temperature within a substance be assumed to be independent of z i.e., there being no heat flow in direction of z , then (14.9) reduces to

$$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots (14.10)$$

which is known as two-dimensional diffusion equation or the equation for two-dimensional flow parallel to x - y plane.

$$\text{COROLLARY 2. Putting } \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0 \text{ in (9) we get } \frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2} \quad \dots (14.11)$$

which is the equation for the one-dimensional flow of heat along a bar.

COROLLARY 3. For steady-state heat flow, u is independent of time i.e., $\frac{\partial u}{\partial t} = 0$ and hence (14.9) reduces to

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots (14.12)$$

which is known as three-dimensional Laplace's equation.

One-Dimensional Diffusion Equation

$$\frac{\partial \mathbf{u}}{\partial t} = h^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} = k \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$$

[A] **Independent Derivation of** $\frac{\partial \mathbf{u}}{\partial t} = k \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

Consider one-dimensional flow of electricity in a long insulated cable and specify the current i and voltage E at any time in the cable by x -coordinate and time-variable t .

The potential drop E in a line-element δx of length at any point x is given by

$$-\delta E = iR \delta x + L \delta x \frac{\partial i}{\partial t} \quad \dots (14.12)$$

where R and L are respectively resistance and induction per unit length.

If C and G are respectively capacitance to earth and conductance per unit length, then we have

$$-\delta i = GE \delta x + C \delta x \frac{\partial E}{\partial t} \quad \dots (14.13)$$

Rewriting (14.12) and (14.13), $\frac{\partial E}{\partial x} + Ri + L \frac{\partial i}{\partial t} = 0 \quad \dots (14.14)$

and $\frac{\partial i}{\partial x} + GE + C \frac{\partial E}{\partial t} = 0 \quad \dots (14.15)$

Differentiating (14.14) w.r.t. x and (14.15) w.r.t. t , we have

$$\frac{\partial^2 E}{\partial x^2} + R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial x \partial t} = 0 \quad \dots (14.16)$$

and $\frac{\partial^2 i}{\partial x \partial t} + G \frac{\partial E}{\partial t} + C \frac{\partial^2 E}{\partial t^2} = 0 \quad \dots (14.17)$

Eliminating $\frac{\partial^2 i}{\partial x \partial t}$ from (14.16) and (14.17), we get

$$\frac{\partial^2 E}{\partial x^2} = CL \frac{\partial^2 E}{\partial t^2} + LG \frac{\partial E}{\partial t} - R \frac{\partial i}{\partial x} \quad \dots (14.18)$$

Again eliminating $\frac{\partial i}{\partial x}$ from (14.15) and 14.18), we find

$$\frac{\partial^2 E}{\partial x^2} = CL \frac{\partial^2 E}{\partial t^2} + (CR + GL) \frac{\partial E}{\partial t} + RGE \quad \dots (14.19)$$

Differentiation of (14.14) w.r.t. 't' and (14.15) w.r.t. 'x' yields

$$\frac{\partial^2 E}{\partial x \partial t} + R \frac{\partial i}{\partial t} + L \frac{\partial^2 i}{\partial t^2} = 0 \quad \dots (14.20)$$

and $\frac{\partial^2 i}{\partial x^2} + G \frac{\partial E}{\partial x} + C \frac{\partial^2 E}{\partial x \partial t} = 0 \quad \dots (14.21)$

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Elimination of $\frac{\partial E}{\partial x}$ and $\frac{\partial^2 E}{\partial x \partial t}$ from (14.14), (14.20) and (14.21) gives

$$\frac{\partial^2 i}{\partial x^2} = CL \frac{\partial^2 i}{\partial t^2} + (CR + GL) \frac{\partial i}{\partial t} + RG i \quad \dots (14.22)$$

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(14.19) and (14.22) follow that E and i satisfy a second order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = CL \frac{\partial^2 u}{\partial t^2} + (CR + GL) \frac{\partial u}{\partial t} + RG u \quad \dots (14.23)$$

which is known as *telegraphy equation*.

If the leakage to the ground is small then $G = 0 = L$ and hence (14.23)

reduces to $\frac{\partial^2 u}{\partial x^2} = CR \frac{\partial u}{\partial t} = \frac{1}{k} \frac{\partial u}{\partial t}$ where $k = \frac{1}{CR}$.

which is one-dimensional diffusion equation.

[B] **Solution of** $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{h}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$ **or** $\mathbf{u}_t = \mathbf{h}^2 \mathbf{u}_{xx}$

The solution of this equation by the method of separation of variables has already been discussed earlier. Here below we discuss the solution in different conditions.

[b₁] (Both the Ends of a Bar at Temperature Zero)

If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $F(x)$ in the bar, then find the temperature at a subsequent time t .

One-dimensional heat equation is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$... (14.24)

we have to find a function $u(x, t)$ satisfying (14.24) with the boundary conditions $u(0, t) = u(l, t) = 0$, $t \geq 0$, l being the length of bar ... (14.25)

and

$$u(x, 0) = F(x), 0 < x < l \quad \dots (14.26)$$

In order to apply the method of separation of variables, let us assume that $u(x, t) = X(x) T(t)$, X and T being respectively the function of x and t alone.

So that $\frac{\partial u}{\partial t} = X \frac{dT}{dt}$ and $\frac{\partial^2 u}{\partial x^2} = T \frac{d^2 X}{dx^2}$.

Their substitution in (14.24) gives

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{h^2 T} \frac{dT}{dt} \quad \dots (14.27)$$

The L.H.S. and R.H.S. of (14.27) are constants because of variables being separated and hence we can write $\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{h^2 T} \frac{dT}{dt} = -\lambda^2$ (constant of separation).

Here $\frac{1}{X} \frac{d^2 X}{dx^2} = -\lambda^2$ i.e., $\frac{d^2 X}{dx^2} + \lambda^2 X = 0$ gives $X = A \cos \lambda x + B \sin \lambda x$... (14.28)

and $\frac{1}{h^2 T} \frac{dT}{dt} = -\lambda^2$ i.e., $\frac{dT}{dt} + h^2 \lambda^2 T = 0$ gives $T = C e^{-\lambda^2 h^2 t}$... (14.29)

In view of condition (14.25) i.e., $u = 0$ at $x = 0$ or (14.28) gives $A = 0$ and λ be chosen such that $\sin \lambda l = 0$ i.e., $\lambda = \frac{n\pi}{l}$, n being an integer.

Hence the solution (14.24) i.e., $u = XT$ takes the form

$$u = B \sin \frac{n\pi}{l} x e^{-\frac{n^2 \pi^2 h^2}{l} t} \quad \dots (14.30)$$

Summing over for all values of n , this becomes

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x e^{-\frac{n^2 \pi^2 h^2}{l} t} \quad \dots (14.31)$$

Applying condition (14.24) i.e., $u(x, 0) = F(x)$ at $t = 0$, we have

$$F(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x \text{ for } 0 < x < l \quad \dots (14.32)$$

$$\text{So that } B_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx \quad \dots (14.33)$$

which is obtained by multiplying (14.32) by $\sin \frac{n\pi x}{l}$ and then integrating from $x = 0$ to $x = l$.

Hence the required solution is

$$u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-\frac{n^2 \pi^2 h^2}{l} t} \sin \frac{n\pi}{l} x \int_0^t F(u) \sin \frac{n\pi}{l} u du \quad \dots (14.33)$$

Deduction: (Insulated Faces)

If instead of the ends of a bar of length l having kept at temperature zero, they are impervious to heat and the initial temperature is the prescribed function $F(x)$ in the bar, then to find the temperature at a subsequent time t , we have the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ or } l \text{ for all } t \quad \dots (14.34)$$

$$u(x, 0) = F(x), 0 < x < l \quad \dots (14.35)$$

Then the solution follows from (14.28) as

$$u = A \cos \lambda x + B \sin \lambda x$$

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which in view of (14.34) requires $B = 0$ and $\sin \lambda x = 0$ i.e., $\lambda = \frac{n\pi}{l}$, $n = 0, 1, 2, 3\dots$

So that the general solution of the one-dimensional diffusion equation will be of the form

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 h^2}{l}t} \cos \frac{n\pi x}{l} \quad \dots (14.36)$$

where B_0 corresponds to $n = 0$.

$$\text{By (14.37), this yields, } F(x) = u(x, 0) = B_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \quad \dots (14.37)$$

from which we can easily find the coefficients

$$A_n = \frac{2}{l} \int_0^l F(x) \cos \frac{n\pi x}{l} dx \quad \dots (14.38)$$

and

$$B_0 = \frac{1}{2} A_0. \quad \dots (14.39)$$

Note 1: The temperature in a slab having initial temperature $F(x)$ and the faces $x = 0, x = \pi$ thermally insulated is given by

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-n^2 h^2 t} \cos nx. \quad \dots (14.40)$$

$$\text{where } A_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \quad \dots (14.41)$$

$$\text{and } B_0 = \frac{1}{2} A_0 = \frac{1}{\pi} \int_0^\pi F(x) dx. \quad \dots (14.42)$$

Note 2: The temperature in a slab having initial temperature $F(x)$ and the faces $x = 0, x = l$ thermally insulated is given by

$$u(x, t) = \frac{1}{l} \int_0^l F(x) dx + \frac{2}{l} \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 h^2}{l}t} \cos \frac{n\pi x}{l} \int_0^l F(x) \frac{n\pi x}{l} dx \quad \dots (14.43)$$

[b₂] (One End of a Bar at Temperature u_0 and Other at Zero Temperature)

If a bar of length l is at a temperature v_0 such that one of its ends $x = 0$ is kept at zero temperature and the other end $x = l$ is kept at temperature u_0 , then find the temperature at any point x of the bar at an instant of time $t > 0$.

or

A rod of length l and thermal conductivity h^2 is maintained at a uniform temperature v_0 . At $t = 0$ the end $x = 0$ is suddenly cooled to 0°C by application of ice and the end $x = l$ is heated to the temperature u_0 by applying steam, the rod being insulated along its length so that no heat can transfer from the sides. Find the temperature of the rod at any point at any time.

$$\text{The equation is } \frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \quad \dots (14.44)$$

With boundary conditions $u(0, t) = 0, u(l, t) = u_0$ for all t ... (14.45)

and $u(x, 0) = v_0$... (14.46)

Let the solution of (14.44) be $u(x, t) = X(x) T(t)$... (14.47)

where X is a function of x alone and T is a function of t alone.

Substituting from (14.47) $\frac{\partial u}{\partial t} = X \frac{dT}{dt}$ and $\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$, in (14.44) we get

$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{h^2 T} \frac{dT}{dt}$ where variables are separated and hence terms on either side are constants.

Now there arise these possibilities:

$$[1] \frac{d^2 X}{dx^2} = 0, \frac{dT}{dt} = 0 \text{ whence the solution is } X = Ax + B, T = C \dots (14.48)$$

$$[2] \frac{d^2 X}{dx^2} = \lambda^2 x, \frac{dT}{dt} = h^2 \lambda^2 T, \text{ the solution being } X = A e^{\lambda x} + B e^{-\lambda x}, T = C e^{h^2 \lambda^2 t} \dots (14.49)$$

$$[3] \frac{d^2 X}{dx^2} = -\lambda^2 x, \frac{dT}{dt} = -h^2 \lambda^2 T, \text{ the solution being } X = A \cos \lambda x + B \sin \lambda x, \\ T = C e^{-h^2 \lambda^2 t}. \dots (14.50)$$

The combined solution in any of the three cases is $u = XT$. But $u = XT$ increases indefinitely with time t so possibility [2] is ruled out since then $u \rightarrow 0$ as $t \rightarrow \infty$. Conclusively the possibilities [1] and [3] determine the solution of (14.44) in the form

$$u(x, t) = u_S(x) + u_T(x, t) \dots (14.51)$$

where $u_S(x)$ is the temperature distribution after a long interval of time when there exists steady state of temperature and $u_T(x, t)$ is the transient effects which die down when the time passes. Consequently there exists uniform temperature after one and $x = 0$ being kept at zero temperature and the end $x = l$ at $u = u_0$ so that

$$u_S(x) = \frac{u_0}{l} x, \text{ whence (14.51) yields } u(x, t) = \frac{u_0}{l} x + u_T(x, t) \dots (14.52)$$

with boundary conditions $u_T(0, t) = 0 = u_T(l, t)$ by (14.45) ... (14.53)

and $u_T(x, 0) =$

$$v_0 - \frac{u_0}{l} x \quad [\text{by (14.46)}] \dots (14.54)$$

Hence the possibility [3] i.e., the solution (14.50) reduces to

$$u_T(x, t) = (A \cos \lambda x + B \sin \lambda x) e^{-h^2 \lambda^2 t} \dots (14.55)$$

whence in view of (14.45), this requires $A = 0$ and $\sin \lambda l = 0$ i.e., $\lambda = \frac{n\pi}{l}$, being an integer.

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We thus obtain a solution

$$u_T(x, t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2 h^2}{l^2} t} \sin \frac{n \pi}{l} x \quad \dots (14.56)$$

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In view of (14.54), this gives $u_T(x, 0) = v_0 - \frac{u_0}{l} x = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi}{l} x$.

$$\begin{aligned} \therefore B_n &= \frac{2}{l} \int_0^l \left(v_0 - \frac{u_0}{l} x \right) \sin \frac{n \pi}{l} x dx \\ &= \frac{2}{n \pi} [v_0 - (-1)^n (v_0 - u_0)] \quad (\text{on integrating by parts}) \end{aligned}$$

Hence the general solution of (14.44) with the help of (14.52) and (14.56) is

$$u(x, t) = \frac{u_0}{l} x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [v_0 - (-1)^n (v_0 - u_0)] e^{-\frac{n^2 \pi^2 h^2}{l^2} t} \sin \frac{n \pi}{l} x \quad \dots (14.57)$$

which gives temperature at any point x of the bar at any time $t > 0$.

Note: If we set $v_0 = 0$, then (14.57) takes the form

$$u(x, t) = \frac{u_0}{l} \left[x + \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n e^{-\frac{n^2 \pi^2 h^2}{l^2} t} \sin \frac{n \pi}{l} x \right] \quad \dots (14.58)$$

[b₃] (Temperature in an Infinite Bar)

If an infinite bar of small cross-section is insulated such that there is no transfer of heat at the surface and the temperature of the bar at $t = 0$ is given by an arbitrary function $F(x)$ of x (taking the bar along x -axis), then find the temperature of the rod at any point of the bar at any time t .

The boundary value problem is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$... (14.59)

With initial condition, $u(x, 0) = F(x)$, $-\infty < x < \infty$... (14.60)

Let the solution be $u(x, t) = X(x) T(t)$... (14.61)

whence (14.59) gives $\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{h^2 T} \frac{dT}{dt} = -\lambda^2$ (say) ... (14.62)

Then the solution of (14.59) is

$$u(x, t) = XT = (A \cos \lambda x + B \sin \lambda x) e^{-h^2 \lambda^2 t} \quad \dots (14.63)$$

Here the arbitrary constants A and B being periodic may be taken as $A = A(\lambda)$, $B = B(\lambda)$ and due to the linearity and homogeneity of the heat equation we may write

$$u(x, t) = \int_0^{\infty} u(x, t, \lambda) d\lambda = \int_0^{\infty} e^{-h^2 \lambda^2 t} [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] d\lambda \quad \dots (14.64)$$

The condition (14.60) claims that

$$u(x, 0) = F(x) = \int_0^{\infty} [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] d\lambda$$

In view of Fourier's integrals we have

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\mu) \cos(\mu\lambda) d\mu \text{ and } B(\lambda) = \frac{1}{\pi} \int_0^{\infty} F(\mu) \sin(\mu\lambda) d\mu$$

so that $u(x, 0) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty F(\mu) \cos \lambda(x - \mu) d\mu \right] d\lambda$

As such (14.64) takes the form

$$\begin{aligned} u(x, t) &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty F(\mu) \cos \lambda(x - \mu) e^{-h^2 \lambda^2 t} d\mu \right] d\lambda \\ &= \frac{1}{\pi} \int_{-\infty}^\infty F(\mu) \left[\int_0^\infty e^{-h^2 \lambda^2 t} \cos \lambda(x - \mu) d\lambda \right] d\mu \dots (14.65) \end{aligned}$$

But we know that $\int_0^\infty e^{-x^2} \cos 2bx dx = \sqrt{\pi} e^{-b^2/2}$

$$\begin{aligned} \text{So that } \int_0^\infty e^{-h^2 \lambda^2 t} \cos \lambda(x - \mu) d\lambda &= \frac{\sqrt{\pi}}{2h\sqrt{t}} e^{-\left(\frac{x-\mu}{2h\sqrt{t}}\right)^2} \\ &= \frac{\sqrt{\pi}}{2h\sqrt{t}} e^{-\frac{(x-\mu)^2}{4h^2 t}} \end{aligned}$$

Hence (14.65) gives

$$u(x, t) = \frac{1}{2h\sqrt{\pi t}} \int_{-\infty}^\infty F(\mu) e^{-\frac{(x-\mu)^2}{4h^2 t}} d\mu \dots (14.66)$$

which gives the required temperature at any point at any time.

Example 14.1: Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ where $u = 0$ for $t = \infty$ and $x = 0$ or l .

Taking $u(x, l) = X(x) T(t)$, the solution of the given equation is

$$X = A \cos \lambda x + B \sin \lambda x, \quad T = C e^{-\lambda^2 t}$$

with boundary conditions, $u(0, t) = 0$ and $u(x, \infty) = 0$.

Hence putting $h = 1$ the required solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2}{l^2} t}$$

Example 14.2: Solve $\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$ under the boundary conditions

$$\theta(0, t) = \theta(l, t) = 0, \quad t > 0 \dots (14.67)$$

$$\text{and} \quad \theta(x, 0) = x, \quad 0 < x < l, \dots (14.68)$$

l being the length of the bar.

Proceeding, we get the required solution on putting $\theta(x, 0) = F(x) = x$

$$\theta(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 h^2 / l^2 t}$$

where

$$B_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^l x \sin \frac{n\pi}{l} x dx$$

$$= -\frac{2l}{n\pi} \cos n\pi$$

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$$= \begin{cases} \frac{2l}{n\pi} & \text{when } n \text{ is odd} \\ \frac{-2l}{n\pi} & \text{when } n \text{ is even} \end{cases}$$

Hence

$$\theta(x, t) = \frac{2l}{\pi} \left[e^{-\pi^2 h^2 t / l^2} \sin \frac{\pi}{l} x - \frac{1}{2} e^{-2^2 \pi^2 h^2 t / l^2} \sin \frac{2\pi}{l} x + \frac{1}{3} e^{-3^2 \pi^2 h^2 t / l^2} \sin \frac{3\pi}{l} x \dots \right]$$

Example 14.3: Find the temperature $u(x, t)$ in a bar of length l , perfectly insulated, and whose ends are kept at temperature zero while the initial temperature is given by

$$F(x) = \begin{cases} x, & 0 < x < l/2 \\ l-x, & l/2 < x < l. \end{cases}$$

The boundary value problem is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$.

With conditions $u(0, t) = u(l, t) = 0$ and $u(x, 0) =$

$$F(x) = \begin{cases} x, & 0 < x < l/2 \\ l-x, & l/2 < x < l. \end{cases}$$

Hence the required solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n F(x) \sin \frac{n\pi x}{l} dx$$

$$\text{where } B_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^{l/2} x \sin \frac{n\pi}{l} x dx + \int_{l/2}^l (l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} = \begin{cases} \frac{4l}{n^2\pi^2} & \text{for } n = 1, 5, 9, \dots \\ 0 & \text{for } n = 2, 4, 6, \dots \\ -\frac{4l}{n^2\pi^2} & \text{for } n = 3, 7, 11, \dots \end{cases}$$

Hence the solution is

$$u(x, t) = \frac{4l}{\pi^2} \left[-\frac{1}{1^2} \sin \frac{\pi x}{l} e^{-h^2 \pi^2 t / l^2} - \frac{1}{3^2} \sin \frac{3\pi x}{l} e^{-9^2 h^2 \pi^2 t / l^2} + \dots \right]$$

Note. Had we considered the case of slab with its ends $x = 0$ and $n = l$ maintained at temperature zero and initial temperature being

$$F(x) = \begin{cases} T_0, & 0 < x < l/2 \\ 0, & l/2 < x < l \end{cases}$$

Then we should have

$$\begin{aligned} B_n &= \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx = \frac{2T_0}{l} \int_0^{l/2} \sin \frac{n\pi x}{l} dx \\ &= \frac{4T_0}{n\pi} \sin^2 \frac{n\pi}{4} \end{aligned}$$

and the solution would be

$$u(x, t) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{n\pi}{4} e^{-n^2\pi^2 t/l^2} \sin \frac{n\pi x}{l}.$$

Example 14.4: Solve $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$, under the boundary conditions $u_x(0, t) = 0 = u_x(\pi, t)$ and $u_x(x, 0) = \sin x$. we have

$$\begin{aligned} u(x, t) &= B_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-n^2\pi^2 h^2 t/l^2} \text{ where } B_0 = \frac{A_0}{2} \text{ and } l = \pi \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx e^{-n^2 h^2 t} \end{aligned}$$

where $A_n = \frac{2}{\pi} \int_0^\pi \sin x \cos nx dx = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{4}{\pi(4m^2 - 1)}, & \text{when } n = 2m \end{cases}$

and $A_n = \frac{2}{\pi} \int_0^\pi \sin x dx = \frac{4}{\pi}$.

Hence the required solution is

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} (4m^2 - 1)^{-1} e^{-4m^2 h^2 t} \cos 2mx.$$

Example 14.5: The face $x = 0$ of a slab is maintained at temperature zero

and heat is supplied at constant rate at the face $x = \pi$, so that $\frac{\partial u}{\partial x} = \mu$

when $x = \pi$. If the initial temperature is zero, show that

$$u(x, t) = \mu x + \sum_{j=1}^{\infty} \frac{(-1)^j}{\left(j - \frac{1}{2}\right)^2} \sin \left(j - \frac{1}{2}\right) x e^{-(j - \frac{1}{2})^2 t}$$

where the unit of time is so chosen that $k = 1$.

Taking $u(x, t)$ as the temperature of the slab, the boundary value problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0. \quad \dots (14.69)$$

with condition $u(0, t) = 0 \quad \dots (14.70)$

$$u(x, 0) = 0 \quad \dots (14.71)$$

and $\frac{\partial}{\partial x} u(\pi, t) = \mu \quad \dots (14.72)$

Applying the method of separation of variables, the solutions of the given equation are

(i) $u = (Ae^{\lambda x} + Be^{-\lambda x}) e^{\lambda^2 t}$

(ii) $u = A_1 + B_1 x$

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$$(iii) u = (A_2 \cos \pi x + B_2 \sin \pi x) e^{-\lambda^2 t}$$

according as the constant of variation is λ^2 or $-\lambda^2$.

Here (i) is inadmissible as $u \rightarrow \infty$ when $t \rightarrow \infty$.

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(ii) alone is inadequate to give the complete solution and hence the complete solution is given by (ii) and (iii) jointly

$$i.e. \quad u(x, t) = u_S(x) + u_T(x, t) \quad \dots (14.73)$$

where $u_S(x)$ is the temperature distribution after a long period of time when the slab has reached the steady state of the temperature distribution and $u_T(x, t)$ denotes the transient effects which die down with the passage of time.

$$\text{From (ii)} \quad u_S(x) = A_1 + B_1 x \quad \dots (14.74)$$

$$\text{and from (iii)} \quad u_T(x, t) = (A_2 \cos \lambda x + B_2 \sin \lambda x) e^{-\lambda^2 t} \quad \dots (14.75)$$

$$\text{Applying (14.70), (14.74) gives } A_1 = 0 \text{ and by (14.72), (14.74) gives } \mu = B_1 \\ \text{so that } u_S = \mu x \quad \dots (14.76)$$

Thus with the help of (14.75) and (14.76), (14.77) reduces to

$$u(x, t) = \mu x + (A_2 \cos \lambda x + B_2 \sin \lambda x) e^{-\lambda^2 t} \quad \dots (14.77)$$

$$\text{Applying (14.70), i.e. } u(0, t) = 0, \text{ we get } A_2 = 0 \quad \dots (14.78)$$

$$\text{Applying (14.74) i.e. } \mu = \frac{\partial}{\partial x} u(\pi, t),$$

$$\text{we have } (\mu + \lambda B_2 \cos \lambda \pi) e^{-\lambda^2 t} = \mu$$

$$i.e. \quad \cos \lambda \pi = 0 \text{ giving } \lambda \pi = (2j - 1) \frac{\pi}{2} \text{ i.e., } \lambda = j - \frac{1}{2} \quad \dots (14.79)$$

$$\text{As such } u_T(x, t) = B_j \sin \left(j - \frac{1}{2} \right) x \cdot e^{\left(j - \frac{1}{2} \right)^2 t} \text{ where we have set } B_j = B_2.$$

Summing over all j , the general solution is

$$u_T(x, t) = \sum_{j=1}^{\infty} B_j \sin \left(j - \frac{1}{2} \right) x e^{-(j - \frac{1}{2})^2 t} \quad \dots (14.80)$$

Hence from (14.73)

$$u(x, t) = \mu x + \sum_{j=1}^{\infty} B_j \sin \left(j - \frac{1}{2} \right) x e^{-(j - \frac{1}{2})^2 t} \quad \dots (14.81)$$

$$\text{Applying the condition (3), } 0 = \mu x + \sum_{j=1}^{\infty} B_j \sin \left(j - \frac{1}{2} \right) x$$

$$i.e. \quad -\mu x = \sum_{j=1}^{\infty} B_j \sin \left(j - \frac{1}{2} \right) x \text{ so that } B_j = \frac{1}{\pi} \int_j^{\pi} (-\mu x) \sin \left(j - \frac{1}{2} \right) x dx$$

$$= \frac{2\mu}{\pi} \frac{(-1)^j}{\left(j - \frac{1}{2} \right)^2}$$

Hence (14.81) reduces to

$$u(x, t) = \mu x + \frac{2\mu}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{\left(j - \frac{1}{2}\right)^2} \sin\left(j - \frac{1}{2}\right) x e^{-(j - \frac{1}{2})^2 t}$$

which is the required relation.

NOTES

Two-Dimensional Diffusion Equation

i.e. $\frac{\partial \mathbf{u}}{\partial t} = \mathbf{h}^2 \left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} \right)$

Consider a thin rectangular plate whose surface is impervious to heat flow and which has an arbitrary function of temperature $F(x, y)$ at $t = 0$. Its four edges say $x = 0, x = a, y = 0, y = b$ are kept at zero temperature. We have to determine the subsequent temperature at a point of the plate as t increases.

The boundary value problem is $\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$... (14.82)

Subject to the conditions for all t , (i) $u(0, y, t) = 0$, (ii) $u(a, y, t) = 0$, (iii) $u(x, 0, t) = 0$, (iv) $u(x, b, t) = 0$ and the initial condition (v) $u(x, y, 0) = F(x, y)$

In order to apply the method of separation of variables, let us assume that

$$u(x, y, t) = X(x) Y(y) T(t) \quad \dots (14.83)$$

where X is a function of x alone, Y is a function of y alone and T is a function of t alone.

From (14.83) we have

$$\frac{\partial u}{\partial t} = XY \frac{dT}{dt}, \frac{\partial^2 u}{\partial x^2} = YT \frac{d^2 X}{dx^2} \text{ and } \frac{\partial^2 u}{\partial y^2} = XT \frac{d^2 Y}{dy^2}$$

Substituting them in (14.82), we find

$$\frac{1}{h^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2}, \text{ after dividing by } XYT \quad \dots (14.84)$$

In (14.84), the variables being separated, we can assume

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\lambda_1^2; \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\lambda_2^2 \text{ and } \frac{1}{h^2 T} \frac{dT}{dt} = -\lambda^2$$

so that $\lambda^2 = \lambda_1^2 + \lambda_2^2$... (14.85)

$$\lambda_1^2 + \lambda_2^2 \quad \dots (14.85)$$

The general solutions of (14.85) are

$$X = A \cos \lambda_1 x + B \sin \lambda_1 x; Y = C \cos \lambda_2 y + D \sin \lambda_2 y;$$

$$T = E e^{-\lambda^2 h^2 t} \quad \dots (14.86)$$

So that with the help of (14.86), (14.83) gives the solution of (14.82) in the form

$$u(x, y, t) = (A \cos \lambda_1 x + B \sin \lambda_1 x) (C \cos \lambda_2 y + D \sin \lambda_2 y) e^{-\lambda^2 h^2 t} \dots (14.87)$$

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In view of condition (i), $0 = u(0, y, t) = A(C \cos \lambda_2 y + D \sin \lambda_2 y) e^{-\lambda^2 h^2 t}$, giving $A = 0$.

In view of condition (ii), we claim $\sin \lambda_1 x = 0$ i.e. $\lambda_1 = \frac{m\pi}{a}$, m being an integer.

Similarly applying conditions (iii) and (iv) to (14.87), we get $C = 0$ and $\lambda^2 = \frac{n\pi}{b}$, n being an integer. As such (14.86) takes the form

$$u_{mn}(x, y, t) = B_{mn} e^{-\lambda_{mn}^2 h^2 t} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

where $\lambda^2 = \lambda_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$.

Summing over all the possible values of m and n , the general solution is

$$u(x, y, t) = \sum_{m, n=1}^{\infty} B_{mn} e^{-\lambda_{mn}^2 h^2 t} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \dots (14.88)$$

where B_{mn} are arbitrary constants to be determined by the condition (v)

$$\text{i.e. } F(x, y) = u(x, y, 0) = \sum_{m, n=1}^{\infty} B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \dots (14.89)$$

Multiplying both sides of (14.89) by $\sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dx dy$ and integrating with regard to x from 0 to a and with regard to y from 0 to b we get on using orthogonality properties of the sines,

$$B_{mn} = \frac{4}{ab} \int_0^a \int_0^b F(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dx dy \dots (14.90)$$

which gives the arbitrary constants of (14.88).

Example 14.6: A rectangular plate bounded by the lines $x = 0, y = 0, x = a, y = b$ has an initial distribution of temperature given by $F(x, y) = B \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. The edges are maintained at zero temperature and the plane faces are impervious to heat. Find the temperature at any point at any time
The general solution is

$$u(x, y, t) = \sum_{m, n=1}^{\infty} B_{mn} e^{-\lambda_{mn}^2 h^2 t} \left\{ \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \right\}$$

$$\begin{aligned} \text{where } B_{mn} &= \frac{4}{ab} \int_0^a \int_0^b F(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dx dy \\ &= \frac{4B}{ab} \int_0^a \int_0^b \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y dx dy \\ &\Leftarrow F(x, y) = B \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \end{aligned}$$

$$\therefore B_{m1} = \frac{4B}{ab} \int_0^a \frac{b}{2} \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx$$

@ $\int_0^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy = \begin{cases} 0, & \text{for } n = 2, 3, 4, \dots \\ b/2, & \text{for } n = 1. \end{cases}$

So that $B_{11} = B$.

$$\text{Also } \lambda_{11}^2 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

Hence the solution is

$$\begin{aligned} u(x, y, t) &= B_{11} e^{-\lambda_{11}^2 h^2 t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \\ &= B e^{-h^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \end{aligned}$$

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Example 14.7: A semi-infinite plate having width π has its faces insulated. The semi-infinite edges are kept at 0°C while the infinite edge is maintained at 100°C . Assuming that the initial temperature is 0°C , find the temperature at any point at any time.

Taking the diffusivity, i.e., $h^2 = 1$, the boundary value problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \dots (14.91)$$

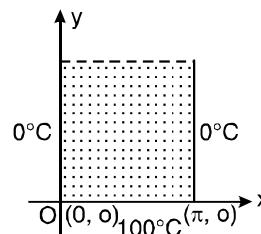


Fig. 14.2

with conditions (i) $u(0, y, t) = 0$

(ii) $u(\pi, y, t) = 0$, (iii) $u(x, y, 0) = 0$;

(iv) $u(x, 0, t) = 100$ and

(v) $|u(x, y, t)| < M$

where $0 < x < \pi$, $y > 0$, $t > 0$.

Taking Laplace transform of (14.91) and assuming $L\{u(x, y, t)\} = U(x, y, t)$,

$$\text{We have } \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = sU \quad [\text{by using condition (iii)}] \quad \dots (14.92)$$

But the finite Fourier sine transform of a function $F(x)$, $0 < x < l$ is defined as

$$f_s(x) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx, \quad n \text{ being an integer} \quad \dots (14.96)$$

Multiplying (14.92) by $\sin nx$ and integrating from 0 to π , we get

$$\int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx \, dx + \int_0^\pi \frac{\partial^2 U}{\partial y^2} \sin nx \, dx = \int_0^\pi sU \sin nx \, dx$$

Setting $\tilde{U} = \int_0^\pi U \sin nx \, dx$, this becomes

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$$-n^2 \tilde{U} + nU(\pi, y, s) \cos n\pi + nU(0, y, s) + \frac{d^2 \tilde{U}}{dy^2} = s\tilde{U} \quad \dots (14.94)$$

But from the Laplace transforms of conditions (i) and (ii), we have

$$U(0, y, s) = 0, U(\pi, y, s) = 0$$

$$\therefore (14.94) \text{ reduces to } \frac{d^2 \tilde{U}}{dy^2} - (n^2 + s)\tilde{U} = 0 \quad \dots (14.95)$$

$$\text{Its solution is } \tilde{U} = Ae^{y\sqrt{n^2+s}} + Be^{-y\sqrt{n^2+s}} \quad \dots (14.96)$$

By condition (v), \tilde{U} being bounded, as $y \rightarrow \infty$, we have $A = 0$ so that (14.97)

$$\text{yields, } \tilde{U} = Be^{-y\sqrt{n^2+s}} \quad \dots (14.98)$$

Applying condition (iv),

$$\tilde{U}(n, 0, s) = \int_0^\pi \frac{100}{s} \sin nx \, dx = \frac{100}{s} \left(\frac{1 - \cos n\pi}{n} \right) \quad \dots (14.99)$$

In (14.98) if we put $y = 0$, we get with the help of (8),

$$B = \tilde{U} = \frac{100}{s} \left(1 - \frac{\cos n\pi}{n} \right)$$

$$\text{Hence } \tilde{U} = \frac{100}{s} \frac{(1 - \cos n\pi)}{n} e^{-y\sqrt{n^2+s}}.$$

Applying Fourier sine inversion formula, we find

$$U = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{100}{s} \left(\frac{1 - \cos n\pi}{n} \right) e^{-y\sqrt{n^2+s}} \sin nx \quad \dots (14.100)$$

$$\text{Now we have } L^{-1} \left\{ e^{-y\sqrt{s}} \right\} = \frac{y}{2\sqrt{\pi t^3}} e^{-\frac{y^2}{4t}}$$

$$\text{so that } L^{-1} \left\{ e^{-y\sqrt{s+n^2}} \right\} = \frac{y}{2\sqrt{\pi t^3}} e^{-\frac{y^2}{4t}} e^{-n^2 t}$$

$$\text{Thus } L^{-1} \left\{ \frac{e^{-y\sqrt{s+n^2}}}{s} \right\} = \int_0^t \frac{y}{2\sqrt{\pi t^3}} e^{-y^2/4t} e^{-n^2 v} dv$$

$$= \frac{2}{\sqrt{\pi}} \int_{y/2\sqrt{t}}^{\infty} e^{-(p^2 + n^2 y^2/4p^2)} dp \text{ where } p^2 = \frac{y^2}{4v}.$$

Hence taking the inverse Laplace transform of (14.100) term by term, we get

$$\begin{aligned} u(x, y, t) &= \\ &\frac{400}{\pi^{3/2}} \sum_{n=1}^{\infty} \left(\frac{1 - \cos n\pi}{n} \right) \sin nx \int_{y/2\sqrt{t}}^{\infty} e^{-(p^2 + n^2 y^2/4p^2)} dp. \end{aligned}$$

Three-Dimensional Diffusion Equation

We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{h^2} \frac{\partial u}{\partial t} \quad \dots (14.101)$$

where $u = u(x, y, z, t)$.

$$\text{Let, } u(x, y, z, t) = X(x) Y(y) Z(z) T(t) \quad \dots (14.102)$$

where X, Y, Z, T being respectively the functions of x, y, z, t alone.

$$\text{From (14.102) we have } \frac{\partial u}{\partial t} = XYZ \frac{dT}{dt}, \frac{\partial^2 u}{\partial x^2} = YZT \frac{d^2 X}{dx^2}, \frac{\partial^2 u}{\partial y^2} = XZT \frac{d^2 Y}{dy^2}$$

and

$$\frac{\partial^2 u}{\partial z^2} = XYT \frac{d^2 Z}{dz^2}$$

Their substitution in (14.101) yields

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{h^2 T} \frac{dT}{dt}$$

$= -\lambda^2$ (say) as variables are separable.

$$\text{Now taking } \frac{1}{X} \frac{d^2 X}{dx^2} = -\lambda_1^2, \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\lambda_2^2, \frac{1}{Z} \frac{d^2 Z}{dz^2} = -\lambda_3^2$$

so that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda^2$, we get the solutions

$$X = A_1 \cos \lambda_1 x + B_1 \sin \lambda_1 x = a \cos (\lambda_1 x + \alpha_{\lambda_1})$$

Similarly $Y = b \cos (\lambda_2 x + \alpha_{\lambda_2})$, $Z = c$ and $(\lambda_3 x + \alpha_{\lambda_3})$ and

$$T = d e^{-\lambda^2 h^2 t} = d e^{-(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) h \lambda^2 t}$$

Hence for all values of t , the general solution of is

$$u(x, y, z, t) = \sum_{\lambda_1=0}^{\infty} \sum_{\lambda_2=0}^{\infty} \sum_{\lambda_3=0}^{\infty} A_{\lambda_1 \lambda_2 \lambda_3} \cos (\lambda_1 x + \alpha_{\lambda_1}) \cos (\lambda_2 x + \alpha_{\lambda_2}) \cos (\lambda_3 x + \alpha_{\lambda_3}) e^{-h^2 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) t}$$

Check Your Progress

1. Define thermal conduction.
2. What is diffusion equation?
3. Give the equation for temperature in a slab that have initial temperature $F(x)$ and the faces $x = 0, x = 1$.
4. How is potential drop E at any time element of length at any point x given?

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14.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

- Thermal conduction is defined as the transport of energy due to random molecular motion across a temperature gradient. It is distinguished from energy transport by convection and molecular work in that it does not involve macroscopic flows or work-performing internal stresses.
- The diffusion equation is a partial differential equation. It describes the behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle.
- The temperature in a slab having initial temperature $F(x)$ and the faces $x=0, x=l$ thermally insulated is given by

$$u(x, t) = \frac{1}{l} \int_0^l F(x) dx + \frac{2}{l} \sum_{n=1}^{\infty} e^{-\frac{n^2 \pi^2 h^2}{l} t} \cos \frac{n \pi x}{l} \int_0^l F(x) \frac{n \pi x}{l} dx$$

- The potential drop E in a line-element $d x$ of length at any point x is given

$$\text{by } -dE = iR dx + L \delta x \frac{\partial i}{\partial t}$$

where R and L are respectively resistance and induction per unit length.

If C and G are respectively capacitance to earth and conductance per unit length, then we have

$$-\delta i = GE \delta x + C \delta x \frac{\partial E}{\partial t}$$

14.4 SUMMARY

- Assuming that the temperature at any point (x, y, z) of a solid at time t is $u(x, y, z, t)$, the thermal conductivity of the solid is K , the density of the solid is r and specific heat is s , the heat equation

$$\frac{\partial u}{\partial t} = h^2 \nabla^2 u$$

where $h^2 = \frac{K}{\rho s}$ (say), k being known as diffusivity, is said to be the equation of diffusion or the Fourier equation of heat flow.

- Heat flows from points at higher temperature to the points at lower temperature and the rate of decrease of temperature at any point varies with the direction. In other words the amount of heat say DH crossing an

element of surface dS in Dt seconds is proportional to the greatest rate of decrease of the temperature u , i.e.,

$$\Delta H = K \Delta S \Delta t \left| \frac{du}{dt} \right|$$

and v the velocity of heat flow is given by

$$v = -K \operatorname{grad} u = -K \cdot \nabla u$$

- Let S be the surface of an arbitrary volume V of the solid. Then the total flux of heat flow across S per unit time is given by

$$H = \iint_S (-K \nabla u) \cdot \hat{n} dS$$

where \hat{n} is the positive outward drawn normal vector to the element dS and the negative sign shows the increase of temperature with the increase of x so that $\frac{\partial u}{\partial x}$ is positive and heat flows towards negative x from points of higher temperature to those of lower temperature, thereby rendering the flux to be negative.

- The quantity of heat entering S per unit time as

$$\begin{aligned} \iint_S (K \nabla u) \cdot \hat{n} dS &= \iiint_V \nabla \cdot (K \nabla u) dV \\ \text{i.e., } \iiint_V \nabla \cdot A dV &= \iint_S A \cdot \hat{n} dS = \iint_S A \cdot dS. \end{aligned}$$

Taking volume element $dV = dx dy dz$, the heat contained in

$$V = \iiint_V \sigma \rho u dV.$$

- For steady-state heat flow, u is independent of time i.e., $\frac{\partial u}{\partial t} = 0$ and hence reduces to

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- Consider one-dimensional flow of electricity in a long insulated cable and specify the current i and voltage E at any time in the cable by x -coordinate and time-variable t .
- The potential drop E in a line-element dx of length at any point x is given

$$\text{by } -dE = iR dx + L \delta x \frac{\partial i}{\partial t} \dots (14.12)$$

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where R and L are respectively resistance and induction per unit length. If C and G are respectively capacitance to earth and conductance per unit length, then we have

$$-\delta i = GE \delta x + C \delta x \frac{\partial E}{\partial t}$$

- If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $F(x)$ in the bar, then find the temperature at a subsequent time t .

- The temperature in a slab having initial temperature $F(x)$ and the faces $x=0, x=p$ thermally insulated is given by

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-n^2 h^2 t} \cos nx.$$

- The temperature in a slab having initial temperature $F(x)$ and the faces $x=0, x=l$ thermally insulated is given by

$$u(x, t) = \frac{1}{l} \int_0^l F(x) dx + \frac{2}{l} \sum_{n=1}^{\infty} e^{-\frac{n^2 \pi^2 h^2}{l} t} \cos \frac{n \pi x}{l} \int_0^l F(x) \frac{n \pi x}{l} dx$$

- If a bar of length l is at a temperature v_0 such that one of its ends $x=0$ is kept at zero temperature and the other end $x=l$ is kept at temperature u_0 , then find the temperature at any point x of the bar at an instant of time $t>0$.

- A rod of length l and thermal conductivity h^2 is maintained at a uniform temperature v_0 . At $t=0$ the end $x=0$ is suddenly cooled to 0°C by application of ice and the end $x=l$ is heated to the temperature u_0 by applying steam, the rod being insulated along its length so that no heat can transfer from the sides. Find the temperature of the rod at any point at any time.

The equation is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0$

14.5 KEY WORDS

- **Diffusion equation:** The diffusion equation is a partial differential equation. It describes the behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle.
- **Thermal conduction:** Thermal conduction is defined as the transport of energy due to random molecular motion across a temperature gradient.

14.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write the diffusion equation of heat flow.
2. Give one dimensional diffusion equation for independent derivation.
3. Give one dimensional diffusion equation for both the ends of a bar at temperature zero.
4. Write about insulated faces.
5. What is temperature in infinite bar?

Long Answer Questions

1. Elaborate on one-dimensional diffusion equation.
2. Explain the two-dimensional diffusion equation.
3. Discuss about three-dimensional diffusion equation.

4. Solve $\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$ under the boundary conditions

$$\theta(0, t) = \theta(l, t) = 0, t > 0 \quad \dots (14.67)$$

$$\text{and} \quad \theta(x, 0) = x, 0 < x < l, \quad \dots (14.68)$$

l being the length of the bar.

5. Find the temperature $u(x, t)$ in a bar of length l , perfectly insulated, and whose ends are kept at temperature zero while the initial temperature is given by

$$F(x) = \begin{cases} x, & 0 < x < l/2 \\ l - x, & l/2 < x < l. \end{cases}$$

6. A rectangular plate bounded by the lines $x = 0, y = 0, x = a, y = b$ has an initial distribution of temperature given by $F(x, y) = B \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. The edges are maintained at zero temperature and the plane faces are impervious to heat. Find the temperature at any point at any time

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14.7 FURTHER READINGS

Sneddon, I. N. 1986. *Elements of Partial Differential Equations*. New York: McGraw Hill Education.

Raisinghania, M. D. 2001. *Advanced Differential Equations*. New Delhi: S. Chand & Company Ltd.

NOTES

- Rao, K. Sankara. 2006. *Introduction to Partial Differential Equations*, 2nd Edition. New Delhi: PHI Learning Pvt. Ltd.
- Sharma, J. N. and K. Singh. 2001. *Partial Differential Equations for Engineers and Scientists*. Chennai: Narosa Publishing House.
- Boyce, W. E. and Richard C. DiPrima. 1986. *Elementary Differential Equations and Boundary Value Problems*. New York: John Wiley and Sons, Inc.
- Ross, S. L. 1984. *Differential Equations*, 3rd Edition. New York: John Wiley and Sons.