Answers to Miscellaneous Problems

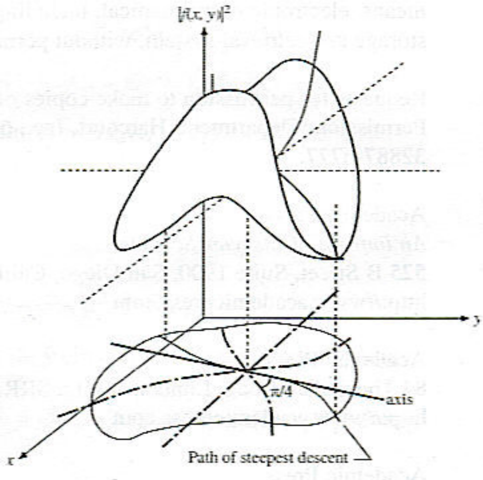
MATHEMATICAL METHODS FOR PHYSICISTS

FIFTH EDITION

George B. Arfken

Miami University

Oxford, Ohio



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1.1.1 If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, $\mathbf{D} = \mathbf{A} - \mathbf{B}$, then $\mathbf{A} = (\mathbf{C} + \mathbf{D})/2$, $\mathbf{B} = (\mathbf{C} - \mathbf{D})/2$.

1.1.2 $A_x = A_y = A_z = 1$.

1.1.3 $A_x = A_y = \sqrt{2}/2$.

1.1.6 From $\mathbf{A} = \mathbf{B}$ we get for the components $A_i = B_i$, $i = 1, 2, 3$.

1.1.7 Add $P_2 = (0, 1, 0)$ to $P_3 - P_1 = (1, -1, 0) - (-1, 0, 2) = (2, -1, -2)$ to get the solution $P_4 = (2, 0, -2)$ in the text. Note that then opposite sides $P_4 - P_3 = (2, 0, -2) - (1, -1, 0) = (1, 1, -2) = P_2 - P_1$ are parallel, as are

$$P_3 - P_1 = (2, -1, -2) = P_4 - P_2 = (2, 0, -2) - (0, 1, 0);$$

and for $P_5 = P_1 + [P_3 - P_2] = (0, -2, 2)$,

$$P_5 - P_1 = (0, -2, 2) - (-1, 0, 2) = (1, -2, 0) = P_3 - P_2,$$
$$P_5 - P_3 = (-1, -1, 2) = P_1 - P_2.$$

1.1.8 The triangle sides are given by

$$\mathbf{AB} = \mathbf{B} - \mathbf{A}, \ \mathbf{BC} = \mathbf{C} - \mathbf{B}, \ \mathbf{CA} = \mathbf{A} - \mathbf{C}$$

with $\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = (\mathbf{B} - \mathbf{A}) + (\mathbf{C} - \mathbf{B}) + (\mathbf{A} - \mathbf{C}) = 0$.

1.1.11 If $\mathrm{v}'_i = \mathrm{v}_i - \mathrm{v}_1$, $\mathrm{r}'_i = \mathrm{r}_i - \mathrm{r}_1$, then $\mathrm{v}'_i = H_0(\mathrm{r}_i - \mathrm{r}_1) = H_0\mathrm{r}'_i$.

1.2.1 (a) Using Eq. (1.9) get

$$(A_x \cos\varphi + A_y \sin\varphi)^2 + (-A_x \sin\varphi + A_y \cos\varphi)^2 = A_x^2 + A_y^2.$$

by multiplying out and using $\cos\varphi^2 + \sin\varphi^2 = 1$.

(b) Decomposing $\hat{\mathbf{x}}' = \hat{\mathbf{x}} \cos\varphi + \hat{\mathbf{y}} \sin\varphi$ into cartesian components, we have $A \cos\alpha' \equiv A_{x'} = \mathbf{A} \cdot \hat{\mathbf{x}} \cos\varphi + \mathbf{A} \cdot \hat{\mathbf{y}} \sin\varphi = A_x \cos\varphi + A_y \sin\varphi = A \cos\alpha \cos\varphi + A \sin\alpha \sin\varphi = A \cos(\alpha - \varphi)$ proving that $\alpha' = \alpha - \varphi$.

1.3.2 From Ex. 1.2.1 we have the component decomposition $\hat{\mathbf{x}}' = \hat{\mathbf{x}} \cos\varphi + \hat{\mathbf{y}} \sin\varphi$. Squaring this we get for arbitrary angle φ, $\hat{\mathbf{x}}'^2 = 1 = \hat{\mathbf{x}}^2 \cos^2\varphi + \hat{\mathbf{y}}^2 \sin^2\varphi + 2\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} \cos\varphi \sin\varphi = 1 + 2\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} \cos\varphi \sin\varphi$, i.e. $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$ provided $\varphi \neq 0, \pi/2$.

1.3.3 (a) The surface is a plane passing through the tip of $\mathbf{a}$ and perpendicular to $\mathbf{a}$.

(b) The surface is a sphere having $\mathbf{a}$ as a diameter: $(\mathbf{r}-\mathbf{a})\cdot\mathbf{r} = (\mathbf{r}-\mathbf{a}/2)^2 - \mathbf{a}^2/4 = 0$.

1.4.1 $\mathbf{A}\cdot\mathbf{B} = -36$. $\mathbf{A}\times\mathbf{B} = -2\hat{\mathbf{x}} + 28\hat{\mathbf{y}} - 18\hat{\mathbf{z}}$.

1.4.2 $A^2 = \mathbf{A}^2 = (\mathbf{B}-\mathbf{C})^2 = B^2 + C^2 - 2BC\cos(\hat{\mathbf{B}}, \hat{\mathbf{C}})$.

1.4.3 $0 = (\mathbf{A}+\mathbf{B})\times(\mathbf{A}+\mathbf{B}) = \mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{A}$.

1.4.5 $\mathbf{P}$ and $\mathbf{Q}$ are antiparallel; $\mathbf{R}$ is perpendicular to both $\mathbf{P}$ and $\mathbf{Q}$.

1.4.7 $(\mathbf{A}\times\mathbf{B})^2 = A^2B^2\sin^2(\hat{\mathbf{A}}, \hat{\mathbf{B}}) = A^2B^2(1-\cos^2(\hat{\mathbf{A}}, \hat{\mathbf{B}})) = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$.

1.4.9 $\mathbf{A} = \mathbf{U}\times\mathbf{V} = 3\hat{\mathbf{y}} - 3\hat{\mathbf{z}}$,
$\mathbf{A}$ (normalized) $= \pm(1/\sqrt{2})(\hat{\mathbf{y}} + \hat{\mathbf{z}})$.

1.4.11 The sides are $\mathbf{C}-\mathbf{A} = (5, 2, 8) - (2, 1, 5) = (3, 1, 3)$ and $\mathbf{B}-\mathbf{A} = (4, 8, 2) - (2, 1, 5) = (2, 7, -3)$. Twice the area is given by their cross product

$$2a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3 & 1 & 3 \\ 2 & 7 & -3 \end{vmatrix} = (-24, 15, 19).$$

Thus $a = \sqrt{24^2 + 15^2 + 19^2}/2 = \sqrt{1162}/2 = 17.04$.

1.4.14 $\angle BAC = 45°$, arc $BC = \pi/3$ radians,
$\angle CBA = 125.3°$, arc $CA = \pi/2$ radians,
$\angle ACB = 35.3°$, arc $AB = \pi/4$ radians.

1.4.15 Cross $\mathbf{A}-\mathbf{B}-\mathbf{C} = 0$ into $\mathbf{A}$ to get $-\mathbf{A}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}$, or $C\sin\beta = B\sin\gamma$, etc.

1.4.16 $\mathbf{B} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$.

1.5.1 Volume $= 18\,\mathrm{cm}^3$.

1.5.4 (a) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0$, A is the plane of B and C. The parallelpiped has zero height above the BC plane and therefore zero volume.

(b) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$.

1.5.5 The BAC-CAB rule gives $\mathbf{L} = m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}) = m[r^2\boldsymbol{\omega} - \mathbf{r}\cdot\boldsymbol{\omega}\mathbf{r}] = mr^2[\boldsymbol{\omega} - \hat{\mathbf{r}}\cdot\boldsymbol{\omega}\hat{\mathbf{r}}]$.

1.5.6 $T = \frac{1}{2}m\left[r^2\omega^2 - (\mathbf{r}\cdot\boldsymbol{\omega})^2\right]$ follows from $(\mathbf{A}\times\mathbf{B})^2 = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$.

1.5.7 Using $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot\mathbf{c}\mathbf{b} - \mathbf{a}\cdot\mathbf{b}\mathbf{c}$, etc. get $[\mathbf{a}\cdot\mathbf{c}\mathbf{b} - \mathbf{a}\cdot\mathbf{b}\mathbf{c}] + [\mathbf{b}\cdot\mathbf{a}\mathbf{c} - \mathbf{b}\cdot\mathbf{c}\mathbf{a}] + [\mathbf{c}\cdot\mathbf{b}\mathbf{a} - \mathbf{c}\cdot\mathbf{a}\mathbf{b}] = 0$.

1.5.8 (a) $\hat{\mathbf{r}} \cdot \mathbf{A}_r = \mathbf{A} \cdot \hat{\mathbf{r}}$.

(b) $\hat{\mathbf{r}} \cdot \mathbf{A}_t = -\hat{\mathbf{r}} \cdot [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{A})] = 0$.

1.5.9 The triple scalar product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ is the volume spanned by the vectors.

1.5.10 $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = -120$,
$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = -60\hat{\mathbf{x}} - 40\hat{\mathbf{y}} + 50\hat{\mathbf{z}}$,
$\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 24\hat{\mathbf{x}} + 88\hat{\mathbf{y}} - 62\hat{\mathbf{z}}$,
$\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = 36\hat{\mathbf{x}} - 48\hat{\mathbf{y}} + 12\hat{\mathbf{z}}$.

1.5.11 $\mathbf{D} \cdot (\mathbf{B} \times \mathbf{C}) = a\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, etc.

1.5.12 $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = [(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}] \cdot \mathbf{D} = [(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}] \cdot \mathbf{D}$
$= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$.

1.6.1 (a) $-3(14)^{-5/2}(\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}})$. (b) $3/196$.

(c) $-1/(14)^{1/2}, -2/(14)^{1/2}, -3/(14)^{1/2}$.

1.6.4 $d\mathbf{F} = \mathbf{F}(\mathbf{r} + d\mathbf{r}, t + dt) - \mathbf{F}(\mathbf{r}, t) = \mathbf{F}(\mathbf{r} + d\mathbf{r}, t + dt) - \mathbf{F}(\mathbf{r}, t + dt) + \mathbf{F}(\mathbf{r}, t + dt) - \mathbf{F}(\mathbf{r}, t) = (d\mathbf{r} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t}\, dt$.

1.6.5 $\nabla(uv) = v\nabla u + u\nabla v$ follows from the product rule of differentiation.

(a) Since $\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v = 0$, ∇u and ∇v are parallel so that $(\nabla u) \times (\nabla v) = 0$, and vice versa.

(b) If $(\nabla u) \times (\nabla v) = 0$, the two-dimensional volume spanned by ∇u and ∇v, also given by the Jacobian

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix},$$

vanishes.

1.7.1 (a) From $\dot{\mathbf{r}} = \omega r(-\hat{\mathbf{x}} \sin \omega t + \hat{\mathbf{y}} \cos \omega t)$, we get
$\mathbf{r} \times \dot{\mathbf{r}} = \hat{\mathbf{z}}\omega r^2(\cos^2 \omega t + \sin^2 \omega t) = \hat{\mathbf{z}}\omega r^2$.

(b) Differentiating $\dot{\mathbf{r}}$ above we get $\ddot{\mathbf{r}} = -\omega^2 r(\hat{\mathbf{x}} \cos \omega t + \hat{\mathbf{y}} \sin \omega t) = -\omega^2 \mathbf{r}$.

1.7.5 The product rule of differentiation in conjunction with $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, etc. gives

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

1.7.6 $$\nabla \cdot (\mathbf{r}r^{-3}) = 3r^{-3} - 3r^{-4}\mathbf{r} \cdot \nabla r = 3r^{-3} - 3r^{-3} = 0$$

for $r > 0$. Since $\mathbf{E} \sim \nabla(1/r)$, $\nabla \cdot \mathbf{E} \sim \nabla^2(1/r) = -4\pi\delta(\mathbf{r})$. At $r = 0$ the charge generates a singularity.

1.8.1 The chain rule gives $\frac{\partial}{\partial x_i'} = \frac{\partial x_k}{\partial x_i'}\frac{\partial}{\partial x_k}$, i.e. the gradient transforms like a covariant vector.

1.8.3 If $\nabla \times \mathbf{A} = 0$, then $\nabla \cdot (\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot (\nabla \times \mathbf{r}) = 0 - 0 = 0$.

1.8.4 From $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ we get $\nabla \cdot (\boldsymbol{\omega} \times \mathbf{r}) = -\boldsymbol{\omega} \cdot (\nabla \times \mathbf{r}) = 0$.

1.8.7 If $\mathbf{L} = -i\mathbf{r} \times \nabla$, then the determinant form of the cross product gives $\mathbf{L}_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$, (in units of $\hbar$), etc.

1.8.9 $[\mathbf{a} \cdot \mathbf{L}, \mathbf{b} \cdot \mathbf{L}] = a_j[L_j, L_k]b_k = i\varepsilon_{jkl}a_jb_kL_l = i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{L}$.

1.8.13 Apply the bac-cab rule to get

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla(A^2) - (\mathbf{A} \cdot \nabla)\mathbf{A}.$$

The factor $1/2$ occurs because ∇ operates only on one $\mathbf{A}$.

1.8.14 $$\nabla(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \nabla(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{r} = \left((\mathbf{A} \times \mathbf{B})_x \frac{\partial x}{\partial x}, ..\right) = \mathbf{A} \times \mathbf{B}.$$

1.8.16 $\mathbf{E}(\mathbf{r}) = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})-\mathbf{p}}{4\pi\varepsilon_0 \mathbf{r}^3}$

1.9.2 From the *BAC-CAB* rule for a triple vector product

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - (\nabla \cdot \nabla)\mathbf{V}.$$

Since ∇ acts on the vector field $\mathbf{V}$, and $\mathbf{a} \cdot \mathbf{c}\,\mathbf{b} = \mathbf{b}\,\mathbf{a} \cdot \mathbf{c}$, we have to write ∇ first.

1.9.3 $\nabla \times (\varphi\nabla\varphi) = \nabla\varphi \times \nabla\varphi + \varphi\nabla \times (\nabla\varphi) = 0 + 0 = 0$.

1.9.7 Using Exercise 1.7.5 $\nabla \cdot (\nabla u \times \nabla v) = (\nabla v) \cdot (\nabla \times \nabla u) - (\nabla u) \cdot (\nabla \times \nabla v) = 0 - 0 = 0$.

1.9.8 $\nabla^2\varphi = \nabla \cdot \nabla\varphi = 0$, and $\nabla \times \nabla\varphi = 0$.

1.9.9 $(\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla) = \mathbf{r} \cdot (\nabla \times (\mathbf{r} \times \nabla)) = \mathbf{r} \cdot [\nabla \cdot \nabla\mathbf{r} - \nabla(\nabla \cdot \mathbf{r})] = \mathbf{r}^2\nabla^2 + \mathbf{r} \cdot \nabla - 3\mathbf{r} \cdot \nabla - r^2\frac{\partial^2}{\partial r^2} = r^2\nabla^2 - 2r\frac{\partial}{\partial r} - r^2\frac{\partial^2}{\partial r^2}$.

1.9.11 Since $\nabla \times \nabla\psi = 0$, $\mathbf{A} \times \mathbf{A} = 0$, and $\mathbf{p} \times \mathbf{A}\psi = -\mathbf{A} \times \mathbf{p}\psi - i(\nabla \times \mathbf{A})\psi$ with $\nabla \times \mathbf{A} = \mathbf{B}$, we have $(\mathbf{p} - e\mathbf{A}) \times (\mathbf{p} - e\mathbf{A})\psi = -e(\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p})\psi = -e(-\mathbf{A} \times \mathbf{p} - i\nabla \times \mathbf{A} + \mathbf{A} \times \mathbf{p})\psi = ie\mathbf{B}\psi$.

1.10.1 (a) $15k$. (b) $15k$. (c) $15k$.

1.10.2 (a) $-\pi$. (b) $+\pi$.

1.10.4 Zero.

1.10.5 1.

1.11.1 For a constant vector $\mathbf{a}$, its divergence is zero. Using Gauss' theorem we have

$$0 = \int_V \nabla \cdot \mathbf{a}\, d\tau = \mathbf{a} \cdot \int_S d\boldsymbol{\sigma},$$

where S is the closed surface of the finite volume V. As $\mathbf{a} \neq 0$ is arbitrary, $\int_S d\boldsymbol{\sigma} = 0$ follows.

1.11.2 From $\nabla \cdot \mathbf{r} = 3$ in Gauss' theorem we have

$$\int_V \nabla \cdot \mathbf{r}\, d\tau = 3 \int_V d\tau = 3V = \int_S \mathbf{r} \cdot d\boldsymbol{\sigma},$$

where V is the volume enclosed by the closed surface S.

1.11.3 Cover the closed surface by small (in general curved) adjacent rectangles S_i whose circumference are formed by four lines L_i each. Then Stokes' theorem gives $\int_S (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \sum_i \int_{S_i} (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \sum_i \int_{L_i} \mathbf{A} \cdot d\mathbf{l} = 0$ because all line integrals cancel each other.

1.11.4 Gauss' theorem gives $0 = \int_V \nabla \cdot \nabla\psi d\tau = \int_S \nabla\psi \cdot d\boldsymbol{\sigma}$.

1.11.5 Apply the corresponding definition of the divergence to the vector $\nabla\varphi$ in conjunction with the mean value theorem for the volume integral.

1.11.7 Apply Gauss' theorem to $\nabla \cdot (\varphi\mathbf{E}) = \nabla\varphi \cdot \mathbf{E} + \varphi\nabla \cdot \mathbf{E} = -\mathbf{E}^2 + \varepsilon_0^{-1}\varphi\rho$, where $\int_{S\to\infty} \varphi\mathbf{E} \cdot d\boldsymbol{\sigma} = 0$.

1.11.9 Using Gauss' theorem we have

$$\begin{aligned}\int \mathbf{H} \cdot \mathbf{B}\, d\tau &= \int \mathbf{H} \cdot (\nabla \times \mathbf{A})\, d\tau \\ &= -\int \nabla \cdot (\mathbf{H} \times \mathbf{A})\, d\tau + \int (\nabla \times \mathbf{H}) \cdot \mathbf{A} \\ &= -\int_{S\to\infty} (\mathbf{H} \times \mathbf{A})\, d\boldsymbol{\sigma} + \int \mathbf{J} \cdot \mathbf{A}\, d\tau,\end{aligned}$$

where the surface integral vanishes.

1.11.10 $\int (v\mathcal{L}u - u\mathcal{L}v)\,d\tau = \int\{v\nabla\cdot(p\nabla)u - u\nabla\cdot(p\nabla)v\}\,d\tau =$
$\int\{p(v\nabla^2 u - u\nabla^2 v) + (v\nabla u - u\nabla v)\cdot\nabla p\}\,d\tau =$
$\int p\nabla\cdot(v\nabla u - u\nabla v)\,d\tau + \int(v\nabla u - u\nabla v)\cdot\nabla p\,d\tau =$
$\int \nabla\cdot\{p(v\nabla u - u\nabla v)\}\,d\tau = \int p(v\nabla u - u\nabla v)\cdot d\sigma.$

1.12.5 Using Stokes' theorem for a loop enclosing the current, we get

$$\int(\nabla\times\mathbf{H})\cdot d\boldsymbol{\sigma} = \int \mathbf{J}\cdot d\boldsymbol{\sigma} = I = \oint \mathbf{H}\cdot d\mathbf{r},$$

where the first surface integral is over the area enclosed by the loop and the second can be limited to the cross section of the wire carrying the current.

1.12.7 The proof is the same as for Ex. 1.11.3, except for $\mathbf{A}\to\mathbf{V}$.

1.12.8 Zero.

1.12.9 This follows from integration by parts shifting ∇ from v to u. The integrated term cancels for a closed loop.

1.12.10 Use the identity of Ex. 1.12.9, i.e. $\oint\nabla(uv)\cdot d\boldsymbol{\lambda} = 0$, and apply Stokes' theorem to $2\int_S u\nabla v\cdot d\boldsymbol{\sigma} = \int(u\nabla v - v\nabla u)\cdot d\boldsymbol{\lambda} = \int_S \nabla\times(u\nabla v - v\nabla u)\cdot d\boldsymbol{\sigma} = 2\int_S(\nabla u\times\nabla v)\cdot d\boldsymbol{\sigma}$.

1.13.2 $\varphi(r) = \frac{Q}{4\pi\varepsilon_0 r},\quad a\le r<\infty,$
$\varphi(r) = \frac{Q}{4\pi\varepsilon_0 a}\left[\frac{3}{2} - \frac{1}{2}\frac{r^2}{a^2}\right],\quad 0\le r\le a.$

1.13.4 The gravitational acceleration in the z-direction relative to the Earth's surface is $-\frac{GM}{(R+z)^2} + \frac{GM}{R^2} \sim 2z\frac{GM}{R^3}$ for $0\le z\ll R$. Thus, the force $F_z = 2z\frac{GmM}{R^3}$. The force $F_x = -x\frac{GmM}{(R+x)^3} \sim -x\frac{GmM}{R^3}$, and $F_y = -y\frac{GmM}{(R+x)^3} \sim -y\frac{GmM}{R^3}$. Integrating $\mathbf{F} = -\nabla V$, yields the potential $V = \frac{GmM}{R^3}(z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2) = \frac{GmMr^2}{2R^3}(3z^2 - r^2) = \frac{GmMr^2}{R^3}P_2(\cos\theta)$.

1.13.7 $\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ for constant $\mathbf{B}$ implies $\mathbf{B} = \nabla\times\mathbf{A} =$
$\frac{1}{2}\mathbf{B}\nabla\cdot\mathbf{r} - \frac{1}{2}\mathbf{B}\cdot\nabla\mathbf{r} = (\frac{3}{2} - \frac{1}{2})\mathbf{B}$.

1.13.10 From the Maxwell-Faraday eq. $\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 = \nabla\times(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t})$ we get $\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t} = -\nabla\varphi$.

1.13.11 With the result of Ex. 1.13.10 we have $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) =$
$q(-\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A})) = q(-\nabla\varphi - \frac{d\mathbf{A}}{dt} + \nabla(\mathbf{A}\cdot\mathbf{v}))$ using the chain rule for $\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \frac{d\mathbf{r}}{dt}\cdot\nabla\mathbf{A}$ with $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and the bac-cab rule for $\mathbf{v}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}$.

1.14.1 Gauss' law is still given by Eq. 1.163.

1.14.2 (a) $\int_V \nabla \cdot \mathbf{E}\, d\tau = \int_S \mathbf{E} \cdot d\boldsymbol{\sigma} = \frac{1}{\varepsilon_0} \int \rho\, d\tau = Q/\varepsilon_0$ by Gauss' theorem and Maxwell-Gauss' eq.

(b) Take $\mathbf{E} = E(r)\hat{\mathbf{r}}$ and write Gauss' law in radial form $q\delta(\mathbf{r})/(4\pi\varepsilon_0) = \nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial r^2 E(r)}{\partial r} = \frac{2}{r}E + \frac{dE}{dr} = \frac{q}{\varepsilon_0}\frac{\delta(r)}{r^2}$. Integrating $E'/E = -2/r$ yields $\ln E = -2\ln r + \ln c$, i.e. $E = c/r^2$. Variation of the constant gives $E' = \frac{c'}{r^2} = q\delta(r)/(4\pi r^2)$, or $c = q/(4\pi\varepsilon_0)$.

1.14.4 Use $\nabla \cdot \mathbf{A} = 0, \mathbf{B} = \mu\mathbf{H}, \mathbf{D} = \varepsilon\mathbf{E}$ with $\partial\mathbf{E}/\partial t = 0$ in $\nabla \times \mathbf{H} = \partial\mathbf{D}/\partial t + \mathbf{J} = \nabla \times (\nabla \times \mathbf{A})/\mu = (\nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A})/\mu = \mathbf{J}$, so that $-\nabla^2\mathbf{A} = \mu\mathbf{J}$ follows.

1.15.1 The mean value theorem gives $\lim_{n\to\infty} \int f(x)\delta_n(x)\, dx = \lim_{n\to\infty} n \int_{-1/2n}^{1/2n} f(x)\, dx = \lim_{n\to\infty} \frac{n}{n} f(\xi_n) = f(0)$, as $-\frac{1}{2n} \le \xi_n \le \frac{1}{2n}$.

1.15.2 $\lim_{n\to\infty} \int f(x)\delta_n(x)dx = \lim_{n\to\infty} n \int_0^\infty f(x)e^{-nx}dx = \lim_{n\to\infty} f(\xi_n) n \int_0^\infty e^{-nx}dx = \lim_{n\to\infty} f(\xi_n) = f(0)$, as $0 \le \xi_n \le \frac{x}{n}$.

1.15.6 $\int_{-\infty}^{\infty} f(x)\delta(a(x - x_1))\, dx = \frac{1}{a}\int_{-\infty}^{\infty} f((y + y_1)/a)\delta(y)\, dy = \frac{1}{a} f(\frac{y_1}{a}) = \frac{1}{a} f(x_1) = \int_{-\infty}^{\infty} f(x)\delta(x - x_1)\, \frac{dx}{a}$.

1.15.9 Integrating by parts we find $\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = -\int_{-\infty}^{\infty} f'(x)\delta(x)\, dx = -f'(0)$.

1.15.11 $\delta(\mathbf{r} - \mathbf{r}') = \dfrac{1}{h_1 h_2 h_3}\, \delta(q_1 - q_1')\delta(q_2 - q_2')\delta(q_3 - q_3')$.

1.15.18 (a)

$$\delta_n(x) = \frac{1}{2\pi} + \frac{2n}{\pi}\sum_{m=1}^{\infty} \frac{\sin(m/2n)}{m}\cos mx.$$

(b)

$$\delta(x - t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mt \cos mx,$$
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(t)\cos mt\, dt = \frac{1}{\pi},\ m = 0, 1, \ldots.$$

$$\frac{n}{\pi\sqrt{\pi}}\int_{-\pi}^{\pi} e^{-n^2x^2}\cos mx\,dx = \frac{1}{\pi\sqrt{\pi}}\int_{-n\pi}^{n\pi} e^{-x^2}\cos\frac{m}{n}x\,dx$$
$$\to \frac{1}{\pi\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}dx = \frac{1}{\pi},$$

as $n \to \infty$.

(c)

$$f(t) = \frac{b_0}{2} + \sum_{m=1}^{\infty} b_m \cos mt, \quad \pi b_m = \int_{-\pi}^{\pi} f(t)\cos mt,$$

and (b) give

$$\int_{-\pi}^{\pi} f(t)\delta(x-t)\,dx$$
$$= \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{m=1}^{\infty} a_m \cos mt \int_{-\pi}^{\pi} f(x)\cos mx\,dx$$
$$= \frac{a_0 b_0 \pi}{2} + \sum_{m=1}^{\infty} a_m b_m \pi \cos mt = f(t).$$

1.16.1 If $0 = \nabla^2\psi = \nabla\cdot\nabla\psi$, then $\nabla\psi$ is solenoidal. Since $\nabla\times\nabla\psi = 0$, $\nabla\psi$ is also irrotational; $\hat{\mathbf{n}}\cdot\nabla\psi$ is given on the boundary from which ψ is obtained by integration.

1.16.2 If $\nabla^2\mathbf{P} = -\mathbf{V}$, then $\mathbf{P} = \frac{1}{4\pi}\int\frac{V(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau$. Using the identity $\nabla\times(\nabla\times\mathbf{P}) = \nabla(\nabla\cdot\mathbf{P}) - \nabla^2\mathbf{P}$ we get $\mathbf{V} = -\nabla\cdot(\nabla\cdot\mathbf{P}) + \nabla\times(\nabla\times\mathbf{P})$.

CHAPTER 2

2.1.1 From

$$d\mathbf{r} = \sum_i \frac{\partial\mathbf{r}}{\partial q_i}\,dq_i$$

we know that the vectors $\frac{\partial\mathbf{r}}{\partial q_i} = h_i\hat{\mathbf{e}}_i$ are pointing in the direction of increasing q_i, i.e. $\hat{\mathbf{e}}_i = \hat{\mathbf{q}}_i$. Since $g_{ij} = \frac{\partial\mathbf{r}}{\partial q_i}\cdot\frac{\partial\mathbf{r}}{\partial q_j}$, orthogonality $(\hat{\mathbf{q}}_i\cdot\hat{\mathbf{q}}_j = \delta_{ij})$ implies $g_{ij} = 0$ for $i \neq j$.

2.1.2 (a) Upon squaring each coordinate differential

$$dx = dr \sin\theta \cos\varphi + r\cos\theta \cos\varphi \, d\theta - r\sin\theta \sin\varphi \, d\varphi,$$
$$dy = dr \sin\theta \sin\varphi + r\cos\theta \sin\varphi \, d\theta + r\sin\theta \cos\varphi \, d\varphi,$$
$$dz = dr \cos\theta - r\sin\theta \, d\theta$$

and then summing their squares, we notice that all cross terms cancel so that we obtain

$$d\mathbf{r}^2 = dr^2 + r^2 \, d\theta^2 + r^2 sin^2\theta \, d\varphi^2.$$

Hence $h_r = 1, h_\theta = r, h_\varphi = r\sin\theta$.

(b) From the last equation of (a) we read off $ds_1 = dr, ds_2 = rd\theta$, $ds_3 = r\sin\theta \, d\varphi$, or from $d\mathbf{r}$ in (a) that

$$\frac{\partial \mathbf{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \hat{\mathbf{r}},$$
$$\frac{\partial \mathbf{r}}{\partial \theta} = r(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = r\hat{\boldsymbol{\theta}},$$
$$\frac{\partial \mathbf{r}}{\partial \varphi} = r(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = r\sin\theta\hat{\boldsymbol{\varphi}}.$$

2.1.3 The u-, v-, z-system is left-handed.

2.1.5 From $d\mathbf{r} = \sum_i \frac{\partial \mathbf{r}}{\partial q_i} dq_i$, the surface element is given by the absolute value (of the 3-component) of the cross product

$$\left(\frac{\partial \mathbf{r}}{\partial q_1} \times \frac{\partial \mathbf{r}}{\partial q_2}\right)_3 dq_1 \, dq_2 = \left(\frac{\partial x_1}{\partial q_1}\frac{\partial x_2}{\partial q_2} - \frac{\partial x_1}{\partial q_2}\frac{\partial x_2}{\partial q_1}\right) dq_1 \, dq_2,$$

which is the standard Jacobian, i.e. the 2×2 determinant spanned by the vectors $\frac{\partial \mathbf{r}}{\partial q_i}$, $i = 1, 2$.

2.1.6 From $dx^2 = dx_\mu dx^\mu = c^2 dt^2 - d\mathbf{r}^2$ we have $g_{00} = +1, g_{11} = -1 = g_{22} = g_{33}$ and $g_{ij} = 0, \ i \neq j$.

2.2.1 From the component definition (projection) $\mathbf{a} = \sum_i \hat{\mathbf{q}}_i \mathbf{a} \cdot \hat{\mathbf{q}}_i \equiv \sum_i a_{q_i} \hat{\mathbf{q}}_i$ and a similar expression for $\mathbf{b}$, get

$$\mathbf{a} \cdot \mathbf{b} = \sum_{ij} \hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j \mathbf{a} \cdot \hat{\mathbf{q}}_i \mathbf{b} \cdot \hat{\mathbf{q}}_j = \sum_i \mathbf{a} \cdot \hat{\mathbf{q}}_i \mathbf{b} \cdot \hat{\mathbf{q}}_i = \sum_i a_{q_i} b_{q_i}$$

using orthogonality, i.e. $\hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j = \delta_{ij}$.

2.2.2 (a) From Eq. (2.17) with $\hat{\mathbf{e}}_1 = \hat{\mathbf{q}}_1$ and $(\hat{\mathbf{e}}_1)_1 = 1$, $(\hat{\mathbf{e}}_1)_2 = 0 = (\hat{\mathbf{e}}_1)_3$ we get $\nabla \cdot \mathbf{e}_1 = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial (h_2 h_3)}{\partial q_1}$.

(b) From Eq. (2.22) with $h_2 V_2 \to 0$, $h_3 V_3 \to 0$, we get $\nabla \times \mathbf{e}_1 = \dfrac{1}{h_1} \left[\mathbf{e}_2 \dfrac{1}{h_3} \dfrac{\partial h_1}{\partial q_3} - \mathbf{e}_3 \dfrac{1}{h_2} \dfrac{\partial h_1}{\partial q_2} \right]$.

2.4.3 From $\boldsymbol{\rho} = (x, y) = \rho(\cos\varphi, \sin\varphi)$ we have $\hat{\boldsymbol{\rho}} = (\cos\varphi, \sin\varphi)$. Differentiating $\hat{\boldsymbol{\rho}}^2 = 1$ we see that $\hat{\boldsymbol{\rho}} \cdot \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \varphi} = 0$. Hence $\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \varphi} \sim \hat{\boldsymbol{\varphi}}$. In components $\frac{\partial \hat{\boldsymbol{\rho}}}{\partial \varphi} = (-\sin\varphi, \cos\varphi) = \hat{\boldsymbol{\varphi}}$. Similarly we obtain $\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} = -\hat{\boldsymbol{\rho}}$, and that all other first derivatives of the circular cylindrical unit vectors with respect to the circular cylindrical coordinates vanish.

2.4.5 (a) $\mathbf{r} = (x, y, z) = (x, y) + z\hat{\mathbf{z}} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$

(b) From Eq. (2.32) we have $\nabla \cdot \mathbf{r} = \frac{1}{\rho} \frac{\partial \rho^2}{\partial \rho} + \frac{\partial z}{\partial z} = 2 + 1 = 3$. From Eq. (2.34) with $V_\rho = \rho$, $V_\varphi = 0$, $V_z = z$ we get $\nabla \times \mathbf{r} = 0$.

2.4.13 (a) $\mathbf{F} = \hat{\varphi} \frac{1}{\rho}$.

(b) $\nabla \times \mathbf{F} = 0$, $\rho \neq 0$.

(c) $\displaystyle\int_0^{2\pi} \mathbf{F} \cdot \hat{\varphi} \rho \, d\varphi = 2\pi$

(d) $\nabla \times F$ is not defined at the origin. A cut line from the origin out to infinity (in any direction) is needed to prevent one from encircling the origin. The scalar potential $\psi = \varphi$ is not single-valued.

2.4.16 Integrating $\mathbf{v} = \hat{\boldsymbol{\varphi}} \rho \omega$ we find $\oint \mathbf{v} \cdot d\mathbf{r} = \rho^2 \omega \int d\varphi = 2\pi\omega\rho^2$. Hence $\oint \mathbf{v} \cdot d\boldsymbol{\lambda}/(\pi\rho^2) = 2\omega = \nabla \times \mathbf{v}|_z$.

2.5.2 (a) From Exercise 2.1.2, i.e. differentiating $\hat{\mathbf{r}}^2 = 1$ we get

$$\frac{\partial \mathbf{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \hat{\mathbf{r}},$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = r(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = r\hat{\boldsymbol{\theta}},$$

$$\frac{\partial \mathbf{r}}{\partial \varphi} = r(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = r\sin\theta\hat{\boldsymbol{\varphi}}.$$

$$\frac{\partial \hat{\mathbf{r}}}{\partial r} = 0, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r} = 0, \qquad \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial r} = 0,$$
$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}}, \qquad \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \theta} = 0,$$
$$\frac{\partial \hat{\mathbf{r}}}{\partial \varphi} = \hat{\boldsymbol{\varphi}} \sin\theta, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \varphi} = \hat{\boldsymbol{\varphi}} \cos\theta,$$
$$\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \varphi} = -\hat{\mathbf{r}} \sin\theta - \hat{\boldsymbol{\theta}} \cos\theta.$$

(b) With ∇ given by

$$\hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi},$$

the alternate derivation of the Laplacian is given by dotting this ∇ into itself. In conjunction with the derivatives of the unit vectors above this gives

$$\begin{aligned}
\nabla \cdot \nabla &= \hat{\mathbf{r}} \cdot \frac{\partial}{\partial r} \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \cdot \frac{1}{r} \frac{\partial \hat{\mathbf{r}}}{\partial \theta} \frac{\partial}{\partial r} + \hat{\boldsymbol{\varphi}} \cdot \frac{1}{r \sin\theta} \frac{\partial \hat{\mathbf{r}}}{\partial \varphi} \frac{\partial}{\partial r} \\
&\quad + \hat{\boldsymbol{\varphi}} \cdot \frac{1}{r \sin\theta} \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \varphi} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \cdot \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \left(\hat{\boldsymbol{\varphi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\tan\theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2}.
\end{aligned}$$

Note that, with

$$\frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) = \frac{\tan\theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},$$

we get the standard result using Ex. 2.5.18 for the radial part.

2.5.3 (a) $\mathbf{v} = \hat{\varphi} \omega r \sin\theta$

(b) $\nabla \times \mathbf{v} = 2\omega$.

2.5.5 Resolving the unit vectors of spherical polar coordinates into cartesian components was accomplished in Ex. 2.5.1 involving an orthogonal matrix. The inverse is the transpose matrix, i.e.

$$\begin{aligned}
\hat{\mathbf{x}} &= \hat{\mathbf{r}} \sin\theta \cos\varphi + \hat{\boldsymbol{\theta}} \cos\theta \cos\varphi - \hat{\boldsymbol{\varphi}} \sin\varphi, \\
\hat{\mathbf{y}} &= \hat{\mathbf{r}} \sin\theta \sin\varphi + \hat{\boldsymbol{\theta}} \cos\theta \sin\varphi + \hat{\boldsymbol{\varphi}} \cos\varphi, \\
\hat{\mathbf{z}} &= \hat{\mathbf{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta.
\end{aligned}$$

2.5.7 $V_\theta, V_\varphi \sim 1/r$.

2.5.9 (a)

$$\mathbf{A} \cdot \nabla \mathbf{r} = A_x \frac{\partial \mathbf{r}}{\partial x} + A_y \frac{\partial \mathbf{r}}{\partial y} + A_z \frac{\partial \mathbf{r}}{\partial z} = \mathbf{A}$$

because

$$\frac{\partial \mathbf{r}}{\partial x} = \hat{\mathbf{x}}, \quad \frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{y}}, \quad \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{z}}.$$

(b) Using $\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}$, $\frac{\partial \hat{\mathbf{r}}}{\partial \varphi} = \sin\theta \hat{\boldsymbol{\varphi}}$ and ∇ in polar coordinates from Ex. 2.5.2 (b) we get

$$\mathbf{A} \cdot \nabla \mathbf{r} = \mathbf{A} \cdot \hat{\mathbf{r}} \frac{\partial \mathbf{r}}{\partial r} + \mathbf{A} \cdot \hat{\boldsymbol{\theta}} \frac{\partial \hat{\mathbf{r}}}{\partial \theta} + \frac{\mathbf{A} \cdot \hat{\boldsymbol{\varphi}}}{\sin\theta} \frac{\partial \hat{\mathbf{r}}}{\partial \varphi}$$
$$= A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\varphi \hat{\boldsymbol{\varphi}} = \mathbf{A}.$$

2.5.18 From (a) $\frac{1}{r^2}\frac{d}{dr}r^2 = \frac{d}{dr} + \frac{2}{r}$ we get (c), and vice versa. From the inner $\frac{d}{dr}r = r\frac{d}{dr} + 1$ in (b) we get $\frac{1}{r}\frac{d^2}{dr^2}r = \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$, hence (c), and vice versa. Here (a) $\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi(r)}{dr}\right]$, (b) $\frac{1}{r}\frac{d^2}{dr^2}[r\psi(r)]$, (c) $\frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr}$.

2.5.20 (a) $\nabla \times \mathbf{F} = 0, \quad r \geq P/2$.

(b) $\oint \mathbf{F} \cdot d\boldsymbol{\lambda} = 0$. This suggests (but does not prove) that the force is conservative.

(c) Potential $= P\cos\theta/r^2$, dipole potential.

2.5.25 (a) (a) follows because ∇^2 is a sum of its plane polar part and its z-part, and $\hat{\mathbf{z}}$ is constant.

(b) This does not work because angular derivatives of the polar unit vectors do not vanish, and these enter into ∇^2, as shown in Ex. 2.5.2.

2.6.2 If $A^0_{ij} = B^0_{ij}$ in one frame of reference, then define a coordinate transformation from that frame to an arbitrary one: $x_i = x_i(x^0_j)$, so that

$$A_{ij} = \frac{\partial x_i}{\partial x^0_\alpha}\frac{\partial x_j}{\partial x^0_\beta}A^0_{\alpha\beta} = \frac{\partial x_i}{\partial x^0_\alpha}\frac{\partial x_j}{\partial x^0_\beta}B^0_{\alpha\beta} = B_{ij}.$$

2.6.5 The four-dimensional fourth-rank Riemann–Christoffel curvature tensor of general relativity, R_{iklm} has $4^4 = 256$ components. The antisymmetry of the first and second pair of indices, $R_{iklm} = -R_{ikml} = -R_{kilm}$, reduces these pairs to 6 values each, i.e. $6^2 = 36$ components. They can be thought of as a 6×6 matrix. The symmetry under exchange of pair indices, $R_{iklm} = R_{lmik}$, reduces this matrix to $6\cdot 7/2 = 21$ components. The Bianchi identity, $R_{iklm} + R_{ilmk} + R_{imkl} = 0$, reduces the independent components to 20 because it represents one constraint. Note that, upon using the permutation symmetries one can always make the first index equal to zero followed by the other indices which are all different from each other.

2.6.6 Zero. Each component has at least one repeated index and is therefore zero.

2.7.2 The contraction of two indices removes two indices, while the derivative adds one, so that $(n+1) - 2 = n - 1$.

2.7.3 The scalar product of the four-vectors $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$, $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$ is the scalar $\partial^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$.

2.8.1 The double summation $K_{ij}A_iB_j$ is a scalar. That K_{ij} is a second-rank tensor follows from the quotient theorem.

2.8.2 Since $K_{ij}A_{jk} = B_{ik}$ is a second-rank tensor the quotient theorem tells us that K_{ij} is a second-rank tensor.

2.8.3 Since the phase $-\mathbf{\Phi} = \mathbf{k}\cdot\mathbf{r} - \omega t$ is a Lorentz scalar and the coordinates $(ct, \mathbf{r})$ form a four-vector, the quotient theorem says that $(\omega/c, \mathbf{k})$ is a four-vector. Alternatively, since ∂^μ is a four-vector and $e^{-i\Phi}$ a scalar, $\partial^\mu e^{-i\Phi} = -ik^\mu e^{i\Phi}$ is a four-vector.

2.9.2 The generalization of the totally antisymmetric ε_{ijk} from three to n dimensions has n indices. Hence the generalized cross product $\varepsilon_{ijk\ldots}A_iB_j$ is an antisymmetric tensor of rank $n - 2 \neq 1$ for $n \neq 3$.

2.9.3 (a) $\delta_{ii} = 1$ (not summed) for each $i = 1, 2, 3$.

(b) $\delta_{ij}\varepsilon_{ijk} = 0$ because δ_{ij} is symmetric in i, j while ε_{ijk} is antisymmetric in i, j.

(c) For each ε in $\varepsilon_{ipq}\varepsilon_{jpq}$ to be non-zero, leaves only one value for i and j, so that $i = j$. Interchanging p and q gives two terms, hence the factor 2.

(d) There are 6 permutations i, j, k of 1, 2, 3 in $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

2.9.4 Given k implies $p \neq q$ for $\varepsilon_{pqk} \neq 0$. For $\varepsilon_{ijk} \neq 0$ requires either $i = p$ and so $j = q$, or $i = q$ and then $j = p$. Hence $\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jp} - \delta_{iq}\delta_{jp}$.

2.9.8 Using Ex. 2.9.4 and $\varepsilon_{pkq} = -\varepsilon_{pqk}$, we get

$$(\mathbf{A} \times (\mathbf{B} \times \mathbf{C}))_q = \varepsilon_{pkq}(\varepsilon_{ijk} B_i C_j) A_p = -(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) A_p B_i C_j$$
$$= -\mathbf{A} \cdot \mathbf{B} C_q + \mathbf{A} \cdot \mathbf{C} B_q.$$

2.9.11 $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $R = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ is a rotation, then

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

2.9.12 If $A_k = \frac{1}{2}\varepsilon_{ijk} B_{ij}$ with $B_{ij} = -B_{ji}$, then

$$2\varepsilon_{mnk} A_k = \varepsilon_{mnk}\varepsilon_{ijk} = (\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}) B_{ij} = B_{mn} - B_{nm} = 2B_{mn}.$$

2.9.13 Using Ex. 2.9.4 we have

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \varepsilon_{ijk}\varepsilon_{mnk} A_i B_j C_m D_n$$
$$= (\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}) A_i B_j C_m D_n = \mathbf{A} \cdot \mathbf{C} \mathbf{B} \cdot \mathbf{D} - \mathbf{A} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{C}.$$

2.10.8
$$\Gamma^1_{22} = -r \quad \Gamma^1_{33} = -r \sin^2\theta$$
$$\Gamma^2_{12} = \Gamma^2_{21} = 1/r \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$
$$\Gamma^3_{13} = \Gamma^3_{31} = 1/r \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

2.10.9 Let s be the proper time on a geodesic and $u^\mu(s)$ the velocity of a mass in free fall. Then the scalar

$$\frac{d}{ds}(V \cdot u) = \frac{dV}{ds} \cdot u + V_\beta \frac{d^2 x^\beta}{ds^2}$$
$$= \left(\partial_\mu V_\alpha \frac{dx^\mu}{ds}\right) u^\alpha - V_\beta \Gamma^\beta_{\alpha\mu} u^\alpha u^\mu = u^\mu u^\alpha \left(\partial_\mu V_\alpha - \Gamma^\beta_{\alpha\mu} V_\beta\right)$$

involves the covariant derivative which is a four-vector by the quotient theorem. Note that the use of the geodesic equation for $d^2 x^\beta / ds^2$ is the key here.

2.10.11 $\Gamma^1_{22} = -\rho, \quad \Gamma^2_{12} = \Gamma^2_{21} = 1/\rho$

2.10.13
$$g_{11} = 2.25 \quad g_{12} = 0.60 \quad g_{13} = 0.30$$
$$g_{22} = 2.72 \quad g_{23} = 0.56 \quad g_{33} = 1.13$$

2.10.14 From the derivation of the geodesic equation we know that

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\alpha} - \partial_\beta g_{\mu\nu}\right).$$

Defining

$$[\mu\nu, k] \equiv g_{k\alpha}\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g_{k\alpha}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\alpha} - \partial_\beta g_{\mu\nu}\right)$$
$$= \frac{1}{2}\delta^{\beta}_{k}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\alpha} - \partial_\beta g_{\mu\nu}\right)$$
$$= \frac{1}{2}\left(\partial_\mu g_{k\nu} + \partial_\nu g_{\mu k} - \partial_k g_{\mu\nu}\right).$$

2.10.15
$$g_{ij;k} = \partial_k g_{ij} - \Gamma^{\alpha}_{ik}g_{\alpha j} - \Gamma^{\alpha}_{jk}g_{i\alpha}$$
$$= \partial_k g_{ij} - \frac{1}{2}g_{j\alpha}g^{\alpha\beta}\left(\partial_i g_{\beta k} + \partial_k g_{\beta i} - \partial_\beta g_{ik}\right)$$
$$-\frac{1}{2}g_{i\alpha}g^{\alpha\beta}\left(\partial_j g_{\beta k} + \partial_k g_{\beta j} - \partial_\beta g_{jk}\right)$$
$$= \partial_k g_{ij} - \frac{1}{2}\left(\partial_i g_{jk} + \partial_k g_{ji} - \partial_j g_{ik}\right)$$
$$-\frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right) \equiv 0.$$

In order to find $g^{ij}_{;k} = 0$ derive covariantly the identity $g_{im}g^{mj} = \delta^j_i$. This gives $0 = g_{im;k}g^{mj} + g_{im}g^{mj}_{;k} = g_{im}g^{mj}_{;k}$. Multiplying this by g^{ni} and using $g^{ni}g_{im} = \delta^n_m$ gives $g^{nj}_{;k} = 0$.

2.10.16 We have from the geodesic equation $du^i = -\Gamma^i_{\alpha\beta}u^\alpha\, dx^\beta$ and from parallel displacing the velocity u^i on it: $\delta u^i = -\Gamma^i_{\alpha\beta}u^\alpha\, dx^\beta$.

2.10.17 The covariant derivative of a vector V^i by parallel transport is given by the limiting procedure $\lim_{dq^j\to 0}\frac{V^i(q^j+dq^j)-V^i(q^j)+\Gamma^i_{nj}V^n dq^j}{dq^j}$.

2.11.3 $\partial_\mu\Phi_{;\nu} = \partial_\nu\partial_\mu\Phi - \Gamma^{\alpha}_{\mu\nu}\partial_\alpha\Phi \equiv \partial_\nu\Phi_{;\mu} = \partial_\mu\partial_\nu\Phi - \Gamma^{\alpha}_{\nu\mu}\partial_\alpha\Phi.$

CHAPTER 3

3.1.1 (a) -1. (b) -11. (c) $9\sqrt{2}/2$.

3.1.2 The determinant of the coefficients is equal to 2. Therefore no nontrivial solution exists.

3.1.3 Given the pair of equations

$$x + 2y = 3, \qquad 2x + 4y = 6.$$

(a) Since the coefficients of the second equation differ from those of the first one just by a factor 2, the determinant of (lhs) coefficients is zero.

(b) Since the inhomogeneous terms on the right-hand side differ by the same factor 2, both numerator determinants also vanish.

(c) It suffices to solve $x + 2y = 3$. Given x, $y = (3 - x)/2$. This is the general solution for arbitrary values of x.

3.1.7 The Gauss elimination yields

$$\begin{aligned}
10x_1 + 9x_2 + 8x_3 + \ 4x_4 + \ \ x_5 \qquad\qquad &= 10\\
x_2 + 2x_3 + \ 3x_4 + \ 5x_5 + \quad 10x_6 &= 5\\
10x_3 + 23x_4 + 44x_5 - \quad 60x_6 &= -5\\
16x_4 + 48x_5 - \quad 30x_6 &= 15\\
48x_5 + \ \ 498x_6 &= 215\\
-\ 11316x_6 &= -4438
\end{aligned}$$

so that $x_6 = 2219/5658 = 0.392$,
$x_5 = (215 - 498x_6)/48 = 0.412$,
$x_4 = (15 + 30x_6 - 48x_5)/16 = 0.437$,
$x_3 = (-5 + 60x_6 - 44x_5 - 23x_4)/10 = -0.966$,
$x_2 = 5 - 10x_6 - 5x_5 - 3x_4 - 2x_3 = -0.359$,
$x_1 = (10 - x_5 - 4x_4 - 8x_3 - 9x_2)/10 = 1.88$.

3.2.1 Writing the product matrices in term of their elements, $\mathsf{AB} = (\sum_m a_{im}b_{mk})$, $\mathsf{BC} = (\sum_n b_{in}c_{nk})$, $(\mathsf{AB})\mathsf{C} = (\sum_n(\sum_m a_{im}b_{mn})c_{nk}) = \sum_{mn} a_{im}b_{mn}c_{nk} = \mathsf{A}(\mathsf{BC}) = (\sum_m a_{im}(\sum_n b_{mn}c_{nk}))$, because products of real and complex numbers are associative the parentheses can be dropped for all matrix elements.

3.2.2 Multiplying out $(\mathsf{A} + \mathsf{B})(\mathsf{A} - \mathsf{B}) = \mathsf{A}^2 + \mathsf{BA} - \mathsf{AB} - \mathsf{B}^2 = \mathsf{A}^2 - \mathsf{B}^2 + [\mathsf{B}, \mathsf{A}]$.

3.2.3 In terms of matrix elements, $\mathsf{A} = (a_{ik})$, we have $(\sum_n a_{in}(c_1r_{1n} + c_2r_{2n})) = c_1(\sum_n a_{in}r_{1n}) + c_2(\sum_n a_{in}r_{2n}) = c_1(\mathsf{A}\mathbf{r}_1)_i + c_2(\mathsf{A}\mathbf{r}_2)_i$.

3.2.4 (a) $(a_1 + ib_1) - (a_2 + ib_2) = a_1 - a_2 + i(b_1 - b_2)$ corresponds to $\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ -(b_1 - b_2) & a_1 - a_2 \end{pmatrix}$, i.e. the correspondence holds for addition and subtraction.
Similarly, it holds for multiplication because first
$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

and matrix multiplication yields $\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & a_1b_2 + a_2b_1 \\ -(a_1b_2 + a_2b_1) & a_1a_2 - b_1b_2 \end{pmatrix}$.

(b) $(a+ib)^{-1} = \frac{a-ib}{a^2+b^2}$ corresponds to $\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

3.2.5 A factor (-1) can be pulled out of each row giving the $(-1)^n$ overall.

3.2.7 $\mathsf{A}^2 = \mathsf{B}^2 = \mathsf{C}^2 = 1$
$\mathsf{AB} = \mathsf{BA} = \mathsf{C}$
$\mathsf{BC} = \mathsf{CB} = \mathsf{A}$
$\mathsf{CA} = \mathsf{AC} = \mathsf{B}$.

3.2.8 $n = 6$.

3.2.9 Expanding the commutators we find $[\mathsf{A}, [\mathsf{B}, \mathsf{C}]] = \mathsf{A}[\mathsf{B}, \mathsf{C}] - [\mathsf{B}, \mathsf{C}]\mathsf{A} = \mathsf{ABC} - \mathsf{ACB} - \mathsf{BCA} + \mathsf{CBA}$, $[\mathsf{B}, [\mathsf{A}, \mathsf{C}]] = \mathsf{BAC} - \mathsf{BCA} - \mathsf{ACB} + \mathsf{CAB}$, $[\mathsf{C}, [\mathsf{A}, \mathsf{B}]] = \mathsf{CAB} - \mathsf{CBA} - \mathsf{ABC} + \mathsf{BAC}$, and subtracting the last double commutator from the second yields the first one, since the BAC and CAB terms cancel.

3.2.12 If $a_{ik} = 0 = b_{ik}$ for $i > k$, then also $\sum_m a_{im}b_{mk} = \sum_{i\leqslant m\leqslant k} a_{im}b_{mk} = 0$, as the sum is empty for $i > k$.

3.2.13 By direct matrix multiplications and additions.

3.2.14 Expanding in cartesian components and multiplying we get $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \sigma_j^2 a_j b_j + \sum_{j\neq k}\sigma_j\sigma_k a_j b_k = \mathbf{a}\cdot\mathbf{b} + i\varepsilon_{jkl}a_jb_k\sigma_l = \mathbf{a}\cdot\mathbf{b}\,1_2 + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$.

3.2.19 If an operator P commutes with J_x and J_y, the x and y components of an angular momentum operator, then $i[\mathsf{P}, \mathsf{J}_z] = [\mathsf{P}, \mathsf{J}_x\mathsf{J}_y - \mathsf{J}_y\mathsf{J}_x] = \mathsf{J}_x[\mathsf{P}, \mathsf{J}_y] + [\mathsf{P}, \mathsf{J}_x]\mathsf{J}_y - \mathsf{J}_y[\mathsf{P}, \mathsf{J}_x] - [\mathsf{P}, \mathsf{J}_y]\mathsf{J}_x = 0$.

3.2.21 If $\hat{\mathbf{A}} = \pm\hat{\mathbf{B}}$, then $\mathbf{B} = \mathsf{T}\mathbf{A}$ is uniquely solved by the real number $T = \pm B/A$, where A, B are the lengths of the corresponding vectors. If $\mathbf{B} = \mathsf{T}\mathbf{A}$ is valid, so is $\mathbf{B}' = \mathbf{B} - b\mathbf{A} = \mathsf{T}\mathbf{A} - b\mathbf{A} = (\mathsf{T} - b)\mathbf{A}$ with $b = \mathbf{A}\cdot\mathbf{B}/A^2$ so that $\mathbf{A}\cdot\mathbf{B}' = 0$. So, let $\mathbf{B} = B\hat{\mathbf{y}}$, $\mathbf{A} = A\hat{\mathbf{x}}$ without loss of generality. Then $\begin{pmatrix} 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$ leads to $B = yA$, or $y = B/A$ and $\mathsf{T} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$ is a solution. But so is $\begin{pmatrix} 0 & x \\ y & z \end{pmatrix}$ with any x, z.

3.2.23 For $i \neq k$ and $a_{ii} \neq a_{kk}$ we get for the product elements $(\mathsf{AB})_{ik} = (\sum_n a_{in}b_{nk}) = (a_{ii}b_{ik}) = (\mathsf{BA})_{ik} = (\sum_n b_{in}a_{nk}) = (b_{ik}a_{kk})$. Hence $b_{ik} = 0$ for $i \neq k$.

3.2.24 $\sum_m a_{im}b_{mk} = a_{ii}b_{ii}\delta_{ik} = \sum_m b_{im}a_{mk}$.

3.2.28 Taking the trace of $\mathsf{A}(\mathsf{BA}) = -\mathsf{A}^2\mathsf{B} = -\mathsf{B}$ yields $-\mathrm{tr}(\mathsf{B}) = \mathrm{tr}(\mathsf{A}(\mathsf{BA})) = \mathrm{tr}(\mathsf{A}^2\mathsf{B}) = \mathrm{tr}(\mathsf{B})$.

3.2.31 If $\mathsf{A}_1\mathsf{A} = 1 = \mathsf{A}\mathsf{A}_2 = 1$, then $\mathsf{A}_2 = (\mathsf{A}_1\mathsf{A})\mathsf{A}_2 = \mathsf{A}_1(\mathsf{A}\mathsf{A}_2) = \mathsf{A}_1$.

3.2.33 Taking the determinant of $\mathsf{A}\mathsf{A}^{-1} = 1$ and using the product theorem yields $\det(\mathsf{A})\det(\mathsf{A}^{-1}) = 1$.

3.2.36 $\mathsf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Note. Assume all matrix elements are real.

3.3.1 If $\mathsf{O}_i^{-1} = \tilde{\mathsf{O}}_i$, $i = 1, 2$, then $(\mathsf{O}_1\mathsf{O}_2)^{-1} = \mathsf{O}_2^{-1}\mathsf{O}_1^{-1} = \tilde{\mathsf{O}}_2\tilde{\mathsf{O}}_1 = \widetilde{\mathsf{O}_1\mathsf{O}_2}$.

3.3.2 Taking the determinant of $\tilde{\mathsf{A}}\mathsf{A} = 1$ and using the product theorem yields $\det(\tilde{\mathsf{A}})\det(\mathsf{A}) = 1 = \det^2(\mathsf{A})$ implying $\det(\mathsf{A}) = \pm 1$.

3.3.5 (a) $\alpha = 70°$, $\beta = 60°$, $\gamma = -80°$.

3.3.8 If $\tilde{\mathsf{A}} = -\mathsf{A}$, $\tilde{\mathsf{S}} = \mathsf{S}$, then $\mathrm{tr}(\widetilde{\mathsf{SA}}) = \mathrm{tr}(\mathsf{SA}) = \mathrm{tr}(\tilde{\mathsf{A}}\tilde{\mathsf{S}}) = -\mathrm{tr}(\mathsf{AS})$.

3.3.9 This follows from the invariance of the characteristic polynomial under similarity transformations, because the trace of the matrix is the coefficient of the λ^{n-1} of the characteristic polynomial. $0 = \det(\mathsf{A} - \lambda 1_n) = \det(\mathsf{S})\det(\mathsf{S}^{-1})\det(\mathsf{A} - \lambda 1_n) = \det(\mathsf{S})\det(\mathsf{A} - \lambda 1_n)\det(\mathsf{S}^{-1}) = \det[\mathsf{S}(\mathsf{A} - \lambda 1_n)\mathsf{S}^{-1}] = \det(\mathsf{SAS}^{-1} - \lambda 1_n)$.

3.3.10 This follows from the invariance of the characteristic polynomial under similarity transformations, because the determinant of the matrix is the coefficient of λ^0 of the characteristic polynomial. For the invariance of the characteristic polynomial see Ex. 3.3.9.

3.3.11 If $\tilde{\mathsf{A}} = -\mathsf{A}$, then $(\mathsf{OAO}^{-1})^{\mathsf{T}} = \mathsf{O}\tilde{\mathsf{A}}\tilde{\mathsf{O}} = -\mathsf{OAO}^{-1}$.

3.3.12 Eq. 4.63 is the most general form.

3.3.15 Using the chain rule of differentiation, $\sum_l \frac{\partial x_l'}{\partial x_i}\frac{\partial x_j}{\partial x_l'} = \delta_{ij}$, $\sum_m \frac{\partial x_m'}{\partial x_n}\frac{\partial x_k}{\partial x_m'} = \delta_{kn}$, we find $\sum_{l,m} S_{lm}' T_{lm}' = \sum_{l,m} \frac{\partial x_l'}{\partial x_i}\frac{\partial x_m'}{\partial x_n} S_{in} \frac{\partial x_j}{\partial x_l'}\frac{\partial x_k}{\partial x_m'} T_{jk} = \sum_{jk} S_{jk}T_{jk}$.

3.4.1 $\det(\mathsf{A}^*) = \sum_{i_k} \varepsilon^*_{i_1 i_2 \ldots i_n} a^*_{1i_1} a^*_{2i_2} \cdots a^*_{ni_n} = \left(\sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}\right)^* = \sum_{i_k} \varepsilon^*_{i_1 i_2 \ldots i_n} a^*_{i_1 1} a^*_{i_2 2} \cdots a^*_{i_n n}$, since ε is real, and $\det(\mathsf{A}) = \det(\tilde{\mathsf{A}})$.

3.4.3 $(\mathsf{AB})^\dagger = \widetilde{\mathsf{A}^*\mathsf{B}^*} = \tilde{\mathsf{B}}^*\tilde{\mathsf{A}}^* = \mathsf{B}^\dagger\mathsf{A}^\dagger$.

3.4.4 As $\mathsf{C}_{jk} = \sum_n S^*_{nj} S_{nk}$, $tr(\mathsf{C}) = \sum_{nj} |S_{nj}|^2$.

3.4.5 If $\mathsf{A}^\dagger = \mathsf{A}$, $\mathsf{B}^\dagger = \mathsf{B}$, then $(\mathsf{AB}+\mathsf{BA})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger + \mathsf{A}^\dagger\mathsf{B}^\dagger = \mathsf{AB}+\mathsf{BA}$, $-i(\mathsf{B}^\dagger\mathsf{A}^\dagger - \mathsf{A}^\dagger\mathsf{B}^\dagger) = i(\mathsf{AB}-\mathsf{BA})$.

3.4.6 If $\mathsf{C}^\dagger \neq \mathsf{C}$, then $(i\mathsf{C}_-)^\dagger \equiv (\mathsf{C}^\dagger - \mathsf{C})^\dagger = \mathsf{C} - \mathsf{C}^\dagger = -i\mathsf{C}^\dagger_-$, i.e. $(\mathsf{C}_-)^\dagger = \mathsf{C}_-$. Similarly $\mathsf{C}^\dagger_+ = \mathsf{C}_+ = \mathsf{C} + \mathsf{C}^\dagger$.

3.4.7 $-i\mathsf{C}^\dagger = (\mathsf{AB}-\mathsf{BA})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger - \mathsf{A}^\dagger\mathsf{B}^\dagger = \mathsf{BA}-\mathsf{AB} = -i\mathsf{C}$.

3.4.8 $(\mathsf{SAS}^\dagger)^\dagger = \mathsf{SA}^\dagger\mathsf{S}^\dagger = \mathsf{SAS}^\dagger$.

3.4.10 $(\mathsf{U}^\dagger)^\dagger = \mathsf{U} = (\mathsf{U}^{-1})^\dagger$.

3.4.13 (a) If $T^{-1} = 1 + \frac{i}{\hbar}\varepsilon H = T^\dagger = 1 + \frac{i}{\hbar}\varepsilon H^\dagger$, then $\mathsf{H} = \mathsf{H}^\dagger$.

(b) This follows from the same equation in (a).

3.4.15 $(\mathsf{U}_1\mathsf{U}_2)^\dagger = \mathsf{U}_2^\dagger\mathsf{U}_1^\dagger = \mathsf{U}_2^{-1}\mathsf{U}_1^{-1} = (\mathsf{U}_1\mathsf{U}_2)^{-1}$.

3.4.20 Since $\gamma_5^2 = 1_4$, $\frac{1}{4}(1_4+\gamma_5)^2 = \frac{1}{4}(1_4 + 2\gamma_5 + 1_4) = \frac{1}{2}(1_4+\gamma_5)$.

3.4.24 Since $\tilde{\mathsf{C}} = -\mathsf{C} = \mathsf{C}^{-1}$, and $\mathsf{C}\gamma^0\mathsf{C}^{-1} = -\gamma^0 = -\tilde{\gamma^0}$, $\mathsf{C}\gamma^2\mathsf{C}^{-1} = -\gamma^2 = -\tilde{\gamma^2}$, $\mathsf{C}\gamma^1\mathsf{C}^{-1} = \gamma^1 = -\tilde{\gamma^1}$, $\mathsf{C}\gamma^3\mathsf{C}^{-1} = \gamma^3 = -\tilde{\gamma^3}$, we have $(\mathsf{C}\gamma^\mu\mathsf{C}^{-1})^\top = \widetilde{\mathsf{C}^{-1}}\tilde{\gamma^\mu}\tilde{\mathsf{C}} = \mathsf{C}\tilde{\gamma^\mu}\mathsf{C}^{-1} = -\tilde{\gamma^\mu}$.

3.4.25 For a rotation about the z-axis $(\cos\theta/2 - i\sigma^{12}\sin\theta/2)\gamma^1(\cos\theta/2 + i\sigma^{12}\sin\theta/2) = \gamma^1\cos\theta + \gamma^2\sin\theta$; for a boost along the x-axis $(\cosh\zeta/2 - i\sigma^{01}\sinh\zeta/2)\gamma^1(\cosh\zeta/2 + i\sigma^{01}\sinh\zeta/2) = \gamma^1\cosh\zeta - \gamma^0\sinh\zeta$.

3.5.5 $\langle jm|\mathbf{J}^2|jm\rangle = \langle jm|J_x^2|jm\rangle + \langle jm|J_y^2|jm\rangle + \langle jm|J_z^2|jm\rangle = |J_x|jm\rangle|^2 + |J_y|jm\rangle|^2 + |J_z|jm\rangle|^2$.

3.5.6 If $\mathsf{A}|x_i\rangle = \lambda_i|x_i\rangle$, then $|x_i\rangle = \lambda_i\mathsf{A}^{-1}|x_i\rangle$ upon multiplying with the inverse matrix. Moving the (non-zero) eigenvalue to the left-hand side proves the claim.

3.5.8 If $\mathsf{SAS}^{-1} = [a_1, \ldots, a_n]$ and $\mathsf{SBS}^{-1} = [b_1, \ldots, b_n]$, then $\mathsf{AB} = \mathsf{S}[a_1, \ldots, a_n]\mathsf{S}^{-1}\mathsf{S}[b_1, \ldots, b_n]\mathsf{S}^{-1} = \mathsf{S}[a_1b_1, \ldots, a_nb_n]\mathsf{S}^{-1} = \mathsf{S}[b_1, \ldots, b_n]\mathsf{S}^{-1}\mathsf{S}[a_1, \ldots, a_n]\mathsf{S}^{-1} = \mathsf{BA}$.

3.5.9 If $\mathsf{U}_1\mathsf{A}\mathsf{U}_1^\dagger = [\lambda_1, \ldots, \lambda_n] = \mathsf{U}_2\mathsf{B}\mathsf{U}_2^\dagger$ with unitary matrices U_i, then $\mathsf{A} = \mathsf{U}_1^\dagger\mathsf{U}_2\mathsf{B}\mathsf{U}_2^\dagger\mathsf{U}_1 = \mathsf{U}_1^\dagger\mathsf{U}_2\mathsf{B}(\mathsf{U}_1^\dagger\mathsf{U}_2)^\dagger$.

3.5.10 For M_x,
$$\begin{aligned} \lambda_1 &= +1, & \mathbf{r}_1 &= (1, +\sqrt{2}, 1)/2 \\ \lambda_2 &= 0, & \mathbf{r}_2 &= (1, 0, -1)/\sqrt{2} \\ \lambda_3 &= -1, & \mathbf{r}_3 &= (1, -\sqrt{2}, 1)/2. \end{aligned}$$

For M_y,
$$\begin{aligned} \lambda_1 &= +1, & \mathbf{r}_1 &= (1, +i\sqrt{2}, -1)/2 \\ \lambda_2 &= 0, & \mathbf{r}_2 &= (1, 0, 1)/\sqrt{2} \\ \lambda_3 &= -1, & \mathbf{r}_3 &= (1, -i\sqrt{2}, -1)/2. \end{aligned}$$

3.5.12 $$\mathrm{I} = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$\begin{aligned} \lambda_1 &= 16, & \mathbf{r}_1 &= (1, -1, 0)/\sqrt{2} \\ \lambda_2 &= 8, & \mathbf{r}_2 &= (0, 0, 1) \\ \lambda_3 &= 8, & \mathbf{r}_3 &= (1, 1, 0)/\sqrt{2}. \end{aligned}$$

3.5.16
$$\begin{aligned} \lambda_1 &= 0, & \mathbf{r}_1 &= (1, 0, -1)/\sqrt{2} \\ \lambda_2 &= 1, & \mathbf{r}_2 &= (0, 1, 0) \\ \lambda_3 &= 2, & \mathbf{r}_3 &= (1, 0, 1)/\sqrt{2}. \end{aligned}$$

3.5.17
$$\begin{aligned} \lambda_1 &= -1, & \mathbf{r}_1 &= (1, -\sqrt{2}, 0)/\sqrt{3} \\ \lambda_2 &= 0, & \mathbf{r}_2 &= (0, 0, 1) \\ \lambda_3 &= 2, & \mathbf{r}_3 &= (\sqrt{2}, 1, 0)/\sqrt{3}. \end{aligned}$$

3.5.18
$$\begin{aligned} \lambda_1 &= -1, & \mathbf{r}_1 &= (1, -2, 1)/\sqrt{6} \\ \lambda_2 &= 1, & \mathbf{r}_2 &= (1, 0, -1)/\sqrt{2} \\ \lambda_3 &= 2, & \mathbf{r}_3 &= (1, 1, 1)/\sqrt{3}. \end{aligned}$$

3.5.19
$$\begin{aligned} \lambda_1 &= -3, & \mathbf{r}_1 &= (1, -\sqrt{2}, 1)/2 \\ \lambda_2 &= 1, & \mathbf{r}_2 &= (1, 0, -1)/\sqrt{2} \\ \lambda_3 &= 5, & \mathbf{r}_3 &= (1, \sqrt{2}, 1)/2. \end{aligned}$$

3.5.20
$$\begin{aligned} \lambda_1 &= 0, & \mathbf{r}_1 &= (0, 1, -1)/\sqrt{2} \\ \lambda_2 &= 1, & \mathbf{r}_2 &= (1, 0, 0) \\ \lambda_3 &= 2, & \mathbf{r}_3 &= (0, 1, 1)/\sqrt{2}. \end{aligned}$$

3.5.21
$$\begin{aligned} \lambda_1 &= -1, & \mathbf{r}_1 &= (0, 1, -\sqrt{2})/\sqrt{3} \\ \lambda_2 &= +1, & \mathbf{r}_2 &= (1, 0, 0) \\ \lambda_3 &= 2, & \mathbf{r}_3 &= (0, \sqrt{2}, 1)/\sqrt{3}. \end{aligned}$$

3.5.22 $\lambda_1 = -\sqrt{2}$, $\mathbf{r}_1 = (1, -\sqrt{2}, 1)/2$
$\lambda_2 = 0$, $\mathbf{r}_2 = (1, 0, -1)/\sqrt{2}$
$\lambda_3 = \sqrt{2}$, $\mathbf{r}_3 = (1, \sqrt{2}, 1)/2$.

3.5.23 $\lambda_1 = 0$, $\mathbf{r}_1 = (0, 1, -1)/\sqrt{2}$
$\lambda_2 = 2$, $\mathbf{r}_2 = (0, 1, 1)/\sqrt{2}$
$\lambda_3 = 2$, $\mathbf{r}_3 = (1, 0, 0)$.

The eigenvectors corresponding to degenerate eigenvalues are *not* unique.

3.5.24 $\lambda_1 = 2$, $\mathbf{r}_1 = (1, 1, 1)/\sqrt{3}$
$\lambda_2 = -1$, $\mathbf{r}_2 = (1, -1, 0)/\sqrt{2}$
$\lambda_3 = -1$, $\mathbf{r}_3 = (1, 1, -2)/\sqrt{6}$.

3.5.25 $\lambda_1 = -1$, $\mathbf{r}_1 = (1, 1, 1)/\sqrt{3}$
$\lambda_2 = 2$, $\mathbf{r}_2 = (1, -1, 0)/\sqrt{2}$
$\lambda_3 = 2$, $\mathbf{r}_3 = (1, 1, -2)/\sqrt{6}$.

3.5.26 $\lambda_1 = 3$, $\mathbf{r}_1 = (1, 1, 1)/\sqrt{3}$
$\lambda_2 = 0$, $\mathbf{r}_2 = (1, -1, 0)/\sqrt{2}$
$\lambda_3 = 0$, $\mathbf{r}_3 = (1, 1, -2)/\sqrt{6}$.

3.5.27 $\lambda_1 = 6$, $\mathbf{r}_1 = (2, 0, 1)/\sqrt{5}$
$\lambda_2 = 1$, $\mathbf{r}_2 = (1, 0, -2)/\sqrt{5}$
$\lambda_3 = 1$, $\mathbf{r}_3 = (0, 1, 0)$.

3.5.28 $\lambda_1 = 2$, $\mathbf{r}_1 = (1, 1, 0)/\sqrt{2}$
$\lambda_2 = 0$, $\mathbf{r}_2 = (1, -1, 0)/\sqrt{2}$
$\lambda_3 = 0$, $\mathbf{r}_3 = (0, 0, 1)$.

3.5.29 $\lambda_1 = 2$, $\mathbf{r}_1 = (1, 0, -\sqrt{3})/2$
$\lambda_2 = 3$, $\mathbf{r}_2 = (0, 1, 0)$
$\lambda_3 = 6$, $\mathbf{r}_3 = (\sqrt{3}, 0, 1)/2$.

3.5.30 (a) $\lambda_1 = 1 + \varepsilon$, $\mathbf{r}_1 = (1/\sqrt{2}, 1/\sqrt{2})$
$\lambda_2 = 1 - \varepsilon$, $\mathbf{r}_2 = (1/\sqrt{2}, -1/\sqrt{2})$

(b) $\lambda_1 = 1 + \varepsilon$, $\mathbf{r}_1 = (1, \varepsilon)$
$\lambda_2 = 1 - \varepsilon$, $\mathbf{r}_2 = (1, -\varepsilon)$

(c) $\cos\theta = (1 - \varepsilon^2)/(1 + \varepsilon^2)$

3.5.33 Since the quadratic form $x^2 + 2xy + 2y^2 + 2yz + z^2 = 1$ defining the surface is obviously positive definite upon writing it as a sum of squares, $(x + y)^2 + (y + z)^2 = 1$, it is an ellipsoid or an ellipse.

Finding the orientation in space amounts to diagonalizing the symmetric 3×3 matrix of coefficients. The characteristic polynomial is $\lambda(1-\lambda)(\lambda-3) = 0$, so that the eigenvalues are $\lambda = 0$ implying an ellipse, and $\lambda = 1$, and 3. For $\lambda = 1$ an eigenvector is $v_1 = (1, 0, -1)$ giving one of its axes, for $\lambda = 3$ an eigenvector is $v_3 = (1, 2, 1)$ giving the other axis. $\mathbf{v_1} \times \mathbf{v_3} = (2, -2, 2)$ is normal to the plane of the ellipse.

3.6.3 The characteristic polynomial is $0 = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 - tr(\mathsf{A})\lambda + \det(\mathsf{A})$. See also Ex. 3.3.9.

3.6.4 If $\mathsf{U}\mathbf{r} = \lambda\mathbf{r}$ with $|\mathbf{r}|^2$, then $1 = \mathbf{r}^\dagger\mathbf{r} = \mathbf{r}^\dagger\mathsf{U}^\dagger\mathsf{U}\mathbf{r} = |\lambda|^2\mathbf{r}^\dagger\mathbf{r} = |\lambda|^2$.

3.6.7 From Ex. 3.6.4 the eigenvalues have $|\lambda| = 1$. If U is Hermitian, then λ is real, hence ± 1.

3.6.8 $\gamma_0\gamma = -\gamma\gamma_0$ gives for the determinants $\det(\gamma_0)\det(\gamma) = (-1)^n \det(\gamma_0)\det(\gamma)$. Hence n is even.

3.6.9 If $\mathsf{A} = \sum_n \lambda_n |\mathbf{x}_n\rangle\langle\mathbf{x}_n|$, then $\mathsf{A}|\mathbf{x}_m\rangle = \lambda_m|\mathbf{x}_m\rangle$. Note that A is unique as it can be constructed from its eigenvalues and eigenvectors.

3.6.14 (a) $\mathsf{A}\tilde{\mathsf{A}} = \frac{1}{5}\begin{pmatrix} 8 & -6 \\ -6 & 17 \end{pmatrix} \qquad \tilde{\mathsf{A}}\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

(b) $\lambda_1 = 1 \quad |g_1> = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, |f_1> = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$

(c) $\lambda_2 = 2, \quad |g_2> = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, |f_2> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

3.6.17 From $\mathsf{A}v_i = \mathsf{A}_i v_i$ obtain $\mathsf{A}^n v_i = (A_i)^n v_i$ for $n = 0, 1, 2, \ldots$. From $\mathsf{B} = \exp(\mathsf{A}) = \sum_{n=0}^{\infty} \mathsf{A}^n/n!$ we then get $\mathsf{B}v_i = \sum_{n=0}^{\infty} \mathsf{A}^n v_i/n! = \sum_{n=0}^{\infty}\{A_i^n/n!\}v_i = (e^{A_i})v_i$.

CHAPTER 4

4.1.1 There are n^2 real matrix elements minus n row (or column) normalizations minus $n(n-1)$ orthogonality constraints, i.e., $n^2 - n - n(n-1)/2 = n(n-1)/2$ independent parameters.

4.1.2 There are n^2 complex matrix elements comprising $2n^2$ parameters, n normalizations, $n(n-1)$ orthogonality constraints, and one determinant $= 1$ constraint, i.e., $2n^2 - n - n(n-1) - 1 = n^2 - 1$.

4.1.3 If $\det(e_1) = 1 = \det(e_2)$, then $\det(e_1 e_2) = \det(e_1)\det(e_2) = 1$, and $e_1 e_2$ is also a 2×2 matrix with complex elements that has an inverse $e_2^{-1}e_1^{-1}$, because $e_1 e_2 e_2^{-1} e_1^{-1} = 1$ with $\det(e_2^{-1}e_1^{-1}) = \det(e_2^{-1})\det(e_1^{-1}) = 1$. Noting that $e_i e_i^{-1} = 1$ implies $1 = \det(1) = \det(e_i)\det(e_i^{-1}) = \det(e_i^{-1})$.

4.1.4 The product of $R_z(\varphi_1)R(\varphi_2) = R_z(\varphi_1 + \varphi_2)$. This is Eq. (4.2). The inverse of $R_z(\varphi)$ is $R_z(-\varphi)$. It is not an invariant subgroup, which can be checked by $R_x(\pi/2)R_z(\varphi)R_x(-\pi/2) \neq R_z(\varphi')$.

4.2.1 (i) Since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is traceless and Hermitian, it is a generator of SU(2), i.e. τ_3. One off-diagonal Pauli matrix is obtained from differentiating at $\varphi = 0$ the 2×2 rotation matrix $R(\varphi)$ of Eq. (4.1) yielding τ_2. The commutator $[\tau_2, \tau_3]$ gives the third one.

(ii) The 3×3 matrices λ_i with τ_i, $i = 1, 2, 3$ in the upper left 2×2 corner and zeros elsewhere are traceless, Hermitian and linearly independent. Hence they are 3 generators of SU(3) corresponding to the SU(2) subgroup of SU(3) that leaves the 3-axis fixed. Two more off-diagonal $\lambda_{4,5}$ are found by considering rotations that leave the 2-axis unchanged. These have zeros in the middle row and column and the elements of τ_1, τ_2 in the corners, respectively. See Eq. (4.61b). $\lambda_{6,7}$ are constructed similarly from rotations that leave the 1-axis unchanged, i.e. they have zeros in the first row and column and the $\tau_{1,2}$ elements in the lower right 2×2 corner. The second diagonal generator λ_8 is chosen with the unit 2×2 matrix in the upper left corner so as to be outside the SU(2) subgroup spanned by the τ_i. For $\mathrm{tr}(\lambda_8) = 0$, its third diagonal element must be -2 and zeros elsewhere in the third row and column. The $\sqrt{3}$ normalization follows from enforcing $\mathrm{tr}(\lambda_8^2) = 2$.

(iii) The Casimir invariant is $\sum_i \lambda_i^2$.

4.2.3 See Ex. 4.2.1 for the transformations that generate the three SU(2) subgroups of SU(3). They leave axis 1, or 2 or 3 unchanged, i.e. are 3×3 matrices with 1 for the diagonal (i,i) element, zeros in the ith row and column elsewhere, and the elements of the general unitary 2×2 matrix of Eq. (4.38) in the remaining places, where $i = 1, 2, 3$ is the fixed axis.

4.2.4 $\exp(ad/dx)\psi(x) = \psi(x + a) = \psi(x) + a\psi'(x) + \cdots$

4.2.6 We apply the generator $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}$ to a two-component wave function $\tilde{\psi} = (\psi_\uparrow, \psi_\downarrow)$ in terms of an infinitesimal coordinate rotation and a corresponding rotation of the wave function components yielding

$$\psi'_\uparrow = \psi_\uparrow(x + y\,d\theta, -x\,d\theta + y) + \frac{1}{2}\sigma_3\,d\theta\psi_\uparrow,$$
$$\psi'_\downarrow = \frac{1}{2}\sigma_3\psi_\downarrow\,d\theta + \psi_\downarrow(x + y\,d\theta, -x\,d\theta + y),$$

where the tilde denotes transposition, $\mathbf{L}$ acts on the spatial coordinates (the argument $\mathbf{r}$ of ψ) and the Pauli matrices on the spin wave function components $\psi_\updownarrow$.

4.3.1 $[J_+, \mathbf{J}^2] = [J_x \pm iJ_y, \mathbf{J}^2] = [J_x, \mathbf{J}^2] \pm i[J_y, \mathbf{J}^2] = 0 \pm i0 = 0.$

4.4.1 We apply $J_+ = J_{1+} + J_{2+}$ to the state $|(j_1 j_2)JM\rangle$ using the $\langle J_+\rangle$ matrix element that we know. This yields

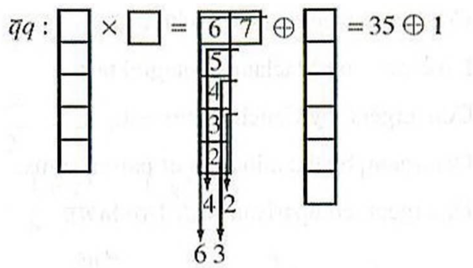
$$\begin{aligned} J_+|(j_1j_2)JM\rangle &= [(J-M)(J+M+1)]^{1/2} \\ &\quad\times \sum_{m_1m_2} C(j_1j_2J, m_1m_2M+1)|j_1m_1\rangle|j_2m_2\rangle \\ &= \sum_{m_1m_2} C(j_1j_2J, m_1m_2M) \\ &\quad\times \{[(j_1-m_1)(j_1+m_1+1)]^{1/2}|j_1m_1+1\rangle|j_2m_2\rangle \\ &\quad+ [(j_2-m_2)(j_2+m_2+1)]^{1/2}|j_1m_1\rangle|j_2m_2+1\rangle\}, \end{aligned}$$

from which we project with $|j_1m_1+1\rangle|j_2m_2\rangle$ to get

$$\begin{aligned} &[(J-M)(J+M+1)]^{1/2}C(j_1j_2J, m_1+1m_2M+1) \\ &\quad= C(j_1j_2J, m_1m_2M)[(j_1-m_1)(j_1+m_1+1)]^{1/2} \\ &\qquad+ C(j_1j_2J, m_1+1m_2-1M)[(j_2-m_2+1)(j_2+m_2)]^{1/2}. \end{aligned}$$

Using this recursion we check that $C(111, 000) = 0, C(111, 101) = 1/\sqrt{2}$, etc. Projecting $|j_1m_1\rangle|j_2m_2+1\rangle$ gives a similar recursion. Using $\mathbf{J}^2 \to J(J+1) = j_1(j_1+1)+j_2(j_2+1)+2m_1m_2+J_{1+}J_{2-}+J_{1-}J_{2+}$ in conjunction with the square-root matrix elements of $J_{i\pm}$ yields a third recursion relation.

4.4.3 The direct product of SU(3)-flavor and SU(2)-spin is obviously contained in SU(6). Since the totally antisymmetric YT = [1, 1, 1, 1, 1, 1] containing a single column of six boxes is 1, the antiquark representation has to correspond to the YT = [1, 1, 1, 1, 1] containing five boxes in a single column. Then



with the dimensions $D = 7!/(6\cdot4\cdot3\cdot2) = 7\cdot5 = 35$ and 1, respectively. The 35 dimensional SU(6) representation contains four SU(3) octets, i.e., three spin-1 octets and one spin-0 octet, and an SU(3) singlet of spin-1.

4.6.2 (c) The transformation properties of zero are indeterminate.

4.6.3 $\mu_0 c \frac{\partial A_\lambda}{\partial x_\lambda} = 0$.

4.7.5 (b) Trace 1 = 2, Trace A = −2
Trace B = Trace C = 0

4.7.6 (b) Trace 1 = 2, Trace B = −2
Trace A = Trace C = 0

4.7.15 (a) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ (b) even

4.7.21 (b) 48 elements.

CHAPTER 5

5.2.5 Convergent for $a_1 - b_1 > 1$. Divergent for $a_1 - b_1 \leq 1$.

5.2.6 (a) Divergent, comparison with harmonic series.

(b) Divergent, by Cauchy ratio test.

(c) Convergent, comparison with $\zeta(p)$, $1 < p < 2$.

(d) Divergent, comparison with $(n+1)^{-1}$.

(e) Divergent, comparison with $\frac{1}{2}(n+1)^{-1}$ or by Maclaurin integral test.

5.2.7 (a) Convergent, comparison with $\zeta(2)$.

(b) Divergent, by Maclaurin integral test.

(c) Convergent, by Cauchy ratio test.

(d) Divergent, by examination of partial sums.

(e) Divergent, comparison with $1/(n \ln n)$.

5.2.11 (b) With $6 \ln 10 = 13.815$, $13.815 < \sum_{n=1}^{10^6} n^{-1} < 14.815$.

5.2.14 Divergent, by Gauss' test.

5.2.20 $\zeta(2) = \dfrac{5}{4} + 2\sum_{n=1}^{\infty} \dfrac{1}{n^2(n+1)(n+2)}$.

5.3.1 (a) Convergent, (b) Divergent.

5.4.2 $a_1 = 3, a_2 = 3, a_3 = 1$.

5.5.3 (a) Convergent for $1 < x < \infty$.

(b) Uniformly convergent for $1 < s \le x < \infty$.

5.6.2 $\cot x = \dfrac{1}{x} - \dfrac{x}{3} - \dfrac{x^3}{45} - \dfrac{2x^5}{945} - \cdots, -\pi < x < \pi$.

5.6.5 (b) $|x - x_0| < x_0$.

5.6.7 $\nabla^2\Phi|_{0,0,0} = \dfrac{24}{a^2}[\Phi - \Phi(0,0,0)]$.

5.6.12 (a) $\nu' = \nu\left\{1 \pm \dfrac{\nu}{c} + \dfrac{\nu^2}{c^2} + \cdots\right\}$.

(b) $\nu' = \nu\left\{1 \pm \dfrac{\nu}{c}\right\}$.

(c) $\nu' = \nu\left\{1 \pm \dfrac{\nu}{c} + \dfrac{1}{2}\dfrac{\nu^2}{c^2} + \cdots\right\}$.

5.6.13 (a) $\frac{\nu_1}{c} = \delta + 1/2\delta^2$.

(b) $\frac{\nu_2}{c} = \delta - 3/2\delta^2 + \cdots$.

(c) $\frac{\nu_3}{c} = \delta - 1/2\delta^2 + \cdots$.

5.6.14 $\frac{w}{c} = 1 - \frac{\alpha^2}{2} - \frac{\alpha^3}{2} + \cdots$.

5.6.15 $x = \frac{1}{2}gt^2 - \frac{1}{8}\frac{g^3t^4}{c^2} + \frac{1}{16}\frac{g^5t^6}{c^4} - \cdots$.

5.6.16 $E = mc^2\left[1 - \frac{\gamma^2}{2n^2} - \frac{\gamma^4}{2n^4}\left(\frac{n}{|k|} - \frac{3}{4}\right) + \cdots\right]$.

5.6.18 The two series have different, nonoverlapping convergence intervals.

5.6.23 Error $\sim \frac{h^2}{12}\left(\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + \frac{\partial^4\psi}{\partial z^4}\right)$.

5.7.1 $P(x) = C\left\{\frac{x}{3} - \frac{x^3}{45} + \cdots\right\}$.

5.7.7 (b) Convergent for $0 \le x < \infty$. The upper limit x does *not* have to be small, but unless it is small the convergence will be slow and the expansion relatively useless.

5.7.11 $\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \cdots, -1 \le x \le 1$.

5.8.1 $dx = a\cos\theta\, d\theta$, $dy = -b\sin\theta\, d\theta$, $dr^2 = dx^2 + dy^2$, $dr = d\theta \times (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}$. Hence the arc is $\int_0^{\pi/2}(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}d\theta = a\int_0^{\pi/2}\sqrt{1 - m\sin^2\theta}\, d\theta$ using $\cos^2\theta = 1 - \sin^2\theta$.

5.9.14 (a) $-1 < x < 1$.

(b) The infinite series will converge for $-2 < x < 2$ but the terms $\sin\pi x$, $\ln(1+x)$, and $\ln(1-x)$ still limit x to $-1 < x < 1$. The advantage of this expression over the preceding one lies in the faster convergence of the infinite series.

5.10.2 (a) $C(x) = \frac{1}{2} + S_1\cos\left(\frac{\pi x^2}{2}\right) - S_2\sin\left(\frac{\pi x^2}{2}\right)$.

(b) $S(x) = \frac{1}{2} + S_1\sin\left(\frac{\pi x^2}{2}\right) + S_2\cos\left(\frac{\pi x^2}{2}\right)$

with

$$S_1 = \frac{1}{\pi x} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (4n+1)}{(\pi x^2)^{2n+1}},$$

$$S_2 = \frac{1}{\pi x} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (4n-1)}{(\pi x^2)^{2n}}.$$

Hint: $C(x) + iS(x) = C(\infty) + iS(\infty) - \int_x^{\infty} \exp[i\pi u^2/2]\, du.$

5.10.6 No, $1/(1+x) = x^{-1}(1+1/x)^{-1} = \sum_{n=0}^{\infty} (-1)^n/x^{n+1}$ converges absolutely for $x > 1$ and $x < -1$.

5.11.4 1.

5.11.5 $\prod_{n=2}^{\infty} \{1 - 2/[n(n+1)]\} = \prod_{n=2}^{\infty} (1-1)/n[1 + 1/(n+1)]$

$$= \prod_{n=2}^{\infty} \frac{n-1}{n} \cdot \frac{n+2}{n+1} = \frac{2}{2 \cdot 3} \prod_{n=2}^{\infty} \frac{n-1}{n} \cdot \frac{n}{n-1} = \frac{1}{3}$$

upon shifting n in the second product down to $n-2$ and correcting for the two first terms.

5.11.6 $\prod_{n=2}^{\infty} (1 - 1/n^2) = \prod_{n=2}^{\infty} (1 - 1/n)(1 + 1/n) = \prod_{n=2}^{\infty} \frac{n-1}{n} \cdot \frac{n+1}{n}$

$$= \frac{1}{2} \prod_{n=2}^{\infty} \frac{n-1}{n} \cdot \frac{n}{n-1} = \frac{1}{2}$$

after shifting n in the second product term down to $n-1$ and correcting for the first missing term.

CHAPTER 6

6.1.1 $(x+iy)^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$

6.1.5 $i^{1/2} = e^{i\pi/4}, e^{-i3\pi/4}.$

6.1.7 $\sum_{n=0}^{N-1} (e^{ix})^n = \frac{1 - e^{iNx}}{1 - e^{ix}} = \frac{e^{iNx/2}}{e^{ix/2}} \frac{e^{iNx/2} - e^{-iNx/2}}{e^{ix/2} - e^{-ix/2}} = e^{i(N-1)x/2}$ $\sin(Nx/2)/\sin(x/2)$. Now take real and imaginary parts to get the result.

6.1.15 (a) $z = n\pi, \ n = 0, \ \pm 1, \ \pm 2, \cdots.$

(b) $z = (n + 1/2)\pi.$

(c) $z = n\pi i.$

(d) $z = (n + 1/2)\pi i.$

6.1.18 $k \to$ attenuation ($k > 0$). The distance for the amplitude to fall by a factor of e is $c/k\omega$.

6.1.19 Since $L_x^\dagger = L_x, \ L_y^\dagger = L_y, \ (L_x + iL_y)^\dagger = L_x - iL_y.$

6.1.22 $\sinh z = -i \sin iz = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2\pi^2}\right),$

$\cosh z = \cos iz = \prod_{n=1}^{\infty} \left(1 + \frac{4z^2}{(2n-1)^2\pi^2}\right).$

6.2.5 (a) $v(x, y) = 3x^2y - y^3, \quad w(z) = z^3.$

(b) $u(x, y) = e^{-y} \cos x, \quad w(z) = e^{iz}.$

6.4.4 0.

6.5.8 $a_{-1} = 1, \quad a_0 = -1/2, \quad a_1 = 1/12.$

6.5.10 (a) $f(z) = \sum_{n=-1}^{\infty} (-1)^{n+1}(z-1)^n, \quad 0 < |z - 1| < 1.$

(b) $f(z) = \sum_{n=-2}^{\infty} (-1)^n (z-1)^{-n}, \quad |z - 1| > 1.$

6.6.1 (a) Ellipses, semimajor axis $(u) = r + r^{-1}$,
semiminor axis $(v) = r - r^{-1}$.

(b) Ellipses, semimajor axis $(v) = r + r^{-1}$,
semiminor axis $(u) = r - r^{-1}$.

6.6.2 (a) $x > 0$. (b) $y > 0$.

6.6.3 (a) $x = c_1 \to$ hyperbola
$y = c_2 \to$ ellipse, semimajor axis along u-axis.

(b) $x = c_1 \to$ hyperbola
$y = c_2 \to$ ellipse, semimajor axis along u-axis.

(c) $x = c_1 \rightarrow$ ellipse, semimajor axis along v-axis
$y = c_2 \rightarrow$ hyperbola.

(d) $x = c_1 \rightarrow$ ellipse, semimajor axis along u-axis
$y = c_2 \rightarrow$ hyperbola.

6.7.2 (a) $z = e^w$, $w = \psi + i\varphi$, and $\rho = e^\psi$.
(b) $z = a \cosh w$, $w = u + iv$.
(c) $z = -(i/2)w^2$, $w = \xi + i\eta$.
(d) $z^* = a \coth(w/2)$, $w = \eta + i\xi$.

The complex conjugate represents a simple reflection of the y-axis and does *not* upset the angle-preserving property of the transformation.

6.7.3 The coordinate lines in the z-plane transform into circles in the w-plane. This is a form of bipolar coordinates.

CHAPTER 7

7.1.2 $f_2(z) = f_2'(z_0)(z - z_0) + \cdots$ implies that $f_1(z)/f_2(z) = f_1(z)/f_2'(z_0)(z-z_0)^{-1} + \cdots = f_1(z_0)/f_2'(z_0)(z-z_0)^{-1} + \cdots$, where the ellipses stand for some regular function at z_0.

7.2.1 (a) $z_0 = \pm ia$, simple poles
$a_{-1} = \pm\dfrac{1}{2ai}$.
(b) $z_0 = \pm ia$, second-order poles
$a_{-1} = \pm\dfrac{1}{4a^3 i}$.
(c) $z_0 = \pm ia$, second-order poles
$a_{-1} = \pm\dfrac{1}{4ai}$.
(d) $z_0 = \pm ia$, simple poles
$a_{-1} = -\dfrac{\sinh(1/a)}{2a}$.
(e) $z_0 = \pm ia$, simple poles
$a_{-1} = 1/2e^{\pm a}$.
(f) $z_0 = \pm ai$, simple poles
$a_{-1} = -i/2e^{\mp a}$.

(g) $z_0 = \pm a$, simple poles
$a_{-1} = \pm\frac{1}{2a}e^{\pm ia}$.

(h) $z_0 = -1$, simple pole
$a_{-1} = e^{-ik\pi}$ for $z = e^{i\pi}$.
$z_0 = 0$ is a branch point.

7.2.9 $-1 < t < 1$, so that the denominator of the integrand will not go to zero. Cf. Eq. 13.64 for an alternate approach.

7.4.4 $I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}$.

7.4.5 $K_\nu(x) \sim \sqrt{\pi/2x}\, e^{-x}$.

CHAPTER 8

8.2.1 (a) $I(t) = I_0 e^{-t/RC}$

(b) $I(0) = 10^{-4}$ amperes. At time $t = 100$ seconds the current is $I(0)\exp(-0.01) = 0.99005\, I(0) = 99.005$ microamperes.

8.2.2 $f(s) = C/\sqrt{s^2+1}$.

8.2.5 $v^{1-n} - v_0^{1-n} = (n-1)\frac{k}{m}t,\ n \neq 1$.

8.2.12 $v(t) = \frac{mg}{b}(1 - e^{-bt/m})$.

8.2.13 $N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$.

8.2.14 $r(t) = r_0(1 - \alpha t)$.

8.3.7 "Neglect end effects" is a physical basis for excluding z dependence.

8.5.7 Let n be a nonnegative integer. Cf. Eq. 13.24.

8.5.8 The infinite series *does* converge for $x = \pm 1$. Hence this imposes no restriction on n. Cf. Ex. 5.2.16. If we demand a polynomial solution then n must be a nonnegative integer. Cf. Eq. 13.83.

8.5.9 For $k = 1$, take n to be a positive odd integer. Cf. Eq. 13.80. Here $a_0 = (-1)^{(n-1)/2}(r+1)$.

8.5.10 Cf. Eq. 13.93. Convergence given in Ex. 5.2.9.

8.5.11 Cf. Eq. 13.112. Convergent for all finite x provided the series exists ($c \neq -n$, a negative integer, in Eq. 13.113, $2-c \neq -n$ in Eq. 13.112).

8.6.19 $y_2(x) = \int^x e^s \frac{ds}{s} = \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$.

8.7.4 (a) $G(\mathbf{r}_1, \mathbf{r}_2) = \frac{\cos k|\mathbf{r}_1 - \mathbf{r}_2|}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}$.

(b) $G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{k}{4\pi} n_0(k|\mathbf{r}_1 - \mathbf{r}_2|)$.

8.7.8 $\varphi(\mathbf{r}_1) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} (-1)^l \frac{a^{2l}}{r_1^{2l+1}} \frac{(2l-1)!!}{(2l)!!} P_{2l}(\cos\theta)$.

CHAPTER 9

9.1.5 (a) $x = \pm 1$, regular singular points.

9.2.3 (b) $\frac{d}{dx}\left[(1-x^2)^{3/2}\frac{dU_n}{dx}\right] + n(n+2)(1-x^2)^{1/2}U_n = 0$.
The necessary boundary conditions are not satisfied.

9.2.6 By integrating by parts the first term of $\int_a^b u_m \frac{d}{dx} p(x) \frac{d}{dx} u_n \, dx + \lambda_n \int_a^b u_m w(x) u_n \, dx = 0$, we obtain

$$u_m p(x) \frac{d}{dx} u_n \Big|_a^b - \int_a^b u'_m p(x) u'_n \, dx + \lambda_n \int_a^b u_m w(x) u_n \, dx = 0.$$

The first term is zero because of the boundary condition, while the third term $\int_a^b u_m w(x) u_n \, dx = \delta_{nm}$ by orthogonality. Hence the orthogonality relation $\int_a^b u'_m p(x) u'_n \, dx = \lambda_n \delta_{nm}$.

9.2.9 (a) Multiply by $(1-x^2)^{\alpha-1/2}$.

9.3.9 $\varphi_1 = \sqrt{2}\, e^{-x}$
$\varphi_2 = -4e^{-x} + 6e^{-2x}$
$\varphi_3 = \sqrt{6}(3e^{-x} - 12e^{-2x} + 10e^{-3x})$.
The Sturm–Liouville theory does not guarantee orthogonality because the required boundary condition at $x = 0$ is *not* satisfied.

9.5.2 (a) $G(x,t) = \begin{cases} \dfrac{\sin x \cos(1-t)}{\cos 1}, & 0 \le x \le t, \\ \dfrac{\sin t \cos(1-x)}{\cos 1}, & t \le x \le 1. \end{cases}$

(b) $G(x,t) = \begin{cases} 1/2e^{x-t}, & -\infty < x < t, \\ 1/2e^{t-x}, & t < x < \infty. \end{cases}$

9.5.8 With $\mathcal{L} = x\dfrac{d^2}{dx^2} + \dfrac{d}{dx} - \dfrac{1}{x}$, $[\lambda\rho(x)y(x) = k^2 xy(x)]$,

$G(x,t) = \begin{cases} \dfrac{x}{2t}(1-t^2), & x < t, \\ \dfrac{t}{2x}(1-x^2), & t < x. \end{cases}$

The integral equation is $y(x) = k^2 \int_0^1 G(x,t)y(t)t\,dt$,

$y(x) = J_1(kx)$ with $J_1(k) = 0$.

If $\mathcal{L}$ is interpreted as $x\frac{d^2}{dx^2} + \frac{d}{dx} + (k^2x^2 - 1)/x$,

$G(x,t) = \begin{cases} \dfrac{\pi}{2}\left\{\dfrac{-N_1(kt) + N_1(k)J_1(kt)}{J_1(k)}\right\} J_1(kx), & 0 \le x < t, \\ \dfrac{\pi}{2}\left\{\dfrac{-N_1(kx) + N_1(k)J_1(kx)}{J_1(k)}\right\} J_1(kt), & t < x \le 1. \end{cases}$

CHAPTER 10

10.1.2 $\dfrac{2^n(s+n)!(n+2s)!(2s+1)!}{2^s s!s!n!(2s+2n+1)!}$.

10.2.5 $\ln(z!) = -\gamma z + \sum_{n=1}^{\infty}\left\{\dfrac{z}{n} - \ln\left(1+\dfrac{z}{n}\right)\right\}$

Expand $\ln\left(1+\dfrac{z}{n}\right)$ by Ex. 5.4.1

10.2.12 $2\left\{1 + \left(\gamma - \frac{3}{2}\right)\alpha^2 Z^2 + \cdots + \left(\frac{\gamma^2}{2} - \frac{5\gamma}{4} + \frac{\zeta(2)}{2} + \frac{1}{8}\right)\alpha^4 Z^4 + \cdots\right\}$.

10.2.13 $(1-\gamma)b + 1/3(\zeta(3) - 1)b^3 + \cdots$.

10.2.19 (a) 1. (b) 0.75.

10.3.2 8.1×10^{67}.

10.3.7 Divergent by Gauss' test.

CHAPTER 11

11.1.14 These are actually Eqs. 11.115 and 11.116.
$C_n(x) = I_n(x) = i^{-n} J_n(ix)$.

11.1.21 $(a^2 + b^2)^{-1/2}$.

11.1.23 (a) The first two zeros of $J_1(x)$ are $x = 3.8317$ and 7.0156, Table 11.1.

(b) $1 - [J_0(3.8317)]^2 = 0.838$.

11.3.8 (a) TM_{01} mode

$$E_z = a_1 J_0\left(\frac{2\pi r}{\lambda_c}\right)\exp[i(\omega t - 2\pi z/\lambda_g)]$$

$$E_r = -i\left(\frac{\lambda_c}{\lambda_g}\right)\left(\frac{\lambda_c}{2\pi}\right) a_1 \frac{d}{dr} J_0\left(\frac{2\pi r}{\lambda_c}\right)\exp[i(\omega t - 2\pi z/\lambda_g)]$$

$$H_\theta = -i\varepsilon_0\omega\left(\frac{\lambda_c}{2\pi}\right)^2 a_1 \frac{d}{dr} J_0\left(\frac{2\pi r}{\lambda_c}\right)\exp[i(\omega t - 2\pi z/\lambda_g)].$$

(b) TE_{01} mode

$$H_z = a_2 J_0\left(\frac{2\pi r}{\lambda_c}\right)\exp[i(\omega t - 2\pi z/\lambda_g)]$$

$$H_r = -i\left(\frac{\lambda_c}{\lambda_g}\right)\left(\frac{\lambda_c}{2\pi}\right) a_2 \frac{d}{dr} J_0\left(\frac{2\pi r}{\lambda_c}\right)\exp[i(\omega t - 2\pi z/\lambda_g)]$$

$$E_\theta = i\mu_0\omega\left(\frac{\lambda_c}{2\pi}\right)^2 a_2 \frac{d}{dr} J_0\left(\frac{2\pi r}{\lambda_c}\right)\exp[i(\omega t - 2\pi z/\lambda g)].$$

11.3.9 TM mode: $\lambda_c = 2.6128\, r_0,\ 1.6398\, r_0,\ 1.2234\, r_0$,
TE mode: $\lambda_c = 3.4126\, r_0,\ 2.0572\, r_0,\ 1.6398\, r_0$ (radius r_0).

11.6.1 In the limit as $z \to \infty$, this integral representation $\to 0$.

11.6.5 (b) $a_{n+1} = \dfrac{i}{2(n+1)}\left[\nu^2 - \dfrac{(2n+1)^2}{2^2}\right] a_n.$

CHAPTER 12

12.1.1 $\varphi(r,\theta) = \dfrac{2q}{4\pi\varepsilon_0 r}\displaystyle\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta), \quad a < r.$

12.1.2 $\varphi(r,\theta) \sim \dfrac{12q}{4\pi\varepsilon_0 r}\left(\dfrac{a}{r}\right)^3 P_3(\cos\theta), \quad r > a.$

12.2.1
$$\mathsf{A}_{\text{even}} = \begin{pmatrix} 1 & 1/3 & 7/35 & 33/231 \\ 0 & 2/3 & 20/35 & 110/231 \\ 0 & 0 & 8/35 & 72/231 \\ 0 & 0 & 0 & 16/231 \end{pmatrix}$$

$$\mathsf{B}_{\text{even}} = \begin{pmatrix} 1 & -1/2 & 3/8 & -5/16 \\ 0 & 3/2 & -30/8 & 105/16 \\ 0 & 0 & 35/8 & -315/16 \\ 0 & 0 & 0 & 231/16 \end{pmatrix}$$

$$\mathsf{A}_{\text{odd}} = \begin{pmatrix} 1 & 3/5 & 27/63 & 143/429 \\ 0 & 2/5 & 28/63 & 182/429 \\ 0 & 0 & 8/63 & 88/429 \\ 0 & 0 & 0 & 16/429 \end{pmatrix}$$

$$\mathsf{B}_{\text{odd}} = \begin{pmatrix} 1 & -3/2 & 15/8 & -35/16 \\ 0 & 5/2 & -70/8 & 315/16 \\ 0 & 0 & 63/8 & -693/16 \\ 0 & 0 & 0 & 429/16 \end{pmatrix}.$$

12.2.3 (b) $(12.25)_{n\to n+1} + x\cdot(12.24) \to (12.27).$

12.2.4 $\varphi^{(3)} = \dfrac{48a^3q}{4\pi\varepsilon_0 r^4}P_3(\cos\theta) + \cdots.$

12.2.9 $P_2(\cos\theta) = (3/4)\cos 2\theta + 1/4.$

12.3.2 $\delta(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{4n+1}{2}\frac{(2n-1)!!}{(2n)!!}P_{2n}(x), \quad -1 \le x \le 1.$

12.3.9 (b) Convergent by Gauss' test.

12.3.15 $\psi(r,\theta) = \dfrac{q}{4\pi\varepsilon_0 a}\displaystyle\sum_{s=0}^{\infty}(-1)^s\frac{(2s)!}{2^{2s}(s!)^2}\left(\frac{r}{a}\right)^{2s}P_{2s}(\cos\theta).$

12.3.16 $E_r(r,\theta) = \dfrac{q}{4\pi\varepsilon_0 r^2}\sum_{s=0}^{\infty}(-1)^s\dfrac{(2s+1)!!}{(2s)!!}\left(\dfrac{a}{r}\right)^{2s} P_{2s}(\cos\theta).$

$E_\theta(r,\theta) = \dfrac{q}{4\pi\varepsilon_0 r^2}\sum_{s=0}^{\infty}(-1)^s\dfrac{(2s-1)!!}{(2s)!!}\left(\dfrac{a}{r}\right)^{2s}\sin\theta\dfrac{d}{d}P_{2s}(\cos\theta).$

$(\cos\theta)r > a$

Note: The derivative of $P_{2s}(\cos\theta)$ leads to associated Legendre functions, Section 12.5.

12.4.12 This contour keeps the function $(t^2-1)^{\nu+1}/(t-z)^{\nu+2}$ (cf. Eq. 12.70) single valued.

12.5.10 $\dfrac{4}{3}\delta_{n,1}.$

12.5.13 (a) $B_r(r,\theta) = \sum\limits_{n=0}^{\infty} d_{2n+1}(2n+1)(2n+2)\dfrac{r^{2n}}{a^{2n+1}}P_{2n+1}(\cos\theta).$

$B_\theta(r,\theta) = -\sum\limits_{n=0}^{\infty} d_{2n+1}(2n+2)\dfrac{r^{2n}}{a^{2n+1}}P^1_{2n+1}(\cos\theta)$

with $d_{2n+1} = (-1)^n\dfrac{\mu_0 I}{2}\dfrac{(2n-1)!!}{(2n+2)!!}.$

12.5.15 (a) $B_z = 2/3\mu_0\omega\sigma a^4/z^3$, where ω is angular velocity, σ is the surface charge density, and a is the radius of the sphere.

(b) $A_\varphi(r,\theta) = 1/3\mu_0\omega\sigma\dfrac{a^4}{r^2}P^1_1(\cos\theta)$

$B_r(r,\theta) = 2/3\mu_0\omega\sigma\dfrac{a^4}{r^3}P_1(\cos\theta)$

$B_\theta(r,\theta) = 1/3\mu_0\omega\sigma\dfrac{a^4}{r^3}P^1_1(\cos\theta), \quad r > a.$

12.6.4 (a) Using the $Y_L^M(\theta,\varphi)$ as given in Table 12.3,

$$\frac{x^2}{r^2} = \frac{\sqrt{4\pi}}{3}Y_0^0 - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 + \sqrt{\frac{2\pi}{15}}(Y_2^2 + Y_2^{-2})$$

$$\frac{y^2}{r^2} = \frac{\sqrt{4\pi}}{3}Y_0^0 - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_2^0 - \sqrt{\frac{2\pi}{15}}(Y_2^2 + Y_2^{-2})$$

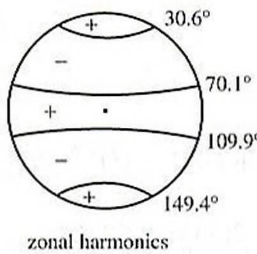
$$\frac{z^2}{r^2} = \frac{\sqrt{4\pi}}{3}Y_0^0 + \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_2^0$$

$$\frac{xy}{r^2} = i\sqrt{\frac{2\pi}{15}}(Y_2^{-2} - Y_2^2)$$

$$\frac{yz}{r^2} = i\sqrt{\frac{2\pi}{15}}(Y_2^{-1} + Y_2^1)$$

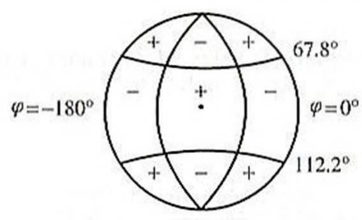
$$\frac{xz}{r^2} = \sqrt{\frac{2\pi}{15}}(Y_2^{-1} - Y_2^1).$$

12.6.9 $Y_{04}^e = P_4(\cos\theta) = 1/8(35\cos^2\theta - 30\cos^2\theta + 3)$



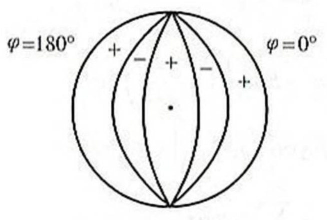
zonal harmonics

$$Y_{24}^e = P_4^2(\cos\theta)\cos 2\varphi = 15/2(7\cos^2\theta - 1)\sin^2\theta\cos 2\varphi$$



tesseral harmonics

$$Y_{44}^e = 105\sin^4\theta\cos 4\varphi$$



sectoral harmonics

12.8.7 $$\psi(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{24}\left[\frac{1}{r_1}\gamma\left(5, \frac{r_1}{a_0}\right) + \frac{1}{a_0}\Gamma\left(4, \frac{r_1}{a_0}\right)\right]\sqrt{4\pi}\,Y_0^0(\theta_1, \varphi_1)$$
$$- \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{120}\left[\frac{a_0^2}{r_1^3}\gamma\left(7, \frac{r_1}{a_0}\right) + \frac{r_1^2}{a_0^3}\Gamma\left(2, \frac{r_1}{a_0}\right)\right]$$
$$\times\sqrt{\frac{4\pi}{5}}\,Y_2^0(\theta_1, \varphi_1).$$

12.8.8 $$\mathbf{A}(r_1, \theta_1) = \hat{\varphi}\frac{\mu_0}{4\pi}\cdot\frac{4\pi\hbar}{96ma_0^5}\left[\frac{a_0^5}{r_1^2}\gamma\left(5, \frac{r_1}{a_0}\right) + a_0^2 r_1\Gamma\left(2, \frac{r_1}{a_0}\right)\right].$$

CHAPTER 13

13.3.1 $X_n(x) = T_n(x)$.

13.3.5 $W(U_n, W_n) = (n+1)/(1-x^2)^{3/2}$.

13.4.1 (b) ${}_2F_1(-1, b, -2; x) = 1 + 1/2\,bx$.

13.4.2 Legendre, Eq. 12.17
Chebyshev I, Eq. 13.69
Chebyshev II, Eq. 13.70

13.5.14 If a is 0 or a negative integer, M does not exist; $W\{M, U\} \to 0$.

CHAPTER 14

14.3.5 $a_0 = \dfrac{2x_0}{\pi}, \quad a_n = \dfrac{2}{\pi}\cdot\dfrac{\sin nx_0}{n}, \quad n \ge 1$
$b_n = 0, \quad n \ge 1.$

14.3.6 $$\psi(r, \varphi) = \frac{4V}{\pi}\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^{2m+1}\frac{\sin(2m+1)\varphi}{2m+1}.$$

14.3.7 (a) $\psi(r, \varphi) = -E_0 r\left(1 - \dfrac{a^2}{r^2}\right)\cos\varphi.$

(b) $\sigma = 2\varepsilon_0 E_0\cos\varphi.$

14.3.10 $$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx, \quad -\pi < x < \pi.$$

14.3.14 $a_0 = \dfrac{\pi}{2}, \qquad a_n = \dfrac{-2}{\pi} \cdot \dfrac{1}{n^2}, \qquad n \geq 1$

$$b_n = \frac{(-1)^{n+1}}{n}, \quad n \geq 1.$$

14.4.5 (a) $\delta_n(x) = \dfrac{1}{2\pi} + \dfrac{2n}{\pi} \displaystyle\sum_{m=1}^{\infty} \dfrac{\sin(m/2n)}{m} \cos mx$. See 1.15.18.

CHAPTER 15

15.3.3 (a) $g_c(\omega) = \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{\sin \omega}{\omega}$.

15.3.5 $g(\omega) = \sqrt{\dfrac{2}{\pi}} \dfrac{ha}{\omega^2} [1 - \cos(\omega/a)]$.

15.3.17 $g(k) = \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{a}{a^2 + k^2}$.

15.3.21 (b) $\displaystyle\int_0^{\infty} \cos(y^2)\, dy = \int_0^{\infty} \sin(y^2)\, dy = \frac{1}{2}\frac{\pi}{2}$.

15.4.4 $\varphi(\mathbf{r}) = \dfrac{\mathrm{Q}}{\mathrm{D}} \cdot \dfrac{e^{-kr}}{4\pi r}$.

15.6.2 $g(p) = \sqrt{2\pi}\hbar\delta(p - p')$, that is, $p = p'$, sharp determination of momentum.

15.9.4 (a) $N_2(t) = nv\sigma_1 N_{10} \dfrac{1 - \exp[-(\lambda_2 + nv\sigma_2)t]}{\lambda_2 + nv\sigma_2}$.

(b) $N_2(1 \text{ year}) = 1.2 \times 10^{15}$ atoms of Eu^{154}.
$N_1(t) \approx N_{10}$.

15.9.5 (a) $N_X(t) = \lambda_I \gamma_I \varphi \sigma_f N_U$

$$\times [(\lambda_I - \lambda_X - \varphi\sigma_X) + (\lambda_X + \varphi\sigma_X)\exp[-\lambda_I t] - \lambda_I \exp[-(\lambda_X + \varphi\sigma_X)t]]/ [\lambda_I(\lambda_X + \varphi\sigma_X)(\lambda_I - \lambda_X - \varphi\sigma_X)] + \gamma_X \varphi\sigma_f N_U \left[\frac{1 - \exp[-(\lambda_X + \varphi\sigma_X)t]}{\lambda_X + \varphi\sigma_X}\right].$$

(b) $N_X(\infty) = \dfrac{(\gamma_I + \gamma_X)\varphi\sigma_f N_U}{\lambda_X + \varphi\sigma_X}$.

(c) $N_X(t) = N_X(0)e^{-\lambda_X t} + N_I(0)\dfrac{\lambda_I}{\lambda_I - \lambda_X}(e^{-\lambda_X t} - e^{-\lambda_I t})$

$$\left.\frac{dN_X(t)}{dt}\right|_{t=0} \approx \gamma_I \varphi \sigma_f N_U, \quad \textit{for } \varphi \gg \lambda_X/\sigma_X.$$

15.10.1 (b) $X(t) = X_0 e^{-(b/2m)t}\left\{\cosh\sigma t + \dfrac{b}{2m\sigma}\sinh\sigma t\right\}$,

$$\sigma^2 = \frac{b^2}{4m^2} - \frac{k}{m}.$$

15.10.2 (c) $X(t) = \dfrac{v_0}{\sigma}e^{-(b/2m)t}\sinh\sigma t, \quad \sigma^2 = \dfrac{b^2}{4m^2} - \dfrac{k}{m}.$

15.10.3 $X(t) = \dfrac{mg}{b}t - \dfrac{m^2 g}{b^2}\left[1 - e^{-(b/m)t}\right]$

$$\frac{dX(t)}{dt} = \frac{mg}{b}[1 - e^{-(b/m)t}].$$

15.10.4 $E(t) = -\dfrac{I_0}{\omega_1 C}e^{-t/2RC}\sin\omega_1 t, \quad \omega_1^2 = \dfrac{1}{LC} - \dfrac{1}{(2RC)^2}.$

This solution is based on the initial conditions $E(0) = 0$ (because the idealized inductance L would have zero DC impedance) and $I_L(0) = I_0$, limited by a resistance in series with the battery or by the internal resistance of the battery. Finally, to be consistent, $q_0 = 0$.

15.11.3 $\dfrac{\cos at - \cos bt}{b^2 - a^2}.$

15.12.4 $F(t) = \cosh kt.$

15.12.10 $F(t) = \dfrac{1}{2a}t\sin at.$

CHAPTER 16

16.1.4 If $a_1 = 0$, the differential equation will be self-adjoint and $K(x, t)$ will be symmetric. Cf. Section 16.4.

16.1.6 $\varphi(x) = \sinh x.$

16.2.6 $\varphi(t) = \delta(t).$

16.2.11 $X(\xi_2) = \int_{-\infty}^{\alpha} e^{-i\xi_1\xi_2} X(\xi_1)\, d\xi_1$. $X(\xi_1)$ is an orthogonalized Hermite polynomial.

16.3.1 (b) $\varphi(x) = \sin x$.

(c) $\varphi(x) = \sinh x$.

16.3.3 $\lambda_1 = i\sqrt{3}/2, \quad \varphi_1(x) = 1 - i\sqrt{3}x.$
$\lambda_2 = -i\sqrt{3}/2, \quad \varphi_2(x) = 1 + i\sqrt{3}x.$

16.3.5 $\lambda_1 = -3/4, \quad y_1(x) = x = P_1(x).$

$$\lambda_2 = \frac{-15 + 9\sqrt{5}}{8}, \quad y_2(x) = P_0(x) + \frac{4}{3}\lambda_2 P_2(x).$$

$$\lambda_3 = \frac{-15 - 9\sqrt{5}}{8}, \quad y_3(x) = P_0(x) + \frac{4}{3}\lambda_3 P_3(x).$$

16.3.7 $\psi(x) = -2$.

16.3.11 $\lambda_1 = 0.7889, \quad \varphi_1 = 1 + 0.5352\,x,$
$\lambda_2 = 15.21, \quad \varphi_2 = 1 - 1.8685\,x$
$(\lambda_1 = 8 - \sqrt{52}, \quad \lambda_2 = 8 + \sqrt{52}).$

16.3.12 $\lambda_1 = 0.7889, \quad \varphi_1(x) = 1 + 0.5352\,x,$
$\lambda_2 = 15.21, \quad \varphi_2(x) = 1 - 1.8685\,x.$

16.4.2 $\varphi(x) = 1/2(3x + 1)$.

16.4.3
$$\lambda_1 = \sqrt{\frac{\sin \pi a}{\pi}}, \quad \varphi_1(x) = \sqrt{(a-1)!}\,x^{-a} + \sqrt{(-a)!}\,x^{a-1},$$
$$\lambda_2 = -\sqrt{\frac{\sin \pi a}{\pi}}, \quad \varphi_2(x) = \sqrt{(a-1)!}\,x^{-a} - \sqrt{(-a)!}\,x^{a-1},$$
$0 < a < 1$.

16.4.4 (a) $y(x) = x \sum_{s=0}^{\infty} \left(\frac{\lambda}{3}\right)^s$.

(b) Convergent for $|\lambda| < 3$. Eq. 16.64 guarantees convergence for $|\lambda| < 1$.

(c) $\lambda = 3, \quad y(x) = x$.

CHAPTER 17

17.1.4 (b) J depends only on the endpoints and is independent of the choice of path.

17.2.1 Eq. 17.34, $y = 1, x = x_0, c_2 = 0$, and Eq. 17.43 set equal to 2π.

$$\begin{cases} x_0 = 0.527\,697 \\ c_1 = 0.825\,517. \end{cases}$$

17.3.3 (a) $m\ddot{r} - mr\dot{\theta}^2 - mr\ \sin^2\theta\dot{\varphi}^2 = 0$

(b) $mr\ddot{\theta} + 2m\dot{r}\dot{\theta} - mr\ \sin\theta\cos\theta\dot{\varphi}^2 = 0$

(c) $mr\sin\theta\ddot{\varphi} + 2m\dot{r}\sin\theta\dot{\varphi} + 2mr\cos\theta\dot{\theta}\dot{\varphi} = 0$

The Second and third terms of (a) correspond to centrifugal force. The second and third terms of (c) may be interpreted as Coriolis forces (with $\dot{\varphi}$ the angular velocity of the rotating coordinate system).

17.3.4 $l\ddot{\theta} - l\sin\theta\cos\theta\dot{\varphi}^2 + g\sin\theta = 0$

$\dfrac{d}{dt}(ml^2\sin^2\theta\dot{\varphi}) = 0$ (conservation of angular momentum.)

17.6.2 $\dfrac{R}{H} = \dfrac{1}{2}$.

17.6.3 Vol. $= 432\ \text{in}^3$.

17.6.5 $p = q, \qquad (p + q)_{\min} = 4f$.

17.7.2 (c) $\omega(t) = \text{Ang. Mom.}/m(\rho_0 - kt)^3$.

CHAPTER 18

18.2.2 See *The Nature of Choas* (T. Mullin, Ed.), Fig. 5.6, p. 107 (referenced in footnote 2, p. 1063 of the text).

18.3.1 See *Chaotic Dynamics: An Introduction*, by G. L. Baker and J.P. Gollub, Fig. 4.9, p. 86 (listed in Additional Readings on p. 1082 of the text).

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