Instructor's Manual

MATHEMATICAL METHODS FOR PHYSICISTS

A Comprehensive Guide

SEVENTH EDITION

George B. Arfken
Miami University
Oxford, OH

Hans J. Weber
University of Virginia
Charlottesville, VA

Frank E. Harris
University of Utah, Salt Lake City, UT;
University of Florida, Gainesville, FL

ELSEVIER

AMSTERDAM • BOSTON • HEIDELBERG • LONDON
NEW YORK • OXFORD • PARIS • SAN DIEGO
SAN FRANCISCO • SINGAPORE • SYDNEY • TOKYO
Academic Press is an imprint of Elsevier

Academic Press is an imprint of Elsevier
225 Wyman Street, Waltham, MA 02451, USA
The Boulevard, Langford Lane, Kidlington, Oxford, OX5 1GB, UK

For information on all Academic Press publications,
visit our website: www.books.elsevier.com

Contents

Chapter 1

Introduction

The seventh edition of *Mathematical Methods for Physicists* is a substantial and detailed revision of its predecessor. The changes extend not only to the topics and their presentation, but also to the exercises that are an important part of the student experience. The new edition contains 271 exercises that were not in previous editions, and there has been a wide-spread reorganization of the previously existing exercises to optimize their placement relative to the material in the text. Since many instructors who have used previous editions of this text have favorite problems they wish to continue to use, we are providing detailed tables showing where the old problems can be found in the new edition, and conversely, where the problems in the new edition came from. We have included the full text of every problem from the sixth edition that was not used in the new seventh edition. Many of these unused exercises are excellent but had to be left out to keep the book within its size limit. Some may be useful as test questions or additional study material.

Complete methods of solution have been provided for all the problems that are new to this seventh edition. This feature is useful to teachers who want to determine, at a glance, features of the various exercises that may not be completely apparent from the problem statement. While many of the problems from the earlier editions had full solutions, some did not, and we were unfortunately not able to undertake the gargantuan task of generating full solutions to nearly 1400 problems.

Not part of this Instructor's Manual but available from Elsevier's on-line web site are three chapters that were not included in the printed text but which may be important to some instructors. These include

- A new chapter (designated 31) on Periodic Systems, dealing with mathematical topics associated with lattice summations and band theory,

- A chapter (32) on Mathieu functions, built using material from two chapters in the sixth edition, but expanded into a single coherent presentation, and

- A chapter (33) on Chaos, modeled after Chapter 18 of the sixth edition but carefully edited.

In addition, also on-line but external to this Manual, is a chapter (designated 1) on Infinite Series that was built by collection of suitable topics from various places in the seventh edition text. This alternate Chapter 1 contains no material not already in the seventh edition but its subject matter has been packaged into a separate unit to meet the demands of instructors who wish to begin their course with a detailed study of Infinite Series in place of the new Mathematical Preliminaries chapter.

Because this Instructor's Manual exists only on-line, there is an opportunity for its continuing updating and improvement, and for communication, through it, of errors in the text that will surely come to light as the book is used. The authors invite users of the text to call attention to errors or ambiguities, and it is intended that corrections be listed in the chapter of this Manual entitled Errata and Revision Status. Errata and comments may be directed to the authors at `harris⟨at⟩qtp.ufl.edu` or to the publisher. If users choose to forward additional materials that are of general use to instructors who are teaching from the text, they will be considered for inclusion when this Manual is updated.

Preparation of this Instructor's Manual has been greatly facilitated by the efforts of personnel at Elsevier. We particularly want to acknowledge the assistance of our Editorial Project Manager, Kathryn Morrissey, whose attention to this project has been extremely valuable and is much appreciated.

It is our hope that this Instructor's Manual will have value to those who teach from *Mathematical Methods for Physicists* and thereby to their students.

Chapter 2

Errata and Revision Status

Last changed: 06 April 2012

Errata and Comments re Seventh Edition text

Page 522	Exercise 11.7.12(a)	This is not a principal-value integral.
Page 535	Figure 11.26	The two arrowheads in the lower part of the circular arc should be reversed in direction.
Page 539	Exercise 11.8.9	The answer is incorrect; it should be $\pi/2$.
Page 585	Exercise 12.6.7	Change the integral for which a series is sought to $\displaystyle\int_0^\infty \frac{e^{-xv}}{1+v^2}\,dv$. The answer is then correct.
Page 610	Exercise 13.1.23	Replace $(-t)^\nu$ by $e^{-\pi i\nu}t^\nu$.
Page 615	Exercise 13.2.6	In the Hint, change Eq. (13.35) to Eq. (13.44).
Page 618	Eq. (13.51)	Change l.h.s. to $B(p+1,q+1)$.
Page 624	After Eq. (13.58)	C_1 can be determined by requiring consistency with the recurrence formula $z\Gamma(z)=\Gamma(z+1)$. Consistency with the duplication formula then determines C_2.
Page 625	Exercise 13.4.3	Replace "(see Fig. 3.4)" by "and that of the recurrence formula".
Page 660	Exercise 14.1.25	Note that $\alpha^2=\omega^2/c^2$, where ω is the angular frequency, and that the height of the cavity is l.

Page 665	Exercise 14.2.4	Change Eq. (11.49) to Eq. (14.44).
Page 686	Exercise 14.5.5	In part (b), change l to h in the formulas for a_{mn} and b_{mn} (denominator and integration limit).
Page 687	Exercise 14.5.14	The index n is assumed to be an integer.
Page 695	Exercise 14.6.3	The index n is assumed to be an integer.
Page 696	Exercise 14.6.7(b)	Change N to Y (two occurrences).
Page 709	Exercise 14.7.3	In the summation preceded by the cosine function, change $(2z)^{2s}$ to $(2z)^{2s+1}$.
Page 710	Exercise 14.7.7	Replace $n_n(x)$ by $y_n(x)$.
Page 723	Exercise 15.1.12	The last formula of the answer should read $P_{2s}(0)/(2s+2) = (-1)^s(2s-1)!!/(2s+2)!!$.
Page 754	Exercise 15.4.10	Insert minus sign before $P_n^1(\cos\theta)$.
Page 877	Exercise 18.1.6	In both (a) and (b), change 2π to $\sqrt{2\pi}$.
Page 888	Exercise 18.2.7	Change the second of the four members of the first display equation to $\left(\frac{x+ip}{\sqrt{2}}\right)\psi_n(x)$, and change the corresponding member of the second display equation to $\left(\frac{x-ip}{\sqrt{2}}\right)\psi_n(x)$.
Page 888	Exercise 18.2.8	Change $x+ip$ to $x-ip$.
Page 909	Exercise 18.4.14	All instances of x should be primed.
Page 910	Exercise 18.4.24	The text does not state that the T_0 term (if present) has an additional factor 1/2.
Page 911	Exercise 18.4.26(b)	The ratio approaches $(\pi s)^{-1/2}$, not $(\pi s)^{-1}$.
Page 915	Exercise 18.5.5	The hypergeometric function should read ${}_2F_1\left(\frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+1; \nu+\frac{3}{2}; z^{-2}\right)$.
Page 916	Exercise 18.5.10	Change $(n-\frac{1}{2})!$ to $\Gamma(n+\frac{1}{2})$.
Page 916	Exercise 18.5.12	Here n must be an integer.
Page 917	Eq. (18.142)	In the last term change $\Gamma(-c)$ to $\Gamma(2-c)$.
Page 921	Exercise 18.6.9	Change b to c (two occurrences).
Page 931	Exercise 18.8.3	The arguments of K and E are m.
Page 932	Exercise 18.8.6	All arguments of K and E are k^2; In the integrand of the hint, change k to k^2.

Page 978	Exercise 20.2.9	The formula as given assumes that $\Gamma > 0$.
Page 978	Exercise 20.2.10(a)	This exercise would have been easier if the book had mentioned the integral representation $J_0(x) = \dfrac{2}{\pi}\displaystyle\int_0^1 \dfrac{\cos xt}{\sqrt{1-t^2}}\,dt$.
Page 978	Exercise 20.2.10(b)	Change the argument of the square root to $x^2 - a^2$.
Page 978	Exercise 20.2.11	The l.h.s. quantities are the transforms of their r.h.s. counterparts, but the r.h.s. quantities are $(-1)^n$ times the transforms of the l.h.s. expressions.
Page 978	Exercise 20.2.12	The properly scaled transform of $f(\mu)$ is $(2/\pi)^{1/2} i^n j_n(\omega)$, where ω is the transform variable. The text assumes it to be kr.
Page 980	Exercise 20.2.16	Change d^3x to d^3r and remove the limits from the first integral (it is assumed to be over all space).
Page 980	Eq. (20.54)	Replace $d\mathbf{k}$ by d^3k (occurs three times)
Page 997	Exercise 20.4.10	This exercise assumes that the units and scaling of the momentum wave function correspond to the formula $\varphi(\mathbf{p}) = \dfrac{1}{(2\pi\hbar)^{3/2}}\displaystyle\int \psi(\mathbf{r})\, e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar}\, d^3r\,.$
Page 1007	Exercise 20.6.1	The second and third orthogonality equations are incorrect. The right-hand side of the second equation should read: N, $p = q = (0 \text{ or } N/2)$; $N/2$, $(p + q = N)$ or $p = q$ but not both; 0, otherwise. The right-hand side of the third equation should read: $N/2$, $p = q$ and $p + q \neq (0 \text{ or } N)$; $-N/2$, $p \neq q$ and $p + q = N$; 0, otherwise.
Page 1007	Exercise 20.6.2	The exponentials should be $e^{2\pi ipk/N}$ and $e^{-2\pi ipk/N}$.
Page 1014	Exercise 20.7.2	This exercise is ill-defined. Disregard it.
Page 1015	Exercise 20.7.6	Replace $(\nu - 1)!$ by $\Gamma(\nu)$ (two occurrences).
Page 1015	Exercise 20.7.8	Change $M(a, c; x)$ to $M(a, c, x)$ (two

		occurrences).
Page 1028	Table 20.2	Most of the references to equation numbers did not get updated from the 6th edition. The column of references should, in its entirety, read: (20.126), (20.147), (20.148), Exercise 20.9.1, (20.156), (20.157), (20.166), (20.174), (20.184), (20.186), (20.203).
Page 1034	Exercise 20.8.34	Note that $u(t-k)$ is the unit step function.
Page 1159	Exercise 23.5.5	This problem should have identified m as the mean value and M as the "random variable" describing individual student scores.

Corrections and Additions to Exercise Solutions

None as of now.

Chapter 3

Exercise Solutions

1. Mathematical Preliminaries

1.1 Infinite Series

1.1.1. (a) If $u_n < A/n^p$ the integral test shows $\sum_n u_n$ converges for $p > 1$.

(b) If $u_n > A/n, \sum_n u_n$ diverges because the harmonic series diverges.

1.1.2. This is valid because a multiplicative constant does not affect the convergence or divergence of a series.

1.1.3. (a) The Raabe test P can be written $1 + \dfrac{(n+1)\ln(1+n^{-1})}{\ln n}$.

This expression approaches 1 in the limit of large n. But, applying the Cauchy integral test,

$$\int \frac{dx}{x \ln x} = \ln \ln x,$$

indicating divergence.

(b) Here the Raabe test P can be written

$$1 + \frac{n+1}{\ln n} \ln\left(1 + \frac{1}{n}\right) + \frac{\ln^2(1+n^{-1})}{\ln^2 n},$$

which also approaches 1 as a large-n limit. But the Cauchy integral test yields

$$\int \frac{dx}{x \ln^2 x} = -\frac{1}{\ln x},$$

indicating convergence.

1.1.4. Convergent for $a_1 - b_1 > 1$. Divergent for $a_1 - b_1 \leq 1$.

1.1.5. (a) Divergent, comparison with harmonic series.

(b) Divergent, by Cauchy ratio test.

(c) Convergent, comparison with $\zeta(2)$.

(d) Divergent, comparison with $(n+1)^{-1}$.

(e) Divergent, comparison with $\frac{1}{2}(n+1)^{-1}$ or by Maclaurin integral test.

1.1.6. (a) Convergent, comparison with $\zeta(2)$.

(b) Divergent, by Maclaurin integral test.

(c) Convergent, by Cauchy ratio test.

(d) Divergent, by $\ln\left(1+\dfrac{1}{n}\right) \sim \dfrac{1}{n}$.

(e) Divergent, majorant is $1/(n \ln n)$.

1.1.7. The solution is given in the text.

1.1.8. The solution is given in the text.

1.1.10. In the limit of large n, $u_{n+1}/u_n = 1 + \dfrac{1}{n} + O(n^{-2})$.

Applying Gauss' test, this indicates divergence.

1.1.11. Let s_n be the absolute value of the nth term of the series.

(a) Because $\ln n$ increases less rapidly than n, $s_{n+1} < s_n$ and $\lim_{n\to\infty} s_n = 0$. Therefore this series converges. Because the s_n are larger than corresponding terms of the harmonic series, this series is not absolutely convergent.

(b) Regarding this series as a new series with terms formed by combining adjacent terms of the same sign in the original series, we have an alternating series of decreasing terms that approach zero as a limit, i.e.,

$$\frac{1}{2n+1} + \frac{1}{2n+2} > \frac{1}{2n+3} + \frac{1}{2n+4},$$

this series converges. With all signs positive, this series is the harmonic series, so it is not aboslutely convergent.

(c) Combining adjacent terms of the same sign, the terms of the new series satisfy

$$2\left(\frac{1}{2}\right) > \frac{1}{2} + \frac{1}{3} > 2\left(\frac{1}{3}\right), \qquad 3\left(\frac{1}{4}\right) > \frac{1}{4} + \frac{1}{5} + \frac{1}{6} > 3\left(\frac{1}{6}\right), \quad \text{etc.}$$

The general form of these relations is

$$\frac{2n}{n^2 - n + 2} > s_n > \frac{2}{n+1}.$$

An upper limit to the left-hand side member of this inequality is $2/(n-1)$. We therefore see that the terms of the new series are decreasing, with limit zero, so the original series converges. With all signs positive, the original series becomes the harmonic series, and is therefore not absolutely convergent.

1.1.12. The solution is given in the text.

1.1.13. Form the nth term of $\zeta(2) - c_1\alpha_1 - c_2\alpha_2$ and choose c_1 and c_2 so that when placed over the common denominator $n^2(n+1)(n+2)$ the numerator will be independent of n. The values of the c_i satisfying this condition are $c_1 = c_2 = 1$, and our resulting expansion is

$$\zeta(2) = \alpha_1 + \alpha_2 + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1_(n+2)} = \frac{5}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2(n+1_(n+2)}\,.$$

Keeping terms through $n = 10$, this formula yields $\zeta(2) \approx 1.6445$; to this precision the exact value is $\zeta(2) = 1.6449$.

1.1.14. Make the observation that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{(2n)^3} = \zeta(3)$$

and that the second term on the left-hand side is $\zeta(3)/8$. Our summation therefore has the value $7\zeta(3)/8$.

1.1.15. (a) Write $\zeta(n) - 1$ as $\sum_{p=2}^{\infty} p^{-n}$, so our summation is

$$\sum_{n=2}^{\infty}\sum_{p=2}^{\infty} \frac{1}{p^n} = \sum_{p=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{p^n}\,.$$

The summation over n is a geometric series which evaluates to

$$\frac{p^{-2}}{1-p^{-1}} = \frac{1}{p^2-p}\,.$$

Summing now over p, we get

$$\sum_{p=2}^{\infty} \frac{1}{p(p-1)} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)} = \alpha_1 = 1\,.$$

(b) Proceed in a fashion similar to part (a), but now the geometric series has sum $1/(p^2+p)$, and the sum over p is now lacking the initial term of α_1, so

$$\sum_{p=2}^{\infty} \frac{1}{p(p+1)} = \alpha_1 - \frac{1}{(1)(2)} = \frac{1}{2}.$$

1.1.16. (a) Write

$$\zeta(3) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^3} - \sum_{n=2}^{\infty} \frac{1}{(n-1)n(n+1)} + \alpha_2' = 1 + \sum_{n=2}^{\infty} \left[\frac{1}{n^3} - \frac{1}{n(n^2-1)}\right] + \frac{1}{4}$$

$$= 1 + \frac{1}{4} - \sum_{n=2}^{\infty} \frac{1}{n^3(n^2-1)}\,.$$

(b) Now use α_2' and $\alpha_4' = \sum_{n=3}^{\infty} \frac{1}{n(n^2-1)(n^2-4)} = \frac{1}{96}$:

$$\zeta(3) = 1 + \frac{1}{2^3} + \sum_{n=3}^{\infty} \frac{1}{n^3} - \sum_{n=3}^{\infty} \frac{1}{n(n^2-1)} + \left[\alpha_2' - \frac{1}{6}\right]$$

$$- \sum_{n=3}^{\infty} \frac{B}{n(n^2-1)(n^2-4)} + B\alpha_4'$$

$$= \frac{29}{24} + \frac{B}{96} + \sum_{n=3}^{\infty} \left[\frac{1}{n^3} - \frac{1}{n(n^2-1)} - \frac{B}{n(n^2-1)(n^2-4)}\right]$$

$$= \frac{29}{24} - \frac{B}{96} + \sum_{n=3}^{\infty} \frac{4-(1+B)n^2}{n(n^2-1)(n^2-4)}\,.$$

The convergence of the series is optimized if we set $B = -1$, leading to the final result

$$\zeta(3) = \frac{29}{24} - \frac{1}{96} + \sum_{n=3}^{\infty} \frac{4}{n(n^2-1)(n^2-4)}\,.$$

(c) Number of terms required for error less than 5×10^{-7}: $\zeta(3)$ alone, 999; combined as in part (a), 27; combined as in part (b), 11.

1.2 Series of Functions

1.2.1. (a) Applying Leibniz' test the series converges uniformly for $\varepsilon \le x < \infty$ no matter how small $\varepsilon > 0$ is.
(b) The Weierstrass M and the integral tests give uniform convergence for $1 + \varepsilon \le x < \infty$ no matter how small $\varepsilon > 0$ is chosen.

1.2.2. The solution is given in the text.

1.2.3. (a) Convergent for $1 < x < \infty$.

(b) Uniformly convergent for $1 < s \le x < \infty$.

1.2.4. From $|\cos nx| \le 1, |\sin nx| \le 1$ absolute and uniform convergence follow for $-s < x < s$ for any $s > 0$.

1.2.5. Since $|\frac{u_{j+2}}{u_j}| \sim |x|^2$, $|x| < 1$ is needed for convergence.

1.2.6. The solution is given in the text.

1.2.7. The solution is given in the text.

1.2.8. (a) For $n = 0, 1, 2, \ldots$ we find

$$\left.\frac{d^{4n+1}\sin x}{dx^{4n+1}}\right|_0 = \cos x|_0 = 1,$$

$$\left.\frac{d^{4n+2}\sin x}{dx^{4n+2}}\right|_0 = -\sin x|_0 = 0,$$

$$\left.\frac{d^{4n+3}\sin x}{dx^{4n+3}}\right|_0 = -\cos x|_0 = -1,$$

$$\left.\frac{d^{4n}\sin x}{dx^{4n}}\right|_0 = \sin x|_0 = 0.$$

Taylor's theorem gives the absolutely convergent series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(b) Similar derivatives for $\cos x$ give the absolutely convergent series

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

1.2.9. $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots, -\pi < x < \pi.$

1.2.10. From $\coth y = \eta_0 = \frac{e^y + e^{-y}}{e^y - e^{-y}} = \frac{e^{2y}+1}{e^{2y}-1}$ we extract $y = \frac{1}{2}\ln\frac{\eta_0+1}{\eta_0-1}$.

To check this we substitute this into the first relation, giving

$$\frac{\dfrac{\eta_0+1}{\eta_0-1}+1}{\dfrac{\eta_0+1}{\eta_0-1}-1} = \eta_0.$$

The series $\coth^{-1}\eta_0 = \sum_{n=0}^{\infty} \frac{(\eta_0)^{-2n-1}}{2n+1}$ follows from Exercise 1.6.1.

1.2.11. (a) Since $\left.\dfrac{d\sqrt{x}}{dx}\right|_0 = \left.\dfrac{1}{2\sqrt{x}}\right|_0$ does not exist, there is no Maclaurin expansion.

(b) $|x - x_0| < x_0$ because the origin must be excluded.

1.2.12. $\displaystyle \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f(x+(x_0-x))}{g(x+(x_0-x)} = \lim_{x\to x_0} \frac{f(x)+(x_0-x)f'(x)+\cdots}{g(x)+(x_0-x)g'(x)+\cdots}$

$\displaystyle = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$, where the intermediate formal expression $\dfrac{f(x+(x_0-x))}{g(x+(x_0-x)}$ may be dropped.

1.2.13. (a) $\displaystyle -\ln\frac{n}{n-1} = \ln\left(1-\frac{1}{n}\right) = -\sum_{\nu=1}^{\infty}\frac{1}{\nu n^{\nu}}.$

Hence $\displaystyle \frac{1}{n} - \ln\frac{n}{n-1} = -\sum_{\nu=2}^{\infty}\frac{1}{\nu n^{\nu}} < 0.$

(b) $\displaystyle \ln\frac{n+1}{n} = \ln(1+\frac{1}{n}) = \sum_{\nu=2}^{\infty}\frac{(-1)^{\nu-1}}{\nu n^{\nu}},\ \frac{1}{n} - \ln\frac{n+1}{n} = \sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}}{\nu n^{\nu}} > 0.$

Summing (a) yields

$$0 > \sum_{m=2}^{n}\frac{1}{m} - \ln\frac{2\cdot 3\cdots n}{1\cdot 2\cdots(n-1)} = \sum_{m=2}^{n}\frac{1}{m} - \ln n \to \gamma - 1.$$

Thus, $\gamma < 1$. Summing (b) yields

$$0 < \sum_{m=2}^{n-1}\frac{1}{m} - \ln\frac{2\cdot 3\cdots n}{1\cdot 2\cdots(n-1)} = \sum_{m=2}^{n-1}\frac{1}{m} - \ln n \to \gamma.$$

Hence $0 < \gamma < 1$.

1.2.14. The solution is given in the text.

1.2.15. The solutions are given in the text.

1.2.16. If $\left|\dfrac{a_{n+1}}{a_n}\right| \to \dfrac{1}{R}$ then $\left|\dfrac{(n+2)a_{n+1}}{(n+1)a_n}\right| \to \dfrac{1}{R}$ and $\left|\dfrac{a_{n+1}/(n+2)}{a_n/(n+1)}\right| \to \dfrac{1}{R}$.

1.3 Binomial Theorem

1.3.1. $P(x) = C\left\{\dfrac{x}{3} - \dfrac{x^3}{45} + \cdots\right\}$.

1.3.2. Integrating termwise $\displaystyle \tan^{-1}1 = \frac{\pi}{4} = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

1.3.3. The solution is given in the text. Convergent for $0 \le x < \infty$. The upper limit x does **not** have to be small, but unless it is small the convergence will be slow and the expansion relatively useless.

1.3.4. $\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \cdots, -1 \le x \le 1.$

1.3.5. The expansion of the integral has the form

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^1 \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

1.3.6. For $m = 1, 2, \ldots$ the binomial expansion gives $(1+x)^{-m/2} = \sum_{n=0}^{\infty} \binom{-m/2}{n} x^n$.

By mathematical induction we show that $\binom{-m/2}{n} = (-1)^n \frac{(m+2n-2)!!}{2^n (m-2)!! n!}$.

1.3.7. (a) $\nu' = \nu\left\{1 \pm \frac{\nu}{c} + \frac{\nu^2}{c^2} + \cdots\right\}$.

(b) $\nu' = \nu\left\{1 \pm \frac{\nu}{c}\right\}$.

(c) $\nu' = \nu\left\{1 \pm \frac{\nu}{c} + \frac{1}{2}\frac{\nu^2}{c^2} + \cdots\right\}$.

1.3.8. (a) $\frac{\nu_1}{c} = \delta + 1/2\delta^2$.

(b) $\frac{\nu_2}{c} = \delta - 3/2\delta^2 + \cdots$.

(c) $\frac{\nu_3}{c} = \delta - 1/2\delta^2 + \cdots$.

1.3.9. $\frac{w}{c} = 1 - \frac{\alpha^2}{2} - \frac{\alpha^3}{2} + \cdots$.

1.3.10. $x = \frac{1}{2}gt^2 - \frac{1}{8}\frac{g^3 t^4}{c^2} + \frac{1}{16}\frac{g^5 t^6}{c^4} - \cdots$.

1.3.11. $E = mc^2\left[1 - \frac{\gamma^2}{2n^2} - \frac{\gamma^4}{2n^4}\left(\frac{n}{|k|} - \frac{3}{4}\right) + \cdots\right]$.

1.3.12. The solution is given in the text.

1.3.13. The two series have different, nonoverlapping convergence intervals.

1.3.14. (a) Differentiating the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x = \exp(-\varepsilon_0/kT)$ yields $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$. Therefore, $\langle \varepsilon \rangle = \frac{\varepsilon_0 x}{1-x} = \frac{\varepsilon_0}{e^{\varepsilon_0/kT} - 1}$.

(b) Expanding $\dfrac{y}{e^y - 1} = 1 + \dfrac{y}{2} + \cdots$ we find

$$\langle\varepsilon\rangle = kT(1 + \frac{\varepsilon_0}{2kT} + \cdots) = kT + \varepsilon_0 + \cdots .$$

1.3.15. (a) $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}, |x| \le 1.$

(b) Writing $x = \tan y$ as $ix = \dfrac{e^{2iy}+1}{e^{2iy}-1}$ we extract $y = -\dfrac{i}{2} \ln \dfrac{1+ix}{1-ix}$.

1.3.16. Start by obtaining the first few terms of the power-series expansion of the expression within the square brackets. Write

$$\frac{2+2\varepsilon}{1+2\varepsilon} = 1 + \frac{1}{1+2\varepsilon} = 2 - 2\varepsilon + (2\varepsilon)^2 - \cdots ,$$

$$\frac{\ln(1+2\varepsilon}{\varepsilon} = \frac{1}{\varepsilon}\left[2\varepsilon - \frac{(2\varepsilon)^2}{2} + \frac{(2\varepsilon)^3}{3} - \cdots\right]$$

$$\left[\frac{2+2\varepsilon}{1+2\varepsilon} - \frac{\ln(1+2\varepsilon}{\varepsilon}\right] = \frac{4}{3}\,\varepsilon^2 + O(\varepsilon^3)\,.$$

Inserting this into the complete expression for $f(\varepsilon)$, the limit is seen to be 4/3.

1.3.17. Let $x = 1/A$, and write $xi_1 = 1 + \dfrac{(1-x)^2}{2x} \ln \dfrac{1-x}{1+x}$.

Expanding the logarithm,

$$\xi_1 = 1 + \frac{(1-x)^2}{2x}\left(-2x - \frac{2x^3}{3} - \cdots\right) = 2x - \frac{4}{3}\,x^2 + \frac{2}{3}\,x^3 - \cdots .$$

The similar expansion of $\xi_2 = \dfrac{2x}{1+2x/3}$ yields

$$\xi_2 = 2x - \frac{4}{3}\,x^2 + \frac{8}{9}\,x^3 - \cdots .$$

Comparing these expansions, we note agreement through x^2, and the x^3 terms differ by $(2/9)x^3$, or $2/9A^3$.

1.3.18. (a) Insert the power-series expansion of $\arctan t$ and carry out the integration. The series for $\beta(2)$ is obtained.

(b) Integrate by parts, converting $\ln x$ into $1/x$ and $1/(1+x^2)$ into $\arctan x$. The integrated terms vanish, and the new integral is the negative of that already treated in part (a).

1.4 Mathematical Induction

1.4.1. Use mathematical induction. First evaluate the claimed expression for the sum for both $n-1$ and n:

$$S_{n-1} = \frac{n-1}{30}(2n-1)n\left[3(n-1)^2 + 3(n-1) - 1\right] = \frac{n^5}{5} - \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$S_n = \frac{n}{30}(2n+1)(n+1)(3n^2+3n-1) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

Next verify that $S_n = S_{n-1} + n^4$. Complete the proof by verifying that $S_1 = 1$.

1.4.2. Use mathematical induction. First, differentiate the Leibniz formula for $n-1$, getting the two terms

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left[\left(\frac{d}{dx}\right)^{j+1} f(x)\right] \left[\left(\frac{d}{dx}\right)^{n-1-j} g(x)\right]$$
$$+ \sum_{j=0}^{n-1} \binom{n-1}{j} \left[\left(\frac{d}{dx}\right)^{j} f(x)\right] \left[\left(\frac{d}{dx}\right)^{n-j} g(x)\right]$$

Now change the index of the first summation to $(j-1)$, with j ranging from 1 to n; the index can be extended to $j=0$ because the binomial coefficient $\binom{n-1}{-1}$ vanishes. The terms then combine to yield

$$\sum_{j=0}^{n} \left[\binom{n-1}{j-1} + \binom{n-1}{j}\right] \left[\left(\frac{d}{dx}\right)^{j} f(x)\right] \left[\left(\frac{d}{dx}\right)^{n-j} g(x)\right]$$

The sum of two binomial coefficients has the value $\binom{n}{j}$, thereby confirming that if the Leibnize formula is correct for $n-1$, it is also correct for n. One way to verify the binomial coefficient sum is to recognize that it is the number of ways j of n objects can be chosen: either $j-1$ choices are made from the first $n-1$ objects, with the nth object the jth choice, or all j choices are made from the first $n-1$ objects, with the nth object remaining unchosen. The proof is now completed by noticing that the Leibniz formula gives a correct expression for the first derivative.

1.5 Operations on Series Expansions of Functions

1.5.1. The partial fraction expansion is

$$\frac{1}{1-t^2} = \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right],$$

with integral

$$\int_{-x}^{x} \frac{dt}{1-t^2} = \frac{1}{2}\Big[\ln(1+x) - \ln(1-x)\Big]\Big|_{-x}^{x} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)_{-x}^{x}.$$

The upper and lower limits give the same result, canceling the factor 1/2.

1.5.2. Start by writing the partial-fraction expansion for $p+1$ using the assumed form of that for p multiplied by an additional factor $1/(n+p+1)$. Thus, we want to see if we can simplify

$$\frac{1}{p\,!}\sum_{j=0}^{p}\binom{p}{j}\frac{(-1)^j}{n+j}\left(\frac{1}{n+p+1}\right)$$

to get the expected formula. Our first step is to expand the two factors containing n into partial fractions:

$$\frac{1}{(n+j)(n+p+1)} = \left(\frac{1}{p+1-j}\right)\left(\frac{1}{n+j} - \frac{1}{n+p+1}\right)$$

Replacing the $1/(n+j)$ term of our original expansion using this result and adding a new $1/(n+p+1)$ term which is the summation of the above result for all j, we reach

$$\sum_{j=0}^{p}\frac{(-1)^j}{n+j}\left[\frac{1}{p\,!}\binom{p}{j}\frac{1}{p+1-j}\right] + \sum_{j=0}^{p}\left[\frac{1}{p\,!}\binom{p}{j}\frac{1}{p+1-j}\right]\frac{(-1)^{j-1}}{n+p+1}$$

Using the first formula supplied in the Hint, we replace each square bracket by the quantity

$$\frac{1}{(p+1)!}\binom{p+1}{j},$$

thereby identifying the first summation as all but the last term of the partial-fraction expansion for $p+1$. The second summation can now be written

$$\frac{1}{(p+1)!}\left[\sum_{j=0}^{p}(-1)^{j-1}\binom{p+1}{j}\right]\frac{1}{n+p+1}.$$

Using the second formula supplied in the Hint, we now identify the quantity within square brackets as

$$\sum_{j=1}^{p+1}(-1)^{j-1}\binom{p+1}{j} - (-1)^p\binom{p+1}{p+1} + (-1)^{-1}\binom{p+1}{0}$$

$$= 1 + (-1)^{p+1} - 1 = (-1)^{p+1},$$

so the second summation reduces to

$$\frac{(-1)^{p+1}}{(p+1)!}\,\frac{1}{n+p+1},$$

as required. Our proof by mathematical induction is now completed by observing that the partial-fraction formula is correct for the case $p=0$.

1.5.3. The formula for $u_n(p)$ follows directly by inserting the partial fraction decomposition. If this formula is summed for n from 1 to infinity, all terms cancel except that containing u_1, giving the result

$$\sum_{n=1}^{\infty} u_n(p) = \frac{u_1(p-1)}{p}.$$

The proof is then completed by inserting the value of $u_1(p-1)$.

1.5.4. After inserting Eq. (1.88) into Eq. (1.87), make a change of summation variable from n to $p=n-j$, with the ranges of j and p both from zero to infinity. Placing the p summation outside, and moving quantities not dependent upon j outside the j summation, reach

$$f(x) = \sum_{p=0}^{\infty}(-1)^p c_p \frac{x^p}{(1+x)^{p+1}} \sum_{j=0}^{\infty} \binom{p+j}{j} \left(\frac{x}{1+x}\right)^j .$$

Using now Eq. (1.71), we identify the binomial coefficient in the above equation as

$$\binom{p+j}{j} = (-1)^j \binom{-p-1}{j},$$

so the j summation reduces to

$$\sum_{j=0}^{\infty} \binom{-p-1}{j} \left(-\frac{x}{1+x}\right)^j = \left(1-\frac{x}{1+x}\right)^{-p-1} = (1+x)^{p+1}.$$

Insertion of this expression leads to the recovery of Eq. (1.86).

1.5.5. Applying Eq. (1.88) to the coefficients in the power-series expansion of $\arctan(x)$, the first 18 a_n (a_0 through a_{17}) are:

$0,\ -1,\ 2,\ -8/3,\ 8/3,\ -28/15,\ 8/15,\ 64/105,\ -64/105, -368/15,$

$1376/315,\ 1376/315,\ -25216/3465,\ 25216/3465,\ -106048/45045,$

$-305792/45045,\ 690176/45045,\ -690176/45045,\ 201472/765765.$

Using these in Eq. (1.87) for $x=1$, the terms through a_{17} yield the approximate value $\arctan(1) \approx 0.785286$, fairly close to the exact value at this precision, 0.785398. For this value of x, the 18th nonzero term in the power series is $-1/35$, showing that a power series for $x=1$ cut off after 18 terms would barely give a result good to two significant figures. The 18-term Euler expansion yields $\arctan(1/\sqrt{3}) \approx 0.523598$, while the exact value at this precision is 0.523599.

1.6 Some Important Series

1.6.1. For $|x| < 1, \ln \frac{1+x}{1-x} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu} \left[(-1)^{\nu-1} + 1\right] = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$.

1.7 Vectors

1.7.1. $A_x = A_y = A_z = 1$.

1.7.2. The triangle sides are given by

$$\mathbf{AB} = \mathbf{B} - \mathbf{A}, \ \mathbf{BC} = \mathbf{C} - \mathbf{B}, \ \mathbf{CA} = \mathbf{A} - \mathbf{C}$$

with $\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = (\mathbf{B} - \mathbf{A}) + (\mathbf{C} - \mathbf{B}) + (\mathbf{A} - \mathbf{C}) = 0$.

1.7.3. The solution is given in the text.

1.7.4. If $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{v}_1, \mathbf{r}'_i = \mathbf{r}_i - \mathbf{r}_1$, are the velocities and distances, respectively, from the galaxy at $\mathbf{r}_1$, then $\mathbf{v}'_i = H_0(\mathbf{r}_i - \mathbf{r}_1) = H_0 \mathbf{r}'_i$ holds, i.e., the same Hubble law.

1.7.5. With one corner of the cube at the origin, the space diagonals of length $\sqrt{3}$ are:

$$\begin{aligned}
(1,0,1) - (0,1,0) &= (1,-1,1), (1,1,1) - (0,0,0) = (1,1,1), \\
(0,0,1) - (1,1,0) &= (-1,-1,1), (1,0,0) - (0,1,1) = (1,-1,-1).
\end{aligned}$$

The face diagonals of length $\sqrt{2}$ are:

$$\begin{aligned}
(1,0,1) - (0,0,0) &= (1,0,1), (1,0,0) - (0,0,1) = (1,0,-1); \\
(1,0,0) - (0,1,0) &= (1,-1,0), (1,1,0) - (0,0,0) = (1,1,0); \\
(0,1,0) - (0,0,1) &= (0,1,-1), (0,1,1) - (0,0,0) = (0,1,1).
\end{aligned}$$

1.7.6. (a) The surface is a plane passing through the tip of $\mathbf{a}$ and perpendicular to $\mathbf{a}$.

(b) The surface is a sphere having $\mathbf{a}$ as a diameter:

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{r} = (\mathbf{r} - \mathbf{a}/2)^2 - \mathbf{a}^2/4 = 0.$$

1.7.7. The solution is given in the text.

1.7.8. The path of the rocket is the straight line

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{v},$$

or in Cartesian coordinates

$$x(t) = 1 + t, \quad y(t) = 1 + 2t, \quad z(t) = 1 + 3t.$$

We now minimize the distance $|\mathbf{r} - \mathbf{r}_0|$ of the observer at the point $\mathbf{r}_0 = (2, 1, 3)$ from $\mathbf{r}(t)$, or equivalently $(\mathbf{r}-\mathbf{r}_0)^2$ =min. Differentiating the rocket path with respect to t yields $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) = \mathbf{v}$. Setting $\frac{d}{dt}(\mathbf{r} - \mathbf{r}_0)^2 = 0$ we obtain the condition

$$2(\mathbf{r} - \mathbf{r}_0) \cdot \dot{\mathbf{r}} = 2[\mathbf{r}_1 - \mathbf{r}_0 + t\mathbf{v}] \cdot \mathbf{v} = 0.$$

Because $\dot{\mathbf{r}} = \mathbf{v}$ is the tangent vector of the line, the geometric meaning of this condition is that the **shortest distance vector through $\mathbf{r}_0$ is perpendicular to the line, or the velocity of the rocket.** Now solving for t yields the ratio of scalar products

$$t = -\frac{(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{v}}{\mathbf{v}^2} = -\frac{(-1, 0, -2) \cdot (1, 2, 3)}{(1, 2, 3) \cdot (1, 2, 3)} = \frac{1 + 0 + 6}{1 + 4 + 9} = \frac{1}{2}.$$

Substituting this parameter value into the rocket path gives the point $\mathbf{r}_s = (3/2, 2, 5/2)$ on the line that is closest to $\mathbf{r}_0$. The shortest distance is $d = |\mathbf{r}_0 - \mathbf{r}_s| = |(-1/2, 1, -1/2)| = \sqrt{2/4 + 1} = \sqrt{3/2}$.

1.7.9. Consider each corner of the triangle to have a unit of mass and be located at $\mathbf{a}_i$ from the origin where, for example,

$$\mathbf{a}_1 = (2, 0, 0),\ \mathbf{a}_2 = (4, 1, 1),\ \mathbf{a}_3 = (3, 3, 2).$$

Then the center of mass of the triangle is

$$\frac{1}{3}(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3) = \mathbf{a}_{cm} = \frac{1}{3}(2 + 4 + 3, 1 + 3, 1 + 2) = \left(3, \frac{4}{3}, 1\right).$$

The three midpoints are located at the point of the vectors

$$\begin{aligned}
\frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2) &= \frac{1}{2}(2 + 4, 1, 1) = \left(3, \frac{1}{2}, \frac{1}{2}\right)\\
\frac{1}{2}(\mathbf{a}_2 + \mathbf{a}_3) &= \frac{1}{2}(4 + 3, 1 + 3, 1 + 2) = \left(\frac{7}{2}, 2, \frac{3}{2}\right)\\
\frac{1}{2}(\mathbf{a}_3 + \mathbf{a}_1) &= \frac{1}{2}(2 + 3, 3, 2) = \left(\frac{5}{2}, \frac{3}{2}, 1\right).
\end{aligned}$$

We start from each corner and end up in the center as follows

$$\begin{aligned}
(2,0,0)+\frac{2}{3}\left[\left(\frac{7}{2},2,\frac{3}{2}\right)-(2,0,0)\right] &= \left(3,\frac{4}{3},1\right)\\
\mathbf{a}_1+\frac{2}{3}\left(\frac{1}{2}(\mathbf{a}_2+\mathbf{a}_3)-\mathbf{a}_1\right) &= \frac{1}{3}(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3),\\
(4,1,1)+\frac{2}{3}\left[\left(\frac{5}{2},\frac{3}{2},1\right)-(4,1,1)\right] &= \left(3,\frac{4}{3},1\right)\\
\mathbf{a}_2+\frac{2}{3}\left(\frac{1}{2}(\mathbf{a}_1+\mathbf{a}_3)-\mathbf{a}_2\right) &= \frac{1}{3}(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3),\\
(3,3,2)+\frac{2}{3}\left[\left(3,\frac{1}{2},\frac{1}{2}\right)-(3,3,2)\right] &= \left(3,\frac{4}{3},1\right)\\
\mathbf{a}_3+\frac{2}{3}\left(\frac{1}{2}(\mathbf{a}_1+\mathbf{a}_2)-\mathbf{a}_3\right) &= \frac{1}{3}(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3).
\end{aligned}$$

1.7.10. $A^2=\mathbf{A}^2=(\mathbf{B}-\mathbf{C})^2=B^2+C^2-2BC\cos\theta$ with θ the angle between $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$.

1.7.11. $\mathbf{P}$ and $\mathbf{Q}$ are antiparallel; $\mathbf{R}$ is perpendicular to both $\mathbf{P}$ and $\mathbf{Q}$.

1.8 Complex Numbers and Functions

1.8.1. (a) $(x+iy)^{-1}=\dfrac{x-iy}{x^2+y^2}$.

(b) $x+iy=re^{i\theta}$ gives

$$(x+iy)^{-1}=\frac{e^{-i\theta}}{r}=\frac{1}{r}(\cos\theta-i\sin\theta)=\frac{x-iy}{r^2}=\frac{x-iy}{x^2+y^2}.$$

1.8.2. If $z=re^{i\theta}$, $\sqrt{z}=\sqrt{r}e^{i\theta/2}=\sqrt{r}(\cos\theta/2+i\sin\theta/2)$. In particular,

$\sqrt{i}=e^{i\pi/4}=\dfrac{1+i}{\sqrt{2}}$ or $\sqrt{i}=e^{-i3\pi/4}$.

1.8.3. $e^{in\theta}=\cos n\theta+i\sin n\theta=(e^{i\theta})^n=(\cos\theta+i\sin\theta)^n=\sum_{\nu=0}^{n}\binom{n}{\nu}\cos^{n-\nu}\theta(i\sin\theta)^\nu$.

Separating real and imaginary parts we have

$$\cos n\theta=\sum_{\nu=0}^{[n/2]}(-1)^\nu\binom{n}{2\nu}\cos^{n-2\nu}\theta\sin^{2\nu}\theta,$$

$$\sin n\theta=\sum_{\nu=0}^{[n/2]}(-1)^\nu\binom{n}{2\nu+1}\cos^{n-2\nu-1}\theta\sin^{2\nu+1}\theta.$$

1.8.4. $\sum_{n=0}^{N-1}(e^{ix})^n = \frac{1-e^{iNx}}{1-e^{ix}} = \frac{e^{iNx/2}}{e^{ix/2}} \frac{e^{iNx/2}-e^{-iNx/2}}{e^{ix/2}-e^{-ix/2}}$

$$= e^{i(N-1)x/2} \sin(Nx/2)/\sin(x/2).$$

Now take real and imaginary parts to get the result.

1.8.5. (a) $\sinh(iz) = \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} = i\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = i\ \sin z.$

All other identities are shown similarly.

(b) $e^{i(z_1+z_2)} = \cos(z_1+z_2) + i\sin(z_1+z_2) = e^{iz_1}e^{iz_2}$

$$= (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$$

$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \sin z_2 \cos z_1).$$

Separating this into real and imaginary parts for real z_1, z_2 proves the addition theorems for real arguments. Analytic continuation extends them to the complex plane.

1.8.6. (a) Using $\cos iy = \cosh y, \sin iy = i\ \sinh y$, etc. and the addition theorem we obtain $\sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$, etc.

(b) $|\sin z|^2 = \sin(x+iy)\sin(x-iy) = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$$= \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y = \sin^2 x + \sinh^2 y, \text{ etc.}$$

1.8.7. (a) Using $\cos iy = \cosh y, \sin iy = i\ \sinh y$, etc. and the addition theorem we obtain $\sinh(x+iy) = \sinh x \cos y + i\cosh x \sin y$, etc.

(b) $|\cosh(x+iy)|^2 = \cosh(x+iy)\cosh(x-iy) = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y = \sinh^2 x + \cos^2 y$, etc.

1.8.8. (a) Using Exercise 1.8.7(a) and rationalizing we get

$$\tanh(x+iy) = \frac{\sinh x \cos y + i\cosh x \sin y}{\cosh x \cos y + i\sinh x \sin y}$$

$$= \frac{\frac{1}{2}\sinh 2x(\cos^2 y + \sin^2 y) + \frac{i}{2}\sin 2y(\cosh^2 x - \sinh^2 x)}{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}$$

$$= \frac{1}{2}\frac{\sinh 2x + i\sin 2y}{\cos^2 y + \sinh^2 x} = \frac{\sinh 2x + \sin 2y}{\cos 2y + \cosh 2x}.$$

(b) Starting from $\frac{\cosh(x+iy)}{\sinh(x+iy)}$ this is similarly proved.

1.8.9. The expansions relevant to this exercise are

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\ln(1 - ix) = -ix + \frac{x^2}{2} + \frac{ix^3}{3} - \cdots$$

$$\ln(1 + ix) = ix + \frac{x^2}{2} - \frac{ix^3}{3} - \cdots$$

The desired identity follows directly by comparing the expansion of $\tan^{-1} x$ with $i/2$ times the difference of the other two expansions.

1.8.10. (a) The cube roots of -1 are -1, $e^{\pi i/3} = 1/2 + i\sqrt{3}/2$, and $e^{-\pi i/3} = 1/2 - i\sqrt{3}/2$, so our answers are -2, $1 + i\sqrt{3}$, and $1 - i\sqrt{3}$.

(b) Write i as $e^{\pi i/2}$; its 1/4 power has values $e^{(\pi i/2+2n\pi)/4}$ for all integer n; there are four distinct values: $e^{i\pi/8} = \cos\pi/8 + i\sin\pi/8$, $e^{5i\pi/8} = \cos 5\pi/8 + i\sin 5\pi/8$, $e^{9i\pi/8} = -e^{i\pi/8}$, and $e^{13i\pi/8} = -e^{5i\pi/8}$.

(c) $e^{i\pi/4}$ has the unique value $\cos\pi/4 + i\sin\pi/4 = (1+i)/\sqrt{2}$.

1.8.11. (a) $(1+i)^3$ has a unique value. Since $1+i$ has magnitude $\sqrt{2}$ and is at an angle of $45° = \pi/4$, $(1+i)^3$ will have magnitude $2^{3/2}$ and argument $3\pi/4$, so its polar form is $2^{3/2}e^{3i\pi/4}$.

(b) Since $-1 = e^{\pi i}$, its 1/5 power will have values $e^{(2n+1)\pi i}$ for all integer n. There will be five distinct values: $e^{k\pi i/5}$ with $k = 1,\ 3,\ 5,\ 7$, and 9.

1.9 Derivatives and Extrema

1.9.1. Expand first as a power series in x, with y kept at its actual value. Then expand each term of the x expansion as a power series in y, regarding x as fixed. The nth term of the x expansion will be

$$\frac{x^n}{n!}\left(\frac{\partial}{\partial x}\right)^n f(x,y)\bigg|_{x=0,y=0}$$

The mth term in the y expansion of the x^n term is therefore

$$\frac{x^n}{n!}\frac{y^m}{m!}\left(\frac{\partial}{\partial y}\right)^m\left(\frac{\partial}{\partial x}\right)^n f(x,y)\bigg|_{x=0,y=0}$$

The coefficient in the above equation can be written

$$\frac{1}{(m+n)!}\frac{(m+n)!}{m!n!} = \frac{1}{(m+n)!}\binom{m+n}{n}.$$

Using the right-hand side of the above equation and collecting together all terms with the same value of $m+n$, we reach the form given in the exercise.

1.9.2. The quantities α_i are regarded as independent of the x_i when the differentiations are applied. Then, expansion of the differential operator raised to the power n will, when combined with t^n, produce terms with a total of n derivatives applied to f, with each term containing a power of each x_i equal to the number of times x_i was differentiated. The coefficient of each distinct term in the expansion of this nth order derivative will be the number of ways that derivative combination occurs in the expansion; the term in which each x_j derivative is applied n_j times occurs in the following number of ways:

$$\frac{n!}{n_1!n_2!\cdots},$$

with the sum of the n_i equal to n. Inserting this formula, we obtain the same result that would be obtained if we expanded, first in x_1, then in x_2, etc.

1.10 Evaluation of Integrals

1.10.1. Apply an integration by parts to the integral in Table 1.2 defining the gamma function, for integer $n > 0$:

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt = \left(\frac{t^n}{n}\right)e^{-t}\Big|_0^\infty + \int_0^\infty \frac{t^n}{n}e^{-x}dx = \frac{\Gamma(n+1)}{n}.$$

Rearranging to $\Gamma(n+1) = n\Gamma(n)$, we apply mathematical induction, noting that if $\Gamma(n) = (n-1)!$, then also $\Gamma(n+1) = n!$. To complete the proof, we directly evaluate the integral $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$, which is 0!.

1.10.2. This integral can also be evaluated using contour integration (see Example 11.8.5). A method motivated by the discussion of this section starts by multiplying the integrand by $e^{-\alpha x}$ and considering the value of this integral when $\alpha = 0$. We can start by differentiating the integral by the parameter α, corresponding to

$$I(\alpha) = \int_0^\infty \frac{\sin x e^{-\alpha x}}{x}\,dx, \qquad I'(\alpha) = -\int_0^\infty e^{-\alpha x}\sin x\,dx = -\frac{1}{\alpha^2+1},$$

where the integral for I' is ientified as having the value found in Example 1.10.4. We now integrate the expression for I', writing it as the indefinite integral

$$I(\alpha) = -\tan^{-1}\alpha + C\,.$$

The value of C is now determined from the value of $I(\infty)$, which from the form of I must be zero. Thus, $C = \tan^{-1}\infty = \pi/2$, and, since $\tan^{-1} 0 = 0$, we find $I(0) = \pi/2$.

1.10.3. Write the integrand as

$$\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^{-x}}{1 + e^{-2x}} = 2(e^{-x} - e^{-3x} + e^{-5x} - \cdots).$$

Now integrate term by term; each integrand is a simple exponential. The result is

$$2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right).$$

The series in parentheses is that discussed in Exercise 1.3.2, with value $\pi/4$. Our integral therefore has value $\pi/2$.

1.10.4. Expand the integrand as a power series in e^{-ax} and integrate term by term:

$$\int_0^\infty \frac{dx}{e^{ax}+1}=\int_0^\infty \left(e^{-ax}-e^{-2ax}+e^{-3ax}-\cdots\right)=\frac{1}{a}-\frac{1}{2a}+\frac{1}{3a}-\cdots$$

After factoring out $(1/a)$, the series that remains is that identified in Eq. (1.53) as $\ln 2$, so our integral has value $\ln(2)/a$.

1.10.5. Integrate by parts, to raise the power of x in the integrand:

$$\int_\pi^\infty \frac{\sin x}{x^2}\,dx=\int_\pi^\infty \frac{\cos x}{x}\,dx\,.$$

Note that the integrated terms vanish. The integral can now be recognized (see Table 1.2) as $-\mathrm{Ci}(\pi)$.

1.10.6. This is a case of the integral $I(\alpha)$ defined in the solution of Exercise 1.10.2, with $\alpha=1$. We therefore have

$$I(\alpha)=\frac{\pi}{2}-\tan^{-1}\alpha;\qquad I(1)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

1.10.7. Write erf as an integral and interchange the order of integration. We get

$$\int_0^x \mathrm{erf}(t)\,dt=\frac{2}{\sqrt{\pi}}\int_0^x dx\int_0^t e^{-u^2}du=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du\int_u^x dt$$

$$=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}(x-u)du\ =x\,\mathrm{erf}(x)-\frac{1}{\sqrt{\pi}}\int_0^x 2ue^{-u^2}du$$

$$=x\,\mathrm{erf}(x)+\frac{1}{\sqrt{\pi}}\left(e^{-x^2}-1\right).$$

1.10.8. Write E_1 as an integral and interchange the order of integration. Now the outer (u) integration must be broken into two pieces:

$$\int_1^x E_1(t)dt=\int_1^x dt\int_t^\infty \frac{e^{-u}}{u}\,du=\int_1^x \frac{e^{-u}}{u}\,du\int_1^u dt+\int_x^\infty \frac{e^{-u}}{u}\,du\int_1^x dt$$

$$=\int_1^x \frac{e^{-u}}{u}(u-1)\,du+\int_x^\infty \frac{e^{-u}}{u}(x-1)\,du$$

$$=e^{-1}-e^{-x}-E_1(1)+E_1(x)+(x-1)E_1(x)$$

$$=e^{-1}-e^{-x}-E_1(1)+xE_1(x).$$

1.10.9. Change the variable of integration to $y = x + 1$, leading to

$$\int_0^\infty \frac{e^{-x}}{x+1}\,dx = \int_1^\infty \frac{e^{-y+1}}{y}\,dy = e\,E_1(1).$$

1.10.10. After the integration by parts suggested in the text, with $[\tan^{-1} x]^2$ differentiated and dx/x^2 integrated, the result is $I(1)$, where

$$I(a) = \int_0^\infty \frac{2\,\tan^{-1} ax}{x(x^2+1)}\,dx$$

We now differentiate $I(a)$ with respect to the parameter a, reaching after a partial-fraction decomposition

$$I'(a) = 2\int_0^\infty \frac{dx}{(x^2+1)(a^2x^2+1)} = \frac{2}{1-a^2}\int_0^\infty \left[\frac{1}{x^2+1} - \frac{a^2}{a^2x^2+1}\right] dx$$

$$= \frac{2}{1-a^2}\left[\frac{\pi}{2} - a^2\left(\frac{\pi}{2a}\right)\right] = \frac{\pi}{1+a}.$$

Integrating with respect to a, we get $I(a) = \pi\,\ln(1+a) + C$, with C set to zero to obtain the correct result $I(0) = 0$. Then, setting $a = 1$, we find $I(1) = \pi\,\ln 2$, as required.

1.10.11. Integrating over one quadrant and multiplying by four, the range of x is $(0, a)$ and, for given x, the range of y is from 0 to the positive y satisfying the equation for the ellipse. Thus,

$$A = 4\int_0^a dx \int_0^{b\sqrt{a^2-x^2}/a} dy = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{4b}{a}\left(\frac{a^2\pi}{4}\right) = \pi ab.$$

1.10.12. Draw the dividing line at $y = 1/2$. Then the contribution to the area for each y between $1/2$ and 1 is $2\sqrt{1-y^2}$, so

$$A = 2\int_{1/2}^1 \sqrt{1-y^2}\,dy = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

A simple explanation of these two terms is that $\pi/3$ is the area of the sector that includes the piece in question, while $\sqrt{3}/4$ is the area of the triangle that is the part of the sector not included in that piece.

1.11 Dirac Delta Function

1.11.1. The mean value theorem gives

$$\lim_{n\to\infty} \int f(x)\delta_n(x)dx = \lim_{n\to\infty} n\int_{-1/2n}^{1/2n} f(x)dx = \lim_{n\to\infty} \frac{n}{n} f(\xi_n) = f(0),$$

as $-\dfrac{1}{2n} \le \xi_n \le \dfrac{1}{2n}$.

1.11.2. Use the elementary integral $\int \frac{dx}{1+x^2} = \arctan z$, thus reaching

$$\int_{-\infty}^{\infty} \frac{dx}{1+n^2x^2} = \frac{\pi}{n}\,.$$

1.11.4. $$\int_{-\infty}^{\infty} f(x)\delta(a(x-x_1))dx = \frac{1}{a}\int_{-\infty}^{\infty} f((y+y_1)/a)\delta(y)dy = \frac{1}{a} f\left(\frac{y_1}{a}\right)$$

$$= \frac{1}{a} f(x_1) = \int_{-\infty}^{\infty} f(x)\delta(x-x_1)\frac{dx}{a}.$$

1.11.5. The left-hand side of this equation is only nonzero in the neighborhood of $x = x_1$, where it is a case of Exercise 1.11.4, and in the neighborhood of $x = x_2$, where it is also a case of Exercise 1.11.4. In both cases, the quantity playing the role of a is $|x_1 - x_2|$.

1.11.7. Integrating by parts we find

$$\int_{-\infty}^{\infty} \delta'(x)f(x)\,dx = -\int_{-\infty}^{\infty} f'(x)\delta(x)dx = -f'(0).$$

1.11.9. (a) Inserting the given form for $\delta_n(x)$ and changing the variable of integration to nx, we obtain a result that is independent of n. The indefinite integral of $1/\cosh^2 x$ is $\tanh(x)$, which approaches $+1$ as $x \to +\infty$ and -1 as $x \to -\infty$, thus confirming the normalization claimed for δ_n. (b) The behavior of $\tanh(x)$ causes the right-hand side of this equation to approach zero for large n and negative x, but to approach $+1$ for large n and positive x.

2. Determinants and Matrices

2.1 Determinants

2.1.1. (a) -1.

(b) -11.

(c) $9/\sqrt{2}$.

2.1.2. The determinant of the coefficients is equal to 2. Therefore no nontrivial solution exists.

2.1.3. Given the pair of equations

$$x + 2y = 3, \qquad 2x + 4y = 6.$$

(a) Since the coefficients of the second equation differ from those of the first one just by a factor 2, the determinant of (lhs) coefficients is zero.

(b) Since the inhomogeneous terms on the right-hand side differ by the same factor 2, both numerator determinants also vanish.

(c) It suffices to solve $x + 2y = 3$. Given x, $y = (3 - x)/2$. This is the general solution for arbitrary values of x.

2.1.4. (a) C_{ij} is the quantity that multiplies a_{ij} in the expansion of the determinant. The sum over i collects the quantities that multiply all the a_{ij} in column j of the determinant.
(b) These summations form determinants in which the same column (or row) appears twice; the determinant is therefore zero,

2.1.5. The solution is given in the text.

2.1.6. If a set of forms is linearly dependent, one of them must be a linear combination of others. Form the determinant of their coefficients (with each row describing one of the forms) and subtract from one row the linear combination of other rows that reduces that row to zero. The determinant (whose value is not changed by the operation) will be seen to be zero.

2.1.7. The Gauss elimination yields

$$\begin{aligned}
10x_1 + 9x_2 + 8x_3 + 4x_4 + x_5 &= 10,\\
x_2 + 2x_3 + 3x_4 + 5x_5 + 10x_6 &= 5,\\
10x_3 + 23x_4 + 44x_5 - 60x_6 &= -5,\\
16x_4 + 48x_5 - 30x_6 &= 15,\\
48x_5 + 498x_6 &= 215,\\
-11316x_6 &= -4438,
\end{aligned}$$

so $x_6 = 2219/5658$, $x_5 = (215 - 498x_6)/48$, $x_4 = (15 + 30x_6 - 48x_5)/16$,

$x_3 = (-5 + 60x_6 - 44x_5 - 23x_4)/10$, $x_2 = 5 - 10x_6 - 5x_5 - 3x_4 - 2x_3$,
$x_1 = (10 - x_5 - 4x_4 - 8x_3 - 9x_2)/10$.

2.1.8. (a) $\delta_{ii} = 1$ (not summed) for each $i = 1, 2, 3$.

(b) $\delta_{ij}\varepsilon_{ijk} = 0$ because δ_{ij} is symmetric in i, j while ε_{ijk} is antisymmetric in i, j.

(c) For each ε in $\varepsilon_{ipq}\varepsilon_{jpq}$ to be non-zero, leaves only one value for i and j, so that $i = j$. Interchanging p and q gives two terms, hence the factor 2.

(d) There are 6 permutations i, j, k of $1, 2, 3$ in $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

2.1.9. Given k implies $p \neq q$ for $\varepsilon_{pqk} \neq 0$. For $\varepsilon_{ijk} \neq 0$ requires either $i = p$ and so $j = q$, or $i = q$ and then $j = p$. Hence $\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jp} - \delta_{iq}\delta_{jp}$.

2.2 Matrices

2.2.1. Writing the product matrices in term of their elements,

$$\mathsf{AB} = (\sum_m a_{im}b_{mk}), \qquad \mathsf{BC} = (\sum_n b_{in}c_{nk}),$$

$$(\mathsf{AB})\mathsf{C} = \left(\sum_n \left(\sum_m a_{im}b_{mn}\right) c_{nk}\right) = \sum_{mn} a_{im}b_{mn}c_{nk}$$
$$= \mathsf{A}(\mathsf{BC}) = \left(\sum_m a_{im}\left(\sum_n b_{mn}c_{nk}\right)\right),$$

because products of real and complex numbers are associative the parentheses can be dropped for all matrix elements.

2.2.2. Multiplying out $(\mathsf{A} + \mathsf{B})(\mathsf{A} - \mathsf{B}) = \mathsf{A}^2 + \mathsf{BA} - \mathsf{AB} - \mathsf{B}^2 = \mathsf{A}^2 - \mathsf{B}^2 + [\mathsf{B}, \mathsf{A}]$.

2.2.3. (a) $(a_1 + ib_1) - (a_2 + ib_2) = a_1 - a_2 + i(b_1 - b_2)$ corresponds to

$$\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ -(b_1 - b_2) & a_1 - a_2 \end{pmatrix},$$

i.e., the correspondence holds for addition and subtraction.
Similarly, it holds for multiplication because first

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

and matrix multiplication yields

$$\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & a_1b_2 + a_2b_1 \\ -(a_1b_2 + a_2b_1) & a_1a_2 - b_1b_2 \end{pmatrix}.$$

(b)

$$(a+ib)^{-1} \longleftrightarrow \begin{pmatrix} a/(a^2+b^2) & -b/(a^2+b^2) \\ b/(a^2+b^2) & a/(a^2+b^2) \end{pmatrix}.$$

2.2.4. A factor (-1) can be pulled out of each row giving the $(-1)^n$ overall.

2.2.5. (a) First we check that

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{pmatrix} = 0.$$

Second, to find the constraints we write the general matrix as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^2 + BC & B(A+D) \\ C(A+D) & BC + D^2 \end{pmatrix} = 0$$

giving $D = -A, D^2 = -BC = A^2$. This implies, if we set $B = b^2, C = -a^2$ without loss of generality, that $A = ab = -D$.

2.2.6. $n = 6$.

2.2.7. Expanding the commutators we find

$$[\mathsf{A}, [\mathsf{B}, \mathsf{C}]] = \mathsf{A}[\mathsf{B}, \mathsf{C}] - [\mathsf{B}, \mathsf{C}]\mathsf{A} = \mathsf{ABC} - \mathsf{ACB} - \mathsf{BCA} + \mathsf{CBA},$$

$$[\mathsf{B}, [\mathsf{A}, \mathsf{C}]] = \mathsf{BAC} - \mathsf{BCA} - \mathsf{ACB} + \mathsf{CAB},$$

$$[\mathsf{C}, [\mathsf{A}, \mathsf{B}]] = \mathsf{CAB} - \mathsf{CBA} - \mathsf{ABC} + \mathsf{BAC},$$

and subtracting the last double commutator from the second yields the first one, since the BAC and CAB terms cancel.

2.2.8. By direct multiplication of the matrices we find $[\mathsf{A}, \mathsf{B}] = \mathsf{AB} = \mathsf{C}, \mathsf{BA} = 0$, etc.

2.2.9. These results can all be verified by carrying out the indicated matrix multiplications.

2.2.10. If $a_{ik} = 0 = b_{ik}$ for $i > k$, then also $\sum_m a_{im} b_{mk} = \sum_{i \le m \le k} a_{im} b_{mk} = 0$, as the sum is empty for $i > k$.

2.2.11. By direct matrix multiplications and additions.

2.2.12. By direct matrix multiplication we verify all claims.

2.2.13. By direct matrix multiplication we verify all claims.

2.2.14. For $i \neq k$ and $a_{ii} \neq a_{kk}$ we get for the product elements

$$(\mathsf{AB})_{ik} = (\sum_n a_{in} b_{nk}) = (a_{ii} b_{ik}) = (\mathsf{BA})_{ik} = (\sum_n b_{in} a_{nk}) = (b_{ik} a_{kk}).$$

Hence $b_{ik} = 0$ for $i \neq k$.

2.2.15. $\sum_m a_{im} b_{mk} = a_{ii} b_{ii} \delta_{ik} = \sum_m b_{im} a_{mk}$.

2.2.16. Since trace ABC = trace BCA, choose one of the foregoing in which two commuting matrices appear adjacent to each other and interchange their order. Then make a cyclic permutation if needed to reach CBA.

2.2.17. Taking the trace, we find from $[\mathsf{M}_i, \mathsf{M}_j] = i\mathsf{M}_k$ that

$$i \text{ trace}(\mathsf{M}_k) = \text{trace}(\mathsf{M}_i\mathsf{M}_j - \mathsf{M}_j\mathsf{M}_i) = \text{trace}(\mathsf{M}_i\mathsf{M}_j) - \text{trace}(\mathsf{M}_i\mathsf{M}_j) = 0.$$

2.2.18. Taking the trace of $\mathsf{A}(\mathsf{BA}) = -\mathsf{A}^2\mathsf{B} = -\mathsf{B}$ yields $-\text{tr}(\mathsf{B}) = \text{tr}(\mathsf{A}(\mathsf{BA})) = \text{tr}(\mathsf{A}^2\mathsf{B}) = \text{tr}(\mathsf{B})$.

2.2.19. (a) Starting from $\mathsf{AB} = -\mathsf{BA}$, multiply on the left by B^{-1} and take the trace. After simplification, we get trace $\mathsf{B} = -$trace B, so trace $\mathsf{B} = 0$.

2.2.20. This is proved in the text.

2.2.21. (a) A unit matrix except that $M_{ii} = k$,

(b) A unit matrix except that $M_{im} = -K$,

(c) A unit matrix except that $M_{ii} = M_{mm} = 0$ and $M_{mi} - M_{im} = 1$.

2.2.22. Same answers as Exercise 2.2.21.

2.2.23. $\mathrm{A}^{-1} = \dfrac{1}{7}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

2.2.24. (a) The equation of part (a) states that T moves people from area j but does not change their total number.

(b) Write the component equation $\sum_j T_{ij} P_j = Q_i$ and sum over i. This summation replaces T_{ij} by unity, leaving that the sum over P_j equals the sum over Q_i, hence conserving people.

2.2.25. The answer is given in the text.

2.2.26. If $\mathsf{O}_i^{-1} = \tilde{\mathsf{O}}_i, i = 1, 2$, then $(\mathsf{O}_1\mathsf{O}_2)^{-1} = \mathsf{O}_2^{-1}\mathsf{O}_1^{-1} = \tilde{\mathsf{O}}_2\tilde{\mathsf{O}}_1 = \widetilde{\mathsf{O}_1\mathsf{O}_2}$.

2.2.27. Taking the determinant of $\tilde{\mathsf{A}}\mathsf{A} = 1$ and using the product theorem yields $\det(\tilde{\mathsf{A}})\det(\mathsf{A}) = 1 = \det^2(\mathsf{A})$ implying $\det(\mathsf{A}) = \pm 1$.

2.2.28. If $\tilde{\mathsf{A}} = -\mathsf{A}, \tilde{\mathsf{S}} = \mathsf{S}$, then trace$(\widetilde{\mathsf{SA}})$ = trace(SA) = trace$(\tilde{\mathsf{A}}\tilde{\mathsf{S}})$ = $-$trace(AS).

2.2.29. From $\tilde{\mathsf{A}} = \mathsf{A}^{-1}$ and $\det(\mathsf{A}) = 1$ we have

$$\mathsf{A}^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \tilde{A} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

This gives $\det(\mathsf{A}) = a_{11}^2 + a_{12}^2 = 1$, hence $a_{11} = \cos\theta = a_{22}, a_{12} = \sin\theta = -a_{21}$, the standard 2×2 rotation matrix.

2.2.30. Because ε is real,

$$\det(\mathsf{A}^*) = \sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a^*_{1i_1} a^*_{2i_2} \cdots a^*_{ni_n} = \left(\sum_{i_k} \varepsilon_{i_1 i_2 \ldots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n} \right)^*$$
$$= (\det \mathsf{A})^* .$$

Because, for any A, $\det(\mathsf{A}) = \det(\tilde{\mathsf{A}})$, $\det(\mathsf{A}^*) = \det(\mathsf{A}^\dagger)$.

2.2.31. If J_x and J_y are real, so also must be their commutator, so the commutation rule requires that J_z be pure imaginary.

2.2.32. $(\mathsf{AB})^\dagger = \widetilde{\mathsf{A}^*\mathsf{B}^*} = \tilde{\mathsf{B}}^*\tilde{\mathsf{A}}^* = \mathsf{B}^\dagger\mathsf{A}^\dagger$.

2.2.33. As $\mathsf{C}_{jk} = \sum_n S^*_{nj} S_{nk}$, trace $(\mathsf{C}) = \sum_{nj} |S_{nj}|^2$.

2.2.34. If $\mathsf{A}^\dagger = \mathsf{A}, \mathsf{B}^\dagger = \mathsf{B}$, then

$$(\mathsf{AB} + \mathsf{BA})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger + \mathsf{A}^\dagger\mathsf{B}^\dagger = \mathsf{AB} + \mathsf{BA},$$

$$-i(\mathsf{B}^\dagger\mathsf{A}^\dagger - \mathsf{A}^\dagger\mathsf{B}^\dagger) = i(\mathsf{AB} - \mathsf{BA}).$$

2.2.35. If $\mathsf{C}^\dagger \neq \mathsf{C}$, then $(i\mathsf{C}_-)^\dagger \equiv (\mathsf{C}^\dagger - \mathsf{C})^\dagger = \mathsf{C} - \mathsf{C}^\dagger = -i\mathsf{C}_-^\dagger$, i.e. $(\mathsf{C}_-)^\dagger = \mathsf{C}_-$. Similarly $\mathsf{C}_+^\dagger = \mathsf{C}_+ = \mathsf{C} + \mathsf{C}^\dagger$.

2.2.36. $-i\mathsf{C}^\dagger = (\mathsf{AB} - \mathsf{BA})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger - \mathsf{A}^\dagger\mathsf{B}^\dagger = \mathsf{BA} - \mathsf{AB} = -i\mathsf{C}$.

2.2.37. $(\mathsf{AB})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger = \mathsf{BA} = \mathsf{AB}$ yields $[\mathsf{A}, \mathsf{B}] = 0$ as the condition, that is, the answer in the text.

2.2.38. $(\mathsf{U}^\dagger)^\dagger = \mathsf{U} = (\mathsf{U}^{-1})^\dagger$.

2.2.39. $(\mathsf{U}_1\mathsf{U}_2)^\dagger = \mathsf{U}_2^\dagger\mathsf{U}_1^\dagger = \mathsf{U}_2^{-1}\mathsf{U}_1^{-1} = (\mathsf{U}_1\mathsf{U}_2)^{-1}$.

2.2.40. Start by noting the relationships $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ if $i \neq j$, and $\sigma_i^2 = \mathbf{1}_2$; see Eq. (2.59); for proof add Eqs. (2.29) and (2.30). Then,

$$(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = (p_x\sigma_1 + p_y\sigma_2 + p_z\sigma_3)^2$$

expands to

$$p_x^2\sigma_1^2 + p_y^2\sigma_2^2 + p_z^2\sigma_3^2 + p_xp_y(\sigma_1\sigma_2 + \sigma_2\sigma_1) + p_xp_z(\sigma_1\sigma_3 + \sigma_3\sigma_1)$$
$$+ p_yp_z(\sigma_1\sigma_2 + \sigma_2\sigma_1) = p_x^2 + p_y^2 + p_z^2 = \mathbf{p}^2 .$$

2.2.41. Writing $\gamma^0 = \sigma_3 \otimes \mathbf{1}$ and $\gamma^i = \gamma \otimes \sigma_i$ $(i = 1,\ 2,\ 3)$, where

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and noting fron Eq. (2.57) that if $\mathsf{C} = \mathsf{A} \otimes \mathsf{B}$ and $\mathsf{C}' = \mathsf{A}' \otimes \mathsf{B}'$ then $\mathsf{CC}' = \mathsf{AA}' \otimes \mathsf{BB}'$,

$$(\gamma^0)^2 = \sigma_3^2 \otimes \mathbf{1}_2^2 = \mathbf{1}_2 \otimes \mathbf{1}_2 = \mathbf{1}_4, \qquad (\gamma^i)^2 = \gamma^2 \otimes \sigma_i^2 = (-\mathbf{1}_2) \otimes \mathbf{1}_2 = -\mathbf{1}_4$$

$$\gamma^0\gamma^i = \sigma_3\gamma \otimes \mathbf{1}_2\sigma_i = \sigma_1 \otimes \sigma_i, \qquad \gamma^i\gamma^0 = \gamma\sigma_3 \otimes \sigma_i\mathbf{1}_2 = (-\sigma_1) \otimes \sigma_i$$

$$\gamma^i\gamma^j = \gamma^2 \otimes \sigma_i\sigma_j \qquad \gamma^j\gamma^i = \gamma^2 \otimes \sigma_j\sigma_i$$

It is obvious from the second line of the above equation set that $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$; from the third line of the equation set we find $\gamma^i\gamma^j + \gamma^j\gamma^i$ is zero if $j \neq i$ because then $\sigma_j\sigma_i = -\sigma_i\sigma_j$.

2.2.42. The anticommutation can be demonstrated by matrix multiplication.

2.2.43. These results can be confirmed by carrying out the indicated matrix operations.

2.2.44. Since $\gamma_5^2 = 1_4$, $\frac{1}{4}(1_4 + \gamma_5)^2 = \frac{1}{4}(1_4 + 2\gamma_5 + 1_4) = \frac{1}{2}(1_4 + \gamma_5)$.

2.2.47. Since $\tilde{\mathsf{C}} = -\mathsf{C} = \mathsf{C}^{-1}$, and

$$\begin{aligned}
\mathsf{C}\gamma^0\mathsf{C}^{-1} &= -\gamma^0 = -\tilde{\gamma}^0, \\
\mathsf{C}\gamma^2\mathsf{C}^{-1} &= -\gamma^2 = -\tilde{\gamma}^2, \\
\mathsf{C}\gamma^1\mathsf{C}^{-1} &= \gamma^1 = -\tilde{\gamma}^1, \\
\mathsf{C}\gamma^3\mathsf{C}^{-1} &= \gamma^3 = -\tilde{\gamma}^3,
\end{aligned}$$

we have $\widetilde{\mathsf{C}\gamma^\mu\mathsf{C}^{-1}} = \tilde{\mathsf{C}}^{-1}\tilde{\gamma}^\mu\tilde{\mathsf{C}} = \mathsf{C}\tilde{\gamma}^\mu\mathsf{C}^{-1} = -\tilde{\gamma}^\mu$.

2.2.48. (a) Written as 2×2 blocks, the matrices α_i and the wave function Ψ are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \text{and} \qquad \Psi = \begin{pmatrix} \Psi_L \\ \Psi_S \end{pmatrix}.$$

In block form, Eq. (2.73) becomes

$$\left[\begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_1 p_x \\ \sigma_1 p_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_2 p_y \\ \sigma_2 p_y & 0 \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} 0 & \sigma_3 p_z \\ \sigma_3 p_z & 0 \end{pmatrix} \right] \begin{pmatrix} \Psi_L \\ \Psi_S \end{pmatrix} = E \begin{pmatrix} \Psi_L \\ \Psi_S \end{pmatrix}$$

The solution is completed by moving the right-hand side of the above equation to the left, written in the form

$$\begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}$$

and combining all the terms by matrix addition.

2.2.49. The requirements the gamma matrices must satisfy are Eqs. (2.74) and (2.75). Use the same process that was illustrated in the solution to Exercise 2.2.41, but now with $\gamma^0 = \sigma_1 \otimes \mathbf{1}_2$.

2.2.50. In the Weyl representation, the matrices α_i and the wave function Ψ, written as 2×2 blocks, take the forms

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Then proceed as in the solution to Exercise 2.2.48, obtaining the matrix equation

$$\left[\begin{pmatrix} 0 & mc^2 \\ mc^2 & 0 \end{pmatrix} + \begin{pmatrix} -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix}\right] \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = E \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$

Here we wrote $\boldsymbol{\sigma}\cdot\mathbf{p}$ for $\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z$.

If m is negligible, this matrix equation becomes two independent equations, one for Ψ_1, and one for Ψ_2. In this limit, one set of solutions will be with $\Psi_2 = 0$ and Ψ_1 a solution to $-\boldsymbol{\sigma}\cdot\mathbf{p}\Psi_1 = E\Psi_1$; a second set of solutions will have zero Ψ_1 and a set of Ψ_2 identical to the previously found set of Ψ_1 but with values of E of the opposite sign.

2.2.51. (a) Form $\mathbf{r}'^\dagger\mathbf{r}' = (\mathsf{U}\mathbf{r})^\dagger\mathsf{U}\mathbf{r} = \mathbf{r}^\dagger\mathsf{U}^\dagger\mathsf{U}\mathbf{r} = \mathbf{r}^\dagger\mathbf{r}$.

(b) If for all $\mathbf{r}$, $\mathbf{r}'^\dagger\mathbf{r} = \mathbf{r}^\dagger\mathsf{U}^\dagger\mathsf{U}\mathbf{r}$, then we must have $\mathsf{U}^\dagger\mathsf{U} = \mathbf{1}$.

3. Vector Analysis

3.1 Review of Basic Properties

(no exercises)

3.2 Vectors in 3-D Space

3.2.1. $\mathbf{P}\times\mathbf{Q} = (P_xQ_y - P_yQ_x)\hat{\mathbf{x}}\times\hat{\mathbf{y}} = (P_xQ_y - P_yQ_x)\hat{\mathbf{z}}$.

3.2.2. $(\mathbf{A}\times\mathbf{B})^2 = A^2B^2\sin^2\theta = A^2B^2(1-\cos^2\theta) = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2$ with θ the angle between $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$.

3.2.3. The vector $\mathbf{P}$ is at an angle θ (in the positive direction) from the x axis, while $\mathbf{Q}$ is at an angle $-\varphi$. The angle between these vectors is therefore $\theta+\varphi$. Both vectors are of unit length. Therefore $\mathbf{P}\cdot\mathbf{Q} = \cos(\theta+\varphi)$ and the z component of $\mathbf{Q}\times\mathbf{P}$ is $\sin(\theta+\varphi)$.

3.2.4. $\mathbf{A} = \mathbf{U}\times\mathbf{V} = -3\hat{\mathbf{y}} - 3\hat{\mathbf{z}}, \mathbf{A}/A = -(\hat{\mathbf{y}}+\hat{\mathbf{z}})/\sqrt{2}$.

3.2.5. If $\mathbf{a}$ and $\mathbf{b}$ both lie in the xy-plane their cross product is in the z-direction. The same is valid for $\mathbf{c}\times\mathbf{d}\sim\hat{\mathbf{z}}$. The cross product of two parallel vectors is zero. Hence $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = 0$.

3.2.6. Cross $\mathbf{A}-\mathbf{B}-\mathbf{C} = 0$ into $\mathbf{A}$ to get $-\mathbf{A}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}$, or $C\sin\beta = B\sin\gamma$, etc.

3.2.7. $\mathbf{B} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$.

3.2.8. (a) $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0$, A is the plane of B and C. The parallelpiped has zero height above the BC plane and therefore zero volume.

(b) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$.

3.2.9. Applying the BAC-CAB rule we obtain

$$[\mathbf{a}\cdot\mathbf{c}\mathbf{b} - \mathbf{a}\cdot\mathbf{b}\mathbf{c}] + [\mathbf{b}\cdot\mathbf{a}\mathbf{c} - \mathbf{b}\cdot\mathbf{c}\mathbf{a}] + [\mathbf{c}\cdot\mathbf{b}\mathbf{a} - \mathbf{c}\cdot\mathbf{a}\mathbf{b}] = 0.$$

3.2.10. (a) $\hat{\mathbf{r}}\cdot\mathbf{A}_r = \mathbf{A}\cdot\hat{\mathbf{r}}$.

(b) $\hat{\mathbf{r}}\cdot\mathbf{A}_t = -\hat{\mathbf{r}}\cdot[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{A})] = 0$.

3.2.11. The scalar triple product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ is the volume spanned by the vectors.

3.2.12. $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = -120$,

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -60\hat{\mathbf{x}} - 40\hat{\mathbf{y}} + 50\hat{\mathbf{z}}$,

$\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 24\hat{\mathbf{x}} + 88\hat{\mathbf{y}} - 62\hat{\mathbf{z}}$,

$\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = 36\hat{\mathbf{x}} - 48\hat{\mathbf{y}} + 12\hat{\mathbf{z}}$.

3.2.13.
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = [(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D} = [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}]\cdot\mathbf{D}$$
$$= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).$$

3.2.14. Using the BAC-CAB rule with $\mathbf{A} \times \mathbf{B}$ as the first vector we obtain
$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D}\mathbf{C} - (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}\mathbf{D}$.

3.2.15. The answer is given in the text.

3.3 Coordinate Transformations

3.3.1. The trigonometric identities follow from the rotation matrix identity

$$\begin{pmatrix} \cos(\varphi_1 + \varphi_2) & \sin(\varphi_1 + \varphi_2) \\ -\sin(\varphi_1 + \varphi_2) & \cos(\varphi_1 + \varphi_2) \end{pmatrix} = \begin{pmatrix} \cos\varphi_2 & \sin\varphi_2 \\ -\sin\varphi_2 & \cos\varphi_2 \end{pmatrix} \begin{pmatrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 \\ -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{pmatrix}.$$

3.3.2. Align the reflecting surfaces with the xy, xz, and yz planes. If an incoming ray strikes the xy plane, the z component of its direction of propagation is reversed. A strike on the xz plane reverses its y component, and a strike on the yz plane reverses its x component. These properties apply for an arbitrary direction of incidence, and together the reverse the propagation direction to the opposite of its incidence orientation.

3.3.3. Because S is orthogonal, its transpose is also its inverse. Therefore

$$(\mathbf{x}')^T = (\mathsf{S}\mathbf{x})^T = \mathbf{x}^T\mathsf{S}^T = \mathbf{x}^T\mathsf{S}^{-1}.$$

Then $(\mathbf{x}')^T\mathbf{y}' = \mathbf{x}^T\mathsf{S}^{-1}\mathsf{S}\mathbf{y} = \mathbf{x}^T\mathbf{y}$.

3.3.4. (a) $\det(\mathsf{S}) = 1$

(b) $\mathbf{a}' = \mathsf{S}\mathbf{a} = \begin{pmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.80 \\ 0.12 \\ 1.16 \end{pmatrix},$

$$\mathbf{b}' = \mathsf{S}\mathbf{b} = \begin{pmatrix} 0.80 & 0.60 & 0.00 \\ -0.48 & 0.64 & 0.60 \\ 0.36 & -0.48 & 0.80 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1.20 \\ 0.68 \\ -1.76 \end{pmatrix},$$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = -1,$$

$$\mathbf{a}' \cdot \mathbf{b}' = \begin{pmatrix} 0.80 & 0.12 & 1.16 \end{pmatrix} \begin{pmatrix} 1.20 \\ 0.68 \\ -1.76 \end{pmatrix} = -1.$$

3.3.5. (a) $\det(\mathsf{S}) = -1$

$$\mathbf{a}' = \mathsf{S}\mathbf{a} = \begin{pmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & -0.80 & 0.36 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.40 \\ -0.16 \\ -0.12 \end{pmatrix},$$

$$\mathbf{b}' = \mathsf{S}\mathbf{b} = \begin{pmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & -0.80 & 0.36 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.80 \\ -1.68 \\ 1.24 \end{pmatrix},$$

$$\mathbf{c}' = \mathsf{S}\mathbf{c} = \begin{pmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & -0.80 & 0.36 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3.60 \\ -0.44 \\ 0.92 \end{pmatrix}.$$

(b) $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{a}' \times \mathbf{b}' = \begin{pmatrix} -0.40 \\ -1.64 \\ -2.48 \end{pmatrix}.$

Compare with

$$\mathsf{S}(\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & -0.80 & 0.36 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.40 \\ 1.64 \\ 2.48 \end{pmatrix}.$$

(c) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 3, \qquad (\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}' = -3.$

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 2 \\ 11 \\ -2 \end{pmatrix}, \qquad \mathbf{a}' \times (\mathbf{b}' \times \mathbf{c}') = \begin{pmatrix} -0.40 \\ -8.84 \\ 7.12 \end{pmatrix}.$

Compare with

$$\mathsf{S}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) = \begin{pmatrix} 0.60 & 0.00 & 0.80 \\ -0.64 & -0.60 & 0.48 \\ -0.48 & -0.80 & 0.36 \end{pmatrix} \begin{pmatrix} 2 \\ 11 \\ -2 \end{pmatrix} = \begin{pmatrix} -0.40 \\ -8.84 \\ 7.12 \end{pmatrix}.$$

(e) Note that S is an improper rotation. The fact that $\mathsf{S}(\mathbf{a} \times \mathbf{b})$ has components of opposite sign to $\mathbf{a}' \times \mathbf{b}'$ shows that $\mathbf{a} \times \mathbf{b}$ is a **pseudovector**. The difference in sign between $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and $(\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}'$ shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is a **pseudoscalar**. The equality of the vectors $\mathsf{S}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))$ and $\mathbf{a}' \times (\mathbf{b}' \times \mathbf{c}')$ shows that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a **vector**.

3.4 Rotations in $I\!R^3$

3.4.1. The Euler rotations defined here differ from those in the text in that the inclination of the polar axis (in amount β, in now about the x_1' axis rather than the x_2' axis. Therefore, to achieve the same polar orientation, we must place the x_1' axis where the x_2' axis was using the text rotation. This requires an additional first rotation of $\pi/2$. After inclining the polar axis, the rotational position is now $\pi/2$ greater (counterclockwise) than from the text rotation, so the third Euler angle must be $\pi/2$ less than its original value.

3.4.2. (a) $\alpha = 70°$, $\beta = 60°$, $\gamma = -80°$.

(b) The answer is in the text.

3.4.3. The angle changes lead to $\cos\alpha \to -\cos\alpha, \sin\alpha \to -\sin\alpha; \cos\beta \to \cos\beta, \sin\beta \to -\sin\beta; \sin\gamma \to -\sin\gamma, \cos\gamma \to -\cos\gamma$. From these we verify that each matrix element of Eq. (3.37) stays the same.

3.4.4. (a) Each of the three Euler rotations is an orthogonal matrix, so their matrix product must also be orthogonal. Therefore its transpose, $\tilde{\mathsf{S}}$, must equal its inverse, S^{-1}.

(b) This equation simply carries out the three Euler rotations in reverse order, each in the opposite direction.

3.4.5. (a) The projection of $\mathbf{r}$ on the rotation axis is not changed by the rotation; it is $(\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$. The portion of $\mathbf{r}$ perpendicular to the rotation axis can be written $\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$. Upon rotation through an angle Φ, this vector perpendicular to the rotation axis will consist of a vector in its original direction $(\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}})\cos\Phi$ plus a vector perpendicular both to it and to $\hat{\mathbf{n}}$ given by $(\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}})\sin\Phi \times \hat{\mathbf{n}}$; this reduces to $\mathbf{r} \times \hat{\mathbf{n}} \sin\Phi$. Adding these contributions, we get the required result.

(b) If $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$, the formula yields

$$\mathbf{r}' = x\cos\Phi\hat{\mathbf{e}}_x + y\cos\Phi\hat{\mathbf{e}}_y + z\cos\Phi\hat{\mathbf{e}}_z + y\sin\Phi\hat{\mathbf{e}}_x - x\sin\Phi\hat{\mathbf{e}}_y + z(1-\cos\Phi)\hat{\mathbf{e}}_z\,.$$

Simplifying, this reduces to

$$\mathbf{r}' = (x\cos\Phi + y\sin\Phi)\hat{\mathbf{e}}_x + (-x\sin\Phi + y\cos\Phi)\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z\,.$$

This corresponds to the rotational transformation given in Eq. (3.35).

(c) Expanding $\mathbf{r}'^2$, recognizing that the second term of $\mathbf{r}'$ is orthogonal to the first and third terms,

$$r'^2 = r^2\cos^2\Phi + (\mathbf{r}\times\hat{\mathbf{n}})\cdot(\mathbf{r}\times\hat{\mathbf{n}})\sin^2\Phi$$
$$+ (\hat{\mathbf{n}}\cdot\mathbf{r})^2(1-\cos\Phi)^2 + 2(\hat{\mathbf{n}}\cdot\mathbf{r})^2\cos\Phi(1-\cos\Phi)\,.$$

Using an identity to make the simplification

$$(\mathbf{r}\times\hat{\mathbf{n}})\cdot(\mathbf{r}\times\hat{\mathbf{n}}) = (\mathbf{r}\cdot\mathbf{r})(\hat{\mathbf{n}}\cdot\mathbf{n}) - (\mathbf{r}\cdot\hat{\mathbf{n}})^2 = r^2 - (\mathbf{r}\cdot\hat{\mathbf{n}})^2\,,$$

we get

$$r'^2 = r^2 + (\mathbf{r}\cdot\hat{\mathbf{n}})^2(-\sin^2\Phi + 1 + \cos^2\Phi - 2\cos^2\Phi) = r^2\,.$$

3.5 Differential Vector Operators

3.5.1. (a) $-3(14)^{-5/2}(\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}})$.

(b) $3/196$.

(c) $-1/(14)^{1/2}, -2/(14)^{1/2}, -3/(14)^{1/2}$.

3.5.2. The solution is given in the text.

3.5.3. From $r_{12} = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$ we obtain

$\boldsymbol{\nabla}_1 r_{12} = \dfrac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}} = \hat{\mathbf{r}}_{12}$ by differentiating componentwise.

3.5.4. $d\mathbf{F} = \mathbf{F}(\mathbf{r}+\mathbf{dr}, t+dt) - \mathbf{F}(\mathbf{r},t) = \mathbf{F}(\mathbf{r}+\mathbf{dr}, t+dt) - \mathbf{F}(\mathbf{r},t+dt)$

$$+\,\mathbf{F}(\mathbf{r},t+dt) - \mathbf{F}(\mathbf{r},t) = (d\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{F} + \frac{\partial\mathbf{F}}{\partial t}\,dt.$$

3.5.5. $\boldsymbol{\nabla}(uv) = v\boldsymbol{\nabla}u + u\boldsymbol{\nabla}v$ follows from the product rule of differentiation.

(a) Since $\boldsymbol{\nabla}f = \dfrac{\partial f}{\partial u}\boldsymbol{\nabla}u + \dfrac{\partial f}{\partial v}\boldsymbol{\nabla}v = 0$, $\boldsymbol{\nabla}u$ and $\boldsymbol{\nabla}v$ are parallel so that $(\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v) = 0$, and vice versa.

(b) If $(\boldsymbol{\nabla}u)\times(\boldsymbol{\nabla}v) = 0$, the two-dimensional volume spanned by $\boldsymbol{\nabla}u$ and $\boldsymbol{\nabla}v$, also given by the Jacobian

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

vanishes.

3.5.6. (a) From $\dot{\mathbf{r}} = \omega r(-\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t)$, we get $\mathbf{r}\times\dot{\mathbf{r}} = \hat{\mathbf{z}}\omega r^2(\cos^2\omega t + \sin^2\omega t) = \hat{\mathbf{z}}\omega r^2$.

(b) Differentiating $\dot{\mathbf{r}}$ above we get $\ddot{\mathbf{r}} = -\omega^2 r(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t) = -\omega^2\mathbf{r}$.

3.5.7. The time derivative commutes with the transformation because the coefficients a_{ij} are constants. Therefore dV_j/dt satisfies the same transformation law as V_j.

3.5.8. The product rule directly implies (a) and (b).

3.5.9. The product rule of differentiation in conjunction with $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, etc. gives

$$\boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) - \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{b}).$$

3.5.10. If $\mathbf{L} = -i\mathbf{r}\times\boldsymbol{\nabla}$, then the determinant form of the cross product gives

$L_z = -i\left(x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}\right)$, (in units of $\hbar$), etc.

3.5.11. Carry out the indicated operations, remembering that derivatives operate on everything to their right in the current expression as well as on the function to which the operator is applied. Therefore,

$$L_xL_y = -\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]\left[z\frac{\partial}{\partial x} - x,\frac{\partial}{\partial z}\right]$$

$$= -\left[y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} - z^2\frac{\partial^2}{\partial y\partial x} - xy\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y\partial z}\right].$$

$$L_yL_x = -\left[z\frac{\partial}{\partial x} - x,\frac{\partial}{\partial z}\right]\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$$

$$= -\left[zy\frac{\partial^2}{\partial x\partial z} - xy\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial x\partial y} + xz\frac{\partial^2}{\partial z\partial y} + x\frac{\partial}{\partial y}\right].$$

Combining the above,

$$L_xL_y - L_yL_x = x\frac{\partial}{\partial y} - u\frac{\partial}{\partial x} = iL_z.$$

3.5.12. $[\mathbf{a}\cdot\mathbf{L}, \mathbf{b}\cdot\mathbf{L}] = a_j[L_j, L_k]b_k = i\varepsilon_{jkl}a_jb_kL_l = i(\mathbf{a}\times\mathbf{b})\cdot\mathbf{L}$.

3.5.13. The stream lines of $\mathbf{b}$ are solutions of the differential equation

$$\frac{dy}{dx} = \frac{b_y}{b_x} = \frac{x}{-y}.$$

Writing this differential equation as $xdx + ydy = 0$, we see that it can be integrated to yield $x^2/2 + y^2/2$ =constant, equivalent to $x^2 + y^2 = C^2$, the equation for a family of circles centered at the coordinate origin. To determine the direction of the stream lines, pick a convenient point on a circle, e.g., the point $(+1, 0)$. Here $b_x = 0$, $b_y = +1$, which corresponds to counterclockwise travel.

3.6 Differential Vector Operators: Further Properties

3.6.1. By definition, $\mathbf{u} \times \mathbf{v}$ is solenoidal if $\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. But we have the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{v}) = v \cdot (\boldsymbol{\nabla} \times u) - u \cdot (\boldsymbol{\nabla} \times v)\,.$$

If a vector $\mathbf{w}$ is irrotational, $\boldsymbol{\nabla} \times \mathbf{w} = 0$, so if $\mathbf{u}$ and $\mathbf{v}$ are both irrotational, the right-hand side of the above equation is zero, proving that $\mathbf{u} \times \mathbf{v}$ is solenoidal.

3.6.2. If $\boldsymbol{\nabla} \times \mathbf{A} = 0$, then $\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{r}) = 0 - 0 = 0$.

3.6.3. From $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ we get $\boldsymbol{\nabla} \cdot (\boldsymbol{\omega} \times \mathbf{r}) = -\boldsymbol{\omega} \cdot (\boldsymbol{\nabla} \times \mathbf{r}) = 0$.

3.6.4. Forming the scalar product of $\mathbf{f}$ with the identity

$$\boldsymbol{\nabla} \times (g\mathbf{f}) = g\boldsymbol{\nabla} \times \mathbf{f} + (\boldsymbol{\nabla} g) \times \mathbf{f} \equiv 0$$

we obtain the result, because the second term of the identity is perpendicular to $\mathbf{f}$.

3.6.5. Applying the BAC-CAB rule naively we obtain $(\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{A} - (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B}$, where $\boldsymbol{\nabla}$ still acts on $\mathbf{A}$ and $\mathbf{B}$. Thus, the product rule of differentiation generates two terms out of each which are ordered so that $\boldsymbol{\nabla}$ acts only on what comes after the operator. That is, $(\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{A} \to \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}$, and similarly for the second term. Hence the four terms.

3.6.6. Write the x components of all the terms on the right-hand side of this equation. We get

$$[(\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{B}]_x = A_z \frac{\partial B_z}{\partial x} - A_x \frac{\partial B_z}{\partial z} - A_x \frac{\partial B_y}{\partial y} + A_y \frac{\partial B_y}{\partial x}\,,$$

$$[(\mathbf{B} \times \boldsymbol{\nabla}) \times \mathbf{A}]_x = B_z \frac{\partial A_z}{\partial x} - B_x \frac{\partial A_z}{\partial z} - B_x \frac{\partial A_y}{\partial y} + B_y \frac{\partial A_y}{\partial x}\,,$$

$$[\mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B})]_x = A_x \frac{\partial B_x}{\partial x} + A_x \frac{\partial B_y}{\partial y} + A_x \frac{\partial B_z}{\partial z}\,,$$

$$[\mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A})]_x = B_x \frac{\partial A_x}{\partial x} + B_x \frac{\partial A_y}{\partial y} + B_x \frac{\partial A_z}{\partial z}\,.$$

All terms cancel except those corresponding to the x component of the left-hand side of the equation.

3.6.7. Apply the BAC-CAB rule to get

$$\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \frac{1}{2}\, \boldsymbol{\nabla}(A^2) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{A}.$$

The factor 1/2 occurs because $\boldsymbol{\nabla}$ operates only on one $\mathbf{A}$.

3.6.8. $\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{A} \times \mathbf{B}) = \hat{\mathbf{e}}_x(\mathbf{A} \times \mathbf{B})_x + \hat{\mathbf{e}}_y(\mathbf{A} \times \mathbf{B})_y + \hat{\mathbf{e}}_z(\mathbf{A} \times \mathbf{B})_z$

$= \mathbf{A} \times \mathbf{B}.$

3.6.9. It suffices to check one Cartesian component; we take x. The x component of the left-hand side of Eq. (3.70) is

$$\frac{\partial}{\partial y}(\boldsymbol{\nabla} \times \mathbf{V})_z - \frac{\partial}{\partial z}(\boldsymbol{\nabla} \times \mathbf{V})_y = \frac{\partial^2 V_y}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial z \partial x}\,.$$

The x component of the right-hand side is

$$\frac{\partial}{\partial x}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] - \left[\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right]\,.$$

After canceling the two right-hand-side occurrences of $\partial^2 V_x/\partial x^2$ these two expressions contain identical terms.

3.6.10. $\boldsymbol{\nabla} \times (\varphi \boldsymbol{\nabla}\varphi) = \boldsymbol{\nabla}\varphi \times \boldsymbol{\nabla}\varphi + \varphi \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi) = 0 + 0 = 0.$

3.6.11. (a) If $\mathbf{F}$ or $\mathbf{G}$ contain an additive constant, it will vanish on application of any component of $\boldsymbol{\nabla}$.

(b) If either vector contains a term $\boldsymbol{\nabla} f$, it will not affect the curl because $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$.

3.6.12. Use the identity $\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$. Taking the curl and noting that the first term on the right-hand side then vanishes, we obtain the desired relation.

3.6.13. Using Exercise 3.5.9,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u \times \boldsymbol{\nabla} v) = (\boldsymbol{\nabla} v) \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} u) - (\boldsymbol{\nabla} u) \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} v) = 0 - 0 = 0.$$

3.6.14. $\boldsymbol{\nabla}^2 \varphi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi = 0$, and $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\varphi = 0$.

3.6.15. From Eq. (3.70), $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\boldsymbol{\nabla}^2 \mathbf{A}$ if $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

3.6.16. Use the identity $\boldsymbol{\nabla}^2(fg) = f\boldsymbol{\nabla}^2 g + g\boldsymbol{\nabla}^2 f + 2(\boldsymbol{\nabla} f) \cdot (\boldsymbol{\nabla} g)$ with $f = g = \Phi$. Then we find

$$\boldsymbol{\nabla}^2 \Psi = \frac{k}{2}\left[2\Phi \boldsymbol{\nabla}^2 \Phi + 2(\boldsymbol{\nabla}\Phi) \cdot (\boldsymbol{\nabla}\Phi)\right]\,,$$

which satisfies the heat conduction equation because $\boldsymbol{\nabla}^2 \Phi = 0$.

3.6.17. Start by forming the matrix $\mathsf{M}\cdot\boldsymbol{\nabla}$. We obtain

$$\mathsf{M}\cdot\boldsymbol{\nabla}=\begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t} & -i\frac{\partial}{\partial z} & i\frac{\partial}{\partial y}\\ i\frac{\partial}{\partial z} & \frac{1}{c}\frac{\partial}{\partial t} & -i\frac{\partial}{\partial x}\\ -i\frac{\partial}{\partial y} & i\frac{\partial}{\partial x} & \frac{1}{c}\frac{\partial}{\partial t}\end{pmatrix}.$$

Apply this matrix to the vector ψ. The result (after multiplication by c) is

$$c\mathsf{M}\cdot\boldsymbol{\nabla}\psi=\begin{pmatrix}\frac{\partial B_x}{\partial t}-\frac{\partial E_y}{\partial z}+\frac{\partial E_z}{\partial y}+i\left[-\frac{1}{c^2}\frac{\partial E_x}{\partial t}-\frac{\partial B_y}{\partial z}+\frac{\partial B_z}{\partial y}\right]\\ \frac{\partial B_y}{\partial t}-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}+i\left[-\frac{1}{c^2}\frac{\partial E_y}{\partial t}-\frac{\partial B_z}{\partial x}+\frac{\partial B_x}{\partial z}\right]\\ \frac{\partial B_z}{\partial t}-\frac{\partial E_x}{\partial y}+\frac{\partial E_y}{\partial x}+i\left[-\frac{1}{c^2}\frac{\partial E_z}{\partial t}-\frac{\partial B_x}{\partial y}+\frac{\partial B_y}{\partial x}\right]\end{pmatrix}.$$

Equating to zero the real and imaginary parts of all components of the above vector, we recover two Maxwell equations.

3.6.18. By direct matrix multiplication we verify this equation.

3.7 Vector Integration

3.7.1. A triangle ABC has area $\frac{1}{2}|\mathbf{B}-\mathbf{A}|\,|\mathbf{C}-\mathbf{A}|\sin\theta$, where θ is the angle between $\mathbf{B}-\mathbf{A}$ and $\mathbf{C}-\mathbf{A}$. This area can be written $|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})|/2$. Expanding,

$$\text{Area } ABC=|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}|/2\,.$$

Applying this formula to OAB, we get just $|\mathbf{A}\times\mathbf{B}|/2$. Continuing to the other three faces, the total area is

$$\text{Area}=\frac{|\mathbf{A}\times\mathbf{B}|+|\mathbf{B}\times\mathbf{C}|+|\mathbf{C}\times\mathbf{A}|+|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}|}{2}\,.$$

3.7.2. Let us parameterize the circle C as $x=\cos\varphi, y=\sin\varphi$ with the polar angle φ so that $dx=-\sin\varphi\,d\varphi$, $dy=\cos\varphi\,d\varphi$. Then the force can be written as $\mathbf{F}=-\hat{\mathbf{x}}\sin\varphi+\hat{\mathbf{y}}\cos\varphi$. The work becomes

$$-\int_C\frac{xdy-ydx}{x^2+y^2}=\int_0^{-\pi}(-\sin^2\varphi-\cos^2\varphi)\,d\varphi=\pi.$$

Here we spend energy. If we integrate counterclockwise from $\varphi=0$ to π we find the value $-\pi$, because we are riding with the force. The work is path dependent which is consistent with the physical interpretation that

$\mathbf{F} \cdot d\mathbf{r} \sim xdy - ydx = L_z$ is proportional to the z-component of orbital angular momentum (involving circulation, as discussed in Section 3.5).

If we integrate along the square through the points $(\pm 1, 0), (0, -1)$ surrounding the circle we find for the clockwise lower half square path

$$
\begin{aligned}
-\int \mathbf{F} \cdot d\mathbf{r} &= -\int_0^{-1} F_y dy|_{x=1} - \int_1^{-1} F_x dx|_{y=-1} - \int_{-1}^{0} F_y dy|_{x=-1} \\
&= \int_0^1 \frac{dy}{1+y^2} + \int_{-1}^{1} \frac{dx}{x^2+(-1)^2} + \int_{-1}^{0} \frac{dy}{(-1)^2+y^2} \\
&= \arctan(1) + \arctan(1) - \arctan(-1) - \arctan(-1) = 4 \cdot \frac{\pi}{4} = \pi,
\end{aligned}
$$

which is consistent with the circular path.

3.7.3. The answer depends upon the path that is chosen. A simple possibility is to move in the x direction from (1,1) to (3,1) and then in the y direction from (3,1) to (3,3). The work is the integral of $\mathbf{F} \cdot d\mathbf{s}$. For the first segment of the path the work is $\int F_x \, dx$; for the second segment it is $\int F_y \, dy$. These correspond to the specific integrals

$$
w_1 = \int_1^3 (x-1)\, dx = \left. \frac{x^2}{2} - x \right|_1^3 = 2, \quad w_2 = \int_1^3 (3+y)\, dy = \left. 3y + \frac{y^2}{2} \right|_1^3 = 10.
$$

3.7.4. Zero.

3.7.5.
$$
\begin{aligned}
\frac{1}{3}\int \mathbf{r} \cdot d\boldsymbol{\sigma} &= \frac{x}{3}\int dydz + \frac{y}{3}\int dzdx + \frac{z}{3}\int dxdy \\
&= \frac{1}{3}\int_0^1 dy \int_0^1 dz + \cdots = \frac{3}{3} = 1.
\end{aligned}
$$

Here the factor x in the first term is constant and therefore outside the integral; it is 0 for one face of the cube and unity for the opposite one. Similar remarks apply to the factors y, z in the other two terms which contribute equally.

3.8 Integral Theorems

3.8.1. For a constant vector $\mathbf{a}$, its divergence is zero. Using Gauss' theorem we have

$$
0 = \int_V \boldsymbol{\nabla} \cdot \mathbf{a} d\tau = \mathbf{a} \cdot \int_S d\boldsymbol{\sigma},
$$

where S is the closed surface of the finite volume V. As $\mathbf{a} \neq 0$ is arbitrary, $\int_S d\boldsymbol{\sigma} = 0$ follows.

3.8.2. From $\nabla \cdot \mathbf{r} = 3$ in Gauss' theorem we have

$$\int_V \nabla \cdot \mathbf{r} d\tau = 3 \int_V d\tau = 3V = \int_S \mathbf{r} \cdot d\boldsymbol{\sigma},$$

where V is the volume enclosed by the closed surface S.

3.8.3. Cover the closed surface by small (in general curved) adjacent rectangles S_i whose circumference are formed by four lines L_i each. Then Stokes' theorem gives

$$\int_S (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \sum_i \int_{S_i} (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \sum_i \int_{L_i} \mathbf{A} \cdot d\mathbf{l} = 0$$

because all line integrals cancel each other.

3.8.4. Apply Gauss' theorem to $\nabla \cdot (\varphi \mathbf{E}) = \nabla \varphi \cdot \mathbf{E} + \varphi \nabla \cdot \mathbf{E} = -\mathbf{E}^2 + \varepsilon_0^{-1} \varphi \rho$, where $\int_{S \to \infty} \varphi \mathbf{E} \cdot d\boldsymbol{\sigma} = 0$.

3.8.5. First, show that $J_i = \nabla \cdot (x\mathbf{J})$ by writing

$$\nabla \cdot (x\mathbf{J}) = x \nabla \cdot \mathbf{J} + (\nabla x) \cdot \mathbf{J} = 0 + \hat{\mathbf{e}}_x \cdot \mathbf{J} = J_x \,.$$

Since $\mathbf{J}$ is zero on the boundary, so is $x\mathbf{J}$, so by Gauss' theorem we have $\int \nabla \cdot (x\mathbf{J}) d\tau = 0$, equivalent to $\int J_x \, d\tau = 0$.

3.8.6. By direct calculation we can find that $\nabla \times t = 2\mathbf{e}_z$. Then, by Stokes' theorem, the line integral has the value $2A$.

3.8.7. (a) As $\mathbf{r} \times d\mathbf{r}/2$ is the area of the infinitesimal triangle, $\oint \mathbf{r} \times d\mathbf{r}$ is twice the area of the loop.

(b) From $d\mathbf{r} = (-\hat{\mathbf{x}} a \sin\theta + \hat{\mathbf{y}} b \cos\theta) d\theta$ and $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ we obtain $\mathbf{r} \times d\mathbf{r} = \hat{\mathbf{z}} ab(\cos^2\theta + \sin^2\theta)$ and $\oint \mathbf{r} \times d\mathbf{r} = \hat{\mathbf{z}} ab \int_0^{2\pi} d\theta = \hat{\mathbf{z}} 2ab\pi$.

3.8.8. We evaluate the surface integral with $\mathbf{P} = \mathbf{r}$. Note that $d\boldsymbol{\sigma} = \hat{\mathbf{e}}_z \, dA$, and that, evaluating components,

$$d\boldsymbol{\sigma} \times \nabla = \left[-\hat{\mathbf{e}}_x \frac{\partial}{\partial y} + \hat{\mathbf{e}}_y \frac{\partial}{\partial x} \right] .$$

Then form $(d\boldsymbol{\sigma} \times \nabla) \times \mathbf{r}$. The x and y components of this expression vanish; the z component is

$$\left(-\frac{\partial}{\partial y} \right) y - \left(\frac{\partial}{\partial x} \right) x = -2 \,.$$

The surface integral then has the value $-2A$, where A is the area of the loop. Note that the alternate form of Stokes' theorem equates this surface integral to $-\oint \mathbf{r} \times d\mathbf{r}$.

3.8.9. This follows from integration by parts shifting $\boldsymbol{\nabla}$ from v to u. The integrated term cancels for a closed loop.

3.8.10. Use the identity of Exercise 3.8.9, i.e. $\oint \boldsymbol{\nabla}(uv)\cdot d\boldsymbol{\lambda} = 0$, and apply Stokes' theorem to

$$2\int_S u\boldsymbol{\nabla}v\cdot d\boldsymbol{\sigma} = \int (u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u)\cdot d\boldsymbol{\lambda}$$
$$= \int_S \boldsymbol{\nabla}\times(u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u)\cdot d\boldsymbol{\sigma} = 2\int_S (\boldsymbol{\nabla}u\times\boldsymbol{\nabla}v)\cdot d\boldsymbol{\sigma}.$$

3.8.11. Starting with Gauss' theorem written as

$$\oint_{\partial V}\mathbf{B}\cdot d\boldsymbol{\sigma} = \int_V \boldsymbol{\nabla}\cdot\mathbf{B}\,d\tau,$$

substitute $\mathbf{B} = \mathbf{a}\times\mathbf{P}$, where $\mathbf{a}$ is a constant vector and $\mathbf{P}$ is arbitrary. The left-hand integrand then becomes $(\mathbf{a}\times\mathbf{P})\cdot d\boldsymbol{\sigma} = (\mathbf{P}\times d\boldsymbol{\sigma})\cdot\mathbf{a}$. The right-hand integrand expands into $\mathbf{P}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) - \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{P})$, the first term of which vanishes because $\mathbf{a}$ is a constant vector. Our Gauss' theorem equation can then be written

$$\mathbf{a}\cdot\oint_{\partial V}\mathbf{P}\times d\boldsymbol{\sigma} = -\mathbf{a}\cdot\int_V \boldsymbol{\nabla}\times\mathbf{P}\,d\tau\,.$$

Rearranging to

$$\mathbf{a}\cdot\left[\oint_{\partial V}\mathbf{P}\times\boldsymbol{\sigma} + \int_V \boldsymbol{\nabla}\times\mathbf{P}\,d\tau\right] = 0,$$

we note that because the constant direction of $\mathbf{a}$ is arbitrary the quantity in square brackets must vanish; its vanishing is equivalent to the relation to be proved.

3.8.12. Start from Stokes' theorem,

$$\int_S (\boldsymbol{\nabla}\times\mathbf{B}\cdot d\boldsymbol{\sigma} = \oint_{\partial S}\mathbf{B}\cdot d\mathbf{r}$$

and substitute $\mathbf{B} = \varphi\,\mathbf{a}$, where $\mathbf{a}$ is a constant vector and φ is an arbitrary scalar function. Because $\mathbf{a}$ is constant, the quantity $\boldsymbol{\nabla}\times\varphi\,\mathbf{a}$ reduces to $(\boldsymbol{\nabla}\varphi)\times\mathbf{a}$, and the left-side integrand is manipulated as follows:

$$(\boldsymbol{\nabla}\varphi)\times\mathbf{a}\cdot d\boldsymbol{\sigma} = (d\boldsymbol{\sigma}\times\boldsymbol{\nabla}\varphi)\cdot\mathbf{a}.$$

The Stokes' theorem formula can then be written

$$\mathbf{a}\cdot\int_S d\boldsymbol{\sigma}\times\boldsymbol{\nabla}\varphi = \mathbf{a}\cdot\oint_{\partial S}\varphi\,d\mathbf{r}.$$

Because $\mathbf{a}$ is arbitrary in direction, the integrals on the two sides of this equation must be equal, proving the desired relation.

3.8.13. Starting from Stokes' theorem as written in the solution to Exercise 3.8.12, set $\mathbf{B} = \mathbf{a} \times \mathbf{P}$. This substitution yields

$$\int_S (\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{P})) \cdot d\boldsymbol{\sigma} = \oint_{\partial S} (\mathbf{a} \times \mathbf{P}) \cdot d\mathbf{r}$$

Applying vector identities and remembering that $\mathbf{a}$ is a constant vector, the left- and right-side integrands can be manipulated so that this equation becomes

$$-\int_S \mathbf{a} \cdot ((d\boldsymbol{\sigma} \times \boldsymbol{\nabla}) \times \mathbf{P}) = \oint_{\partial S} (\mathbf{P} \times d\mathbf{r}) \cdot \mathbf{a}.$$

Bringing $\mathbf{a}$ outside the integrals and rearranging, we reach

$$\mathbf{a} \cdot \left[\int_S (d\boldsymbol{\sigma} \times \boldsymbol{\nabla}) \times \mathbf{P} - \oint_{\partial S} d\mathbf{r} \times \mathbf{P} \right] = 0\,.$$

Since the direction of $\mathbf{a}$ is arbitrary, the quantity within the square brackets vanishes, thereby confirming the desired relation.

3.9 Potential Theory

3.9.1. The solution is given in the text.

3.9.2. $\varphi(r) = \dfrac{Q}{4\pi\,\varepsilon_0 r}, \quad a \le r < \infty,$

$$\varphi(r) = \frac{Q}{4\pi\,\varepsilon_0 a}\left[\frac{3}{2} - \frac{1}{2}\frac{r^2}{a^2}\right], \quad 0 \le r \le a.$$

3.9.3. The gravitational acceleration in the z-direction relative to the Earth's surface is

$$-\frac{GM}{(R+z)^2} + \frac{GM}{R^2} \sim 2z\frac{GM}{R^3} \quad \text{for } 0 \le z \ll R.$$

Thus, $F_z = 2z\dfrac{GmM}{R^3}$, and

$$F_x = -x\frac{GmM}{(R+x)^3} \sim -x\frac{GmM}{R^3}, \qquad F_y = -y\frac{GmM}{(R+x)^3} \sim -y\frac{GmM}{R^3}.$$

Integrating $\mathbf{F} = -\boldsymbol{\nabla}V$ yields the potential

$$V = \frac{GmM}{R^3}\left(z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) = \frac{GmMr^2}{2R^3}(3z^2 - r^2) = \frac{GmMr^2}{R^3}P_2(\cos\theta).$$

3.9.4. The answer is given in the text.

3.9.5. The answer is given in the text.

3.9.6. $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ for constant $\mathbf{B}$ implies

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{2}\mathbf{B}\boldsymbol{\nabla} \cdot \mathbf{r} - \frac{1}{2}\mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{r} = \left(\frac{3}{2} - \frac{1}{2}\right)\mathbf{B}.$$

3.9.7. (a) This is proved in Exercise 3.6.14.

(b) $2\boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times (u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u) = \boldsymbol{\nabla}u \times \boldsymbol{\nabla}v - \boldsymbol{\nabla}v \times \boldsymbol{\nabla}u = 2\boldsymbol{\nabla}u \times \boldsymbol{\nabla}v.$

3.9.8. If $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda$, then

$$\mathbf{B}' = \boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\Lambda = \mathbf{B}$$

because $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Lambda = 0$, and

$$\oint \mathbf{A}' \cdot d\mathbf{r} = \oint \mathbf{A} \cdot d\mathbf{r} + \oint \boldsymbol{\nabla}\Lambda \cdot d\mathbf{r} = \oint \mathbf{A} \cdot d\mathbf{r}$$

because $\int_a^b \boldsymbol{\nabla}\Lambda \cdot d\mathbf{r} = \Lambda|_a^b = 0$ for $b = a$ in a closed loop.

3.9.9. Using Green's theorem as suggested in the problem and the formula for the Laplacian of $1/r$ (where r is the distance from $\mathbf{P}$), the volume integral of Green's theorem reduces to

$$\int_V (-\varphi)\,\boldsymbol{\nabla}^2\left(\frac{1}{r}\right) d\tau = \int_V (-\varphi)\,[-4\pi\delta(\mathbf{r})]\,d\tau = 4\pi\varphi(\mathbf{P})\,.$$

The surface integrals, for a sphere of radius a centered at $\mathbf{P}$, are

$$\int_S \left[\frac{1}{a}\boldsymbol{\nabla}\varphi - \varphi\,\boldsymbol{\nabla}\left(\frac{1}{r}\right)\right] d\boldsymbol{\sigma}\,.$$

Using $\boldsymbol{\nabla}(1/r) = -\hat{\mathbf{e}}_r/r^2$, the second term of the surface integral yields 4π times $\langle\varphi\rangle$, the average of φ on the sphere. The first surface-integral term vanishes by Gauss' theorem because $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi$ vanishes everywhere within the sphere. We thus have the final result $4\pi\varphi_0 = 4\pi\langle\varphi\rangle$.

3.9.10. Use $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \varepsilon\mathbf{E}$ with $\dfrac{\partial \mathbf{E}}{\partial t} = 0$ in

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})/\mu = (\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{A} - \boldsymbol{\nabla}^2\mathbf{A})/\mu = \mathbf{J}$$

so that $-\boldsymbol{\nabla}^2\mathbf{A} = \mu\mathbf{J}$ follows.

3.9.11. Start from Maxwell's equation for $\boldsymbol{\nabla} \times \mathbf{B}$ and substitute for the fields $\mathbf{B}$ and $\mathbf{E}$ in terms of the potentials $\mathbf{A}$ and φ. The relevant equations are

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad \mathbf{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\nabla\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \mu_0\mathbf{J}$$

Next manipulate the left-hand side using Eqs. (3.70) and (3.109):

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A} = -\nabla^2 \mathbf{A} - \nabla \left(\frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right).$$

Inserting this result for $\nabla \times (\nabla \times \mathbf{A})$, the terms in $\partial\varphi/\partial t$ cancel and the desired formula is obtained.

3.9.12. Evaluate the components of $\nabla \times \mathbf{A}$.

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y}$$

$$= -\frac{\partial}{\partial y} \left[\int_{x_0}^{x} B_y(x, y_0, z)\, dx - \int_{y_0}^{y} B_x(x, y, z)\, dy \right] = 0 + B_x(x, y, z)\,,$$

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\frac{\partial}{\partial z} \int_{y_0}^{y} B_z(x, y, z)\, dy$$

$$+ \frac{\partial}{\partial x} \left[\int_{x_0}^{x} B_y(x, y_0, z)\, dx - \int_{y_0}^{y} B_x(x, y, z)\, dy \right]$$

$$= -\int_{y_0}^{y} \frac{\partial B_z}{\partial z}\, dy + B_y(x, y_0, z) - \int_{y_0}^{y} \frac{\partial B_x}{\partial x}\, dy\,.$$

The evaluation of $(\nabla \times \mathbf{A})_y$ is now completed by using the fact that $\nabla \cdot \mathbf{B} = 0$, so we continue to

$$(\nabla \times \mathbf{A})_y = \int_{y_0}^{y} \frac{\partial B_y}{\partial y}\, dy + B_y(x, y_0, z) = B_y(x, y, z),$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y} \int_{y_0}^{y} B_z(x, y, z)\, dy = B_z(x, y, z).$$

3.10 Curvilinear Coordinates

3.10.1. (a) In the xy-plane different u, v values describe a family of hyperbolas in the first and third quadrants with foci along the diagonal $x = y$ and asymptotes given by $xy = u = 0$, i.e. the x- and y-axes, and orthogonal hyperbolas with foci along the x-axis with asymptotes given by $v = 0$, i.e. the lines $x \pm y$. The values $z =$constant describe a family of planes parallel to the xy-plane.

(c) For $u =$const. and $v =$const. we get from $x^2 - y^2 = v, xdx - ydy = 0$, or $dx/dy = y/x, dy/dx = x/y$. Thus, on the x-axis these hyperbolas have a vertical tangent. Similarly $xy = u =$const. gives $xdy + ydx = 0$, or $dy/dx = -y/x$. The product of these slopes is equal to -1, which proves orthogonality. Alternately, from $ydx + xdy = du, 2xdx + 2ydy = dv$ we get by squaring and adding that $(x^2 + y^2)(dx^2 + dy^2) = du^2 + dv^2/4$. Here, the mixed terms $dudv, dxdy$ drop out, proving again orthogonality.

(d) The uvz-system is left-handed. This follows from the negative Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{x^2 + y^2}.$$

To prove this, we differentiate the hyperbolas with respect to u and v giving, respectively,

$$\begin{aligned} y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} &= 1, \quad y\frac{\partial x}{\partial v} + x\frac{\partial y}{\partial v} = 0, \\ x\frac{\partial x}{\partial u} - y\frac{\partial y}{\partial u} &= 0, \quad x\frac{\partial x}{\partial v} - y\frac{\partial y}{\partial v} = \frac{1}{2}. \end{aligned}$$

Solving for the partials we obtain

$$\frac{\partial x}{\partial u} = \frac{y}{x^2 + y^2} = \frac{y}{x}\frac{\partial y}{\partial u}, \qquad \frac{\partial x}{\partial v} = \frac{x}{2(x^2 + y^2)} = -\frac{x}{y}\frac{\partial y}{\partial u}.$$

From these we find the Jacobian given above. The coordinate vectors are

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = \frac{\partial x}{\partial u}\left(1, \frac{x}{y}\right), \qquad \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = \frac{\partial x}{\partial v}\left(1, -\frac{y}{x}\right).$$

3.10.2. These elliptical cylinder coordinates can be parameterized as

$$x = c\cosh u \cos v, \qquad y = c\sinh u \sin v, \qquad z = z,$$

(using c instead of a). As we shall see shortly, the parameter $2c > 0$ is the distance between the foci of ellipses centered at the origin of the x, y-plane and described by different values of u =const. Their major and minor half-axes are respectively $a = c\cosh u$ and $b = c\sinh u$. Since

$$\frac{b}{a} = \tanh u = \sqrt{1 - \frac{1}{\cosh^2 u}} = \sqrt{1 - \varepsilon^2},$$

the eccentricity $\varepsilon = 1/\cosh u$, and the distance between the foci $2a\varepsilon = 2c$, proving the statement above. As $u \to \infty, \varepsilon \to 0$ so that the ellipses become circles. As $u \to 0$, the ellipses become more elongated until, at $u = 0$, they shrink to the line segment between the foci. Different values of v =const. describe a family of hyperbolas. To show orthogonality of the ellipses and hyperbolas we square and add the coordinate differentials

$$dx = c\sinh u \cos v du - c\cosh u \sin v dv,$$

$$dy = c\cosh u \sin v du + c\sinh u \cos v dv,$$

to obtain

$$dx^2 + dy^2 = c^2(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v)(du^2 + dv^2)$$
$$= c^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2).$$

Since there is no cross term $dudv$, these coordinates are locally orthogonal. Differentiating the ellipse and hyperbola equations with respect to u and v we can determine $\partial x/\partial u, \ldots$, just as in Exercise 3.10.1, and obtain the coordinate vectors $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$.

3.10.3. From the component definition (projection) $\mathbf{a} = \sum_i \hat{\mathbf{q}}_i \mathbf{a} \cdot \hat{\mathbf{q}}_i \equiv \sum_i a_{q_i} \hat{\mathbf{q}}_i$ and a similar expression for $\mathbf{b}$, get

$$\mathbf{a} \cdot \mathbf{b} = \sum_{ij} \hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j \mathbf{a} \cdot \hat{\mathbf{q}}_i \mathbf{b} \cdot \hat{\mathbf{q}}_j = \sum_i \mathbf{a} \cdot \hat{\mathbf{q}}_i \mathbf{b} \cdot \hat{\mathbf{q}}_i = \sum_i a_{q_i} b_{q_i}$$

using orthogonality, i.e. $\hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j = \delta_{ij}$.

3.10.4. (a) From Eq. (3.141) with $\hat{\mathbf{e}}_1 = \hat{\mathbf{q}}_1$ and $(\hat{\mathbf{e}}_1)_1 = 1$, $(\hat{\mathbf{e}}_1)_2 = (\hat{\mathbf{e}}_1)_3 = 0$, we get

$$\boldsymbol{\nabla} \cdot \hat{\mathbf{e}}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial (h_2 h_3)}{\partial q_1}.$$

(b) From Eq. (3.143) with $h_2 V_2 \to 0$, $h_3 V_3 \to 0$, we get

$$\boldsymbol{\nabla} \times \hat{\mathbf{e}}_1 = \frac{1}{h_1} \left[\hat{\mathbf{e}}_2 \frac{1}{h_3} \frac{\partial h_1}{\partial q_3} - \hat{\mathbf{e}}_3 \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \right].$$

3.10.5. This problem assumes that the unit vectors $\hat{\mathbf{q}}_i$ are orthogonal. From $d\mathbf{r} = \dfrac{\partial \mathbf{r}}{\partial q_i} dq_i$ we see that the $\dfrac{\partial \mathbf{r}}{\partial q_i}$ are tangent vectors in the directions $\hat{\mathbf{e}}_i = \hat{\mathbf{q}}_i$ with lengths h_i. This establishes the first equation of this problem.

Writing (for any i)

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = \frac{1}{h_i^2} \left[\frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_i} \right] = \frac{1}{h_i^2} \left[\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2 \right] = 1\,,$$

we confirm the formula for h_i.

If we now differentiate $h_i \hat{\mathbf{e}}_i = \partial \mathbf{r}/\partial q_i$ with respect to q_j (with $j \neq i$) and note that the result is symmetric in i and j, we get

$$\frac{\partial (h_i \hat{\mathbf{e}}_i)}{\partial q_j} = \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_j} = \frac{\partial (h_j \hat{\mathbf{e}}_j)}{\partial q_i}\,.$$

Expanding the differentiations of the left and right members of this equation and equating the results,

$$\frac{\partial h_i}{\partial q_j} \hat{\mathbf{e}}_i + h_i \frac{\partial \hat{\mathbf{e}}_i}{\partial q_j} = \frac{\partial h_j}{\partial q_i} \hat{\mathbf{e}}_j + h_j \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}.$$

Since $\partial \hat{\mathbf{e}}_i / \partial q_j$ must be a vector in the $\hat{\mathbf{e}}_j$ direction, we are able to establish the second equation of the exercise.

To prove the last relation, we differentiate $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = 1$ and $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0$ with respect to q_i. We find

$$\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} = 0, \qquad \frac{\partial \hat{\mathbf{e}}_i}{\partial q_i} \cdot \hat{\mathbf{e}}_j = -\hat{\mathbf{e}}_i \cdot \frac{\partial \hat{\mathbf{e}}_j}{\partial q_i}\,.$$

These equations show that $\partial \hat{\mathbf{e}}_i / \partial q_i$ has no component in the $\hat{\mathbf{e}}_i$ direction and that its components in the $\hat{\mathbf{e}}_j$ directions are $-\hat{\mathbf{e}}_i \cdot \partial \hat{\mathbf{e}}_j / \partial q_i$. Using the second formula to write these derivatives in terms of the h_i, we reach the final equation of this exercise.

3.10.6. The solution is given in the text.

3.10.7. The solution is given in the text.

3.10.8. Using the formulas from Exercise 3.10.5, with $h_\rho = h_z = 1$ and $h_\varphi = \rho$, nonzero terms only result if the h_i being differentiated is h_φ, and then only if differentiated with respect to ρ. These conditions cause all the first derivatives of the unit vectors to vanish except for the two cases listed in the exercise; those cases are straightforward applications of the formulas.

3.10.9. The formula given in the exercise is incorrect because it neglects the φ-dependence of $\hat{\mathbf{e}}_\rho$. When this is properly included, instead of $\partial V_\rho / \partial \rho$ we get $\rho^{-1} \partial(\rho V_\rho)/\partial \rho$.

3.10.10. (a) $\mathbf{r} = (x, y, z) = (x, y) + z\hat{\mathbf{z}} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$.

(b) From Eq. (3.148) we have

$$\boldsymbol{\nabla} \cdot \mathbf{r} = \frac{1}{\rho}\frac{\partial \rho^2}{\partial \rho} + \frac{\partial z}{\partial z} = 2 + 1 = 3.$$

From Eq. (3.150) with $V_\rho = \rho$, $V_\varphi = 0, V_z = z$ we get $\boldsymbol{\nabla} \times \mathbf{r} = 0$.

3.10.11. (a) The points x, y, z and $-x, -y, -z$ have the same value of ρ, values of z of opposite sign, and if $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, then $-x$ and $-y$ must have a value of φ displaced from the original φ value by π.

(b) A unit vector $\hat{\mathbf{e}}_z$ will always be in the same (the $+z$) direction, but the change by π in φ will cause the $\hat{\mathbf{e}}_\rho$ unit vector to change sign under inversion. The same is true of $\hat{\mathbf{e}}_\varphi$.

3.10.12. The solution is given in the text.

3.10.13. The solution is given in the text.

3.10.14. Using $V_z \equiv 0$ we obtain

$$\nabla \times \mathbf{V}|_\rho = \frac{1}{\rho}\frac{\partial(\rho V_\varphi(\rho,\varphi))}{\partial z} = 0,$$

$$\nabla \times \mathbf{V}|_\varphi = \frac{1}{\rho}\frac{\partial(V_\rho(\rho,\varphi))}{\partial z} = 0,$$

$$\nabla \times \mathbf{V}|_z = \frac{1}{\rho}\left(\frac{\partial(\rho V_\varphi(\rho,\varphi))}{\partial \rho} - \frac{\partial V_\rho(\rho,\varphi)}{\partial \varphi}\right).$$

3.10.15. The solution is given in the text.

3.10.16. (a) $\mathbf{F} = \hat{\varphi}\frac{1}{\rho}$.

(b) $\nabla \times \mathbf{F} = 0,\ \rho \neq 0$.

(c) $\displaystyle\int_0^{2\pi} \mathbf{F}\cdot\hat{\varphi}\rho d\varphi = 2\pi$.

(d) $\nabla \times \mathbf{F}$ is not defined at the origin. A cut line from the origin out to infinity (in any direction) is needed to prevent one from encircling the origin. The scalar potential $\psi = \varphi$ is not single-valued.

3.10.17. The solution is given in the text.

3.10.18. The solution is given in the text.

3.10.19. Resolving the unit vectors of spherical polar coordinates into Cartesian components was accomplished in Exercise 3.10.18 involving an orthogonal matrix. The inverse is the transpose matrix, i.e.

$$\begin{aligned}
\hat{\mathbf{x}} &= \hat{\mathbf{r}}\sin\theta\cos\varphi + \hat{\boldsymbol{\theta}}\cos\theta\cos\varphi - \hat{\boldsymbol{\varphi}}\sin\varphi,\\
\hat{\mathbf{y}} &= \hat{\mathbf{r}}\sin\theta\sin\varphi + \hat{\boldsymbol{\theta}}\cos\theta\sin\varphi + \hat{\boldsymbol{\varphi}}\cos\varphi,\\
\hat{\mathbf{z}} &= \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta.
\end{aligned}$$

3.10.20. (a) The transformation between Cartesian and spherical polar coordinates is not represented by a constant matrix, but by a matrix whose components depend upon the value of $\mathbf{r}$. A matrix equation of the indicated type has no useful meaning because the components of B depend upon both $\mathbf{r}$ and $\mathbf{r}'$.

(b) Using the fact that both the Cartesian and spherical polar coordinate systems are orthogonal, the transformation matrix between a Cartesian-component vector $\mathbf{A}$ and its spherical-polar equivalent $\mathbf{A}'$ must have the form $\mathbf{A}' = \mathsf{U}\mathbf{A}$, with

$$\mathsf{U} = \begin{pmatrix} \hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_y & \hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_y & \hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\varphi\cdot\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_\varphi\cdot\hat{\mathbf{e}}_y & \hat{\mathbf{e}}_\varphi\cdot\hat{\mathbf{e}}_z \end{pmatrix}$$

Using the data in Exercise 3.10.19, we have

$$\mathsf{U} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

Note that both $\mathbf{A}$ and $\mathbf{A}'$ are associated with the same point, whose angular coordinates are (θ, φ). To check orthogonality, transpose and check the product $\mathsf{U}^T\mathsf{U}$. We find $\mathsf{U}^T\mathsf{U} = \mathbf{1}$.

3.10.21. One way to proceed is to first obtain the transformation of a vector $\mathbf{A}$ to its representation $\mathbf{A}''$ in cylindrical coordinates. Letting V be the transformation matrix satisfying $\mathbf{A}'' = \mathsf{V}\mathbf{A}$, with

$$\mathsf{V} = \begin{pmatrix} \hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_\varphi \cdot \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z \end{pmatrix}$$

Using data given in the answer to Exercise 3.10.6, V evaluates to

$$\mathsf{V} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $\mathbf{A}$ and $\mathbf{A}''$ are associated with the same point, which has angular coordinate φ. We now convert from spherical polar to cylindrical coordinates in two steps, of which the first is from spherical polar to Cartesian coordinates, accomplished by the transformation U^T, the inverse of the transformation U of Exercise 3.10.20(b). We then apply transformation V to convert to cylindrical coordinates. The overall transformation matrix W is then the matrix product $\mathsf{V}\mathsf{U}^T$. Thus,

$$\mathsf{W} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

The inverse of this transformation is represented by the transpose of W.

3.10.22. (a) Differentiating $\hat{\mathbf{r}}^2 = 1$ we get

$$\frac{\partial \mathbf{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \hat{\mathbf{r}},$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = r(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = r\hat{\boldsymbol{\theta}},$$

$$\frac{\partial \mathbf{r}}{\partial \varphi} = r(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = r\sin\theta\hat{\boldsymbol{\varphi}}.$$

(b) With $\boldsymbol{\nabla}$ given by

$$\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

the alternate derivation of the Laplacian is given by dotting this $\boldsymbol{\nabla}$ into itself. In conjunction with the derivatives of the unit vectors above this gives

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \hat{\mathbf{r}}\cdot\frac{\partial}{\partial r}\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\cdot\frac{1}{r}\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\frac{\partial}{\partial r} + \hat{\boldsymbol{\varphi}}\cdot\frac{1}{r\sin\theta}\frac{\partial\hat{\mathbf{r}}}{\partial\varphi}\frac{\partial}{\partial r}$$

$$+ \hat{\boldsymbol{\varphi}}\cdot\frac{1}{r\sin\theta}\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\varphi}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\cdot\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\tan\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Note that, with

$$\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = \frac{\tan\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

we get the standard result using Exercise 3.10.34 for the radial part.

3.10.23. The solution is given in the text.

3.10.24. $V_\theta, V_\varphi \sim 1/r$.

3.10.25. (a) Since $r = \sqrt{x^2+y^2+z^2}$, changes of sign in x, y, and z leave r unchanged. Since $z \to -z$, $\cos\theta$ changes sign, converting θ into $\pi - \theta$. Sign changes in x and y require that both $\sin\varphi$ and $\cos\varphi$ change sign; this requires that φ change to $\varphi \pm \pi$.

(b) Since the coordinate point is after inversion on the opposite side of the polar axis, increases in r or φ correspond to displacements in directions opposite to their effect before inversion. Both before and after inversion, an increase in θ is in a direction tangent to the same circle of radius r that passes through both the north and south poles of the coordinate system. The two tangent directions are parallel because they are at opposite points of the circle, and both are in the southerly tangent direction. They are therefore in the same direction.

3.10.26. (a) $\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{r} = A_x\dfrac{\partial\mathbf{r}}{\partial x} + A_y\dfrac{\partial\mathbf{r}}{\partial y} + A_z\dfrac{\partial\mathbf{r}}{\partial z} = \mathbf{A}$ because

$$\frac{\partial\mathbf{r}}{\partial x} = \hat{\mathbf{x}}, \qquad \frac{\partial\mathbf{r}}{\partial y} = \hat{\mathbf{y}}, \qquad \frac{\partial\mathbf{r}}{\partial z} = \hat{\mathbf{z}}.$$

(b) Using $\dfrac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}}$, $\dfrac{\partial\hat{\mathbf{r}}}{\partial\varphi} = \sin\theta\,\hat{\boldsymbol{\varphi}}$ and $\boldsymbol{\nabla}$ in polar coordinates from Exercise 3.10.22 we get

$$\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{r} = \mathbf{A}\cdot\hat{\mathbf{r}}\frac{\partial\mathbf{r}}{\partial r} + \mathbf{A}\cdot\hat{\boldsymbol{\theta}}\frac{\partial\hat{\mathbf{r}}}{\partial\theta} + \frac{\mathbf{A}\cdot\hat{\boldsymbol{\varphi}}}{\sin\theta}\frac{\partial\hat{\mathbf{r}}}{\partial\varphi}$$

$$= A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}} = \mathbf{A}.$$

3.10.27. The solution is given in the text.

3.10.28. The solution is given in the text.

3.10.29. From Exercise 3.10.32 and using the Cartesian decomposition in Exercise 3.10.18 $\hat{\theta}_z = -\sin\theta$ we get

$$L_z = -i\,\frac{\sin\theta}{\sin\theta}\frac{\partial}{\partial\varphi}.$$

3.10.30. Use Exercise 3.10.32 to get this result.

3.10.31. Solving this problem directly in spherical coordinates is somewhat challenging. From the definitions of the unit vectors, one can establish

$$\frac{\partial\hat{\mathbf{e}}_\theta}{\partial\varphi} = \cos\theta\hat{\mathbf{e}}_\varphi\,, \qquad \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\varphi} = -\sin\theta\,\hat{\mathbf{e}}_r - \cos\theta\hat{\mathbf{e}}_\theta\,,$$

$$\frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\hat{\mathbf{e}}_r, \qquad \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\theta} = 0\,,$$

$$\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\varphi, \quad \hat{\mathbf{e}}_\theta\times\hat{\mathbf{e}}_\varphi = \hat{\mathbf{e}}_r, \quad \hat{\mathbf{e}}_\varphi\times\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta.$$

We now write $\mathbf{L}\times\mathbf{L}$ and expand it into its four terms, which we process individually. When a unit vector is to be differentiated, the differentiation should be carried out before evaluating the cross product. This first term only has a contribution when the second $\hat{\mathbf{e}}_\theta$ is differentiated:

$$-\left(\frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(\frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) = -\left(\frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\right)\times\left(\frac{\partial\hat{\mathbf{e}}_\theta}{\partial\varphi}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$= -(\hat{\mathbf{e}}_\theta\times\hat{\mathbf{e}}_\varphi)\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}\,.$$

Next we process

$$-\left(\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\right)\left(\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\right) = -\hat{\mathbf{e}}_\varphi \times \left(\frac{\partial \hat{\mathbf{e}}_\varphi}{\partial\theta}\right)\frac{\partial}{\partial\theta} - (\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\varphi)\frac{\partial^2}{\partial\theta^2} = 0\,.$$

Then

$$\left(\frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\left(\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\right) = \frac{\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta}\frac{\hat{\mathbf{e}}_\theta}{\sin\theta} \times (-\hat{\mathbf{e}}_r \sin\theta - \hat{\mathbf{e}}_\theta \cos\theta)\frac{\partial}{\partial\theta}$$

$$= \frac{\hat{\mathbf{e}}_r}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} + \hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\,.$$

Finally,

$$\left(\hat{\mathbf{e}}_\varphi \frac{\partial}{\partial\theta}\right)\left(\frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) =$$

$$\frac{-\hat{\mathbf{e}}_r}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} - (\hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\theta)\left(-\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi} + (\hat{\mathbf{e}}_\varphi \times (-\hat{\mathbf{e}}_r)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\,.$$

Several of the terms in the above expressions cancel. The remaining terms correspond to $i\mathbf{L}$.

3.10.32. (a) Using

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

and $\mathbf{r} = r\hat{\mathbf{r}}$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{\theta}}$, we find

$$\mathbf{L} = -i(\mathbf{r} \times \boldsymbol{\nabla}) = -i\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\theta} - \hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}.$$

(b) Using Eq. (2.44), $\hat{\boldsymbol{\theta}}_z = -\sin\theta$ we find

$$L_z = -i\frac{\partial}{\partial\varphi},$$

and from $\hat{\boldsymbol{\theta}}_x = \cos\theta\cos\varphi$, $\hat{\boldsymbol{\varphi}}_x = -\sin\varphi$ we get

$$L_x = i\sin\varphi\frac{\partial}{\partial\theta} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi};$$

from $\hat{\boldsymbol{\theta}}_y = \cos\theta\sin\varphi$, $\hat{\boldsymbol{\varphi}}_y = \cos\varphi$ we get

$$L_y = -i\cos\varphi\frac{\partial}{\partial\theta} + i\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}.$$

(c) Squaring and adding gives the result.

3.10.33. (a) Using

$$\hat{\mathbf{r}} \times \mathbf{r} = 0 \quad \text{and} \quad \boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} - i\frac{\mathbf{r} \times \mathbf{L}}{r^2}$$

and the BAC-CAB rule we get

$$-i\mathbf{r} \times \boldsymbol{\nabla} = -\frac{1}{r^2}\mathbf{r} \times (\mathbf{r} \times \mathbf{L}) = -\frac{1}{r^2}(\mathbf{r} \cdot \mathbf{L}\mathbf{r} - r^2\mathbf{L}) = \mathbf{L}$$

because $\mathbf{L} \cdot \mathbf{r} = 0$.

(b) It suffices to verify the x-component of this equation. Substituting the formula for $\mathbf{L}$, the result to be proved is

$$\boldsymbol{\nabla} \times (\mathbf{r} \times \boldsymbol{\nabla}) = \mathbf{r}\boldsymbol{\nabla}^2 - \boldsymbol{\nabla}\left(1 + \mathbf{r} \cdot \boldsymbol{\nabla}\right) .$$

The x-component of the left-hand side expands into

$$\Big[\boldsymbol{\nabla} \times (\mathbf{r} \times \boldsymbol{\nabla})\Big]_x = \frac{\partial}{\partial y}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= x\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x} - y\frac{\partial^2}{\partial y \partial x} - \frac{\partial}{\partial x} - z\frac{\partial^2}{\partial z \partial x} + x\frac{\partial^2}{\partial z^2} .$$

The x-component of the right-hand side is

$$x\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] - \frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right] .$$

The left- and right-hand sides simplify to identical expressions.

3.10.34. From (a) $\dfrac{1}{r^2}\dfrac{d}{dr}r^2 = \dfrac{d}{dr} + \dfrac{2}{r}$ we get (c), and vice versa.

From the inner $\dfrac{d}{dr}r = r\dfrac{d}{dr} + 1$ in (b) we get $\dfrac{1}{r}\dfrac{d^2}{dr^2}r = \dfrac{1}{r}\dfrac{d}{dr} + \dfrac{d^2}{dr^2} + \dfrac{d}{dr}$, hence (c), and vice versa.

3.10.35. (a) $\nabla \times \mathbf{F} = 0, \quad r \geq P/2$.

(b) $\displaystyle\oint \mathbf{F} \cdot d\lambda = 0$. This suggests (but does not prove) that the force is conservative.

(c) Potential $= P\cos\theta/r^2$, dipole potential.

3.10.36. Solutions are given in the text.

3.10.37. $\mathbf{E}(\mathbf{r}) = \dfrac{3\hat{\mathbf{r}}(\mathbf{p} \cdot \hat{\mathbf{r}}) - \mathbf{p}}{4\pi\,\varepsilon_0 \mathrm{r}^3}$.

4. Tensors and Differential Forms

4.1 Tensor Analysis

4.1.1. This is a special case of Exercise 4.1.2 with $B^0_{ij} = 0$.

4.1.2. If $A^0_{ij} = B^0_{ij}$ in one frame of reference, then define a coordinate transformation from that frame to an arbitrary one: $x_i = x_i(x^0_j)$, so that

$$A_{ij} = \frac{\partial x_i}{\partial x^0_\alpha}\frac{\partial x_j}{\partial x^0_\beta}A^0_{\alpha\beta} = \frac{\partial x_i}{\partial x^0_\alpha}\frac{\partial x_j}{\partial x^0_\beta}B^0_{\alpha\beta} = B_{ij}.$$

4.1.3. Make a boost in the z-direction. If $A_z = A'_z = A^0 = 0$, then $A'^0 = 0$ in the boosted frame by the Lorentz transformation, etc.

4.1.4. Since

$$T'_{12} = \frac{\partial x'_i}{\partial x_1}\frac{\partial x'_k}{\partial x_2}T_{ik} = \cos\theta\sin\theta\; T_{11} + \cos^2\theta\; T_{12}\sin^2\theta\; T_{21} - \sin\theta\cos\theta\; T_{22}$$

we find $T'_{12} = T_{12}$ for a rotation by π, but $T_{12} = -T_{21}$ for a rotation by $\pi/2$. Isotropy demands $T_{21} = 0 = T_{12}$. Similarly all other off-diagonal components must vanish, and the diagonal ones are equal.

4.1.5. The four-dimensional fourth-rank Riemann–Christoffel curvature tensor of general relativity, R_{iklm} has $4^4 = 256$ components. The antisymmetry of the first and second pair of indices, $R_{iklm} = -R_{ikml} = -R_{kilm}$, reduces these pairs to 6 values each, i.e. $6^2 = 36$ components. They can be thought of as a 6×6 matrix. The symmetry under exchange of pair indices, $R_{iklm} = R_{lmik}$, reduces this matrix to $6 \cdot 7/2 = 21$ components. The Bianchi identity, $R_{iklm} + R_{ilmk} + R_{imkl} = 0$, reduces the independent components to 20 because it represents one constraint. Note that, upon using the permutation symmetries one can always make the first index equal to zero followed by the other indices which are all different from each other.

4.1.6. Each component has at least one repeated index and is therefore zero.

4.1.7. As the gradient transforms like a vector, it is clear that the gradient of a tensor field of rank n is a tensor of rank $n + 1$.

4.1.8. The contraction of two indices removes two indices, while the derivative adds one, so $(n + 1) - 2 = n - 1$.

4.1.9. The scalar product of the four-vectors

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \quad \text{and} \quad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$$

is the scalar $\partial^2 = \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$.

4.1.10. The double summation $K_{ij}A_iB_j$ is a scalar. That K_{ij} is a second-rank tensor follows from the quotient theorem.

4.1.11. Since $K_{ij}A_{jk} = B_{ik}$ is a second-rank tensor the quotient theorem tells us that K_{ij} is a second-rank tensor.

4.2 Pseudotensors, Dual Tensors

4.2.1. The direct product $\varepsilon_{ijk}C_{lm}$ is a tensor of rank 5. Contracting 4 indices leaves a tensor of rank 1, a vector. Inverting gives $C_{jk} = \varepsilon_{jki}C_i$, a tensor of rank 2.

4.2.2. The generalization of the totally antisymmetric ε_{ijk} from three to n dimensions has n indices. Hence the generalized cross product $\varepsilon_{ijk\ldots}A_iB_j$ is an antisymmetric tensor of rank $n-2 \neq 1$ for $n \neq 3$.

4.2.3. The solution is given in the text.

4.2.4. (a) As each δ_{ij} is isotropic, their direct product must be isotropic as well. This is valid for any order of the indices. The last statement implies (b) and (c).

4.2.5. The argument relating to Eq. (4.29) holds in two dimensions, too, with $\delta'_{ij} = \det(a)a_{ip}a_{jq}\delta_{pq}$. No contradiction arises because ε_{ij} is antisymmetric while δ_{ij} is symmetric.

4.2.6. $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $R = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ is a rotation, then

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

4.2.7. If $A_k = \frac{1}{2}\varepsilon_{ijk}B_{ij}$ with $B_{ij} = -B_{ji}$, then

$$2\varepsilon_{mnk}A_k = \varepsilon_{mnk}\varepsilon_{ijk} = (\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni})B_{ij} = B_{mn} - B_{nm} = 2B_{mn}.$$

4.3 Tensors in General Coordinates

4.3.1. The vector $\boldsymbol{\varepsilon}^i$ is completely specified by its projections onto the three linearly independent $\boldsymbol{\varepsilon}_k$, i.e., by the requirements that $\boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}_j = \delta^i_j$. Taking the form given in the exercise, we form

$$\boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}_i = \frac{(\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k) \cdot \boldsymbol{\varepsilon}_i}{(\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k) \cdot \boldsymbol{\varepsilon}_i} = 1\,, \qquad \boldsymbol{\varepsilon}^i \cdot \varepsilon_j = \frac{(\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k) \cdot \boldsymbol{\varepsilon}_j}{(\boldsymbol{\varepsilon}_j \times \boldsymbol{\varepsilon}_k) \cdot \boldsymbol{\varepsilon}_i} = 0\,,$$

the zero occurring because the three vectors in the scalar triple product are not linearly independent. The above equations confirm that $\boldsymbol{\varepsilon}_i$ is the contravariant version of $\boldsymbol{\varepsilon}_i$.

4.3.2. (a) From the defining formula, Eq. (4.40), the orthogonality of the $\boldsymbol{\varepsilon}_i$ implies that $g_{ij} = 0$ when $i \neq j$.

(b) See the answer to part (c).

(c) From Eq. (4.46) with the $\boldsymbol{\varepsilon}_i$ orthogonal, the $\boldsymbol{\varepsilon}^i$ must also be orthogonal and have magnitudes that are the reciprocals of the $\boldsymbol{\varepsilon}_i$. Then, from Eq. (4.47), the g^{ii} must be the reciprocals of the g_{ii}.

4.3.3. This exercise assumes use of the Einstein summation convention. Inserting the definitions of the $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}^i$ and evaluating the scalar products, we reach

$$(\boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}^j)(\boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}_k) =$$

$$\left(\frac{\partial q^i}{\partial x}\frac{\partial q^j}{\partial x} + \frac{\partial q^i}{\partial y}\frac{\partial q^j}{\partial y} + \frac{\partial q^i}{\partial z}\frac{\partial q^j}{\partial z}\right)\left(\frac{\partial x}{\partial q^j}\frac{\partial x}{\partial q^k}\frac{\partial q^i}{\partial x}\frac{\partial q^j}{\partial x} + \frac{\partial y}{\partial q^j}\frac{\partial y}{\partial q^k} + \frac{\partial z}{\partial q^j}\frac{\partial z}{\partial q^k}\right)$$

The term of the product arising from the first term of each factor has the form

$$\frac{\partial q^i}{\partial x}\frac{\partial q^j}{\partial x}\frac{\partial x}{\partial q^j}\frac{\partial x}{\partial q^k} = \frac{\partial q^i}{\partial x}\frac{\partial x}{\partial q^k}\sum_j \frac{\partial x}{\partial q^j}\frac{\partial q^j}{\partial x} = \frac{\partial q^i}{\partial x}\frac{\partial x}{\partial q^k},$$

where we have noted that the j summation is the chain-rule expansion for $\partial x/\partial x$, which is unity. The products arising from the second terms and third terms of both factors have analogous forms, and the sum of these "diagonal" terms is also a chain-rule expansion:

$$\frac{\partial q^i}{\partial x}\frac{\partial x}{\partial q^k} + \frac{\partial q^i}{\partial y}\frac{\partial y}{\partial q^k} + \frac{\partial q^i}{\partial z}\frac{\partial z}{\partial q^k} = \frac{\partial q^i}{\partial q^k} = \delta^i_k .$$

The remaining terms of the original product expression all reduce to zero; we illustrate with

$$\frac{\partial q^i}{\partial x}\frac{\partial q^j}{\partial x}\frac{\partial y}{\partial q^j}\frac{\partial y}{\partial q^k} = \frac{\partial q^i}{\partial x}\frac{\partial y}{\partial q^k}\sum_j \frac{\partial y}{\partial q^j}\frac{\partial q^j}{\partial x} .$$

Here the j summation is the chain-rule expansion of $\partial y/\partial x$ and therefore vanishes.

4.3.4. Starting from Eq. (4.54), $\Gamma^m_{jk} = \boldsymbol{\varepsilon}^m \cdot (\partial \boldsymbol{\varepsilon}_k/\partial q^j)$, we see that a proof that $\partial \boldsymbol{\varepsilon}_k/\partial q^j = \partial \boldsymbol{\varepsilon}_j/\partial q^k$ would also demonstrate that $\Gamma^m_{jk} = \Gamma^m_{kj}$. From the definition of $\boldsymbol{\varepsilon}_k$, we differentiate with respect to q^j, reaching

$$\frac{\partial \boldsymbol{\varepsilon}_k}{\partial q^j} = \frac{\partial^2 x}{\partial q^j \partial q^k}\,\hat{\mathbf{e}}_x + \frac{\partial^2 y}{\partial q^j \partial q^k}\,\hat{\mathbf{e}}_y + \frac{\partial^2 z}{\partial q^j \partial q^k}\,\hat{\mathbf{e}}_z .$$

Because the coordinates are differentiable functions the right-hand side of this equation is symmetric in j and k, indicating that j and k can be interchanged without changing the value of the left-hand side of the equation.

4.3.5. The covariant metric tensor is diagonal, with nonzero elements $g_{ii} = h_i^2$, so $g_{\rho\rho} = g_{zz} = 1$ and $g_{\varphi\varphi} = \rho^2$. The contravariant metric tensor is also diagonal, with nonzero elements that are the reciprocals of the g_{ii}. Thus, $g^{\rho\rho} = g^{zz} = 1$ and $g^{\varphi\varphi} = 1/\rho^2$.

4.3.6. Let s be the proper time on a geodesic and $u^\mu(s)$ the velocity of a mass in free fall. Then the scalar

$$\begin{aligned}\frac{d}{ds}(V \cdot u) = \frac{dV}{ds} \cdot u + V_\beta \frac{d^2x^\beta}{ds^2} &= \left(\partial_\mu V_\alpha \frac{dx^\mu}{ds}\right) u^\alpha - V_\beta \Gamma^\beta_{\alpha\mu} u^\alpha u^\mu \\ &= u^\mu u^\alpha \left(\partial_\mu V_\alpha - \Gamma^\beta_{\alpha\mu} V_\beta\right)\end{aligned}$$

involves the covariant derivative which is a four-vector by the quotient theorem. Note that the use of the geodesic equation for d^2x^β/ds^2 is the key here.

4.3.7. For this exercise we need the identity

$$\frac{\partial g_{ik}}{\partial q^j} = \boldsymbol{\varepsilon}_i \cdot \frac{\partial \boldsymbol{\varepsilon}_k}{\partial q^j} + \boldsymbol{\varepsilon}_k \cdot \frac{\partial \boldsymbol{\varepsilon}_i}{\partial q^j},$$

which can be proved by writing $g_{ik} = \boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_k$ and differentiating.

We now write

$$\begin{aligned}\frac{\partial V_i}{\partial q^j} = \frac{\partial(g_{ik}V^k)}{\partial q^j} &= g_{ik}\frac{\partial V^k}{\partial q^j} + V^k \frac{\partial g_{ik}}{\partial q^j} \\ &= g_{ik}\frac{\partial V^k}{\partial q^j} + V^k \boldsymbol{\varepsilon}_i \cdot \frac{\partial \boldsymbol{\varepsilon}_k}{\partial q^j} + V^k \boldsymbol{\varepsilon}_k \cdot \frac{\partial \boldsymbol{\varepsilon}_i}{\partial q^j} \\ &= g_{ik}\frac{\partial V^k}{\partial q^j} + V^k g_{il} \boldsymbol{\varepsilon}^l \cdot \frac{\partial \boldsymbol{\varepsilon}_k}{\partial q^j} + V_k \boldsymbol{\varepsilon}^k \cdot \frac{\partial \boldsymbol{\varepsilon}_i}{\partial q^j} \\ &= g_{ik}\frac{\partial V^k}{\partial q^j} + g_{il} V^k \Gamma^l_{kj} + V_k \Gamma^k_{ij}\,.\end{aligned}$$

In the above we have used the metric tensor to raise or lower indices and the relation $A_k B^k = A^k B_k$, and have identified Christoffel symbols using the definition in Eq. (4.54).

The last line of the above series of equations can be rearranged to the form constituting a solution to the exercise.

4.3.8. $\Gamma^1_{22} = -\rho$, $\Gamma^2_{12} = 1/\rho$.

4.3.9. All but three of the covariant derivative components of a contravariant vector $\mathbf{V}$ are of the form

$$V^i_{;j} = \frac{\partial V^i}{\partial q^j}\,.$$

The remaining three components are

$$V^{\varphi}_{;\varphi} = \frac{\partial V^{\varphi}}{\partial \varphi} + \frac{V^{\rho}}{\rho}, \quad V^{\varphi}_{;\rho} = \frac{\partial V^{\varphi}}{\partial \rho} + \frac{V^{\varphi}}{\rho}, \quad V^{\rho}_{;\varphi} = \frac{\partial V^{\rho}}{\partial \varphi} - \rho V^{\varphi}.$$

4.3.10.
$$\begin{aligned} g_{ij;k} &= \partial_k g_{ij} - \Gamma^{\alpha}_{ik} g_{\alpha j} - \Gamma^{\alpha}_{jk} g_{i\alpha} \\ &= \partial_k g_{ij} - \frac{1}{2} g_{j\alpha} g^{\alpha\beta} \left(\partial_i g_{\beta k} + \partial_k g_{\beta i} - \partial_\beta g_{ik}\right) \\ &\qquad\qquad - \frac{1}{2} g_{i\alpha} g^{\alpha\beta} \left(\partial_j g_{\beta k} + \partial_k g_{\beta j} - \partial_\beta g_{jk}\right) \\ &= \partial_k g_{ij} - \frac{1}{2}\left(\partial_i g_{jk} + \partial_k g_{ji} - \partial_j g_{ik}\right) - \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right) \equiv 0. \end{aligned}$$

In order to find $g^{ij}_{;k} = 0$ take the covariant derivative of the identity $g_{im} g^{mj} = \delta^j_i$. This gives $0 = g_{im;k} g^{mj} + g_{im} g^{mj}_{;k} = g_{im} g^{mj}_{;k}$. Multiplying this by g^{ni} and using $g^{ni} g_{im} = \delta^n_m$ gives $g^{nj}_{;k} = 0$.

4.3.11. To start, note that the contravariant V^k are, in the notation of Eq. (3.157), V_r, V_θ/r, $V_\varphi/r\sin\theta$ and that $[\det)g)]^{1/2} = r^2 \sin\theta$. Then the tensor derivative formula, Eq. (4.69), evaluates straightforwardly to Eq. (3.157).

4.3.12. $\partial_\mu \Phi_{;\nu} = \partial_\nu \partial_\mu \Phi - \Gamma^{\alpha}_{\mu\nu} \partial_\alpha \Phi \equiv \partial_\nu \Phi_{;\mu} = \partial_\mu \partial_\nu \Phi - \Gamma^{\alpha}_{\nu\mu} \partial_\alpha \Phi.$

4.4 Jacobians

4.4.1. $\boldsymbol{\nabla}(uv) = v\boldsymbol{\nabla} u + u \boldsymbol{\nabla} v$ follows from the product rule of differentiation.

(a) Since $\boldsymbol{\nabla} f = \dfrac{\partial f}{\partial u}\boldsymbol{\nabla} u + \dfrac{\partial f}{\partial v}\boldsymbol{\nabla} v = 0$, $\boldsymbol{\nabla} u$ and $\boldsymbol{\nabla} v$ are parallel,

so that $(\boldsymbol{\nabla} u) \times (\boldsymbol{\nabla} v) = 0$.

(b) If $(\boldsymbol{\nabla} u) \times (\boldsymbol{\nabla} v) = 0$, the two-dimensional volume spanned by $\boldsymbol{\nabla} u$ and $\boldsymbol{\nabla} v$, also given by the Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

vanishes.

4.4.3. (a) The direct computation of $\partial(x,y)/\partial(u,v)$ requires derivatives of x and y with respect to u and v. To get these derivatives, it is convenient to get explicit formulas for x and y in terms of u and v: these formulas are

$x = uv/(v+1)$, $y = u/(v+1)$. Now,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{v}{v+1} & \dfrac{u}{(v+1)^2} \\ \dfrac{1}{v+1} & \dfrac{-u}{(v+1)^2} \end{vmatrix}$$

$$= -\frac{uv}{(v+1)^3} - \frac{u}{(v+1)^3} = -\frac{u}{(v+1)^2}\,.$$

(b) Here we first need J^{-1}, computed as follows:

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \dfrac{1}{y} & -\dfrac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2} - \frac{1}{y} = -\frac{x+y}{y^2}.$$

Taking the reciprocal to obtain J, and rewriting in terms of u and v (the form usually needed if J is to be inserted into an integral over u and v), we get

$$J = -\frac{y^2}{x+y} = -\left(\frac{u}{v+1}\right)^2 \left(\frac{1}{u}\right) = -\frac{u}{(v+1)^2},$$

in agreement with the answer to part (a).

4.5 Differential Forms

4.5.1. The results for $*1$ and $*(dt \wedge dx_1 \wedge dx_2 \wedge dx_3)$ were explicitly discussed in Example 4.5.2, as was the value of $*dx_1$. The results for $*dx_i$ ($i \neq 1$) correspond in sign, since the ordering dx_i, dt, dx_j, dx_k with i, j, k cyclic has the same parity as dx_1, dt, dx_2, dx_3. For $*dt$, dt followed by the other differentials produces a standard ordering, and dt (the only differential in the expression being starred) has the metric tensor element $g_{tt} = +1$. This confirms the value given for $*dt$.

Example 4.5.2 derived a value for $*(dt \wedge dx_1)$. Corresponding results for $*(dt \wedge dx_i)$ hold because the cyclic ordering of dx_i, dx_j, dx_k causes the permutation to standard order to be the same for all i.

To verify that $*(dx_j \wedge dx_k) = dt \wedge dx_i$, note that the ordering dx_j, dx_k, dt, dx_i is an even permutation of the standard order, and that both g_{jj} and g_{kk} are -1, together producing no sign change.

Turning now to $*(dx_1 \wedge dx_2 \wedge dx_3)$, we note that dx_1, dx_2, dx_3, dt is an odd permutation of the standard order, but the quantity being starred is associated with three negative diagonal metric tensor elements. The result therefore has positive sign, as shown in Eq. (4.82).

The final case to be considered is $*(dt \wedge dx_i \wedge dx_j)$. Note that dt, dx_i, dx_j, dx_k is an even permutation of the standard order, and that dt, dx_i, dx_j contribute two minus signs from metric tensor elements. The overall sign is therefore plus, as shown in Eq. (4.82).

4.5.2. Since the force field is constant, the work associated with motion in the x direction will have the form $a_x\,dx$, where a_x is a constant. Similar statements apply to motion in y and z. Thus, the work w is described by the 1-form

$$w + \frac{a}{3}\,dx + \frac{b}{2}\,dy + c\,dz\,.$$

4.6 Differentiating Forms

4.6.1. (a) $d\omega_1 = dx \wedge dy + dy \wedge dx = 0$.

(b) $d\omega_2 = dx \wedge dy - dy \wedge dx = 2\,dx \wedge dy$.

(c) $d(dx \wedge dy) = d(dx) \wedge dy - dx \wedge d(dy) = 0$.

4.6.2. $d\omega_3 = y\,dx \wedge dz + xdy \wedge dz + z\,dx \wedge dy + xdz \wedge dy - z\,dy \wedge dx - zdy \wedge dx$

$= 2y\,dx \wedge dz + 2z\,dx \wedge dy$.

$d(d\omega_3) = 2\,dy \wedge dx \wedge dz + 2\,dz \wedge dx \wedge dy = 0$.

4.6.3. (a) $(xdy - ydx) \wedge (xy\,dz + xz\,dy - yz\,dx) = x^2y\,dy \wedge dz$

$- xy^2\,dx \wedge dz + x^2z\,dy \wedge dy - xyz\,dx \wedge dy - xyz\,dy \wedge dx + y^2z\,dx \wedge dx$

$= x^2y\,dy \wedge dz - xy^2\,dx \wedge dz$.

Apply d: $d(x^2y\,dy \wedge dz) - d(xy^2\,dx \wedge dz) = 2xy\,dx \wedge dy \wedge dz$

$+ x^2\,dy \wedge dy \wedge dz - y^2\,dx \wedge dx \wedge dz - 2xy\,dy \wedge dx \wedge dz$

$= 4xy\,dx \wedge dy \wedge dz$.

(b) $d\omega_2 \wedge \omega_3 - \omega_2 \wedge d\omega_3 = 2\,dx \wedge dy \wedge (xy\,dz + xz\,dy - yz\,dx)$

$- (x\,dy - y\,dx) \wedge (2y\,dx \wedge dz + 2z\,dx \wedge dy)$

$= 2xy\,dx \wedge dy \wedge dz - 2xy\,dy \wedge dx \wedge dz$

$= 4xy\,dx \wedge dy \wedge dz$.

4.7 Integrating Forms

4.7.1. Let $dx = a_1du + a_2dv + a_3dw$, $dy = b_1du + b_2dv + b_3dw$, and $dz = c_1du + c_2dv + c_3dw$. Then,

$$dx\wedge dy\wedge dz = (a_1du+a_2dv+a_3dw)\wedge(b_1du+b_2dv+b_3dw)\wedge(c_1du+c_2dv+c_3dw).$$

Expanding the right-hand side, discarding terms with duplicate differentials, and arranging the wedge products to standard order with the necessary sign assignments, we reach

$$dx\wedge dy\wedge dz = (a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1)du\wedge dv\wedge dw\,.$$

We recognize the coefficient of $du \wedge dv \wedge dw$ as the determinant

$$J = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

To complete the idenfication of J as a Jacobian, note that $a_1 = \partial x/\partial u$, $a_2 = \partial x/\partial v$, and so on, and therefore $J = \partial(x, y, z)/\partial(u, v, w)$.

4.7.2. See Example 4.6.2.

4.7.3. $ydx + xdy$: Closed; exact because it is $d(xy)$.

$\dfrac{ydx + xdy}{x^2 + y^2}$: Not closed because

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y}\frac{y}{x^2 + y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \qquad \frac{\partial B}{\partial x} = \frac{\partial}{\partial x}\frac{x}{x^2 + y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

$[\ln(xy) + 1]dx + \dfrac{x}{y}\,dy$: Closed because $\partial A/\partial y = 1/y = \partial B/\partial x$.

It is exact, being $d(x \ln xy)$.

$\dfrac{-ydx + xdy}{x^2 + y^2}$:

Noting that this is similar to a previous differential form of this exercise except for the sign of the dx term, we see that this form is closed. It is exact, being $d\ \tan(y/x)$.

$f(z)dz = f(x + iy)(dx + idy)$:

$\partial A/\partial y = \partial B/\partial x = if'(z)$. It is closed; also exact because A and B can be obtained as derivatives of the indefinite integral $\int f(z)\,dz$.

5. Vector Spaces

5.1 Vectors in Function Spaces

5.1.1. Using orthogonality the $\langle\phi_n|f\rangle = a_n = \int_a^b w(x)f(x)\phi_n(x)dx$ are derived from f and therefore unique.

5.1.2. If $f(x) = \sum_i c_i\phi_i(x) = \sum_j c'_j\phi_j$, then $\sum_i (c_i - c'_i)\phi_i = 0$. Let $c_k - c'_k \neq 0$ be the first non-zero term. Then

$$\phi_k = -\frac{1}{c_k - c'_k}\sum_{i>k}(c_i - c'_i)\phi_i$$

would say that ϕ_k is not linearly independent of the $\phi_i, i > k$, which is a contradiction.

5.1.3. For $f(x) = \sum\limits_{i=0}^{n-1} c_i x^i$ we have

$$b_j = \int_0^1 x^j f(x)dx = \sum_i c_i \int_0^1 x^{i+j}dx = \sum_{i=0}^{n-1}\frac{c_i}{i+j+1} = A_{ji}c_i.$$

This results also from minimizing the mean square error

$$\int_0^1 \left[f(x) - \sum_{i=0}^{n-1} c_i x^i\right]^2 dx$$

upon varying the c_i.

5.1.4. From

$$0 = \frac{\partial}{\partial c_l}\int_a^b [F(x) - \sum_{n=0}^{m} c_n\phi_n(x)]^2 w(x)dx$$

$$= 2\int_a^b \left[F(x) - \sum_{n=0}^{m} c_n\phi_n(x)\right]\phi_n w(x)\,dx$$

we obtain $c_n = \int_a^b F(x)\phi_n(x)w(x)dx$.

5.1.5. (a) and (b)

$$\int_{-\pi}^{\pi} f(x)^2 dx = \left(\frac{h}{2}\right)^2 2\pi \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)}$$

$$\times \int_{-\pi}^{\pi} \sin(2m+1)x \sin(2n+1)x\, dx$$

$$= \frac{4h^2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{h^2\pi}{2}\,.$$

Using $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\zeta(2)$, we reach $\zeta(2) = \dfrac{\pi^2}{6}$.

5.1.6. $|\langle f|g\rangle|^2 = \langle f^2\rangle\langle g^2\rangle - \dfrac{1}{2}\displaystyle\int_a^b \int_a^b |f(x)g(y) - f(y)g(x)|^2 dx dy$ implies

$|\langle f|g\rangle|^2 \le \langle f^2\rangle\langle g^2\rangle$ because the double integral is nonnegative.

5.1.7. The φ_j are assumed to be orthonormal. Expanding I, we have

$$I = \langle f|f\rangle - \sum_i a_i^* \langle\varphi_i|f\rangle - \sum_i a_i \langle f|\varphi_i\rangle + \sum_{ij} a_i^* a_j \langle\varphi_i|\varphi_j\rangle \ge 0\,.$$

Using the relation $a_i = \langle\varphi_i|f\rangle$ and the orthonormality condition $\langle\varphi_i|\varphi_j\rangle = \delta_{ij}$,

$$I = \langle f|f\rangle - \sum_i a_i^* a_i - \sum_i a_i a_i^* + \sum_i a_i^* a_i = \langle f|f\rangle - \sum_i |a_i|^2 \ge 0\,.$$

5.1.8. The expansion we need is

$$\sin\pi x = \sum_i \frac{\langle\varphi_i|\sin\pi x\rangle}{\langle\varphi_i|\varphi_i\rangle}\,\varphi_i(x)\,.$$

The necessary integrals are

$$\langle\varphi_0|\varphi_0\rangle = \int_0^1 dx = 1, \qquad \langle\varphi_1|\varphi_1\rangle = \int_0^1 (2x-1)^2\, dx = \frac{1}{3},$$

$$\langle\varphi_2|\varphi_2\rangle = \int_0^1 (6x^2 - 6x + 1)^2\, dx = \frac{1}{5}, \qquad \langle\varphi_3|\varphi_3\rangle = \frac{1}{7}$$

$$\langle\varphi_0|f\rangle = \int_0^1 \sin\pi x\, dx = \frac{2}{\pi}, \qquad \langle\varphi_1|f\rangle = \int_0^1 (2x-1)\sin\pi x\, dx = 0,$$

$$\langle\varphi_2|f\rangle = \frac{2}{\pi} - \frac{24}{\pi^3}, \qquad \langle\varphi_3|f\rangle = 0$$

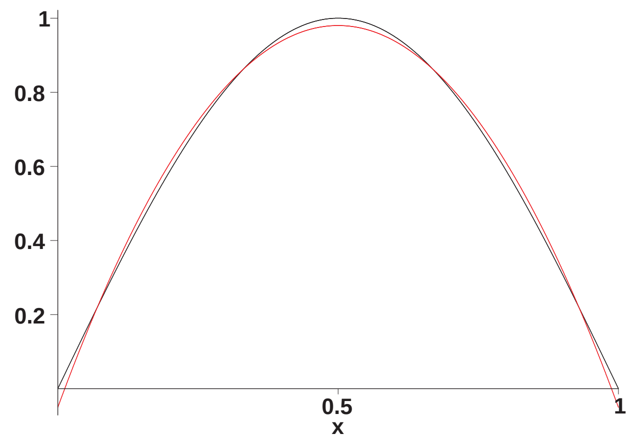

Figure 5.1.8. Red line is approximation through φ_3, black line is exact.

$$\sin \pi x = \frac{2/\pi}{1}\varphi_0 + \frac{2/\pi - 24/\pi^3}{1/5}\varphi_2 + \cdots = 0.6366 - 0.6871(6x^2 - 6x + 1) + \cdots .$$

This series converges fairly rapidly. See Fig. 5.1.8.

5.1.9. $e^{-x} = a_0 L_0(x) + a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x) + \cdots$,

$$a_i = \int_0^\infty L_i(x) e^{-2x}\, dx .$$

By integration we find $a_0 = 1/2$, $a_1 = 1/4$, $a_2 = 1/8$, $a_3 = 1/16$. Thus,

$$e^{-x} = \frac{1}{2}(1) + \frac{1}{4}(1 - x) + \frac{1}{8}\,\frac{2 - 4x + x^2}{2} + \frac{1}{16}\,\frac{6 - 18x + 9x^2 - x^3}{6} + \cdots .$$

This expansion when terminated after L_3 fails badly beyond about $x = 3$. See Fig. 5.1.9.

5.1.10. The forms $\sum_i |\varphi_i\rangle\langle\varphi_i|$ and $\sum_j |\chi_j\rangle\langle\chi_j|$ are resolutions of the identity. Therefore

$$|f\rangle = \sum_{ij} |\chi_j\rangle\langle\chi_j|\varphi_i\rangle\langle\varphi_i|f\rangle .$$

The coefficients of f in the φ basis are $a_i = \langle\varphi_i|f\rangle$, so the above equation is equivalent to

$$f = \sum_j b_j \chi_j, \quad \text{with} \quad b_j = \sum_i \langle\chi_j|\varphi_i\rangle a_i .$$

5.1.11. We assume the unit vectors are orthogonal. Then,

$$\sum_j |\hat{\mathbf{e}}_j\rangle\langle\hat{\mathbf{e}}_j|\mathbf{a}\rangle = \sum_j (\hat{\mathbf{e}}_j \cdot \mathbf{a})\, \hat{\mathbf{e}}_j .$$

This expression is a component decomposition of $\mathbf{a}$.

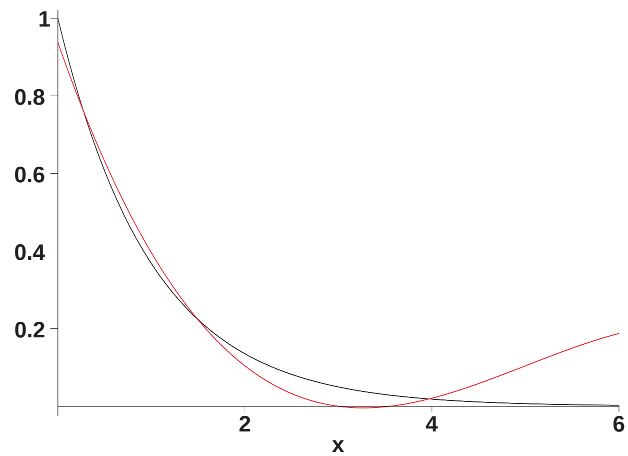

Figure 5.1.9. Red line is approximation through L_3, black line is exact.

5.1.12. The scalar product $\langle \mathbf{a}|\mathbf{a}\rangle$ must be positive for every nonzero vector in the space. If we write $\langle \mathbf{a}|\mathbf{a}\rangle$ in the form $(a_1 - a_2)^2 + (k-1)a_2^2$, this condition will be violated for some nonzero $\mathbf{a}$ unless $k > 1$.

5.2 Gram-Schmidt Orthogonalization

5.2.1. The solution is given in the text. Note that $a_{10} = -1/2,\ a_{20} = -1/3,\ a_{21} = -1/2,\ a_{30} = -1/4,\ a_{31} = -9/20,\ a_{32} = -1/4$.

5.2.2. The solution is given in the text. Note that $a_{10} = -1,\ a_{20} = -2,\ a_{21} = 4$.

5.2.3. The solution is given in the text. Note that $a_{10} = -2,\ a_{20} = -6,\ a_{21} = -6\sqrt{2}$.

5.2.4. The solution is given in the text. Note that $a_{10} = 0,\ a_{20} = -1/2,\ a_{21} = 0$.

5.2.5. Relying without comment on the integral formulas in Exercise 13.3.2, we compute first

$$\langle x^0|x^0\rangle = \int_{-1}^{1} (1-x^2)^{-1/2}\,dx = \pi,$$

$$\langle x^1|x^1\rangle = \langle x^0|x^2\rangle = \int_{-1}^{1} x^2(1-x^2)^{-1/2}\,dx = \pi/2,$$

$$\langle x^2|x^2\rangle = \int_{-1}^{1} x^4(1-x^2)^{-1/2}\,dx = 3\pi/8,$$

$$\langle x^0|x^1\rangle = \langle x^2|x^1\rangle = 0\,.$$

Note that some integrals are zero by symmetry.

The polynomial T_0 is of the form $c_0 x^0$, with c_0 satisfying

$$\langle c_0 x^0 | c_0 x^0 \rangle = |c_0|^2 \langle x^0 | x^0 \rangle = \pi,$$

so $c_0 = 1$ and $T_0 = 1$. By symmetry, the polynomial T_1, which in principle is a linear combination of x^0 and x^1, must actually be an odd function that depends only on x^1, so is of the form $c_1 x$. It is automatically orthogonal to T_0, and c_1 must satisfy

$$\langle c_1 x | c_1 x \rangle = |c_1|^2 \langle x | x \rangle = \frac{\pi}{2}\,.$$

Because $\langle x|x\rangle = \pi/2$, we have $c_1 = 1$ and $T_1 = x$.

The determination of T_1 is a bit less trivial. T_2 will be an even function of x, and will be of the general form

$$T_2 = c_2 \left[x^2 - \frac{\langle T_0 | x^2 \rangle}{\langle T_0 | T_0 \rangle}\, T_0 \right] = c_2 \left[x^2 - \frac{\pi/2}{\pi} T_0 \right] = c_2 \left[x^2 - \frac{1}{2} \right].$$

The constant c_2 is now determined from the normalization condition:

$$\langle T_2 | T_2 \rangle = |c_2|^2 \left\langle \left(x^2 - \frac{1}{2} \right) \middle| \left(x^2 - \frac{1}{2} \right) \right\rangle$$

$$= |c_2|^2 \left(\frac{3\pi}{8} - \frac{\pi}{2} + \frac{1}{4}\,\pi \right) = |c_2|^2\, \frac{\pi}{8} = \frac{\pi}{2}\,,$$

from which we find $c_2 = 2$ and $T_2 = 2x^2 - 1$.

5.2.6. From the formula given in the Hint, we have

$$\langle x^0 | x_0 \rangle = \int_{-1}^{1} (1 - x^2)^{1/2}\, dx = \frac{\pi}{2}\,,$$

$$\langle x^0 | x^2 \rangle = \langle x^1 | x^1 \rangle = \frac{\pi}{8}\,,$$

$$\langle x^2 | x^2 \rangle = \frac{\pi}{16}\,.$$

Taking $U_0 = c_0 x^0$, we find $|c_0|^2 \langle x^0 | x^0 \rangle = \pi/2$, so $c_0 = 1$ and $U_0 = 1$. The U_n have even/odd symmetry, so $U_1 = c_1 x$, and $|c_1|^2 \langle x|x\rangle = \pi/2$, so $|c_1|^2 \pi/8 = \pi/2$, and $c_1 = 2$, $U_1 = 2x$.

Finally,

$$U_2 = c_2 \left[x^2 - \frac{\langle U_0 | x^2 \rangle}{\langle U_0 | U_0 \rangle}\, U_0 \right] = c_2 \left[x^2 - \frac{\pi/8}{\pi/2}\, U_0 \right] = c_2 \left[x^2 - \frac{1}{4} \right].$$

We determine c_2 from

$$\langle U_2|U_2\rangle = |c_2|^2 \left\langle \left(x^2 - \frac{1}{4}\right) \middle| \left(x^2 - \frac{1}{4}\right) \right\rangle = \frac{|c_2|^2\pi}{32} = \frac{\pi}{2}\,,$$

so $c_2 = 4$, $U_2 = 4x^2 - 1$.

5.2.7. The solution is given in the text. Note that $a_{10} = -1/\sqrt{\pi}$.

5.2.8. Let the orthonormalized vectors be denoted $\mathbf{b}_i$. First, Make $\mathbf{b}_1$ a normalized version of $\mathbf{c}_1$: $\mathbf{b}_1 = \mathbf{c}_1/\sqrt{3}$. Then obtain $\overline{b}_2$ (denoting $\mathbf{b}_2$ before normalization) as

$$\overline{\mathbf{b}}_2 = \mathbf{c}_2 - (\mathbf{b}_1 \cdot \mathbf{c}_2)\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - (4\sqrt{3}) \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix}.$$

Normalizing, $\mathbf{b}_2 = \sqrt{3/2}\,\overline{\mathbf{b}}_2$. Finally, form

$$\overline{\mathbf{b}}_3 = \mathbf{c}_3 - (\mathbf{b}_1 \cdot \mathbf{c}_3)\mathbf{b}_1 - (\mathbf{b}_2 \cdot \mathbf{c}_3)\mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - (\sqrt{3}) \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$- (\sqrt{3}/2) \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}.$$

Normalizing, $\mathbf{b}_3 = \sqrt{2}\,\overline{\mathbf{b}}_3$. Collecting our answers, the orthonormal vectors are

$$\mathbf{b}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}.$$

5.3 Operators

5.3.1. For arbitrary φ and ψ within our Hilbert space, and an arbitrary operator A,

$$\langle\varphi|A\,\psi\rangle = \langle A^\dagger\varphi|\psi\rangle = \langle\psi|A^\dagger\varphi\rangle^* = \langle(A^\dagger)^\dagger\psi|\varphi\rangle^* = \langle\varphi|(A^\dagger)^\dagger\psi\rangle.$$

Since the first and last expressions in this chain of equations are equal for all A, ψ, and φ, we may conclude that $(A^\dagger)^\dagger = A$.

5.3.2. $$\langle\psi_2|V^\dagger(U^\dagger\psi_1)\rangle = \int (V\psi_2)^*(U^\dagger\psi_1)d^3r$$

$$= \int (UV\psi_2)^*\psi_1 d^3r = \langle\psi_2|(UV)^\dagger\psi_1\rangle.$$

5.3.3. (a) $(\mathsf{A}_1)_{ij} = \langle x_i|A_1|x_j\rangle$. A corresponding formula holds for A_2. Computing for each i and j, we find

$$\mathsf{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{A}_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) $\psi = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

(c) $(\mathsf{A}_1 - \mathsf{A}_2)\psi = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \chi$.

Check: $A_1\psi = x_1 - 2x_2 + 3x_3$; $A_2\psi = -2x_1 - x_2$;
$(A_1 - A_2)\psi = 3x_1 - x_2 + 3x_3$.

5.3.4. (a) First compute $A\mathcal{P}_n$ ($\mathcal{P}_n$ are the normalized polynomials).

$$A\mathcal{P}_0 = 0, \qquad A\mathcal{P}_1 = \sqrt{3/2}\,x = \mathcal{P}_1$$

$$A\mathcal{P}_2 = \sqrt{5/2}(3x^2) = 2\mathcal{P}_2 + \sqrt{5}\,\mathcal{P}_0,$$

$$A\mathcal{P}_3 = \sqrt{7/2}\left(\frac{15}{2}x^3 - \frac{3}{2}x\right) = 3\mathcal{P}_3 + \sqrt{21}\,\mathcal{P}_1.$$

Using the above and noting that our basis is the $\mathcal{P}_n$, we construct

$$\mathsf{A} = \begin{pmatrix} 0 & 0 & \sqrt{5} & 0 \\ 0 & 1 & 0 & \sqrt{21} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Note: We built the matrix of A directly from the expansions. An alternate and equally valid approach would be to identify the matrix elements as scalar products.

(b) To expand x^3 we need $\langle\mathcal{P}_3|x^3\rangle = 2\sqrt{14}/35$ and $\langle\mathcal{P}_1|x^3\rangle = \sqrt{6}/5$; the coefficients of $\mathcal{P}_2$ and $\mathcal{P}_0$ vanish because x^3 is odd. From the above data, we get $x^3 = (2\sqrt{14}/35)\mathcal{P}_3(x) + (\sqrt{6}/5)\mathcal{P}_1(x)$. Thus, the column vector representing x^3 is

$$x^3 \longleftrightarrow \begin{pmatrix} 0 \\ \sqrt{6}/5 \\ 0 \\ 2\sqrt{14}/35 \end{pmatrix}.$$

(c) $$\mathsf{A}x^3 = \begin{pmatrix} 0 & 0 & \sqrt{5} & 0 \\ 0 & 1 & 0 & \sqrt{21} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{6}/5 \\ 0 \\ 2\sqrt{14}/35 \end{pmatrix} = \begin{pmatrix} 0 \\ 3\sqrt{6}/5 \\ 0 \\ 6\sqrt{14}/35 \end{pmatrix}.$$

Inserting the explicit forms of $\mathcal{P}_1$ and $\mathcal{P}_3$, we find

$$Ax^3 = (3\sqrt{6}/5)(\sqrt{3/2}\,x) + (6\sqrt{14}/35)\sqrt{7/2}\,(\frac{5}{2}\,x^3 - \frac{3}{2}\,x) = 3x^3,$$

in agreement with the directly computed value.

5.4 Self-Adjoint Operators

5.4.1. (a) $(A + A^\dagger)^\dagger = A + A^\dagger$, $[i(A - A^\dagger)]^\dagger = -i(A^\dagger - A) = i(A - A^\dagger)$.

(b) $A = \frac{1}{2}(A + A^\dagger) - \frac{i}{2}i(A - A^\dagger)$.

5.4.2. $(AB)^\dagger = B^\dagger A^\dagger = BA = AB$ if and only if $[B, A] = 0$.

5.4.3. $(AB - BA)^\dagger = (iC)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -iC^\dagger = BA - AB = -iC$.

5.4.4. If $\mathcal{L}^\dagger = \mathcal{L}$ then $\langle\psi|\mathcal{L}^2\psi\rangle = \langle\psi|\mathcal{L}^\dagger(\mathcal{L}\psi)\rangle = \langle\mathcal{L}\psi|\mathcal{L}\psi\rangle = \int_a^b |\mathcal{L}\psi(x)|^2 dx \ge 0$.

5.4.5. (a) For the normalization of $\varphi_3 = Cz/r = C\cos\theta$, we need the following integral:

$$\left\langle \frac{z}{r}\middle|\frac{z}{r}\right\rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \cos^2\theta = 2\pi\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{4\pi}{3}.$$

The normalized form of φ_3 is therefore $\sqrt{3/4\pi}(z/r)$. To check orthogonality, we need integrals such as

$$\left\langle \frac{x}{r}\middle|\frac{y}{r}\right\rangle = \int_0^{2\pi} \cos\varphi\sin\varphi\, d\varphi \int_0^{\pi} \sin^3\theta\, d\theta\,.$$

The φ integral vanishes; one easy way to see this is to note that $\cos\varphi\sin\varphi = \sin(2\varphi)/2$; the φ integral is over two complete periods of this function. An appeal to symmetry confirms that all the other normalization and orthgonality integrals have similar values.

(b) It is useful to note that $\partial(1/r)/\partial x = -x/r^3$; similar expressions are obtained if x is replaced by y or z. Now,

$$L_z\varphi_1 = L_z\left(\frac{x}{r}\right) = -i\left[x\frac{\partial x/r}{\partial y} - y\frac{\partial x/r}{\partial x}\right] = i\frac{y}{r} = i\varphi_2.$$

Because L_z is antisymmetric in x and y, we also have

$$L_z\varphi_2 = L_z\left(\frac{y}{r}\right) = -i\frac{x}{r} = -i\varphi_1$$

Finally,

$$L_z\varphi_3 = L_z\left(\frac{z}{r}\right) = -i\left[x\frac{\partial(1/r)}{\partial y} - y\frac{\partial(1/r)}{\partial x}\right] = 0.$$

Combining the above into a matrix representation of L_z,

$$\mathsf{L}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similar processes (or cyclic permutation of x, y, z) lead to the matrix representations

$$\mathsf{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \mathsf{L}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}.$$

(c) Form the matrix operations corresponding to $\mathsf{L}_x\mathsf{L}_y - \mathsf{L}_y - \mathsf{L}_x$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Carrying out the matrix multiplication and subtraction, the result is i times the matrix of L_z.

5.5 Unitary Operators

5.5.1. (a) (1) The column vector representing $f(\theta,\varphi)$ is $\mathbf{c} = \begin{pmatrix} 3 \\ 2i \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

(2) $$\mathbf{c}' = \begin{pmatrix} -1/\sqrt{2} & -i/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -i\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2i \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 5/\sqrt{2} \\ -i/\sqrt{2} \\ i/\sqrt{2} \\ 1 \end{pmatrix}.$$

(3) Check: $\sum_i c_i'\chi_i' = \sum_i c_i\chi_i$.

(b) Form $\mathsf{U}^\dagger$ and verify that $\mathsf{U}\,\mathsf{U}^\dagger = \mathbf{1}$:

$$\begin{pmatrix} -1/\sqrt{2} & -i/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 \\ i/\sqrt{2} & i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -i/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{1}.$$

5.5.2. (a) The ith column of U describes φ_i in the new basis. Thus,

$$\mathsf{U} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(b) The transformation is a counterclockwise rotation of the coordinate system about the y axis; this corresponds to the Euler angles $\alpha = 0$, $\beta = \pi/2$, $\gamma = 0$. The above U is reproduced when these angles are substituted into Eq. (3.37).

(c) $\mathbf{c} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

$$\mathsf{U}\,\mathbf{c} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}.$$

This vector corresponds to $f' = -x - 3y + 2z$, which is consistent with application of the relevant basis transformation to f.

5.5.3. Since the matrix U for the transformation of Exercise 5.5.2 is unitary, the inverse transformation has matrix $\mathsf{U}^\dagger$, which is

$$\mathsf{U}^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Multiplying, we find that $\mathsf{U}\mathsf{U}^\dagger = \mathbf{1}$.

5.5.4. (a) $\mathsf{U}f = \begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos\theta & i\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos\theta + 3i\sin\theta \\ -3\cos\theta + i\sin\theta \\ -2 \end{pmatrix}$,

$$\mathsf{V}(\mathsf{U}f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & i\sin\theta \\ 0 & \cos\theta & -i\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta + 3i\sin\theta \\ -3\cos\theta + i\sin\theta \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta + 3i\sin\theta \\ -3\cos^2\theta + i\sin\theta(cos\theta - 2) \\ 3\cos^2\theta + i\sin\theta(\cos\theta + 2) \end{pmatrix}.$$

The above indicates that $f(x) = (\cos\theta + 3i\sin\theta)\chi_1 + (-3cos^2\theta + i\sin\theta(\cos\theta - 2))\chi_2 + (3\cos^2\theta + i\sin\theta(\cos\theta + 2))\chi_3$.

(b) $\mathsf{U}\mathsf{V} = \begin{pmatrix} i\sin\theta & \cos^2\theta & i\sin\theta\cos\theta \\ -\cos\theta & i\sin\theta\cos\theta & -\sin^2\theta \\ 0 & \cos\theta & -i\sin\theta \end{pmatrix}$

$$\mathsf{VU} = \begin{pmatrix} i\sin\theta & \cos\theta & 0 \\ -\cos^2\theta & i\sin\theta\cos\theta & i\sin\theta \\ -\cos^2\theta & i\sin\theta\cos\theta & -i\sin\theta \end{pmatrix}.$$

Using the above, we find

$$\mathsf{UV}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos^2\theta + i\sin\theta(3-2\cos\theta) \\ 2\sin^2\theta - 3\cos\theta + i\sin\theta\cos\theta \\ \cos\theta + 2i\sin\theta \end{pmatrix},$$

$$\mathsf{VU}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos\theta + 3i\sin\theta \\ -3\cos^2\theta + i\sin\theta(cos\theta - 2) \\ 3\cos^2\theta + i\sin\theta(\cos\theta + 2) \end{pmatrix}.$$

Only $\mathsf{VU}f$ gives the correct result that we found in part (a).

5.5.5. (a) The normalized versions of the P_n, denoted $\mathcal{P}_n$, are $\mathcal{P}_n = \sqrt{(2n+1)/2}\; P_n$. The normalized versions of the F_n, denoted $\mathcal{F}_n$, are $\mathcal{F}_0 = \sqrt{5/2}\; F_0$, $\mathcal{F}_1 = \sqrt{3/2}\; F_1$, and $\mathcal{F}_2 = \sqrt{1/8}\; F_2$.
(b) The transformation matrix U has elements $u_{ij} = \langle \mathcal{F}_i | \mathcal{P}_j \rangle$. For example,

$$u_{02} = \int_{-1}^{1} \mathcal{F}_0\, \mathcal{P}_2\, dx = \frac{\sqrt{5}}{2} \int_{-1}^{1} (x^2)(1)\, dx = \frac{\sqrt{5}}{3}.$$

The complete transformation matrix is

$$\mathsf{U} = \begin{pmatrix} \sqrt{5}/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{pmatrix}.$$

(c) V has elements $v_{ij} = \langle \mathcal{P}_i | \mathcal{F}_j \rangle$. Thus,

$$\mathsf{V} = \begin{pmatrix} \sqrt{5}/3 & 0 & -2/3 \\ 0 & 1 & 0 \\ 2/3 & 0 & \sqrt{5}/3 \end{pmatrix}.$$

(d) By matrix multiplication we can verify that $\mathsf{UV} = \mathbf{1}$, showing that $\mathsf{V} = \mathsf{U}^{-1}$. Since V is also $\mathsf{U}^\dagger$, we can also conclude that U and V are unitary.

(e) $$f(x) = \frac{8\sqrt{2}}{3}\mathcal{P}_0(x) - \sqrt{6}\,\mathcal{P}_1(x) + \frac{2\sqrt{10}}{3}\mathcal{P}_2(x)$$

$$= \frac{4\sqrt{10}}{3}\mathcal{F}_0(x) - \sqrt{6}\,\mathcal{F}_1(x) - \frac{2\sqrt{2}}{3}\mathcal{F}_2(x).$$

Letting $\mathbf{c}$ and $\mathbf{c}'$ be the vectors describing the expansions of $f(x)$ respectively in the $\mathcal{P}_n$ and the $\mathcal{F}_n$ bases,

$$\mathbf{c} = \begin{pmatrix} 8\sqrt{2}/3 \\ -\sqrt{6} \\ 2\sqrt{10}/3 \end{pmatrix}, \qquad \mathbf{c}' = \begin{pmatrix} 4\sqrt{10}/3 \\ -\sqrt{6} \\ -2\sqrt{2}/3 \end{pmatrix},$$

we check that

$$\mathbf{c}' = \mathsf{U}\mathbf{c}, \quad \text{i.e.,} \quad \begin{pmatrix} 4\sqrt{10}/3 \\ -\sqrt{6} \\ -2\sqrt{2}/3 \end{pmatrix} = \begin{pmatrix} \sqrt{5}/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{pmatrix} \begin{pmatrix} 8\sqrt{2}/3 \\ -\sqrt{6} \\ 2\sqrt{10}/3 \end{pmatrix}.$$

5.6 Transformations of Operators

5.6.1. (a) The first column of S_x shows the result of its operation on α; the second column describes $S_x\beta$. Similar observations apply to S_y amd S_z. We get

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) (1) Check that $\langle\alpha+\beta|\alpha-\beta\rangle = 0$. Expanding, we have $\langle\alpha|\alpha\rangle - \langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle - \langle\beta|\beta\rangle = 1 + 0 + 0 - 1 = 0$.

(2) A similar expansion shows that $\langle\alpha+\beta|\alpha+\beta\rangle = 2$, so a proper value of C is $1/\sqrt{2}$. The same result is obtained for $\langle\varphi_2'|\varphi_2'\rangle$.

(3) The matrix elements of the transformation are $u_{ij} = \langle\varphi_i'|\varphi_j\rangle$. These evaluate to

$$\mathsf{U} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(c) In the transformed basis, the matrix of an operator S becomes $\mathsf{S}' = \mathsf{U}\,\mathsf{S}\,\mathsf{U}^{-1}$. Noting that $\mathsf{U}^{-1} = \mathsf{U}$, we compute

$$\mathsf{S}_x' = \mathsf{U}\,\mathsf{S}_x\mathsf{U}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S_z\,.$$

Similar operations for S_y and S_z yield

$$\mathsf{S}_y' = \mathsf{U}\,\mathsf{S}_y\mathsf{U}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -S_y\,.$$

$$\mathsf{S}_z' = \mathsf{U}\,\mathsf{S}_z\mathsf{U}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S_x\,.$$

5.6.2. (a) Apply L_x to the φ_i: $L_x\varphi_1 = 0$, $L_x\varphi_2 = i\varphi_3$, and $L_x\varphi_3 = -i\varphi_2$. Therefore the matrix of L_x for this basis is

$$\mathsf{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

(b) Form $\mathsf{UL}_x\mathsf{U}^{-1}$. U is unitary; this can be checked by verifying that $\mathsf{UU}^\dagger = \mathbf{1}$. Thus,

$$\mathsf{UL}_x\mathsf{U}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(c) The new basis functions have coefficients (in terms of the original basis) that are the columns of $\mathsf{U}^\dagger$. Reading them out, we have

$$\varphi_1' = Cxe^{-r^2}, \qquad \varphi_2' = \frac{C}{\sqrt{2}}(y+iz)e^{-r^2}, \qquad \varphi_3' = \frac{C}{\sqrt{2}}(y-iz)e^{-r^2}.$$

Applying L_x,

$$L_x\varphi_1' = 0\,,$$

$$L_x\varphi_2' = L_x\frac{C}{\sqrt{2}}\,(y+iz)e^{-r^2} = \frac{C}{\sqrt{2}}\,[iz+i(-iy)]e^{-r^2} = \frac{C}{\sqrt{2}}\,(y+iz)e^{-r^2},$$

$$L_x\varphi_3' = \frac{C}{\sqrt{2}}\,(y-iz)e^{-r^2} = \frac{C}{\sqrt{2}}\,[iz-i(-iy)]e^{-r^2} = -\frac{C}{\sqrt{2}}\,(y-iz)e^{-r^2}.$$

5.6.3. Define D_1, D_2, D_3 as the determinants formed from the overlap matrix elements of the first, the first two, and all three basis functions. Letting $s_{ij} = \langle\chi_i|\chi_j\rangle$ be the elements of this overlap matrix, $D_1 = S_{11}$, $D_2 = S_{11}S_{22} - S_{12}S_{21}$ etc. By substitution into the formulas obtained as in Section 5.2, we find the systematic formulas for the φ_i:

$$\varphi_1 = \frac{\chi_1}{\sqrt{D_1}} \qquad \varphi_2 = \frac{\chi_2 - \dfrac{S_{12}\chi_1}{D_1}}{\sqrt{D_2/D_1}}\,,$$

$$\varphi_3 = \frac{\chi_3 - \dfrac{S_{13}\chi_1}{D_1} - \dfrac{D_1S_{23}\chi_2}{D_2} + \dfrac{S_{12}S_{23}\chi_1}{D_2} + \dfrac{S_{21}S_{13}\chi_2}{D_2} - \dfrac{S_{12}S_{21}S_{13}\chi_1}{D_1D_2}}{\sqrt{D_3/D_2}}\,.$$

Comparing with the matrix T as defined in Example 5.6.1, we see that its jth column consists of the coefficients of the χ_i in the formula for φ_j. From the above formulas, we find

$$\mathsf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix},$$

with

$$T_{11} = \frac{1}{\sqrt{D_1}}, \qquad T_{12} = -\frac{S_{12}}{\sqrt{D_1 D_2}}, \qquad T_{22} = \sqrt{\frac{D_1}{D_2}}\,,$$

$$T_{13} = \frac{-D_2 S_{13} + D_1 S_{12} S_{23} - S_{12} S_{21} S_{13}}{D_1 \sqrt{D_2 D_3}}\,, \qquad T_{23} = \frac{-D_1 S_{23} + S_{21} S_{13}}{\sqrt{D_2 D_3}}\,,$$

$$T_{33} = \sqrt{\frac{D_2}{D_3}}\,.$$

5.7 Invariants

5.7.1. Replace x by $\mathsf{x}' = \mathsf{U}\mathsf{x}\mathsf{U}^{-1}$ and $\mathsf{p}' = \mathsf{p}$ by $\mathsf{U}\mathsf{p}\mathsf{U}^{-1}$, so

$$[\mathsf{x}', \mathsf{p}'] = \mathsf{x}'\mathsf{p}' - \mathsf{p}'\mathsf{x}' = \left(\mathsf{U}\mathsf{x}\mathsf{U}^{-1}\right)\left(\mathsf{U}\mathsf{p}\mathsf{U}^{-1}\right) - \left(\mathsf{U}\mathsf{p}\mathsf{U}^{-1}\right)\left(\mathsf{U}\mathsf{x}\mathsf{U}^{-1}\right)$$

$$= \mathsf{U}(\mathsf{x}\mathsf{p} - \mathsf{p}\mathsf{x})\mathsf{U}^{-1} = i\mathsf{U}\mathsf{U}^{-1} = i\mathbf{1}\,.$$

5.7.2. We need

$$\boldsymbol{\sigma}_1' = \mathsf{U}\boldsymbol{\sigma}_1\mathsf{U}^\dagger = \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}, \qquad \boldsymbol{\sigma}_2' = \mathsf{U}\boldsymbol{\sigma}_2\mathsf{U}^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\boldsymbol{\sigma}_3' = \mathsf{U}\boldsymbol{\sigma}_3\mathsf{U}^\dagger = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Now form

$$\boldsymbol{\sigma}_1'\boldsymbol{\sigma}_2' = \begin{pmatrix} i\cos 2\theta & -i\sin 2\theta \\ -i\sin 2\theta & -i\cos 2\theta \end{pmatrix} = i\boldsymbol{\sigma}_3'\,,$$

$$\boldsymbol{\sigma}_2'\boldsymbol{\sigma}_1' = \begin{pmatrix} -i\cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & i\cos 2\theta \end{pmatrix} = -i\boldsymbol{\sigma}_3'\,.$$

From the above, $\boldsymbol{\sigma}_1'\boldsymbol{\sigma}_2' - \boldsymbol{\sigma}_2'\boldsymbol{\sigma}_1' = 2i\boldsymbol{\sigma}_3'$.

5.7.3. (a) From the equations $L_x\varphi_1 = 0$, $L_x\varphi_2 = i\varphi_3$, $L_x\varphi_3 = -i\varphi_2$, we see that L_x applied to any function in the space spanned by φ_1, φ_2, φ_3 a function that remains within that space. The above equations correspond to the action on the φ basis of the matrix

$$\mathsf{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

(b) $L_x(\varphi_1 + i\varphi_2) = 0 + i(i\varphi_3) = -\varphi_3 = -ze^{-r^2}$.

(c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

If this equation is transformed by U, the quantities in it become

$$\mathsf{L}'_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ i/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix},$$

and the transformed matrix equation is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ i/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}.$$

(d) The φ' are those linear combinations of the φ with coefficients that are complex conjugates of the corresponding row of U, and are $\varphi'_1 = \varphi_1 = x\,e^{-r^2}$, $\varphi'_2 = (\varphi_2 + i\varphi_3)/\sqrt{2} = (y+iz)e^{-r^2}/\sqrt{2}$, $\varphi'_3 = (\varphi_2 - i\varphi_3)/\sqrt{)}2 = (y-iz)e^{-r^2}/\sqrt{2}$.

(e) The matrix equation is equivalent to

$$L_x\left(\left[x + \frac{i}{\sqrt{2}}\left(\frac{y+iz}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}\left(\frac{y-iz}{\sqrt{2}}\right)\right] e^{-r^2}\right)$$
$$= \left[\frac{i}{\sqrt{2}}\left(\frac{y+iz}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}\left(\frac{y-iz}{\sqrt{2}}\right)\right] e^{-r^2},$$

which simplifies to

$$L_x\left[(x+iy)e^{-r^2}\right] = -ze^{-r^2}.$$

This is a result that was proved in part (b).

5.8 Summary—Vector Space Notation

(no exercises)

6. Eigenvalue Problems

6.1 Eigenvalue Equations

(no exercises)

6.2 Matrix Eigenvalue Problems

The solutions to matrix eigenvalue problems consist of the eigenvalues λ_i, and associated with each a normalized eigenvector $\mathbf{r}_i$. The eigenvectors corresponding to degenerate eigenvalues are **not** unique.

6.2.1. $\lambda_1 = 0, \quad \mathbf{r}_1 = (1, 0, -1)/\sqrt{2}$

$\lambda_2 = 1, \quad \mathbf{r}_2 = (0, 1, 0)$

$\lambda_3 = 2, \quad \mathbf{r}_3 = (1, 0, 1)/\sqrt{2}.$

6.2.2. $\lambda_1 = -1, \quad \mathbf{r}_1 = (1, -\sqrt{2}, 0)/\sqrt{3}$

$\lambda_2 = 0, \quad \mathbf{r}_2 = (0, 0, 1)$

$\lambda_3 = 2, \quad \mathbf{r}_3 = (\sqrt{2}, 1, 0)/\sqrt{3}.$

6.2.3. $\lambda_1 = -1, \quad \mathbf{r}_1 = (1, -2, 1)/\sqrt{6}$

$\lambda_2 = 1, \quad \mathbf{r}_2 = (1, 0, -1)/\sqrt{2}$

$\lambda_3 = 2, \quad \mathbf{r}_3 = (1, 1, 1)/\sqrt{3}.$

6.2.4. $\lambda_1 = -3, \quad \mathbf{r}_1 = (1, -\sqrt{2}, 1)/2$

$\lambda_2 = 1, \quad \mathbf{r}_2 = (1, 0, -1)/\sqrt{2}$

$\lambda_3 = 5, \quad \mathbf{r}_3 = (1, \sqrt{2}, 1)/2.$

6.2.5. $\lambda_1 = 0, \quad \mathbf{r}_1 = (0, 1, -1)/\sqrt{2}$

$\lambda_2 = 1, \quad \mathbf{r}_2 = (1, 0, 0)$

$\lambda_3 = 2, \quad \mathbf{r}_3 = (0, 1, 1)/\sqrt{2}.$

6.2.6. $\lambda_1 = -1, \quad \mathbf{r}_1 = (0, 1, -\sqrt{2})/\sqrt{3}$

$\lambda_2 = +1, \quad \mathbf{r}_2 = (1, 0, 0)$

$\lambda_3 = 2, \quad \mathbf{r}_3 = (0, \sqrt{2}, 0)/\sqrt{3}.$

6.2.7. $\lambda_1 = -\sqrt{2}, \quad \mathbf{r}_1 = (1, -\sqrt{2}, 1)/2$

$\lambda_2 = 0, \quad \mathbf{r}_2 = (1, 0, -1)/\sqrt{2}$

$\lambda_3 = \sqrt{2}, \quad \mathbf{r}_3 = (1, \sqrt{2}, 1)/2.$

6.2.8. $\lambda_1 = 0, \quad \mathbf{r}_1 = (0, 1, -1)/\sqrt{2}$

$\lambda_2 = 2, \quad \mathbf{r}_2 = (0, 1, 1)/\sqrt{2}$

$\lambda_3 = 2, \quad \mathbf{r}_3 = (1, 0, 0).$

6.2.9. $\lambda_1 = 2, \quad \mathbf{r}_1 = (1,1,1)/\sqrt{3}$
$\lambda_2 = -1, \quad \mathbf{r}_2 = (1,\ -1,\ 0)/\sqrt{2}$
$\lambda_3 = -1, \quad \mathbf{r}_3 = (1,1,\ -2)/\sqrt{6}.$

6.2.10. $\lambda_1 = -1, \quad \mathbf{r}_1 = (1,1,1)/\sqrt{3}$
$\lambda_2 = 2, \quad \mathbf{r}_2 = (1,\ -1,\ 0)/\sqrt{2}$
$\lambda_3 = 2, \quad \mathbf{r}_3 = (1,1,\ -2)/\sqrt{6}.$

6.2.11. $\lambda_1 = 3, \quad \mathbf{r}_1 = (1,1,1)/\sqrt{3}$
$\lambda_2 = 0, \quad \mathbf{r}_2 = (1,-1,0)/\sqrt{2}$
$\lambda_3 = 0, \quad \mathbf{r}_3 = (1,1,-2)/\sqrt{6}.$

6.2.12. $\lambda_1 = 6, \quad \mathbf{r}_1 = (2,0,1)/\sqrt{5}$
$\lambda_2 = 1, \quad \mathbf{r}_2 = (1,0,-2)/\sqrt{5}$
$\lambda_3 = 1, \quad \mathbf{r}_3 = (0,1,0).$

6.2.13. $\lambda_1 = 2, \quad \mathbf{r}_1 = (1,1,0)/\sqrt{2}$
$\lambda_2 = 0, \quad \mathbf{r}_2 = (1,-1,0)/\sqrt{2}$
$\lambda_3 = 0, \quad \mathbf{r}_3 = (0,0,1).$

6.2.14. $\lambda_1 = 2, \quad \mathbf{r}_1 = (1,0,-\sqrt{3})/2$
$\lambda_2 = 3, \quad \mathbf{r}_2 = (0,1,0)$
$\lambda_3 = 6, \quad \mathbf{r}_3 = (\sqrt{3},0,1)/2.$

6.2.15. Since the quadratic form $x^2 + 2xy + 2y^2 + 2yz + z^2 = 1$ defining the surface is obviously positive definite upon writing it as a sum of squares, $(x + y)^2 + (y + z)^2 = 1$, it is an ellipsoid or an ellipse. Finding the orientation in space amounts to diagonalizing the symmetric 3×3 matrix of coefficients. The characteristic polynomial is $\lambda(1 - \lambda)(\lambda - 3) = 0$, so that the eigenvalues are $\lambda = 0$ implying an ellipse, and $\lambda = 1$, and 3. For $\lambda = 1$ an eigenvector is $v_1 = (1,0,-1)$ giving one of its axes, for $\lambda = 3$ an eigenvector is $v_3 = (1,2,1)$ giving the other axis. $\mathbf{v_1} \times \mathbf{v_3} = (2,-2,2)$ is normal to the plane of the ellipse.

6.3 Hermitian Eigenvalue Problems

(no exercises)

6.4 Hermitian Matrix Diagonalization

6.4.1. This follows from the invariance of the characteristic polynomial under similarity transformation.

6.4.2. The orthonormality of the eigenvectors implies that the transformation matrix U diagonalizing our matrix H is unitary. Since the diagonal matrix is made up by the real eigenvectors, it is Hermitian, and so is the transformed matrix H.

6.4.3. Assume that a unitary matrix U causes the real nonsymmetric matrix A to be diagonal, i.e., that $\mathsf{UAU}^T = \mathsf{D}$, a diagonal matrix. If we apply the inverse transformation to D, to recover A, we would have $\mathsf{A} = \mathsf{U}^T\mathsf{DU}$. But this form for A is symmetric: $(\mathsf{U}^T\mathsf{DU})^T = \mathsf{U}^T\mathsf{DU}$.

6.4.4. First, note that L_x^2 has the same eigenvectors as L_x, with eigenvalues that are the squares of the (real) L_x eigenvalues. Therefore, L_x^2 (and for the same reason, L_y^2 and L_z^2) have only nonnegative eigenvalues. Second, for vectors $|\mathbf{x}\rangle$ of unit length, the expectation value $\langle \mathbf{x}|L_x^2|\mathbf{x}\rangle$ will be real and have as its smallest possible value the smallest eigenvalue of L_x^2. Proof of this statement is the topic of Exercise 6.5.5. Similar statements are true for L_y^2 and L_z^2, so $\langle \mathbf{x}|L_x^2 + L_y^2 + L_z^2|\mathbf{x}\rangle$ must always be nonnegative. We therefore may conclude that all the eigenvalues of $L_x^2 + L_y^2 + L_z^2$ are nonnegative.

6.4.5. If $\mathsf{A}|x_i\rangle = \lambda_i|x_i\rangle$, then $|x_i\rangle = \lambda_i \mathsf{A}^{-1}|x_i\rangle$ upon multiplying with the inverse matrix. Moving the (non-zero) eigenvalue to the left-hand side proves the claim.

6.4.6. (a) If A is singular, its determinant is zero. If A is transformed to diagonal form, its determinant is seen to be the product of its eigenvalues, so a zero determinant indicates that at least one eigenvalue is zero. The eigenvector corresponding to a zero eigenvalue will have the property that $\mathsf{A}|\mathbf{v}\rangle = 0$.

(b) If $\mathsf{A}|\mathbf{v}\rangle = 0$, then $|\mathbf{v}\rangle$ is an eigenvector with eigenvalue zero, the determinant of A will be zero, and A will be singular.

6.4.7. If $\mathsf{U}_1\mathsf{AU}_1^\dagger = [\lambda_1, \cdots, \lambda_n] = \mathsf{U}_2\mathsf{BU}_2^\dagger$ with unitary matrices U_i, then

$$\mathsf{A} = \mathsf{U}_1^\dagger \mathsf{U}_2 B \mathsf{U}_2^\dagger \mathsf{U}_1 = \mathsf{U}_1^\dagger \mathsf{U}_2 B (\mathsf{U}_1^\dagger \mathsf{U}_2)^\dagger .$$

6.4.8.

$$\text{For } \mathrm{M}_x, \quad \begin{aligned} &\lambda_1 = +1, && \mathbf{r}_1 = (1, +\sqrt{2}, 1)/2 \\ &\lambda_2 = 0, && \mathbf{r}_2 = (1, 0, -1)/\sqrt{2} \\ &\lambda_3 = -1, && \mathbf{r}_3 = (1, -\sqrt{2}, 1)/2. \end{aligned}$$

$$\text{For } \mathrm{M}_y, \quad \begin{aligned} &\lambda_1 = +1, && \mathbf{r}_1 = (1, +i\sqrt{2}, -1)/2 \\ &\lambda_2 = 0, && \mathbf{r}_2 = (1, 0, 1)/\sqrt{2} \\ &\lambda_3 = -1, && \mathbf{r}_3 = (1, -i\sqrt{2}, -1)/2. \end{aligned}$$

6.4.9. (a) Form $a'_{ij} = \langle \varphi_i \cos\theta - \varphi_j \sin\theta | \mathsf{A} | \varphi_i \sin\theta + \varphi_j \cos\theta\rangle$.

Using the fact that $\langle\varphi_\mu|\mathsf{A}|\varphi_\nu\rangle = a_{\mu\nu}$ and remembering that $a_{\nu\mu} = a_{\mu\nu}$, we expand the expression for a'_{ij} and set it equal to zero, getting

$$(a_{ii} - a_{jj})\sin\theta\cos\theta + a_{ij}(\cos^2\theta - \sin^2\theta) = 0\,.$$

Using the trigonometric double-angle formulas and rearranging, we reach

$$\tan 2\theta = \frac{2a_{ij}}{a_{jj} - a_{ii}}\,.$$

(b) Since only the basis functions φ_i and φ_j are altered by the Jacobi transformation, all matrix elements of A not involving i and not involving j remain unchanged.

(c) Proceeding as in part (a), we find

$$a'_{ii} = a_{ii}\cos^2\theta + a_{jj}\sin^2\theta - 2a_{ij}\sin\theta\cos\theta,$$

$$a'_{jj} = a_{ii}\sin^2\theta + a_{jj}\sin^2\theta + 2a_{ij}\sin\theta\cos\theta.$$

Forming $a'_{ii} + a'_{jj}$, the a_{ij} terms cancel and, using the identity $\sin^2\theta + \cos^2\theta = 1$, the remaining terms are seen to add to $a_{ii} + a_{jj}$, as required.

(d) Form the squares of $a'_{\mu i}$ and $a'_{\mu j}$:

$$a'_{\mu i} = a_{\mu i}\cos\theta - a_{\mu j}\sin\theta,$$

$$(a'_{\mu i})^2 = (a_{\mu i})^2\cos^2\theta + (a_{\mu j})^2\sin^2\theta - 2a_{\mu i}a_{\mu j}\sin\theta\cos\theta,$$

$$a'_{\mu j} = a_{\mu j}\cos\theta + a_{\mu i}\sin\theta,$$

$$(a'_{\mu j})^2 = (a_{\mu i})^2\sin^2\theta + (a_{\mu j})^2\cos^2\theta + 2a_{\mu i}a_{\mu j}\sin\theta\cos\theta\,.$$

Thus, $a^2_{\mu i} + a^2_{\mu j}$ is not changed by the transformation, and the sum of the squares of the off-diagonal elements has been changed only by the replacement of a_{ij} and a_{ji} by zero, a net decrease of $2(a_{ij})^2$.

6.5 Normal Matrices

6.5.1. The solution is given in the text.

6.5.2. The characteristic polynomial is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 - \mathrm{trace}(\mathsf{A})\lambda + \det(\mathsf{A}) = 0.$$

6.5.3. If $\mathsf{U}\mathbf{r} = \lambda\mathbf{r}$ with $|\mathbf{r}|^2 = 1$, then $1 = \mathbf{r}^\dagger\mathbf{r} = \mathbf{r}^\dagger\mathsf{U}^\dagger\mathsf{U}\mathbf{r} = |\lambda|^2\mathbf{r}^\dagger\mathbf{r} = |\lambda|^2$.

6.5.4. Choose a coordinate system in which the rotation is about the z-axis, and transform our rotation matrix to these coordinates. This transformation will not change the trace of the rotation matrix. Now the rotation matrix will have the form

$$\mathsf{U} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the trace of U is obviously $1 + 2\cos\varphi$.

6.5.5. Expand $|\mathbf{y}\rangle$ in the eigenvectors: $|\mathbf{y}\rangle = \sum_i c_i|\mathbf{x}_i\rangle$.

Then note that, because $|\mathbf{y}\rangle$ is of unit magnitude and the $|\mathbf{x}_i\rangle$ are orthonormal,

$$\langle\mathbf{y}|\mathbf{y}\rangle = \sum_{ij} c_i^* c_j \langle\mathbf{x}_i|\mathbf{x}_j\rangle = \sum_i |c_i^2| = 1\,.$$

Moreover, because the $|\mathbf{x}_i\rangle$ are orthonormal eigenvectors,

$$\langle\mathbf{y}|\mathsf{A}|\mathbf{y}\rangle = \sum_{ij} c_i^* c_j \langle\mathbf{x}_i|\mathsf{A}|\mathbf{x}_j\rangle = \sum_i |c_i^2|\lambda_i\,.$$

Lower and upper bounds for this expression can now be obtained by replacing λ_i by the smallest or the largest eigenvalue, after which the $|c_i^2|$ can be summed (yielding unity).

6.5.6. From Exercise 6.5.3 the eigenvalues have $|\lambda| = 1$. If U is Hermitian, then λ is real, hence ± 1.

6.5.7. If γ_μ and γ_ν anticommute, $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$. Take the determinants of the two sides of this equation:

$$\det(\gamma_\mu\gamma_\nu) = \det(\gamma_\mu)\det(\gamma_\nu) = \det(-\gamma_\nu\gamma_\mu) = (-1)^n \det(\gamma_\nu)\det(\gamma_\mu).$$

Here we have used the fact that the determinant is not a linear operator, and that the determinant of $-A$ is $(-1)^n \det(A)$, where n is the dimension of the determinant. Since the γ are unitary, they cannot be singular, and the anticommutation leads to an inconsistency unless n is even, making $(-1)^n = +1$.

6.5.8. Expand $|\mathbf{y}\rangle$ in the eigenfunctions: $\mathsf{A}|\mathbf{y}\rangle = \sum_n c_n \mathsf{A}|\mathbf{x}_n\rangle = \sum_n \lambda_n c_n |\mathbf{x}_n\rangle$, with $c_n = \langle\mathbf{x}_n|\mathbf{y}\rangle$. We get the same result from the eigenvector form of A:

$$A|\mathbf{y}\rangle = \sum_n \lambda_n |\mathbf{x}_n\rangle\langle\mathbf{x}_n|\mathbf{y}\rangle = \sum_n \lambda_n c_n |\mathbf{x}_n\rangle\,.$$

6.5.9. The solution is given in the text.

6.5.10. Write

$$\langle \mathbf{v}_i|\mathsf{A}|\mathbf{u}_j\rangle = \lambda_j\langle \mathbf{v}_i|\mathbf{u}_j\rangle$$

$$= \langle \mathsf{A}^\dagger \mathbf{v}_i|\mathbf{u}_j\rangle = \lambda_i^*\langle \mathbf{v}_i|\mathbf{u}_j\rangle\,.$$

Subtracting the right-hand side of the second line from that of the first line,

$$(\lambda_j - \lambda_i^*)\langle \mathbf{v}_i|\mathbf{u}_j\rangle = 0\,,$$

from which we conclude that $\langle \mathbf{v}_i|\mathbf{u}_j\rangle = 0$ unless $\lambda_i^* = \lambda_j$.

6.5.11. (a) and (b) Apply $\tilde{\mathsf{A}}$ to the first equation and A to the second:

$$\tilde{\mathsf{A}}\mathsf{A}|\mathbf{f}_n\rangle = \lambda_n\tilde{\mathsf{A}}|\mathbf{g}_n\rangle = \lambda_n^2|\mathbf{f}_n\rangle\,,$$

$$\mathsf{A}\tilde{\mathsf{A}}|\mathbf{g}_n\rangle = \lambda_n\mathsf{A}|\mathbf{f}_n\rangle = \lambda_n^2|\mathbf{g}_n\rangle\,.$$

(c) Because A is real, $\tilde{\mathsf{A}}\mathsf{A}$ and $\mathsf{A}\tilde{\mathsf{A}}$ are both self-adjoint (Hermitian), and therefore have eigenvectors that have real eigenvalues and form an orthogonal set.

6.5.12. If the given formula for A gives the required result for every member of an orthogonal set it is a valid expression for A. Apply the formula to an $|\mathbf{f}_j\rangle$ of arbitrary j. Because the $\mathbf{f}_n\rangle$ are orthonormal, the result reduces to $\lambda_j|\mathbf{g}_j\rangle$.

6.5.13. (a) $\mathsf{A}\tilde{\mathsf{A}} = \dfrac{1}{5}\begin{pmatrix} 8 & -6 \\ -6 & 17 \end{pmatrix} \qquad \tilde{\mathsf{A}}\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$,

(b) $\lambda_1 = 1 \quad |g_1>= \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad |\mathrm{f}_1>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

(c) $\lambda_2 = 2, \quad |g_2>= \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad |\mathrm{f}_2>= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

6.5.14. Disregard this exercise (it is ill-defined).

6.5.15. (a) Take the adjoint of U; because H is self-adjoint, the result is $\mathsf{U}^\dagger = \exp(-ia\mathsf{H})$. Note that an exponential can be interpreted as its power-series expansion and the adjoint taken termwise, thus validating the processing applied to the exponent. The result shows that $\mathsf{U}^\dagger = \mathsf{U}^{-1}$.

(b) Form $\mathsf{U}\mathsf{U}^T = \exp(ia\mathsf{H})\exp(-ia\mathsf{H})$. Because H commutes with itself, this product reduces to $\exp(ia\mathsf{H} - ia\mathsf{H}) = \mathbf{1}$. Note that exponentials can be combined in this way only if the exponents commute.

(c) If H is diagonalized by a similarity transformation, the zero trace implies that the sum of its eigenvalues λ_n is zero. Then U, which is also diagonal, will have diagonal elements $\exp(ia\lambda_n)$, and its determinant, which will then be the product of its diagonal elements, will be $\exp(ia\sum\lambda_n) = \exp(0) = +1$.

(d) Conversely, in a basis in which H and U are diagonal, a unit determinant for U implies an exponential in which $\exp(ia\sum\lambda_n) = 1$; this condition does not quite imply that trace $\mathsf{H} = 0$, but only that $a(\text{trace } \mathsf{H}) = 0$ is an integer multiple of 2π.

6.5.16. From $\mathsf{A}v_i = A_i v_i$ we obtain $\mathsf{A}^n = A_i^n v_i$ for $i = 0, 1, 2, \dots$.

From $\mathsf{B} = \exp(\mathsf{A}) = \sum\limits_{n=0}^{\infty} \mathsf{A}^n/n!$ we get

$$\mathsf{B} = \sum_{n=0}^{\infty} \mathsf{A}^n v_i/n! = \sum_{n=0}^{\infty} \left[A_i^n/n!\right] v_i = (e^{A_i})v_i .$$

6.5.17. For any operator A, the eigenvalues of A^2 are the squares of the eigenvalues of A.

6.5.18. Inserting the indicated expansion and using the orthogonality property of the eigenvectors,

$$\langle \mathbf{x}|\mathsf{A}|\mathbf{x}\rangle = \lambda_1\langle \mathbf{x}_1|\mathbf{x}_1\rangle + \sum_{i=2}^{n} |\delta_i|^2\lambda_n\langle \mathbf{x}_i|\mathbf{x}_i\rangle \,,$$

$$\langle \mathbf{x}|\mathbf{x}\rangle = \langle \mathbf{x}_1|\mathbf{x}_1\rangle + \sum_{i=2}^{n} |\delta_i|^2\langle \mathbf{x}_i|\mathbf{x}_i\rangle \,.$$

Because all λ_i for $i > 1$ are smaller than λ_1,

$$\langle \mathbf{x}|\mathsf{A}|\mathbf{x}\rangle < \lambda_1\left(\langle \mathbf{x}_1|\mathbf{x}_1\rangle + \sum_{i=2}^{n} |\delta_i|^2\langle \mathbf{x}_i|\mathbf{x}_i\rangle\right) \,,$$

so

$$\frac{\langle \mathbf{x}|\mathsf{A}|\mathbf{x}\rangle}{\langle \mathbf{x}|\mathbf{x}\rangle} < \lambda_1 \,.$$

The error when this ratio is used to approximate λ_1 is approximately

$$\frac{1}{\langle \mathbf{x}|\mathbf{x}\rangle}\sum_{i=2}^{n}(\lambda_1 - \lambda_i)|\delta_i|^2\langle \mathbf{x}_i|\mathbf{x}_i\rangle \,,$$

which is of order $|\delta_i|^2$.

6.5.19. (a) Letting x_1 and x_2 be the displacements of the two moveable masses, each of the same mass m, measured from their equilibrium positions (with the positive direction for both x_i the same), the equations of motion are

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)\,,$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)\,.$$

In a normal mode of oscillation $x_i = X_i e^{i\omega t}$, with the same angular frequency ω for both masses. Inserting these expressions,

$$-\frac{m\omega^2}{k} X_1 = -2X_1 + X_2\,,$$

$$-\frac{m\omega^2}{k} X_2 = X_1 - 2X_2\,.$$

These equations are equivalent to the matrix equation

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

with $\lambda = m\omega^2/k$.

(b) This is an eigenvalue equation which has solutions only if

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0\,,$$

with eigenvalues $\lambda = 1$ and $\lambda = 3$.

(c) For $\lambda = 1$, the equation solution is $X_1 = X_2$, corresponding to the two masses moving, in phase, back and forth. For $\lambda = 3$, the equation solution is $X_1 = -X_2$, corresponding to a periodic motion in which the masses oscillate relative to each other.

6.5.20. Relying on the proof that a normal matrix A and its adjoint have the same eigenvectors $\mathbf{x}_j$,

$$\langle \mathbf{x}_j|\mathsf{A}|\mathbf{x}_j\rangle = \lambda_j \langle \mathbf{x}_j|\mathbf{x}_j\rangle$$

$$= \langle \mathsf{A}^\dagger \mathbf{x}_j|\mathbf{x}_j\rangle = \mu_j^* \langle \mathbf{x}_j|\mathbf{x}_j\rangle\,,$$

where μ_j is the eigenvalue of $\mathsf{A}^\dagger$ corresponding to $\mathbf{x}_j$. We see that $\mu_j^* = \lambda_j$. Since A and $\mathsf{A}^\dagger$ have common eigenvectors,

$$(\mathsf{A} + \mathsf{A}^\dagger)|\mathbf{x}_j\rangle = (\lambda_j + \lambda_j^*)|\mathbf{x}_j\rangle = 2\Re e\,\lambda_j\,.$$

Likewise, $\mathsf{A} - \mathsf{A}^\dagger$ has eigenvalues $\lambda_j - \lambda_j^*$, or $2i\,\Im m\ \lambda_j$.

6.5.21. (a) Using Eq. (3.37), the matrix U of the rotation is

$$\mathsf{U} = \begin{pmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}.$$

U has the following eigenvalues and eigenvectors:

$$\lambda_1 = 1 \qquad \mathbf{r}_1 = \sqrt{2/3}\,\hat{\mathbf{e}}_x + \sqrt{1/3}\,\hat{\mathbf{e}}_z$$

$$\lambda_2 = \frac{1}{2}(-1 + i\sqrt{3}) \qquad \mathbf{r}_2 = -\sqrt{1/6}\,\hat{\mathbf{e}}_x + i\sqrt{1/2}\,\hat{\mathbf{e_y}} + \sqrt{1/3}\,\hat{\mathbf{e}}_z$$

$$\lambda_3 = \frac{1}{2}(-1 - i\sqrt{3}) \qquad \mathbf{r}_3 = -\sqrt{1/6}\,\hat{\mathbf{e}}_x - i\sqrt{1/2}\,\hat{\mathbf{e_y}} + \sqrt{1/3}\,\hat{\mathbf{e}}_z$$

From these data we see that the rotation of the coordinate axes corresponding to U is equivalent to a single rotation about φ_1 by an angle given as the phase of λ_2 (the angle it makes with the real axis), which is $120°$.

7. Ordinary Differential Equations

7.1 Introduction

(no exercises)

7.2 First-Order Equations

7.2.1. (a) Separating the variables obtain $I(t) = I_0 e^{-t/RC}$, where I_0 is the integration constant.

(b) Here $\Omega = 10^6$ Ohm, then $I_0 = 10^{-4}$ Amp, $RC = 10^4$ sec and at $t = 100$, $e^{-t/RC} = e^{-0.01} \approx 0.99$. Thus, $I = 0.99 \times 10^{-4}$ Amp. The time 100 sec is only 1% of the time constant RC.

7.2.2. Separating variables obtain

$$\ln f(s) = \int \frac{f'(s)}{f(s)} ds = -\int \frac{sds}{s^2+1} = -\frac{1}{2}\ln(s^2+1) + \ln C,$$

implying $f(s) = \dfrac{C}{\sqrt{s^2+1}}$.

7.2.3. $\displaystyle\int_{N_0}^{N} \frac{dN}{N^2} = -\int_0^t kdt = -kt = -\frac{1}{N} + \frac{1}{N_0}$.

Thus, $N = \dfrac{N_0}{1+t/\tau}$, $\tau = (kN_0)^{-1}$.

7.2.4. (a) Set $A_0 = A(0), B_0 = B(0)$. Separating variables and using a partial fraction expansion obtain

$$\alpha \int dt = \alpha t = \int \frac{dC}{(A_0 - C)(B_0 - C)} =$$

$$\frac{1}{B_0 - A_0} \int \left(\frac{1}{A_0 - C} - \frac{1}{B_0 - C} \right) dC.$$

Thus $\ln \dfrac{A_0 - C}{B_0 - C} = (A_0 - B_0)\alpha t + \ln \dfrac{A_0}{B_0}$.

Rewrite this as $C(t) = \dfrac{A_0 B_0 [e^{(A_0 - B_0)\alpha t} - 1]}{A_0 e^{(A_0-B_0)\alpha t} - B_0}$. Then $C(0) = 0$.

(b) From $\displaystyle\int \frac{dC}{(A_0 - C)^2} = \alpha t$ get $\dfrac{1}{A_0 - C} = \alpha t + \dfrac{1}{A_0}$,

which yields $C(t) = \dfrac{\alpha A_0^2 t}{1 + \alpha A_0 t}$. Again $C(0) = 0$.

7.2.5. The values $n < 0$ are unphysical as the acceleration diverges.

The case $n = 0$ gives

$$m[v - v(0)] = -kt, \quad v(t) = v(0) - kt/m, \quad x(t) = x(0) + v(0)t - kt^2/2m\,.$$

The case $n = 1$ gives

$$v(t) = v(0)e^{-kt/m}, \qquad x(t) = x(0) + \frac{mv(0)}{k}\left(1 - e^{-kt/m}\right).$$

For $n \neq 0, 1, 2$ and $n > 0$ we integrate to get

$$\frac{v^{1-n} - v(0)^{1-n})}{1-n} = \int \frac{dv}{v^n} = -\frac{k}{m}t,$$

$$v(t) = v(0)\left[1 + (n-1)\frac{kt}{m}v(0)^{n-1}\right]^{1/(1-n)}.$$

Integrating again gives

$$x(t) = x(0) + \frac{mv(0)^{2-n}}{(2-n)k}\left[1 - \left(1 + (n-1)\frac{kt}{m}v(0)^{n-1}\right)^{(n-2)/(n-1)}\right].$$

The case $n = 2$ leads to

$$\dot{x} = \frac{v(0)}{1 + \alpha t}, \qquad x(t) = x(0) + \frac{m}{k}\ln\left(1 + \frac{kv(0)t}{m}\right).$$

7.2.6. The substitution $u = y/x$, or $y = xu$, corresponds to $dy = xdu + udx$, and our ODE assumes the form

$$x\,du + u\,dx = g(u)\,dx\,, \quad \text{or} \quad x\,du = [g(u) - u]\,dx\,,$$

which is separable.

7.2.7. If $\dfrac{\partial\varphi}{\partial x} = P(x, y)$ then $\varphi(x, y) = \displaystyle\int_{x_0}^{x} P(X, y)dX + \alpha(y)$ follows.

Differentiating this and using $\dfrac{\partial\varphi}{\partial y} = Q(x, y)$ we obtain

$$Q(x, y) = \frac{d\alpha}{dy} + \int_{x_0}^{x} \frac{\partial P(X, y)}{\partial y}\,dX,$$

so

$$\frac{d\alpha}{dy} = Q(x, y) - \int_{x_0}^{x} \frac{\partial Q(X, y)}{\partial X}\,dX.$$

So $\dfrac{d\alpha}{dy} = Q(x_0, y)$ and $\alpha(y) = \displaystyle\int_{y_0}^{y} Q(x_0, Y)\, dY$. Thus

$$\varphi(x, y) = \int_{x_0}^{x} P(X, y) dX + \int_{y_0}^{y} Q(x_0, Y) dY.$$

From this we get $\dfrac{\partial \varphi}{\partial x} = P(x, y)$ and

$$\frac{\partial \varphi}{\partial y} = \int_{x_0}^{x} \frac{\partial P(X, y)}{\partial y} dX + Q(x_0, y) = \int_{x_0}^{x} \frac{\partial Q(X, y)}{\partial X} dX + Q(x_0, y) = Q(x, y).$$

7.2.8. See proof of Exercise 7.2.7.

7.2.9. For $\alpha\, dy + \alpha(py - q)\, dx = 0$ to be exact requires

$$\frac{\partial \alpha}{\partial x} = \frac{\partial}{\partial y} \alpha(x)(py - q) = \alpha\, p,$$

which is Eq. (7.14).

7.2.10. For $f(x)dx + g(x)h(y)dy = 0$ to be exact requires

$$\frac{\partial f(x)}{\partial y} = 0 = \frac{\partial g(x)h(y)}{\partial x} = h(y) \frac{\partial g(x)}{\partial x}, \quad \text{i.e.,} \quad g = \text{const.}$$

7.2.11. $y' = -pe^{-\int^x p dt} \left[\displaystyle\int^x e^{\int^s p(t)dt} q(s) ds + C \right] + e^{-\int^x p dt} e^{\int^x p dt} q(x)$

implies $y' + p(x)y(x) = q(x)$.

7.2.12. Separating variables we get

$$-\frac{bt}{m} = \ln(g - \frac{b}{m} v) - \ln(A \frac{b}{m})$$

with A an integration constant. Exponentiating this we obtain

$$v(t) = \frac{mg}{b} - Ae^{-bt/m}, \quad \text{thus} \quad v_0 = v(0) = \frac{mg}{b} - A.$$

Hence $v(t) = \left(v_0 - \dfrac{mg}{b} \right) e^{-bt/m} + \dfrac{mg}{b}$. Set $v_0 = 0$ here.

The velocity dependent resistance force opposes the gravitational acceleration implying the relative minus sign.

7.2.13. Solve first for N_1, which is separable and has the general solution

$$N_1 = Ce^{-\lambda_1 t}. \quad \text{Since } N_1(0) = N_0, \text{ we have} \quad N_1(t) = N_0 e^{-\lambda_1 t}.$$

Substitute this result into the equation for N_2, which is now an inhomogeneous equation in which N_2 is the only unknown. We look for a particular integral of the inhomogeneous equation, guessing the form of the solution to be $N_2 = A\exp(-\lambda_1 t)$. Thus,

$$\frac{dN_2}{dt}+\lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t} \quad \text{becomes} \quad -A\lambda_1 e^{-\lambda_1 t}+A\lambda_2 e^{-\lambda_1 t} = \lambda_1 N_0 e^{-\lambda_1 t}\,,$$

confirming that with a proper choice of A our guess will work. We find that $A = \lambda_1 N_0/(\lambda_2 - \lambda_1)$.

To this particular integral we must add the multiple of the solution to the homogeneous equation that is needed to satisfy the condition $N_2(0) = 0$. The homogeneous equation has solution $e^{-\lambda_2 t}$, so our complete solution is

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

7.2.14. We have $dV/dt = -C4\pi r^2$ with $V = 4\pi r^3/3$ the volume and C a positive constant. So $dr/dt = -C$ and $r(t) = r_0 - Ct$.

7.2.15. (a) Separating variables, $dv/v = -a\,dt$ yields

$$\ln\frac{v}{v_0} = -at, \qquad v = v_0 e^{-at}.$$

(b) $dv/v + a\,dt = 0$ yields

$$\varphi(t, v) = \ln v + at.$$

$\varphi(t, v) = \ln v_0$ =const. is equivalent to (a).

(c) Substituting into the form of solution written in Exercise 7.2.11 with $q = 0$, $p = a$ we get $v(t) = Ce^{-at}$. Setting $t = 0$ we identify C as c_0.

7.2.16. Separating variables as in Example 7.2.1 we get the velocity

$$v(t) = v_0 \tanh\left[\frac{t}{T} + \tanh^{-1}\left(\frac{v_i}{v_0}\right)\right]$$

for $v_i \ge 0$.

7.2.17. This ODE is isobaric, and becomes separable under the substitution $v = xy$. Removing x via this substitution, the ODE becomes

$$(vy - y)\left(\frac{dv}{y} - \frac{v\,dy}{y^2}\right) + \frac{v\,dy}{y} = 0\,.$$

This equation separates into

$$-\left(\frac{v-1}{v^2-2v}\right)dv+\frac{dy}{y} = 0\,, \quad \text{with integral} \quad -\frac{1}{2}\ln(v^2-2v)+\ln y = \ln C\,.$$

Exponentiating, we get

$$\frac{y^2}{v^2-2v} = C, \qquad \text{or} \qquad \frac{y}{x^2y-2x} = C\,.$$

7.2.18. This ODE is homogeneous, so we substitute $y = vx$, obtaining initially

$$(x^2 - v^2x^2e^v)dx + (x^2 + x^2v)e^v(x\,dv + v\,dx) = 0\,.$$

This rearranges to

$$\frac{dx}{x} + \frac{(1+v)e^v\,dv}{1+ve^v} = 0 \quad \text{with integral} \quad \ln x + \ln(1+ve^v) = \ln C\,.$$

Thus,

$$x(1+ve^v) = C\,, \qquad \text{or} \qquad x + ye^{y/x} = C\,.$$

7.3 ODEs with Constant Coefficients

7.3.1. Try solution e^{mx}. The condition on m is $m^3 - 2m^2 - m + 2 = 0$, with roots $m = 2$, $m = 1$, $m = -1$. The general solution to the ODE is therefore $c_1e^{2x} + c_2e^x + c_3e^{-x}$.

7.3.2. Try solution e^{mx}. The condition on m is $m^3 - 2m^2 + m - 2 = 0$, with roots $m = 2$, $m = i$, $m = -i$. The solutions e^{ix} and e^{-ix} can be expressed in terms of the real quantities $\sin x$ and $\cos x$, so the general solution to the ODE is $c_1e^{2x} + c_2\sin x + c_3\cos x$.

7.3.3. Try solution e^{mx}. The condition on m is $m^3 - 3m + 2 = 0$, with roots $m = 1$, $m = 1$, $m = -2$. Two independent solutions for $m = 1$ are e^x and xe^x, so the general solution to the ODE is $c_1e^x + c_2xe^x + c_3e^{-2x}$.

7.3.4. Try solution e^{mx}. The condition on m is $m^2 + 2m + 2 = 0$, with roots $m = -1 + i$ and $m = -1 - i$. We can combine $e^{(-1+i)x}$ and $e^{(-1-i)x}$ to form $e^{-x}\sin x$ and $e^{-x}\cos x$, so the general solution to the ODE is $c_1e^{-x}\sin x + c_2e^{-x}\cos x$.

7.4 Second-Order Linear ODEs

7.4.1. For $P(x) = -\dfrac{2x}{1-x^2}$, $\quad Q(x) = \dfrac{l(l+1)}{1-x^2}$,

$(1 \mp x)P$ and $(1 \mp x)^2Q$ are regular at $x = \pm 1$, respectively. So these are regular singularities.

As $z \to 0$, $2z - \dfrac{2/z}{1-\frac{1}{z^2}} = 2(z + \dfrac{z}{1-z^2})$ is regular, and $\dfrac{Q(z^{-1})}{z^4} = \dfrac{l(l+1)}{z^2(z^2-1)} \sim z^{-2}$ diverges. So ∞ is a regular singularity.

7.4.2. For $P = \dfrac{1-x}{x}$, $\quad Q = \dfrac{n}{x}$, $\quad x = 0$ is a regular singularity.

For $z \to 0$, $\dfrac{2z - P(z^{-1})}{z^2} = \dfrac{z+1}{z^2} \sim 1/z^2$ diverges more rapidly than $1/z$, so ∞ is an irregular singularity.

7.4.3. Writing the Chebyshev equation in the form

$$y'' + \left(\frac{x}{1-x^2}\right) y' + \left(\frac{n^2}{1-x^2}\right) y = 0\,,$$

we see that the coefficients of y' and y become singular (for finite x) only at $x = \pm 1$ and that each singularity is first order, so the ODE has regular singularities at these points. At infinity, we apply the criterion given after Eq. (7.22). For the present ODE,

$$\frac{2x - P(x^{-1})}{x^2} = \frac{2}{x} - \frac{1/x}{x^2 - 1},$$

$$\frac{Q(x^{-1})}{x^4} = \frac{n^2}{x^4 - x^2}.$$

These have, at $x = 0$, singularities that are respectively of first and second order, indicating that the ODE has a regular singularity at infinity.

7.4.4. Hermite's ODE (as given in Table 7.1) has no coefficients that are singular at finite x, and therefore is regular for all finite x. At infinity,

$$\frac{2x - P(x^{-1})}{x^2} = \frac{2}{x} + \frac{2}{x^3}$$

has a singularity of order 3 at $x = 0$, so the ODE will have an irregular singularity at infinity.

7.4.5. $x(1-x)\dfrac{d^2}{dx^2} + [c - (a+b+1)x]\dfrac{d}{dx} - ab \to (1-x^2)\dfrac{d^2}{dx^2} - 2x\dfrac{d}{dx} + l(l+1)$

because

$$\frac{d^2}{dx^2} \to 4\frac{d^2}{dx^2}, \qquad \frac{d}{dx} \to -2\frac{d}{dx}, \qquad -ab \to l(l+1),$$

$$x(1-x) \to \frac{1-x}{2}\left(1 - \frac{1-x}{2}\right) = \frac{1}{4}(1-x^2),$$

$$c - (a+b+1)x \to 1 - (l+2-l)\frac{1-x}{2} = x.$$

7.5 Series Solutions—Frobenius' Method

7.5.1. If initial conditions are $y(x_0) = y_0, y'(x_0) = y'_0$ are given, the solutions' Taylor expansions are identical provided x_0 is no worse than a regular singularity. The factor x^k from the indicial equation does not affect this.

7.5.2. Under the translation $x_1 = x - x_0$, $d/dx_1 = d/dx$, etc. the ODE is invariant and $y(x - x_0) = y(x_1)$ has the same Maclaurin expansion as $y(x)$ at $x = 0$. As a result, the recursion relations for the coefficients and the indicial equation stay the same.

7.5.3. If $a_1k(k+1) = 0$ with $a_1 \neq 0$, then $k = 0$ or $k = -1$.

(a) $k = 0$ sets $a_1k(k+1) = 0$ where a_1 remains undetermined.

(b) If $k = 1$ then the indicial equation $a_1k(k+1) = 0$ requires $a_1 = 0$.

7.5.4. The two indicial equations for Legendre's ODE are $k(k-1)a_0 = 0$ and $k(k+1)a_1 = 0$. For Bessel's ODE they are $(k^2 - n^2)a_0 = 0$ and $[(k+1)^2 - n^2)]a_1 = 0$. For Hermite's ODE they are the same as Legendre's. The rest of the solution is given in the text.

7.5.5. Compare with Eq. (18.120). Convergent for $|x| < 1$, also at $x = 1$ for $c > a + b$ and at $x = -1$ for $c > a + b - 1$.

7.5.6. Compare with Eq. (18.136). Convergent for all finite x provided the series exists [$c \neq -n$, a negative integer, in Eq. (18.137), $2 - c \neq -n$ in Eq. (18.136)].

7.5.7. The point $\xi = 0$ is a regular singularity of the ODE. The trial solution $\sum_j a_j \xi^{k+j}$ yields the given indicial equation. For $k = m/2, a_0 \neq 0$ and non-negative m we set the coefficient of the term ξ^{k+1} to zero. This gives $a_1 = -\alpha a_0/(m+1)$. Setting the coefficient of ξ^{k+2} to zero gives the second given term, etc.

7.5.8. Substituting $\sum\limits_{j=0}^{\infty} a_j \eta^{j+k}$ and its derivatives into

$$\frac{d}{d\eta}(1-\eta^2)\frac{du}{d\eta} + \alpha u + \beta u^2 = 0,$$

we obtain the recursion relation

$$a_{j+2}(j+k+2)(j+k+1) - a_j[(j+k)(j+k+1) - \alpha] + \beta a_{j-2} = 0.$$

For $j = -2$, $a_{-2} = 0 = a_{-4}$ by definition and the indicial equation $k(k-1)a_0 = 0$ comes out, i.e. $k = 0$ or $k = 1$ for $a_0 \neq 0$.

For $j = -1$ with $a_{-3} = 0 = a_{-1}$ we have $a_1k(k+1) = 0$. If $k = 1$, then $a_1 = 0$ implying $a_3 = 0 = a_5$.

For $j = 0, k = 0$ we get $2a_2 = -a_0\alpha$ and $6a_2 = a_0(2-\alpha)$ for $k = 1$.

For $j = 1, k = 0$ we find $6a_3 = 12a_3 = a_1(6-\alpha)$ for $k = 1$.

Finally, for $j = 2, k = 1$ we have $20a_4 - (12-\alpha)a_2 + \beta a_0 = 0$, giving the expansion listed in the problem set.

7.5.9. Substituting

$$\psi = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

and setting $A' = \dfrac{2mA}{\hbar^2}$, $E' = \dfrac{2mE}{\hbar^2}$, $V = \dfrac{A}{x}e^{-ax}$, $A < 0$, $a > 0$, we obtain

$$2a_2 + 6a_3x + \cdots + \left[-A' + (E' + aA')x - \frac{1}{2}A'a^2x^2 + \cdots\right](a_1 + a_2x + \cdots) = 0,$$

where the coefficients of all powers of x vanish. This implies

$$a_0 = 0, \quad 2a_2 = A'a_1, \quad 6a_3 + a_1(E' + aA') - A'a_2 = 0, \quad \text{etc.}$$

Thus, we get the given series.

7.5.10. Even though the point $x = 0$ is an essential singularity we try substituting

$$\sum_{j=0}^{\infty} a_j x^{j+k}, \quad y' = \sum_{j=0}^{\infty} a_j (j+k) x^{j+k-1}, \quad y'' = \sum_{j=0}^{\infty} a_j (j+k)(j+k-1) x^{j+k}$$

into our ODE we obtain the recursion relation

$$a_j[(j+k)(j+k-1) - 2] + a_{j+1}(j+k+1) = 0.$$

For $j = -1, a_{-1} = 0$ by definition, so $k = 0$ for $a_0 \neq 0$ is the indicial equation. For $j = 0, -2a_0 + a_1 = 0$, and for $j = 1, -2a_1 + 2a_2 = 0$, while $j = 2$ yields $a_3 = 0$, etc. Hence our solution is $y = a_0(1 + 2x + 2x^2)$, and this is readily verified to be a solution.

7.5.11. Writing the solution to the ODE as $\dfrac{e^x}{\sqrt{2\pi x}}\, f(x)$,

we find that $f(x)$ satisfies the ODE $x^2 f'' + 2x^2 f' + f/4 = 0$. Substituting into this ODE the series expansion $f(x) = b_0 + b_1/x + b_2/x^2 + \cdots$, we find that the b_n satisfy the recurrence formula

$$b_{n+1} = \frac{n(n+1) + \frac{1}{4}}{2n+2}\, b_n\,,$$

which, with the initial value $b_0 = 1$, we can use to obtain the coefficients in the asymptotic expansion. The first two coefficients are $b_1 = (1/4)/2 = 1/8$ and $b_2 = (2 + \frac{1}{4})b_1/4 = 9/128$.

7.6 Other Solutions

7.6.1. $a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$ implies $a = b = c = 0$.

7.6.2. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent, geometry tells us that their volume $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \neq 0$, and vice versa.

7.6.3. Using $y_n = \dfrac{x^n}{n!}, \quad y_n' = \dfrac{x^{n-1}}{(n-1)!}$, etc. for $n = 0, 1, \ldots, N$ we get

$$W_1 = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1, \quad W_2 = \begin{vmatrix} 1 & x & \frac{x^2}{2} \\ 0 & 1 & x \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = W_1 = 1,$$

and, continuing, $W_2 = \cdots = W_N = 1$.

7.6.4. If $W = y_1 y_2' - y_1' y_2 = 0$, then $\dfrac{y_1'}{y_1} = \dfrac{y_2'}{y_2}$.

Integrating gives $\ln y_1 = \ln y_2 + \ln C$. Hence $y_1 = C y_2$, and vice versa.

7.6.5. If the Wronskian $W(x)$ is written as a Taylor series at x_0, all of its coefficients must be zero.

7.6.6. The answer is given in the text.

7.6.7. φ_2' does not exist at $x = 0$..

7.6.8. These functions are related by $2y_1(x) - y_2(x) - \dfrac{1}{y_2(x)} = 0$, which is nonlinear.

7.6.9. $P_n Q_n' - P_n' Q_n = W(x) = A_n e^{-\int^x P dt} = \dfrac{A_n}{1-x^2}$

because $-\displaystyle\int^x P dt = \int^x \frac{2t}{1-t^2}\,dt = -\ln(1-x^2)$.

7.6.10. Assuming there to be three linearly independent solutions, construct their Wronskian. It will be identically zero.

7.6.11. From $\left(\dfrac{d}{dx} p(x) \dfrac{d}{dx} + q(x)\right) u = 0$ we have

(a) $\displaystyle\int \frac{dW}{W} = -\int^x \frac{p'}{p} dx = \ln\frac{1}{p} + \ln C = \ln W.$

Hence $W = \dfrac{W(a)}{p(x)}$ with $W(a) = C$.

(b) $y_2 = W(a) y_1 \displaystyle\int^x \frac{ds}{p(s) y_1(s)^2}$ follows from $W(y_1, y_2) = y_1^2(x) \dfrac{d}{dx}\dfrac{y_2(x)}{y_1(x)}$.

7.6.12. Using $y = zE$, $E = e^{-\frac{1}{2}\int^x P dt}$,

$$y' = z'E - \frac{1}{2} zPE, \qquad y'' = z''E - Pz'E - \frac{z}{2}P'E + \frac{z}{4}P^2 E,$$

we obtain

$$y'' + Py' + Q = E\left[z'' - \frac{z}{2}P' - \frac{z}{4}P^2 + Qz\right] = 0.$$

7.6.13. Since $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r} + \dfrac{L^2}{r^2}$ we have

$$-\frac{1}{2}\int^r P\,dt = -\int^r \frac{dr}{r} = -\ln r, \qquad e^{-\frac{1}{2}\int^r P dt} = \frac{1}{r},$$

so that $\varphi(r) = \psi(r)/r$. Equivalently $\nabla^2 \varphi(r) = \dfrac{1}{r}\dfrac{d^2}{dr^2}(r\varphi) + \dfrac{L^2}{r^2}\,\varphi$.

7.6.14. Defining $E_1 = \int^x \frac{e^{-\int^s P dt}}{y_1(s)^2} ds, \quad E(x) = e^{-\int^x P dt}$ and using

$$y_2 = y_1(x)E_1, \qquad y_2' = y_1' E_1 + \frac{E}{y_1}, \qquad y_2'' = y_1'' E_1 - \frac{PE}{y_1},$$

we obtain

$$y_2'' + Py_2' + Qy_2 = E_1(y_1'' + Py_1' + Qy_1) = 0.$$

7.6.15. Changing the lower limit from a to b changes the integral that multiplies y_2 by a constant:

$$\int_a^s P\,dt = \int_b^s P\,dt + \int_a^b P\,dt$$

and via

$$\int_a^x \frac{e^{-\int_a^s P\,dt}}{y_1^2(s)}\,ds = e^{-\int_a^b P dt} \int_b^x \frac{e^{-\int_b^s P\,dt}}{y_1^2(s)}\,ds + \int_a^b \frac{e^{-\int_a^s\,dt}}{y_1^2(s)}\,ds$$

adds a constant to y_2.

7.6.16. Using $-\int^r \frac{dr}{r} = -\ln r, \quad e^{-\int^r P\,dt} = \frac{1}{r}, \quad \int^r \frac{ds}{s \cdot s^{2m}} = -\frac{r^{-2m}}{2m}$, we have

$$y_2 = -r^m \frac{r^{-2m}}{2m} = -\frac{1}{2mr^m}.$$

7.6.17. As $P = 0$, $y_1 = \sin x$, $e^{-\int^x P\,dt}$ =const. and $y_2 = \sin x \int^x \frac{ds}{\sin^2 s} = \sin x \cot x = \cos x$.

Using the series expansions with $p_{-1} = 0 = q_{-2}$ gives the indicial equation $k(k-1) = 0$. Thus $k = \alpha = 1 = n$ and

$$y_2(x) = y_1(x) \left[c_1 \ln x + \sum_{j=0, j\neq 1}^{\infty} \frac{c_j}{j-1} x^{j-1} \right].$$

Substituting these y_2, y_2', y_2'' into the classical harmonic oscillator ODE yields

$$2y_2' \left(\frac{c_1}{x} + \sum_{j=0, j\neq 1}^{\infty} c_j x^{j-2} \right) + y_1 \left(-\frac{c_1}{x^2} + \sum_{j=0, j\neq 1}^{\infty} c_j (j-2) x^{j-3} \right) = 0.$$

The Taylor series for $y_1 = \sin x$ gives $2c_1 - c_1 = c_1 = 0$ for the coefficient of $1/x$. Thus, y_2 does not contain a term proportional to $\ln x$.

7.6.18. Since Bessel's ODE is invariant under $n \to -n$ we expect and verify that $J_{-n}(x)$, defined by its Taylor series, is a solution along with $J_n(x)$. From the lowest power series coefficients we obtain

$$W(J_n, J_{-n}) = \frac{A_n}{x} = -2\frac{\sin \pi n}{\pi x} \neq 0,$$

so that they are independent if $n \neq$ integer. This is Eq. (14.67).

The standard series $y = \sum_{j=0}^{\infty} a_j x^{j+k}$ leads to the indicial equation

$$[k(k-1) + k - N^2]a_0 = 0.$$

For $a_0 \neq 0$ we obtain $k = \pm N, N \geq 0$. The roots are $\alpha = N, n = 2N$, consistent with

$$p_j = \delta_{j,-1}, q_j = \delta_{j0} + N^2\delta_{j,-2}, n - 2\alpha = p_{-1} - 1, \alpha(\alpha - n) = q_{-2}.$$

The second solution is

$$y_2 = y_1(x)\sum_{j=0}^{\infty} c_j \int^x x_1^{j-n-1}dx_1.$$

If $n \neq$integer there is no $\ln x$ term in y_2. Since $n = 2N$, if N is neither an integer nor half of an odd integer, there is no logarithmic term in y_2.

It remains for us to show that when $N =$half an odd integer there is no $\ln x$ term in y_2. Since $W_N(x) \neq 0$ for $N \neq$integer, this is clear from our first part.

7.6.19. (a) If $y_1 = 1$ for $\alpha = 0$ then $\int^x P dt = -x^2$,and

$$y_2 = \int^x e^{s^2} ds, \qquad y_2' = e^{x^2}, \qquad y_2'' = 2xe^{x^2}.$$

Hence $y_2'' - 2xy_2' = 0$. Integrating the power series for e^{s^2} yields

$$y_2(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)j!} = \sum_{j=0}^{\infty} a_{2j}x^{2j+1}$$

with

$$\frac{a_{j+2}}{a_j} = \frac{2(j+1)}{(j+2)(j+3)}, \qquad j \text{ even},$$

which is the recursion for the $k = 1$ case of Exercise 8.3.3 (a) for $\alpha = 0$.

(b) If $\alpha = 1$ then $y_1 = x$ is a solution of the ODE, as is easily verified, and

$$y_2 = x\int^x \frac{e^{s^2}}{s^2} ds.$$

Integrating the power series yields

$$y_2(x) = \sum_{j=0}^{\infty} a_{2j} x^{2j} = -1 + \sum_{j=1}^{\infty} \frac{x^{2j}}{(2j-1)j!}$$

with

$$\frac{a_{j+2}}{a_j} = \frac{2(j-1)}{(j+1)(j+2)}, \qquad j \text{ even},$$

which is the recursion for $k = 0$ of Exercise 8.3.3 (a) for $\alpha = 1$.

7.6.20. For $n = 0$, $y_1 = 1$ is verified to be a solution of Laguerre's ODE where $P(x) = \frac{1}{x} - 1$. As $\int^x P dt = \ln x - x$,

$$y_2(x) = \int^x \frac{e^x}{x} dx = \ln x + 1 + \frac{x}{2} + \cdots .$$

$$= \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}.$$

7.6.21. (a) See the solution of Exercise 7.6.20.

(b) $y_2' = \frac{e^x}{x}$, $y_2'' = \frac{e^x}{x} - \frac{e^x}{x^2} = y_2' - \frac{y_2'}{x}$.

Hence $y_2'' + \left(\frac{1}{x} - 1\right) y_2' = 0$.

(c) $y_2 = \int^x \frac{e^s}{s}\, ds = \sum_{n=1}^{\infty} \frac{1}{n!} \int^x s^{n-1}\, ds = \ln x + \sum_{n=0}^{\infty} \frac{x^n}{n!n}$,

$y_2' = \frac{e^x}{x} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}$, and $y_2'' = -\frac{1}{x^2} + \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!n}$ imply

$$y_2'' + \left(\frac{1}{x} - 1\right) y_2' = \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} \left(\frac{1}{n} + \frac{1}{n(n-1)} - \frac{1}{n-1}\right) = 0.$$

7.6.22. (a) The coefficient $P(x_1)$ is the coefficient of y' when the ODE is written in a form such that the coefficient of y'' is unity; thus, $P(x) = -x/(1-x^2)$, and therefore $\int P(x)\, dx = \ln(1-x^2)/2$. Then the formula of Eq. (7.67) becomes (for $n = 0$, $y_1 = 1$),

$$y_2(x) = \int^x e^{-\ln(1-x_2^2)/2}\, dx_2 = \int (1-x^2)^{-1/2}\, dx = \sin^{-1} x\,.$$

(b) Letting $v = y'$, our ODE becomes $(1 - x^2)v' - xv = 0$, which is separable, of the form

$$\frac{dv}{v} = \frac{x\,dx}{1 - x^2}\,, \quad \text{with integral} \quad \ln v = -\frac{1}{2}\ln(1 - x^2)\,.$$

Exponentiating both sides, and then writing y_2 as the integral of v, we reach the same integral as in part (a).

7.6.23. The value of $\exp(-\int P\,dx)$ has the same value as in Exercise 7.6.22, namely $(1 - x^2)^{-1/2}$. Therefore our solution y_2 (for $n = 1$, $y_1 = x$) is

$$y_2 = x\int^x \frac{du}{u^2(1 - u^2)^{1/2}} = -(1 - x^2)^{1/2}\,.$$

7.6.24. Rescale the ODE by multiplying by $2m/\hbar^2$ so that

$$E' = 2mE/\hbar^2, \qquad b'_{-1} = 2mb_{-1}/\hbar^2, \text{ etc.}$$

The indicial equation has roots

$$\frac{-(p_{-1} - 1) \mp \sqrt{(p_{-1} - 1)^2 - 4q_{-2}}}{2}\,,$$

with $p_{-1} = 0$ and $q_{-2} = -l(l + 1)$. The root for the regular solution is $\alpha_1 = l + 1$ and that of the irregular solution is $\alpha_2 = -l$. Since $P(r) = 0$ we have

$$y_2(r) = y_1(r)\int^r \frac{ds}{y_1(s)^2}.$$

This leads to $y_2(r) \sim r^{-l}[1 + O(r)]$ as well.

7.6.25. $y_2' = y_1 f$ implies

$$y_2' = y_1' f + y_1 f', \qquad y_2'' = y_1'' f + 2y_1' f' + y_1 f'',$$

and so

$$y_2'' + Py_2' + Qy_2 = f(y_1'' + Py_1' + Qy_1) + Py_1 f' + 2y_1' f' + y_1 f'' = 0.$$

Thus $f''y_1 + f'(2y_1' + Py_1) = 0$.

Separating variables and integrating yields

$$\ln f' = -2\ln y_1 - \int^x P\,dt, \quad f' = \frac{1}{y_1(x)^2}e^{-\int^x P dt}, \text{ and } f \text{ as given.}$$

7.6.26. (a) From $y_1 = a_0 x^{(1+\alpha)/2}$, we have

$$y_1' = \frac{a_0}{2}(1 + \alpha)x^{(\alpha-1)/2}, \qquad y_1'' = \frac{a_0}{4}(\alpha^2 - 1)x^{(\alpha-3)/2}.$$

Hence
$$y_1'' + \frac{1-\alpha^2}{4x^2}y_1 = \frac{a_0}{4}x^{(\alpha-3)/2}(\alpha^2-1)(1-1) = 0.$$
Similarly,
$$y_2 = a_0 x^{(1-\alpha)/2}, \quad y_2' = \frac{a_0}{2}(1-\alpha)x^{-(\alpha+1)/2}, \quad y_2'' = \frac{a_0}{4}(\alpha^2-1)x^{-(\alpha+3)/2}.$$
Hence
$$y_2'' + \frac{1-\alpha^2}{4x^2}y_2 = \frac{a_0}{4}x^{-(\alpha+3)/2}(\alpha^2-1)(1-1) = 0.$$
Alternatively, a solution $y \sim x^p$ leads to $p(p-1)+(1-\alpha^2)/4 = 0$ with the roots $p = (1\pm\alpha)/2$.

(b) $y_{10} = a_0 x^{1/2}$, $P = 0$ give $\int^x Pdt = 0$, $e^{-\int^x Pdt} = 1$. Hence
$$y_{20} = a_0 x^{1/2}\int^x \frac{ds}{a_0^2 s} = \frac{1}{a_0}x^{1/2}\ln x.$$

(c) L'Hôpital's rule gives
$$\lim_{\alpha\to 0}\frac{y_1-y_2}{\alpha} = \lim_{\alpha\to 0}\frac{x^{(\alpha+1)/2}-x^{(-\alpha+1)/2}}{\alpha} = \frac{1}{2}\,x^{1/2}\ln x.$$

7.7 Inhomogeneous Linear ODEs

7.7.1. Denoting
$$E_1 = \int^x \frac{y_1 F ds}{W(y_1,y_2)}, \qquad E_2 = \int^x \frac{y_2 F ds}{W(y_1,y_2)},$$
we check that
$$y_p' = y_2'E_1 - y_1'E_2 + \frac{y_2y_1F}{W} - \frac{y_1y_2F}{W} = y_2'E_1 - y_1'E_2, y_p'' =$$
$$y_2''E_1 - y_1''E_2 + \frac{F}{W}(y_2'y_1 - y_2y_1') = y_2''E_1 - y_1''E_2 + F.$$
Hence
$$y_p'' + Py_p' + Qy_p = E_1(y_2''+Py_2'+Qy_2) - E_2(y_1''+Py_1'+Qy_1) + F = F.$$
This is the generalization of the variation of the constant method of solving inhomogeneous first-order ODEs to second-order ODEs.

If we seek a particular solution of the form $y_p(x) = y_1(x)v(x)$ with $y_1(x)$ a solution of the homogeneous ODE $y'' + Py + Qy = 0$, then v obeys
$$\frac{d}{dx}(y_1^2 v') + Py_1^2 v' = y_1 F,$$

from which there follows

$$\frac{d}{dx}(y_1^2 v' e^{\int^x P(t)\,dt}) = y_1(x)F(x)e^{\int^x P(t)\,dt} = \frac{y_1 F}{W(y_1, y_2)}.$$

Integrating this gives

$$\frac{y_1^2 v'}{W} = \int^x \frac{y_1(s)F(s)}{W(y_1(s), y_2(s))}\,ds.$$

Rewriting this as

$$v'(x) = \frac{d}{dx}\frac{y_2(x)}{y_1(x)}\int^x \frac{y_1(s)F(s)}{W(y_1(s), y_2(s))}\,ds$$

and integrating a second time yields

$$v(x) = \frac{y_2(x)}{y_1(x)}\int^x \frac{y_1(s)F(s)}{W(y_1(s), y_2(s))}\,ds - \int^x \frac{y_2(s)F(s)}{W(y_1(s), y_2(s))}\,ds.$$

Hence the desired y_p.

7.7.2. We need the general solution to the related homogeneous equation and a particular integral of the complete inhomogeneous ODE. The homogeneous equation $y'' + y = 0$ has solutions $y_1 = \cos x$ and $y_2 = \sin x$. We might be able to guess a particular integral ($y = 1$) but we can also use the method of variation of parameters. This method assumes a particular integral of the form $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, and leads to the two equations

$$u_1' y_1 + u_2' y_2 = u_1' \cos x + u_2' \sin x = 0\,,$$

$$u_1' y_1' + u_2' y_2' = -u_1' \sin x + u_2' \cos x = 1\,.$$

These equations have solution $u_1' = -\sin x$, $u_1' = \cos x$; these can be integrated to obtain $u_1 = \cos x$, $u_2 = \sin x$. Inserting these into the expression for $y(x)$, we get $y(x) = \cos^2 x + \sin^2 x = 1$. The general solution to the original ODE is therefore $c_1 \cos x + c_2 \sin x + 1$.

7.7.3. Following the strategy and notation of the answer to Exercise 7.7.2, we find $y_1 = \cos 2x$, $y_2 = \sin 2x$, from which we find $u_1' = -e^x \sin(2x)/2$ and $u_2' = e^x \sin(2x)/2$. We integrate these expressions to find $u_1 = (e^x/10)(2\cos 2x - \sin 2x)$, $u_2 = (e^x/10)(\cos 2x + 2\sin 2x)$, so $y = u_1 y_1 + u_2 y_2 = e^x/5$. The original ODE has general solution $c_1 \cos 2x + c_2 \sin 2x + e^x/5$.

7.7.4. Following the strategy and notation of the answer to Exercise 7.7.2, we find $y_1 = e^x$, $y_2 = e^{2x}$, from which we find $u_1' = -e^{-x}\sin x$ and $u_2' = e^{-2x}\sin x$. We integrate these expressions to find $u_1 = (e^{-x}/2)(\cos x + \sin x)$, $u_2 = -(e^{-2x}/5)(\cos x + 2\sin x)$, so $y = u_1 y_1 + u_2 y_2 = (3\cos x + \sin x)/10$. The original ODE has general solution $c_1 e^x + c_2 e^{2x} + (3\cos x + \sin x)/10$.

7.7.5. Following the strategy and notation of the answer to Exercise 7.7.2, we find by inspection $y_1 = x + 1$; using the Wronskian method we get the second solution $y_2 = e^x$. Remembering that the inhomogeneous term is to be determined when the original ODE is in standard form (coefficient of y'' equal to 1) we set up the equations for the u'_i and find $u'_1 = -1$, $u'_2 = (x+1)e^{-x}$, so $u_1 = -x$ and $u_2 = -(x+2)e^{-x}$. Thus, $y = u_1y_1 + u_2y_2 = -(x^2 + 2x + 2)$. We can, without generating an error, remove from y the $2x + 2$ since it is just $2y_1$. Thus, the original ODE has general solution $c_1(x + 1) + c_2e^x - x^2$.

7.8 Nonlinear Differential Equations

7.8.1. A more general solution to this Riccati equation is $y = 2+u$, where u is a general solution to the Bernoulli equation $u' = 3u + u^2$. See Eq. (7.104). In the notation of Eq. (7.101), $p = 3$, $q = 1$, and $n = 2$, and the Bernoulli equation has solution $u = 1/v$, where v is a solution of $v'+3v = -1$, namely $v = Ce^{-3x} + \frac{1}{3}$. Therefore $u = 3/(Ce^{-3x} - 1)$ and $y = 2+3/(Ce^{-3x} - 1)$.

7.8.2. A more general solution to this Riccati equation is $y = x^2 + u$, where u is a general solution to the Bernoulli equation $u' = u^2/x^3 + u/x$. See Eq. (7.104). In the notation of Eq. (7.101), $p = 1/x$, $q = 1/x^3$, and $n = 2$, and the Bernoulli equation has solution $u = 1/v$, where v is a solution of $v' + v = -1/x^3$, namely $v = (Cx+1)/x^2$. Therefore $u = x^2/(Cx+1)$ and

$$y = x^2 + \frac{x^2}{Cx+1} = \frac{Cx^3 + 2x^2}{Cx+1}\,.$$

7.8.3. This ODE corresponds to Eq. (7.101) with $p = -x$, $q = x$, and $n = 3$. Thus, with $u = y^{-2}$, Eq. (7.102) becomes $u' - 2xu = -2x$. The homogeneous equation for u has solution e^{x^2}, and from the method of variation of parameters or by inspection, a particular integral of the inhomogeneous equation is $u = 1$. Thus the general solution for u is $u = Ce^{x^2} + 1$. Since $y = u^{-1/2}$, the general solution for y is $y = 1/\sqrt{Ce^{x^2} + 1}$.

7.8.4. (a) The general solution comes from $y'' = 0$, and therefore has the form $y = ax + b$. However, not all values of a and b lead to solutions of the original Clairaut equation. Substituting into $y = xy' + (y')^2$, we find $ax + b = xa + a^2$, which shows that y is a solution only if $b = a^2$.
(b) The singular solution comes from $2y' = -x$, which integrates to $y = -x^2/4 + C$. Substituting into $y = xy' + (y')^2$, we get $-x^2/4 + C = x(-x/2) + x^2/4$, which shows that this y is a solution only if $C = 0$.

The singular and a general solution coincide only if $-x_0^2/4 = ax_0+a^2$, the solution to which is $x_0 = -2a$. At x_0, both solutions have slope a, so the singular solution is tangent to each instance of the general solution and is therefore referred to as their envelope.

8. Sturm-Liouville Theory

8.1 Introduction

(no exercises)

8.2 Hermitian Operators

8.2.1. Using $\psi(x) = e^{-x/2}y(x)$ in the ODE $xy'' + (1-x)y' + ny = 0$ gives the equivalent self-adjoint ODE

$$\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) + (n + \frac{1}{2} - \frac{x}{4})\psi = e^{-x/2}\left[xy'' + (1-x)y' + ny\right] = 0.$$

The weight function $w = e^{-x}$ and the interval are also obvious from the orthogonality relation, Eq. (18.55). Note also that

$$\frac{d}{dx}\left(xe^{-x}y'(x)\right) = e^{-x}[(1-x)y' + xy'']$$

from which $p(x) = xe^{-x}$ follows. Note that multiplying the wave function by $e^{-x/2}$ and the ODE by e^{-x} leads to the same results.

8.2.2. Using $\psi_n(x) = e^{-x^2/2}H_n(x)$ in the ODE $H_n'' - 2xH_n' + 2nH_n = 0$ gives the equivalent self-adjoint Hermite ODE

$$\psi_n'' + (2n + 1 - x^2)\psi_n = e^{-x^2/2}[H_n'' - 2xH_n' + 2nH_n] = 0.$$

The weight function $w = e^{-x^2}$ and the interval are obvious from the orthogonality relation in Eq. (18.11). Note that multiplying the wave function by $e^{-x^2/2}$ and the ODE by e^{-x^2} leads to the same results.

8.2.3. The Chebyshev ODE in Table 7.1 is that whose polynomial solutions are the Type I Chebyshev polynomials T_n. Multiplying the ODE $(1-x^2)T_n'' - xT_n' + n^2T_n = 0$ by $(1-x^2)^{-1/2}$, we obtain the equivalent self-adjoint ODE

$$\frac{d}{dx}\left[(1-x^2)^{1/2}\frac{dT_n}{dx}\right] + n^2(1-x^2)^{-1/2}T_n = 0\,.$$

The coefficient of T_n has the functional form of the scalar-product weighting function.

8.2.4. (a) For Legendre's ODE $p(x) = 1 - x^2$, which is zero for $x = \pm 1$. Thus $x = \pm 1$ can be the endpoints of the interval. Since polynomial solutions of the ODE will be finite and have finite derivatives at $x = \pm 1$, then

$$v^*pu'|_{x=\pm 1} = \mp 1\frac{n(n+1)}{2}(1-x^2)|_{x=\pm 1}v^*(\pm 1)u(\pm 1) = 0,$$

and Sturm-Liouville boundary conditions will be satisfied for the interval $[-1, 1]$.

(b) We consider here the Chebyshev polynomials $T_n(x)$. When the Chebyshev ODE is written in self-adjoint form, the coefficient $p(x)$ is $(1-x^2)^{1/2}$, which is zero at $x = \pm 1$. Therefore the Sturm-Liouville boundary conditions are satisfied at $x = \pm 1$ because the polynomials remain finite there and have finite derivatives.

(c) For Hermite's ODE $p(x) = e^{-x^2} \to 0$ only for $x \pm \infty$., and $p(x)$ goes to zero faster than any polynomial. Thus, the boundary conditions are satisfied for the interval $(-\infty, \infty)$.

(d) For Laguerre's ODE $p(x) = xe^{-x}$ is zero for $x = 0$ and $p(x)$ goes to zero as $x \to \infty$ faster than ay polynomial, so the boundary conditions are satisfied for the interval $[0, \infty)$.

8.2.5. If $u_2 = Cu_1$, then $Hu_2 = C(Hu_1) = \lambda_1 Cu_1 = \lambda_1 u_2$, i.e., $\lambda_1 = \lambda_2$. Thus, two linearly dependent eigenfunctions cannot have different eigenvalues.

8.2.6. (a) Use integration by parts, integrating the factor x and writing the result as $(x^2-1)/2$, and differentiating the logarithms. This yields

$$\int_{-1}^{1} \frac{x}{2} \ln \frac{1+x}{1-x}\, dx = \frac{x^2-1}{4} \ln \frac{1+x}{1-x}\bigg|_{-1}^{1} - \int_{-1}^{1} \frac{x^2-1}{4}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} dx = 1,$$

the integrated term being zero. Alternatively, we can expand $Q_0(x)$ as a power series and then integrate $xQ_0(x)$ term by term. We get

$$\int_{-1}^{1} P_1 Q_0\, dx = \lim_{\epsilon\to 0} \sum_{\nu=0}^{\infty} \int_{-1+\epsilon}^{1-\epsilon} \frac{x^{2\nu+2}}{2\nu+1}\, dx = 2\sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)(2\nu+3)} = 1 \neq 0.$$

(b) The necessary boundary conditions are violated because Q_0 is singular at $x = \pm 1$.

8.2.7. Dividing $(1-x^2)y'' - xy' + n^2 y = 0$ by $(1-x^2)^{1/2}$ puts the Chebyshev ODE in self-adjoint form with $p(x) = (1-x^2)^{1/2}$, $q(x) = 0$, $w(x) = (1-x^2)^{-1/2}$, $\lambda = n^2$. The boundary condition

$$p(v^* u' - v'^* u)\bigg|_{-1}^{1} = 0$$

is not satisfied when $u = T_0(x)$ and $v = V_1(x)$. In this particular case, $u' = 0$, $u = 1$, and v' is an odd function which becomes infinite at $x = \pm 1$ at a rate that is proportional to $1/p(x)$. The result is that the Sturm-Liouville boundary condition is not satisfied.

8.2.8. By integrating by parts the first term of

$$\int_a^b u_m \frac{d}{dx} p(x) u_n'\, dx + \lambda_n \int_a^b u_m w(x) u_n\, dx = 0,$$

we obtain

$$u_m p(x) u'_n|_a^b - \int_a^b u'_m p u'_n \, dx + \lambda_n \int_a^b u_m w(x) u_n \, dx = 0.$$

The first term is zero because of the boundary condition, while the third term reduces to $\lambda_n \delta_{nm}$ by orthogonality. Hence the orthogonality relation

$$\int_a^b u'_m p u'_n dx = \lambda_n \delta_{mn}.$$

8.2.9. If $\psi_n = \sum_{i=1}^{n-1} a_i \psi_i$ then $A\psi_n = \lambda_n \psi_n = \sum_{i=1}^{n-1} a_i \lambda_i \psi_i$. Comparing both expansions, $a_i \lambda_i / \lambda_n = a_i$, i.e., $\lambda_i = \lambda_n$ for those i for which $a_i \neq 0$. This contradicts our hypothesis.

8.2.10. (a) Multiply by $(1-x^2)^{\alpha-1/2}$.

8.3 ODE Eigenvalue Problems

8.3.1. Using $y = \sum_{j=0}^{\infty} a_j x^{j+k}$ to solve $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ yields

$$a_{j+2} = \frac{(j+k)(j+k+1) - n(n+1)}{(j+k+2)(j+k+1)} \, a_j.$$

(a) For $j = -2$, $a_{-2} = 0$ sets up the indicial equation $k(k-1)a_0 = 0$, with solutions $k = 0$ and $k = 1$ for $a_0 \neq 0$.

(b) The case $k = 0$ gives the recursion formula

$$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+2)(j+1)} \, a_j.$$

Hence $y(x)$ has even parity.

(c) If $k = 1$ then we get the recursion formula

$$a_{j+2} = \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)} \, a_j.$$

Hence $y(x)$ has odd parity.

(d) If the numerator of either recursion formula is always nonzero, then the ratio $a_{j+2}/a_j \to 1$ as $j \to \infty$, implying divergence for $x = 1$. Both the above series also diverge at $x = -1$.

(e) If n is a non-negative integer one of the two series of cases (b) and (c) breaks off at $j = n$, generating in case (b) the Legendre polynomials containing even powers of x, and in case (c) the Legendre polynomials containing odd powers of x.

8.3.2. If the Hermite ODE is multiplied through by $\exp(-x^2)$, it becomes

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0 \quad \longrightarrow \quad \Big[e^{-x^2}y'\Big]' + 2\alpha e^{-x^2}y = 0,$$

a manifestly self-adjoint ODE. This eigenvalue problem will be Hermitian if the weight factor $\exp(-x^2)$ is included in the scalar product and the ODE is solved subject to Sturm-Liouville boundary conditions. The requirement that the scalar product exist will necessarily mean that the boundary terms must vanish at $x = \pm\infty$, thereby defining a Hermitian problem.

8.3.3. (a) The trial solution $\sum_j a_j x^{k+j}$ yields the recursion formula

$$a_{j+2} = \frac{2(k+j-\alpha)\,a_j}{(k+j+1)(k+j+2)}.$$

For $k = 0$, $a_0 \neq 0$, $a_1 = 0$ we get the given y_{even}.

For $k = 1$, $a_0 \neq 0$, $a_1 = 0$ we get the given y_{odd}.

(b) For $j \gg \alpha, k$ the recursion yields $a_{j+2}/a_j \to 2/j$, just like the coefficients of e^{x^2}, viz. $(j/2)!/(\frac{j}{2}+1)! \to 1/(\frac{j}{2}+1)$.

(c) If $\alpha =$non-negative integer, then the series break off.

8.3.4. Let n be a non-negative integer. Then the ODE is Eq. (18.44) and its solutions are given in Eqs. (18.46), (18.53) and Table 18.2. The trial solution $\sum_j a_j^{(n)} x^{k+j}$ yields the recursion formula

$$a_{j+2}^{(n)} = \frac{(k+j-n)a_j^{(n)}}{(k+j+1)^2}.$$

For $k = 0$ and n a non-negative integer the series breaks off.

8.3.5. The infinite series **does** converge for $x = \pm 1$. Hence this imposes no restriction on n. Compare with Exercise 1.2.6. If we demand a polynomial solution then n must be a nonnegative integer.

8.3.6. For $k = 1$, take n to be a positive odd integer. Compare with Eq. (18.98). Here $a_0 = (-1)^{(n-1)/2}(r+1)$.

8.4 Variation Method

8.4.1. (a) The normalization integral is $4\alpha^3 \displaystyle\int_0^\infty x^2 e^{-2\alpha x}\,dx = 4\alpha^3 \left(\frac{2!}{(2\alpha)^3}\right) = 1.$

(b) $\langle x^{-1}\rangle = 4\alpha^3 \displaystyle\int_0^\infty x e^{-2\alpha x}\,dx = 4\alpha^3 \left(\frac{1!}{(2\alpha)^2}\right) = \alpha.$

(c) $\dfrac{d^2\psi}{dx^2} = 2\alpha^{3/2}(\alpha^2 x - 2\alpha)$, and therefore

$$\left\langle \frac{d^2}{dx^2} \right\rangle = 4\alpha^3 \int_0^\infty (\alpha^2 x^2 - 2\alpha x)e^{-2\alpha x}\, dx = \alpha^2 - 2\alpha^2 = -\alpha^2.$$

(d) For general α,

$$W(\alpha) = \left\langle \psi \left| -\frac{1}{2}\,\frac{d^2}{dx^2} - \frac{1}{x} \right| \psi \right\rangle = \frac{\alpha^2}{2} - \alpha\,.$$

The value of α that minimizes $W(\alpha)$ is obtained by setting $dW/d\alpha = 0$; the result is $\alpha = 1$, from which we find $W(1) = -1/2$.

8.5 Summary, Eigenvalue Problems

(no exercises)

9. Partial Differential Equations

9.1 Introduction

(no exercises)

9.2 First-Order Equations

9.2.1. Introduce variables $s = x + 2y$, $t = 2x - y$. Then

$$\frac{\partial\psi}{\partial x} + 2\frac{\partial\psi}{\partial y} = 5\left(\frac{\partial\psi}{\partial s}\right)_t ,$$

and our PDE becomes an ODE in s with parametric dependence on t:

$$5\,\frac{d\psi}{ds} + t\psi = 0\,, \quad \text{so} \quad \ln\psi = -\frac{ts}{5} + C(t) \quad \text{or} \quad \psi = f(t)e^{-ts/5}\,,$$

where $f(t)$ is arbitrary. In terms of x and y, the general solution of this PDE is

$$\psi(x,y) = f(2x-y)\,e^{-(2x^2-2y^2+3xy)/5}\,.$$

This solution assumes a somewhat simpler form if we multiply the exponential by $\exp(-2t^2/5) = \exp(-[8x^2+2y^2-8xy]/5)$ (incorporating the change into f), reaching $\psi(x,y) = f(2x-y)\exp(-2x^2+xy)$.

9.2.2. Following a procedure similar to that in the solution to Exercise 9.2.1, set $s = x - 2y$ and $t = 2x + y$, and note that $x + y = (3t - s)/5$. The PDE reduces to

$$5\,\frac{d\psi}{ds} + \frac{3t-s}{5} = 0\,.$$

This ODE has solution $\psi = (s-3t)^2/50 + f(t) = (x+y)^2/2 + f(2x+y)$, with f arbitrary.

9.2.3. Here $s = x + y - z$; t and u can be $t = x - y$, $u = x + y + 2z$. The PDE reduces to $3\,d\psi/ds = 0$, with solution $\psi = f(t,u) = f(x-y, x+y+2z)$, with f arbitrary.

9.2.4. Here $s = x + y + z$, take $t = x - y$, $u = x + y - 2z$. The PDE reduces to

$$3\frac{d\psi}{ds} = t\,, \quad \text{with solution} \quad \psi = \frac{ts}{3} + C\,.$$

The solution can be simplified by subtracting $tu/3$ and making the observation that $t(s-u)/3 = tz$. We then have $\psi = z(x-y) + f(x-y, x+y-2z)$, with f arbitrary.

9.2.5. (a) It is useful to note that

$$dv = 2x\,dx - 2y\,dy \longrightarrow 2x\left(\frac{\partial x}{\partial u}\right)_v - 2y\left(\frac{\partial y}{\partial_u}\right)_v = 0 \longrightarrow \left(\frac{\partial y}{\partial u}\right)_v = \frac{x}{y}\left(\frac{\partial x}{\partial u}\right)_v .$$

Then we form

$$\left(\frac{\partial\psi}{\partial u}\right)_v = \left[\left(\frac{\partial\psi}{\partial x}\right)_y + \frac{x}{y}\left(\frac{\partial\psi}{\partial y}\right)_x\right]\left(\frac{\partial x}{\partial u}\right)_v = 0\,,$$

where the right-hand side of this equation vanishes because the quantity within the square brackets is zero according to the PDE. We now effectively have an ODE in u with solution $\psi = f(v) = f(x^2 - y^2)$, with f arbitrary.

(b) The lines of constant v are characteristics of this equation; they differ from our earlier examples in that they are not straight lines, but curves defined by $x^2 - y^2 =$ constant.

9.2.6. Define u and v as in Exercise 9.2.5, and from $du = x\,dy + y\,dx$ find

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{y}{x}\left(\frac{\partial x}{\partial v}\right)_u\,.$$

Now we form

$$\left(\frac{\partial\psi}{\partial v}\right)_u = \left[\left(\frac{\partial\psi}{\partial x}\right)_y - \frac{y}{x}\left(\frac{\partial\psi}{\partial y}\right)_x\right]\left(\frac{\partial x}{\partial v}\right)_u = 0\,,$$

where the quantity within square brackets vanishes by virtue of the PDE. Integrating the resulting ODE, we get $\psi = f(u) = f(xy)$, with f arbitrary.

9.3 Second-Order Equations

9.3.1. It may be easiest to multiply out the factored expression for $\mathcal{L}$ and start from $\mathcal{L}f = af_{xx} + 2bf_{xy} + cf_{yy}$, where the subscripts identify differentiations. Then, using the definitions of ξ and η, we have

$$f_x = c^{1/2} f_\xi\,, \quad f_y = c^{-1/2}(-b\,f_\xi + f_\eta)\,, \quad f_{xx} = c\,f_{\xi\xi}$$

$$f_{xy} = -b\,f_{\xi\xi} + f_{\xi\eta}\,, \quad f_{yy} = c^{-1}(b^2 f_{\xi\xi} - 2bf_{\xi\eta} + f_{\eta\eta})\,.$$

Substituting into the original expression for $\mathcal{L}$, we get

$$\mathcal{L}f = (ac - b^2)f_{\xi\xi} + f_{\eta\eta}\,.$$

9.4 Separation of Variables

9.4.1. $(\boldsymbol{\nabla}^2 + k^2)(a_1\psi_1 + a_2\psi_2) = a_1\boldsymbol{\nabla}^2\psi_1 + a_1k^2\psi_1 + a_2\boldsymbol{\nabla}^2\psi_2 + a_2k^2\psi_2$.

9.4.2. If $\psi = R(\rho)\Phi(\varphi)Z(z)$ then

$$\left(\frac{1}{R\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho} + f(\rho) + k^2\right) + \frac{1}{\rho^2}\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + g(\varphi)\right) + \left(\frac{1}{Z}\frac{d^2Z}{dz^2} + h(z)\right) = 0$$

leads to the separated ODEs

$$\frac{d^2Z}{dz^2} + h(z)Z = n^2 Z,$$

$$\frac{d^2\Phi}{d\varphi^2} + g(\varphi)\Phi = -m^2\Phi,$$

$$\rho\frac{d}{d\rho}\rho\frac{dR}{d\rho} + [(n^2 + f(\rho) + k^2)\rho^2 - m^2]R = 0.$$

9.4.3. Writing $\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$, and noting that

$$\mathbf{L}^2 Y(\theta,\varphi) = l(l+1)Y(\theta,\varphi),$$

$$\mathbf{L}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\,,$$

we have

$$(\boldsymbol{\nabla}^2 + k^2)\psi(r,\theta,\varphi) = \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\mathbf{L}^2}{r^2} + k^2\right)R(r)Y(\theta,\varphi),$$

$$\frac{d}{dr}r^2\frac{dR}{dr} + (k^2r^2 - l(l+1))R = 0.$$

The order in which variables are separated doesn't matter.

9.4.4. Separating

$$\frac{1}{R}\frac{d}{dr}r^2\frac{dR}{dr} + (k^2 + f(r))r^2 = \frac{\mathbf{L}^2}{r^2} - g(\theta) - \frac{h(\varphi)}{\sin^2\theta} = l(l+1)$$

implies

$$\frac{d}{dr}r^2\frac{dR}{dr} + [(k^2 + f(r))r^2 - l(l+1)]R = 0,$$

$$-\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial P}{\partial\theta} - P[(g(\theta) + l(l+1)]\sin^2\theta + m^2P = 0,$$

$$\frac{d^2\Phi}{d\varphi^2} + h(\varphi)\Phi = -m^2\Phi.$$

9.4.5. $\psi = Ae^{i\mathbf{k}\cdot\mathbf{r}}$ obtained from separating the Cartesian coordinates gives

$$\boldsymbol{\nabla}\psi = i\mathbf{k}\psi, \qquad \boldsymbol{\nabla}^2\psi = -k^2\psi, \quad \text{with}$$

$$k_x = \frac{\pi}{a}n_x, \ldots, \qquad E = \hbar^2k^2/2m, \qquad k^2 = \pi^2\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right).$$

The case $n_x = n_y = n_z = 1$ gives the answer in the text.

9.4.6. Writing

$$\mathbf{L}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\text{with} \quad \frac{\partial^2 e^{im\varphi}}{\partial\varphi^2} = -m^2 e^{im\varphi} \quad \text{for} \quad \psi \sim e^{im\varphi}P_l^m$$

gives the ODE for the associated Legendre polynomials.

9.4.7. (a) $b\psi'' - \dfrac{mk}{\hbar^2}x^2\psi = -\dfrac{2mE}{\hbar^2}\psi$ becomes $\alpha^2\dfrac{d^2\psi}{d\xi^2} - \alpha^4 x^2\psi = -\alpha^2\lambda\psi$.

(b) $\psi(\xi) = y(\xi)e^{-\xi^2/2}$ implies

$$\psi' = y'e^{-\xi^2/2} - \xi y e^{-\xi^2/2},$$

$$\psi'' = y''e^{-\xi^2/2} - 2\xi y' e^{-\xi^2/2} - ye^{-\xi^2/2} + \xi^2 y e^{-\xi^2/2},$$

$$\text{and} \quad e^{-\xi^2/2}(y'' + \lambda y - 2\xi y' - y) = 0,$$

which is Hermite's ODE for y.

9.5 Laplace and Poisson Equations

9.5.1. (a) $\nabla^2\dfrac{1}{r} = -\dfrac{2}{r}\dfrac{1}{r^2} - \dfrac{d}{dr}\dfrac{1}{r^2} = 0 \quad$ for $r > 0$.

See Example 3.6.1. Or use

$$\nabla^2 f(r) = \frac{1}{r}\frac{d^2}{dr^2}\Big[rf(r)\Big] \quad \text{for } f(r) = r,\ r > 0.$$

For $r = 0$ there is a singularity described by

$$\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r}).$$

(b) In spherical polar coordinates $z = r\cos\theta$, so $\psi_2 = \dfrac{1}{2r}\ln\dfrac{1+\cos\theta}{1-\cos\theta}$.
For $r \neq 0$,

$$\begin{aligned}
\nabla^2\psi_2 &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_2}{\partial\theta}\right) \\
&= \frac{1}{2r^3\sin\theta}\frac{d}{d\theta}\left[\sin\theta\left(-\frac{\sin\theta}{1+\cos\theta} - \frac{\sin\theta}{1-\cos\theta}\right)\right] \\
&= -\frac{1}{2r^3\sin\theta}\frac{d}{d\theta}\left(\frac{2\sin^2\theta}{1-\cos^2\theta}\right) = 0.
\end{aligned}$$

For $r = 0$,

$$\int_{r\le R} \nabla^2 \psi_2 d^3 r = \oint \hat{\mathbf{r}} \cdot \nabla \psi_2 = -\frac{R^2}{2R^2} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \ \ln \frac{1+\cos\theta}{1-\cos\theta}\, d\theta$$

$$= -\pi \int_{-1}^{1} \ln \frac{1+z}{1-z}\, dz = 0$$

because $\ln \dfrac{1+z}{1-z}$ is odd in z.

9.5.2. $\nabla^2 \dfrac{\partial}{\partial z}\psi = \dfrac{\partial}{\partial z}\nabla^2\psi = 0$ because $\left[\nabla^2, \dfrac{\partial}{\partial z}\right] = 0.$

9.5.3. Taking ψ to be the difference of two solutions with the same Dirichlet boundary conditions, we have for ψ a Laplace equation with $\psi = 0$ on an entire closed boundary. That causes the left-hand side of Eq. (9.88) to vanish; the first integral on the right-hand side also vanishes, so the remaining integral must also be zero. This integral cannot vanish unless $\nabla\psi$ vanishes everywhere, which means that ψ can only be a constant.

9.6 Wave Equation

9.6.1. The most general solution with $\psi(x, 0) = \sin x$ is $\psi = A \sin(x - ct) + (1 - A)\sin(x+ct)$; for this solution $\partial\psi/\partial t$ evaluated at $t = 0$ is $(1-2A)c \cos x$. The condition on $\partial\psi/\partial t$ requires that we set $(1 - 2A)c = 1$, or $A = (c - 1)/2c$. Thus,

$$\psi(x, t) = \left(\frac{c-1}{2c}\right)\sin(x - ct) + \left(\frac{c+1}{2c}\right)\sin(x + ct)$$

$$= \sin x \cos ct + c^{-1} \cos x \sin ct\, .$$

9.6.2. Given a general solution of the form $f(x - ct) + g(x + ct)$ we require $f(x) + g(x) = \delta(x)$ and $f'(x) - g'(x) = 0$, i.e., $f'(x) = g'(x)$. This second condition leads to $g(x) = f(x) +$ constant, and the first condition then yields $f(x) = g(x) = \delta(x)/2$. Therefore,

$$\psi(x, t) = \frac{1}{2}\Big[\delta(x - ct) + \delta(x - ct)\Big]\, .$$

9.6.3. By a process similar to that for Exercise 9.6.2, we have

$$\psi(x, t) = \frac{1}{2}\Big[\psi_0(x - ct) + \psi_0(x + ct)\Big]\, .$$

9.6.4. The functions $f(x - ct)$ and $g(x + ct)$ with t derivatives equal to $\sin x$ at $t = 0$ are (apart from a constant) $c^{-1} \cos(x - ct)$ and $-c^{-1} \cos(x + ct)$.

Thus, we must have $\psi(x,0) = c^{-1}[A\cos(x-ct) - (1-A)\cos(x+ct)]$; to make $\psi(x,0) = 0$ we take $A = 1/2$, so

$$\psi(x,t) = \frac{1}{2c}\Big[\cos(x-ct) - \cos(x+ct)\Big] = \frac{1}{c}\,\sin x \sin ct\,.$$

9.7 Heat Flow, or Diffusion PDE

9.7.1. From

$$\frac{1}{KT}\frac{dT}{dt} = \frac{1}{Rr^2}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{\mathbf{L}^2 Y}{Yr^2} = -\alpha^2,$$

it follows that

$$\frac{dT}{dt} = -\alpha^2 KT,$$

$$\mathbf{L}^2 Y = l(l+1)Y,$$

$$\frac{d}{dr}r^2\frac{dR}{dr} + \alpha^2 r^2 R = l(l+1)R.$$

By spherical symmetry $l = 0$, $Y = Y_{00} = 1/\sqrt{4\pi}$. So $l = m = 0$.

9.7.2. Without z, φ-dependence and denoting $\lambda = \kappa/\sigma\rho$ we have $\dfrac{\partial\psi}{\partial t} = \lambda\boldsymbol{\nabla}^2\psi$ with $\psi = P(\rho)T(t)$ so that

$$\frac{1}{\lambda T}\frac{dT}{dt} = -\alpha^2 = \frac{1}{R\rho}\frac{d}{d\rho}\rho\frac{dP}{d\rho}.$$

Hence

$$\frac{dT}{dt} = -\lambda\alpha^2 T,$$

$$\frac{d}{d\rho}\rho\frac{dP}{d\rho} + \alpha^2\rho P = 0 = \rho\frac{d^2P}{d\rho^2} + \frac{dP}{d\rho} + \alpha^2\rho P.$$

9.7.3. Equation (9.114) applies to this problem, as it is for the 1-D boundary condition that $\psi \to 0$ at $x = \pm\infty$ and is written in terms of the temperature distribution at $t = 0$. Thus, with $\psi_0 = A\delta(x)$, we have

$$\psi(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} A\,\delta(x - 2a\xi\sqrt{t})\,e^{-\xi^2}\,d\xi\,.$$

Using the relation $\int \delta(at-b)f(t)\,dt = a^{-1}f(b/a)$, we find

$$\psi(x,t) = \frac{A}{2a\sqrt{\pi t}}\,e^{-x^2/4a^2t}\,.$$

This has the expected properties: at $t = 0$ it is zero everywhere except at $x = 0$; it approaches zero everywhere at $t \to \infty$; for all t the integral of ψ over x is A.

9.7.4. This problem becomes notationally simpler if the coordinates of the ends of the rod are placed at $-L/2$ and $L/2$, with the end at $-L/2$ kept at $T = 0$ and the end at $L/2$ kept at $T = 1$. We write the initial temperature distribution in terms of the spatial eigenfunctions of the problem as

$$\psi_0 = \frac{x}{L} + \frac{1}{2} - \sum_j c_j \varphi_j(x)\,,$$

where the j summation is the expansion of $x/L+1/2$ in the eigenfunctions of nonzero ω. This mode of organization makes explicit that all the terms in the j sum must decay exponentially in t, leaving in the large-t limit the steady-state temperature profile that connects the fixed temperatures at the ends of the rod.

The φ_j must be chosen subject to the boundary condition that they vanish at $x = \pm L/2$; those representing the expansion of $1/2$ must be cosine functions, while those for the expansion of x/L must be sine functions. Specifically,

$$\begin{array}{lll} \text{Expansion of } 1/2: & \varphi_j = \cos j\pi x/L, & j \text{ odd} \\ \text{Expansion of } x/L: & \varphi_j = \sin j\pi x/L, & j \text{ even} \end{array}$$

Making use of the orthogonality properties of these functions and changing the indices to account for the restriction to odd and even values, we have

$$\frac{1}{2} = \sum_{j=0}^{\infty} c_{2j+1} \cos\frac{(2j+1)\pi x}{L}, \qquad c_{2j+1} = \frac{2(-1)^j}{\pi(2j+1)}\,,$$

$$\frac{x}{L} = \sum_{j=1}^{\infty} c_{2j} \sin\frac{2j\pi x}{L}, \qquad c_{2j} = \frac{(-1)^{j+1}}{\pi j}\,.$$

To form $\psi(x,t)$ we now attach to each term in the summations the decaying exponential factor shown in Eq. (9.101) and append the time-independent terms corresponding to $\omega = 0$:

$$\psi(x,t) = \frac{x}{L} + \frac{1}{2} - \sum_{j=0}^{\infty} \frac{2(-1)^j}{\pi(2j+1)} \cos\frac{(2j+1)\pi x}{L}\; e^{-t[(2j+1)\pi a/L]^2}$$

$$- \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{\pi j} \sin\frac{2j\pi x}{L}\; e^{-t(2j\pi a/L)^2}\,.$$

9.8 Summary

(no exercises)

10. Green's Functions

10.1 One-Dimensional Problems

10.1.1. The general solution to $-d^2y/dx^2 = 0$ is $y = c_1x + c_0$; a solution u with $u(0) = 0$ is $u(x) = x$; a solution v with $v'(1) = 0$ is $v(x) = 1$. Thus the form of the Green's function must be $G(x,t) = Ax$ for $x < t$ and $G(x,t) = At$ for $x > t$. To find A we note that for the given $\mathcal{L}$, $p = -1$ and that $A = p(t)[uv' - u'v] = (-1)[0-1] = +1$. We recover the required formula for $G(x,t)$.

10.1.2. (a) $$G(x,t) = \begin{cases} -\dfrac{\sin x\cos(1-t)}{\cos 1}, & 0 \le x \le t, \\[2ex] -\dfrac{\sin t\cos(1-x)}{\cos 1}, & t \le x \le 1. \end{cases}$$

(b) $$G(x,t) = \begin{cases} -e^{x-t}/2, & -\infty < x < t, \\ -e^{t-x}/2, & t < x < \infty. \end{cases}$$

10.1.3. Our expression for $y(x)$ is $y(x) = \displaystyle\int_0^x \sin(x-t)f(t)\,dt$. Its derivatives are

$$y'(x) = \sin(x-x)f(x) + \int_0^x \cos(x-t)f(t)\,dt = \int_0^x \cos(x-t)f(t)\,dt.$$

$$y''(x) = \cos(x-x)f(x) - \int_0^x \sin(x-t)f(t)\,dt = f(x) - y(x)\,.$$

This equation shows that $y(x)$ satisfies Eq. (10.24) and the formulas for $y(0)$ and $y'(0)$ show that both vanish.

10.1.4. The solutions to the homogeneous ODE of this exercise (that with $f(x) = 0$) are $y_1(x) = \sin(x/2)$ and $y_2(x) = \cos(x/2)$. To satisfy the boundary condition at $x = 0$ we take $G(x,t) = \sin(x/2)h_1(t)$ for $x < t$; to satisfy the boundary condition at $x = \pi$ we take $G(x,t) = \cos(x/2)h_2(t)$ for $x > t$. To achieve continuity at $x = t$ we take $h_1(t) = A\cos(t/2)$ and $h_2(t) = A\sin(t/2)$. The value of A must cause $\partial G/\partial x$ to have a discontinuous jump of -1 at $x = t$ (the coefficient p of the ODE is -1). The difference in those derivatives is

$$\frac{\partial G(x,t)}{\partial x}\Big|_{x=t+} - \frac{\partial G(x,t)}{\partial x}\Big|_{x=t-} =$$

$$-\frac{A}{2}\sin(x/2)\sin(t/2) - \frac{A}{2}\cos(x/2)\cos(t/2) \longrightarrow -\frac{A}{2} = -1,$$

so $A = 2$.

10.1.5. With $\mathcal{L} = x\dfrac{d^2}{dx^2} + \dfrac{d}{dx} + \dfrac{k^2x^2 - 1}{x}$, $\mathcal{L} = 0$ has solutions $J_1(kx)$ and $Y_1(kx)$.

We use $J_1(kx)$ as a solution that vanishes at $x = 0$; we form a linear combination of $J_1(kx)$ and $Y_1(kx)$ that vanishes at $x = 1$. The constant $\pi/2$ comes from an evaluation of the Wronskian of these two solutions, most easily evaluated from their asymptotic forms, see Eqs. (14.140) and (14.141).

$$G(x,t) = \begin{cases} \dfrac{\pi}{2}\left[Y_1(kt) - \dfrac{Y_1(k)J_1(kt)}{J_1(k)}\right] J_1(kx), & 0 \le x < t, \\[2ex] \dfrac{\pi}{2}\left[Y_1(kx) - \dfrac{Y_1(k)J_1(kx)}{J_1(k)}\right] J_1(kt), & t < x \le 1. \end{cases}$$

10.1.6. $\mathcal{L}$ is the operator defining the Legendre equation. This equation has singular points at $x = \pm 1$ and there is only one solution that is finite at these points. Hence $u(x)v(t) = v(x)u(t)$ and it is not possible to obtain a discontinuity in the derivative at $x = t$.

10.1.7. The homogeneous equation corresponding to this ODE can be solved by integrating once to get $y' + ky = C$ and then rearranging to the form $dy = (C - ky)dx$. We identify the general solution to this equation as $y(t) = C(1 - he^{-kt})$. Letting the Green's function be written in the form $G(t,u)$, we note that the only solution for $0 \le t < u$ that satisfies the boundary conditions $y(0) = y'(0) = 0$ is the trivial solution $y(t) = 0$. For $u < t < \infty$ there is no boundary condition at $t = \infty$, so $G(t,u)$ can have the general form $G(t,u) = C(u)(1 - h(u)e^{-kt})$, with $C(u)$ and $h(u)$ determined by the connection conditions at $t = u$. Continuity at $t = u$ leads to $1 - h(u)e^{-ku} = 0$, or $h(u) = e^{ku}$, so $G(t,u)$ has for $t > u$ the more explicit form

$$G(t,u) = C(u)\left(1 - e^{-k(t-u)}\right).$$

To determine $C(u)$ from the discontinuity in the derivative of $G(t,u)$ we must first find the quantity p when the homogeneous ODE is written in self-adjoint form. That value of p is e^{kt}, and our ODE is modified to

$$\frac{d}{dt}\left[e^{kt}\psi'(t)\right] = e^{kt}f(t).$$

We now determine $C(u)$ from

$$\frac{\partial}{\partial t}\left[C(u)\left(1 - e^{-k(t-u)}\right)\right]_{t=u} = k\,C(u) = \frac{1}{p(u)} = e^{-ku}.$$

The final form for our Green's function is therefore

$$G(t,u) = \begin{cases} 0, & 0 \le t < u, \\[1ex] \dfrac{e^{-ku} - e^{-kt}}{k}, & t > u, \end{cases}$$

and the inhomogeneous equation has the solution

$$\psi(t) = \int_0^t G(t,u)e^{ku} f(u)\, du\, .$$

Note that we would have gotten the same overall result if we had simply taken p to be the coefficient of y'' in the original equation and not multiplied $f(u)$ by the factor needed to make the ODE self-adjoint.

Finally, with $f(t) = e^{-t}$, we compute

$$\psi(t) = \frac{1}{k}\int_0^t \left(e^{-ku} - e^{-kt}\right) e^{(k-1)u}\, du = \frac{1}{k}\left[1 - \frac{1}{k-1}\left(ke^{-t} - e^{-kt}\right)\right].$$

10.1.8. The answer is given in the text.

10.1.9. The answer is given in the text.

10.1.10. The answer is given in the text.

10.1.11. The answer is given in the text.

10.1.12. If $a_1 = 0$, the differential equation will be self-adjoint and $K(x,t)$ will be symmetric. Cf. Section 21.4.

10.1.13. Start by finding the Green's function of the ODE without the V_0 term. The truncated ODE has solutions $e^{\pm kr}$. A solution satisfying the boundary condition at $r = 0$ is $e^{kr} - e^{-kr}$, equivalent (except for a factor 2) to $\sinh kr$. A solution satisfying the boundary condition at $r = \infty$ is e^{-kr}. The Wronksian of $\sinh kr$ and e^{-kr} is $-k$, so the Green's function is

$$G(r,t) = \begin{cases} -\dfrac{1}{k}\, e^{-kt} \sinh kr, & 0 \le r < t, \\ -\dfrac{1}{k}\, e^{-kr} \sinh kt, & t < r < \infty\, . \end{cases}$$

We now treat our ODE as an inhomogeneous equation whose right-hand side is $-V_0 e^{-r} y(r)/r$. Using the Green's function to form its solution, we obtain the integral equation given in the text. Note that $G(r,t)$ of the exercise is -1 times the Green's function.

10.2 Problems in Two and Three Dimensions

10.2.1. This problem was solved in Example 10.2.1.

10.2.2. The operator $\mathcal{L}$ is Hermitian if, for all $\varphi(\mathbf{r})$ and $\psi(\mathbf{r})$ satisfying the boundary conditions, $\langle\varphi|\mathcal{L}\psi\rangle = \langle\mathcal{L}\varphi|\psi\rangle$. To show this, use the identity $f\boldsymbol{\nabla}\cdot\mathbf{U} = \boldsymbol{\nabla}\cdot(f\mathbf{U}) - \boldsymbol{\nabla} f\cdot\mathbf{U}$, recognize that one of the integrals is zero because from

Gauss' theorem it is equivalent to a surface integral on the boundary, and write

$$\langle\varphi|\mathcal{L}\psi\rangle = \int_V \boldsymbol{\nabla}\cdot[\varphi^*\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\psi)]\,d\tau - \int_V \boldsymbol{\nabla}\varphi^*\cdot(\rho\boldsymbol{\nabla}\psi)d\tau = -\int_V \rho\boldsymbol{\nabla}\varphi^*\cdot\boldsymbol{\nabla}\psi\,d\tau$$

$$\langle\mathcal{L}\varphi|\psi\rangle = \int_V \boldsymbol{\nabla}\cdot[\psi\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\varphi^*)]\,d\tau - \int_V \boldsymbol{\nabla}\psi\cdot(\rho\boldsymbol{\nabla}\varphi^*)d\tau = -\int_V \rho\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\varphi^*\,d\tau$$

The fact that these two integrals are identical confirms that $\mathcal{L}$ is Hermitian.

10.2.3. Using the Fourier transform of the Green's function and of the delta function we find

$$\int G(\mathbf{r}_1,\mathbf{r}_2)d^3r_2 = \int \frac{d^3p}{(2\pi)^3}\int \frac{e^{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)}}{k^2-p^2}d^3r_2 = \int d^3p\frac{e^{i\mathbf{p}\cdot\mathbf{r}_1}}{k^2-p^2}\delta(\mathbf{p}) = \frac{1}{k^2}.$$

10.2.4. Using Example 20.3.3 we have $(\boldsymbol{\nabla}^2+k^2)\displaystyle\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2-p^2} = \delta(\mathbf{r}-\mathbf{r}')$.

10.2.5. $G(\mathbf{r}_1,\mathbf{r}_2) = -\dfrac{\cos k|\mathbf{r}_1-\mathbf{r}_2|}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}$.

10.2.6. Integrate the equation for the modified Helmholtz Green's function over a sphere of radius a, using Gauss' theorem to avoid the necessity of evaluating the Laplacian at the origin, where it is singular. We must have:

$$\int_{V_a}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}G(r_{12})d\tau - k^2\int_{V_a}G(r_{12})\,d\tau =$$

$$\int_{\partial V_a}\boldsymbol{\nabla}G(r_{12})\cdot d\boldsymbol{\sigma} - k^2\int_{V_a}G(r_{12})\,d\tau = \int_{V_a}\delta(r_{12})d\tau = 1.$$

Using the form given for $G(r_{12})$ and recognizing the spherical symmetry, the integrals become

$$\int_{\partial V_a}\boldsymbol{\nabla}G(r_{12})\cdot d\boldsymbol{\sigma} = 4\pi a^2\left[\frac{ke^{-kr_{12}}}{4\pi r_{12}} + \frac{e^{-kr_{12}}}{4\pi r_{12}^2}\right]_{r_{12}=a} = (ka+1)e^{-ka}$$

$$\int_{V_a}G(r_{12})\,d\tau = -\int_0^a \frac{e^{-kr_{12}}}{4\pi r_{12}}\,4\pi r_{12}^2\,dr_{12} = \frac{-1+(1+ka)e^{-ka}}{k^2}.$$

Inserting these results, we verify the initial equation of this problem solution.

10.2.7. The answer is given in the text.

11. Complex Variable Theory

11.1 Complex Variables and Functions

(no exercises)

11.2 Cauchy-Riemann Conditions

11.2.1. $f(z) = x$ implies $u = x$, $v = 0$, $\dfrac{\partial u}{\partial x} = 1 \neq \dfrac{\partial v}{\partial y} = 0$. Hence f is not analytic.

11.2.2. This follows from Exercise 6.2.4.

11.2.3. (a) $w = f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$.

(b) $w = f(z) = e^{iz} = e^{i(x+iy)} = e^{-y}(\cos x + i \sin x)$.

11.2.4. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x}$ implies $\dfrac{\partial u}{\partial x} = 0$. Similarly, $\dfrac{\partial u}{\partial y} = 0$ follows.

Therefore, both u and v must be constants, and hence $w_1 = w_2 =$constant.

11.2.5. Write $1/(x+iy)$ as $u + iv$ with $u = x/(x^2+y^2)$, $v = -y/(x^2+y^2)$ and check that the Cauchy-Riemann equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}\,, \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}\,,$$

$$\frac{\partial v}{\partial y} = -[\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}\,, \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}\,.$$

11.2.6. Write $f = u + iv$; the derivative in the direction $a\,dx + ib\,dy$ is

$$f' = \frac{a\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)dx + b\left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)dy}{a\,dx + ib\,dy}\,.$$

Inserting the Cauchy-Riemann equations to make all the derivatives with respect to x, we get

$$f' = \frac{a\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)dx + b\left(-\dfrac{\partial v}{\partial x} + i\dfrac{\partial u}{\partial x}\right)dy}{a\,dx + ib\,dy}\,.$$

$$= \frac{\dfrac{\partial u}{\partial x}(a\,dx + ib\,dy) + i\,\dfrac{\partial v}{\partial x}\,(a\,dx + ib\,dy)}{a\,dx + ib\,dy} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\,.$$

The derivative has the same value as in the x direction.

11.2.7. The real and imaginary parts of a analytic function must satisfy the Cauchy-Riemann equations for an arbitrary orientation of the coordinate system. Take one coordinate direction to be in the direction of $\hat{r}$ and the other in the direction of $\hat{\theta}$, and note that the derivatives of displacement in these directions are respectively $\partial/\partial r$ and $r^{-1}\,\partial/\partial\theta$. Noting also that the real and imaginary parts of $Re^{i\Theta}$ are respectively $R\cos\Theta$ and $R\sin\Theta$, the Cauchy-Riemann equations take the form

$$\frac{\partial R\cos\Theta}{\partial r} = \frac{\partial R\sin\Theta}{r\,\partial\theta}, \qquad \frac{\partial R\cos\Theta}{r\partial\theta} = -\frac{\partial R\sin\Theta}{\partial r}\,.$$

Carrying out the differentiations and rearranging, these equations become

$$\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\Theta}{\partial\theta} = \tan\Theta\left[R\,\frac{\partial\Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right],$$

$$\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\Theta}{\partial\theta} = -\cot\Theta\left[R\,\frac{\partial\Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right].$$

Multiplying together the left-hand sides of both these equations and setting the result equal to the product of the right-hand sides, we get

$$\left[\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\Theta}{\partial\theta}\right]^2 = -\left[R\,\frac{\partial\Theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right]^2.$$

The quantities in square brackets are real, so the above equation is equivalent to the requirement that they must both vanish. These relations are the Cauchy-Riemann equations in polar coordinates.

11.2.8. Differentiating the first Cauchy-Riemann equation from Exercise 11.2.7 with respect to θ and rearranging, we get

$$\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} = \frac{1}{rR}\frac{\partial^2 R}{\partial r\partial\theta} - \frac{1}{r^2R}\frac{\partial R}{\partial\theta}\frac{\partial\Theta}{\partial\theta} = \frac{1}{rR}\frac{\partial^2 R}{\partial r\partial\theta} + \frac{1}{R}\frac{\partial R}{\partial r}\frac{\partial\Theta}{\partial r},$$

where we reached the last member of the above equation by substituting from the polar Cauchy-Riemann equations. Differentiating the second Cauchy-Riemann equation with respect to r and simplifying, we get after rearrangement

$$\frac{\partial^2\Theta}{\partial r^2} = -\frac{1}{R}\frac{\partial\Theta}{\partial r}\frac{\partial R}{\partial r} + \frac{1}{r^2R}\frac{\partial R}{\partial\theta} - \frac{1}{rR}\frac{\partial^2 R}{\partial r\partial\theta}.$$

We also need, from the second Cauchy-Riemann equation,

$$\frac{1}{r}\frac{\partial\Theta}{\partial r} = -\frac{1}{r^2R}\frac{\partial R}{\partial\theta}.$$

Adding together the three foregoing equations, the left-hand sides combine to give the Laplacian operator, while the right-hand sides cancel to give zero.

11.2.9. (a) $f'(z) = \dfrac{\cos z}{z} - \dfrac{\sin z}{z^2}$.

Analytic everywhere except at infinity. Note that $f(z)$ approaches a finite limit at $z = 0$ and has Taylor expansion $1 - z^2/3! + \cdots$. At $z = 0$ the formula for f' must be interpreted as its limit (which is zero).

(b) $f'(z) = \dfrac{-2z}{(z^2+1)^2}$.

Analytic everywhere except at $z = i$ and $z = -i$ (becomes infinite at those values of z).

(c) $f'(z) = -\dfrac{1}{z^2} + \dfrac{1}{(z+1)^2}$.

Analytic everywhere except at $z = 0$ and $z = -1$.

(d) $f'(z) = \dfrac{e^{-1/z}}{z^2}$. Analytic everywhere except at $z = 0$.

(e) $f'(z) = 2z - 3$. Analytic everywhere except at $z = \infty$.

(f) $f'(z) = \dfrac{1}{\cos^2 z}$.

Analytic everywhere except at infinity and at the zeros of $\cos z$, which are at $(n + \frac{1}{2})\pi$, for n any positive or negative integer or zero.

(g) $f'(z) = \dfrac{1}{\cosh^2 z}$.

Analytic everywhere except at infinity and at the zeros of $\cosh z$, which are at $(n + \frac{1}{2})i\pi$, for n any positive or negative integer or zero.

11.2.10. (a) For all finite z except $z = 0$. Even though $z^{1/2}$ is zero at $z = 0$, this function does not have a well-defined derivative there.

(b) For all finite z except $z = 0$.

(c) From the formula $\tan^{-1} z = \dfrac{1}{2i} \ln\left(\dfrac{1+iz}{1-iz}\right)$,

we identify singularities at $z = \pm i$; at these points $\tan^{-1} z$ has no derivative.

(d) From the formula $\tanh^{-1} z = \dfrac{1}{2} \ln\left(\dfrac{1+z}{1-z}\right)$,

we identify singularities at $z = \pm 1$; at these points, $\tanh^{-1} z$ has no derivative.

11.2.11. (a) Since $f'(z)$ is independent of direction, compute it for an infinitesimal displacement in the x direction. We have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\,,$$

where the last member was obtained using a Cauchy-Riemann equation. Now identify $\partial u/\partial x$ as $(\boldsymbol{\nabla} u)_x = V_x$ and $\partial u/\partial y$ as $(\boldsymbol{\nabla} u)_y = V_y$ to obtain $f' = V_x - iV_y$.

(b) Use the fact that the real and imaginary parts of an analytic function each satisfy the Laplace equation. Therefore $\boldsymbol{\nabla}\cdot\mathbf{V} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} u = 0$.

(c) $\displaystyle \boldsymbol{\nabla}\times\mathbf{V} = \frac{\partial}{\partial x}\,V_y - \frac{\partial}{\partial y}\,V_x = \frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 u}{\partial y\partial x} = 0.$

11.2.12. Equate the derivatives of $f(z) = u + iv$ with respect to z^* in the x and y directions:

$$\frac{\dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x}}{dx} = \frac{\dfrac{\partial u}{\partial y} + i\,\dfrac{\partial v}{\partial y}}{-i\,dy}\,.$$

This yields equations similar to the Cauchy-Riemann equations, but with opposite signs. The derivative with respect to z^* does not exist unless these equations are satisfied. The only way to satisfy both the Cauchy-Riemann equations and their sign-reversed analogs is to have all the derivatives in these equations vanish, equivalent to the requirement that f be a constant.

11.3 Cauchy's Integral Theorem

11.3.1. $\displaystyle \int_{z_2}^{z_1} f(z)dz = -\int_{z_1}^{z_2}\Big[(udx - vdy) + i(vdx + udy)\Big].$

11.3.2. $\displaystyle \left|\int_C f(z)dz\right| \le \int_C |f(z)|\;ds \le |f|_{\max}L$ with L the length of the path C.

11.3.3. (a) In terms of x and y,

$$F = 4z^2 - 3iz = 4(x^2 - y^2) + 3y + (8xy - 3x)i\,.$$

On the straight-line path, x and y are related by $y = -7x + 25$, so F has the two representations

$$F_1(x) = -192x^2 + 1379x - 2425 + (-56x^2 + 197x)i\,,$$

$$F_2(y) = \frac{-192y^2 - 53y + 2500}{49} + \frac{(-8y^2 + 203y - 75)i}{7}\,.$$

Integrating,

$$\int_{3+4i}^{4-3i} F(z)\,dz = \int_{3+4i}^{4-3i} F(z)(dx+idy) = \int_3^4 F_1(x)\,dx + i\int_4^{-3} F_2(y)\,dy$$

$$= \left(\frac{67}{2} - \frac{7i}{6}\right) + \left(-\frac{49}{6} - \frac{469i}{2}\right) = \frac{76-707i}{3}\,.$$

(b) To integrate on the circle $|z| = 5$, use the polar representation $z = 5e^{i\theta}$. The starting point of the integral is at $\theta_1 = \tan^{-1}(4/3)$ and its end point is at $\theta_2 = \tan^{-1}(-3/4)$. F can now be written $F_3(\theta) = 4(5^2e^{2i\theta}) - 3i(5e^{i\theta})$. The integral then takes the form

$$\int_{\theta_1}^{\theta_2} F_3(\theta)(5ie^{i\theta})\,d\theta = \int_{\theta_1}^{\theta_2} (500ie^{3i\theta} + 75e^{2i\theta})d\theta$$

$$= \frac{500}{3}\left(e^{3i\theta_2} - e^{3i\theta_1}\right) - \frac{75i}{2}\left(e^{2i\theta_2} - e^{2i\theta_1}\right).$$

These expressions simplify, because

$$e^{3i\theta_1} = \frac{-117+44i}{125}, \qquad e^{3i\theta_2} = \frac{-44-117i}{125},$$

$$e^{2i\theta_1} = \frac{-7+24i}{25}, \qquad e^{2i\theta_2} = \frac{7-24i}{25},$$

and we get

$$\int_{\theta_1}^{\theta_2} F_3(\theta)(5ie^{i\theta})\,d\theta = \frac{76-707i}{3}\,,$$

the same result as in part (a).

Note that this integral is far easier if we integrate directly in z:

$$\int_{3+4i}^{4-3i} (4z^2 - 3iz)\,dz = \left[\frac{4z^3}{3} - \frac{3iz^2}{2}\right]_{3+4i}^{4-3i} = \frac{76-707i}{3}\,.$$

11.3.4. The integrand is an analytic function for all finite z, so its integral between the given endpoints can be deformed in any way without changing its value. We may therefore evaluate $F(z)$ using the indefinite integral $F(z) = (\sin 2z)/2 + C$, so $F(\pi i) = \sin(2\pi i)/2 - \sin(2\pi[1+i])/2$. This expression can be simplified using the formula $\sin(a+b) = \sin a\cos b + \cos a \sin b$; the second term reduces to $-\sin(2\pi i)/2$, and we get $F(\pi i) = 0$.

11.3.5. (a) To integrate around the unit circle, set $x = \cos\theta$, $y = \sin\theta$, $dz = ie^{i\theta}d\theta = i(\cos\theta + i\sin\theta)d\theta$, and (because the integration is clockwise) integrate from $\theta = 0$ to $\theta = -2\pi$. The integral will vanish because every term contains one odd power of either $\sin\theta$ or $\cos\theta$ and the integral is over an interval of length 2π.

(b) For the square, taking first the horizontal lines at $y = 1$ and $y = -1$, we note that for any given x the integrand has the same value on both lines, but the dz values are equal and opposite; these portions of the contour integral add to zero. Similar remarks apply to the vertical line segments at $x = \pm 1$, giving an overall result of zero.

These integrals are equal because of symmetry, and not because of analyticity; the integrand is not analytic.

11.3.6. $\displaystyle\int_C z^* dz = \int_0^1 x dx + \int_0^1 (1 - iy) i dy = \frac{1}{2} + i + \frac{1}{2} = 1 + i,$

whereas

$$\int_{C'} z^* dz = \int_0^1 (-iy) dy + \int_0^1 (-i + x) dx = -\frac{i}{2} + \frac{1}{2} - i = \frac{1}{2} - \frac{3}{2}\, i\,.$$

11.3.7. Since the contour is assumed to be a circle of radius greater than unity, it will surround both the points $z = 0$ and $z = -1$ for which the integrand becomes infinite. Using Cauchy's integral theorem, deform the contour (without changing the value of the integral) until the upper and lower arcs of the circle touch each other at some point between $z = 0$ and $z = -1$, and then further deform the left-hand and right-hand loops of the integral to convert them into separate circles surrounding these two values of z. Finally, write the contour integral as

$$\oint \left[\frac{1}{z} - \frac{1}{z+1}\right] dz$$

and expand into two separate integrals (each over both circles). For the circle about $z = 0$ only the first integral contributes; for the circle about $z = -1$ only the second integral contributes. Because of the minus sign in the partial fraction expansion, these integrals will be equal and opposite and therefore sum to zero.

Note that if the original contour was a circle of radius less than unity, only the first of the two partial fractions would be within the contour, so Cauchy's integral theorem tells us that the integral of the second partial fraction must vanish and we do not know until reading Section 11.3 how to evaluate the nonzero integral of the first partial fraction.

11.4 Cauchy's Integral Formula

11.4.1. $\displaystyle\frac{1}{2\pi i}\oint_{C(0)} z^{m-n-1} dz = \begin{cases} 1 & m = n, \\ 0 & \text{otherwise,} \end{cases}$ which is δ_{mn} by definition.

11.4.2. 0.

11.4.3. First, note that

$$\oint \frac{f'(z)\,dz}{z-z_0} = 2\pi i f'(z_0)\,,$$

where the contour surrounds z_0. This formula is legitimate since f' must be analytic because f is. Now apply Eq. (11.32) to identify

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)\,dx}{(z-z_0)^2}\,.$$

11.4.4. Differentiating with respect to z_0,

$$f^{(n+1)}(z_0) = \frac{n!}{2\pi i}\int \frac{(n+1)f(z)}{(z-z_0)^{n+2}}\,dz$$

is the step from n to $n+1$ in a proof by mathematical induction.

11.4.6. The detailed description of the contour is irrelevant; what is important is that it encloses the point $z=0$. Using Eq. (11.33), this integral evaluates to $\displaystyle \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\,e^{iz}\bigg|_{z=0} = -\pi i\,.$

11.4.7. This integral is a case of Eq. (11.33). We need the second derivative of $\sin^2 z - z^2$, evaluated at $z=a$; it is $2\cos 2a - 2$. Thus, our integral is $(2\pi i/2!)(2\cos 2a - 2) = 2\pi i(\cos 2a - 1)$.

11.4.8. Make a partial fraction decomposition of the integrand. We have

$$\frac{1}{z(2z+1)} = \frac{1}{z} - \frac{2}{2z+1} = \frac{1}{z} - \frac{1}{z+\frac{1}{2}}\,.$$

Both denominators are of the form $z-a$ with a within the unit circle, and the integrals of the partial fractions are cases of Cauchy's formula with the respective functions $f(z)=1$ and $f(z)=-1$. Therefore the value of the integral is zero.

11.4.9. After the partial fraction decomposition we have

$$\oint f(z)\left[\frac{1}{z} - \frac{2}{2z+1} - \frac{2}{(2z+1)^2}\right]dz$$

$$= \oint \frac{f(z)\,dz}{z} - \oint \frac{f(z)\,dz}{z+\frac{1}{2}} - \oint \frac{\frac{1}{2}f(z)\,dz}{(z+\frac{1}{2})^2}\,.$$

Each integral is now a case of Cauchy's formula (in one case, for a derivative). Termwise evaluation yields

$$2\pi i f(0) - 2\pi i f(-\tfrac{1}{2}) - \pi i\, f'(-\tfrac{1}{2}).$$

11.5 Laurent Expansion

11.5.1. The solution is given in the text.

11.5.2. From

$$\frac{d}{dz}(1+z)^m\bigg|_0 = m(1+z)^{m-1}|_0 = m,$$

$$\frac{d^2}{dz^2}(1+z)^m\bigg|_0 = m(m-1),$$

$$\frac{d^\nu}{dz^\nu}(1+z)^m\bigg|_0 = m(m-1)\cdots(m-\nu+1),$$

Taylor's theorem yields for $|z|<1$

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2}z^2 + \cdots = \sum_{n=0}^{\infty}\binom{m}{n}z^n\,.$$

11.5.3. $\displaystyle \left|\frac{f(z_0)}{z_0}\right|^n = \left|\frac{1}{2\pi i}\int_{C_1(0)}\left(\frac{f(z)}{z}\right)^n \frac{dz}{(z-z_0)}\right| \le \frac{1}{1-|z_0|}$ implies

$$|f(z_0)| \le \frac{|z_0|}{(1-|z_0|)^{1/n}}\,.$$

For $n\to\infty$ this yields $|f(z_0)| \le |z_0|$.

11.5.4. $z^n f(z) = \sum\limits_{\nu=0}^{\infty} a_\nu z^\nu$ is analytic and real for real z. Hence a_ν are real.

11.5.5. $(z-z_0)^N f(z)$ is analytic and its power series is unique.

11.5.6. Make the Taylor series expansion of e^z and divide by z^2:

$$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}\,.$$

11.5.7. One way to proceed is to write $z=(z-1)+1$ and $e^z = e\cdot e^{z-1}$. Expanding the exponential, we have

$$e\left(1+\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} = \frac{e}{z-1} + e\sum_{n=0}^{\infty}\left(\frac{n+2}{n+1}\right)\frac{(z-1)^n}{n!}\,.$$

11.5.8. Expand $e^{1/z}$ in powers of $1/z$, then multiply by $z-1$:

$$(z-1)\,e^{1/z} = (z-1)\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = z - \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)\frac{z^{-n}}{n!}\,.$$

11.6 Singularities

11.6.1. Truncating

$$e^{1/z} \approx 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots + \frac{1}{n!z^n} = z_0,$$

multiplying this by z^n and solving the resulting nth-order polynomial yields n different solutions $z = z_j$, $j = 1, 2, \ldots, n$. Then we let $n \to \infty$.

11.6.2. If the branch points at ± 1 are linked by a cut line one cannot have a path around them separately, only around both. As a result $w(z)$ remains single-valued. The phases are shown in Example 11.6.4.

11.6.3. $f_2(z) = f_2'(z_0)(z - z_0) + \cdots$ implies that

$$f_1(z)/f_2(z) = f_1(z)/f_2'(z_0)(z-z_0)^{-1} + \cdots = f_1(z_0)/f_2'(z_0)(z-z_0)^{-1} + \cdots,$$

where the ellipses stand for some function that is regular at z_0.

11.6.4. With the branch cuts of Example 11.6.4 and with $\sqrt{z^2-1}$ chosen (as in that example) to be on the branch that gives it positive values for large real z, its value at $z = i$ (where the angles shown in Fig. 11.12 are $\varphi = 3\pi/4$, $\theta = \pi/4$ and $r_1 = r_2 = \sqrt{2}$) is

$$f(i) = (\sqrt{2}\, e^{\pi i/8})(\sqrt{2}\, e^{3\pi i/8}) = 2e^{\pi i/2} = 2i\,.$$

For all points in the upper half-plane, both the branch cuts of Example 11.6.4 and Exercise 11.6.2 will yield the same angle assignments and therefore, once the branches are chosen so $f(i)$ has the same value, the two function definitions will agree. However, points in the lower half-plane will have different angle assignments in the two branch-cut schemes: In Example 11.6.4, both φ and θ will have values in the lower half-plane that are both reached by counterclockwise (or both by clockwise) rotation. But in Exercise 11.6.2, points in the lower half-plane are reached by counterclockwise rotation in φ and clockwise rotation in θ. This changes the sum of the two angles by an amount 2π; half of this (because of the square root) produces a sign change. Thus, the two function definitions are opposite in sign in the lower half-plane.

11.6.5. The first two terms both have fractional powers of z and therefore indicate the existence of a branch point at $z = 0$. The original value of both terms cannot be recovered simultaneously until the number of circuits around the branch point is the smallest common multiple of 3 and 4, i.e., 12. There is also a third-order pole at $z = 3$ and a second-order branch point at $z = 2$. To determine the singularity structure at infinity, replace z by $1/w$ and check for singularities at $w = 0$. All three terms exhibit branching at $w = 0$; since the smallest common multiple of 2, 3, and 4 is 12, the branch point at infinity will be of order 12.

11.6.6. Write $z^2 + 1 = (z - i)(z + i) = \left(r_1 e^{i\varphi}\right)\left(r_2 e^{i\theta}\right)$. At $z = 0$, $\varphi = -\pi/2$, $\theta = \pi/2$, and $r_1 = r_2 = 1$. The most general possibility for the argument of $z^2 + 1$ st $z = 0$ is therefore $\varphi + \theta + 2n\pi = 2n\pi$. Since we are to be on the branch of $\ln(z^2 + 1)$ that is $-2\pi i$ at $z = 0$, we must take $n = -1$, and for all points on this branch its argument must be $\varphi + \theta - 2\pi$. If we now move to $z = -2 + i$, φ becomes $-\pi$ while θ becomes $3\pi/4$. The argument of $z^2 + 1$ at this point is therefore $-\pi + 3\pi/4 - 2\pi = -9\pi/4$. Thus, $F(i - 2) = \ln|z^2 + 1| - 9i\pi/4 = \ln(4\sqrt{2}) - 9i\pi/4$.

11.6.7. The solution is given in the text.

11.6.8. $(1 + z)^m = e^{m\ln(1+z)} = (1 + r^2 + 2r\cos\theta)^{m/2} e^{im\arg(1+z)}$

has a branch point at $z = -1$. A cut line is drawn from -1 to $-\infty$ along the negative real axis, and $|z| < 1$ is the convergence region. An additional phase $e^{2\pi imn}$ is present for branches other than the standard binomial expansion of Exercise 11.5.2 (n an integer).

11.6.9. The extra phase $e^{2\pi imn}$ mentioned in the solution of Exercise 11.6.8 multiplies each coefficient of the Taylor expansion.

11.6.10. (a) $f(z) = \displaystyle\sum_{n=-1}^{\infty} (-1)^{n+1}(z-1)^n, \quad 0 < |z - 1| < 1.$

(b) $f(z) = \displaystyle\sum_{n=-2}^{\infty} (-1)^n (z-1)^{-n}, \quad |z - 1| > 1.$

11.6.11. (a) This representation of $f(z)$ diverges when $\Re(z) \le 0$. However, $f(z)$ can be analytically continued to all the remainder of the finite z-plane except for singularities at $z = 0$ and all negative integers.

(b) $f_1(z) = \displaystyle\int_0^\infty e^{-zt}dt = -\frac{1}{z}e^{-zt}\Big|_{t=0}^{\infty} = \frac{1}{z}$, provided $\Re(z) > 0$.

(c) $\displaystyle\frac{1}{z} = \frac{1}{z - i + i} = \frac{1}{i}\frac{1}{1 - i(z-i)} = -i\sum_{n=0}^{\infty}[-i(z-i)]^n, \quad |z - i| < 1.$

11.7 Calculus of Residues

11.7.1. (a) $z_0 = \pm ia$, simple poles

$$a_{-1} = \pm\frac{1}{2ai}.$$

(b) $z_0 = \pm ia$, second-order poles

$$a_{-1} = \pm\frac{1}{4a^3 i}.$$

(c) $z_0 = \pm ia$, second-order poles

$$a_{-1} = \pm\frac{1}{4ai}.$$

(d) $z_0 = \pm ia$, simple poles

$$a_{-1} = -\frac{\sinh(1/a)}{2a}.$$

(e) $z_0 = \pm ia$, simple poles

$a_{-1} = 1/2e^{\pm a}$.

(f) $z_0 = \pm ai$, simple poles

$a_{-1} = -i/2e^{\mp a}$.

(g) $z_0 = \pm a$, simple poles

$$a_{-1} = \pm\frac{1}{2a}e^{\mp ia}.$$

(h) $z_0 = -1$, simple poles

$a_{-1} = e^{-ik\pi}$ for $z = e^{i\pi}$.

$z_0 = 0$ is a branch point.

11.7.2. Start by making a partial fraction expansion on the integrand:

$$\frac{\pi\cot\pi z}{z(z+1)} = \frac{\pi\cot\pi z}{z} - \frac{\pi\cot\pi z}{z+1}\,.$$

For the residue at $z = 0$, note that because $\cot z$ is a odd function of z, the first term when expanded in powers of z will contain only even powers and therefore has zero residue. Inserting the power series expansion of $\cot z$ into the second term, we get

$$-\frac{\pi}{z+1}\left(\frac{1}{\pi z} + O(z)\right), \quad \text{with residue } -1.$$

At $z = -1$, we use the periodicity of the cotangent to replace $\cot\pi z$ by $\cot\pi(z+1)$, after which we note that if expanded about $z = -1$ the second term will have only even powers of $z + 1$ and have zero residue. The first term expands to

$$\frac{\pi}{z}\left(\frac{1}{\pi(z+1)} + O(z+1)\right), \quad \text{also with residue } -1.$$

11.7.3. Divide the principal-value integral into three parts and combine the last two into a single integral (possible because both involve the same value of ε):

$$\mathrm{Ei}(x) = \int_{-\infty}^{-x}\frac{e^t}{t}\,dt + \int_{-x}^{-\varepsilon}\frac{e^t}{t}\,dt + \int_{\varepsilon}^{x}\frac{e^t}{t}\,dt = \int_{-\infty}^{-x}\frac{e^t}{t}\,dt + \int_{\varepsilon}^{x}\frac{e^t - e^{-t}}{t}\,dt\,.$$

In the last member of this equation both the integrals are convergent (the integrand of the second integral has a Taylor series expansion in t).

11.7.4. Break the integral into its two parts and for each part make a binomial expansion of the denominator in a form that will converge for the region of integration:

$$\begin{aligned}
⨍ \frac{x^{-p}}{1-x}\,dx &= \lim_{\varepsilon\to 0}\left[\int_0^{1-\varepsilon} \frac{x^{-p}}{1-x}\,dx + \int_{1+\varepsilon}^{\infty} \frac{x^{-p}}{1-x}\,dx\right] \\
&= \lim_{\varepsilon\to 0}\left[\int_0^{1-\varepsilon} \sum_{n=0}^{\infty} x^{n-p}\,dx - \int_{1+\varepsilon}^{\infty} \sum_{n=0}^{\infty} x^{-n-p-1}\,dx\right] \\
&= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\left[\frac{(1-\varepsilon)^{n-p+1}}{n-p+1} - \frac{(1+\varepsilon)^{-n-p-1}}{n+p}\right].
\end{aligned}$$

Combining terms appropriately and taking the limit, we finally arrive at

$$⨍ \frac{x^{-p}}{1-x}\,dx = -\frac{1}{p} - \sum_{n=1}^{\infty} \frac{2p}{p^2-n^2} = -\pi\cot p\,\pi\,,$$

where the final step is carried out by invoking Eq. (11.81) with $p\pi$ substituted for z in that equation.

11.7.5. An attempt to use $\sin z$ directly in Eq. (11.88) cannot be carried out because we would need to insert $f(0)=0$ into that equation. However, if our function is $\sin z/z$, then we can proceed with $f(0)=1$, $f'(0)=0$, and with the zeros of f at $n\pi$ $(n\neq 0)$.

11.7.6. The observations we are starting from are sufficient to enable the application of Rouché's theorem to conclude that every polynomial $\sum_{m=0}^{n} a_m z^m$ has n zeros within the region bounded by some sufficiently large $|R|$.

11.7.7. We start by noting that $f(z) = z^6 + 10$ has all its six zeros on a circle about $z=0$ of radius $10^{1/6}$, which is between 1 and 2. Next we note that $|f(z)| > |-4z^3|$ for all z within the circle $|z|=2$. Therefore $F(z) = z^6 - 4z^3 + 10$ has, like $f(z)$, no zeros inside the circle $|z|=1$ and six zeros inside $|z|=2$, and therefore also outside $|z|=1$.

11.7.8. Applying Eq (11.79) to $f(z) = \sec z$, we note that $f(0)=1$, and that $f(z)$ has poles at $(n+\frac{1}{2})\pi$ for all integer n. The residue of $f(z)$ at $(n+\frac{1}{2})\pi$ is

$$\lim_{z\to(n+\frac{1}{2})\pi} \frac{z-(n+\frac{1}{2})\pi}{\cos z} = \frac{1}{-\sin[(n+\frac{1}{2})\pi]} = (-1)^{n+1}\,.$$

Thus,

$$\sec z = 1 + \sum_{n=-\infty}^{\infty} (-1)^{n+1}\left(\frac{1}{z-(n+\frac{1}{2})\pi} + \frac{1}{(n+\frac{1}{2})\pi}\right).$$

The terms of the summation not containing z can be brought to the form

$$\sum_{-\infty}^{\infty} \frac{(-1)^{n+1}}{(n+\frac{1}{2})\pi} = -\frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) = -\frac{4}{\pi}\left(\frac{\pi}{4}\right) = -1\,.$$

This cancels the $+1$ from $f(0)$; the z-containing terms of $+n$ and $-n$ can be combined over a common denominator to obtain the successive terms of the summation in Eq. (11.82).

Treating now $\csc z$, it is convenient to consider the expansion of $f(z) = \csc z - z^{-1}$, which is regular at $z = 0$, with the value $f(0) = 0$. The function $f(z)$ has poles at $z = n\pi$ for all nonzero integers n, with residues $(-1)^n$. The pole expansion of $f(z)$ is therefore

$$f(z) = \csc z - \frac{1}{z} = 0 + \sum_{n=1}^{\infty} (-1)^n \left[\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right) + \left(\frac{1}{z + n\pi} - \frac{1}{n\pi}\right)\right]$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - (n\pi)^2}\,,$$

a result clearly equivalent to Eq. (11.83).

11.7.9. It is clear that the function f has a simple pole at $z = 0$ and zeros at $z = 1$ and $z = 2$. From $f = (z^2 - 3z + 2)/z = z - 3 + (2/z)$, we find $f' = 1 - (2/z^2)$ and (making a partial-fraction decomposition)

$$\frac{f'}{f} = \frac{z^2 - 2}{z(z-2)(z-1)} = -\frac{1}{z} + \frac{1}{z-2} + \frac{1}{z-1}.$$

Therefore, the integral of f'/f on any contour that encloses $z = 0$ will have from that source a contribution $-2\pi i$, while the integral will have contributions $2\pi i$ for each of $z = 1$ and $z = 2$ that are enclosed. These observations are consistent with the formula

$$\oint f'(z) f(z)\, dz = 2\pi i (N_f - P_f)\,.$$

11.7.10. Integrating over the upper half circle we obtain

$$\int (z - z_0)^{-m} dz = \int_0^{\pi} \frac{ire^{i\theta} d\theta}{r^m e^{im\theta}} = \frac{r^{1-m}}{1-m} e^{i(1-m)\theta}\Big|_0^{\pi}$$

$$= \frac{r^{1-m}}{1-m}\Big[(-1)^{1-m} - 1\Big] = \begin{cases} 0, & m \text{ odd}, \neq 1 \\ -\dfrac{2r^{1-m}}{1-m}, & m \text{ even}. \end{cases}$$

For $m = 1$, $\quad i\displaystyle\int_0^{\pi} d\theta = i\pi = \frac{1}{2}\int_0^{2\pi} d\theta.$

11.7.11. (a) This follows for $\delta \to 0, \Lambda \to \infty$ from

$$\int_{-\Lambda+x_0}^{x_0-\delta} \frac{dx}{x-x_0} + \int_{x_0+\delta}^{\Lambda+x_0} \frac{dx}{x-x_0} = \ln(x-x_0)\Big|_{-\Lambda+x_0}^{x_0-\delta} + \ln(x-x_0)\Big|_{x_0+\delta}^{\Lambda+x_0}$$

$$= \ln\left(\frac{-\delta}{-\Lambda}\right) + \ln\frac{\Lambda}{\delta} = 0$$

and

$$\int_{-\Lambda+x_0}^{x_0-\delta} \frac{dx}{(x-x_0)^3} + \int_{x_0+\delta}^{\Lambda+x_0} \frac{dx}{(x-x_0)^3}$$

$$= -\frac{1}{2(x-x_0)^2}\Big|_{-\Lambda+x_0}^{x_0-\delta} - \frac{1}{2(x-x_0)^2}\Big|_{x_0+\delta}^{\Lambda+x_0} = -\frac{1}{2\delta^2} + \frac{1}{2\Lambda^2} - \frac{1}{2\Lambda^2} + \frac{1}{2\delta^2} = 0.$$

(b) $\displaystyle\int_{C(x_0)} \frac{dz}{z-x_0} = \int_0^\pi \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\pi, \int_{C(x_0)} \frac{dz}{(z-x_0)^3}$

$$= \int_0^\pi \frac{ire^{i\theta}d\theta}{r^3e^{3i\theta}} = -\frac{e^{-2i\theta}}{2r^2}\Big|_0^\pi = 0\,.$$

11.7.12. (a) The integral should not have been designated as a principal value.

Irrespective of the sign of s, the integrand will have a pole in the upper half-plane, with a residue that for small ε will approach unity. If $s > 0$, integrate over the entire real axis and close the contour with a large semicircle in the upper half-plane, where the complex exponential becomes small. The semicircle does not contribute to the integral, and the contour encloses the pole, so our formula for $u(s)$ will be equal to the residue, namely unity. However, if $s < 0$, close the contour with a semicircle in the lower half-plane, thereby causing the semicircle not to contribute to the integral. But now the pole is not within the contour, so our expression for $u(s)$ will evaluate to zero.

(b) If $s > 0$, consider a contour integral that includes the principal-value integral, a large semicircle in the upper half-plane, and a small clockwise semicircle that connects the two pieces of the principal-value integral by passing above the pole at $z = 0$. This contour encloses no singularities and therefore evaluates to zero. Thus, the principal-value integral will be equal to that over the small semicircle (traversed counterclockwise). The integral over the small semicircle will contribute πi times the (unit) residue at the pole, so

$$u(s) = \frac{1}{2} + \frac{1}{2\pi i}\,\pi i = 1\,.$$

If $s < 0$, we must close the contour in the lower half-plane. If we still connect the pieces of the principal-value integral by a small semicircle passing above the pole at $z = 0$, our contour integral will now encircle (in the negative direction) the pole (with unit residue), and the sum of all the contributions to the principal-value integral will now be $\pi i - 2\pi i$, leading to $u(s) = 0$.

11.8 Evaluation of Definite Integrals

11.8.1. This is a case of Example 11.8.1; to make the correspondence exact, bring a factor $1/a$ outside the integral and note that

$$\frac{1}{a}\int_0^{2\pi}\frac{d\theta}{a \pm (b/a)\cos\theta} = \frac{2\pi}{a\sqrt{1-b^2/a^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}\,.$$

The integral containing the sine instead of the cosine can be regarded as having the same integrand, but for an interval of length 2π ranging from $-\pi/2$ to $3\pi/2$. Since the integrand is periodic (with period 2π), this shifted interval yields the same value for the integral.

If $|b| > |a|$, there are singularities on the integration path and the integral does not exist.

11.8.2. For the special case $b = 1$, differentiate the formula of Exercise 11.8.1 with respect to a. Changing the upper limit of the integral from 2π to π and therefore dividing the result by 2, we have

$$\frac{d}{da}\int_a^{\pi}\frac{d\theta}{a+\cos\theta} = \frac{\pi}{(a^2-1)^{1/2}} \quad\longrightarrow\quad -\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2} = -\frac{\pi a}{(a^2-1)^{3/2}}\,,$$

equivalent to the required result.

11.8.3. For $a = 1+t^2$, $b = 2t$ we have $0 \le (1-t)^2 = a-b$ and $1+t^2-2t\cos\theta = a - b\cos\theta$ in Exercise 11.8.1 with $a^2-b^2 = (1-t^2)^2$. Hence the result.

If $|t| > 1$, then the integral equals $2\pi/(t^2-1)$.

If $t = 1$ the denominator has a singularity at $\theta = 0$ and 2π; if $t = -1$ there is a singularity at $\theta = \pi$. In both these cases the integral does not exist.

11.8.4. Introduce the complex variable $z = e^{i\theta}$; integration from 0 to 2π in θ corresponds in z to a closed counterclockwise contour around the unit circle. Then write $\cos\theta = (z+z^{-1})/2$, $\cos 3\theta = (z^3+z^{-3})/2$, $d\theta = dz/iz$, and

$$\int_0^{\infty}\frac{\cos 3\theta\,d\theta}{5-4\cos\theta} = \frac{1}{2}\oint\frac{z^3+z^{-3}\,dz}{iz[5-2(z+z^{-1})]} = \frac{i}{2}\oint\frac{(z^6+1)\,dz}{z^3(2z^2-5z+2)}$$

$$= \frac{i}{2}\oint\frac{(z^6+1)\,dz}{z^3(z-2)(2z-1)}\,.$$

The integrand of this contour integral has a pole of order 3 at $z = 0$ and simple poles at $z = 1/2$ and $z = 2$. The poles at 0 and 1/2 are enclosed by the contour. The residue at the pole of order 3 is

$$\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{z^6+1}{(z-2)(2z-1)}\right] = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{1}{3(z-2)} - \frac{2}{3(2z-1)}\right]$$

$$= \frac{1}{2!}\lim_{z\to 0}\left[\frac{1}{3(z-2)^3} - \frac{8}{3(2z-1)^3}\right] = \frac{21}{8}\,.$$

In the above analysis we took advantage of the fact that z^6 does not get differentiated enough to permit it to contribute to the residue at $z = 0$, and we simplified the differentiation by first making a partial fraction decomposition.

The residue at the pole at $z = -1/2$ is

$$\frac{2^{-6}+1}{(-2)^{-3}(-\frac{1}{2}+2)2} = -\frac{65}{24}\,,$$

so

$$\int_0^\infty \frac{\cos 3\theta\, d\theta}{5 - 4\cos\theta} = \frac{i}{2}\, 2\pi i\left(\frac{21}{8} - \frac{65}{24}\right) = \frac{\pi}{12}\,.$$

11.8.5. $$2\int_0^\pi \cos^{2n}\theta\, d\theta = \int_0^{2\pi}\cos^{2n}\theta\, d\theta = -i\oint\frac{dz}{z}\left[\frac{1}{2}(z+\frac{1}{z})\right]^{2n}$$

$$= -2^{-2n} i\oint\frac{dz}{z^{2n+1}}(z^2+1)^{2n} = -2^{-2n} i\binom{2n}{n}2\pi i$$

$$= \frac{2\pi}{2^{2n}}\binom{2n}{n} = \frac{2\pi(2n)!}{2^{2n}n!^{\,2}} = \frac{2\pi(2n-1)!!}{(2n)!!},$$

using $(z^2+1)^{2n} = \sum_{m=0}^{2n}\binom{2n}{n}z^{2m}$ in conjunction with Cauchy's integral.

11.8.6. Substituting $e^{\pm 2\pi i/3} = -\frac{1}{2} \pm i\sqrt{3}/2$ and solving for I, the result follows immediately.

11.8.7. At large $|z|$, the integrand of the contour integral of Example 11.8.8 asymptotically approaches $1/z^{2-p}$. Since $0 < p < 1$, this power of z is more negative than -1 so the large circle makes no contribution to the integral. At small $|z|$, the denominator of the integrand approaches unity, so the integrand is basically of the form z^p. Writing the integral over the small circle in polar coordinates, it becomes

$$\int_{2\pi}^0 (re^{i\theta})^p(ire^{i\theta})\, d\theta,$$

which has r dependence r^{p+1} and vanishes in the limit of small r.

To reconcile Eqs. (11.115) and (11.116), multiply Eq. (11.115) through by $e^{-\pi i}$, reaching after minor rearrangement

$$\left(\frac{e^{-p\pi i} - e^{p\pi i}}{2i}\right) 2i\, I = 2\pi i \left(\frac{e^{-p\pi i/2} - e^{p\pi i/2}}{2i}\right).$$

The quantities in large parentheses can now be identified as (minus) the sine functions that appear in Eq. (11.116).

11.8.8. The integral has no singularity at $x = 0$ because

$$\frac{\cos bx - \cos ax}{x^2} = \frac{1}{2}(a^2 - b^2) + O(x^2).$$

Next write the integral as shown below, note that its integrand is an even function of x, and break the integral into two parts, in one of which replace bx by x and in the other replace ax by x:

$$I = \int_{-\infty}^{\infty} \frac{(\cos bx - 1) - (\cos ax - 1)}{x^2}\, dx = 2(b - a) \int_0^{\infty} \frac{\cos x - 1}{x^2}\, dx\,.$$

Now replace $\cos x - 1$ by $-2\sin^2(x/2)$ and continue as follows:

$$I = 4(a - b) \int_0^{\infty} \frac{\sin^2(x/2)}{x^2}\, dx = 2(a - b) \int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx\,.$$

This integral, which is the topic of Exercise 11.8.9, has value $\pi/2$, so $I = (a - b)\pi$.

11.8.9. The answer in the text is incorrect; the correct value of the integral in the text is π.

The integral has value $2I$, where

$$I = \int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx\,.$$

Integrating by parts,

$$I = -\frac{\sin^2 x}{x}\bigg|_0^{\infty} + \int_0^{\infty} \frac{2\sin x \cos x}{x}\, dx = \int_0^{\infty} \frac{\sin 2x}{x}\, dx = \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}\,.$$

11.8.10. Write $\sin x = (e^{ix} - e^{-ix})/2i$, and write the integral we require as follows:

$$I = \int_0^{\infty} \frac{x \sin x}{x^2 + 1}\, dx = \frac{1}{2i} \int_0^{\infty} \frac{xe^{ix}}{x^2 + 1}\, dx - \frac{1}{2i} \int_0^{\infty} \frac{xe^{-ix}}{x^2 + 1}\, dx$$

$$= \frac{1}{2i} \int_0^{\infty} \frac{xe^{ix}}{x^2 + 1}\, dx + \frac{1}{2i} \int_{-\infty}^{0} \frac{(-x)e^{ix}}{(-x)^2 + 1}(-dx)$$

$$= \frac{1}{2i} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + 1}\, dz\,.$$

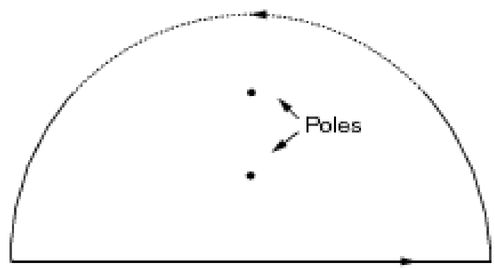

Figure 11.8.10. Contour encloses pole at $z = i$.

Because the above integral has an integrand that contains a complex exponential and approaches zero for large $|z|$ as z^{-1}, we do not change its value if we close its contour by a large semicircle in the upper half-plane (where the complex exponential becomes negligible). Thus, we consider

$$I = \frac{1}{2i} \oint \frac{ze^{iz}}{z^2+1}\, dz \,,$$

where the contour is that of Fig. 11.8.10 of this manual. The integrand has two poles, at $z = i$ and $z = -i$, both of first order, but only the pole at $z = i$ lies within the contour. Writing $z^2+1 = (z+i)(z-i)$, we identify the residue at $z = i$ as $ie^i(i)/2i = 1/2e$. Therefore,

$$I = \frac{1}{2i}\,(2\pi i)\frac{1}{2e} = \frac{\pi}{2e}\,.$$

11.8.11. Using $\displaystyle\int_0^\omega \sin xt\, dx = -\frac{\cos xt}{t}\bigg|_0^\omega$ we integrate by parts:

$$\int_{-\infty}^{\infty} \frac{1-\cos\omega t}{\omega^2} d\omega = -\frac{1-\cos\omega t}{\omega}\bigg|_{-\infty}^{\infty} + t\int_{-\infty}^{\infty} \frac{\sin\omega t}{\omega} d\omega$$

$$= t\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} d\omega = \pi t\,.$$

Note that the integrated term vanishes.

11.8.12. (a) Using $\cos x = (e^{ix} + e^{-ix})/2$ we integrate e^{ix} along the real axis and over a half circle in the upper half-plane and e^{-ix} over a half circle in the lower half-plane getting, by the residue theorem,

$$\frac{1}{2\pi i}\int_{\text{u.h.c.}} \frac{e^{iz}}{z^2+a^2}\, dz = \frac{e^{-a}}{2ia}, \qquad \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+a^2}\, dx = \frac{\pi}{a}\, e^{-a},$$

and

$$\frac{1}{2\pi i}\int_{\text{l.h.c.}}\frac{e^{iz}}{z^2+a^2}\,dz = -\frac{e^{-a}}{(-2ia)}, \qquad \int_{-\infty}^{\infty}\frac{e^{-ix}}{x^2+a^2}\,dx = \frac{\pi}{a}\,e^{-a}.$$

Combining the above, we obtain the answer in the text.

For $\cos kx$ we rescale:

$$\int_{-\infty}^{\infty}\frac{\cos kx}{x^2+a^2}\,dx = k\int_{-\infty}^{\infty}\frac{\cos x\,dx}{x^2+k^2a^2}\,.$$

(b) This result is obtained similarly.

11.8.13. Because the integrand is even, change the lower limit of the integral to $x=0$ and multiply the result by 2. Then substitute $\sin x = (e^{ix}-e^{-ix})/2i$ and rearrange to a principal-value integral:

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \int_{0+}^{\infty}\frac{e^{ix}\,dx}{ix} + \int_{0+}^{\infty}\frac{e^{-ix}\,dx}{ix}$$

$$= \int_{0+}^{\infty}\frac{e^{ix}\,dx}{ix} + \int_{-\infty}^{0-}\frac{e^{ix}\,dx}{ix} = \frac{1}{i}\,⨍_{-\infty}^{\infty}\frac{e^{ix}\,dx}{x}\,.$$

Consider now the integral of e^{iz}/z over the contour of Fig. 11.28. The small semicircle passes above the pole at $z=0$, at which the residue is unity. Thus, the contour encloses no singularities, so the contour integral vanishes. Since the closure of the integral in the upper half-plane makes no contribution, the principal-value integral (along the real axis) plus the contribution from the small clockwise semicircle (namely, $-\pi i$) must add to zero. Thus,

$$\frac{1}{i}\,⨍_{-\infty}^{\infty}\frac{e^{ix}\,dx}{x} = \frac{1}{i}\,\pi i = \pi.$$

11.8.14. For $p>1$ we integrate over the real axis from $-R$ to R and a half circle in the upper plane, where $e^{-p\Im t}|\sin t| \to 0$ for $R\to\infty$ and the integral over the half circle vanishes. Since there are no singularities, the residue theorem gives zero for the loop integral. Thus $\int_{-\infty}^{\infty}\frac{\sin t}{t}e^{ipt}dt = 0$. For $0<p<1$ we use $e^{ipt} = \cos pt + i\sin pt$ and notice that the integral over the imaginary part vanishes as $\sin(-pt) = -\sin pt$. Finally, we use $2\sin t\cos pt = \sin(1+p)t + \sin(1-p)t$. Each integral including $\sin(1\pm p)t$ yields π.

11.8.15. Differentiating

$$\int_0^{\infty}\frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}\bigg|_0^{\infty} = \frac{\pi}{2a}$$

we get

$$\frac{d}{da}\int_0^{\infty}\frac{dx}{a^2+x^2} = \int_0^{\infty}\frac{(-2a)dx}{(a^2+x^2)^2} = -\frac{\pi}{2a^2}\,,$$

equivalent to the answer in the text.

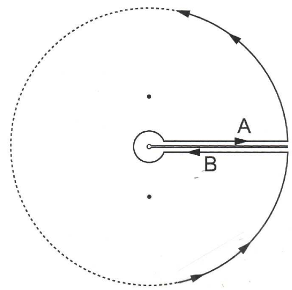

Figure 11.8.17. Contour surrounds branch cut and two poles.

11.8.16. We factor $1 + x^4 = (x^2 - i)(x^2 + i)$ and use the partial fraction expansion

$$\frac{x^2}{1+x^4} = \frac{1}{2}\left(\frac{1}{x^2-i} + \frac{1}{x^2+i}\right).$$

Applying Exercise 11.8.15 for each term with $a = e^{\pm\pi/4}$ we obtain for the integral

$$\frac{2\pi}{4}\left(e^{-i\pi/4} + e^{i\pi/4}\right) = \pi\cos\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

11.8.17. We approach this problem by considering the integral

$$\oint \frac{z^p \ln z}{z^2+1}\, dz$$

on an appropriate contour. The integrand has a branch point at the origin and we choose a contour of the form shown in Fig. 11.8.17 of this manual, corresponding to making the branch cut along the positive real axis and assigning $\theta = 0$ as the argument of points on the real axis just above the cut. Points on the positive real axis just below the cut will therefore have argument 2π. The contour consists of four pieces: (1) A line from 0+ to infinity above the cut; the integral on this line is equal to the integral we wish to evaluate:

$$I = \int_0^\infty \frac{x^p \ln x}{x^2+1}\, dx\,.$$

(2) A 360° counterclockwise arc at large $|z|$ to reach $x = +\infty$ just below the cut (this makes no contribution to the integral); (3) A line from $x = +\infty$ to $x = 0+$ below the cut, whose contribution to the contour integral will be discussed shortly, and (4) A clockwise 360° arc at small $|z|$ to close the contour (this also make no contribution because $\lim_{z\to 0} z^p \ln z = 0$).

On the line below the cut, $z^p = (xe^{2\pi i})^p = x^p e^{2\pi i p}$, and $\ln z = \ln x + 2\pi i$.

Taking these observations into account,

$$\int_{\infty-\varepsilon i}^{0+} \frac{z^p \ln z}{z^2+1}\,dz = -\int_0^\infty \frac{x^p e^{2\pi i p}[\ln(x)+2\pi i]}{x^2+1}\,dx$$

$$= -e^{2\pi i p} I - 2\pi i e^{2\pi i p}\int_0^\infty \frac{x^p}{x^2+1}\,dx\,.$$

The integral on the last line was the topic of Example 11.8.8, where it was shown to have the value $\pi/2\cos(p\pi/2)$. Putting all this information together, we have

$$\oint \frac{z^p \ln z}{z^2+1}\,dz = I - e^{2\pi i p} I - \frac{i\pi^2 e^{2\pi i p}}{\cos(p\pi/2)}\,,$$

We now evaluate the contour integral using the residue theorem. The integrand has first-order poles at $i = e^{\pi i/2}$ and $-i = e^{3\pi i/2}$, and we must use these representations to obtain the correct arguments for the pole locations. Both poles lie within the contour; the residues are

$$\text{Residue } (z=i) = \frac{e^{\pi i p/2}(\pi i/2)}{2i}\,, \quad \text{Residue } (z=-i) = \frac{e^{3\pi i p/2}(3\pi i/2)}{-2i}\,.$$

Setting the contour integral to its value from the residue theorem,

$$I\left(1-e^{2\pi i p}\right) - \frac{i\pi^2 e^{2\pi i p}}{\cos(p\pi/2)} = \frac{i\pi^2}{2}\left(e^{\pi i p/2} - 3e^{3\pi i p/2}\right)\,.$$

Multiplying through by $e^{-\pi i p}$ and rearranging slightly,

$$I\sin p\pi = \frac{\pi^2}{4}\left(3e^{\pi i p/2} - e^{-\pi i p/2} - \frac{2\,e^{\pi i p}}{\cos(p\pi/2)}\right)\,.$$

Writing all the complex exponentials in the above equation as trigonometric functions and using identities to make all the trigonometric functions have argument $p\pi/2$, the above equation reduces to

$$2I\sin(p\pi/2)\cos(p\pi/2) = \frac{\pi^2}{4}\left(\frac{2\sin^2(p\pi/2)}{\cos(p\pi/2)}\right)\,, \quad \text{or} \quad I = \frac{\pi^2\sin(p\pi/2)}{4\cos^2(p\pi/2)}\,.$$

11.8.18. (a) See solution of Exercise 12.4.3.

(b) It is useful to form the integral

$$I_C = \oint \frac{\ln^3 z}{1+z^2}\,dz$$

over the contour of Fig. 11.8.17 of this Manual. The integrand has a branch point at $z=0$ and we are making a cut along the positive real axis. It also has simple poles at $z = i = e^{i\pi/2}$ and at $z = -i = e^{3i\pi/2}$; we must use these exponential forms when computing the residues at the two poles. The small and large arcs of the contour do not contribute to the integral; for the segment from $x = 0+$ to infinity aboe the branch cut $\ln^3 z$ can be represented as $\ln^3 x$; for the segment from infinity to $x = 0+$ we must write $\ln^3 z = (\ln x + 2\pi i)^3$.

Keeping in mind that $\ln i = \pi i/2$ and that $\ln(-i) = 3\pi i/2$, we note that the residue of the integrand at $z = i$ is $(\pi i/2)^3/2i$ and that the residue at $z = -i$ is $(3\pi i/2)^3/(-2i)$.

Now we write

$$I_C = \int_0^\infty \frac{\ln^3 x}{1+x^2}\,dx - \int_0^\infty \frac{(\ln x + 2\pi i)^3}{1+x^2}\,dx = 2\pi i \left(\frac{\pi i}{2}\right)^3 \left(\frac{1-3^3}{2i}\right) = \frac{13 i\pi^4}{4}\,.$$

Expanding the left-hand side of the above equation,

$$-6\pi i \int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx + 12\pi^2 \int_0^\infty \frac{\ln x}{1+x^2}\,dx + 8\pi^3 i \int_0^\infty \frac{dx}{1+x^2} = \frac{13 i\pi^4}{4}\,.$$

Our present interest is in the imaginary part of this equation, which is

$$-6\pi \int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx + 8\pi^3 \int_0^\infty \frac{dx}{1+x^2} = \frac{13\pi^4}{4}\,.$$

The first of the two integrals on the left-hand side is that whose value we seek; the second is an elementary integral with value $\pi/2$. The equation therefore reduces to

$$\int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx = \frac{-8\pi^3(\pi/2) + 13\pi^4/4}{-6\pi} = \frac{\pi^3}{8}\,.$$

11.8.19. Use the symmetry of the integrand to extend the integral from $-\infty$ to ∞, and write the logarithm as a sum of two terms:

$$I = \int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \frac{1}{2}\left[\int_{-\infty}^\infty \frac{\ln(x+i)}{x^2+1}\,dx + \int_{-\infty}^\infty \frac{\ln(x-i)}{x^2+1}\,dx\right].$$

The first integrand has a branch point at $x = -i$ and its integral can be evaluated by the residue theorem using a contour that is closed in the upper half-plane (causing the branch point to lie outside the contour). The second integrand has a branch point at $x = i$ and its integral can be evaluated using a contour that is closed in the lower half-plane. We must choose branches for the logarithms that are consistent with their sum being real; a simple way to do this is to assign branches in a way such that the arguments of both $x - i$ and $x + i$ approach zero at large positive x.

The contour for the first integral encloses a first-order pole at $z = i$; the residue there is $\ln(2i)/2i$, with the logarithm on the branch such that $\ln i = \pi i/2$. The residue therefore evaluates to $(\ln 2)/2i + \pi/4$. The contour for the second integral encloses a first-order pole at $z = -i$, with residue $\ln(-2i)/(-2i)$, with the logarithm on the branch such that $\ln(-i) = -\pi i/2$. This residue is therefore $-(\ln 2)/2i + \pi/4$. Noting that the contour for the second integral circles the pole in the clockwise (mathematically negative) direction, application of the residue theorem leads to

$$I = \frac{1}{2}\, 2\pi i \left(\frac{\ln 2}{2i} + \frac{\pi}{4} + \frac{\ln 2}{2i} - \frac{\pi}{4} \right) = \pi \ln 2\,.$$

11.8.20. Use the contour of Fig. 11.26. The small and large circles do not contribute to the integral; the path element A evaluates to I, our integral, while the path element B contributes $-e^{2\pi i a} I$. The contour encloses a second-order pole at $z = -1$, at which the residue is

$$\frac{d}{dz}\, z^a \bigg|_{z=-1} = a \left(e^{i\pi(a-1)} \right) = -a e^{i\pi a}\,.$$

Therefore,

$$\left(1 - e^{2\pi i a}\right) I = 2\pi i \left(-a e^{i\pi a}\right)\,, \quad \text{or} \quad e^{-\pi i a} - e^{\pi i a} = -2\pi i a\,,$$

or $I \sin \pi a = \pi a$, equivalent to the result we are to prove.

11.8.21. Start by finding the zeros of the denominator:

$$z^2 = \cos 2\theta \pm \sqrt{cos^2 2\theta - 1} = e^{\pm 2i\theta}\,.$$

From this we find $z = \pm e^{\pm i\theta}$. Thus the integrand has (for most values of θ) four first-order poles, two in the upper half-plane and two in the lower half-plane. For simplicity we consider only the case that $e^{i\theta}$ and $-e^{-i\theta}$ are distinct and in the upper half-plane, and we take a contour that includes the real axis and a large semicircle in the upper half-plane. The large semicircle does not contribute to the integral. Writing our integral in the form

$$I = \oint \frac{z^2\, dz}{(z - e^{i\theta})(z + e^{-i\theta})(z + e^{i\theta})(z - e^{-i\theta})}\,,$$

application of the residue theorem yields

$$I = 2\pi i \left[\frac{e^{2i\theta}}{(e^{i\theta} + e^{-i\theta})(2e^{i\theta})(e^{i\theta} - e^{-i\theta})} + \frac{e^{-2i\theta}}{(-e^{-i\theta} - e^{i\theta})(-e^{-i\theta} + e^{i\theta})(-2e^{-i\theta})} \right]$$

$$= 2\pi i \left[\frac{e^{i\theta}}{8i \sin\theta \cos\theta} + \frac{e^{-i\theta}}{8i \sin\theta \cos\theta} \right] = \frac{\pi}{2 \sin\theta}\,.$$

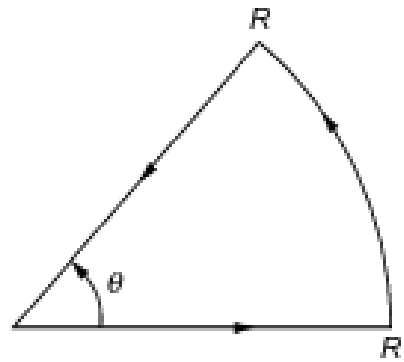

Figure 11.8.22. Sector contour.

11.8.22. If L denotes the triangular path of Fig. 11.8.22 of this manual with the angle θ set to $2\pi/n$, we find for $R \to \infty$,

$$I_L = \oint_L \frac{dz}{1+z^n} = (1 - e^{2\pi i/n}) \int_0^\infty \frac{dx}{1+x^n} \, .$$

The contour encloses a simple pole at $z_0 = e^{\pi i/n}$; we can find the residue there as the limit

$$\text{Residue} = \lim_{z \to z_0} \frac{z - z_0}{1+z^n} \, .$$

Using l'Hôpital's rule to evaluate the limit, we find the residue to be z_0^{1-n}/n, or, since $z_0^{-n} = -1$, the residue is $e^{i\pi/n}/n$. Thus,

$$I_L = (1 - e^{2\pi i/n}) \int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi i e^{i\pi/n}}{n} \, ,$$

which rearranges to

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n} \frac{2i\, e^{i\pi/n}}{(1 - e^{2\pi i/n})} = \frac{\pi}{n \sin(\pi/n)} \, .$$

11.8.23. This problem is similar to Exercise 11.8.21 except for the absence of the factor x^2 in the numerator of the integrand. This change causes the last line of the solution of Exercise 11.8.21 to be changed to

$$I = 2\pi i \left[\frac{e^{-i\theta}}{8i \sin\theta \cos\theta} + \frac{e^{i\theta}}{8i \sin\theta \cos\theta} \right] = \frac{\pi}{2 \sin\theta} \, ,$$

which is the same answer as that found for Exercise 11.8.21.

11.8.24. Use the contour in Fig. 11.8.17 of this manual. Letting I be the integral we want, path element A contributes I to the contour integral, while path

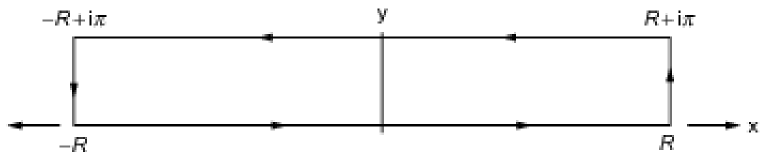

Figure 11.8.25. Lower line is on real axis; upper line is at $y = \pi$.

element B contributes $-e^{-2\pi ia}I$ to the contour integral. The value of the contour integral is $2\pi i$ times the residue of the integrand at $z = -1 = e^{\pi i}$, which is $e^{-\pi ia}$. Therefore,

$$I\left(1 - e^{-2\pi ia}\right) = 2\pi i e^{-\pi ia},$$

which rearranges to $I = \pi/\sin \pi a$.

11.8.25. Write $\cosh bx = (e^{bx} + e^{-bx})/2$, so

$$I = \int_0^\infty \frac{\cosh bx}{\cosh x}\,dx = \frac{1}{2}\int_0^\infty \frac{e^{bx}}{\cosh x}\,dx + \frac{1}{2}\int_0^\infty \frac{e^{-bx}}{\cosh x}\,dx$$
$$= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{bx}}{\cosh x}\,dx\,.$$

Evaluate this integral by considering it on a contour that takes account of the periodicity of $\cosh z$ (it has period 2π in the imaginary direction). Take the contour to be that shown in Fig. 11.8.25 of this manual. This contour consists of four line segments: (1) From $-R$ to R along the real axis, in the limit of large R; the integral of this segment is I; (2) From R to $R + i\pi$; this segment makes no contribution to the integral; (3) From $R + i\pi$ to $-R + i\pi$ on a line parallel to the real axis; the travel is toward negative x but the denominator is $\cosh(x + i\pi) = -\cosh x$, and (noting that the numerator is $e^{b(x+\pi i)}$) this segment evaluates to $+Ie^{ib\pi}$; (4) From $-R+i\pi$ to $-R$; the integral on this segment is zero. Combining the above,

$$\frac{1}{2}\oint \frac{e^{bz}}{\cosh z}dz = I\left(1 + e^{ib\pi}\right)\,.$$

We now evaluate the contour integral using the residue theorem. The integrand has poles at the zeros of $\cosh z$; the only pole within the contour is at $z = \pi i/2$, with residue (evaluated by l'Hôpital's rule)

$$\lim_{z\to\pi i/2} \frac{(z - \pi i/2)e^{bz}}{\cosh z} = \frac{e^{ib\pi/2}}{\sinh(\pi i/2)} = -ie^{ib\pi/2}\,.$$

We therefore have

$$I\left(1 + e^{ib\pi}\right) = \frac{1}{2}(2\pi i)\left(-ie^{ib\pi/2}\right)\,.$$

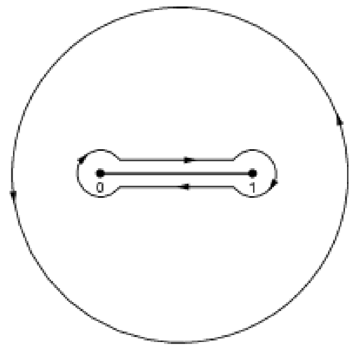

Figure 11.8.27. Contours for Exercise 11.8.27.

Solving for I,

$$I = \frac{\pi e^{ib\pi/2}}{1 + e^{ib\pi}} = \frac{\pi}{2\cos(\pi b/2)} .$$

11.8.26. Consider the integral $\oint e^{-z^2}\, dz$ on the sector contour in Fig. 11.8.22 of this manual, with θ set to $\pi/4$ (45°). This contour encloses no singularities, and the arc at infinity does not contribute to the integral. We therefore have

$$\oint e^{-z^2}\, dz = \int_0^\infty e^{-x^2}\, dx - \int_0^\infty e^{-ir^2} e^{i\pi/4}\, dr = 0 .$$

The x integration evaluates to $\sqrt{\pi}/2$, and we can write the integrand of the last integral in terms of its real and imaginary parts:

$$\frac{\sqrt{\pi}}{2} - \int_0^\infty \left[\cos r^2 - i \sin r^2\right] \frac{1+i}{\sqrt{2}}\, dr = 0 .$$

Letting I_c and I_s respectively stand for $\int_0^\infty \cos^2 r\, dr$ and $\int_0^\infty \sin^2 r\, dr$, the real and imaginary parts of the above equation take the forms

$$\frac{\sqrt{\pi}}{2} - \frac{I_c}{\sqrt{2}} - \frac{I_s}{\sqrt{2}} = 0, \qquad \frac{I_c}{\sqrt{2}} - \frac{I_s}{\sqrt{2}} = 0,$$

from which we deduce $I_c = I_s$ and $\sqrt{\pi}/2 = 2I_c/\sqrt{2}$, which reduce to the stated answers.

11.8.27. Consider

$$\oint \frac{dz}{z^{2/3}(1-z)^{1/3}} ,$$

with the contour the two closed curves shown in Fig. 11.8.27 of this manual. Note that together these curves enclose a region in which the integrand is analytic, so the line integrals on the two curves are equal and opposite. We pick a branch of the integrand which is real and positive above the branch cut, so the integral on the straight line above the cut from 0 to 1 is the integral we seek,

$$I = \int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}}\,.$$

The small circular arcs around $z = 0$ and $z = 1$ do not contribute to the integral; in polar coordinates about these respective points the singular factors are of order $r^{-2/3}$ and $r^{-1/3}$, while $dz = ire^{i\theta}\,d\theta$. On the straight line from 1 to 0 below the branch cut, we still have $z^{2/3} = x^{2/3}$, but instead of $(1-x)^{1/3}$ we have $e^{-2\pi i/3}(1-x)^{1/3}$, the minus sign in the exponent arising because the branch point at $z = 1$ was circled clockwise. On this line below the cut, the integrand is therefore $e^{2\pi i/3}/(x^2 - x^3)^{1/3}$, and this segment of the integral can be identified as $-e^{2\pi i/3}I$ (the new minus sign because the integration is toward negative x).

On the large circle the integrand becomes $1/(-1)^{1/3}z$, and we must select the proper branch for $(-1)^{1/3}$. To do so, note that at large real positive z, $z^{2/3}$ remains $x^{2/3}$, but $(1-x)^{1/3}$ becomes $|1-x|^{1/3}e^{-\pi i/3}$, so the entire integrand asymptotically becomes $1/e^{-\pi i/3}z$. Since the integral of $1/z$ around any circle (counterclockwise) is $2\pi i$, our integral over the large circle will have the value $2\pi i\, e^{\pi i/3}$.

Combining the above,

$$\oint \frac{dz}{z^{2/3}(1-z)^{1/3}} = I - e^{2\pi i/3}I + 2\pi i e^{\pi i/3} = 0\,,$$

which we can solve for I to obtain

$$I = \frac{-2\pi i e^{\pi i/3}}{1 - e^{2\pi i/3}} = \frac{\pi}{\sin(\pi i/3)} = \frac{2\pi}{\sqrt{3}}\,.$$

11.8.28. The key to this problem is the evaluation of the difference between the values of $\tan^{-1} az$ on the two sides of its branch cut in the upper half-plane. Referring to Fig. 11.8.28a of this manual and Eq. (1.137), define $r_1 = |z - a^{-1}i|$, $r_2 = |z + a^{-1}i|$, $\theta_1 = \arg(z - a^{-1}i)$, $\theta_2 = \arg(z + a^{-1}i)$, write

$$\tan^{-1} az = \frac{1}{2i}\Big[\ln r_1 + i\theta_1 - \ln r_2 - i\theta_2 + \pi i\Big]\,,$$

and note that for z just to the right of the cut in the upper half-plane the values of θ_1 and θ_2 are both $+\pi/2$, while for a corresponding value of z just to the left of the cut $\theta_1 = -3\pi/2$ and θ_2 remains $+\pi/2$. We therefore see that

$$\tan^{-1} az(\text{left of cut}) - \tan^{-1} az(\text{right of cut}) = -\pi\,.$$

We are now ready to evaluate the integral

$$\oint \frac{\tan^{-1} az}{z(z^2 + b^2)}\,dz$$

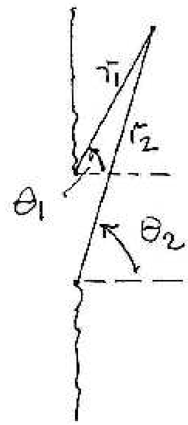

Figure 11.8.28a. Arguments and moduli of singular factors.

for the contour of Fig. 11.8.28b of this manual. The small arc around the branch point at a^{-1} and the large arcs at infinity do not contribute to the integral; the line along the real axis evaluates to I, the integral we seek. Writing $z = iy$ on segments B and B$'$ of the contour, and combining these segments so that the difference of the arctangents can be replaced by $-\pi$, we reach

$$\oint \frac{\tan^{-1} az}{z(z^2+b^2)}\, dz = I - \pi \int_{a^{-1}}^{\infty} \frac{i\, dy}{iy(b^2-y^2)} = I - \frac{\pi}{2b^2} \ln(1-a^2b^2)\,,$$

where we have evaluated the y integral, which is elementary.

The contour integral encloses a region in which the integrand is analytic everywhere except for a first-order pole at $z = ib$, at which its residue is

$$\frac{1}{ib(2ib)}\ \frac{1}{2i} \ln\left(\frac{1+i(iab)}{1-i(iab)}\right) = \frac{i}{4b^2} \ln\left(\frac{1-ab}{1+ab}\right).$$

Finally, from

$$I - \frac{\pi}{2b^2} \ln(1-a^2b^2) = 2\pi i\ \frac{i}{4b^2} \ln\left(\frac{1-ab}{1+ab}\right)$$

we solve for I, getting the relatively simple result $I = (\pi/b^2)\ln(1+ab)$.

There is no singularity at $z = 0$ because for small z, $\tan^{-1} az \approx az$.

11.9 Evaluation of Sums

11.9.1. Here $g(z)$ has, about z_0, the Laurent expansion $b_0(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \cdots$, while $f(z)$ has a Taylor series $f(z_0) + f'(z_0)(z-z_0) + \cdots$. Multiplying these expansions, the only term singular at $z = z_0$ is $b_0 f(z_0)(z-z_0)^{-1}$, which corresponds to a simple pole with residue $b_0 f(z_0)$.

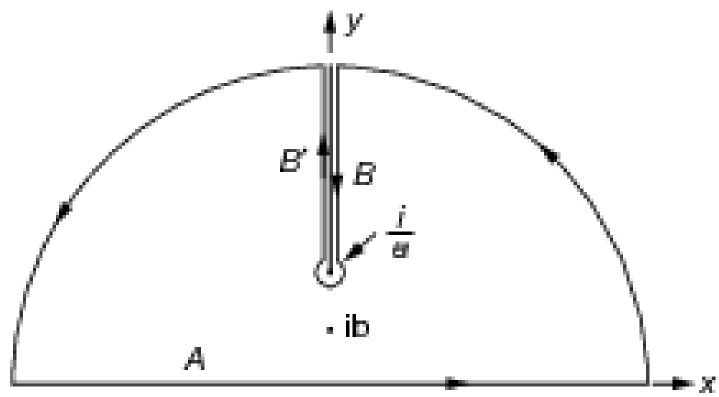

Figure 11.8.28b. Contour avoiding branch cut.

11.9.2. The limiting behavior for I_N that is needed for the contour-integral evaluation of sums is produced by the behavior of $f(z)$ for large $|z|$, which in applicable cases becomes small rapidly enough that the integrand of I_N becomes negligible on its entire contour and therefore evaluates to zero. To see that $\cot \pi z$ remains of order of magnitude unity for large $|z|$ and does not affect this analysis, write it in terms of exponentials:

$$|\cot \pi z| = \left| \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right| .$$

Because each of these exponentials occurs in both the numerator and denominator, this expression will remain of order unity except where the denominator approaches zero (i.e., near the poles of $\cot \pi z$).

11.9.3. Defining $S = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$, note that $\frac{1}{8} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^3} = 2S$.

Therefore $2S = \frac{1}{8} \sum (\text{residue of } z^{-3}\pi \sec \pi z \text{ at } z = 0)$.

This residue is $\frac{1}{2!} \frac{d^2}{dz^2} \pi \sec \pi z \Big|_{z=0} = \frac{\pi^3}{2}$, and $S = \frac{1}{16} \frac{\pi^3}{2} = \frac{\pi^3}{32}$.

11.9.4. This summation is most easily done by decomposing the summand into partial fractions:

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}.$$

If it is desired to use the contour-integral method for this summation, consider (for nonzero a)

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)(n+a+2)},$$

$$S(-a) = \sum_{n=1}^{\infty} \frac{1}{(n-a)(n-a+2)} = \sum_{n=-\infty}^{-3} \frac{1}{(n+a)(n+a+2)},$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+a+2)} =$$

$$S(a) + S(-a) + \frac{1}{(-2+a)a} + \frac{1}{(-1+a)(1+a)} + \frac{1}{a(a+2)}.$$

The summation in the last of the above equations can now be evaluated as minus the sum of the residues of $\pi \cot \pi z/[(z+a)(z+a+2)]$ at $z = -a$ and $z = -a-2$. These residues are respectively $\pi \cot(-\pi a)/2$ and $\pi \cot(-\pi a - 2\pi)/(-2)$. Invoking the periodicity of the cotangent, these are seen to add to zero.

Finally, setting the right-hand side of the last above equation to zero and then taking the limit $a \to 0$, we find

$$2S - 1 + \lim_{a\to 0} \frac{1}{a}\left[\frac{1}{a-2} + \frac{1}{a+2}\right] = 2S - \frac{3}{2} = 0, \qquad \text{or } S = \frac{3}{4}.$$

11.9.5. Our summation S is minus the residue of $\pi \csc \pi z$ at $z = -a$. This residue is

$$\pi \frac{d}{dz} \csc \pi z\Big|_{-a} = -\frac{\pi^2 \cos(-\pi a)}{\sin^2(-\pi a)}, \quad \text{so } S = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}.$$

11.9.6. (a) Note that our summation S is 1/2 times the result of extending the indicated summation to $-\infty$. Writing

$$S = \frac{1}{8}\sum_{-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^2} = \frac{1}{8}\left(\text{residue of } \frac{\pi \tan \pi z}{z^2} \text{ at } z = 0\right).$$

This residue is $\displaystyle \frac{\pi^2}{\cos^2 \pi z}\Big|_{z=0} = \pi^2$, so $S = \dfrac{\pi^2}{8}$.

(b) Write $\displaystyle S = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \zeta(2) - \frac{1}{4}\zeta(2) = \frac{3}{4}\zeta(2) = \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}$.

11.9.7. The summation is of the form

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{1}{\cosh(n+\frac{1}{2})\pi} = \frac{1}{2}\sum_{-\infty}^{\infty} \frac{(-1)^n}{2n+1} \frac{1}{\cosh(n+\frac{1}{2})\pi}.$$

Thus,

$$2S = \sum \left(\text{residues of } \frac{\pi \sec \pi z}{2z \cosh \pi z} \text{ at the singularities of } \frac{1}{2z \cosh \pi z} \right).$$

These singularities occur at $z = 0$ and and $z_n = (n + \frac{1}{2})i$, for all integer n from $-\infty$ to ∞. The residue at $z = 0$ is $\pi/2$. The residues at $z = z_n$ can be calculated using l'Hôpital's rule:

$$\text{residue at } z_n = \lim_{z \to z_n} \frac{\pi(z - z_n)}{2z \cos \pi z \cosh \pi z} = \frac{\pi}{2\pi z_n \cos \pi z_n \sinh \pi z_n}$$

$$= \frac{1}{(2n+1)i \cos[(n+\frac{1}{2})\pi i][i(-1)^n]}$$

$$= -\frac{(-1)^n}{(2n+1)\cosh(n+\frac{1}{2})\pi}.$$

When these residues are summed over n, the result is $-2S$, so we finally reach

$$2S = -2S + \frac{\pi}{2}, \quad \text{or} \quad S = \frac{\pi}{8}.$$

11.9.8. The summand of our sum S is even, so $\displaystyle\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n \sin n\varphi}{n^3} = 2S$,

where the prime indicates omission of the term $n = 0$. Therefore, since $\sin \varphi z / z^3$ has only a pole at $z = 0$,

$$2S = -(\text{residue of } \pi \csc \pi z \sin \varphi z / z^3 \text{ at } z = 0).$$

The pole at $z = 0$ is of third order, so the residue we seek is

$$\lim_{z \to 0} \frac{\pi}{2!} \frac{d^2}{dz^2} \left(\frac{\sin \varphi z}{\sin \pi z} \right) = \frac{\pi}{2} \lim_{z \to 0} \left[\frac{(\pi^2 - \varphi^2) \sin \varphi z}{\sin \pi z} \right.$$

$$\left. + \frac{2\pi \cos \pi z}{\sin^3 \pi z} (\pi \sin \varphi z \cos \pi z - \varphi \cos \varphi z \sin \pi z) \right] = \frac{\varphi(\pi^2 - \varphi^2)}{6}.$$

From this result we get $S = \dfrac{\varphi}{12}(\varphi^2 - \pi^2)$.

11.10 Miscellaneous Topics

11.10.1. $f^*(z^*) = f(z)$ is equivalent to

$$u(x, -y) - iv(x, -y) = u(x, y) + iv(x, y),$$

which in turn implies

(a) $u(x, -y) = u(x, y)$ and (b) $v(x, -y) = -v(x, y)$.

11.10.2. Given that $f(z) = \sum_n a_n z^n$, with a_n real, then

$$f(z^*) = \sum_n a_n (z^*)^n = \left[\sum_n a_n z^n\right]^* .$$

(a) If $f(z) = z^n$, then $f(z^*) = (z^*)^n = (z^n)^*$, as predicted.

(b) If $f(z) = \sin z = \sum_{n=0}^{\infty} \frac{z^n}{(2n+1)!}$, then $f(z^*) = [f(z)]^*$.

But if $f(z) = iz$, then $f(z^*) = iz^*$ while $[f(z)]^* = -iz^*$.

11.10.3. (a) $f^*(x) = -f(x)$ implies $(if(x))^* = -i(-f(x)) = if(x)$, i.e., that $if(x)$ is real and its power series expansion will have real coefficients a_n. Thus, $if(z)$ meets the conditions of Exercise 11.10.2, so $(if(z))^* = if(z^*)$ and therefore also $f^*(z) = -f(z^*)$.

(b) $f(z) = iz = ix - y$, $f(z^*) = iz^* = ix + y$, and

$f^*(z) = (iz)^* = -i(x - iy) = -ix - y$.

11.10.4. $|z|^2 = r^2 = x^2 + y^2$.

(a) $w_1(z) = z + \frac{1}{z}$ yields

$$w_1 = u_1 + iv_1 = z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2} = x\left(1 + \frac{1}{r^2}\right) + iy\left(1 - \frac{1}{r^2}\right).$$

Parameterizing the circle as $x = r\cos\theta$, $y = r\sin\theta$ we obtain

$$u_1 = r\cos\theta\left(1 + \frac{1}{r^2}\right), \qquad v_1 = r\sin\theta\left(1 - \frac{1}{r^2}\right).$$

For $r \neq 1$,

$$\frac{u_1^2}{r^2\left(1 + \frac{1}{r^2}\right)^2} + \frac{v_1^2}{r^2\left(1 - \frac{1}{r^2}\right)^2}$$

are ellipses centered at the origin. For $r \to 1$, $v_1 \to 0$, the ellipses flatten to the u_1−axis.

(b) This map leads to

$$z - \frac{1}{z} = x\left(1 - \frac{1}{r^2}\right) + iy\left(1 + \frac{1}{r^2}\right) = u_2 + iv_2.$$

The curves

$$u_2 = r\left(1 - \frac{1}{r^2}\right)\cos\theta, \qquad v_2 = r\left(1 + \frac{1}{r^2}\right)\sin\theta$$

are ellipses for $r \neq 1$ and flatten to the v_2-axis for $r \to 1$ because $u_2 \to 0$.

11.10.5. (a) $x > 0$.

(b) $y > 0$.

11.10.6. (a) From $1/z = (x - iy)/(x^2 + y^2)$, or

$$u = x/(x^2 + y^2)\,, \qquad v = -y/(x^2 + y^2)\,,$$

we obtain

$$x = u/(u^2 + v^2)\,, \qquad y = -v/(u^2 + v^2)\,.$$

Substituting these expressions into the equation $(x-a)^2+(y-b)^2-r^2 = 0$, we initially have

$$\left(\frac{u}{u^2+v^2} - a\right)^2 + \left(\frac{-v}{u^2+v^2} - b\right)^2 - r^2 = 0\,.$$

Expanding this expression, clearing the denominators by multiplying through by $(u^2 + v^2)^2$, cancelling common factors, and completing the squares on the terms involving u and those involving v, we ultimately reach

$$(u - A)^2 + (v - B)^2 = R^2, \quad \text{with}$$

$$A = \frac{-a}{r^2 - a^2 - b^2}, \quad B = \frac{b}{r^2 - a^2 - b^2}, \quad R = \frac{r}{r^2 - a^2 - b^2}\,.$$

(b) The transformation produced a circle whose center is at $A + iB$. The center before transformation was at $a + ib$. The transformation of $a + ib$ is to $(a - ib)/(a^2 + b^2)$, which is not at $A + iB$.

11.10.7. If two curves in the z plane pass through a point z_0, one in the direction $dz_1 = e^{i\theta_1}ds$ and the other in the direction $dz_2 = e^{i\theta_2}ds$, the angle from the first curve to the second will be $\theta_2 - \theta_1$. If these curves are mapped into the w plane, with $w = f(z)$, then

$$dw_1 = f'(z_0)e^{i\theta_1}ds \quad \text{and} \quad dw_2 = f'(z_0)e^{i\theta_2}ds\,.$$

Writing $f'(z_0)$ in polar form as $|f'(z_0)|e^{i\varphi}$, we see that

$$dw_1 = |f'(z_0)|e^{i(\theta_1+\varphi)}ds, \qquad dw_2 = |f'(z_0)|e^{i(\theta_2+\varphi)}ds\,,$$

and, because $f'(z_0)$ was assumed nonzero, the angle from the first curve to the second in the w plane will also be $\theta_2 - \theta_1$. That is why mappings by an analytic function are termed **conformal**.

12. Further Topics in Analysis

12.1 Orthogonal Polynomials

12.1.1. We express the derivative in the Rodrigues formula for H_n as a contour integral, and then form the sum $g(x,t) = H_n(x)t^n/n!$,

$$g(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint \frac{e^{-z^2}}{(z-x)^{n+1}} \frac{t^n}{n!}\, dz\,.$$

The contour must enclose the point $z = x$; there are no other singularities at finite z to be avoided. Next we interchange the summation and integration and evaluate the sum, which is

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(z-x)^{n+1}} = \frac{1}{z-x+t}\,.$$

Inserting this result,

$$g(x,t) = \frac{e^{x^2}}{2\pi i} \oint \frac{e^{-z^2}\, dz}{z-x+t} = e^{x^2}\, e^{-(x-t)^2} = e^{-t^2+2xt}\,.$$

12.1.2. (a) Since the Laguerre ODE has the form $xy'' + (1-x)y + ny = 0$, we calculate the weight function w as

$$w = \frac{1}{x} \exp\left[\int^x \frac{1-t}{t}\, dt\right] = e^{-x}\,.$$

Then form

$$L_n(x) = \text{constant} \cdot \frac{1}{w} \left(\frac{d}{dx}\right)^n (x^n\, w) = \text{constant} \cdot e^x \left(\frac{d}{dx}\right)^n \left(x^n\, e^{-x}\right).$$

The constant is assiged the value $1/n!$ to produce the Laguerre polynomials at the conventional scaling.

(b) We obtain the generating function by using a contour integral to represent the differentiation in the Rodrigues formula:

$$g(x,t) = \sum_{n=0}^{\infty} L_n(x)t^n = \sum_{n=0}^{\infty} \frac{e^x}{n!} \frac{n!}{2\pi i} \oint \frac{z^n e^{-z} t^n}{(z-x)^{n+1}}\, dz\,.$$

The contour must enclose the point $z = x$; there are no other finite singularities. We now interchange the summation and integration and evaluate the sum:

$$\sum_{n=0}^{\infty} \frac{(zt)^n}{(z-x)^{n+1}} = \frac{1}{z(1-t)-x}\,.$$

Then we have

$$g(x,t) = \frac{e^x}{2\pi i} \oint \frac{e^{-z}\, dz}{z(1-t)-x} = \frac{e^x}{1-t} e^{-x/(1-t)} = \frac{e^{-tx/(1-t)}}{1-t}\,.$$

12.1.3. The three terms on the left-hand side of Eq. (12.11) respectively correspond to

(1) Applying all $n+1$ differentiations to wp^n,

(2) Applying n differentiations to wp^n and one to p, and

(3) Note that $n-1$ differentiations of wp^n and two of p; the two right-hand terms respectively correspond to (i) all $n+1$ differentiations to wp^n, and (ii) n differentiations of wp^n and one of $(n-1)p' + q$.

To reach Eq. (12.12), we combined or canceled similar terms and inserted y_n from Eq. (12.9) in the term involving n-fold differentiation of wp^n. Note that we are not assuming that y_n is a solution to our ODE; at this point we treat it solely as a defined quantity.

12.1.4. The unnumbered identity following Eq. (12.12) is confirmed by regarding its left-hand side as involving a two-fold differentiation of the nth derivative of wp^n. Equation (12.13) is obtained by recognizing y_n and using Eq. (12.6) to evaluate the derivatives of w^{-1}. Equation (12.15) can be interpreted as involving a single differentiation of the n-fold derivative of wp^n.

12.1.5. Using the Cauchy integral for the power series for the generating functions yields the results.

12.1.6. The solution is given in the text.

12.1.7. Differentiate the generating-function formula with respect to t:

$$\frac{2x-2t}{(1-2xt+t^2)^2} = \sum_{n=0}^{\infty} nU_n(x)t^{n-1}\,.$$

Multiply both sides of this equation by $1-2xt+t^2$, then identify the left-hand side as $2x-2t$ times the generating-function expansion. The resulting equation has the form

$$2x\sum_{n=0}^{\infty} U_n(x)t^n - 2\sum_{n=0}^{\infty} U_n(x)t^{n+1} =$$

$$\sum_{n=0}^{\infty} nU_n(x)t^{n-1} - 2x\sum_{n=0}^{\infty} nU_n(x)t^n + \sum_{n=0}^{\infty} nU_n(x)t^{n+1}\,.$$

Combining similar terms and collecting the coefficient of each power of t, we find

$$U_{n-1}(x) - 2xU_n(x) + U_{n+1}(x) = 0\,.$$

12.2 Bernoulli Numbers

12.2.1. Eq. (12.32): Multiply numerator and denominator of the fraction on the right-hand side by e^t and then put both right-hand terms over a common denominator.

Eq. (12.46): Multiply numerator and denominator of the right-hand side by e^t.

12.2.2. Using the power series for $\dfrac{x}{e^x - 1}\, e^{xs}$ we get

$$\sum_{n=0}^{\infty} B_n(s)\frac{x^n}{n!} = \left(1 - \frac{x}{2}\right)\left(1 + xs + \frac{x^2 s^2}{2} + \cdots\right)$$

$$+ \sum_{n=1}^{\infty} B_{2n}\frac{x^{2n}}{(2n)!}\left(1 + xs + \frac{x^2 s^2}{2} + \cdots\right)$$

$$= 1 + \left(xs - \frac{x}{2}\right) + \frac{1}{2}\,x^2\left(s^2 - s + \frac{1}{6}\right) + \cdots .$$

Reading off coefficients of $x^n/n!$ we find

$$B_0(s) = 1, \qquad B_1(s) = s - \frac{1}{2}, \qquad B_2(s) = s^2 - s + \frac{1}{6}, \qquad \text{etc.}$$

12.2.3. Checking first the identity we then use it to get from the power series

$$x\tan x = x\cot x - 2x\cot 2x = \sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2x)^{2n}}{(2n)!}\,\left(1 - 2^{2n}\right) .$$

12.3 Euler-Maclaurin Integration Formula

12.3.1. (a) $\displaystyle\sum_{m=1}^{n} m = \int_0^n m dm + \int_0^n (x - [x])dx$

$$= \frac{1}{2}n^2 + \sum_{m=0}^{n-1}\int_m^{m+1}\Big(x - [x]\Big)\,dx = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1),$$

because

$$\int_m^{m+1}(x - m)\,dx = \int_0^1 y\,dy = \frac{1}{2}\,.$$

(b) $\displaystyle\sum_{m=1}^{n} m^2 = \int_0^n m^2\,dm + 2\sum_{m=0}^{n-1}\int_m^{m+1}(x - m)x\,dx$

$$= \frac{1}{3}\, n^3 + 2 \sum_{m=0}^{n-1} \int_m^{m+1} \left[(x-m)^2 + m(x-m)\right]\, dx$$

$$= \frac{1}{3}\, n^3 + 2 \sum_{m=0}^{n-1} \left[\int_0^1 y^2\, dy + m \int_0^1 y\, dy\right]$$

$$= \frac{1}{3}\, n^3 + \frac{2}{3}\, n + \frac{1}{2}\, n(n-1) = \frac{1}{6}\, n(n+1)(2n+1).$$

(c) Omitting steps but working as before we obtain

$$\sum_{m=1}^{n} m^3 = \int_0^n m^3\, dm + 3 \int_0^n \Big(x - [x]\Big)\, x^2\, dx$$

$$= \frac{1}{4}\, n^4 + 3 \sum_{m=0}^{n-1} \left[\int_0^1 y^3\, dy + 2m \int_0^1 y^2\, dy + m^2 \int_0^1 y\, dy\right]$$

$$= \frac{n^2}{4}\, (n+1)^2.$$

(d) $$\sum_{m=1}^{n} m^4 = \int_0^n m^4\, dm + 4 \int_0^n \Big(x - [x]\Big)\, x^3\, dx$$

$$= \frac{n}{30}\, (n+1)(2n+1)(3n^2+3n-1).$$

12.3.2. The solution is given in the text.

12.4 Dirichlet Series

12.4.1. Simplifying the formula given in the exercise, we have

$$\zeta(2n) = \frac{\pi^{2n}\, 2^{2n-1}}{(2n)!}\, |B_n|\, .$$

The B_n can be read out of Table 12.2.

12.4.2. Make the substitution $1 - x = e^{-t}$. The limits $x = 0$ and $x = 1$ correspond respectively to $t = 0$ and $t = \infty$. The integral becomes

$$I = \int_0^\infty \frac{[\ln e^{-t}]^2}{1 - e^{-t}}\, e^{-t}\, dt = \int_0^\infty \frac{t^2 e^{-t}}{1 - e^{-t}}\, dt\, .$$

Now expand the denominator as a geometric series and make the further change of variable to $u = nt$:

$$I = \sum_{n=1}^{\infty} \int_0^\infty t^2 e^{-nt}\, dt = \sum_{n=1}^{\infty} \frac{1}{n^3} \int u^2 e^{-u}\, du = \sum_{n=1}^{\infty} \frac{2!}{n^3} = 2\zeta(3)\, .$$

12.4.3. For convergence, we split up the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$ and substitute $y = 1/x, dy = -dx/x^2$ in the first integral. This gives

$$\int_0^1 \frac{\ln^2 y}{1+y^2}dy = \int_1^\infty \frac{\ln^2 x}{1+x^{-2}}\frac{dx}{x^2} = \int_1^\infty \frac{\ln^2 x}{1+x^2}dx.$$

Upon substituting $x = e^t$, $dx = e^t dt$ and using the geometric series for $(1+e^{-2t})^{-1}$, we obtain

$$\int_0^\infty \frac{\ln^2 x}{1+x^2}dx = 2\int_0^\infty \frac{t^2 dt}{e^t(1+e^{-2t})} = 2\sum_{n=0}^\infty (-1)^n \int_0^\infty t^2 e^{-(2n+1)t}dt$$

$$= 2\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3}\int_0^\infty t^2 e^{-t}dt = 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3},$$

using $\int_0^\infty t^2 e^{-t}dt = \Gamma(3) = 2.$

12.4.4. Starting from the definition, rearrange $\beta(2)$ as follows:

$$\beta(2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots$$

$$= 2\left[1 + \frac{1}{5^2} + \frac{1}{9^2} + \cdots\right] - \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$= 2\sum_{k=1}^\infty \frac{1}{(4k-3)^2} - \frac{\pi^2}{8},$$

where we have reecognized the last sum as the known series $\lambda(2)$.

12.4.5. (a) Insert a series expansion for $\ln(1+x)$:

$$I = \int_0^1 \frac{\ln(1+x)}{x}\,dx = \sum_{n=1}^\infty \int_0^1 \frac{(-1)^{n+1}x^{n-1}}{n}\,dx = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}$$

$$= \frac{1}{1^2} - \frac{2}{2^2} + \frac{1}{3^2} - \cdots = \left[\frac{1}{1^2} + \frac{2}{2^2} + \cdots\right] - 2\left[\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right]$$

$$= \zeta(2) - 2\left(\frac{1}{2^2}\right)\zeta(2) = \frac{1}{2}\,\zeta(2)\,.$$

(b) Note that the answer in the text is missing a minus sign. Use the series expansion for $\ln(1-x)$:

$$I = \int_0^1 \frac{\ln(1-x)}{x}\,dx = -\sum_{n=1}^\infty \int_0^1 \frac{x^{n-1}}{x}\,dx = -\sum_{n=1}^\infty \frac{1}{n^2} = -\zeta(2)\,.$$

12.4.6. Starting from the summation (from $s = 2$ to $s = n$) of $2^{-s}\zeta(s)$, insert the expansion of the zeta function (using p as the expansion index) and then perform the summation over s. A convenient way to organize the process is to write the terms of $2^{-s}\zeta(s)$ in a two-dimensional array:

$$\begin{array}{llll}
+\dfrac{1}{2^2} & +\dfrac{1}{4^2} & +\dfrac{1}{6^2} & +\dfrac{1}{8^2} \\[2ex]
+\dfrac{1}{2^3} & +\dfrac{1}{4^3} & +\dfrac{1}{6^3} & +\dfrac{1}{8^3} \\[2ex]
\cdots & \cdots & \cdots & \cdots \\[2ex]
+\dfrac{1}{2^n} & +\dfrac{1}{4^n} & +\dfrac{1}{6^n} & +\dfrac{1}{8^n}
\end{array}$$

Now sum the entries in each vertical column; they form finite geometric series:

$$\frac{\dfrac{1}{2^2}-\dfrac{1}{2^{n+1}}}{1-\dfrac{1}{2}} \qquad \frac{\dfrac{1}{4^2}-\dfrac{1}{4^{n+1}}}{1-\dfrac{1}{4}} \qquad \frac{\dfrac{1}{6^2}-\dfrac{1}{6^{n+1}}}{1-\dfrac{1}{6}} \qquad \frac{\dfrac{1}{8^2}-\dfrac{1}{8^{n+1}}}{1-\dfrac{1}{8}}$$

Collecting now the first term of each of these column sums, we get

$$\frac{1/2^2}{1-\frac{1}{2}} + \frac{1/4^2}{1-\frac{1}{4}} + \cdots = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \cdots = \ln 2\,.$$

The second terms of the column sums can be identified as

$$-\frac{1}{2^{n+1}(1-\frac{1}{2})} - \frac{1}{4^{n+1}(1-\frac{1}{4})} - \cdots = -\sum_{p=1}^{\infty}(2p)^{-n-1}\left[1-\frac{1}{2p}\right]^{-1}.$$

Putting everything together, we get

$$\sum_{s=2}^{n} 2^{-s}\zeta(s) = \ln 2 - \sum_{p=1}^{\infty}(2p)^{-n-1}\left[1-\frac{1}{2p}\right]^{-1},$$

equivalent to the stated answer for the exercise.

12.4.7. This problem can be approached in a way similar to the solution of Exercise 12.4.6. If $\zeta(2s)$ is expanded (with expansion index p) and the terms of each p are summed over s, we find

$$\sum_{s=1}^{n} 4^{-2s}\zeta(2s) = \left[\frac{1}{4^2-1} + \frac{1}{8^2-1} + \cdots\right] - \sum_{p=1}^{\infty}(4p)^{-2n-2}\left[1-\frac{1}{(4p)^2}\right]^{-1}.$$

The summation in square brackets expands into

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\cdots\right)=\frac{1-\pi/4}{2}\,,$$

leading to the stated answer for the exercise.

12.5 Infinite Products

12.5.1. Writing $\ln P = \sum_{n=1}^{\infty} \ln(1 \pm a_n)$,

insert the power-series expansion of the logarithm. As a_n becomes small (a necessary condition for convergence), only the leading term (linear in a_n) remains significant. Thus, P will be finite only if the sum of the a_n converges.

12.5.2. Expand $1/(1+b/n)$ and examine the leading terms of $(1+a/n)/(1+b/n)$, which are of the form $1+(a-b)/n+O(n^{-2})$. The tests series for convergence will be the (divergent) harmonic series unless $a = b$.

12.5.3. Form $2\sin x\cos x$, using for each its infinite product formula. The result (with factors of two inserted in a way that does not change the value of the expression) is

$$2\sin x\cos x = 2x\prod\left(1-\frac{4x^2}{(2n)^2\pi^2}\right)\left(1-\frac{4x^2}{(2n-1)^2\pi^2}\right)\,.$$

Each term of the above infinite product corresponds to two consecutive terms of the expansion of $\sin 2x$, consistent with the relation $\sin 2x = 2\sin x\cos x$.

12.5.4. 1.

12.5.5. $$\prod_{n=2}^{\infty}\{1-2/[n(n+1)]\} = \prod_{n=2}^{\infty}(1-1)/n[1+1/(n+1)]$$
$$= \prod_{n=2}^{\infty}\frac{n-1}{n}\cdot\frac{n+2}{n+1} = \frac{2}{2\cdot 3}\prod_{n=2}^{\infty}\frac{n-1}{n}\cdot\frac{n}{n-1} = \frac{1}{3}$$

upon shifting n in the second product down to $n-2$ and correcting for the two first terms.

12.5.6. $$\prod_{n=2}^{\infty}(1-1/n^2) = \prod_{n=2}^{\infty}(1-1/n)\,(1+1/n) = \prod_{n=2}^{\infty}\frac{n-1}{n}\cdot\frac{n+1}{n}$$
$$= \frac{1}{2}\prod_{n=2}^{\infty}\frac{n-1}{n}\cdot\frac{n}{n-1} = \frac{1}{2}$$

after shifting n in the second product term down to $n-1$ and correcting for the first missing term.

12.5.7. Write $1 + z^p = \dfrac{1 - z^{2p}}{1 - z^p}$.

When this is inserted into the infinite product the numerators cancel against the even powers in the denominator, leaving

$$\frac{1}{(1-z)(1-z^3)(1-z^5)\cdots},$$

as found by Euler.

12.5.8. Expand the exponential in powers of x/r. The leading terms that are significant for large r are $1 - x^2/2r^2$. Since $\sum_r x^2/2r^2$ converges for all x, so also does the infinite product.

12.5.9. Find the indefinite integral of $\cot t$, by integrating the expansion given in Eq. (12.35) and, alternatively, as its closed-form expression:

$$\begin{aligned}\int^x \cot t\, dt &= \ln x + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \frac{x^{2n}}{2n} + C \\ &= \ln \sin x\,.\end{aligned}$$

The constant of integration is zero, since $\lim_{x\to 0}(\sin x/x) = 1$ and $\ln 1 = 0$. Thus, the explicit form for the coefficients a_n is

$$a_0 = 0, \qquad a_{2n+1} = 0, \qquad a_{2n} = (-1)^n \frac{2^{2n} B_{2n}}{2n(2n)!}, \quad n \ge 1.$$

12.5.10. The key to this problem is to recognize that $d \ln \sin x/dx = \cot x$. Taking the logarithm of the infinite product formula for $\sin x$ and expanding the logarithm, we get

$$\ln \sin x = \ln x - \sum_{m,n=1}^{\infty} \frac{1}{m} \left(\frac{x}{n\pi}\right)^{2m}.$$

Differentiating, and then multiplying by x, we reach

$$x \cot x = 1 - \sum_{m,n=1}^{\infty} 2 \left(\frac{x}{n\pi}\right)^{2m}.$$

12.6 Asymptotic Series

12.6.1. (a) $C(x) = \dfrac{1}{2} + S_1 \cos\left(\dfrac{\pi x^2}{2}\right) - S_2 \sin\left(\dfrac{\pi x^2}{2}\right)$.

(b) $S(x) = \dfrac{1}{2} + S_1 \sin\left(\dfrac{\pi x^2}{2}\right) + S_2 \cos\left(\dfrac{\pi x^2}{2}\right)$

with

$$S_1 = \frac{1}{\pi x}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(4n+1)}{(\pi x^2)^{2n+1}},$$

$$S_2 = \frac{1}{\pi x}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(4n-1)}{(\pi x^2)^{2n}}.$$

Hint: $C(x) + iS(x) = C(\infty) + iS(\infty) - \int_x^\infty \exp[i\pi u^2/2]\,du.$

12.6.2. Consider the repeated application of an integration by parts to the integral representation of $\mathrm{Ci}(x) + i\mathrm{si}(x)$. Letting $I_0 = e^{it}$ and $D_0 = 1/t$, the first integration by parts yields

$$-\int_x^\infty \frac{e^{it}}{t}\,dt = D_0(x)I_1(x) + \int_x^\infty D_1(t)I_1(t)\,dt\,,$$

where $I_n = \int I_{n-1}(t)\,dt$ and $D_n = dD_{n-1}/dt$. Continuing,

$$-\int_x^\infty \frac{e^{it}}{t}\,dt = D_0(x)I_1(x) - D_1(x)I_2(x) + \cdots\,,$$

where $I_n = (-i)^n e^{it}$ and $D_n = (-1)^n n!/t^{n+1}$. Proceeding through N steps, we reach

$$\mathrm{Ci}(x) + i\mathrm{si}(x) = e^{ix}\sum_{n=0}^{N}\frac{(-i)^{n+1}\,n!}{x^{n+1}}\,.$$

Writing $e^{ix} = \cos(x) + i\sin(x)$ and identifying the real and imaginary parts of the eight-hand side of the above equation, we get the formulas in Eqs. (12.93) and (12.94).

12.6.3. As suggested in the Hint, we consider the integral $\displaystyle\int_x^\infty e^{-t^2}\,dt$.

To facilitate repeated integration by parts, we multiply the factor e^{-t^2} by $t\,dt$ and then integrate, thereby requiring that we divide the remainder if the integrand by t before differentiating it. Our scheme is therefore to define $I_0 = e^{-t^2}$ and $D_0 = 1$, with $I_n = \int t\,I_{n-1}\,dt$ and $D_n = d[D_{n-1}/t]/dt$. This partial integration scheme corresponds to

$$\int_x^\infty e^{-t^2}\,dt = -\frac{I_1(x)D_0(x)}{x} + \frac{I_2(x)D_1(x)}{x} - \cdots\,,$$

where

$$I_n(x) = \frac{(-1)^n}{2^n}\,e^{-t^2}\,, \qquad D_n(x) = \frac{(-1)^n(2n-1)!!}{t^{2n}}\,.$$

Substitution of these quantities leads to the expected result.

12.6.4. In the limit of large n, the ratio of the $(n+1)$th term of P to its nth term is (in relevant part)

$$\frac{\text{term } n+1}{\text{term } n} \approx \frac{(4n+3)^2(4n+1)^2}{(2n+1)(2n+2)(8z)^2},$$

the value of which approaches constant $\times\, n^2/z^2$. Since this ratio increases without limit, the series can only be asymptotic. A similar analysis applies to the function Q.

12.6.5. For $|x| > 1$, $\dfrac{1}{1+x} = \dfrac{1}{x(1+1/x)} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{x^{n+1}}$ converges, so is not an asymptotic series.

12.6.6. Writing

$$-\gamma = \left[\int_1^n \frac{dx}{x} - \sum_{s=1}^{n} \frac{1}{s}\right] + \left[\int_n^\infty \frac{dx}{x} - \sum_{s=n+1}^{\infty} \frac{1}{s}\right],$$

and apply the Euler-Maclaurin formula to the quantity in the second set of square brackets. Noting that the derivatives in the Euler-Maclaurin formula vanish at $x = \infty$ and that

$$f^{(2k-1)}(n) = \frac{(-1)^{2k-1}(2k-1)!}{n^{2k}},$$

the second bracketed quantity is identified as

$$\int_n^\infty \frac{dx}{x} - \sum_{s=n+1}^{\infty} \frac{1}{s} = \frac{1}{2n} + \sum_{k=1}^{N} \frac{B_{2k}}{(2k)!}\, f^{(2k-1)}(n) = \frac{1}{2n} - \sum_{k=1}^{N} \frac{B_{2k}}{(2k)n^{2k}},$$

equivalent to the desired result.

12.6.7. The answer given in the text is incorrect; it applies when the denominator of the integral is $1+v^2$, not the specified $(1+v^2)^2$.

Applying a binomial expansion to the denominator, the integral is asymptotically represented by the series

$$\sum_{n=0}^{N} \binom{-2}{n} \int_0^\infty v^{2n} e^{-xv}\, dv = \sum_{n=0}^{N} \frac{(-1)^n (n+1)(2n)!}{x^{2n+1}}.$$

12.7 Method of Steepest Descents

12.7.2. Substituting $z = x/s, dz = dx/s$ we have

$$\int_0^s \cos x^2 dx = s \int_0^1 \cos(s^2 z^2)\, dz,$$

and the corresponding result is valid for the sine integral. Now we replace $s^2 \to s$ and apply the saddle point method to

$$I = \int_0^1 e^{isz^2} dz = \int_0^1 \left[\cos(sz^2) + i\sin(sz^2)\right] dz.$$

With $f(z) = iz^2$, $f'(z) = 2iz$, $f''(z) = 2i$, we have a saddle point at $z = 0$ and $\alpha = \pi/2 - \pi/4$. Thus

$$I = \frac{\sqrt{2\pi}\, e^{i\pi/4}}{|2is|^{1/2}} = \sqrt{\frac{\pi}{2s}}\,(i+1).$$

This implies

$$\int_0^s \cos x^2 dx \sim \sqrt{\frac{\pi}{2}} \sim \int_0^s \sin x^2 dx.$$

12.7.3. Eq. (12.109) is valid for $\Re(s) > 0$ and, therefore, this asymptotic result is valid for large $\Re(s) > 0$.

12.8 Dispersion Relations

12.8.1. The answer is given in the text.

12.8.2. The integral over the small semicircle evaluates to $f(x_0)/2$, so we have

$$\frac{f(x_0)}{2} = \frac{1}{2\pi i} ⨍_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx\,,$$

equivalent to the answer we seek.

13. Gamma Function

13.1 Definitions, Properties

13.1.1. $\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt = -e^{-t}t^z\Big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}\,dt = z\Gamma(z).$

13.1.2. (a) In terms of factorials,

$$\frac{(s+n)!\,(n+2s)!\,(2n+1)!}{s!\,n!\,n!\,(2s+2n+1)!}\,.$$

(b) Using Pochhammer symbols, this expression can be written

$$\frac{(n+1)_{2s}(s+1)_n}{(2n+2)_{2s}(1)_n}\,.$$

13.1.3. Substituting $t^2 = u$, $2t\,dt = du$ we get for $\Re e(z) > 0$

$$\Gamma(z) = 2\int_0^\infty e^{-u^2}u^{2z-1}du\,.$$

(b) Substituting $\ln(1/t) = u$, $dt = -e^{-u}\,du$ we get

$$\Gamma(z) = \int_0^1 \left(\ln\frac{1}{u}\right)^{z-1} du\,.$$

Note that $t = 0$ corresponds to $u \to \infty$ and $t = 1$ to $u = 0$.

13.1.4. The expectation value of v^n, $\langle v^n\rangle$, is given by

$$\langle v^n\rangle = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty e^{-mv^2/2kT}\,v^{n+2}\,dv$$

$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left(\frac{m}{2kT}\right)^{-(n+3)/2}\int_0^\infty e^{-u^2}\,u^{n+2}\,du\,.$$

Making a change of variable in the u integral to $x = u^2$, that integral becomes

$$\int_0^\infty e^{-u^2}\,u^{n+2}\,du = \frac{1}{2}\int_0^\infty e^{-x}\,x^{(n+1)/2}\,dx = \frac{1}{2}\,\Gamma\left(\frac{n+3}{2}\right)\,.$$

To bring this expression to the form given as the answer in the text, replace $2/\sqrt{\pi}$ by $1/\Gamma(3/2)$.

13.1.5. For $k > -1$,

$$-\int_0^1 x^k \ln x\,dx = -\int_{-\infty}^0 e^{(k+1)t}t\,dt = (k+1)^{-2}\int_0^\infty e^{-t}t\,dt$$

$$= \frac{\Gamma(2)}{(k+1)^2} = \frac{1}{(k+1)^2}\,,$$

using the substitution $x = e^t$, $dx = e^t\,dt$.

13.1.6. $\displaystyle\int_0^\infty e^{-x^4} dx = \frac{1}{4}\int_0^\infty e^{-t} t^{-3/4}\, dt = \frac{\Gamma(1/4)}{4} = \Gamma(5/4)\,,$

where we have made the substitution $t = x^4$, $dt = 4x^3\, dx$.

13.1.7. Write $\Gamma(ax) = \dfrac{\Gamma(1+ax)}{ax}$ and $\Gamma(x) = \dfrac{\Gamma(1+x)}{x}$, after which both gamma functions approach the limit unity, and we are left with the easily reducible form x/ax.

13.1.8. The denominator of Eq. (13.1) shows that $\Gamma(z)$ has simple poles at $z = 0$, -1, -2, $\cdots$.

To find the residues, divide the Euler integral into two parts, integrating from 0 to 1, and from 1 to ∞. For the integral from 0 to 1 insert the power-series expansion of e^{-t}.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^1 \sum_{n=0}^\infty \frac{(-t)^n}{n!} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt$$

The integral from 1 to ∞ exhibits no singularities and we need not consider it further. Evaluating the integral from 0 to 1, we get

$$\int_0^1 \sum_{n=0}^\infty \frac{(-t)^n}{n!} t^{z-1} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)}\,,$$

which displays first-order poles at all negative integers $z = -n$ with respective residues $(-1)^n/n!$.

13.1.9. From Fig. 13.1 we see qualitatively that, for negative z, the lobes of $\Gamma(z)$ move closer to the horizontal axis as $-z$ increases in magnitude. To prove that the line representing any nonzero value of k will have an infinite number of intersections with the curve for $\Gamma(z)$, we need to show that its positive minima and negative maxima (near the half-integer values of $-z$) become arbitrarily small for large negative values of z. Using Eq. (13.23) for $z = 2n + \frac{1}{2}$, with n a positive integer, we find that

$$\Gamma(-2n + \tfrac{1}{2}) = \frac{\pi}{\Gamma(2n + \frac{1}{2})} \to 0$$

as $n \to \infty$ and is positive because $\sin(\pi/2) = +1$, while, for $x = 2n + \frac{3}{2}$,

$$\Gamma(-2n - \tfrac{1}{2}) = -\frac{\pi}{\Gamma(2n + \frac{3}{2})} \to 0$$

as $n \to \infty$ and is negative because $\sin(3\pi/2) = -1$. See also Exercise 13.1.14(a).

13.1.10. In both parts of this exercise, make a change of integration variable to $ax^2 = u$, with $dx = du/2u^{1/2}$. Then,

(a) $\displaystyle\int_0^\infty x^{2s+1}\, e^{-ax^2}\, dx = \frac{1}{2a^{s+1}} \int_0^\infty u^s\, e^{-u}\, du = \frac{s!}{2a^{s+1}}\,.$

(b) $\displaystyle\int_0^\infty x^{2s}\, e^{-ax^2}\, dx = \frac{1}{2a^{s+1/2}} \int_0^\infty u^{s-1/2}\, e^{-u}\, du$

$$= \frac{\Gamma(s+\frac{1}{2})}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}\,.$$

13.1.11. The answer is given in the text.

13.1.12. The answer is given in the text.

13.1.13. The coefficient of $\cos(n-2k)\theta$ in the expansion has the form

$$2\,\frac{(2n-1)!!}{(2n)!!}\,\frac{1\cdot 3\cdots(2k-1)}{1\cdot 2\cdots k}\,\frac{n(n-1)\cdots(n-k+1)}{(2n-1)(2n-3)\cdots(2n-2k+1)}$$

$$= 2\,\frac{(2n-1)!!}{(2n)!!}\,\frac{(2k-1)!!}{k!}\,\frac{n!\,(2n-2k-1)!!}{(n-k)!\,(2n-1)!!}\,.$$

Cancelling where possible, and changing notation to s and m, where $n = 2s+1$ and $n-2k = 2m+1$ (so $k = s-m$), we get

$$P_{2s+1}(\cos\theta) = \sum_{m=0}^{s} \frac{(2s-2m-1)!!(2s+2m+1)!!}{2^{2s}(s-m)!\,(s+m+1)!}\,\cos(2m+1)\theta\,.$$

13.1.14. Using the identities

$$\Gamma(\tfrac{1}{2}+n) = \Gamma(\tfrac{1}{2})\left[\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2n-1)}{2}\right],$$

$$\Gamma(\tfrac{1}{2}-n) = \frac{\Gamma(\frac{1}{2})}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)},$$

we form

$$\Gamma(\tfrac{1}{2}+n)\Gamma(\tfrac{1}{2}-n) = \Gamma(\tfrac{1}{2})^2\,(-1)^n = (-1)^n\pi\,.$$

13.1.15. Within the region of convergence of the Euler integral,

$$\left[\int_0^\infty t^{x+iy+1}\, e^{-t}\, dt\right]^* = \int_0^\infty t^{x-iy+1}\, e^{-t}\, dt\,.$$

By analytic continuation this relation extends to all nonsingular values of $\Gamma(z)$.

13.1.16. Letting z and z^* respectively stand for $\alpha + i\beta$ and $\alpha - i\beta$, and using the infinite-product formula, Eq. (13.15),

$$\frac{1}{\Gamma(z)\Gamma(z^*)} = zz^* e^{\gamma(z+z^*)} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)\left(1 + \frac{z^*}{n}\right) e^{-(z+z^*)/n} .$$

Writing $z + z^* = 2\alpha$ and $zz^* = \alpha^2 + \beta^2$ and identifying much of the above equation as similar to the product form of $1/\Gamma(\alpha)^2$, we find

$$\frac{1}{\Gamma(z)\Gamma(z^*)} = \frac{1}{\Gamma(\alpha)^2}\left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \frac{\left(1 + \dfrac{\alpha+i\beta}{n}\right)\left(1 + \dfrac{\alpha-i\beta}{n}\right)}{\left(1 + \dfrac{\alpha}{n}\right)^2} .$$

The argument of the infinite product simplifies to the form given in the exercise, so we reach

$$\frac{1}{\Gamma(z)\Gamma(z^*)} = \frac{1}{\Gamma(\alpha)^2}\left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) \prod_{n=1}^{\infty} \left[1 + \frac{\beta^2}{(n+\alpha)^2}\right] .$$

We now notice that the factor preceding the infinite product is exactly what we would get if we evaluated the product argument for $n = 0$. We therefore remove this factor and change the product lower limit to $n = 0$. Our formula is then entirely equivalent to that in the exercise.

13.1.17. As a first step, examine $|\Gamma(1 + ib)|$. Using Exercise 13.1.16,

$$|\Gamma(1+ib)|^{-2} = \prod_{n=0}^{\infty} \left[1 + \frac{b^2}{(n+1)^2}\right] .$$

Comparing with the infinite-product representation of $\sin x$, Eq. (12.77), the above product is identified as $\sin(i\pi b)/i\pi b = \sinh(\pi b)/\pi b$, so

$$|\Gamma(1+ib)|^2 = \frac{\pi b}{\sinh \pi b} .$$

We now use the functional relation $\Gamma(z + 1) = z\Gamma(z)$ for each of the two factors in $|\Gamma(1 + ib)|^2 = \Gamma(1 + ib)\Gamma(1 - ib)$, thereby reaching

$$\begin{aligned} |\Gamma(n+1+ib)|^2 &= [(1+ib)(2+ib)\cdots(n+ib)] \\ &\qquad \times [(1-ib)(2-ib)\cdots(n-ib)]\,|\Gamma(1+ib)|^2 \\ &= (1+b^2)(2^2+b^2)\cdots(n^2+b^2)\,\frac{\pi b}{\sinh \pi b} , \end{aligned}$$

equivalent to the result we seek.

13.1.18. Referring to the solution of Exercise 13.1.16, we see that $\Gamma(x+iy)$ is reached from $\Gamma(x)$ by multiplying the latter by an infinite series of factors each of which is smaller than unity.

13.1.19. Using the formula of Exercise 13.1.16 with $\alpha = 1/2$,

$$|\Gamma(\tfrac{1}{2}+iy)|^{-2} = \frac{1}{\pi}\prod_{n=0}^{\infty}\left[1+\frac{y^2}{(n+\frac{1}{2})^2}\right].$$

Comparing with the infinite-product expansion of $\cos x$ in Eq. (12.77), we identify the product here as $\cos i\pi y = \cosh \pi y$. Inserting this and taking the reciprocal, we confirm the desired answer.

13.1.20. (a) The mean is obtained from the integral $\displaystyle\int x\, f(x)\, dx$.

Writing this integral and making a change of variable to $y = x - \mu$, we get

$$\langle x\rangle = \frac{1}{\sigma(2\pi)^{1/2}}\int_{-\infty}^{\infty} x\, e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{\sigma(2\pi)^{1/2}}\int_{-\infty}^{\infty}(y+\mu)e^{-y^2/2\sigma^2}\, dy\,.$$

The y in the integrand can be dropped because, by symmetry, it makes no net contribution to the integral. The remainder of the expression now contains an integral of the form treated in Exercise 13.1.10(b); it simplifies to $\langle x\rangle = \mu$.

(b) To continue, we need to evaluate

$$\langle x^2\rangle = \frac{1}{\sigma(2\pi)^{1/2}}\int_{-\infty}^{\infty} x^2\, e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{\sigma(2\pi)^{1/2}}\int_{-\infty}^{\infty}(y+\mu)^2 e^{-y^2/2\sigma^2}\, dy\,.$$

We now expand $(y+\mu)^2$, drop the linear (odd) term, and evaluate the integrals using Exercise 13.1.10(b). The result is $\langle x^2\rangle = \sigma^2+\mu^2$. Therefore,

$$\langle x^2\rangle - \langle x\rangle^2 = (\sigma^2+\mu^2)-\mu^2 = \sigma^2\,, \quad \text{so} \quad (\langle x^2\rangle - \langle x\rangle^2)^{1/2} = \sigma\,.$$

13.1.21. (a) Here $\displaystyle\langle x\rangle = \int_0^{\infty}\frac{x^{\alpha}}{\beta^{\alpha}\Gamma(\alpha)}\, e^{-x/\beta}\, dx = \frac{\beta}{\Gamma(\alpha)}\int_0^{\infty} u^{\alpha}e^{-u}\, du$

$$= \frac{\beta\,\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\beta\,.$$

(b) For σ^2, we need $\displaystyle\langle x^2\rangle = \int_0^{\infty}\frac{x^{\alpha+1}}{\beta^{\alpha}\Gamma(\alpha)}\,,$

which by the same technique as used for part (a) is found to have the value

$$\langle x^2\rangle = \frac{\beta^2\,\Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2\,.$$

Thus, $\langle x^2\rangle - \langle x\rangle^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$.

13.1.22. Referring to the first part of the solution to Exercise 13.1.17, we have

$$|\Gamma(1+i\gamma)|^2 = \frac{\pi\gamma}{\sinh\pi\gamma}\,.$$

Multiplying by $e^{-\pi\gamma}$ and writing $\sinh\pi\gamma$ as exponentials, the result is immediate.

13.1.23. This problem would be specified more precisely if, instead of $(-t)^\nu$, it had contained $e^{-\pi i\nu}t^\nu$.

Starting from Eq. (13.30), consider a contour that starts at $+\infty+\varepsilon\, i$, continues (segment A) nearly to the origin, which it circles counterclockwise (B), then returning (segment C) to $+\infty-\varepsilon\, i$. For suitable values of ν, part B of the contour will make a negligible contribution, while part A will contribute $-\Gamma(\nu+1)$. On part C, $\arg t = 2\pi$, and that segment will make a contribution $e^{2\pi i\nu}\Gamma(\nu+1)$. All together, these contributions to Eq. (13.30) confirm its right-hand side. Then, multiplying both sides of that equation by $e^{-\pi i\nu}$, its right-hand side becomes $2i\sin\nu\pi$, and its left hand side is consistent with the value with which we replaced $(-t)^\nu$.

13.2 Digamma and Polygamma Functions

13.2.1. The answer is given in the text.

13.2.2. (a) Use Eq. (12.38) to rewrite $\zeta(2n)$ in terms of the Bernoulli numbers:

$$\begin{aligned}\ln\Gamma(x+1) &= -\gamma x + \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}x^n \\ &= -\gamma x - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}x^{2n+1} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{4n(2n)!}(2\pi x)^{2n}\,.\end{aligned}$$

The sum involving the Bernoulli numbers closely resembles the expansion for $\cot\pi x$, Eq. (12.35), differing therefrom primarily by the factor $4n$ in the denominator. This observation indicates that our Bernoulli sum will be related to

$$\int cot\pi x\,dx = -\pi^{-1}\ln\sin\pi x\,.$$

The precise relationship needed here is

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{4n(2n)!}(2\pi x)^{2n} = \frac{1}{2}\,\ln\frac{\pi x}{\sin\pi x}\,,$$

which can be verified by differentiating both sides and invoking Eq. (12.35). Substituting this expression for the Bernoulli sum, we reach the answer in the text, in which there is a remaining summation of the zeta functions of odd argument. That series has the range of convergence $-1 < x < 1$.

(b) This formula exhibits better convergence than that of part (a). The replacement of $\zeta(2n+1)$ by $\zeta(2n+1)-1$ is equivalent to adding the series

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) - x\,.$$

We therefore also subtract the right-hand side of this equation from the formula of part (a).

The infinite series of part (b) will converge for $-2 < x < 2$ but the terms $\sin \pi x$, $\ln(1+x)$, and $\ln(1-x)$ still limit x to $-1 < x < 1$.

13.2.3. For $x = n$ a positive integer, $\psi(n+1) = \sum_{r=1}^{\infty}(\frac{1}{r} - \frac{1}{r+n}) - \gamma = -\gamma + \sum_{r=1}^{n} \frac{1}{r}$.

13.2.4. Expanding $\frac{1}{z+n} = \frac{1}{n}\sum_{\nu=0}^{\infty}\left(-\frac{z}{n}\right)^{\nu}$ in a geometric series we obtain

$$\frac{d}{dz} \ln z! = \psi(z+1) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

$$= -\gamma - \sum_{\nu=1}^{\infty}(-z)^{\nu} \sum_{n=1}^{\infty} \frac{1}{n^{\nu+1}} = -\gamma - \sum_{n=2}^{\infty}(-z)^{n-1}\zeta(n).$$

Interchanging the summations is justified by absolute and uniform convergence for $|z| \le 1 - \varepsilon$, with $\varepsilon > 0$ arbitrarily small.

13.2.5. (a) Using

$$\ln(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{z^n}{n}\,,$$

we add $\ln(1+z)$ to both sides of Eq. (13.44):

$$\ln\Gamma(z+1) + \ln(1+z) = -\gamma z + \sum_{n=2}^{\infty}(-1)^n\,\frac{z^n}{n}\,\zeta(n) + \left[z - \sum_{n=2}^{\infty}(-1)^n, \frac{z^n}{n}\right]$$

$$= z(1-\gamma) + \sum_{n=2}^{\infty}(-1)^m\Big[\zeta(m) - 1\Big]\,\frac{z^n}{n}\,.$$

This is the relation we were asked to confirm.

(b) Since $\zeta(n) - 1 \sim 1/2^n$ for $n \to \infty$ the radius of convergence is $R = 2$.

13.2.6. Using Eq. (13.44), rather than the equation suggested in the *Hint*, we obtain

$$\ln\Big[\Gamma(1+z)\Gamma(1-z)\Big] = \sum_{n=2}^{\infty}(-1)^n\zeta(n)\left[\frac{z^n}{n}+\frac{(-z)^n}{n}\right] = 2\sum_{n=2}^{\infty}\zeta(2n)\frac{z^{2n}}{2n}\,.$$

But, from Eq. (13.23), multiplied by z to change $\Gamma(z)$ to $\Gamma(1+z)$,

$$\ln\Big[\Gamma(1+z)\Gamma(1-z)\Big] = \frac{\pi z}{\sin \pi z}\,,$$

thereby establishing the relation to be proved.

13.2.7. The logarithm of the Weierstrass infinite-product form, after combining z and $\Gamma(z)$ to make $\Gamma(z+1)$, is

$$\ln\Gamma(z+1) = -\gamma z + \sum_{n=1}^{\infty}\left[\frac{z}{n} - \ln\Big(1+\frac{z}{n}\Big)\right]\,.$$

Now expand $\ln\Big(1+\dfrac{z}{n}\Big)$ in powers of z, reaching

$$\ln\Gamma(z+1) = -\gamma z + \sum_{n=1}^{\infty}\sum_{\nu=2}^{\infty}\frac{(-1)^\nu}{\nu}\,\frac{z^\nu}{n^\nu}$$

$$= -\gamma z + \sum_{\nu=2}^{\infty}(-1)^\nu\frac{z^\nu}{\nu}\sum_{n=1}^{\infty}\frac{1}{n^\nu}$$

$$= -\gamma z + \sum_{\nu=2}^{\infty}(-1)^\nu\frac{z^\nu}{\nu}\,\zeta(\nu)\,,$$

which is Eq. (13.44).

13.2.8. First form, using Eq. (13.38) and expanding into partial fractions, most terms of which cancel,

$$\psi(z+2)-\psi(z+1) = \sum_{m=1}^{\infty}\frac{z+1}{m(m+z+1)} - \sum_{m=1}^{\infty}\frac{z}{m(m+z)}$$

$$= \sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{m+z+1} - \frac{1}{m} + \frac{1}{m+z}\right] = \frac{1}{z+1}\,.$$

Now differentiate this result m times:

$$\frac{d^m}{dz^m}\Big[\psi(z+2)-\psi(z+1)\Big] = \frac{d^m}{dz^m}\left(\frac{1}{z+1}\right) = \frac{(-1)^m\,m!}{(z+1)^{m+1}}\,.$$

13.2.9. (a) We have $(a)_n = \dfrac{(a+n-1)!}{(a-1)!} = \dfrac{\Gamma(a+n)}{\Gamma(a)}$.

(b) $$\frac{d(a)_n}{da} = \frac{1}{\Gamma(a)}\frac{d\Gamma(a+n)}{da} - \frac{\Gamma(a+n)}{\Gamma(a)^2}\frac{d\Gamma(a)}{da}$$
$$= \frac{\Gamma(a+n)}{\Gamma(a)}\Big[\psi(a+n)-\psi(a)\Big] = (a)_n\Big[\psi(a+n)-\psi(a)\Big].$$

13.2.10. Setting $z=0$ in the solution to Exercise 13.2.4 we confirm $\psi(1) = -\gamma$.

Setting $z=0$ in Eq. (13.41), the summation in that equation becomes $\zeta(m+1)$, as also written in Eq. (13.43). Evaluation from that equation gives the results for $\psi^{(1)}(1)$ and $\psi^{(2)}(1)$.

13.2.11. (a) One way to proceed is to start by integrating the subject integral by parts. The integral is convergent at $r=0$ but to avoid divergences in some of the steps to be taken we change its lower limit to ε and later take the limit $\varepsilon \to 0$. Thus,

$$\int_\varepsilon^\infty e^{-r}\ln r\,dr = -e^{-r}\ln r\Big|_\varepsilon^\infty + \int_\varepsilon^\infty \frac{e^{-r}}{r}\,dr = e^{-\varepsilon}\ln\varepsilon + E_1(\varepsilon).$$

Inserting the expansion for E_1 from Eq. (13.83) and noting that the entire summation in that equation is $O(\varepsilon)$, we have

$$\int_\varepsilon^\infty e^{-r}\ln r\,dr = e^{-\varepsilon}\ln\varepsilon - \gamma - \ln\varepsilon + O(\varepsilon).$$

Expanding $e^{-\varepsilon} = 1-\varepsilon+\cdots$, we see that in the limit $\varepsilon\to 0$ the only nonvanishing contribution is $-\gamma$. (Note that $\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon = 0$.)

(b) This part is most easily approached after solving part (c).

(c) Introduce the notation

$$I_n = \int_0^\infty r^n e^{-r}\ln r\,dr.$$

Integrate I_n by parts, differentiating $r^n\ln r$ and integrating e^{-r}.

$$I_n = \Big[-r^n\ln r\,e^{-r}\Big]_0^\infty + \int_0^\infty r^{n-1}e^{-r}(n\ln r+1)\,dr$$
$$= 0 + nI_{n-1} + \int_0^\infty r^{n-1}e^{-r}\,dr = n\,I_{n-1} + (n-1)!\,.$$

From part (a) we have $I_0 = -\gamma$. Then the integral of part (b) is

$$I_1 = 0! + 1\,I_0 = 1-\gamma\,.$$

13.2.12. Defining $x = \alpha^2 Z^2$ and $z = 2(1-x)^{1/2}$, we need the first few terms in the Maclaurin series (in powers of x) of $\Gamma(z+1)$. To start, when $x = 0$, $z = 2$, so $\Gamma(z+1) = \Gamma(3) = 2$. From the definition of the digamma function, we also have

$$\frac{d\Gamma(z+1)}{dz} = \Gamma(z+1)\psi(z+1)\,,$$

$$\frac{d\Gamma(z+1)}{dx} = \Gamma(z+1)\psi(z+1)\,\frac{dz}{dx} = \Gamma(z+1)\psi(z+1)\,\left[-(1-x)^{-1/2}\right]\,,$$

which at $x = 0$ has the value $-\Gamma(3)\psi(3) = -2(-\gamma + \frac{3}{2})$.

Continuing to the second derivative,

$$\begin{aligned}\frac{d^2\Gamma(z+1)}{dx^2} &= \frac{d}{dz}\left[\Gamma(z+1)\psi(z+1)\right]\,\left[-(1-x)^{-1/2}\right]^2 \\ &\qquad + \Gamma(z+1)\psi(z+1)\frac{d}{dx}\left[-(1-x)^{-1/2}\right] \\ &= \Gamma(z+1)\left(\left[\psi(z+1)\right]^2 + \psi^{(1)}(z+1)\right)\left[-(1-x)^{-1/2}\right]^2 \\ &\qquad + \Gamma(z+1)\psi(z+1)\left[-\frac{(1-x)^{-3/2}}{2}\right]\,.\end{aligned}$$

At $x = 0$, $z = 2$, $\psi^{(1)}(z+1) = \zeta(2) - \frac{5}{4}$, and

$$\frac{d^2\Gamma(z+1)}{dx^2} = 2\left[\gamma^2 - \frac{5\gamma}{2} + \zeta(2) + \frac{1}{4}\right]\,.$$

Forming the Maclaurin series,

$$\Gamma[2(1-\alpha^2 Z^2)^{1/2}+1] = 2+(2\gamma-3)\alpha^2 Z^2+\left(\gamma^2 - \frac{5\gamma}{2} + \zeta(2) + \frac{1}{4}\right)\alpha^4 Z^4+\cdots\,.$$

13.2.13. One way to obtain the argument of a complex quantity is to identify it as the imaginary part of its logarithm. Using Eq. (13.44) for $z = ib$, we have

$$\ln\Gamma(1+ib) = -\gamma ib + \sum_{n=2}^{\infty}(-1)^n\,\frac{(ib)^n}{n}\,\zeta(n)\,.$$

The imaginary part of this expression is $-\gamma\, b + \dfrac{\zeta(3)\, b^3}{3} - \cdots$.

13.2.14. From Eqs. (13.38) and (13.40),

$$\psi(n+1) = -\gamma + \sum_{m=1}^{\infty}\frac{n}{m(n+m)} = -\gamma + \sum_{m=1}^{n}\frac{1}{m}\,.$$

(a) Taking $n = 1$ in the above equation, the infinite sum is that whose value is sought; we identify its value as that of the finite sum on the right-hand side, here, 1.

(b) Writing $n^2 - 1 = (n-1)(n+1)$, this summation is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}\,,$$

which is (1/2) times the summation associated with $\psi(2)$, and thereby has the value $(1 + \frac{1}{2})/2 = 3/4$.

13.2.15. $\psi(a+1) - \psi(b+1) = \sum_{n=1}^{\infty}\left(\frac{1}{b+n} - \frac{1}{a+n}\right) = (a-b)\sum_{n=1}^{\infty}\frac{1}{(a+n)(b+n)}.$

13.3 The Beta Function

13.3.1. Expanding all the beta functions, and using Eq. (13.2) to make the gamma functions have similar arguments, we reduce these expressions to identities:

(a) $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} + \frac{b\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}\,.$

(b) $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \left(\frac{a+b}{b}\right)\frac{b\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}\,.$

(c) $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \frac{a\Gamma(a)(b-1)^{-1}\Gamma(b)}{\Gamma(a+b)}\,.$

(d) $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,\frac{\Gamma(a+b)\Gamma(c)}{\Gamma(a+b+c)} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c}$. This is symmetric in a, b, and c, so the presumed relation must be correct.

13.3.2. (a) This is a case of Eq. (13.50); the "2" in that equation compensates for the range of integration, in this exercise $(-1, +1)$. The values of p and q in the equation are $p = n - \frac{1}{2}$, $q = \frac{1}{2}$, so the integral has the value

$$B(n+\tfrac{1}{2}, \tfrac{3}{2}) = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+2)} = \frac{\sqrt{\pi}\,(2n-1)!!}{2^n}\,\frac{\sqrt{\pi}}{2}\,\frac{1}{(n+1)!}\,,$$

equivalent to the desired answer.

(b) This problem is similar, reducing to $B(n+\frac{1}{2}, \frac{1}{2})$.

13.3.3. This is a case of Eq. (13.50), with $p = -\frac{1}{2}$, $q = n$, and thereby reduces to

$$B(\tfrac{1}{2}, n+1) = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} = \frac{\sqrt{\pi}\,n!}{\sqrt{\pi}\,(2n+1)!!/2^{n+1}} = \frac{2(2^n\,n!)}{(2n+1)!!}\,.$$

13.3.4. Setting $x = \cos\theta$, we identify $1 + x$ as $2\cos^2\chi$ and $1 - x$ as $2\sin^2\chi$, where $\chi = \theta/2$. We then write dx as $-\sin\theta\,d\theta = -4\sin\chi\cos\chi\,d\chi$. The integration range, from -1 to 1 in x, is $\pi/2$ to 0 in χ. With these changes, the integral under study becomes

$$\int_{-1}^{1}(1+x)^a(1-x)^b\,dx = 2^{a+b+2}\int_0^{\pi/2}\cos^{2a+1}\chi\sin^{2b+1}\chi\,d\chi\,,$$

which is a case of Eq. (13.47) with $p = a + 1$ and $q = b + 1$. The integral therefore has value $B(a+1, b+1)/2$; when the power of 2 multiplying the integral is taken into account, we obtain the answer in the text.

13.3.5. Make a change of the variable of integration to make the integration limits zero and one: $u = (x-t)/(z-t)$. Then $x-t = (z-t)u$, $z-x = (z-t)(1-u)$, $dx = (z-t)\,du$, and the integral becomes

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha} = \int_0^1 \frac{du}{(1-u)^{1-\alpha}u^\alpha}\,,$$

which is a case of Eq. (13.49) with $p = -\alpha$ and $= \alpha - 1$. Therefore, using Eq. (13.23),

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha} = B(1-\alpha,\alpha) = \frac{\Gamma(1-\alpha)\,\Gamma(\alpha)}{\Gamma(1)} = \frac{\pi}{\sin\pi\alpha}\,.$$

13.3.6. Writing this integral with limits defining the triangular integration region, and using Eq. (13.49),

$$\int_0^1 dx\,x^p\int_0^{1-x}y^q\,dy = \int_0^1 dx\,\frac{x^p(1-x)^{q+1}}{q+1} = \left(\frac{1}{q+1}\right)B(p+1,q+2)$$

$$= \frac{B(p+1,q+1)}{p+q+2}\,,$$

where the last step uses the identity of Exercise 13.3.1(b).

13.3.8. The integrals at issue here are cases of Eq. (13.47), which for the integral of part (b) with general n has $p = (n+1)/2$, $q = 1/2$, and

$$\int_0^{\pi/2}\cos^n\theta\,d\theta = \frac{1}{2}\,B\left(\frac{n+1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{n+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}\,.$$

(a) For $n = 1/2$, $\displaystyle\int_0^{\pi/2}\cos^{1/2}\theta\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(3/4)}{\Gamma(5/4)}\,,$

which can be converted into the listed answer using the reflection formula, Eq. (13.23).

(b) For n odd, the gamma function in the denominator can be written

$$\Gamma\left(\frac{n}{2}+1\right) = \frac{\sqrt{\pi}}{2}\,\frac{n!!}{2^{(n+1)/2}}\,,$$

and the net overall power of 2 can be used to convert $\left(\dfrac{n-1}{2}\right)!$ to $(n-1)!!$, thereby reaching the listed answer.

For n even, a similar process can be used to convert the gamma function of half-integer argument to the more convenient form given as the listed answer.

13.3.9. Make the substitution $x^2 = y$, so $dx = dy/2y^{1/2}$, and the integral becomes

$$\int_0^1 (1-x^4)^{-1/2}\,dx = \frac{1}{2}\int_0^1 (1-y^2)^{-1/2}\,y^{-1/2}\,dy\,.$$

This is a case of Eq. (13.50) with $p = -3/4$ and $q = -1/2$, and the integral has the value $B(\frac{1}{4},\frac{1}{2})/4$. This can be brought to the form in the text using Eq. (13.23); the potential advantage in doing so is that then, only one gamma function of fractional argument enters a numerical evaluation.

13.3.10. This problem can be approached by expanding $\cos(z\cos\theta)$ in powers of z, resulting in a series each term of which contains a trigonometric integral in θ. The expansion yields

$$J_\nu(z) = \frac{2}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-1)^n z^{2n}}{(2n)!}\int_0^{\pi/2} \sin^{2\nu}\theta\cos^{2n}\theta\,d\theta\,.$$

This integral is a case of Eq. (13.47) with value

$$\int_0^{\pi/2} \sin^{2\nu}\theta\cos^{2n}\theta\,d\theta = \frac{1}{2}\,B(\nu+\tfrac{1}{2},n+\tfrac{1}{2}) = \frac{1}{2}\,\frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\Gamma(n+\nu+1)}\,.$$

When this is combined with the other factors in the representation of $J_\nu(z)$, the standard Bessel series definition is obtained.

13.3.11. These integrals are cases of Eq. (13.50); since the present integrands are even functions of x, the fact that the integration range is from -1 to 1 instead of 0 to 1 is compensated by the factor of 2 multiplying the integral in Eq. (13.50).

(a) This integral is $\Big[(2m-1)!!\Big]^2 B(\frac{1}{2},m+1) = \Big[(2m-1)!!\Big]^2 \dfrac{\Gamma(\frac{1}{2})\,\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}$

$$= \Big[(2m-1)!!\Big]^2 \frac{\sqrt{\pi}\,2^{m+1}\,m!}{\sqrt{\pi}\,(2m+1)!!} = (2m-1)!!\,\frac{2\,(2m)!!}{2m+1} = \frac{2\,(2m)!}{2m+1}\,.$$

(b) This is $\Big[(2m-1)!!\Big]^2 B(\frac{1}{2}, m)$.

A reduction similar to that of part (a) leads to the listed answer.

13.3.12. These integrals are cases of Eq. (13.50).

(a) $\displaystyle\int_0^1 x^{2p+1}(1-x^2)^{-1/2}\,dx = \frac{1}{2}\,B(p+1, \tfrac{1}{2}) = \frac{\Gamma(p+1)\,\Gamma(\frac{1}{2})}{2\Gamma(p+\frac{3}{2})}$,

which reduces to the listed answer.

(b) $\displaystyle\int_0^1 x^{2p}(1-x^2)^q\,dx = \frac{1}{2}\,B(p+\tfrac{1}{2}, q+1)$.

This reduces in a way similar to part (a).

13.3.13. Change the integration variable to $y = Ax^n/E$, so $dx = (x/ny)\,dy$, and thereby make the integral a case of Eq.!(13.49):

$$\tau = 2\sqrt{\frac{2m}{E}}\int_0^1 \frac{(x/ny)\,dy}{(1-y)^{1/2}} = \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{1/n}\int_0^1 y^{(1/n)-1}(1-y)^{-1/2}\,dy$$

$$= \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{1/n} B(1/n, 1/2)\,.$$

This beta function has value $\Gamma\left(\dfrac{1}{n}\right)\sqrt{\pi}\Big/\Gamma\left(\dfrac{1}{n}+\dfrac{1}{2}\right)$; when inserted we confirm the answer given for τ.

13.3.14. (a) Write the potentially singular part of the n dependence of τ as

$$\frac{\frac{1}{n}\Gamma(1/n)}{\Gamma(\frac{1}{n}+\frac{1}{2})} = \frac{\Gamma(1+1/n)}{\Gamma(\frac{1}{n}+\frac{1}{2})} \to \frac{\Gamma(1)}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi}}\,.$$

Combining with the other factors, the limit simplifies to

$$\tau \to 2\sqrt{\frac{2m}{E}}\,.$$

(b) In the limit of large n the potential is zero between the turning points (which are then at $x = \pm 1$) and infinite elsewhere; the integral for τ reduces to

$$\tau_\infty = 2\sqrt{2m}\int_0^1 E^{-1/2}\,dx = 2\sqrt{\frac{2m}{E}}\,.$$

(c) At infinite n the particle will be moving with kinetic energy $mv^2/2 = E$, so $v = \sqrt{2E/m}$. At this velocity, the time to travel 4 units of distance (one period) will be $4/v = 4\sqrt{m/2E} = 2\sqrt{2m/E}$.

13.3.15. Following the Hint, let $\sinh^2 x = u$, $2\sinh x \cosh x\, dx = du$, and the integral becomes (using the identity $\cosh^2 x = \sinh^2 x + 1$)

$$\int_0^\infty \frac{\sinh^\alpha x}{\cosh^\beta x}\, dx = \int_0^\infty \frac{u^{\alpha/2}}{(1+u)^{\beta/2}} \frac{du}{2[u(1+u)]^{1/2}} = \frac{1}{2}\int_0^\infty \frac{u^{(\alpha-1)/2}\, du}{(1+u)^{(\beta+1)/2}}\,.$$

This is a case of Eq. (13.51) with $p = (\alpha-1)/2$, $p+q+2 = (\beta+1)/2$, corresponding to $q = (\beta-\alpha)/2 - 1$. The integral of this exercise therefore has the value

$$\int_0^\infty \frac{\sinh^\alpha x}{\cosh^\beta x}\, dx = \frac{1}{2}\, B\left(\frac{\alpha+1}{2}, \frac{\beta-\alpha}{2}\right)\,.$$

13.3.16. The integrals occurring here are cases of Eq. (13.49).

(a) $$\langle x\rangle = \int_0^1 x\, f(x)\, dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 x^\alpha (1-x)^{\beta-1}\, dx$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}\,.$$

(b) $$\langle x^2\rangle = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\, dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \frac{\Gamma(\alpha+2)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$
$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}\,.$$

We now form

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{\alpha+\beta)^2} = \frac{\alpha\beta}{(\alpha+\beta)^2)(\alpha+\beta+1)}\,.$$

13.3.17. The integrals of this exercise are cases of Eq. (13.47); both are for $p = 1/2$; the numerator has $q = n + \frac{1}{2}$, while the denominator has $q = n+1$. The ratio of the two integrals is

$$\frac{B(\frac{1}{2}, n+\frac{1}{2})}{B(\frac{1}{2}, n+1)} = \frac{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)} = \frac{[\Gamma(n+\frac{1}{2})]^2}{(n!)^2}\,(n+\tfrac{1}{2})\,.$$

To reach the Wallis formula, write $\Gamma(n+\frac{1}{2})$ as $\sqrt{\pi}\,(2n-1)!!/2^n$ and write $n+\frac{1}{2} = (2n+1)/2$. Then

$$\frac{B(\frac{1}{2}, n+\frac{1}{2})}{B(\frac{1}{2}, n+1)} = \frac{\pi}{2} \frac{(2n-1)!!\,(2n+1)!!}{2^n n!\,2^n n!} = \frac{\pi}{2} \frac{(2n-1)!!\,(2n+1)!!}{(2n)!!\,(2n)!!}\,.$$

We now set this result (in the limit of infinite n) equal to unity and solve for $\pi/2$.

13.4 Stirling's Series

13.4.1. The answer is given in the text.

13.4.2. Use Eq. (13.60) with $z = 52$; it is not necessary to keep any of the terms with negative powers of z. Compute the logarithm, then exponentiate. The result is 8.1×10^{67}.

13.4.3. Keeping only terms that do not vanish in the limit of large z, the logarithm of the gamma function recurrence formula can be written

$$\ln\Gamma(z+1) - \ln z - \ln\Gamma(z) = C_2 + (z+\tfrac{1}{2})\ln z + (C_1-1)z - \ln z$$
$$- \big[C_2 + (z-\tfrac{1}{2})\ln(z-1) + (C_1-1)(z-1)\big] = 0\,.$$

The above simplifies immediately to

$$(z-\tfrac{1}{2})\ln z - (z-\tfrac{1}{2})\ln(z-1) + C_1 - 1 = 0\,.$$

Noting now that

$$\ln(z-1) = \ln z + \ln\left(1-\frac{1}{z}\right) = \ln z - \frac{1}{z} - O(z^{-2}),$$

our equation further simplifies in the limit of large z to

$$(z-\tfrac{1}{2})\ln z - (z-\tfrac{1}{2})\left(\ln z - \frac{1}{z}\right) + C_1 - 1 = 0 \longrightarrow C_1 = 0\,,$$

confirming the asserted value $C_1 = 0$.

We now prepare to use the duplication formula by writing

$$\ln\Gamma(z+1) \sim C_2 + (z+\tfrac{1}{2})\ln z - z$$

$$\ln\Gamma(z+\tfrac{1}{2}) \sim C_2 + z\ln(z-\tfrac{1}{2}) - (z-\tfrac{1}{2})$$

$$\ln\Gamma(2z+1) \sim C_2 + (2z+\tfrac{1}{2})\ln 2z - 2z$$

Substitute the above into the Legendre duplication formula

$$\ln\Gamma(z+1) + \ln\Gamma(z+\tfrac{1}{2}) = \frac{1}{2}\ln\pi - 2z\ln 2 + \ln\Gamma(2z+1)\,.$$

Many terms now cancel; to complete the cancellation we need to expand $\ln(z-\frac{1}{2})$ in a way similar to our earlier treatment of $\ln(z-1)$. After this further simplification we get

$$C_2 = \frac{1}{2}\ln 2\pi\,,$$

the other required result.

13.4.4. Because $\ln x$ is a monotone increasing function,

$$\ln n < \int_n^{n+1} \ln x\,dx < \ln(n+1)\,.$$

Since $\ln(n!) = \ln 1 + \ln 2 + \cdots + \ln n$, $\ln(n!)$ will lie between the two integrals of the present exercise.

13.4.5. Using Stirling's asymptotic formula we find that

$$\frac{\Gamma(p+1/2)}{\Gamma(p+1)} \sim \sqrt{e}\left(\frac{p+1/2}{p+1}\right)^{p+1/2} \Big/ \sqrt{p+1} \sim \frac{\text{constant}}{\sqrt{p}}$$

for $p \gg 1$. Hence the series diverges.

13.4.6. As n is increased, the asymptotic formula given by Stirling's series (truncated before some negative power of the expansion argument) can be brought arbitrarily close to the infinite-n limit. Including all the terms that do not go to zero at large n, our current expression has the asymptotic limit

$$\ln\left[x^{b-a}\,\frac{\Gamma(x+a+1)}{\Gamma(x+b+1)}\right] \sim (b-a)\ln x + (x+a+\tfrac{1}{2})\ln(x+a) - (x+a)$$

$$- (x+b+\tfrac{1}{2})\ln(x+b) + (x+b) = (b-a)\ln x + (a-b)\ln x\,.$$

To simplify this, write $\ln(x+a) = \ln x + \ln\left(1+\dfrac{a}{x}\right) \approx \ln x + \dfrac{a}{x} + \cdots$, and the same for $\ln(x+b)$. We can then verify that all the terms that survive at large x add to zero, so the limit we seek is $\exp(0) = 1$.

13.4.7. Write this expression in terms of factorials so that Stirling's formula can be used. It is convenient to work with logarithms of the factors.

$$L = \frac{(2n-1)!!}{(2n)!!}\,n^{1/2} = \frac{(2n)!\,n^{1/2}}{2^{2n}[n!]^2}\,.$$

Then

$$\ln L \sim \frac{\ln(2\pi)}{2} + (2n+\tfrac{1}{2})\ln(2n) - 2n + \frac{\ln n}{2}$$

$$- 2n\ln 2 - \ln(2\pi) - 2(n+\tfrac{1}{2})\ln n + 2n + \cdots\,.$$

This simplifies to $-\ln \pi/2$, consistent with the listed answer.

13.4.8. (a) Using Stirling's formula, dropping all terms of scaling lower than N, we have $N! \approx N\ln N - N$ and then (for arbitrary n_i but subject to the condition that $\sum_i n_i = N$)

$$S = k\ln W = k\left[N\ln N - N - \sum_{i=1}^{M}(n_i\ln n_i - n_i)\right] = k\sum_{i=1}^{M} n_i(\ln N - \ln n_i).$$

Introducing the notation $p_i = n_i/N$, this equation becomes

$$S = -Nk \sum_{i=1}^{M} p_i \ln p_i \,.$$

If the number of states M is fixed, this expression scales as N and is therefore extensive. Note that the individual terms of scaling greater than N have combined in a way that makes S an extensive quantity.

(b) We must maximize S subject to the constraint $P = \sum_i p_i = 1$. We proceed by obtaining an unconstrained maximum of $S - \lambda P$, after which we set λ (called a Lagrange multiplier) to a value consistent with the constraint. For details, see Section 22.3. We have for each state i

$$\frac{\partial}{\partial p_i}\Big[W - \lambda P\Big] = \ln p_i - 1 - \lambda = 0 \,,$$

indicating that the extremum of S is reached when all the p_i are the same; since there are M p_i, each must have the value $1/M$. Inserting these p_i values into the formula for S, we get

$$S = -Nk \sum_{i=1}^{M} \frac{1}{M}\, \ln(1/M) = Nk \ln M \,.$$

13.5 Riemann Zeta Function

13.5.1. Starting from the equation given as a starting point, divide both sides by $2ie^{\pi iz}$, converting the parenthesized quantities containing complex exponentials into sine functions. Then replace each sine function by its equivalent as given by the reflection formula. Eq. (13.23). These steps proceed as follows:

$$\frac{e^{2\pi iz} - 1}{2ie^{\pi iz}} = \sin \pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

$$\frac{e^{3\pi iz/2} - e^{\pi iz/2}}{2ie^{\pi iz}} = \sin(\pi z/2) = \frac{\pi}{\Gamma(z/2)\Gamma(1-\frac{1}{2}\,z)}$$

$$\frac{\pi}{\Gamma(z)\Gamma(1-z)}\,\zeta(z) = \frac{2^z\pi^z}{\Gamma(z)}\,\frac{\pi}{\Gamma(z/2)\Gamma(1-\frac{1}{2}\,z)}\,\zeta(1-z)\,.$$

Next, use the Legendre duplication formula to replace $\Gamma(1-z)$:

$$\Gamma(1-z) = \frac{\Gamma\left(\dfrac{1-z}{2}\right)\Gamma\left(1-\dfrac{z}{2}\right)}{2^z\;\pi^{1/2}}$$

After this result is inserted into the equation preceding it, a rearrangement without further analysis yields the desired formula.

13.5.2. Integrating by parts, using $\dfrac{d}{dx}\left(\dfrac{1}{e^x-1}\right) = -\dfrac{e^x}{(e^x-1)^2}$, we obtain

$$\int_0^\infty \frac{x^n e^x}{(e^x-1)^2}\,dx = -\left.\frac{x^n}{e^x-1}\right|_0^\infty + n\int_0^\infty \frac{x^{n-1}}{e^x-1}\,dx.$$

Here the integrated term vanishes and the second term contains the integral of Eq. (13.62) and has value $n!\zeta(n)$.

13.5.3. We treat only the limiting cases $T \to \infty$ and $T \to 0$.

(a) For $T \to \infty$, we need the value of the integral when the upper limit is small. Expanding the denominator and keeping only the leading terms,

$$\int_0^x \frac{x^5\,dx}{(e^x-1)(1-e^{-x})} \approx \int_0^x \frac{x^5\,dx}{(x)(x)} = \int_0^x x^3\,dx = \frac{x^4}{4}\,.$$

Setting $x = \Theta/T$, we get $\rho \approx C\left(\dfrac{T^5}{\Theta^6}\right)\dfrac{1}{4}\left(\dfrac{\Theta}{T}\right)^4 = \dfrac{C}{4}\,\dfrac{T}{\Theta^2}\,.$

(b) The upper integration limit is now infinity. Start with an integration by parts, to bring the integrand to a form that can be identified, using Eq. (13.62), as leading to a zeta function.

$$I = \int_0^\infty \frac{x^5\,e^x\,dx}{(e^x-1)^2} = \left[\frac{-x^5}{e^x-1}\right]_0^\infty + \int_0^\infty \frac{5x^4\,dx}{e^x-1} = 5\Gamma(5)\,\zeta(5) = 5!\,\zeta(5)\,.$$

Therefore, $\rho \approx 5!\,\zeta(5)\,C\,\dfrac{T^5}{\Theta^6}\,.$

13.5.4. The denominator of the integrand (with a factor t in the numerator) is the generating function for the Bernoulli numbers, so we can introduce that expansion and integrate termwise.

$$\int_0^x \frac{t^n\,dt}{e^t-1} = \int_0^x t^{n-1}\,dt \sum_{p=0}^\infty \frac{B_p t^p}{p!} = \sum_{p=0}^\infty \frac{x^{n+p}}{n+p}\,\frac{B_p}{p!}\,.$$

Using the facts that $B_0 = 1$, $B_1 = -1/2$, and that the B_p of odd $p > 1$ vanish, we can bring the above expansion to the form given in the text.

13.5.5. The integral in this expression is a case of Eq. (13.62) with $z = 4$. It therefore has the value $\Gamma(4)\,\zeta(4) = 3!\,\zeta(4)$.

13.5.6. Summing $\displaystyle\int_0^\infty e^{-nx}x^{s-1}dx = \frac{\Gamma(s)}{n^s}$ over the positive integers n we obtain

$$\sum_{n=1}^\infty \int_0^\infty e^{-nx}x^{s-1}\,dx = \zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx$$

for $\Re e(s) > 1$.

13.5.7. Make a binomial expansion of the denominator of the integrand as given in the exercise:

$$\frac{1}{e^x+1} = \frac{e^{-x}}{1+e^{-x}} = \sum_{n=1}^{\infty}(-1)^{n+1}e^{-nx}\,.$$

Insert this into the integral of this exercise and integrate termwise:

$$\int_0^\infty \frac{x^s\,dx}{e^x+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty x^s e^{-nx}\,dx = \left[\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{s+1}}\right]\Gamma(s+1)\,.$$

Referring to Eq. (12.62), we identify the sum over n as the Dirichlet series $\eta(s+1)$, with sum $(1-2^{-s})\,\zeta(s+1)$, so

$$\int_0^\infty \frac{x^s\,dx}{e^x+1} = (1-2^{-s})\,\zeta(s+1)\,\Gamma(s+1)\,,$$

equivalent to the result to be proved.

13.5.8. Changing the integration variable to $t = x/kT$,

$$\rho_\nu = \frac{4\pi(kT)^4}{h^3}\int_0^\infty \frac{x^3\,dx}{e^x+1}\,.$$

This integration is a case of Exercise 13.5.7, and ρ_ν therefore has the value

$$\rho_\nu = \frac{4\pi(kT)^4}{h^3}\,3!(1-2^{-3})\,\zeta(4) = \frac{4\pi(kT)^4}{h^3}\left(\frac{7}{8}\right)\frac{\pi^4}{90} = \frac{7\pi^5}{30h^3}\,(kT)^4\,.$$

13.5.9. Use the binomial theorem to expand the denominator of the integrand and then integrate termwise.

$$(-1)^{n+1}\int_0^\infty \frac{t^n\,e^{-zt}\,dt}{1-e^{-t}} = (-1)^{n+1}\int_0^\infty t^n\,e^{-zt}\sum_{p=0}^{\infty}e^{-pt}\,dt$$

$$= (-1)^{n+1}\sum_{p=0}^{\infty}\int_0^\infty t^n\,e^{-(z+p)t}\,dt = (-1)^{n+1}\sum_{p=0}^{\infty}\frac{n!}{(z+p)^{n+1}}\,.$$

To identify this summation as a polygamma function we need to change the indexing to move the lower summation limit from 0 to 1. We then have

$$(-1)^{n+1}n!\sum_{p=1}^{\infty}\frac{1}{(z-1+p)^{n+1}}\,,$$

which corresponds to Eq. (13.41) for $z-1$.

13.5.10. The alternating series for $\zeta(z)$, Eq. (13.68), converges for all $\Re e\, z > 0$, and provides a definition for that entire region except at $z = 1$, where the factor multiplying the series becomes singular. The reflection formula, Eq. (13.67), extends the analyticity to $\Re e\, z \le 0$ (except for the point $z = 0$).

Returning to the point $z = 1$, we show it to be a simple pole with residue $+1$ by taking the limit, applying l'Hôpital's rule:

$$\lim_{z\to 1} \frac{(z-1)}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = \frac{1}{\ln 2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{\ln 2} \ln 2 = 1\,.$$

Finally, we establish $\zeta(0)$ as regular by taking the limit

$$\zeta(0) = \lim_{z\to 0} \frac{\pi^{z-1/2}\Gamma((1-z)/2)}{\Gamma(z/2)} \zeta(1-z) = \pi^{-1/2}\Gamma(1/2) \lim_{z\to 0} \frac{\zeta(1-z)}{\Gamma(z/2)}$$

$$= \lim_{z\to 0} \frac{1}{(1-z)-1} \frac{1}{\Gamma(z/2)} = \lim_{z\to 0} \frac{1}{-z\,\Gamma(z/2)}$$

$$= -\lim_{z\to 0} \frac{1}{2\,\Gamma(1+z/2)} = -\frac{1}{2}\,.$$

13.6 Other Related Functions

13.6.1. Integrate by parts the integral defining $\gamma(a,x)$:

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\, dt = \frac{t^a}{a}(-e^t)\Big|_0^x + \frac{1}{a}\int_0^x t^a\, e^{-t}\, dt$$

$$= \frac{x^a\, e^{-x}}{a} + \frac{1}{a}\int_0^x t^a\, e^{-t}\, dt\,.$$

Further integrations by parts leads to the series

$$\gamma(a,x) = \frac{x^a\, e^{-x}}{a} + \frac{x^{a+1}\, e^{-x}}{a(a+1)} + \cdots$$

$$= e^{-x} \sum_{n=0}^{\infty} x^{a+n} \frac{\Gamma(a)}{\Gamma(a+n+1)}\,.$$

When a is an integer, this reduces to the answer in the text.

13.6.2. (a) Starting with the case $m = 1$,

$$\frac{d}{dx}\Big[x^{-a}\gamma(a,x)\Big] = -ax^{-a-1}\gamma(a,x) + x^{-a}x^{a-1}e^{-x}\,.$$

Using the formula of Exercise 13.6.3(a), the above simplifies to

$$\frac{d}{dx}\Big[x^{-a}\gamma(a,x\Big]\,) = -x^{-a-1}\gamma(a+1,x)\,.$$

Applying this result m times in succession yields the formula to be proved.

(b) Start with

$$\frac{d}{dx}\Big[e^{x}\gamma(a,x)\Big] = e^{x}\gamma(a,x) + x^{a-1}e^{-x}\,.$$

Substitute the formula of Exercise 13.6.3(a), with a changed to $a-1$, thereby reaching

$$\frac{d}{dx}\Big[e^{x}\gamma(a,x)\Big] = (a-1)e^{x}\gamma(a-1,x)\,.$$

Applying this result m times in succession, we get

$$\frac{d^m}{dx^m}\Big[e^{x}\gamma(a,x)\Big] = (a-1)(a-2)\cdots(a-m)\;e^{x}\gamma(a-m,x)\,,$$

equivalent to the formula given in the text.

13.6.3. (a) The integration by parts exhibited as the first equation in the solution to Exercise 13.6.1 is equivalent to the formula to be proved, as it can be written

$$\gamma(a,x) = \frac{x^a e^{-x}}{a} + \frac{\gamma(a+1,x)}{a}\,.$$

(b) This result can be proved via an integration by parts. It can also be confirmed by adding together the formulas of parts (a) and (b) of this exercise. Applying Eq. (13.74), the addition yields the familiar functional relation $\Gamma(a+1) = a\Gamma(a)$.

13.6.4. In Eq. (13.73) defining $\Gamma(a,x)$, change the integration variable to u, with $t = u + x$ and the integral now for u from zero to infinity. Then

$$\Gamma(a,x) = \int_0^\infty (u+x)^{a-1}e^{-u-x}\,du = x^{a-1}e^{-x}\int_0^\infty \Big(1+\frac{u}{x}\Big)^{a-1} e^{-u}\,du\,.$$

We now introduce the binomial expansion for $(\cdots)^{a-1}$; because the expansion does not converge for all u (for nonintegral a), further steps will lead to an asymptotic formula. We get

$$\Gamma(a,x) \sim x^{a-1}e^{-x}\sum_{n=0}^{\infty}\binom{a-1}{n}\frac{1}{x^n}\int_0^\infty u^n e^{-u}\,du\,.$$

The integral evaluates to $n!$ and yields the required answer when combined with the binomial coefficient.

The alternate form involving a Pochhammer symbol follows immediately from the formula $\Gamma(a) = (a-n)_n\Gamma(a-n)$.

13.6.5. The ratio of successive terms is

$$\frac{\text{term } n-1}{\text{term } n} = \frac{n\,x}{n-q}\,\frac{p+n}{p+n-1}\,.$$

This ratio approaches x as a limit for large n, so the series converges for $x < 1$. For $x = 1$ the ratio test is indeterminate, and we resort to the Gauss test. The expansion of this ratio in powers of $1/n$ is $1 + (q+1)/n + \cdots$; since the coefficient of $1/n$ is larger than unity this series converges at $x = 1$.

13.6.6. Starting from

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt\,,$$

note that

$$E_1(ix) = \int_{ix}^\infty \frac{e^{-t}}{t}\,dt = \int_x^\infty \frac{e^{-it}}{it}\,d(it) = \int_x^\infty \frac{e^{-it}}{t}\,dt$$

$$= \int_x^\infty \frac{\cos t - i\sin t}{t}\,dt = -\text{Ci}(x) + i\,\text{si}(x).$$

This is the answer to part (c). Replacing i by $-i$ in this formula, to get

$$E_1(-ix) = -\text{Ci}(x) - i\,\text{si}(x)\,,$$

we can form $E_1(ix) - E_1(-ix)$ to obtain the answer to part (a) or add these quantities to prove the result of part (b).

13.6.7. (a) For small x, the leading term in the series expansion of $\gamma(a,x)$ is, from Eq. (13.76), x^a/a. Use this initial term for $\gamma(3,2r)$ and rewrite $\Gamma(2,2r)$ as $\Gamma(2) - \gamma(2,2r)$ so that we can use the initial term of $\gamma(2,2r)$. We then have

$$\frac{\gamma(3,2r)}{2r} + \Gamma(2,2r) = \frac{\gamma(3,2r)}{2r} + \Gamma(2) - \gamma(2,2r) = \frac{(2r)^2}{3} + 1 - \frac{(2r)^2}{2}$$

$$= 1 - \frac{2r^2}{3}\,.$$

This result corresponds to the answer we require.

(b) For large r, $\Gamma(2,2r)$ becomes negligible, while $\gamma(3,2r)$ approaches $\Gamma(3) = 2!$. Therefore, as required,

$$\frac{\gamma(3,2r)}{2r} + \Gamma(2,2r) \to \frac{2!}{2r} = \frac{1}{r}\,.$$

13.6.8. (a) For small x, the leading term in the series expansion of $\gamma(a,x)$ is, from Eq. (13.76), x^a/a. Use this initial term for $\gamma(5,r)$ and $\gamma(7,r)$ and rewrite

$\Gamma(4,r) = \Gamma(4) - \gamma(4,r)$and $r^2\Gamma(2,r) = r^2\Gamma(2) - r^2\gamma(2,r)$. We then have (to terms through order r^2)

$$\frac{1}{r}\,\gamma(5,r) + \Gamma(4,r) = O(r^4) + 3! - O(r^4) \approx 6\,,$$

$$\frac{1}{r^3}\,\gamma(7,r) + r^2\Gamma(2,4) = O(r^4) + 1!\,r^2 + O(r^4) \approx r^2\,.$$

When these expressions are substituted into the form for $V(r)$ we recover the answer in the text.

At large x, $\gamma(a,r) \approx \Gamma(a)$ and $\Gamma(a,r)$ goes to zero as e^{-r}. Therefore

$$\frac{1}{r}\,\gamma(5,r) + \Gamma(4,r) \approx \frac{4!}{r} = \frac{24}{r}\,,$$

$$\frac{1}{r^3}\,\gamma(7,r) + r^2\Gamma(2,4) \approx \frac{6!}{r^3} = \frac{120\cdot 6}{r^3}\,.$$

When these expressions are substituted into the form for $V(r)$ we recover the answer in the text.

13.6.9. This is shown in Eqs. (13.81), (13.82) and (13.73).

13.6.10. Write the formula given in the exercise as

$$E_1(z) = \int_0^\infty \frac{e^{-z(1+t)}}{1+t}\,dt$$

and make a change of variable to $u = z(t+1)$, with $dt = du/z$. The range of u is (z,∞), and our formula becomes

$$E_1(z) = \int_z^\infty \frac{e^{-u}}{u}\,du\,,$$

corresponding to the defining equation for E_1, Eq. (13.79).

13.6.11. Integrating by parts,

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\,dt = -\frac{e^{-xt}}{xt^n}\Bigg|_1^\infty - \frac{n}{x}\int_1^\infty \frac{e^{-xt}}{t^{n+1}}\,dt \quad = \frac{e^{-x}}{x} - \frac{n}{x}\,E_{n+1}(x)\,.$$

Rearranging, we reach the desired expression:

$$E_{n+1}(x) = \frac{e^{-x}}{n} - \frac{x}{n}\,E_n(x)\,.$$

13.6.12. $E_n(0) = \displaystyle\int_1^\infty \frac{dt}{t^n} = \frac{t^{1-n}}{1-n}\Bigg|_1^\infty = \frac{1}{n-1}, \qquad n > 1.$

13.6.13. (a) Bring the integral representation of si(x) to a more convenient term for our present purpose by writing it as

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = \int_0^x \frac{\sin t}{t}\,dt - \int_0^\infty \frac{\sin t}{t}\,dt = \int_0^x \frac{\sin t}{t}\,dt - \frac{\pi}{2}\,.$$

Now introduce the Maclaurin series for $\sin t$ and integrate termwise. The result is the answer in the text.

(b) Insert the expansion of $E_1(x)$, Eq. (13.83), into the expression for Ci(x) in Eq. (13.87):

$$\mathrm{Ci}(x) = -\frac{1}{2}\left[-\gamma - \ln(xe^{i\pi/2}) - \sum_{n=1}^\infty \frac{(-ix)^n}{n\cdot n!}\right]$$

$$- \frac{1}{2}\left[-\gamma - \ln(xe^{-i\pi/2}) - \sum_{n=1}^\infty \frac{(+ix)^n}{n\cdot n!}\right]$$

$$= \gamma + \ln x + \sum_{p=1}^\infty \frac{(-1)^p x^{2p}}{2p\,(2p)!}\,.$$

13.6.14. Expanding $\cos t$ in the integral and integrating termwise, we get

$$\int_0^x \frac{1-\cos t}{t}\,dt = -\sum_{n=1}^\infty \frac{(-1)^n x^{2n}}{(2n)(2n)!}\,,$$

The summation in the above equation is that found in part (b) of Exercise 13.6.13, and can therefore be identified as $\gamma + \ln x - \mathrm{Ci}(x)$.

13.6.15. Insert the relation connecting the incomplete gamma functions to the identity of part (a) of Exercise 13.6.2. We have initially

$$\frac{d^m}{dx^m}x^{-a}\Gamma(a) - \frac{d^m}{dx^m}x^{-a}\Gamma(a,x) =$$

$$(-1)^m x^{-a-m}\Gamma(a+m) - (-1)^m x^{-a-m}\Gamma(a+m,x)\,.$$

If the identity of part (a) is also to apply for $\Gamma(a,x)$, it is then necessary that

$$\frac{d^m}{dx^m}x^{-a}\Gamma(a) = (-1)^m x^{-a-m}\Gamma(a+m)\,.$$

The differentiation of x^{-a} provides the factors necessary to convert $\Gamma(a)$ into $\Gamma(a+m)$, so the formula is proved. A similar approach can be used to verify that part (b) of Exercise 13.6.2 also applies for $\Gamma(a,x)$.

13.6.16. The formula indicates that n must be a nonnegative integer. Start by expanding the denominator in the integrand:

$$\frac{1}{e^t - 1} = \sum_{k=1}^{\infty} e^{-kt}\,.$$

Then make the substitution $t = u + x$; after these steps we have

$$\int_x^{\infty} \frac{t^n\,dt}{e^t - 1} = \sum_{k=1}^{\infty} \int_0^{\infty} e^{-k(u+x)}(u+x)^n\,du\,.$$

Next introduce the binomial expansion for $(u+x)^n$. We get

$$\int_x^{\infty} \frac{t^n\,dt}{e^t - 1} = \sum_{k=1}^{\infty} e^{-kx} \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \int_0^{\infty} u^j e^{-ku}\,du\,.$$

The u integral evaluates to $j!/k^{j+1}$; insertion of this value into the above equation leads directly to the problem answer.

14. Bessel Functions

14.1 Bessel Functions of the First Kind

14.1.1. The product

$$g(x,t)g(x,-t) = e^{(x/2)(t-1/t-t+1/t)} = 1 = \sum_{m,n} J_m(x)t^m J_n(x)(-t)^n$$

has zero coefficients of t^{m+n} for $m = -n \neq 0$. This yields

$$1 = \sum_{n=-\infty}^{\infty} J_n^2(x) = J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x)\,,$$

using $(-1)^n J_{-n} = J_n$. For real x the inequalities follow.

14.1.2. The Bessel function generating function satisfies the indicated relation.

(a) Therefore, using Eq. (14.2),

$$\sum_{n=-\infty}^{\infty} J_n(u+v)t^n = \sum_{\nu=-\infty}^{\infty} J_\nu(u)t^\nu \sum_{\mu=-\infty}^{\infty} J_\mu(v)t^\mu\,.$$

Equating the coefficients of t^n on the two sides of this equation, which for the right-hand side involves terms for which $\mu = n - \nu$, so

$$J_n(u+v) = \sum_{\nu=-\infty}^{\infty} J_\nu(u)J_{n-\nu}(v)\,.$$

(b) Applying the above formula for $n = 0$, note that for $|\nu| \neq 0$, the summation contains the two terms $J_\nu(u)J_{-\nu}(v)$ and $J_{-\nu}(u)J_\nu(v)$. But because for any x, $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$, both these terms are equal, with value $(-1)^\nu J_\nu(u)J_\nu(v)$. Combining them yields the answer to this part of the exercise.

14.1.3. The generating function remains unchanged if we change the signs of both x and t,and therefore

$$\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} J_n(-x)(-t)^n = (-1)^n J_n(-x)t^n\,.$$

For this equation to be satisfied it is necessary that, for all n, $J_n(x) = (-1)^n J_n(-x)$.

14.1.4. (a) $\dfrac{d}{dx}\left[x^n J_n(x)\right] = nx^{n-1}J_n(x) + x^n J_n'(x) = \dfrac{x^n}{2}\left[\dfrac{2n}{x}J_n(x) + 2J_n'(x)\right]\,.$

Replace $(2n/x)J_n(x)$ using Eq. (14.7) and $2J_n'(x)$ using Eq. (14.8):

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{x^n}{2}\left[J_{n-1}(x) + J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= x^n J_{n-1}(x)\,.$$

(b) $$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -nx^{-n-1}J_n(x) + x^{-n}J_n'(x)$$

$$= \frac{x^{-n}}{2}\left[-\frac{2n}{x}J_n(x) + 2J_n'(x)\right].$$

Replace $-(2n/x)J_n(x)$ using Eq. (14.7) and $2J_n'(x)$ using Eq. (14.8):

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = \frac{x^{-n}}{2}\left[-J_{n-1}(x) - J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= -x^{-n}J_{n+1}(x)\,.$$

(c) Start from Eq. (14.8) with n replaced by $n+1$, and use Eq. (14.7) to replace $J_{n+2}(x)$ by its equivalent in terms of J_{n+1} and J_n:

$$2J_{n+1}'(x) = J_n(x) - Jn + 2(x) = J_n(x) - \frac{2(n+1)}{x}J_{n+1}(x) + J_n(x)\,.$$

Collecting terms and dividing through by 2 yields the desired result.

14.1.5. In the generating function for the J_n as given in Eq. (14.2), make the substitution $t = ie^{i\varphi}$, leading (with x replaced by ρ) to the formula

$$e^{(\rho/2)(ie^{i\varphi}-1/ie^{i\varphi})} = e^{i\rho\cos\varphi} = \sum_{m=-\infty}^{\infty} J_m(\rho)\left[ie^{i\varphi}\right]^m,$$

equivalent to the required expansion.

14.1.6. Set $\varphi = 0$ in the plane wave expansion of Exercise 14.1.5 and separate into real and imaginary parts. This yields

(a) $$e^{ix} = \sum_{m=-\infty}^{\infty} i^m J_m(x), \cos x = J_0(x) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(x)\,;$$

(b) $$\sin x = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(x)\,,$$

using $J_{-2m-1} = -J_{2m+1}, i^{-2m-1} = -(-1)^m i$.

14.1.7. Following the procedure outlined in the hint, we have after step (b)

$$t\sum_{n=-\infty}^{\infty} t^{n-1}J_n - 1(x) + t^{-1}\sum_{n=-\infty}^{\infty} t^{n+1}J_n + 1(x) = \sum_{n=-\infty}^{\infty}\frac{2n}{x}t^n J_n(x)\,.$$

Writing as in part (c) and separating variables in the resulting ODE:

$$\frac{dg}{g} = \frac{x}{2}(1 + t^{-2})\,dt\,,$$

with solution

$$\ln g = \frac{x}{2}\left(t - \frac{1}{t}\right) + C_0(x) \qquad \longrightarrow \qquad g = C(x)\,e^{(x/2)(t-1/t)}\,,$$

where $C(x) = \exp(C_0(x))$ is an integration constant. The coefficient of t^0 can be found by expanding $e^{xt/2}$ and $e^{-x/2t}$ separately, multiplying the expansions together, and extracting the t^0 term:

$$e^{xt/2}e^{-x/2t} = \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \frac{t^n}{n!}\sum_{m=0}^{\infty}\left(\frac{x}{2}\right)^m \frac{t^{-m}}{m!} \longrightarrow \sum_{n=0}^{\infty}\frac{)-1)^n}{n!\,n!}\left(\frac{x}{2}\right)^{2n} t^0 + \cdots\,.$$

This is the series expansion of $J_0(x)$, so we set $C(x) = 1$.

14.1.8. Write out the term containing $x^{\nu+2s+1}$ in $J_{\nu\pm1}(x)$, $(2\nu/x)J_\nu(x)$, and $2J'_\nu(x)$:

$$J_{\nu-1}(x) = \cdots + \frac{(-1)^{s+1}}{(s+1)!\Gamma(s+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2s+1} + \cdots$$

$$J_{\nu+1}(x) = \cdots + \frac{(-1)^{s}}{s!\Gamma(s+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2s+1} + \cdots$$

$$\left(\frac{2\nu}{x}\right)J_{\nu}(x) = \cdots + \frac{(-1)^{s+1}\nu}{(s+1)!\Gamma(s+\nu+2)}\left(\frac{x}{2}\right)^{\nu+2s+1} + \cdots$$

$$2J'_\nu(x) = \sum_{s=0}^{\infty}\frac{(-1)^s(\nu+2s)}{s!\Gamma(s+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2s-1}$$

$$= \cdots + \frac{(-1)^{s+1}(\nu+2s+2)}{(s+1)!\Gamma(s+\nu+2)}\left(\frac{x}{2}\right)^{\nu+2s+1} + \cdots$$

Note that in several of the above formulas we redefined the summation index s so that corresponding powers of x were associated with the same index value.

Combining the corresponding powers of x, the recurrence formulas are easily confirmed.

14.1.9. Introduce the power-series expansions of the Bessel functions and then integrate over θ. The first integral assumes the form

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,n!}\,\frac{x^{2n}}{2^{2n}}\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!\,2^n n!}\,\frac{(2n)!!}{(2n+1)!!}\,x^{2n}\,.$$

This simplifies to

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!},$$

which is the power-series expansion of $\sin x/x$.

The second integral can be written

$$\frac{1-\cos x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \frac{x^{2n+1}}{2^{2n+1}} \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!\, 2^n n!(2n+2)} \frac{(2n)!!}{(2n+1)!!} x^{2n+1}.$$

This simplifies to

$$\frac{1-\cos x}{x} = \frac{(-1)^n x^{2n+1}}{(2n+2)!},$$

which is the power-series expansion of the left-hand side.

14.1.10. To use mathematical induction, assume the formula for J_n is valid for index value n and then verify that, under that assumption, it is also valid for index value $n+1$. Proceed by applying Eq. (14.11) with the J_n on its left-hand side given the assumed form:

$$-x^{-n}J_{n+1}(x) = \frac{d}{dx}\left[x^{-n}J_n(x)\right] = (-1)^n \frac{d}{dx}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)\right]$$

$$= (-1)^n x \left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

This equation easily rearranges to

$$J_{n+1}(x) = (-1)^{n+1}x^{n+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} J_0(x),$$

confirming the verification. To complete the proof by induction, we must show that the formula of this exercise is valid for $n=0$; for that case it is trivial.

14.1.11. We consider for now only the zeros of $J_n(x)$ for $x > 0$. Other cases can be treated by obvious extensions of the method to be used here. There must be at least one zero of $J_n'(x)$ between two consecutive zeros of $J_n(x)$, and, by Eq. (14.10) this implies that at least one zero of $J_{n-1}(x)$ lies in this interval. From Eq. (14.11) we may in a similar fashion conclude that at least one zero of $J_n(x)$ lies between two consecutive zeros of J_{n-1}. For these observations to be mutually consistent the zeros of J_n and J_{n-1} must alternate, i.e., there must be exactly one zero of J_{n-1} between two consecutive zeros of J_n.

14.1.12. Rewrite the integral of this exercise in terms of the integration variable $x = ur$:

$$I = \frac{1}{u^2}\int_0^u \left(1 - \frac{x^2}{u^2}\right) x J_0(x)\, dx\,.$$

Then note that by Eq. (14.10) $xJ_0(x) = [xJ_1(x)]'$, and integrate by parts. The integration that remains can also be rewritten using Eq. (14.10):

$$\begin{aligned} I &= \frac{1}{u^2}\left[xJ_1(x)\left(1 - \frac{x^2}{u^2}\right)\right]_0^u + \frac{1}{u^2}\int_0^u \left(\frac{2x}{u^2}\right) xJ_1(x)\, dx \\ &= 0 + \frac{2}{u^4}\int_0^u \left[x^2 J_2(x)\right]'\, dx = \frac{2}{u^4}\, u^2 J_2(u)\,, \end{aligned}$$

equivalent to the answer we seek.

14.1.13. Write $f(\theta)$ as

$$f(\theta) = -\frac{ik}{2\pi}\int_0^R \rho\, d\rho \int_0^{2\pi} d\varphi \Big[\cos(k\rho\sin\theta\sin\varphi) + i\sin(k\rho\sin\theta\sin\varphi)\Big]\,.$$

From Eqs. (14.18) and (14.19) with $n = 0$, note that the integral of the cosine has value $2\pi J_0(k\rho\sin\theta)$, and the integral of the sine vanishes. We now make a change of variable from ρ to $x = k\rho\sin\theta$, and then note, applying Eq. (14.10), that $xJ_0(x) = [xJ_1(x)]'$, so

$$\begin{aligned} f(\theta) &= -\frac{i}{k\sin^2\theta}\int_0^{kR\sin\theta} xJ_0(x)\, dx = -\frac{i}{k\sin^2\theta}\Big[xJ_1(x)\Big]_0^{kR\sin\theta} \\ &= -\frac{iR}{\sin\theta}\, J_1(kR\sin\theta)\,. \end{aligned}$$

We now form $|f(\theta)|^2$, obtaining the desired result.

14.1.14. (a) We perform operations similar to those used to obtain Eq. (14.13). To do so it will be convenient to have a formula similar to that of Eq. (14.12). Adding or subtracting the two recurrence formulas of this exercise, we establish

$$C_n(x) = C'_{n\pm1}(x) \pm \frac{n \pm 1}{x}\, C_{n\pm1}(x)\,.$$

Now we form $x^2 C''_n$ as $x^2/2$ times the derivative of the second recurrence formula of this exercise, xC'_n from that second recurrence formula times $x/2$, and $n^2 C_n$ by multiplying the first recurrence formula by $nx/2$. In this way we reach

$$x^2 C''_n + xC'_n - n^2 C_n = \frac{x^2}{2}\left[C'_{n-1} - \frac{n-1}{x}\, C_{n-1} + C'_{n+1} + \frac{n+1}{x}\, C_{n+1}\right]\,.$$

Using the formula derived earlier in this problem solution, the right-hand side of the above equation simplifies to $x^2 C_n$, so we have

$$x^2 C_n''(x) + x^2 C_n'(x) - (x^2 + n^2)\, C_n(x) = 0\,.$$

This is the linear ODE we seek.

(b) The ODE found in part (a) becomes the Bessel equation if we make the change of variable $t = ix$. This substitution causes

$$x^2(d^2/dx^2) \to t^2(d^2/dt^2) \quad \text{and} \quad x(d/dx) \to t(d/dt), \quad \text{but} \quad x^2 \to -t^2,$$

so

$$t^2 \frac{d^2}{dt^2} C_n(it) + t\, \frac{d}{dt} C_n(it) + (t^2 - n^2)\, C_n(it) = 0\,.$$

14.1.15. (a) Using the Schlaefli integral representation and writing only the integrand, we have

$$J_\nu'(x) \longrightarrow \frac{1}{2}\left(t - \frac{1}{t}\right) \frac{e^{(x/2)(t-1/t)}}{t^{\nu+1}}\,,$$

$$J_\nu''(x) \longrightarrow \frac{1}{4}\left(t - \frac{1}{t}\right)^2 \frac{e^{(x/2)(t-1/t)}}{t^{\nu+1}}\,,$$

$$\begin{aligned} &x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) \\ &\qquad \longrightarrow \left[\frac{x^2}{4}\left(t - \frac{1}{t}\right)^2 + \frac{x}{2}\left(t - \frac{1}{t}\right) + x^2 - \nu^2\right] \frac{e^{(x/2)(t-1/t)}}{t^{\nu+1}} \\ &\qquad \longrightarrow \left[\frac{x^2}{4}\left(t + \frac{1}{t}\right)^2 + \frac{x}{2}\left(t - \frac{1}{t}\right) - \nu^2\right] \frac{e^{(x/2)(t-1/t)}}{t^{\nu+1}}\,. \end{aligned}$$

Evaluating the derivative in Eq. (14.38), again writing only the integrand,

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{e^{(x/2)(t-1/t)}}{t^\nu}\left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right)\right]\right\} \\ &\qquad = \frac{e^{(x/2)(t-1/t)}}{t^\nu}\left\{\left(-\frac{\nu}{t}\right)\left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right)\right] + \frac{x}{2}\left(1 - \frac{1}{t^2}\right)\right. \\ &\qquad\qquad \left. + \frac{x}{2}\left(1 + \frac{1}{t^2}\right)\left[\nu + \frac{x}{2}\left(t + \frac{1}{t}\right)\right]\right\} \end{aligned}$$

These two expressions are now easily shown to be equal, permitting us to proceed to the analysis following Eq. (14.38).

The representation given in the text as an integral over s can be reached from the Schlaefli integral by changing the integration variable to $s = xt/2$. Then $ds = (x/2)\,dt$. The contour in s is the same as the contour for t.

(b) Make the change of variable $t = e^{i\theta}$. Then $dt = ie^{i\theta}$; because n is integral, there is no cut and the integral is a counterclockwise traverse of the unit circle; the limits on θ are 0 and 2π. The exponential now becomes $(x/2)(e^{i\theta} - e^{-i\theta}) = ix\sin\theta$. Also, $dt/t^{n+1} = ie^{-in\theta}\,d\theta$. With these changes, we recover the first integral of part (b) for J_n.

Make now a further change of variable to $\theta' = (\pi/2) - \theta$; then $\sin\theta = \cos\theta'$ and $-n\theta = -n(\pi/2) + n\theta'$. Noting that $e^{-n\pi/2} = i^{-n}$, we obtain the final formula of part (b).

14.1.16. The contour consists of three parts: (1) $z = e^{-i\pi+u}$, with u ranging from $+\infty$ to zero; (2) $z = e^{i\theta}$, with θ ranging from $-\pi$ to π; and (3) $z = e^{i\pi+u}$, with u ranging from zero to $+\infty$. The first contour integral of Exercise 14.1.15 is therefore the sum of the following three integrals:

$$\text{Range (1):} \quad \frac{1}{2\pi i}\int_\infty^0 \frac{e^{(x/2)(e^{-i\pi+u}-e^{i\pi-u})}}{(e^{-i\pi+u})^{\nu+1}}\, e^{-i\pi+u}\,du\,,$$

$$\text{Range (2):} \quad \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{(x/2)(e^{i\theta}-e^{-i\theta})}}{e^{i\theta(\nu+1)}}\, ie^{i\theta}\,d\theta\,,$$

$$\text{Range (3):} \quad \frac{1}{2\pi i}\int_0^\infty \frac{e^{(x/2)(e^{i\pi+u}-e^{-i\pi-u})}}{(e^{i\pi+u})^{\nu+1}}\, e^{i\pi+u}\,du\,.$$

Introducing trigonometric and hyperbolic functions where appropriate, and adding together the three contributions to the overall contour integral, designated I, we reach

$$I = \frac{e^{i\nu\pi}}{2\pi i}\int_\infty^0 e^{-x\sinh u-\nu u}\,du + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta-i\nu\theta}\,d\theta$$
$$+\frac{e^{-i\nu\pi}}{2\pi i}\int_0^\infty e^{-x\sinh u-\nu u}\,du\,.$$

The first and third integrals can now be combined to yield

$$\frac{1}{\pi}\int_0^\infty e^{-vu-x\sinh u}\left[\frac{-e^{i\nu\pi}+e^{-i\nu\pi}}{2i}\right]du\,,$$

which reduces to
$$-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-vu-x\sinh u}\,du\,.$$

The second integral can be expanded into real and imaginary parts. Recognizing symmetry, the imaginary part is seen to vanish, while the real part can be written as twice an integral over half the original range:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(x\sin\theta-\nu\theta)+i\sin(x\sin\theta-\nu\theta)\right]d\theta$$

$$=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-\nu\theta)\,d\theta\,.$$

Putting together the above results, we obtain Bessel's integral.

14.1.17. (a) Expand $\cos(x\sin\theta)$:

$$J_\nu(x)=\frac{2}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\int_0^{\pi/2}\sin^{2k}\theta\cos^{2\nu}\theta\,d\theta$$

$$=\frac{2}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\frac{\Gamma(k+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{2\,\Gamma(k+\nu+1)}\,.$$

Writing $\Gamma(k+\frac{1}{2})=\pi^{1/2}(2k-1)!!/2^k$, substituting for the double factorial from Eq. (1.76) and simplifying,

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2k},$$

the series expansion of $J_nu(x)$.

(b) Change the integration variable to $\chi=\pi/2-\theta$ and therefore change $\sin\theta$ and $\cos\theta$ respectively to $\cos\chi$ and $\sin\chi$. The integrand is now symmetric about $\pi/2$ so the integration range can be extended to $(0,\pi)$ and the result then divided by 2. This establishes the first formula of part (b).

The second formula follows because the real part of $e^{\pm ix\cos\theta}$ is $\cos(x\cos\theta$ and the integral of the imaginary part, $\pm\sin(x\cos\theta)$, vanishes by symmetry.

The last formula of part (b) follows directly from the change of integration variable to $p=\cos\theta$, taking note that $dp=-\sin\theta\,d\theta$.

14.1.18. (a) Differentiate the integral representation of this exercise with respect to x. Differentiation of the factor $(x/2)^\nu$ returns the original integral, but multiplied by ν/x. Differentiation of the x dependence within the integral causes the integrand to be multiplied by $-x/2t$. That factor causes the expression to represent $-J_{\nu+1}$.

(b) Differentiation of the integral representation with respect to x causes us to reach

$$J'_\nu(x)=\frac{1}{2\pi i}\int_C\frac{1}{2}\left(t-\frac{1}{t}\right)t^{-\nu-1}\,e^{(x/2)(t-1/t)}\,dt\,.$$

Expanding the integrand into its two terms, we identify them respectively as $J_{\nu-1}/2$ and $J_{\nu+1}/2$, confirming the desired result.

14.1.19. Differentiating the integral representation of $J_n(x)$, we get

$$J_n'(x) = \frac{1}{\pi}\int_0^\pi sin(n\theta - x\sin\theta)\sin\theta\, d\theta\,.$$

In the integral representations of $J_{n\pm1}(x)$, introduce the trigonometric formulas

$$\cos[(n\pm1)\theta - x\cos\theta] = \cos\theta\cos(n\theta - x\sin\theta) \mp \sin\theta\sin(n\theta - x\sin\theta)\,.$$

When we form $J_{n-1} - J_{n+1}$, the $\cos\theta$ terms cancel and the $\sin\theta$ terms add, giving the desired result.

14.1.20. Write $J_0(bx)$ as its series expansion, and integrate termwise, recognizing the integrals as factorials. Considering for the moment the case $a > b > 0$, we have

$$\int_0^\infty e^{-ax}J_0(bx)\,dx = \sum_{n=0}^\infty \frac{(-1)^n b^{2n}}{2^{2n}n!\,n!}\int_0^\infty e^{-ax}x^{2n}\,dx = \sum_{n=0}^\infty \frac{(-1)^n b^{2n}}{2^{2n}n!\,n!}\frac{(2n)!}{a^{2n+1}}$$

$$= \frac{1}{a}\sum_{n=0}^\infty \frac{(-1)^n(2n)!}{(2n)!!(2n)!!}\left(\frac{b^2}{a^2}\right)^n = \frac{1}{a}\sum_{n=0}^\infty \frac{(-1)^n(2n-1)!!}{(2n)!!}\left(\frac{b^2}{a^2}\right)^n .$$

We now identify the final form of the summation as the binomial expansion

$$\left(1+\frac{b^2}{a^2}\right)^{-1/2} = \sum_{n=0}^\infty \binom{-1/2}{n}\left(\frac{b^2}{a^2}\right)^n ,$$

where, from Eq. (1.74), the binomial coefficient has the value

$$\binom{-1/2}{n} = \frac{(-1)^n(2n-1)!!}{(2n)!!}\,.$$

Inserting the value of the summation and multiplying it by $1/a$, we obtain

$$\int_0^\infty e^{-ax}J_0(bx)\,dx = \frac{1}{(a^2+b^2)^{1/2}}\,.$$

This result can now be analytically continued to the entire region for which the integral representation converges.

14.1.21. Expand the integrand:

$$F(\theta) = \cos(x\sin\theta - n\theta) = \cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)sin(n\theta)\,.$$

Compare the above with the result if θ is replaced by $2\pi - \theta$:

$$F(2\pi-\theta) = \cos(x\sin[2\pi-\theta])\cos(n[2\pi-\theta]) + \sin(x\sin[2\pi-\theta])sin(n[2\pi-\theta]).$$

Now note that $\sin(2\pi-\theta) = -\sin\theta$, $\cos(n[2\pi-\theta]) = \cos(n\theta)$, and $\sin(n[2\pi-\theta]) = -\sin(n\theta)$, so the above equation becomes

$$F(2\pi-\theta) = \cos(-x\sin\theta)\cos(n\theta) - \sin(-x\sin\theta)\sin(n\theta)$$

$$= \cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta) = F(\theta)\,.$$

This relation causes the integral from π to 2π to be equal to the integral from 0 to π, thereby confirming the desired result.

14.1.22. (a) The minima occur at the zeros of $J_1[(2\pi a/\lambda)\sin\alpha]$. The first two zeros of $J_1(x)$ are $x = 3.8317$ and 7.0156. See Table 14.1.

(b) The contribution to the intensity for Bessel-function argument x in the region $(0, x_0)$ is (because the aperture is circular and the element of area is proportional to x)

$$\text{Intensity } (0, x_0) \sim \int_0^{x_0}\left[\frac{J_1(x)}{x}\right]^2 x\,dx = \int_0^{x_0} J_1(x)^2\,\frac{dx}{x}\,.$$

The total intensity is the integral of Φ^2 over the entire diffraction pattern, which if a/λ is small can be approximated by setting $x_0 = \infty$. The integral in question is elementary, with value

$$\int_0^{x_0}[J_1(x)]^2\,\frac{dx}{x} = -\frac{1}{2}\Big[J_0(x)^2 + J_1(x)^2\Big]_0^{x_0} \sim 1 - J_0(x_0)^2 - J_1(x_0)^2\,.$$

Using the above expression, the total intensity of the diffraction pattern corresponds to unity, while that out to the first zero of J_1 will correspond to $1 - J_0(3.8317)^2 = 0.838$. This, therefore, is the fraction of the intensity in the central maximum.

14.1.23. In the first integral, replace $J_2(x)/x$ by $(-J_1(x)/x)'$:

$$2\int_0^{2ka}\frac{J_2(x)}{x}\,dx = -2\left[\frac{J_1(x)}{x}\right]_0^{2ka} = -\frac{J_1(2ka)}{ka} + 1\,,$$

where the $+1$ results from the lower limit because $\lim_{x\to 0} J_1(x)/x = 1/2$. We now rewrite the second integral as

$$-\frac{1}{2ka}\int_0^{2ka} J_2(x)\,dx = -\frac{1}{2ka}\int_0^{2ka}\left[J_0(x) - 2J_1'(x)\right]\,dx$$

$$= -\frac{1}{2ka}\int_0^{2ka} J_0(x)\,dx + \frac{J_1(2ka)}{ka}\,.$$

Combining these forms of the two integrals, the J_1 terms cancel, leaving the result given for part (b) of the exercise. To reach the result for part (a), replace J_0 in the integrand of the answer for part (b) by $2J_1' + J_2$, then replace J_2 by $2J_3' + J_4$, and continue indefinitely, to reach

$$T = 1 - \frac{1}{ka}\int_0^{2ka} \left[J_1'(x) + J_3'(x) + \cdots\right] dx\,.$$

The integrals evaluate to $J_1(2ka) - J_1(0) + J_3(2ka) - J_3(0) + \cdots$; since all these $J_n(x)$ vanish at $x = 0$, we recover the answer given for part (a).

14.1.24. Solve by the method of separation of variables, taking $U = P(\rho)\Phi(\varphi)T(t)$. The separated equations are

$$\frac{1}{v^2}\frac{d^2T}{dt^2} = -k^2\,, \qquad \frac{d^2\Phi}{d\varphi^2} = -m^2\,, \qquad \frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} - \frac{m^2}{\rho^2}P = -k^2 P\,.$$

The t equation has solution $b_1e^{i\omega t} + b_2e^{-i\omega t}$, with $k^2 = \omega^2/v^2$. The φ equation has solution $c_1e^{im\varphi} + c_2e^{-im\varphi}$, with m an integer to assure continuity at all φ. The ρ equation is a Bessel ODE in the variable $k\rho$, with solution $J_m(k\rho)$ that is nonsingular everywhere on the membrane. The function $J_m(k\rho)$ must vanish at $\rho = a$, so the points ka must be zeros of J_m.

The exercise asks for the allowable values of k, to which the foregoing provides an answer. More relevant is that these values of k determine the values of ω that are the oscillation frequencies of the membrane. If α_{mn} is the nth zero of J_m, then $k_n = \alpha_{mn}/a$ and $\omega_n = \alpha_{mn}v/a$.

14.1.25. This problem seeks periodic solutions at angular frequency ω, with time dependence $e^{\pm\omega t}$; we then have $\alpha^2 = \omega^2/c^2$. Solving by the method of separation of variables, write $B_z = P(\rho)\Phi(\varphi)Z(z)$. The separated equations are

$$\frac{d^2Z}{dz^2} = -g^2\,, \quad \frac{d^2\Phi}{d\varphi^2} = -m^2\,, \quad \frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left[(\alpha^2 - g^2) - \frac{m^2}{\rho^2}\right]P = 0\,.$$

The z equation has solution $C\sin(p\pi/l)$, where p must be a positive integer in order to satisfy the boundary conditions at $z = 0$ and $z = l$. The corresponding values of g^2 are $p^2\pi^2/l^2$. The φ equation has solutions $e^{\pm im\varphi}$, with m an integer to assure continuity at all φ. The ρ equation can be written as a Bessel ODE of order m in the variable $k\rho$, where $k^2 = \alpha^2 - g^2$; it will only have solutions with $dP/d\rho$ zero on a finite boundary if $k^2 > 0$, and a zero derivative will then occur at $\rho = a$ if $ka = \beta_{mn}$, where β_{mn} is the nth positive zero of J_m.

With these values of k, we can solve for ω, getting the result given in the text.

14.1.26. In order for a wave guide to transmit electromagnetic waves it must be consistent with solutions of Maxwell's equations that do not decay exponentially in the z direction (the axial direction of the wave guide). From

Example 14.1.2 we see that the boundary conditions on the cylindrical surface of the wave guide (at radius a) require that the traveling-wave TM solutions be of the form

$$E_z = J_m(\alpha_{mj}\rho/a)e^{\pm im\varphi}e^{ilz}e^{-i\omega t}\,,$$

with $(\omega^2/c^2) - l^2 = (\alpha_{mj}/a)^2$ and α_{mj} the jth positive zero of J_m. It is necessary that l be real to avoid a decay of E_z as z increases, so the minimum possible value of ω/c consistent with an oscillatory solution in the TM mode characterized by m and j is $\omega/c = \alpha_{mj}/a$. Since $\omega = 2\pi\nu$, where ν is the frequency of the electromagnetic oscillation, we have $\nu_{\min}(m,j) = \alpha_{mj}/2\pi a$.

14.1.28. Rewrite the integrand as $x^{m-n-1}[x^{n+1}J_n(x)]$ and integrate by parts, differentiating the first factor and using Eq. (14.10) to integrate the second factor. The result is

$$\int_0^a x^m J_n(x)\,dx = x^{m-n-1}\,x^{n+1}J_{n+1}(x)\Big|_0^a - (m-n-1)\int_0^a x^{m-1}J_{n+1}(x)\,dx\,.$$

Whether or not $m \ge n$, this process can be continued until the only integration is that of $J_{m+n}(x)$. We may then use Eq. (14.8) to write

$$\int_0^a J_{n+m}(x)\,dx = -2\int_0^a J'_{n+m-1}(x)\,dx + \int_0^a J_{n+m-2}(x)\,dx$$

$$= -2J_{n+m-1}(x)\Big|_0^a + \int_0^a J_{n+m-2}(x)\,dx\,,$$

continuing until the only unintegrated quantity is either J_0 or J_1.

(a) If $n+m$ is odd, the final integration is

$$\int_0^a J_1(x)\,dx = -\int_0^a J'_0(x)\,dx = 1 - J_0(a)\,.$$

(b) If $n+m$ is even, the final integration is $\int_0^a J_0(x)\,dx$; this integral cannot be written as a finite linear combination of $a^p J_q(a)$.

14.1.29. Write $yJ_0(y) = [yJ_1(y)]'$ and integrate by parts, simplifying the result using the fact that the upper integration limit is a zero of J_0, then replacing

J_1 by $-J_0'$ and integrating by parts a second time.

$$\int_0^{\alpha_0}\left(1-\frac{y}{\alpha_0}\right)J_0(y)y\,dy = \left(1-\frac{y}{\alpha_0}\right)yJ_1(y)\Big|_0^{\alpha_0} + \frac{1}{\alpha_0}\int_0^{\alpha_0} yJ_1(y)\,dy$$

$$= -\frac{1}{\alpha_0}\int_0^{\alpha_0} yJ_0'(y)\,dy$$

$$= -\frac{yJ_0(y)}{\alpha_0}\Bigg|_0^{\alpha_0} + \frac{1}{\alpha_0}\int_0^{\alpha_0} J_0(y)\,dy$$

$$= \frac{1}{\alpha_0}\int_0^{\alpha_0} J_0(y)\,dy\,.$$

14.2 Orthogonality

14.2.1. Write the Bessel equation of order ν, with solutions $J_\nu(k\rho)$ and $J_\nu(k'\rho)$ as

$$\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right) + \left(k^2\rho - \frac{\nu^2}{\rho}\right)J_\nu(k\rho) = 0\,,$$

$$\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k'\rho)}{d\rho}\right) + \left(k'^2\rho - \frac{\nu^2}{\rho}\right)J_\nu(k'\rho) = 0\,,$$

and form

$$\int_0^a J_v(k'\rho)\,\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right)\,d\rho$$

$$= -k^2\int_0^a J_\nu(k'\rho)J_\nu(k\rho)\,\rho\,d\rho + \nu^2\int_0^a J_\nu(k'\rho)J_\nu(k\rho)\,\rho^{-1}\,d\rho\,,$$

$$\int_0^a J_v(k\rho)\,\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k'\rho)}{d\rho}\right)\,d\rho$$

$$= -k'^2\int_0^a J_\nu(k\rho)J_\nu(k'\rho)\,\rho\,d\rho + \nu^2\int_0^a J_\nu(k\rho)J_\nu(k'\rho)\,\rho^{-1}\,d\rho\,.$$

Subtract the first of these two equations from the second, reaching

$$(k^2 - k'^2)\int_0^a J_\nu(k\rho)J_\nu(k'\rho)\,\rho\,d\rho = \int_0^a J_v(k\rho)\,\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k'\rho)}{d\rho}\right)\,d\rho$$

$$-\int_0^a J_v(k'\rho)\,\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right)\,d\rho\,.$$

The first of the two integrals on the right hand side can be converted via two integrations by parts into (plus) the second integral, so they cancel, leaving only the integrated boundary terms, which are

$$\left[J_\nu(k\rho)\rho \frac{dJ_\nu(k'\rho)}{d\rho} - \rho \frac{dJ_\nu(k\rho)}{d\rho} J_\nu(k'\rho) \right]_0^a$$

$$= \rho \left[J_\nu(k\rho) k' J'_\nu(k'\rho) - k J'_\nu(k\rho) J_\nu(k'\rho) \right] \Big|_0^a .$$

Note the factors k and k' that arise because now the derivatives are taken with regard to the function arguments ($k\rho$ or $k'\rho$). The terms from the boundary $\rho = 0$ vanish; those from $\rho = a$ constitute the value of the first Lommel integral.

To evaluate the second Lommel integral, start from the equation at the bottom of page 662 of the text, which is the result of applying l'Hôpital's rule to the indeterminate form obtained when we divide the first Lommel formula by $k^2 - k'^2$ and take the limit $k' \to k$. Note that

$$\frac{d}{dk'} \Big[k' J'_\nu(k'a) \Big] = \frac{1}{a} \frac{d}{dk'} \left(k' \frac{dJ_\nu(k'a)}{dk'} \right) = -\frac{1}{a} \left(a^2 k' - \frac{\nu^2}{k'} \right) J_\nu(k'a) \,,$$

a result of the same type as the first formulas of this exercise solution (but now with k' the variable). Substituting into the equation on page 662, and setting $k' = k$, we obtain

$$\int_0^a \rho \left[J_\nu(k\rho) \right]^2 \, d\rho = \frac{J_\nu(ka) \left(-a^2 k + \dfrac{\nu^2}{k} \right) J_\nu(ka) - k a^2 \left[J'_\nu(ka) \right]^2}{-2k} .$$

This expression reduces to the value given for the second Lommel integral.

14.2.2. (a) From Exercise 14.2.1, with $k = \beta_{\nu m}/a$ and $k' = \beta_{\nu n}/a$, we have (for $m \neq n$, and therefore $k \neq k'$)

$$\int_0^a J_\nu(ka)) J_\nu(k'a) \rho \, d\rho = \frac{a}{k^2 - k'^2} \left[k' J_\nu(ka) J'_v(k'a) - k J'_\nu(ka) J_\nu(k'a) \right] .$$

But $ka = \beta_{\nu m}$ and therefore $J'_\nu(ka) = 0$, and $k'a = \beta_{\nu n}$ and therefore $J'_\nu(k'a) = 0$, so the right-hand side of the above equation vanishes, establishing the result of part (a).

(b) This normalization integral is a case of the second Lommel integral of Exercise 14.2.1. In the value given for that integral, the $J'_\nu(ka)$ term vanishes because $ka = \beta_{\nu m}$, leaving only the second term, which is the desired answer.

14.2.3. This result is proved in Exercise 14.2.1.

14.2.4. The equation referenced in this exercise should have been Eq. (14.44).

Pure imaginary roots can be excluded because when z is pure imaginary, all terms of the power-series expansion have the same sign and therefore cannot sum to zero. Because all coefficients in the power-series expansion are real, complex roots must occur in complex-conjugate pairs. If there were such a pair, the orthogonality integral would involve $|J_\nu|^2$ and could not be zero; hence a contradiction.

14.2.5. (a) This is an expansion in functions that are orthogonal, but not normalized. The coefficients are therefore given as

$$c_{\nu m} = \frac{\langle J_\nu(\alpha_{\nu m}\rho/a)|f(\rho)\rangle}{\langle J_\nu(\alpha_{\nu m}\rho/a)|J_\nu(\alpha_{\nu m}\rho/a)\rangle}\,.$$

The normalization integral in the denominator has the value given in Eq. (14.46).

(b) This is also an expansion in functions that are orthogonal, but not normalized; see Exercise 14.2.2. A formula similar to that of part (a) applies, but the Bessel function arguments are $\beta_{\nu m}\rho/a$. The normalization integral in the denominator has the value given in the solution to Exercise 14.2.2.

14.2.6. Take our cylinder to have radius a and to have end caps at $z = \pm h$. The potential satisfies Laplace's equation, which has the separated-variable form given in Example 14.2.1, at Eqs. (14.49)–(14.51). As in the example, the ODE for $P(\rho)$ has solutions that are Bessel functions $J_m(l\rho)$, with l chosen to make $J_m(la) = 0$; thus, l must have one of the values α_{mj}/a, where j refers to the jth positive zero of J_m.

The general solution for Z_l is a linear combination of $e^{\pm lz}$; that consistent with the symmetry of the present problem is $\cosh lz$. The most general solution that vanishes on the curved surface and satisfies the problem symmetry is

$$\sum_{mj} c_{mj} J_m(\alpha_{mj}\rho/a)\, e^{im\varphi} \cosh(\alpha_{mj}z/a)\,.$$

We must now choose the coefficients c_{mj} so as to reproduce the potential $\psi(\rho,\varphi)$ at $z = h$ (and, by symmetry, also at $z = -h$). This requirement corresponds to

$$\psi(\rho,\varphi) = \sum_{mj} c_{mj} J_m(\alpha_{mj}\rho/a)\, e^{im\varphi} \cosh(\alpha_{mj}h/a)\,.$$

Exploiting the orthogonality of the $\Phi_m(\varphi)$ and of the Bessel functions (as in Example 14.2.1), we find

$$c_{mj} = \left[\pi a^2 \cosh(\alpha_{mj}h/a) J_{m+1}^2(\alpha_{mj})\right]^{-1}$$
$$\times \int_0^{2\pi} d\varphi \int_0^a \psi(\rho,\varphi) J_m(\alpha_{mj}\rho/a) e^{-im\varphi} \rho\, d\rho\,.$$

14.2.7. Substitute the Bessel series for $f(x)$ into the integral for the Parseval relation and invoke orthogonality of the Bessel functions:

$$\int_0^1 [f(x)]^2\, dx = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} a_n a_{n'} \int_0^1 J_m(\alpha_{mn})x) J_m(\alpha_{mn'}x)\, x\, dx$$
$$= \sum_{n=1}^{\infty} a_n^2 \int_0^1 \left[J_m(\alpha_{mn})\right]^2\, dx\,.$$

Now invoking Eq. (14.46) with $a = 1$, we recover the desired result.

14.2.8. Following the hint, we write

$$x^m = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn}(x)\,.$$

We now evaluate the coefficients a_n:

$$a_n = \frac{\displaystyle\int_0^1 x^m J_m(\alpha_{mn}(x)\, x\, dx}{\displaystyle\frac{1}{2}\left[J_{m+1}(\alpha_{mn})\right]^2}\,.$$

The integrand of the numerator can be identified as a derivative using Eq. (14.10), which in the present context can be written

$$\frac{d}{d(\alpha x)}\left[(\alpha x)^{m+1} J_{m+1}(\alpha x)\right] = (\alpha x)^{m+1} J_m(\alpha x) \longrightarrow$$
$$\frac{1}{\alpha}\frac{d}{dx}\left[x^{m+1} J_{m+1}(\alpha x)\right] = x^{m+1} J_m(\alpha x)\,.$$

Inserting this expression and thereby evaluating the integral, we get

$$a_n = \frac{J_{m+1}(\alpha_{mn})/\alpha_{mn}}{\left[J_{m+1}(\alpha_{mn})\right]/2} = \frac{2}{\alpha_{mn} J_{m+1}(\alpha_{mn})}\,.$$

Next, also as suggested by the hint, we form the Parseval integral

$$\int_0^1 x^m\, x^m\, x\, dx = \frac{1}{2m+2}$$
$$= \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} a_n a_{n'} \int_0^1 J_m(\alpha_{mn}x) J_m(\alpha_{mn'}x)\, x\, dx$$
$$= \sum_{n=1}^{\infty} a_n^2 \int_0^1 \left[J_m(\alpha_{mn}x)\right]^2 x\, dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left[J_{m+1}(\alpha_{mn}x)\right]^2 .$$

Finally. we insert the expression previously found for a_n, reaching

$$\frac{1}{2(m+1)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{\alpha_{mn}^2}\,,$$

equivalent to the answer we seek.

14.3 Neumann Functions

14.3.1. Use the recursion relations, Eqs. (14.7) and (14.8), which are obeyed for both positive and negative ν. Changing ν to $-\nu$, we have

$$J_{-\nu-1}(x) + J_{-\nu+1}(x) = -\frac{2\nu}{x} J_{-\nu}(x)\,,$$

$$J_{-\nu-1}(x) - J_{-\nu+1}(x) = -\frac{2\nu}{x} J'_{\nu}(x)\,.$$

Hence

$$Y_{\nu+1} + Y_{\nu-1} = \frac{\cos(\nu+1)\pi J_{\nu+1} - J_{-\nu-1}}{\sin(\nu+1)\pi} + \frac{\cos(\nu-1)\pi J_{\nu-1} - J_{-\nu+1}}{\sin(\nu-1)\pi}$$
$$= \frac{\cos\nu\pi(J_{\nu+1}+J_{\nu-1})}{\sin\nu\pi} + \frac{J_{-\nu-1}+J_{-\nu+1}}{\sin\nu\pi}$$
$$= \frac{\cos\nu\pi(2\nu/x)J_{\nu}}{\sin\nu\pi} + \frac{(-2\nu/x)J_{-\nu}}{\sin\nu\pi} = \frac{2\nu}{x}Y_{\nu}\,.$$

The second recursion is proved similarly.

14.3.2. Proceed by mathematical induction using the recursion formula

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x}\,Y_n(x),$$

which is valid for both positive and negative n. Assume that

$$Y_{-k}(x) = (-1)^k Y_k(x)$$

for $k = n$ and $k = n - 1$. Then use the above recurrence formula to form $Y_{-n-1}(x)$ from $Y_{-n}(x)$ and $Y_{-n+1}(x)$:

$$Y_{-n-1} = \frac{(-2n)}{x} Y_{-n} - Y_{-n+1} = \frac{(-2n}{x}(-1)^n Y_n(x) - (-1)^{n-1} Y_{n-1}$$

$$= (-1)^{n+1}\left[\frac{2n}{x} Y_n - Y_{n-1}\right] = (-1)^{n+1} Y_{n+1} \,.$$

To complete the proof we need to establish the two starting values $Y_0(x) = (-1)^0 Y_0(x)$ (which is trivial), and $Y_1(x) = -Y_{-1}(x)$, which follows directly from the recurrence formula first mentioned above with $n = 0$.

14.3.3. From Exercise 14.3.2 we know that $Y_{-1}(x) = -Y_1(x)$. Using this result, the second formula in Exercise 14.3.1 with $n = 0$ yields $-2J_1(x) = 2J_0'(x)$, equivalent to the result we seek.

14.3.4. The left-hand side of the formula of this exercise is the Wronskian W of the two solutions.

For an ODE in the form $y'' + P(x)y' + Q(x)y = 0$, we found in Section 7.6 that its Wronskian has the form

$$W(x) = A \exp\left(-\int^x P(x_1)\, dx_1\right) ,$$

where A is independent of x. Applying this formula to Bessel's equation, for which $P(x) = x^{-1}$, we have

$$\int^x P(x_1)\, dx_1 = \ln x \,, \qquad \exp(-\ln x) = \frac{1}{x} \,,$$

and the constant A in the Wronskian formula may depend upon the specific solutions X and Z and on the index ν.

14.3.5. In principle we need to begin by verifying that the left-hand side of the first of the two formulas given for this exercise is actually a Wronskian. From Eq. (14.12), with (1) $n = -\nu + 1$ and the minus sign of the symbol $\pm$, and (2) $n = \nu - 1$ and the plus sign of the symbol $\pm$, we get

$$(1) \quad J_{-\nu+1} = -J'_{-\nu} + \frac{-\nu}{x} J_{-\nu} \,, \qquad (2) \quad J_{\nu-1} = J'_\nu + \frac{\nu}{x} J_\nu \,.$$

Inserting these expressions into the first formula of the exercise and noting that two terms cancel, we have

$$J_\nu J_{-\nu+1} + J_{-\nu} J_{\nu-1} = -J_\nu J'_{-\nu} - J_{-\nu} J'_\nu \,,$$

which is indeed a Wronskian, as J_ν and $J_{-\nu}$ are solutions of the same Bessel equation.

We now continue by observing that from Exercise 14.3.4 we know that these Wronskian formulas must have right-hand sides whose x dependence is $1/x$. We may determine the coefficient of $1/x$ by examining the leading term in a power-series expansion of the left-hand side. For the first formula, we need

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} + \cdots .$$

Since the leading power of x in $J_\nu(x)$ is x^ν, the leading power for small x (the lowest power) will come only from the second term of the Wronskian; we get

$$J_{-\nu}(x)J_{\nu-1}(x) = \frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)}\,\frac{x^{\nu-1}}{2^{\nu-1}\Gamma(\nu)} = \frac{2}{x\Gamma(\nu)\Gamma(1-\nu)} .$$

We now replace the product of gamma functions using the reflection formula, Eq. (13.23), reaching the result given in the text.

Inserting the definition of Y_ν into the second Wronskian formula of the exercise and noting that two terms cancel, we get

$$J_\nu Y'_\nu - J'_\nu Y_\nu = \frac{-J_\nu J'_{-\nu} + J'_\nu J_{-\nu}}{\sin \nu\pi} ,$$

which can be further simplified by using Eq. (14.67) to replace the numerator by $2\sin\nu\pi/\pi x$. The result is the answer given in the text.

14.3.6. Since the power-series expansions of $J_\nu(x)$ involve powers of x that increase in steps of 2 and the leading power is x^{-1}, the coefficient of x^0 will vanish and we need only to confirm the vanishing of the coefficient of x^1. To confirm this we will need to keep two terms in the expansions of the J_ν and J'_ν, and keep only the second terms of their products. The expansions of J_ν and $J'_{-\nu}$ are

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1)} - \frac{(x/2)^{\nu+2}}{1!\Gamma(\nu+2)} + \cdots ,$$

$$J'_{-\nu}(x) = -\frac{\nu}{x}\,\frac{(x/2)^{-\nu}}{\Gamma(1-\nu)} + \frac{\nu-2}{x}\,\frac{(x/2)^{-\nu+2}}{1!\Gamma(2-\nu)} + \cdots .$$

Showing explicitly only the x^1 term in the product of these functions,

$$J_\nu J'_{-\nu} = \cdots + \frac{x}{4}\left[\frac{1}{\Gamma(\nu+1)}\,\frac{\nu-2}{\Gamma(2-\nu)} + \frac{\nu}{\Gamma(1-\nu)}\,\frac{1}{\Gamma(\nu+2)}\right] + \cdots .$$

The other product in the Wronskian can be obtained by replacing ν in the above expression by $-\nu$, yielding

$$J_{-\nu} J'_{\nu} = \cdots + \frac{x}{4}\left[\frac{1}{\Gamma(1-\nu)}\,\frac{-\nu-2}{\Gamma(\nu+2)} - \frac{\nu}{\Gamma(\nu+1)}\,\frac{1}{\Gamma(2-\nu)}\right] + \cdots .$$

The x^1 contribution to the Wronskian becomes

$$J_\nu J'_{-\nu} - J_{-\nu} J'_\nu = \cdots + \frac{x}{4}\left[\frac{2\nu-2}{\Gamma(\nu+1)\Gamma(2-\nu)} + \frac{2\nu+2}{\Gamma(1-\nu)\Gamma(\nu+2)}\right] + \cdots .$$

The coefficient of x^1 reduces to zero.

14.3.7. Let y denote the integral of this exercise. We wish to verify that

$$I = x^2 y'' + xy' + x^2 y = -x^2 \int_0^\infty \cos(x\cosh t)\cosh^2 t\, dt$$

$$- x\int_0^\infty \sin(x\cosh t)\cosh t\, dt + x^2 \int_0^\infty \cos(x\cosh t)\, dt = 0 .$$

Integrate by parts the second term of the above expression, differentiating $\sin(x\cosh t)$ and integrating $\cosh t$. The integrated boundary terms vanish, and we have

$$-x\int_0^\infty \sin(x\cosh t)\cosh t\, dt = x\int_0^\infty \cos(x\cosh t)(x\sinh t)(\sinh t)\, dt$$

$$= x^2 \int_0^\infty cos(x\cosh t)\sinh^2 t\, dt .$$

Combining this result with the other two terms, we have

$$x^2 \int_0^\infty \cos(x\cosh t)\left[-\cosh^2 t + \sinh^2 t + 1\right]\, dt .$$

The integrand is identically zero.

14.3.8. Starting from the power-series expansion of $J_\nu(x)$, we get

$$\frac{\partial J_\nu}{\partial \nu} = \sum_{s=0}^\infty \frac{(-1)^s}{s!\Gamma(\nu+s+1)}\left(\frac{x}{2}\right)^{2s+\nu} \ln\left(\frac{x}{2}\right) - \sum_{s=0}^\infty \frac{(-1)^s}{s!}\left(\frac{x}{2}\right)^{2s+\nu} W ,$$

where W is shorthand for an expression that can be written in terms of the gamma and digamma functions:

$$W = \frac{1}{[\Gamma(\nu+s+1)]^2}\,\frac{d\Gamma(\nu+s+1)}{d\nu} = \frac{\psi(\nu+s+1)}{\Gamma(\nu+s+1)} .$$

The term of the above equation containing $\ln(x/2)$ is just $J_\nu(x)\ln(x/2)$, and in the limit that ν is a positive integer, the first equation above simplifies to

$$\left(\frac{\partial J_\nu}{\partial \nu}\right)_{\nu=n} = J_n \ln(x/2) - \sum_{s=0}^\infty \frac{(-1)^s \psi(n+s+1)}{s!(s+n)!}\left(\frac{x}{2}\right)^{2s+n} .$$

Similar processing of $(\partial J_{-\nu}/\partial\nu$ produces

$$\left(\frac{\partial J_{-\nu}}{\partial\nu}\right)_{\nu=n} = -J_{-n}\ln(x/2) + \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left(\frac{x}{2}\right)^{2s-n}\lim_{k\to s-n}\frac{\psi(k+1)}{\Gamma(k+1)}\,.$$

The ratio ψ/Γ is written as a limit because it is an indeterminate form for negative k.

To proceed further we divide the summation in $\partial J_{-\nu}/\partial\nu$ into parts $s<n$ and $s\ge n$, in the latter part changing the summation index to $t=s-n$ and the associated range to $t=[0,\infty]$. For the first part of the summation we insert the relationship (given in the Hint) $\psi(-m)/\Gamma(-m)\to(-1)^{m+1}m!$. In addition, we write $J_{-n}=(-1)^nJ_n$. With these adjustments, we reach

$$\left(\frac{\partial J_{-\nu}}{\partial\nu}\right)_{\nu=n} = (-1)^{n+1}J_n\ln(x/2)$$
$$+\sum_{s=0}^{n-1}\frac{(-1)^s}{s!}\left(\frac{x}{2}\right)^{2s-n}(-1)^{n-s}(n-s-1)!$$
$$+\sum_{t=0}^{\infty}\frac{(-1)^{t+n}}{(t+n)!}\left(\frac{x}{2}\right)^{2t+n}\frac{\psi(t+1)}{t!}\,,$$

which can be further simplified to

$$(-1)^{n+1}\left(\frac{\partial J_{-\nu}}{\partial\nu}\right)_{\nu=n} = J_n\ln(x/2) - \sum_{s=0}^{n-1}\frac{(n-s-1)!}{s!}\left(\frac{x}{2}\right)^{2s-n}$$
$$-\sum_{s=0}^{\infty}\frac{(-1)^s\psi(s+1)}{s!(s+n)!}\left(\frac{x}{2}\right)^{2s+n}\,.$$

We now add our final derivative expressions and divide by π, thereby obtaining Eq. (14.61).

14.3.9. We need both the equation given in the exercise and a similar formula for $J_{-\nu}$ which we can obtain by differentiating the solution $J_{-\nu}$ to Bessel's ODE:

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{-\nu}}{\partial\nu}\right) + x\frac{d}{dx}\left(\frac{\partial J_{-\nu}}{\partial\nu}\right) + (x^2-\nu^2)\frac{\partial J_{-\nu}}{\partial\nu} = 2\nu J_{-\nu}\,.$$

If we now form the sum of the equation given in the exercise and $(-1)^{n+1}$ times the equation given above, we get

$$x^2\frac{d^2}{dx^2}Y_n + x\frac{d}{dx}Y_n + (x^2-n^2)Y_n = 2n\left[J_n + (-1)^{n+1}J_{-n}\right]\,.$$

The right-hand side of this equation vanishes because $J_{-n}=(-1)^nJ_n$, showing that Y_n is, as claimed, a solution to Bessel's ODE.

14.4 Hankel Functions

14.4.1. Parts (a) through (e) of this exercise are easily proved using the Wronskian formula, Eq. (14.70):

$$J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x) = \frac{2}{\pi x}\,,$$

together with the definitions of the Hankel functions, Eqs. (14.76) and (14.77). As an example, the formula of part (a) can be written

$$J_\nu(J'_\nu + iY'_\nu) - J'_\nu(J_\nu + iY_\nu) = i(J_\nu Y'_\nu - J'_\nu Y\nu) = i\left(\frac{2}{\pi x}\right)\,.$$

Similar processes prove parts (b) through (e).

For parts (f) and (g) we need the relationship in Eq. (14.71). As an example, write the formula of part (g) as follows:

$$J_{\nu-1}(J_\nu + iY_\nu) - J_\nu(J_{\nu-1} + iY_{\nu-1} = i(J_{\nu-1}Y_\nu - J_\nu Y_{\nu-1}) = i\left(-\frac{2}{\pi x}\right)\,.$$

The formula of part (f) is proved similarly.

14.4.2. The solutions to both parts of this problem are justified by the discussion on pages 676 and 677 of the text.

14.4.3. The substitution $s = e^{i\pi}/t$ maps $t = 0+$ into $s = \infty\, e^{i\pi}$, $t = i$ into $s = i$, and $t = \infty\, e^{i\pi}$ into $s = 0+$. Thus, the s and t contours are identical except that they are traversed in opposite directions, which can be compensated by introducing a minus sign. Substituting into Eq. (14.90), we have

$$H^{(1)}_\nu(x) = -\frac{1}{\pi i}\int_{C_1} e^{(x/2)(s-1/s)}\frac{ds/s^2}{(e^{i\pi}/s)^{\nu+1}} = \frac{e^{-\nu\pi i}}{\pi i}\int_{C_1}\frac{e^{(x/2)(s-1/s)}\,ds}{s^{-\nu+1}}$$

$$= e^{-\nu\pi i}H^{(1)}_{-\nu}(x)\,.$$

14.4.5. Changing the integration variable by the substitution $t = e^\gamma$, the integrals of this problem have the integrand shown.

(a) The point $t = 0$, approached from the first quadrant, can be transformed into $\gamma = -\infty\, e^{+\pi i}$, and (for positive x) the integrand becomes negligible at that integration limit. The point $t = \infty\, e^{\pi i}$ transforms into $\gamma = +\infty + \pi i$ and the integrand of the contour integral will remain analytic if we get to that endpoint on the path in the γ-plane shown as C_3 in Fig. 14.11.

(b) Here $t = 0$ is approached from the fourth quadrant, and $t = \infty\, e^{-\pi i}$ transforms into $\gamma = +\infty - \pi i$, so the path in the γ-plane is C_4.

14.4.6. (a) In Eq. (14.90) for $n = 0$, make a change of variable defined by $t = ie^s = e^{s+i\pi/2}$, causing $(x/2)(t - t^{-1}$ to become $(x/2)(ie^t - 1/ie^t) = ix\cosh t$. Then note that $dt/t = ie^s\,ds/ie^s = ds$, so the integration assumes the form

$$H_0^{(1)}(x) = \frac{1}{\pi i}\int_{C'} e^{ix\cosh s}\,ds\,.$$

To determine the contour C', note that $t = 0$ (approached from positive t) corresponds to $s = -\infty - i\pi/2$, while $t = -\infty$ corresponds to $s = \infty + i\pi/2$. The contour is in the upper half-plane because $s = 0$ corresponds to $t = i$.

(b) Since $\cosh s = \cosh(-s)$ and each point s of the contour in the right half-plane corresponds to a point $-s$ on the contour of the left half-plane, we can restrict the integral to the right half-plane and multiply by 2.

14.4.7. (a) Since J_0 is the real part of $H_0^{(1)}$ (when x is real), we need only take the real part of the integral given for $H_0^{(1)}$.

(b) Make a change of variable to $t = \cosh s$, $dt = \sinh s\,ds$. But

$$\sinh s = \sqrt{\cosh^2 s - 1} = \sqrt{t^2 - 1}\,,$$

and we obtain the integral representation required here.

14.4.8. (a) $Y_0(x)$ is the imaginary part of the integral representation given for $H_0^{(1)}(x)$. Writing

$$H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^\infty \left[\cos(x\cosh s) + i\sin(x\cosh s)\right]\,ds,$$

we identify the imaginary part as the formula given in part (a).

(b) Change the integration variable to $t = \cosh s$. Then $dt = \sinh s\,ds$. Since $\sinh^2 s = \cosh^2 s - 1 = t^2 - 1$, we have $ds = dt/\sqrt{t^2-1}$. This substitution leads to the integral of part (b). The lower limit $t = 1$ corresponds to $s = 0$.

14.5 Modified Bessel Functions

14.5.1. In the generating function formula, change x to ix and t to $-it$; that formula then becomes

$$e^{(x/2)(t+1/t)} = \sum_{n=-\infty}^{\infty} J_n(ix)(-it)^n = \sum_{n=-\infty}^{\infty} i^{-n} J_n(ix)t^n = \sum_{n=-\infty}^{\infty} I_n(x)t^n\,.$$

14.5.2. (a) Using the expansion of Exercise 14.1.5 with $\varphi = \pi/2$, we have, setting $\rho = ix$,

$$e^{-x\cos(\pi/2)} = 1 = \sum_{n=-\infty}^{\infty} i^m J_m(ix)\,e^{im\pi/2} = \sum_{n=-\infty}^{\infty}(-1)^m J_m(ix)\,.$$

Because $J_{-m}(ix) = (-1)^m J_m(ox)$, the summands for $+m$ and $-m$ cancel when m is odd but are equal when m is even and nonzero, so we have

$$1 = J_0(ix) + 2\sum_{m=1}^{\infty} J_{2m}(ix) = I_0(x) + 2\sum_{m=1}^{\infty}(-1)^m I_{2m}(x)\,.$$

(b) A process similar to that for part (a), but with $\varphi = 0$ and $\rho = -ix$, yields the expansion in the text for e^x.

(c) Inserting $-x$ for x in part (b) and noting that I_n has the parity of $(-1)^n$, the result is immediate.

(d) Using the result of Exercise 14.1.6, with $\rho = ix$, we have

$$\cos ix = \cosh x = J_0(ix) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(ix) = I_0(x) + 2\sum_{n=1}^{\infty} I_{2n}(x)\,.$$

(e)A process for $\sin ix = i\sinh x$ similar to that employed in part (d) leads to the answer in the text.

14.5.3. (a) The integrand of the integral of this exercise has a pole at $t = 0$ whose residue is the coefficient of t^n in the expansion of the exponential, namely $I_n(x)$. Thus the contour integral and the factor preceding it, $1/2\pi i$, yield the required result.

(b) A procedure similar to that developed at Eqs. (14.38)–(14.40) confirms that the integral representation is that of I_ν.

14.5.4. Expand the exponential in the integrand of the first expression for $I_\nu(z)$ in a power series; the odd powers vanish upon integration; the even powers lead to

$$I_\nu(z) = \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{\nu+2k}\int_0^\pi \cos^{2k}\theta\,\sin^{2\nu}\theta\,d\theta\,.$$

The integral is the beta function $B(k+\frac{1}{2},\nu+\frac{1}{2})$, so

$$I_\nu(z) = \frac{1}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{\nu+2k}\frac{\Gamma(k+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+k+1)}\,.$$

If we now write

$$\Gamma(k+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,(2k)!}{2^{2k}k!}\,,$$

the right-hand side of the equation for $I_\nu(z)$ reduces to its power-series expansion.

The second expression for $I_\nu(z)$ can be obtained from the first by the substitution $p = \cos\theta$ in the integral; then $\sin\theta = (1-p^2)^{1/2}$.

The third expression for $I_\nu(z)$ can be obtained from the first expression if we remember that in the first expression we could drop the odd powers of z from the power-series expansion. That corresponds to replacing $\exp(\pm z\cos\theta)$ by $\cosh(z\cos\theta)$. After making this replacement we note that the integral from $\pi/2$ to π is equal to that from 0 to $\pi/2$, so we can use the latter integration range and append a factor 2, thereby confirming the given result.

14.5.5. (a) When Laplace's equation is written in separated-variable form in cylindrical coordinates, the φ equation has periodic solutions $a_m \sin m\varphi + b_m \cos m\varphi$, with m required to be an integer so that the solutions will be continuous and differentiable for all φ. The separation constant of the φ equation is $-m^2$.

The solutions that are needed for the z equation must be zero at $z = 0$ and $z = h$; these solutions are of the form $\sin n\pi z/h$, with n a positive integer; the separation constant of the z equation is therefore $-n^2\pi^2/h^2 \equiv -k_n^2$. With these separation constants, the ρ equation becomes

$$\rho^2 P'' + \rho P - (\rho^2 k_n^2 + m^2)P = 0\,,$$

showing that P must be a solution of the modified Bessel equation, of the form $I_m(k_n\rho)$. We must choose the solution to be I_m so that it will be regular at $\rho = 0$. The most general solution of the Laplace equation satisfying the φ and z boundary conditions is therefore a linear combination of the solutions we have found, i.e., the form shown in the exercise.

(b) The unique solution also satisfying the boundary condition at $\rho = a$ can be found using the fact that the set of $\sin k_n z$ are orthogonal, as is the set of functions $\sin m\varphi$ and $\cos m\varphi$. Thus, if we regard $V(\varphi, z)$ as the expansion

$$V(\varphi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(k_n a)(a_{mn}\sin m\varphi + b_{mn}\cos m\varphi)\sin k_n z\,,$$

the coefficients a_{mn} and b_{mn} are obtained by the type of expression written as the answer to this problem. Because the functions of z and φ are not normalized, we must divide the integral in the answer to part (b) by the normalization integral for $\sin k_n z$ (which is $h/2$) and by that of $\{\sin/\cos\}m\varphi$, which is π for $m \neq 0$ and 2π for $m = 0$. After this division, we get the answer shown in the text (after making a typographical correction to change l into h).

14.5.6. Write the definition of K_ν from Eq. (14.106) and insert the definition of

Y_ν from Eq. (14.57):

$$\begin{aligned}
K_\nu(x) &= \frac{\pi}{2}\, i^{\nu+1}\Big[J_\nu(ix) + iY_\nu(ix)\Big] \\
&= \frac{\pi}{2}\, i^{\nu+1}\left[J_\nu(ix) + \frac{i\cos\nu\pi}{\sin\nu\pi}J_\nu(ix) - \frac{iJ_{-\nu}(ix)}{\sin\nu\pi}\right] \\
&= \frac{\pi}{2\sin\nu\pi}\, i^{\nu+1}\Big[ie^{-i\nu\pi}J_\nu(ix) - iJ_{-\nu}(ix)\Big] \\
&= \frac{\pi}{2\sin\nu\pi}\Big[I_{-\nu}(x) + i^{\nu+2}e^{-i\nu\pi}I_\nu(x)\Big]\,.
\end{aligned}$$

Noting that $e^{-i\nu\pi} = i^{-2\nu}$, we obtain the required result.

14.5.7. Here use the recurrence relations for I_v, Eq. (14.104) and (14.105), and note that $\sin(\nu\pm 1)\pi = -\sin\nu\pi$. The verification is then straightforward.

14.5.8. The K_ν recurrence formulas differ from those for I_ν only in the signs of their right-hand members. If these formulas are rewritten in terms of $\mathcal{K}_\nu$, the right-hand sides will acquire a factor $e^{\pi\nu i}$ while the terms on the left-hand sides will have additional factors $e^{\pi(\nu\pm1)i} = -e^{\pi\nu i}$. These additions change only the relative signs of the two sides of the equations.

14.5.9. We have

$$\begin{aligned}
K_0(x) = \frac{\pi i}{2}\,H_0^{(1)}(ix) &= \frac{\pi i}{2}\left[\frac{2i}{\pi}\,\ln(ix) + 1 + \frac{2i}{\pi}\,(\gamma - \ln 2) + \cdots\right] \\
&= -\ln x - \ln i + \frac{\pi i}{2} - (\gamma - \ln 2) + \cdots\,.
\end{aligned}$$

Agreement with Eq. (14.110) is only achieved if we use the principal branch of $\ln x$ and take $\ln i$ to be on the branch with value $\pi i/2$.

14.5.10. Proof of the second formula for $K_\nu(z)$ is the topic of the subsection that starts on page 690 of the text, where the validity of that integral representation is discussed in detail. The first formula for $K_\nu(z)$ can be obtained from the second by the substitution $\rho = \cosh t$.

14.5.11. Write $K_\nu(x) = (\pi/2)i^{\nu+1}H_\nu^{(1)}(ix)$, $I_\nu(x) = i^{-\nu}J_\nu(ix)$. Thus,

$$I_\nu(x)K_\nu'(x) - I_\nu'(x)K_\nu(x) = \frac{i\pi}{2}\left[J_\nu(ix)iH_\nu^{(1)\prime}(ix) - iJ_\nu'(ix)H_\nu^{(1)}(ix)\right]\,.$$

Note the factors i that accompany the derivatives. Because notations like $J_\nu'(ix)$ refer to derivatives with respect to the Bessel function argument (here ix), a derivative with respect to x generates an additional factor i.

Now, using the Wronskian formula for J_ν and $H_\nu^{(1)}$ from Exercise 14.4.1(a) replace the quantity within square brackets by $i[2i/\pi(ix)]$, obtaining the final result

$$I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = \frac{i\pi}{2}\,\frac{2i}{\pi x} = -\frac{1}{x}\,.$$

14.5.12. The coefficient in the axial Green's function is a constant, and we can evaluate it using any convenient value of its argument $k\rho$. Let's take $k\rho$ small enough that we can use the limiting forms given in Eq. (14.100) for I_m and in Eqs. (14.110) and (14.111) for K_m. For positive integers m, we have (for small x)

$$I_m = \frac{x^m}{2^m m!} + \cdots\,, \qquad I'_m = \frac{x^{m-1}}{2^m(m-1)!} + \cdots\,,$$

$$K_m = 2^{m-1}(m-1)!\,x^{-m} + \cdots\,, \quad K'_m = -2^{m-1}m!\,x^{-m-1} + \cdots\,.$$

From the above data we get $K'_m I_m - K_m I'_m = -x^{-1}$; taking $x = k\rho$ and multiplying by $p = k\rho$, we find the coefficient to be -1.

The index value $m = 0$ is a special case. The formula for I_m still applies; we get $I_0 = 1 + \cdots$, $I'_0 = 0 + \cdots$. But $K_0 = -\ln x + \cdots$ and $K'_0 = -x^{-1}$. We still get $K'_0 I_0 - K_0 I'_0 = -x^{-1}$, leading to the coefficient value -1.

14.5.13. Start from the integral representation, Eq. (14.113); multiply by $\cos xu$ and integrate with respect to x. Simplification occurs because the integral over x defines a Dirac delta function. Thus,

$$\int_0^\infty \cos(zu)K_0(z)\,dz = \int_0^\infty \cos(zu)\,du \int_0^\infty \frac{\cos(zt)\,dt}{\sqrt{t^2+1}}$$

$$= \int_0^\infty \frac{dt}{\sqrt{t^2+1}} \int_0^\infty \cos(zu)\cos(zt)\,dz$$

$$= \int_0^\infty \frac{dt}{\sqrt{t^2+1}}\,\frac{\pi}{2}\,\delta(t-u) = \frac{\pi}{2}\,\frac{1}{\sqrt{u^2+1}}\,.$$

Now set $u = x/y$ and $z = yt$, so $zu = xt$ and $dz = y\,dt$. Then the first and last members of the above equation set translate into

$$\int_0^\infty \cos(xt)\,K_0(yt)\,y\,dt = \frac{\pi}{2}\,\frac{y}{\sqrt{x^2+y^2}}\,,$$

which is equivalent to the relation to be proved.

14.5.14. In this exercise n is assumed to be an integer.

Starting from the generating function for I_n given in Exercise 14.5.1, divide by t^{n+1} and integrate in t (regarded as a complex variable) over the unit circle, thereby obtaining a Schlaefli integral representation for I_n. Then write $t = e^{i\theta}$ and take the range of θ as $(-\pi, \pi)$. These steps lead to

$$I_n(x) = \frac{1}{2\pi i}\oint \frac{e^{(x/2)(t+1/t)}}{t^{n+1}}\,dt = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{(x/2)(e^{i\theta}+e^{-i\theta})}}{e^{(n+1)i\theta}}\,ie^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{x\cos\theta}}{e^{ni\theta}}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}(\cos n\theta - i\sin n\theta)\,d\theta\,.$$

Expanding the integrand into two terms, that involving $\sin n\theta$ vanishes due to its odd symmetry, and the integration range of the $\cos n\theta$ term can be changed to $(0, \pi)$ with insertion of a factor 2. These steps lead to the claimed result.

14.5.15. Substituting into the modified Bessel equation and carrying out the indicated differentiations, we get, letting y stand for $K_0(z)$,

$$0 = z^2y'' + zy' - z^2y = \int_0^\infty e^{-z\cosh t}\left[z^2\cosh^2 t - z\cosh t - z^2\right]\,dt$$

$$= \int_0^\infty e^{-z\cosh t}\left[z^2\sinh^2 t - z\cosh t\right]\,dt\,.$$

Now integrate by parts the $z\cosh t$ term of the integral, differentiating $e^{-z\cosh t}$ and integrating $z\cosh t$. The integrated terms vanish, and the resultant integration cancels the $z^2\sinh^2 t$ term of the original integral. We thereby attain the desired zero result.

14.6 Asymptotic Expansions

14.6.1. The function relevant for the saddle-point behavior is $w = (x/2)(t + 1/t)$. Saddle points are where $w' = 0$, namely $t = \pm x$. We cannot go through $t = -1$ because there is a cut there; we therefore consider only the saddle point at $t = +1$. Here, $w = x$ and $w'' = x$. Assuming x to be positive, $\arg(x) = 0$, and the angle θ needed for the steepest-descents formula is, from Eq. (12.106), $\pi/2$. The slowly-varying quantity $g(t)$ is $t^{-\nu-1}$; at $t = 1$, $g(t) = 1$. Inserting these data into the steepest-descents formula, Eq. (12.108), we find

$$I_\nu(x) \sim \left(\frac{1}{2\pi i}\right) e^x e^{i\pi/2}\sqrt{\frac{2\pi}{x}} = \frac{e^x}{\sqrt{2\pi x}}\,.$$

14.6.2. This problem can be solved by deforming the integration contour as needed to pass through a saddle point and use the steepest descents method. Taking $w = -(x/2)(s + 1/s)$, there is a saddle point at $s = 1$, where $w = -x$ and $w'' = -x/s^3 = -x$. Assuming x to be positive, $\arg(x) = \pi$, and the angle θ needed for the steepest-descents formula is, from Eq. (12.106), zero. The slowly-varying quantity $g(s)$ is $s^{1-\nu}$; at $s = 1$, $g(s) = 1$. Inserting these data into the steepest-descents formula, Eq. (12.108), we find

$$K_\nu(x) \sim \frac{1}{2}\, e^{-x} \sqrt{\frac{2\pi}{|-x|}} = \sqrt{\frac{\pi}{2x}}\, e^{-x}\,.$$

14.6.3. The modified Bessel ODE of order ν for $y(z)$ is $x^2 y'' + xy' - (z^2+\nu^2)y = 0$. Letting y first stand for the integral representation of $I_n(z)$ and noting that, though not stated in the exercise, n is assumed to be a nonnegative integer, write

$$z^2 y'' - z^2 y = \frac{z^2}{\pi}\int_0^\pi e^{z\cos t}(\cos^2 t - 1)\cos(nt)\,dt = -\frac{z^2}{\pi}\int_0^\pi e^{z\cos t}\sin^2 t\,dt$$

$$zy' = \frac{z}{\pi}\int_0^\pi e^{z\cos t}\cos t\cos(nt)\,dt\,.$$

The remaining term of the ODE is processed by carrying out two successive integrations by parts, with $\cos nt$ or $\sin nt$ integrated and the remainder of the integrand differentiated. the result is

$$\begin{aligned}
-n^2 y &= -\frac{n^2}{\pi}\int_0^\pi e^{z\cos t}\cos(nt)\,dt \\
&= -\frac{n^2}{\pi}\left[\frac{\sin(nt)}{n}\, e^{z\cos t}\right]_0^\pi - \frac{nz}{\pi}\int_0^\pi e^{z\cos t}\sin t\sin(nt)\,dt \\
&= -\frac{nz}{\pi}\left[\left(-\frac{cos(nt)}{n}\right) e^{z\cos t}\sin t\right]_0^\pi \\
&\qquad\qquad - \frac{z}{\pi}\int_0^\pi e^{z\cos t}\left[\cos t - z\sin^2 t\right]\cos(nt)\,dt \\
&= -\frac{z}{\pi}\int_0^\pi e^{z\cos t}\left[\cos t - z\sin^2 t\right]\cos(nt)\,dt\,.
\end{aligned}$$

In writing these equations we have used the fact that the integrated endpoint terms all vanish. Here is where it is necessary that n be an integer.

Both integrals in the final expression for $-n^2 y$ cancel against the other terms of the ODE, indicated that the ODE is satisfied.

The demonstration for the integral representation of $K_\nu(z)$ proceeds upon entirely similar lines. Because the upper integration limit is infinity, the endpoint terms in the integrations by parts vanish whether or not ν is an integer. The vanishing occurs because (for nonzero z) the factor $e^{-z\cosh t}$ approaches zero faster than finite powers of the hyperbolic functions diverge.

14.6.4. (a) Differentiate the integral representation for $K_0(z)$, Eq. (14.128), to obtain

$$\frac{dK_0(z)}{dz} = -\int_1^\infty xe^{-zx}(x^2-1)^{1/2}\,dx\,.$$

Integrate this expression by parts, integrating $x/(x^2-1)^{1/2}$ and differentiating e^{-zx}. The integrated term vanishes, and the integral can be identified as the integral representation of $-K_1(z)$.

(b) To find the small-z behavior of K_1, change the integration variable from x to $u = zx$, after which the integral representation for K_1 takes the form

$$K_1(z) = \frac{\pi^{1/2}}{\Gamma(3/2)}\left(\frac{z}{2}\right)\int_z^\infty e^{-u}\left(\frac{u^2}{z^2}-1\right)^{1/2}\frac{du}{z} = \frac{1}{z}\int_z^\infty e^{-u}(u^2-z^2)^{1/2}\,du\,.$$

In the limit of small z, the integral becomes 1!, and $K_1(z) \approx 1/z$. In this limit the indefinite integral of K_1 is therefore $\ln z + C$, and $K_0(z) \longrightarrow -\ln z + C$, which is the scaling identified in Eq. (14.110).

14.6.5. When $r \neq 0$, the quotient of two factorials occurring in Eq. (14.132) is

$$(\nu+r-\tfrac{1}{2})(\nu+r-\tfrac{3}{2})\cdots(\nu+\tfrac{1}{2})(\nu-\tfrac{1}{2})\cdots(\nu-r+\tfrac{3}{2})(\nu-r+\tfrac{1}{2}) =$$

$$(\nu+r-\tfrac{1}{2})(\nu-r+\tfrac{1}{2})(\nu+r-\tfrac{3}{2})(\nu-r+\tfrac{1}{2})\cdots(\nu+\tfrac{1}{2})(\nu-\tfrac{1}{2}) =$$

$$\left(\frac{4\nu^2-(2r-1)^2}{4}\right)\left(\frac{4\nu^2-(2r-3)^2}{4}\right)\cdots\left(\frac{4\nu^2-1}{4}\right)\,.$$

The term of any given r will therefore contain in its denominator 4^r which combines with the remaining factors of the summation in Eq. (14.132), $1/[r!(2z)^r]$, to give the result shown in Eq. (14.133).

14.6.6. (a) The modified Bessel ODE of order ν for $y(z)$ is

$$\mathcal{L}(y) = z^2y'' + zy' - z^2y - \nu^2y = 0\,.$$

Inserting the form given for $y(z)$, we get initially

$$z^2y'' = \nu(\nu+1)y + 2\nu z^{\nu+1}\int\left(-te^{-zt}\right)(t^2-1)^{\nu-1/2}\,dt$$

$$+\, z^{\nu+2}\int\left(t^2e^{-zt}\right)(t^2-1)^{\nu-1/2}\,dt\,,$$

$$zy' = \nu y + z^{\nu+1}\int\left(-te^{-zt}\right)(t^2-1)^{\nu-1/2}\,dt\,,$$

and, after cancellations and minor rearrangement, we obtain

$$\mathcal{L}(y) = z^{\nu+2}\int e^{-zt}(t^2-1)^{\nu+1/2}\,dt - (2\nu+1)z^{\nu+1}\int e^{-zt}\,t\,(t^2-1)^{\nu-1/2}\,dt\,.$$

We now integrate the first integral of $\mathcal{L}$ by parts, integrating e^{-zt} and differentiating the remainder of the integrand. The new integral obtained in this way cancels against the second integral of $\mathcal{L}$, leaving only the endpoint term

$$\left[-\frac{e^{-zt}}{z}\,(t^2-1)^{\nu+1/2}\right]_{z_1}^{z_2}.$$

If this term is zero (i.e., has the same value at both endpoints), Bessel's modifierd ODE will be satisfied.

14.6.7. We only need the initial term of each asymptotic expansion.

(a) From Eq. (14.144),

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\,\cos\left[x-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right],$$

we get the following products of cosine functions which are simplified using trigonometric identities:

$$\begin{aligned}
J_\nu J_{-\nu-1}+J_{-\nu}J_{\nu+1} &\sim \frac{2}{x\pi}\cos\left[x-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]\cos\left[x+\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right] \\
&\quad+\frac{2}{x\pi}\cos\left[x+\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}\right]\cos\left[x-\left(\nu+\frac{3}{2}\right)\frac{\pi}{2}\right] \\
&\sim \frac{1}{x\pi}\left[\cos 2x+\cos\left(\nu+\frac{1}{2}\right)\pi+\cos(2x-\pi)+\cos\left(\nu+\frac{1}{2}\right)\pi\right] \\
&\sim -\frac{2\sin\pi\nu}{x\pi}
\end{aligned}$$

using the cosine addition theorem.

In Part (b), N should be replaced by Y.

Parts (b), (c), (d) and (e) are proved similarly, using the asymptotic forms for K_ν and $H_\nu^{(2)}$ in Eqs. (14.126) and (14.127), and for I_ν and Y_ν the leading terms of Eqs. (14.141) and (14.143):

$$I_\nu \sim \sqrt{\frac{1}{2\pi z}}\;e^z, \qquad Y_\nu \sim \sqrt{\frac{2}{\pi z}}\,\sin\left[z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right].$$

We also need the derivatives I'_ν and K'_ν, obtained by keeping the most divergent term when the leading terms of I_ν and K_ν are differentiated:

$$I'_\nu \sim I_\nu\,, \qquad K'_\nu \sim -K_\nu\,.$$

14.6.8. The Green's function for an outgoing wave with no finite boundary must have the form $C\,H_0^{(1)}(k(|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|)$, because this function is an outgoing-wave solution of the homogeneous Helmholtz equation that is circularly symmetric in $\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ and satisfied wherever $\boldsymbol{\rho}_1 \neq \boldsymbol{\rho}_2$. To find the proportionality constant C, we evaluate the integral (in $\boldsymbol{\rho}_1$) of $(\boldsymbol{\nabla}_1^2 + k^2)G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ over the area enclosed by a circle of radius a centered at $\boldsymbol{\rho}_2$. Transforming the integral of $\boldsymbol{\nabla}_1^2$ to a line integral over the circular perimeter and letting r stand for $|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|$, our computation takes the form

$$\int_0^a (\boldsymbol{\nabla}_1^2 + k^2)C\, H_0^{(1)}(kr)(2\pi r)\, dr = 2\pi a C \frac{d}{dr} H_0^{(1)}(kr)\Big|_{r=a} + k^2 C \int_0^a H_0^{(1)}(kr)(2\pi r)\, dr\,.$$

Because at small r

$$H_0^{(1)}(kr) = \frac{2i}{\pi} \ln kr + \text{constant} + \cdots\,,$$

the integral of $H_0^{(1)}(kr)$ over the circular area vanishes, but the radial derivative term has the following small-r limit:

$$2\pi a C \left(\frac{2i}{\pi a}\right) = 4iC\,, \qquad \text{corresponding to} \qquad C = -\frac{i}{4}\,.$$

14.6.9. Substitute $-ix$ for z in Eq. (14.134), writing $-ix = e^{-\pi i/2}x$ so as to be on the same branch of K_ν as that taken for real arguments as its definition. Then

$$\sqrt{\frac{\pi}{2(-ix)}} = \sqrt{\frac{\pi}{2x}}\;\; e^{\pi i/4}$$

and the verification of Eq.)14.138) becomes immediate.

14.6.10. (a) This is verified by direct substitution.

(b) $\quad a_{n+1} = \dfrac{i}{2(n+1)}\left[v^2 - \dfrac{(2n+1)^2}{2^2}\right]a_n.$

(c) From Eq. (14.125), we have

$$a_0 = \sqrt{\frac{2}{\pi}}\;\; e^{-i(\nu+1/2)(\pi/2)}\,.$$

14.6.11. The answer is given in the text.

14.6.13. The answer is given in the text.

14.7 Spherical Bessel Functions

14.7.1. Here we essentially reverse the process that was used in solving Exercise 14.6.5, writing for the term in P or Q whose largest odd integer squared was $(2s-1)^2$,

$$\frac{(4\nu^2-1^2)}{4}\,\frac{(4\nu^2-3^2)}{4}\cdots=\cdots(\nu+\tfrac{1}{2})(\nu-\tfrac{1}{2})\cdots=\frac{\Gamma(\nu+s+\frac{1}{2})}{\Gamma(\nu-s+\frac{1}{2})}\,.$$

Since these quantities are needed for $\nu=n+\frac{1}{2}$, the ratio of gamma functions becomes a ratio of factorials: $(n+s)!/(n-s)!$. The sign alternation in P and Q and the presence of the i multiplying Q are both accounted for by the factor i^s in Eq. (14.162).

14.7.2. Using the definitions in Eqs. (14.57) and (14.151),

$$y_n(x)=\sqrt{\frac{\pi}{2x}}\,Y_{n+1/2}(x)=\sqrt{\frac{\pi}{2x}}\,\frac{\cos(n+1/2)\pi J_{n+1/2}(x)-J_{-n-1/2}(x)}{\sin(n+1/2)\pi}$$

$$=(-1)^{n+1}\sqrt{\frac{\pi}{2x}}\,J_{-n-1/2}(x)\,,$$

where we have simplified the formula using the relationships $\sin(n+\frac{1}{2})\pi=(-1)^n$ and $\cos(n+\frac{1}{2})\pi=0$.

14.7.3. Start from Eq. (14.140) for the expansion of $J_\nu(z)$. Application to $j_n(z)$ is as follows:

$$j_n(z)=\sqrt{\frac{\pi}{2z}}J_{n+1/2}(z)$$

$$=\frac{1}{z}\left[P_{n+1/2}(z)\cos\left(z-(n+1)\,\frac{\pi}{2}\right)-Q_{n+1/2}(z)\sin\left(z-(n+1)\,\frac{\pi}{2}\right)\right]$$

$$=\frac{1}{z}\left[P_{n+1/2}(z)\sin\left(z-\frac{n\pi}{2}\right)+Q_{n+1/2}(z)\cos\left(z-\frac{n\pi}{2}\right)\right]\,.$$

The expansion of $P_{n+1/2}(z)$ is that given in Eq. (14.135), but for half-integer ν the series terminates. The individual factors in the numerators of the terms of P_ν involve $\mu=(2n+1)^2$ and are of forms $(2n+1)^2-(2j+1)^2$, which factor into $(2n-2j)(2n+2j+2)$. We now see that the series for P_ν terminates when $2j$ reaches $2n$. Taking all the above into account, and noting that each term has four more linear factors than its predecessor, the series for $P_{n+1/2}$ can be written

$$P_{n+1/2}(z)=\sum_k\frac{(-1)^k\,(2n+4k)!!}{(2k)!(2z)^{2k}2^{4k}(2n-4k)!!}$$

$$=\sum_k\frac{(-1)^k(n+2k)!}{(2k)!(2z)^{2k}(n-2k)!}\,.$$

The second line of the above equation is reached by writing the double factorials $(2p)!!=2^p p\,!$. The lower limit of the k summation is $k=0$; the upper limit is the largest value of k for which the term is nonzero; from the denominator of the final expression it is clear that contributions are restricted to k such that $n - 2k \geq 0$.

A similar analysis yields the formula for $Q_{n+1/2}$, which, written first using double factorials, is

$$Q_{n+1/2}(z) = \sum_k \frac{(-1)^k\,(2n+4k+2)!!}{(2k+1)!(2z)^{2k+1}2^{4k+2}(2n-4k-2)!!}$$

$$= \sum_k \frac{(-1)^k(n+2k+1)!}{(2k+1)!(2z)^{2k+1}(n-2k-1)!}\,,$$

thereby completing the formula for $j_n(z)$. Note that the second term of the formula in the text (arising from $Q_{n+1/2}$) is incorrect; the power of $2z$ should be $2s+1$.

14.7.4. Since $\nu = n + 1/2$, the integral contains an integer power of $(1-p^2)$ and therefore it will expand into integrals of forms

$$\int_{-1}^{1} p^{2k}(\cos xp \pm i \sin xp)\,dp\,.$$

The imaginary parts of these integrals vanish due to symmetry; via repeated integrations by parts (differentiating p^{2k}) the real parts will have integrated terms dependent on $\cos x$ and/or $\sin x$. When the integrations by parts have reduced the power of p to zero, the final integral will also involve a trigonometric function.

14.7.5. The functions J_ν, Y_ν, and $H^{(i)}\nu$ all satisfy the same recurrence formulas, and all are related to the corresponding spherical Bessel functions in identical ways, so a proof for j_n can be extended to y_n and $h_n^{(i)}$. From the J_ν recurrence formula, Eq. (14.7), written for $\nu = n + \frac{1}{2}$,

$$\sqrt{\frac{2x}{\pi}}\;j_{n-1}(x) + \sqrt{\frac{2x}{\pi}}\;j_{n+1}(x) = \frac{2(n+\frac{1}{2}}{x}\;\sqrt{\frac{2x}{\pi}}\;j_n(x)\,,$$

which easily simplifies to the first recurrence formula of this exercise.

The second recurrence formula is a bit less trivial, since

$$j_n'(x) = -\frac{1}{2x}\,j_n(x) + \sqrt{\frac{\pi}{2x}}\;J'_{n+\frac{1}{2}}(x)\,.$$

Using Eq. (14.8), with $J_{n-\frac{1}{2}}$ and $J_{n+\frac{3}{2}}$ written in terms of $j_{n\pm1}$, and also using the newly found recurrence formula to rewrite j_n in terms of $j_{n\pm1}$, our formula for j_n' reduces to the result shown in the exercise.

14.7.6. Assume the validity of the formula given for $j_n(x)$ with $n = k$, and then use Eq. (14.172) to obtain a formula for $j_{k+1}(x)$. We have

$$j_{k+1}(x) = -x^k \frac{d}{dx}\left[x^{-k} j_k(x)\right] = -x^k \frac{d}{dx}\left[(-1)^k \left(\frac{1}{x}\frac{d}{dx}\right)^k \left(\frac{\sin x}{x}\right)\right]$$

$$= (-1)^{k+1} x^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right)\left[\left(\frac{1}{x}\frac{d}{dx}\right)^k \left(\frac{\sin x}{x}\right)\right],$$

which is the assumed formula for $k+1$. To complete the proof by mathematical induction, we need a starting value. For $k = 0$ the assumed formula is simply the explicit form for $j_0(x)$.

14.7.7. Since each spherical Bessel function is proportional to a conventional Bessel function divided by $x^{1/2}$ and since Wronskian formulas are quadratic in the Bessel functions, all spherical Bessel Wronskians must be proportional to conventional Bessel Wronskians divided by x, i.e., proportional to $1/x^2$. We can determine the proportionality constant most easily from the behavior at $x \to 0$ or at $x \to \infty$.

For small x, the limiting behavior of j_n and y_n is given by Eqs. (14.177) and (14.178). Differentiating these expressions gives values for j_n' and y_n'. The four results we need are

$$j_n(x) \approx \frac{x^n}{(2n+1)!!}, \qquad j_n'(x) \approx \frac{n\,x^{n-1}}{(2n+1)!!},$$

$$y_n(x) \approx -\frac{(2n-1)!!}{x^{n+1}}, \qquad y_n'(x) \approx \frac{(n+1)(2n-1)!!}{x^{n+2}}.$$

Then our Wronskian takes the form

$$j_n(x)y_n'(x) - j_n'(x)y_n(x) \approx$$

$$\frac{x^n}{(2n+1)!!}\frac{(n+1)(2n-1)!!}{x^{n+2}} + \frac{nx^{n-1}}{(2n+1)!!}\frac{(2n-1)!!}{x^{n+1}}$$

$$= \frac{1}{x^2}\left(\frac{n+1}{2n+1} + \frac{n}{2n+1}\right) = \frac{1}{x^2},$$

the result in the text.

14.7.8. Writing the Wronskian in the notation $W(y_1, y_2) = y_1 y_2' - y_1' y_2$, we note that from Exercise 14.7.7,

$$W(j_n(x), y_n(x)) = 1/x^2.$$

Moreover,

$$W(y_2, y_1) = -W(y_1, y_2) \quad \text{and} \quad W(y, y) = 0.$$

Therefore, suppressing arguments x,

$$W(h_n^{(1)}, h_n^{(2)}) = W(j_n + iy_n, j_n - iy_n) = -2iW(j_n, y_n) = -\frac{2i}{x^2}\,.$$

14.7.9. Introduce a power-series expansion of $\cos(z\cos\theta)$ in the integrand of Poisson's representation of j_n:

$$j_n(z) = \frac{z^n}{2^{n+1}n!}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}\int_0^\pi \cos^{2k}\theta\sin^{2n+1}\theta\,d\theta\,.$$

The integral can now be recognized as the beta function $B(k+\frac{1}{2}, n+1)$. (To make this identification, we can start from Eq. (13.47), remove the factor 2, and extend the upper integration limit to π, because in the present case the cosine occurs to an even integer power.)

The beta function can be written

$$B(k+\tfrac{1}{2}, n+1) = \frac{\sqrt{\pi}\,(2k)!}{2^{2k}k!}\,\frac{n!}{\Gamma(n+k+\frac{3}{2})}\,.$$

Substituting this form for the integral, we reach

$$j_n(z) = \sqrt{\frac{\pi}{2z}}\sum_{k=0\infty}\frac{(-1)^k}{k!\,\Gamma(n+k+\frac{3}{2})}\left(\frac{z}{2}\right)^{n+2k+1/2}\,.$$

This is the correct power-series expansion of $j_n(z)$.

14.7.10. From the definition, form $k_n(x) = \dfrac{2^{n+1}n!}{\pi x^{n+1}}\displaystyle\int_0^\infty \frac{\cos xt}{(t^2+1)^{n+1}}\,dt\,.$

Integrate the t integral by parts, integrating the factor $\cos xt$ and differentiating the factor $(t^2+1)^{-n-1}$. We then have

$$k_n(x) = \frac{2^{n+1}n!}{\pi x^{n+1}}\int_0^\infty \frac{2(n+1)t\sin tx}{x(t^2+1)^{n+2}}\,dt = \frac{2^{n+2}(n+1)!}{\pi x^{n+1}}\int_0^\infty \frac{t^2 j_0(tx)}{(t^2+1)^{n+2}}\,dt\,.$$

14.7.11. Write Bessel's ODE in self-adjoint form for $J_\mu(x)$ and $J_\nu(x)$, and multiply the J_μ equation by J_ν and the J_ν equation by J_μ:

$$J_\nu(x)[xJ'_\mu(x)]' + xJ_\nu(x)J_\mu(x) = \frac{\mu^2}{x}J_\nu(x)J_\mu(x)\,,$$

$$J_\mu(x)[xJ'_\nu(x)]' + xJ_\mu(x)J_\nu(x) = \frac{\nu^2}{x}J_\mu(x)J_\nu(x)\,.$$

Subtract the second of these equations from the first and integrate from $x=0$ to $x=\infty$:

$$(\mu^2-\nu^2)\int_0^\infty J_\mu(x)J_\nu(x)\,\frac{dx}{x} = \int_0^\infty J_\nu[xJ'_\mu]'\,dx - \int_0^\infty J_\mu[xJ'_\nu]'\,dx\,.$$

Integrate each right-hand-side integral by parts, integrating the explicit derivative and differentiating the other Bessel function. The resultant integrals are equal in magnitude but opposite in sign; all that remains are the endpoint terms:

$$(\mu^2-\nu^2)\int_0^\infty J_\mu(x)J_\nu(x)\,\frac{dx}{x} = xJ_\nu(x)J'_\mu\Big|_0^\infty - xJ_\mu(x)J'_\nu(x)\Big|_0^\infty .$$

The endpoint at zero makes no contribution to either term; that at infinity is finite because the asymptotic limit of each Bessel function contains a factor $x^{-1/2}$. From the asymptotic forms

$$J_\mu(x)\sim\sqrt{\frac{2}{\pi x}}\,\cos\left(x-(\mu+\tfrac{1}{2})\frac{\pi}{2}\right), \quad J'_\mu(x)\sim-\sqrt{\frac{2}{\pi x}}\,\sin\left(x-(\mu+\tfrac{1}{2})\frac{\pi}{2}\right),$$

we have

$$(\mu^2-\nu^2)\int_0^\infty J_\mu(x)J_\nu(x)\,\frac{dx}{x} =$$

$$\frac{2}{\pi}\Big[-\cos\left(x-(\nu+\tfrac{1}{2})\,\frac{\pi}{2}\right)\sin\left(x-(\mu+\tfrac{1}{2})\,\frac{\pi}{2}\right)$$

$$+\cos\left(x-(\mu+\tfrac{1}{2})\,\frac{\pi}{2}\right)\sin\left(x-(\nu+\tfrac{1}{2})\,\frac{\pi}{2}\right)\Big]$$

$$=\frac{2}{\pi}\,\sin\left[x-(\nu+\tfrac{1}{2})\,\frac{\pi}{2}-x+(\mu+\tfrac{1}{2})\,\frac{\pi}{2}\right]=\frac{2}{\pi}\,\sin[(\mu-\nu)\pi/2]\,.$$

This equation rearranges to the answer in the text.

14.7.12. The integral under consideration vanishes due to symmetry if $m-n$ is odd. If $m-n$ is even, then we identify the integral as proportional to that evaluated in Exercise 14.7.11:

$$\int_0^\infty J_{m+1/2}J_{n+1/2}\,\frac{dx}{x}=\frac{2}{\pi}\,\frac{\sin[(m-n)\pi/2]}{m^2-n^2}\,.$$

Because $m-n$ is even, the sine function in the above formula is zero, confirming the desired result.

14.7.13. Consider the result in Exercise 14.7.11 in the limit $\mu\to\nu$. Introducing the leading term in the power-series expansion of $\sin[(\nu-\nu)\pi/2]$, we get

$$\lim_{\mu\to\nu}\int_0^\infty J_\mu(x)J_\nu(x)\,\frac{dx}{x}=\lim_{\mu\to\nu}\frac{2}{\pi}\,\frac{(\mu-\nu)\pi/2}{(\mu-\nu)(\mu+\nu)}=\frac{1}{2\nu}\,.$$

Use the above result to evaluate

$$\int_0^\infty [j_n(x)]^2\,dx=\frac{\pi}{2}\int_0^\infty\left[J_{n+1/2}\right]^2\,\frac{dx}{x}=\frac{\pi}{2(2n+1)}\,.$$

Extending the integration range to $-\infty$ (and thereby multiplying the result by 2), we reach the result given in the text.

14.7.14. The integrals given for $x(s)$ and $y(s)$ follow directly from making the substitution $v^2 = u$ into $x(t)$ and $y(t)$, and then identifying $u^{-1}\cos u$ as $j_{-1}(u)$ and $u_{-1}\sin u$ as $j_0(u)$.

Rewrite the expansion formulas in terms of J_n and cancel the constant factors $\sqrt{\pi/2}$. We then have

$$\int_0^s J_{-1/2}(u)\,du = 2\sum_{n=0}^{\infty} J_{2n+1/2}(s)\,,$$

$$\int_0^s J_{1/2}(u)\,du = 2\sum_{n=0}^{\infty} J_{2n+3/3}(s)\,.$$

Now differentiate both sides of these equations with respect to s, reaching

$$J_{-1/2}(s) = 2\sum_{n=0}^{\infty} J'_{2n+1/2}(s)\,,$$

$$J_{1/2}(s) = 2\sum_{n=0}^{\infty} J'_{2n+3/2}(s)\,.$$

Now use Eq. (14.8) to replace $2J'_\nu(x)$ by $J_{\nu-1}(x) - J_{\nu+1}(x)$. When the sums are evaluated, everything cancels except the initial instance of $J_{\nu-1}$, confirming these equations.

14.7.15. For a standing-wave solution with time dependence $e^{i\omega t}$, the wave equation becomes a spherical Bessel equation with radial solutions (regular at the origin) $j_m(\omega r/v)$. Here m can be any nonnegative integer and v is the velocity of sound. The solutions satisfying a Neumann boundary condition at $r = a$ will have a vanishing value of $j'_m(\omega a/v)$, and the minimum value of ω that meets this condition will correspond to the smallest zero of j'_m for any m. Consulting the list of zeros of j' given in Table 14.2, we see that the smallest zero is for $m = 1$, which occurs at $b_{11} = 2.0816$. Note that the smallest zero of j'_0 is larger; that is because j_0 has a maximum at $r = 0$ while j_1 has no extrema at arguments smaller than b_{11}.

Writing $\omega = 2\pi\nu$, where ν is the oscillation frequency, we find

$$\frac{2\pi\nu a}{v} = b_{11}\,, \quad \text{or} \quad \nu = \frac{b_{11}}{2\pi}\,\frac{v}{a}\,.$$

Note also that the wavelength given in the answer to this problem corresponds to unconstrained waves of frequency ν.

14.7.16. (a) The power-series expansion of $x^{-1/2}J_{n+\frac{1}{2}}(ix)$ will start with $(ix)^n$ and continue with powers $n+2s$. Therefore, $i_n(x)$ will have parity $(-1)^n$.

(b) Ir is clear from the explicit forms shown in Eq. (14.196) that $k_n(x)$ has no definite parity.

14.7.17. We check the formula for $n = 0$. The relevant quantities are

$$i_0(x) = \frac{\sinh x}{x}\,, \qquad i_0'(x) = \frac{\cosh x}{x} - \frac{\sinh x}{x^2}\,,$$

$$k_0(x) = \frac{e^{-x}}{x}\,, \qquad k_0'(x) = -e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right).$$

Inserting these into the Wronskian,

$$\begin{aligned} i_0(x)k_0'(x) - i_0'(x)k_0(x) &= -\sinh x\; e^{-x}\left(\frac{1}{x^2} + \frac{1}{x^3}\right) \\ &\qquad - \frac{e^{-x}\cosh x}{x^2} + \frac{e^{-x}\sinh x}{x^3} \\ &= -\frac{e^{-x}}{x^2}(\cosh x + \sinh x) = -\frac{1}{x^2}\,. \end{aligned}$$

15. Legendre Functions

15.1 Legendre Polynomials

15.1.1. Differentiate Eq. (15.25), obtaining

$$(1-x^2)P_n''(x) - 2xP_n'(x) = nP_{n-1}'(x) - nP_n(x) - nxP_n'(x)\,.$$

Use Eq. (15.24) to replace $nP_{n-1}'x$ by $-n^2P_n(x)+nxP_n'(x)$. After removing canceling terms, what remains is the Legendre ODE.

15.1.2. Expand $(-2xt+t^2)^n$ in Eq. (15.12), obtaining

$$g(x,t) = \sum_{n=0}^{\infty} \binom{-1/2}{n} \sum_{j=0}^{n} \binom{n}{j} t^{2j}(-2xt)^{n-j}\,.$$

Now change the summation variable n to $m = n+j$; the range of m will be from zero to infinity, but the range of j will now only include values no larger than $m/2$. Also writing the binomial coefficient involving $-1/2$ using a Pochhammer symbol, we reach

$$g(x,t) = \sum_{m=0}^{\infty} \sum_{j=0}^{[m/2]} \frac{(-\frac{1}{2})_{m-j}}{(m-j)!}\, \frac{(m-j)!}{(m-2j)!j!}\, t^m(-2x)^{m-2j}\,.$$

The coefficient of t^m in this expression for $g(x,t)$ is $P_m(x)$. Inserting

$$(-\tfrac{1}{2})_{m-j} = \frac{(-1)^{m-j}(2m-2j)!}{2^{2m-2j}(m-j)!}\,,$$

we obtain the required formula.

15.1.3. Start by writing P_n'' in series form, using the notation of Exercise 15.1.2. It is convenient to change the summation index to k', with k replaced by $k'-1$, so that the power of x in the summand will be $n-2k'$. This change causes P_n'' to take the form

$$\begin{aligned}
P_n''(x) &= \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!x^{n-2k-2}}{2^n k!(n-k)!(n-2k-2)!} \\
&= \sum_{k'} \frac{(-1)^{k'}(2n-2k')!x^{n-2k'}}{2^n(k')!(n-k')!(n-2k')!}\Big[-2k'(2n-2k'+1)\Big]\,.
\end{aligned}$$

We have organized the k' summation in a way that retains the factors present in the original summation for P_n. Formally the k' summation ranges from $k'=1$ to $k'=[n/2]+1$, but the presence of an extra factor k' enables us to extend the lower limit to $k'=0$ and the factorial $(n-2k')!$ in the denominator causes the contribution at $k'=[n/2]+1$ to vanish.

When we use the above form in the work that follows we will remove the prime from the summation index.

We are now ready to write all the terms in the Legendre ODE. Only the term we have already processed causes a change in the indicated powers of x.

$$P_n'' - x^2 P_n'' - 2xP_n' + n(n+1)P_n = \sum_{k=0}^{[n/2]} \left[\frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n k!(n-k)!(n-2k)!} \right]$$

$$\times \Big[-2k(2n-2k+1) - (n-2k)(n-2k-1) - 2(n-2k) + n(n+1) \Big] .$$

The quantity in the last set of square brackets vanishes, confirming that the expansion satisfies the Legendre ODE.

15.1.4. If we set $P^*(x) = P(y)$, where $y = 2x - 1$ then

$$\int_0^1 P_n^*(x) P_m^*(x)\, dx = \frac{1}{2} \int_{-1}^1 P_n(y) P_m(y)\, dy = \frac{1}{2}\, \frac{2}{2n+1}\, \delta_{nm} .$$

This equation confirms the orthogonality and normalization of the $P_n^*(x)$.

(a) Replacing x by $2x - 1$ in the P_n recurrence formula, Eq. (15.18), we find

$$(n+1)P_{n+1}^*(x) - (2n+1)(2x-1)P_n^*(x) + nP_{n-1}^*(x) = 0 .$$

(b) By examination of the first few P_n^*, we guess that they are given by the general formula

$$P_n^*(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} x^k .$$

This formula is easily proved by mathematical induction, using the recurrence formula. First, we note from the explicit formula for P_n^* that it gives correct results for $P_0^*(x)$ and $P_1^*(x)$. For $n > 1$, the recurrence formula for $P_{n+1}^*(x)$ yields

$$P_{n+1}^*(x) = \frac{1}{n+1}\,(2n+1)(2x-1) \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k}\, x^k$$

$$- \frac{n}{n+1} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k-1}{k} (-1)^{n-1-k}\, x^k .$$

Collecting the contributions for each power of x, we confirm that the formula for $P_{n+1}^*(x)$ is correct, thereby completing the proof. Finally, the form of the explicit formula for P_n^* shows that the coefficients of all the x^k are products of binomial coefficients, which reduce to integers.

15.1.5. $$\mathsf{A}_{\text{even}} = \begin{pmatrix} 1 & 1/3 & 7/35 & 33/231 \\ 0 & 2/3 & 20/35 & 110/231 \\ 0 & 0 & 8/35 & 72/231 \\ 0 & 0 & 0 & 16/231 \end{pmatrix}$$

$$\mathsf{B}_{\text{even}} = \begin{pmatrix} 1 & -1/2 & 3/8 & -5/16 \\ 0 & 3/2 & -30/8 & 105/16 \\ 0 & 0 & 35/8 & -315/16 \\ 0 & 0 & 0 & 231/16 \end{pmatrix}$$

$$\mathsf{A}_{\text{odd}} = \begin{pmatrix} 1 & 3/5 & 27/63 & 143/429 \\ 0 & 2/5 & 28/63 & 182/429 \\ 0 & 0 & 8/63 & 88/429 \\ 0 & 0 & 0 & 16/429 \end{pmatrix}$$

$$\mathsf{B}_{\text{odd}} = \begin{pmatrix} 1 & -3/2 & 15/8 & -35/16 \\ 0 & 5/2 & -70/8 & 315/16 \\ 0 & 0 & 63/8 & -693/16 \\ 0 & 0 & 0 & 429/16 \end{pmatrix}.$$

15.1.6. $$2t\frac{\partial g(t,x)}{\partial t} + g = \frac{1-t^2}{(1-2xt+t^2)^{3/2}}$$
$$= \sum_{n=0}^{\infty}(2nP_n t^n + P_n t^n) = \sum_{n=0}^{\infty}(2n+1)P_n t^n.$$

15.1.7. (a) Substituting

$$nP_{n-1}(x) = (2n+1)xP_n(x) - (n+1)P_{n+1}(x)$$

from Eq. (15.18) into Eq.(15.25) yields

$$(1-x^2)P_n'(x) = (2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nxP_n(x)$$
$$= (n+1)P_n(x) - (n+1)P_{n+1}(x),$$

i.e. Eq. (15.26).

(b) $(15.24)_{n\to n+1}$+$x\cdot$(15.23)$\to$ (15.26).

15.1.8. For $n=1$, we establish $P_1'(1) = (1\cdot 2)/2 = 1$ as the first step of a proof by mathematical induction. Now assuming $P_n'(1) = n(n+1)/2$ and using Eq. (15.23),

$$P_{n+1}'(1) = (n+1)P_n(1) + P_n'(1) = (n+1) + \frac{n}{2}(n+1) = \frac{1}{2}(n+1)(n+2)\,,$$

which proves our assumed formula for $n+1$.

15.1.9. For a proof by mathematical induction we start by verifying that $P_0(-x) = P_0(x) = 1$ and that $P_1(-x) = -P_1(x) = -x$. We then need to show that if $P_m(-x) = (-1)^m P_m(x)$ for $m = n-1$ and $m = n$, the relationship also holds for $m = n+1$. Applying Eq. (15.18) with x replaced by $-x$ we have

$$-(2n+1)xP_n(-x) = (n+1)P_{n+1}(-x) + nP_{n-1}(-x) \quad \longrightarrow$$

$$(-1)^{n+1}(2n+1)xP_n(x) = (n+1)P_{n+1}(-x) + (-1)^{n-1}nP_{n-1}(x).$$

We have used our assumed relationship for P_n and P_{n-1}. Since this last equation has to agree with Eq. (15.18) we conclude that $P_{n+1}(-x) = (-1)^{n+1}P_{n+1}(x)$. This completes the proof.

15.1.10. $P_2(\cos\theta) = \dfrac{3\cos 2\theta + 1}{4}$.

15.1.11. Derivation of this formula is presented in the footnote referenced just after Eq. (15.22).

15.1.12. The solution is given in the text. There is a misprint in the answer: For $n = 2s+1$ a correct version of the second form of the answer is $(-1)^s(2s-1)!!/(2s+2)!!$.

15.1.13. The sum contains only powers x^i with $i < n$. If n is even, the smallest value of r is $(n/2)+1$; if n is odd, it is $(n-1)/2+1 = (n+1)/2$. In either case, $i_{\max} = 2n - 2r_{\min} < n$.

15.1.14. Expand $x^m = \displaystyle\sum_{l\le m} a_l P_l(x)$.

Now orthogonality gives $\displaystyle\int_{-1}^{1} x^m P_n\,dx = 0, \quad m < n.$

15.1.15. Following the directions in the exercise to use a Rodrigues formula and perform integrations by parts,

$$F_n = \int_{-1}^{1} x^n P_n\,dx = \frac{1}{2^n n!}\int_{-1}^{1} x^n \left(\frac{d}{dx}\right)^n (x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\left[x^n \left(\frac{d}{dx}\right)^{n-1}(x^2-1)^n\Big|_{-1}^{1} - \int_{-1}^{1} nx^{n-1}\left(\frac{d}{dx}\right)^{n-1}(x^2-1)^n\,dx\right].$$

The integrated terms vanish; a second integration by parts (for which the integrated terms also vanish) yields

$$F_n = +\frac{1}{2^n n!}\int_{-1}^{1} n(n-1)x^{n-2}\left(\frac{d}{dx}\right)^{n-2}(x^2-1)^n\,dx\,.$$

Further integrations by parts until the differentiation within the integral has been completely removed lead to

$$F_n = \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} n!\,(x^2-1)^n\, dx = 2^{-n}\left[2\int_0^1 (1-x^2)^n\, dx\right].$$

This integral (including the premultiplier "2") is of the form given in Eq. (13.50), where it is identified as the beta function $B(1/2, n+1)$, with value

$$B(1/2, n+1) = \frac{2^{n+1} n!}{(2n+1)!!}\,,$$

causing F to have the value claimed.

15.1.16. Introduce the Rodrigues formula for the Legendre polynomial and integrate by parts $2n$ times to remove the derivatives from the Rodrigues formula. The boundary terms vanish, so

$$I = \int_{-1}^{1} x^{2r} P_{2n}(x) = \frac{1}{2^{2n}(2n)!}\int_{-1}^{1} x^{2r}\left(\frac{d}{dx}\right)^{2n} (x^2-1)^{2n}\, dx$$

$$= \frac{(2r)!}{2^{2n}(2n)!(2r-2n)!}\int_{-1}^{1} x^{2r-2n}(1-x^2)^{2n}\, dx\,.$$

The integral is a beta function with value

$$\int_{-1}^{1} x^{2r-2n}(1-x^2)^{2n}\, dx = 2^{4n+1}(2n)!\,\frac{(r+n)!(2r-2n)!}{(r-n)!(2r+2n+1)!}\,.$$

Substituting this into I, we get after cancellation

$$I = \frac{2^{2n+1}(2r)!(r+n)!}{(r-n)!(2r+2n+1)!}\,.$$

15.1.17. Using integration by parts, orthogonality and mathematical induction as in Exercise 15.1.15, we obtain the solutions given in the text.

15.1.18. A continuous and differentiable function that is zero at both ends of an interval must have a derivative that is zero at some point within the interval. A continuous and differentiable function that is zero at both ends of an interval and at p intermediate points must have a derivative with $p+1$ zeros within the overall interval.

Applying these observations to $(x^2-1)^n$ with $n>0$, its first derivative must have one zero at a point intermediate to ± 1. If $n>1$, this derivative will also be zero at the endpoints, so the second derivative of $(x^2-1)^n$ must have two intermediate zeros. Continuing to the $(n-1)$th derivative (which will be zero at the endpoints and at $n-1$ intermediate points), we find that the nth derivative (that relevant for P_n) will have n zeros between -1 and $+1$.

15.2 Orthogonality

15.2.1. For $n > m$

$$\int_{-1}^{1} P_m P_n\,dx = \frac{1}{2^{m+n} m!n!}\int_{-1}^{1}\left(\frac{d}{dx}\right)^m (x^2-1)^m \left(\frac{d}{dx}\right)^n (x^2-1)^n\,dx = 0$$

because

$$\left(\frac{d}{dx}\right)^{n+m} (x^2-1)^m = 0$$

and, upon integrating by parts, the integrated terms vanish.

For $n = m$, the repeated integrations by parts yield

$$\frac{(-1)^n}{2^{2n} n!n!}\int_{-1}^{1}(x^2-1)^n \left(\frac{d}{dx}\right)^{2n}(x^2-1)^n\,dx = \frac{(2n)!}{2^{2n}n!n!}\int_{-1}^{1}(1-x^2)^n\,dx$$

$$= \frac{(2n)!}{2^{2n}n!n!}B(1/2, n+1) = \frac{(2n)!}{2^{2n}n!n!}\,\frac{2^{n+1}n!}{(2n+1)!!} = \frac{2}{2n+1}\,.$$

The beta function enters the solution because the integral can be identified as a case of Eq. (13.50).

15.2.2. The space spanned by $\{x^j\}$, $j = 0,\ 1,\ \cdots,\ n$ is identical with the space spanned by $\{P_j(x)\}$ for the same value of n. If the Gram-Schmidt process as described in this problem has produced functions $\varphi_j(x)$ that are respectively proportional to $P_j(x)$ through $j = n-1$, then the remainder of the space (known formally as the **orthogonal complement**) is one-dimensional and $\varphi_n(x)$ must be proportional to $P_n(x)$. To complete the proof, we need only observe that if $n = 0$ we have $P_0(x) = x^0$.

15.2.3. $\delta(x) = \displaystyle\sum_{l=0}^{\infty}(-1)^l\frac{(4l+1)(2l-1)!!}{2(2l)!!}P_{2l}(x), \quad -1 \le x \le 1.$

15.2.4. Insert the expansions to be verified, and then note that the expansion of $f(x)$ in Legendre polynomials takes the form

$$f(x) = \sum_n a_n P_n(x), \quad \text{where} \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx\,.$$

$$\int_{-1}^{1} f(x)\delta(1-x)dx = \sum_{n=0}^{\infty}\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx = \sum_{n=0}^{\infty} a_n$$

$$= \sum_{n=0}^{\infty} a_n P_n(1) = f(1).$$

$$\int_{-1}^{1} f(x)\delta(1+x)dx = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx = \sum_{n=0}^{\infty}(-1)^n a_n$$

$$= \sum_{n=0}^{\infty} a_n P_n(-1) = f(-1).$$

15.2.5. $(A^2 + 2A\cos\theta + 1)^{-1/2} = \dfrac{1}{A}\left(1 + \dfrac{2}{A}\cos\theta + \dfrac{1}{A^2}\right)^{-1/2}$

$$= \frac{1}{A}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(-\frac{1}{A}\right)^n;$$

$$\langle\cos\psi\rangle = \frac{1}{2A}\sum_{n=0}^{\infty}\left(-\frac{1}{A}\right)^n \int_0^{\pi} (A\cos\theta + 1)\, P_n(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{1}{2A}\left(2 - \frac{2}{3}\right) = \frac{2}{3A}.$$

15.2.6. This follows from Eq. (15.40).

15.2.7. 5If $f(x) = \sum_{n=0}^{\infty} a_n P_n$, then

$$\int_{-1}^{1} [f(x)]^2\, dx = \sum_{m,n=0}^{\infty} a_m a_n \int_{-1}^{1} P_m P_n\, dx = \sum_{n=0}^{\infty} \frac{2a_n^2}{2n+1}.$$

15.2.8. (a) Expand $f(x)$ in a Legendre series $\sum_{n=0}^{\infty} a_n P_n(x)$. This gives

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)\, dt = \frac{2n+1}{2}\left[\int_0^1 P_n(t)\, dt - \int_{-1}^{0} P_n(t)\right] dt\,.$$

For n even, $a_n = 0$ but for $n = 2s+1$

$$a_{2s+1} = (4s+3)\int_0^1 P_{2s+1}(t)dt = \frac{(4s+3)P_{2s}(0)}{2s+2} = \frac{(4s+3)(-1)^s(2s-1)!!}{(2s+2)!!}$$

See Exercise 15.1.12. We now use the result of Exercise 15.2.7 to obtain the answer given in the text.

(b) Using Stirling's asymptotic formula for the ratio

$$\frac{(2n-1)!!}{(2n+2)!!} = \frac{(2n-1)!}{2^{2n}(n-1)!(n+1)!} \approx \frac{1}{2n\sqrt{n\pi}}\,,$$

we see that the terms of the series approach zero as $1/n^2$.

(c) The sum of the first ten terms of the series is $1.943\cdots$, as compared to the exact value, 2.

15.2.9. Because P'_n is a polynomial of degree $n-1$ with the parity of $n-1$, the integral I of this problem will vanish unless $n-m$ is an odd integer. Choose n to be the larger of the two indices. Then use Eq. (15.26) to write

$$I = \int_{-1}^{1} x(1-x^2)P'_n P'_m \, dx = \int_{-1}^{1} xP'_m \left[(n+1)xP_n - (n+1)P_{n+1}\right] dx \,.$$

From this equation we note that in the integrand, P_n is multiplied by a polynomial in x of degree $m+1$ and that P_{n+1} is multiplied by one of degree m. The integral I will vanish by orthogonality if $n-m>1$, and if $n=m+1$ only the P_n term of the integrand will contribute to the integral. To evaluate the nonzero case of this integral we therefore set $m=n-1$ and continue by replacing P'_{n-1} using Eq. (15.24). With these changes we now have

$$I = \int_{-1}^{1} \left[P'_{n-2} + (n-1)P_{n-1}\right] (n+1)xP_n \, dx \,.$$

In the integrand of this expression the term arising from P'_{n-2} consists of a polynomial of degree $n-2$ multiplying P_n and therefore does not contribute to the integral. We are left with

$$I = (n^2-1) \int_{-1}^{1} P_{n-1} x P_n \, dx \,,$$

which we can evaluate invoking orthogonality, using Eq. (15.18) to express xP_n in terms of P_{n+1} and P_{n-1} and getting the normalization constant from Eq. (15.38). Thus,

$$I = (n^2-1) \int_{-1}^{1} P_{n-1} \frac{(n+1)P_{n+1} + nP_{n-1}}{2n+1} \, dx = \frac{n(n^2-1)}{2n+1} \, \frac{2}{2n-1} \,.$$

The answer given for $m=n+1$ can be obtained from that already given by the substitution $n \to n+1$.

15.2.10. Because $P_{2n}(x)$ is an even function of x,

$$P_{2n}(\cos[\pi-\theta]) = P_{2n}(-\cos\theta) = P_{2n}(\cos\theta) \,,$$

so data for $\theta > \pi/2$ add no new information.

15.2.11. With the conducting sphere centered at $r=0$, the previously uniform electric field E_0 (assumed to be in the z direction) will be the value for large r of $-\partial V/\partial z$, where V is the electrostatic potential. The charge density σ on the conducting sphere (of radius r_0) is given by $-\varepsilon_0 \, \partial V/\partial r$; and the induced dipole moment of the sphere is the coefficient of $\cos\theta/4\pi\varepsilon_0 r^2$ in the region external to the sphere.

The potential V must be a solution to Laplace's equation with symmetry about the z-axis, and it can therefore be described by an expansion of the form

$$V = a_0 + \sum_{n=1}^{\infty} \left[a_n r^n + \frac{b_n}{r^{n+1}} \right] P_n(\cos\theta) .$$

The value of a_0 is irrelevant; we set it to zero. At large r, V approaches $-E_0\, z = -E_0\, r\cos\theta = -E_0\, r\, P_1(\cos\theta)$, thereby showing that $a_1 = -E_0$ and all other $a_n = 0$ The b_n terms become negligible at large r so they are not determined by the large-r limit of V.

The condition that the sphere be an equipotential leads to the conclusion that all the b_n other than b_1 vanish, and that

$$a_1 r_0 + b_1/r_0^2 = 0, \qquad \text{so} \quad b_1 = E_0 r_0^3 .$$

The potential is therefore

$$V = E_0 \left(\frac{r_0^3}{r^2} - r \right) \cos\theta .$$

(a) The induced charge density is

$$\sigma = -\varepsilon_0 E_0 \cos\theta \frac{d}{dr} \left(\frac{r_0^3}{r^2} - r \right) = -\varepsilon_0 E_0 \cos\theta(-2-1) = 3\varepsilon_0 E_0 \cos\theta .$$

(b) The coefficient of $\cos\theta/4\pi\varepsilon_0 r^2$ is $4\pi\varepsilon_0 b_1 = 4\pi\varepsilon_0(E_0 r_0^3)$.

15.2.12. For the region $r < a$, the potential must be described by a series of the form

$$V(r,\theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta) .$$

The coefficients c_n can be determined by making them yield a potential that is correct on the polar axis $\theta = 0$, where the potential is easy to calculate. On the axis, $r = z$ and $P_n(\cos\theta) = P_n(1) = 1$, so

$$V(z,0) = \sum_{n=0}^{\infty} c_n z^n .$$

The point z on the polar axis is at the same distance $\sqrt{z^2 + a^2}$ from every point on the charged ring. Letting q be the total charge on the ring, by direct computation we find

$$V(z,0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{z^2+a^2}} = \frac{q}{4\pi\varepsilon_0 a} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{z}{a}\right)^{2n}$$

$$= \frac{q}{4\pi\varepsilon_0 a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} \left(\frac{z}{a}\right)^{2n} ,$$

where we have written $V(z,0)$ as its power-series expansion valid for $z < a$ and used Eq. (1.74) to obtain an explicit formula for the binomial coefficients. Comparing this expansion with that from the Legendre series, we note that $c_n = 0$ for all odd n, and

$$c_{2n} = \left(\frac{q}{4\pi\varepsilon_0}\right) \frac{(-1)^n(2n-1)!!}{(2n)!!\,a^{2n+1}}\,.$$

15.2.13. Compute $E_r = -(\partial\psi/\partial r)$, $E_\theta = (\sin\theta/r)(\partial\psi/\partial\cos\theta)$.

(a) For $r > a$,

$$E_r(r,\theta) = \frac{q}{4\pi\varepsilon_0 r^2}\sum_{s=0}^{\infty}(-1)^s\frac{(2s+1)!!}{(2s)!!}\left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta),$$

$$E_\theta(r,\theta) = \frac{q}{4\pi\varepsilon_0 r^2}\sum_{s=0}^{\infty}(-1)^s\frac{(2s-1)!!}{(2s)!!}\left(\frac{a}{r}\right)^{2s} \sin\theta\,\frac{dP_{2s}(\cos\theta)}{d\cos\theta}.$$

(b) For $r < a$,

$$E_r(r,\theta) = \frac{q}{4\pi\varepsilon_0 a^2}\sum_{s=1}^{\infty}(-1)^{s-1}\frac{(2s-1)!!}{(2s-2)!!}\left(\frac{r}{a}\right)^{2s-1} P_{2s}(\cos\theta)$$

$$E_\theta(r,\theta) = \frac{q}{4\pi\varepsilon_0 a^2}\sum_{s=1}^{\infty}(-1)^s\frac{(2s-1)!!}{(2s)!!}\left(\frac{r}{a}\right)^{2s-1} \sin\theta\,\frac{dP_{2s}(\cos\theta)}{d\cos\theta}.$$

The derivative of P_{2s} leads to associated Legendre functions, Section 15.4.

15.2.14. The answer is given in the text.

15.2.15. Writing the potential in a form valid for $r < a$,

$$V = \sum_{n=0}^{\infty} c_n \left(\frac{r}{a}\right)^n P_n(\cos\theta)\,,.$$

determine the coefficients c_n by requiring that at $r = a$ they yield $V = V_0$ for $\theta < \pi/2$ and $V = -V_0$ for $\theta > \pi/2$. Because the distribution of V is an odd function of $\cos\theta$, all c_{2n} must vanish, and c_{2n+1} can be found using the formulas for orthogonal expansions. Setting $\cos\theta = x$, we have

$$c_{2n+1} = \frac{2(2n+1)+1}{2}\left[V_0\int_0^1 P_{2n+1}(x)\,dx - V_0\int_{-1}^0 P_{2n+1}(x)\,dx\right]$$

$$= (4n+3)V_0\int_0^1 P_{2n+1}(x)\,dx\,.$$

This integral, which was the topic of Exercise 15.1.12, has the value $P_{2n}(0)/(2n+2)$, so

$$V = V_0 \sum_{n=0}^{\infty} \frac{4n+3}{2n+2} P_{2n}(0) \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta)\,.$$

The second form of the answer is obtained using Eq. (15.11) to introduce an explicit formula for $P_{2n}(0)$.

15.2.16. The answer is given in the text.

15.2.17. If $|f\rangle = \sum_n a'_n|\varphi_n\rangle$, then projecting yields $|\varphi_s\rangle\langle\varphi_s|f\rangle = a'_s|\varphi_s\rangle$.

15.2.18. The answer is given in AMS-55.

15.2.19. The answer is given in the text.

15.2.20. The answer is given in the text.

15.2.21. The answer is given in the text.

15.2.22. The answer is given in the text.

15.2.24. The expansion $e^{ikr\cos\gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\gamma)$

involves the angular solutions $P_n(\cos\gamma)$ and radial solutions $j_n(kr)$ of Helmholtz's PDE for which the plane wave on the left-hand side of the equation is also a solution. Setting $\cos\gamma = x$ and using orthogonality we have

$$\int_{-1}^{1} e^{ikrx} P_n(x)\,dx = \frac{2a_n}{2n+1}\, j_n(kr)\,,$$

which implies, setting $k=1$,

$$\left(\frac{d}{dr}\right)^n \int_{-1}^{1} e^{irx} P_n(x) dx = a_n \left(\frac{d}{dr}\right)^n j_n(r) = \int_{-1}^{1} (ix)^n e^{irx} P_n(x)\,dx.$$

Using Eq. (14.177)

$$\left(\frac{d}{dr}\right)^n j_n(r)\bigg|_{r=0} = \frac{n!}{(2n+1)!!}\,,$$

we have

$$\frac{2a_n}{2n+1}\,\frac{n!}{(2n+1)!!} = i^n \int_{-1}^{1} x^n P_n(x)\,dx = \frac{2i^n n!}{(2n+1)!!}.$$

This gives $a_n = i^n(2n+1)$.

15.2.25. Differentiating the Rayleigh equation with respect to kr, we get

$$i\cos\gamma\, e^{ikr\cos\gamma} = i\sum_{n=0}^{\infty} a_n j_n(kr)\cos\gamma P_n(\cos\gamma) = \sum_{n=0}^{\infty} a_n j_n'(kr) P_n(\cos\gamma)\,.$$

Now, replacing $\cos\gamma$ by x, insert Eq. (15.18),

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)\,,$$

in the left-hand member of the differentiated equation and Eq. (14.170),

$$(2n+1)j_n' = nj_{n-1} - (n+1)j_{n+1}\,,$$

in the right member, thereby reaching

$$i\sum_{n=0}^{\infty} a_n j_n(kr)\left[\frac{n}{2n+1}\,P_{n-1}(x) + \frac{n+1}{2n+1}\,P_{n+1}(x)\right]$$
$$= \sum_{n=0}^{\infty} a_n\left[\frac{n}{2n+1}\,j_{n-1}(kr) - \frac{n+1}{2n+1}\,j_{n+1}(kr)\right]P_n(x)\,.$$

Shifting the index definitions in the left-hand side of this equation, it can be written

$$i\sum_{n=0}^{\infty} a_{n+1}\left(\frac{n+1}{2n+3}\right) j_{n+1}(kr)P_n(x) + i\sum_{n=0}^{\infty} a_{n-1}\left(\frac{n}{2n-1}\right) j_{n-1}(kr)P_n(x)$$
$$= \sum_{n=0}^{\infty} a_n\left[\frac{n}{2n+1}\,j_{n-1}(kr) - \frac{n+1}{2n+1}\,j_{n+1}(kr)\right]P_n(x)\,.$$

Comparing coefficients of like terms yields the two equations

$$i\,a_{n+1}\,\frac{n+1}{2n+3} = -a_n\,\frac{n+1}{2n+1},$$
$$i\,a_{n-1}\,\frac{n}{2n-1} = a_n\,\frac{n}{2n+1}\,.$$

These equations are mutually consistent, indicative of the validity of the Rayleigh expansion, and correspond to the explicit formula

$$a_n = i^n(2n+1)a_0\,.$$

Setting $kr = 0$ in the Rayleigh formula, we complete the verification by noting that $j_n(0) = \delta_{n0}$ and that therefore $a_0 = 1$.

15.2.26. Starting from the solution to Exercise 15.2.24 (with $\cos\gamma$ renamed μ), multiply by $P_m(\mu)$ and integrate, thereby taking advantage of orthogonality. We get

$$\int_{-1}^{1} e^{ikr\mu}\, P_m(\mu)\, d\mu = \sum_{n=0}^{\infty} i^n(2n+1)j_n(kr)\int_{-1}^{1} P_n\mu)P_m(\mu\, d\mu$$

$$= i^m(2m+1)j_m(kr)\left(\frac{2}{2m+1}\right).$$

This equation rearranges into the required answer.

15.2.27. Write the integral I that is the starting point for this problem using the Rodrigues formula for the Legendre polynomial and with $t=\cos\theta$:

$$I = \frac{(-i)^n}{2}\int_{-1}^{1} e^{izt}\,\frac{1}{2^n\, n!}\left(\frac{d}{dt}\right)^n (t^2-1)^n\, dt\,.$$

Integrate by parts n times, thereby removing the indicated derivatives and differentiating the exponential. The boundary terms vanish, so we get

$$I = \frac{(-i)^n}{2}\,\frac{(-iz)^n}{2^n\, n!}\int_{-1}^{1} e^{izt}(t^2-1)^n\, dt\,.$$

Next we write $e^{izt}=\cos zt + i\sin zt$ and note that the part of the integral containing the sine vanishes due to its odd parity. Thus,

$$I = \frac{z^n}{2^{n+1}\, n!}\int_{-1}^{1}\cos zt(1-t^2)^n\, dt = \frac{z^n}{2^{n+1}\, n!}\int_0^{\pi}\cos(z\cos\theta)\sin^{2n+1}\theta\, d\theta\,.$$

This last form is Poisson's integral representation of $j_n(z)$, which was derived in Exercise 14.7.9.

15.3 Physical Interpretation, Generating Function

15.3.1. $V = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{|\mathbf{r}+a\hat{\mathbf{z}}|} - \dfrac{2}{r} + \dfrac{1}{|\mathbf{r}-a\hat{\mathbf{z}}|}\right)$

$$= \frac{q}{4\pi\varepsilon_0 r}\left(\left[1-\frac{2a}{r}\cos\theta+\left(\frac{a}{r}\right)^2\right]^{-1/2} - 2 + \left[1+\frac{2a}{r}\cos\theta+\left(\frac{a}{r}\right)^2\right]^{-1/2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{a}{r}\right)^l - 2 + \sum_{l=0}^{\infty}P_l(\cos\theta)\left(-\frac{a}{r}\right)^l\right]$$

$$= \frac{q}{2\pi\varepsilon_0 r}\sum_{l=1}^{\infty}P_{2l}\left(\frac{a}{r}\right)^{2l}.$$

15.3.2. $$V = \frac{q}{4\pi\varepsilon_0 r}\left(-\left[1+\frac{4a}{r}\cos\theta+\left(\frac{2a}{r}\right)^2\right]^{-1/2}+\left[1-\frac{4a}{r}\cos\theta+\left(\frac{2a}{r}\right)^2\right]^{-1/2}\right.$$
$$\left.+2\left[1+\frac{2a}{r}\cos\theta+\left(\frac{a}{r}\right)^2\right]^{-1/2}-2\left[1-\frac{a}{r}\cos\theta+\left(\frac{a}{r}\right)^2\right]^{-1/2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[-\sum_{l=0}^{\infty}P_l(\cos\theta)\left(-\frac{2a}{r}\right)^l+\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{2a}{r}\right)^l\right.$$
$$\left.+2\sum_{l=0}^{\infty}P_l(\cos\theta)\left(-\frac{a}{r}\right)^l-2\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{a}{r}\right)^l\right]$$
$$= \frac{q}{4\pi\varepsilon_0 r}\left[2\sum_{l=0}^{\infty}P_{2l+1}\left(\frac{2a}{r}\right)^{2l+1}-4\sum_{l=0}^{\infty}P_{2l+1}\left(\frac{a}{r}\right)^{2l+1}\right]$$
$$= \frac{q}{\pi\varepsilon_0 r}\sum_{l=1}^{\infty}P_{2l+1}(\cos\theta)\left(\frac{a}{r}\right)^{2l+1}(2^{2l}-1)\,.$$

15.3.3. $$V = \frac{q}{4\pi\varepsilon_0}\left(r^2-2ar\cos\theta+a^2\right)^{-1/2} = \frac{q}{4\pi\varepsilon_0 a}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{r}{a}\right)^l\,.$$

15.3.4. $$\mathbf{E} = -\boldsymbol{\nabla}\phi = -\frac{2aq}{4\pi\varepsilon_0}\boldsymbol{\nabla}\frac{\cos\theta}{r^2}.$$

Using the gradient in polar coordinates from Section 3.10 gives the result given in the text.

15.3.5. Using $\dfrac{\partial}{\partial z} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta}$,

we obtain

$$\frac{\partial}{\partial z}\frac{P_l}{r^{l+1}} = -\frac{l+1}{r^{l+2}}\cos\theta P_l + \frac{\sin^2\theta}{r^{l+2}}P_l'(\cos\theta) = -\frac{l+1}{r^{l+2}}P_{l+1}(\cos\theta),$$

in conjunction with Eq. (15.26),

$$(1-x^2)P_n'(x)-(n+1)xP_n(x) = -(n+1)P_{n+1}(x)\,.$$

15.3.6. A dipole at $z=a$ may be generated by opposite charges $\pm q$ at $z=a\pm\delta$ for $\delta\to 0$. Expanding the difference of Coulomb potentials in Legendre polynomials yields

$$\frac{q}{4\pi\varepsilon_0 r}\sum_{l=0}^{\infty}\left[\left(\frac{a+\delta}{r}\right)^l-\left(\frac{a+\delta}{r}\right)^l\right]P_l(\cos\theta).$$

For $\delta \to 0$, this becomes

$$\frac{2q\delta}{4\pi\varepsilon_0 r} \sum_{l=1}^{\infty} \frac{\partial}{\partial a} \left(\frac{a}{r}\right)^l P_l(\cos\theta) \longrightarrow \frac{p^{(1)}}{4\pi\varepsilon_0 ra} \sum_{l=1}^{\infty} l \left(\frac{a}{r}\right)^l P_l(\cos\theta),$$

with $2q\delta$ approaching the finite limit $p^{(1)}$. For small a the leading term,

$$\frac{p^{(1)}}{4\pi\varepsilon_0} \frac{P_1(\cos\theta)}{r^2}$$

cancels against the point dipole at the origin. The next term,

$$\frac{2ap^{(1)}}{4\pi\varepsilon_0} \frac{P_2(\cos\theta)}{r^3},$$

is the point quadrupole potential at the origin corresponding to the limit $a \to 0$ but with $2ap^{(1)}$ approaching the nonzero limit $p^{(2)}$.

15.3.7. $\varphi^{(3)} = \dfrac{48a^3q}{4\pi\varepsilon_0 r^4} P_3(\cos\theta) + \cdots$.

15.3.8. A charge q and its image charge q' are placed as shown in Fig. 15.3.8 of this manual. The respective distances between the charges and a point P on a sphere of radius r_0 are r_1 and r_2 when P is at the angle θ shown in the figure. The charge q is at a distance a from the center of the sphere, its image q' is at the distance $a' = r_0^2/a$ from the sphere center, and $q' = -qr_0/a$. Our task is to show that the potentials produced at P from the two charges add to zero.

What we need to prove is that $q/r_1 = -q'/r_2$, i.e., $q^2r_2^2 = q'^2r_1^2$, equivalent to $a^2r_2^2 = r_0^2r_1^2$.

Now use the law of cosines and the geometry of Fig. 15.3.8 to write

$$r_1^2 = r_0^2 + a^2 - 2r_0 a\cos\theta\,,$$

$$r_2^2 = r_0^2 + a'^2 - 2r_0 a'\cos\theta\,.$$

Then form $r_0^2r_1^2$ and $a^2r_2^2$, replacing a' by r_0^2/a. The results are

$$r_0^2r_1^2 = r_0^4 + a^2r_0^2 - 2r_0^3 a\cos\theta,$$

$$a^2r_2^2 = a^2r_0^2 + a^2\left(\frac{r_0^2}{a^2}\right) - 2a^2r_0\left(\frac{r_0^2}{a}\right)\cos\theta\,.$$

These two expressions are clearly equal, completing our proof.

15.4 Associated Legendre Equation

15.4.1. Assuming a power series solution $\mathcal{P} = \sum_j a_j x^{k+j}$ to Eq. (15.72), we obtain

$$(1-x^2)\sum_{j=0}^{\infty} a_j(k+j)(k+j-1)x^{k+j-2} - 2x(m+1)\sum_{j=0}^{\infty} a_j(k+j)x^{k+j-1}$$
$$+ [\lambda - m(m+1)]\sum_{j=0}^{\infty} a_j x^{k+j} = 0\,.$$

For this equation to be satisfied for all x the coefficient of each power of x must individually vanish. From the coefficient of x^{k-2} we obtain the indicial equation $k(k-1)a_0 = 0$. Since a_0 was assumed not to vanish, the indicial equation has solutions $k = 0$ and $k = 1$. Taking $k = 0$ and shifting the summation indices to exhibit equal powers of j, the above equation takes the form

$$\sum_{j=0}^{\infty}\Big[(j+2)(j+1)a_{j+2} - j(j-1)a_j - 2(m+1)ja_j + [\lambda - m(m+1)]a_j\Big]x^j = 0\,.$$

Since the coefficient of each x^j must vanish, we have the recurrence formula given as Eq. (15.73). Just as for the Legendre equation, this recurrence relation leads to an infinite series that diverges at $x = \pm 1$, so we must choose a value of λ that causes the right-hand side of Eq. (15.73) to vanish for some j. The value needed for λ is $(j+m)(j+m+1)$, as is easily verified. It is customary to identify $j + m = l$, so we can write $\lambda = l(l+1)$.

15.4.2. We start from the recurrence formula, Eq. (15.87), that connects associated Legendre functions of the same l but differing m:

$$P_l^{m+1}(x) + \frac{2mx}{\sqrt{1-x^2}}\,P_l^m + (l+m)(l-m+1)P_l^{m-1}(x) = 0\,.$$

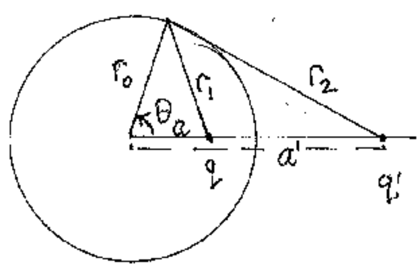

Figure 15.3.8. Image charge geometry.

Using expressions for P_2^2 and P_1^1 from Table 15.3, we properly reproduce $P_2(x) = (3x^2 - 1)/2$. Continuing, we get $P_2^{-1} = x\sqrt{1-x^2}/2$ and $P_2^{-2} = (1-x^2)/8$. These are related to the corresponding functions with $+m$ as required by Eq. (15.81).

15.4.3. Taking P_l^{-m} first (with $m \ge 0$), and taking note of the *Hint*, we will need to evaluate

$$\left(\frac{d}{dx}\right)^{l-m} (x-1)^l (x+1)^l \,.$$

Leibniz's formula gives this derivative as a sum (over j) of all the ways j of the differentiations can be applied to the first factor, with $l - m - j$ differentiations applied to the second factor. Appending to the Leibniz formula the remaining factors in P_l^{-m}, we have

$$P_l^{-m} = (x-1)^{-m/2}(-1)^{-m/2}(x+1)^{-m/2}$$

$$\times \sum_{j=0}^{l-m} \binom{l-m}{j} \frac{l!}{(l-j)!}\,(x-1)^{l-j}\,\frac{l!}{(m+j)!}\,(x-1)^{m+j}$$

$$= (-1)^{-m/2} \sum_{j=0}^{l-m} \frac{(l-m)!\,l!\,l!\,(x-1)^{l-j-m/2}(x+1)^{j+m/2}}{j!\,(l-m-j)!\,(l-j)!\,(m+j)!} \,.$$

We now apply a similar procedure to P_l^m. However, this time the total number of differentiations exceeds l, so the j summation must reflect the fact that neither factor can be differentiated more than l times. We have

$$P_l^m = (x-1)^{m/2}(-1)^{m/2}(x+1)^{m/2}$$

$$\times \sum_{j=m}^{l} \binom{l+m}{j} \frac{l!}{(l-j)!}\,(x-1)^{l-j}\,\frac{l!}{(j-m)!}\,(x-1)^{j-m}$$

$$= (-1)^{m/2} \sum_{j=m}^{l} \frac{(l+m)!\,l!\,l!\,(x-1)^{l-j+m/2}(x+1)^{j-m/2}}{j!\,(l+m-j)!\,(l-j)!\,(j-m)!} \,.$$

Next we replace the summation index j by $k + m$; this causes P_l^m to assume the form

$$P_l^m = (-1)^{m/2} \sum_{k=0}^{l-m} \frac{(l+m)!\,l!\,l!\,(x-1)^{l-k-m/2}(x+1)^{k+m/2}}{(m+k)!\,(l-k)!\,(l-m-k)!\,k!} \,.$$

Comparing the final forms of the expressions for P_l^m and P_l^{-m}, we confirm that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!}\,P_l^m(x) \,.$$

15.4.4. (a) From Eq. (15.88) with $x = 0$ and $m = 1$,

$$P^1_{l+1}(0) = -\frac{l+1}{l}\, P^1_{l-1}(0)\,.$$

Starting from $P^1_1(0) = -1$, we have $P^1_3(0) = +3/2$, $P^1_5(0) = -3\cdot 5/2\cdot 4$, or in general $P^1_{2l+1}(0) = (-1)^{l+1}(2l+1)!!/(2l)!!$. The functions $P^1_{2l}(0)$ vanish because they have odd parity.

(b) At $x = 0$, the generating function for $m = 1$ has the form

$$g_1(0,t) = \frac{-1}{(1+t^2)^{3/2}} = -\sum_{s=0}^{\infty}\binom{-3/2}{s} t^{2s} \sum_{s=0}^{\infty} P^1_{2s+1}(0)t^{2s}\,.$$

From this equation we see that $P^1_{2s+1}(0) = -\binom{-3/2}{s} = (-1)^{s+1}\,\dfrac{(2s+1)!!}{(2s)!!}$.

15.4.5. Using the generating function,

$$g_m(0,t) = \frac{(-1)^m(2m-1)!!}{(1+t^2)^{m+1/2}}$$

$$= (-1)^m(2m-1)!!\sum_{s=0}^{\infty}\binom{-m-1/2}{s} t^{2s} = \sum_{s=0}^{\infty} P^m_{2s+m}(0)t^{2s}\,,$$

we read out

$$P^m_{2s+m}(0) = (-1)^m(2m-1)!!\binom{-m-1/2}{s}$$

$$= (-1)^m(2m-1)!!\frac{(-1)^s(2s+2m-1)!!}{(2s)!!(2m-1)!!}\,,$$

which simplifies to the given answer for $l+m$ even. $P^m_l(0)$ vanishes due to its odd parity when $l+m$ is odd.

15.4.6. The field has only r and θ components, with $E_r = -\partial\psi/\partial r$ and $E_\theta = -(1/r)\partial\psi/\partial\theta$. Thus,

$$E_r = -\frac{2q}{4\pi\varepsilon_0}\left[P_1(\cos\theta)\,\frac{-2a}{r^3} + P_3(\cos\theta)\,\frac{-4a^3}{r^5} + \cdots\right],$$

in agreement with Eq. (15.130). Then,

$$E_\theta = -\frac{2q}{4\pi\varepsilon_0 r^2}\left[\frac{a}{r}\,\frac{dP_1(\cos\theta)}{d\theta} + \frac{a^3}{r^3}\,\frac{dP_3(\cos\theta)}{d\theta} + \cdots\right]$$

$$= -\frac{2q}{4\pi\varepsilon_0 r^2}\left[\frac{a}{r}\,P^1_1(\cos\theta) + \frac{a^3}{r^3}\,P^1_3(\cos\theta) + \cdots\right],$$

in agreement with Eq. (15.131). The second line of the above equation was obtained by substituting $dP_1/d\theta = -\sin\theta = P_1^1(\cos\theta)$ and $dP_3/d\theta = P_3^1(\cos\theta)$.

15.4.7. Using the Rodrigues formula and noting that the $2l$th derivative of $(x^2-1)^l$ is $(2l)!$, we have

$$P_l^l(x) = \frac{(-1)^l(1-x^2)^{l/2}(2l)!}{2^l l!} = (-1)^l(2l-1)!!(1-x^2)^{l/2}\,,$$

equivalent to the given answer when $x = \cos\theta$.

15.4.8. There are many ways to prove this formula. An approach that relies on the Rodrigues formulas and the fact that P_l^m satisfies the associated Legendre ODE is the following, in which $\mathcal{D}$ is used as shorthand for the operator d/dx. From the ODE, written in self-adjoint form, we have initially

$$\mathcal{D}(1-x^2)\mathcal{D}(1-x^2)^{m/2}\mathcal{D}^m P_l + \left[l(l+1) - \frac{m^2}{1-x^2}\right](1-x^2)^{m/2}\mathcal{D}^m P_l = 0\,.$$

Evaluating all the derivatives except those that apply only to P_n, we reach

$$\begin{aligned}&(1-x^2)^{m/2+1}\mathcal{D}^{m+2}P_l - m(1-x^2)^{m/2}\mathcal{D}^m P_l \\ &\quad - 2(m+1)x(1-x^2)^{m/2}\mathcal{D}^{m+1}P_l + m^2x^2(1-x^2)^{m/2-1}\mathcal{D}^m P_l \\ &\qquad + \left[l(l+1) - \frac{m^2}{1-x^2}\right](1-x^2)^{m/2}\mathcal{D}^m P_l = 0\,.\end{aligned}$$

Rewriting this equation using the Rodrigues formulas (remembering that they contain a factor $(-1)^m$), we get

$$P_l^{m+2} - mP_l^m + \frac{2(m+1)x}{(1-x^2)^{1/2}}P_l^{m+1} + \frac{m^2x^2}{1-x^2}P_l^m + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0\,.$$

Combining the P_l^m terms, we have

$$P_l^{m+2} + \frac{2(m+1)x}{(1-x^2)^{1/2}}P_l^{m+1} + \left[l(l+1) - m(m+1)\right]P_l^m = 0\,.$$

If m is replaced by $m-1$ we recover the formula as presented in the text.

15.4.9. The answer is given in the text.

15.4.10. The formula given here needs a minus sign to be consistent with the sign conventions used throughout the text.

15.4.11. These integrals can be rewritten using the relation

$$\frac{dP(\cos\theta)}{d\theta} = -\sin\theta\,\frac{dP(x)}{dx} = (1-x^2)^{1/2}\,\frac{dP(x)}{dx}\,,$$

where $x = \cos\theta$. Moreover, $\displaystyle\int_0^\pi \cdots \sin\theta\, d\theta \longrightarrow \int_{-1}^1 \cdots dx$.

(a) The first integral then assumes the form

$$I = \int_{-1}^1 \left[(1-x^2)\frac{dP_l^m(x)}{dx}\frac{dP_{l'}^m(x)}{dx} + \frac{m^2}{1-x^2}\,P_l^m(x)P_{l'}^m(x)\right] dx\,.$$

Integrate the first term by parts, differentiating $(1-x^2)\,dP_l^m/dx$ and integrating $dP_{l'}^m/dx$. The boundary terms vanish and we get

$$I = \int_{-1}^1 \left(-\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m(x)}{dx}\right] + \frac{m^2}{1-x^2}\,P_l^m(x)\right) P_{l'}^m\, dx\,.$$

The first term of the integrand is the differential-operator part of the associated Legendre ODE, so we replace it by the properly signed remainder of that ODE:

$$I = \int_{-1}^1 \left(\left[l(l+1) - \frac{m^2}{1-x^2}\right] P_l^m(x) + \frac{m^2}{1-x^2}\,P_l^m(x)\right) P_{l'}^m\, dx\,.$$

We now cancel the m^2 terms and identify what is left as an orthogonality integral, with the result

$$I = l(l+1)\left(\frac{2}{2l+1}\right)\frac{(l+m)!}{(l-m)!}\,\delta_{ll'}\,.$$

(b) This integral assumes the form

$$\int_{-1}^1 \left[P_l^1(x)\,\frac{dP_{l'}^1(x)}{dx} + P_{l'}^1(x)\,\frac{dP_l^1(x)}{dx}\right] dx = \int_{-1}^1 \frac{d}{dx}\left[P_l^1(x)P_{l'}^1(x)\right]\, dx\,.$$

This integrates to $P_l^1(x)P_{l'}^1(x)$, a quantity that vanishes at $x = \pm 1$, so the value of the integral is zero.

15.4.12. Rewrite this integral using P_l^m with $m = 1$:

$$I = \int_{-1}^1 x(1-x^2)P_n'(x)P_m'(x)\, dx = \int_{-1}^1 xP_n^1(x)P_m^1(x)\, dx\,.$$

Use Eq. (15.88) to bring I to the form

$$I = \int_{-1}^1 \left[\frac{n}{2n+1}P_{n+1}^1(x) + \frac{n+1}{2n+1}P_{n-1}^1(x)\right] P_m^1(x)\, dx\,.$$

If $m = n+1$ the first term is nonzero; with the help of Eq. (15.104) we get

$$\frac{n}{2n+1}\int_{-1}^1 P_{n+1}^1(x)P_{n+1}^1(x)\, dx = \frac{n}{2n+1}\,\frac{2}{2n+3}\,\frac{(n+2)!}{n!}\,.$$

The only other nonzero case is when $m = n-1$; a similar analysis confirms the answer given in the text.

15.4.13. $\dfrac{4}{3}\,\delta_{n,1}$.

15.4.14. Use the $\langle\ \rangle$ notation to denote a scalar product with unit weight, and write the self-adjoint ODE as

$$\mathcal{L}P_l^m = \left(l(l+1) - \frac{m^2}{1-x^2}\right) P_l^m\,.$$

We then write

$$\langle\, P_l^m |\, \mathcal{L}P_l^k \rangle = \left\langle P_l^m \,\middle|\, \left(l(l+1) - \frac{k^2}{1-x^2}\right) P_l^k \right\rangle,$$

$$\langle\, P_l^k |\, \mathcal{L}P_l^m \rangle = \left\langle P_l^k \,\middle|\, \left(l(l+1) - \frac{m^2}{1-x^2}\right) P_l^m \right\rangle.$$

Because $\mathcal{L}$ is self-adjoint and the Legendre functions are real, $\langle P_l^k|\mathcal{L}P_l^m\rangle = \langle P_l^m|\mathcal{L}P_l^k\rangle$, and the above equations can be rewritten

$$\langle\, P_l^m |\, \mathcal{L}P_l^k \rangle = \left\langle P_l^m \,\middle|\, l(l+1) - \frac{k^2}{1-x^2} \,\middle|\, P_l^k \right\rangle,$$

$$\langle\, P_l^m |\, \mathcal{L}P_l^k \rangle = \left\langle P_l^m \,\middle|\, l(l+1) - \frac{m^2}{1-x^2} \,\middle|\, P_l^k \right\rangle.$$

If we now subtract the second of these equations from the first, we get

$$(m^2 - k^2)\left\langle P_l^m \,\middle|\, \frac{1}{1-x^2} \,\middle|\, P_l^k \right\rangle = 0\,,$$

showing that functions with $m \neq k$ are orthogonal on $(-1,1)$ with weight $1/(1-x^2)$.

15.4.15. The answer is given in the text.

15.4.16. (a) $B_r(r,\theta) = \displaystyle\sum_{n=0}^{\infty} d_{2n+1}(2n+1)\,(2n+2)\frac{r^{2n}}{a^{2n+1}}\,P_{2n+1}(\cos\theta)$.

$$B_\theta(r,\theta) = -\sum_{n=0}^{\infty} d_{2n+1}(2n+2)\frac{r^{2n}}{a^{2n+1}}\,P_{2n+1}^1(\cos\theta)$$

with $d_{2n+1} = (-1)^n \dfrac{\mu_0 I}{2}\,\dfrac{(2n-1)!!}{(2n+2)!!}$.

15.4.17. The answer is given in the text.

15.4.18. The answer is given in the text.

15.4.19. Let ω be the angular velocity of the rotating sphere, σ its surface charge density, and a the radius of the sphere.

(a) $\mathbf{B}$ is in the z direction, with $B_z = \dfrac{2\mu_0\omega\sigma}{3}\,\dfrac{a^4}{3z^3}$.

(b) $A_\varphi(r,\theta) = \dfrac{\mu_0\omega\sigma}{3}\,\dfrac{a^4}{r^2}\,P_1^1(\cos\theta)\,,$

$$B_r(r,\theta) = \frac{2\mu_0\omega\sigma}{3}\,\frac{a^4}{r^3}\,P_1(\cos\theta)\,,$$

$$B_\theta(r,\theta) = \frac{\mu_0\omega\sigma}{3}\,\frac{a^4}{r^3}\,P_1^1(\cos\theta)\,.$$

15.4.20. The answer is given in the text.

15.5 Spherical Harmonics

15.5.1. Starting from the relations

$$P_l^m(-\cos\theta) = (-1)^{l+m}P_l^m(\cos\theta)\,, \qquad e^{im(\pi+\varphi)} = (-1)^m e^{im\varphi}\,,$$

we find $Y_l^m(\pi-\theta,\pi+\varphi) = (-1)^l Y_l^m(\theta,\varphi)$.

15.5.2. Since $Y_l^m(\theta,\varphi)$ contains the factor $\sin^m\theta$, $Y_l^m(0,\varphi) = 0$ for $m \neq 0$. Also

$$Y_l^0(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta) \quad \text{with} \quad P_l(1) = 1.$$

Hence $Y_l^m(0,\varphi) = \sqrt{\dfrac{2l+1}{4\pi}}\,\delta_{m0}$.

15.5.3. $Y_l^m\left(\dfrac{\pi}{2},0\right) = P_l^m(0)\left[\dfrac{2l+1}{4\pi}\,\dfrac{(l-m)!}{(l+m)!}\right]^{1/2}$.

Using Eq. (15.4.5) to substitute for $P_l^m(0)$, we get the value zero unless $l+m$ is even. For even $l+m$, we get

$$Y_l^m\left(\frac{\pi}{2},0\right) = (-1)^{(l+m)/2}\,\frac{(l+m-1)!!}{(l-m)!!}\left[\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2},$$

which can be brought to the form shown in the text as the answer to this problem.

15.5.4. Substituting the delta-function formula:

$$\int_{-\pi}^{\pi} f(\varphi_2)\delta(\varphi_1-\varphi_2)\,d\varphi_2 = \sum_{m=-\infty}^{\infty} e^{im\varphi_1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi_2)e^{-im\varphi_2}\,d\varphi_2$$

$$= \sum_m c_m e^{im\varphi_1} \quad \text{with} \quad c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi_2)e^{-im\varphi_2}\,d\varphi_2\,.$$

Since the c_m are the coefficients in the expansion of $f(\varphi)$ in the orthogonal functions $\chi_m(\varphi) = e^{im\varphi}$, this sum reduces to $f(\varphi_1)$.

15.5.5. We can demonstrate the validity of the closure relation by verifying that it gives correct results when used with an arbitrary function $f(\theta, \varphi)$. Multiplying the assumed relation by $f(\theta_1, \varphi_1)$ and integrating,

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left(\int \left[Y_l^m(\theta_1, \varphi_1) \right]^* f(\theta_1, \varphi_1) \, d\Omega_1 \right) Y_l^m(\theta_2, \varphi_2)$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_l^m(\theta_2 \varphi_2) = f(\theta_2, \varphi_2) \, ,$$

where we have observed that that the Ω_1 integral is that which defines the coeficients a_{lm} of the spherical harmonic expansion. Since this result obtains for arbitrary $f(\theta, \varphi)$ and is that required of the delta function, our verification is complete.

15.5.6. $Y_{04}^e = P_4(\cos\theta) = \dfrac{35\cos^2\theta - 30\cos^2\theta + 3}{8}$

$Y_{24}^e = P_4^2(\cos\theta)\cos 2\varphi = \dfrac{15(7\cos^2\theta - 1)\sin^2\theta\cos 2\varphi}{2}$

$Y_{44}^e = 105\sin^4\theta\cos 4\varphi$

15.5.7. The angular average of Y_l^m vanishes except for $l = m = 0$ by orthogonality of Y_l^m and $Y_0^0 = 1/\sqrt{4\pi}$. Thus, when $f(r, \theta, \varphi)$ is written as its Laplace expansion, only the term $a_{00}Y_0^0$ will contribute to the average. This implies

$$\langle f \rangle_{\text{sphere}} = \frac{1}{4\pi} \int f(r, \theta, \varphi) \, d\cos\theta \, d\varphi = \frac{a_{00}}{\sqrt{4\pi}} = a_{00} Y_0^0 \, .$$

But setting $r = 0$ in the Laplace series, we also have $f(0, 0, 0) = a_{00}Y_0^0$, so $\langle f \rangle_{\text{sphere}} = f(0, 0, 0)$.

15.6 Legendre Functions of the Second Kind

15.6.1. First, note that $Q_0(x)$ is odd; changing the sign of x interchanges the numerator and denominator of the logarithm defining Q_0. Next, note that $Q_1(x)$ is even. Then, the recurrence formula yielding Q_n for $n \geq 2$ causes $Q_{n+2}(x)$ to have the same parity as $Q_n(x)$. Thus, in general Q_n has a parity opposite to that of n.

15.6.2. First verify by explicit computation that the formulas give correct results for Q_0 and Q_1. For Q_0, the first summation contains a single term equal to x, while the second sum starts with $s = 1$. Together they yield

$$Q_0 = x + \sum_{s=1}^{\infty} \frac{x^{2s+1}}{2s+1}\,,$$

which is the power-series expansion of $Q_0(x)$. A similar check confirms $Q_1(x)$.

We next check that the formulas of parts (a) and (b) are consistent with the recurrence formula. We need to examine both

$$2nQ_{2n} - (4n-1)xQ_{2n-1} + (2n-1)Q_{2n-2} = 0 \quad \text{and}$$

$$(2n+1)Q_{2n+1} - (4n+1)xQ_{2n} + 2nQ_{2n-1} = 0\,.$$

Each check involves three parts: (1) Coefficients of powers of x that arise from the first (finite) summations only; (2) Coefficients that arise from the second (infinite) summation only; and (3) coefficients for values of s near n that do not fall into the previous two cases. The verification is straightforward but tedious. It may be useful to organize the terms in a fashion similar to that illustrated in the solution to Exercise 15.6.3.

15.6.3. We use the formula scaled as in part (b) so that a check of Q_0, Q_1, and the recurrence formula verifies both the general formula and its scaling. For Q_0, we get

$$Q_0(x) = \sum_{s=0}^{\infty} \frac{(2s)!x^{-2s-1}}{(2s)!!(2s+1)!!} = \sum_{s=0}^{\infty} \frac{x^{-2s-1}}{2s+1} = \frac{1}{2}\ln\left(\frac{1+x^{-1}}{1-x^{-1}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)\,,$$

which is the form for Q_0 which is standard for $|x| > 1$. The verification of $Q_1 = xQ_0 - 1$ is similar.

To complete the proof we now write the three terms of the recurrence formula as summations and equate the coefficients of individual powers of

x. We have

$$(2l+1)xQ_l = \sum_{s=0}^{\infty} \left[\frac{(l+2s)!\, x^{-2s-l}}{(2s)!!(2l+2s+1)!!} \right] (2l+1)\,,$$

$$lQ_{l-1} = l \sum_{s=0}^{\infty} \frac{(l+2s-1)!\, x^{-2s-l}}{(2s)!!(2l-2s-1)!!}$$

$$= \sum_{s=0}^{\infty} \left[\frac{(l+2s)!\, x^{-2s-l}}{(2s)!!(2l+2s+1)!!} \right] \frac{l(2l+2s+1)}{l+2s}\,,$$

$$(l+1)Q_{l+1} = (l+1) \sum_{s=0}^{\infty} \frac{(l+2s+1)!\, x^{-2s-l-2}}{(2s)!!(2l+2s+3)!!}\,.$$

To make the exponents correspond for the same index values we replace s by $s'-1$ in the Q_{l+1} summation; formally the s' sum then starts from $s'=1$ but the summand vanishes for $s'=0$ so we can without error use zero as the lower summation limit. With these changes, we have

$$(l+1)Q_{l+1} = \sum_{s'=0}^{\infty} \left[\frac{(l+2s')!\, x^{-2s'-l}}{(2s')!!(2l+2s'+1)!!} \right] \frac{(l+1)(2s')}{l+2s'}\,.$$

Forming now

$$(l+1)Q_{l+1} - (2l+1)xQ_l + lQ_{l-1}\,,$$

we find that the coefficient of each power of x evaluates to zero.

15.6.4. (a) Apply the recurrence formula, Eq. (15.18), to both P_n and Q_n:

$$n\left[P_n(x)Q_{n-1}(x)\right] = (2n-1)xP_{n-1}(x)Q_{n-1}(x) - (n-1)P_{n-2}(x)Q_{n-1}(x)\,,$$

$$n\left[P_{n-1}(x)Q_n(x)\right] = (2n-1)xP_{n-1}(x)Q_{n-1}(x) - (n-1)P_{n-1}(x)Q_{n-2}(x)\,.$$

Using the above, we find

$$n\left[P_n(x)Q_{n-1}(x) - P_{n-1}(x)Q_n(x)\right] = (n-1)\left[P_{n-1}(x)Q_{n-2}(x) - P_{n-2}(x)Q_{n-1}(x)\right]\,.$$

Applying repeatedly, we finally get to

$$n\left[P_n(x)Q_{n-1}(x) - P_{n-1}(x)Q_n(x)\right] = \left[P_1(x)Q_0(x) - P_0(x)Q_1(x)\right]\,.$$

(b) From $P_0 = 1$, $P_1 = x$, $Q_0 = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $Q_1 = \dfrac{x}{2}\ln\left(\dfrac{1+x}{1-x}\right) - 1$,

we directly verify $P_1Q_0 - P_0Q_1 = 1$.

16. Angular Momentum

16.1 Angular Momentum Operators

16.1.1. (a) and (b) are Eqs.(16.30) and (16.31) applied to $j = L, m = M$.

16.1.2. This problem is interpreted as requiring that the form given for $L_\pm$ be shown to convert Y_l^l as given by Eq. (15.137) into Y_l^m as given by that equation (and not by the use of general operator formulas).

We consider explicitly L_+. The procedure for L_- is similar. Applying L_+ to Y_l^m, noting that

$$\frac{dP_l^m(\cos\theta)}{d\theta} = -\sin\theta\,(P_l^m)'\,, \qquad i\cot\theta\,\frac{d}{d\varphi}\,e^{im\varphi} = -m\cot\theta\,e^{im\varphi}\,,$$

and using Eqs. (15.87) and (15.91), written here as

$$-m\cot\theta\,P_l^m = \frac{1}{2}P_l^{m+1} + \frac{(l+m)(l-m+1)}{2}\,P_l^{m-1}\,,$$

$$-\sin\theta\,(P_l^m)' = \frac{1}{2}P_l^{m+1} - \frac{(l+m)(l-m+1)}{2}\,P_l^{m-1}\,,$$

we obtain

$$L_+Y_l^m = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]\,P_l^m\,e^{im\varphi}$$

$$= \sqrt{(l-m)(l+m+1)}\,\sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m-1)!}{(l+m+1)!}}\,P_l^{m+1}e^{(m+1)\varphi}$$

$$= \sqrt{(l-m)(l+m+1)}\,Y_l^{m+1}\,.$$

The corresponding result for L_- is

$$L_-Y_l^m = \sqrt{(l+m)(l-m+1)}\,Y_l^{m-1}\,.$$

Continuing now,

(a) Apply L_- k times to Y_l^l. We get

$$(L_-)^kY_l^l = \Big[(2l)(2l-1)\cdots(2l-k+1)(1)(2)\cdots(k)\Big]^{1/2}Y_l^{l-k}$$

$$= \sqrt{\frac{(2l)!\,k!}{(2l-k)!}}\,Y_l^{l-k}\,.$$

Setting $l - k = m$, we recover the answer to part (a).

(b) Applying L_+ k times to Y_l^{-l}, we get by a similar procedure the expected result.

16.1.3. The equation of this exercise, written in Dirac notation, is

$$\left\langle Y_L^M \middle| L_- \, L_+ Y_L^M \right\rangle = \left\langle L_+ Y_L^M \middle| L_+ Y_L^M \right\rangle .$$

This equation is valid because L_x and L_y are Hermitian, so $(L_-)^\dagger = L_+$.

16.1.4. (a) Insert $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$ into the formula for $\mathbf{J}^2$ and expand, maintaining the operator order in all terms. After cancellation we reach $J_x^2 + J_y^2 + J_z^2$.

(b) One way to proceed is to start by building the operator L_+L_-:

$$\begin{aligned} L_+L_- &= -e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right] e^{-i\varphi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right] \\ &= -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \cot^2\theta\frac{\partial^2}{\partial\varphi^2} - i\frac{\partial}{\partial\varphi} . \end{aligned}$$

Then L_-L_+ is obtained from the above by changing the sign of i, so $(L_+L_- - L_-L_+)/2$ is simply the first three terms of the above expression. We also need an expression for $L_z^2 = -\partial^2/\partial\varphi^2$, and can then write

$$\begin{aligned} \mathbf{L}^2 &= -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \cot^2\theta\frac{\partial^2}{\partial\varphi^2} - \frac{\partial^2}{\partial\varphi^2} \\ &= -\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} , \end{aligned}$$

where the second line of the above equation results from application of a trigonometric identity.

Now we can apply this operator to the θ and φ dependence of the spherical harmonics of $l = 2$ from Table 15.4.

$$\begin{aligned} Y_2^2 : \quad \mathbf{L}^2 \sin^2\theta e^{2i\varphi} &= (-2\cos^2\theta + 2\sin^2\theta)e^{2i\varphi} - \cot\theta(2\sin\theta\cos\theta)e^{2i\varphi} + 4\,e^{2i\varphi} \\ &= (2\sin^2\theta - 4\cos^2\theta + 4)e^{2i\varphi} = 6\sin^2\theta\, e^{2i\varphi} \end{aligned}$$

$$\begin{aligned} Y_2^1 : \quad \mathbf{L}^2 \sin\theta\cos\theta e^{i\varphi} &= 4\sin\theta\cos\theta\, e^{i\varphi} - \cot\theta(\cos^2\theta - \sin^2\theta)e^{i\varphi} - \frac{\sin\theta\cos\theta}{\sin^2\theta}(-e^{i\varphi}) \\ &= \sin\theta\cos\theta\left[4 - \frac{\cos^2\theta}{\sin^2\theta} + 1 + \frac{1}{\sin^2\theta}\right] e^{i\varphi} = 6\sin\theta\cos\theta\, e^{i\varphi} \end{aligned}$$

$$\begin{aligned} Y_2^0 : \quad \mathbf{L}^2(3\cos^2\theta - 1) &= 6(\cos^2\theta - \sin^2\theta) + 6\cot\theta(\cos\theta\sin\theta) \\ &= 12\cos^2\theta - 6\sin^2\theta = 6(3\cos^2\theta - 1). \end{aligned}$$

The evaluations for Y_2^{-M} are similar to those for the corresponding $+M$.

16.1.5. This problem is interpreted as referring to general angular momentum eigenfunctions, with $L^2\psi_{LM} = L(L+1)\psi_{LM}$ and $L_z\psi_{LM} = M\psi_{LM}$, and with the effect of L_+ and L_- as shown in Eq. (16.25). From these equations we proceed as in Exercise 16.1.2, getting (in the present notation) the formulas of that exercise.

16.1.6. Use mathematical induction, assuming the equations of this exercise to be valid for $n-1$. We apply L_+ to the $(L_+)^{n-1}$ equation. Treating the two terms of L_+ individually,

$$\text{Term 1:}\;\; e^{i\varphi}\frac{\partial}{\partial\theta}(-1)^{n-1}\sin^{M+n-1}\theta\left(\frac{d}{d\cos\theta}\right)^{n-1}\sin^{-M}\theta\,\Theta_{LM}\,e^{i(M+n-1)\varphi}$$

$$= e^{i(M+n)\varphi}(-1)^{n-1}(M+n-1)\sin^{M+n-2}\theta\cos\theta\left(\frac{d}{d\cos\theta}\right)^{n-1}\sin^{-M}\theta\,\Theta_{LM}$$

$$+\,e^{i(M+n)\varphi}(-1)^{n-1}\sin^{M+n-1}\theta\,(-\sin\theta)\left(\frac{d}{d\cos\theta}\right)^{n}\sin^{-M}\theta\,\Theta_{LM}\,,$$

$$\text{Term 2:}\;\; -(M+n-1)e^{i(M+n)\varphi}(-1)^{n-1}\cot\theta\,\sin^{M+n-1}\theta$$

$$\times\left(\frac{d}{d\cos\theta}\right)^{n-1}\sin^{-M}\theta\,\Theta_{LM}$$

Term 2 cancels against the first part of Term 1, leaving

$$(-1)^n\sin^{M+n}\theta\left(\frac{d}{d\cos\theta}\right)^{n}\sin^{-M}\theta\,\Theta_{LM}\,e^{i(M+n)\varphi}\,.$$

This is the formula for $(L_+)^n$. Since the formula is clearly valid for $n=0$, the proof by mathematical induction is complete.

A similar proof can be developed for the $(L_-)^n$ equation.

16.1.7. Start from Y_0^0 and compare the result of applying $(L_+)^M$ and that obtained by applying $(L_-)^M$, using the formulas in Exercise 16.1.6.

$$(L_+)^M Y_L^0 = (-1)^M e^{iM\varphi}\sin^M\theta\,\frac{d^M\,Y_L^0}{d\cos\theta^M}\,,$$

$$(L_-)^M Y_L^0 = e^{-iM\varphi}\sin^M\theta\,\frac{d^M\,Y_L^0}{d\cos\theta^M}\,.$$

These repeated applications of $L_\pm$ produce Y_L^M and Y_L^{-M} with equal scale factors, as can be seen by examination of the formulas in Exercise 16.1.1. We also note that the expressions differ only by a factor $(-1)^M$ and the sign of the exponent in $e^{\pm iM\varphi}$, as required to obtain the desired answer.

16.1.8. (a) $L_+Y_1^0 = e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]\sqrt{\frac{3}{4\pi}}\,\cos\theta = \sqrt{\frac{3}{4\pi}}\,e^{i\varphi}(-\sin\theta)$

$$= \sqrt{2}\,Y_1^1\,.$$

(b) $L_-Y_1^0 = -e^{-i\varphi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right]\sqrt{\frac{3}{4\pi}}\,\cos\theta = -\sqrt{\frac{3}{4\pi}}\,e^{-i\varphi}(-\sin\theta)$

$$= \sqrt{2}\,Y_1^{-1}\,.$$

16.2 Angular Momentum Coupling

16.2.1. We apply $J_+ = J_{1+} + J_{2+}$ to the state $|(j_1j_2)JM\rangle$ using the $\langle J_+\rangle$ matrix element that we know. This yields

$$J_+|JM\rangle = [(J-M)(J+M+1)]^{1/2}\sum_{m_1m_2} C(j_1j_2J|m_1,m_2,M+1)|j_1m_1\rangle|j_2m_2\rangle$$

$$= \sum_{m_1m_2} C(j_1j_2J|m_1m_2M)\Big\{[(j_1-m_1)(j_1+m_1+1)]^{1/2}|j_1,m_1+1\rangle|j_2m_2\rangle$$

$$+ [(j_2-m_2)(j_2+m_2+1)]^{1/2}|j_1m_1\rangle|j_2,m_2+1\rangle\Big\},$$

from which we project with $|j_1,m_1+1\rangle|j_2m_2\rangle$ to get

$$[(J-M)(J+M+1)]^{1/2}C(j_1j_2J|m_1{+}1,m_2,M{+}1)$$

$$= C(j_1j_2J|m_1m_2M)[(j_1-m_1)(j_1+m_1+1)]^{1/2}$$

$$+ C(j_1j_2J|m_1{+}1,m_2{-}1,M)[(j_2-m_2+1)(j_2+m_2)]^{1/2}.$$

To avoid introducing additional indexing symbols we have used the fact that the projection corresponds, for the first right-hand term, to reducing the sum to a single term in which m_1 and m_2 have the index values shown in the sum, while for the second term, the sum becomes a single term with m_1 replaced by m_1+1 and with m_2 replaced by m_2-1.

Using this recursion we check that $C(111|000) = 0$, $C(111|101) = 1/\sqrt{2}$, etc. Projecting $|j_1m_1\rangle|j_2,m_2+1\rangle$ gives a similar recursion. Using $\mathbf{J}^2 \to J(J{+}1) = j_1(j_1{+}1){+}j_2(j_2{+}1){+}2m_1m_2{+}J_{1+}J_{2-}{+}J_{1-}J_{2+}$ in conjunction with the matrix elements of $J_{i\pm}$ given in Eqs. (16.30) and (16.31) yields a third recursion relation.

16.2.2. The formula of this exercise is that for angular momentum coupling, so the result must be a spherical tensor of rank J.

16.2.3. Denote the p states p_+, p_0, p_- and the spin states α, β. Note that $j_+ = l_+ + s_+$ and $j_- = l_- + s_-$, so we will need, applying Eqs. (16.30) and

(16.31):

$$l_+p_+ = 0, \quad l_+p_0 = \sqrt{2}\,p_+, \quad l_+p_- = \sqrt{2}\,p_0,$$

$$l_-p_+ = \sqrt{2}\,p_0, \quad l_-p_0 = \sqrt{2}\,p_-, \quad l_-p_- = 0,$$

$$s_+\alpha = 0, \quad s_+\beta = \alpha, \quad s_-\alpha = \beta, \quad s_-\beta = 0.$$

The m_l, m_s state of maximum $m = m_l + m_s$ is $p_+\alpha$, so this is a state with $j = m = 3/2$. Applying j_-, we form the $(\frac{3}{2}, m)$ states of smaller m:

$$j = \tfrac{3}{2},\ m = \tfrac{1}{2}: \quad (l_- + s_-)p_+\alpha = \sqrt{2}\,p_0\alpha + p_+\beta,$$

$$j = \tfrac{3}{2},\ m = -\tfrac{1}{2}: \quad (l_- + s_-)(\sqrt{2}\,p_0\alpha + p_+\beta) = 2p_-\alpha + \sqrt{2}\,p_0\beta + \sqrt{2}\,p_0\beta$$
$$= 2(p_-\alpha + \sqrt{2}\,p_0\beta),$$

$$j = \tfrac{3}{2},\ m = -\tfrac{3}{2}: \quad 2(p_-\beta + 2p_-\beta) = 6p_-\beta\,.$$

These states are not yet normalized. Each state with $j = \frac{1}{2}$ will be orthogonal to the $j = \frac{3}{2}$ state of the same m, so we have (also unnormalized)

$$j = \tfrac{1}{2},\ m = \tfrac{1}{2}: \quad p_0\alpha - \sqrt{2}\,p_+\beta, \qquad j = \tfrac{1}{2},\ m = -\tfrac{1}{2}: \quad p_0\beta - \sqrt{2}\,p_-\alpha\,.$$

Normalizing all these states and using the conventional labeling:

$$^2p_{3/2}: \quad m = \tfrac{3}{2}, \quad p_+\alpha, \qquad m = \tfrac{1}{2}, \quad \sqrt{2/3}\,p_0\alpha + \sqrt{1/3}\,p_+\beta,$$
$$m = -\tfrac{1}{2}, \quad \sqrt{2/3}\,p_0\beta + \sqrt{1/3}\,p_-\alpha, \qquad m = -\tfrac{3}{2}, \quad p_-\beta,$$

$$^2p_{1/2}: \quad m = \tfrac{1}{2}, \quad \sqrt{1/3}\,p_0\alpha - \sqrt{2/3}\,p_+\beta,$$
$$m = -\tfrac{1}{2}, \quad \sqrt{1/3}\,p_0\beta - \sqrt{2/3}\,p_-\alpha\,.$$

16.2.4. The p states are designated as in Exercise 16.2.3 and the spin states are given the (nonstandard) designations $[\frac{3}{2}]$, $[\frac{1}{2}]$, etc. The definitions of $j_\pm$ are as in Exercise 16.2.3 and the behavior of $l_\pm$ is as listed there; we need the additional relationships

$$s_-[\tfrac{3}{2}] = \sqrt{3}\,[\tfrac{1}{2}], \quad s_-[\tfrac{1}{2}] = 2[-\tfrac{1}{2}], \quad s_-[-\tfrac{1}{2}] = \sqrt{3}\,[-\tfrac{3}{2}], \quad s_-[-\tfrac{3}{2}] = 0.$$

The largest possible value of $m = m_l + m_s$ is $\frac{5}{2}$, so that the (j, m) state of largest j and m is $(\frac{5}{2},\frac{5}{2}) = p_+[\frac{3}{2}]$. Applying j_-, we reach

$$j=\tfrac{5}{2},\ m=\tfrac{3}{2},\qquad \sqrt{2}\,p_0[\tfrac{3}{2}] + \sqrt{3}\,p_+[\tfrac{1}{2}],$$

$$j=\tfrac{5}{2},\ m=\tfrac{1}{2},\qquad 2p_-[\tfrac{3}{2}] + 2\sqrt{6}\,p_0[\tfrac{1}{2}] + 2\sqrt{3}\,p_+[-\tfrac{1}{2}],$$

$$j=\tfrac{5}{2},\ m=-\tfrac{1}{2},\qquad 6\sqrt{3}\,p_-[\tfrac{1}{2}] + 6\sqrt{6}\,p_0[-\tfrac{1}{2}] + 6p_+[-\tfrac{3}{2}]$$

$$j=\tfrac{5}{2},\ m=-\tfrac{3}{2},\qquad 24\sqrt{2}\,p_0[-\tfrac{3}{2}] + 24\sqrt{3}\,p_o[-\tfrac{1}{2}]$$

$$j=\tfrac{5}{2},\ m=-\tfrac{5}{2},\qquad 120p_-[-\tfrac{3}{2}].$$

We next need to identify the state $(\frac{3}{2},\frac{3}{2})$ as that of $m=\frac{3}{2}$ that is orthogonal to $\sqrt{2}\,p_0[\frac{3}{2} + \sqrt{3}\,p_+[\frac{1}{2}]$. It is $(\frac{3}{2},\frac{3}{2}) = \sqrt{3}\,p_0[\frac{3}{2}] - \sqrt{2}\,p_+[\frac{1}{2}]$. Applying j_-, we now generate the states of $j=\frac{3}{2}$ of smaller m:

$$j=\tfrac{3}{2},\ m=\tfrac{1}{2},\qquad \sqrt{6}\,p_-[\tfrac{3}{2}] + p_0[\tfrac{1}{2}] - 2\sqrt{2}\,p_+[-\tfrac{1}{2}],$$

$$j=\tfrac{3}{2},\ m=-\tfrac{1}{2},\qquad 4\sqrt{2}\,p_-[\tfrac{1}{2}] - 2p_0[-\tfrac{1}{2}] - 2\sqrt{6}\,p_+[-\tfrac{3}{2}],$$

$$j=\tfrac{3}{2},\ m=-\tfrac{3}{2},\qquad 6\sqrt{2}\,p_-[-\tfrac{1}{2}] - 6\sqrt{3}\,p_0[-\tfrac{3}{2}].$$

Finally, we construct the states with $j=\frac{1}{2}$, starting from the state $(\frac{1}{2},\frac{1}{2})$ that is orthogonal to both $(\frac{5}{2},\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2})$. Writing this state as $p_-[\frac{3}{2}] + b\,p_0[\frac{1}{2} + c\,p_+[-\frac{1}{2}]$, the orthogonality requirement is

$$\left\langle p_-[\tfrac{3}{2}] + b\,p_0[\tfrac{1}{2}] + c\,p_+[-\tfrac{1}{2}] \middle| p_-[\tfrac{3}{2}] + \sqrt{6}\,p_0[\tfrac{1}{2}] + \sqrt{3}\,p_+[-\tfrac{1}{2}] \right\rangle =$$

$$1 + \sqrt{6}\,b + \sqrt{3}\,c = 0,$$

$$\left\langle p_-[\tfrac{3}{2}] + b\,p_0[\tfrac{1}{2}] + c\,p_+[-\tfrac{1}{2}] \middle| \sqrt{6}\,p_-[\tfrac{3}{2}] + p_0[\tfrac{1}{2}] - 2\sqrt{2}\,p_+[-\tfrac{1}{2}] \right\rangle =$$

$$\sqrt{6} + b - 2\sqrt{2}\,c = 0\,.$$

These equations have solution $b = -\sqrt{2/3}\,$, $c = \sqrt{1/3}\,$, so

$$(\tfrac{1}{2},\tfrac{1}{2}) = p_-[\tfrac{3}{2}] - \sqrt{2/3}\,p_0[\tfrac{1}{2}] + \sqrt{1/3}\,p_+[-\tfrac{1}{2}]\,.$$

Operating on this with j_-, we get

$$(\tfrac{1}{2},-\tfrac{1}{2}) = p_+[-\tfrac{3}{2}] - \sqrt{2/3}\,p_0[-\tfrac{1}{2}] + \sqrt{1/3}\,p_0[\tfrac{1}{2}]\,.$$

Collecting the above results and normalizing, our final result is:

$$
\begin{aligned}
{}^4p_{5/2}: \quad & m=\tfrac{5}{2}, \quad p_+[\tfrac{3}{2}],\\
& m=\tfrac{3}{2}, \quad \sqrt{2/5}\,p_0[\tfrac{3}{2}]+\sqrt{3/5}\,p_+[\tfrac{1}{2}],\\
& m=\tfrac{1}{2}, \quad \sqrt{1/10}\,p_-[\tfrac{3}{2}]+\sqrt{3/5}\,p_0[\tfrac{1}{2}]+\sqrt{3/10}\,p_+[-\tfrac{1}{2}],\\
& m=-\tfrac{1}{2}, \quad \sqrt{3/10}\,p_-[\tfrac{1}{2}]+\sqrt{3/5}\,p_0[-\tfrac{1}{2}]+\sqrt{1/10}\,p_+[-\tfrac{3}{2}],\\
& m=-\tfrac{3}{2}, \quad \sqrt{2/5}\,p_0[-\tfrac{3}{2}]+\sqrt{3/5}\,p_-[-\tfrac{1}{2}],\\
& m=-\tfrac{5}{2}, \quad p_-[-\tfrac{3}{2}],\\
{}^4p_{3/2}: \quad & m=\tfrac{3}{2}, \quad \sqrt{3/5}\,p_0[\tfrac{3}{2}]-\sqrt{2/5}p_+[\tfrac{1}{2}],\\
& m=\tfrac{1}{2}, \quad \sqrt{2/5}\,p_-[\tfrac{3}{2}]+\sqrt{1/15}\,p_0[\tfrac{1}{2}]-\sqrt{8/15}\,p_+[-\tfrac{1}{2}],\\
& m=-\tfrac{1}{2}, \quad \sqrt{8/15}\,p_-[\tfrac{1}{2}]-\sqrt{1/15}\,p_0[-\tfrac{1}{2}]-\sqrt{2/5}\,p_+[-\tfrac{2}{5}],\\
& m=-\tfrac{3}{2}, \quad \sqrt{2/5}\,p_-[-\tfrac{1}{2}]-\sqrt{3/5}\,p_0[-\tfrac{3}{2}],\\
{}^4p_{1/2}: \quad & m=\tfrac{1}{2}, \quad \sqrt{1/2}\,p_-[\tfrac{3}{2}]-\sqrt{1/3}\,p_0[\tfrac{1}{2}]+\sqrt{1/6}\,p_+[-\tfrac{1}{2}],\\
& m=-\tfrac{1}{2}, \quad \sqrt{1/2}\,p_+[-\tfrac{3}{2}]-\sqrt{1/3}\,p_0[-\tfrac{1}{2}]+\sqrt{1/6}\,p_-[\tfrac{1}{2}].
\end{aligned}
$$

16.2.5. (a) Writing the three-particle states in the m_p, m_n, m_e basis, putting the states of the same $M = m_p + m_n + m_e$ in the same row, we construct the diagram

$$
\begin{array}{llll}
M=3/2 & p_\alpha n_\alpha e_\alpha & & \\
M=1/2 & p_\alpha n_\alpha e_\beta & p_\alpha n_\beta e_\alpha & p_\beta n_\alpha e_\alpha \\
M=-1/2 & p_\alpha n_\beta e_\beta & p_\beta n_\alpha e_\beta & p_\beta n_\beta e_\alpha \\
M=-3/2 & p_\beta n_\beta e_\beta & &
\end{array}
$$

The diagram shows that there is one set of states with $J = 3/2$ (a **quartet**) and two additional sets of states with $J = 1/2$ (**doublets**).

(b) Finding the states reached by coupling the proton and neutron spins is the same as the problem discussed in Exercise 16.2.2; writing the results of that exercise in a notation reflecting the current situation, we have a nuclear triplet,

$$(pn)_+ = p_\alpha n_\alpha, \qquad (pn)_0 = \sqrt{1/2}\,(p_\alpha n_\beta + p_\beta n_\alpha), \qquad (pn)_- = p_\beta n_\beta,$$

and a nuclear singlet: $(pn)_s = \sqrt{1/2}(p_\alpha n_\beta - p_\beta n_\alpha)$. Coupling the triplet nuclear state with the electron spin produces a quartet state and a doublet state. Finding the quartet and doublet states is the same problem as

Exercise 16.2.3; the results are

$$(\tfrac{3}{2},\tfrac{3}{2}):\quad (pn)_+e_\alpha,\qquad (\tfrac{3}{2},\tfrac{1}{2}):\quad \sqrt{2/3}\,(pn)_0e_\alpha+\sqrt{1/3}\,(pn)_+e_\beta,$$

$$(\tfrac{3}{2},-\tfrac{1}{2}):\quad \sqrt{2/3}\,(pn)_0e_\beta+\sqrt{1/3}\,(pn)_-e_\alpha,\qquad (\tfrac{3}{2},-\tfrac{3}{2}):\quad (pn)_-e_\beta\,,$$

$$(\tfrac{1}{2},\tfrac{1}{2}):\quad \sqrt{1/3}\,(pn)_0e_\alpha-\sqrt{2/3}\,(pn)_+e_\beta,$$

$$(\tfrac{1}{2},-\tfrac{1}{2}):\quad \sqrt{1/3}\,(pn)_0e_\beta-\sqrt{2/3}\,(pn)_-e_\alpha\,.$$

The singlet nuclear state has no spin angular momentum, so coupling it with the electron spin produces the doublet (which we mark with a prime)

$$(\tfrac{1}{2},\tfrac{1}{2})':\quad (pn)_se_\alpha,\qquad (\tfrac{1}{2},-\tfrac{1}{2})':\quad (pn)_se_\beta.$$

All these states can now be expanded into weighted sums of the (m_p, m_n, m_e) states. We get

$$(\tfrac{3}{2},\tfrac{3}{2}):\quad p_\alpha n_\alpha e_\alpha$$

$$(\tfrac{3}{2},\tfrac{1}{2}):\quad \sqrt{1/3}\,(p_\alpha n_\beta e_\alpha+p_\beta n_\alpha e_\alpha+p_\alpha n_\alpha e_\beta)$$

$$(\tfrac{3}{2},-\tfrac{1}{2}):\quad \sqrt{1/3}\,(p_\alpha n_\beta e_\beta+p_\beta n_\alpha e_\beta+p_\beta n_\beta e_\alpha)$$

$$(\tfrac{3}{2},-\tfrac{3}{2}):\quad p_\beta n_\beta e_\beta$$

$$(\tfrac{1}{2},\tfrac{1}{2}):\quad \sqrt{1/6}\,(p_\alpha n_\beta e_\alpha+p_\beta n_\alpha e_\alpha)-\sqrt{2/3}\,p_\alpha n_\alpha e_\beta$$

$$(\tfrac{1}{2},-\tfrac{1}{2}):\quad \sqrt{1/6}\,(p_\alpha n_\beta e_\beta+p_\beta n_\alpha e_\beta)-\sqrt{2/3}\,p_\beta n_\beta e_\alpha$$

$$(\tfrac{1}{2},\tfrac{1}{2})':\quad \sqrt{1/2}\,(p_\alpha n_\beta e_\alpha-p_\beta n_\alpha e_\alpha)$$

$$(\tfrac{1}{2},-\tfrac{1}{2})':\quad \sqrt{1/2}\,(p_\alpha n_\beta e_\beta-p_\beta n_\alpha e_\beta\,.$$

(c) This part of the exercise is the same as part (b) except that the roles of the neutron and electron are interchanged. Thus, replace the (pn) states by corresponding (pe) states and change e_α and e_β to n_α and n_β. After these changes, one can expand the resulting states. The quartet states are the same in both coupling schemes, but the doublets found here, marked with multiple primes,

$$(\tfrac{1}{2},\tfrac{1}{2})'':\quad \sqrt{1/6}\,(p_\alpha n_\alpha e_\beta+p_\beta n_\alpha e_\alpha)-\sqrt{2/3}\,p_\alpha n_\beta e_\alpha$$

$$(\tfrac{1}{2},-\tfrac{1}{2})'':\quad \sqrt{1/6}\,(p_\alpha n_\beta e_\beta+p_\beta n_\beta e_\alpha)-\sqrt{2/3}\,p_\beta n_\alpha e_\beta$$

$$(\tfrac{1}{2},\tfrac{1}{2})''':\quad \sqrt{1/2}\,(p_\alpha n_\alpha e_\beta-p_\beta n_\alpha e_\alpha)$$

$$(\tfrac{1}{2},-\tfrac{1}{2})''':\quad \sqrt{1/2}\,(p_\alpha n_\beta e_\beta-p_\beta n_\beta e_\alpha\,.$$

differ from those of part (b).

(d) The difference in the two coupling schemes is only in the doublet states; those associated with the triplet nuclear state of part (b) are the states of most interest since they correspond to observed states of the deuterium atom. One way to show that these doublets span the same space is to write those from part (c) as linear combinations of those from part (b). We illustrate for $M = +\frac{1}{2}$:

$$(\tfrac{1}{2},\tfrac{1}{2})'' = -\frac{1}{2}\,(\tfrac{1}{2},\tfrac{1}{2}) - \frac{\sqrt{3}}{2}\,(\tfrac{1}{2},\tfrac{1}{2})', \qquad (\tfrac{1}{2},\tfrac{1}{2})''' = -\frac{\sqrt{3}}{2}(\tfrac{1}{2},\tfrac{1}{2}) + \frac{1}{2}(\tfrac{1}{2},\tfrac{1}{2})'\,.$$

16.3 Spherical Tensors

16.3.2. The set of Y_l^m for given l is closed under rotation. This must be the case, because L^2, a scalar, has a value that is independent of the orientation of the coordinate system.

16.3.3. This problem is a special case of Eq. (16.53) with $\Omega_1 = \Omega_2$. The quantity A is shown in Eq. (16.55) to be rotationally invariant (i.e., spherically symmetric).

16.3.4. This formula is derived at Eq. (16.66).

16.3.5. Note that this problem uses a hybrid unit system and is neither in MKS units (it lacks the factor $1/4\pi\varepsilon$) nor in the hartree unit system common in electronic structure computations (having a value of e not set to unity).

To solve the problem, use the Laplace expansion, Eq. (16.66), and note that the only term that survives upon integration is that with $l = 0$, for which the product of the two spherical harmonics becomes $1/4\pi$. Therefore, after performing the angular integrations for both $\mathbf{r}_1$ and $\mathbf{r}_2$, what remains is

$$E = \left(\frac{3e}{4\pi R^3}\right)^2 (4\pi)^2 \int_0^R r_1^2 dr_1 \int_0^R r_2^2 dr_2 \frac{1}{r_>}\,,$$

where $r_>$ is the larger of r_1 and r_2. To evaluate the integral, break it into the two regions $r_1 < r_2$ and $r_1 > r_2$:

$$E = \frac{9e^2}{R^6}\left[\int_0^R r_1\,dr_1 \int_0^{r_1} r_2^2\,dr_2 + \int_0^R r_2\,dr_2 \int_0^{r_2} r_1^2\,dr_1\right]$$

$$= \frac{18e^2}{R^6}\int_0^R r_1\,dr_1 \int_0^{r_1} r_2^2\,dr_2\,.$$

Note that we have written the integrations in a way that shows them to be equal. The double integral evaluates to $R^5/15$, so $E = 6e^2/5R$.

16.3.6. Note that this problem uses a hybrid unit system and is neither in MKS units (it lacks the factor $1/4\pi\varepsilon$) nor in the hartree unit system common in electronic structure computations (having a value of e not set to unity).

This problem proceeds in a way similar to Exercise 16.3.5. It is convenient to change the integration variables to $x = 2Zr_1/a_0$ and $y = 2Zr_2/a_0$, and the integral we seek can then be written

$$V = \frac{e^2 Z}{a_0} \int_0^\infty x^2 e^{-x}\, dx \int_x^\infty y e^{-y}\, dy\,.$$

The y integral has the value $(x+1)e^{-x}$, so we have

$$V = \frac{e^2 Z}{a_0} \int_0^\infty e^{-2x}(x^3 + x^2)\, dx = \frac{e^2 Z}{a_0}\left[\frac{3!}{2^4} + \frac{2!}{2^3}\right] = \frac{5e^2 Z}{8a_0}\,.$$

16.3.7. This problem is presented in MKS units, with q denoting the electron charge. As in Exercises 16.3.5 and 16.3.6, only the spherically symmetric term of the Laplace expansion survives the integration, and we have

$$V(\mathbf{r}_1) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{\pi a_0^3}\left[4\pi \int_0^{r_1} \frac{r_2^2}{r_1}\, e^{-2r_2/a_0}\, dr_2 + 4\pi \int_{r_1}^\infty r_2\, e^{-2r_2/a_0}\, dr_2\right].$$

Changing the integration variable to $x = 2r_2/a_0$, we get

$$V(\mathbf{r}_1) = \frac{q}{4\pi\varepsilon_0}\,\frac{4}{a_0^3}\left[\left(\frac{a_0}{2}\right)^3 \frac{1}{r_1}\int_0^{2r_1/a_0} x^2 e^{-x}\, dx + \left(\frac{a_0}{2}\right)^2 \int_{2r_1/a_0}^\infty x\, e^{-x}\, dx\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{2r_1}\gamma(3, 2r_1/a_0) + \frac{1}{a_0}\Gamma(2, 2r_1/a_0)\right].$$

We have identified the integrals as incomplete gamma functions. Alternatively, because the first arguments of these functions are integers, they can be written entirely in terms of elementary functions.

16.3.8.

$$\psi(r_1) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{24}\left[\frac{1}{r_1}\gamma\left(5, \frac{r_1}{a_0}\right) + \frac{1}{a_0}\Gamma\left(4, \frac{r_1}{a_0}\right)\right]\sqrt{4\pi}\, Y_0^0(\theta_1, \varphi_1)$$

$$-\frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{120}\left[\frac{a_0^2}{r_1^3}\gamma\left(7, \frac{r_1}{a_0}\right) + \frac{r_1^2}{a_0^3}\Gamma\left(2, \frac{r_1}{a_0}\right)\right]\sqrt{\frac{4\pi}{5}}\, Y_2^0(\theta_1, \varphi_1).$$

16.3.9. (a) Place this alleged delta function into the integral of an arbitrary $f(\Omega_2) = f(\theta_2, \varphi_2)$:

$$\sum_{l=0}^\infty \sum_{m=-l}^{l} Y_l^m(\Omega_1) \int Y_l^m(\Omega_2)^* f(\Omega_2)\, d\Omega_2 = \sum_{l=0}^\infty \sum_{m=-l}^{l} Y_l^m(\Omega_1)\, c_{lm}\,,$$

where we now recognize c_{lm} as the coefficient of Y_l^m in the Laplace expansion of $f(\Omega)$. The summations will therefore yield $Y_l^m(\Omega_1)$; this result is the defining property of the delta function.

(b) Use Eq. (16.57) to replace the summation over m with its equivalent in terms of $P_l(\cos\chi)$, where χ is the angle between Ω_1, i.e., (θ_1, φ_1), and Ω_2, i.e., (θ_2, φ_2).

16.3.10. (a) Replace Y_0^0 by $1/\sqrt{4\pi}$; there remains only a normalization integral.

(b) Replace Y_1^0 by $\sqrt{3/4\pi}\,\cos\theta$ and use Eq. (15.150), after which the integral reduces due to the orthonormality of the remaining factors.

(c) and (d) Replace Y_1^1 by $-\sqrt{3/8\pi}\,\sin\theta\, e^{i\varphi}$ and use Eq. (15.151).

16.3.11. (a) Use the recurrence formula for the Legendre polynomials, Eq. (15.18), to convert xP_L into a linear combination of P_{L+1} and P_{L-1}. Then invoke orthogonality and use the normalization of P_N, given in Eq. (15.38).

(b) Apply the recurrence formula twice, converting x^2P_L into a linear combination of P_L and $P_{L\pm 2}$ and simplify using orthogonality and normalization.

16.3.12. (a) From the recurrence formula, Eq. (15.18), xP_n is a linear combination of only P_{n-1} and P_{n+1}, so these are the only nonvanishing coefficients in its expansion (which is unique).

16.4 Vector Spherical Harmonics

16.4.1. The answers are given in the text.

16.4.2. The parity is the same as that of the spherical harmonic, whose parity is controlled by its value of the lower index L (which is the second index of the vector spherical harmonic). Therefore the parity of $\mathbf{Y}_{LLM}$ is $(-1)^L$, while $\mathbf{Y}_{L,L+1,M}$ and $\mathbf{Y}_{L,L-1,M}$ have parity $(-1)^{L+1}$.

16.4.3. The orthogonality integral is

$$\int \mathbf{Y}^*_{JLM_J}\cdot\mathbf{Y}_{J'L'M'_J}\,d\Omega =$$

$$\sum_m\sum_{\mu,\mu'} C(L1J|\mu m M_J)C(L'1J'|\mu' m M'_J)\int (Y_L^\mu)^* Y_{L'}^{\mu'}\,d\Omega$$

$$= \sum_{m,\mu} C(L1J|\mu m M_J)C(L1J'|\mu m M'_J)\delta_{LL'}\,.$$

At least one of the Clebsch-Gordan coefficients is zero unless $M_J = M_{J'}$, we must have $\mu = M_j - m$, and the sum of the squares of the Clebsch-

Gordan coefficients is $\delta_{JJ'}$. Thus,

$$\int \mathbf{Y}^*_{JLM_J} \cdot \mathbf{Y}_{J'L'M'_J}\, d\Omega = \delta_{JJ'}\, \delta_{LL'}\, \delta_{M_J M_{J'}} \,.$$

16.4.5. Write

$$\sum_M \mathbf{Y}^*_{LLM} \mathbf{Y}_{LLM} = \sum_{\mu m M} C(L1L|\mu m M)^2 (Y_L^\mu)^* Y_L^\mu \,.$$

We have used the condition $\mu + m = M$ on the Clebsch-Gordan coefficients to set equal the upper indices of the Y_L^μ. The sum over M of the squares of the Clebsch-Gordan coefficients (for any fixed μ) yields unity, and we are left with

$$\sum_M \mathbf{Y}^*_{LLM} \mathbf{Y}_{LLM} = \sum_\mu (Y_L^\mu)^* Y_L^\mu \;= \frac{2L+1}{4\pi} \,,$$

where we have recognized the final sum over μ as a special case of Eq. (16.57) for $\Omega_1 = \Omega_2$ (thereby making it a case for which $\cos\gamma = 1$).

16.4.6. The cross product is orthogonal to both its vectors, so the dot product in the integrand yields zero.

17. Group Theory

17.1 Introduction to Group Theory

17.1.1. From Table 17.2 of the text, we see that for any element R of the Vierergruppe, $R^2 = I$, so we cannot reach all elements of the group as powers of any one element; that means the group is not cyclic. Moreover, the table shows that for any elements R and R', $RR' = R'R$, so the group is abelian.

17.1.2. (a) The group operation here is the successive application of two permutations. (1) Successive permutations result in a permutation, so the set of permutations is closed under the group operation. (2) We get the same result if we carry out three permutations in order, i.e., as $c * (b * a)$, or first identify the permutation $(c*b)$ and make the successive permutations corresponding to $(c * b) * a$. (3) The identity I is the "permutation" that does not change the ordering of the objects; $I * a$ and $a * I$ are both A. (4) The inverse of a permutation is clearly also a permutation; it restores the original order.

(b) Name the permutations I (no reordering), P_{12} (interchange objects 1 and 2, leaving 3 in its original position), P_{13}, P_{23}, P_{123} (move 1 to the original position of 2, 2 to the original position of 3, 3 to the original position of 1), so $P_{123}abc = cab$, and P_{321} (move 3 to the original position of 2, 2 to the original position of 1, 1 to the original position of 3), so $P_{321}abc = bca$. Then build the group multiplication table shown in Table 17.1 of this manual by noting that, for example, $P_{12}P_{13}abc = P_{12}cba = bca = P_{321}abc$, so $P_{12}P_{13} = P_{321}$. From this and other successive operations, we can identify all elements of the group multiplication table.

(c) The multiplication table for this group S_3 can be put into one-one correspondence with that in Table 17.1 of the text for D_3, if we identify P_{12} with C_2, P_{13} with C_2', P_{23} with C_2'', P_{321} with C_3, and P_{123} with C_3'. Since the correspondence is one-to-one, the groups are isomorphic. The

Table 17.1 Multiplication table, permutations of three objects

S_3	I	P_{12}	P_{13}	P_{23}	P_{123}	P_{321}
I	I	P_{12}	P_{13}	P_{23}	P_{123}	P_{321}
P_{12}	P_{12}	I	P_{321}	P_{123}	P_{23}	P_{13}
P_{13}	P_{13}	P_{123}	I	P_{321}	P_{12}	P_{23}
P_{23}	P_{23}	P_{321}	P_{123}	I	P_{13}	P_{12}
P_{123}	P_{123}	P_{13}	P_{23}	P_{12}	P_{321}	I
P_{321}	P_{321}	P_{23}	P_{12}	P_{13}	I	P_{123}

correspondence, however, is not unique; we could have associated any one of P_{12}, P_{13}, or P_{23} with C_2, with possible changes in the identifications of P_{123} and P_{321}.

17.1.3. Suppose that b and c are different elements of our group, and that $ab = ac$. Multiply each side of this equation on the left by a^{-1} (which must exist, since a is a member of a group). We then get $b = c$, which contradicts our initial assumption. Thus, all the elements aI, a^2, ab, $\ldots$ must be distinct, and therefore constitute a permutation of the group elements.

17.1.4. We need to verify that the set of xh_ix^{-1} satisfy the group conditions.

1. The product of two elements is a group member:

$$(xh_ix^{-1})(xh_jx^{-1}) = xh_ih_jx^{-1} = xh_kx^{-1}\,,$$

where $h_k = h_ih_j$ is a member oif H.

2. Since all the multiplications involved are associative, the insertion of x and x^{-1} cannot affect the associativity.

3. xIx^{-1} is the unit element of our conjugate subgroup.

4. Direct multiplication shows that $xh_i^{-1}x^{-1}$ is the inverse of xh_ix^{-1}.

17.1.5. (a) Because our group is abelian, $ab = c$ implies $ba = c$. Taking the inverse of this last expression, we have $a^{-1}b^{-1} = c^{-1}$, showing that the set of inverses of the group elements forms a group isomorphic with the original group.

(b) If the two groups are isomorphic with $a \leftrightarrow a^{-1}$, then $ab = c$ implies $a^{-1}b^{-1} = c^{-1}$; taking the inverse of this equation, we reach $ba = c$, showing that the original group is abelian. The isomorphism further implies that the group of inverses must also be abelian.

17.1.6. (a) A 90° positive rotation of the cubic crystal (about the z-axis of a right-handed system) causes $x \longrightarrow y$ and $y \longrightarrow -x$. Applying this transformation to the atom at (la, ma, na) causes it to now be located at $(-ma, la, na)$, which is a point that contained another atom before the rotation.

(b) The positive x-axis of the crystal can be placed in any one of six orientations (the $\pm x$, $\pm y$, or $\pm z$ directions of a fixed-space axial system). Then the positive y-axis of the crystal can be placed (applying a rotation) in any one of the four directions perpendicular to the direction chosen for the crystal x-axis. Finally, the positive z-axis of the crystal can be placed in one of the two directions perpendicular to the crystal x- and y-axes (applying a reflection if necessary). Thus, the number of possible orientations is $6 \cdot 4 \cdot 2 = 48$, and that will be the dimension of the cubic point group.

17.1.7. (a) Point A of the hexagonal tiling is a three-fold axis (rotation $2\pi/3$, which is $120°$). The 2×2 matrix transforming a point (x, y) by this rotation is

$$C_3 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

(b) Point B us a six-fold axis (rotation $\pi/3$, which is $60°$). Its 2×2 matrix is

$$C_6 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

17.2 Representation of Groups

17.2.1. Because the $\mathsf{U}^K(a)$ are members of a representation, $\mathsf{U}^K(aa^{-1})$ must be equal to $\mathsf{U}^K(a)\mathsf{U}^K(a^{-1})$. But $\mathsf{U}^K(aa^{-1}) = \mathsf{U}^K(I)$, which is a unit matrix, which in turn means that $\mathsf{U}^K(a^{-1}) = \left[\mathsf{U}^K(a)\right]^{-1}$.

17.2.2. Since the matrix of I is a unit matrix, the group operations Ia, aI, Ib, bI, Ic, and cI are consistent with the corresponding matrix products. Since $\mathsf{A} = -\mathsf{I}$, it is easy to verify that $\mathsf{AB} = \mathsf{BA} = -\mathsf{B} = \mathsf{C}$ and that $\mathsf{AC} = \mathsf{CA} = -\mathsf{C} = \mathsf{B}$, all of which are consistent with the group multiplication table. By matrix multiplication we also find that $\mathsf{BC} = \mathsf{CB} = \mathsf{A}$, completing our check of the representation.

17.2.3. The matrix $\mathsf{U} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ transforms B and C to

$$\mathsf{B}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathsf{C}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The transformation does not change I or A. Because all four representation matrices are now block diagonal (the two blocks are 1×1), the elements U_{11} of the transformed matrices form a one-dimensional representation, as do the U_{22} elements.

17.2.4. (a) Apply the determinant product theorem: If $AB = C$, then also $\det(A)\det(B) = \det(C)$.

(b) From the representation

$$U(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U(C_3) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$U(C_3^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \quad U(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$U(\sigma') = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad U(\sigma'') = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

we take determinants: $U(I) = U(C_3) = U(C_3^2) = 1$, $U(\sigma) = U(\sigma') = U(\sigma'') = -1$.

17.2.5. To verify the representation property we note first that repeated application of r, starting from $r^0 = 1$, produces $1, r, r^2$, etc. It is also necessary that $r^n = 1$. From the form given for r, we have $r^n = \exp(2\pi i s)$ which, because s is an integer, evaluates to unity. Finally, note that $r^m r^k = r^{m+k}$ and that if $m + k \geq n$ we can divide the result by unity (in the form r^n) to obtain a element $r^{m'}$ with m' in the range $(0, n-1)$.

17.2.6. This group, D_4, has eight elements, which we denote I, C_4, C_2, C_4^3, σ_x, σ_y, σ_d, and $\sigma_{d'}$. I is the identity, C_4, $C_2 = C_4^2$, and C_4^3 are rotations, σ_x is the reflection $x \leftrightarrow -x$, σ_y ia $y \leftrightarrow -y$, σ_d is a reflection about the line $x = y$, while σ_d' is a reflection about the line $x = -y$. The group multiplication table is

D_4	I	C_4	C_2	C_4^3	σ_x	σ_y	σ_d	σ_d'
I	I	C_4	C_2	C_4^3	σ_x	σ_y	σ_d	σ_d'
C_4	C_4	C_2	C_4^3	I	σ_d'	σ_d	σ_x	σ_y
C_2	C_2	C_4^3	I	C_4	σ_y	σ_x	σ_d'	σ_d
C_4^3	C_4^3	I	C_4	C_2	σ_d	σ_d'	σ_y	σ_x
σ_x	σ_x	σ_d	σ_y	σ_d'	I	C_2	C_4	C_4^3
σ_y	σ_y	σ_d'	σ_x	σ_d	C_2	I	C_4^3	C_4
σ_d	σ_d	σ_y	σ_d'	σ_x	C_4^3	C_4	I	C_2
σ_d'	σ_d'	σ_x	σ_d	σ_y	C_4	C_4^3	C_2	I

A 2×2 irreducible representation of this group is

$$U(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U(C_4) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U(C_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$U(C_4^3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U(\sigma_x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U(\sigma_y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$U(\sigma_d) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(\sigma_d') = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

17.3 Symmetry and Physics

17.3.1. Referring to Fig. 17.2 of the text, the basis functions of the present problem are p orbitals centered at each vertex of the triangle and oriented perpendicular to the plane of the triangle. Since all these orbitals are of opposite sign above and below this plane, they cannot be used to construct a representation that does not change sign when the triangle is turned over.

Denoting the individual orbitals φ_i, we can form the linear combination $\psi_0 = \varphi_1 + \varphi_2 + \varphi_3$, and it is clear that ψ_1 will be invariant with respect to I and the rotations C_3 and C_3', but will change sign under the operations C_2, C_2', and C_2'', and will therefore form a basis for the one-dimensional representation we call A_2. There are no other linear combinations of the basis functions that remain invariant (except for a possible sign change) under all the operations of D_3, so the two members of our basis space independent of ψ_0 must be associated with an irreducible representation of dimension 2. Example 17.3.2 provides a clue as to how to proceed. We try $\psi_1 = (\varphi_1 - \varphi_3)/\sqrt{2}$, $\psi_2 = (-\varphi_1 + 2\varphi_2 - \varphi_3)/\sqrt{6}$, and with Fig. 17.2 at hand, develop relationships such as the following, which may be hard to find but are easily checked:

$$C_3\psi_1 = (\varphi_2 - \varphi_1)/\sqrt{2} = -\frac{1}{2}\varphi_1 + \frac{\sqrt{3}}{2}\varphi_2,$$

$$C_3\psi_2 = (-\varphi_2 + 2\varphi_3 - \varphi_1) = -\frac{\sqrt{3}}{2}\varphi_1 - \frac{1}{2}\varphi_2.$$

These relationships can be expressed as the matrix equation

$$\mathsf{U}(C_3)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \text{with} \quad \mathsf{U}(C_3) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Some of the operations yield diagonal matrices, such as

$$I\psi_1 = \psi_1, \quad I\psi_2 = \psi_2, C_2'\psi_1 = -\psi_1, \quad C_2'\psi_2 = -\psi_2,$$

with

$$\mathsf{U}(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{U}(C_2') = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the representation of D_3 called E and discussed in Example 17.2.1.

17.4 Discrete Groups

17.4.1. (a) Since the Vierergruppe is abelian, all expressions of the form gag^{-1} reduce to a, so every element is in a class by itself. There are therefore four classes.

(b) There are four irreducible representations. The only way the dimensionality theorem, Eq. (17.10), can be satisfied is for each irreducible representation to be 1×1.

(c) There will be one irreducible representation whose characters are all unity (usually called A_1). The orthogonality theorem and the fact that all elements are their own inverses means that all other representations must have two classes (here, elements) with character $+1$ and two with character -1. There are three distinct ways to assign the minus signs (which cannot be assigned to I), leading to the following character table:

	I	A	B	C
A_1	1	1	1	1
A_2	1	1	-1	-1
A_3	1	-1	1	-1
A_4	1	-1	-1	1

17.4.2. (a) Denoting the permutation in the problem statement P_{123}, indicating that it is the cycle $1 \to 2 \to 3 \to 1$, and writing P_{ij} for the permutation that interchanges i and j, the six members of the D_3 group have the 3×3 representation

$$U(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U(P_{123}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$U(P_{132}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U(P_{12}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U(P_{13}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U(P_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) The reduction is to the direct sum of a 1×1 and a 2×2 representation. The reduction is accomplished by applying the same transformation $VU(P)V^T$ to all members of the representation, where V is a unitary matrix and all the $VU(P)V^T$ have the same block structure. A matrix V that accomplishes the reduction is

$$V = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

We check by transforming some of the group elements:

$$VU(P_{12})V^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix},$$

$$VU(P_{13})V^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$VU(P_{123})V^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

All the transformed matrices have the form of a 1×1 block followed by a 2×2 block. The block of dimension 1 is the A_1 irreducible representation; that of dimension 2 is the E representation. Note that different transformations can produce this representation with different assignments of the matrices to group operations of the same class, and possibly even with the rows and columns in a permuted order. Comparing with the solution to Exercise 17.2.4, we see that the V we have used makes P_{123} correspond to C_3^2, with P_{13} corresponding to σ.

17.4.3. (a) There are five classes, hence five irreducible representations. The group has eight elements, so the squares of the dimensions of the irreducible representations must add to 8. The only possibility is to have four representations of dimension 1 and one of dimension 2.

(b) Since the characters of C_4 for the representations of dimension 1 are all ±1, those of C_2 can only be $+1$. The characters for I must all be equal to the dimension of the representation, and one representation, usually called A_1, must have all its characters equal to 1. Then, applying Eq. (17.9), the character of C_2 for representation E must be -2 and the remaining characters for representation E must all vanish. This leaves us needing to assign one $+1$ and two (-1)s to the remaining items in the columns $2C_4$, $2C_2'$, and $2C_2''$; this can be done in three different ways. The result is the following character table (in which two of the representations are, for reasons we do not discuss, conventionally labeled B_1 and B_2. (The last row is not part of the table but is relevant to Exercise 17.4.4.)

D_4	I	C_2	$2C_4$	$2C_2'$	$2C_2''$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	1	-1	1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0
Γ	8	0	0	4	0

17.4.4. We start by finding the characters of the representation spanned by the eight functions; we do so by determining how many of the eight functions remain unchanged when we perform an operation on a member of each class. For the class containing I all eight basis functions remain unchanged, so $\Gamma(I) = 8$. Taking next C_4, under which $x \to y$ and $y \to -x$, no basis function remains unchanged, so $\Gamma(C_4) = 0$. For C_2 under which $x \to -x$ and $x \to -y$, no function remains unchanged, so $\Gamma(C_2) = 0$. For C_2', x^2y, $-x^2y$, y^3, and $-y^3$ remain unchanged, so $\Gamma(C_2') = 4$. And for C_2'', no function remains unchanged, so $\Gamma(C_2'') = 0$. These data are appended to the D_4 character table generated in the solution to Exercise 17.4.3.

Applying Eq. (17.12) successively for each irreducible representation, we find $\Gamma = 2A_1 \oplus 2B_1 \oplus 2E$. This result can be checked by adding (with appropriate coefficients) entries from the character table.

17.4.5. (a) Denote the four orbitals x, y, $-x$, $-y$, where these names indicate the locations of their centers. We identify the rows and columns of our representation matrices as corresponding to the basis in the above-given order. Thus, the matrix of C_4, in which $x \to y$, $y \to -x$, $-x \to -y$, $-y \to x$, is

$$C_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix of I, in which all basis functions remain unchanged, and that of C_2, in which $x \leftrightarrow -x$ and $y \leftrightarrow -y$, are

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Next we take the member of the σ_v class that corresponds to reflection about the y axis. For this operation, $y \to y$ and $-y \to -y$, but $x \leftrightarrow -x$. And finally, we take the σ_d operation that is a reflection over a plane containing the line $x = y$. For this operation, $x \leftrightarrow y$ and $-x \leftrightarrow -y$. These operations are represented by

$$\sigma_v = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_d = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Taking the traces of the above matrices, we have $\Gamma(I) = 4$, $\Gamma(C_4) = \Gamma(C_2) = \Gamma(\sigma_d) = 0$, and $\Gamma(\sigma_v) = 2$.

(b) Applying Eq. (17.12), using the above Γ and the characters in the table included in this exercise, we find $\Gamma = A_1 \oplus B_1 \oplus E$.

(c) At arbitrary scale, $\psi_{A_1} = (x) + (y) + (-x) + (-y)$. We note that ψ_{B_1} must change sign on application of C_4 and σ_d, but not σ_v, and therefore will have the form $\psi_{B_1} = (x) - (y) + (-x) - (-y)$. The two-dimensional space orthogonal to these basis functions will be spanned by a basis for E. One choice is $\chi_1 = (x) + (y) - (-x) - (-y)$, $\chi_2 = (x) - (y) - (-x) + (-y)$.

17.4.6. (a) Label the basis functions located at $(x, 0)$ $p_x(x)$, $p_y(x)$, where the subscript denotes the orientation of the p orbital and the parenthesized argument denotes its location. We assume that the positive lobe of the p_x orbital points toward positive x, no matter where it is located. Similar remarks apply to the p_y orbitals. Note that if $p_x(x)$ is subjected to the C_4 operation that brings it to location y, p_x is thereby transformed to p_y, so $C_4 p_x(x) = p_y(y)$. As another example, the operation σ_v that is reflection about the y axis converts $p_x(x)$ into $-p_x(-x)$ but changes $p_y(x)$ into $p_y(-x)$. this same σ_v converts $p_x(y)$ into $-p_x(y)$ but leaves $p_y(y)$ unchanged. Using the principles illustrated by the above, and designating the rows and columns of the representation matrices to correspond, in order, to $p_x(x)$, $p_y(x)$, $p_x(y)$, $p_y(y)$, $p_x(-x)$, $p_y(-x)$, $p_x(-y)$, $p_y(-y)$, we develop the representation

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_v = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_d = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) From the above matrices, the characters of the reducible basis are $\Gamma(I) = 8$, $\Gamma(C_4) = \Gamma(C_2) = \Gamma(\sigma_v) = \Gamma(\sigma_d) = 0$. Using Eq. (17.12), we find $\Gamma = A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus 2E$.

(c) All the representations of dimension 1 have basis functions that are invariant under C_2. Those named A must also be invariant under C_4, while those named B must change sign under C_4. These observations suffice to identify the following symmetry bases:

$$\psi_{A_1} = p_x(x) + p_y(y) - p_x(-x) - p_y(-y)$$

$$\psi_{A_2} = p_y(x) - p_x(y) - p_y(-x) + p_x(-y)$$

$$\psi_{B_1} = p_x(x) - p_y(y) - p_x(-x) + p_y(-y)$$

$$\psi_{B_2} = p_y(x) + p_x(y) - p_y(-x) - p_x(-y)$$

Both members of each E basis must change sign under C_2 and be transformed into each other under C_4. There are two independent sets of basis functions that satisfy this requirement. Denoting one set (θ_1, θ_2) and the other (χ_1, χ_2), they can be

$$\theta_1 = p_x(x) + p_x(y) + p_x(-x) + p_x(-y)$$

$$\theta_2 = p_y(x) + p_y(y) + p_y(-x) + p_y(-y)$$

$$\chi_1 = p_x(x) - p_x(y) + p_x(-x) - p_x(-y)$$

$$\chi_2 = p_y(x) - p_y(y) + p_y(-x) - p_y(-y)$$

17.5 Direct Products

17.5.1. (a) Squaring the characters for E, we have $\Gamma(I) = \Gamma(C_2) = 4$, $\Gamma(C_4) = \Gamma(\sigma_v) = \Gamma(\sigma_d) = 0$. Applying Eq. (17.12) successively for each irreducible representation, using Table 17.4 of the text, $E \otimes E = A_1 \oplus A_2 \oplus B_1 \oplus B_2$; there is no E representation in this direct product.

(b) Some operations (defined as in Figure 17.6 of the text): $C_4\varphi_1 = \varphi_2$, $C_4\varphi_2 = -\varphi_1$, $C_2\varphi_1 = -\varphi_1$, $C_2\varphi_2 = -\varphi_2$, $\sigma_v\varphi_1 = -\varphi_1$, $\sigma_v\varphi_2 = \varphi_2$, $\sigma_d\varphi_1 = -\varphi_2$, $\sigma_d\varphi_2 = -\varphi_1$.

(c) $\varphi_{A_1} = x_1x_2 + y_1y_2, \qquad \varphi_{A_2} = x_1y_2 - y_1x_2,$

$\varphi_{B_1} = x_1x_2 - y_1y_2, \qquad \varphi_{B_2} = x_1y_2 + y_1x_2.$

17.6 Symmetric Group

17.6.1. (a) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(b) even.

17.6.2. (a) The four members of C_4 are C_4, $C_4^2 = C_2$, and C_4^3. By considering the cyclic permutations of a four-component vector, we get the representation

$$U(I) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U(C_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$U(C_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad U(C_4^3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b) The *Note* provides an answer to this question.

17.6.3. (a) Because the group must be symmetric in the treatment of its elements, all permutations of the same cycle structure will be in the same class. The basic combinatorial fact we need is that the number of distinct permutations corresponding to a cycle of n objects (linked in any order) is $(n-1)!$. The possible cycle structures are:

(1) $(i)(j)(k)(l)$, the identity permutation, which forms a one-member class.

(2) $(i,j)(k)(l)$, which describes a single permutation. But (i,j) can be chosen in six ways, so this is a six-member class.

(3) $(i,j)(k,l)$. This also describes a single permutation, but we can assign four objects to this cycle structure in three different ways, so we have a three-member class; note that $(i,j)(k,l)$ is the same as $(k,l)(i,j)$.

(4) $(i,j,k)(l)$; this cycle structure describes 2! permutations, and can be set up from four objects in four different ways (that is the number of ways to choose the object to be left out). This class therefore has eight members;

(5) (i,j,k,l); this cycle structure describes 3! permutations and can be set up in only one way, defining a six-member class.

Since there are five classes, there are five irreducible representations.

(b) A_1 is the completely symmetric representation, corresponding to all permutations leaving a single basis function unaltered. Its characters are all $+1$. A_2 is the completely antisymmetric representation, with characters of $+1$ for the even permutations and -1 for the odd permutations.

(c) We now know that three of the five irreducible representations have dimensions 1, 1, and 2, whose squares sum to 6. The group S_4 has 24 elements, so the two remaining irreducible representations must be of dimensions n_4 and n_5, with $n_4^2+n_5^2 = 18$. This equation can only be satisfied if $n_4 = n_5 = 3$. Representations of dimension 3 are customarily labeled T, so our roster of irreducible representations is A_1, A_2, E, T_1, and T_2.

(d) Setting up a character table and inserting the information we presently have (the complete characters of A_1 and A_2, the characters of I in all representations, and the zeros from the *Hint*), the partially complete table is the following:

D_4	I	$6P_{12}$	$3P_{12}P_{34}$	$8P_{123}$	$6P_{1234}$
A_1	1	1	1	1	1
A_2	1	-1	1	1	-1
E	2	0			0
T_1	3				
T_2	3				

The remainder of the table can now be filled in with signed integers that cause the orthogonality conditions to be satisfied. The column $8P_{123}$ can only be completed by adding a single ± 1; the orthogonality can only be maintained if in that column $\Gamma(E) = -1$ and $\Gamma(T_1) = \Gamma(T_2) = 0$. The columns $6P_{12}$ and $6P_{1234}$ can only be completed properly if T_1 and T_2 are assigned $+1$ and -1 (in opposite order in the two columns). Which column gets the $+1$ for T_1 is not material because the choice only determines the relative meanings of T_1 and T_2. There is now only one consistent choice for the remaining characters; the complete table takes the form

D_4	I	$6P_{12}$	$3P_{12}P_{34}$	$8P_{123}$	$6P_{1234}$
A_1	1	1	1	1	1
A_2	1	-1	1	1	-1
E	2	0	2	-1	0
T_1	3	-1	-1	0	1
T_2	3	1	-1	0	-1

17.7 Continuous Groups

17.7.1. It is convenient to use the generators of the $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ groups to identify the subgroup structure of the latter. As pointed out when writing Eq. (17.54), the Pauli matrices $\boldsymbol{\sigma}_i$ form a set of generators for $\mathsf{SU}(2)$, and they remain generators if they are expanded to 3×3 matrices by inserting a zero row and column before, after, or between the two rows and columns.

We now search for $\mathsf{SU}(2)$ generators that can be formed from the eight generators of $\mathsf{SU}(3)$. One such set consists of $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and $\boldsymbol{\lambda}_3$. A second set consists of $\boldsymbol{\lambda}_4$, $\boldsymbol{\lambda}_5$, and $(\sqrt{3}\,\boldsymbol{\lambda}_8 + \boldsymbol{\lambda}_3)/2$. A third set is $\boldsymbol{\lambda}_6$, $\boldsymbol{\lambda}_7$, and $(\sqrt{3}\,\boldsymbol{\lambda}_8 - \boldsymbol{\lambda}_3)/2$.

17.7.2. To prove that the matrices $\mathsf{U}(n)$ form a group one needs to verify that they satisfy the group postulates:

(1) They form a set that is closed under the group operation (matrix multiplication), i.e., that the product of two unitary matrices is also unitary. Let U and V be unitary, so $\mathsf{U}^\dagger = \mathsf{U}^{-1}$ and $\mathsf{U}^\dagger = \mathsf{U}^{-1}$. Then

$$(\mathsf{UV})^\dagger = \mathsf{V}^\dagger\mathsf{U}^\dagger = \mathsf{V}^{-1}\mathsf{U}^{-1} = (\mathsf{UV})^{-1}\,,$$

showing that UV is also unitary.

(2) The group operation is associative. Matrix multiplication satisfies this requirement.

(3) There is an identity element; it is the unit matrix $\mathbf{1}_n$.

(4) Each element has an inverse; we have an explicit rule for constructing it; it is the matrix adjoint.

$\mathsf{SU}(n)$ is the subset of $\mathsf{U}(n)$ hose members have determinant $+1$; to identify that it is a subgroup we need to verify that it satisfies the group postulates:

(1) From the determinant product theorem, if U and V have determinant $+1$, so does UV. We have a closed subset.

(2) Matrix multiplication is associative.

(3) $\mathsf{SU}(n)$ includes the identity element.

(4) Again invoking the determinant product theorem, if U has determinant $+1$, so does U^{-1}.

17.7.3. With the Euler angles defined as in Section 3.4, the coordinate rotation (α,β,γ) is defined by a matrix product of the form given in Eq. (3.36), but with the matrices those defined by Eq. (17.56). Thus,

$$\mathsf{U}(\alpha,\beta,\gamma) = \left(e^{i\gamma\boldsymbol{\sigma}_3/2}\right)\left(e^{i\beta\boldsymbol{\sigma}_2/2}\right)\left(e^{i\alpha\boldsymbol{\sigma}_3/2}\right)$$

Using Eq. (17.57) to write explicit forms for these matrices, we get

$$\mathsf{U}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}\begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix}\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

Performing the matrix multiplications,

$$\mathsf{U}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{i(\alpha+\gamma)/2}\cos(\beta/2) & e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ -e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{-i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}$$

17.7.4. (a) Starting from $(I,Y) = (\frac{3}{2},1) = uuu$, repeatedly apply $I_- = I_-(1) + I_-(2) + I_-(3)$ until a further application would produce a zero result. This operator decreases I by 1 and changes u to d. Thus,

$$\begin{aligned} (I,Y) = \quad (\tfrac{3}{2},1): &\quad uuu \\ (\tfrac{1}{2},1): &\quad duu + udu + uud \\ (-\tfrac{1}{2},1): &\quad 2(ddu + dud + udd) \\ (-\tfrac{3}{2},1): &\quad 6ddd \end{aligned}$$

Returning to uuu, (1) apply once V_-, which decreases I by 1/2, decreases Y by 1, and changes u to s, and then (2) apply I_- repeatedly until a zero is produced:

$$\begin{aligned} (I,Y) = \quad (1,0): &\quad suu + usu + uus \\ (0,0): &\quad sdu + sud + dsu + usd + dus + uds \\ (-1,0): &\quad 2(sdd + dsd + dds) \end{aligned}$$

We continue, applying V_-, then I_-, and finally V_-, after which the three-quark subspace is exhausted. These steps correspond to

$$\begin{aligned} (I,Y) = \quad (\tfrac{1}{2},-1): &\quad 2(ssu + sus + uss) \\ (-\tfrac{1}{2},-1): &\quad 2(ssd + sds + dss) \\ (0,-2): &\quad 6sss \end{aligned}$$

(b) The three-quark subspace with $(I,Y) = (\frac{1}{2},1)$ is spanned by uud, udu, and duu. One vector in this subspace is a member of the representation **10**; vectors orthogonal to the member of **10** must belong to other representations. The vectors ψ_1 and ψ_2 are orthogonal both to $uud+udu+duu$ and to each other.

(c) The following chart shows how, starting from ψ_1, we can make the five additional members of **8** listed in this part of the exercise.

$$\begin{aligned}
& \psi_1(\tfrac{1}{2},1): && udu-duu \\
I_-\psi_1(\tfrac{1}{2},1) = {}& \psi_1(-\tfrac{1}{2},1): && udd-dud \\
V_-\psi_1(-\tfrac{1}{2},1) = {}& \psi_1(-1,0): && sdd-dsd \\
U_-\psi_1(-1,0) = {}& \psi_1(-\tfrac{1}{2},-1): && sds-dss \\
I_+\psi_1(-\tfrac{1}{2},-1) = {}& \psi_1(\tfrac{1}{2},-1): && sus-uss \\
V_+\psi_1(\tfrac{1}{2},-1) = {}& \psi_1(1,0): && suu-usu
\end{aligned}$$

(d) Each of the following operations produces a function with $(I,Y) = (0,0)$:

$$\begin{aligned}
V_-\psi_1(\tfrac{1}{2},1) &= V_-(udu-duu) = sdu+uds-dsu-dus \\
U_-\psi_1(-\tfrac{1}{2},1) &= U_-(udd-dud) = usd+uds-sud-dus \\
I_+\psi_1(-1,0) &= I_+(sdd-dsd) = sud+sdu-usd-dsu \\
V_+\psi_1(-\tfrac{1}{2},-1) &= V_+(sds-dss) = uds+sdu-dus-dsu \\
U_+\psi_1(\tfrac{1}{2},-1) &= U_+(sus-uss) = dus+sud-uds-usd \\
I_-\psi_1(1,0) &= I_-(suu-usu) = sdu+sud-dsu-usd
\end{aligned}$$

The fourth of the above expressions is the same as the first; the fifth is (-1) times the second; the sixth is the same as the third, and the third is equal to the first minus the second. So there are two members of the octet at $(Y,I) = (0,0)$, namely $sdu+uds-dsu-dus$ and $usd+uds-sud-dus$. These functions are not orthonormal, but can be made so: in orthonormal form, they are $(sdu+uds-dsu-dus)/2$ and $(sdu-uds-dsu+dus-2usd+2sud)/\sqrt{12}$.

(e) Repeating the steps of parts (c) and (d) with ψ_2, we find

$$\begin{aligned}
\psi_2(\tfrac{1}{2},1) &= 2uud-udu-duu \\
\psi_2(-\tfrac{1}{2},1) &= dud+udd-2ddu \\
\psi_2(-1,0) &= dsd+sdd-2dds \\
\psi_2(-\tfrac{1}{2},-1) &= 2ssd-sds-dss \\
\psi_2(\tfrac{1}{2},-1) &= 2ssu-sus-uss \\
\psi_2(1,0) &= usu+suu-2uus
\end{aligned}$$

The two $(0,0)$ members of the ψ_2 octet are (after orthonormalization) $(uds+dus-dsu-sdu)/2$ and $(uds-2usd+dus+dsu-2sud+sdu)/\sqrt{12}$. It is obvious by inspection that the ψ_1 and ψ_2 functions are linearly independent.

(f) The subspace $(Y,I) = (0,0)$ is spanned by the six quark products uds, usd, dus, dsu, sud, sdu. In the earlier parts of this exercise we found one

$(0,0)$ function in representation **10**, and two from each of the representations **8**. In normalized form, they are

$$\begin{aligned}
\varphi_1 &= (uds + usd + dus + dsu + sud + sdu)/\sqrt{6} \\
\varphi_2 &= (uds - dus - dsu + sdu)/2 \\
\varphi_3 &= (uds + 2usd - dus + dsu - 2sud - sdu)/\sqrt{12} \\
\varphi_4 &= (uds + dus - dsu - sdu)/2 \\
\varphi_5 &= (uds - 2usd + dus + dsu - 2sud + sdu)/\sqrt{12}.
\end{aligned}$$

The function from the $(Y, I) = (0, 0)$ subspace that is orthgonal to all these functions belongs to representation **1**, and can be found from the above by the Gram-Schmidt process. Carrying out that process, we find $\varphi_6 = (uds - usd - dus + dsu + sud - sdu)/\sqrt{6}$. The orthogonality to φ_i, $i = 1$ to 5, is easily checked.

17.8 Lorentz Group

17.8.1. A reference frame moving at infinitestimal velocity $c\delta\rho$ at an angle θ from the x-axis in the xy plane will cause the x, y, and x_0 $(= ct)$ coordinates to transform linearly to

$$x' = x - \cos\theta(\delta\rho)x_0, \qquad y' = y - \sin\theta(\delta\rho)x_0, \qquad x_0' = x_0 - a(\delta\rho)x - b(\delta\rho)y\,,$$

where a and b are determined by requiring $x^2 + y^2 - x_0^2$ to remain constant to first order in $\delta\rho$. Writing $2xdx + 2ydy - 2x_0dx_0 = 0$ and inserting the differentials, we have

$$-2x\cos\theta(\delta\rho)x_0 - 2y\sin\theta(\delta\rho)x_0 + 2x_0a(\delta\rho)x + 2x_0b(\delta\rho)y = 0,$$

from which we find $a = \cos\theta$, $b = \sin\theta$. The 4×4 matrix equation for this linear transformation is

$$\begin{pmatrix} x_0' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -\delta\rho\cos\theta & -\delta\rho\sin\theta & 0 \\ -\delta\rho\cos\theta & 1 & 0 & 0 \\ -\delta\rho\sin\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}.$$

If the matrix in the above equation is written $\mathbf{1}_4 + (\delta\rho)i\mathsf{S}$, S can be identified as the generator of a boost in the direction defined by θ and will have the form given in the exercise.

17.8.2. (a) $\mathsf{U} = e^{i\rho\mathsf{S}} = e^{-\rho\mathsf{M}}$, where M is the matrix S of Exercise 17.8.1 without the prefactor i. If we calculate powers of M we find

$$\mathsf{M} = \begin{pmatrix} 0 & \cos\theta & \sin\theta & 0 \\ \cos\theta & 0 & 0 & 0 \\ \sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{M}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\theta & \cos\theta\sin\theta & 0 \\ 0 & \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathsf{M}^3 = \mathsf{M}$, so all odd powers of M are equal to M, while all even nonzero powers of M are equal to M^2. This enables us to reduce $e^{-\rho\mathsf{M}}$ as follows:

$$e^{-\rho\mathsf{M}} = \mathbf{1}_4 + \mathsf{M}\left(-\frac{\rho}{1!} - \frac{\rho^3}{3!} - \cdots\right) + \mathsf{M}^2\left(\frac{\rho^2}{2!} + \frac{\rho^4}{4!} + \cdots\right)$$

$$= \mathbf{1}_4 - \mathsf{M}\sinh\rho + \mathsf{M}^2\big(\cosh\rho - 1\big)\,.$$

Insertion of the explicit forms for M and M^2 lead directly to U as given in the exercise.

(b) Rotation to align the boost with the x-axis, followed by an x boost, and then the inverse rotation corresponds to forming $\mathsf{R}^{-1}\mathsf{U}_x\mathsf{R}$, where $\mathsf{R}^{-1} = \mathsf{R}^T$ and

$$\mathsf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{U}_x = \begin{pmatrix} \cosh\rho & -\sinh\rho & 0 & 0 \\ -\sinh\rho & \cosh\rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above data to carry out the necessary matrix multiplications, the confirmation of part (a) is immediate.

17.8.3. The transformation matrices are

$$\mathsf{U}_x = \begin{pmatrix} \cosh\rho & -\sinh\rho & 0 & 0 \\ -\sinh\rho & \cosh\rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{U}_y = \begin{pmatrix} \cosh\rho'' & 0 & -\sinh\rho'' & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\rho'' & 0 & \cosh\rho'' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The successive boosts correspond to the matrix product $\mathsf{U}_y\mathsf{U}_x$:

$$\mathsf{U}_y\mathsf{U}_x = \begin{pmatrix} \cosh\rho''\cosh\rho' & -\cosh\rho''\sinh\rho' & -\sinh\rho'' & 0 \\ -\sinh\rho' & \cosh\rho' & 0 & 0 \\ -\sinh\rho''\cosh\rho' & \sinh\rho''\sinh\rho' & \cosh\rho'' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is not symmetric, and unless either ρ' or ρ'' is zero it cannot correspond to any case of the matrix of Exercise 17.8.2, and cannot represent a pure boost.

17.9 Lorentz Covariance of Maxwell's Equations

17.9.1. Form the transformed electromagnetic tensor as the matrix product

$$\mathsf{F}' = \mathsf{UFU}\,,$$

and bring to final form by using the identities $\beta c = v$ and $\gamma^2(1-\beta^2) = 1$. Then

$$F'_{21} = E'_x, \qquad F'_{31} = E'_y, \qquad F'_{41} = E'_z,$$

in agreement with Eq. (17.82). The transformed components of **B** are obtained from

$$B'_x = c^{-1}F'_{43}, \qquad B'_y = c^{-1}F'_{24}, \qquad B'_z = c^{-1}F'_{32};$$

they agree with Eq. (17.83).

17.9.2. It is convenient to introduce the direction cosines of **v**; denote the angles between **v** and the coordinate axes χ_1, χ_2, χ_3. Then the generalization of M and M^2 (see the solution to Exercise 17.8.2) to an arbitrary direction of v are

$$\mathsf{M} = \begin{pmatrix} 0 & \cos\chi_1 & \cos\chi_2 & \cos\chi_3 \\ \cos\chi_1 & 0 & 0 & 0 \\ \cos\chi_2 & 0 & 0 & 0 \\ \cos\chi_3 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{M}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\chi_1 & \cos\chi_1\cos\chi_2 & \cos\chi_1\cos\chi_3 \\ 0 & \cos\chi_1\cos\chi_2 & \cos^2\chi_2 & \cos\chi_2\cos\chi_3 \\ 0 & \cos\chi_1\cos\chi_3 & \cos\chi_2\cos\chi_3 & \cos^2\chi_3 \end{pmatrix}.$$

From these, form $U = \mathbf{1}_4 - \mathsf{M}\sinh\rho + \mathsf{M}^2(\cosh\rho - 1)$, and use the notations $\cosh\rho = \gamma$, $\sinh\rho = \beta\gamma$. The sum $\sum_i \cos^2\chi_i = 1$ is also used to simplify U. The result is

$$\begin{pmatrix} \gamma & -\beta\gamma\cos\chi_1 & -\beta\gamma\cos\chi_2 & -\beta\gamma\cos\chi_3 \\ -\beta\gamma\cos\chi_1 & 1+(\gamma-1)\cos^2\chi_1 & (\gamma-1)\cos\chi_1\cos\chi_2 & (\gamma-1)\cos\chi_1\cos\chi_3 \\ -\beta\gamma\cos\chi_2 & (\gamma-1)\cos\chi_1\cos\chi_2 & 1+(\gamma-1)\cos^2\chi_2 & (\gamma-1)\cos\chi_2\cos\chi_3 \\ -\beta\gamma\cos\chi_3 & (\gamma-1)\cos\chi_1\cos\chi_3 & (\gamma-1)\cos\chi_2\cos\chi_3 & 1+(\gamma-1)\cos^2\chi_3 \end{pmatrix}.$$

Finally, form the matrix $\mathsf{F}' = U\mathsf{F}U$. This operation produces a relatively complicated matrix that can be simplified to give the desired result. It becomes easier to find simplification steps if one identifies the subexpressions corresponding to E_v and B_v (the magnitudes of the projections of **E** and **B** on **v**); the formulas for these quantities are

$$E_v = \cos\chi_1 E_x + \cos\chi_2 E_y + \cos\chi_3 E_z, \quad B_v = \cos\chi_1 B_x + \cos\chi_2 B_y + \cos\chi_3 B_z.$$

Another helpful hint is to recognize that $\gamma^2(1-\beta^2) = 1$ and at some point to write $\beta = v/c$.

17.10 Space Groups

(no exercises)

18. More Special Functions

18.1 Hermite Functions

18.1.1. It is convenient to use the formula in Eq. (18.4) for H_n', but it was not one of the relationships whose validity was to be assumed. However, Eq. (18.4) can be derived from the ODE, assuming the validity for all x of the basic recurrence formula, Eq. (18.3). Proceed by forming

$$(H_{n+1} - 2xH_n + 2nH_{n-1})'' - 2x(H_{n+1} - 2xH_n + 2nH_{n-1})'$$
$$+ 2n(H_{n+1} - 2xH_n + 2nH_{n-1}) = 0\,,$$

then expand the parenthesized derivatives and use the Hermite ODE to cancel as many terms as possible. In this way we reach

$$-2H_{n+1} + 4xH_n + 4nH_{n-1} - 4H_n' = 0\,,$$

equivalent, again using Eq. (18.3), to

$$H_n' = 2nH_{n-1}\,.$$

(a) Now differentiate $g(x,t)$ with respect to x and use the above derivative formula:

$$\frac{\partial g}{\partial x} = \sum_{n=0}^{\infty} H_n'(x)\,\frac{t^n}{n!} = \sum_{n=0}^{\infty} 2nH_{n-1}\,\frac{t^n}{n!} = 2t\,g(x,t)\,.$$

(b) This is a separable first-order equation with solution

$$g(x,t) = e^{2tx} f(t)\,.$$

(c) Find $f(t)$ by setting $x = 0$ and using $H_{2n}(0) = (-1)^n(2n)!/n!$ and $H_{2n+1}(0) = 0$. We get

$$g(x,0) = f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} = e^{-t^2}\,.$$

(d) The final result is $g(x,t) = e^{2xt-t^2}$.

18.1.2. The connection of these starting points can, of course, be accomplished in many ways.

Starting with the ODE, one can derive a Rodrigues formula as shown in Eq. (18.8), and therefrom, as in Eq. (12.18), a Schlaefli integral representation. Applying Eq. (12.18), we can use the Schlaefli integral to obtain a generating function, which can in turn be applied, e.g., by developing recurrence formulas and then using them as in the text after Eq. (18.7), to recover the Hermite ODE. Since these steps form a closed loop, we can regard any of the four relationships as a valid starting point.

From the Rodrigues formula we can also, by repeated integrations by parts, establish the orthogonality of the Hermite polynomials on $(-\infty, \infty)$ and note the associated weighting factor; conversely, the weighting factor suffices to determine the ODE that yields the orthogonal polynomials. These last relationships mean that we can begin an analysis from any of the five listed starting points.

18.1.3. From Eq. (18.9) we see that

$$i^{-n}(2x)^{-n}H_n(ix) = \sum_{s=0}^{[n/2]} \frac{n!(4x^2)^{-s}}{(n-2s)!s!}$$

$$\geq \sum_{s=0}^{[n/2]} (-1)^s \frac{n!(4x^2)^{-s}}{(n-2s)!s!} = (2x)^{-n}H_n(x)$$

because the first sum has only positive terms. Taking the absolute values of the first and last members of the above relation and multiplying through by $|(2x)^n|$ we obtain the desired inequality.

18.1.4. The solution is given in the text.

18.1.5. The solution is given in the text.

18.1.6. The answers in the text are incorrect. In each answer, replace 2π by $(2\pi)^{1/2}$.

(a) Using the generating function, form

$$\int_{-\infty}^{\infty} e^{-x^2/2} g(x,t)\,dx = \int_{-\infty}^{\infty} e^{-t^2+2tx-x^2/2}\,dx = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} H_n(x)\,\frac{t^n}{n!}\,dx\,.$$

Now convert the central member of this equation to obtain a power series in t, by first completing the square in the exponent and performing the x integration, and then expanding the resultant function of t. Setting $y = x - 2t$ we get

$$\int_{-\infty}^{\infty} e^{-t^2+2tx-x^2/2}\,dx = \int_{-\infty}^{\infty} e^{-(x-2t)^2/2+t^2}\,dx = e^{t^2} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= (2\pi)^{1/2} e^{t^2} = (2\pi)^{1/2} \sum_{m=0}^{\infty} \frac{t^{2m}}{m!}\,.$$

We now equate the coefficients of equal powers of t in the last two equations, noting from the second of these equations that the integrals involving H_n of odd n vanish, and that those of even n correspond to the equation

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, \frac{H_{2m}(x)}{(2m)!}\, dx = \frac{(2\pi)^{1/2}}{m!}\,.$$

This formula is equivalent to the corrected form of the answer.

(b) This part of the exercise can be treated in a way similar to that of part (a). The relevant equation set is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} e^{-x^2/2} x H_n(x)\, dx = \int_{-\infty}^{\infty} e^{-(x-2t)^2/2+t^2}\, x\, dx$$

$$= e^{t^2} \int_{-\infty}^{\infty} e^{-y^2/2}(y+2t)\, dy = (2\pi)^{1/2} \sum_{m=0}^{\infty} \frac{2\, t^{2m+1}}{m!}\,.$$

The term of the integral containing a linear factor y vanishes due to its odd symmetry. The left-hand side of this equation must have zero coefficients for even n, while for odd n (set to $2m+1$) we get

$$\frac{1}{(2m+1)!} \int_{-\infty}^{\infty} e^{-x^2/2}\, x\, H_{2m+1}(x)\, dx = (2\pi)^{1/2}\, \frac{2}{m!} = (2\pi)^{1/2}\, \frac{2(m+1)}{(m+1)!}\,,$$

which easily rearranges into the corrected form of the answer.

18.1.7. (a) The solution is given in the text.

(b) Defining $z + x = w$, $z^2 - x^2 = w^2 - 2xw$ we have

$$H_n(x) = \frac{n!}{2\pi i} \oint \frac{e^{-w^2+2xw}}{w^{n+1}}\, dw\,, \quad H_n'(x) = 2\frac{n!}{2\pi i} \oint \frac{e^{-w^2+2xw}}{w^n}\, dw\,,$$

$$H_n''(x) = 4\frac{n!}{2\pi i} \oint \frac{e^{-w^2+2xw}}{w^{n-1}}\, dw\,,$$

and

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = \frac{n!}{2\pi i} \oint \frac{e^{-w^2+2xw}}{w^{n+1}} (2n - 4xw + 4w^2)\, dw$$

$$= -2\frac{n!}{2\pi i} \oint \frac{d}{dw} \left(\frac{e^{-w^2+2xw}}{w^n} \right) dw = 0\,,$$

where a zero value is obtained because the start and end points of the contour coincide and the quantity being differentiated is analytic at all points of the contour.

18.2 Applications of Hermite Functions

18.2.1. Combining the recursion formulas in Eqs. (18.3) and (18.4), we have

$$H_{n+1}(x) = (2x - \frac{d}{dx})H_n(x)\,.$$

This formula suggests that we use mathematical induction, since it shows that if the formula of this exercise is true for $H_n(x)$, it is also true for $H_{n+1}(x)$. To complete a proof we need to verify that the formula is valid for $n = 0$, i.e., that

$$H_1(x) = 2xH_0(x) - H_0'(x).$$

Since $H_0(x) = 1$ and $H_1(x) = 2x$, this equation is clearly satisfied.

18.2.2. Introduce the expansion $x^m = \sum_{i=1}^{m} a_i H_i(x)$. Then, for $m < n$, using orthogonality we have

$$\int_{-\infty}^{\infty} e^{-x^2} x^m H_n(x)\,dx = \ .$$

18.2.3. Using Eq. (18.3) to replace $xH_n(x)$ by $\frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)$, then invoking orthogonality and inserting the normalization integral from Eq. (18.15),

$$\int_{-\infty}^{\infty} e^{-x^2} xH_n(x)H_m(x)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x)H_m(x)\,dx + n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x)H_m(x)\,dx$$

$$= \left[\frac{1}{2}\,\delta_{n+1,m} + n\delta_{n-1,m}\right]\pi^{1/2}2^m m! = 2^n n!\,\pi^{1/2}\left[(n+1)\delta_{m,n+1} + \frac{1}{2}\,\delta_{m,n-1}\right].$$

18.2.4. Using $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ we get

$$I_2 = \int_{-\infty}^{\infty} x^2[H_n]^2 e^{-x^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty}[H_{n+1} + 2nH_{n-1}]^2 e^{-x^2}\,dx\,.$$

Expanding, discarding terms that vanish due to orthogonality, and using the normalization integral, Eq. (18.15),

$$I_2 = \frac{1}{4}\int_{-\infty}^{\infty}[H_{n+1}]^2 e^{-x^2} dx + n^2\int_{-\infty}^{\infty}[H_{n-1}]^2 e^{-x^2} dx$$

$$= 2^{n-1}\sqrt{\pi}\,n!(n+1+n) = 2^{n-1}\sqrt{\pi}\,n!(2n+1)\,.$$

18.2.5. Applying Eq. (18.3) twice, we get

$$x^2 H_n(x) = \frac{1}{4}\, H_{n+2}(x) + \left(\frac{2n+1}{2}\right) H_n(x) + n(n-1)H_{n-2}(x)\,.$$

Substituting this form into the integral of this exercise, invoking orthogonality and using the normalization integral, Eq. (18.15),

$$\int_{-\infty}^{\infty} x^2 e^{-{}^2} H_n(x) H_m(x)\, dx = \frac{1}{4}\left(\pi^{1/2} 2^{n+2}(n+2)!\right)\delta_{n+2,m}$$

$$+\left(\frac{2n+1}{2}\right)\left(\pi^{1/2}2^n n!\right)\delta_{nm} + n(n-1)\left(\pi^{1/2}2^{n-2}(n-2)!\right)\delta_{n-2,m}\,,$$

equivalent to the answer in the text.

18.2.6. The product $x^r H_n(x)$ is a polynomial of degree $n+r$ and therefore its expansion in Hermite polynomials cannot involve any H_m with index $m > n+r$. If $n+r < n+p$ (i.e., if $p > r$), the integral of the present exercise must vanish due to orthogonality.

The other case under consideration here is $p = r$, for which we need to prove

$$\int_{-\infty}^{\infty} x^r e^{-x^2} H_n(x) H_{n+r}(x)\, dx = 2^n \pi^{1/2}(n+r)!\,.$$

Using mathematical induction, we start by assuming the above equation to be valid for some $r-1$ and, subject to that assumption, prove its validity for r. Write

$$\int_{-\infty}^{\infty} x^r e^{-x^2} H_n(x) H_{n+r}(x)\, dx =$$

$$\int_{-\infty}^{\infty} x^{r-1} e^{-x^2} H_n(x)\left[(n+r)H_{n+r-1}(x) + \frac{1}{2}\, H_{n+r+1}(x)\right] dx\,,$$

where we have used the recurrence formula, Eq. (18.3), to convert xH_{n+r} into a linear combination of $H_{n+r\pm1}$. The second term of the integrand leads to a vanishing integral because $r+1 > r-1$, while the first term corresponds to the integral under study for $r-1$. Inserting the assumed result, we get

$$\int_{-\infty}^{\infty} x^r e^{-x^2} H_n(x) H_{n+r}(x)\, dx = (n+r)2^n \pi^{1/2}(n+r-1)!\,,$$

which is the correct formula for index value r. To complete the proof, we must verify the formula for $r = 0$, which is just the normalization formula, Eq. (18.15).

18.2.7. The signs of the operators ip in these equations should be reversed, and the quantity $\psi_n(x)$ should be inserted immediately following the operators $(x \pm ip)/\sqrt{2}$.

Noting first that $(d/dx)e^{-x^2/2} = -xe^{-x^2/2}$, the expressions of this exercise reduce to

$$a\psi_n(x) = \frac{e^{-x^2}}{(2^{n+1}n!\,\pi^{1/2})^{1/2}}\; H_n'(x)\,,$$

$$a^\dagger\psi_n(x) = \frac{e^{-x^2}}{(2^{n+1}n!\,\pi^{1/2})^{1/2}}\; [2xH_n(x) - H_n'(x)]\,.$$

We now replace $H_n'(x)$ by $2nH_{n-1}(x)$ by applying Eq. (18.4), and in the second of the two above equations we also use the recurrence formula, Eq. (18.3), to replace $2xH_n - 2nH_n$ by H_{n+1}. When we write the coefficients of $H_{n\pm1}$ in forms that include the constant factors in the definitions of $\psi_{n\pm1}$, we obtain the required answers.

18.2.8. (a) Since p is conventionally taken to be $-i\,d/dx$, the first member of this equation should read $x - ip$.

To verify an operator identity we must show that the two operators involved produce identical results when applied to an arbitrary function. Applying the operator on the right-hand side to an arbitrary differentiable function $f(x)$, we have

$$-e^{x^2/2}\frac{d}{dx}\left[e^{-x^2/2}f(x)\right] = \left[x - \frac{d}{dx}\right] f(x)\,,$$

and note that the result is identical to that of applying the left-hand-side operator to $f(x)$.

(b) Using the second equation of Exercise 18.2.7 n times, we note that

$$a^\dagger\psi_0(x) = 1^{1/2}\psi_1(x), \quad \left[a^\dagger\right]^2\psi_0(x) = [1\cdot 2]^{1/2}\psi_2(x), \quad \cdots\,, \text{ or}$$

$$\left[a^\dagger\right]^n\psi_0(x) = (n!)^{1/2}\psi_n(x)\,.$$

Noting that

$$\psi_0(x) = \pi^{-1/4}e^{-x^2/2} \quad \text{and} \quad [a^\dagger]^n = 2^{-n/2}\left(x - \frac{d}{dx}\right)^n\,,$$

we confirm the formula given in the text.

18.3 Laguerre Functions

18.3.1. Using Leibniz' rule,

$$(uv)^{(n)} = \sum_{m=0}^{n}\binom{n}{m}u^{(m)}v^{(n-m)}\,,$$

we get

$$L_n = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) = \frac{e^x}{n!}\sum_{m=0}^{n}\binom{n}{m}\left(\frac{n!}{(n-m)!}x^{n-m}\right)(-1)^{n-m}e^{-x}$$

$$= \sum_{m=0}^{n}\frac{(-1)^{n-m}n!\,x^{n-m}}{m!\,(n-m)!(n-m)!}\,,$$

which is Eq. (18.53).

18.3.2. (a) The value of $L_n'(0)$ is the coefficient of x in the power series, and $L_n''(0)$ is twice the coefficient of x^2.

(b) Differentiating the recurrence formula, Eq. (18.51) and rearranging, we get (then setting $x = 0$ and $L_n(0) = 1$)

$$(n+1)\left[L_{n+1}'(0) - L_n'(0)\right] = n\left[L_n'(0) - L_{n-1}'(0)\right] - 1\,.$$

Using this equation iteratively to further reduce the index values on its right-hand side, we reach

$$(n{+}1)\left[L_{n+1}'(0) - L_n'(0)\right] = 1\,[L_1'(0) - L_0'(0)] - n = [-1-0] - n = -(n{+}1)\,.$$

Therefore, for all n we have $L_{n+1}'(0) - L_n'(0) = -1$; since $L_0'(0) = 0$, this proves that $L_n'(0) = -n$.

A similar process can be applied for $L_n''(0)$.

18.3.3. It is convenient to solve this problem using a generating function for L_n^k that is obtained by differentiating that for $L_{n+k}(x)$ k times with respect to x. Referring to Eq. (18.49) for the generating function and Eq. (18.58) for the definition of L_n^k, we have

$$\frac{t^k e^{-xt/(1-t)}}{(1-t)^{k+1}} = \sum_n L_n^k(x)t^{n+k}, \quad \text{equivalent to} \quad \frac{e^{-xt/(1-t)}}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} L_n^k(x)t^n.$$

We next use a product of two generating functions of this type to form orthogonality/normalization integrals for associated Laguerre functions. Let

$$J = \int_0^\infty x^k e^{-x}\left[\frac{e^{-xt/(1-t)}}{(1-t)^{k+1}}\right]\left[\frac{e^{-xu/(1-u)}}{(1-u)^{k+1}}\right]dx$$

$$= \sum_{nm} t^n u^m \int_0^\infty x^k e^{-x} L_n^k(x) L_m^k(x)\,dx\,.$$

We now evaluate J as given in terms of the generating functions, starting this process by collecting the exponentials into the form e^{-Ax}, with

$$A = 1 + \frac{t}{1-t} + \frac{u}{1-u} = \frac{1-tu}{(1-t)(1-u)}\,.$$

We then change the integration variable to $y = Ax$, obtaining

$$J = \frac{A^{-k-1}}{(1-t)^{k+1}(1-u)^{k+1}} \int_0^\infty y^k e^{-y}\, dy = \frac{k!}{(1-tu)^{k+1}}\,.$$

Then we expand J as a binomial series:

$$J = k! \sum_p \binom{-k-1}{p} (-tu)^p = \sum_p \frac{(k+p)!}{p!}\,(tu)^p\,.$$

From this expression for J we get

$$\sum_p \frac{(k+p)!}{p!}\,(tu)^p = \sum_{nm} t^n u^m \int_0^\infty x^k e^{-x} L_n^k(x) L_m^k(x)\, dx\,.$$

Comparing the coefficients of like powers of t and u, we find

$$\int_0^\infty x^k e^{-x} L_n^k(x) L_m^k(x)\, dx = \frac{(k+n)!}{n!}\,\delta_{mn}\,.$$

18.3.4. The solution is given in the text.

18.3.5. The solution is given in the text.

18.3.6. When the hint is inserted in the integral, each term can be evaluated using Eq. (18.71). This yields

$$\int_0^\infty e^{-x} x^{k+1} L_n^k(x) L_n^k(x) dx = (2n+k+1)\frac{(n+k)!}{k!}\,.$$

18.3.7. (a) For $x \to \infty$, where the x^{-2}, x^{-1} terms are negligible, solve $y'' - y/4 = 0$ and obtain as a solution the negative exponential $y = e^{-x/2} = A(x)$.

(b) For $0 < x \ll 1$, solve

$$y'' - \frac{k^2-1}{4x^2}y = 0\,,$$

which has the regular solution $y = x^{(k+1)/2} = B(x)$.

Then $C(x) = L_n^k(x)$ will follow.

18.3.8. The solution is given in the text.

18.3.9. The solution is given in the text.

18.3.10. Use mathematical induction. To develop an appropriate formula, write

$$H_n(xy) = \frac{1}{2(n+1)} H'_{n+1}(xy) = \frac{1}{2(n+1)y}\,\frac{dH_{n+1}(xy)}{dx}\,,$$

which we now insert into our integral and integrate by parts. The result is

$$\int_{-\infty}^{\infty} x^n e^{-x^2} H_n(xy)\,dx = \frac{1}{2(n+1)y}\int_{-\infty}^{\infty} x^n e^{-x^2}\,\frac{dH_{n+1}(xy)}{dx}\,dx$$

$$= -\int_{-\infty}^{\infty}(nx^{n-1}-2x^{n+1})e^{-x^2}H_{n+1}(xy)\,dx\,.$$

Now convert the term containing x^{n-1} to a more useful form using the Hermite recurrence formula:

$$nx^{n-1}e^{-x^2}H_{n+1}(xy) = 2nx^n y e^{-x^2}H_n(xy) - 2n^2x^{n-1}H_{n-1}(xy)\,.$$

Making this substitution, after minor rearrangement we reach

$$(2n+1)y\int_{-\infty}^{\infty} x^n e^{-x^2}H_n(xy)\,dy = n^2\int_{-\infty}^{\infty} x^{n-1}e^{-x^2}H_{n-1}(xy)\,dy$$

$$+\int_{-\infty}^{\infty} x^{n+1}e^{-x^2}H_{n+1}(xy)\,dy\,.$$

We now substitute into the above equation the assumed identification of these integrals with P_n, finding that these integrals obey the Legendre polynomial recurrence formula, Eq. (15.18). They are therefore valid representations of these polynomials if they also give correct results for $n=0$ and $n=1$.

For $n=0$ our integral reduces to an error integral of value $\sqrt{\pi}$, consistent with the assumed integral representation. For $n=1$, our integral has the form

$$\int_{-\infty}^{\infty} xe^{-x^2}(2xy)\,dx = \sqrt{\pi}\,y = \sqrt{\pi}\,P_1(y)\,,$$

also consistent with our assumed formula. This completes the proof.

18.4 Chebyshev Polynomials

18.4.1. For $x = 1$, $x = -1$, and $x = 0$ the generating function $g(x,t)$ for T_n becomes

$$g(1,t) = \frac{1-t^2}{1-2t+t^2} = \frac{1+t}{1-t} = 1 + 2(t + t^2 + t^3 + \cdots) = 1 + \sum_1^\infty 2t^n,$$

$$g(-1,t) = \frac{1-t^2}{1+2t+t^2} = \frac{1-t}{1+t} = 1 + 2(-t + t^2 - t^3 + \cdots)$$

$$= 1 + \sum_1^\infty (-1)^n 2t^n,$$

$$g(0,t) = \frac{1-t^2}{1+t^2} = 1 + 2(-t^2 + t^4 - t^6 + \cdots) = 1 + \sum_1^\infty (-1)^n 2t^{2n}.$$

Comparing the above with $g(x,t) = T_0(x) + \sum_{n=1}^\infty 2T_n(x)t^n$, we see that

$T_n(1) = 1$, $T_n(-1) = (-1)^n$, $T_{2n}(0) = (-1)^n$, $T_{2n+1}(0) = 0$.

18.4.2. For $x = 1$, $x = -1$, and $x = 0$ the generating function $g(x,t)$ for U_n becomes

$$g(1,t) = \frac{1}{1-2t+t^2} = \frac{1}{(1-t)^2} = \frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{d}{dt}(1 + t + t^2 + t^3 + \cdots)$$

$$= 0 + 1 + 2t + 3t^2 + \cdots = \sum_{n=0}^\infty (n+1)t^n\,,$$

$$g(-1,t) = \frac{1}{1+2t+t^2} = \frac{1}{(1+t)^2} = -\frac{d}{dt}\left(\frac{1}{1+t}\right) = -\frac{d}{dt}(1 - t + t^2 - \cdots)$$

$$= 0 + 1 - 2t + 3t^2 - \cdots = \sum_{n=0}^\infty (-1)^n(n+1)t^n\,,$$

$$g(0,t) = \frac{1}{1+t^2} = \sum_{n=0}^\infty (-1)^n t^{2n}\,.$$

Comparing with $g(x,t) = \sum_{n=0}^\infty U_n(x)t^n$, we see that

$U_n(1) = n+1$, $U_n(-1) = (-1)^n(n+1)$, $U_{2n}(0) = (-1)^n$, $U_{2n+1}(0) = 0$.

18.4.3. $X_n(x) = T_n(x)$.

18.4.4. Using Eq. (18.109), evaluate the terms of the ODE for V_n in terms of U_{n-1} and its derivatives, with the aim of showing that the ODE suggested for

V_n is automatically satisfied, given the ODE given for U_{n-1}. Specifically, we have

$$V_n(x) = (1-x^2)^{1/2} U_{n-1}(x)\,,$$

$$V_n'(x) = (1-x^2)^{1/2} U_{n-1}'(x) - \frac{x}{(1-x^2)^{1/2}}\, U_{n-1}(x)\,,$$

$$V_n''(x) = (1-x^2)^{1/2} U_{n-1}''(x) - \frac{2x}{(1-x^2)^{1/2}} U_{n-1}'(x) - \frac{‘1}{(1-x^2)^{3/2}} U_{n-1}(x)\,.$$

Using the above, form

$$(1-x^2)V_n''(x) - xV_n'(x) + n^2 V_n(x) = (1-x^2)^{3/2} U_{n-1}''(x)$$

$$- 3x(1-x^2)^{1/2} U_{n-1}'(x) + (n^2-1)(1-x^2)^{1/2} U_{n-1}(x)$$

$$= (1-x^2)^{1/2}\Big[(1-x^2)U_{n-1}''(x) - 3xU_{n-1}'(x) + (n-1)(n+1)U_{n-1}(x)\Big]\,.$$

The expression within the square brackets vanishes because of the equation satisfied by $U_{n-1}(x)$, confirming that V_n satisfies the suggested ODE.

18.4.5. Writing the ODE for T_n and V_n in self-adjoint form, we obtain (calling the dependent variable y)

$$\Big[(1-x^2)^{1/2} y'\Big]' + \frac{n^2 y}{(1-x^2)^{1/2}} = [p(x)y']' + q(x)y = 0\,,$$

we use the fact that the Wronskian of any two solutions to this ODE has the general form $A/p(x)$, in this case

$$W(T_n, V_n) = T_n(x)V_n'(x) - T_n'(x)V_n(x) = \frac{A_n}{(1-x^2)^{1/2}}\,.$$

To find the value of A_n, evaluate the Wronskian at $x=0$, where its value will be A_n. From Eq. (18.95), we note that $T_n'(0) = nT_{n-1}(0)$. From Eq. (18.109) we identify $V_n(0) = U_{n-1}(0)$; differentiating Eq. (18.109) we also find $V_n'(0) = U_{n-1}'(0)$, which using Eq. (18.96), can be written $V_n'(0) = nU_{n-2}(0)$. Our Wronskian can therefore (for $x=0$) be written

$$T_n(0)V_n'(0) - T_n'(0)V_n(0) = n\Big[T_n(0)U_{n-2}(0) - T_{n-1}(0)U_{n-1}(0)\Big] = A_n\,.$$

All the function values on the left-hand side of this equation are given in Eq. (18.100); the quantity in square brackets evaluates to -1 for both even and odd n, so

$$W(T_n, V_n) = \frac{A_n}{(1-x^2)^{1/2}} = \frac{n(-1)}{(1-x^2)^{1/2}}\,.$$

18.4.6. We know that $T_n(x)$ satisfies the ODE

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n = 0\,,$$

and the exercise gives $W_n(x)$ in terms of T_{n+1}. We can verify the asserted ODE for W_n by rewriting it as an equation in T_{n+1} and comparing with the above ODE. To do so, we need

$$W_n(z) = (1-x^2)^{-1/2}T_{n+1}(x)\,,$$

$$W_n'(x) = \frac{1}{(1-x^2)^{1/2}}T_{n+1}'(x) + \frac{x}{(1-x^2)^{3/2}}T_{n+1}(x)\,,$$

$$W_n''(x) = \frac{1}{(1-x^2)^{1/2}}T_{n+1}''(x) + \frac{2x}{(1-x^2)^{3/2}}T_{n+1}'(x) + \frac{1+2x^2}{(1-x^2)^{5/2}}T_{n+1}(x)\,.$$

Using the above, form

$$(1-x^2)W_n''(x) - 3xW_n'(x) + n(n+2)W_n(x) = (1-x^2)^{1/2}T_{n+1}''(x)$$

$$-\frac{x}{(1-x^2)^{1/2}}T_{n+1}'(x) + \frac{1+n(n+2)}{(1-x^2)^{1/2}}T_{n+1}(x)$$

$$= (1-x^2)^{-1/2}\Big[(1-x^2)T_{n+1}''(x) - xT_{n+1}'(x) + (n+1)^2T_{n+1}(x)\Big]\,.$$

The expression in square brackets vanishes because of the equation satisfied by $T_{n+1}(x)$, confirming the ODE proposed for W_n.

18.4.7. Compare with the method of solution to Exercise 18.4.5. The ODE for U_n and W_n is, in self-adjoint form,

$$\Big[(1-x^2)^{3/2}y'\Big]' + n(n+2)(1-x^2)^{1/2}y = 0\,,$$

so the Wronskian has the functional form

$$W(U_n,W_n) = U_n(x)W_n'(x) - U_n'(x)W_n(x) = \frac{A_n}{(1-x^2)^{3/2}}\,.$$

At $x=0$, $U_n'(0) = (n+1)U_{n-1}(0)$, and $U_n(0)$ is given in Eq. (18.100). In addition, $W_n(0) = T_{n+1}(0)$, $W_n'(0) = (n+1)T_n(0)$, with $T_n(0)$ also given in Eq. (18.100). Thus

$$U_n(0)W_n'(0) - U_n'(0)W_n(0) = (n+1)\Big[U_n(0)T_n(0) - U_{n-1}(0)T_{n+1}(0)\Big] = n+1\,.$$

Note that we get the same result for both even and odd n, and that it fixes the constant A_n as $n+1$. Therefore,

$$W(U_n,W_n) = \frac{n+1}{(1-x^2)^{3/2}}\,.$$

18.4.8. Work with the ODE for T_0 in the form given in Eq. (18.104), but written as $y'' = 0$ to imply that we will search for the general solution. Thus,

$$\frac{d^2y}{d\theta^2} = 0\,, \quad \text{with solutions} \quad y = c_0 \quad \text{and} \quad y = c_1\theta\,.$$

The solution $y = c_0$ is just $c_0T_0(x)$; since $x = \cos\theta$, the solution $y = c_1\theta$ is equivalent to $y = c_1 \arccos x$. Since $\pi/2 - \arccos x$ is a linear combination of these two solutions, it is also a solution, more compactly written as $\arcsin x$.

18.4.9. Write the proposed recurrence relation for V_n with V_n written in terms of U_{n-1} according to Eq, (18.109). Because U_n and T_n satisfy the same recurrence formula, and it is independent of n, so also does V_n.

18.4.10. For $T_n(x)$, using the expansion in Eq. (18.114), we write first the entire ODE except the single term $1T''_n(x)$, and then that remaining term:

$$-x^2T''_n(x) - xT'_n(x) + x^2T_n(x) = \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m-1)!}{m!\,(n-2m)!}\,(2x)^{n-2m}$$
$$\times\Big[-(n-2m)(n-2m-1)-(n-2m)+n^2\Big]\,,$$

$$T''_n(x) = \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m-1)!}{m!\,(n-2m)!}\,\frac{(2x)^{n-2m}}{x^2}\,(n-2m)(n-2m-1)\,.$$

To obtain a final expression as a single power series, we rewrite the second of the two above equations with the summation index m changed to $m-1$, thereby obtaining

$$T''_n(x) = \frac{n}{2}\sum_{m=1}^{[n/2+1]}\frac{(-1)^{m-1}(n-m)!}{(m-1)!\,(n-2m)!}\,\frac{(2x)^{n-2m+2}}{x^2}\,(n-2m+2)(n-2m+1)\,.$$

When this form of T''_n is added to the remainder of the ODE, the coefficient of each power of x is found to vanish, thereby showing that the series for T_n satisfies the ODE. To verify its scale, set $x = 0$, thereby causing the entire summation to be zero if n is odd, and to the single term $m = n/2$ if n is even. That single term of the sum evaluates to

$$(-1)^{n/2}\,\frac{(n/2-1)!}{(n/2)!} = (-1)^{n/2}\left(\frac{2}{n}\right)\,,$$

consistent with $T_{2n}(0) = (-1)^n$, the value given in Eq. (18.100).

The series expansion of $U_n(x)$ is verified in a similar fashion.

18.4.11. The answer is given in the text.

18.4.12. The answer is given in the text.

18.4.13. (a) The equations for T_n and T_m can be written

$$\Big[(1-x^2)^{1/2}T_n'(x)\Big]' = -\frac{n^2T_n(x)}{(1-x^2)^{1/2}}\,,$$

$$\Big[(1-x^2)^{1/2}T_m'(x)\Big]' = -\frac{m^2T_m(x)}{(1-x^2)^{1/2}}\,.$$

Multiplying the first of these equations by $m^2T_m(x)$ and the second by $n^2T_n(x)$, then subtracting the first from the second and integrating from -1 to 1, we reach

$$n^2\int_{-1}^{1}T_n(x)\Big[(1-x^2)^{1/2}T_m'(x)\Big]'\,dx - m^2\int_{-1}^{1}T_m(x)\Big[(1-x^2)^{1/2}T_n'(x)\Big]'\,dx = 0.$$

Note that the right-hand-side integrals are convergent and their cancellation is therefore legitimate.

Now integrate the integrals in the above equation by parts, integrating the explicit derivatives and differentiating the other Chebyshev polynomials. The presence of the factor $(1-x^2)^{1/2}$ causes all the endpoint terms to cancel, and we are left with

$$(m^2-n^2)\int_{-1}^{1}T_m'(x)T_n'(x)(1-x^2)^{1/2}\,dx = 0\,.$$

The left-hand side of this equation is zero whether or not $m=n$, but only if $m\neq n$ does it show that the integral vanishes.

(b) Differentiating Eq. (18.105), we get, noting that $x=\cos\theta$ and comparing the result with Eq. (18.107),

$$\frac{dT_n(x)}{dx} = \frac{dT_n(\theta}{d\theta}\,\frac{d\theta}{dx} = (-n\sin n\theta)\left(-\frac{1}{\sin\theta}\right) = \frac{n\sin n\theta}{\sin\theta} = nU_{n-1}(x)\,.$$

18.4.14. Note that all instances of x and dx in the orthogonality integral should have been primed, i.e., written x' or dx'.

The shift compresses the original T_n into half the original range; this would decrease the orthogonality integral by a factor of 2. However,

$$\frac{1}{(1-x^2)^{1/2}} \longrightarrow \frac{1}{2\sqrt{x(1-x)}}\,,$$

so the weight factor given in the exercise is twice that of the original T_n. These factors of 2 cancel, so the orthogonality condition given for the shifted T_n has the same value as that given for the T_n.

18.4.15. (a) The expansion of x^m cannot include any Chebyshev polynomial of degree higher than m. Therefore if $n > m$ this integral must vanish due to orthogonality.

(b) The Chebyshev polynomial T_m has the parity of m; i.e., the $T_m(x)$ of even m are even functions of x; those of odd m are odd functions of x. Therefore the integral will vanish due to symmetry unless n and m have the same parity, meaning that the integral will vanish if $m+n$ is odd.

18.4.16. (a) Introducing the Rodrigues formula for T_n,

$$I_{mn} = \frac{(-1)^n \pi^{1/2}}{2^n \Gamma(n+\frac{1}{2})} \int_{-1}^{1} x^m \left(\frac{d}{dx}\right)^n (1-x^2)^{n+1/2}\, dx\,.$$

Integrating by parts n times, and noting that the endpoint integrated terms vanish, we reach

$$I_{mn} = \frac{\pi^{1/2}}{2^n \Gamma(n+\frac{1}{2})} \frac{m!}{(m-n)!} \int_{-1}^{1} x^{m-n}(1-x^2)^{n+1/2}\, dx$$

$$= \frac{\pi^{1/2}}{2^n \Gamma(n+\frac{1}{2})} \frac{m!}{(m-n)!} B\left(\frac{m-n+1}{2}, n+\frac{1}{2}\right)\,.$$

Here B is a beta function; the integral was evaluated using Eq. (13.50). Inserting the value of B in terms of gamma functions, we reach

$$I_{mn} = \frac{\pi^{1/2} m!}{2^n (m-n)!} \frac{\Gamma\left(\dfrac{m-n+1}{2}\right)}{\left(\dfrac{m-n}{2}\right)!}\,,$$

which when written in terms of double factorials can be seen equivalent to the answer in the text.

(b) From Eq. (18.105), write I_{mn} in terms of θ as

$$I_{mn} = \int_0^{\pi} \cos^m\theta \cos n\theta\, d\theta\,.$$

This integral can be evaluated by writing it entirely in terms of complex exponentials, expanding the mth power by the binomial theorem, and noting that nearly all the resulting integrals cancel. Alternatively, it can be recognized as a case of Formula 3.631(9) in Gradshteyn & Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. (Academic Press, 2000). The result can be brought to the same form as in part (a).

18.4.17. (a) Use the trigonometric form of U_n given in Eq. (18.107), and expand $\sin(n+1)\theta$:

$$|U_n| = \left|\frac{\sin(n+1)\theta}{\sin\theta}\right| = \left|\frac{\sin n\theta \cos\theta}{\sin\theta} + \cos n\theta\right| \le \left|\frac{\sin n\theta}{\sin\theta}\right| + 1 = |U_{n-1}| + 1\,.$$

Applying this result successively to U_{n-1}, $U_{n-2}, \cdots$, we conclude that $|U_n| \le |U_0| + n = n+1$, as required.

(b) From the result of Exercise 18.4.13(b), we know that

$$\left|\frac{d}{dx}\,T_n(x)\right| = n\,|U_{n-1}|\ .$$

Applying the result from part (a), $n\,|U_{n-1}| \le n^2$, as given in the text.

18.4.18. (a) From Eqs. (18.107) and (18.109), write $V_n = \sin n\theta$. This form is clearly bounded by ± 1.

(b) Equation (18.110) shows that W_n becomes infinite at $x = \pm 1$, so it is clearly unbounded on the specified range.

18.4.19. (a) Writing as trigonometric integrals, with $x = \cos\theta$, $dx = -\sin\theta\, d\theta$, and T_n given by Eq. (18.105),

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}\,dx = \int_0^{\pi} \cos m\theta \cos n\theta\, d\theta\,.$$

Evaluating these integrals leads to the results given in Eq. (18.116).

(b) Using Eq. (18.106),

$$\int_{-1}^{1} V_m(x)V_n(x)(1-x^2)^{-1/2}\,dx = \int_0^{\pi} \sin m\theta \sin n\theta\, d\theta\,,$$

consistent with the formulas in Eq. (18.117).

(c) Using Eq. (18.107),

$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}\,dx = \int_0^{\pi} \sin(m+1)\theta \sin(n+1)\theta\, d\theta\,.$$

Note that the negative powers of $\sin\theta$ cancel against the weighting function and the representation of dx. This integral leads to the formulas in Eq. (18.118).

(d) Using Eq. (18.108),

$$\int_{-1}^{1} W_m(x)W_n(x)(1-x^2)^{1/2}\,dx = \int_0^{\pi} \cos(m+1)\theta \cos(m+1)\theta\, d\theta\,,$$

consistent with Eq. (18.119).

18.4.20. (a) Using the techniques in parts (a) and (b) of Exercise 18.4.19, we get

$$\int_{-1}^{1} T_m(x)V_n(x)(1-x^2)^{-1/2}\,dx = \int_0^{\pi} \cos m\theta \sin n\theta\, d\theta\,,$$

which for many values of $m \neq n$ is nonzero. (The interval $(0, \pi)$ is not an interval of orthogonality for this function set.)

(b) Orthogonality is not obtained for all U_m and W_n. See the solution to part (a).

18.4.21. (a) Using Eq. (18.105), this recurrence formula is equivalent to

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta \,.$$

Writing

$$\cos(n \pm 1)\theta = \cos\theta\cos n\theta \mp \sin\theta\sin n\theta \,,$$

the verification is immediate.

(b) Since the equation of this part can be written

$$\cos(m+n)\theta + \cos(m-n)\theta = 2\cos m\theta\cos n\theta \,,$$

it can be confirmed by the approach of part (a) using the formulas for $\cos(m \pm n)\theta$.

18.4.22. Equation (18.91), the generating function for the T_n, can also be used to develop a formula involving the U_n. In particular,

$$\frac{1-t^2}{1-2xt+t^2} = T_0 + 2\sum_{n=1}^{\infty} T_n(x)t^n \,,$$

$$= (1-t^2)\sum_{n=0}^{\infty} U_n(x)t^n \,.$$

Equating the coefficients of equal powers of t in these two expansions, we get, for $n \geq 2$,

$$U_n(x) - U_{n-2}(x) = 2T_n(x) \,.$$

To reach the form given in the text, use the recurrence formula, Eq. (18.93), to replace $U_{n-2}(x)$ by $2xU_{n-1}(x) - U_n(x)$, and then divide through by 2.

To derive the second formula of this exercise, start from Eq. (18.95):

$$(1-x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x) \,,$$

and use the result from Exercise 18.4.13(b) to replace T_n' by nU_{n-1}. This yields

$$(1-x^2)U_{n-1}(x) = -xT_n(x) + T_{n-1}(x) = xT_n(x) - T_{n+1}(x) \,,$$

where the last equality was obtained using the recurrence formula for T_n, Eq. (18.92). Replacing n by $n+1$ leads to the formula in the text.

18.4.23. (a) Differentiate the trigonometric form for V_n, Eq. (18.106). The result is

$$\frac{dV_n(x)}{dx} = \frac{dV_n(\theta)}{d\theta}\,\frac{d\theta}{dx} = (n\cos n\theta)\left(-\frac{1}{\sin\theta}\right) = -n\,\frac{T_n(x)}{\sqrt{1-x^2}}\,.$$

(b) Using Eq. (18.109), we write the Rodrigues formula for V_n by multiplying that for U_{n-1} i Eq. (18.103) by $(1-x^2)^{1/2}$. Thus,

$$V_n = \frac{(-1)^{n-1}n\pi^{1/2}}{2^n\Gamma(n+\frac{1}{2})}\,\frac{d^{n-1}}{dx^{n-1}}\left[(1-x^2)^{n-1/2}\right].$$

Differentiating, multiplying by $(1-x^2)^{1/2}$, and changing the sign, we identify the result as n times the Rodrigues formula for T_n.

18.4.24. Making the binomial expansion of x^k, and combining terms with the same value of the binomial coeffiient,

$$x^k = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^k = \frac{1}{2^k}\sum_{n=0}^{k}\binom{k}{n}e^{in\theta}e^{-i(k-n)\theta}$$

$$= \frac{1}{2^k}\sum_{0\le n<k/2}\binom{k}{n}\left[e^{i(2n-k)\theta}+e^{i(k-2n)\theta}\right] + \begin{cases}\dfrac{1}{2^k}\dbinom{k}{k/2}, & k \text{ even},\\[1ex] 0\,, & k \text{ odd}.\end{cases}$$

The exponentials in the sum combine to form $2\cos(k-2n)\theta = 2T_{k-2n}(x)$, yielding a final result similar to that shown in the text. However, the text did not make clear that when k is even, the final term of the expansion is

$$\frac{1}{2^k}\binom{k}{k/2} = \frac{1}{2^k}\binom{k}{k/2}T_0(x)\,,$$

i.e., different by a factor 2 from the other terms of the finite expansion.

18.4.25. (a) Here we need the Chebyshev expansion of $\sin\theta$. Letting c_l be the coefficient of T_l in the expansion, we have

$$c_0 = \frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = -\frac{\cos\theta}{\pi}\bigg|_0^\pi = \frac{2}{\pi}\,.$$

For nonzero l, we have

$$c_l = \frac{2}{\pi}\int_0^\pi \sin\theta\,\cos(l\theta)\,d\theta\,.$$

This integral vanishes for odd l; for even l it can be evaluated by subjecting it to two integrations by parts, twice differentiating the factor $\cos(l\theta)$; this results in an integral proportional to the original one plus an endpoint term. Alternatively, the integral can be evaluated by table lookup. The result is (for even nonzero l)

$$c_l = -\frac{4}{\pi}\,\frac{1}{l^2-1}\,,$$

which corresponds to the answer in the text when l is replaced by $2s$.

(b) This function is odd and its expansion therefore contains only Chebyshev polynomials of odd order. The expansion coefficients c_l can be computed as

$$c_l = \left(\frac{2}{\pi}\right) 2\int_0^{\pi/2} \cos(l\theta)\,d\theta = \frac{4}{\pi l}\,\sin(l\pi/2)\,.$$

This sine function is $+1$ for $l = 1,\ 5, \cdots$ and -1 for $l = 3,\ 7, \cdots$; changing the index from l to $2s+1$ we can write

$$c_{2s+1} = \frac{4}{\pi}\,\frac{(-1)^s}{2s+1}\,,$$

consistent with the answer in the text.

18.4.26. (a) The Legendre expansion of $|x|$ is of the form $|z| = \sum_s c_{2s}P_{2s}(x)$, where

$$c_0 = \int_0^1 x\,dx = \frac{1}{2}$$

and, for nonzero s,

$$c_{2s} = (4s+1)\int_0^1 xP_{2s}(x)\,dx = \int_0^1 \Big[(2s+1)P_{2s+1}(x) + 2sP_{2s-1}(x)\Big]\,dx$$

$$= \int_0^1 \left[\frac{(2s+1)[P'_{2s+2}(x) - P'_{2s}(x)]}{4s+3} + \frac{2s[P'_{2s}(x) - P'_{2s-2}(x)]}{4s-1}\right] dx\,.$$

The two steps in the forgoing analysis were the use of the basic Legendre recurrence formula, Eq. (15.18), and the derivative recurrence, Eq. (15.22). The reason for taking these steps is that the integration is now trivial; the upper integration limit cancels, as all $P_n(1)$ are unity; the lower limit involves values of $P_n(0)$ which can be obtained either as a coefficient in Eq. (15.14) or from the answer to Exercise 15.1.12. The result we need is

$$P_{2s}(0) = (-1)^s\,\frac{(2s-1)!!}{(2s)!!}\,.$$

Thus,

$$c_{2s} = -\frac{2s+1}{4s+3}(-1)^{s+1}\frac{(2s+1)!!}{(2s+2)!!} + \left[\frac{2s+1}{4s+3} - \frac{2s}{4s-1}\right](-1)^s\,\frac{(2s-1)!!}{(2s)!!}$$

$$+ \frac{2s}{4s-1}(-1)^{s-1}\frac{(2s-3)!!}{(2s-2)!!}\,.$$

This expression simplifies to the form given in the Exercise.

The expansion of $|x|$ in Chebyshev polynomials is most easily done using their trigonometric forms, in which the subrange $x > 0$ corresponds to $0 \le \theta \le \pi/2$. Letting c_{2s} be the coefficient of T_{2s}, we require

$$c_0 = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta\, d\theta = \frac{2}{\pi}\,,$$

and for nonzero s,

$$c_{2s} = \frac{4}{\pi}\int_0^{\pi/2} \cos\theta \cos 2s\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \left[\cos(2s+1)\theta + \cos(2s-1)\theta\right]\, d\theta$$

$$= \frac{2}{\pi}\left[\frac{\sin(s+\frac{1}{2})\pi}{2s+1} + \frac{\sin(s-\frac{1}{2})\pi}{2s-1}\right] = \frac{2}{\pi}\,(-1)^s\left[\frac{1}{2s+1} - \frac{1}{2s-1}\right]$$

$$= \frac{4}{\pi}\,\frac{(-1)^{s+1}}{4s^2-1}\,.$$

(b) The limiting ratio of the coefficients is given incorrectly in the text. The correct value of the ratio in the limit of large s is $(\pi s)^{-1/2}$.

Foe general s, the ratio of the coefficient of T_{2s} to that of P_{2s} is

$$\text{Ratio} \;= \frac{\dfrac{4}{\pi}\,\dfrac{1}{4s^2-1}}{\dfrac{(2s-3)!!}{(2s+2)!!}\,(4s+1)} \approx \frac{1}{\pi s}\,\frac{(2s+2)!!}{(2s+1)!!}\,.$$

The ratio of double factorials can be treated by writing them in terms of factorials and then using Stirling's formula. We have

$$\ln\left[\frac{(2s+2)!!}{(2s+1)!!}\right] = \ln\left[\frac{2^{2s+2}[(s+1)!]^2}{(2s+2)!}\right]$$

$$= (2s+2)\ln 2 + 2\ln\Gamma(s+2) - \ln\Gamma(2s+3)$$

$$= (2s+2)\ln 2 + 2\left[\frac{1}{2}\,\ln 2\pi + \left(s+\frac{3}{2}\right)\ln(s+1) - (s+1)\right]$$

$$- \left[\frac{1}{2}\,\ln 2\pi + \left(2s+\frac{5}{2}\right)\ln(2s+2) - (2s+2)\right]$$

$$= \frac{1}{2}\,\ln\pi + \frac{1}{2}\ln(s+1)\,.$$

The final result in the limit of large s is therefore

$$\text{Ratio} \;\approx (\pi s)^{-1}\,(\pi s)^{1/2} = (\pi s)^{-1/2}.$$

18.4.27. Form

$$\int_{-1}^{1} |x|^2 (1-x^2)^{-1/2}\, dx = \int_0^{\pi} \cos^2\theta\, d\theta = \frac{\pi}{2}\,,$$

and then also calculate the same quantity using the expansion of $|x|$ in Chebyshev polynomials from Exercise 18.4.26. We have

$$\frac{\pi}{2} = \int_{-1}^{1} |x|^2 (1-x^2)^{-1/2}\, dx$$

$$= \int_{-1}^{1} \left[\frac{2}{\pi} + \frac{4}{\pi}\sum_{s=1}^{\infty}(-1)^{s+1}\frac{1}{4s^2-1}\,T_{2s}(x)\right]^2 (1-x^2)^{-1/2}\, dx\,.$$

Using the orthogonality of the T_n and the values of their normalization integrals, the above equation reduces to

$$\frac{\pi}{2} = \frac{4}{\pi^2}\,(\pi) + \frac{16}{\pi^2}\sum_{s=1}^{\infty}\frac{1}{(4s^2-1)^2}\left(\frac{\pi}{2}\right)\,.$$

When this equation is multiplied through by $\pi/4$ we get the result given in the text.

18.4.28. (a) Taking $x = \cos\theta$, this equation is seen equivalent to

$$\theta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\,T_{2n+1}(\cos\theta)\,.$$

To confirm it we therefore need to develop the Chebyshev expansion of θ. The coefficient of T_0, c_0, is

$$c_0 = \frac{1}{\pi}\int_0^{\pi}\theta\, T_0(\cos\theta)\, d\theta = \frac{\pi^2}{2}\,.$$

The coefficients of T_l for nonzero l can be developed via an integration by parts (for which the integrated endpoint terms vanish). We have

$$c_l = \frac{2}{\pi}\int_0^{\pi}\theta\ \cos(l\theta)\, d\theta = -\frac{2}{\pi l}\int_0^{\pi}\sin(l\theta)\, d\theta = \frac{2}{\pi l^2}\cos(l\theta)\Big|_0^{\pi}$$

$$= \begin{cases} -\dfrac{4}{\pi l^2}\,, & l \text{ odd},\\ 0, & l \text{ even}.\end{cases}$$

Making a change of the index variable from l to $2n+1$, the expansion becomes

$$\theta = \frac{\pi}{2}\,T_0(\cos\theta) - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\,T_{2n+1}(\cos\theta)\,,$$

equivalent to the answer in the text.

(b) We note that $\sin^{-1} x = \sin^{-1}(\cos\theta) = \pi/2 - \theta$. This observation leads immediately to the expansion in the text.

18.5 Hypergeometric Functions

18.5.1. (a) If c is integral and other than 1, either $(c)_n$ or $(2-c)_n$ vanishes for some n, so that one of the two hypergeometric series has infinite terms and cannot represent a function. If $c = 1$, both series become identical, so then there is also only one series solution.

(b) If $c = -2$ and $a = -1$, and the series ${}_2F_1(-1, b; -2; x)$ is deemed to terminate when the zero in the numerator is reached, we get

$$ {}_2F_1(-1, b; -2; x) = 1 + \frac{1}{2}\, bx\,, $$

which can be confirmed as a solution to the hypergeometric ODE for $a = -1$, $c = -2$. Since we still get $x^3 {}_2F_1(2, b+3; 4; x)$ as another series solution, we see that for these values of a and c both solutions can be written in terms of hypergeometric series.

18.5.2. These recurrence relations are those in Eqs. (15.18), (18.92), and (18.93).

18.5.3. It is somewhat easier to work backward from the answers than to derive them. A derivation could use the following notions: (1) A series in x that terminates after x^n can be obtained by placing $-n$ within one of the numerator Pochhammer symbols; (2) If the mth term of the expansion involves $m!$, it can be obtained as $(1)_m$; (3) If the mth term involves $(n+m)!/(n-m)!$, it can be obtained as $(-1)^m(-n)_m(n+1)_m$; (4) If the mth term involves $(2m)!$, it can be obtained as $2^m m!\,(2m-1)!!$, with the double factorial generated from a construction such as $2^m(1/2)_m$.

(a) Start from Eq. (18.14), and write $T_{2n}(x)$ as an ascending power series:

$$ T_{2n}(x) = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m 2^{2m} n(n+m-1)!}{(n-m)!(2m)!} (x^2)^m\,. $$

Rearrange the coefficient within the sum to the form

$$ \frac{2^{2m}\big[(-n)(-n+1)\cdots(-n+m-1)\big]\big[(n)(n+1)\cdots(n+m-1)\big]}{(2^m m!)\;\, 2^m \;\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2m-1}{2}\right)} $$

$$ = \frac{(-n)_m (n)_m}{m!\left(\frac{1}{2}\right)_m}\,, $$

thereby identifying the summation as a hypergeometric function.

Parts (b), (c), and (d) are transformed in a similar fashion.

18.5.4. The representation of part (a) has a leading factor that was obtained while solving Exercise 18.5.3. To obtain the leading factors of the other representations, it suffices to check the coefficient of x^0, which for all the functions ${}_2F_1$ is unity. From Eq. (18.114), the coefficient of x^0 in $x^{-1}T_{2n+1}(x)$ is

$$\frac{2n+1}{2}(-1)^n \frac{n!}{n!\,1!}\,2 = (-1)^n(2n+1)\,.$$

The other leading factors are checked in the same way.

18.5.5. The formula for Q_ν given in this exercise is incorrect; the third argument of the hypergeometric function should be $\nu + \frac{3}{2}$.

The series in inverse powers for $Q_l(x)$ is given in a convenient form in Exercise 15.6.3; it is

$$Q_l(x) = x^{-l-1}\sum_{s=0}^{\infty}\frac{(l+2s)!}{(2s)!!(2l+2s+1)!!}\,x^{-2s}\,.$$

The hypergeometric series with which this expansion is to be compared is, from Eq. (18.121),

$$Q_l(x) = \frac{\pi^{1/2}l!\,x^{-l-1}}{\Gamma\left(l+\frac{3}{2}\right)2^{l+1}}\sum_{s=0}^{\infty}\frac{\left(\dfrac{l+1}{2}\right)_s\left(\dfrac{l+2}{2}\right)_s}{s!\,\left(l+\dfrac{3}{2}\right)_s}\,x^{-2s}\,.$$

An initial step toward the verification is to make the identification

$$2^{l+s+1}\Gamma(l+\tfrac{3}{2})\left(l+\tfrac{3}{2}\right)_s = \pi^{1/2}(2l+2s+1)!!\,.$$

We also note that $2^s\,s! = (2s)!!$ and that

$$l!\,\left(\frac{l+1}{2}\right)_s\left(\frac{l+2}{2}\right)_s = 2^{-2s}(l+2s)!\,.$$

Inserting these relationships, the two forms for Q_l are brought into correspondence.

18.5.6. Introduce a binomial expansion in the integral for B_x and perform the integration in t termwise. The result is

$$B_x(p,q) = \sum_{k=0}^{\infty}(-1)^k\binom{q-1}{k}\frac{x^{p+k}}{p+k} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(q)}{k!\,\Gamma(q-k)\,(p+k)}\,x^{p+k}\,.$$

From Eq. (18.121), the proposed hypergeometric series is

$$B_x(p,q) = p^{-1}x^p \sum_{k=0}^{\infty} \frac{(p)_k(1-q)_k}{k!\,(p+1)_k}\, x^k\,.$$

Noting that $(p)_k/(p+1)_k = p/(p+k)$ and that

$$(1-q)_k = (1-q)(2-q)\cdots(k-q) = \frac{(-1)^k\Gamma(q)}{\Gamma(q-k)}\,,$$

the verification is straightforward.

18.5.7. Introduce a binomial expansion for $(1-tz)^{-a}$ and integrate termwise, identifying the integrals as beta functions according to Eq. (13.49):

$$\begin{aligned}
{}_2F_1(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k z^k \int_0^1 t^{b-1}(1-t)^{c-b-1} t^k\, dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k B(b+k, c-b)\, z^k\,.
\end{aligned}$$

Evaluating the beta function and using

$$\binom{-a}{k} = \frac{(-1)^k (a)_k}{k!}\,,$$

we arrive at

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, \frac{\Gamma(b+k)}{\Gamma(b)}\, \frac{\Gamma(c)}{\Gamma(c+k)}\, z^k\,.$$

Since $\Gamma(b+k)/\Gamma(b) = (b)_k$ and $\Gamma(c+k)/\Gamma(c) = (c)_k$, the standard expansion of the hypergeometric function is recovered.

The integral diverges unless the powers of both t and $1-t$ are greater than -1. Hence the condition $c > b > 0$.

18.5.8. If we set $z = 1$, we have, using the integral representation of Exercise 18.5.7,

$$\begin{aligned}
{}_2F_1(a,b;c;1) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\, dt \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\, B(b, c-b-a)\,.
\end{aligned}$$

Inserting the value of the beta function, we obtain the desired result.

18.5.9. Use the integral representation of Exercise 18.5.7, with z replaced by $-x/(1-x)$. Then, noting that

$$(1-tz)^{-a} \longrightarrow \left(1+\frac{tx}{1-x}\right)^{-a} = (1-x)^a \Big[1-(1-t)x\Big]^{-a},$$

and changing the integration variable to $u = 1-t$, the integral representation assumes the form

$${}_2F_1\left(a,b;c;\frac{-x}{1-x}\right) = \frac{\Gamma(c)(1-x)^a}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 (1-u)^{b-1}u^{c-b-1}(1-ux)^{-a}\,du$$

$$= (1-x)^a\ {}_2F_1(a,c-b;c;x)\,,$$

equivalent to what we are to prove.

18.5.10. To conform to the notation adopted in the text, $(n-\frac{1}{2})!$ should be written $\Gamma(n+\frac{1}{2})$.

A simple approach is to use Eq. (12.9); for the ODE satisfied by T_n, $p(x) = (1-x^2)$, and the weight function for orthogonality is $w(x) = (1-x^2)^{-1/2}$. These parameter values make $wp^n = (1-x^2)^{n-1/2}$ and the Rodrigues formula therefore has the form

$$T_n(x) = c_n(1-x^2)^{1/2}\left(\frac{d}{dx}\right)^n (1-x^2)^{n-1/2}\,.$$

The coefficients c_n must now be chosen to reproduce the T_n at their conventional scaling. It is convenient to use the values $T_n(1) = 1$ to set the scaling. All terms of the n-fold differentiation in the Rodrigues formula will contain a net positive power of $1-x^2$ and therefore vanish at $x=1$ unless all the n differentiations are applied to the factor $(1-x^2)^{n-1/2}$ (and none to the factors $-2x$ that are produced by earlier differentiations). If the n differentiations are applied in this way, they will produce a final result containing $(1-x^2)^{-1/2}$, which will cancel against the factor $(1-x^2)^{1/2}$ preceding the derivative. The differentiation also produces a coefficient

$$\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right) = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(1/2)} = \frac{\Gamma)n+\frac{1}{2})}{\sqrt{\pi}}\,,$$

and a factor $(-2x)^n$ from the derivative of $1-x^2$. When we set $x=1$,

$$T_n(1) = 1 = c_n(-1)^n\,2^n\,\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\,,$$

from which we obtain the value of c_n shown in the exercise.

18.5.11. This summation has a form corresponding to a hypergeometric function, except that the extent of the ν summation does not explicitly extend to infinity. However, all terms with $\nu > \min(m,n)$ vanish because of the vanishing of one or both of the Pochhammer symbols, so formally the summation can be extended to infinity without changing its value. Referring to Eq. (18.121), we identify the summation as

$$ {}_2F_1\left(-m,-n;\frac{1-m-n}{2};\frac{a^2}{2(a^2-1)}\right). $$

18.5.12. Write the formula of this exercise in the form

$$ {}_2F_1(-n,b;c;1) = \frac{\Gamma(c-b+n}{\Gamma(c-b)}\,\frac{\Gamma(c)}{\Gamma(c+n)}. $$

This form is what is obtained if we evaluate ${}_2F_1(-n,b;c;1)$ using the formula of Exercise 18.5.8.

18.6 Confluent Hypergeometric Functions

18.6.1. The power-series expansion of the error function, Eq. (13.91), is

$$ \operatorname{erf}(x) = \frac{2}{\pi^{1/2}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}. $$

The form given in the exercise corresponds to the expansion

$$ \frac{2x}{\pi^{1/2}}\sum_{n=0}^{\infty}\frac{(-1)^n (1/2)_n x^{2n}}{(3/2)_n\,n!}. $$

Noting that

$$ \frac{(\frac{1}{2})_n}{(\frac{3}{2})_n} = \frac{1/2}{n+\frac{1}{2}} = \frac{1}{2n+1}, $$

the confluent hypergeometric representation is confirmed.

18.6.2. From the definitions in Exercise 12.6.1, we can identify

$$ C(x) + is(x) = \int_0^x e^{i\pi u^2/2}\,du. $$

Making a change of variable to $y = e^{-i\pi/4}(\pi/2)^{1/2}u$, this integral becomes

$$ \begin{aligned} C(x) + is(x) &= \left(\frac{2}{\pi}\right)^{1/2} e^{i\pi/4}\int_0^{xe^{-i\pi/4}\sqrt{\pi/2}} e^{-y^2}\,dy \\ &= \frac{e^{i\pi/4}}{2^{1/2}}\ \operatorname{erf}\left(xe^{-i\pi/4}\sqrt{\frac{\pi}{2}}\right). \end{aligned} $$

Using the confluent hypergeometric representation of the error function in Exercise 18.6.1, this expression can be written

$$C(x) + is(x) = \frac{e^{i\pi/4}}{2^{1/2}}\,\frac{2}{\pi^{1/2}}\left(xe^{-i\pi/4}\sqrt{\frac{\pi}{2}}\right) M\left(\frac{1}{2},\frac{3}{2},-\left[xe^{-i\pi/4}\sqrt{\frac{\pi}{2}}\right]^2\right)$$

$$= x\,M\left(\frac{1}{2},\frac{3}{2},\frac{i\pi x^2}{2}\right).$$

18.6.3. Starting from the formula for y in the exercise,

$$y' = -\frac{ay}{x} + \frac{ae^{-x}}{x},$$

$$y'' = \frac{ay}{x^2} - \frac{ay'}{x} - \frac{ae^{-x}}{x^2} - \frac{ae^{-x}}{x}$$

$$= \frac{a(a+1)y}{x^2} - \frac{a(a+1)e^{-x}}{x^2} - \frac{ae^{-x}}{x},$$

and therefore

$$xy'' = \frac{a(a+1)y}{x} - \frac{a(a+1)e^{-x}}{x} - ae^{-x},$$

$$(a+1)y' = -\frac{a(a+1)y}{x} + \frac{a(a+1)e^{-x}}{x},$$

$$xy' = -ay + ae^{-x},$$

$$ay = ay\,.$$

Adding these equations together, we form the ODE relevant to this exercise: $xy'' + (a+1+x)y' + ay = 0$.

18.6.4. From Eq. (14.131),

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\,\frac{e^{-z}}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-t}\,t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2} dt\,.$$

Change the integration variable to $y = t/2z$, reaching

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\,\frac{e^{-z}}{\Gamma(\nu+\frac{1}{2})}(2z)^{\nu+1/2}\int_0^\infty e^{z-2zy}y^{\nu-1/2}(1+y)^{\nu-1/2}\,dy\,.$$

The integral is now in the form corresponding to the integral representation of $U(\nu+1/2, 2\nu+1, 2z)$ given in Eq. (18.145), and the formula for $K_\nu(z)$ reduces to that given in the text.

18.6.5. Using the formulas from Section 13.6, we write

$$\mathrm{Ci}(x) + i\mathrm{si}(x) = -\int_x^\infty \frac{e^{it}}{t}\,dt = -\int_{-ix}^\infty \frac{e^{-y}}{y}\,dy = -E_1(-ix)\,.$$

Note now that, using Eq. (18.145),

$$U(1,1,x) = \frac{1}{\Gamma(1)}\int_0^\infty \frac{e^{-xt}}{1+t}\,dt = e^x\int_0^\infty \frac{e^{-x(t+1)}}{t+1}\,dt = e^x\int_1^\infty \frac{e^{-xt}}{t}\,dt$$
$$= e^x E_1(x)\,.$$

Using this formula, we write $-E_1(-ix)$ in terms of U, obtaining the desired result.

18.6.6. (a) Because the confluent hypergeometric function has argument $u = x^2$, the corresponding ODE is

$$u\,\frac{d^2y}{du^2} + (c-u)\,\frac{dy}{du} - ay = 0\,,$$

and if $y = y(x)$ the derivatives in this equation take the form

$$\frac{dy}{du} = \frac{dy}{dx}\,\frac{dx}{du} = \frac{1}{2x}\,\frac{dy}{dx}\,,$$

$$\frac{d^2y}{du^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{du}\right)^2 + \frac{dy}{dx}\,\frac{d^2x}{du^2} = \frac{1}{4x^2}\,\frac{d^2y}{dx^2} - \frac{1}{4x^3}\,\frac{dy}{dx}\,.$$

Now, with $c = 3/2$, $a = -n$, and $y = H_{2n+1}(x)/x$, we have

$$-ay = \frac{nH_{2n+1}}{x}\,,$$

$$(c-u)\,\frac{dy}{du} = \left(\frac{3}{2} - x^2\right)\frac{1}{2x}\left(\frac{H'_{2n+1}}{x} - \frac{H_{2n+1}}{x^2}\right),$$

$$u\,\frac{d^2y}{du^2} = \frac{1}{4}\left[\frac{H''_{2n+1}}{x} - \frac{3H'_{2n+1}}{x^2} + \frac{3H_{2n+1}}{x^3}\right].$$

Forming the confluent hypergeometric ODE by adding these terms together and then multiplying through by $4x$, we reach

$$H''_{2n+1} - 2xH'_{2n+1} + 2(2n+1)H_{2n+1} = 0\,,$$

confirming that $H_{2n+1}(x)/x$ is a solution to the specified confluent hypergeometric equation (in x^2) if H_{2n+1} is a solution of index $2n+1$ to the Hermite ODE.

(b) The parameter value $a = -n$ shows that our confluent hypergeometric function will be a polynomial (in x^2) of degree n, so H_{2n+1} must be a polynomial (in x) of degree $2n+1$. Since $M(-n,3/2,0)=1$, this part of the exercise is designed to show that Eq. (18.149) yields H_{2n+1} at its agreed-upon scale. From Eq. (18.9), changing n to $2n+1$ and examining the term with $s=n$, we have

$$\text{Term containing } x^1 \;=\frac{(-1)^n(2n+1)!}{1!\,n!}\,2x\,,$$

consistent with the scale of Eq. (18.149).

18.6.7. Use the confluent hypergeometric representation of $L_n^m(x)$ in Eq. (18.151) to rewrite the equation of this exercise in terms of Laguerre functions. The relevant parameter values are $a=-n$ and $c=m+1$. We get

$$(m+n+1)\,\frac{(n+1)!\;m!}{(n+m+1)!}\,L_{n+1}^m-(2n+m+1-x)\,\frac{n!\;m!}{(n+m)!}\,L_n^m$$
$$+\,n\,\frac{(n-1)!\,m!}{(n+m-1)!}\,L_{n-1}^m=0\,.$$

Dividing through by $n!\,m!/(n+m)!$, we get the recurrence formula given in Eq. (18.66):

$$(n+1)\,L_{n+1}^m(x)-(2n+m+1-x)\,L_n^m(x)+(n+m)\,L_{n-1}^k(x)=0\,.$$

18.6.8. (a) Use the integral representation in Eq. (18.144) and make a change of the integration variable to $s=1-t$. We get

$$\begin{aligned}M(a,c,x)&=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}\,dt\\&=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{x(1-s)}(1-s)^{a-1}s^{c-a-1}\,ds\\&=\frac{\Gamma(c)\,e^x}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-xs}s^{c-a-1}(1-s)^{a-1}\,ds\\&=e^xM(c-a,c,-x)\,.\end{aligned}$$

(b) This part of the exercise uses Eq. (18.142). Note that Eq. (18.142) contains a misprint; the quantity $\Gamma(-c)$ in its last term should be changed to $\Gamma(2-c)$.

If in Eq. (18.142) we replace a by $a'=a+1-c$ and c by $c'=2-c$ and in addition multiply both terms within the square brackets by x^{1-c}, the square bracket will remain unchanged except for an overall sign change. But the factor $\sin\pi c$ becomes $\sin\pi c'=\sin(2\pi-\pi c)$, so its sign changes also and we have $x^{1-c}U(a',c',x)=U(a,c,x)$, as required.

18.6.9. (a) On the right-hand side of the equation of this exercise, change b to c (two occurrences).

In the integral representation, Eq. (18.144), differentiation with respect to x will cause the power of t to be incremented, but does not change the power of $1-t$. This change corresponds to a unit increase in both a and c in the confluent hypergeometric function. Since the gamma functions preceding the integral have not changed, we need to increment the arguments of $\Gamma(a)$ and $\Gamma(c)$ and compensate for these changes through multiplication by a/c. A second derivative will cause a similar index shift, but this time the compensation factor will be $(a+1)/(c+1)$. The generalization to arbitrary derivatives corresponds to the formula in the text.

(b) This formula can be derived by a procedure similar to that used in part (a). The negative exponential generates a sign change with each differentiation.

18.6.10. Our procedure will be to verify that the integral representations satisfy the confluent hypergeometric ODE, identify the representation as M or U, and confirm its scale either at $x=0$ or asymptotically at large x.

(a) Dropping for the moment the constant factors preceding the integral, we consider the effect upon the representation of the operations that correspond to the ODE. The differentiations to produce M'' will introduce an additional factor t^2 in the integrand, while the differentiation to produce M' will introduce a factor t. Then, assuming the validity of the integral representation, our ODE corresponds to

$$\int_0^1 e^{xt}\left[xt^{a+1}(1-t)^{c-a-1}+(c-x)t^a(1-t)^{c-a-1}-at^{a-1}(1-t)^{c-a-1}\right]dt=0\,.$$

We now perform integrations by parts on those terms that contain x, integrating e^{xt} and differentiating the remainder of the integrand. By choosing these terms, we eliminate all x dependence from the integrand except for the single positive factor e^{xt}. The result is that the entire quantity multiplying e^{xt} now vanishes, and the endpoint integrated terms vanish as well. Thus, the ODE is satisfied for the integral representation of $M(a,c,x)$.

(b) A procedure similar to that given for the integral representation of M can also be carried out for U; the main difference is that the minus sign in e^{-xt} and the presence of $1+t$ rather than $1-t$ cause some sign differences, but all terms still cancel. The negative exponential also causes vanishing of the endpoint integrated terms at $x=\infty$, so the ODE is also satisfied for the integral representation of $U(a,c,x)$.

The definition of $M(a,c,x)$ as presented in the text yields a result that can become large for large x (for suitably chosen parameter values) and is regular at $x=0$; in fact, its value at $x=0$ is unity. The text does not show that the formula given for $U(a,c,x)$, Eq. (18.142), has particular properties at $x=0$ and $x=\infty$, but a more detailed analysis (see the additional readings) establishes that U as defined in that equation is singular at $x=0$ and approaches zero for a range of parameter values at $x=\infty$. With these facts available, we can conclude that because the integral representation given for M is nonsingular at $x=0$, it cannot contain an admixture of U. Moreover, the representation given for U cannot contain M because it vanishes at large x for all parameter values for which the integral converges.

Our final task is to confirm that these integral representations are properly scaled. The representation for M reduces at $x=0$ to a beta function, and the quantity premultiplying the integral is just the inverse of that beta function, leading to $M(a,c,0)=1$. The factor multiplying the integral for U is that needed for correct asymptotic behavior; for a proof see the additional readings.

18.6.11. This procedure was used to solve Exercise 18.6.8.

18.6.12. This formula was derived as a step in the solution of Exercise 18.6.5.

18.6.13. (a) For $M(a,c,x)$, change the integration variable to $u=1-t$ and develop the integrand as a power series in u. Thus, formally (irrespective of convergence) we have

$$M(a,c,x) \sim \frac{\Gamma(c)\,e^x}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-xu}u^{c-a-1}(1-u)^{a-1}\,du$$

$$= \frac{\Gamma(c)\,e^x}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}\binom{a-1}{n}(-1)^n\int_0^1 e^{-ux}u^{c-a-1+n}\,du\,.$$

Changing the integration variable to $v=xu$ and assuming that x is large enough that the upper limit for integration in v can without significant error be changed to $v=\infty$, we have

$$M(a,c,x) \sim \frac{\Gamma(c)\,e^x}{\Gamma(a)\Gamma(c-a)x^{c-a}}\sum_{n=0}^{n_{\max}}\binom{a-1}{n}\frac{(-1)^n}{x^n}\int_0^\infty v^{c-a-1+n}e^{-v}\,dv$$

$$\approx \frac{\Gamma(c)\,e^x}{\Gamma(a)\Gamma(c-a)x^{c-a}}\sum_{n=0}^{n_{\max}}\binom{a-1}{n}\frac{(-1)^n}{x^n}\,\Gamma(c-a+n)\,.$$

To reconcile this answer with that given in the text, note that

$$(-1)^n\binom{a-1}{n} = \frac{(1-a)(2-a)\cdots(n-a)}{n!} \quad \text{and}$$

$$\frac{\Gamma(c-a+n)}{\Gamma(c-a)} = (c-a)\cdots(c-a+n-1)\,.$$

(b) A treatment similar to that used in part(a) confirms the answer given in the text.

18.6.14. When a linear second-order ODE is written in self-adjoint form as $[p(x)y']'+q(x)y = 0$, the Wronskian of any two linearly independent solutions of the ODE must be proportional to $1/p$. For the confluent hypergeometric equation $p(x) = x^c e^{-x}$; this can be found by methods discussed in Chapter 7 and checked by direct evaluation. The proportionality constant, which may depend upon the parameters a and c, may be determined from the behavior of M and U at any convenient value of x.

Using the asymptotic values of M and U from Exercise 18.6.13 and the leading terms of their derivatives, we have

$$M(a,c,x) \sim \frac{\Gamma(c)}{\Gamma(a)}\,\frac{e^x}{x^{c-a}} \sim M'(a,c,x)\,,$$

$$U(a,c,x) \sim x^{-a}\,, \qquad U'(a,c,x) \sim -\frac{a}{x^{a+1}}\,.$$

Noting that MU' becomes negligible relative to $M'U$, we identify the Wronskian as

$$\text{Wronskian}(M,U) = -M'U = -\frac{\Gamma(c)}{\Gamma(a)}\,\frac{e^x}{x^{c-a}}\,x^{-a}\,,$$

equivalent to the formula in the text.

If a is zero or a negative integer, M does not exist, and the Wronskian evaluates to zero.

18.6.15. The Coulomb wave equation is of the form of Eq. (18.153), so its regular solutions will be Whittaker functions $M_{k\mu}$ of appropriate indices and argument. To convert the term $-1/4$ of Eq. (18.153) into the $+1$ of the Coulomb equation, the argument of the Whittaker function needs to be $2ir$. This will cause the y'' term of the ODE to be multiplied by $-1/4$. We also need to replace x by $2ir$ in the coefficient of $M_{k\mu}$.

To complete the correspondence of these two ODEs, we set $k = i\eta$ and $\mu = L + 1/2$. Writing $M_{k\mu}$ in terms of $M(a,c,x)$, we get

$$M_{i\eta,L+1/2}(2ir) = e^{-ir}(2ir)^{L+1}M(L+1-i\eta, 2L+2, 2ir)\,.$$

18.6.16. (a) Insert the confluent hypergeometric representation of the Laguerre function. The demonstration is straightforward.

(b) We need a solution with the opposite sign of n^2, i.e., with n replaced by in. As seen in Exercise 18.6.15, this also corresponds to the replacement of α by $i\alpha$.

18.6.17. The answers are given in the text.

18.7 Dilogarithm

18.7.1. Since this series, with $z = 1$, is convergent, with value $\zeta(2)$, the magnitude of its sum for all z of unit magnitude will be no greater than $\zeta(2)$ (and equal to $\zeta(2)$ only if all its terms have the same phase). Thus, the series is convergent for all z on the unit circle.

18.7.2. Using Eq. (18.161) with $z = \frac{1}{2}$, we get

$$\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}\,.$$

18.7.3. The multiple values arise from

$$\ln^2 2 = (\ln 2 + 2\pi n i)^2 = \ln^2 2 + 4n\pi i \ln 2 - 4n^2\pi^2\,,$$

, so

$$\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{\ln^2 2}{2} + 2n^2\pi^2 - 2n\pi i \ln 2\,,$$

where n can be any positive or negative integer or zero. Note that different n lead to different values of both the real and imaginary parts of $\mathrm{Li}_2(1/2)$.

18.7.4. The principal branch of Li_2 is usually defined to be the result given by the power series expansion and its analytic continuation, with a branch cut extending just below the positive real axis from 1 to infinity. This means that in using Eq. (18.161) we take $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$ and seek to verify that $\mathrm{Li}_2(0) = 0$. The verification depends upon the fact that $\lim_{z\to 0} \ln z \ln(1-z) = 0$, which can be proved by expanding $\ln(1-z)$ as $-z - z^2/2 - \cdots$ and noting that the leading term for small z, $-z \ln z$, approaches the limit zero.

18.7.5. Rewrite Eq. (18.163), with z replaced by $(1 + y^{-1})/2$. Then

$$\frac{z}{z-1} = \frac{\dfrac{y^{-1}+1}{2}}{\dfrac{y^{-1}-1}{2}} = \frac{1+y}{1-y}\,, \qquad 1 - z = 1 - \left(\frac{1+y^{-1}}{2}\right) = \frac{1-y^{-1}}{2}\,,$$

and we have

$$\mathrm{Li}_2\left(\frac{1+y^{-1}}{2}\right) + \mathrm{Li}_2\left(\frac{1+y}{1-y}\right) = -\frac{1}{2}\ln^2\left(\frac{1-y^{-1}}{2}\right)\,,$$

equivalent to the relationship to be proved.

18.7.6. Transform the Li_2 functions to forms in which their arguments are real and in the range $(-\infty, +1)$. The function needing transformation is $\mathrm{Li}_2(\zeta_j)$, since ζ_j can be larger than $+1$. Using Eq. (18.161) for each j, the three j values together contribute $\pi^2/2$ (which cancels the $-\pi^2/2$ in the original form. Within the summation, we replace $\mathrm{Li}_2(\zeta_j)$ with $-\mathrm{Li}_2(1-\zeta_j) - \ln\zeta_j \ln(1-\zeta_j)$. The final result is

$$-\frac{32\pi^3}{\alpha_1\alpha_2\alpha_3}\sum_{j=1}^{3}\Big[\mathrm{Li}_2(-\zeta_j) + \mathrm{Li}_2(1-\zeta_j) + \ln\zeta_j \ln(1+\zeta_j)\Big].$$

18.8 Elliptic Integrals

18.8.1. Writing $ds = (dx^2 + dy^2)^{1/2}$ as the differential of distance along the path, we compute for the ellipse at the point described by parameter θ

$$\frac{ds}{d\theta} = \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{1/2} = \left[a^2\cos^2\theta + b^2\sin^2\theta\right]^{1/2}.$$

The path length for the first quadrant is therefore the integral

$$\int_0^{\pi/2}\left[a^2\cos^2\theta + b^2\sin^2\theta\right]^{1/2} d\theta = \int_0^{\pi/2}\left[a^2 + (b^2-a^2)\sin^2\theta\right]^{1/2}$$

$$= a\int_0^{\pi/2}(1 - m\sin^2\theta)^{1/2}\,d\theta\,,$$

where $m = (a^2 - b^2)/a^2$. This rearrangement is only appropriate if $a > b$, since we want our elliptic integral to be in the standard form with $0 < m < 1$. If $b > a$, we could interchange the roles of these parameters by changing the integration variable to $\pi/2 - \theta$ and taking a factor b outside the square root.

Completing the analysis for the current case, we identify the elliptic integral as $E(m)$, thereby confirming the answer in the text.

18.8.2. Expand the integrand in the trigonometric form of $E(m)$:

$$E(m) = \int_0^{\pi/2}(1 - m\sin^2\theta)^{1/2}\,d\theta$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\binom{1/2}{n}(-1)^n m^n \int_0^{\pi/2}\sin^{2n}\theta\,d\theta\,.$$

Then use the formulas (valid for $n > 0$)

$$\binom{1/2}{n} = (-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}\,, \qquad \int_0^{\pi/2}\sin^{2n}\theta\,d\theta = \frac{\pi}{2}\,\frac{(2n-1)!!}{(2n)!!}$$

to bring $E(m)$ to the form given in the text.

18.8.3. Here the arguments of K and E are implicitly assumed to be m.

Form the difference $K = E$ showing explicitly the first two terms of their power-series expansions:

$$K(m) = \frac{\pi}{2}\left[1 + \left(\frac{1!!}{2!!}\right)^2 m + \cdots\right],$$

$$E(m) = \frac{\pi}{2}\left[1 - \left(\frac{1!!}{2!!}\right)^2 m - \cdots\right],$$

so

$$K(m) - E(m) = \frac{\pi}{2}\left[2\left(\frac{1!!}{2!!}\right)^2 m + \cdots\right] = \frac{\pi}{4}\, m + \cdots .$$

Dividing by m and taking the limit $m \to 0$, we get the desired result.

18.8.4. Rewrite the denominator of the integrand of the expression for A_φ as

$$a^2 + \rho^2 + z^2 - 2a\rho\cos\alpha = a^2 + \rho^2 + 2a\rho + z^2 - 2a\rho(\cos\alpha + 1)\,.$$

Then define $\theta = \alpha/2$, write $\cos\alpha + 1 = 2\cos^2\theta$, and thereby convert A_φ to the form

$$A_\varphi = \frac{a\mu_0 I}{2\pi(a^2 + \rho^2 + 2a\rho + z^2)^{1/2}} \int_0^\pi \frac{\cos\alpha\, d\alpha}{(1 - k^2\cos^2\theta)^{1/2}}\,,$$

where k^2, as defined in the exercise, is

$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2}\,.$$

Further simplification and a change of the integration variable to θ bring us to

$$A_\varphi = \frac{\mu_0 I k}{2\pi}\left(\frac{a}{\rho}\right)^{1/2} \int_0^{\pi/2} \frac{(2\cos^2\theta - 1)\, d\theta}{(1 - k^2\cos^2\theta)^{1/2}}\,.$$

Now bring the integrand to a more convenient form by identifying

$$2\cos^2\theta - 1 = -\frac{2}{k^2}\,(1 - k^2\cos^2\theta) + \frac{2}{k^2} - 1\,,$$

reaching

$$A_\varphi = \frac{\mu_0 I k}{2\pi}\left(\frac{a}{\rho}\right)^{1/2} \int_0^{\pi/2} d\theta \Bigg[-\frac{2}{k^2}\,(1 - k^2\cos^2\theta)^{1/2} + \left(\frac{2}{k^2} - 1\right)(1 - k^2\cos^2\theta)^{-1/2}\Bigg]\,.$$

Finally, we note that because the range of integration is $(0, \pi/2)$ we can replace $\cos\theta$ by $\sin\theta$ without changing the value of the integral, so the two terms of the integrand can be identified with $E(k^2)$ and $K(k^2)$, thereby obtaining the result in the text.

18.8.5. Here $E(k^2)$ and $K(k^2)$ need to be expanded in power series, and the answer given in the text suggests that we must keep explicit terms in the expansions though k^4. We therefore write

$$f(k^2) = k^{-2}\left[(2-k^2)\,\frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}+\cdots\right)\right.$$
$$\left. -\,2\,\frac{\pi}{2}\left(1-\frac{k^2}{4}-\frac{3k^4}{64}-\cdots\right)\right]$$

The leading term in the above expression (that of lowest order in k) is $\pi k^2/16$.

18.8.6. In this exercise, all instances of E and K without arguments refer respectively to $E(k^2)$ and $K(k^2)$.

(a) Starting from the trigonometric form for $E(k^2)$, differentiate, getting

$$\frac{dE(k^2)}{dk} = \int_0^{\pi/2} \frac{-k\sin^2\theta}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta\,.$$

Simplify by rewriting the numerator of the integrand:

$$-k\sin^2\theta = \frac{(1-k^2\sin^2\theta)}{k} - \frac{1}{k}\,,$$

after which we get

$$\frac{dE(k^2)}{dk} = \frac{1}{k}\int_0^{\pi/2} d\theta\left[(1-k^2\sin^2\theta)^{1/2} - \frac{1}{(1-k^2\sin^2\theta)^{1/2}}\right]$$
$$= \frac{E(k^2)-K(k^2)}{k}\,.$$

(b) Note that the formula in the hint contains a misprint: in the integrand, k should be replaced by k^2.

Before solving this problem, we follow the hint and establish the equation it provides. We do so by expanding the integrand of the hint equation in power series, then multiplying the expansion by $1-k^2$ and organizing the result in powers of k, and finally identifying that power series as $E(k^2)$.

Thus,

$$I = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-3/2}\, d\theta$$

$$= \int_0^{\pi/2} d\theta + \sum_{n=1}^{\infty} (-1)^n k^{2n} \binom{-3/2}{n} \int_0^{\pi/2} \sin^{2n}\theta\, d\theta\,.$$

Using now the formulas (valid for $n \ge 1$)

$$\binom{-3/2}{n} = \frac{(-1)^n (2n+1)!!}{(2n)!!}\,, \qquad \int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{(2n-1)!!}{(2n)!!}\,\frac{\pi}{2}\,,$$

we bring I to the form

$$I = \frac{\pi}{2}\left[1 + \sum_{n=1}^{\infty} k^{2n} \frac{(2n+1)!!(2n-1)!!}{(2n)!!(2n)!!}\right]\,.$$

Next we write $(1-k^2)I$, grouping terms with equal powers of k:

$$(1-k^2)I =$$

$$\frac{\pi}{2}\left(1 + \sum_{n=1}^{\infty} k^{2n}\left[\frac{(2n+1)!!(2n-1)!!}{(2n)!!(2n)!!} - \frac{(2n-1)!!(2n-3)!!}{(2n-2)!!(2n-2)!!}\right]\right)$$

$$= \frac{\pi}{2}\left(1 + \sum_{n=1}^{\infty} k^{2n}\frac{(2n-1)!!(2n-3)!!}{(2n)!!(2n)!!}\Big[(2n+1)(2n-1) - (2n)(2n)\Big]\right)$$

$$= \frac{\pi}{2}\left(1 - \sum_{n=1}^{\infty} k^{2n}\frac{(2n-1)!!(2n-3)!!}{(2n)!!(2n)!!}\right)\,,$$

which is equivalent to $E(k^2)$.

Proceeding now to the solution of part (b), we write

$$\frac{dK(k^2)}{dk} = \int_0^{\pi/2} \frac{k\sin^2\theta}{(1-k^2\sin^2\theta)^{3/2}}\, d\theta\,.$$

Replacing $k\sin^2\theta$ as in part (a) of this problem, we have

$$\frac{dK(k^2)}{dk} = \frac{1}{k}\int_0^{\pi/2}\left[-\frac{1}{(1-k^2\sin^2\theta)^{1/2}} + \frac{1}{(1-k^2\sin^2\theta)^{3/2}}\right] d\theta$$

$$= \frac{1}{k}\Big[[-K(k^2) + I\Big]\,,$$

where I is the integral in the hint. Writing $I = E(k^2)/(1-k^2)$, we retrieve the answer to this problem.

19. Fourier Series

19.1 General Properties

19.1.1. By orthogonality

$$0 = \frac{\partial \Delta_p}{\partial a_n} = -2\int_0^{2\pi}\left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos nx\,dx$$

$$= -2\int_0^{2\pi} f(x)\cos nx\,dx + 2\pi a_n,$$

$$0 = \frac{\partial \Delta_p}{\partial b_n} = -2\int_0^{2\pi}\left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin nx\,dx$$

$$= -2\int_0^{2\pi} f(x)\sin nx\,dx + 2\pi b_n.$$

19.1.2. Substituting $\alpha_n \cos\theta_n = a_n, \alpha_n \sin\theta_n = b_n$ into Eq. (19.1) we have

$$a_n \cos nx + b_n \sin nx = \alpha_n(\cos\theta_n \cos nx + \sin\theta_n \sin nx) = \alpha_n \cos(nx - \theta_n).$$

19.1.3. The exponential Fourier series can be real only if, for each n, $c_n e^{inx} + c_{-n}e^{-inx}$ is real. Expanding the complex exponential,

$$c_n e^{inx} + c_{-n}e^{-inx} = (c_n + c_{-n})\cos nx + i(c_n - c_{-n})\sin nx\,.$$

This expression will be real if $c_n + c_{-n}$ is real and $c_n - c_{-n}$ is pure imaginary. Writing $c_n = a_n + ib_n$ with a_n and b_n real, these two conditions on the c_n take the form

$$b_n + b_{-n} = 0\,, \qquad a_n - a_{-n} = 0\,,$$

equivalent to a requirement that $c_{-n} = c_n^*$.

19.1.4. Expand $f(x)$ in a Fourier series. Then if

$$\int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]^2 dx = \frac{1}{2}a_0^2\pi + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2) < \infty$$

is absolutely convergent, it is necessary that

$$\lim_{n\to\infty} a_n \to 0, \qquad \lim_{n\to\infty} b_n \to 0.$$

19.1.5. By parity, $a_n = 0, n \ge 0$, while

$$b_n = \frac{1}{2\pi}\left[\int_0^\pi (\pi - x)\sin nx\,dx - \int_{-\pi}^0 (\pi + x)\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{n}\cos nx + \frac{x}{n}\cos nx\right]_0^\pi - \frac{1}{2\pi n}\int_0^\pi \cos nx\,dx$$

$$- \frac{1}{2\pi n}\int_{-\pi}^0 \cos nx\,dx - \frac{1}{2\pi}\left[-\frac{\pi}{n}\cos nx - \frac{x}{n}\cos nx\right]_{-\pi}^0$$

$$= \frac{1}{n} - \left.\frac{\sin nx}{2\pi n^2}\right|_0^\pi - \left.\frac{\sin nx}{2\pi n^2}\right|_{-\pi}^0 = \frac{1}{n}.$$

19.1.6. Writing $\sin nx$ in terms of complex exponentials, this summation becomes

$$S = \sum_{n=1}^\infty (-1)^{n+1}\frac{\sin nx}{n} = \frac{1}{2i}\left[\sum_{n=1}^\infty (-1)^{n+1}\frac{e^{inx}}{n} - \sum_{n=1}^\infty (-1)^{n+1}\frac{e^{-inx}}{n}\right].$$

These summations correspond to the expansion of $\ln(1 + e^{\pm ix})$, so

$$S = \frac{1}{2i}\Big[\ln(1 + e^{ix}) - \ln(1 + e^{-ix})\Big] = \frac{1}{2i}\ln\left(\frac{1 + e^{ix}}{1 + e^{-ix}}\right) = \frac{1}{2i}\ln e^{ix}.$$

We need the principal value of this logarithm so that $S = 0$ when $x = 0$. Thus, $S = ix/2i = x/2$.

19.1.7. By parity, $a_n = 0$ while

$$b_n = \frac{1}{4}\left[\int_0^\pi \sin nx\,dx - \int_{-\pi}^0 \sin nx\,dx\right] = -\frac{1}{4}\left(\left.\frac{\cos nx}{n}\right|_0^\pi - \left.\frac{\cos nx}{n}\right|_{-\pi}^0\right)$$

$$= -\frac{1}{4}\big[2(-1)^n - 2\big] = \frac{1 - (-1)^n}{2n} = \begin{cases} 1/n, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

19.1.8. (a) Write $2\cos\theta/2$ in terms of complex exponentials and rearrange:

$$2\cos\theta/2 = e^{i\theta/2} + e^{-i\theta/2} = e^{-i\theta/2}\left[1 + e^{i\theta}\right].$$

Take the logarithm of both sides of this equation and use the expansion of $\ln(1 + e^{i\theta})$:

$$\ln\left(2\cos\frac{\theta}{2}\right) = \frac{-i\theta}{2} + \sum_{n=1}^\infty (-1)^{n+1}\frac{e^{in\theta}}{n}.$$

Since the left-hand side of this equation is real, it must be equal to the real part of the right-hand side (the imaginary part of the right-hand must be zero; this does not concern us here but provides another route to the solution of Exercise 19.1.6). Thus,

$$\ln\left(2\cos\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{\cos n\theta}{n}\,.$$

(b) Here we have $2\sin\dfrac{\theta}{2} = i\left(e^{i\theta/2} - e^{-i\theta/2}\right) = -ie^{-i\theta/2}\left(1 - e^{i\theta}\right).$

Taking the logarithm, noting that the factor preceding the parentheses on the right-hand side has the purely imaginary logarithm $-i(\pi+\theta)/2$, we get

$$\ln\left(2\sin\frac{\theta}{2}\right) = -\frac{(\pi+\theta)i}{2} + \ln(1-e^{i\theta})\,.$$

Introducing a power series for the logarithmic term and equating the real parts of the two sides of this equation, we reach our desired answer:

$$\ln\left(2\sin\frac{\theta}{2}\right) = -\sum_{n=1}^{\infty}\frac{\cos n\theta}{n}\,.$$

19.1.9. The solution is given in the text.

19.1.10. The solution is given in the text.

19.1.11. $$\int_{-\pi}^{\pi} f(\varphi_1)\delta(\varphi_1-\varphi)\,d\varphi_1 = \sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{-im\varphi}\int_{-\pi}^{\pi} f(\varphi_1)e^{im\varphi_1}\,d\varphi_1$$
$$= \sum_m f_{-m}e^{-im\varphi} = f(\varphi).$$

19.1.12. Integrating Example 19.1.1 yields

$$\int_0^x x\,dx = \frac{1}{2}x^2 = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\Big|_0^x = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(\cos nx - 1).$$

For $x=\pi$ we obtain

$$\frac{\pi^2}{2} = 2\zeta(2) + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

Hence $\dfrac{\pi^2}{2}\left(\dfrac{1}{2}-\dfrac{1}{3}\right) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$

19.1.13. (a) Using orthogonality gives

$$\int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]^2 dx = \left(\frac{a_0}{2}\right)^2\frac{2\pi}{\pi} + \frac{\pi}{\pi}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

(b) $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{x^5}{5\pi}\bigg|_{-\pi}^{\pi} = \frac{2}{5}\pi^4 = \frac{2\pi^4}{9} + 4^2\zeta(4).$

Hence $\displaystyle \zeta(4) = \frac{\pi^4}{4^2}\left(\frac{2}{5} - \frac{2}{9}\right) = \frac{4\pi^4}{2^3\cdot 3^2\cdot 5}.$

(c) $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} dx = 2 = \left(\frac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$

This checks with $\displaystyle \zeta(2) = \frac{1}{2^2}\,\zeta(2) + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$

19.1.14. For $0 < x < \pi$,

$$\sum_{n=1}^{\infty}\int_0^x \frac{\sin nx}{n}\,dx = -\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}\bigg|_0^x = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

$$= \int_0^x \left(\frac{\pi}{2} - \frac{x}{2}\right) dx = \frac{x\pi}{2} - \frac{x^2}{4}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{1}{4}(\pi - x)^2 - \frac{\pi^2}{12}.$$

For $-\pi < x < 0$, the claim is proved similarly.

19.1.15. (a) $\displaystyle \int_0^x \psi_{2s-1}(x)\,dx = \sum_{n=1}^{\infty}\int_0^x \frac{\cos nx}{n^{2s-1}}\,dx = \sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s}}\bigg|_0^x$

$$= \sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s}} = \psi_{2s}(x)\,.$$

(b) $\displaystyle \int_0^x \psi_{2s}(x)\,dx = \sum_{n=1}^{\infty}\int_0^x \frac{\sin nx}{n^{2s}}\,dx = -\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s+1}}\bigg|_0^x$

$$= -\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s+1}} + \sum_{n=1}^{\infty}\frac{1}{2s+1} = -\psi_{2s+1}(x) + \zeta(2s+1)\,.$$

This equation rearranges into the required result.

19.1.16. Make the partial fraction decomposition

$$\frac{1}{n^2(n+1)} = \frac{1}{n+1} - \frac{1}{n} + \frac{1}{n^2}\,.$$

Then

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2(n+1)} = \sum_{n=1}^{\infty} \frac{\cos nx}{n+1} - \sum_{n=1}^{\infty} \frac{\cos nx}{n} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$= f(x) - \psi_1(x) + \varphi_2(x),$$

equivalent to the required form.

19.2 Applications of Fourier Series

19.2.1. The Fourier expansion of the present problem is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right].$$

The expansions of the first few x-containing terms are

$$\frac{\sin x}{1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\,,$$

$$\frac{\sin 3x}{3} = x - \frac{3^2 x^3}{3!} + \frac{3^4 x^5}{5!} - \cdots\,,$$

$$\frac{\sin 5x}{5} = x - \frac{5^2 x^3}{3!} + \frac{5^4 x^5}{5!} - \cdots\,.$$

Collecting the coefficients of x, x^3, $\cdots$, we find

$$\text{Coefficient of } x = 1 + 1 + 1 + \cdots\,,$$

$$\text{Coefficient of } x^3 = -\frac{1}{3!}\left[1 + 3^2 + 5^2 + \cdots\right], \quad \text{etc.}$$

These expressions diverge.

19.2.2. $\delta(x) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\sum_{n=1}^{\infty} \cos nx\,, \qquad -\pi \le x \le \pi.$

19.2.3. Solution is given in the text.

19.2.4. $\displaystyle\int_0^{\pi} \delta(x-y)\,dt = \int_0^{\pi}\left[\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos n(x-t)\right] dt$

$$= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin n(x-t)}{n}\Bigg|_0^{\pi} = \frac{1}{2} + \frac{1}{\pi}\sum_{1}^{\infty} \frac{1-(-1)^n}{n}\,\sin nx$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = \begin{cases} 1, & \text{for } 0 < x < \pi, \\ 1, & \text{for } -\pi < x < 0. \end{cases}$$

19.2.5. Subtract the series on line 5 of Table 19.1 from the series on line 4 of that table; the result is

$$\sum_{n=0}^{\infty} \frac{\cos nx}{x}\,[1-(-1)^n] = 2\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{2n+1}$$

$$= -\ln\left[\sin\frac{|x|}{2}\right] + \ln\left[\cos\frac{x}{2}\right] = \ln\left[\cot\frac{|x|}{2}\right].$$

Because the cosine is an even function, $\cos(x/2) = \cos(|x|/2)$.

19.2.6. Solution is given in the text.

19.2.7. The cosine terms of the expansion all vanish because $f(x)$ has odd parity.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = -\frac{x\cos nx}{n\pi}\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx\,dx = -\frac{2(-1)^n}{n}\,.$$

19.2.8. The cosine terms of the expansion all vanish because $f(x)$ has odd parity.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}\left(-\frac{\pi+x}{2}\right)\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi-x}{2}\right)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi-x}{2}\right)\sin nx\,dx = \int_0^{\pi}\sin nx\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx\,.$$

The second of these integrals is half the formula for b_n in the solution to Exercise 19.2.7, while the first integrates to $(1-\cos n\pi)/n = [1-(-1)^n]/n$. The final result is $b_n = 1/n$.

19.2.9. Solution is given in the text.

19.2.10. $a_0 = \dfrac{2x_0}{\pi}$, $\quad a_n = \dfrac{2}{\pi}\dfrac{\sin nx_0}{n}$, $\quad n \ge 1$, $\quad b_n = 0$, $\quad n \ge 1$.

19.2.11. $\psi(r,\varphi) = \dfrac{4V}{\pi}\displaystyle\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^{2m+1}\frac{\sin(2m+1)\varphi}{2m+1}\,.$

19.2.12. (a) $\psi(r,\varphi) = -E_0 r\left(1-\dfrac{a^2}{r^2}\right)\cos\varphi\,.$

(b) $\sigma = 2\varepsilon_0 E_0\cos\varphi$.

19.2.13. (a) $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} x\cos nx\,dx = \frac{x\sin nx}{n\pi}\Big|_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nx\,dx$

$$= \frac{\cos nx}{n^2\pi}\Big|_0^{\pi} = \frac{(-1)^n-1}{n^2\pi},$$

$$b_n = \frac{1}{\pi}\int_0^\pi x \sin nx\, dx = -\left.\frac{x\cos nx}{n\pi}\right|_0^\pi + \frac{1}{n\pi}\int_0^\pi \cos nx\, dx = \frac{(-1)^{n-1}}{n},$$

$$a_0 = \frac{1}{\pi}\int_0^\pi x\, dx = \frac{\pi}{2}.$$

Thus,

$$\left\{\begin{array}{ll} x, & 0 < x < \pi \\ 0, & -\pi < x < 0 \end{array}\right\} = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^\infty \frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^\infty \frac{(-1)^n \sin nx}{n}.$$

(b) is the above at $x = 0$.

19.2.14. Integrating

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{\sin(2n-1)x}{2n-1} = \left\{\begin{array}{ll} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{array}\right.$$

yields

$$\frac{1}{2}\int_0^x dx - \frac{2}{\pi}\sum_{n=1}^\infty \frac{\cos(2n-1)x|_0^x}{(2n-1)^2}$$

$$= \sum_{n=1}^\infty (-1)^{n-1}\frac{\sin nx}{n} + \frac{2}{\pi}\frac{\pi^2}{8} - \frac{2}{\pi}\sum_{n=1}^\infty \frac{\cos(2n-1)x}{(2n-1)^2} = \left\{\begin{array}{ll} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{array}\right.$$

19.2.15. (a) $\displaystyle \delta_n(x) = \frac{1}{2\pi} + \frac{2n}{\pi}\sum_{m=1}^\infty \frac{\sin(m/2n)}{m}\cos\ mx.$

19.2.16. $\displaystyle \int_{-\pi}^{\pi} f(x)\,\delta_n(x)\, dx = \int_{-\pi}^{\pi} f(x)\left[\frac{1}{2\pi} + \frac{2n}{\pi}\sum_{m=1}^\infty \frac{\sin(m/2n)}{m}\cos mx\right] dx$

$$= \frac{a_0}{2} + \frac{2n}{\pi}\sum_{m=1}^\infty \frac{\sin(m/2n)}{m}\int_{-\pi}^{\pi} f(x)\cos mx\, dx$$

$$\to \frac{a_0}{2} + \sum_{m=1}^\infty \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx = f(0) \text{ in the limit } n \to \infty.$$

19.2.17. (a) The coefficient b_n is given as the integral

$$b_n = \frac{2}{L}\int_0^L \delta(x-a)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\sin\left(\frac{n\pi a}{L}\right).$$

(b) Integration of the left-hand side of the delta-function formula from 0 to x yields unity if a is within the range of the integration and zero otherwise, producing the step function shown in the exercise. A corresponding integration of the right-hand side gives the listed result:

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi a}{L}\right)\left(\frac{L}{n\pi}\right)\left[1 - \cos\left(\frac{n\pi x}{L}\right)\right].$$

(c) Referring to the first entry in Table 19.1, the first summation of part (b) evaluates to

$$\frac{2}{\pi}\left(\frac{\pi - \pi a/L}{2}\right) = 1 - \frac{a}{L}.$$

Since $f(x) = 0$ on the interval $0 < x < a$ and 1 elsewhere, this expression gives the average value of $f(x)$, as claimed.

19.2.18. Calculation of the Fourier coefficients with this $f(x)$ is equivalent to integrating with $f(x) = 1$ over the range $a \le x \le L$. Thus,

$$a_0 = \frac{2}{L}\int_a^L dx = 2\left(1 - \frac{a}{L}\right).$$

$$a_n = \frac{2}{L}\int_a^L \cos\left(\frac{n\pi x}{L}\right) dx = -\frac{2}{n\pi}\sin\left(\frac{n\pi a}{L}\right), \qquad n > 0.$$

Inserting this into the formula for the Fourier cosine series,

$$f(x) = \left(1 - \frac{a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

The correspondence is seen to be exact when we use the observation developed in part (c) of Exercise 19.2.17.

19.2.19. (a) Solution is given in the text.

(b) Differentiation of the solution to part (a) leads directly to the stated result for part (b).

19.2.20. Solution is given in the text.

19.2.21. Solution is given in the text.

19.3 Gibbs Phenomenon

19.3.2. Using the guidance provided in the exercise, write

$$s_n(x) = \frac{2h}{\pi}\sum_{p=1}^{n}\frac{\sin(2p-1)x}{2p-1} = \frac{2h}{\pi}\int_0^x \sum_{p=1}^{n}\cos(2p-1)y\,dy$$

$$= \frac{2h}{\pi}\int_0^x \frac{\sin 2ny}{2\sin y}\,dy \approx \frac{h}{\pi}\int_0^x \frac{\sin 2ny}{y}\,dy = \frac{h}{\pi}\int_0^{2nx}\frac{\sin\xi}{\xi}\,d\xi,$$

which reaches its maximum value at $2nx = \pi$, with the integral then given by Eq. (19.41).

19.3.3. Solution is given in the text.

20. Integral Transforms

20.1 Introduction

(no exercises)

20.2 Fourier Transform

20.2.1. (a) If $f(x)$ is real, then $f(x) = f^*(x)$, and

$$g^*(\omega) = \left[\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx\right]^* = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = g(-\omega)\,.$$

We must also prove the converse, namely that if $g^*(\omega) = g(-\omega)$, then $f(x)$ is real. So, making no assumption as to the reality of $f(x)$, the condition $g^*(\omega) = g(-\omega)$ is equivalent to

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f^*(x)e^{-i\omega x}\,dx = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

for all ω. That can only be true if $f(x) = f^*(x)$; one way to see this is to multiply the above equation by $e^{i\omega t}$, with t arbitrary, and integrate in ω from $-\infty$ to ∞, thereby forming $2\pi\delta(t-x)$. Then the x integration yields $f^*(t) = f(t)$.

(b) A proof can be along the lines of that for part (a). The sign change in going from $g(-\omega)$ to $g^*(\omega)$ is compensated by that between $f(x)$ and $f^*(x)$.

20.2.2. (a) $g_c(\omega) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^1 \cos\omega x\,dx = \sqrt{\dfrac{2}{\pi}}\dfrac{\sin\omega}{\omega}\,.$

(b) The equation written here is just the inverse cosine transform of g_c and therefore has to yield $f(x)$.

(c) For all x such that $|x| \neq 1$ the integral of this part is $(\pi/2)f(x)$, in agreement with the answer in the text. For $x = 1$, the present integral can be evaluated as

$$\int_0^{\infty}\frac{\sin\omega\cos\omega}{\omega}\,d\omega = \frac{1}{2}\int_0^{\infty}\frac{\sin 2\omega}{\omega}\,d\omega = \frac{1}{2}\int\frac{\sin u}{u}\,du = \frac{1}{2}\,\frac{\pi}{2} = \frac{\pi}{4}\,.$$

The u integral is that in Eq. (11.107).

20.2.3. (a) Integrating by parts twice we obtain

$$\int_0^{\infty} e^{-ax}\cos\omega x\,dx = -\frac{1}{a}\,e^{-ax}\cos\omega x\Big|_0^{\infty} - \frac{\omega}{a}\int_0^{\infty} e^{-ax}\sin\omega x\,dx$$

$$= \frac{1}{a} - \frac{\omega}{a}\left[-\frac{1}{a}e^{-ax}\sin\omega x\Big|_0^{\infty} + \frac{\omega}{a}\int_0^{\infty} e^{-ax}\cos\omega x\,dx\right].$$

Now we combine the integral on the right-hand side with that on the left giving

$$\left(1+\frac{\omega^2}{a^2}\right)\int_0^\infty e^{-ax}\cos\omega x\,dx = \frac{1}{a},$$

or

$$g_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\omega^2}\,.$$

A similar process yields the formula for $g_s(\omega)$.

(b) The second integral of this part can be written as half the real part of an integral involving $e^{i\omega x}$:

$$I = \int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}\,d\omega = \frac{1}{2}\,\Re e\int_{-\infty}^\infty \frac{e^{i\omega x}}{\omega^2+a^2}\,dx\,.$$

We now replace ω by a complex variable z and employ a contour along the real axis and closed by an arc of infinite radius in the upper half-plane (on which there is no contribution to the integral). This contour encloses a pole at $z = ia$; the other pole of the integrand, $z = -ia$, is external to the contour. Therefore,

$$I = \frac{1}{2}\Re e\oint \frac{e^{ixz}}{z^2+a^2}\,dz = \Re e\left[\frac{1}{2}\,2\pi i\ \times\ (\text{Residue of integrand at } z = ia)\right].$$

This residue has the value $e^{-ax}/2ia$, so $I = \dfrac{\pi}{2a}\,e^{-ax}$.

The first integral of this part can now be obtained easily by differentiating the above result with respect to x.

20.2.4. $g(\omega) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{ha}{\omega^2}\left[1-\cos\left(\dfrac{\omega}{a}\right)\right].$

20.2.5. Write the Fourier integral representation of the delta sequence:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega x}\,d\omega\,\lim_{n\to\infty}\int_{-\infty}^\infty \delta_n(t)e^{i\omega t}\,dt = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega x}\,d\omega\,.$$

We have used the fact that the integral over t reduces to unity, the value of $e^{i\omega t}$ at $t = 0$.

20.2.6. See solution to Exercise 20.2.5. The same result will be obtained for any valid delta sequence.

20.2.7. The solution is given in the text.

20.2.8. The solution is given in the text.

20.2.9. The answer to this problem depends upon the sign of Γ and it is assumed here that $\Gamma > 0$.

The integrand of this problem has a simple pole at

$$a = \frac{E_0 - i\Gamma/2}{\hbar}\,, \quad \text{with residue} \quad -\exp(iat)\,,$$

where a is located in the lower half of the ω-plane.

The integral in question can be converted into one with a closed contour by connecting the points $\pm\infty$ by a large arc; if $t > 0$ a suitable arc will be clockwise, in the lower half-plane, as $e^{-i\omega t}$ becomes negligible when ω has a large negative imaginary part. The contour will then enclose the pole in the mathematically negative direction, so the contour integral (and also our original integral) has the value $+\exp(iat)$, corresponding to the answer for $t > 0$ in the text. If, however, $t < 0$, the contour must be closed in the upper half-plane, the pole is not encircled, and both the contour integral and our original integral will vanish. These observations confirm the answer in the text for $t < 0$.

20.2.10. (a) A natural approach to this problem is to use the integral representation

$$J_0(ay) = \frac{2}{\pi}\int_0^1 \frac{\cos ayt}{\sqrt{1-t^2}}\,dt\,.$$

However, all mention of this representation was inadvertently omitted from the present edition, so it may be easier for readers to proceed by recognizing the case $n = 0$ of Exercise 20.2.11 as a starting point. In that transform pair, replacement of t by ay causes the replacement of x by x/a and multiplication of the expression in x by a^{-1}, thereby obtaining the desired answer.

If we use the integral representation provided above, we would form the integral producing the transform and interchange the order of the two integrals:

$$[J_0(ay)]^T(x) = \frac{1}{\sqrt{2\pi}}\,\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\int_{-\infty}^{\infty} \cos ayt\; e^{ixy}\,dy\,.$$

Replacing $\cos ayt$ by $(e^{iayt} + e^{-iayt})/2$, we identify the y integral in terms of delta functions, reaching

$$[J_0(ay)]^T(x) = \frac{1}{\sqrt{2\pi}}\,\frac{2}{\pi}\,\frac{2\pi}{a}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\left[\frac{\delta(t+x/a) + \delta(t-x/a)}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\begin{cases} \dfrac{1}{\sqrt{a^2-x^2}}\,, & |x| < a, \\ 0\,, & |x| > a. \end{cases}$$

(b) The expression in x is incorrect; the quantity within the square root should be $x^2 - a^2$.

Use the second integral representation given in Eq. (14.63) and proceed as in the second approach given above for part (a).

(c) Here use the integral representation given in Eq. (14.113), and proceed as in the earlier parts of this exercise.

(d) $I_0(ay)$ diverges exponentially at large y and does not have a Fourier transform.

20.2.11. Strictly speaking, these expressions are not Fourier transforms **of each other**. While $i^n J_n(t)$ is the transform of the expression opposite it, the transform of $i^n J_n$ is $(-1)^n$ times the Chebyshev expression.

To establish the transform relationship, start by writing the transform of the right-hand expression in the angular variable θ, where $x = \cos\theta$:

$$\left[\sqrt{\frac{2}{\pi}}\, T_n(x)(1-x^2)^{-1/2}\right]^T = \frac{1}{\pi}\int_{-1}^{1} T_n(x)e^{itx}(1-x^2)^{-1/2}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos n\theta\, e^{it\cos\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi} \frac{e^{i(t\cos\theta+n\theta)} + e^{i(t\cos\theta-n\theta)}}{2}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(t\cos\theta+n\theta)}\,d\theta\,.$$

We now identify this integral as the integral representation of J_n that was given in Exercise 14.1.15(b), with value $2\pi i^n J_n(t)$, and therefore

$$\left[\sqrt{\frac{2}{\pi}}\, T_n(x)(1-x^2)^{-1/2}\right]^T = i^n J_n(t)\,.$$

20.2.12. The transform as given in the text is improperly scaled. Its correct value is $(2/\pi)^{1/2} i^n j_n(\omega)$, where ω is the transform variable. The exercise also assumes the transform variable to be kr.

The Fourier transform of $f(\mu)$ is

$$f^T(\mu) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{i\omega\mu} P_n(\mu)\,d\mu\,.$$

This integral was evaluated in Exercise 15.2.26, where it was shown to have the value $2i^n j_n(\omega)$. Inserting this result into the formula for the transform, we verify its value (as corrected).

20.2.13. (a) Consider the integral

$$I = \int_0^{\infty} z^{-1/2} e^{izt}\,dz = \int_0^{\infty} x^{-1/2}(\cos xt + i\sin xt)\,dx\,.$$

Make a change of the integration variable to $u = -izt$ and assume $t > 0$; the range of the integration in u is $(0, i\infty)$, and the integral becomes

$$I = e^{\pi i/4} t^{-1/2} \int_0^{i\,\infty} u^{-1/2} e^{-u}\, du\,.$$

Deform the contour to go along the real axis from zero to infinity, and then along a counterclockwise loop at large $|u|$ to the imaginary axis. The large arc does not contribute to the integral, but the path along the real axis evaluates to $\Gamma(1/2) = \sqrt{\pi}$. Thus,

$$I = t^{1/2}\, \frac{1+i}{\sqrt{2}}\, \sqrt{\pi}\,,$$

from which we verify both the integrals of this part of the exercise.

(b) In the first integral of part (a), make a change of the integration variable to $y^2 = xt$, so $x^{-1/2}dx = 2t^{-1/2}dy$, and that integral becomes

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{-1/2} \cos xt\, dx = \sqrt{\frac{2}{\pi}}\;\; 2t^{-1/2} \int_0^\infty \cos y^2\, dy = t^{-1/2}\,,$$

which rearranges to $\displaystyle\int_0^\infty \cos(y^2)\, dy = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$

The same result is obtained for $\displaystyle\int_0^\infty \sin(y^2)\, dy\,.$

20.2.14. We must evaluate

$$f^T = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r^2}\, d^3r\,.$$

Use spherical polar coordinates with axis in the direction of $\mathbf{k}$. After changing the integration variable θ into $t = \cos\theta$, integrate over φ and t, reaching

$$f_T = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\varphi \int_0^\infty dr \int_0^\pi \sin\theta\, e^{ikr\cos\theta}\, d\theta$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{2\sin kr}{kr}\, dr = \frac{1}{(2\pi)^{1/2}}\, \frac{\pi}{k} = \frac{1}{k}\sqrt{\frac{\pi}{2}}\,.$$

20.2.15. Write the formula for $F(u,v)$ in polar coordinates, setting $x = r\cos\theta$, $y = r\sin\theta$, $u = \rho\cos\theta'$, $v = \rho\sin\theta'$, with $f = f(r)$ and $F = F(\rho)$:

$$F(\rho) = \frac{1}{2\pi} \int_0^\infty r\, dr \int_0^{2\pi} d\theta f(r)\, e^{i\rho r(\cos\theta\cos\theta' + \sin\theta\sin\theta')}$$

$$= \int_0^\infty r\, f(r)\, dr\, \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\rho r\cos(\theta-\theta')}\,.$$

We may replace $\theta - \theta'$ by θ without changing the value of the θ integral, after which it (including the premultiplier $1/2\pi$) can be identified as the integral representation of $J_0(\rho r)$ given as Eq. (14.20). This reduces $f(\rho)$ to the Hankel transform given in the text. A similar procedure can be used to verify the inverse Hankel transform.

20.2.16. Change d^3x in this exercise to d^3r and remove the integration limits of the d^3r integral (they are understood to be the entire 3-D space).

Write the integral of this exercise in spherical polar coordinates and make the change of variable $\cos\theta = t$:

$$\frac{1}{(2\pi)^{3/2}} \int_0^\infty r^2 f(r)\,dr \int_0^{2\pi} d\varphi \int_{-1}^1 e^{ikrt}\,dt$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty r^2 f(r)\,dr \frac{e^{ikr} - e^{-ikr}}{ikr} = \frac{1}{(2\pi)^{1/2}} \int_0^\infty r^2 f(r) \frac{2\sin kr}{kr}\,,$$

which rearranges to the answer in the text.

20.3 Properties of Fourier Transforms

20.3.1. The following relationships can be established using the methods for the solution of Exercise 20.3.2.

$$\Big[f(t-a)\Big]^T(\omega) = e^{i\omega t} g(\omega)\,, \qquad \Big[f(\alpha t)\Big]^T(\omega) = \frac{1}{\alpha}\, g(\alpha^{-1}\omega)\,,$$

$$\Big[f(-t)\Big]^T(\omega) = g(-\omega)\,, \qquad \Big[f^*(-t)\Big]^T(\omega) = g^*(\omega)\,.$$

20.3.2. (a) $$\Big[f(\mathbf{r}-\mathbf{R})\Big]^T(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}-\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

$$= \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{R})} d^3r = e^{i\mathbf{k}\cdot\mathbf{R}} g(\mathbf{k})\,.$$

(b) $$\Big[f(\alpha\mathbf{r})\Big]^T = \frac{1}{(2\pi)^{3/2}} \int f(\alpha\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

$$= \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}/\alpha} \alpha^{-3} d^3r = \frac{1}{\alpha^3}\, g(\alpha^{-1}\mathbf{k}).$$

$$\Big[f(-\mathbf{r})\Big]^T(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(-\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

$$= \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot(-\mathbf{r})} d^3r = g(-\mathbf{k}).$$

$$\Big[f^*(-\mathbf{r})\Big]^T(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f^*(-\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r = \left[\frac{1}{(2\pi)^{3/2}} \int f(-\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r\right]^*$$

$$= \left[\frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{-i\mathbf{k}\cdot(-\mathbf{r})} d^3r \right]^* = g^*(\mathbf{k}).$$

20.3.3. Applying Green's theorem, Eq. (3.85) and recognizing that its surface terms vanish here, the formal expression for the transform of the Laplacian becomes

$$\left[\nabla^2 f(\mathbf{r}) \right]^T = \frac{1}{(2\pi)^{3/2}} \int \nabla^2 f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) \nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r\,.$$

The easiest way to evaluate $\nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}}$ is to do so in Cartesian coordinates, writing it as $\nabla^2 e^{ik_x x} e^{ik_y y} e^{ik_z z}$, which reduces to $(-k_x^2 - k_y^2 - k_z^2) e^{ik_x x} e^{ik_y y} e^{ik_z z}$, or $-k^2 e^{i\mathbf{k}\cdot\mathbf{r}}$. When this expression is inserted into the above integral, it is seen to be equivalent to $-k^2 f^T(\mathbf{k}) = -k^2 g(\mathbf{k})$.

20.3.4. We manipulate the transform of $f'(t)$ by integrating by parts, as follows:

$$\left[f'(t) \right]^T (\omega) = \frac{1}{(2\pi)^{1/2}} \int f'(t) e^{i\omega t}\, dt$$

$$= -\frac{1}{(2\pi)^{1/2}} \int f(t)(i\omega) e^{i\omega t}\, dt = -i\omega f^T(\omega) = -i\omega g(\omega)\,.$$

Higher derivatives can be reached by multiple integrations by parts. Each derivative generates a factor $-i\omega$ in the transform.

20.3.5. Differentiating the formula for $g(\omega)$ n times with respect to ω,

$$\frac{d^n}{d\omega^n} g(\omega) = \frac{1}{(2\pi)^{1/2}} \int f(t) \frac{d^n}{d\omega^n}\, e^{i\omega t}\, dt = \frac{1}{(2\pi)^{1/2}} \int f(t)(it)^n\, e^{i\omega t}\, dt$$

$$= \frac{i^n}{(2\pi)^{1/2}} \int t^n f(t)\, e^{i\omega t}\, dt = i^n \left[t^n f(t) \right]^T (\omega)\,.$$

20.3.6. Letting $g(t)$ be the Fourier transform of $\varphi(x)$, using Eq. (20.56) to identify $[\varphi(x)'']^T = -t^2 g(t)$, and noting from Eq. (20.14) that $[\delta(x)]^T = (2\pi)^{-1/2}$, our ODE transforms into

$$Dt^2\, g(t) + K^2 D\, g(t) = \frac{Q}{\sqrt{2\pi}}\,,$$

an algebraic equation with solution $g(t) = \dfrac{Q}{D\sqrt{2\pi}} \dfrac{1}{t^2 + K^2},.$

To recover $\varphi(x)$, we need the inverse transform of $g(t)$. Noting from Eq. (20.13) that $2KDg(t)/Q$ is the transform of $e^{-K|x|}$ with K then assumed to be positive, we get the answer given in the text.

20.4 Fourier Convolution Theorem

20.4.1. (a) Form the expression on the right-hand side of the formula to be proved, inserting the definitions of the sine transforms.

$$\int_0^\infty F_s(s)G_s(s)\cos xs\,ds$$

$$= \frac{2}{\pi}\int_0^\infty ds\int_0^\infty dy\,g(y)\sin sy\int_0^\infty dt\,f(t)\sin st\cos sx$$

$$= \frac{2}{\pi}\int_0^\infty g(y)\,dy\int_0^\infty f(t)\,dt\int_0^\infty \sin sy\sin st\cos sx\,ds\,.$$

Now apply a trigonometric addition formula, enabling the identification of the s integral in terms of delta functions as shown in Exercise 20.2.7.

$$\int_0^\infty \sin sy\sin st\cos sx\,ds = \int_0^\infty \sin st\left[\frac{\sin s(y+x)+\sin s(y-x)}{2}\right]ds$$

$$= \frac{\pi}{4}\Big[\delta(t-y-x)+\delta(t-y+x)\Big]\,.$$

Inserting this value for the s integral, we then integrate over t and obtain the formula in the text.

(b) A similar treatment of the Fourier cosine convolution formula leads to

$$\int_0^\infty F_c(s)G_c(s)\cos xs\,ds$$

$$= \frac{2}{\pi}\int_0^\infty g(y)\,dy\int_0^\infty f(t)\,dt\int_0^\infty \cos sy\cos st\cos sx\,ds$$

and to the delta-function formula

$$\int_0^\infty \cos sy\cos st\cos sx\,ds = \frac{\pi}{4}\Big[\delta(t-y-x)+\delta(t-y+x)\Big]\,.$$

Keeping in mind that f is an even function, we recover the answer in the text.

20.4.2. Insert the definitions of the Fourier sine transforms into the left-hand side of the Parseval formula, and identify the t integral as a delta function (see

Exercise 20.2.7).

$$\frac{2}{\pi}\int_0^\infty dt\int_0^\infty dx\, f(x)\sin tx\int_0^\infty dy\, g(y)\sin ty$$

$$=\int_0^\infty f(x)\,dx\int_0^\infty g(y)\,dy\,\frac{2}{\pi}\int_0^\infty \sin tx\sin ty\,dt$$

$$=\int_0^\infty f(x)\,dx\int_0^\infty g(y)\,dy\,\delta(x-y)=\int_0^\infty f(x)g(x)\,dx\,.$$

Proof of the Parseval formula for the cosine transforms is similar.

20.4.3. (a) Compute

$$F(t)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{itx}\,dx=\frac{1}{\sqrt{2\pi}}\,\frac{e^{ita}-e^{-ita}}{it}=\frac{1}{\sqrt{2\pi}}\,\frac{2\sin at}{t}\,,$$

equivalent to the answer in the text.

(b) The Parseval relation, applied to $F(t)$ and $F^*(t)$, with $a=1$, yields

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\int_{-1}^{1}[f(x)]^2\,dx=2\,.$$

Minor rearrangement shows that $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 t}{t^2}\,dt=\pi$.

20.4.4. (a) Let $\varphi(\mathbf{k})$ be the Fourier transform of $\psi(\mathbf{r})$, and let $\hat\rho(\mathbf{k})$ be the Fourier transform of $\rho(\mathbf{r})$. Using Eq. (20.53), Poisson's equation becomes

$$-k^2\varphi(\mathbf{k})=-\frac{\hat\rho(\mathbf{k})}{\varepsilon_0}\,,\quad\text{and}\quad \varphi(\mathbf{k})=\frac{\hat\rho(\mathbf{k})}{\varepsilon_0\,k^2}\,.$$

(b) One could now carry out the inverse transform directly:

$$\psi(\mathbf{r})=\frac{1}{(2\pi)^{3/2}\varepsilon_0}\int\frac{\hat\rho(\mathbf{k})}{k^2}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3k\,.$$

It may be more instructive to use the convolution theorem, with

$$F(\mathbf{k})=\hat\rho(\mathbf{k})/\varepsilon_0,\quad f(\mathbf{r})=\rho(\mathbf{r})/\varepsilon_0,\quad G(\mathbf{k})=\frac{1}{k^2},\quad g(\mathbf{r})=\frac{(2\pi)^{3/2}}{4\pi r}\,,$$

where g was obtained using Eq. (20.42). We have

$$\psi(\mathbf{r})=F*G=\frac{1}{(2\pi)^{3/2}\varepsilon_0}\int\rho(\mathbf{r}')\,\frac{(2\pi)^{3/2}}{4\pi}\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'$$

$$=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'\,.$$

This is a confirmation that Poisson's equation is consistent with Coulomb's law.

20.4.5. (a) Compute

$$F(t) = \frac{1}{\sqrt{2\pi}}\left[\int_0^2 \left(1-\frac{x}{2}\right) e^{itx}\,dx + \int_{-2}^0 \left(1+\frac{x}{2}\right) e^{itx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{e^{2it}}{2t^2} + \frac{2it+1}{2t^2} - \frac{e^{-2it}}{2t^2} - \frac{2it-1}{2t^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{2it}-2+e^{-2it}}{-2t^2}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{\sin t}{t}\right)^2 .$$

(b) From the Parseval relation,

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^2 dx = \int_{-2}^{2}[f(x)]^2\,dx = 2\int_0^2\left(1-x+\frac{x^2}{4}\right)dx = \frac{4}{3} .$$

This equation simplifies to the desired answer.

20.4.6. Setting $h(x) = f(x) - g(x)$, we have $H(t) = F(t) - G(t)$, and Parseval's relation gives

$$\int_{-\infty}^{\infty} h(x)h^*(x)\,dx = \int_{-\infty}^{\infty} H(t)H^*(t)\,dt ,$$

which is the result we need.

20.4.7. (a) Use the cosine-transform Parseval relation

$$\int_0^{\infty}[G_c(\omega)]^2\,d\omega = \int_0^{\infty}[g(t)]^2\,dt ,$$

with

$$g(t) = e^{-at} , \qquad G_c(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{a}{\omega^2+a^2} .$$

We get

$$\frac{2\,a^2}{\pi}\int_0^{\infty}\left(\frac{1}{\omega^2+a^2}\right)^2 d\omega = \int_0^{\infty} e^{-2at}\,dt = \frac{1}{2a} .$$

Solving for the ω integral and doubling the result, as the range asked for in the text is $(-\infty,\infty)$, we get $\pi/2a^3$.

(b) Proceed as in part (a), but use the sine-transform Parseval relation, with

$$g(t) = e^{-at} , \qquad G_s(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\omega}{\omega^2+a^2} .$$

This leads to

$$\frac{2}{\pi}\int_0^{\infty}\left(\frac{\omega}{\omega^2+a^2}\right)^2 d\omega = \int_0^{\infty} e^{-2at}\,dt = \frac{1}{2a} ,$$

from which the ω integral is found to have the value $\pi/2a$.

20.4.8. The solution is given in the text.

20.4.9. The intent of this problem is to use Fourier convolution methods to write this interaction integral in what may be a more convenient form. A direct-space integral describing the interaction energy is

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{R}-\mathbf{A})}{|\mathbf{r}-\mathbf{C}|}\, d^3r = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{R})}{|\mathbf{C}-\mathbf{A}-\mathbf{R}|}\, d^3r\,.$$

Applying the convolution theorem as given in Eq. (20.72), and noting that $[1/r]^T = (2\pi)^{-3/2}(4\pi/k^2)$, we get

$$V = \frac{1}{(2\pi)^{3/2}\varepsilon_0} \int \frac{\rho^T(\mathbf{k})}{k^2}\, e^{-i\mathbf{k}\cdot\mathbf{R}_{AC}}\, d^3k\,,$$

where $\mathbf{R}_{AC} = \mathbf{C} - \mathbf{A}$.

20.4.10. This problem assumes that the momentum wave function is defined (including its scale) as

$$\varphi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\mathbf{r})\, e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar}\, d^3r\,.$$

(a) Apply $i\hbar\boldsymbol{\nabla}_p$ to both sides of the above equation. When the p_x component of the gradient is applied to $\exp(-i(xp_x + yp_y + zp_z)/\hbar)$ within the integrand, the result is $(-ix/\hbar)\exp(-i(xp_x + yp_y + zp_z)/\hbar)$, so application of the entire gradient causes the integrand to be multiplied by $-i\mathbf{r}/\hbar$, thereby producing the result in the text.

(b) Two successive applications of the gradient in momentum space produce two factors $\mathbf{r}$, as shown in the text. This result can be construed either as involving a scalar product $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ (and correspondingly, $\mathbf{r}\cdot\mathbf{r}$), or as the creation of a dyadic (tensor) quantity.

20.4.11. Apply the Fourier transform operator (defined as in the solution of Exercise 20.4.10) to both sides of the Schrödinger equation. With the scaling in use here,

$$\left[\boldsymbol{\nabla}^2\psi\right]^T(\mathbf{p}) = -\frac{p^2}{\hbar^2}\,\varphi(\mathbf{p})\,,$$

and the term $V(\mathbf{r})\psi$ can be expanded in a Maclaurin series with each term treated as in Exercise 20.4.10.

20.5 Signal-Processing Applications

20.5.1. The potential across a capacitor for a current I that is periodic at angular frequency ω is $\int^t (I/C)e^{i\omega t} = I/i\omega C$. Using Kirchhoff's equation we have $V_{\text{in}} = RI + I/i\omega C$ and $V_{\text{out}} = I/i\omega C$, so

$$\varphi(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{I/i\omega C}{IR + I/i\omega C} = \frac{1}{1 + i\omega RC}\,.$$

Since $\varphi(\omega)$ decreases as ω increases, this is a low-pass filter.

20.5.2. The potential across an inductor for a current I that is periodic at angular frequency ω is $LdI/dt = i\omega LI$. Thus, applying Kirchhoff's equation, $V_{\text{in}} = i\omega LI + IR$ and $V_{\text{out}} = IR$, so

$$\varphi(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{IR}{IR + i\omega LI} = \frac{1}{1 + i\omega L/R}\,.$$

Since $\varphi(\omega)$ decreases as ω increases, this is a low-pass filter.

20.5.3. Using the potential across a capacitor in the form given in the solution to Exercise 20.5.1, we write Kirchhoff's equation for each of the two independent loops in the present circuit, obtaining

$$V_{\text{in}} = \frac{I_1 + I_2}{i\omega C_1} + I_1 R_1\,, \qquad I_2 R_2 + \frac{I_2}{i\omega C_2} - I_1 R_1 = 0\,, \qquad V_{\text{out}} = \frac{I_2}{i\omega C_2}\,.$$

Before computing the transfer function it is convenient to solve for I_1 in terms of I_2:

$$I_1 = \frac{I_2}{R_1}\left(R_2 + \frac{1}{i\omega C_2}\right), \qquad I_1 + I_2 = \frac{I_2}{R_1}\left(R_1 + R_2 + \frac{1}{i\omega C_2}\right).$$

Now,

$$\varphi(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{I_2/i\omega C_2}{(I_2/i\omega C_1 R_1)\left(R_1 + R_2 + \frac{1}{i\omega C_2}\right) + I_2\left(R_2 + \frac{1}{i\omega C_2}\right)}$$

$$= \frac{i\omega R_1 C_1}{1 + i\omega[R_1 C_1 + (R_1 + R_2)C_2] - \omega^2 R_1 R_2 C_1 C_2}\,.$$

This form for $\varphi(\omega)$ becomes small for both small and large ω.

20.5.4. The transfer function describes the circuit functionality in the absence of loading, i.e., in the limit that negligible current flows between the output terminals. For a second circuit element not to affect the transfer function of the first, it must not load the first circuit. Here that means that the current through R_2 must be much smaller than the current through R_1; this can be assured if $R_2 \gg R_1$.

20.6 Discrete Fourier Transform

20.6.1. The range of p and q for this problem, though not stated, is assumed to be integers satisfying $0 \le p, q < N$.

The second and third orthogonality equations as given in the text are incorrect. Corrected versions of these equations are:

$$\sum_{k=0}^{N-1}\cos\left(\frac{2\pi pk}{N}\right)\cos\left(\frac{2\pi qk}{N}\right)=\begin{cases} N, & p=q=(0 \text{ or } N/2) \\ N/2, & (p+q=N) \text{ or } p=q \text{ but not both} \\ 0, & \text{otherwise}, \end{cases}$$

$$\sum_{k=0}^{N-1}\sin\left(\frac{2\pi pk}{N}\right)\sin\left(\frac{2\pi qk}{N}\right)=\begin{cases} N/2, & p=q \text{ and } p+q\neq(0 \text{ or } N) \\ -N/2, & p\neq q \text{ and } p+q=N \\ 0, & \text{otherwise}. \end{cases}$$

All these orthogonality equations depend upon the relationships

$$\sum_{k=0}^{N-1}\cos\left(\frac{2\pi pk}{N}\right)=\sum_{k=0}^{N-1}\sin\left(\frac{2\pi pk}{N}\right)=0\,.$$

The first formula is valid for nonzero integers p that are not multiples of N; the second is valid for all integral p. These relationships become obvious when identified as the real and imaginary parts of summations of the type discussed at Eq. (20.118). Alternatively, note that when plotted on a complex plane, sums of $\exp(2\pi ipk/N)$ form closed figures (N-sided regular polygons) and therefore their real and imaginary parts each evaluate to zero.

The first summation can be written

$$\sum_{k=0}^{N-1}\cos\left(\frac{2\pi pk}{N}\right)\sin\left(\frac{2\pi qk}{N}\right)=\frac{1}{2}\sum_{k=0}^{N-1}\left[\sin\left(\frac{2\pi[q+p]k}{N}\right)+\sin\left(\frac{2\pi[q-p]k}{N}\right)\right]\,.$$

Both these sums vanish for all integral p and q.

The second summation can be written

$$\sum_{k=0}^{N-1}\cos\left(\frac{2\pi pk}{N}\right)\cos\left(\frac{2\pi qk}{N}\right)=\frac{1}{2}\sum_{k=0}^{N-1}\left[\cos\left(\frac{2\pi[q+p]k}{N}\right)+\cos\left(\frac{2\pi[q-p]k}{N}\right)\right]\,.$$

The first summation term on the right-hand side leads to a vanishing contribution unless $q+p$ is zero or N; in those cases all summands are unity and they together contribute N to the sum. The second summation term makes no contribution unless $q=p$, in which case it also contributes N to the sum. Thus, the combined contributions of these terms (including the premultiplier $1/2$) yield the orthogonality relations, as corrected above.

The third summation can be written

$$\sum_{k=0}^{N-1}\cos\left(\frac{2\pi pk}{N}\right)\cos\left(\frac{2\pi qk}{N}\right)=\frac{1}{2}\sum_{k=0}^{N-1}\left[\cos\left(\frac{2\pi[q-p]k}{N}\right)-\cos\left(\frac{2\pi[q+p]k}{N}\right)\right]\,.$$

A treatment similar to that of the second summation now leads to different results than were obtained there because of the presence of a minus sign for the last summation term. The results correspond to the corrected orthogonality formula.

20.6.2. The formulas in this exercise are incorrect. They lack a factor N in the denominator of the exponent; the exponentials should read $\exp(\pm 2\pi ipk/N)$.

Multiply the formula given for F_p by $e^{-2\pi ipj/N}$, where j is an integer in the range $0 \le j < N$, and then divide by $N^{1/2}$ and sum over p. Calling the result g_j, we have

$$g_j = \frac{1}{N}\sum_{p=0}^{N-1} e^{-2\pi ipj/N} \sum_{k=0}^{N-1} f_k\, e^{2\pi ipk/N} = \frac{1}{N}\sum_{k=0}^{N-1} f_k \sum_{p=0}^{N-1} e^{2\pi ip(k-j)/N}\,.$$

Using Eq. (20.120), the p summation reduces to $N\delta_{kj}$, showing that the formula for g_j reduces to f_j.

20.6.3. (a) Write the formula for F_{N-p}, and simplify its exponential by removing a factor unity in the form $e^{2\pi ikN/N}$:

$$F_{N-p} = \frac{1}{N^{1/2}}\sum_{k=0}^{N-1} f_k\, e^{2\pi ik(N-p)/N} = \frac{1}{N^{1/2}}\sum_{k=0}^{N-1} f_k\, e^{-2\pi ikp/N}\,.$$

Because f_k is real (so $f_k^* = f_k$), complex conjugation of F_{N-p} changes the above expression only by changing the sign of the exponent, thereby yielding the standard expression for F_p.

(b) Using the same expression for F_{n-p} as in the equation of part (a), complex conjugation still changes the sign of the exponent, but also replaces f_k by $f_k^* = -f_k$. Thus, $F_{N-p}^* = -F_p$.

20.7 Laplace Transforms

20.7.1. If $F(t) = \sum_{n=0}^{\infty} a_n t^n$, then $f(s) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-st} t^n dt = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}}$.

Hence for $s \to \infty$, $sf(s) \to a_0$, and for $t \to 0$, $F(t) \to a_0$.

20.7.2. This problem is ill-defined.

20.7.3. From Table 20.1, we find

$$\frac{\dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}}{b^2-a^2} = \frac{s}{(s^2+a^2)(s^2+b^2)}\,.$$

20.7.4. (a) The argument of $\mathcal{L}^{-1}$ has the partial-fraction expansion

$$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right).$$

Each term of the above expansion can be identified as a transform corresponding to Formula 4 of Table 20.1. Replacing each transform by its inverse leads to the answer in the text.

(b) The argument of $\mathcal{L}^{-1}$ for this part has the partial-fraction expansion

$$\frac{s}{(s+a)(s+b)} = \frac{1}{a-b}\left(\frac{a}{s+a} - \frac{b}{s+b}\right),$$

leading to the result in the text.

20.7.5. (a) Make a partial-fraction expansion of the argument of $\mathcal{L}^{-1}$. Do not further factor into quantities linear in s. The result,

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right),$$

produces factors that correspond to Formula 9 of Table 20.1.

(b) This argument of $\mathcal{L}^{-1}$ has partial-fraction expansion

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right),$$

with terms that also correspond to Formula 9 of Table 20.1.

20.7.6. The notational conventions of the text indicate that the two instances of $(\nu-1)!$ in this exercise should be written $\Gamma(\nu)$.

The hint suggests writing $s^{-\nu}$ as the transform integral

$$s^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-ts}t^{\nu-1}\,dt\,.$$

(a) Use the above to form

$$\int_0^\infty \frac{\cos s}{s^\nu}\,ds = \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}\,dt \int_0^\infty \cos s\; e^{-ts}\,ds$$

$$= \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}\left(\frac{t}{t^2+1}\right)dt\,.$$

We were able to perform the s integral because (in a different notation than we usually use) it is the integral defining the Laplace transform of $\cos s$. The remaining integral over t is shown at the end of this problem solution to have the value

$$I_\nu = \int_0^\infty \frac{t^\nu}{t^2+1}\,dt = \frac{\pi}{2\cos(\nu\pi/2)},$$

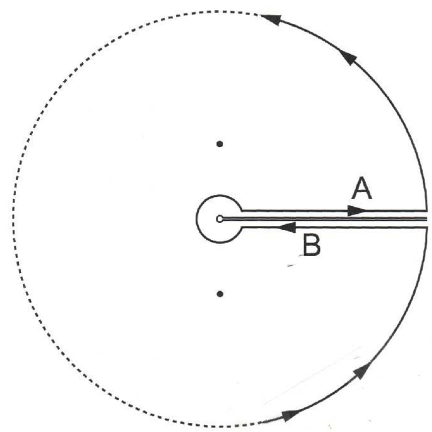

Figure 20.7.6. Contour for Exercise 20.7.6.

so the integral of part (a) evaluates to $I_\nu/\Gamma(\nu)$. This result agrees with the answer in the text.

(b) This integral can be treated in a way similar to that of part (a). The only difference is that the integral containing $\sin s$ leads to an overall power $t^{\nu-1}$ instead of t^ν. Therefore the final result will be $I_{\nu-1}/\Gamma(\nu)$. Since $\cos([\nu-1]\pi/2) = \sin(\nu\pi/2)$, this result can be brought to the form given in the text.

The restrictions on ν are needed as otherwise the integrals we were to evaluate would diverge.

The integral I_ν is most conveniently evaluated by contour integration, using the contour shown in Fig. 20.7.6. Segment A of the contour has contribution I_ν, while segment B of the contour contributes $-e^{2\pi i\nu}I_\nu$. The remainder of the contour makes no contribution to the contour integral. We therefore write

$$\begin{aligned} I_\nu\left(1 - e^{2\pi i\nu}\right) &= 2\pi i(\text{sum of residues at } t = \pm i) \\ &= 2\pi i\left[\frac{e^{\pi i\nu/2}}{2i} + \frac{e^{3\pi i\nu/2}}{-2i}\right], \end{aligned}$$

which can be manipulated to

$$I_\nu e^{\pi i\nu}\left(e^{-\pi i\nu} - e^{\pi i\nu}\right) = \pi\, e^{\pi i\nu}\left(e^{\pi i\nu/2} - e^{\pi i\nu/2}\right),$$

after which the exponentials can be identified as trigonometric functions. With use of the identity $\sin(\pi\nu) = 2\sin(\pi\nu/2)\cos(\pi\nu/2)$, further simplification leads to the result given above.

20.7.7. Make a change of integration variable from t to $u = ts$ in

$$\int_0^\infty e^{-st} t^n \, dt = s^{-n-1} \int_0^\infty e^{-u} u^n \, du \, .$$

Since $n \geq 0$, the power of s cannot exceed -1.

$\mathcal{L}\{\delta(t)\} = 1 = s^0$, but this is not in conflict with our earlier demonstration because $\delta(t)$ does not have a power-series expansion.

20.7.8. Change the semicolon between arguments of M to a comma to be consistent with the notational conventions of the text and many reference works.

Write the series expansion of $M(a,c,x)$ and apply the Laplace transform operator to it termwise.

$$\mathcal{L}\{M(a,c,x)\} = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n \, n!} \, \mathcal{L}\{x^n\} = \sum_{n=0}^{\infty} \frac{(a)_n \, n!}{(c)_n \, n!} \, \frac{1}{s^{n+1}} \, .$$

Since the $n!$ in the numerator can be written $(1)_n$, the summation contains the Pochhammer symbols needed for ${}_2F_1(a,1;c;s^{-1})$. However, we need to append a factor $1/s$ to make the power of s correct.

20.8 Properties of Laplace Transforms

20.8.1. Use the fact that

$$\frac{d^2}{dt^2} \cos kt = -k^2 \cos kt \, .$$

Using the formula for the transform of a second derivative and taking the Laplace transform of both sides of this equation,

$$\mathcal{L}\left\{\frac{d^2}{dt^2} \cos kt\right\} = s^2 \mathcal{L}\{\cos kt\} - s\Big[\cos kt\Big]_{t=0} - \left[\frac{d}{dt} \cos kt\right]_{t=0}$$

$$= s^2 \mathcal{L}\{\cos kt\} - s = -k^2 \mathcal{L}\{\cos kt\} \, .$$

Solving this equation, we get $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$.

20.8.2. The solution is given in the text.

20.8.3. The solution is given in the text.

20.8.4. (a) $\quad N_2(t) = \varphi \sigma_1 N_1(0) \dfrac{1 - \exp[-(\lambda_2 + \varphi \sigma_2)t]}{\lambda_2 + \varphi \sigma_2}$.

(b) $N_2(1 \text{ year}) = 1.2 \times 10^{15}$ atoms of Eu^{154}.

$N_1(1 \text{ year}) = 10^{20} - 1.2 \times 10^{15} \approx 10^{20}$ atoms. This justifies the assumption $N_1(t) = N_1(0)$.

20.8.5. (a) $N_{\rm Xe}(t) =$

$$\lambda_{\rm I}\gamma_{\rm I}\varphi\sigma_f N_{\rm U}\frac{(\lambda_{\rm I}-\lambda_{\rm Xe}-\varphi\sigma_{\rm Xe})+(\lambda_{\rm Xe}+\varphi\sigma_{\rm Xe})e^{-\lambda_{\rm I}t}-\lambda_{\rm I}e^{-(\lambda_{rmXe}+\varphi\sigma_{\rm Xe})t}}{\lambda_{\rm I}(\lambda_{\rm Xe}+\varphi\sigma_{\rm Xe})\,(\lambda_{\rm I}-\lambda_{\rm Xe}-\varphi\sigma_{\rm Xe})}$$

$$+\gamma_{\rm Xe}\varphi\sigma_f N_{\rm U}\left[\frac{1-e^{-(\lambda_{\rm Xe}+\varphi\sigma_{\rm Xe})t}}{\lambda_{\rm Xe}+\varphi\sigma_{\rm Xe}}\right].$$

(b) $N_{\rm Xe}(\infty) = \dfrac{(\gamma_{\rm I}+\gamma_{\rm Xe})\varphi\sigma_f N_{\rm U}}{\lambda_{\rm Xe}+\varphi\sigma_{\rm Xe}}.$

(c) $N_{\rm Xe}(t) = N_{\rm Xe}(0)\,e^{-\lambda_{\rm Xe}t} + N_{\rm Xe}(0)\dfrac{\lambda_{\rm I}}{\lambda_{\rm I}-\lambda_{\rm Xe}}(e^{-\lambda_{\rm Xe}t}-e^{-\lambda_{\rm I}t}),$

$$\left.\frac{dN_{\rm Xe}(t)}{dt}\right|_{t=0} \approx \gamma_{\rm I}\varphi\sigma_f N_{\rm U}, \quad \text{for } \varphi \gg \lambda_{\rm Xe}/\sigma_{\rm Xe}.$$

20.8.6. (a) The solution is given in the text.

(b) $X(t) = X_0 e^{-(b/2m)t}\left\{\cosh\sigma t + \dfrac{b}{2m\sigma}\sinh\sigma t\right\}$, where

$\sigma^2 = \dfrac{b^2}{4m^2} - \dfrac{k}{m}$. See Example 20.8.5.

20.8.7. (a) and (b) Solutions are given in the text.

(c) $X(t) = \dfrac{v_0}{\sigma}e^{-(b/2m)t}\sinh\sigma t, \quad \sigma^2 = \dfrac{b^2}{4m^2} - \dfrac{k}{m}\,.$

20.8.8. Take the Laplace transform of the equation of motion.

$$ms^2x(s) = \frac{mg}{s} - bsx(s)\,.$$

Solve for $x(s)$,

$$x(s) = \frac{mg}{s^2(ms+b)}\,,$$

and take the inverse transform:

$$X(t) = \frac{mg}{b}t - \frac{m^2g}{b^2}\left(1-e^{-bt/m}\right) = \frac{m^2g}{b^2}\left(\frac{b}{m}t - 1 + e^{-bt/m}\right).$$

Differentiating, $\dfrac{dX(t)}{dt} = \dfrac{mg}{b}\left(1-e^{-(b/m)t}\right).$

20.8.9. $E(t) = -\dfrac{I_0}{\omega_1 C}e^{-t/2RC}\sin\omega_1 t, \quad \omega_1^2 = \dfrac{1}{LC} - \dfrac{1}{(2RC)^2}\,.$

This solution is based on the initial conditions $E(0) = 0$ (because the idealized inductance L would have zero DC impedance) and $I_L(0) = I_0$, limited by a resistance in series with the battery or by the internal resistance of the battery. Finally, to be consistent, $q_0 = 0$.

20.8.10. Start from Eq. (14.20), $J_0(t) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{it\cos\theta}\, d\theta\,.$

form the Laplace transform, interchange the two integrations, and evaluate the integral over t:

$$\mathcal{L}\{J_0(t)\} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} dt\, e^{-st+it\cos\theta} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{s - i\cos\theta}\,.$$

The integral over θ can be evaluated by writing it as a contour integral in the variable $z = e^{i\theta}$ around the unit circle. Compare with Example 11.8.1. Using the result of that example, namely

$$\int_0^{2\pi}\frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}\,,$$

we obtain

$$\mathcal{L}\{J_0(t)\} = \frac{1}{2\pi s}\int_0^{2\pi}\frac{d\theta}{1-(i/s)\cos\theta} = \frac{1}{2\pi s}\,\frac{2\pi}{\sqrt{1-(i/s)^2}}\,,$$

which simplifies to the required result.

20.8.11. Set $\mathcal{L}\{J_n(t)\} = g_n(s)$. From Exercise 20.8.10, we have $g_0(s) = \dfrac{1}{\sqrt{s^2+1}}\,.$

Using the formula $J_1 = -J_0'$ of Eq. (14.9) and Eq. (20.147) to transform J_0', we also have

$$g_1(s) = -\left[\frac{s}{\sqrt{s^2+1}} - 1\right] = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}\,.$$

Next take the transform of the recurrence formula, Eq. (14.8):

$$g_{n+1} = g_{n-1} - 2\mathcal{L}\{J_n'\} = g_{n-1} - 2sg_n\,, \qquad n \ge 1\,,$$

where we have used the formula for the transform of a derivative, Eq. (20.147), and restricted its use to n values for which $J_n(0) = 0$. Using the recurrence formula for $n = 1$, we get a result that can be simplified to

$$g_2(s) = \frac{(\sqrt{s^2+1}-s)^2}{\sqrt{s^2+1}}\,.$$

The results for g_0, g_1, and g_2 suggest the general formula

$$g_n(s) = \frac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}\,.$$

This suggestion can be confirmed by mathematical induction; we already have confirmed it for $n = 0$, 1, and 2. To confirm it for general n we need only show it to be consistent with the recurrence formula. Substitution of the suggested form into the g_n recurrence formula is found to lead to an algebraic identity.

20.8.12. Setting $I = \int_0^\infty e^{-kz} k J_1(ka)\, dk$, we first note that

$$I = -\frac{d}{dz}\int_0^\infty e^{-kz} K_1(ka)\, dk = +\frac{d}{dz}\int_0^\infty e^{-kz}\frac{d}{d(ka)}\Big[J_0(ka)\Big]\, dk\,.$$

The last member of this equation was reached by the use of Eq. (14.9).

We next integrate by parts and then simplify the result by inserting the value of the Laplace transform of J_0 given in Table 20.1:

$$I = \frac{1}{a}\frac{d}{dz}\left[e^{-kz}J_0(ka)\Big|_{k=0}^{\infty} + z\int_0^\infty e^{-kz}J_0(ka)\, dk\right]$$

$$= \frac{1}{a}\frac{d}{dz}\left[1 + \frac{z}{(z^2+a^2)^{1/2}}\right]\,.$$

Evaluating the z derivative, we recover the answer in the text.

20.8.13. Since $I_0(at) = J_0(iat)$, we may obtain its Laplace transform from the solution to Exercise 20.8.10 by application of Eq. (20.156), with a replaced by ia. We get

$$\mathcal{L}\{J_0(iat)\} = \frac{1}{ia}\left[\left(\frac{s}{ia}\right)^2 + 1\right]^{-1/2} = \frac{1}{\sqrt{s^2-a^2}}\,.$$

20.8.14. (a) Use Eq. (20.184), the formula for the integral of a transform, taking $F(t) = \sin at$, and (from Table 20.1) $f(s) = a/(s^2+a^2)$.

$$\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\int_s^\infty \frac{a\, ds}{s^2+a^2} = \frac{1}{a}\int_{s/a}^\infty \frac{du}{u^2+1}$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right)\,.$$

(b) The integral defining the transform diverges at $t = 0$.

(c) A procedure similar to that used in part (a) leads to

$$\mathcal{L}\left\{\frac{\sinh at}{at}\right\} = \int_s^\infty \frac{ds}{s^2-a^2}\,.$$

Make the partial fraction decomposition

$$\frac{1}{s^2-a^2} = \frac{1}{2a}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]\,.$$

Although the insertion of this expression creates two divergent integrals, their divergences (at $s = \infty$) cancel, and we get

$$\mathcal{L}\left\{\frac{\sinh at}{at}\right\} = \frac{1}{2}\Big[-\ln(s-a) + \ln(s+a)\Big]\,,$$

equivalent to the answer in the text.

(d) The integral defining the transform diverges at $t = 0$.

20.8.15. Let $f(s)$ denote the Laplace transform of $F(t)$. To deal with the transform of the Laguerre ODE, we need the following transforms, which involve the use of Eqs. (20.147), (20.148), and (20.174).

$$\mathcal{L}\{F'(t)\} = s\,f(s) - F(0)\,,$$

$$\mathcal{L}\{t\,F'(t)\} = -\frac{d}{ds}\,\mathcal{L}\{F'(t)\} = -f(s) - s\,f'(s)\,,$$

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)\,,$$

$$\mathcal{L}\{t\,F''(t)\} = -\frac{d}{ds}\,\mathcal{L}\{F''(t)\} = -2s\,f(s) - s^2 f'(s) + F(0)\,.$$

Combining these to form the transform of the Laguerre equation,

$$\mathcal{L}\{tF''(t) + F'(t) - tF'(t) + nF(t)\} = s(1-s)f'(s) + (n-s+1)f(s) = 0\,.$$

This is a separable homogeneous first-order ODE, which can be written, using a partial fraction decomposition, as

$$\frac{df}{f} = -\left(\frac{n-s+1}{s(1-s)}\right)ds = -\left(\frac{n}{1-s} + \frac{n+1}{s}\right)ds\,.$$

Integrating the ODE, and taking the antilogarithm of the solution, we reach after a few steps

$$f(s) = \frac{C}{s}\left(1 - \frac{1}{s}\right)^n\,.$$

Checking this solution for $n = 0$ (with $C = 1$), we have $f(s) = s^{-1}$, corresponding to $F_0(t) = 1$, the correct value of $L_0(t)$. Continuing to $n = 1$, we have $f(s) = s^{-1} - s^{-2}$, corresponding to $F_1(t) = 1 - t$, the correct value of $L_1(t)$.

20.8.16. Here is a proof using mathematical induction. Writing $g_n(s) = \mathcal{L}\{L_n(at)\}$, we find by direct evaluation (use Table 20.1 if necessary)

$$g_0(s) = \mathcal{L}\{1\} = \frac{1}{s}, \qquad g_1(s) = \mathcal{L}\{1 - at\} = \frac{1}{s} - \frac{a}{s^2} = \frac{s-a}{s^2}\,,$$

thereby establishing the formula of this exercise for $n = 0$ and $n = 1$. We now show that for $n \ge 1$ the value of g_{n+1} is consistent with the Laguerre polynomial recurrence formula, Eq. (18.51), using the assumed formulas for g_n and g_{n-1}. Taking the Laplace transform of that equation, we get

$$(n+1)g_{n+1}(s) = (2n+1)g_n(s) - ng_{n-1}(s) + \mathcal{L}\{(-at)L_n(at)\}\,.$$

The transform of $-tL_n(at)$ can be computed using Eq. (20.174) and the assumed form of g_n. We get

$$\mathcal{L}\{(-at)L_n(at)\} = a\,g_n' = a\left[\frac{n(s-a)^{n-1}}{s^{n+1}} - \frac{(n+1)(s-a)^n}{s^{n+2}}\right].$$

Substituting the assumed forms for all the quantities on the right-hand side of the recurrence formula and simplifying, we confirm that g_{n+1} also has the assumed form, thereby completing the proof.

20.8.17. Form the Laplace transform, interchange the two integrations, and evaluate the integral over t:

$$\mathcal{L}\{E_1(t)\} = \int_1^\infty dx \int_0^\infty dt \frac{e^{-xt-ts}}{x} = \int_1^\infty \frac{dx}{x(x+s)}\,.$$

Make a partial fraction decomposition of the x integrand; this results in integrals that are individually divergent at large x but with a difference that is a finite limit. Thus,

$$\mathcal{L}\{E_1(t)\} = \lim_{R\to\infty}\int_1^R \frac{1}{s}\left[\frac{1}{x} - \frac{1}{x+s}\right] dx$$

$$= \frac{1}{s}\lim_{R\to\infty}\Big[\ln R - \ln(R+s) + \ln(s+1)\Big] = \frac{1}{s}\,\ln(s+1)\,,$$

because $\ln R - \ln(R+s) = -\ln\left(1+\frac{s}{R}\right) = -\frac{s}{R} + \cdots \longrightarrow 0.$

20.8.18. (a) This is the case $s = 0$ of Eq. (20.184).

(b) Taking $F(t) = \sin t$, then, from Table 20.1, $f(s) = 1/(s^2+1)$, and the formula of this exercise indicates that

$$\int_0^\infty \frac{F(t)}{t}\,dt = \int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty f(s)\,ds = \int_0^\infty \frac{ds}{s^2+1}\,ds = \frac{\pi}{2}\,.$$

20.8.19. (a) See Exercise 20.8.14(a).

(b) Write $\text{si}(x) = -\int_t^\infty \frac{\sin x}{x}\,dx = -\frac{\pi}{2} + \int_0^t \frac{\sin x}{x}\,dx\,.$

Take the Laplace transform of both sides of this equation, using Formula 3 of Table 20.2 and the result of part (a) for the transform of the integral. We get

$$\mathcal{L}\{\text{si}(t)\} = -\frac{\pi}{2s} + \frac{1}{s}\,\cot^{-1}s = -\frac{1}{s}\,\tan^{-1}(s)\,.$$

20.8.20. Taking note of the periodicity of $F(t)$, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t, dt = \sum_{n=0}^\infty \int_{na}^{(n+1)a} e^{-st}F(t)\,dt$$

$$= \sum_{n=0}^\infty e^{-nas}\int_0^a e^{-st}F(t)\,dt\,.$$

Performing the summation,

$$\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-as}}\int_0^a e^{-st}F(t)\,dt\,.$$

20.8.21. The solution is given in the text.

20.8.22. (a) Writing $\cosh at = \cos(-iat)$, using

$$\cos at\cos(-iat) = \frac{\cos(1-i)at + \cos(1+i)at}{2}$$

and noting that $(1+i)^2 = 2i$, $(1-i)^2 = -2i$, we can use the formula for the transform of $\cos kt$ to obtain

$$\mathcal{L}\{\cosh at\cos at\} = \frac{1}{2}\left[\frac{s}{s^2+(1-i)^2a^2} + \frac{s}{s^2+(1+i)^2a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2-2ia^2} + \frac{s}{s^2+2ia^2}\right],$$

which simplifies to the formula given in the text.

(b) Use an approach similar to that of part (a), but here

$$\sin at\cos(-iat) = \frac{\sin(1-i)at + \sin(1+i)at}{2}\,.$$

Using the transform of $\sin kt$, we now have

$$\mathcal{L}\{\cosh at\sin at\} = \frac{1}{2}\left[\frac{(1-i)a}{s^2+(1-i)^2a^2} + \frac{(1+i)a}{s^2+(1+i)^2a^2}\right],$$

which simplifies to the result given in the text.

Parts (c) and (d) are handled in ways similar to those used for parts (a) and (b).

20.8.23. The formulas of this exercise can be obtained by evaluating the transforms on their right-hand sides. Terms of the forms $\sin at$ or $\cos at$ have transforms given in Table 20.1; terms of the forms $t\sin at$ or $t\cos at$ can be obtained from the sine and cosine transforms by differentiating them (see Formula 7, Table 20.2):

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}\,,$$

$$\mathcal{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}\,.$$

The evaluations are straightforward.

20.8.24. Start from the integral representation of $K_0(z)$ obtained by specializing Eq. (14.128):

$$K_0(z) = \int_1^\infty \frac{e^{-zx}}{(x^2-1)^{1/2}}\,dx\,.$$

In the above, change z to ks and change the integration variable to $t = kx$. Our integral representation then becomes

$$K_0(ks) = \int_k^\infty \frac{e^{-st}}{(t^2-k^2)^{1/2}}\,dt\,.$$

This equation becomes equivalent to that in the text if we introduce a unit step function that permits our changing the lower integration limit to zero.

20.9 Laplace Convolution Theorem

20.9.1. Apply the Laplace convolution theorem with $F(t)$, $f(s)$, $G(t) = 1$, and $g(s) = 1/s$:

$$f(s)g(s) = \frac{f(s)}{s} = \mathcal{L}\{G * F\}\,,$$

where

$$G * F = \int_0^t G(t-z)F(z)\,dz = \int_0^t F(z)\,dz\,.$$

20.9.2. (a) Write the formula for $F * G$, then move a factor t^{a+b+1} outside the integral and change the integration variable to z/t:

$$F * G = \int_0^t F(t-z)G(z)\,dz = \int_0^t (t-z)^a z^b\,dz$$

$$= t^{a+b+1}\int_0^t \left(1-\frac{z}{t}\right)^a \left(\frac{z}{t}\right)^b d\left(\frac{z}{t}\right) = t^{a+b+1}\int_0^1 (1-y)^a\, y^b\,dy\,,$$

equivalent to the answer in the text.

(b) The functions F and G have the respective Laplace transforms

$$f(s) = \frac{\Gamma(a+1)}{s^{a+1}}\,, \qquad g(s) = \frac{\Gamma(b+1)}{s^{b+1}}\,.$$

We next form their product and identify it by inspection as a transform:

$$f(s)g(s) = \frac{\Gamma(a+1)\Gamma(b+1)}{s^{a+b+2}} = \mathcal{L}\left\{\frac{\Gamma(a+1)\Gamma(b+1)t^{a+b+1}}{\Gamma(a+b+2)}\right\}.$$

Finally, we take the inverse transform of $f(s)g(s)$ and equate it to the convolution $F * G$; the result is

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \frac{\Gamma(a+1)\Gamma(b+1)t^{a+b+1}}{\Gamma(a+b+2)} = t^{a+b+1}\int_0^1 (1-y)^a\, y^b\, dy\,.$$

This equation is equivalent to the relationship to be proved.

20.9.3. The two factors in the transform product and their respective inverse transforms are

$$f(s) = \frac{s}{s^2+a^2}, \quad F(t) = \cos at, \quad g(s) = \frac{1}{s^2+b^2}, \quad G(t) = \frac{\sin bt}{b}\,.$$

We now apply the convolution theorem:

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(t-z)G(z)\,dz = \frac{1}{b}\int_0^t \cos a(t-z)\sin bz\,dz\,.$$

Using a trigonometric identity, this becomes

$$\begin{aligned}
\mathcal{L}^{-1}\{f(s)g(s)\} &= \frac{1}{2b}\int_0^t [\sin(at-az+bz) - \sin(at-az-bz)]\,dz \\
&= \left[\frac{1}{2b(a-b)}\cos(at-az+bz) - \frac{1}{2b(a+b)}(\cos at - az - bz)\right]_{z=0}^{t} \\
&= \frac{\cos bt - \cos at}{a^2-b^2}\,,
\end{aligned}$$

equivalent to a result found by other methods in Exercise 20.7.3.

20.9.4. The solution is given in the text.

20.10 Inverse Laplace Transform

20.10.1. Application of $\mathcal{L}^{-1}$ to the integral representation of $f(s)$ can be moved into the integral over z, where it converts $(s-z)^{-1}$ into e^{zt}, thereby producing the Bromwich formula.

20.10.2. Make the insertion suggested in the problem statement, and interchange the order of the two integrations:

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st} f(s)\,ds = \int_0^\infty F(z)\,dz\,\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{s(t-z)}\,ds\,.$$

Next make the substitution $s = \beta + iu$, with $ds = idu$; the s integral is thereby changed to

$$\int_{\beta-i\infty}^{\beta+i\infty} e^{s(t-z)}\, ds = e^{\beta(t-z)} \int_{-\infty}^{\infty} e^{iu(t-z)}\, i\, du = 2\pi i\, e^{\beta(t-z)}\delta(t-z)$$

$$= 2\pi i\, \delta(t-z)\,.$$

We have identified the u integral as the delta function representation in Eq. (20.20).

Use of this delta-function formula brings us to

$$\int_{\beta-i\infty}^{\beta+i\infty} e^{s(t-z)}\, ds = \int_0^{\infty} F(z)\, \delta(t-z)\, dz = F(t)\,,$$

which is the Bromwich integral for the inverse Laplace transform.

20.10.3. Let $f_1(s)$ And $f_2(s)$ be the respective Laplace transforms of $F_1(t)$ and $F_2(t)$, and use the Bromwich integral for the inverse transform of the product $f_1(s)f_2(s)$:

$$\mathcal{L}^{-1}\{f_1(s)f_2(s)\}(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{ts} f_1(s) f_2(s)\, ds$$

$$= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{ts}\, ds \int_0^{\infty} e^{-xs} F_1(x)\, dx \int_0^{\infty} e^{-ys} F_2(y)\, dy\,.$$

Collect the factors that depend upon s, move the s integral to the right of the other integrations, and make a change of variable to u, with $s = \beta + iu$, so the s integral becomes, with the aid of Eq. (20.20),

$$\int_{\beta-i\infty}^{\beta+i\infty} e^{(t-x-y)s}\, ds = e^{\beta(t-x-y)} \int_{-\infty}^{\infty} e^{iu(t-x-y)}\, i\, du = 2\pi i \delta(t-x-y)\,.$$

Using this delta-function integral, we now have

$$\mathcal{L}^{-1}\{f_1(s)f_2(s)\}(t) = \int_0^{\infty} F_1(x)\, dx \int_0^{\infty} F_2(y)\delta(t-x-y)\, dy$$

$$= \int_0^{\infty} F_1(x) F_2(t-x)\, dx\,,$$

which is the Laplace convolution theorem.

20.10.4. (a) The partial fraction expansion required here is

$$\frac{s}{s^2-k^2} = \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right].$$

Using Table 20.1 to invert the individual terms of this expansion,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \frac{1}{2}(e^{kt}+e^{-kt}) = \cosh kt.$$

(b) Starting from the Bromwich integral

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{se^{st}}{s^2-k^2}\,ds\,,$$

close the contour with a counterclockwise arc at infinite s that does not contribute to the integral. The contour then encloses simple poles at $s=k$ and $s=-k$, with respective residues $e^{kt}/2$ and $e^{-kt}/2$, leading to the same result as in part (a).

20.10.5. (a) The partial fraction expansion required here is

$$\frac{k^2}{s(s^2+k^2)} = \frac{1}{s} - \frac{s}{s^2+k^2}\,.$$

Using Table 20.1, this can be identified as the transform of $1-\cos kt$.

(b) We seek $\mathcal{L}^{-1}\{f(s)g(s)\}$, where $f(s)=1/x$ and $g(s)=k^2/(s^2+k^2)$ are the transforms of $F(t)=1$ and $G(t)=k\sin kt$. Using the convolution theorem,

$$\mathcal{L}^{-1}\{f(s)g(s)\} = F*G = \int_0^t k\sin kz\,dz = -\cos kz\Big|_0^t\,,$$

which evaluates to the desired result.

(c) Form the Bromwich integral, noting that its vertical path in the complex plane must fall to the right of all singularities in the integrand, which are ar $s=0$ and $s=\pm ik$. We can close the contour of the integral with an arc at infinity in the left half-plane, so we have

$$F(t) = (L)\left\{\frac{k^2}{s(s^2+k^2)}\right\} = \frac{1}{2\pi i}\oint\frac{k^2e^{st}\,ds}{s(s-ik)(s+ik)}\,,$$

where the contour surrounds the three singularities of s. Applying the residue theorem,

$$F(t) = \sum(\text{residues}) = \frac{k^2e^0}{(-ik)(ik)} + \frac{k^2e^{ikt}}{(ik)(2ik)} + \frac{k^2e^{-ikt}}{(-ik)(-2ik)}\,,$$

which simplifies to the required result.

20.10.6. Close the contour for the Bromwich integral as shown in Fig. 20.23 of the text. The contour encloses no singularities, and neither the large nor the small circular arc makes a contribution to the contour integral. However, the two horizontal segments above and below the branch cut are nonzero and do not cancel; in fact, because the branch point at $s = 0$ is of order two, the sum of the contribution of these segments is two times the contribution of either one. Taking the segment below the branch cut, s there has the value $re^{-i\pi}$, $ds = -dr$, $e^{ts} = e^{-tr}$, $s^{-1/2} = r^{-1/2}e^{+i\pi/2} = ir^{-1/2}$, the integration is from $r = 0$ to $r = \infty$, and

$$\mathcal{L}^{-1}\{s^{-1/2}\} = -2\left(\frac{1}{2\pi i}\right)\int_0^\infty (ir^{-1/2})e^{-tr}(-dr)$$

$$= \frac{1}{\pi}\int_0^\infty r^{-1/2}e^{-tr}\,dr = \frac{1}{\pi t^{1/2}}\,\Gamma(1/2) = \frac{1}{(\pi t)^{1/2}}\,.$$

20.10.7. As indicated by Fig. 20.24 in the text, the Bromwich integral $F(t)$ for this problem is equal to a contour integral (in the mathematically positive direction) along a closed path that is adjacent to the singular points and the vertical branch cut that connects them. In the right-hand vertical segment of this closed path, the integrand must (by continuous deformation of the Bromwich integrand) be on the branch for which $\sqrt{1+s^2}$ is positive; on the other side of the branch cut, this quantity will be negative. Since the circular arcs at $\pm i$ do not contribute to the contour integral, its value must be

$$F(t) = 2\left[\frac{1}{2\pi i}\int_{-i}^{i}\frac{e^{st}}{\sqrt{1+s^2}}\,ds\right]\,.$$

Now set $s = iy$, so $ds = idy$, leading to

$$F(t) = \frac{1}{\pi i}\int_{-1}^{1}\frac{e^{iyt}}{\sqrt{1-y^2}}\,i\,dy = \frac{1}{\pi}\int_0^1\frac{e^{iyt}+e^{-iyt}}{\sqrt{1-y^2}}\,dy = \frac{2}{\pi}\int_0^1\frac{\cos yt}{\sqrt{1-y^2}}\,dy\,.$$

This is an integral representation of $J_0(t)$. See the solution to Exercise 20.2.10.

20.10.8. (a) Carry out a binomial expansion on $f(s) = (s^2 - a^2)^{-1/2}$, then invert termwise to obtain $F(t)$.

$$f(s) = \frac{1}{s}\left(1 - \frac{a^2}{s^2}\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{a^{2n}}{s^{2n+1}}\,,$$

and, using Table 20.1 to identify the inversions,

$$F(t) = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{a^{2n}t^{2n}}{(2n)!}\,.$$

Inserting $\displaystyle\binom{-1/2}{n} = \frac{(-1)^n(2n-1)!!}{2^n\,n!}$,

$$F(t) = \sum_{n=0}^{\infty}\frac{(2n-1)!!\,a^{2n}t^{2n}}{2^n\,n!\,(2n)!} = \sum_{n=0}^{\infty}\frac{1}{n!\,n!}\left(\frac{at}{2}\right)^{2n} = I_0(at)\,.$$

Figure 20.10.8. Contour for Exercise 20.10.8.

(b) The Bromwich integral for this problem is equal to a closed contour integral of the form

$$I = \frac{1}{2\pi i}\oint \frac{e^{tz}\,dz}{(z-a)^{1/2}(z+a)^{1/2}}\,,$$

where the contour, shown in Fig. 20.10.8, surrounds a branch cut that connects $z=-a$ and $z=a$. When $z=x+iy$ is on the real axis, with $x>a$, we are on the branch of the integrand for which both $(x+a)^{1/2}$ and $(x-a)^{1/2}$ are real and positive (i.e., have zero arguments). Above the branch cut but to the right of $x=-a$, $(x+a)^{1/2}$ remains positive, but $(x-a)^{1/2}$ becomes $+i(a-x)^{1/2}$. Below the branch cut, $(x+a)^{1/2}$ is still positive, but $(x-a)^{1/2}=-i(a-x)^{1/2}$. When all this is taken into account and we note (1) that these square roots occur in the denominator of the integrand and (2) the direction of the integration path, we find

$$I = \frac{1}{2\pi i}\,2\int_{-a}^{a}\frac{ie^{tx}\,dx}{\sqrt{a^2-x^2}} = \frac{1}{\pi}\int_0^a \frac{(e^{tx}+e^{-tx})}{\sqrt{a^2-x^2}}\,dx = \frac{2}{\pi}\int_0^a \frac{\cosh tx}{\sqrt{a^2-x^2}}\,dx\,.$$

Changing the integration variable to $u=x/a$, we bring I to the form

$$I = \frac{2}{\pi}\int_0^1 \frac{\cosh atu}{\sqrt{1-u^2}}\,du\,,$$

which is an integral representation of $I_0(at)$.

(c) In the new variable z, the points $z=\beta\pm i\infty$ correspond to $s=a\beta/2\pm i\infty$, $ds=(a/2)(1-z^{-2})\,dz$, and $(s^2-a^2)^{1/2}=(a/2)(z-z^{-1})$. The Bromwich integral for this problem can therefore be written

$$F(t) = \frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty}\frac{e^{(a/2)(z+1/z)t}}{(a/2)(z-1/z)}\,\frac{a}{2}\,(1-z^{-2})\,dz$$

$$= \frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty}\frac{e^{(a/2)(z+1/z)t}}{z}\,dz\,.$$

We may close the contour in z by an arc at large $|z|$ in the left half-plane, thereby creating a closed contour containing as its only singularity a simple pole at $z = 0$. $F(t)$ will be the residue at this singularity. To obtain it, write

$$e^{(a/2)(z+1/z)t} = e^{azt/2}e^{at/2z} = \sum_{n=0}^{\infty}\frac{(azt/2)^n}{n!}\sum_{m=0}^{\infty}\frac{(at/2z)^m}{m!}\,.$$

The coefficient of z^0, which is the residue we seek, is the contribution to the above expression from terms for which $m = n$. Therefore,

$$F(t) = \text{residue} = \sum_{n=0}^{\infty}\frac{(at/2)^{2n}}{n!\,n!} = I_0(at)\,.$$

20.10.9. Start from Eqs. (12.79) and (12.80),

$$E_1(t) = \int_1^{\infty}\frac{e^{-ty}}{y}\,dy = -\gamma - \ln t - \sum_{n=1}^{\infty}\frac{(-1)^n t^n}{n\,n!}\,.$$

Our task is to show that

$$\mathcal{L}\{-\gamma - \ln t\} = \mathcal{L}\left\{E_1(t) + \sum_{n=1}^{\infty}\frac{(-1)^n t^n}{n\,n!}\right\}$$

$$= \mathcal{L}\{E_1(t)\} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n\,s^{n+1}} = \frac{\ln s}{s}\,.$$

We have taken the transform of the power series termwise, using a formula from Table 20.1.

Now form the transform of $E_1(t)$ using its integral representation and interchange the two integrations:

$$\mathcal{L}\{E_1(t)\} = \int_0^{\infty}dt\,e^{-ts}\int_1^{\infty}\frac{e^{-ty}}{y}\,dy = \int_1^{\infty}\frac{dy}{y}\int_0^{\infty}e^{-t(s+y)}\,dt = \int_1^{\infty}\frac{dy}{y(y+s)}\,.$$

Apply a partial fraction decomposition to the y integral; this produces two integrals that are individually divergent, but the divergences cancel. We get

$$\mathcal{L}\{E_1(t)\} = \frac{1}{s}\int_1^{\infty}\left[\frac{1}{y} - \frac{1}{y+s}\right]dy = \frac{1}{s}\int_1^{s+1}\frac{dy}{y} = \frac{\ln(s+1)}{s}\,.$$

For our present purposes we write this result as an expansion:

$$\mathcal{L}\{E_1(t)\} = \frac{\ln(s+1)}{s} = \frac{\ln s}{s} + \frac{1}{s}\ln\left(1+\frac{1}{s}\right) = \frac{\ln s}{s} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n\,s^{n+1}}\,.$$

Inserting this result into the equation for $\mathcal{L}\{-\gamma - \ln t\}$, we find that the summations in inverse powers of s cancel, leaving the desired result.

20.10.10. The Bromwich integral can be converted without changing its value into a closed contour integral with an arc at infinity encircling the left half-plane.

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{2\pi i}\oint \frac{se^{st}}{(s^2+a^2)^2}\,ds\,,$$

where the contour encloses the singularities of $e^{st}f(s)$, which consist of two second-order poles at the points $s = ia$ and $s = -ia$. Applying the residue theorem,

$$\mathcal{L}^{-1}\{f(s)\} = \frac{d}{ds}\left[\frac{se^{st}}{(s+ia)^2}\right]_{s=ia} + \frac{d}{ds}\left[\frac{se^{st}}{(s-ia)^2}\right]_{s=-ia}$$

$$= e^{iat}\left[\frac{1+iat}{-4a^2} - \frac{2ia}{(-4a^2)(2ia)}\right] + e^{-iat}\left[\frac{1-iat}{-4a^2} - \frac{-2ia}{(-4a^2)(-2ia)}\right].$$

The above expression simplifies to $\mathcal{L}^{-1}\{f(s)\} = \dfrac{t}{2a}\sin at$.

20.10.11. The Bromwich integral is converted without changing its value into a closed contour integral with an arc at infinity encircling the left half-plane. Applying the residue theorem, noting that there are simple poles at the zeros s_i of $h(s)$, we get

$$F(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}g(s)}{h(s)}\,ds = \sum_i \frac{g(s_i)}{h'(s_i)}\,e^{s_i t}\,.$$

20.10.12. The Bromwich integral for this problem,

$$\mathcal{L}\{s^{-2}e^{-ks}\} = \frac{1}{2\pi i}\int_{\beta-\infty i}^{\beta+\infty i} \frac{e^{s(t-k)}}{s^2}\,ds\,,$$

yields the value zero if $t < k$, as under that condition its contour can be closed by an arc to the right at large $|s|$ and the contour then includes no singularities. If $t > k$, the contour can be closed by an arc to the left at large $|s|$, enclosing a second-order pole at $s = 0$ with residue $t - k$ (an easy way to obtain the residue is to look at the linear term when the exponential is expanded in a power series). Thus,

$$\mathcal{L}\{s^{-2}e^{-ks}\} = \begin{cases} 0, & t < k, \\ t-k, & t > k\,. \end{cases}$$

Both these cases can be incorporated into a single formula by appending a unit step function that is zero for $t < k$ and unity for $t > k$.

20.10.13. (a) Use the partial fraction identity

$$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right).$$

Then invert by inspection, using entry 4 of Table 20.1:

$$F(t) = \frac{e^{-bt} - e^{-at}}{a-b}.$$

(b) Recognize $f(s) = (s+a)^{-1}$ and $g(s) = (s+b)^{-1}$ as the transforms of $F(t) = e^{-at}$ and $G(t) = e^{-bt}$. The product $f(s)g(s)$ is the transform of the convolution $F * G$, so this convolution is the inverse transform we seek. Evaluating it,

$$F * G = \int_0^t e^{-a(t-z)-bz}\,dz = e^{-at}\left.\frac{e^{(a-b)z}}{a-b}\right|_0^t = \frac{e^{-bt} - e^{-at}}{a-b}.$$

(c) Using Exercise 20.10.11,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{(s+a)(s+b)}\,ds = \frac{e^{-at}}{-a+b} + \frac{e^{-bt}}{-b+a}.$$

21. Integral Equations

21.1 Introduction

21.1.1. (a) Integrate $y'' = y$ from $x = 0$ to $x = x$ and set $y'(0) = 1$:

$$\int_0^x y''(x)\,dx = y'(x) - y'(0) = \int_0^x y(t)\,dt$$

$$\longrightarrow \quad y'(x) = 1 + \int_0^x y(t)\,dt\,.$$

Integrate again from 0 to x, setting $y(0) = 0$:

$$y(x) - y(0) = y(x) = \int_0^x du + \int_0^x du \int_0^u y(t)\,dt$$

$$= x + \int_0^x dt\, y(t) \int_t^x du = x + \int_0^x (x - t)\, y(t)\,dt\,.$$

(b) Same ODE, but with $y'(0) = -1$ and $y(0) = 1$. The first integration yields

$$y'(x) = -1 + \int_0^x y(t)\,dt\,.$$

Integrating again,

$$y(x) - 1 = -x + \int_0^x du \int_0^u y(t)\,t = -x + \int_0^x dt\, y(t) \int_t^x du$$

$$= -x + \int_0^x (x - t)\, y(t)\,dt\,.$$

This rearranges into the answer in the text.

21.1.2. (a) From the integral equation, $y(x) = \displaystyle\int_0^x (x - t)y(t)\,dt + x$, we see that $y(0) = 0$. Differentiating,

$$y'(x) = \int_0^x y(t)\,dt + 1\,,$$

also showing that $y'(0) = 1$. Differentiating again,

$$y''(x) = y(x)\,.$$

(b) The integral equation, $y(x) = \displaystyle\int_0^x (x - t)y(t)\,dt - x + 1$, shows that $y(0) = 1$. Differentiating,

$$y'(x) = \int_0^x y(t)\,dt - 1 \quad \text{and} \quad y''(x) = y(x)\,.$$

The first of these equations also shows that $y'(0) = -1$.

21.1.3. $\varphi(x) = \sinh x$.

21.2 Some Special Methods

21.2.1. Letting $F(t)$, $K(t)$, and $\Phi(t)$ be the Fourier transforms of $f(x)$, $k(x)$, and $\varphi(x)$, the integral equation becomes

$$\Phi(t) = F(t) + \lambda\sqrt{2\pi}\, K(t)\Phi(t)\,,$$

where we have used the fact that the integral is of the form of a convolution. Solving for $\Phi(t)$, we get

$$\Phi(t) = \frac{F(t)}{1 - \lambda\sqrt{2\pi}\, K(t)}\,.$$

We now take the inverse Fourier transform to reach

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(t)\, e^{-ixt}\, dt}{1 - \lambda\sqrt{2\pi}\, K(t)}\,.$$

21.2.2. (a) Take the Laplace transform of this integral equation, with $F(s)$, $K(s)$, and $\Phi(s)$ the transforms of $f(x)$, $k(x)$, and $\varphi(x)$. We get

$$F(s) = K(s)\Phi(s)\,,$$

where we have identified the integral as a convolution. Solving for $\Phi(s)$ and using the Bromwich formula for the inverse transform,

$$\varphi(x) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{F(s)}{K(s)}\, e^{xs}\, dx\,.$$

(b) Take the Laplace transform of this integral equation, with $F(s)$, $K(s)$, and $\Phi(s)$ the transforms of $f(x)$, $k(x)$, and $\varphi(x)$. We get

$$\Phi(s) = F(s) + \lambda K(s)\Phi(s)\,,$$

where we have identified the integral as a convolution. Solving for $\Phi(s)$ and using the Bromwich formula for the inverse transform,

$$\varphi(x) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{F(s)}{1 - \lambda K(s)} e^{xs}\, ds\,.$$

21.2.3. (a) The integral in this equation is of the form of a convolution; since the general form of a Laplace transform convolution is

$$\int_0^x f(x - t)\varphi(t)\, dt\,,$$

we note that $f(x) = -x$ and its transform $F(s)$ is $-1/s^2$. Since the Laplace transform of x is $1/s^2$, the integral equation transforms to

$$\Phi(s) = \frac{1}{s^2} - \frac{\Phi(s)}{s^2}, \quad \text{so} \;\; \Phi(s) = \frac{s^{-2}}{1+s^{-2}} = \frac{1}{s^2+1}\,.$$

From a table of Lasplace transforms, $\Phi(s)$ is identified as the transform of $\sin x$.

(b) A treatment similar to that of part (a) yields $\Phi(s) = \dfrac{1}{s^2-1}\,.$ This is the Laplace transform of $\sinh x$.

21.2.4. The convolution formula for the Fourier cosine transform can be written in the form

$$\frac{1}{2}\int_{-\infty}^{\infty} g(y)f(x-y)\,dy = \int_0^{\infty} F_c(s)G_c(s)\,\cos xs\,ds\,,$$

where the subscript c denotes the cosine transform and it is assumed that f and g are even functions of their arguments. Consider the integral equation

$$f(x) = \int_{-\infty}^{\infty} k(x-y)\varphi(y)\,dy\,,$$

where $k(x-y)$, $f(x)$, and $\varphi(y)$ are assumed to be even functions. Applying the convolution formula,

$$f(x) = 2\int_0^{\infty} K_c(s)\Phi_c(s)\cos xs\,ds\,,$$

and then taking the cosine transform, we reach

$$F_c(\omega) = 2\sqrt{\frac{\pi}{2}}\;K_c(\omega)\Phi_c(\omega)\,.$$

Solving for Φ:

$$\Phi(\omega) = \frac{F_c(\omega)}{\sqrt{2\pi}\;K_c(\omega)}\,,$$

Taking the inverse transform,

$$\varphi(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{F_c(\omega)}{\sqrt{2\pi}\;K_c(\omega)}\cos x\omega\,d\omega = \frac{1}{\pi}\int_0^{\infty}\frac{F_c(\omega)}{K_c(\omega)}\,\cos x\omega\,d\omega\,.$$

21.2.5. $\varphi(t) = \delta(t)$.

21.2.6. Following the procedure suggested for this problem,

$$\int_0^z f(x)(z-x)^{\alpha-1}\,dx = \int_0^z dx\int_0^x \frac{(z-x)^{\alpha-1}}{(x-t)^{\alpha}}\,\varphi(t)\,dt$$

$$= \int_0^z \varphi(t)\,dt\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{\alpha}} = \frac{\pi}{\sin\pi\alpha}\int_0^z \varphi(t)\,dt\,.$$

This last integral over x was evaluated using the formula given in the *Note* attached to this problem.

We next need to differentiate the first and last members of the above equation with respect to z to obtain an explicit formula for φ. The left-hand member becomes an indeterminate form when differentiated; the indeterminacy can be resolved by first carrying out an integration by parts:

$$\int_0^z f(x)(z-x)^{\alpha-1}\,dx = \frac{f(0)}{\alpha}\,z^\alpha + \frac{1}{\alpha}\int_0^z f'(x)(z-x)^\alpha\,dz\,.$$

Now we differentiate with respect to z, obtaining

$$\varphi(z) = \frac{\sin\pi\alpha}{\pi}\left[f(0)z^{\alpha-1} + \int_0^z f'(x)(z-x)^{\alpha-1}\,dx\right].$$

21.2.7. Using the solution to Exercise 21.2.6, we have $f(0)=1$, $f'(x)=0$, so $\varphi(z)$ has the value that was given. The solution can be checked by inserting φ into the integral equation. The resulting integral is elementary.

21.2.8. Use the generating function formula for the Hermite polynomials to identify

$$e^{-(x-t)^2} = e^{-t^2}\sum_{n=0}^\infty \frac{H_n(t)x^n}{n!}\,.$$

Also write the Maclaurin expansion of $f(x)$, so our Fredholm equation takes the form

$$\sum_{n=0}^\infty f^{(n)}(0)\,\frac{x^n}{n!} = \sum_{n=0}^\infty \frac{x^n}{n!}\int_{-\infty}^\infty e^{-t^2}H_n(t)\varphi(t)\,dt\,,$$

equivalent to the set of formulas (for $n=0,\ 1,\ \cdots$)

$$f^{(n)}(0) = \int_{-\infty}^\infty e^{-t^2}H_n(t)\,\varphi(t)\,dt\,.$$

We now recognize the integral over t as proportional to the coefficient a_n in the Hermite polynomial expansion of φ:

$$\varphi(x) = \sum_{n=0}^\infty a_nH_n(x)\,,\qquad a_n = \frac{1}{2^n\,n!\,\pi^{1/2}}\int_{-\infty}^\infty e^{-t^2}H_n(t)\,\varphi(t)\,dt\,.$$

In terms of $f^{(n)}(0)$, this is

$$\varphi(x) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{f^{(n)}(0)}{2^n\,n!}\,H_n(x)\,.$$

21.2.9. The denominator in the integral equation corresponds to the generating function for the Legendre polynomials; introducing that expansion,

$$f(x) = \sum_{n=0}^{\infty} \left[\int_{-1}^{1} P_n(t)\varphi(t)\,dt \right] x^n .$$

Inserting the Legendre polynomial expansion $\varphi(t) = \sum_n a_n P_n(t)$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{2\,a_n}{2n+1}\, x^n .$$

Proceeding now to the cases at hand,

(a) If $f(x) = x^{2s}$, all a_n except a_{2s} vanish, and

$$1 = \frac{2\,a_{2s}}{2(2s)+1}, \quad \text{or} \quad a_{2s} = \frac{4s+1}{2} .$$

This in turn means that $\varphi(x) = \dfrac{4s+1}{2}\, P_{2s}(x)$.

(b) This case is similar, but with $2s$ replaced by $2s+1$.

The final result is $\varphi(x) = \dfrac{4s+3}{2}\, P_{2s+1}(x)$.

21.2.10. $\lambda_1 = i\sqrt{3}/2, \quad \varphi_1(x) = 1 - i\sqrt{3}x.$

$\lambda_2 = -i\sqrt{3}/2, \quad \varphi_2(x) = 1 + i\sqrt{3}x.$

21.2.11. This integral equation has a separable kernel; write $\cos(x-t) = \cos x \cos t + \sin x \sin t$, so the right-hand side of the equation must be a linear combination of $\sin x$ and $\cos x$. Inserting $\varphi(t) = A\cos t + B\sin t$ and evaluating the integral, it becomes $\pi(A\cos x + B\sin x)$, so the integral equation is satified for arbitrary A and B with $\lambda = 1/\pi$.

21.2.12. $\lambda_1 = -3/4, \quad y_1(x) = x = P_1(x).$

$\lambda_2 = \dfrac{-15+9\sqrt{5}}{8}, \quad y_2(x) = P_0(x) + \dfrac{4}{3}\,\lambda_2 P_2(x).$

$\lambda_3 = \dfrac{-15-9\sqrt{5}}{8}, \quad y_3(x) = P_0(x) + \dfrac{4}{3}\,\lambda_3 P_2(x).$

21.2.13. We note that $\psi(x)$ must be proportional to $\cos x$. But then the t integral has $\sin t \cos t$ as its integrand. But $\sin t$ is symmetric about $\pi/2$, while $\cos t$ is antisymmetric; the integral vanishes, forcing $\psi(x) = 0$.

21.2.14. $\lambda_1 = 0.7889, \quad \varphi_1 = 1 + 0.5352\, x,$

$\lambda_2 = 15.21, \quad \varphi_2 = 1 - 1.8685\, x$

$(\lambda_1 = 8 - \sqrt{52}, \quad \lambda_2 = 8 + \sqrt{52}).$

21.2.15. $\lambda_1 = 0.7889, \quad \varphi_1(x) = 1 + 0.5352\ x,$

$\lambda_2 = 15.21, \quad \varphi_2(x) = 1 - 1.8685\ x.$

21.2.16. (a) Inserting the expansion of the kernel into the integral, we find that it can be written

$$\sum_{i=1}^{n} M_i(x) \int_a^b N_i(t)\varphi(t)\, dt = \sum_{i=1}^{n} M_i(x) C_i\,,$$

where C_i is the (presently unknown) constant value of the t integral. We therefore have an inconsistency unless $f(x)$ is a linear combination of only the functions $M_i(x)$.

(b) The condition that $\psi(x)$ be orthogonal to all $N_i(x)$ causes its addition to a solution not to affect the value of the integral; of course, any function $\psi(x)$ to be added must also be a linear combination of the $M_i(x)$.

21.3 Neumann Series

21.3.1. (a) Solution is given in the text.

(b) $\varphi(x) = \sin x$.

(c) $\varphi(x) = \sinh x$.

21.3.2. $\psi(x) = -2$.

21.3.3. (a) Directly from the integral equation, $\varphi(0) = 1$. Differentiating,

$$\varphi'(x) = \lambda^2 \int_0^x \varphi(t)\, dt\,,$$

from which we deduce $\varphi'(0) = 0$. Differentiating again,

$$\varphi''(x) = \lambda^2 \varphi(x)\,.$$

This ODE has general solution $\varphi = A \sinh \lambda x + B \cosh \lambda x$; the boundary conditions require $A = 0$, $B = 1$.

(b) The Neumann series for this problem is

$$\varphi(x) = 1 + \lambda^2 \int_0^x (x - x_1)\, dx_1 + \lambda^2 \int_0^x (x - x_2)\, dx_2 \lambda^2 \int_0^{x_1} (x_2 - x_1)\, dx_1 + \cdots$$

$$+ \lambda^2 \int_0^x (x - x_n)\, dx_n \lambda \int_0^{x_n} (x_n - x_{n-1}\, dx_{n-1} \cdots + \lambda^2 \int_0^{x_2} (x_2 - x_1)\, dx_1 + \cdots\,.$$

We can find the iterated integral through x_n by mathematical induction. By inspection we guess that its value will be $(\lambda x)^{2n}/(2n)!$. Assuming this to be correct for the integral through x_{n-1}, the integral through x_n will be

$$\lambda^2 \int_0^x (x - x_n) \frac{(\lambda x_n)^{2n-2}}{(2n-2)!}\, dx_n = \lambda^{2n} \int_0^x \frac{(x\, x_n^{2n-2} - x_n^{2n-1})}{(2n-2)!}\, dx_n$$

$$= \lambda^{2n} \left[\frac{1}{2n-1} + \frac{1}{2n} \right] \frac{x^2 n}{(2n-2)!} = \frac{(\lambda x)^{2n}}{(2n)!} .$$

To complete the proof we note that the general result applies for $n = 1$.

This is the series expansion of $\cosh \lambda x$.

(c) Taking Laplace transforms of the individual terms of the integral equation, noting that the integral is a convolution in which $x-t$ corresponds to $f(y)$ for $y = x - t$ and that the transform of f is $1/s^2$, and also observing that the transform of unity is $1/s$:

$$\Phi(s) = \frac{1}{s} + \lambda^2 \frac{\Phi(s)}{s^2} .$$

Solving for Φ, we find

$$\Phi(s) = \frac{1/s}{1 - \lambda^2/s^2} = \frac{s}{s^2 - \lambda^2} .$$

This is the transform of $\cosh \lambda x$.

21.3.4. Assume $U(t, t_0) = U(t-t_0)$, since the result is expected to be independent of the zero from which t is measured. Then assume that U can be expanded in a power series

$$U(t - t_0) = \sum_{n=0}^{\infty} c_n (t - t_0)^n .$$

Setting $V(t_1) = V_0$ and inserting the expansion of $U(t_1 - t_0)$, we get

$$\sum_{n=0}^{\infty} c_n (t-t_0)^n = 1 - \frac{iV_0}{\hbar} \sum_{n=0}^{\infty} c_n \int_{t_0}^{t} (t-t_0)^n\, dt = 1 - \frac{iV_0}{\hbar} \sum_{n=0}^{\infty} \frac{c_n (t - t_0)^{n+1}}{n+1} .$$

Equating equal powers of $t - t_0$ in the first and last members of this equation, we find

$$c_0 = 1, \qquad c_{n+1} = -\frac{iV_0}{\hbar} \frac{c_n}{(n+1)}, \quad n = 0,\ 1,\ \cdots .$$

This recurrence formula can be solved:

$$c_n = \frac{1}{n!} \left(-\frac{iV_0}{\hbar} \right)^n .$$

These are the coefficients in the expansion of $\exp \left[-\frac{i}{\hbar} (t - t_0) V_0 \right]$.

21.4 Hilbert-Schmidt Theory

21.4.1. Let $\varphi_n(x)$ be an eigenfunction with eigenvalue λ_n. Multiply the Fredholm equation for $\varphi_n(x)$ by $\varphi_m^*(x)$ and integrate:

$$\int \varphi_m^*(x)\varphi_n(x)\,dx = \lambda_n \int_a^b dx \int_a^b dt\,\varphi_m^*(x)K(x,t)\varphi_n(t)$$

$$= \frac{\lambda_n}{\lambda_m^*}\int_a^b dt\left[\lambda_m^*\int_a^b K^*(t,x)\varphi_m^*(x)\,dx\right]\varphi_n(t)$$

$$= \frac{\lambda_n}{\lambda_m^*}\int_a^b dt\,\varphi_m^*(t)\varphi_n(t)\,.$$

We have used the self-adjoint property of $K(x,t)$ to make the x integral correspond to the Fredholm equation for $\varphi_m^*(t)$. Using Dirac notation, the above equation can be written

$$\langle\varphi_m|\varphi_n\rangle = \frac{\lambda_n}{\lambda_m^*}\langle\varphi_m|\varphi_n\rangle, \quad \text{or} \quad (\lambda_m^* - \lambda_n)\langle\varphi_m|\varphi_n\rangle = 0\,.$$

If $m = n$, the scalar product must be nonzero, so we must have $\lambda_n^* - \lambda_n = 0$, showing that λ_n is real. If $m \neq n$ and $\lambda_m \neq \lambda_n$, then the scalar product must vanish, indicating orthogonality.

21.4.2. (a) Referring to the answer to Exercise 21.2.12, we see that y_1 is orthogonal to y_2 and y_3 by symmetry. To check the orthogonality of y_2 and y_3, form

$$\langle y_2|y_3\rangle = \langle P_0|P_0\rangle + \frac{16}{9}\,\lambda_2\lambda_3\langle P_2|P_2\rangle\,,$$

where we have omitted terms that vanish because the Legendre polynomial P_0 is orthogonal to P_2. Using $\langle P_0|P_0\rangle = 2$ and $\langle P_2|P_2\rangle = 2/5$ and substituting the values of λ_2 and λ_3, we obtain the desired zero result.

(b) Referring to the answer to Exercise 21.2.14, we can check that

$$\int_0^1 (1 + 0.5352x)(1 - 1.8685x)\,dx = 0\,.$$

21.4.3. $\varphi(x) = \dfrac{3x+1}{2}$.

21.4.4. $\lambda_1 = \sqrt{\dfrac{\sin\pi a}{\pi}}, \quad \varphi_1(x) = \sqrt{\Gamma(a)}\,x^{-a} + \sqrt{\Gamma(1-a)}\,x^{a-1},$

$$0 < a < 1$$

$\lambda_2 = -\sqrt{\dfrac{\sin\pi a}{\pi}}, \quad \varphi_2(x) = \sqrt{\Gamma(a)}\,x^{-a} - \sqrt{\Gamma(1-a)}\,x^{a-1},$

21.4.5. (a) $y(x) = x\sum_{s=0}^{\infty}\left(\frac{\lambda}{3}\right)^s.$

(b) Convergent for $|\lambda| < 3$. Eq. (21.55) assures convergence for $|\lambda| < 1$.

(c) $\lambda = 3, \quad y(x) = x.$

21.4.6. The normalized eigenfunctions of this problem are $\varphi_1(x) = \cos x/\sqrt{\pi}$ and $\varphi_2(x) = \sin x/\sqrt{\pi}$. Writing

$$K(x,t) = \cos(x-t) = \pi\varphi_1(x)\varphi_1(t) + \pi\varphi_2(x)\varphi_2(t)),$$

and noting that $\pi = 1/\lambda_1 = 1/\lambda_2$, we recover the formula that is to be verified.

21.4.7. Solution is given in the text.

21.4.8. Expand $\varphi(x) = \sum_i a_i\varphi_i(x), \quad f(x) = \sum_i b_i\varphi_i(x),$ and write

$$K(x,t) = \sum_i \frac{\varphi_i(x)\varphi_i(t)}{\lambda_i}.$$

We get

$$\sum_i a_i\varphi_i(x) = \sum_i b_i\varphi_i(x) + \sum_i \frac{\lambda}{\lambda_i}\varphi(x)\int_a^b \varphi_i(t)\sum_j a_j\varphi_j(t)\,dt$$

$$= \sum_i b_i\,\varphi_i(x) + \sum_i \frac{\lambda}{\lambda_i}\,a_i\,\varphi_i(x).$$

From the coefficients of φ_i we have

$$a_i = b_i + \frac{\lambda}{\lambda_i} \quad \text{or} \quad a_i = \frac{b_i\lambda_i}{\lambda_i - \lambda},$$

corresponding to $\varphi = \sum_i \frac{b_i\lambda_i}{\lambda_i - \lambda}\,\varphi_i(x)$.

22. Calculus of Variations

22.1 Euler Equation

22.1.1. Expand the alternate expression for the Euler equation:

$$\frac{\partial f}{\partial x}-\frac{d}{dx}\left[f-y_x\,\frac{\partial f}{\partial y_x}\right]=\frac{\partial f}{\partial x}-\left[\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y_x}\,y_{xx}+\frac{\partial f}{\partial y}y_x\right]+y_{xx}\,\frac{\partial f}{\partial y_x}-y_x\,\frac{d}{dx}\,\frac{\partial f}{\partial y_x}$$

$$=-y_x\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\,\frac{\partial f}{\partial y_x}\right].$$

If the Euler equation in its original form is satisfied, then the part of the alternate equation within square brackets vanishes, and the alternate form of the present exercise is also satisfied.

22.1.2. Letting y and y_x stand respectively for $y(x,0)$ and $y_x(x,0)$, we need for the first two terms of the expansion

$$J(0)=\int_{x_1}^{x_2} f(y,y_x,x)\,dx\,,$$

$$J'(0)=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\,\frac{\partial y}{\partial \alpha}+\frac{\partial f}{\partial y_x}\,\frac{\partial y_x}{\partial \alpha}\right]\,dx\bigg|_{\alpha=0}.$$

Now carry out an integration by parts on the second term of the integrand of J', using the fact that $dy_x/d\alpha = d^2y/dxd\alpha$:

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y_x}\right)\frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right)\,dx=\left(\frac{\partial f}{\partial y_x}\right)\left(\frac{\partial y}{\partial \alpha}\right)\bigg|_{x_1}^{x_2}-\int_{x_1}^{x_2}\left(\frac{\partial y}{\partial \alpha}\right)\,\frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right)\,dx\,.$$

The integrated terms vanish because y is fixed at the endpoints. When the transformed integral is inserted into the expression for J', we get

$$J'(0)=\int_{x_1}^{x_2}\left(\frac{\partial y}{\partial \alpha}\right)\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\,\frac{\partial f}{\partial y_x}\right]\,dx=0\,.$$

Our expansion is now $J(\alpha) = J(0) + J'(0)\alpha + O(\alpha^2)$, and J will be stationary at $\alpha = 0$ only if $J'(0) = 0$. But since the dependence of y upon α is arbitrary and $\partial y/\partial\alpha$ can be nonzero anywhere within the integration interval, $J'(0)$ can only be made to vanish if the quantity within square brackets in the above equation is zero.

22.1.3. Generalizing the procedure carried out in Eqs. (22.9) through (22.12), we define

$$\frac{\partial y(x,\alpha)}{\partial \alpha}=\eta(x),\qquad \frac{\partial y_x(x,\alpha)}{\partial \alpha}=\eta_x(x),\qquad \frac{\partial y_{xx},\alpha)}{\partial \alpha}=\eta_{xx}(x),$$

then write

$$\frac{dJ(\alpha)}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\, \eta(x) + \frac{\partial f}{\partial y_x}\, \eta_x(x) + \frac{\partial f}{\partial y_{xx}}\, \eta_{xx}(x) \right) dx\,.$$

We now integrate by parts the η_x term and (twice) the η_{xx} term of the integral. All the integrated terms vanish, and the new integrals can be collected into

$$\frac{dJ(\alpha)}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y_{xx}} \right] \eta(x)\, dx = 0\,.$$

22.1.4. If $f(y, y_x, x) = f_1(x,y) + f_2(x,y)y_x$, then

(a) $\dfrac{\partial f}{\partial y} = \dfrac{\partial f_1}{\partial y} + \dfrac{\partial f_2}{\partial y} y_x = \dfrac{d}{dx}\dfrac{\partial f}{\partial y_x} = \dfrac{d}{dx} f_2 = \dfrac{\partial f_2}{\partial x} + y_x \dfrac{\partial f_2}{\partial y}$

implies $\dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x}$.

(b) $\displaystyle\int_a^b (f_1 + f_2 y_x) dx = \int_a^b (f_1 dx + f_2 dy)$ is independent of the choice of path, i.e. depends only on the endpoints because of (a).

22.1.5. If $f = f(x,y)$ then

(a) $\dfrac{\partial f}{\partial y} = \dfrac{d}{dx}\dfrac{\partial f}{\partial y_x} = 0$, so that f is independent of y.

(b) There is no information on $f(x)$.

22.1.6. (a) Differentiate the equation $c_1 \cosh(x_0/c_1)$ as follows:

$$dc_1 \, \cosh\left(\frac{x_0}{c_1}\right) + c_1 \, \sinh\left(\frac{x_0}{c_1}\right) \left[\frac{dx_0}{c_1} - \frac{x_0}{c_1^2}\, dc_1 \right] = 0\,.$$

Substitute $\cosh(x_0/c_1) = 1/c_1$ and $\sinh(x_0/c_1) = 1/x_0$, reaching

$$\frac{dc_1}{c_1} + \frac{c_1}{x_0}\left[\frac{dx_0}{c_1} - \frac{x_0}{c_1^2}\, dc_1 \right] = 0\,.$$

Rearrange this expression to solve for dx_0/dc_1:

$$\frac{dx_0}{dc_1} = x_0 \left(\frac{1}{c_1} - \frac{1}{c_1} \right) = 0.$$

(b) Dividing the above expressions for $\cosh(x_0/c_1)$ and $\sinh(x_0/c_1)$, we find that under the conditions of part (a) we have

$$\frac{\cosh(x_0/c_1)}{\sinh(x_0/c_1)} = \frac{x_0}{c_1} = \coth(x_0/c_1)\,.$$

(c) Solve the equation of part (b) to obtain $x_0/c_1 \approx 1.199679$; then obtain x_0 from $x_0 = 1/\sinh(1.199679) = 0.662743$ and $c_1 = x_0/1.199679 = 0.5523434$.

22.1.7. The condition that the shallow curve and Goldschmidt solution have equal area corresponds to the equation

$$\pi c_1^2 \left[\sinh\left(\frac{2x_0}{c_1}\right) + \frac{2x_0}{c_1}\right] = 2\pi\,.$$

Using also the relation $1/c_1 = \cosh(x_0/c_1)$, the above equation can be brought to the form

$$\sinh(2w) + 2w = 2\cosh^2 w, \qquad \text{where} \quad w = \frac{x_0}{c_1}\,.$$

This is a transcendental equation in a single variable, with solution $w = 0.63923$. Then, $c_1 = 1/\cosh w = 0.82551$, and $x_0 = wc_1 = 0.5277$.

22.1.8. Taking the second derivative of J,

$$\frac{\partial^2 J}{\partial \alpha^2} = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha}\left[\frac{\partial f}{\partial y_x}\,\eta'(x)\right] dx = \int_{x_1}^{x_2} \frac{\partial^2 f}{\partial y_x^2}\,[\eta'(x)]^2\,dx\,,$$

where we have omitted terms that vanish because in this problem f does not depend on y. Here

$$f = (1+y_x^2)^{1/2}\,, \qquad \frac{\partial^2 f}{\partial y_x^2} = \frac{1}{(1+y_x^2)^{3/2}}\,,$$

and we see that the integrand in the integral for $\partial^2 J/\partial y_x^2$ is everywhere nonnegative. Since the original formulation of this problem included the tacit assumption that $x_2 > x_1$, we see that our second derivative will be positive, indicating that the stationary J will be a minimum.

22.1.9. (a) Assuming that $y(x_1)$ and $y(x_2)$ are fixed, one can choose a function y that assumes arbitrarily large positive values for a significant portion of the range (x_1, x_2), thereby making J large and positive without limit. Alternatively, one can choose a y that assume arbitrarily large negative values, thereby making J be large and negative without limit.

(b) With $f = y^2$, a minimum in J can be achieved by setting f to zero, with discontinuities in f to reach the fixed values at x_1 and x_2.

22.1.10. (a) Use the alternate form of the Euler equation $f - y_x(\partial f/\partial y_x) = \text{constant}$. We have

$$J = \int_{(-1,1)}^{(1,1)} e^y \sqrt{1+y_x^2}\,dx\,,$$

$$f - y_x\,\frac{\partial f}{\partial y_x} = e^y\sqrt{1+y_x^2} - \frac{e^y y_x^2}{\sqrt{1+y_x^2}}\,dx = C, \quad \text{or} \quad \frac{e^y}{\sqrt{2+y_x^2}} = C\,.$$

Solve for y_x and integrate: $y_x = \pm\sqrt{C^2 e^{2y} - 1}$,

$$dx = \pm\frac{dy}{\sqrt{C^2 e^{2y} - 1}}, \qquad x = \pm \tan^{-1}\sqrt{C^2 e^{2y} - 1} + C' .$$

The symmetry of the problem requires that $C' = 0$, and C must be set to make $x = 1$ when $y = 1$. This condition is equivalent to $1 = \tan^{-1}\sqrt{C^2 e^2 - 1}$, or $C^2 e^2 - 1 = \pi^2/16$. Substituting into the formula for x, we have

$$x = \tan^{-1}\left[e^{2(y-1)}\left(1 + \frac{\pi^2}{16}\right) - 1\right]^{1/2} .$$

(b) This problem is not straightforward. If y_0 is not sufficiently less than 1, the optimum path will be discontinuous, consisting of a straight-line segment from $(-1, 1)$ to $(-1, y_0)$ followed by a straight line (requiring zero travel time) from $(-1, y_0)$ to $(1, y_0)$ and then a straight line from there to $(1, 1)$. The travel time for this path will be $a(1 - y_0)^2$. This is clearly not a physically relevant situation.

Looking now for continuous paths, we note that the travel time will be

$$J = \int a(y - y_0)\sqrt{1 + y_x^2}\, dx ,$$

$$f - y_x \frac{\partial f}{\partial y_x} = \frac{a(y - y_0)}{\sqrt{1 + y_x^2}} = C^{-1} ,$$

where we have used the alternate form of the Euler equation. Solving for y_x and integrating, we get

$$dx = \pm\frac{dy}{\sqrt{C^2 a^2 (y - y_0)^2 - 1}} , \qquad x = \pm\frac{1}{Ca}\cosh^{-1}\left[Ca(y - y_0)\right] .$$

We have not shown a constant of integration; the symmetry of the problem requires that it be zero. Rearranging the equation for x, we bring it to the form

$$\frac{\cosh Cax}{Ca} = y - y_0 .$$

We now attempt to find a solution that passes through the point $(1, 1)$. Such a solution must have a value of Ca that satisfies $(Ca)^{-1}\cosh Ca = 1 - y_0$. However, for real Ca the left-hand side of this equation is always greater than approximately 1.5089, indicating that no optimum continuous path exists unless y_0 is less than about -0.5089.

Assuming a value of y_0 permitting an optimum continuous path, the travel time on that path is given by

$$t = \int_{-1}^{1} a(y-y_0)\sqrt{1+y_x^2}\,dx = 2a\int_0^1 Ca(y-y_0)^2\,dx$$

$$= 2a\int_{y_{\min}}^{1} \frac{Ca(y-y_0)^2}{y_x}\,dy = 2a\int_{y_{\min}}^{1} \frac{Ca(y-y_0)^2}{\sqrt{C^2a^2(y-y_0)^2-1}}\,dy$$

$$= 2a\left\{\frac{1-y_0}{2Ca}\sqrt{C^2a^2(1-y_0)^2-1}+\frac{1}{2}\cosh^{-1}[Ca(1-y_0)]\right\}$$

$$= a\left\{\frac{1-y_0}{Ca}\sqrt{C^2a^2(1-y_0)^2-1}+Ca\right\}.$$

where we have changed the integration variable from x to y and have introduced $y_{\min}$, the value of y at $x = 0$. We will need to determine Ca and $y_{\min}$ from the boundary condition at $(1,1)$. Let's first make a computation for $y_0 = -0.5089$, for which $Ca = 1.1997$ and $y_{\min} = y_0 + 1/Ca$ is 0.3245. We get $t = 3.098a$, but the discontinuous solution is $t_{\text{dis}} = 2.28a$, so the continuous solution does not yield a global minimum time. For $y_0 = -0.60$, we find $t = 2.706a$ vs. $t_{\text{dis}} = 2.56a$. At $y_0 = -0.70$, $t = 3.648a$ vs. $t_{\text{dis}} = 2.89a$; the continuous solution now yields the global minimum time. Finally, at $y_0 = -1$, $t = 2.73a$ and $t_{\text{dis}} = 4.0a$.

22.1.11. The solution is given in the text.

22.1.12. Assign the half-plane $y > 0$ index of refraction n_1 and the half-plane $y < 0$ index of refraction n_2. The velocity of light in a region of index of refraction n is c/n. Consider a light ray traveling from $(0,1)$ to $(d,-1)$, in the upper half-plane at angle θ_1 relative to the direction normal to the boundary between the half-planes, and in the lower half-plane at angle θ_2 from that normal. The time of travel for the light ray is

$$t = \frac{n_1}{c\cos\theta_1} + \frac{n_2}{c\cos\theta_2}$$

where θ_1 and θ_2 are related by $\tan\theta_1 + \tan\theta_2 = d$. Fermat's principle is that the path be such that t is a minimum. From the equation for t and the constraint equation, we have

$$c\,dt = \frac{n_1\sin\theta_1}{\cos^2\theta_1}\,d\theta_1 + \frac{n_2\sin\theta_2}{\cos^2\theta_2}\,d\theta_2 = 0, \qquad \frac{d\theta_1}{\cos^2\theta_1} + \frac{d\theta_2}{\cos^2\theta_2} = 0\,.$$

Using the second of these equations to write $d\theta_2$ in terms of $d\theta_1$, we are left with

$$(n_1\sin\theta_1 - n_2\sin\theta_2)\,\frac{d\theta_1}{\cos^2\theta_1} = 0\,.$$

This equation is satisfied only if $n_1\sin\theta_1 = n_2\sin\theta_2$

22.1.13. The solution is given in the text.

22.1.14. The solution is given in the text.

22.1.15. As in free fall $y \sim gt^2, v \sim gt \sim \sqrt{y}$ in

$$\int dt = \int \frac{ds}{v} = \int \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx = \text{minimum}.$$

As there is no x-dependence, $F - y' \dfrac{\partial F}{\partial y'} = 1/c$ =const. using Eq. (22.19). This yields

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} = \frac{1}{c}, \text{ or } c^2 = y(1+y'^2), \text{ or } y'^2 = \frac{c^2-y}{y},$$

with parametric solution $y = \dfrac{c^2}{2}(1-\cos u)$, $x = \pm c^2(u - \sin u)/2$. These parametric equations describe a cycloid.

22.2 More General Variations

22.2.1. (a) $\delta \displaystyle\int Ldt = 0$, $L = m(\dot{x}^2 + \dot{y}^2)/2$ lead to $m\ddot{x} = 0 = \ddot{y}$. So $x(t)$, $y(t)$ are linear in the time.

(b) x =const. and y =const. give $\int Ldt = 0$, while a straight line (in t) gives $\int Ldt$ =constant$\neq 0$.

22.2.2. Assuming $T(\dot{x}), V(x)$ a stable equilibrium with $\dot{x}_i$ =constant gives

$\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_i} = 0 = \dfrac{\partial L}{\partial x_i}$. Hence $\partial V/\partial x_i = 0$ from $L = T - V$.

22.2.3. (a) $m\ddot{r} - mr\dot{\theta}^2 - mr\ \sin^2\theta\dot{\varphi}^2 = 0$

(b) $mr\ddot{\theta} + 2m\dot{r}\dot{\theta} - mr\ \sin\theta\ \cos\ \theta\dot{\varphi}^2 = 0$

(c) $mr\sin\theta\ddot{\varphi} + 2m\dot{r}\ \sin\theta\dot{\varphi} + 2mr\ \cos\ \theta\dot{\theta}\dot{\varphi} = 0$

The second and third terms of (a) correspond to centrifugal force. The second and third terms of (c) may be interpreted as Coriolis forces (with $\dot{\varphi}$ the angular velocity of the rotating coordinate system).

22.2.4. $l\ddot{\theta} - l\sin\theta\cos\theta\dot{\varphi}^2 + g\ \sin\theta = 0\,,$

$\dfrac{d}{dt}(ml^2\sin^2\theta\dot{\varphi}) = 0$ (conservation of angular momentum).

22.2.5. The independent variable is t; the dependent variables are the components of position x_i; the Lagrangian depends also on the time derivatives of the x_i, often written $\dot{x}_i$. Write $v^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$ and $V(\mathbf{r}) = V(x_1, x_2, x_3)$.

The equations of motion can be derived from $\delta \int L\,dt = 0$. The Euler equations for this relativistic Lagrangian are, for each component i,

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0\,.$$

We have $\partial L/\partial x_i = -\partial V/\partial x_i = F_i$, and $\partial L/\partial \dot{x}_i = m_0 \dot{x}_i/\sqrt{1 - v^2/c^2}$. Thus, the Euler equations become

$$F_i - \frac{d}{dt}\left(\frac{m_0\,\dot{x}_i}{\sqrt{1 - v^2/c^2}}\right).$$

22.2.6. The solution is given in the text.

22.2.7. (a) From $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} = \dfrac{\partial L}{\partial q_i}$, $\dfrac{dL}{dt} = \dfrac{\partial L}{\partial q_i}\dot{q}_i + \dfrac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$,

where sums over repeated indices are implied, we get $\dfrac{d}{dt}(\dot{q}_i \dfrac{\partial L}{\partial \dot{q}_i} - L) = 0$.

(b) $\dot{q}_i \dfrac{\partial L}{\partial \dot{q}_i} - L = \dot{q}_i p_i - (T - V) = 2T - T + V = T + V = H$, $H =$const. follows from $\dfrac{dH}{dt} = 0$, i.e. (a).

22.2.8. From the Lagrange density $\mathcal{L} = \dfrac{\rho}{2}\,u_t^2 - \dfrac{\tau}{2}\,u_x^2$, the Euler equation yields

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} = \frac{\partial \mathcal{L}}{\partial u} = 0 = \rho\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \tau\frac{\partial}{\partial x}\frac{\partial u}{\partial x}.$$

22.2.10. We require $\delta J = \delta \int L\,dx\,dy\,dz\,dt = 0$. We need to write L in terms of the dependent variables and their derivatives; to keep the development more compact we adopt the following nonstandard notations: φ_x for $\partial\varphi/\partial x$ etc., $(A_i)_x$ for $\partial A_i/\partial x$ etc. Note that the index within the parentheses denotes the component of $\mathbf{A}$ while the outer index indicates a derivative. Using the equations $\mathbf{E} = -\boldsymbol{\nabla}\varphi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, we write

$$\begin{aligned} L = {} & \\ & \frac{\varepsilon_0}{2}\Big[\varphi_x^2 + \varphi_y^2 + \varphi_z^2 + (A_x)_t^2 + (A_y)_t^2 + (A_z)_t^2 + 2\varphi_x(A_x)_t + 2\varphi_y(A_y)_t + 2\varphi_z(A_z)_t\Big] \\ & - \frac{1}{2\mu_0}\Big[(A_z)_y^2 + (A_y)_z^2 - 2(A_z)_y(A_y)_z + (A_z)_x^2 + (A_x)_z^2 - 2(A_z)_x(A_x)_z \\ & \quad + (A_x)_y^2 + (A_y)_x^2 - 2(A_x)_y(A_y)_x\Big] - \rho\varphi + J_xA_x + J_yA_y + J_zA_z\,. \end{aligned}$$

The Euler equation for φ is

$$\frac{\partial L}{\partial \varphi} - \frac{d}{dx}\left(\frac{\partial L}{\partial \varphi_x}\right) - \frac{d}{dy}\left(\frac{\partial L}{\partial \varphi_y}\right) - \frac{d}{dz}\left(\frac{\partial L}{\partial \varphi_z}\right)$$

$$= -\rho - \varepsilon_0 \left[\frac{d}{dx}(\varphi_x + (A_x)_t) + \frac{d}{dy}(\varphi_y + (A_y)_t) + \frac{d}{dz}(\varphi_z + (A_z)_t)\right]$$

$$= \rho - \varepsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = 0\,, \quad \text{equivalent to} \quad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}\,.$$

The Euler equation for A_x is

$$\frac{\partial L}{\partial A_x} - \frac{d}{dy}\left(\frac{\partial L}{\partial (A_x)_y}\right) - \frac{d}{dz}\left(\frac{\partial L}{\partial (A_x)_z}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial (A_x)_t}\right)$$

$$= J_x - \frac{1}{\mu_0}\left[\frac{d}{dy}\big((A_x)_y - (A_y)_x\big) + \frac{d}{dz}\big((A_x)_z - (A_z)_x\big)\right] - \varepsilon_0 \frac{d}{dt}\big((A_x)_t + \varphi_x\big)$$

$$= J_x - \frac{1}{\mu_0}\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right] + \varepsilon_0 \frac{d}{dt} E_x = J_x - \frac{1}{\mu_0}[\boldsymbol{\nabla} \times \mathbf{B}]_x + \varepsilon_0 \frac{dE_x}{dt} = 0\,.$$

Multiplying this equation through by μ_0 and combining it with similar equations for A_y and A_z to make a vector equation, we get

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{d\mathbf{E}}{dt}\,.$$

22.3 Constrained Minima/Maxima

22.3.1. Differentiating $\dfrac{2mE}{\hbar^2} - \lambda \pi R^2 H$ with respect to R and H we obtain

$$-2\frac{2.048^2}{R^3} - 2\pi\lambda R H = 0, \quad -\frac{2\pi^2}{H^3} - \lambda \pi R^2 = 0.$$

Equating $\pi\lambda$ from both equations yields $\dfrac{R}{H} = \dfrac{2.048}{\pi\sqrt{2}}$.

22.3.2. Let the parallelepiped have dimensions $a \times a \times c$. The length plus girth is $D = c + 4a$; the volume is $V = a^2 c$. We carry out an unconstrained minimization of $a^2 c - \lambda(c + 4a)$: Differentiating with respect to a and c we obtain the equations

$$2ac - 4\lambda = 0, \qquad a^2 - \lambda = 0\,,$$

from which we find $\lambda = a^2$, $c = 2a$. From the constraint equation $c + 4a = 36$, we now find $a = 6''$, $c = 12''$. The maximum volume is 432 in^3.

22.3.3. The volume $V = abc$ is to be minimized subject to the constraint $\varphi(a, b, c) = B$. We now carry out an unconstrained minimization of $V + \lambda\varphi$, differentiating this quantity with respect to a, b, and c. The result is

$$bc - \frac{2\pi^2\lambda}{a^3} = 0, \quad ac - \frac{2\pi^2\lambda}{b^3} = 0, \quad ab - \frac{2\pi^2\lambda}{c^3} = 0.$$

These equations can be rearranged into $a^2 = b^2 = c^2 = 2\pi^2\lambda/abc$. We therefore get the unsurprising result $a = b = c$.

22.3.4. $p = q$, $(p + q)_{\min} = 4f$.

22.3.5. If c, d are half the rectangle sides, differentiating $4cd - \lambda\left(\frac{c^2}{a^2} + \frac{d^2}{b^2} - 1\right)$ with respect to c and d yields $4d - 2\frac{\lambda c}{a^2} = 0$, $4c - 2\frac{\lambda d}{b^2} = 0$, or $\frac{d}{c} = \frac{\lambda}{2a^2}$, $\frac{c}{d} = \frac{\lambda}{2b^2}$, $\frac{c}{a} = \frac{d}{b}$, from which the ellipse equation implies $\frac{c}{a} = \frac{1}{\sqrt{2}} = \frac{d}{b}$. Hence $\frac{4cd}{\pi ab} = \frac{2}{\pi}$.

22.3.6. If the parallelepiped coordinates are $x = \pm a', y = \pm b', z = \pm c'$, differentiate $a'b'c' - \lambda\left(\frac{a'^2}{a^2} + \frac{b'^2}{b^2} + \frac{c'^2}{c^2}\right)$ with respect to a', b', c'. This yields $8b'c' = 2\frac{\lambda a'}{a^2}$, $4a^2b'c' = \lambda a'$ and by symmetry $4b^2a'c' = \lambda b'$, $4c^2a'b' = \lambda c'$. Taking ratios we find $\frac{a^2}{b^2}\frac{b'}{a'} = \frac{a'}{b'}$, or $\frac{a'}{b'} = \frac{a}{b}$, i.e., $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$. Substituting this into the ellipsoid equation yields $3\frac{a'^2}{a^2} = 1$, $a' = a/\sqrt{3}$, etc. Hence $\frac{8a'b'c'}{4\pi abc/3} = \frac{2}{\pi\sqrt{3}}$.

22.4 Variation with Constraints

22.4.1. The solution is given in the text.

22.4.2. (c) $\omega(t) = \frac{L}{m(\rho_0 - kt)^3}$ where L is the angular momentum.

22.4.3. The integrals H and K, that respectively describe the potential energy and the length of the cable in terms of the vertical position $y(x)$ of the cable at points x, are proportional to

$$H = \int_{x_1}^{x_2} y\sqrt{1 + y_x^2}\, dx \quad \text{and} \quad K = \int_{x_1}^{x_2} \sqrt{1 + y_x^2}\, dx\,.$$

The quantity to be made stationary is

$$J = H + \lambda K = \int_{x_1}^{x_2} f(y, y_x)\, dx, \qquad f(y, y_x) = (y - \lambda)\sqrt{1 + y_x^2}\,.$$

The Euler equation for this problem is

$$f - y_x \frac{\partial f}{\partial y_x} = C_1 \quad \text{or} \quad (y - \lambda)\sqrt{1 + y_x^2} - \frac{y_x^2 (y - \lambda)}{\sqrt{1 + y_x^2}} = C_1\,,$$

which simplifies to

$$y - \lambda = C1\sqrt{1 + y_x^2}\,.$$

Solving for y_x and integrating,

$$y_x = \frac{dy}{dx} = \left[\left(\frac{y - \lambda}{C_1}\right)^2 - 1\right]^{1/2}, \qquad C_1 \cosh^{-1}\left(\frac{y - \lambda}{C_1}\right) = x - C_2$$

which rearranges to

$$y - \lambda = C_1 \cosh\left(\frac{x - C_2}{C_1}\right).$$

The constants λ, C_1, and C_2 must now be chosen so that $y(x)$ passes through the points (x_1, y_1) and (x_2, y_2) and that the integral K evaluates to the cable length L. The conditions on the endpoints correspond to

$$y_1 - \lambda = C_1 \cosh\left(\frac{x_1 - C_2}{C_1}\right), \qquad y_2 - \lambda = C_1 \cosh\left(\frac{x_2 - C_2}{C_1}\right).$$

The cable length satisfies

$$\begin{aligned} L &= \int_{x_1}^{x_2} \frac{y - \lambda}{C_1}\, dx = \int_{x_1}^{x_2} \cosh\left(\frac{x - C_2}{C_1}\right) dx \\ &= C_1 \left[\sinh\left(\frac{x_2 - C_2}{C_1}\right) - \sinh\left(\frac{x_2 - C_2}{C_1}\right)\right]. \end{aligned}$$

From the above we form

$$L^2 - (y_2 - y_1)^2 = C_1^2 \left[-2 + 2\cosh\left(\frac{x_2 - x_1}{C_1}\right)\right] = 4C_1^2 \sinh^2\left(\frac{x_2 - x_1}{2C_1}\right).$$

This is a transcendental equation in the single unknown C_1 and can easily be solved numerically for any given input. Using this value for C_1, the equation for L then contains only the undetermined quantity C_2, which can be obtained by numerical methods. Finally, the equation for y_1 (or y_2) can be solved for λ.

The curve $y(x)$ solving this problem is known as a **catenary**.

22.4.4. The gravitational potential per unit volume of water at vertical position y relative to a zero at y_0 is $\rho g(y - y_0)$, where ρ is the mass density and g is the acceleration of gravity. The total gravitational potential energy of a cylindrical shell of water of radius r, thickness dr, base at $y = y_0$, and surface at $y(r)$ is $\rho g[y(r) - y_0]^2(2\pi r)\,dr/2$. The centrifugal force per unit volume of water a distance r from the axis of rotation, under a rotational angular velocity ω, is $\rho\omega^2 r$, and its rotational potential energy (relative to a position on the axis of rotation) is $-\rho\omega^2 r^2/2$. Therefore the total rotational potential energy of the cylindrical shell described above is $-\rho\omega^2 r^2(y - y_0)(2\pi r)/2$. Based on the above analysis, the total potential energy E and volume V of a column of water of radius a, rotating at angular velocity ω, with a base at y_0 and surface at $y(r)$, are

$$E = 2\pi\rho \int_0^a dr \left[\frac{gy(r)^2 r}{2} - \frac{\omega y(r) r^3}{2}\right], \qquad V = 2\pi \int_0^a y(r) r\, dr\,.$$

Discarding common factors, we want to minimize E for constant V, corresponding to

$$\delta J = \int_0^a f(y, r)\, dr = 0\,, \qquad f = gy^2 r - \omega^2 y r^3 - 2\lambda y r\,.$$

The Euler equation is

$$\frac{\partial f}{\partial y} = 0\,, \quad \text{or} \quad 2gry - \omega^2 r^3 - 2\lambda r = 0,$$

and its solution is

$$y = \frac{\omega^2 r^2 - 2\lambda}{2g}\,.$$

If we set $\lambda = 0$ our water column will have its surface at $y = 0$ at the axis ($r = 0$) and for $r = a$ the surface will be at $y = \omega^2 a^2/2g$. The surface will be a paraboloid of revolution.

22.4.5. Define a curve parametrically, as $x(t)$, $y(t)$. Then, for a closed curve, for which $x(t_2) = x(t_1)$ and $y(t_2) = y(t_1)$, the area A and the perimeter L are given by the following integrals:

$$A = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - \dot{x}y)\, dt\,, \qquad L = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt\,,$$

where we have written the line integral for the area in a more or less symmetric form and have used the dot notation for derivatives with respect to t, even though t is not really a time variable.

(a) For maximum area at fixed perimeter we therefore consider variation of

$$J = \int_{t_1}^{t_2} f(x, y, \dot{x}, \dot{y})\, dt, \quad \text{with} \quad f = \frac{1}{2}(x\dot{y} - \dot{x}y) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}\,,$$

leading to the two Euler equations

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = \frac{\dot{y}}{2} - \frac{d}{dt}\left(-\frac{y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0\,,$$

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = -\frac{\dot{x}}{2} - \frac{d}{dt}\left(\frac{x}{2} + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0\,.$$

Integrating these equations with respect to t,

$$y - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1\,, \qquad x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2\,.$$

Moving the constants C_1 and C_2 to the left-hand sides of these equations, then squaring both equations and adding them together, we reach

$$(x - C_2)^2 + (y - C_1)^2 = \frac{\lambda^2 \dot{y}^2}{\dot{x}^2 + \dot{y}^2} + \frac{\lambda^2 \dot{x}^2}{\dot{x}^2 + \dot{y}^2} = \lambda^2.$$

This is the equation of a circle.

(b) If we divide the expression for J by λ, its variation can be interpreted as describing an extremum in the perimeter with a constraint of fixed area. The solution will be the same as that already obtained, so the closed curve of minimum perimeter will be a circle.

22.4.6. Inserting $\varphi = \varphi + \delta\varphi$, defining $S = \displaystyle\int_a^b \varphi^2(x)\, dx$, and using the symmetry of $K(x,t)$:

$$\delta(J - \lambda S) = \int_a^b dx \delta\varphi(x) \left[2 \int_a^b K(x,t)\varphi(t)\, dt - 2\lambda\varphi(x)\right]\,,$$

where we retain only terms that are first-order in $\delta\varphi$. Since the above equation must be satisfied for arbitrary $\delta\varphi$, the integrand of the x integral must vanish, leading to the required integral equation.

22.4.7. (a) Here $F[\ ]$ refers to J as defined in Eq. (22.100); in the notation most frequently used in quantum mechanics,

$$F = \frac{\langle y|H|y\rangle}{\langle y|y\rangle}\,,$$

where here $H = -(d^2/dx^2)$ and we can use the Rayleigh-Ritz method to find an approximate function y. Noting that the trial function $y_{\text{trial}} = 1 - x^2$ satisfies the boundary conditions, we insert it into F, obtaining

$$F = \frac{-\displaystyle\int_0^1 (1 - x^2)(1 - x^2)''\, dx}{\displaystyle\int_0^1 (1 - x^2)^2\, dx} = \frac{4/3}{8/15} = \frac{5}{2} = 2.5\,.$$

(b) The exact value is $(\pi/2)^2 = 2.467$.

22.4.8. Using the notation of the solution to Exercise 22.4.7, we require

$$F = \frac{-\displaystyle\int_0^1 (1-x^n)(1-x^n)''\,dx}{\displaystyle\int_0^1 (1-x^n)^2\,dx} = \frac{n(n-1)\displaystyle\int_0^1 \left(x^{n-2}-x^{2n-2}\right)dx}{n\displaystyle\int_0^1 \left(x^{2n}-2x^n+1\right)dx}$$

$$= \frac{n^2/(2n-1)}{2n^2/(2n+1)(n+1)} = \frac{(2n+1)(n+1)}{2(2n-1)}\,.$$

Setting $dF/dn = 0$, we find $n = (1\pm\sqrt{6})/2 = 1.7247$ or -0.7247. We reject the negative n value because it does not satisfy the boundary condition at $x = 0$. From $n = 1.7247$ we evaluate $F = 2.4747$, much closer to the exact value 2.467 than was the approximation of Exercise 22.4.7 (which was 2.50).

22.4.9. In the usual quantum-mechanics notation, in this problem

$$H = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}, \quad \langle H\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}, \qquad \psi = 1 - \left(\frac{r}{a}\right)^2.$$

For spherically symmetric functions, the scalar product has definition

$$\langle\chi|\varphi\rangle = 4\pi\int_0^a r^2\chi(r)^*\varphi(r)\,dr\,.$$

We have $H\psi = 6/a^2$, so $\langle\psi|H|\psi\rangle = \dfrac{24\pi}{a^2}\displaystyle\int_0^a \left(r^2 - \frac{r^4}{a^2}\right)dr = \frac{16\pi a}{5}$.

Also, $\langle\psi|\psi\rangle = 4\pi\displaystyle\int_0^a \left(r^2 - \frac{2r^4}{a^2} + \frac{r^6}{a^4}\right)dr = \frac{32\pi a^3}{105}$.

Therefore, $\langle H\rangle = \dfrac{16\pi a/5}{32\pi a^3/105} = \dfrac{21}{2a^2} = \dfrac{10.5}{a^2}$.

The exact ψ is the spherical Bessel function $j_0(\pi r/a)$; note that

$$-\boldsymbol{\nabla}^2 j_0(\pi r/a) = (\pi/a)^2 j_0(\pi r/a),$$

showing that the exact eigenvalue is $\pi^2/a^2 = 9.87/a^2$.

22.4.10. Normalizing, and changing the integration variable to $u = x/a$,

$$N^2\int_0^a \left(1-\frac{x^2}{a^2}\right)^2 dx = N^2 a\int_0^1 (1-u^2)^2\,du = N^2 a\left(\frac{8}{15}\right) = 1$$

gives $N^2 = 15/8a$. The expectation value of H is

$$N^2\left[-\int_0^a\left(1-\frac{x^2}{a^2}\right)\frac{d^2}{dx^2}\left(1-\frac{x^2}{a^2}\right)dx+\int_0^a\left(1-\frac{x^2}{a^2}\right)x^2\left(1-\frac{x^2}{a^2}\right)dx\right]$$

$$=\frac{15}{4a^2}\int_0^1 *1-u^2)\,du+\frac{15a^2}{8}\int_0^1 u^2(1-u^2)^2\,du=\frac{5}{2a^2}+\frac{a^2}{7}\,.$$

Differentiating with respect to a yields $-\dfrac{5}{a^3}+\dfrac{2a}{7}=0$, i.e., $a=(35/2)^{1/4}$. This value of a causes the energy to have expectation value $2\sqrt{10/7}$, which is quite far from the exact value $\lambda = 1$. The large error is indicative of the fact that we chose a trial function that cannot for any value of the parameter a be a good approximation to the exact eigenfunction.

22.4.11. Let u_1 be an eigenfunction of the entire Schrödinger equation for some nonzero l. Note that $\langle u_1|\mathcal{L}|u_1\rangle$ must be at least as large as E_0, and that the centrifugal operator will have a positive expectation value for any function. (To prove this, multiply the wave function in each half of the scalar product by the square root of the centrifugal term, giving us an expression of the form $\langle f|f\rangle$.) The sum of these two terms will be larger than E_0.

23. Probability and Statistics

23.1 Probability: Definitions, Simple Properties

23.1.1. (a) $\dfrac{52/2}{52} = \dfrac{1}{2}$.

(b) $\dfrac{2}{52} = \dfrac{1}{26}$.

(c) $\dfrac{1}{52}$.

23.1.2. (a) $\left(\dfrac{4}{52}\right)^2 = \dfrac{1}{13^2}$.

(b) $\dfrac{4}{52}\dfrac{3}{51} = \dfrac{1}{13 \cdot 17}$.

23.1.3. (a) $\dfrac{1+2}{6^2} = \dfrac{1}{12}$.

(b) $\dfrac{(1+2)+2+2}{6^2} = \dfrac{7}{6^2}$.

23.1.4. There are three events: (2,2,2), (4,1,1), (1,2,3) whose probabilities sum up to $\dfrac{1+3+3\cdot 2}{6^3} = \dfrac{5}{3 \cdot 6^2}$.

23.1.5. $P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B) = P(A)P(B \cap C|A)$

$= P(B)P(C|B)P(A|B \cap C)$.

23.1.6. Maxwell-Boltzmann: $k^{N_1+\cdots+N_k}$,

Fermi-Dirac: $\prod_i \binom{k+N_i-1}{N_i}$,

Bose-Einstein: $\prod_i \binom{k}{N_i}$.

23.1.7. If $A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset$ then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

If $A \cap B \neq \emptyset, B \cap C \neq \emptyset, A \cap C \neq \emptyset$, but $A \cap B \cap C = \emptyset$, then

$$P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C).$$

www.elsolucionario.net

In general,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C)$$
$$- P(A \cap C) - P(A \cap B) + P(A \cap B \cap C).$$

23.1.8. To a good approximation $1/p$. For $p = 3$ compare $1/3$ with $33/100$; for $p = 5$ compare $1/5$ with $20/100$; for $p = 7$ compare $1/7$ with $14/100$; or $[100/p]/100 \approx 1/p$ with $[x]$ the largest integer below x.

23.1.9. Maxwell-Boltzmann: 3^2, Fermi-Dirac: $\binom{3}{2}$, Bose-Einstein: $\binom{4}{2}$.

23.2 Random Variables

23.2.1. The equally probable mutually exclusive two-card draws are:

$$(1,2), \quad (1,3), \quad (1,4), \quad (2,3), \quad (2,4), \quad (3,4)\,.$$

Each of these pairs has an equal probability of being drawn in either order, so we can make computations based on the six listed possibilities. Adding the numbers on the cards, 3, 4, 6, and 7 each occur once, 5 occurs twice. A single occurrence corresponds to $P = 1/6$, two occurrences to $P = 2/6 = 1/3$. The mean computed directly from the six events is

$$(3+4+5+5+6+7)/6 = 5\,.$$

To obtain the variance, we can compute

$$\frac{(3-5)^2 + (4-5)^2 + 2(5-5)^2 + (6-5)^2 + (7-5)^2}{6} = \frac{10}{6} = \frac{5}{3}.$$

23.2.2.

$$\langle X \rangle + c = \int_{-\infty}^{\infty} (x+c) f(x)\,dx,$$

$$\sigma^2(X) = \int_{-\infty}^{\infty} (x - \langle X \rangle)^2 f(x)\,dx,$$

$$\langle cX \rangle = c \int_{-\infty}^{\infty} x f(x)\,dx = c\langle X \rangle,$$

$$c^2\sigma^2(X) = \int_{-\infty}^{\infty} (cx - \langle cX \rangle)^2 f(x)\,dx.$$

23.2.3. Expanding the integral in Eq. (23.27) and identifying the three resultant integrals as respectively $\langle X^2 \rangle$, $\langle X \rangle$, and unity, we get

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\langle X \rangle \int_{-\infty}^{\infty} x f(x)\,dx + \langle X \rangle^2 \int_{-\infty}^{\infty} f(x)\,dx$$
$$= \langle X^2 \rangle - 2\langle X \rangle^2 + \langle X \rangle^2 = \langle X^2 \rangle - \langle X \rangle^2.$$

23.2.4. $$\frac{1}{N}\sum_{j=1}^{N} v_j = \bar{v} = \frac{1}{N}\sum_{j=1}^{N}\frac{x_j}{t_j} = \frac{1}{N}\sum_{j=1}^{N}\frac{\bar{x}+\Delta x_j}{\bar{t}+\Delta t_j} = \frac{\bar{x}}{\bar{t}}\,\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1+\dfrac{\Delta x_j}{\bar{x}}}{1+\dfrac{\Delta t_j}{\bar{t}}}\right).$$

Hence $\left|\bar{v}-\dfrac{\bar{x}}{\bar{t}}\right| \le \left|\dfrac{\bar{x}}{\bar{t}}\right|\;\dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\left[\frac{|\Delta x_j|}{\bar{x}}+\frac{|\Delta t_j|}{\bar{t}}\right] \ll \left|\dfrac{\bar{x}}{\bar{t}}\right|.$

23.2.5. Since X has the same definition as in the example, its expectation value $\langle X\rangle = 10/13$ and variance $\sigma^2(X) = 80/13^2$ are the same as found there. To compute the other quantities, we need to develop the probability distribution. We have for $f(x,y)$:

$$f(0,0) = \left(\frac{2}{13}\right)^2 = \frac{4}{13^2} \qquad f(1,1) = 2\left(\frac{5}{13}\right)\left(\frac{6}{13}\right) = \frac{60}{13^2}$$

$$f(2,0) = \left(\frac{5}{13}\right)^2 = \frac{25}{13^2} \qquad f(1,0) = 2\left(\frac{5}{13}\right)\left(\frac{2}{13}\right) = \frac{20}{13^2}$$

$$f(0,2) = \left(\frac{6}{13}\right)^2 = \frac{36}{13^2} \qquad f(0,1) = 2\left(\frac{6}{13}\right)\left(\frac{2}{13}\right) = \frac{24}{13^2}$$

Then we can compute

$$\sigma^2(Y) = \left(\frac{12}{13}\right)^2\left[\frac{4+25+20}{13^2}\right] + \left(\frac{1}{13}\right)^2\left[\frac{60+24}{13^2}\right] + \left(\frac{14}{13}\right)^2\frac{36}{13^2} = \frac{84}{13^2}\,.$$

Next we compute the covariance:

$$\begin{aligned}\mathrm{cov}(X,Y) = {} & \left(-\frac{10}{13}\right)\left(-\frac{12}{13}\right)\left(\frac{4}{13^2}\right) + \left(\frac{16}{13}\right)\left(-\frac{12}{13}\right)\left(\frac{25}{13^2}\right) \\ & + \left(-\frac{10}{13}\right)\left(\frac{14}{13}\right)\left(\frac{36}{13^2}\right) + \left(\frac{3}{13}\right)\left(\frac{1}{13}\right)\left(\frac{60}{13^2}\right) + \left(\frac{3}{13}\right)\left(-\frac{12}{13}\right)\left(\frac{20}{13^2}\right) \\ & + \left(-\frac{10}{13}\right)\left(\frac{1}{13}\right)\left(\frac{24}{13^2}\right) = -\frac{10140}{13^4} = -\frac{60}{13^2}\,.\end{aligned}$$

Finally, we obtain the correlation:

$$\frac{\mathrm{cov}(X,Y)}{\sigma(X)\sigma(Y)} = -\frac{60}{\sqrt{80\cdot 84}} = -0.732.$$

23.2.6. The mean free path is $\displaystyle\int_0^\infty e^{-x/f}\,dx = -\,f e^{-x/f}\Big|_0^\infty = f\,,$

or normalizing the probability density as $p(x)\,dx = (e^{-x/f}/f)\,dx$, we have

$$\int_0^\infty x e^{-x/f}\,\frac{dx}{f} = \int_0^\infty x e^{-x} dx = f.$$

For $l > 3f$, the probability is

$$\int_{3f}^{\infty} e^{-x/f}\,\frac{dx}{f} = \int_{3}^{\infty} e^{-x}dx = e^{-3} = 5\%.$$

23.2.7. $x(t) = A\cos\omega t\, dx = -A\omega\sin\omega t\, dt$,

$$\frac{dx}{v} = dt = \frac{dx}{\omega\sqrt{A^2 - x^2}}; \qquad \int_{-A}^{A}\frac{dx}{\sqrt{A^2 - x^2}} = \arcsin\frac{x}{A}\Big|_{-A}^{A} = \pi.$$

Hence the probability density is $p(x)\,dx = \dfrac{dx}{\pi\sqrt{A^2 - x^2}}$.

23.3 Binomial Distribution

23.3.1. The probability distribution is

$$f(X = x) = \binom{n}{x}\frac{1}{2^n}.$$

The sample space corresponds to $x = 0, 1, \ldots, n$. A typical event has x heads up and $n - x$ down. Using the formulas from Example 23.3.2,

$$\langle X\rangle = \frac{n}{2}, \qquad \sigma^2(X) = \frac{n}{4}.$$

23.3.2. Plot $f(X = x) = \binom{6}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{6-x}$, $x = 0, 1\ldots, 6$ as a function of x.

23.3.3. Compute this probability as unity minus the sum of the probabilities of exactly zero, one, and two defective nails, i.e.,

$$P = 1 - \binom{100}{0}(0.97)^{100} - \binom{100}{1}(0.97)^{99}(0.03) - \binom{100}{2}(0.97)^{98}(0.03)^2$$

$$= 0.58 \quad (58\%).$$

23.3.4. When the cards are put back at random places the probabilities are

$$\text{All red: } \frac{1}{2^4}. \quad \text{All hearts: } \frac{1}{4^4}. \quad \text{All honors: } \left(\frac{5}{13}\right)^4.$$

When the cards are not put back, these probabilities become

$$\text{Red: } \frac{1}{2}\frac{25}{51}\frac{24}{50}\frac{23}{49}. \quad \text{Hearts: } \frac{1}{4}\frac{12}{51}\frac{11}{50}\frac{10}{49}. \quad \text{Honors: } \frac{5}{13}\frac{19}{51}\frac{18}{50}\frac{17}{49}.$$

23.3.5. Form $\langle e^{tX}\rangle = (pe^t + q)^n$, $\quad \dfrac{\partial}{\partial t}\langle e^{tX}\rangle\Big|_{t=0} = \langle X\rangle = np.$

23.4 Poisson Distribution

23.4.1. From the data or the problem statement, we determine the average number of particles λ per time interval:

$$n = \sum_i n_i = 2608, \quad \lambda = \frac{1}{n}\sum_i i n_i = \frac{203 + 2\cdot 383 + \cdots + 10\cdot 16}{2608} = 3.87,$$

yielding $e^{-\lambda} = 0.0209$. From the Poisson distribution, $np_i = \dfrac{\lambda^i}{i!}e^{-\lambda}$. Evaluating np_i and tabulating,

i	n_i	np_i
0	57	55.7
1	203	210.8
2	383	407.8
3	525	525.7
4	532	508.4
5	408	393.3
6	273	253.5
7	139	140.1
8	45	67.7
9	27	29.1
10	16	11.3

23.4.2. $\sigma^2 = \langle X^2\rangle - \langle X\rangle^2$.

From the mean value, $\langle X\rangle^2 = \mu^2$.

If we write

$$\frac{n^2}{n!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}\,,$$

we can evaluate $\langle X^2\rangle$ as follows:

$$\langle X^2\rangle = e^{-\mu}\sum_{n=1}^{\infty}\frac{n^2\mu^n}{n!} = e^{-\mu}\sum_{n=2}^{\infty}\frac{\mu^n}{(n-2)!} + e^{-\mu}\sum_{n=1}^{\infty}\frac{\mu^n}{(n-1)!}$$

$$= \mu^2 e^{-\mu}\sum_{n=2}^{\infty}\frac{\mu^{n-2}}{(n-2)!} + \mu e^{-\mu}\sum_{n=1}^{\infty}\frac{\mu^{n-1}}{(n-1)!} = \mu^2 + \mu\,.$$

Therefore, $\sigma^2 = \mu^2 + \mu - \mu^2 = \mu$.

23.4.3. The parameter μ of the Poisson distribution (based on 5000 counts in 2400 seconds) is $\mu = 5000/2400 = 2.083$. The probability of n counts in a one-second interval is $\mu^n e^{-\mu}/n!$; for $n = 2$ and $n = 5$ we have

$$P(2) = \frac{\mu^2}{2!}\,e^{-\mu} = 0.270, \qquad P(5) = \frac{\mu^5}{5!}\,e^{-\mu} = 0.041.$$

23.4.4. Take the time unit to be 10 s, thereby making $\mu = 1$. The probability of three counts in one time interval is $(\mu^3/3!)e^{-\mu} = 1/6\,e = 0.061$.

23.4.5. The solution is given in the text.

23.4.6. This is a binomial distribution problem. We compute the probability of one or more hits as unity minus the probability that all five shots miss the target. This is

$$P = 1 - (0.80)^5 = 0.67.$$

23.5 Gauss' Normal Distribution

23.5.1. Use Eq. (23.56) with $\mu = 0$, which means that $\langle X \rangle = 0$. We therefore need to show that $\langle X^2 \rangle = \sigma^2$ when σ is the symbol occurring in Eq. (23.56). We start by verifying that the probability density is properly normalized. Setting $u = x/(\sigma\sqrt{2})$, we recognize the u integration as an error function (Section 13.6), so

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2}(\sigma\sqrt{2}\,du)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du = 1\,.$$

In a similar way we now compute

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2}\,du = \frac{2\sigma^2}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2}\right) = \sigma^2\,,$$

completing the proof. The value of the u integral can be obtained by integrating the error-function integral by parts, differentiating e^{-u^2} and integrating du.

23.5.2. In Eq. (23.61), write

$$(pn + v)^{s+1/2} = (pn)^{s+1/2}(1 + v/pn)^{s+1/2},$$

$$(qn - v)^{n-s+1/2} = (qn)^{n-s+1/2}(1 - v/qn)^{n-s+1/2}\,,$$

cancel the exponential (whose argument is zero), and combine the powers of p, q, and n. We then get Eq. (23.62) when we replace s by $pn + v$ and $n - s$ by $qn - v$.

23.5.3. By taking the logarithm the quantity $(1 + v/pn)^{-(pn+v+1/2)}$ becomes

$$-\left(pn + v + \frac{1}{2}\right) \ln\left(1 + \frac{v}{pn}\right) = -\left(pn + v + \frac{1}{2}\right)\left(\frac{v}{pn} - \frac{v^2}{2p^2n^2} + \cdots\right),$$

and a similar treatment can be applied to $(1-v/qn)^{-(qn-v+1/2)}$. Equation (23.63) results when the terms linear and quadratic in v are collected.

23.5.4. $P(|X - \langle X \rangle| > 4\sigma) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_{4/\sqrt{2}}^{\infty} e^{-x^2} dx = 6.3 \cdot 10^{-5} \approx 10^{-3}$.

This can be compared to

$$P_{\text{Cheby}}(|X - \langle X \rangle| > 4\sigma) \le \frac{1}{16} = 0.0625,$$

23.5.5. This problem has been worded incorrectly. Reinterpret it to identify m as the mean of the student scores and M as an individual student score. Then an A grade corresponds to a score higher than $m + 3\sigma/2$, etc. The probabilities we need are the following:

$$P_A = P_F = P\left(M - m > \frac{3\sigma}{2}\right) = \frac{1}{\sqrt{\pi}} \int_{3/2\sqrt{2}}^{\infty} e^{-x^2} dx = \frac{1}{2}\,\text{erfc}\left(\frac{3}{2\sqrt{2}}\right) = 6.7\%\,;$$

$$P_B = P_D = P\left(\frac{\sigma}{2} < M - m < \frac{3\sigma}{2}\right) = \frac{1}{\sqrt{\pi}} \int_{1/2\sqrt{2}}^{3/2\sqrt{2}} e^{-x^2} dx$$

$$= \frac{1}{2}\,\text{erf}\left(\frac{3}{2\sqrt{2}}\right) - \frac{1}{2}\,\text{erf}\left(\frac{1}{2\sqrt{2}}\right) = 24.2\%\,;$$

$$P_C = P\left(|x - m| < \frac{\sigma}{2}\right) = \text{erf}\left(\frac{1}{2\sqrt{2}}\right) = 38.3\%\,;$$

with percentages (reflecting round-off error) A: 6.7%, B: 24.2%, C: 38.3%, D: 24.2%, F: 6.7%.

A redesign to bring the percentage of As to 5% would require us to find a value of $k_A\sigma$ such that $P_A = \text{erfc}(k_A/\sqrt{2})/2 = 0.05$; it is $k_A = 1.645$. With this value of k_A, a value of k_B that makes

$$P_B = \frac{\text{erf}(k_A/\sqrt{2}) - \text{erf}(k_B/\sqrt{2})}{2} = 0.25$$

is $k_B = 0.524$. Then $P_C = \text{erf}(k_B/\sqrt{2}) = 0.40$.

23.6 Transformations of Random Variables

23.6.1. The addition theorem states $\displaystyle\sum_{n=1}^{n} X_i$ has mean value $n\bar{x}$, and $\dfrac{1}{n}\displaystyle\sum_{n=1}^{n} X_i$ has mean value $\bar{x}$. Since $\sigma^2 \displaystyle\sum_{n=1}^{n} X_i = n\sigma^2$, and translation does not change the variance, we obtain the σ^2 claim.

23.6.2. $\langle 2X - 1 \rangle = 2\langle X \rangle - 1 = 2 \cdot 29 - 1 = 57\,,$

$\sigma^2(2X - 1) = \sigma^2(2X) = 2\sigma^2 = 2 \cdot 9 = 18\,,$

$\sigma(2X-1) = 3\sqrt{2}\,.$

$\langle 3X+2\rangle = \langle 3X\rangle + 2 = 3\langle X\rangle + 2 = 87 + 2 = 89\,,$

$\sigma(3X) = 3\sqrt{3}.$

23.6.3. Set $I = \dfrac{1}{\sigma\sqrt{2\pi}}\displaystyle\int_{m-r}^{m+r} e^{-(x-m)^2/2\sigma^2}\,dx = \frac{1}{2}\,.$

Simplify this integral to

$$I = \frac{1}{\sigma\sqrt{2\pi}}\int_{-r}^{r} e^{-x^2/2\sigma^2}\,dx = \frac{2}{\sqrt{\pi}}\int_0^{r/\sigma\sqrt{2}} e^{-x^2}\,dx$$

$$= \operatorname{erf}\left(\frac{r}{\sigma\sqrt{2}}\right) = \frac{1}{2}\,.$$

The value of r that satisfies this equation is $r = 0.477\sigma\sqrt{2} = 0.674\sigma$.

23.6.4. $\langle X\,Y\rangle = \displaystyle\int\!\!\int f(x,y)\,dx\,dy = \int f(x)\,dx \int g(y)\,dy = \langle X\rangle\,\langle Y\rangle.$

23.6.5. $\langle f(X,Y)\rangle = \displaystyle\int\!\!\int f(x,y)P(x)Q(y)\,.dx\,dy$

$$= \int\!\!\int \left[f(0,0) + x\,\frac{\partial f}{\partial x}\bigg|_{(0,0)} + y\,\frac{\partial f}{\partial y}\bigg|_{(0,0)} + \cdots \right] P(x)Q(y)\,dx\,dy$$

$$= f(0,0) + \langle X\rangle\,\frac{\partial f}{\partial x}\bigg|_{(0,0)} + \langle Y\rangle\,\frac{\partial f}{\partial y}\bigg|_{(0,0)} + \cdots\,.$$

A corresponding expansion of the covariance takes the form

$$\operatorname{cov}\, f(X,Y) = \int\!\!\int (x-\langle X\rangle)(y-\langle Y\rangle) f(x,y)P(x)Q(y)\,dx\,dy$$

$$= \int\!\!\int (x-\langle X\rangle)(y-\langle Y\rangle)\left[f(0,0) + \frac{\partial f}{\partial x}\bigg|_{(0,0)} x + \frac{\partial f}{\partial y}\bigg|_{(0,0)} y \right.$$

$$+ \frac{\partial^2 f}{\partial x^2}\bigg|_{(0,0)}\frac{x^2}{2} + \frac{\partial^2 f}{\partial x\partial y}\bigg|_{(0,0)} xy + \frac{\partial^2 f}{\partial y^2}\bigg|_{(0,0)}\frac{y^2}{2}$$

$$\left. + \cdots \right] P(x)Q(y)\,dx\,dy\,.$$

Of the terms explicitly shown, all but the xy term vanish because the double integral can be separated to contain as a factor

$$\int (x-\langle X\rangle)P(x)\,dx \quad \text{or} \quad \int (y-\langle Y\rangle)Q(y)\,dy\,.$$

Thus, the lowest-order contribution to the covariance is

$$\begin{aligned}\text{cov } f(X,Y) &= \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} \int\int (x-\langle X\rangle)(y-\langle Y\rangle) xyP(x)Q(y)\,dx\,dy \\ &= \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} \int (x^2 - x\langle X\rangle)P(x)\,dx \int (y^2 - y\langle Y\rangle)Q(y)\,dy \\ &= \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} \sigma^2(X)\sigma^2(Y)\,.\end{aligned}$$

Hence the correlation, to lowest order, is $\dfrac{\text{cov} f(X,Y)}{\sigma^2(X)\sigma^2(Y)} \approx \left.\dfrac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)}$.

23.6.6. $\sigma^2(aX+bY) = \int\int \Big[(ax+by) - \langle aX+bY\rangle\Big]^2 f(x,y)\,dx\,dy$

$= a^2\sigma^2(X) + b^2\sigma^2(Y) + 2ab\big[\langle XY\rangle - \langle X\rangle\langle Y\rangle\big]\,.$

When X and Y are independent then $\sigma^2(aX+bY) = a^2\sigma^2(X) + b^2\sigma^2(Y)$.

23.6.7. A normally distributed Gaussian variable with mean μ and variance σ^2 has Fourier transform, and equivalently $\langle e^{itX}\rangle$, given by

$$f^T(t) = \frac{1}{\sqrt{2\pi}}\, e^{it\mu}\, e^{-t^2\sigma^2/2}, \qquad \langle e^{itX}\rangle = e^{it\mu}\, e^{-t^2\sigma^2/2}\,.$$

If a random variable Y is the sum of two Gaussian variables X_1 and X_2 with respective means μ_1, μ_2 and variances σ_1^2, σ_2^2, it is useful to form

$$\langle e^{itY}\rangle = \langle e^{it(X_1+X_2)}\rangle = \langle e^{itX_1}\rangle\langle e^{itX_2}\rangle = e^{it(\mu_1+\mu_2)} e^{-t^2(\sigma_1^2+\sigma_2^2)}.$$

This equation shows that Y is described by a Gauss normal distribution with mean $\mu_1+\mu_2$ and variance $\sigma_1^2+\sigma_2^2$.

23.6.8. The Fourier transform of $g(p,\sigma;y)$ is

$$[g(p,\sigma)]^T(t) = \frac{1}{\sqrt{2\pi}\,(2\sigma^2)^p\Gamma(p)} \int_0^\infty y^{p-1} e^{iyt} e^{-y/2\sigma^2}\,dy\,.$$

Noting now that $e^{iyt}\,e^{-y/2\sigma^2} = e^{-y(1/2\sigma^2 - it)}$, we change the variable of integration to $u = (1-2it\sigma^2)y/2\sigma^2$, converting the integral to

$$\int_0^\infty y^{p-1} e^{iyt} e^{-y/2\sigma^2}\,dy = \left(\frac{2\sigma^2}{1-2it\sigma^2}\right)^p \int_0^\infty u^{p-1} e^{-u} du$$

The integration path (from zero to infinity through complex values) can be (as indicated above) deformed to the real line without changing its value because σ^2 is positive.

Recognizing the u integral as $\Gamma(p)$ and inserting the value of the y integral, we get

$$[g(p,\sigma)]^T(t) = \frac{1}{\sqrt{2\pi}}\,(1-2it\sigma^2)^{-p}\,.$$

23.7 Statistics

23.7.1. If $C = AB$ then $\sigma^2(AB) = A^2\sigma^2(B) + B^2\sigma^2(A)$, and

$$\frac{\sigma^2(C)}{C^2} = \frac{\sigma^2(A)}{A^2} + \frac{\sigma^2(B)}{B^2}\,.$$

23.7.2. $\bar{x} = \dfrac{1}{4}(6.0 + 6.5 + 5.9 + 6.2) = 6.15,$

$$\bar{x}' = \frac{1}{5}(4\bar{x} + 6.1) = 6.14;$$

$$\sigma^2 = \frac{1}{4}(0.15^2 + 0.35^2 + 0.25^2 + 0.05^2) = 0.0525,$$

$$\sigma'^2 = \frac{1}{5}(0.14^2 + 0.36^2 + 0.24^2 + 0.06^2 + 0.04^2) = 0.0424.$$

Since x_6 is close to $\bar{x}$, $\bar{x}' \approx \bar{x}$ and σ^2 decreases a bit.

23.7.3. The problem is to be solved using the data from Example 23.7.2, but with the uncertainties given in the example associated with the t_j rather than with the y_j.

Reversing the roles of y and t, we first compute the expectation value of dt/dy using

$$N = \frac{1\cdot 0.8}{0.1^2} + \frac{2\cdot 1.5}{0.05^2} + \frac{3\cdot 3}{0.2^2} = 1505, \qquad D = \frac{0.8^2}{0.1^2} + \frac{1.5^2}{0.05^2} + \frac{3^2}{0.2^2} = 1189,$$

from which we find

$$\left\langle \frac{dt}{dy} \right\rangle = \frac{N}{D} = 1.266, \qquad \sigma_b^2 = \frac{1}{D} = 0.000841\,.$$

From the square root of σ_b^2 we get $\sigma_b = 0.029$, so our chi-square fit for dt/dy is 1.166 ± 0.029.

We can compare the reciprocal of this result with that obtained for dy/dt in Example 23.7.2, where we found $dy/dt = 0.782{\pm}0.023$: $1/1.266 = 0.790$, and the reciprocals of 1.266 ± 0.029 are 0.772 and 0.808, spanning an interval of width 0.036, not too different from $2 \times 0.023 = 0.046$.

To find the 95% confidence interval, we compute $A = 4.3\sigma_b/\sqrt{3}$ (compare with Example 23.7.2). We obtain $A = 0.072$, meaning that at this confidence level $dt/dy = 1.266 \pm 0.072$.

23.7.4. The sample mean is $\overline{X} = (6.0+6.5+5.9+6.1+6.2+6.1)/6 = 6.133$. The sample standard deviation (based on five degrees of freedom) is

$$\sqrt{\frac{(6.0-6.133)^2+(6.5-6.133)^2+\cdots+(6.1-6.133)^2}{5}} = 0.2066\,.$$

Then, using Table 23.3, with $p = 0.95$ for the 90% confidence level and $p = 0.975$ for the 95% confidence interval, we obtain (based on $n = 5$) $C_{90} = 2.02$ and $C_{95} = 2.57$. These translate into

$$\text{For 90\% confidence:} \qquad 6.133 \pm \frac{(2.02)(0.2066)}{\sqrt{5}} = 6.133 \pm 0.187,$$

$$\text{For 95\% confidence:} \qquad 6.133 \pm \frac{(2.57)(0.2066)}{\sqrt{5}} = 6.133 \pm 0.237.$$

Chapter 4

Correlation of Sixth and Seventh Edition Exercises

The following two tables indicate:

(1) The source of the exercises in the Seventh Edition; "new" indicates that the exercise was not in the Sixth Edition;

(2) The locations at which Sixth-Edition exercises can be found in the Seventh Edition; "unused" indicates that the exercise was not used in the Seventh Edition.

Source of Exercises in Seventh Edition

7th	6th	7th	6th	7th	6th
1.1.1	5.2.1	1.3.9	5.6.14	1.9.1	new
1.1.2	5.2.2	1.3.10	5.6.15	1.9.2	new
1.1.3	5.2.4	1.3.11	5.6.16	1.10.1	new
1.1.4	5.2.5	1.3.12	5.6.17	1.10.2	new
1.1.5	5.2.6	1.3.13	5.6.18	1.10.3	new
1.1.6	5.2.7	1.3.14	5.6.19	1.10.4	new
1.1.7	5.2.8	1.3.15	5.6.20	1.10.5	new
1.1.8	5.2.12	1.3.16	5.7.15	1.10.6	new
1.1.9	5.2.13	1.3.17	5.7.16	1.10.7	new
1.1.10	5.2.14	1.3.18	5.7.17	1.10.8	new
1.1.11	new	1.4.1	new	1.10.9	new
1.1.12	5.2.22	1.4.2	new	1.10.10	new
1.1.13	5.2.20	1.5.1	new	1.10.11	new
1.1.14	new	1.5.2	new	1.10.12	new
1.1.15	5.4.3	1.5.3	new	1.11.1	1.15.1
1.1.16	5.2.21	1.5.4	new	1.11.2	1.15.3
1.2.1	5.5.1	1.5.5	new	1.11.3	1.15.5
1.2.2	5.5.2	1.6.1	5.4.1	1.11.4	1.15.6
1.2.3	5.5.3	1.7.1	1.1.2	1.11.5	1.15.7
1.2.4	5.5.4	1.7.2	1.1.8	1.11.6	1.15.8
1.2.5	5.2.15	1.7.3	1.1.9	1.11.7	1.15.9
1.2.6	5.2.16	1.7.4	1.1.11	1.11.8	1.15.10
1.2.7	5.2.17	1.7.5	1.1.12	1.11.9	1.15.13
1.2.8	5.6.1	1.7.6	1.3.3	2.1.1	3.1.1
1.2.9	5.6.2	1.7.7	1.3.5	2.1.2	3.1.2
1.2.10	5.6.4	1.7.8	1.3.6	2.1.3	3.1.3
1.2.11	5.6.5	1.7.9	1.4.1	2.1.4	3.1.5
1.2.12	5.6.8	1.7.10	1.4.2	2.1.5	3.1.6
1.2.13	5.6.9	1.7.11	1.4.5	2.1.6	new
1.2.14	5.6.21	1.8.1	6.1.1	2.1.7	3.1.7
1.2.15	5.7.9	1.8.2	6.1.5	2.1.8	2.9.3
1.2.16	5.7.13	1.8.3	6.1.6	2.1.9	2.9.4
1.3.1	5.7.1	1.8.4	6.1.7	2.2.1	3.2.1
1.3.2	5.7.6	1.8.5	6.1.9	2.2.2	3.2.2
1.3.3	5.7.7	1.8.6	6.1.10	2.2.3	3.2.4
1.3.4	5.7.11	1.8.7	6.1.11	2.2.4	3.2.5
1.3.5	new	1.8.8	6.1.14	2.2.5	3.2.6
1.3.6	5.6.11	1.8.9	new	2.2.6	3.2.8
1.3.7	5.6.12	1.8.10	new	2.2.7	3.2.9
1.3.8	5.6.13	1.8.11	new	2.2.8	3.2.10

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
2.2.9	3.2.11	2.2.49	new	3.5.13	new
2.2.10	3.2.12	2.2.50	new	3.6.1	1.8.2
2.2.11	3.2.13	2.2.51	3.4.26	3.6.2	1.8.3
2.2.12	3.2.15	3.2.1	1.4.6	3.6.3	1.8.4
2.2.13	3.2.18	3.2.2	1.4.7	3.6.4	1.8.5
2.2.14	3.2.23	3.2.3	1.4.8	3.6.5	1.8.11
2.2.15	3.2.24	3.2.4	1.4.9	3.6.6	1.8.12
2.2.16	3.2.25	3.2.5	1.4.10	3.6.7	1.8.13
2.2.17	3.2.26	3.2.6	1.4.15	3.6.8	1.8.14
2.2.18	3.2.28	3.2.7	1.4.16	3.6.9	1.8.15
2.2.19	3.2.30	3.2.8	1.5.4	3.6.10	1.9.1
2.2.20	3.2.32	3.2.9	1.5.7	3.6.11	1.9.3
2.2.21	3.2.34	3.2.10	1.5.8	3.6.12	1.9.4
2.2.22	3.2.35	3.2.11	1.5.9	3.6.13	1.9.5
2.2.23	3.2.36	3.2.12	1.5.10	3.6.14	1.9.7
2.2.24	3.2.38	3.2.13	1.5.12	3.6.15	1.9.8
2.2.25	3.2.39	3.2.14	1.5.13	3.6.16	1.9.12
2.2.26	3.3.1	3.2.15	1.5.18	3.6.17	1.9.13
2.2.27	3.3.2	3.3.1	3.3.16	3.6.18	3.2.16
2.2.28	3.3.8	3.3.2	1.1.10	3.6.18	3.2.17
2.2.29	3.3.12	3.3.3	3.3.13	3.7.1	1.4.13
2.2.30	3.4.1	3.3.4	new	3.7.2	1.10.2
2.2.31	3.4.2	3.3.5	new	3.7.3	1.10.3
2.2.32	3.4.3	3.4.1	3.3.4	3.7.4	1.10.4
2.2.33	3.4.4	3.4.2	3.3.5	3.7.5	1.10.5
2.2.34	3.4.5	3.4.3	3.3.6	3.8.1	1.11.1
2.2.35	3.4.6	3.4.4	3.3.7	3.8.2	1.11.2
2.2.36	3.4.7	3.4.5	2.5.4	3.8.3	1.11.3
2.2.37	3.4.9	3.5.1	1.6.1	3.8.4	1.11.7
2.2.38	3.4.10	3.5.2	1.6.2	3.8.5	1.11.8
2.2.39	3.4.15	3.5.3	1.6.3	3.8.6	1.12.1
2.2.40	new	3.5.4	1.6.4	3.8.7	1.12.2
2.2.41	new	3.5.5	1.6.5	3.8.8	1.12.3
2.2.42	3.4.16	3.5.6	1.7.1	3.8.9	1.12.9
2.2.43	3.4.19	3.5.7	1.7.2	3.8.10	1.12.10
2.2.44	3.4.20	3.5.8	1.7.3	3.8.11	new
2.2.45	3.4.22	3.5.9	1.7.5	3.8.12	new
2.2.46	3.4.23	3.5.10	1.8.7	3.8.13	new
2.2.47	3.4.24	3.5.11	1.8.8	3.9.1	1.13.1
2.2.48	new	3.5.12	1.8.9	3.9.2	1.13.2

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
3.9.3	1.13.4	3.10.31	2.5.15	4.5.1	new
3.9.4	1.13.5	3.10.32	2.5.16	4.5.2	4.8.2
3.9.5	1.13.6	3.10.33	2.5.17	4.6.1	new
3.9.6	1.13.7	3.10.34	2.5.18	4.6.2	new
3.9.7	1.13.8	3.10.35	2.5.20	4.6.3	new
3.9.8	1.13.9	3.10.36	2.5.21	4.7.1	new
3.9.9	1.14.3	3.10.37	1.8.16	4.7.2	4.8.5
3.9.10	1.14.4	4.1.1	2.6.1	4.7.3	4.8.11
3.9.11	new	4.1.2	2.6.2	5.1.1	10.4.1
3.9.12	new	4.1.3	2.6.3	5.1.2	10.4.2
3.10.1	2.1.3	4.1.4	2.6.4	5.1.3	10.4.3
3.10.2	2.1.4	4.1.5	2.6.5	5.1.4	10.4.4
3.10.3	2.2.1	4.1.6	2.6.6	5.1.5	10.4.5
3.10.4	2.2.2	4.1.7	2.7.1	5.1.6	10.4.7
3.10.5	2.2.3	4.1.8	2.7.2	5.1.7	new
3.10.6	2.4.1	4.1.9	2.7.3	5.1.8	new
3.10.7	2.4.2	4.1.10	2.8.1	5.1.9	new
3.10.8	2.4.3	4.1.11	2.8.2	5.1.10	new
3.10.9	2.4.4	4.2.1	2.9.1	5.1.11	new
3.10.10	2.4.5	4.2.2	2.9.2	5.1.12	new
3.10.11	2.4.6	4.2.3	2.9.7	5.2.1	10.3.2
3.10.12	2.4.7	4.2.4	2.9.9	5.2.2	10.3.3
3.10.13	2.4.8	4.2.5	2.9.10	5.2.3	10.3.4
3.10.14	2.4.10	4.2.6	2.9.11	5.2.4	10.3.5
3.10.15	2.4.12	4.2.7	2.9.12	5.2.5	10.3.6
3.10.16	2.4.13	4.3.1	2.10.3	5.2.6	10.3.7
3.10.17	2.4.15	4.3.2	2.10.5	5.2.7	10.3.8
3.10.18	2.5.1	4.3.3	new	5.2.8	new
3.10.19	2.5.5	4.3.4	new	5.3.1	new
3.10.20	new	4.3.5	2.10.6	5.3.2	10.1.13
3.10.21	new	4.3.6	2.10.9	5.3.3	new
3.10.22	2.5.2	4.3.7	2.10.10	5.3.4	new
3.10.23	2.5.3	4.3.8	2.10.11	5.4.1	10.1.12
3.10.24	2.5.7	4.3.9	2.10.12	5.4.2	10.1.14
3.10.25	2.5.8	4.3.10	2.10.15	5.4.3	10.1.15
3.10.26	2.5.9	4.3.11	2.11.2	5.4.4	10.1.16
3.10.27	2.5.10	4.3.12	2.11.3	5.4.5	new
3.10.28	2.5.12	4.4.1	1.6.5	5.5.1	new
3.10.29	2.5.13	4.4.2	2.1.5	5.5.2	new
3.10.30	2.5.14	4.4.3	new	5.5.3	new

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
5.5.4	new	6.5.9	3.6.10	7.4.5	9.4.3
5.5.5	new	6.5.10	3.6.11	7.5.1	9.5.1
5.6.1	new	6.5.11	3.6.12	7.5.2	9.5.2
5.6.2	new	6.5.12	3.6.13	7.5.3	9.5.3
5.6.3	new	6.5.13	3.6.14	7.5.4	9.5.4
5.7.1	new	6.5.14	3.6.15	7.5.5	9.5.10
5.7.2	new	6.5.15	3.4.12	7.5.6	9.5.11
5.7.3	new	6.5.15	3.6.16	7.5.7	9.5.12
6.2.1	3.5.16	6.5.16	3.6.17	7.5.8	9.5.13
6.2.2	3.5.17	6.5.17	3.6.18	7.5.9	9.5.14
6.2.3	3.5.18	6.5.18	3.6.19	7.5.10	9.5.16
6.2.4	3.5.19	6.5.19	3.6.20	7.5.11	9.5.17
6.2.5	3.5.20	6.5.20	3.6.21	7.5.12	9.5.18
6.2.6	3.5.21	6.5.21	new	7.5.13	9.5.19
6.2.7	3.5.22	7.2.1	9.2.1	7.6.1	9.6.1
6.2.8	3.5.23	7.2.2	9.2.2	7.6.2	9.6.2
6.2.9	3.5.24	7.2.3	9.2.3	7.6.3	9.6.3
6.2.10	3.5.25	7.2.4	9.2.4	7.6.4	9.6.4
6.2.11	3.5.26	7.2.5	9.2.5	7.6.5	9.6.5
6.2.12	3.5.27	7.2.6	9.2.6	7.6.6	9.6.6
6.2.13	3.5.28	7.2.7	9.2.7	7.6.7	9.6.7
6.2.14	3.5.29	7.2.8	9.2.8	7.6.8	9.6.8
6.2.15	3.5.33	7.2.9	9.2.9	7.6.9	9.6.9
6.4.1	3.5.2	7.2.10	9.2.10	7.6.10	9.6.10
6.4.2	3.5.3	7.2.11	9.2.11	7.6.11	10.1.4
6.4.3	3.5.4	7.2.12	9.2.12	7.6.12	9.6.11
6.4.4	3.5.5	7.2.13	9.2.13	7.6.13	9.6.12
6.4.5	3.5.6	7.2.14	9.2.14	7.6.14	9.6.13
6.4.6	3.5.7	7.2.15	9.2.15	7.6.15	9.6.14
6.4.7	3.5.9	7.2.16	9.2.18	7.6.16	9.6.15
6.4.8	3.5.10	7.2.17	new	7.6.17	9.6.16
6.4.9	new	7.2.18	new	7.6.18	9.6.17
6.5.1	3.6.2	7.3.1	new	7.6.19	9.6.18
6.5.2	3.6.3	7.3.2	new	7.6.20	9.6.19
6.5.3	3.6.4	7.3.3	new	7.6.21	9.6.20
6.5.4	3.6.5	7.3.4	new	7.6.22	9.6.21
6.5.5	3.6.6	7.4.1	9.4.1	7.6.23	9.6.22
6.5.6	3.6.7	7.4.2	9.4.2	7.6.24	9.6.23
6.5.7	3.6.8	7.4.3	new	7.6.25	9.6.24
6.5.8	3.6.9	7.4.4	new	7.6.26	9.6.26

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
7.7.1	9.6.25	9.5.1	9.3.10	11.2.10	new
7.7.2	new	9.5.2	9.3.11	11.2.11	6.2.10
7.7.3	new	9.5.3	new	11.2.12	6.2.12
7.7.4	new	9.6.1	new	11.3.1	6.3.1
7.7.5	new	9.6.2	new	11.3.2	6.3.2
7.8.1	new	9.6.3	new	11.3.3	new
7.8.2	new	9.6.4	new	11.3.4	new
7.8.3	new	9.7.1	9.3.6	11.3.5	new
7.8.4	new	9.7.2	9.3.7	11.3.6	6.3.3
8.2.1	10.1.1	9.7.3	new	11.3.7	6.3.4
8.2.2	10.1.2	9.7.4	new	11.4.1	6.4.2
8.2.3	10.1.3	10.1.1	10.5.1	11.4.2	6.4.4
8.2.4	10.1.10	10.1.2	10.5.2	11.4.3	6.4.5
8.2.5	10.2.1	10.1.3	new	11.4.4	6.4.6
8.2.6	10.2.3	10.1.4	new	11.4.5	6.4.7
8.2.7	10.2.4	10.1.5	10.5.8	11.4.6	new
8.2.8	10.2.6	10.1.6	10.5.9	11.4.7	new
8.2.9	10.2.7	10.1.7	new	11.4.8	new
8.2.10	10.2.9	10.1.8	10.5.10	11.4.9	new
8.3.1	9.5.5	10.1.9	10.5.11	11.5.1	6.5.1
8.3.2	new	10.1.10	10.5.12	11.5.2	6.5.2
8.3.3	9.5.6	10.1.11	16.1.2	11.5.3	6.5.3
8.3.4	9.5.7	10.1.12	16.1.4	11.5.4	6.5.4
8.3.5	9.5.8	10.1.13	9.7.7	11.5.5	6.5.9
8.3.6	9.5.9	10.2.1	new	11.5.6	new
8.4.1	new	10.2.2	new	11.5.7	new
9.2.1	new	10.2.3	9.7.2	11.5.8	new
9.2.2	new	10.2.4	9.7.3	11.6.1	6.6.5
9.2.3	new	10.2.5	9.7.4	11.6.2	6.7.4
9.2.4	new	10.2.6	new	11.6.3	6.6.2
9.2.5	new	10.2.7	9.7.6	11.6.4	new
9.2.6	new	11.2.1	6.2.2	11.6.5	new
9.3.1	new	11.2.2	6.2.3	11.6.6	new
9.4.1	9.3.1	11.2.3	6.2.5	11.6.7	6.7.5
9.4.2	9.3.2	11.2.4	6.2.6	11.6.8	6.7.7
9.4.3	9.3.3	11.2.5	new	11.6.9	6.7.8
9.4.4	9.3.4	11.2.6	new	11.6.10	6.5.10
9.4.5	9.3.5	11.2.7	6.2.8	11.6.11	6.5.11
9.4.6	9.3.8	11.2.8	6.2.9	11.7.1	7.1.1
9.4.7	9.3.9	11.2.9	new	11.7.2	new

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
11.7.3	new	11.9.3	new	12.5.9	new
11.7.4	new	11.9.4	new	12.5.10	5.11.7
11.7.5	new	11.9.5	new	12.6.1	5.10.2
11.7.6	new	11.9.6	new	12.6.2	5.10.3
11.7.7	new	11.9.7	new	12.6.3	5.10.4
11.7.8	new	11.9.8	new	12.6.4	5.10.5
11.7.9	new	11.10.1	6.5.5	12.6.5	5.10.6
11.7.10	7.1.3	11.10.2	6.5.6	12.6.6	5.10.7
11.7.11	7.1.4	11.10.3	6.5.7	12.6.7	5.10.8
11.7.12	7.1.5	11.10.4	6.7.1	12.7.1	new
11.8.1	7.1.7	11.10.5	6.7.2	12.7.2	7.3.2
11.8.2	7.1.8	11.10.6	new	12.7.3	7.3.6
11.8.3	7.1.9	11.10.7	new	12.8.1	7.2.1
11.8.4	new	12.1.1	new	12.8.2	7.2.2
11.8.5	7.1.10	12.1.2	new	12.8.3	7.2.3
11.8.6	new	12.1.3	new	12.8.4	7.2.4
11.8.7	new	12.1.4	new	12.8.5	7.2.5
11.8.8	7.1.11	12.1.5	7.1.6	12.8.6	7.2.6
11.8.9	7.1.12	12.1.6	5.6.10	12.8.7	7.2.7
11.8.10	new	12.1.7	new	12.8.8	7.2.8
11.8.11	7.1.13	12.2.1	new	13.1.1	8.1.1
11.8.12	7.1.14	12.2.2	5.9.2	13.1.2	8.1.2
11.8.13	7.1.15	12.2.3	5.9.1	13.1.3	8.1.4
11.8.14	7.1.16	12.3.1	5.9.5	13.1.4	8.1.5
11.8.15	7.1.20	12.3.2	5.10.11	13.1.5	8.1.6
11.8.16	7.1.21	12.4.1	5.9.6	13.1.6	8.1.7
11.8.17	new	12.4.2	5.9.11	13.1.7	8.1.8
11.8.18	7.1.17	12.4.3	5.9.12	13.1.8	8.1.9
11.8.19	new	12.4.4	5.9.15	13.1.9	8.1.10
11.8.20	7.1.18	12.4.5	5.9.10	13.1.10	8.1.11
11.8.21	7.1.26	12.4.6	5.9.17	13.1.11	8.1.14
11.8.22	7.1.24	12.4.7	5.9.18	13.1.12	8.1.15
11.8.23	7.1.25	12.5.1	5.11.1	13.1.13	8.1.16
11.8.24	7.1.19	12.5.2	5.11.2	13.1.14	8.1.17
11.8.25	new	12.5.3	5.11.3	13.1.15	8.2.7
11.8.26	7.1.22	12.5.4	5.11.4	13.1.16	8.1.19
11.8.27	new	12.5.5	5.11.5	13.1.17	8.1.20
11.8.28	new	12.5.6	5.11.6	13.1.18	8.1.21
11.9.1	new	12.5.7	5.11.8	13.1.19	8.1.22
11.9.2	new	12.5.8	5.11.9	13.1.20	8.1.23

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
13.1.21	8.1.24	13.4.6	8.3.8	14.1.12	11.1.12
13.1.22	8.1.25	13.4.7	8.3.9	14.1.13	11.1.13
13.1.23	8.1.26	13.4.8	new	14.1.14	11.1.14
13.2.1	5.9.13	13.5.1	new	14.1.15	11.1.16
13.2.2	5.9.14	13.5.2	5.9.8	14.1.16	11.1.17
13.2.3	8.2.1	13.5.3	5.9.9	14.1.17	11.1.18
13.2.4	8.2.2	13.5.4	5.9.16	14.1.18	11.1.19
13.2.5	8.2.3	13.5.5	8.2.14	14.1.19	11.1.20
13.2.6	8.2.4	13.5.6	8.2.15	14.1.20	11.1.21
13.2.7	8.2.5	13.5.7	8.2.17	14.1.21	new
13.2.8	8.2.6	13.5.8	8.2.16	14.1.22	11.1.23
13.2.9	8.2.8	13.5.9	8.2.18	14.1.23	11.1.24
13.2.10	8.2.9	13.5.10	8.2.22	14.1.24	11.1.25
13.2.11	8.2.11	13.6.1	8.5.1	14.1.25	11.1.26
13.2.12	8.2.12	13.6.2	8.5.2	14.1.26	new
13.2.13	8.2.13	13.6.3	8.5.3	14.1.27	11.1.27
13.2.14	8.2.19	13.6.4	new	14.1.28	11.1.29
13.2.15	8.2.20	13.6.5	5.2.18	14.1.29	11.1.30
13.3.1	8.4.2	13.6.6	new	14.2.1	11.2.1
13.3.2	8.4.3	13.6.7	8.5.4	14.2.2	11.2.3
13.3.3	8.4.4	13.6.8	8.5.5	14.2.3	11.2.4
13.3.4	8.4.5	13.6.9	8.5.6	14.2.4	11.2.5
13.3.5	8.4.6	13.6.10	8.5.7	14.2.5	11.2.6
13.3.6	8.4.7	13.6.11	8.5.8	14.2.6	11.2.7
13.3.7	8.4.8	13.6.12	8.5.9	14.2.7	11.2.9
13.3.8	8.4.9	13.6.13	8.5.10	14.2.8	11.2.10
13.3.9	8.4.10	13.6.14	8.5.11	14.2.9	11.2.11
13.3.10	8.4.11	13.6.15	8.5.12	14.3.1	11.3.1
13.3.11	8.4.12	13.6.16	new	14.3.2	11.3.2
13.3.12	8.4.13	14.1.1	11.1.1	14.3.3	11.3.3
13.3.13	8.4.14	14.1.2	11.1.2	14.3.4	11.3.4
13.3.14	8.4.15	14.1.3	11.1.3	14.3.5	11.3.5
13.3.15	8.4.16	14.1.4	new	14.3.6	11.3.6
13.3.16	8.4.17	14.1.5	11.1.4	14.3.7	11.3.7
13.3.17	8.4.18	14.1.6	11.1.5	14.3.8	new
13.4.1	8.3.1	14.1.7	11.1.6	14.3.9	new
13.4.2	8.3.2	14.1.8	11.1.7	14.3.10	11.3.11
13.4.3	8.3.4	14.1.9	11.1.8	14.4.1	11.4.1
13.4.4	8.3.6	14.1.10	11.1.10	14.4.2	11.4.2
13.4.5	8.3.7	14.1.11	11.1.11	14.4.3	new

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
14.4.4	11.4.3	14.7.7	11.7.6	15.2.12	12.3.15
14.4.5	11.4.4	14.7.8	11.7.7	15.2.13	12.3.16
14.4.6	11.4.5	14.7.9	11.7.8	15.2.14	12.3.17
14.4.7	11.4.6	14.7.10	new	15.2.15	12.3.19
14.4.8	11.4.7	14.7.11	11.7.9	15.2.16	12.3.20
14.5.1	11.5.1	14.7.12	11.7.10	15.2.17	12.3.21
14.5.2	11.5.2	14.7.13	11.7.11	15.2.18	12.3.22
14.5.3	11.5.3	14.7.14	11.7.13	15.2.19	12.3.23
14.5.4	11.5.4	14.7.15	11.7.14	15.2.20	12.3.24
14.5.5	11.5.5	14.7.16	11.7.16	15.2.21	12.3.25
14.5.6	11.5.6	14.7.17	11.7.20	15.2.22	12.3.26
14.5.7	11.5.7	15.1.1	new	15.2.23	12.3.27
14.5.8	11.5.8	15.1.2	new	15.2.24	12.4.7
14.5.9	new	15.1.3	12.1.8	15.2.25	12.4.8
14.5.10	11.5.9	15.1.4	new	15.2.26	12.4.9
14.5.11	11.5.10	15.1.5	12.2.1	15.2.27	12.4.10
14.5.12	new	15.1.6	12.2.2	15.3.1	12.1.1
14.5.13	11.5.11	15.1.7	12.2.3	15.3.2	12.1.2
14.5.14	11.5.14	15.1.8	12.2.7	15.3.3	12.1.3
14.5.15	11.5.15	15.1.9	12.2.8	15.3.4	12.1.4
14.5.16	11.5.18	15.1.10	12.2.9	15.3.5	12.2.5
14.6.1	7.3.4	15.1.11	12.3.7	15.3.6	12.1.5
14.6.2	7.3.5	15.1.12	12.3.8	15.3.7	12.2.4
14.6.3	11.5.13	15.1.13	12.4.1	15.3.8	12.1.6
14.6.4	new	15.1.14	12.4.3	15.4.1	new
14.6.5	new	15.1.15	12.4.4	15.4.2	new
14.6.6	11.6.2	15.1.16	12.4.5	15.4.3	12.5.1
14.6.7	11.6.3	15.1.17	12.4.6	15.4.4	12.5.2
14.6.8	new	15.1.18	12.4.13	15.4.5	12.5.3
14.6.9	11.6.4	15.2.1	12.4.2	15.4.6	new
14.6.10	11.6.5	15.2.2	12.3.1	15.4.7	12.5.4
14.6.11	7.3.1	15.2.3	12.3.2	15.4.8	12.5.5
14.6.12	7.3.3	15.2.4	12.3.3	15.4.9	12.5.6
14.6.13	11.6.6	15.2.5	12.3.4	15.4.10	12.5.7
14.7.1	new	15.2.6	12.3.5	15.4.11	12.5.8
14.7.2	11.7.1	15.2.7	12.3.6	15.4.12	12.5.9
14.7.3	11.7.2	15.2.8	12.3.9	15.4.13	12.5.10
14.7.4	11.7.3	15.2.9	12.3.10	15.4.14	12.5.11
14.7.5	11.7.4	15.2.10	12.3.12	15.4.15	12.5.12
14.7.6	11.7.5	15.2.11	12.3.13	15.4.16	12.5.13

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
15.4.17	1.8.17	16.4.1	12.11.1	18.1.2	13.1.2
15.4.18	12.5.14	16.4.2	12.11.2	18.1.3	13.1.4
15.4.19	12.5.15	16.4.3	12.11.3	18.1.4	13.1.5
15.4.20	12.5.16	16.4.4	12.11.4	18.1.5	13.1.6
15.5.1	12.6.1	16.4.5	12.11.5	18.1.6	13.1.7
15.5.2	12.6.2	16.4.6	12.11.6	18.1.7	13.1.13
15.5.3	12.6.3	17.1.1	new	18.2.1	13.1.3
15.5.4	12.6.5	17.1.2	new	18.2.2	13.1.8
15.5.5	12.6.6	17.1.3	4.7.4	18.2.3	13.1.9
15.5.6	12.6.9	17.1.4	4.7.8	18.2.4	13.1.10
15.5.7	12.6.10	17.1.5	4.7.9	18.2.5	13.1.11
15.6.1	12.10.1	17.1.6	4.7.21	18.2.6	13.1.12
15.6.2	12.10.3	17.1.7	4.7.20	18.2.7	13.1.14
15.6.3	12.10.4	17.2.1	new	18.2.8	13.1.15
15.6.4	12.10.6	17.2.2	new	18.3.1	13.2.1
16.1.1	12.6.7	17.2.3	new	18.3.2	13.2.2
16.1.2	12.6.8	17.2.4	4.7.10	18.3.3	13.2.4
16.1.3	12.7.1	17.2.5	4.7.17	18.3.4	13.2.5
16.1.4	new	17.2.6	4.7.18	18.3.5	13.2.6
16.1.5	12.7.2	17.3.1	new	18.3.6	13.2.7
16.1.6	12.7.3	17.4.1	new	18.3.7	13.2.8
16.1.7	12.7.4	17.4.2	4.7.13	18.3.8	13.2.9
16.1.8	12.7.5	17.4.3	new	18.3.9	13.2.10
16.2.1	4.4.1	17.4.4	new	18.3.10	13.2.21
16.2.2	4.4.2	17.4.5	new	18.4.1	new
16.2.3	new	17.4.6	new	18.4.2	new
16.2.4	new	17.5.1	new	18.4.3	13.3.1
16.2.5	new	17.6.1	4.7.15	18.4.4	13.3.2
16.3.1	4.4.4	17.6.2	4.7.14	18.4.5	13.3.3
16.3.2	12.8.1	17.6.3	new	18.4.6	13.3.4
16.3.3	12.8.2	17.7.1	4.2.3	18.4.7	13.3.5
16.3.4	12.8.3	17.7.2	new	18.4.8	13.3.6
16.3.5	12.8.4	17.7.3	new	18.4.9	13.3.7
16.3.6	12.8.5	17.7.4	new	18.4.10	13.3.8
16.3.7	12.8.6	17.8.1	new	18.4.11	13.3.9
16.3.8	12.8.7	17.8.2	new	18.4.12	13.3.10
16.3.9	12.8.9	17.8.3	new	18.4.13	13.3.12
16.3.10	12.9.1	17.9.1	new	18.4.14	new
16.3.11	12.9.2	17.9.2	new	18.4.15	13.3.13
16.3.12	12.9.3	18.1.1	13.1.1	18.4.16	13.3.14

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
18.4.17	13.3.15	18.6.17	13.5.17	19.2.12	14.3.7
18.4.18	13.3.16	18.7.1	new	19.2.13	14.3.14
18.4.19	13.3.17	18.7.2	new	19.2.14	14.4.4
18.4.20	13.3.18	18.7.3	new	19.2.15	14.4.5
18.4.21	13.3.19	18.7.4	new	19.2.16	14.4.6
18.4.22	13.3.20	18.7.5	new	19.2.17	14.4.7
18.4.23	13.3.21	18.7.6	new	19.2.18	14.4.8
18.4.24	13.3.22	18.8.1	5.8.1	19.2.19	14.4.9
18.4.25	13.3.27	18.8.2	5.8.2	19.2.20	14.4.10
18.4.26	13.3.28	18.8.3	5.8.3	19.2.21	14.4.11
18.4.27	13.3.29	18.8.4	5.8.4	19.3.1	14.5.1
18.4.28	13.3.30	18.8.5	5.8.5	19.3.2	14.5.2
18.5.1	13.4.1	18.8.6	5.8.6	19.3.3	14.5.4
18.5.2	13.4.2	19.1.1	14.1.1	20.2.1	15.3.1
18.5.3	13.4.3	19.1.2	14.1.2	20.2.2	15.3.3
18.5.4	13.4.4	19.1.3	14.1.3	20.2.3	15.3.4
18.5.5	13.4.5	19.1.4	14.1.4	20.2.4	15.3.5
18.5.6	13.4.6	19.1.5	14.1.5	20.2.5	15.3.6
18.5.7	13.4.7	19.1.6	14.1.6	20.2.6	15.3.7
18.5.8	13.4.8	19.1.7	14.1.7	20.2.7	15.3.8
18.5.9	13.4.9	19.1.8	14.1.9	20.2.8	15.3.9
18.5.10	13.3.11	19.1.9	14.2.2	20.2.9	15.3.10
18.5.11	new	19.1.10	14.2.3	20.2.10	15.3.11
18.5.12	13.4.10	19.1.11	14.3.12	20.2.11	15.3.18
18.6.1	13.5.1	19.1.12	14.4.1	20.2.12	15.3.19
18.6.2	13.5.2	19.1.13	14.4.2	20.2.13	15.3.21
18.6.3	13.5.3	19.1.14	14.4.13	20.2.14	new
18.6.4	13.5.4	19.1.15	14.4.14	20.2.15	15.1.1
18.6.5	13.5.5	19.1.16	14.4.15	20.2.16	15.3.20
18.6.6	13.5.6	19.2.1	14.3.8	20.3.1	new
18.6.7	13.5.7	19.2.2	14.3.10	20.3.2	new
18.6.8	13.5.8	19.2.3	14.3.1	20.3.3	new
18.6.9	13.5.9	19.2.4	14.4.3	20.3.4	new
18.6.10	13.5.10	19.2.5	new	20.3.5	new
18.6.11	13.5.11	19.2.6	14.3.1	20.3.6	15.4.3
18.6.12	13.5.12	19.2.7	14.3.2	20.4.1	15.5.1
18.6.13	13.5.13	19.2.8	14.3.3	20.4.2	15.5.3
18.6.14	13.5.14	19.2.9	14.3.4	20.4.3	15.5.5
18.6.15	13.5.15	19.2.10	14.3.5	20.4.4	15.5.6
18.6.16	13.5.16	19.2.11	14.3.6	20.4.5	15.5.7

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
20.4.6	15.5.8	20.8.20	15.10.16	21.2.14	16.3.11
20.4.7	15.5.9	20.8.21	15.10.17	21.2.15	16.3.12
20.4.8	15.6.8	20.8.22	15.10.18	21.2.16	16.3.15
20.4.9	new	20.8.23	15.10.19	21.2.17	16.2.11
20.4.10	15.6.10	20.8.24	15.10.20	21.3.1	16.3.1
20.4.11	15.6.11	20.9.1	15.11.1	21.3.2	16.3.7
20.5.1	new	20.9.2	15.11.2	21.3.3	16.3.9
20.5.2	new	20.9.3	15.11.3	21.3.4	16.3.10
20.5.3	new	20.9.4	15.11.4	21.4.1	16.4.1
20.5.4	new	20.10.1	15.12.1	21.4.2	16.4.8
20.6.1	14.6.1	20.10.2	15.12.2	21.4.3	16.4.2
20.6.2	14.6.3	20.10.3	15.12.3	21.4.4	16.4.3
20.6.3	14.6.4	20.10.4	15.12.4	21.4.5	16.4.4
20.7.1	15.8.1	20.10.5	15.12.5	21.4.6	16.4.5
20.7.2	15.8.2	20.10.6	15.12.6	21.4.7	16.4.6
20.7.3	15.8.3	20.10.7	15.12.7	21.4.8	16.4.7
20.7.4	15.8.4	20.10.8	15.12.8	22.1.1	17.1.1
20.7.5	15.8.5	20.10.9	15.12.9	22.1.2	17.1.2
20.7.6	15.8.7	20.10.10	15.12.10	22.1.3	17.1.3
20.7.7	15.8.8	20.10.11	15.12.11	22.1.4	17.1.4
20.7.8	15.8.9	20.10.12	15.12.12	22.1.5	17.1.5
20.8.1	15.9.1	20.10.13	15.12.13	22.1.6	new
20.8.2	15.9.2	21.1.1	16.1.1	22.1.7	17.2.1
20.8.3	15.9.3	21.1.2	16.1.3	22.1.8	17.2.2
20.8.4	15.9.4	21.1.3	16.1.6	22.1.9	17.2.3
20.8.5	15.9.5	21.1.4	16.1.7	22.1.10	17.2.4
20.8.6	15.10.1	21.2.1	16.2.1	22.1.11	17.2.5
20.8.7	15.10.2	21.2.2	16.2.2	22.1.12	17.2.6
20.8.8	15.10.3	21.2.2	16.2.3	22.1.13	17.2.7
20.8.9	15.10.4	21.2.3	16.2.4	22.1.14	17.2.8
20.8.10	15.10.5	21.2.4	16.2.5	22.1.15	17.2.12
20.8.11	15.10.6	21.2.5	16.2.6	22.2.1	17.3.1
20.8.12	15.10.7	21.2.6	16.2.7	22.2.2	17.3.2
20.8.13	15.10.9	21.2.7	16.2.8	22.2.3	17.3.3
20.8.14	15.10.10	21.2.8	16.2.9	22.2.4	17.3.4
20.8.15	15.10.11	21.2.9	16.2.10	22.2.5	17.3.5
20.8.16	15.10.12	21.2.10	16.3.3	22.2.6	17.3.6
20.8.17	15.10.13	21.2.11	16.3.4	22.2.7	17.3.7
20.8.18	15.10.14	21.2.12	16.3.5	22.2.8	17.4.1
20.8.19	15.10.15	21.2.13	16.3.8	22.2.9	17.4.2

Source of Exercises in Seventh Edition (continued)

7th	6th	7th	6th	7th	6th
22.2.10	17.5.1	23.4.1	19.4.1	33.1.2	18.2.2
22.3.1	17.6.1	23.4.2	19.4.2	33.1.3	18.2.3
22.3.2	17.6.3	23.4.3	19.4.3	33.1.4	18.2.4
22.3.3	17.6.4	23.4.4	19.4.4	33.1.5	18.2.5
22.3.4	17.6.5	23.4.5	19.4.5	33.1.6	18.2.6
22.3.5	17.6.6	23.4.6	19.4.6	33.1.7	18.2.7
22.3.6	17.6.7	23.5.1	new	33.1.8	18.2.9
22.3.7	17.6.9	23.5.2	new	33.1.9	18.2.10
22.4.1	17.7.1	23.5.3	new	33.1.10	18.3.2
22.4.2	17.7.2	23.5.4	19.5.1	33.2.1	new
22.4.3	17.7.3	23.5.5	19.5.3	33.2.2	18.4.1
22.4.4	17.7.4	23.6.1	19.5.2	33.3.1	new
22.4.5	17.7.5	23.6.2	19.5.4	33.3.2	new
22.4.6	17.7.7	23.6.3	19.5.5	33.3.3	new
22.4.7	17.8.2	23.6.4	19.2.2	33.4.1	new
22.4.8	17.8.3	23.6.5	19.2.5	33.4.2	new
22.4.9	17.8.4	23.6.6	19.2.6	33.4.3	18.4.2
22.4.10	17.8.5	23.6.7	new	33.5.1	18.4.3
22.4.11	17.8.6	23.6.8	new	33.5.2	new
23.1.1	19.1.1	23.7.1	19.6.1	33.5.3	new
23.1.2	19.1.2	23.7.2	19.6.2	33.5.4	new
23.1.3	19.1.3	23.7.3	19.6.3	33.5.5	new
23.1.4	19.1.4	23.7.4	new	33.5.6	18.4.2
23.1.5	19.1.5	31.1.1	new		
23.1.6	19.1.6	31.1.2	new		
23.1.7	19.1.7	31.1.3	new		
23.1.8	19.1.8	31.2.1	new		
23.1.9	19.1.9	31.2.2	new		
23.2.1	new	31.2.3	new		
23.2.2	19.2.1	31.3.1	new		
23.2.3	new	31.3.2	new		
23.2.4	19.2.3	31.3.3	new		
23.2.5	19.2.4	32.1.1	new		
23.2.6	19.2.7	32.2.1	new		
23.2.7	19.2.8	32.2.2	new		
23.3.1	19.3.1	32.2.3	new		
23.3.2	19.3.2	32.2.4	new		
23.3.3	19.3.3	32.2.5	new		
23.3.4	19.3.4	32.2.6	new		
23.3.5	19.3.5	33.1.1	18.2.1		

New Locations of Sixth Edition Exercises

6th	7th	6th	7th	6th	7th
1.1.1	unused	1.5.2	unused	1.8.12	3.6.6
1.1.2	1.7.1	1.5.3	unused	1.8.13	3.6.7
1.1.3	unused	1.5.4	3.2.8	1.8.14	3.6.8
1.1.4	unused	1.5.5	unused	1.8.15	3.6.9
1.1.5	unused	1.5.6	unused	1.8.16	3.10.37
1.1.6	unused	1.5.7	3.2.9	1.8.17	15.4.17
1.1.7	unused	1.5.8	3.2.10	1.8.18	unused
1.1.8	1.7.2	1.5.9	3.2.11	1.8.19	unused
1.1.9	1.7.3	1.5.10	3.2.12	1.9.1	3.6.10
1.1.10	3.3.2	1.5.11	unused	1.9.2	unused
1.1.11	1.7.4	1.5.12	3.2.13	1.9.3	3.6.11
1.1.12	1.7.5	1.5.13	3.2.14	1.9.4	3.6.12
1.2.1	unused	1.5.14	unused	1.9.5	3.6.13
1.2.2	unused	1.5.15	unused	1.9.6	unused
1.3.1	unused	1.5.16	unused	1.9.7	3.6.14
1.3.2	unused	1.5.17	unused	1.9.8	3.6.15
1.3.3	1.7.6	1.5.18	3.2.15	1.9.9	unused
1.3.4	unused	1.6.1	3.5.1	1.9.10	unused
1.3.5	1.7.7	1.6.2	3.5.2	1.9.11	unused
1.3.6	1.7.8	1.6.3	3.5.3	1.9.12	3.6.16
1.3.7	unused	1.6.4	3.5.4	1.9.13	3.6.17
1.4.1	1.7.9	1.6.5	3.5.5	1.10.1	unused
1.4.2	1.7.10	1.6.5	4.4.1	1.10.2	3.7.2
1.4.3	unused	1.7.1	3.5.6	1.10.3	3.7.3
1.4.4	unused	1.7.2	3.5.7	1.10.4	3.7.4
1.4.5	1.7.11	1.7.3	3.5.8	1.10.5	3.7.5
1.4.6	3.2.1	1.7.4	unused	1.10.6	unused
1.4.7	3.2.2	1.7.5	3.5.9	1.11.1	3.8.1
1.4.8	3.2.3	1.7.6	unused	1.11.2	3.8.2
1.4.9	3.2.4	1.8.1	unused	1.11.3	3.8.3
1.4.10	3.2.5	1.8.2	3.6.1	1.11.4	unused
1.4.11	unused	1.8.3	3.6.2	1.11.5	unused
1.4.12	unused	1.8.4	3.6.3	1.11.6	unused
1.4.13	3.7.1	1.8.5	3.6.4	1.11.7	3.8.4
1.4.14	unused	1.8.6	unused	1.11.8	3.8.5
1.4.15	3.2.6	1.8.7	3.5.10	1.11.9	unused
1.4.16	3.2.7	1.8.8	3.5.11	1.11.10	unused
1.4.17	unused	1.8.9	3.5.12	1.12.1	3.8.6
1.4.18	unused	1.8.10	unused	1.12.2	3.8.7
1.5.1	unused	1.8.11	3.6.5	1.12.3	3.8.8

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
1.12.4	unused	1.15.20	unused	2.5.7	3.10.24
1.12.5	unused	1.15.21	unused	2.5.8	3.10.25
1.12.6	unused	1.15.22	unused	2.5.9	3.10.26
1.12.7	unused	1.15.23	unused	2.5.10	3.10.27
1.12.8	unused	1.15.24	unused	2.5.11	unused
1.12.9	3.8.9	1.16.1	unused	2.5.12	3.10.28
1.12.10	3.8.10	1.16.2	unused	2.5.13	3.10.29
1.13.1	3.9.1	2.1.1	unused	2.5.14	3.10.30
1.13.2	3.9.2	2.1.2	unused	2.5.15	3.10.31
1.13.3	unused	2.1.3	3.10.1	2.5.16	3.10.32
1.13.4	3.9.3	2.1.4	3.10.2	2.5.17	3.10.33
1.13.5	3.9.4	2.1.5	4.4.2	2.5.18	3.10.34
1.13.6	3.9.5	2.1.6	unused	2.5.19	unused
1.13.7	3.9.6	2.2.1	3.10.3	2.5.20	3.10.35
1.13.8	3.9.7	2.2.2	3.10.4	2.5.21	3.10.36
1.13.9	3.9.8	2.2.3	3.10.5	2.5.22	unused
1.13.11	unused	2.2.4	unused	2.5.23	unused
1.14.1	unused	2.4.1	3.10.6	2.5.24	unused
1.14.2	unused	2.4.2	3.10.7	2.5.25	unused
1.14.3	3.9.9	2.4.3	3.10.8	2.6.1	4.1.1
1.14.4	3.9.10	2.4.4	3.10.9	2.6.2	4.1.2
1.15.1	1.11.1	2.4.5	3.10.10	2.6.3	4.1.3
1.15.2	unused	2.4.6	3.10.11	2.6.4	4.1.4
1.15.3	1.11.2	2.4.7	3.10.12	2.6.5	4.1.5
1.15.4	unused	2.4.8	3.10.13	2.6.6	4.1.6
1.15.5	1.11.3	2.4.9	unused	2.7.1	4.1.7
1.15.6	1.11.4	2.4.10	3.10.14	2.7.2	4.1.8
1.15.7	1.11.5	2.4.11	unused	2.7.3	4.1.9
1.15.8	1.11.6	2.4.12	3.10.15	2.8.1	4.1.10
1.15.9	1.11.7	2.4.13	3.10.16	2.8.2	4.1.11
1.15.10	1.11.8	2.4.14	unused	2.8.3	unused
1.15.11	unused	2.4.15	3.10.17	2.9.1	4.2.1
1.15.12	unused	2.4.16	unused	2.9.2	4.2.2
1.15.13	1.11.9	2.4.17	unused	2.9.3	2.1.8
1.15.14	unused	2.5.1	3.10.18	2.9.4	2.1.9
1.15.15	unused	2.5.2	3.10.22	2.9.5	unused
1.15.16	unused	2.5.3	3.10.23	2.9.6	unused
1.15.17	unused	2.5.4	3.4.5	2.9.7	4.2.3
1.15.18	unused	2.5.5	3.10.19	2.9.8	unused
1.15.19	unused	2.5.6	unused	2.9.9	4.2.4

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
2.9.10	4.2.5	3.2.6	2.2.5	3.3.3	unused
2.9.11	4.2.6	3.2.7	unused	3.3.4	3.4.1
2.9.12	4.2.7	3.2.8	2.2.6	3.3.5	3.4.2
2.9.13	unused	3.2.9	2.2.7	3.3.6	3.4.3
2.9.14	unused	3.2.10	2.2.8	3.3.7	3.4.4
2.10.1	unused	3.2.11	2.2.9	3.3.8	2.2.28
2.10.2	unused	3.2.12	2.2.10	3.3.9	unused
2.10.3	4.3.1	3.2.13	2.2.11	3.3.10	unused
2.10.4	unused	3.2.15	2.2.12	3.3.11	unused
2.10.5	4.3.2	3.2.16	3.6.18	3.3.12	2.2.29
2.10.6	4.3.5	3.2.17	3.6.18	3.3.13	3.3.3
2.10.7	unused	3.2.18	2.2.13	3.3.14	unused
2.10.8	unused	3.2.19	unused	3.3.15	unused
2.10.9	4.3.6	3.2.20	unused	3.3.16	3.3.1
2.10.10	4.3.7	3.2.21	unused	3.3.17	unused
2.10.11	4.3.8	3.2.22	unused	3.3.18	unused
2.10.12	4.3.9	3.2.23	2.2.14	3.4.1	2.2.30
2.10.13	unused	3.2.24	2.2.15	3.4.2	2.2.31
2.10.14	unused	3.2.25	2.2.16	3.4.3	2.2.32
2.10.15	4.3.10	3.2.26	2.2.17	3.4.4	2.2.33
2.10.16	unused	3.2.27	unused	3.4.5	2.2.34
2.10.17	unused	3.2.28	2.2.18	3.4.6	2.2.35
2.11.1	unused	3.2.29	unused	3.4.7	2.2.36
2.11.2	4.3.11	3.2.30	2.2.19	3.4.8	unused
2.11.3	4.3.12	3.2.31	unused	3.4.9	2.2.37
3.1.1	2.1.1	3.2.32	2.2.20	3.4.10	2.2.38
3.1.2	2.1.2	3.2.33	unused	3.4.12	6.5.15
3.1.3	2.1.3	3.2.34	2.2.21	3.4.13	unused
3.1.4	unused	3.2.35	2.2.22	3.4.14	unused
3.1.5	2.1.4	3.2.36	2.2.23	3.4.15	2.2.39
3.1.6	2.1.5	3.2.37	unused	3.4.16	2.2.42
3.1.7	2.1.7	3.2.38	2.2.24	3.4.17	unused
3.1.8	unused	3.2.39	2.2.25	3.4.18	unused
3.1.9	unused	3.2.40	unused	3.4.19	2.2.43
3.1.10	unused	3.2.41	unused	3.4.20	2.2.44
3.2.1	2.2.1	3.2.42	unused	3.4.21	unused
3.2.2	2.2.2	3.2.43	unused	3.4.22	2.2.45
3.2.3	unused	3.2.44	unused	3.4.23	2.2.46
3.2.4	2.2.3	3.3.1	2.2.26	3.4.24	2.2.47
3.2.5	2.2.4	3.3.2	2.2.27	3.4.25	unused

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
3.4.26	2.2.51	3.6.5	6.5.4	4.5.2	unused
3.4.27	unused	3.6.6	6.5.5	4.5.3	unused
3.4.28	unused	3.6.7	6.5.6	4.6.1	unused
3.4.29	unused	3.6.8	6.5.7	4.6.2	unused
3.5.1	unused	3.6.9	6.5.8	4.6.3	unused
3.5.2	6.4.1	3.6.10	6.5.9	4.6.4	unused
3.5.3	6.4.2	3.6.11	6.5.10	4.6.5	unused
3.5.4	6.4.3	3.6.12	6.5.11	4.6.6	unused
3.5.5	6.4.4	3.6.13	6.5.12	4.6.7	unused
3.5.6	6.4.5	3.6.14	6.5.13	4.6.8	unused
3.5.7	6.4.6	3.6.15	6.5.14	4.6.9	unused
3.5.9	6.4.7	3.6.16	6.5.15	4.6.10	unused
3.5.10	6.4.8	3.6.17	6.5.16	4.6.11	unused
3.5.11	unused	3.6.18	6.5.17	4.6.12	unused
3.5.12	unused	3.6.19	6.5.18	4.6.13	unused
3.5.13	unused	3.6.20	6.5.19	4.6.14	unused
3.5.14	unused	3.6.21	6.5.20	4.7.1	unused
3.5.15	unused	4.1.1	unused	4.7.2	unused
3.5.16	6.2.1	4.1.2	unused	4.7.3	unused
3.5.17	6.2.2	4.1.3	unused	4.7.4	17.1.3
3.5.18	6.2.3	4.1.4	unused	4.7.5	unused
3.5.19	6.2.4	4.1.5	unused	4.7.6	unused
3.5.20	6.2.5	4.2.1	unused	4.7.7	unused
3.5.21	6.2.6	4.2.2	unused	4.7.8	17.1.4
3.5.22	6.2.7	4.2.3	17.7.1	4.7.9	17.1.5
3.5.23	6.2.8	4.2.4	unused	4.7.10	17.2.4
3.5.24	6.2.9	4.2.5	unused	4.7.11	unused
3.5.25	6.2.10	4.2.6	unused	4.7.12	unused
3.5.26	6.2.11	4.3.1	unused	4.7.13	17.4.2
3.5.27	6.2.12	4.3.2	unused	4.7.14	17.6.2
3.5.28	6.2.13	4.4.1	16.2.1	4.7.15	17.6.1
3.5.29	6.2.14	4.4.2	16.2.2	4.7.16	unused
3.5.30	unused	4.4.3	unused	4.7.17	17.2.5
3.5.31	unused	4.4.4	16.3.1	4.7.18	17.2.6
3.5.32	unused	4.4.5	unused	4.7.19	unused
3.5.33	6.2.15	4.4.6	unused	4.7.20	17.1.7
3.6.1	unused	4.4.7	unused	4.7.21	17.1.6
3.6.2	6.5.1	4.4.8	unused	4.7.22	unused
3.6.3	6.5.2	4.4.9	unused	4.8.1	unused
3.6.4	6.5.3	4.5.1	unused	4.8.2	4.5.2

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
4.8.3	unused	5.4.1	1.6.1	5.7.7	1.3.3
4.8.4	unused	5.4.2	unused	5.7.8	unused
4.8.5	4.7.2	5.4.3	1.1.15	5.7.9	1.2.15
4.8.6	unused	5.4.4	unused	5.7.10	unused
4.8.7	unused	5.5.1	1.2.1	5.7.11	1.3.4
4.8.8	unused	5.5.2	1.2.2	5.7.12	unused
4.8.9	unused	5.5.3	1.2.3	5.7.13	1.2.16
4.8.10	unused	5.5.4	1.2.4	5.7.14	unused
4.8.11	4.7.3	5.6.1	1.2.8	5.7.15	1.3.16
4.8.12	unused	5.6.2	1.2.9	5.7.16	1.3.17
4.8.13	unused	5.6.2	unused	5.7.17	1.3.18
4.8.14	unused	5.6.3	unused	5.7.18	unused
4.8.15	unused	5.6.4	1.2.10	5.7.19	unused
5.1.1	unused	5.6.5	1.2.11	5.8.1	18.8.1
5.1.2	unused	5.6.6	unused	5.8.2	18.8.2
5.2.1	1.1.1	5.6.7	unused	5.8.3	18.8.3
5.2.2	1.1.2	5.6.8	1.2.12	5.8.4	18.8.4
5.2.3	unused	5.6.9	1.2.13	5.8.5	18.8.5
5.2.4	1.1.3	5.6.10	12.1.6	5.8.6	18.8.6
5.2.5	1.1.4	5.6.10	unused	5.8.7	unused
5.2.6	1.1.5	5.6.11	1.3.6	5.8.8	unused
5.2.7	1.1.6	5.6.12	1.3.7	5.8.9	unused
5.2.8	1.1.7	5.6.13	1.3.8	5.9.1	12.2.3
5.2.9	unused	5.6.14	1.3.9	5.9.2	12.2.2
5.2.10	unused	5.6.15	1.3.10	5.9.3	unused
5.2.11	unused	5.6.16	1.3.11	5.9.4	unused
5.2.12	1.1.8	5.6.17	1.3.12	5.9.5	12.3.1
5.2.13	1.1.9	5.6.18	1.3.13	5.9.6	12.4.1
5.2.14	1.1.10	5.6.19	1.3.14	5.9.7	unused
5.2.15	1.2.5	5.6.20	1.3.15	5.9.8	13.5.2
5.2.16	1.2.6	5.6.21	1.2.14	5.9.9	13.5.3
5.2.17	1.2.7	5.6.22	unused	5.9.10	12.4.5
5.2.18	13.6.5	5.6.23	unused	5.9.11	12.4.2
5.2.19	unused	5.6.24	unused	5.9.12	12.4.3
5.2.20	1.1.13	5.7.1	1.3.1	5.9.13	13.2.1
5.2.21	1.1.16	5.7.2	unused	5.9.14	13.2.2
5.2.22	1.1.12	5.7.3	unused	5.9.15	12.4.4
5.3.1	unused	5.7.4	unused	5.9.16	13.5.4
5.3.2	unused	5.7.5	unused	5.9.17	12.4.6
5.3.3	unused	5.7.6	1.3.2	5.9.18	12.4.7

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
5.9.19	unused	6.1.16	unused	6.5.9	11.5.5
5.9.20	unused	6.1.17	unused	6.5.10	11.6.10
5.10.1	unused	6.1.18	unused	6.5.11	11.6.11
5.10.2	12.6.1	6.1.19	unused	6.6.1	unused
5.10.3	12.6.2	6.1.20	unused	6.6.2	11.6.3
5.10.4	12.6.3	6.1.21	unused	6.6.3	unused
5.10.5	12.6.4	6.1.22	unused	6.6.4	unused
5.10.6	12.6.5	6.1.23	unused	6.6.5	11.6.1
5.10.7	12.6.6	6.1.24	unused	6.7.1	11.10.4
5.10.8	12.6.7	6.1.25	unused	6.7.2	11.10.5
5.10.9	unused	6.1.26	unused	6.7.3	unused
5.10.10	unused	6.2.1	unused	6.7.4	11.6.2
5.10.11	12.3.2	6.2.2	11.2.1	6.7.5	11.6.7
5.11.1	12.5.1	6.2.3	11.2.2	6.7.6	unused
5.11.2	12.5.2	6.2.5	11.2.3	6.7.7	11.6.8
5.11.3	12.5.3	6.2.6	11.2.4	6.7.8	11.6.9
5.11.4	12.5.4	6.2.7	unused	6.8.1	unused
5.11.5	12.5.5	6.2.8	11.2.7	6.8.2	unused
5.11.6	12.5.6	6.2.9	11.2.8	6.8.3	unused
5.11.6	unused	6.2.10	11.2.11	7.1.1	11.7.1
5.11.7	12.5.10	6.2.11	unused	7.1.2	unused
5.11.8	12.5.7	6.2.12	11.2.12	7.1.3	11.7.10
5.11.9	12.5.8	6.3.1	11.3.1	7.1.4	11.7.11
5.11.10	unused	6.3.2	11.3.2	7.1.5	11.7.12
6.1.1	1.8.1	6.3.3	11.3.6	7.1.6	12.1.5
6.1.2	unused	6.3.4	11.3.7	7.1.7	11.8.1
6.1.3	unused	6.4.2	11.4.1	7.1.8	11.8.2
6.1.4	unused	6.4.4	11.4.2	7.1.9	11.8.3
6.1.5	1.8.2	6.4.5	11.4.3	7.1.10	11.8.5
6.1.6	1.8.3	6.4.6	11.4.4	7.1.11	11.8.8
6.1.7	1.8.4	6.4.7	11.4.5	7.1.12	11.8.9
6.1.8	unused	6.4.8	unused	7.1.13	11.8.11
6.1.9	1.8.5	6.5.1	11.5.1	7.1.14	11.8.12
6.1.9	unused	6.5.2	11.5.2	7.1.15	11.8.13
6.1.10	1.8.6	6.5.3	11.5.3	7.1.16	11.8.14
6.1.11	1.8.7	6.5.4	11.5.4	7.1.17	11.8.18
6.1.12	unused	6.5.5	11.10.1	7.1.18	11.8.20
6.1.13	unused	6.5.6	11.10.2	7.1.19	11.8.24
6.1.14	1.8.8	6.5.7	11.10.3	7.1.20	11.8.15
6.1.15	unused	6.5.8	unused	7.1.21	11.8.16

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
7.1.22	11.8.26	8.1.19	13.1.16	8.3.6	13.4.4
7.1.23	unused	8.1.20	13.1.17	8.3.7	13.4.5
7.1.24	11.8.22	8.1.21	13.1.18	8.3.8	13.4.6
7.1.25	11.8.23	8.1.22	13.1.19	8.3.9	13.4.7
7.1.26	11.8.21	8.1.23	13.1.20	8.3.10	unused
7.1.27	unused	8.1.24	13.1.21	8.3.11	unused
7.1.28	unused	8.1.25	13.1.22	8.3.12	unused
7.2.1	12.8.1	8.1.26	13.1.23	8.4.1	unused
7.2.2	12.8.2	8.1.27	unused	8.4.2	13.3.1
7.2.3	12.8.3	8.1.28	unused	8.4.3	13.3.2
7.2.4	12.8.4	8.1.29	unused	8.4.4	13.3.3
7.2.5	12.8.5	8.1.30	unused	8.4.5	13.3.4
7.2.6	12.8.6	8.2.1	13.2.3	8.4.6	13.3.5
7.2.7	12.8.7	8.2.2	13.2.4	8.4.7	13.3.6
7.2.8	12.8.8	8.2.3	13.2.5	8.4.8	13.3.7
7.2.9	unused	8.2.4	13.2.6	8.4.9	13.3.8
7.3.1	14.6.11	8.2.5	13.2.7	8.4.10	13.3.9
7.3.2	12.7.2	8.2.6	13.2.8	8.4.11	13.3.10
7.3.3	14.6.12	8.2.7	13.1.15	8.4.12	13.3.11
7.3.4	14.6.1	8.2.8	13.2.9	8.4.13	13.3.12
7.3.5	14.6.2	8.2.9	13.2.10	8.4.14	13.3.13
7.3.6	12.7.3	8.2.10	unused	8.4.15	13.3.14
7.3.7	unused	8.2.11	13.2.11	8.4.16	13.3.15
8.1.1	13.1.1	8.2.12	13.2.12	8.4.17	13.3.16
8.1.2	13.1.2	8.2.13	13.2.13	8.4.18	13.3.17
8.1.3	unused	8.2.14	13.5.5	8.4.19	unused
8.1.4	13.1.3	8.2.15	13.5.6	8.4.20	unused
8.1.5	13.1.4	8.2.16	13.5.8	8.5.1	13.6.1
8.1.6	13.1.5	8.2.17	13.5.7	8.5.2	13.6.2
8.1.7	13.1.6	8.2.18	13.5.9	8.5.3	13.6.3
8.1.8	13.1.7	8.2.19	13.2.14	8.5.4	13.6.7
8.1.9	13.1.8	8.2.20	13.2.15	8.5.5	13.6.8
8.1.10	13.1.9	8.2.21	unused	8.5.6	13.6.9
8.1.11	13.1.10	8.2.22	13.5.10	8.5.7	13.6.10
8.1.12	unused	8.2.23	unused	8.5.8	13.6.11
8.1.13	unused	8.3.1	13.4.1	8.5.9	13.6.12
8.1.14	13.1.11	8.3.2	13.4.2	8.5.10	13.6.13
8.1.15	13.1.12	8.3.3	unused	8.5.11	13.6.14
8.1.16	13.1.13	8.3.4	13.4.3	8.5.12	13.6.15
8.1.17	13.1.14	8.3.5	unused	8.5.13	unused

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
8.5.14	unused	9.5.6	8.3.3	9.7.2	10.2.3
8.5.15	unused	9.5.7	8.3.4	9.7.3	10.2.4
8.5.16	unused	9.5.8	8.3.5	9.7.4	10.2.5
9.2.1	7.2.1	9.5.9	8.3.6	9.7.5	unused
9.2.2	7.2.2	9.5.10	7.5.5	9.7.6	10.2.7
9.2.3	7.2.3	9.5.11	7.5.6	9.7.7	10.1.13
9.2.4	7.2.4	9.5.12	7.5.7	9.7.8	unused
9.2.5	7.2.5	9.5.13	7.5.8	9.7.9	unused
9.2.6	7.2.6	9.5.14	7.5.9	9.7.10	unused
9.2.7	7.2.7	9.5.16	7.5.10	9.7.11	unused
9.2.8	7.2.8	9.5.17	7.5.11	9.7.12	unused
9.2.9	7.2.9	9.5.18	7.5.12	9.7.13	unused
9.2.10	7.2.10	9.5.19	7.5.13	9.7.14	unused
9.2.11	7.2.11	9.6.1	7.6.1	9.7.15	unused
9.2.12	7.2.12	9.6.2	7.6.2	9.7.16	unused
9.2.13	7.2.13	9.6.3	7.6.3	9.7.17	unused
9.2.14	7.2.14	9.6.4	7.6.4	9.7.18	unused
9.2.15	7.2.15	9.6.5	7.6.5	9.7.19	unused
9.2.16	unused	9.6.6	7.6.6	9.7.20	unused
9.2.17	unused	9.6.7	7.6.7	10.1.1	8.2.1
9.2.18	7.2.16	9.6.8	7.6.8	10.1.2	8.2.2
9.3.1	9.4.1	9.6.9	7.6.9	10.1.3	8.2.3
9.3.2	9.4.2	9.6.10	7.6.10	10.1.4	7.6.11
9.3.3	9.4.3	9.6.11	7.6.12	10.1.5	unused
9.3.4	9.4.4	9.6.12	7.6.13	10.1.6	unused
9.3.5	9.4.5	9.6.13	7.6.14	10.1.7	unused
9.3.6	9.7.1	9.6.14	7.6.15	10.1.8	unused
9.3.7	9.7.2	9.6.15	7.6.16	10.1.9	unused
9.3.8	9.4.6	9.6.16	7.6.17	10.1.10	8.2.4
9.3.9	9.4.7	9.6.17	7.6.18	10.1.11	unused
9.3.10	9.5.1	9.6.18	7.6.19	10.1.12	5.4.1
9.3.11	9.5.2	9.6.19	7.6.20	10.1.13	5.3.2
9.4.1	7.4.1	9.6.20	7.6.21	10.1.14	5.4.2
9.4.2	7.4.2	9.6.21	7.6.22	10.1.15	5.4.3
9.4.3	7.4.5	9.6.22	7.6.23	10.1.16	5.4.4
9.5.1	7.5.1	9.6.23	7.6.24	10.1.17	unused
9.5.2	7.5.2	9.6.24	7.6.25	10.1.18	unused
9.5.3	7.5.3	9.6.25	7.7.1	10.1.19	unused
9.5.4	7.5.4	9.6.26	7.6.26	10.1.20	unused
9.5.5	8.3.1	9.7.1	unused	10.1.21	unused

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
10.1.22	unused	10.5.8	10.1.5	11.1.31	unused
10.2.1	8.2.5	10.5.9	10.1.6	11.1.32	unused
10.2.3	8.2.6	10.5.10	10.1.8	11.2.1	14.2.1
10.2.4	8.2.7	10.5.11	10.1.9	11.2.2	unused
10.2.5	unused	10.5.12	10.1.10	11.2.3	14.2.2
10.2.6	8.2.8	10.5.13	unused	11.2.4	14.2.3
10.2.7	8.2.9	10.5.14	unused	11.2.5	14.2.4
10.2.8	unused	10.5.15	unused	11.2.6	14.2.5
10.2.9	8.2.10	10.5.16	unused	11.2.7	14.2.6
10.2.10	unused	10.5.17	unused	11.2.8	unused
10.2.11	unused	11.1.1	14.1.1	11.2.9	14.2.7
10.2.12	unused	11.1.2	14.1.2	11.2.10	14.2.8
10.2.13	unused	11.1.3	14.1.3	11.2.11	14.2.9
10.2.14	unused	11.1.4	14.1.5	11.3.1	14.3.1
10.3.1	unused	11.1.5	14.1.6	11.3.2	14.3.2
10.3.2	5.2.1	11.1.6	14.1.7	11.3.3	14.3.3
10.3.3	5.2.2	11.1.7	14.1.8	11.3.4	14.3.4
10.3.4	5.2.3	11.1.8	14.1.9	11.3.5	14.3.5
10.3.5	5.2.4	11.1.9	unused	11.3.6	14.3.6
10.3.6	5.2.5	11.1.10	14.1.10	11.3.7	14.3.7
10.3.7	5.2.6	11.1.11	14.1.11	11.3.8	unused
10.3.8	5.2.7	11.1.12	14.1.12	11.3.9	unused
10.3.9	unused	11.1.13	14.1.13	11.3.10	unused
10.4.1	5.1.1	11.1.14	14.1.14	11.3.11	14.3.10
10.4.2	5.1.2	11.1.15	unused	11.4.1	14.4.1
10.4.3	5.1.3	11.1.16	14.1.15	11.4.2	14.4.2
10.4.4	5.1.4	11.1.17	14.1.16	11.4.3	14.4.4
10.4.5	5.1.5	11.1.18	14.1.17	11.4.4	14.4.5
10.4.6	unused	11.1.19	14.1.18	11.4.5	14.4.6
10.4.7	5.1.6	11.1.20	14.1.19	11.4.6	14.4.7
10.4.8	unused	11.1.21	14.1.20	11.4.7	14.4.8
10.4.10	unused	11.1.22	unused	11.5.1	14.5.1
10.4.11	unused	11.1.23	14.1.22	11.5.2	14.5.2
10.5.1	10.1.1	11.1.24	14.1.23	11.5.3	14.5.3
10.5.2	10.1.2	11.1.25	14.1.24	11.5.4	14.5.4
10.5.3	unused	11.1.26	14.1.25	11.5.5	14.5.5
10.5.4	unused	11.1.27	14.1.27	11.5.6	14.5.6
10.5.5	unused	11.1.28	unused	11.5.7	14.5.7
10.5.6	unused	11.1.29	14.1.28	11.5.8	14.5.8
10.5.7	unused	11.1.30	14.1.29	11.5.9	14.5.10

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
11.5.10	14.5.11	11.7.25	unused	12.3.15	15.2.12
11.5.11	14.5.13	11.7.26	unused	12.3.16	15.2.13
11.5.12	unused	11.7.27	unused	12.3.17	15.2.14
11.5.13	14.6.3	12.1.1	15.3.1	12.3.18	unused
11.5.14	14.5.14	12.1.2	15.3.2	12.3.19	15.2.15
11.5.15	14.5.15	12.1.3	15.3.3	12.3.20	15.2.16
11.5.16	unused	12.1.4	15.3.4	12.3.21	15.2.17
11.5.17	unused	12.1.5	15.3.6	12.3.22	15.2.18
11.5.18	14.5.16	12.1.6	15.3.8	12.3.23	15.2.19
11.6.1	unused	12.1.7	unused	12.3.24	15.2.20
11.6.2	14.6.6	12.1.8	15.1.3	12.3.25	15.2.21
11.6.3	14.6.7	12.1.9	unused	12.3.26	15.2.22
11.6.4	14.6.9	12.2.1	15.1.5	12.3.27	15.2.23
11.6.5	14.6.10	12.2.2	15.1.6	12.4.1	15.1.13
11.6.6	14.6.13	12.2.3	15.1.7	12.4.2	15.2.1
11.6.7	unused	12.2.4	15.3.7	12.4.3	15.1.14
11.7.1	14.7.2	12.2.5	15.3.5	12.4.4	15.1.15
11.7.2	14.7.3	12.2.6	unused	12.4.5	15.1.16
11.7.3	14.7.4	12.2.7	15.1.8	12.4.6	15.1.17
11.7.4	14.7.5	12.2.8	15.1.9	12.4.7	15.2.24
11.7.5	14.7.6	12.2.9	15.1.10	12.4.8	15.2.25
11.7.6	14.7.7	12.2.10	unused	12.4.9	15.2.26
11.7.7	14.7.8	12.2.11	unused	12.4.10	15.2.27
11.7.8	14.7.9	12.2.12	unused	12.4.11	unused
11.7.9	14.7.11	12.2.13	unused	12.4.12	unused
11.7.10	14.7.12	12.2.14	unused	12.4.13	15.1.18
11.7.11	14.7.13	12.3.1	15.2.2	12.5.1	15.4.3
11.7.12	unused	12.3.2	15.2.3	12.5.2	15.4.4
11.7.13	14.7.14	12.3.3	15.2.4	12.5.3	15.4.5
11.7.14	14.7.15	12.3.4	15.2.5	12.5.4	15.4.7
11.7.15	unused	12.3.5	15.2.6	12.5.5	15.4.8
11.7.16	14.7.16	12.3.6	15.2.7	12.5.6	15.4.9
11.7.17	unused	12.3.7	15.1.11	12.5.7	15.4.10
11.7.18	unused	12.3.8	15.1.12	12.5.8	15.4.11
11.7.19	unused	12.3.9	15.2.8	12.5.9	15.4.12
11.7.20	14.7.17	12.3.10	15.2.9	12.5.10	15.4.13
11.7.21	unused	12.3.11	unused	12.5.11	15.4.14
11.7.22	unused	12.3.12	15.2.10	12.5.12	15.4.15
11.7.23	unused	12.3.13	15.2.11	12.5.13	15.4.16
11.7.24	unused	12.3.14	unused	12.5.14	15.4.18

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
12.5.15	15.4.19	12.11.2	16.4.2	13.2.13	unused
12.5.16	15.4.20	12.11.3	16.4.3	13.2.14	unused
12.5.17	unused	12.11.4	16.4.4	13.2.15	unused
12.5.18	unused	12.11.5	16.4.5	13.2.16	unused
12.5.19	unused	12.11.6	16.4.6	13.2.17	unused
12.6.1	15.5.1	13.1.1	18.1.1	13.2.18	unused
12.6.2	15.5.2	13.1.2	18.1.2	13.2.19	unused
12.6.3	15.5.3	13.1.3	18.2.1	13.2.20	unused
12.6.4	unused	13.1.4	18.1.3	13.2.21	18.3.10
12.6.5	15.5.4	13.1.5	18.1.4	13.2.22	unused
12.6.6	15.5.5	13.1.6	18.1.5	13.3.1	18.4.3
12.6.7	16.1.1	13.1.7	18.1.6	13.3.2	18.4.4
12.6.8	16.1.2	13.1.8	18.2.2	13.3.3	18.4.5
12.6.9	15.5.6	13.1.9	18.2.3	13.3.4	18.4.6
12.6.10	15.5.7	13.1.10	18.2.4	13.3.5	18.4.7
12.7.1	16.1.3	13.1.11	18.2.5	13.3.6	18.4.8
12.7.2	16.1.5	13.1.12	18.2.6	13.3.7	18.4.9
12.7.3	16.1.6	13.1.13	18.1.7	13.3.8	18.4.10
12.7.4	16.1.7	13.1.14	18.2.7	13.3.9	18.4.11
12.7.5	16.1.8	13.1.15	18.2.8	13.3.10	18.4.12
12.8.1	16.3.2	13.1.16	unused	13.3.11	18.5.10
12.8.2	16.3.3	13.1.17	unused	13.3.12	18.4.13
12.8.3	16.3.4	13.1.18	unused	13.3.13	18.4.15
12.8.4	16.3.5	13.1.19	unused	13.3.14	18.4.16
12.8.5	16.3.6	13.1.20	unused	13.3.15	18.4.17
12.8.6	16.3.7	13.1.21	unused	13.3.16	18.4.18
12.8.7	16.3.8	13.1.22	unused	13.3.17	18.4.19
12.8.8	unused	13.1.23	unused	13.3.18	18.4.20
12.8.9	16.3.9	13.2.1	18.3.1	13.3.19	18.4.21
12.9.1	16.3.10	13.2.2	18.3.2	13.3.20	18.4.22
12.9.2	16.3.11	13.2.3	unused	13.3.21	18.4.23
12.9.3	16.3.12	13.2.4	18.3.3	13.3.22	18.4.24
12.9.4	unused	13.2.5	18.3.4	13.3.23	unused
12.10.1	15.6.1	13.2.6	18.3.5	13.3.24	unused
12.10.2	unused	13.2.7	18.3.6	13.3.25	unused
12.10.3	15.6.2	13.2.8	18.3.7	13.3.26	unused
12.10.4	15.6.3	13.2.9	18.3.8	13.3.27	18.4.25
12.10.5	unused	13.2.10	18.3.9	13.3.28	18.4.26
12.10.6	15.6.4	13.2.11	unused	13.3.29	18.4.27
12.11.1	16.4.1	13.2.12	unused	13.3.30	18.4.28

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
13.4.1	18.5.1	14.2.3	19.1.10	14.6.4	20.6.3
13.4.2	18.5.2	14.3.1	19.2.3	14.6.5	unused
13.4.3	18.5.3	14.3.1	19.2.6	14.6.6	unused
13.4.4	18.5.4	14.3.2	19.2.7	14.7.1	unused
13.4.5	18.5.5	14.3.3	19.2.8	14.7.2	unused
13.4.6	18.5.6	14.3.4	19.2.9	14.7.3	unused
13.4.7	18.5.7	14.3.5	19.2.10	15.1.1	20.2.15
13.4.8	18.5.8	14.3.6	19.2.11	15.1.2	unused
13.4.9	18.5.9	14.3.7	19.2.12	15.1.3	unused
13.4.10	18.5.12	14.3.8	19.2.1	15.1.4	unused
13.5.1	18.6.1	14.3.9	unused	15.3.1	20.2.1
13.5.2	18.6.2	14.3.10	19.2.2	15.3.2	unused
13.5.3	18.6.3	14.3.12	19.1.11	15.3.3	20.2.2
13.5.4	18.6.4	14.3.13	unused	15.3.4	20.2.3
13.5.5	18.6.5	14.3.14	19.2.13	15.3.5	20.2.4
13.5.6	18.6.6	14.3.15	unused	15.3.6	20.2.5
13.5.7	18.6.7	14.3.16	unused	15.3.7	20.2.6
13.5.8	18.6.8	14.3.17	unused	15.3.8	20.2.7
13.5.9	18.6.9	14.4.1	19.1.12	15.3.9	20.2.8
13.5.10	18.6.10	14.4.2	19.1.13	15.3.10	20.2.9
13.5.11	18.6.11	14.4.3	19.2.4	15.3.11	20.2.10
13.5.12	18.6.12	14.4.4	19.2.14	15.3.12	unused
13.5.13	18.6.13	14.4.5	19.2.15	15.3.13	unused
13.5.14	18.6.14	14.4.6	19.2.16	15.3.14	unused
13.5.15	18.6.15	14.4.7	19.2.17	15.3.15	unused
13.5.16	18.6.16	14.4.8	19.2.18	15.3.16	unused
13.5.17	18.6.17	14.4.9	19.2.19	15.3.17	unused
13.6.1	unused	14.4.10	19.2.20	15.3.18	20.2.11
13.6.2	unused	14.4.11	19.2.21	15.3.19	20.2.12
14.1.1	19.1.1	14.4.12	unused	15.3.20	20.2.16
14.1.2	19.1.2	14.4.13	19.1.14	15.3.21	20.2.13
14.1.3	19.1.3	14.4.14	19.1.15	15.4.1	unused
14.1.4	19.1.4	14.4.15	19.1.16	15.4.2	unused
14.1.5	19.1.5	14.5.1	19.3.1	15.4.3	20.3.6
14.1.6	19.1.6	14.5.2	19.3.2	15.4.4	unused
14.1.7	19.1.7	14.5.3	unused	15.4.5	unused
14.1.8	unused	14.5.4	19.3.3	15.5.1	20.4.1
14.1.9	19.1.8	14.6.1	20.6.1	15.5.3	20.4.2
14.2.1	unused	14.6.2	unused	15.5.4	unused
14.2.2	19.1.9	14.6.3	20.6.2	15.5.5	20.4.3

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
15.5.6	20.4.4	15.10.10	20.8.14	16.2.4	21.2.3
15.5.7	20.4.5	15.10.11	20.8.15	16.2.5	21.2.4
15.5.8	20.4.6	15.10.12	20.8.16	16.2.6	21.2.5
15.5.9	20.4.7	15.10.13	20.8.17	16.2.7	21.2.6
15.6.1	unused	15.10.14	20.8.18	16.2.8	21.2.7
15.6.2	unused	15.10.15	20.8.19	16.2.9	21.2.8
15.6.3	unused	15.10.16	20.8.20	16.2.10	21.2.9
15.6.4	unused	15.10.17	20.8.21	16.2.11	21.2.17
15.6.5	unused	15.10.18	20.8.22	16.3.1	21.3.1
15.6.6	unused	15.10.19	20.8.23	16.3.2	unused
15.6.7	unused	15.10.20	20.8.24	16.3.3	21.2.10
15.6.8	20.4.8	15.10.21	unused	16.3.4	21.2.11
15.6.9	unused	15.10.22	unused	16.3.5	21.2.12
15.6.10	20.4.10	15.11.1	20.9.1	16.3.6	unused
15.6.11	20.4.11	15.11.2	20.9.2	16.3.7	21.3.2
15.6.12	unused	15.11.3	20.9.3	16.3.8	21.2.13
15.7.1	unused	15.11.4	20.9.4	16.3.9	21.3.3
15.8.1	20.7.1	15.12.1	20.10.1	16.3.10	21.3.4
15.8.2	20.7.2	15.12.2	20.10.2	16.3.11	21.2.14
15.8.3	20.7.3	15.12.3	20.10.3	16.3.12	21.2.15
15.8.4	20.7.4	15.12.4	20.10.4	16.3.13	unused
15.8.5	20.7.5	15.12.5	20.10.5	16.3.14	unused
15.8.6	unused	15.12.6	20.10.6	16.3.15	21.2.16
15.8.7	20.7.6	15.12.7	20.10.7	16.3.16	unused
15.8.8	20.7.7	15.12.8	20.10.8	16.3.17	unused
15.8.9	20.7.8	15.12.9	20.10.9	16.3.18	unused
15.9.1	20.8.1	15.12.10	20.10.10	16.3.19	unused
15.9.2	20.8.2	15.12.11	20.10.11	16.4.1	21.4.1
15.9.3	20.8.3	15.12.12	20.10.12	16.4.2	21.4.3
15.9.4	20.8.4	15.12.13	20.10.13	16.4.3	21.4.4
15.9.5	20.8.5	16.1.1	21.1.1	16.4.4	21.4.5
15.10.1	20.8.6	16.1.2	10.1.11	16.4.5	21.4.6
15.10.2	20.8.7	16.1.3	21.1.2	16.4.6	21.4.7
15.10.3	20.8.8	16.1.4	10.1.12	16.4.7	21.4.8
15.10.4	20.8.9	16.1.5	unused	16.4.8	21.4.2
15.10.5	20.8.10	16.1.6	21.1.3	17.1.1	22.1.1
15.10.6	20.8.11	16.1.7	21.1.4	17.1.2	22.1.2
15.10.7	20.8.12	16.2.1	21.2.1	17.1.3	22.1.3
15.10.8	unused	16.2.2	21.2.2	17.1.4	22.1.4
15.10.9	20.8.13	16.2.3	21.2.2	17.1.5	22.1.5

New Locations of Sixth Edition Exercises (continued)

6th	7th	6th	7th	6th	7th
17.2.1	22.1.7	17.7.7	22.4.6	19.2.5	23.6.5
17.2.2	22.1.8	17.8.1	unused	19.2.6	23.6.6
17.2.3	22.1.9	17.8.2	22.4.7	19.2.7	23.2.6
17.2.4	22.1.10	17.8.3	22.4.8	19.2.8	23.2.7
17.2.5	22.1.11	17.8.4	22.4.9	19.3.1	23.3.1
17.2.6	22.1.12	17.8.5	22.4.10	19.3.2	23.3.2
17.2.7	22.1.13	17.8.6	22.4.11	19.3.3	23.3.3
17.2.8	22.1.14	17.8.7	unused	19.3.4	23.3.4
17.2.9	unused	17.8.8	unused	19.3.5	23.3.5
17.2.10	unused	18.2.1	33.1.1	19.4.1	23.4.1
17.2.11	unused	18.2.2	33.1.2	19.4.2	23.4.2
17.2.12	22.1.15	18.2.3	33.1.3	19.4.3	23.4.3
17.3.1	22.2.1	18.2.4	33.1.4	19.4.4	23.4.4
17.3.2	22.2.2	18.2.5	33.1.5	19.4.5	23.4.5
17.3.3	22.2.3	18.2.6	33.1.6	19.4.6	23.4.6
17.3.4	22.2.4	18.2.7	33.1.7	19.4.7	unused
17.3.5	22.2.5	18.2.8	unused	19.5.1	23.5.4
17.3.6	22.2.6	18.2.9	33.1.8	19.5.2	23.6.1
17.3.7	22.2.7	18.2.10	33.1.9	19.5.3	23.5.5
17.4.1	22.2.8	18.2.11	unused	19.5.4	23.6.2
17.4.2	22.2.9	18.3.1	unused	19.5.5	23.6.3
17.5.1	22.2.10	18.3.2	33.1.10	19.6.1	23.7.1
17.6.1	22.3.1	18.4.1	33.2.2	19.6.2	23.7.2
17.6.2	unused	18.4.2	33.4.3	19.6.3	23.7.3
17.6.3	22.3.2	18.4.2	33.5.6	19.6.4	unused
17.6.4	22.3.3	18.4.3	33.5.1		
17.6.5	22.3.4	18.4.4	unused		
17.6.6	22.3.5	19.1.1	23.1.1		
17.6.7	22.3.6	19.1.2	23.1.2		
17.6.8	unused	19.1.3	23.1.3		
17.6.9	22.3.7	19.1.4	23.1.4		
17.6.10	unused	19.1.5	23.1.5		
17.6.11	unused	19.1.6	23.1.6		
17.6.12	unused	19.1.7	23.1.7		
17.7.1	22.4.1	19.1.8	23.1.8		
17.7.2	22.4.2	19.1.9	23.1.9		
17.7.3	22.4.3	19.2.1	23.2.2		
17.7.4	22.4.4	19.2.2	23.6.4		
17.7.5	22.4.5	19.2.3	23.2.4		
17.7.6	unused	19.2.4	23.2.5		

Chapter 5

Unused Sixth Edition Exercises

These exercises from the Sixth Edition were not used in the Seventh, for any one of several reasons. A few were adjudged unclear or not sufficiently relevant. Others were made unnecesssary because their subject matter was explicitly discussed in the text, or became redundant because of the selection of similar problems. And some interesting and worthy problems had to be moved on-line due to space limitations of the book.

There has been no attempt to edit these exercises. The exercise numbers, equations, sections, figures, and other exercises referred to are from the Sixth Edition. We supply this material because some instructors may find it useful.

1.1.1 Show how to find $\mathbf{A}$ and $\mathbf{B}$, given $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$.

1.1.3 Calculate the components of a unit vector that lies in the xy-plane and makes equal angles with the positive directions of the x- and y-axes.

1.1.4 The velocity of sailboat A relative to sailboat B, $\mathbf{v}_{\text{rel}}$, is defined by the equation $\mathbf{v}_{\text{rel}} = \mathbf{v}_A - \mathbf{v}_B$, where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_B$ is the velocity of B. Determine the velocity of A relative to B if

$$\begin{aligned}\mathbf{v}_A &= 30 \text{ km/hr east}\\ \mathbf{v}_B &= 40 \text{ km/hr north.}\end{aligned}$$

ANS. $\mathbf{v}_{\text{rel}} = 50$ km/hr, 53.1° south of east.

1.1.5 A sailboat sails for 1 hr at 4 km/hr (relative to the water) on a steady compass heading of 40° east of north. The sailboat is simultaneously carried along by a current. At the end of the hour the boat is 6.12 km from its starting point. The line from its starting point to its location lies 60° east of north. Find the x (easterly) and y (northerly) components of the water's velocity.

ANS. $v_{\text{east}} = 2.73$ km/hr, $v_{\text{north}} \approx 0$ km/hr.

1.1.6 A vector equation can be reduced to the form $\mathbf{A} = \mathbf{B}$. From this show that the one vector equation is equivalent to **three** scalar equations. Assuming the validity of Newton's second law, $\mathbf{F} = m\mathbf{a}$, as a **vector** equation, this means that a_x depends only on F_x and is independent of F_y and F_z.

1.1.7 The vertices A, B, and C of a triangle are given by the points $(-1, 0, 2)$, $(0, 1, 0)$, and $(1, -1, 0)$, respectively. Find point D so that the figure $ABCD$ forms a plane parallelogram.

ANS. $(0, -2, 2)$ or $(2, 0, -2)$.

1.2.1 (a) Show that the magnitude of a vector $\mathbf{A}$, $A = (A_x^2 + A_y^2)^{1/2}$, is independent of the orientation of the rotated coordinate system,

$$(A_x^2 + A_y^2)^{1/2} = (A_x'^2 + A_y'^2)^{1/2}$$

independent of the rotation angle φ.
This independence of angle is expressed by saying that A is **invariant** under rotations.

(b) At a given point (x, y), $\mathbf{A}$ defines an angle α relative to the positive x-axis and α' relative to the positive x'-axis. The angle from x to x' is φ. Show that $\mathbf{A} = \mathbf{A}'$ defines the **same** direction in space when expressed in terms of its primed components as in terms of its unprimed components; that is,

$$\alpha' = \alpha - \varphi.$$

1.2.2 Prove the orthogonality condition $\sum_i a_{ji} a_{ki} = \delta_{jk}$. As a special case of this, the direction cosines of Section 1.1 satisfy the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

a result that also follows from Eq. (1.6).

1.3.1 Two unit magnitude vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ are required to be either parallel or perpendicular to each other. Show that $\mathbf{e}_i \cdot \mathbf{e}_j$ provides an interpretation of Eq. (1.18), the direction cosine orthogonality relation.

1.3.2 Given that (1) the dot product of a unit vector with itself is unity and (2) this relation is valid in all (rotated) coordinate systems, show that $\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}}' = 1$ (with the primed system rotated 45° about the z-axis relative to the unprimed) implies that $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$.

1.3.4 The interaction energy between two dipoles of moments $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ may be written in the vector form

$$V = -\frac{\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{r^3} + \frac{3\,(\boldsymbol{\mu}_1 \cdot \mathbf{r})\,(\boldsymbol{\mu}_2 \cdot \mathbf{r})}{r^5}$$

and in the scalar form

$$V = \frac{\mu_1 \mu_2}{r^3}\,(2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\varphi)\,.$$

Here θ_1 and θ_2 are the angles of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ relative to $\mathbf{r}$, while φ is the azimuth of $\boldsymbol{\mu}_2$ relative to the $\boldsymbol{\mu}_1 - \mathbf{r}$ plane (Fig. 1.11). Show that these two forms are equivalent.
Hint: Equation (12.178) will be helpful.

1.3.7 Prove the law of cosines from the triangle with corners at the point of $\mathbf{C}$ and $\mathbf{A}$ in Fig. 1.10 and the projection of vector $\mathbf{B}$ onto vector $\mathbf{A}$.

1.4.3 Starting with $\mathbf{C} = \mathbf{A} + \mathbf{B}$, show that $\mathbf{C} \times \mathbf{C} = 0$ leads to

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

1.4.4 Show that

(a) $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = A^2 - B^2$,

(b) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B}) = 2\mathbf{A} \times \mathbf{B}$.
The distributive laws needed here,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C},$$

and

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C},$$

may easily be verified (if desired) by expansion in Cartesian components.

1.4.11 The coordinates of the three vertices of a triangle are $(2, 1, 5), (5, 2, 8)$, and $(4, 8, 2)$. Compute its area by vector methods, its center and medians. Lengths are in centimeters.
Hint. See Exercise 1.4.1.

1.4.12 The vertices of parallelogram $ABCD$ are $(1, 0, 0), (2, -1, 0), (0, -1, 1)$, and $(-1, 0, 1)$ in order. Calculate the vector areas of triangle ABD and of triangle BCD. Are the two vector areas equal?

ANS. $\text{Area}_{ABD} = -\frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}})$.

1.4.14 Find the sides and angles of the spherical triangle ABC defined by the three vectors

$$\begin{aligned}\mathbf{A} &= (1,0,0), \\ \mathbf{B} &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \\ \mathbf{C} &= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).\end{aligned}$$

Each vector starts from the origin (Fig. 1.14).

1.4.17 Define a cross product of two vectors in two-dimensional space and give a geometrical interpretation of your construction.

1.4.18 Find the shortest distance between the paths of two rockets in free flight. Take the first rocket path to be $\mathbf{r} = \mathbf{r}_1 + t_1\mathbf{v}_1$ with launch at $\mathbf{r}_1 = (1,1,1)$, velocity $\mathbf{v}_1 = (1,2,3)$ and time parameter t_1 and the second rocket path as $\mathbf{r} = \mathbf{r}_2 + t_2\mathbf{v}_2$ with $\mathbf{r}_2 = (5,2,1)$, $\mathbf{v}_2 = (-1,-1,1)$ and time parameter t_2. Lengths are in kilometers, velocities in kilometers per hour.

1.5.1 One vertex of a glass parallelepiped is at the origin (Fig. 1.18). The three adjacent vertices are at $(3,0,0), (0,0,2)$, and $(0,3,1)$. All lengths are in centimeters. Calculate the number of cubic centimeters of glass in the parallelepiped using the triple scalar product.

1.5.2 Verify the expansion of the triple vector product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

by direct expansion in Cartesian coordinates.

1.5.3 Show that the first step in Eq. (1.43), which is

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2,$$

is consistent with the BAC-CAB rule for a triple vector product.

1.5.5 The orbital angular momentum $\mathbf{L}$ of a particle is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$, where $\mathbf{p}$ is the linear momentum. With linear and angular velocity related by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, show that

$$\mathbf{L} = mr^2 \left[\boldsymbol{\omega} - \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \boldsymbol{\omega}\right)\right].$$

Here $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{r}$-direction. For $\mathbf{r} \cdot \boldsymbol{\omega} = 0$ this reduces to $\mathbf{L} = I\boldsymbol{\omega}$, with the moment of inertia I given by mr^2. In Section 3.5 this result is generalized to form an inertia tensor.

1.5.6 The kinetic energy of a single particle is given by $T = \frac{1}{2}mv^2$. For rotational motion this becomes $\frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r})^2$. Show that

$$T = \frac{1}{2}m\left[r^2\omega^2 - (\mathbf{r} \cdot \boldsymbol{\omega})^2\right].$$

For $\mathbf{r} \cdot \boldsymbol{\omega} = 0$ this reduces to $T = \frac{1}{2}I\omega^2$, with the moment of inertia I given by mr^2.

1.5.11 Vector $\mathbf{D}$ is a linear combination of three noncoplanar (and nonorthogonal) vectors:

$$\mathbf{D} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}.$$

Show that the coefficients are given by a ratio of triple scalar products,

$$a = \frac{\mathbf{D} \cdot \mathbf{B} \times \mathbf{C}}{\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}}, \quad \text{and so on.}$$

1.5.14 For a **spherical** triangle such as pictured in Fig. 1.14 show that

$$\frac{\sin A}{\sin \overline{BC}} = \frac{\sin B}{\sin \overline{CA}} = \frac{\sin C}{\sin \overline{AB}}.$$

Here $\sin A$ is the sine of the included angle at A, while $\overline{BC}$ is the side opposite (in radians).

1.5.15 Given

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \quad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \quad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}},$$

and $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0$, show that

(a) $\mathbf{x} \cdot \mathbf{y}' = \delta_{xy}, (\mathbf{x}, \mathbf{y} = \mathbf{a}, \mathbf{b}, \mathbf{c})$,

(b) $\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}' = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})^{-1}$,

(c) $\mathbf{a} = \dfrac{\mathbf{b}' \times \mathbf{c}'}{\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}'}$.

1.5.16 If $\mathbf{x} \cdot \mathbf{y}' = \delta_{xy}, (\mathbf{x}, \mathbf{y} = \mathbf{a}, \mathbf{b}, \mathbf{c})$, prove that

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}.$$

(This is the converse of Problem 1.5.15.)

1.5.17 Show that any vector $\mathbf{V}$ may be expressed in terms of the reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ (of Problem 1.5.15) by

$$\mathbf{V} = (\mathbf{V} \cdot \mathbf{a})\,\mathbf{a}' + (\mathbf{V} \cdot \mathbf{b})\,\mathbf{b}' + (\mathbf{V} \cdot \mathbf{c})\,\mathbf{c}'.$$

1.7.4 In Chapter 2 it will be seen that the unit vectors in non-Cartesian coordinate systems are usually functions of the coordinate variables, $\mathbf{e}_i = \mathbf{e}_i(q_1, q_2, q_3)$ but $|\mathbf{e}_i| = 1$. Show that either $\partial\mathbf{e}_i/\partial q_j = 0$ or $\partial\mathbf{e}_i/\partial q_j$ is orthogonal to $\mathbf{e}_i$.
Hint. $\partial\mathbf{e_i}^2/\partial q_j = 0$.

1.7.6 The electrostatic field of a point charge q is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{\hat{\mathbf{r}}}{r^2}.$$

Calculate the divergence of $\mathbf{E}$. What happens at the origin?

1.8.1 Show, by rotating the coordinates, that the components of the curl of a vector transform as a vector.
Hint. The direction cosine identities of Eq. (1.46) are available as needed.

1.8.6 If (a) $\mathbf{V} = \hat{\mathbf{x}}V_x(x,y) + \hat{\mathbf{y}}V_y(x,y)$ and (b) $\boldsymbol{\nabla}\times\mathbf{V} \neq 0$, prove that $\boldsymbol{\nabla}\times\mathbf{V}$ is perpendicular to $\mathbf{V}$.

1.8.10 For $\mathbf{A} = \hat{\mathbf{x}}A_x(x,y,z)$ and $\mathbf{B} = \hat{\mathbf{x}}B_x(x,y,z)$ evaluate each term in the vector identity

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})$$

and verify that the identity is satisfied.

1.8.18 The velocity of a two-dimensional flow of liquid is given by

$$\mathbf{V} = \hat{\mathbf{x}}u(x,y) - \hat{\mathbf{y}}v(x,y).$$

If the liquid is incompressible and the flow is irrotational, show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the Cauchy-Riemann conditions of Section 6.2.

1.8.19 The evaluation in this section of the four integrals for the circulation omitted Taylor series terms such as $\partial V_x/\partial x$, $\partial V_y/\partial y$ and all second derivatives. Show that $\partial V_x/\partial x$, $\partial V_y/\partial y$ cancel out when the four integrals are added and that the second-derivative terms drop out in the limit as $dx \to 0$, $dy \to 0$.
Hint. Calculate the circulation per unit area and then take the limit $dx \to 0$, $dy \to 0$.

1.9.2 Show that the identity

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V}) = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{V} - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{V}$$

follows from the *BAC*-*CAB* rule for a triple vector product. Justify any alteration of the order of factors in the *BAC* and *CAB* terms.

1.9.6 From the Navier-Stokes equation for the steady flow of an incompressible viscous fluid we have the term

$$\nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})],$$

where $\mathbf{v}$ is the fluid velocity. Show that this term vanishes for the special case

$$\mathbf{v} = \hat{\mathbf{x}} v(y, z).$$

1.9.9 With ψ a scalar (wave) function, show that

$$(\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla)\psi = r^2 \nabla^2 \psi - r^2 \frac{\partial^2 \psi}{\partial r^2} - 2r \frac{\partial \psi}{\partial r}.$$

(This can actually be shown more easily in spherical polar coordinates, Section 2.5.)

1.9.10 In a (nonrotating) isolated mass such as a star, the condition for equilibrium is

$$\nabla P + \rho \nabla \varphi = 0.$$

Here P is the total pressure, ρ is the density, and φ is the gravitational potential. Show that at any given point the normals to the surfaces of constant pressure and constant gravitational potential are parallel.

1.9.11 In the Pauli theory of the electron, one encounters the expression

$$(\mathbf{p} - e\mathbf{A}) \times (\mathbf{p} - e\mathbf{A})\psi,$$

where ψ is a scalar (wave) function. $\mathbf{A}$ is the magnetic vector potential related to the magnetic induction $\mathbf{B}$ by $\mathbf{B} = \nabla \times \mathbf{A}$. Given that $\mathbf{p} = -i\nabla$, show that this expression reduces to $ie\mathbf{B}\psi$. Show that this leads to the orbital g-factor $g_L = 1$ upon writing the magnetic moment as $\boldsymbol{\mu} = g_L \mathbf{L}$ in units of Bohr magnetons and $\mathbf{L} = -i\mathbf{r} \times \nabla$. See also Exercise 1.13.7.

1.10.1 The force field acting on a two-dimensional linear oscillator may be described by

$$\mathbf{F} = -\hat{\mathbf{x}} kx - \hat{\mathbf{y}} ky.$$

Compare the work done moving against this force field when going from $(1, 1)$ to $(4, 4)$ by the following straight-line paths:

(a) $(1, 1) \to (4, 1) \to (4, 4)$

(b) $(1, 1) \to (1, 4) \to (4, 4)$

(c) $(1, 1) \to (4, 4)$ along $x = y$.
This means evaluating

$$-\int_{(1,1)}^{(4,4)} \mathbf{F} \cdot d\mathbf{r}$$

along each path.

1.10.6 Show, by expansion of the surface integral, that

$$\lim_{\int d\tau \to 0} \frac{\int_s d\boldsymbol{\sigma} \times \mathbf{V}}{\int d\tau} = \boldsymbol{\nabla} \times \mathbf{V}.$$

Hint. Choose the volume $\int d\tau$ to be a differential volume $dx\,dy\,dz$.

1.11.4 Over some volume V let ψ be a solution of Laplace's equation (with the derivatives appearing there continuous). Prove that the integral over any closed surface in V of the normal derivative of ψ ($\partial\psi/\partial n$, or $\boldsymbol{\nabla}\psi \cdot \mathbf{n}$) will be zero.

1.11.5 In analogy to the integral definition of gradient, divergence, and curl of Section 1.10, show that

$$\boldsymbol{\nabla}^2 \varphi = \lim_{\int d\tau \to 0} \frac{\int \boldsymbol{\nabla}\varphi \cdot d\boldsymbol{\sigma}}{\int d\tau}.$$

1.11.6 The electric displacement vector $\mathbf{D}$ satisfies the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$, where ρ is the charge density (per unit volume). At the boundary between two media there is a surface charge density σ (per unit area). Show that a boundary condition for $\mathbf{D}$ is

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma.$$

$\mathbf{n}$ is a unit vector normal to the surface and out of medium 1.
Hint. Consider a thin pillbox as shown in Fig. 1.29.

1.11.9 The creation of a **localized** system of steady electric currents (current density $\mathbf{J}$) and magnetic fields may be shown to require an amount of work

$$W = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B}\, d\tau.$$

Transform this into

$$W = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A}\, d\tau.$$

Here $\mathbf{A}$ is the magnetic vector potential: $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$.
Hint. In Maxwell's equations take the displacement current term $\partial\mathbf{D}/\partial t = 0$. If the fields and currents are localized, a bounding surface may be taken far enough out so that the integrals of the fields and currents over the surface yield zero.

1.11.10 Prove the generalization of Green's theorem:

$$\iiint_V (v\mathcal{L}u - u\mathcal{L}v)\, d\tau = \oint_{\partial V} p\,(v\boldsymbol{\nabla}u - u\boldsymbol{\nabla}v) \cdot d\boldsymbol{\sigma}.$$

Here $\mathcal{L}$ is the self-adjoint operator (Section 10.1),

$$\mathcal{L} = \boldsymbol{\nabla} \cdot [p(\mathbf{r})\boldsymbol{\nabla}] + q(\mathbf{r})$$

and p, q, u, and v are functions of position, p and q having continuous first derivatives and u and v having continuous second derivatives.
Note. This generalized Green's theorem appears in Section 9.7.

1.12.4 In steady state the magnetic field $\mathbf{H}$ satisfies the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}$, where $\mathbf{J}$ is the current density (per square meter). At the boundary between two media there is a surface current density $\mathbf{K}$. Show that a boundary condition on $\mathbf{H}$ is

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}.$$

$\mathbf{n}$ is a unit vector normal to the surface and out of medium 1.
Hint. Consider a narrow loop perpendicular to the interface as shown in the Fig. 1.31

1.12.5 From Maxwell's equations, $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}$, with $\mathbf{J}$ here the current density and $\mathbf{E} = 0$. Show from this that

$$\oint \mathbf{H} \cdot d\mathbf{r} = I,$$

where I is the net electric current enclosed by the loop integral. These are the differential and integral forms of Ampère's law of magnetism.

1.12.6 A magnetic induction $\mathbf{B}$ is generated by electric current in a ring of radius R. Show that the **magnitude** of the vector potential $\mathbf{A}$ ($\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$) at the ring can be

$$|\mathbf{A}| = \frac{\varphi}{2\pi R},$$

where φ is the total magnetic flux passing through the ring.
Note. $\mathbf{A}$ is tangential to the ring and may be changed by adding the gradient of a scalar function.

1.12.7 Prove that

$$\int_S \boldsymbol{\nabla} \times \mathbf{V} \cdot d\boldsymbol{\sigma} = 0$$

if S is a closed surface.

1.12.8 Evaluate $\oint \mathbf{r} \cdot d\mathbf{r}$ (Exercise 1.10.4) by Stokes' theorem.

1.13.3 The usual problem in classical mechanics is to calculate the motion of a particle given the potential. For a uniform density (ρ_0), nonrotating massive sphere, Gauss' law of Section 1.14 leads to a gravitational force on a unit mass m_0 at a point r_0 produced by the attraction of the mass at $r \le r_0$. The mass at $r > r_0$ contributes nothing to the force.

(a) Show that $\mathbf{F}/m_0 = -(4\pi G\rho_0/3)\mathbf{r}$, $0 \le r \le a$, where a is the radius of the sphere.

(b) Find the corresponding gravitational potential, $0 \le r \le a$.

(c) Imagine a vertical hole running completely through the center of the Earth and out to the far side. Neglecting the rotation of the Earth and assuming a uniform density $\rho_0 = 5.5$ gm/cm^3, calculate the nature of the motion of a particle dropped into the hole. What is its period?
Note. $\mathbf{F} \propto \mathbf{r}$ is actually a very poor approximation. Because of varying density, the approximation $\mathbf{F}$ = constant along the outer half of a radial line and $\mathbf{F} \propto \mathbf{r}$ along the inner half is a much closer approximation.

1.13.10 With $\mathbf{E}$ the electric field and $\mathbf{A}$ the magnetic vector potential, show that $[\mathbf{E} + \partial\mathbf{A}/\partial t]$ is irrotational and that therefore we may write

$$E = -\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial t}.$$

1.13.11 The total force on a charge q moving with velocity $\mathbf{v}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Using the scalar and vector potentials, show that

$$\mathbf{F} = q\left[-\boldsymbol{\nabla}\varphi - \frac{d\mathbf{A}}{dt} + \boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{v})\right].$$

Note that we now have a total time derivative of $\mathbf{A}$ in place of the partial derivative of Exercise 1.13.10.

1.14.1 Develop Gauss' law for the two-dimensional case in which

$$\varphi = -q\frac{\ln \rho}{2\pi\varepsilon_0}, \quad \mathbf{E} = -\boldsymbol{\nabla}\varphi = q\frac{\hat{\boldsymbol{\rho}}}{2\pi\varepsilon_0\rho}.$$

Here q is the charge at the origin or the line charge per unit length if the two-dimensional system is a unit thickness slice of a three-dimensional (circular cylindrical) system. The variable ρ is measured radially outward from the line charge. $\hat{\boldsymbol{\rho}}$ is the corresponding unit vector (see Section 2.4).

1.14.2 (a) Show that Gauss' law follows from Maxwell's equation

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Here ρ is the usual charge density.

(b) Assuming that the electric field of a point charge q is spherically symmetric, show that Gauss' law implies the Coulomb inverse square expression

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}.$$

1.15.2 Verify that the sequence $\delta_n(x)$, based on the function

$$\delta_n(x) = \begin{cases} 0, & x < 0, \\ ne^{-nx} & x > 0, \end{cases}$$

is a delta sequence (satisfying Eq. (1.177)). Note that the singularity is at $+0$, the positive side of the origin.
Hint. Replace the upper limit (∞) by c/n, where c is large but finite, and use the mean value theorem of integral calculus.

1.15.4 Demonstrate that $\delta_n = \sin nx/\pi x$ is a delta distribution by showing that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \frac{\sin nx}{\pi x}\, dx = f(0).$$

Assume that $f(x)$ is continuous at $x = 0$ and vanishes as $x \to \pm\infty$.
Hint. Replace x by y/n and take $\lim n \to \infty$ **before** integrating.

1.15.11 Show that in spherical polar coordinates $(r, \cos\theta, \varphi)$ the delta function $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ becomes

$$\frac{1}{r_1^2}\delta(r_1 - r_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\varphi_1 - \varphi_2).$$

Generalize this to the curvilinear coordinates (q_1, q_2, q_3) of Section 2.1 with scale factors h_1, h_2, and h_3.

1.15.12 A rigorous development of Fourier transforms [1] includes as a theorem the relations

$$\lim_{a\to\infty} \frac{2}{\pi} \int_{x_1}^{x_2} f(u + x) \frac{\sin ax}{x}\, dx$$
$$= \begin{cases} f(u+0) + f(u-0), & x_1 < 0 < x_2 \\ f(u+0), & x_1 = 0 < x_2 \\ f(u-0), & x_1 < 0 = x_2 \\ 0 & x_1 < x_2 < 0 \text{ or } 0 < x_1 < x_2. \end{cases}$$

Verify these results using the Dirac delta function.

1.15.14 Show that the unit step function $u(x)$ may be represented by

$$u(x) = \frac{1}{2} + \frac{1}{2\pi i} P \int_{-\infty}^{\infty} e^{ixt} \frac{dt}{t},$$

where P means Cauchy principal value (Section 7.1).

[1] I. N. Sneddon, *Fourier Transforms*. New York: McGraw-Hill (1951).

1.15.15 As a variation of Eq. (1.174), take

$$\delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt-|t|/n}\,dt.$$

Show that this reduces to $(n/\pi)1/(1+n^2x^2)$, Eq. (1.173), and that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1.$$

Note. In terms of integral transforms, the initial equation here may be interpreted as either a Fourier exponential transform of $e^{-|t|/n}$ or a Laplace transform of e^{ixt}.

1.15.16 (a) The Dirac delta function representation given by Eq. (1.189)

$$\delta(x-t) = \sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(t)$$

is often called the **closure relation**. For an orthonormal set of real functions, φ_n, show that closure implies completeness, that is, Eq. (1.190) follows from Eq. (1.189).
Hint. One can take

$$F(x) = \int F(t)\delta(x-t)\,dt.$$

(b) Following the hint of part (a) you encounter the integral $\int F(t)\varphi_n(t)\,dt$. How do you know that this integral is finite?

1.15.17 For the finite interval $(-\pi,\pi)$ write the Dirac delta function $\delta(x-t)$ as a series of sines and cosines: $\sin nx, \cos nx, n = 0,1,2,\ldots$. Note that although these functions are orthogonal, they are not normalized to unity.

1.15.18 In the interval $(-\pi,\pi)$, $\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$.

(a) Write $\delta_n(x)$ as a Fourier cosine series.

(b) Show that your Fourier series agrees with a Fourier expansion of $\delta(x)$ in the limit as $n\to\infty$.

(c) Confirm the delta function nature of your Fourier series by showing that for any $f(x)$ that is finite in the interval $[-\pi,\pi]$ and continuous at $x=0$,

$$\int_{-\pi}^{\pi} f(x)[\text{Fourier expansion of } \delta_\infty(x)]\,dx = f(0).$$

1.15.19 (a) Write $\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$ in the interval $(-\infty,\infty)$ as a Fourier integral and compare the limit $n\to\infty$ with Eq. (1.192c).

(b) Write $\delta_n(x) = n\exp(-nx)$ as a Laplace transform and compare the limit $n \to \infty$ with Eq. (1.194).

Hint. See Eqs. (15.22) and (15.23) for (a) and Eq. (15.212) for (b).

1.15.20 (a) Show that the Dirac delta function $\delta(x-a)$, expanded in a Fourier sine series in the half-interval $(0, L)$, $(0 < a < L)$, is given by

$$\delta(x-a) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Note that this series actually describes

$$-\delta(x+a) + \delta(x-a) \text{ in the interval } (-L, L).$$

(b) By integrating both sides of the preceding equation from 0 to x, show that the cosine expansion of the square wave

$$f(x) = \begin{cases} 0, & 0 \le x < a \\ 1, & a < x < L, \end{cases}$$

is, for $0 \le x < L$,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi a}{L}\right) \cos\left(\frac{n\pi x}{L}\right).$$

(c) Verify that the term

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi a}{L}\right) \quad \text{is} \quad \langle f(x)\rangle \equiv \frac{1}{L}\int_0^L f(x)\,dx.$$

1.15.21 Verify the Fourier cosine expansion of the square wave, Exercise 1.15.20(b), by direct calculation of the Fourier coefficients.

1.15.22 We may define a sequence

$$\delta_n(x) = \begin{cases} n, & |x| < 1/2n, \\ 0, & |x| > 1/2n. \end{cases}$$

(This is Eq. (1.171).) Express $\delta_n(x)$ as a Fourier integral (via the Fourier integral theorem, inverse transform, etc.). Finally, show that we may write

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk.$$

1.15.23 Using the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2),$$

show that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk.$$

Note. Remember that $\delta(x)$ is defined in terms of its behavior as part of an integrand — especially Eqs. (1.177) and (1.188).

1.15.24 Derive sine and cosine representations of $\delta(t-x)$ that are comparable to the exponential representation, Eq. (1.192*c*).

ANS. $\frac{2}{\pi}\int_0^\infty \sin\omega t \sin\omega x\,d\omega,\ \frac{2}{\pi}\int_0^\infty \cos\omega t\cos\omega x\,d\omega.$

1.16.1 Implicit in this section is a proof that a function $\psi(\mathbf{r})$ is uniquely specified by requiring it to (1) satisfy Laplace's equation and (2) satisfy a complete set of boundary conditions. Develop this proof explicitly.

1.16.2 (a) Assuming that $\mathbf{P}$ is a solution of the vector Poisson equation, $\nabla_1^2\mathbf{P}(\mathbf{r}_1) = -\mathbf{V}(\mathbf{r}_1)$, develop an alternate proof of Helmholtz's theorem, showing that $\mathbf{V}$ may be written as

$$\mathbf{V} = -\nabla\varphi + \nabla\times\mathbf{A},$$

where

$$\mathbf{A} = \nabla\times\mathbf{P},$$

and

$$\varphi = \nabla\cdot\mathbf{P}.$$

(b) Solving the vector Poisson equation, we find

$$\mathbf{P}(\mathbf{r}_1) = \frac{1}{4\pi}\int_V \frac{\mathbf{V}(\mathbf{r}_2)}{r_{12}}d\tau_2.$$

Show that this solution substituted into φ and $\mathbf{A}$ of part (a) leads to the expressions given for φ and $\mathbf{A}$ in Section 1.16.

2.1.1 Show that limiting our attention to orthogonal coordinate systems implies that $g_{ij} = 0$ for $i \neq j$ (Eq. (2.7)).

Hint. Construct a triangle with sides ds_1, ds_2, and ds_2. Equation (2.9) must hold regardless of whether $g_{ij} = 0$. Then compare ds^2 from Eq. (2.5) with a calculation using the law of cosines. Show that $\cos\theta_{12} = g_{12}/\sqrt{g_{11}g_{22}}$.

2.1.2 In the spherical polar coordinate system, $q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$. The transformation equations corresponding to Eq. (2.1) are

$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\quad z = r\cos\theta.$$

(a) Calculate the spherical polar coordinate scale factors: h_r, h_θ, and h_φ.

(b) Check your calculated scale factors by the relation $ds_i = h_i\, dq_i$.

2.1.6 In Minkowski space we define $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_0 = ct$. This is done so that the metric interval becomes $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ (with $c =$ velocity of light). Show that the metric in Minkowski space is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We use Minkowski space in Sections 4.5 and 4.6 for describing Lorentz transformations.

2.2.4 Derive

$$\boldsymbol{\nabla}\psi = \hat{\mathbf{q}}_1 \frac{1}{h_1}\frac{\partial\psi}{\partial q_1} + \hat{\mathbf{q}}_2 \frac{1}{h_2}\frac{\partial\psi}{\partial q_2} + \hat{\mathbf{q}}_3 \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}$$

by direct application of Eq. (1.97),

$$\boldsymbol{\nabla}\psi = \lim_{\int d\tau \to 0} \frac{\int \psi\, d\boldsymbol{\sigma}}{\int d\tau}.$$

Hint. Evaluation of the surface integral will lead to terms like $(h_1 h_2 h_3)^{-1}\, (\partial/\partial q_1)(\hat{\mathbf{q}}_1 h_2 h_3)$. The results listed in Exercise 2.2.3 will be helpful. Cancellation of unwanted terms occurs when the contributions of all three pairs of surfaces are added together.

2.4.9 Solve Laplace's equation, $\boldsymbol{\nabla}^2\psi = 0$, in cylindrical coordinates for $\psi = \psi(\rho)$.

ANS. $\quad \psi = k \ln \dfrac{\rho}{\rho_0}.$

2.4.11 For the flow of an incompressible viscous fluid the Navier-Stokes equations lead to

$$-\boldsymbol{\nabla} \times (\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})) = \frac{\eta}{\rho_0} \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \times \mathbf{v}).$$

Here η is the viscosity and ρ_0 is the density of the fluid. For axial flow in a cylindrical pipe we take the velocity $\mathbf{v}$ to be

$$\mathbf{v} = \hat{\mathbf{z}} v(\rho).$$

From Example 2.4.2,

$$\boldsymbol{\nabla} \times (\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})) = 0$$

for this choice of $\mathbf{v}$.

Show that

$$\nabla^2(\nabla \times \mathbf{v}) = 0$$

leads to the differential equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d^2v}{d\rho^2}\right) - \frac{1}{\rho^2}\frac{dv}{d\rho} = 0$$

and that this is satisfied by

$$v = v_0 + a_2\rho^2.$$

2.4.14 A transverse electromagnetic wave (TEM) in a coaxial waveguide has an electric field $\mathbf{E} = \mathbf{E}(\rho,\varphi)e^{i(kz-\omega t)}$ and a magnetic induction field of $\mathbf{B} = \mathbf{B}(\rho,\varphi)e^{i(kz-\omega t)}$. Since the wave is transverse, neither $\mathbf{E}$ nor $\mathbf{B}$ has a z component. The two fields satisfy the **vector** Laplacian equation

$$\begin{aligned} \nabla^2\mathbf{E}(\rho,\varphi) &= 0 \\ \nabla^2\mathbf{B}(\rho,\varphi) &= 0. \end{aligned}$$

(a) Show that $\mathbf{E} = \hat{\boldsymbol{\rho}}E_0(a/\rho)e^{i(kz-\omega t)}$ and $\mathbf{B} = \hat{\boldsymbol{\varphi}}B_0(a/\rho)e^{i(kz-\omega t)}$ are solutions. Here a is the radius of the inner conductor and E_0 and B_0 are constant amplitudes.

(b) Assuming a vacuum inside the waveguide, verify that Maxwell's equations are satisfied with

$$B_0/E_0 = k/\omega = \mu_0\varepsilon_0(\omega/k) = 1/c.$$

2.4.16 The linear velocity of particles in a rigid body rotating with angular velocity ω is given by

$$\mathbf{v} = \hat{\boldsymbol{\varphi}}\rho\omega.$$

Integrate $\oint \mathbf{v}\cdot d\boldsymbol{\lambda}$ around a circle in the xy-plane and verify that

$$\frac{\oint \mathbf{v}\cdot d\boldsymbol{\lambda}}{\text{area}} = \nabla\times\mathbf{v}|_z.$$

2.4.17 A proton of mass m, charge $+e$, and (asymptotic) momentum $p = mv$ is incident on a nucleus of charge $+Ze$ at an impact parameter b. Determine the proton's distance of closest approach.

2.5.6 The direction of one vector is given by the angles θ_1 and φ_1. For a second vector the corresponding angles are θ_2 and φ_2. Show that the cosine of the included angle γ is given by

$$\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2).$$

See Fig. 12.15.

2.5.11 A particle m moves in response to a central force according to Newton's second law,

$$m\ddot{\mathbf{r}} = \hat{\mathbf{r}} f(r).$$

Show that $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c}$, a constant, and that the geometric interpretation of this leads to Kepler's second law.

2.5.19 One model of the solar corona assumes that the steady-state equation of heat flow,

$$\boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T) = 0,$$

is satisfied. Here, k, the thermal conductivity, is proportional to $T^{5/2}$. Assuming that the temperature T is proportional to r^n, show that the heat flow equation is satisfied by $T = T_0 (r_0/r)^{2/7}$.

2.5.22 A magnetic vector potential is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}.$$

Show that this leads to the magnetic induction $\mathbf{B}$ of a point magnetic dipole with dipole moment $\mathbf{m}$.

$$\begin{aligned} ANS. \quad \text{for } \mathbf{m} &= \hat{\mathbf{z}} m, \\ \boldsymbol{\nabla} \times \mathbf{A} &= \hat{\mathbf{r}} \frac{\mu_0}{4\pi} \frac{2m \cos\theta}{r^3} + \hat{\boldsymbol{\theta}} \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^3}. \end{aligned}$$

Compare Eqs. (12.133) and (12.134)

2.5.23 At large distances from its source, electric dipole radiation has fields

$$\mathbf{E} = a_E \sin\theta \frac{e^{i(kr-\omega t)}}{r} \hat{\boldsymbol{\theta}}, \qquad \mathbf{B} = a_B \sin\theta \frac{e^{i(kr-\omega t)}}{r} \hat{\boldsymbol{\varphi}}.$$

Show that Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{and} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

are satisfied, if we take

$$\frac{a_E}{a_B} = \frac{\omega}{k} = c = (\varepsilon_0 \mu_0)^{-1/2}.$$

Hint. Since r is large, terms of order r^{-2} may be dropped.

2.5.24 The magnetic vector potential for a uniformly charged rotating spherical shell is

$$\mathbf{A} = \begin{cases} \hat{\boldsymbol{\varphi}} \dfrac{\mu_0 a^4 \sigma \omega}{3} \cdot \dfrac{\sin\theta}{r^2}, & r > a \\ \hat{\boldsymbol{\varphi}} \dfrac{\mu_0 a \sigma \omega}{3} \cdot r \cos\theta, & r < a. \end{cases}$$

(a = radius of spherical shell, σ= surface charge density, and ω= angular velocity.) Find the magnetic induction $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

$$
\begin{aligned}
ANS. \quad B_r(r,\theta) &= \frac{2\mu_0 a^4\sigma\omega}{3}\cdot\frac{\cos\theta}{r^3}, \quad r>a\\
B_\theta(r,\theta) &= \frac{\mu_0 a^4\sigma\omega}{3}\cdot\frac{\sin\theta}{r^3}, \quad r>a\\
\mathbf{B} &= \hat{\mathbf{z}}\frac{2\mu_0 a\sigma\omega}{3}, \quad r<a.
\end{aligned}
$$

2.5.25 (a) Explain why ∇^2 in plane polar coordinates follows from ∇^2 in circular cylindrical coordinates with $z =$ constant.

(b) Explain why taking ∇^2 in spherical polar coordinates and restricting θ to $\pi/2$ does **not** lead to the plane polar form of ∇.

Note.

$$\nabla^2(\rho,\varphi) = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}.$$

2.8.3 The exponential in a plane wave is $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$. We recognize $x^\mu = (ct, x_1, x_2, x_3)$ as a prototype vector in Minkowski space. If $\mathbf{k}\cdot\mathbf{r}-\omega t$ is a scalar under Lorentz transformations (Section 4.5), show that $k^\mu = (\omega/c, k_1, k_2, k_3)$ is a vector in Minkowski space.

Note. Multiplication by $\hbar$ yields $(E/c, \mathbf{p})$ as a vector in Minkowski space.

2.9.5 (a) Express the components of a cross-product vector $\mathbf{C}$, $\mathbf{C} = \mathbf{A}\times\mathbf{B}$, in terms of ε_{ijk} and the components of $\mathbf{A}$ and $\mathbf{B}$.

(b) Use the antisymmetry of ε_{ijk} to show that $\mathbf{A}\cdot\mathbf{A}\times\mathbf{B} = 0$.

ANS. (a) $C_i = \varepsilon_{ijk}A_jB_k$.

2.9.6 (a) Show that the inertia tensor (matrix) may be written

$$I_{ij} = m(x_ix_j\delta_{ij} - x_ix_j)$$

for a particle of mass m at (x_1, x_2, x_3).

(b) Show that

$$I_{ij} = -M_{il}M_{lj} = -m\varepsilon_{ilk}x_k\varepsilon_{ljm}x_m,$$

where $M_{il} = m^{1/2}\varepsilon_{ilk}x_k$. This is the contraction of two second-rank tensors and is identical with the matrix product of Section 3.2.

2.9.8 Expressing cross products in terms of Levi-Civita symbols (ε_{ijk}), derive the *BAC-CAB* rule, Eq. (1.55).

Hint. The relation of Exercise 2.9.4 is helpful.

2.9.13 Show that the vector identity

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

(Exercise 1.5.12) follows directly from the description of a cross product with ε_{ijk} and the identity of Exercise 2.9.4.

2.9.14 Generalize the cross product of two vectors to n-dimensional space for $n = 4, 5, \ldots$. Check the consistency of your construction and discuss concrete examples. See Exercise 1.4.17 for the case $n = 2$.

2.10.1 Equations (2.115) and (2.116) use the scale factor h_i, citing Exercise 2.2.3. In Section 2.2 we had restricted ourselves to orthogonal coordinate systems, yet Eq. (2.115) holds for nonorthogonal systems. Justify the use of Eq. (2.115) for nonorthogonal systems.

2.10.2 (a) Show that $\boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}_j = \delta^i_j$.

(b) From the result of part (a) show that

$$F^i = \mathbf{F} \cdot \boldsymbol{\varepsilon}^i \qquad \text{and} \qquad F_i = \mathbf{F} \cdot \boldsymbol{\varepsilon}_i.$$

2.10.4 Prove that the contravariant metric tensor is given by

$$g^{ij} = \boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}^j.$$

2.10.7 Transform the right-hand side of Eq. (2.129),

$$\boldsymbol{\nabla}\psi = \frac{\partial \psi}{\partial q^i}\boldsymbol{\varepsilon}^i,$$

into the $\mathbf{e}_i$ basis, and verify that this expression agrees with the gradient developed in Section 2.2 (for orthogonal coordinates).

2.10.8 Evaluate $\partial \boldsymbol{\varepsilon}_i / \partial q^j$ for spherical polar coordinates, and from these results calculate Γ^k_{ij} for spherical polar coordinates.

Note. Exercise 2.5.2 offers a way of calculating the needed partial derivatives. Remember,

$$\boldsymbol{\varepsilon}_1 = \hat{\mathbf{r}} \qquad \text{but} \quad \boldsymbol{\varepsilon}_2 = r\hat{\boldsymbol{\theta}} \qquad \text{and} \qquad \boldsymbol{\varepsilon}_3 = r \sin\theta \hat{\boldsymbol{\varphi}}.$$

2.10.13 A triclinic crystal is described using an oblique coordinate system. The three covariant base vectors are

$$\begin{aligned} \boldsymbol{\varepsilon}_1 &= 1.5\hat{\mathbf{x}}, \\ \boldsymbol{\varepsilon}_2 &= 0.4\hat{\mathbf{x}} + 1.6\hat{\mathbf{y}}, \end{aligned}$$

$$\boldsymbol{\varepsilon}_3 = 0.2\hat{\mathbf{x}} + 0.3\hat{\mathbf{y}} + 1.0\hat{\mathbf{z}}.$$

(a) Calculate the elements of the covariant metric tensor g_{ij}.

(b) Calculate the Christoffel three-index symbols, Γ^k_{ij}. (This is a "by inspection" calculation.)

(c) From the cross-product form of Exercise 2.10.3 calculate the contravariant base vector $\boldsymbol{\varepsilon}^3$.

(d) Using the explicit forms $\boldsymbol{\varepsilon}^3$ and $\boldsymbol{\varepsilon}_i$, verify that $\boldsymbol{\varepsilon}^3 \cdot \boldsymbol{\varepsilon}_i = \delta^3{}_i$.

Note. If it were needed, the contravariant metric tensor could be determined by finding the inverse of g_{ij} or by finding the $\boldsymbol{\varepsilon}^i$ and using $g^{ij} = \boldsymbol{\varepsilon}^i \cdot \boldsymbol{\varepsilon}^j$.

2.10.14 Verify that

$$[ij, k] = \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right\}.$$

Hint. Substitute Eq. (2.135) into the right-hand side and show that an identity results.

2.10.16 Show that parallel displacement $\delta dq^i = d^2 q^i$ along a geodesic. Construct a geodesic by parallel displacement of δdq^i.

2.10.17 Construct the covariant derivative of a vector V^i by parallel transport starting from the limiting procedure

$$\lim_{dq^j \to 0} \frac{V^i(q^j + dq^j) - V^i(q^j)}{dq^j}$$

.

2.11.1 Verify Eq. (2.160),

$$\frac{\partial g}{\partial q^k} = g g^{im} \frac{\partial g_{im}}{\partial q^k},$$

for the specific case of spherical polar coordinates.

3.1.4 Express the **components** of $\mathbf{A} \times \mathbf{B}$ as 2×2 determinants. Then show that the dot product $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$ yields a Laplacian expansion of a 3×3 determinant. Finally, note that two rows of the 3×3 determinant are identical and hence that $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$.

3.1.8 Solve the linear equations $\mathbf{a} \cdot \mathbf{x} = c$, $\mathbf{a} \times \mathbf{x} + \mathbf{b} = 0$ for $\mathbf{x} = (x_1, x_2, x_3)$ with constant vectors $\mathbf{a} \neq 0, \mathbf{b}$ and constant c.

ANS. $\mathbf{x} = \frac{c}{a^2}\mathbf{a} + (\mathbf{a} \times \mathbf{b})/a^2$.

3.1.9 Solve the linear equations $\mathbf{a} \cdot \mathbf{x} = d, \mathbf{b} \cdot \mathbf{x} = e, \mathbf{c} \cdot \mathbf{x} = f$, for $\mathbf{x} = (x_1, x_2, x_3)$ with constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and constants d, e, f such that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \neq 0$.

ANS. $[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{x} = d(\mathbf{b} \times \mathbf{c}) + e(\mathbf{c} \times \mathbf{a}) + f(\mathbf{a} \times \mathbf{b})$.

3.1.10 Express in vector form the solution (x_1, x_2, x_3) of $\mathbf{a}x_1 + \mathbf{b}x_2 + \mathbf{c}x_3 + \mathbf{d} = 0$ with constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ so that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \neq 0$.

3.2.3 Show that matrix A is a **linear operator** by showing that

$$\mathsf{A}(c_1\mathbf{r}_1 + c_2\mathbf{r}_2) = c_1\mathsf{A}\mathbf{r}_1 + c_2\mathsf{A}\mathbf{r}_2.$$

It can be shown that an $n \times n$ matrix is the **most general** linear operator in an n-dimensional vector space. This means that every linear operator in this n-dimensional vector space is equivalent to a matrix.

3.2.7 Given the three matrices

$$\mathsf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

find all possible products of A, B, and C, two at a time, including squares. Express your answers in terms of A, B, and C, and 1, the unit matrix. These three matrices, together with the unit matrix, form a representation of a mathematical group, the **vierergruppe** (see Chapter 4).

3.2.19 An operator P commutes with J_x and J_y, the x and y components of an angular momentum operator. Show that P commutes with the third component of angular momentum, that is, that

$$[\mathsf{P}, \mathsf{J}_z] = 0.$$

Hint. The angular momentum components must satisfy the commutation relation of Exercise 3.2.15(a).

3.2.20 The L^+ and L^- matrices of Exercise 3.2.15 are **ladder operators** (see Chapter 4): L^+ operating on a system of spin projection m will raise the spin projection to $m+1$ if m is below its maximum. L^+ operating on $m_{\max}$ yields zero. L^- reduces the spin projection in unit steps in a similar fashion. Dividing by $\sqrt{2}$, we have

$$\mathsf{L}^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{L}^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that

$$\mathsf{L}^+|-1\rangle = |0\rangle, \mathsf{L}^-|-1\rangle = \text{null column vector},$$
$$\mathsf{L}^+|0\rangle = |1\rangle, \mathsf{L}^-|0\rangle = |-1\rangle,$$
$$\mathsf{L}^+|1\rangle = \text{null column vector}, \mathsf{L}^-|1\rangle = |0\rangle,$$

where

$$|-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

represent states of spin projection $-1, 0$, and 1, respectively.
Note. Differential operator analogs of these ladder operators appear in Exercise 12.6.7.

3.2.21 Vectors $\mathbf{A}$ and $\mathbf{B}$ are related by the tensor T,

$$\mathbf{B} = \mathsf{T}\mathbf{A}.$$

Given $\mathbf{A}$ and $\mathbf{B}$, show that there is **no unique solution** for the components of T. This is why vector division $\mathbf{B}/\mathbf{A}$ is undefined (apart from the special case of $\mathbf{A}$ and $\mathbf{B}$ parallel and T then a scalar).

3.2.22 We might ask for a vector $\mathbf{A}^{-1}$, an inverse of a given vector $\mathbf{A}$ in the sense that

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = 1.$$

Show that this relation does not suffice to define $\mathbf{A}^{-1}$ uniquely; $\mathbf{A}$ would then have an infinite number of inverses.

3.2.27 (a) The operator trace replaces a matrix A by its trace; that is,

$$\text{trace}(\mathsf{A}) = \sum_i a_{ii}.$$

Show that trace is a **linear** operator.

(b) The operator det replaces a matrix A by its determinant; that is,

$$\det(\mathsf{A}) = \text{determinant of } \mathsf{A}.$$

Show that det is **not** a linear operator.

3.2.29 With $|x>$ an N-dimensional column vector and $<y|$ an N-dimensional row vector, show that

$$\text{trace}(|x\rangle\langle y|) = \langle y|x\rangle.$$

Note. $|x\rangle\langle y|$ means direct product of column vector $|x\rangle$ with row vector $\langle y|$. The result is a square $N \times N$ matrix.

3.2.31 If a matrix has an inverse, show that the inverse is unique.

3.2.33 Show that $\det \mathsf{A}^{-1} = (\det \mathsf{A})^{-1}$.
Hint. Apply the product theorem of Section 3.2.
Note. If $\det \mathsf{A}$ is zero, then A has no inverse. A is singular.

3.2.37 (a) Rewrite Eq. (2.4) of Chapter 2 (and the corresponding equations for dy and dz) as a single matrix equation

$$|dx_k\rangle = \mathsf{J}|dq_j\rangle.$$

J is a matrix of derivatives, the **Jacobian** matrix. Show that

$$\langle dx_k|dx_k\rangle = \langle dq_i|\mathsf{G}|dq_j\rangle,$$

with the metric (matrix) G having elements g_{ij} given by Eq. (2.6).

(b) Show that

$$\det(\mathsf{J})dq_1dq_2dq_3 = dxdydz,$$

with $\det(\mathsf{J})$ the usual Jacobian.

3.2.40 Exercise 3.1.7 may be written in matrix form

$$\mathsf{A}\mathbf{X} = \mathbf{C}.$$

Find A^{-1} and calculate $\mathbf{X}$ as $\mathsf{A}^{-1}\mathbf{C}$.

3.2.41 (a) Write a **subroutine** that will multiply **complex** matrices. Assume that the complex matrices are in a general rectangular form.

(b) Test your subroutine by multiplying pairs of the Dirac 4×4 matrices, Section 3.4.

3.2.42 (a) Write a subroutine that will call the complex matrix multiplication subroutine of Exercise 3.2.41 and will calculate the commutator bracket of two complex matrices.

(b) Test your complex commutator bracket subroutine with the matrices of Exercise 3.2.16.

3.2.43 **Interpolating polynomial** is the name given to the $(n-1)$-degree polynomial determined by (and passing through) n points, (x_i, y_i) with all the x_i distinct. This interpolating polynomial forms a basis for numerical quadratures.

(a) Show that the requirement that an $(n-1)$-degree polynomial in x passes through each of the n points (x_i, y_i) with all x_i distinct leads to n simultaneous equations of the form

$$\sum_{j=0}^{n-1} a_j x_i^j = y_i, \quad i = 1, 2, \ldots, n.$$

(b) Write a computer program that will read in n data points and return the n coefficients a_j. Use a subroutine to solve the simultaneous equations if such a subroutine is available.

(c) Rewrite the set of simultaneous equations as a matrix equation

$$\mathsf{X}\mathbf{A} = \mathbf{Y}.$$

(d) Repeat the computer calculation of part (b), but this time solve for vector $\mathbf{A}$ by inverting matrix X (again, using a subroutine).

3.2.44 A calculation of the values of electrostatic potential inside a cylinder leads to

$$\begin{array}{ll} V(0.0) = 52.640 & V(0.6) = 25.844 \\ V(0.2) = 48.292 & V(0.8) = 12.648 \\ V(0.4) = 38.270 & V(1.0) = 0.0. \end{array}$$

The problem is to determine the values of the argument for which $V = 10, 20, 30, 40$, and 50. Express $V(x)$ as a series $\sum_{n=0}^{5} a_{2n}x^{2n}$. (Symmetry requirements in the original problem require that $V(x)$ be an even function of x.) Determine the coefficients a_{2n}. With $V(x)$ now a known function of x, find the root of $V(x) - 10 = 0, 0 \le x \le 1$. Repeat for $V(x) - 20$, and so on.

$$ANS. \quad \begin{aligned} a_0 &= 52.640 \\ a_2 &= -\,117.676 \\ V(0.6851) &= 20. \end{aligned}$$

3.3.3 If A is orthogonal and $\det \mathsf{A} = +1$, show that $(\det \mathsf{A})a_{ij} = C_{ij}$, where C_{ij} is the **cofactor** of a_{ij}. This yields the identities of Eq. (1.46), used in Section 1.4 to show that a cross product of vectors (in three-space) is itself a vector.
Hint. Note Exercise 3.2.32.

3.3.9 Show that the trace of a matrix remains invariant under similarity transformations.

3.3.10 Show that the determinant of a matrix remains invariant under similarity transformations.
Note. Exercises (3.3.9) and (3.3.10) show that the trace and the determinant are independent of the Cartesian coordinates. They are characteristics of the matrix (operator) itself.

3.3.11 Show that the property of antisymmetry is invariant under orthogonal similarity transformations.

3.3.14 Show that the sum of the squares of the elements of a matrix remains invariant under orthogonal similarity transformations.

3.3.15 As a generalization of Exercise 3.3.14, show that

$$\sum_{jk} S_{jk}T_{jk} = \sum_{l,m} S'_{lm}T'_{lm},$$

where the primed and unprimed elements are related by an orthogonal similarity transformation. This result is useful in deriving invariants in electromagnetic theory (compare Section 4.6).
Note. This product $M_{jk} = \sum S_{jk}T_{jk}$ is sometimes called a **Hadamard product**. In the framework of tensor analysis, Chapter 2, this exercise becomes a double contraction of two second-rank tensors and therefore is clearly a scalar (invariant).

3.3.17 A column vector $\mathbf{V}$ has components V_1 and V_2 in an initial (unprimed) system. Calculate V'_1 and V'_2 for a

(a) rotation of the coordinates through an angle of θ **counterclockwise**,

(b) rotation of the vector through an angle of θ **clockwise**.

The results for parts (a) and (b) should be identical.

3.3.18 Write a subroutine that will test whether a real $N \times N$ matrix is symmetric. Symmetry may be defined as

$$0 \leq |a_{ij} - a_{ji}| \leq \varepsilon,$$

where ε is some small tolerance (which allows for truncation error, and so on in the computer).

3.4.8 Show that a Hermitian matrix remains Hermitian under unitary similarity transformations.

3.4.11 A particular similarity transformation yields

$$\begin{aligned} \mathsf{A}' &= \mathsf{U}\mathsf{A}\mathsf{U}^{-1}, \\ \mathsf{A}'^{\dagger} &= \mathsf{U}\mathsf{A}^{\dagger}\mathsf{U}^{-1}. \end{aligned}$$

If the adjoint relationship is preserved ($\mathsf{A}^{\dagger'} = \mathsf{A}^{'\dagger}$) and $\det \mathsf{U} = 1$, show that U must be unitary.

3.4.13 An operator $T(t+\varepsilon, t)$ describes the change in the wave function from t to $t+\varepsilon$. For ε real and small enough so that ε^2 may be neglected,

$$T(t+\varepsilon, t) = 1 - \frac{i}{\hbar}\varepsilon \mathsf{H}(t).$$

(a) If T is unitary, show that H is Hermitian.

(b) If H is Hermitian, show that T is unitary.

Note. When $\mathsf{H}(t)$ is independent of time, this relation may be put in exponential form - Exercise 3.4.12.

3.4.14 Show that an alternate form,

$$T(t+\varepsilon, t) = \frac{1 - i\varepsilon \mathsf{H}(t)/2\hbar}{1 + i\varepsilon \mathsf{H}(t)/2\hbar},$$

agrees with the T of part (a) of Exercise 3.4.13, neglecting ε^2, and is exactly unitary (for H Hermitian).

3.4.17 Use the four-dimensional Levi-Civita symbol $\varepsilon_{\lambda\mu\nu\rho}$ with $\varepsilon_{0123} = -1$ (generalizing Eqs. (2.93) in Section 2.9 to four dimensions) and show that (i) $2\gamma_5\sigma_{\mu\nu} = -i\varepsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}$ using the summation convention of Section 2.6 and (ii) $\gamma_\lambda\gamma_\mu\gamma_\nu = g_{\lambda\mu}\gamma_\nu - g_{\lambda\nu}\gamma_\mu + g_{\mu\nu}\gamma_\lambda + i\varepsilon_{\lambda\mu\nu\rho}\gamma^\rho\gamma_5$. Define $\gamma_\mu = g_{\mu\nu}\gamma^\nu$ using $g^{\mu\nu} = g_{\mu\nu}$ to raise and lower indices.

3.4.18 Evaluate the following traces: (see Eq. (3.123) for the notation)

(i) $\text{trace}(\gamma \cdot a\gamma \cdot b) = 4a \cdot b$,
(ii) $\text{trace}(\gamma \cdot a\gamma \cdot b\gamma \cdot c) = 0$,
(iii) $\text{trace}(\gamma \cdot a\gamma \cdot b\gamma \cdot c\gamma \cdot d) = 4(a \cdot bc \cdot d - a \cdot cb \cdot d + a \cdot db \cdot c)$,
(iv) $\text{trace}(\gamma_5\gamma \cdot a\gamma \cdot b\gamma \cdot c\gamma \cdot d) = 4i\varepsilon_{\alpha\beta\mu\nu}a^\alpha b^\beta c^\mu d^\nu$.

3.4.21 Show that

$$\boldsymbol{\alpha} \times \boldsymbol{\alpha} = 2i\boldsymbol{\sigma} \otimes 1_2,$$

where $\boldsymbol{\alpha} = \gamma_0\boldsymbol{\gamma}$ is a vector

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3).$$

Note that if $\boldsymbol{\alpha}$ is a polar vector (Section 2.4), then $\boldsymbol{\sigma}$ is an axial vector.

3.4.25 Let $x'_\mu = \Lambda_\mu^\nu x_\nu$ be a rotation by an angle θ about the 3-axis,

$$x'_0 = x_0, \quad x'_1 = x_1 \cos\theta + x_2 \sin\theta,$$
$$x'_2 = -x_1 \sin\theta + x_2 \cos\theta, \quad x'_3 = x_3.$$

Use $R = \exp(i\theta\sigma^{12}/2) = \cos\theta/2 + i\sigma^{12}\sin\theta/2$ (see Eq. (3.170b)) and show that the γ's transform just like the coordinates x^μ, that is, $\Lambda_\mu^\nu\gamma_\nu = R^{-1}\gamma_\mu R$. (Note that $\gamma_\mu = g_{\mu\nu}\gamma^\nu$ and that the γ^μ are well defined only up to a similarity transformation.) Similarly, if $x' = \Lambda x$ is a boost (pure Lorentz transformation) along the 1-axis, that is,

$$x'_0 = x_0 \cosh\zeta - x_1 \sinh\zeta, \quad x'_1 = -x_0 \sinh\zeta + x_1 \cosh\zeta,$$
$$x'_2 = x_2, \quad x'_3 = x_3,$$

with $\tanh\zeta = v/c$ and $B = \exp(-i\zeta\sigma^{01}/2) = \cosh\zeta/2 - i\sigma^{01}\sinh\zeta/2$ (see Eq. (3.170b)), show that $\Lambda_\mu^\nu\gamma_\nu = B\gamma_\mu B^{-1}$.

3.4.27 Write a subroutine that will test whether a complex $n \times n$ matrix is self-adjoint. In demanding equality of matrix elements $a_{ij} = a_{ij}^\dagger$, allow some small tolerance ε to compensate for truncation error of the computer.

3.4.28 Write a subroutine that will form the adjoint of a complex $M \times N$ matrix.

3.4.29 (a) Write a subroutine that will take a complex $M \times N$ matrix A and yield the product $\mathsf{A}^\dagger\mathsf{A}$.
Hint. This subroutine can call the subroutines of Exercises 3.2.41 and 3.4.28.

(b) Test your subroutine by taking A to be one or more of the Dirac matrices, Eq. (3.124).

3.5.1 (a) Starting with the orbital angular momentum of the ith element of mass,

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i = m_i\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i),$$

derive the inertia matrix such that $\mathbf{L} = \mathsf{I}\boldsymbol{\omega}$, $|\mathbf{L}\rangle = \mathsf{I}|\boldsymbol{\omega}\rangle$.

Figure 3.6. Mass sites for inertia tensor.

(b) Repeat the derivation starting with kinetic energy

$$T_i = \frac{1}{2} m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 \quad \left(T = \frac{1}{2} \langle \boldsymbol{\omega} | \mathsf{I} | \boldsymbol{\omega} \rangle \right).$$

3.5.11 Show that the inertia matrix for a single particle of mass m at (x, y, z) has a zero determinant. Explain this result in terms of the invariance of the determinant of a matrix under similarity transformations (Exercise 3.3.10) and a possible rotation of the coordinate system.

3.5.12 A certain rigid body may be represented by three point masses:

$$m_1 = 1 \text{ at } (1, 1, -2), \; m_2 = 2 \text{ at } (-1, -1, 0), \text{ and } m_3 = 1 \text{ at } (1, 1, 2).$$

(a) Find the inertia matrix.

(b) Diagonalize the inertia matrix, obtaining the eigenvalues and the principal axes (as orthonormal eigenvectors).

3.5.13 Unit masses are placed as shown in Fig. 3.6 (6th edition).

(a) Find the moment of inertia matrix.

(b) Find the eigenvalues and a set of orthonormal eigenvectors.

(c) Explain the degeneracy in terms of the symmetry of the system.

$$ANS. \; \mathsf{I} = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix} \quad \begin{array}{l} \lambda_1 = 2 \\ \mathbf{r}_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \\ \lambda_2 = \lambda_3 = 5. \end{array}$$

3.5.14 A mass $m_1 = 1/2$ kg is located at (1, 1, 1) (meters), a mass $m_2 = 1/2$ kg is at $(-1, -1, -1)$. The two masses are held together by an ideal (weightless, rigid) rod.

(a) Find the inertia tensor of this pair of masses.

(b) Find the eigenvalues and eigenvectors of this inertia matrix.

(c) Explain the meaning, the physical significance of the $\lambda = 0$ eigenvalue. What is the significance of the corresponding eigenvector?

(d) Now that you have solved this problem by rather sophisticated matrix techniques, explain how you could obtain

(1) $\lambda = 0$ and $\lambda = ?$ - by inspection (that is, using common sense).

(2) $\mathbf{r}_{\lambda=0} = ?$ - by inspection (that is, using freshman physics).

3.5.15 Unit masses are at the eight corners of a cube $(\pm1, \pm1, \pm1)$. Find the moment of inertia matrix and show that there is a triple degeneracy. This means that so far as moments of inertia are concerned, the cubic structure exhibits spherical symmetry.

3.5.30 (a) Determine the eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}.$$

Note that the eigenvalues are degenerate for $\varepsilon = 0$ but that the eigenvectors are orthogonal for all $\varepsilon \neq 0$ and $\varepsilon \to 0$.

(b) Determine the eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ \varepsilon^2 & 1 \end{pmatrix}.$$

Note that the eigenvalues are degenerate for $\varepsilon = 0$ and that for this (nonsymmetric) matrix the eigenvectors ($\varepsilon = 0$) do **not** span the space.

(c) Find the cosine of the angle between the two eigenvectors as a function of ε for $0 \leq \varepsilon \leq 1$.

3.5.31 (a) Take the coefficients of the simultaneous linear equations of Exercise 3.1.7 to be the matrix elements a_{ij} of matrix A (symmetric). Calculate the eigenvalues and eigenvectors.

(b) Form a matrix R whose columns are the eigenvectors of A, and calculate the triple matrix product $\tilde{\mathsf{R}}\mathsf{A}\mathsf{R}$.

ANS. $\lambda = 3.33163$.

3.5.32 Repeat Exercise 3.5.31 by using the matrix of Exercise 3.2.39.

3.6.1 Show that every 2×2 matrix has two eigenvectors and corresponding eigenvalues. The eigenvectors are not necessarily orthogonal and may be degenerate. The eigenvalues are not necessarily real. BAD PROBLEM!

4.1.1 Show that an $n \times n$ orthogonal matrix has $n(n-1)/2$ independent parameters.
Hint. The orthogonality condition, Eq. (3.71), provides constraints.

4.1.2 Show that an $n \times n$ unitary matrix has $n^2 - 1$ independent parameters.
Hint. Each element may be complex, doubling the number of possible parameters. Some of the constraint equations are likewise complex and count as two constraints.

4.1.3 The special linear group $\mathsf{SL}(2)$ consists of all 2×2 matrices (with complex elements) having a determinant of $+1$. Show that such matrices form a group.
Note. The $\mathsf{SL}(2)$ group can be related to the full Lorentz group in Section 4.4, much as the $\mathsf{SU}(2)$ group is related to $\mathsf{SO}(3)$.

4.1.4 Show that the rotations about the z-axis form a subgroup of $\mathsf{SO}(3)$. Is it an invariant subgroup?

4.1.5 Show that if R, S, T are elements of a group G so that $RS = T$ and $R \to (r_{ik}), S \to (s_{ik})$ is a representation according to Eq. (4.7), then

$$(r_{ik})(s_{ik}) = \left(t_{ik} = \sum_n r_{in} s_{nk} \right),$$

that is, group multiplication translates into matrix multiplication for any group representation.

4.2.1 (i) Show that the Pauli matrices are the generators of $\mathsf{SU}(2)$ without using the parameterization of the general unitary 2×2 matrix in Eq. (4.38). (ii) Derive the eight independent generators λ_i of $\mathsf{SU}(3)$ similarly. Normalize them so that $\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{ij}$. Then determine the structure constants of $\mathsf{SU}(3)$.
Hint. The λ_i are traceless and Hermitian 3×3 matrices.
(iii) Construct the quadratic Casimir invariant of $\mathsf{SU}(3)$.
Hint. Work by analogy with $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$ of $\mathsf{SU}(2)$ or $\mathbf{L}^2$ of $\mathsf{SO}(3)$.

4.2.2 Prove that the general form of a 2×2 unitary, unimodular matrix is

$$\mathsf{U} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

with $a^* a + b^* b = 1$.

4.2.4 A **translation** operator $T(a)$ converts $\psi(x)$ to $\psi(x+a)$,

$$T(a)\psi(x) = \psi(x+a).$$

In terms of the (quantum mechanical) linear momentum operator $p_x = -id/dx$, show that $T(a) = \exp(iap_x)$, that is, that p_x is the generator of translations.
Hint. Expand $\psi(x+a)$ as a Taylor series.

4.2.5 Consider the general $\mathsf{SU}(2)$ element Eq. (4.38) to be built up of three Euler rotations: (i) a rotation of $a/2$ about the z-axis, (ii) a rotation of $b/2$ about the new x-axis, and (iii) a rotation of $c/2$ about the new z-axis. (All rotations are counterclockwise.) Using the Pauli $\boldsymbol{\sigma}$ generators, show that these rotation angles are determined by

$$\begin{aligned} a &= \quad \xi - \zeta + \tfrac{\pi}{2} &&= \alpha + \tfrac{\pi}{2} \\ b &= \quad 2\eta &&= \beta \\ c &= \quad \xi + \zeta - \tfrac{\pi}{2} &&= \gamma - \tfrac{\pi}{2}. \end{aligned}$$

Note. The angles a and b here are not the a and b of Eq. (4.38).

4.2.6 Rotate a nonrelativistic wave function $\tilde{\psi} = (\psi_\uparrow, \psi_\downarrow)$ of spin 1/2 about the z-axis by a small angle $d\theta$. Find the corresponding generator.

4.3.1 Show that (a) $[J_+, \mathbf{J}^2] = 0$, (b) $[J_-, \mathbf{J}^2] = 0$.

4.3.2 Derive the root diagram of $\mathsf{SU}(3)$ in Fig. 4.6b from the generators λ_i in Eq. (4.61).
Hint. Work out first the $\mathsf{SU}(2)$ case in Fig. 4.6a from the Pauli matrices.

4.4.3 When the spin of quarks is taken into account, the $\mathsf{SU}(3)$ flavor symmetry is replaced by the $\mathsf{SU}(6)$ symmetry. Why? Obtain the Young tableau for the antiquark configuration $\bar{q}$. Then decompose the product $q\bar{q}$. Which $\mathsf{SU}(3)$ representations are contained in the nontrivial $\mathsf{SU}(6)$ representation for mesons?
Hint. Determine the dimensions of all YT.

4.4.5 Assuming that $D^j(\alpha,\beta,\gamma)$ is unitary, show that

$$\sum_{m=-l}^{l} Y_l^{m*}(\theta_1,\varphi_1) Y_l^m(\theta_2,\varphi_2)$$

is a scalar quantity (invariant under rotations). This is a spherical tensor analog of a scalar product of vectors.

4.4.6 (a) Show that the α and γ dependence of $\mathsf{D}^j(\alpha,\beta,\gamma)$ may be factored out such that

$$\mathsf{D}^j(\alpha,\beta,\gamma) = \mathsf{A}^j(\alpha)\mathsf{d}^j(\beta)\mathsf{C}^j(\gamma).$$

(b) Show that $\mathsf{A}^j(\alpha)$ and $\mathsf{C}^j(\gamma)$ are diagonal. Find the explicit forms.

(c) Show that $\mathsf{d}^j(\beta) = \mathsf{D}^j(0, \beta, 0)$.

4.4.7 The angular momentum-exponential form of the Euler angle rotation operators is

$$\begin{aligned} \mathsf{R} &= \mathsf{R}_{z''}(\gamma)\mathsf{R}_{y'}(\beta)\mathsf{R}_z(\alpha) \\ &= \exp(-i\gamma J_{z''})\exp(-i\beta J_{y'})\exp(-i\alpha J_z). \end{aligned}$$

Show that in terms of the original axes

$$\mathsf{R} = \exp(i\alpha J_z)\exp(-i\beta J_y)\exp(-i\gamma J_z).$$

Hint. The R operators transform as matrices. The rotation about the y'-axis (second Euler rotation) may be referred to the original y-axis by

$$\exp(-i\beta J_{y'}) = \exp(-i\alpha J_z)\exp(-i\beta J_y)\exp(i\alpha J_z).$$

4.4.8 Using the Wigner-Eckart theorem, prove the decomposition theorem for a spherical vector operator $\langle j'm'|T_{1m}|jm\rangle = \frac{\langle jm'|\mathbf{J}\cdot\mathbf{T}_1|jm\rangle}{j(j+1)}\delta_{jj'}$.

4.4.9 Using the Wigner-Eckart theorem, prove the factorization
$\langle j'm'|J_M \mathbf{J}\cdot\mathbf{T_1}|jm\rangle = \langle jm'|J_M|jm\rangle\delta_{j'j}\langle jm|\mathbf{J}\cdot\mathbf{T_1}|jm\rangle$.

4.5.1 Two Lorentz transformations are carried out in succession: v_1 along the x-axis, then v_2 along the y-axis. Show that the resultant transformation (given by the product of these two successive transformations) **cannot** be put in the form of a single Lorentz transformation.
Note. The discrepancy corresponds to a rotation.

4.5.2 Rederive the Lorentz transformation, working entirely in the real space (x^0, x^1, x^2, x^3) with $x^0 = x_0 = ct$. Show that the Lorentz transformation may be written $\mathsf{L}(\mathbf{v}) = \exp(\rho\sigma)$, with

$$\sigma = \begin{pmatrix} 0 & -\lambda & -\mu & -\nu \\ -\lambda & 0 & 0 & 0 \\ -\mu & 0 & 0 & 0 \\ -\nu & 0 & 0 & 0 \end{pmatrix}$$

and λ, μ, ν the direction cosines of the velocity $\mathbf{v}$.

4.5.3 Using the matrix relation, Eq. (4.136), let the rapidity ρ_1 relate the Lorentz reference frames (x'^0, x'^1) and (x^0, x^1). Let ρ_2 relate (x''^0, x''^1) and (x'^0, x'^1). Finally, let ρ relate (x''^0, x''^1) and (x^0, x^1). From $\rho = \rho_1 + \rho_2$ derive the Einstein velocity addition law

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.$$

4.6.1 (a) Show that every four-vector in Minkowski space may be decomposed into an ordinary three-space vector and a three-space scalar. Examples: $(ct, \mathbf{r}), (\rho, \rho\mathbf{v}/c), (\varepsilon_0\varphi, c\varepsilon_0\mathbf{A}), (E/c, \mathbf{p}), (\omega/c, \mathbf{k})$.
Hint. Consider a rotation of the three-space coordinates with time fixed.

(b) Show that the converse of (a) is **not** true - every three-vector plus scalar does **not** form a Minkowski four-vector.

4.6.2 (a) Show that

$$\partial^\mu j_\mu = \partial \cdot j = \frac{\partial j_\mu}{\partial x_\mu} = 0.$$

(b) Show how the previous tensor equation may be interpreted as a statement of continuity of charge and current in ordinary three-dimensional space and time.

(c) If this equation is known to hold in all Lorentz reference frames, why can we not conclude that j_μ is a vector?

4.6.3 Write the Lorentz gauge condition (Eq. (4.143)) as a tensor equation in Minkowski space.

4.6.4 A gauge transformation consists of varying the scalar potential φ_1 and the vector potential $\mathbf{A}_1$ according to the relation

$$\begin{aligned} \varphi_2 &= \varphi_1 + \frac{\partial \chi}{\partial t}, \\ \mathbf{A}_2 &= \mathbf{A}_1 - \boldsymbol{\nabla}\chi. \end{aligned}$$

The new function χ is required to satisfy the homogeneous wave equation

$$\boldsymbol{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0.$$

Show the following:

(a) The Lorentz gauge relation is unchanged.

(b) The new potentials satisfy the same inhomogeneous wave equations as did the original potentials.

(c) The fields $\mathbf{E}$ and $\mathbf{B}$ are unaltered.

The invariance of our electromagnetic theory under this transformation is called gauge invariance.

4.6.5 A charged particle, charge q, mass m, obeys the Lorentz covariant equation

$$\frac{dp^\mu}{d\tau} = \frac{q}{\varepsilon_0 mc}F^{\mu\nu}p_\nu,$$

where p^ν is the four-momentum vector $(E/c; p^1, p^2, p^3)$. τ is the proper time; $d\tau = dt\sqrt{1 - v^2/c^2}$, a Lorentz scalar. Show that the explicit space-time forms are

$$\frac{dE}{dt} = q\mathbf{v} \cdot \mathbf{E}; \ \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

4.6.6 From the Lorentz transformation matrix elements (Eq. (4.158)) derive the Einstein velocity addition law

$$u' = \frac{u - v}{1 - (uv/c^2)} \quad \text{or} \quad u = \frac{u' + v}{1 + (u'v/c^2)},$$

where $u = cdx^3/dx^0$ and $u' = cdx'^3/dx'^0$.
Hint. If $\mathsf{L}_{12}(v)$ is the matrix transforming system 1 into system 2, $\mathsf{L}_{23}(u')$ the matrix transforming system 2 into system 3, $\mathsf{L}_{13}(u)$ the matrix transforming system 1 directly into system 3, then $\mathsf{L}_{13}(u) = \mathsf{L}_{23}(u')\mathsf{L}_{12}(v)$. From this matrix relation extract the Einstein velocity addition law.

4.6.7 The dual of a four-dimensional second-rank tensor $\mathbf{B}$ may be defined by $\tilde{\mathbf{B}}$, where the elements of the dual tensor are given by

$$\tilde{B}^{ij} = \frac{1}{2!}\varepsilon^{ijkl}B_{kl}.$$

Show that $\tilde{\mathbf{B}}$ transforms as

(a) a second-rank tensor under rotations,

(b) a pseudotensor under inversions.

Note. The tilde here does **not** mean transpose.

4.6.8 Construct $\tilde{\mathbf{F}}$, the dual of $\mathbf{F}$, where $\mathbf{F}$ is the electromagnetic tensor given by Eq. (4.153).

$$ANS. \ \ \tilde{\mathbf{F}}^{\mu\nu} = \varepsilon_0 \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{pmatrix}.$$

This corresponds to

$$\begin{aligned} c\mathbf{B} &\to -\mathbf{E}, \\ \mathbf{E} &\to c\mathbf{B}. \end{aligned}$$

This transformation, sometimes called a **dual transformation,** leaves Maxwell's equations in vacuum ($\rho = 0$) invariant.

4.6.9 Because the quadruple contraction of a fourth-rank pseudotensor and two second-rank tensors $\varepsilon_{\mu\lambda\nu\sigma}F^{\mu\lambda}F^{\nu\sigma}$ is clearly a pseudoscalar, evaluate it.

ANS. $\quad -8\varepsilon_0^2 c\mathbf{B}\cdot\mathbf{E}$.

4.6.10 (a) If an electromagnetic field is purely electric (or purely magnetic) in one particular Lorentz frame, show that **E** and **B** will be orthogonal in other Lorentz reference systems.

(b) Conversely, if **E** and **B** are orthogonal in one particular Lorentz frame, there exists a Lorentz reference system in which **E** (or **B**) vanishes. Find that reference system.

4.6.11 Show that $c^2\mathbf{B}^2 - \mathbf{E}^2$ is a Lorentz scalar.

4.6.12 Since (dx^0, dx^1, dx^2, dx^3) is a four-vector, $dx_\mu dx^\mu$ is a scalar. Evaluate this scalar for a moving particle in two different coordinate systems: (a) a coordinate system fixed relative to you (lab system), and (b) a coordinate system moving with a moving particle (velocity v relative to you). With the time increment labeled $d\tau$ in the particle system and dt in the lab system, show that

$$d\tau = dt\sqrt{1 - v^2/c^2}.$$

τ is the proper time of the particle, a Lorentz invariant quantity.

4.6.13 Expand the scalar expression

$$-\frac{1}{4\varepsilon_0}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\varepsilon_0}j_\mu A^\mu$$

in terms of the fields and potentials. The resulting expression is the Lagrangian density used in Exercise 17.5.1.

4.6.14 Show that Eq. (4.157) may be written

$$\varepsilon_{\alpha\beta\gamma\delta}\frac{\partial F^{\alpha\beta}}{\partial x_\gamma} = 0.$$

4.7.1 Show that the matrices $1, \mathsf{A}, \mathsf{B}$, and C of Eq. (4.165) are reducible. Reduce them.
Note. This means transforming A and C to diagonal form (by the same unitary transformation).
Hint. A and C are anti-Hermitian. Their eigenvectors will be orthogonal.

4.7.2 Possible operations on a crystal lattice include A_π (rotation by π), m (reflection), and i (inversion). These three operations combine as

$$\begin{aligned} A_\pi^2 &= m^2 = i^2 = 1, \\ A_\pi \cdot m &= i, \qquad m \cdot i = A_\pi, \quad \text{and} \quad i \cdot A_\pi = m. \end{aligned}$$

Show that the group $(1, A_\pi, m, i)$ is isomorphic with the vierergruppe.

4.7.3 Four possible operations in the xy-plane are:

$$1.\ \text{no change} \begin{cases} x \to x \\ y \to y \end{cases}$$

$$2.\ \text{inversion} \begin{cases} x \to -x \\ y \to -y \end{cases}$$

$$3.\ \text{reflection} \begin{cases} x \to -x \\ y \to y \end{cases}$$

$$4.\ \text{reflection} \begin{cases} x \to x \\ y \to -y. \end{cases}$$

(a) Show that these four operations form a group.
(b) Show that this group is isomorphic with the vierergruppe.
(c) Set up a 2×2 matrix representation.

4.7.5 Using the 2×2 matrix representation of Exercise 3.2.7 for the vierergruppe,

(a) Show that there are four classes, each with one element.
(b) Calculate the character (trace) of each class. Note that two different classes may have the same character.
(c) Show that there are three two-element subgroups. (The unit element by itself always forms a subgroup.)
(d) For any one of the two-element subgroups show that the subgroup and a single coset reproduce the original vierergruppe.

Note that subgroups, classes, and cosets are entirely different.

4.7.6 Using the 2×2 matrix representation, Eq. (4.165), of C_4,

(a) Show that there are four classes, each with one element.
(b) Calculate the character (trace) of each class.
(c) Show that there is one two-element subgroup.
(d) Show that the subgroup and a single coset reproduce the original group.

4.7.7 Prove that the number of distinct elements in a coset of a subgroup is the same as the number of elements in the subgroup.

4.7.11 Explain how the relation

$$\sum_i n_i^2 = h$$

applies to the vierergruppe ($h = 4$) and to the dihedral group D_3 with $h = 6$.

4.7.12 Show that the subgroup $(\mathsf{1}, \mathsf{A}, \mathsf{B})$ of D_3 is an invariant subgroup.

4.7.16 The elements of the dihedral group D_n may be written in the form

$$\mathsf{S}^{\lambda}\mathsf{R}_z^{\mu}(2\pi/n), \qquad \begin{array}{l} \lambda = 0, 1 \\ \mu = 0, 1, \ldots, n-1, \end{array}$$

where $\mathsf{R}_z(2\pi/n)$ represents a rotation of $2\pi/n$ about the n-fold symmetry axis, whereas S represents a rotation of π about an axis through the center of the regular polygon and one of its vertices.
For $\mathsf{S} = \mathsf{E}$ show that this form may describe the matrices $\mathsf{A}, \mathsf{B}, \mathsf{C}$, and D of D_3.
Note. The elements R_z and S are called the generators of this finite group. Similarly, i is the generator of the group given by Eq. (4.164).

4.7.19 The permutation group of four objects contains $4! = 24$ elements. From Exercise 4.7.18, D_4, the symmetry group for a square, has far fewer than 24 elements. Explain the relation between D_4 and the permutation group of four objects.

4.7.22 (a) From the D_3 multiplication table of Fig. 4.18 construct a similarity transform table showing xyx^{-1}, where x and y each range over all six elements of D_3:.

(b) Divide the elements of D_3 into classes. Using the 2×2 matrix representation of Eqs. (4.169) - (4.172) note the trace (character) of each class.

4.8.1 Evaluate the 1-form $adx + 2bdy + 4cdz$ on the line segment PQ with $P = (3, 5, 7), Q = (7, 5, 3)$.

4.8.3 Evaluate the flow described by the 2-form $dxdy + 2dydz + 3dzdx$ across the oriented triangle PQR with corners at

$$P = (3, 1, 4), Q = (-2, 1, 4), R = (1, 4, 1).$$

4.8.4 Are the points, in this order,

$$(0, 1, 1), \quad (3, -1, -2), \quad (4, 2, -2), \quad (-1, 0, 1)$$

coplanar, or do they form an oriented volume (right-handed or left-handed)?

4.8.6 Describe the electric field by the 1-form $E_1dx + E_2dy + E_3dz$ and the magnetic induction by the 2-form $B_1dydz + B_2dzdx + B_3dxdy$. Then formulate Faraday's induction law in terms of these forms.

4.8.7 Evaluate the 1-form

$$\frac{xdy}{x^2+y^2} - \frac{ydx}{x^2+y^2}$$

on the unit circle about the origin oriented counterclockwise.

4.8.8 Find the pullback of $dxdz$ under $x = u\cos v, y = u - v, z = u\sin v$.

4.8.9 Find the pullback of the 2-form $dydz + dzdx + dxdy$ under the map $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$.

4.8.10 Parameterize the surface obtained by rotating the circle $(x-2)^2 + z^2 = 1, y = 0$, about the z-axis in a counterclockwise orientation, as seen from outside.

4.8.12 Show that $\sum_{i=1}^{n} x_i^2 = a^2$ defines a differentiable manifold of dimension $D = n - 1$ if $a \neq 0$ and $D = 0$ if $a = 0$.

4.8.13 Show that the set of orthogonal 2×2 matrices form a differentiable manifold, and determine its dimension.

4.8.14 Determine the value of the 2-form $Adydz + Bdzdx + Cdxdy$ on a parallelogram with sides $\mathbf{a}, \mathbf{b}$.

4.8.15 Prove Lorentz invariance of Maxwell's equations in the language of differential forms.

5.1.1 Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

Hint. Show (by mathematical induction) that $s_m = m/(2m+1)$.

5.1.2 Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Find the partial sum s_m and verify its correctness by mathematical induction.

Note. The method of expansion in partial fractions, Section 15.8, offers an alternative way of solving Exercises 5.1.1 and 5.1.2.

5.2.3 Show that the complete d'Alembert ratio test follows directly from Kummer's test with $a_i = 1$.

5.2.9 Determine the range of convergence for Gauss's **hypergeometric series**

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{1!\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \cdots .$$

Hint. Gauss developed his test for the specific purpose of establishing the convergence of this series.

ANS. Convergent for $-1 < x < 1$ and $x = \pm 1$ if $\gamma > \alpha + \beta$.

5.2.10 A pocket calculator yields

$$\sum_{n=1}^{100} n^{-3} = 1.202\ 007.$$

Show that

$$1.202056 \leq \sum_{n=1}^{\infty} n^{-3} \leq 1.202\ 057.$$

Hint. Use integrals to set upper and lower bounds on $\sum_{n=101}^{\infty} n^{-3}$.

Note. A more exact value for summation of $\zeta(3) = \sum\limits_{n=1}^{\infty} n^{-3}$ is 1.202 056 903$\cdots$; $\zeta(3)$ is known to be an irrational number, but it is not linked to known constants such as $e, \pi, \gamma, \ln 2$.

5.2.11 Set upper and lower bounds on $\sum_{n=1}^{1{,}000{,}000} n^{-1}$, assuming that

(a) the Euler-Mascheroni constant is known.

$$ANS.\ 14.392726 < \sum_{n=1}^{1{,}000{,}000} n^{-1} < 14.392\ 727.$$

(b) The Euler-Mascheroni constant is unknown.

5.2.19 Show that the following series is convergent.

$$\sum_{s=0}^{\infty} \frac{(2s-1)!!}{(2s)!!(2s+1)}.$$

Note. $(2s-1)!! = (2s-1)(2s-3)\cdots 3\cdot 1$ with $(-1)!! = 1$; $(2s)!! = (2s)(2s-2)\cdots 4\cdot 2$ with $0!! = 1$. The series appears as a series expansion of $\sin^{-1}(1)$ and equals $\pi/2$, and $\sin^{-1} x \equiv \arcsin x \neq (\sin x)^{-1}$.

5.3.1 (a) From the electrostatic two-hemisphere problem (Exercise 12.3.20) we obtain the series

$$\sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s+2)!!}.$$

Test it for convergence.

(b) The corresponding series for the surface charge density is

$$\sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s)!!}.$$

Test it for convergence.

The !! notation is explained in Section 8.1 and Exercise 5.2.19.

5.3.2 Show by direct numerical computation that the sum of the first 10 terms of

$$\lim_{x\to 1}\ln(1+x) = \ln 2 = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-1}$$

differs from ln 2 by less than the eleventh term: $\ln 2 = 0.69314\ 71806\cdots$.

5.3.3 In Exercise 5.2.9 the hypergeometric series is shown convergent for $x = \pm 1$, if $\gamma > \alpha + \beta$. Show that there is conditional convergence for $x = -1$ for γ down to $\gamma > \alpha + \beta - 1$.
Hint. The asymptotic behavior of the factorial function is given by Stirling's series, Section 8.3.

5.4.2 Determine the values of the coefficients a_1, a_2, and a_3 that will make $(1 + a_1x + a_2x^2 + a_3x^3)\ln(1+x)$ converge as n^{-4} . Find the resulting series.

5.4.4 Write a program that will rearrange the terms of the alternating harmonic series to make the series converge to 1.5. Group your terms as indicated in Eq. (5.61). List the first 100 successive partial sums that just climb above 1.5 or just drop below 1.5, and list the new terms included in each such partial sum.

ANS.

n	1	2	3	4	5
s_n	1.5333	1.0333	1.5218	1.2718	1.5143

5.6.2 Derive a series expansion of cot x in increasing powers of x by dividing $\cos x$ by $\sin x$.
Note. The resultant series that starts with $1/x$ is actually a Laurent series (Section 6.5). Although the two series for $\sin x$ and $\cos x$ were valid for all x, the convergence of the series for cot x is limited by the zeros of the denominator, $\sin x$ (see Analytic Continuation in Section 6.5).

5.6.3 The Raabe test for $\sum_n (n\ln n)^{-1}$ leads to

$$\lim_{n\to\infty} n\left[\frac{(n+1)\ln(n+1)}{n\ln n} - 1\right].$$

Show that this limit is unity (which means that the Raabe test here is indeterminate).

5.6.6 Let x be an approximation for a zero of $f(x)$ and Δx be the correction. Show that by neglecting terms of order $(\Delta x)^2$,

$$\Delta x = -\frac{f(x)}{f'(x)}.$$

This is Newton's formula for finding a root. Newton's method has the virtues of illustrating series expansions and elementary calculus but is very treacherous.

5.6.7 Expand a function $\Phi(x, y, z)$ by Taylor's expansion about (0,0,0) to $\mathcal{O}(a^3)$. Evaluate $\bar{\Phi}$, the average value of Φ, averaged over a small cube of side a centered on the origin and show that the Laplacian of Φ is a measure of deviation of Φ from $\Phi(0, 0, 0)$.

5.6.22 You have a function $y(x)$ tabulated at equally spaced values of the argument

$$\begin{cases} y_n = y(x_n) \\ x_n = x + nh. \end{cases}$$

Show that the linear combination

$$\frac{1}{12h}\{-y_2 + 8y_1 - 8y_{-1} + y_{-2}\}$$

yields

$$y_0' - \frac{h^4}{30} y_0^{(5)} + \cdots .$$

Hence this linear combination yields y_0' if $(h^4/30)y_0^{(5)}$ and higher powers of h and higher derivatives of $y(x)$ are negligible.

5.6.23 In a numerical integration of a partial differential equation, the three-dimensional Laplacian is replaced by

$$\begin{aligned} \nabla^2\psi(x, y, z) \to \quad & h^{-2} \quad [\psi(x + h, y, z) + \psi(x - h, y, z) \\ & + \quad \psi(x, y + h, z) + \psi(x, y - h, z) + \psi(x, y, z + h) \\ & + \quad \psi(x, y, z - h) - 6\psi(x, y, z)]. \end{aligned}$$

Determine the error in this approximation. Here h is the step size, the distance between adjacent points in the x-, y-, or z-direction.

5.6.24 Using double precision, calculate e from its Maclaurin series.
Note. This simple, direct approach is the best way of calculating e to high accuracy. Sixteen terms give e to 16 significant figures. The reciprocal factorials give very rapid convergence.

5.7.2 The depolarizing factor L for an oblate ellipsoid in a uniform electric field parallel to the axis of rotation is

$$L = \frac{1}{\varepsilon_0}(1 + \zeta_0^2)(1 - \zeta_0 \cot^{-1}\zeta_0),$$

where ζ_0 defines an oblate ellipsoid in oblate spheroidal coordinates (ξ, ζ, φ). Show that

$$\lim_{\zeta_0\to\infty} L = \frac{1}{3\varepsilon_0} \quad \text{(sphere)}, \qquad \lim_{\zeta_0\to 0} L = \frac{1}{\varepsilon_0} \quad \text{(thin sheet)}.$$

5.7.3 The depolarizing factor (Exercise 5.7.2) for a prolate ellipsoid is

$$L = \frac{1}{\varepsilon_0}(\eta_0^2 - 1)\left(\frac{1}{2}\eta_0 \ln \frac{\eta_0 + 1}{\eta_0 - 1} - 1\right).$$

Show that

$$\lim_{\eta_0 \to \infty} L = \frac{1}{3\varepsilon_0} \quad \text{(sphere)}, \qquad \lim_{\eta_0 \to 0} L = 0 \quad \text{(long needle)}.$$

5.7.4 The analysis of the diffraction pattern of a circular opening involves

$$\int_0^{2\pi} \cos(c \cos\varphi) d\varphi.$$

Expand the integrand in a series and integrate by using

$$\int_0^{2\pi} \cos^{2n}\varphi d\varphi = \frac{(2n)!}{2^{2n}(n!)^2} \cdot 2\pi, \qquad \int_0^{2\pi} \cos^{2n+1}\varphi d\varphi = 0.$$

The result is 2π times the Bessel function $J_0(c)$.

5.7.5 Neutrons are created (by a nuclear reaction) inside a hollow sphere of radius R. The newly created neutrons are uniformly distributed over the spherical volume. Assuming that all directions are equally probable (isotropy), what is the average distance a neutron will travel before striking the surface of the sphere? Assume straight-line motion and no collisions.

(a) Show that

$$\bar{r} = \tfrac{3}{2}R \int_0^1 \int_0^\pi \sqrt{1 - k^2 \sin^2\theta}\, k^2 \; dk \sin\theta \; d\theta.$$

(b) Expand the integrand as a series and integrate to obtain

$$\bar{r} = R\left[1 - 3\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)(2n+3)}\right].$$

(c) Show that the sum of this infinite series is $1/12$, giving $\bar{r} = \frac{3}{4}R$.

Hint. Show that $s_n = 1/12 - [4(2n+1)(2n+3)]^{-1}$ by mathematical induction. Then let $n \to \infty$.

5.7.8 Derive the series expansion of the incomplete beta function

$$\begin{aligned} B_x(p,q) &= \int_0^x t^{p-1}(1-t)^{q-1} dt \\ &= x^p \left\{ \frac{1}{p} + \frac{1-q}{p+1}x + \cdots + \frac{(1-q)\cdots(n-q)}{n!(p+n)}x^n + \cdots \right\} \end{aligned}$$

for $0 \le x \le 1, p > 0$, and $q > 0$ (if $x = 1$).

5.7.10 Neutron transport theory gives the following expression for the inverse neutron diffusion length of k:

$$\frac{a-b}{k}\tanh^{-1}\left(\frac{k}{a}\right) = 1.$$

By series inversion or otherwise, determine k^2 as a series of powers of b/a. Give the first two terms of the series.

$$ANS.\;\; k^2 = 3ab\left(1 - \frac{4}{5}\frac{b}{a}\right).$$

5.7.12 A function $f(z)$ is represented by a **descending** power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \quad R \le z < \infty.$$

Show that this series expansion is unique; that is, if $f(z) = \sum_{n=0}^{\infty} b_n z^{-n}$, $R \le z < \infty$, then $a_n = b_n$ for all n.

5.7.14 Assuming that $f(x)$ may be expanded in a power series about the origin, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with some nonzero range of convergence. Use the techniques employed in proving uniqueness of series to show that your assumed series is a Maclaurin series:

$$a_n = \frac{1}{n!} f^{(n)}(0).$$

5.7.18 Calculate π (double precision) by each of the following arc tangent expressions:

$$\begin{aligned}
\pi &= 16\tan^{-1}(1/5) - 4\tan^{-1}(1/239) \\
\pi &= 24\tan^{-1}(1/8) + 8\tan^{-1}(1/57) + 4\tan^{-1}(1/239) \\
\pi &= 48\tan^{-1}(1/18) + 32\tan^{-1}(1/57) - 20\tan^{-1}(1/239).
\end{aligned}$$

Obtain 16 significant figures. Verify the formulas using Exercise 5.6.2.
Note. These formulas have been used in some of the more accurate calculations of π.[162]

5.7.19 An analysis of the Gibbs phenomenon of Section 14.5 leads to the expression

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi} d\xi.$$

(a) Expand the integrand in a series and integrate term by term. Find the numerical value of this expression to four significant figures.

[216] D. Shanks and J. W. Wrench, Computation of π to 100 000 decimals. *Math. Comput.* **16**: 76 (1962).

(b) Evaluate this expression by the Gaussian quadrature if available.

ANS. 1.178980.

5.8.7 (a) Write a function subroutine that will compute $E(m)$ from the series expansion, Eq. (5.137).

(b) Test your function subroutine by using it to calculate $E(m)$ over the range $m = 0.0(0.1)0.9$ and comparing the result with the values given by AMS-55 (see footnote 4 for this reference).

5.8.8 Repeat Exercise 5.8.7 for $K(m)$.
Note. These series for $E(m)$, Eq. (5.137), and $K(m)$, Eq. (5.136), converge only very slowly for m near 1. More rapidly converging series for $E(m)$ and $K(m)$ exist. See Dwight's Tables of Integrals:[183] No. 773.2 and 774.2. Your computer subroutine for computing E and K probably uses polynomial approximations: AMS-55, Chapter 17 (see footnote 4 for this reference).

5.8.9 A simple pendulum is swinging with a maximum amplitude of θ_M. In the limit as $\theta_M \to 0$, the period is 1 s. Using the elliptic integral, $K(k^2), k = \sin(\theta_M/2)$, calculate the period T for $\theta_M = 0\ (10^\circ)\ 90^\circ$.
Caution. Some elliptic integral subroutines require $k = m^{1/2}$ as an input parameter, not m itself. **Check values**.

θ_M	10°	50°	90°
T(sec)	1.00193	1.05033	1.18258

5.8.10 Calculate the magnetic vector potential $\mathbf{A}(\rho, \varphi, z) = \hat{\boldsymbol{\varphi}} A_\varphi(\rho, \varphi, z)$ of a circular current loop (Exercise 5.8.4) for the ranges $\rho/a = 2, 3, 4$, and $z/a = 0, 1, 2, 3, 4$.
Note. This elliptic integral calculation of the magnetic vector potential may be checked by an associated Legendre function calculation, Example 12.5.1.

Check value. For $\rho/a = 3$ and $z/a = 0$; $A_\varphi = 0.029023\mu_0 I$.

5.9.3 Show that $B_n'(s) = nB_{n-1}(s), n = 1, 2, 3, \ldots$
Hint. Differentiate Eq. (5.158).

5.9.4 Show that

$$B_n(1) = (-1)^n B_n(0).$$

Hint. Go back to the generating function, Eq. (5.158), or Exercise 5.9.2.

[318] H. B. Dwight, *Tables of Integrals and Other Mathematical Data*. New York: Macmillan (1947).

5.9.7 Planck's blackbody radiation law involves the integral

$$\int_0^\infty \frac{x^3 dx}{e^x - 1}.$$

Show that this equals $6\zeta(4)$. From Exercise 5.9.6,

$$\zeta(4) = \frac{\pi^4}{90}.$$

Hint. Make use of the gamma function, Chapter 8.

5.9.19 Write a function subprogram ZETA(N) that will calculate the Riemann zeta function for integer argument. Tabulate $\zeta(s)$ for $s = 2, 3, 4, \dots, 20$. Check your values against Table 5.3 and AMS-55, Chapter 23 (see footnote 4 for this reference).
Hint. If you supply the function subprogram with the known values of $\zeta(2), \zeta(3)$, and $\zeta(4)$, you avoid the more slowly converging series. Calculation time may be further shortened by using Eq. (5.170).

5.9.20 Calculate the logarithm (base 10) of $|B_{2n}|, n = 10, 20, \dots, 100$.
Hint. Program $\zeta(n)$ as a function subprogram, Exercise 5.9.19.

Check values. $\log|B_{100}| = 78.45$
$\log|B_{200}| = 215.56.$

5.10.1 Stirling's formula for the logarithm of the factorial function is

$$\ln(x!) = \frac{1}{2}\ln 2\pi + \left(x + \frac{1}{2}\right)\ln x - x - \sum_{n=1}^{N} \frac{B_{2n}}{2n(2n-1)} x^{1-2n}.$$

The B_{2n} are the Bernoulli numbers (Section 5.9). Show that Stirling's formula is an **asymptotic** expansion.

5.10.9 Calculate partial sums of $e^x E_1(x)$ for $x = 5, 10$, and 15 to exhibit the behavior shown in Fig. 5.11. Determine the width of the throat for $x = 10$ and 15, analogous to Eq. (5.183).

ANS. Throat width: $n = 10$, 0.000051
$n = 15$, 0.0000002.

5.10.10 The knife-edge diffraction pattern is described by

$$I = 0.5 I_0 \left\{[C(u_0) + 0.5]^2 + [S(u_0) + 0.5]^2\right\},$$

where $C(u_0)$ and $S(u_0)$ are the Fresnel integrals of Exercise 5.10.2. Here I_0 is the incident intensity and I is the diffracted intensity; u_0 is proportional to the distance away from the knife edge (measured at right angles to the incident beam). Calculate I/I_0 for u_0 varying from -1.0 to $+4.0$ in steps of 0.1. Tabulate your results and, if a plotting routine is available, plot them.

Check value. $u_0 = 1.0$, $I/I_0 = 1.259226$.

5.11.6 Prove that

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$

5.11.10 Calculate $\cos x$ from its infinite product representation, Eq. (5.211), using (a) 10, (b) 100, and (c) 1000 factors in the product. Calculate the absolute error. Note how slowly the partial products converge - making the infinite product quite unsuitable for precise numerical work.

ANS. For 1000 factors, $\cos\pi = -1.00051$.

6.1.2 The complex quantities $a = u+iv$ and $b = x+iy$ may also be represented as two-dimensional vectors, $\mathbf{a} = \hat{\mathbf{x}}u + \hat{\mathbf{y}}v$, $\mathbf{b} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. Show that

$$a^*b = \mathbf{a}\cdot\mathbf{b} + i\hat{\mathbf{z}}\cdot\mathbf{a}\times\mathbf{b}.$$

6.1.3 Prove algebraically that for complex numbers,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Interpret this result in terms of two-dimensional vectors.
Prove that

$$|z-1| < |\sqrt{z^2-1}| < |z+1|, \quad \text{for} \quad \Re(z) > 0.$$

6.1.4 We may define a complex conjugation operator K such that $Kz = z^*$. Show that K is not a linear operator.

6.1.8 For $-1 < p < 1$ prove that

(a) $$\sum_{n=0}^{\infty} p^n \cos nx = \frac{1-p\cos x}{1-2p\cos x + p^2},$$

(b) $$\sum_{n=0}^{\infty} p^n \sin nx = \frac{p\sin x}{1-2p\cos x + p^2}.$$

These series occur in the theory of the Fabry-Perot interferometer.

6.1.9 Assume that the trigonometric functions and the hyperbolic functions are defined for complex argument by the appropriate power series

$$\begin{aligned}
\sin z &= \sum_{n=1,\text{odd}}^{\infty} (-1)^{(n-1)/2}\frac{z^n}{n!} = \sum_{s=0}^{\infty}(-1)^s\frac{z^{2s+1}}{(2s+1)!},\\
\cos z &= \sum_{n=0,\text{even}}^{\infty} (-1)^{n/2}\frac{z^n}{n!} = \sum_{s=0}^{\infty}(-1)^s\frac{z^{2s}}{(2s)!},\\
\sinh z &= \sum_{n=1,\text{odd}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty}\frac{z^{2s+1}}{(2s+1)!},\\
\cosh z &= \sum_{n=0,\text{even}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty}\frac{z^{2s}}{(2s)!}.
\end{aligned}$$

(a) Show that

$$\begin{aligned} i\sin z &= \sinh iz, \quad \sin iz = i\sinh z,\\ \cos z &= \cosh iz, \quad \cos iz = \cosh z. \end{aligned}$$

(b) Verify that familiar functional relations such as

$$\begin{aligned} \cosh z \quad &= \frac{e^z + e^{-z}}{2},\\ \sin(z_1 + z_2) \quad &= \sin z_1 \cos z_2 + \sin z_2 \cos z_1, \end{aligned}$$

still hold in the complex plane.

6.1.12 Prove that
(a) $|\sin z| \geq |\sin x|$ (b) $|\cos z| \geq |\cos x|$.

6.1.13 Show that the exponential function e^z is periodic with a pure imaginary period of $2\pi i$.

6.1.15 Find all the zeros of
(a) $\sin z$, (b) $\cos z$, (c) $\sinh z$, (d) $\cosh z$.

6.1.16 Show that

(a) $\sin^{-1} z = -i\ln(iz \pm \sqrt{1-z^2})$,
(b) $\cos^{-1} z = -i\ln(z \pm \sqrt{z^2-1})$,
(c) $\tan^{-1} z = \dfrac{i}{2}\ln\left(\dfrac{i+z}{i-z}\right)$,
(d) $\sinh^{-1} z = \ln(z + \sqrt{z^2+1})$,
(e) $\cosh^{-1} z = \ln(z + \sqrt{z^2-1})$,
(f) $\tanh^{-1} z = \dfrac{1}{2}\ln\left(\dfrac{1+z}{1-z}\right)$.

Hint. 1. Express the trigonometric and hyperbolic functions in terms of exponentials. 2. Solve for the exponential and then for the exponent.

6.1.17 In the quantum theory of photoionization we encounter the identity

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = \exp(-2b\cot^{-1} a),$$

in which a and b are real. Verify this identity.

6.1.18 A plane wave of light of angular frequency ω is represented by

$$e^{i\omega(t-nx/c)}.$$

In a certain substance the simple real index of refraction n is replaced by the complex quantity $n - ik$. What is the effect of k on the wave? What does k correspond to physically? The generalization of a quantity from real to complex form occurs frequently in physics. Examples range from the complex Young's modulus of viscoelastic materials to the complex (optical) potential of the "cloudy crystal ball" model of the atomic nucleus.

6.1.19 We see that for the angular momentum components defined in Exercise 2.5.14

$$L_x - iL_y \neq (L_x + iL_y)^*.$$

Explain why this occurs.

6.1.20 Show that the **phase** of $f(z) = u + iv$ is equal to the imaginary part of the logarithm of $f(z)$. Exercise 8.2.13 depends on this result.

6.1.21 (a) Show that $e^{\ln z}$ always equals z.

(b) Show that $\ln e^z$ does not always equal z.

6.1.22 The infinite product representations of Section 5.11 hold when the real variable x is replaced by the complex variable z. From this, develop infinite product representations for
(a) $\sinh z$, (b) $\cosh z$.

6.1.23 The equation of motion of a mass m **relative to a rotating coordinate system** is

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right) - m\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right).$$

Consider the case $\mathbf{F} = 0, \mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, with ω constant. Show that the replacement of $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ by $z = x + iy$ leads to

$$\frac{d^2 z}{dt^2} + i2\omega\frac{dz}{dt} - \omega^2 z = 0.$$

Note. This ODE may be solved by the substitution $z = fe^{-i\omega t}$.

6.1.24 Using the complex arithmetic available in FORTRAN, write a program that will calculate the complex exponential e^z from its series expansion (definition). Calculate e^z for $z = e^{in\pi/6}, n = 0, 1, 2, \ldots, 12$. Tabulate the phase angle $(\theta = n\pi/6), \Re z, \Im z, \Re(e^z), \Im(e^z), |e^z|$, and the phase of e^z.

Check value. $n = 5, \theta = 2.61799, \Re(z) = -0.86602,$
$\Im z = 0.50000, \Re(e^z) = 0.36913, \Im(e^z) = 0.20166,$
$|e^z| = 0.42062$, phase$(e^z) = 0.50000$.

6.1.25 Using the complex arithmetic available in FORTRAN, calculate and tabulate $\Re(\sinh z), \Im(\sinh z), |\sinh z|$, and phase$(\sinh z)$ for $x = 0.0(0.1)1.0$ and $y = 0.0(0.1)1.0$.
Hint. Beware of dividing by zero when calculating an angle as an arc tangent.

Check value. $z = 0.2 + 0.1i, \Re(\sinh z) = 0.20033,$
$\Im(\sinh z) = 0.10184, |\sinh z| = 0.22473,$
phase$(\sinh z) = 0.47030$.

6.1.26 Repeat Exercise 6.1.25 for $\cosh z$.

6.2.1 The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.

(a) Assuming that the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

Solutions of Laplace's equation such as $u(x, y)$ and $v(x, y)$ are called **harmonic** functions.

(b) Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

and give a geometric interpretation.
Hint. The technique of Section 1.6 allows you to construct vectors normal to the curves $u(x, y) = c_i$ and $v(x, y) = c_j$.

6.2.7 The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $f^*(z^*)$ is also analytic.

6.2.11 A proof of the Schwarz inequality (Section 10.4) involves minimizing an expression,

$$f = \psi_{aa} + \lambda\psi_{ab} + \lambda^*\psi_{ab}^* + \lambda\lambda^*\psi_{bb} \geq 0.$$

The ψ are integrals of products of functions; ψ_{aa} and ψ_{bb} are real, ψ_{ab} is complex and λ is a complex parameter.

(a) Differentiate the preceding expression with respect to λ^*, treating λ as an independent parameter, independent of λ^*. Show that setting the derivative $\partial f/\partial\lambda^*$ equal to zero yields

$$\lambda = -\frac{\psi_{ab}^*}{\psi_{bb}}.$$

(b) Show that $\partial f/\partial\lambda = 0$ leads to the same result.

(c) Let $\lambda = x + iy$, $\lambda^* = x - iy$. Set the x and y derivatives equal to zero and show that again

$$\lambda = -\frac{\psi_{ab}^*}{\psi_{bb}}.$$

This independence of λ and λ^* appears again in Section 17.7.

6.4.8 Using the Cauchy integral formula for the nth derivative, convert the following Rodrigues formulas into the corresponding so-called Schlaefli integrals.

(a) Legendre:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

ANS. $\displaystyle \frac{(-1)^n}{2^n} \cdot \frac{1}{2\pi i}\oint \frac{(1 - z^2)^n}{(z - x)^{n+1}}dz$

(b) Hermite:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

(c) Laguerre:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}).$$

Note. From the Schlaefli integral representations one can develop generating functions for these special functions. Compare Sections 12.4, 13.1, and 13.2.

6.5.8 Develop the first three nonzero terms of the Laurent expansion of

$$f(z) = (e^z - 1)^{-1}$$

about the origin. Notice the resemblance to the Bernoulli number-generating function, Eq. (5.144) of Section 5.9.

6.6.1 The function $f(z)$ expanded in a Laurent series exhibits a pole of order m at $z = z_0$. Show that the coefficient of $(z - z_0)^{-1}, a_{-1}$, is given by

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]_{z=z_0},$$

with

$$a_{-1} = [(z - z_0) f(z)]_{z=z_0},$$

when the pole is a simple pole ($m = 1$). These equations for a_{-1} are extremely useful in determining the residue to be used in the residue theorem of Section 7.1.
Hint. The technique that was so successful in proving the uniqueness of power series, Section 5.7, will work here also.

6.6.3 In analogy with Example 6.6.1, consider in detail the phase of each factor and the resultant overall phase of $f(z) = (z^2 + 1)^{1/2}$ following a contour similar to that of Fig. 6.16 but encircling the new branch points.

6.6.4 The Legendre function of the second kind, $Q_\nu(z)$, has branch points at $z = \pm 1$. The branch points are joined by a cut line along the real (x)-axis.

(a) Show that $Q_0(z) = \frac{1}{2} \ln((z+1)/(z-1))$ is single-valued (with the real axis $-1 \le x \le 1$ taken as a cut line).

(b) For real argument x and $|x| < 1$ it is convenient to take

$$Q_0(x) = \tfrac{1}{2} \ln \tfrac{1+x}{1-x}.$$

Show that

$$Q_0(x) = \tfrac{1}{2}[Q_0(x + i0) + Q_0(x - i0)].$$

Here $x + i0$ indicates that z approaches the real axis from above, and $x - i0$ indicates an approach from below.

6.7.3 Discuss the transformations

(a) $w(z) = \sin z$, (c) $w(z) = \sinh z$,
(b) $w(z) = \cos z$, (d) $w(z) = \cosh z$.

Show how the lines $x = c_1, y = c_2$ map into the w-plane. Note that the last three transformations can be obtained from the first one by appropriate translation and/or rotation.

6.7.6 An integral representation of the Bessel function follows the contour in the t-plane shown in Fig. 6.24. Map this contour into the θ-plane with $t = e^{\theta}$. Many additional examples of mapping are given in Chapters 11, 12, and 13.

6.8.1 Expand $w(x)$ in a Taylor series about the point $z = z_0$, where $f'(z_0) = 0$. (Angles are not preserved.) Show that if the first $n-1$ derivatives vanish but $f^{(n)}(z_0) \neq 0$, then angles in the z-plane with vertices at $z = z_0$ appear in the w-plane multiplied by n.

6.8.2 Develop the transformations that create each of the four cylindrical coordinate systems:

(a) Circular cylindrical: $x = \rho \cos \varphi$, $y = \rho \sin \varphi$.

(b) Elliptic cylindrical: $x = a \cosh u \cos v$, $y = a \sinh u \sin v$.

(c) Parabolic cylindrical: $x = \xi\eta$, $y = \frac{1}{2}(\eta^2 - \xi^2)$.

(d) Bipolar: $$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}, \qquad y = \frac{a \sin \xi}{\cosh \eta - \cos \xi}.$$

Note. These transformations are not necessarily analytic.

6.8.3 In the transformation
$$e^z = \frac{a - w}{a + w},$$
how do the coordinate lines in the z-plane transform? What coordinate system have you constructed?

7.1.2 Locate the singularities and evaluate the residues of each of the following functions.

(a) $z^{-n}(e^z - 1)^{-1}, \quad z \neq 0$,

(b) $\dfrac{z^2 e^z}{1 + e^{2z}}$.

(c) Find a closed-form expression (that is, not a sum) for the sum of the finite-plane singularities.

(d) Using the result in part (c), what is the residue at $|z| \to \infty$?

www.elsolucionario.net

Hint. See Section 5.9 for expressions involving Bernoulli numbers. Note that Eq. (5.144) cannot be used to investigate the singularity at $z \to \infty$, since this series is only valid for $|z| < 2\pi$.

7.1.23 Several of the Bromwich integrals, Section 15.12, involve a portion that may be approximated by

$$I(y) = \int_{a-iy}^{a+iy} \frac{e^{zt}}{z^{1/2}} dz.$$

Here a and t are positive and finite. Show that

$$\lim_{y\to\infty} I(y) = 0.$$

7.1.27 Apply the techniques of Example 7.1.5 to the evaluation of the improper integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 - \sigma^2}.$$

(a) Let $\sigma \to \sigma + i\gamma$.

(b) Let $\sigma \to \sigma - i\gamma$.

(c) Take the Cauchy principal value.

7.1.28 The integral in Exercise 7.1.17 may be transformed into

$$\int_0^{\infty} e^{-y} \frac{y^2}{1+e^{-2y}} dy = \frac{\pi^3}{16}.$$

Evaluate this integral by the Gauss-Laguerre quadrature and compare your result with $\pi^3/16$.

ANS. Integral = 1.93775 (10 points).

7.2.9 Show that

$$\delta(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t(t-x)}$$

is a valid representation of the delta function in the sense that

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0).$$

Assume that $f(x)$ satisfies the condition for the existence of a Hilbert transform.
Hint. Apply Eq. (7.84) twice.

www.elsolucionario.net

7.3.7 Assume $H_\nu^{(1)}(s)$ to have a negative power-series expansion of the form

$$H_\nu^{(1)}(s) = \sqrt{\frac{2}{\pi s}} e^{i(s-\nu(\pi/2)-\pi/4)} \sum_{n=0}^{\infty} a_{-n} s^{-n},$$

with the coefficient of the summation obtained by the method of steepest descent. Substitute into Bessel's equation and show that you reproduce the asymptotic series for $H_\nu^{(1)}(s)$ given in Section 11.6.

8.1.3 Show that, as $s - n \to$ negative integer,

$$\frac{(s-n)!}{(2s-2n)!} \to \frac{(-1)^{n-s}(2n-2s)!}{(n-s)!}.$$

Here s and n are integers with $s < n$. This result can be used to avoid negative factorials, such as in the series representations of the spherical Neumann functions and the Legendre functions of the second kind.

8.1.12 (a) Develop recurrence relations for $(2n)!!$ and for $(2n+1)!!$.

(b) Use these recurrence relations to calculate (or to define) $0!!$ and $(-1)!!$.

ANS. $0!! = 1, \quad (-1)!! = 1.$

8.1.13 For s a nonnegative integer, show that

$$(-2s-1)!! = \frac{(-1)^s}{(2s-1)!!} = \frac{(-1)^s 2^s s!}{(2s)!}.$$

item[**8.1.18**] From one of the definitions of the factorial or gamma function, show that

$$|(ix)!|^2 = \frac{\pi x}{\sinh \pi x}.$$

8.1.27 Write a function subprogram $FACT(N)$ (fixed-point independent variable) that will calculate $N!$. Include provision for rejection and appropriate error message if N is negative.
Note. For small integer N, direct multiplication is simplest. For large N, Eq. (8.55), Stirling's series would be appropriate.

8.1.28 (a) Write a function subprogram to calculate the double factorial ratio $(2N-1)!!/(2N)!!$. Include provision for $N = 0$ and for rejection and an error message if N is negative. Calculate and tabulate this ratio for $N = 1(1)100$.

(b) Check your function subprogram calculation of $199!!/200!!$ against the value obtained from Stirling's series (Section 8.3).

ANS. $\dfrac{199!!}{200!!} = 0.056348.$

8.1.29 Using either the FORTRAN-supplied GAMMA or a library supplied subroutine for $x!$ or $\Gamma(x)$, determine the value of x for which $\Gamma(x)$ is a minimum ($1 \le x \le 2$) and this minimum value of $\Gamma(x)$. Notice that although the minimum value of $\Gamma(x)$ may be obtained to about six significant figures (single precision), the corresponding value of x is much less accurate. Why this relatively low accuracy?

8.1.30 The factorial function expressed in integral form can be evaluated by the Gauss-Laguerre quadrature. For a 10-point formula the resultant $x!$ is theoretically exact for x an integer, 0 up through 19. What happens if x is not an integer? Use the Gauss-Laguerre quadrature to evaluate $x!$, $x = 0.0(0.1)2.0$. Tabulate the absolute error as a function of x.

Check value. $x!_{\text{exact}} - x!_{\text{quadrature}} = 0.00034 \quad \text{for} \quad x = 1.3.$

8.2.10 Derive the polygamma function recurrence relation

$$\psi^{(m)}(1+z) = \psi^{(m)}(z) + (-1)^m m!/z^{m+1}, \quad m = 0, 1, 2, \ldots.$$

8.2.21 Verify the contour integral representation of $\zeta(s)$,

$$\zeta(s) = -\frac{(-s)!}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz.$$

The contour C is the same as that for Eq. (8.35). The points $z = \pm 2n\pi i$, $n = 1, 2, 3, \ldots$, are all excluded.

8.2.23 Using the complex variable capability of FORTRAN calculate $\Re(1+ib)!$, $\Im(1+ib)!$, $|(1+ib)!|$ and phase $(1+ib)!$ for $b = 0.0(0.1)1.0$. Plot the phase of $(1+ib)!$ versus b.
Hint. Exercise 8.2.3 offers a convenient approach. You will need to calculate $\zeta(n)$.

8.3.3 By integrating Eq. (8.51) from $z-1$ to z and then letting $z \to \infty$, evaluate the constant C_1 in the asymptotic series for the digamma function $\psi(z)$.

8.3.5 By direct expansion, verify the doubling formula for $z = n + \frac{1}{2}$; n is an integer.

8.3.10 Calculate the binomial coefficient $\binom{2n}{n}$ to six significant figures for $n = 10$, 20, and 30. Check your values by

(a) a Stirling series approximation through terms in n^{-1},
(b) a double precision calculation.

ANS. $\binom{20}{10} = 1.84756 \times 10^5$, $\binom{40}{20} = 1.37846 \times 10^{11}$, $\binom{60}{30} = 1.18264 \times 10^{17}$.

8.3.11 Write a program (or subprogram) that will calculate $\log_{10}(x!)$ directly from Stirling's series. Assume that $x \geq 10$. (Smaller values could be calculated via the factorial recurrence relation.) Tabulate $\log_{10}(x!)$ versus x for $x = 10(10)300$. Check your results against AMS-55 (see Additional Readings for this reference) or by direct multiplication (for $n = 10$, 20, and 30).

Check value. $\log_{10}(100!) = 157.97$.

8.3.12 Using the complex arithmetic capability of FORTRAN, write a subroutine that will calculate $\ln(z!)$ for complex z based on Stirling's series. Include a test and an appropriate error message if z is too close to a negative real integer. Check your subroutine against alternate calculations for z real, z pure imaginary, and $z = 1 + ib$ (Exercise 8.2.23).

Check values.
$$\begin{aligned} |(i0.5)!| &= 0.82618 \\ \text{phase } (i0.5)! &= -0.24406. \end{aligned}$$

8.4.1 Derive the doubling formula for the factorial function by integrating $(\sin 2\theta)^{2n+1} = (2\sin\theta\cos\theta)^{2n+1}$ (and using the beta function).

8.4.19 Tabulate the beta function $B(p, q)$ for p and $q = 1.0(0.1)2.0$ independently.
Check value. $B(1.3, 1.7) = 0.40774$.

8.4.20 (a) Write a subroutine that will calculate the incomplete beta function $B_x(p, q)$. For $0.5 < x \leq 1$ you will find it convenient to use the relation
$$B_x(p, q) = B(p, q) - B_{1-x}(q, p).$$

(b) Tabulate $B_x(\frac{3}{2}, \frac{3}{2})$. Spot check your results by using the Gauss-Legendre quadrature.

8.5.13 (a) Write a subroutine that will calculate the incomplete gamma functions $\gamma(n, x)$ and $\Gamma(n, x)$ for n a positive integer. Spot check $\Gamma(n, x)$ by Gauss-Laguerre quadratures.

(b) Tabulate $\gamma(n, x)$ and $\Gamma(n, x)$ for $x = 0.0(0.1)1.0$ and $n = 1$, 2, 3.

8.5.14 Calculate the potential produced by a $1S$ hydrogen electron (Exercise 8.5.4) (Fig. 8.10). Tabulate $V(r)/(q/4\pi\varepsilon_0 a_0)$ for $x = 0.0(0.1)4.0$. Check your calculations for $r \ll 1$ and for $r \gg 1$ by calculating the limiting forms given in Exercise 8.5.4.

8.5.15 Using Eqs. (5.182) and (8.75), calculate the exponential integral $E_1(x)$ for
(a) $x = 0.2(0.2)1.0$, (b) $x = 6.0(2.0)10.0$.
Program your own calculation but check each value, using a library subroutine if available. Also check your calculations at each point by a Gauss-Laguerre quadrature.

You'll find that the power-series converges rapidly and yields high precision for small x. The asymptotic series, even for $x = 10$, yields relatively poor accuracy.

Check values. $E_1(1.0) = 0.219384$
$E_1(10.0) = 4.15697 \times 10^{-6}$.

8.5.16 The two expressions for $E_1(x)$, (1) Eq. (5.182), an asymptotic series and (2) Eq. (8.75), a convergent power series, provide a means of calculating the Euler-Mascheroni constant γ to high accuracy. Using double precision, calculate γ from Eq. (8.75), with $E_1(x)$ evaluated by Eq. (5.182).
Hint. As a convenient choice take x in the range 10 to 20. (Your choice of x will set a limit on the accuracy of your result.) To minimize errors in the alternating series of Eq. (8.75), accumulate the positive and negative terms separately.

ANS. For $x = 10$ and "double precision," $\gamma = 0.57721566$.

9.2.16 Bernoulli's equation,

$$\frac{dy}{dx} + f(x)y = g(x)y^n,$$

is nonlinear for $n \neq 0$ or 1. Show that the substitution $u = y^{1-n}$ reduces Bernoulli's equation to a linear equation. (See Section 18.4.)

ANS. $\dfrac{du}{dx} + (1-n)f(x)u = (1-n)g(x)$.

9.2.17 Solve the linear, first-order equation, Eq. (9.25), by assuming $y(x) = u(x)v(x)$, where $v(x)$ is a solution of the corresponding homogeneous equation $[q(x) = 0]$. This is the method of **variation of parameters** due to Lagrange. We apply it to second-order equations in Exercise 9.6.25.

9.7.1 Verify Eq. (9.168),

$$\int (v\mathcal{L}_2 u - u\mathcal{L}_2 v)d\tau_2 = \int p(v\boldsymbol{\nabla}_2 u - u\boldsymbol{\nabla}_2 v)\cdot d\boldsymbol{\sigma}_2.$$

9.7.5 The homogeneous Helmholtz equation

$$\boldsymbol{\nabla}^2\varphi + \lambda^2\varphi = 0$$

has eigenvalues λ_i^2 and eigenfunctions φ_i. Show that the corresponding Green's function that satisfies

$$\boldsymbol{\nabla}^2 G(\mathbf{r}_1, \mathbf{r}_2) + \lambda^2 G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

may be written as

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{i=1}^{\infty} \frac{\varphi_i(\mathbf{r}_1)\varphi_i(\mathbf{r}_2)}{\lambda_i^2 - \lambda^2}.$$

An expansion of this form is called a **bilinear** expansion. If the Green's function is available in **closed** form, this provides a means of generating functions.

9.7.8 A charged conducting ring of radius a (Example 12.3.3) may be described by

$$\rho(\mathbf{r}) = \frac{q}{2\pi a^2}\delta(r-a)\delta(\cos\theta).$$

Using the known Green's function for this system, Eq. (9.187), find the electrostatic potential.
Hint. Exercise 12.6.3 will be helpful.

9.7.9 Changing a separation constant from k^2 to $-k^2$ and putting the discontinuity of the first derivative into the z-dependence, show that

$$\frac{1}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} e^{im(\varphi_1-\varphi_2)} J_m(k\rho_1)J_m(k\rho_2)e^{-k|z_1-z_2|}dk.$$

Hint. The required $\delta(\rho_1 - \rho_2)$ may be obtained from Exercise 15.1.2.

9.7.10 Derive the expansion

$$\begin{aligned}\frac{\exp[ik|\mathbf{r}_1 - \mathbf{r}_2|]}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} &= ik\sum_{l=0}^{\infty}\left\{\begin{array}{l} j_l(kr_1)h_l^{(1)}(kr_2),\ r_1 < r_2 \\ \\ j_l(kr_2)h_l^{(1)}(kr_1),\ r_1 > r_2 \end{array}\right\} \\ &\times \sum_{m=-l}^{l} Y_l^m(\theta_1,\varphi_1)Y_l^{m*}(\theta_2,\varphi_2).\end{aligned}$$

Hint. The left side is a known Green's function. Assume a spherical harmonic expansion and work on the remaining radial dependence. The spherical harmonic closure relation, Exercise 12.6.6, covers the angular dependence.

9.7.11 Show that the modified Helmholtz operator Green's function

$$\frac{\exp(-k|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}$$

has the spherical polar coordinate expansion

$$\frac{\exp(-k|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} = k\sum_{l=0}^{\infty} i_l(kr_<)k_l(kr_>)\sum_{m=-l}^{l} Y_l^m(\theta_1,\varphi_1)Y_l^{m*}(\theta_2,\varphi_2).$$

Note. The modified spherical Bessel functions $i_l(kr)$ and $k_l(kr)$ are defined in Exercise 11.7.15.

9.7.12 From the spherical Green's function of Exercise 9.7.10, derive the plane-wave expansion

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\gamma),$$

where γ is the angle included between $\mathbf{k}$ and $\mathbf{r}$. This is the Rayleigh equation of Exercise 12.4.7.
Hint. Take $\mathbf{r}_2 \gg \mathbf{r}_1$ so that

$$|\mathbf{r}_1 - \mathbf{r}_2| \to r_2 - \mathbf{r}_{20} \cdot \mathbf{r}_1 = r_2 - \frac{\mathbf{k}\cdot\mathbf{r}_1}{k}.$$

Let $r_2 \to \infty$ and cancel a factor of e^{ikr_2}/r_2.

9.7.13 From the results of Exercises 9.7.10 and 9.7.12, show that

$$e^{ix} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(x).$$

9.7.14 (a) From the circular cylindrical coordinate expansion of the Laplace Green's function (Eq. (9.197)), show that

$$\frac{1}{(\rho^2+z^2)^{1/2}} = \frac{2}{\pi}\int_0^{\infty} K_0(k\rho)\cos kz dk.$$

This same result is obtained directly in Exercise 15.3.11.

(b) As a special case of part (a) show that

$$\int_0^{\infty} K_0(k) dk = \frac{\pi}{2}.$$

9.7.15 Noting that

$$\psi_k(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

is an eigenfunction of

$$(\boldsymbol{\nabla}^2 + k^2)\psi_k(\mathbf{r}) = 0$$

(Eq. (9.206)), show that the Green's function of $\mathcal{L} = \boldsymbol{\nabla}^2$ may be expanded as

$$\frac{1}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\frac{d^3k}{k^2}.$$

9.7.16 Using Fourier transforms, show that the Green's function satisfying the nonhomogeneous Helmholtz equation

$$(\boldsymbol{\nabla}^2 + k_0^2) G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

is

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}}{k^2 - k_0^2} d^3k,$$

in agreement with Eq. (9.213).

9.7.17 The basic equation of the scalar Kirchhoff diffraction theory is

$$\psi(\mathbf{r}_1) = \frac{1}{4\pi}\int_{S_2}\left[\frac{e^{ikr}}{r}\boldsymbol{\nabla}\psi(\mathbf{r}_2) - \psi(\mathbf{r}_2)\boldsymbol{\nabla}\left(\frac{e^{ikr}}{r}\right)\right]\cdot d\boldsymbol{\sigma}_2,$$

where ψ satisfies the homogeneous Helmholtz equation and $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Derive this equation. Assume that $\mathbf{r}_1$ is interior to the closed surface S_2.
Hint. Use Green's theorem.

9.7.18 The Born approximation for the scattered wave is given by Eq. (9.203b) (and Eq. (9.211)). From the asymptotic form, Eq. (9.199),

$$f_k(\theta,\varphi)\frac{e^{ikr}}{r} = -\frac{2m}{\hbar^2}\int V(\mathbf{r}_2)\frac{e^{ik|\mathbf{r}-\mathbf{r}_2|}}{4\pi|\mathbf{r}-\mathbf{r}_2|}e^{i\mathbf{k}_0\cdot\mathbf{r}_2}d^3r_2.$$

For a scattering potential $V(\mathbf{r}_2)$ that is independent of angles and for $r \gg r_2$, show that

$$f_k(\theta,\varphi) = -\frac{2m}{\hbar^2}\int_0^\infty r_2 V(\mathbf{r}_2)\frac{\sin(|\mathbf{k}_0-\mathbf{k}|r_2)}{|\mathbf{k}_0-\mathbf{k}|}dr_2.$$

Here $\mathbf{k}_0$ is in the $\theta = 0$ (original z-axis) direction, whereas $\mathbf{k}$ is in the (θ,φ) direction. The magnitudes are equal: $|\mathbf{k}_0| = |\mathbf{k}|$; m is the reduced mass.
Hint. You have Exercise 9.7.12 to simplify the exponential and Exercise 15.3.20 to transform the three-dimensional Fourier exponential transform into a one-dimensional Fourier sine transform.

9.7.19 Calculate the scattering amplitude $f_k(\theta,\varphi)$ for a mesonic potential $\mathbf{V}(r) = V_0(e^{-\alpha r}/\alpha r)$.
Hint. This particular potential permits the Born integral, Exercise 9.7.18, to be evaluated as a Laplace transform.

$$\textit{ANS. } f_k(\theta,\varphi) = -\frac{2mV_0}{\hbar^2\alpha}\frac{1}{\alpha^2+(\mathbf{k}_0-\mathbf{k})^2}.$$

9.7.20 The mesonic potential $V(r) = V_0(e^{-\alpha r}/\alpha r)$ may be used to describe the Coulomb scattering of two charges q_1 and q_2. We let $\alpha \to 0$ and $V_0 \to 0$ but take the ratio V_0/α to be $q_1q_2/4\pi\varepsilon_0$. (For Gaussian units omit the $4\pi\varepsilon_0$.) Show that the differential scattering cross section $d\sigma/d\Omega = |f_k(\theta,\varphi)|^2$ is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{q_1q_2}{4\pi\varepsilon_0}\right)^2\frac{1}{16E^2\sin^4(\theta/2)}, \quad E = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m}.$$

It happens (coincidentally) that this Born approximation is in exact agreement with both the exact quantum mechanical calculations and the classical Rutherford calculation.

10.1.5 $U_n(x)$, the Chebyshev polynomial (type II) satisfies the ODE, Eq. (13.1),

$$(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0.$$

(a) Locate the singular points that appear in the finite plane, and show whether they are regular or irregular.

(b) Put this equation in self-adjoint form.

(c) Identify the complete eigenvalue.

(d) Identify the weighting function.

10.1.6 For the very special case $\lambda = 0$ and $q(x) = 0$ the self-adjoint eigenvalue equation becomes

$$\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] = 0,$$

satisfied by

$$\frac{du}{dx} = \frac{1}{p(x)}.$$

Use this to obtain a "second" solution of the following:

(a) Legendre's equation,
(b) Laguerre's equation,
(c) Hermite's equation.

ANS (a) $u_2(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$,

(b) $u_2(x) - u_2(x_0) = \int_{x_0}^{x} e^t \frac{dt}{t}$,

(c) $u_2(x) = \int_0^x e^{t^2}\, dt$.

These second solutions illustrate the divergent behavior usually found in a second solution.
Note. In all three cases $u_1(x) = 1$.

10.1.7 Given that $\mathcal{L}u = 0$ and $g\mathcal{L}u$ is self-adjoint, show that for the adjoint operator $\bar{\mathcal{L}}$, $\bar{\mathcal{L}}(gu) = 0$.

10.1.8 For a second-order differential operator $\mathcal{L}$ that is self-adjoint show that

$$\int_a^b [y_2\mathcal{L}y_1 - y_1\mathcal{L}y_2]dx = p(y_1'y_2 - y_1y_2')\mid_a^b .$$

10.1.9 Show that if a function ψ is required to satisfy Laplace's equation in a finite region of space and to satisfy Dirichlet boundary conditions over the entire closed bounding surface, then ψ is unique.
Hint. One of the forms of Green's theorem, Section 1.11, will be helpful.

10.1.11 Within the framework of quantum mechanics (Eqs. (10.26) and following), show that the following are Hermitian operators:

(a) momentum $\mathbf{p} = -i\hbar\boldsymbol{\nabla} \equiv -i\dfrac{h}{2\pi}\boldsymbol{\nabla}$

(b) angular momentum $\mathbf{L} = -i\hbar\mathbf{r} \times \boldsymbol{\nabla} \equiv -i\frac{h}{2\pi}\mathbf{r} \times \boldsymbol{\nabla}$.

Hint. In Cartesian form $\mathbf{L}$ is a linear combination of noncommuting Hermitian operators.

10.1.17 A quantum mechanical expectation value is defined by

$$\langle A \rangle = \int \psi^*(x) A\psi(x) dx,$$

where A is a linear operator. Show that demanding that $\langle A \rangle$ be real means that A must be Hermitian - with respect to $\psi(x)$.

10.1.18 From the definition of adjoint, Eq. (10.27), show that $A^{\dagger\dagger} = A$ in the sense that $\int \psi_1^* A^{\dagger\dagger}\psi_2 d\tau = \int \psi_1^* A\psi_2 d\tau$. The adjoint of the adjoint is the original operator.
Hint. The functions ψ_1 and ψ_2 of Eq. (10.27) represent a class of functions. The subscripts 1 and 2 may be interchanged or replaced by other subscripts.

10.1.19 The Schrödinger wave equation for the deuteron (with a Woods-Saxon potential) is

$$-\frac{\hbar^2}{2M}\boldsymbol{\nabla}^2\psi + \frac{V_0}{1+\exp[(r-r_0)/a]}\psi = E\psi.$$

Here $E = -2.224$ MeV, a is a "thickness parameter," 0.4×10^{-13} cm. Expressing lengths in fermis (10^{-13} cm) and energies in million electron volts (MeV), we may rewrite the wave equation as

$$\frac{d^2}{dr^2}(r\psi) + \frac{1}{41.47}\left[E - \frac{V_0}{1+\exp((r-r_0)/a)}\right](r\psi) = 0.$$

E is assumed known from experiment. The goal is to find V_0 for a specified value of r_0 (say, $r_0 = 2.1$). If we let $y(r) = r\psi(r)$, then $y(0) = 0$ and we take $y'(0) = 1$. Find V_0 such that $y(20.0) = 0$. (This should be $y(\infty)$, but $r = 20$ is far enough beyond the range of nuclear forces to approximate infinity.)

ANS. For $a = 0.4$ and $r_0 = 2.1$ fm, $V_0 = -34.159$ MeV.

10.1.20 Determine the nuclear potential well parameter V_0 of Exercise 10.1.19 as a function of r_0 for $r = 2.00(0.05)2.25$ fermis. Express your results as a power law

$$|V_0| r_0^{\nu} = k.$$

Determine the exponent ν and the constant k. This power-law formulation is useful for accurate interpolation.

10.1.21 In Exercise 10.1.19 it was assumed that 20 fermis was a good approximation to infinity. Check on this by calculating V_0 for $r\psi(r) = 0$ at (a) $r = 15$, (b) $r = 20$, (c) $r = 25$, and (d) $r = 30$. Sketch your results. Take $r_0 = 2.10$ and $a = 0.4$ (fermis).

10.1.22 For a quantum particle moving in a potential well, $V(x) = \frac{1}{2}m\omega^2 x^2$, the Schrödinger wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x),$$

or

$$\frac{d^2\psi(z)}{dz^2} - z^2\psi(z) = -\frac{2E}{\hbar\omega}\psi(z),$$

where $z = (m\omega/\hbar)^{1/2}x$. Since this operator is even, we expect solutions of definite parity. For the initial conditions that follow, integrate out from the origin and determine the minimum constant $2E/\hbar\omega$ that will lead to $\psi(\infty) = 0$ in each case. (You may take $z = 6$ as an approximation of infinity.)

(a) For an even eigenfunction,

$$\psi(0) = 1, \qquad \psi'(0) = 0.$$

(b) For an odd eigenfunction,

$$\psi(0) = 0, \qquad \psi'(0) = 1.$$

Note. Analytical solutions appear in Section 13.1.

10.2.2 (a) The vectors $\mathbf{e}_n$ are orthogonal to each other: $\mathbf{e}_n \cdot \mathbf{e}_m = 0$ for $n \neq m$. Show that they are linearly independent.

(b) The functions $\psi_n(x)$ are orthogonal to each other over the interval $[a, b]$ and with respect to the weighting function $w(x)$. Show that the $\psi_n(x)$ are linearly independent.

10.2.5 (a) Show that the first derivatives of the Legendre polynomials satisfy a self-adjoint differential equation with eigenvalue $\lambda = n(n+1) - 2$.

(b) Show that these Legendre polynomial derivatives satisfy an orthogonality relation

$$\int_{-1}^{1} P'_m(x)P'_n(x)(1-x^2)dx = 0, \qquad m \neq n.$$

Note. In Section 12.5, $(1-x^2)^{1/2}P'_n(x)$ will be labeled an associated Legendre polynomial, $P_n^1(x)$.

10.2.8 (a) Show that the Liouville substitution

$$u(x) = v(\xi)[p(x)w(x)]^{-1/4}, \quad \xi = \int_a^x [\frac{w(t)}{p(t)}]^{1/2} dt$$

transforms

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}u\right] + [\lambda w(x) - q(x)]u(x) = 0$$

into

$$\frac{d^2 v}{d\xi^2} + [\lambda - Q(\xi)]v(\xi) = 0,$$

where

$$Q(\xi) = \frac{q(x(\xi))}{w(x(\xi))} + [p(x(\xi))w(x(\xi))]^{-1/4} \frac{d^2}{d\xi^2}(pw)^{1/4}.$$

(b) If $v_1(\xi)$ and $v_2(\xi)$ are obtained from $u_1(x)$ and $u_2(x)$, respectively, by a Liouville substitution, show that $\int_a^b w(x)u_1u_2dx$ is transformed into $\int_0^c v_1(\xi)v_2(\xi)d\xi$ with $c = \int_a^b [\frac{w}{p}]^{1/2} dx$.

10.2.10 With $\mathcal{L}$ **not** self-adjoint,

$$\mathcal{L}u_i + \lambda_i w u_i = 0$$

and

$$\bar{\mathcal{L}}v_j + \lambda_j w v_j = 0.$$

(a) Show that

$$\int_a^b v_j \mathcal{L} u_i dx = \int_a^b u_i \bar{\mathcal{L}} v_j dx,$$

provided

$$u_i p_0 v_j' \Big|_a^b = v_j p_0 u_i' \Big|_a^b$$

and

$$u_i(p_1 - p_0')v_j \Big|_a^b = 0.$$

(b) Show that the orthogonality integral for the eigenfunctions u_i and v_j becomes

$$\int_a^b u_i v_j w \ dx = 0 \qquad (\lambda_i \neq \lambda_j).$$

10.2.11 In Exercise 9.5.8 the series solution of the Chebyshev equation is found to be convergent for all eigenvalues n. Therefore n is **not** quantized by the argument used for Legendre's (Exercise 9.5.5). Calculate the sum of the indicial equation $k = 0$ Chebyshev series for $n = v = 0.8, 0.9$, and 1.0 and for $x = 0.0(0.1)0.9$.

Note. The Chebyshev series recurrence relation is given in Exercise 5.2.16.

10.2.12 (a) Evaluate the $n = \nu = 0.9$, indicial equation $k = 0$ Chebyshev series for $x = 0.98, 0.99$, and 1.00. The series converges very slowly at $x = 1.00$. You may wish to use double precision. Upper bounds to the error in your calculation can be set by comparison with the $\nu = 1.0$ case, which corresponds to $(1 - x^2)^{1/2}$.

(b) These series solutions for eigenvalue $\nu = 0.9$ and for $\nu = 1.0$ are obviously **not** orthogonal, despite the fact that they satisfy a self-adjoint eigenvalue equation with different eigenvalues. From the behavior of the solutions in the vicinity of $x = 1.00$ try to formulate a hypothesis as to why the proof of orthogonality does not apply.

10.2.13 The Fourier expansion of the (asymmetric) square wave is given by Eq. (10.38). With $h = 2$, evaluate this series for $x = 0(\pi/18)\pi/2$, using the first (a) 10 terms, (b) 100 terms of the series.
Note. For 10 terms and $x = \pi/18$, or 10°, your Fourier representation has a sharp hump. This is the Gibbs phenomenon of Section 14.5. For 100 terms this hump has been shifted over to about 1°.

10.2.14 The **symmetric** square wave

$$f(x) = \begin{cases} 1, & |x| < \dfrac{\pi}{2} \\ -1, & \dfrac{\pi}{2} < |x| < \pi \end{cases}$$

has a Fourier expansion

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos(2n+1)x}{2n+1}.$$

Evaluate this series for $x = 0(\pi/18)\pi/2$ using the first
(a) 10 terms, (b) 100 terms of the series.
Note. As in Exercise 10.2.13, the Gibbs phenomenon appears at the discontinuity. This means that a Fourier series is not suitable for precise numerical work in the vicinity of a discontinuity.

10.3.1 Rework Example 10.3.1 by replacing $\varphi_n(x)$ by the conventional Legendre polynomial, $P_n(x)$.

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}.$$

Using Eqs. (10.47a), and (10.49a), construct $P_0, P_1(x)$, and $P_2(x)$.

ANS. $P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{3}{2}x^2 - \frac{1}{2}$.

10.3.9 Form an orthogonal set over the interval $0 \le x < \infty$, using $u_n(x) = e^{-nx}, n = 1, 2, 3, \ldots$. Take the weighting factor, $w(x)$, to be unity. These functions are solutions of $u_n'' - n^2 u_n = 0$, which is clearly already in Sturm-Liouville (self–adjoint) form. Why doesn't the Sturm-Liouville theory guarantee the orthogonality of these functions?

10.4.6 Differentiate Eq. (10.79),

$$\langle\psi|\psi\rangle = \langle f|f\rangle + \lambda\langle f|g\rangle + \lambda^*\langle g|f\rangle + \lambda\lambda^*\langle g|g\rangle,$$

with respect to λ^* and show that you get the Schwarz inequality, Eq. (10.78).

10.4.8 If the functions $f(x)$ and $g(x)$ of the Schwarz inequality, Eq. (10.78), may be expanded in a series of eigenfunctions $\varphi_i(x)$, show that Eq. (10.78) reduces to Eq. (10.76) (with n possibly infinite).

Note the description of $f(x)$ as a vector in a function space in which $\varphi_i(x)$ corresponds to the unit vector $\mathbf{e}_1$.

10.4.10 A normalized wave function $\psi(x) = \sum_{n=0}^{\infty} a_n\varphi_n(x)$. The expansion coefficients a_n are known as probability amplitudes. We may define a density matrix ρ with elements $\rho_{ij} = a_i a_j^*$. Show that

$$(\rho^2)_{ij} = \rho_{ij},$$

or

$$\rho^2 = \rho.$$

This result, by definition, makes ρ a projection operator.
Hint: Use

$$\int \psi^*\psi dx = 1.$$

10.4.11 Show that

(a) the operator

$$|\varphi_i(x)\rangle\langle\varphi_i(t)|$$

operating on

$$f(t) = \sum_j c_j|\varphi_j(t)\rangle$$

yields $\quad c_i|\varphi_i(x)\rangle.$

(b) $\displaystyle\sum_i |\varphi_i(x)\rangle\langle\varphi_i(x)| = 1.$

This operator is a **projection operator** projecting $f(x)$ onto the ith coordinate, selectively picking out the ith component $c_i|\varphi_i(x)\rangle$ of $f(x)$.

Hint. The operator operates via the well-defined inner product.

10.5.3 Find the Green's function for the operators

(a) $\mathcal{L}y(x) = \dfrac{d}{dx}\left(x\dfrac{dy(x)}{dx}\right).$

$$\textit{ANS.}\ \text{(a)}\ G(x,t) = \begin{cases} -\ln\ t, & 0 \le x < t, \\ -\ln\ x, & t < x \le 1. \end{cases}$$

(b) $\mathcal{L}y(x) = \dfrac{d}{dx}\left(x\dfrac{dy(x)}{dx}\right) - \dfrac{n^2}{x}y(x)$, with $y(0)$ finite and $y(1) = 0$.

$$ANS.\ \text{(b)}\ G(x,t) = \begin{cases} \dfrac{1}{2n}\left[\left(\dfrac{x}{t}\right)^n - (xt)^n\right], & 0 \le x < t, \\ \dfrac{1}{2n}\left[\left(\dfrac{t}{x}\right)^n - (xt)^n\right], & t < x \le 1. \end{cases}$$

The combination of operator and interval specified in Exercise 10.5.3(a) is pathological, in that one of the endpoints of the interval (zero) is a singular point of the operator. As a consequence, the integrated part (the surface integral of Green's theorem) does not vanish. The next four exercises explore this situation.

10.5.4 (a) Show that the particular solution of

$$\frac{d}{dx}\left[x\frac{d}{dx}y(x)\right] = -1$$

is $y_P(x) = -x$.

(b) Show that

$$y_P(x) = -x \neq \int_0^1 G(x,t)(-1)dt,$$

where $G(x,t)$ is the Green's function of Exercise 10.5.3(a).

10.5.5 Show that Green's theorem, Eq. (1.104) in one dimension with a Sturm-Liouville type operator $(d/dt)p(t)(d/dt)$ replacing $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$, may be rewritten as

$$\begin{aligned} &\int_a^b \left[u(t)\frac{d}{dt}\left(p(t)\frac{dv(t)}{dt}\right) - v(t)\frac{d}{dt}\left(p(t)\frac{du(t)}{dt}\right)\right] dt \\ = \ &[u(t)p(t)\frac{dv(t)}{dt} - v(t)p(t)\frac{du(t)}{dt}]\Big|_a^b . \end{aligned}$$

10.5.6 Using the one-dimensional form of Green's theorem of Exercise 10.5.5, let

$$v(t) = y(t) \qquad \text{and} \qquad \frac{d}{dt}\left(p(t)\frac{dy(t)}{dt}\right) = -f(t)$$

$$u(t) = G(x,t) \qquad \text{and} \qquad \frac{d}{dt}\left(p(t)\frac{\partial G(x,t)}{\partial t}\right) = -\delta(x-t).$$

Show that Green's theorem yields

$$\begin{aligned} y(x) \ &= \int_a^b G(x,t)f(t)dt \\ &\quad +[G(x,t)p(t)\frac{dy(t)}{dt} - y(t)p(t)\frac{\partial}{\partial t}G(x,t)]\Big|_{t=a}^{t=b} . \end{aligned}$$

10.5.7 For $p(t) = t, y(t) = -t$,

$$G(x,t) = \begin{cases} -\ln t, & 0 \le x < t \\ -\ln x, & t < x \le 1, \end{cases}$$

verify that the integrated part does not vanish.

10.5.13 In the Fredholm equation,

$$f(x) = \lambda^2 \int_a^b G(x,t)\varphi(t)dt,$$

$G(x,t)$ is a Green's function given by

$$G(x,t) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(t)}{\lambda_n^2 - \lambda^2}.$$

Show that the solution is

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{\lambda_n^2 - \lambda^2}{\lambda^2} \varphi_n(x) \int_a^b f(t)\varphi_n(t)dt.$$

10.5.14 Show that the Green's function integral transform operator

$$\int_a^b G(x,t)[\quad]dt$$

is equal to $-\mathcal{L}^{-1}$, in the sense that

(a) $\mathcal{L}_x \int_a^b G(x,t)y(t)dt = -y(x)$,

(b) $\int_a^b G(x,t)\mathcal{L}_t y(t)dt = -y(x)$.

Note. Take $\mathcal{L}y(x) + f(x) = 0$, Eq. (10.92).

10.5.15 Substitute Eq. (10.87), the eigenfunction expansion of Green's function, into Eq. (10.88) and then show that Eq. (10.88) is indeed a solution of the inhomogenous Helmholtz equation (10.82).

10.5.16 (a) Starting with a one-dimensional inhomogeneous differential equation, (Eq. (10.89)), assume that $\psi(x)$ and $\rho(x)$ may be represented by eigenfunction expansions. Without any use of the Dirac delta function or its representations, show that

$$\psi(x) = \sum_{n=0}^{\infty} \frac{\int_a^b \rho(t)\varphi_n(t)dt}{\lambda_n - \lambda} \varphi_n(x).$$

Note that (1) if $\rho = 0$, no solution exists unless $\lambda = \lambda_n$ and (2) if $\lambda = \lambda_n$, no solution exists unless ρ is orthogonal to φ_n. This same behavior will reappear with integral equations in Section 16.4.

(b) Interchanging summation and integration, show that you have constructed the Green's function corresponding to Eq. (10.90).

10.5.17 The eigenfunctions of the Schrödinger equation are often complex. In this case the orthogonality integral, Eq. (10.40), is replaced by

$$\int_a^b \varphi_i^*(x)\varphi_j(x)w(x)dx = \delta_{ij}.$$

Instead of Eq. (1.189), we have

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=0}^{\infty} \varphi_n(\mathbf{r}_1)\varphi_n^*(\mathbf{r}_2).$$

Show that the Green's function, Eq. (10.87), becomes

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} \frac{\varphi_n(\mathbf{r}_1)\varphi_n^*(\mathbf{r}_2)}{k_n^2 - k^2} = G^*(\mathbf{r}_2, \mathbf{r}_1).$$

11.1.9 Show that

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}dt.$$

This integral is a Fourier cosine transform (compare Section 15.3). The corresponding Fourier sine transform,

$$J_0(x) = \frac{2}{\pi}\int_1^{\infty} \frac{\sin xt}{\sqrt{t^2-1}}\,dt,$$

is established in Section 11.4 (Exercise 11.4.6), using a Hankel function integral representation.

11.1.15 A particle (mass m) is contained in a right circular cylinder (pillbox) of radius R and height H. The particle is described by a wave function satisfying the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi(\rho,\varphi,z) = E\psi(\rho,\varphi,z)$$

and the condition that the wave function go to zero over the surface of the pillbox. Find the lowest (zero point) permitted energy.

$$\textit{ANS. } E = \frac{\hbar^2}{2m}\left[\left(\frac{z_{pq}}{R}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right],$$
$$E_{\min} = \frac{\hbar^2}{2m}\left[\left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right],$$

where z_{pq} is the qth zero of J_p and the index p is fixed by the azimuthal dependence.

11.1.22 Using trigonometric forms, verify that

$$J_0(br) = \frac{1}{2\pi}\int_0^{2\pi} e^{ibr\sin\theta} d\theta.$$

11.1.28 A thin conducting disk of radius a carries a charge q. Show that the potential is described by

$$\varphi(r,z) = \frac{q}{4\pi\varepsilon_0 a}\int_0^\infty e^{-k|z|} J_0(kr)\frac{\sin ka}{k} dk,$$

where J_0 is the usual Bessel function and r and z are the familiar cylindrical coordinates.
Note. This is a difficult problem. One approach is through Fourier transforms such as Exercise 15.3.11. For a discussion of the physical problem see Jackson (*Classical Electrodynamics* in Additional Readings).

11.1.31 The circular aperature diffraction amplitude Φ of Eq. (11.34) is proportional to $f(z) = J_1(z)/z$. The corresponding single slit diffraction amplitude is proportional to $g(z) = \sin z/z$.

(a) Calculate and plot $f(z)$ and $g(z)$ for $z = 0.0(0.2)12.0$.
(b) Locate the two lowest values of $z(z > 0)$ for which $f(z)$ takes on an extreme value. Calculate the corresponding values of $f(z)$.
(c) Locate the two lowest values of $z(z > 0)$ for which $g(z)$ takes on an extreme value. Calculate the corresponding values of $g(z)$.

11.1.32 Calculate the electrostatic potential of a charged disk $\varphi(r,z)$ from the integral form of Exercise 11.1.28. Calculate the potential for $r/a = 0.0(0.5)2.0$ and $z/a = 0.25(0.25)1.25$. Why is $z/a = 0$ omitted? Exercise 12.3.17 is a spherical harmonic version of this same problem.

11.2.2 Show that

$$\int_0^a \left[J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]^2 \rho\, d\rho = \frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu m})]^2, \quad \nu > -1.$$

Here $\alpha_{\nu m}$ is the mth zero of J_ν.
Hint. With $\alpha_{\nu n} = \alpha_{\nu m} + \varepsilon$, expand $J_\nu[(\alpha_{\nu m} + \varepsilon)\rho/a]$ about $\alpha_{\nu m}\rho/a$ by a Taylor expansion.

11.2.8 For the continuum case, show that Eqs. (11.51) and (11.52) are replaced by

$$\begin{aligned} f(\rho) &= \int_0^\infty a(\alpha) J_\nu(\alpha\rho)\, d\alpha, \\ a(\alpha) &= \alpha\int_0^\infty f(\rho) J_\nu(\alpha\rho)\rho\, d\rho. \end{aligned}$$

Hint. The corresponding case for sines and cosines is worked out in Section 15.2. These are Hankel transforms. A derivation for the special case $\nu = 0$ is the topic of Exercise 15.1.1.

11.3.8 A cylindrical wave guide has radius r_0. Find the nonvanishing components of the electric and magnetic fields for

(a) TM_{01}, transverse magnetic wave $(H_z = H_\rho = E_\varphi = 0)$,
(b) TE_{01}, transverse electric wave $(E_z = E_\rho = H_\varphi = 0)$.

The subscripts 01 indicate that the longitudinal component (E_z or H_z) involves J_0 and the boundary condition is satisfied by the **first** zero of J_0 or J_0'.
Hint. All components of the wave have the same factor: $\exp i(kz - \omega t)$.

11.3.9 For a given mode of oscillation the **minimum** frequency that will be passed by a circular cylindrical wave guide (radius r_0) is

$$\nu_{\min} = \frac{c}{\lambda_c},$$

in which λ_c is fixed by the boundary condition

$$\begin{aligned} J_n\left(\frac{2\pi r_0}{\lambda_c}\right) &= 0 \text{ for } \mathrm{TM}_{nm} \text{ mode,} \\ J_n'\left(\frac{2\pi r_0}{\lambda_c}\right) &= 0 \text{ for } \mathrm{TE}_{nm} \text{ mode.} \end{aligned}$$

The subscript n denotes the order of the Bessel function and m indicates the zero used. Find this cutoff wavelength λ_c for the three TM and three TE modes with the longest cutoff wavelengths. Explain your results in terms of the graph of J_0, J_1, and J_2 (Fig. 11.1).

11.3.10 Write a program that will compute successive roots of the Neumann function $N_n(x)$, that is α_{ns}, where $N_n(\alpha_{ns}) = 0$. Tabulate the first five roots of N_0, N_1, and N_2. Check your values for the roots against those listed in AMS-55 (see Additional Readings of Chapter 8 for the full ref.).

Check value. $\alpha_{12} = 5.42968$.

11.5.12 (a) Verify that

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\cos\theta)d\theta$$

satisfies the modified Bessel equation, $\nu = 0$.
(b) Show that this integral contains no admixture of $K_0(x)$, the irregular second solution.
(c) Verify the normalization factor $1/\pi$.

11.5.16 Show that

$$e^{ax} = I_0(a)T_0(x) + 2\sum_{n=1}^{\infty} I_n(a)T_n(x), \quad -1 \le x \le 1.$$

$T_n(x)$ is the nth-order Chebyshev polynomial, Section 13.3.
Hint. Assume a Chebyshev series expansion. Using the orthogonality and normalization of the $T_n(x)$, solve for the coefficients of the Chebyshev series.

11.5.17 (a) Write a double precision subroutine to calculate $I_n(x)$ to 12-decimal-place accuracy for $n = 0, 1, 2, 3, \ldots$ and $0 \le x \le 1$. Check your results against the 10-place values given in AMS-55, Table 9.11, see Additional Readings of Chapter 8 for the reference.
(b) Referring to Exercise 11.5.16, calculate the coefficients in the Chebyshev expansions of $\cosh x$ and of $\sinh x$.

11.6.1 In checking the normalization of the integral representation of $K_\nu(z)$ (Eq. (11.122)), we assumed that $I_\nu(z)$ was not present. How do we know that the integral representation (Eq. (11.122)) does not yield $K_\nu(z) + \varepsilon I_\nu(z)$ with $\varepsilon \ne 0$?

11.6.7 (a) Using the asymptotic series (partial sums) $P_0(x)$ and $Q_0(x)$ determined in Exercise 11.6.6, write a function subprogram FCT(X) that will calculate $J_0(x), x$ real, for $x \ge x_{\min}$.
(b) Test your function by comparing it with the $J_0(x)$ (tables or computer library subroutine) for $x = x_{\min}(10)x_{\min} + 10$.

Note. A more accurate and perhaps simpler asymptotic form for $J_0(x)$ is given in AMS-55, Eq. (9.4.3), see Additional Readings of Chapter 8 for the reference.

11.7.12 Set up the orthogonality integral for $j_L(kr)$ in a sphere of radius R with the boundary condition

$$j_L(kR) = 0.$$

The result is used in classifying electromagnetic radiation according to its angular momentum.

11.7.15 Defining the spherical modified Bessel functions (Fig. 11.16) by

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x), \qquad k_n(x) = \sqrt{\frac{2}{\pi x}} K_{n+1/2}(x),$$

show that

$$i_0(x) = \frac{\sinh x}{x}, \qquad k_0(x) = \frac{e^{-x}}{x}.$$

Note that the numerical factors in the definitions of i_n and k_n are not identical.

11.7.17 Show that the spherical modified Bessel functions satisfy the following relations:

(a)
$$\begin{aligned} i_n(x) &= i^{-n} j_n(ix), \\ k_n(x) &= -i^n h_n^{(1)}(ix), \end{aligned}$$

(b)
$$\begin{aligned} i_{n+1}(x) &= x^n \frac{d}{dx}(x^{-n} i_n), \\ k_{n+1}(x) &= -x^n \frac{d}{dx}(x^{-n} k_n), \end{aligned}$$

(c)
$$\begin{aligned} i_n(x) &= x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sinh x}{x}, \\ k_n(x) &= (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{-x}}{x}. \end{aligned}$$

11.7.18 Show that the recurrence relations for $i_n(x)$ and $k_n(x)$ are

(a)
$$\begin{aligned} i_{n-1}(x) - i_{n+1}(x) &= \frac{2n+1}{x} i_n(x), \\ n i_{n-1}(x) + (n+1) i_{n+1}(x) &= (2n+1) i_n'(x), \end{aligned}$$

(b)
$$\begin{aligned} k_{n-1}(x) - k_{n+1}(x) &= -\frac{2n+1}{x} k_n(x), \\ n k_{n-1}(x) + (n+1) k_{n+1}(x) &= -(2n+1) k_n'(x). \end{aligned}$$

11.7.19 Derive the limiting values for the spherical modified Bessel functions

(a) $$i_n(x) \approx \frac{x^n}{(2n+1)!!}, \qquad k_n(x) \approx \frac{(2n-1)!!}{x^{n+1}}, \quad x \ll 1.$$

(b) $$i_n(x) \sim \frac{e^x}{2x}, \qquad k_n(x) \sim \frac{e^{-x}}{x}, \quad x \gg \frac{1}{2} n(n+1).$$

11.7.21 A quantum particle of mass M is trapped in a "square" well of radius a. The Schrödinger equation potential is

$$V(r) = \begin{cases} -V_0, & 0 \le r < a \\ 0, & r > a. \end{cases}$$

The particle's energy E is negative (an eigenvalue).

(a) Show that the radial part of the wave function is given by $j_l(k_1 r)$ for $0 \le r < a$ and $k_l(k_2 r)$ for $r > a$. (We require that $\psi(0)$ be finite and $\psi(\infty) \to 0$.) Here $k_1^2 = 2M(E+V_0)/\hbar^2, k_2^2 = -2ME/\hbar^2$, and l is the angular momentum (n in Eq. (11.139)).

(b) The boundary condition at $r = a$ is that the wave function $\psi(r)$ and its first derivative be continuous. Show that this means

$$\left.\frac{(d/dr) j_l(k_1 r)}{j_l(k_1 r)}\right|_{r=a} = \left.\frac{(d/dr) k_l(k_2 r)}{k_l(k_2 r)}\right|_{r=a}.$$

This equation determines the energy eigenvalues.

Note. This is a generalization of Example 10.1.2.

11.7.22 The quantum mechanical radial wave function for a scattered wave is given by

$$\psi_k = \frac{\sin(kr + \delta_0)}{kr},$$

where k is the wave number, $k = \sqrt{2mE/\hbar}$, and δ_0 is the scattering phase shift. Show that the normalization integral is

$$\int_0^\infty \psi_k(r)\psi_{k'}(r)r^2\, dr = \frac{\pi}{2k}\delta(k - k').$$

Hint. You can use a sine representation of the Dirac delta function. See Exercise 15.3.8.

11.7.23 Derive the spherical Bessel function closure relation

$$\frac{2a^2}{\pi}\int_0^\infty j_n(ar)j_n(br)r^2\, dr = \delta(a - b).$$

Note. An interesting derivation involving Fourier transforms, the Rayleigh plane-wave expansion, and spherical harmonics has been given by P. Ugincius, *Am. J. Phys.* **40**: 1690 (1972).

11.7.24 (a) Write a subroutine that will generate the spherical Bessel functions, $j_n(x)$, that is, will generate the numerical value of $j_n(x)$ given x and n.
Note. One possibility is to use the explicit known forms of j_0 and j_1 and to develop the higher index j_n, by repeated application of the recurrence relation.

(b) Check your subroutine by an independent calculation, such as Eq. (11.154). If possible, compare the machine time needed for this check with the time required for your subroutine.

11.7.25 The wave function of a particle in a sphere (Example 11.7.1) with angular momentum l is $\psi(r,\theta,\varphi) = Aj_l((\sqrt{2ME})r/\hbar)Y_l^m(\theta,\varphi)$. The $Y_l^m(\theta,\varphi)$ is a spherical harmonic, described in Section 12.6. From the boundary condition $\psi(a,\theta,\varphi) = 0$ or $j_l((\sqrt{2ME})a/\hbar) = 0$ calculate the 10 lowest-energy states. Disregard the m degeneracy ($2l + 1$ values of m for each choice of l). Check your results against AMS-55, Table 10.6, see Additional Readings of Chapter 8 for the reference.
Hint. You can use your spherical Bessel subroutine and a root-finding subroutine.

Check values.
$$\begin{aligned} j_l(\alpha_{ls}) &= 0, \\ \alpha_{01} &= 3.1416 \\ \alpha_{11} &= 4.4934 \\ \alpha_{21} &= 5.7635 \\ \alpha_{02} &= 6.2832. \end{aligned}$$

11.7.26 Let Example 11.7.1 be modified so that the potential is a finite V_0 outside $(r > a)$.

(a) For $E < V_0$ show that

$$\psi_{\text{out}}(r,\theta,\varphi) \sim k_l\left(\frac{r}{\hbar}\sqrt{2M(V_0 - E)}\right).$$

(b) The new boundary conditions to be satisfied at $r = a$ are

$$\begin{aligned}\psi_{\text{in}}(a,\theta,\varphi) &= \psi_{\text{out}}(a,\theta,\varphi),\\ \frac{\partial}{\partial r}\psi_{\text{in}}(a,\theta,\varphi) &= \frac{\partial}{\partial r}\psi_{\text{out}}(a,\theta,\varphi)\end{aligned}$$

or

$$\left.\frac{1}{\psi_{\text{in}}}\frac{\partial\psi_{\text{in}}}{\partial r}\right|_{r=a} = \left.\frac{1}{\psi_{\text{out}}}\frac{\partial\psi_{\text{out}}}{\partial r}\right|_{r=a}.$$

For $l = 0$ show that the boundary condition at $r = a$ leads to

$$f(E) = k\left\{\cot ka - \frac{1}{ka}\right\} + k'\left\{1 + \frac{1}{k'a}\right\} = 0,$$

where $k = \sqrt{2ME}/\hbar$ and $k' = \sqrt{2M(V_0 - E)}/\hbar$.

(c) With $a = 4\pi\epsilon_0\hbar^2/Me^2$ (Bohr radius) and $V_0 = 4Me^4/2\hbar^2$, compute the possible bound states $(0 < E < V_0)$.
Hint. Call a root-finding subroutine after you know the approximate location of the roots of

$$f(E) = 0, \quad (0 \le E \le V_0).$$

(d) Show that when $a = 4\pi\epsilon_0\hbar^2/Me^2$ the minimum value of V_0 for which a bound state exists is $V_0 = 2.4674Me^4/2\hbar^2$.

11.7.27 In some nuclear stripping reactions the differential cross section is proportional to $j_l(x)^2$, where l is the angular momentum. The location of the maximum on the curve of experimental data permits a determination of l, if the location of the (first) maximum of $j_l(x)$ is known. Compute the location of the first maximum of $j_1(x), j_2(x)$, and $j_3(x)$. *Note*. For better accuracy look for the first zero of $j_l'(x)$. Why is this more accurate than direct location of the maximum?

12.1.7 Prove that

$$P_n(\cos\theta) = (-1)^n\frac{r^{n+1}}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

Hint. Compare the Legendre polynomial expansion of the generating function ($a \to \Delta z$, Fig. 12.1) with a Taylor series expansion of $1/r$, where z dependence of r changes from z to $z - \Delta z$ (Fig. 12.7).

12.1.9 The Chebyshev polynomials (type II) are generated by (Eq. (13.93), Section 13.3)

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n.$$

Using the techniques of Section 5.4 for transforming series, develop a series representation of $U_n(x)$.

$$\textit{ANS. } U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k}.$$

12.2.6 From

$$P_L(\cos\theta) = \frac{1}{L!}\frac{\partial^L}{\partial t^L}(1-2t\cos\theta+t^2)^{-1/2}\,|_{t=0}$$

show that

$$P_L(1) = 1, \quad P_L(-1) = (-1)^L.$$

12.2.10 Write a program that will generate the coefficients a_s in the polynomial form of the Legendre polynomial

$$P_n(x) = \sum_{s=0}^{n} a_s x^s.$$

12.2.11 (a) Calculate $P_{10}(x)$ over the range $[0,1]$ and plot your results.
(b) Calculate precise (at least to five decimal places) values of the five positive roots of $P_{10}(x)$. Compare your values with the values listed in AMS-55, Table 25.4. (For the complete reference see Additional Readings of Chapter 8.)

12.2.12 (a) Calculate the **largest** root of $P_n(x)$ for $n = 2(1)50$.
(b) Develop an approximation for the largest root from the hypergeometric representation of $P_n(x)$ (Section 13.4) and compare your values from part (a) with your hypergeometric approximation. Compare also with the values listed in AMS-55, Table 25.4. (For the complete reference see Additional Readings of Chapter 8.) References).

12.2.13 (a) From Exercise 12.2.1 and AMS-55, Table 22.9 develop the 6×6 matrix B that will transform a series of even order Legendre polynomials through $P_{10}(x)$ into a power series $\sum_{n=0}^{5}\alpha_{2n}x^{2n}$.
(b) Calculate A as B^{-1}. Check the elements of A against the values listed in AMS-55, Table 22.9. (For the complete reference see additional Readings of Chapter 8.)
(c) By using matrix multiplication, transform some even power-series $\sum_{n=0}^{5}\alpha_{2n}x^{2n}$ into a Legendre series.

12.2.14 Write a subroutine that will transform a finite power series $\sum_{n=0}^{N} a_n x^n$ into a Legendre series $\sum_{n=0}^{N} b_n P_n(x)$. Use the recurrence relation, Eq. (12.17), and follow the technique outlined in Section 13.3 for a Chebyshev series.

12.3.11 The amplitude of a scattered wave is given by

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\exp[i\delta_l]\sin\delta_l P_l(\cos\theta).$$

Here θ is the angle of scattering, l is the angular momentum eigenvalue, $\hbar k$ is the incident momentum, and δ_l is the phase shift produced by the central potential that is doing the scattering. The total cross section is $\sigma_{\text{tot}} = \int |f(\theta)|^2 d\Omega$. Show that

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l.$$

12.3.14 A charge q is displaced a distance a along the z-axis from the center of a spherical cavity of radius R.

(a) Show that the electric field averaged over the volume $a \le r \le R$ is zero.
(b) Show that the electric field averaged over the volume $0 \le r \le a$ is

$$\mathbf{E} = \hat{\mathbf{z}}E_z = -\hat{\mathbf{z}}\frac{q}{4\pi\varepsilon_0 a^2} \quad \text{(SI units)} = -\hat{\mathbf{z}}\frac{nqa}{3\varepsilon_0},$$

where n is the number of such displaced charges per unit volume. This is a basic calculation in the polarization of a dielectric.
Hint. $\mathbf{E} = -\nabla\varphi$.

12.3.18 From the result of Exercise 12.3.17 calculate the potential of the disk. Since you are violating the condition $r > a$, justify your calculation.
Hint. You may run into the series given in Exercise 5.2.9.

12.4.11 By direct evaluation of the Schlaefli integral show that $P_n(1) = 1$.

12.4.12 Explain why the contour of the Schlaefli integral, Eq. (12.69), is chosen to enclose the points $t = z$ and $t = 1$ when $n \to \nu$, not an integer.

12.5.17 A nuclear particle is in a spherical square well potential $V(r,\theta,\varphi) = 0$ for $0 \le r < a$ and ∞ for $r > a$. The particle is described by a wave function $\psi(r,\theta,\varphi)$ which satisfies the wave equation

$$-\frac{\hbar^2}{2M}\nabla^2\psi + V_0\psi = E\psi, \quad r < a,$$

and the boundary condition

$$\psi(r = a) = 0.$$

Show that for the energy E to be a minimum there must be no angular dependence in the wave function; that is, $\psi = \psi(r)$.
Hint. The problem centers on the boundary condition on the radial function.

12.5.18 (a) Write a subroutine to calculate the numerical value of the associated Legendre function $P_N^1(x)$ for given values of N and x.
Hint. With the known forms of P_1^1 and P_2^1 you can use the recurrence relation Eq. (12.92) to generate P_N^1, $N > 2$.
(b) Check your subroutine by having it calculate $P_N^1(x)$ for $x = 0.0(0.5)$ 1.0 and $N = 1(1)10$. Check these numerical values against the known values of $P_N^1(0)$ and $P_N^1(1)$ and against the tabulated values of $P_N^1(0.5)$.

12.5.19 Calculate the magnetic vector potential of a current loop, Example 12.5.1. Tabulate your results for $r/a = 1.5(0.5)5.0$ and $\theta = 0°(15°)90°$. Include terms in the series expansion, Eq. (12.137), until the absolute values of the terms drop below the leading term by a factor of 10^5 or more.
Note. This associated Legendre expansion can be checked by comparison with the elliptic integral solution, Exercise 5.8.4.

Check value. For $r/a = 4.0$ and $\theta = 20°$,
$A_\varphi/\mu_0 I = 4.9398 \times 10^{-3}$.

12.6.4 (a) Express the elements of the quadrupole moment tensor $x_i x_j$ as a linear combination of the spherical harmonics Y_2^m (and Y_0^0).
Note. The tensor $x_i x_j$ is reducible. The Y_0^0 indicates the presence of a scalar component.
(b) The quadrupole moment tensor is usually defined as

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbf{r})d\tau,$$

with $\rho(\mathbf{r})$ the charge density. Express the components of $(3x_i x_j - r^2\delta_{ij})$ in terms of $r^2 Y_2^M$.
(c) What is the significance of the $-r^2\delta_{ij}$ term?
Hint. Compare Sections 2.9 and 4.4.

12.8.8 The electric current density produced by a $2P$ electron in a hydrogen atom is

$$\mathbf{J} = \hat{\boldsymbol{\varphi}}\frac{q\hbar}{32ma_0^5}e^{-r/a_0}r\sin\theta.$$

Using

$$\mathbf{A}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}d^3r_2,$$

find the magnetic vector potential produced by this hydrogen electron. *Hint.* Resolve into Cartesian components. Use the addition theorem to eliminate γ, the angle included between $\mathbf{r}_1$ and $\mathbf{r}_2$.

12.9.4 Show that Eq. (12.199) is a special case of Eq. (12.190) and derive the reduced matrix element $\langle Y_{L_1} \| Y_1 \| Y_L \rangle$.

$$\textit{ANS.}\ \langle Y_{L_1} \| Y_1 \| Y_L \rangle = (-1)^{L_1+1-L} C(1LL_1|000) \frac{\sqrt{3(2L+1)}}{4\pi}.$$

12.10.2 From Eqs. (12.212) and (12.213) show that

$$\text{(a)}\ P_{2n}(x) = \frac{(-1)^n}{2^{2n-1}} \sum_{s=0}^{n} (-1)^s \frac{(2n+2s-1)!}{(2s)!(n+s-1)!(n-s)!} x^{2s}.$$

$$\text{(b)}\ P_{2n+1}(x) = \frac{(-1)^n}{2^{2n}} \sum_{s=0}^{n} (-1)^s \frac{(2n+2s+1)!}{(2s+1)!(n+s)!(n-s)!} x^{2s+1}.$$

Check the normalization by showing that one term of each series agrees with the corresponding term of Eq. (12.8).

12.10.5 Verify that the Legendre functions of the second kind, $Q_n(x)$, satisfy the same recurrence relations as $P_n(x)$, both for $|x| < 1$ and for $|x| > 1$:

$$\begin{aligned} (2n+1)xQ_n(x) &= (n+1)Q_{n+1}(x) + nQ_{n-1}(x), \\ (2n+1)Q_n(x) &= Q'_{n+1}(x) - Q'_{n-1}(x). \end{aligned}$$

item[**12.10.7**]

(a) Write a subroutine that will generate $Q_n(x)$ and Q_0 through Q_{n-1} based on the recurrence relation for these Legendre functions of the second kind. Take x to be within $(-1, 1)$ - excluding the endpoints. *Hint.* Take $Q_0(x)$ and $Q_1(x)$ to be known.

b) Test your subroutine for accuracy by computing $Q_{10}(x)$ and comparing with the values tabulated in AMS-55 (for a complete reference, see Additional Readings of Chapter 8).

13.1.16 (a) Show that the simple oscillator Hamiltonian (from Eq. (13.38)) may be written as

$$\mathcal{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 = \frac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}).$$

Hint. Express E in units of $\hbar\omega$.

(b) Using the creation-annihilation operator formulation of part (a), show that

$$\mathcal{H}\psi(x) = (n + \tfrac{1}{2})\psi(x).$$

This means the energy eigenvalues are $E = (n+\frac{1}{2})(\hbar\omega)$, in agreement with Eq. (13.40).

13.1.17 Write a program that will generate the coefficients a_s, in the polynomial form of the Hermite polynomial $H_n(x) = \sum_{s=0}^{n} a_s x^s$.

13.1.18 A function $f(x)$ is expanded in a Hermite series:

$$f(x) = \sum_{n=0}^{\infty} a_n H_n(x).$$

From the orthogonality and normalization of the Hermite polynomials the coefficient a_n is given by

$$a_n = \frac{1}{2^n \pi^{1/2} n!} \int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2}\, dx.$$

For $f(x) = x^8$ determine the Hermite coefficients a_n by the Gauss-Hermite quadrature. Check your coefficients against AMS-55, Table 22.12 (for the reference, see footnote 4 in Chapter 5 or the General References at book's end).

13.1.19 (a) In analogy with Exercise 12.2.13, set up the matrix of even Hermite polynomial coefficients that will transform an even Hermite series into an even power series:

$$\mathsf{B} = \begin{pmatrix} 1 & -2 & 12 & \cdots \\ 0 & 4 & -48 & \cdots \\ 0 & 0 & 16 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Extend B to handle an even polynomial series through $H_8(x)$.

(b) Invert your matrix to obtain matrix A, which will transform an even power series (through x^8) into a series of even Hermite polynomials. Check the elements of A against those listed in AMS-55 (Table 22.12, in General References at book's end).

(c) Finally, using matrix multiplication, determine the Hermite series equivalent to $f(x) = x^8$.

13.1.20 Write a subroutine that will transform a finite power series, $\sum_{n=0}^{N} a_n x^n$, into a Hermite series, $\sum_{n=0}^{N} b_n H_n(x)$. Use the recurrence relation, Eq. (13.2).

Note. Both Exercises 13.1.19 and 13.1.20 are faster and more accurate than the Gaussian quadrature, Exercise 13.1.18, if $f(x)$ is available as a power series.

13.1.21 Write a subroutine for evaluating Hermite polynomial matrix elements of the form

$$M_{pqr} = \int_{-\infty}^{\infty} H_p(x) H_q(x) x^r e^{-x^2}\, dx,$$

using the 10-point Gauss-Hermite quadrature (for $p+q+r \leq 19$). Include a parity check and set equal to zero the integrals with odd-parity integrand. Also, check to see if r is in the range $|p-q| \leq r$. Otherwise $M_{pqr} = 0$. Check your results against the specific cases listed in Exercises 13.1.9, 13.1.10, 13.1.11, and 13.1.12.

13.1.22 Calculate and tabulate the normalized linear oscillator wave functions

$$\psi_n(x) = 2^{-n/2}\pi^{-1/4}(n!)^{-1/2}H_n(x)\exp\left(-\frac{x^2}{2}\right) \quad \text{for} \quad x = 0.0(0.1)5.0$$

and $n = 0(1)5$. If a plotting routine is available, plot your results.

13.1.23 Evaluate $\int_{-\infty}^{\infty} e^{-2x^2} H_{N_1}(x) \cdots H_{N_4}(x)dx$ in closed form.
Hint. $\int_{-\infty}^{\infty} e^{-2x^2} H_{N_1}(x)H_{N_2}(x)H_{N_3}(x)dx = \frac{1}{\pi}2^{(N_1+N_2+N_3-1)/2}$
$\cdot\,\Gamma(s-N_1)\Gamma(s-N_2)\Gamma(s-N_3)$, $s = (N_1+N_2+N_3+1)/2$ or
$\int_{-\infty}^{\infty} e^{-2x^2} H_{N_1}(x)H_{N_2}(x)dx = (-1)^{(N_1+N_2-1)/2}2^{(N_1+N_2-1)/2}$
$\cdot\,\Gamma((N_1+N_2+1)/2)$ may be helpful. Prove these formulas (see Gradshteyn and Ryshik, no. 7.375 on p. 844, in the Additional Readings).

13.2.3 From the generating function derive the Rodrigues representation

$$L_n^k(x) = \frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+k}).$$

13.2.11 The hydrogen wave functions, Eq. (13.91), are mutually orthogonal as they should be, since they are eigenfunctions of the self-adjoint Schrödinger equation

$$\int \psi^*_{n_1L_1M_1}\psi_{n_2L_2M_2}r^2drd\Omega = \delta_{n_1n_2}\delta_{L_1L_2}\delta_{M_1M_2}.$$

Yet the radial integral has the (misleading) form

$$\int_0^\infty e^{-\alpha r/2}(\alpha r)^L L^{2L+1}_{n_1-L-1}(\alpha r)e^{-\alpha r/2}(\alpha r)^L L^{2L+1}_{n_2-L-1}(\alpha r)r^2dr,$$

which **appears** to match Eq. (13.83) and not the associated Laguerre orthogonality relation, Eq. (13.79). How do you resolve this paradox?

ANS. The parameter α is dependent on n. The first three α, previously shown, are $2Z/n_1a_0$. The last three are $2Z/n_2a_0$. For $n_1 = n_2$ Eq. (13.83) applies. For $n_1 \neq n_2$ neither Eq. (13.79) nor Eq. (13.83) is applicable.

13.2.12 A quantum mechanical analysis of the Stark effect (parabolic coordinate) leads to the ODE

$$\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right) + \left(\frac{1}{2}E\xi + L - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2\right)u = 0.$$

Here F is a measure of the perturbation energy introduced by an external electric field. Find the unperturbed wave functions ($F = 0$) in terms of associated Laguerre polynomials.

ANS. $u(\xi) = e^{-\varepsilon\xi/2}\xi^{m/2}L_p^m(\varepsilon\xi)$, with $\varepsilon = \sqrt{-2E} > 0$,
$p = L/\varepsilon - (m+1)/2$, a nonnegative integer.

13.2.13 The wave equation for the three-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2M}\nabla^2\psi + \tfrac{1}{2}M\omega^2 r^2\psi = E\psi.$$

Here ω is the angular frequency of the corresponding classical oscillator. Show that the radial part of ψ (in spherical polar coordinates) may be written in terms of associated Laguerre functions of argument (βr^2), where $\beta = M\omega/\hbar$.
Hint. As in Exercise 13.2.8, split off radial factors of r^l and $e^{-\beta r^2/2}$. The associated Laguerre function will have the form $L_{1/2(n-l-1)}^{l+1/2}(\beta r^2)$.

13.2.14 Write a computer program that will generate the coefficients a_s in the polynomial form of the Laguerre polynomial $L_n(x) = \sum_{s=0}^n a_s x^s$.

13.2.15 Write a computer program that will transform a finite power series $\sum_{n=0}^N a_n x^n$ into a Laguerre series $\sum_{n=0}^N b_n L_n(x)$. Use the recurrence relation, Eq. (13.62).

13.2.16 Tabulate $L_{10}(x)$ for $x = 0.0(0.1)30.0$. This will include the 10 roots of L_{10}. Beyond $x = 30.0, L_{10}(x)$ is monotonic increasing. If graphic software is available, plot your results.

Check value. Eighth root = 16.279.

13.2.17 Determine the 10 roots of $L_{10}(x)$ using root-finding software. You may use your knowledge of the approximate location of the roots or develop a search routine to look for the roots. The 10 roots of $L_{10}(x)$ are the evaluation points for the 10-point Gauss-Laguerre quadrature. Check your values by comparing with AMS-55, Table 25.9. (For the reference, see footnote 4 in Chapter 5 or the General References at book's end.)

13.2.18 Calculate the coefficients of a Laguerre series expansion ($L_n(x), k = 0$) of the exponential e^{-x}. Evaluate the coefficients by the Gauss-Laguerre quadrature (compare Eq. (10.64)). Check your results against the values given in Exercise 13.2.6.
Note. Direct application of the Gauss-Laguerre quadrature with
$f(x)L_n(x)e^{-x}$ gives poor accuracy because of the extra e^{-x}. Try a change of variable, $y = 2x$, so that the function appearing in the integrand will be simply $L_n(y/2)$.

13.2.19 (a) Write a subroutine to calculate the Laguerre matrix elements

$$M_{mnp} = \int_0^\infty L_m(x)L_n(x)x^p e^{-x}\,dx.$$

Include a check of the condition $|m-n| \le p \le m+n$. (If p is outside this range, $M_{mnp} = 0$. Why?)

Note. A 10-point Gauss-Laguerre quadrature will give accurate results for $m+n+p \le 19$.

(b) Call your subroutine to calculate a variety of Laguerre matrix elements. Check M_{mn1} against Exercise 13.2.7.

13.2.20 Write a subroutine to calculate the numerical value of $L_n^k(x)$ for specified values of n, k, and x. Require that n and k be nonnegative integers and $x \ge 0$.

Hint. Starting with known values of L_0^k and $L_1^k(x)$, we may use the recurrence relation, Eq. (13.75), to generate $L_n^k(x), n = 2, 3, 4, \dots$.

13.2.22 Write a program to calculate the normalized hydrogen radial wave function $\psi_{nL}(r)$. This is ψ_{nLM} of Eq. (13.91), omitting the spherical harmonic $Y_L^M(\theta, \varphi)$. Take $Z = 1$ and $a_0 = 1$ (which means that r is being expressed in units of Bohr radii). Accept n and L as input data. Tabulate $\psi_{nL}(r)$ for $r = 0.0(0.2)R$ with R taken large enough to exhibit the significant features of ψ. This means roughly $R = 5$ for $n = 1, R = 10$ for $n = 2$, and $R = 30$ for $n = 3$.

13.3.23 (a) Calculate and tabulate the Chebyshev functions $V_1(x), V_2(x)$, and $V_3(x)$ for $x = -1.0(0.1)1.0$.

(b) A second solution of the Chebyshev differential equation, Eq. (13.100), for $n = 0$ is $y(x) = \sin^{-1} x$. Tabulate and plot this function over the same range: $-1.0(0.1)1.0$.

13.3.24 Write a computer program that will generate the coefficients a_s in the polynomial form of the Chebyshev polynomial $T_n(x) = \sum_{s=0}^n a_s x^s$.

13.3.25 Tabulate $T_{10}(x)$ for 0.00(0.01)1.00. This will include the five positive roots of T_{10}. If graphics software is available, plot your results.

13.3.26 Determine the five positive roots of $T_{10}(x)$ by calling a root–finding subroutine. Use your knowledge of the approximate location of these roots from Exercise 13.3.25 or write a search routine to look for the roots. These five positive roots (and their negatives) are the evaluation points of the 10-point Gauss-Chebyshev quadrature method.

Check values $x_k = \cos[(2k-1)\pi/20], \quad k = 1, 2, 3, 4, 5.$

13.6.1 For the simple pendulum ODE of Section 5.8, apply Floquet's method and derive the properties of its solutions similar to those marked by bullets before Eq. (13.186).

13.6.2 Derive a Mathieu-function analog for the Rayleigh expansion of a plane wave for $\cos(k\cos\eta\cos\theta)$ and $\sin(k\cos\eta\cos\theta)$.

14.1.8 Calculate the sum of the finite Fourier sine series for the sawtooth wave, $f(x) = x, (-\pi, \pi)$, Eq. (14.21). Use 4-, 6-, 8-, and 10-term series and $x/\pi = 0.00(0.02)1.00$. If a plotting routine is available, plot your results and compare with Fig. (14.1).

14.2.1 The boundary conditions (such as $\psi(0) = \psi(l) = 0$) may suggest solutions of the form $\sin(n\pi x/l)$ and eliminate the corresponding cosines.

(a) Verify that the boundary conditions used in the Sturm-Liouville theory are satisfied for the interval $(0, l)$. Note that this is only half the usual Fourier interval.

(b) Show that the set of functions $\varphi_n(x) = \sin(n\pi x/l), n = 1, 2, 3, \ldots$, satisfies an orthogonality relation

$$\int_0^l \varphi_m(x)\varphi_n(x)dx = \frac{l}{2}\delta_{mn}, \quad n > 0.$$

14.3.9 (a) Show that the Fourier expansion of $\cos ax$ is

$$\begin{aligned}\cos ax &= \frac{2a\sin a\pi}{\pi}\left\{\frac{1}{2a^2} - \frac{\cos x}{a^2-1^2} + \frac{\cos 2x}{a^2-2^2} - \cdots\right\},\\ a_n &= (-1)^n\frac{2a\sin a\pi}{\pi(a^2-n^2)}.\end{aligned}$$

(b) From the preceding result show that

$$a\pi\cot a\pi = 1 - 2\sum_{p=1}^{\infty}\zeta(2p)a^{2p}.$$

This provides an alternate derivation of the relation between the Riemann zeta function and the Bernoulli numbers, Eq. (5.151).

14.3.13 (a) Using

$$f(x) = x^2, \quad -\pi < x < \pi,$$

show that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} = \eta(2).$$

(b) Using the Fourier series for a triangular wave developed in Exercise 14.3.4, show that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8} = \lambda(2).$$

(c) Using

$$f(x) = x^4, \quad -\pi < x < \pi,$$

show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = \zeta(4), \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720} = \eta(4).$$

(d) Using

$$f(x) = \begin{cases} x(\pi - x), & 0 < x < \pi, \\ x(\pi + x), & \pi < x < 0, \end{cases}$$

derive

$$f(x) = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin nx}{n^3}$$

and show that

$$\sum_{n=1,3,5,\dots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32} = \beta(3).$$

(e) Using the Fourier series for a square wave, show that

$$\sum_{n=1,3,5,\dots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} = \beta(1).$$

This is Leibniz' formula for π, obtained by a different technique in Exercise 5.7.6.
Note. The $\eta(2), \eta(4), \lambda 2), \beta(1)$, and $\beta(3)$ functions are defined by the indicated series. General definitions appear in Section 5.9.

14.3.15 A symmetric triangular pulse of adjustable height and width is described by

$$f(x) = \begin{cases} a(1 - x/b), & 0 \le |x| \le b \\ 0, & b \le |x| \le \pi. \end{cases}$$

(a) Show that the Fourier coefficients are

$$a_0 = \frac{ab}{\pi}, \quad a_n = \frac{2ab}{\pi(nb)^2}(1 - \cos nb).$$

Sum the finite Fourier series through $n = 10$ and through $n = 100$ for $x/\pi = 0(1/9)1$. Take $a = 1$ and $b = \pi/2$.

(b) Call a Fourier analysis subroutine (if available) to calculate the Fourier coefficients of $f(x), a_0$ through a_{10}.

14.3.16 (a) Using a Fourier analysis subroutine, calculate the Fourier cosine coefficients a_0 through a_{10} of

$$f(x) = [1 - \left(\frac{x}{\pi}\right)^2]^{1/2}, \quad x \in [-\pi, \pi].$$

(b) Spot-check by calculating some of the preceding coefficients by direct numerical quadrature.

Check values. $a_0 = 0.785, a_2 = 0.284$.

14.3.17 Using a Fourier analysis subroutine, calculate the Fourier coefficients through a_{10} and b_{10} for

(a) a full-wave rectifier, Example 14.3.2,
(b) a half-wave rectifier, Exercise 14.3.1. Check your results against the analytic forms given (Eq. (14.41) and Exercise 14.3.1).

14.4.12 Find the charge distribution over the interior surfaces of the semicircles of Exercise 14.3.6.
Note. You obtain a divergent series and this Fourier approach fails. Using conformal mapping techniques, we may show the charge density to be proportional to $\csc\theta$. Does $\csc\theta$ have a Fourier expansion?

14.5.3 Evaluate the finite step function series, Eq. (14.73), $h = 2$, using 100, 200, 300, 400, and 500 terms for $x = 0.0000(0.0005)0.0200$. Sketch your results (five curves) or, if a plotting routine is available, plot your results.

14.6.2 Equation (14.84) exhibits orthogonality summing over time points. Show that we have the same orthogonality summing over frequency points

$$\frac{1}{2N}\sum_{p=0}^{2N-1}(e^{i\omega_p t_m})^* e^{i\omega_p t_k} = \delta_{mk}.$$

14.6.5 Given $N = 2, T = 2\pi$, and $f(t_k) = \sin t_k$,

(a) find $F(\omega_p), p = 0, 1, 2, 3$, and
(b) reconstruct $f(t_k)$ from $F(\omega_p)$ and exhibit the aliasing of $\omega_1 = 1$ and $\omega_3 = 3$.

ANS. (a) $F(\omega_p) = (0, i/2, 0, -i/2)$
(b) $f(t_k) = \frac{1}{2}\sin t_k - \frac{1}{2}\sin 3t_k$.

14.6.6 Show that the Chebyshev polynomials $T_m(x)$ satisfy a discrete orthogonality relation

$$\begin{aligned}\tfrac{1}{2}T_m(-1)T_n(-1) \quad &+ \quad \sum_{s=1}^{N-1} T_m(x_s)T_n(x_s) + \tfrac{1}{2}T_m(1)T_n(1) \\ &= \begin{cases} 0, & m \neq n \\ N/2, & m = n \neq 0 \\ N, & m = n = 0. \end{cases}\end{aligned}$$

Here, $x_s = \cos\theta_s$, where the $(N+1)\theta_s$ are equally spaced along the θ-axis:

$$\theta_s = \frac{s\pi}{N}, \quad s = 0, 1, 2, \ldots, N.$$

14.7.1 Determine the nonleading coefficients $\beta^{(n)}_{n+2}$ for se_1. Derive a suitable recursion relation.

14.7.2 Determine the nonleading coefficients $\beta^{(n)}_{n+4}$ for ce_0. Derive the corresponding recursion relation.

14.7.3 Derive the formula for ce_1, Eq. (14.155), and its eigenvalue, Eq.(14.156).

15.1.2 Assuming the validity of the Hankel transform-inverse transform pair of equations

$$\begin{aligned} g(\alpha) &= \int_0^\infty f(t)J_n(\alpha t)t\ dt, \\ f(t) &= \int_0^\infty g(\alpha)J_n(\alpha t)\alpha\ d\alpha, \end{aligned}$$

show that the Dirac delta function has a Bessel integral representation

$$\delta(t-t') = t\int_0^\infty J_n(\alpha t)J_n(\alpha t')\alpha\ d\alpha.$$

This expression is useful in developing Green's functions in cylindrical coordinates, where the eigenfunctions are Bessel functions.

15.1.3 From the Fourier transforms, Eqs. (15.22) and 15.23), show that the transformation

$$\begin{aligned} t &\rightarrow \ln x \\ i\omega &\rightarrow \alpha - \gamma \end{aligned}$$

leads to

$$G(\alpha) = \int_0^\infty F(x)x^{\alpha-1}\ dx$$

and

$$F(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} G(\alpha)x^{-\alpha}\ d\alpha.$$

These are the Mellin transforms. A similar change of variables is employed in Section 15.12 to derive the inverse Laplace transform.

15.1.4 Verify the following Mellin transforms:

$$\text{(a)}\quad \int_0^\infty x^{\alpha-1}\sin(kx)\ dx = k^{-\alpha}(\alpha-1)!\ \sin\frac{\pi\alpha}{2}, \quad -1 < \alpha < 1.$$

$$\text{(b)}\quad \int_0^\infty x^{\alpha-1}\cos(kx)\ dx = k^{-\alpha}(\alpha-1)!\ \cos\frac{\pi\alpha}{2}, \quad 0 < \alpha < 1.$$

Hint. You can force the integrals into a tractable form by inserting a convergence factor e^{-bx} and (after integrating) letting $b \rightarrow 0$. Also, $\cos kx + i\sin kx = \exp ikx$.

15.3.2 Let $F(\omega)$ be the Fourier (exponential) transform of $f(x)$ and $G(\omega)$ be the Fourier transform of $g(x) = f(x+a)$. Show that

$$G(\omega) = e^{-ia\omega} F(\omega).$$

15.3.12 A calculation of the magnetic field of a circular current loop in circular cylindrical coordinates leads to the integral

$$\int_0^\infty \cos kz \; k \; K_1(ka) dk.$$

Show that this integral is equal to

$$\frac{\pi a}{2(z^2+a^2)^{3/2}}.$$

Hint. Try differentiating Exercise 15.3.11(c).

15.3.13 As an extension of Exercise 15.3.11, show that
(a) $\displaystyle\int_0^\infty J_0(y)dy = 1$, (b) $\displaystyle\int_0^\infty N_0(y)dy = 0$, (c) $\displaystyle\int_0^\infty K_0(y)dy = \frac{\pi}{2}$.

15.3.14 The Fourier integral, Eq. (15.18), has been held meaningless for $f(t) = \cos \alpha t$. Show that the Fourier integral can be extended to cover $f(t) = \cos \alpha t$ by use of the Dirac delta function.

15.3.15 Show that

$$\int_0^\infty \sin ka \; J_0(k\rho) dk = \begin{cases} (a^2-\rho^2)^{-1/2}, & \rho < a, \\ 0, & \rho > a. \end{cases}$$

Here a and ρ are positive. The equation comes from the determination of the distribution of charge on an isolated conducting disk, radius a. Note that the function on the right has an **infinite** discontinuity at $\rho = a$.
Note. A Laplace transform approach appears in Exercise 15.10.8.

15.3.16 The function $f(r)$ has a Fourier exponential transform

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \, d^3 r = \frac{1}{(2\pi)^{3/2} k^2}.$$

Determine $f(\mathbf{r})$.
Hint. Use spherical polar coordinates in k-space.

$$\textit{ANS. } f(\mathbf{r}) = \frac{1}{4\pi r}.$$

15.3.17 (a) Calculate the Fourier exponential transform of $f(x) = e^{-a|x|}$.
(b) Calculate the inverse transform by employing the calculus of residues (Section 7.1).

15.4.1 The one-dimensional Fermi age equation for the diffusion of neutrons slowing down in some medium (such as graphite) is

$$\frac{\partial^2 q(x,\tau)}{\partial x^2} = \frac{\partial q(x,\tau)}{\partial \tau}.$$

Here q is the number of neutrons that slow down, falling below some given energy per second per unit volume. The Fermi age, τ, is a measure of the energy loss.

If $q(x,0) = S\delta(x)$, corresponding to a plane source of neutrons at $x = 0$, emitting S neutrons per unit area per second, derive the solution

$$q = S\frac{e^{-x^2/4\tau}}{\sqrt{4\pi\tau}}.$$

Hint. Replace $q(x,\tau)$ with

$$p(k,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} q(x,\tau)e^{ikx}\,dx.$$

This is analogous to the diffusion of heat in an infinite medium.

15.4.2 Equation (15.41) yields

$$g_2(\omega) = -\omega^2 g(\omega)$$

for the Fourier transform of the second derivative of $f(x)$. The condition $f(x) \to 0$ for $x \to \pm\infty$ may be relaxed slightly. Find the least restrictive condition for the preceding equation for $g_2(\omega)$ to hold.

$$ANS.\ \left[\frac{df(x)}{dx} - i\omega f(x)\right] e^{i\omega x}\Big|_{-\infty}^{\infty} = 0.$$

15.4.4 For a point source at the origin the three-dimensional neutron diffusion equation becomes

$$-D\,\nabla^2\varphi(\mathbf{r}) + K^2 D\varphi(\mathbf{r}) = Q\delta(\mathbf{r}).$$

Apply a three-dimensional Fourier transform. Solve the transformed equation. Transform the solution back into $\mathbf{r}$-space.

15.4.5 (a) Given that $F(\mathbf{k})$ is the three-dimensional Fourier transform of $f(\mathbf{r})$ and $F_1(\mathbf{k})$ is the three-dimensional Fourier transform of $\nabla f(\mathbf{r})$, show that

$$F_1(\mathbf{k}) = (-i\mathbf{k})F(\mathbf{k}).$$

This is a three-dimensional generalization of Eq. (15.40).

(b) Show that the three-dimensional Fourier transform of $\nabla \cdot \nabla f(\underline{r})$ is

$$F_2(\mathbf{k}) = (-i\mathbf{k})^2 F(\mathbf{k}).$$

Note. Vector $\mathbf{k}$ is a vector in the transform space. In Section 15.6 we shall have $\hbar\mathbf{k} = \mathbf{p}$, linear momentum.

item[**15.5.2**] $F(\rho)$ and $G(\rho)$ are the Hankel transforms of $f(r)$ and $g(r)$, respectively (Exercise 15.1.1). Derive the Hankel transform Parseval relation:

$$\int_0^\infty F^*(\rho)G(\rho)\rho\, d\rho = \int_0^\infty f^*(r)g(r)rdr.$$

15.5.4 Starting from Parseval's relation (Eq. (15.54)), let $g(y) = 1, 0 \le y \le \alpha$, and zero elsewhere. From this derive the Fourier inverse transform (Eq. (15.23)).
Hint. Differentiate with respect to α.

15.6.1 The function $e^{i\mathbf{k}\cdot\mathbf{r}}$ describes a plane wave of momentum $\mathbf{p} = \hbar\mathbf{k}$ normalized to unit density. (Time dependence of $e^{-i\omega t}$ is assumed.) Show that these plane-wave functions satisfy an orthogonality relation

$$\int (e^{i\mathbf{k}\cdot\mathbf{r}})^* e^{i\mathbf{k}'\cdot\mathbf{r}} dx\, dy\, dz = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}').$$

15.6.2 An infinite plane wave in quantum mechanics may be represented by the function

$$\psi(x) = e^{ip'x/\hbar}.$$

Find the corresponding momentum distribution function. Note that it has an infinity and that $\psi(x)$ is not normalized.

15.6.3 A linear quantum oscillator in its ground state has a wave function

$$\psi(x) = a^{-1/2}\pi^{-1/4}e^{-x^2/2a^2}.$$

Show that the corresponding momentum function is

$$g(p) = a^{1/2}\pi^{-1/4}\hbar^{-1/2}e^{-a^2p^2/2\hbar^2}.$$

15.6.4 The nth excited state of the linear quantum oscillator is described by

$$\psi_n(x) = a^{-1/2}2^{-n/2}\pi^{-1/4}(n!)^{-1/2}e^{-x^2/2a^2}H_n(x/a),$$

where $H_n(x/a)$ is the nth Hermite polynomial, Section 13.1. As an extension of Exercise 15.6.3, find the momentum function corresponding to $\psi_n(x)$.
Hint. $\psi_n(x)$ may be represented by $(\hat{a}^\dagger)^n\psi_0(x)$, where $\hat{a}^\dagger$ is the raising operator, Exercise 13.1.14 to 13.1.16.

15.6.5 A free particle in quantum mechanics is described by a plane wave

$$\psi_k(x,t) = e^{i[kx-(\hbar k^2/2m)t]}.$$

Combining waves of adjacent momentum with an amplitude weighting factor $\varphi(k)$, we form a wave packet

$$\Psi(x,t) = \int_{-\infty}^{\infty} \varphi(k) e^{i[kx-(\hbar k^2/2m)t]}\, dk.$$

(a) Solve for $\varphi(k)$ given that

$$\Psi(x,0) = e^{-x^2/2a^2}.$$

(b) Using the known value of $\varphi(k)$, integrate to get the explicit form of $\Psi(x,t)$. Note that this wave packet diffuses, or spreads out, with time.

$$\textit{ANS.}\quad \Psi(x,t) = \frac{e^{-\left\{x^2/2[a^2+(i\hbar/m)t]\right\}}}{[1+(i\hbar t/ma^2)]^{1/2}}.$$

Note. An interesting discussion of this problem from the evolution operator point of view is given by S. M. Blinder, Evolution of a Gaussian wavepacket, *Am. J. Phys.* **36**: 525 (1968).

15.6.6 Find the time-dependent momentum wave function $g(k,t)$ corresponding to $\Psi(x,t)$ of Exercise 15.6.5. Show that the momentum wave packet $g^*(k,t)g(k,t)$ is **independent** of time.

15.6.7 The deuteron, Example 10.1.2, may be described reasonably well with a Hulthén wave function

$$\psi(\mathbf{r}) = \frac{A}{r}[e^{-\alpha r} - e^{-\beta r}],$$

with A, α, and β constants. Find $g(\mathbf{p})$, the corresponding momentum function.
Note. The Fourier transform may be rewritten as Fourier sine and cosine transforms or as a Laplace transform, Section 15.8.

15.6.9 Check the normalization of the hydrogen momentum wave function

$$g(\mathbf{p}) = \frac{2^{3/2}}{\pi} \frac{a_0^{3/2}\hbar^{5/2}}{(a_0^2 p^2 + \hbar^2)^2}$$

by direct evaluation of the integral

$$\int g^*(\mathbf{p}) g(\mathbf{p}) d^3p.$$

15.6.12 The one-dimensional time-independent Schrödinger wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

For the special case of $V(x)$ an analytic function of x, show that the corresponding momentum wave equation is

$$V\left(i\hbar\frac{d}{dp}\right)g(p) + \frac{p^2}{2m}g(p) = Eg(p).$$

Derive this momentum wave equation from the Fourier transform, Eq. (15.62), and its inverse. Do not use the substitution $x \to i\hbar(d/dp)$ directly.

15.7.1 Derive the convolution

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\Phi(t-\tau)d\tau.$$

15.8.6 The electrostatic potential of a charged conducting disk is known to have the general form (circular cylindrical coordinates)

$$\Phi(\rho, z) = \int_0^{\infty} e^{-k|z|}J_0(k\rho)f(k)dk,$$

with $f(k)$ unknown. At large distances ($z \to \infty$) the potential must approach the Coulomb potential $Q/4\pi\varepsilon_0 z$. Show that

$$\lim_{k\to 0} f(k) = \frac{q}{4\pi\varepsilon_0}.$$

Hint. You may set $\rho = 0$ and assume a Maclaurin expansion of $f(k)$ or, using e^{-kz}, construct a delta sequence.

15.10.8 The electrostatic potential of a point charge q at the origin in circular cylindrical coordinates is

$$\frac{q}{4\pi\varepsilon_0}\int_0^{\infty} e^{-kz}J_0(k\rho)dk = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{(\rho^2+z^2)^{1/2}}, \quad \Re(z) \geq 0.$$

From this relation show that the Fourier cosine and sine transforms of $J_0(k\rho)$ are

$$\text{(a)} \quad \sqrt{\frac{\pi}{2}}F_c\left\{J_0(k\rho)\right\} = \int_0^{\infty} J_0(k\rho)\cos k\zeta\, dk = \begin{cases} (\rho^2-\zeta^2)^{-1/2}, & \rho > \zeta, \\ 0, & \rho < \zeta. \end{cases}.$$

$$\text{(b)} \quad \sqrt{\frac{\pi}{2}}F_s\left\{J_0(k\rho)\right\} = \int_0^{\infty} J_0(k\rho)\sin k\zeta\, dk = \begin{cases} 0, & \rho > \zeta, \\ (\rho^2-\zeta^2)^{-1/2}, & \rho < \zeta, \end{cases}.$$

Hint. Replace z by $z + i\zeta$ and take the limit as $z \to 0$.

15.10.21 The Laplace transform

$$\int_0^\infty e^{-xs} x J_0(x) dx = \frac{s}{(s^2+1)^{3/2}}$$

may be rewritten as

$$\frac{1}{s^2}\int_0^\infty e^{-y} y J_0(\frac{y}{s})\ dy = \frac{s}{(s^2+1)^{3/2}},$$

which is in Gauss-Laguerre quadrature form. Evaluate this integral for $s = 1.0, 0.9, 0.8, \ldots$, decreasing s in steps of 0.1 until the relative error rises to 10 percent. (The effect of decreasing s is to make the integrand oscillate more rapidly per unit length of y, thus decreasing the accuracy of the numerical quadrature.)

15.10.22 (a) Evaluate

$$\int_0^\infty e^{-kz} k J_1(ka) dk$$

by the Gauss-Laguerre quadrature. Take $a = 1$ and $z = 0.1(0.1)1.0$.

(b) From the analytic form, Exercise 15.10.7, calculate the absolute error and the relative error.

16.1.5 Verify that $\int_a^x \int_a^x f(t)dt\ dx = \int_a^x (x-t)f(t)dt$ for all $f(t)$ (for which the integrals exist).

16.3.2 Solve the equation

$$\varphi(x) = x + \tfrac{1}{2}\int_{-1}^1 (t+x)\varphi(t)dt$$

by the separable kernel method. Compare with the Neumann method solution of Section 16.3.

ANS. $\varphi(x) = \frac{1}{2}(3x-1)$.

16.3.6 If the separable kernel technique of this section is applied to a Fredholm equation of the first kind (Eq. (16.1)), show that Eq. (16.76) is replaced by

$$\mathbf{c} = \mathsf{A}^{-1}\mathbf{b}.$$

In general the solution for the unknown $\varphi(t)$ is **not** unique.

16.3.13 The integral equation

$$\varphi(x) = \lambda \int_0^1 J_0(\alpha x t)\varphi(t)dt, \quad J_0(\alpha) = 0,$$

is approximated by

$$\varphi(x) = \lambda \int_0^1 [1 - x^2t^2]\varphi(t)dt.$$

Find the minimum eigenvalue λ and the corresponding eigenfunction $\varphi(t)$ of the approximate equation.

ANS. $\lambda_{\min} = 1.112486, \quad \varphi(x) = 1 - 0.303337x^2.$

16.3.14 You are given the integral equation

$$\varphi(x) = \lambda \int_0^1 \sin \pi xt \varphi(t)dt.$$

Approximate the kernel by

$$K(x,t) = 4xt(1 - xt) \approx \sin \pi xt.$$

Find the positive eigenvalue and the corresponding eigenfunction for the approximate integral equation.
Note. For $K(x,t) = \sin \pi xt, \lambda = 1.6334$.

ANS. $\lambda = 1.5678, \quad \varphi(x) = x - 0.6955x^2$
$(\lambda_+ = \sqrt{31} - 4,\ \lambda_- = -\sqrt{31} - 4).$

16.3.16 Using numerical quadrature, convert

$$\varphi(x) = \lambda \int_0^1 J_0(\alpha xt)\varphi(t)dt, \quad J_0(\alpha) = 0,$$

to a set of simultaneous linear equations.

(a) Find the minimum eigenvalue λ.
(b) Determine $\varphi(x)$ at discrete values of x and plot $\varphi(x)$ versus x. Compare with the approximate eigenfunction of Exercise 16.3.13.

ANS. (a) $\lambda_{\min} = 1.14502.$

16.3.17 Using numerical quadrature, convert

$$\varphi(x) = \lambda \int_0^1 \sin \pi xt \varphi(t)dt$$

to a set of simultaneous linear equations.

(a) Find the minimum eigenvalue λ.
(b) Determine $\varphi(x)$ at discrete values of x and plot $\varphi(x)$ versus x. Compare with the approximate eigenfunction of Exercise 16.3.14.

ANS. (a) $\lambda_{\min} = 1.6334$.

16.3.18 Given a homogeneous Fredholm equation of the second kind

$$\lambda\varphi(x) = \int_0^1 K(x,t)\varphi(t)dt.$$

(a) Calculate the largest eigenvalue λ_0. Use the 10-point Gauss-Legendre quadrature technique. For comparison the eigenvalues listed by Linz are given as λ_{exact}.
(b) Tabulate $\varphi(x_k)$, where the x_k are the 10 evaluation points in $[0,1]$.
(c) Tabulate the ratio

$$\frac{1}{\lambda_0\varphi(x)}\int_0^1 K(x,t)\varphi(t)dt \quad \text{for } x = x_k.$$

This is the test of whether or not you really have a solution.

(a) $K(x,t) = e^{xt}$.

ANS. $\lambda_{\text{exact}} = 1.35303$.

(b) $K(x,t) = \begin{cases} \frac{1}{2}x(2-t), & x < t, \\ \frac{1}{2}t(2-x), & x > t. \end{cases}$

ANS. $\lambda_{\text{exact}} = 0.24296$.

(c) $K(x,t) = |x-t|$.

ANS. $\lambda_{\text{exact}} = 0.34741$.

(d) $K(x,t) = \begin{cases} x, & x < t, \\ t, & x > t. \end{cases}$

ANS. $\lambda_{\text{exact}} = 0.40528$.

Note. (1) The evaluation points x_i of Gauss-Legendre quadrature for $[-1,1]$ may be **linearly** transformed into $[0,1]$,

$$x_i[0,1] = \tfrac{1}{2}(x_i[-1,1] + 1).$$

Then the weighting factors A_i are reduced in proportion to the length of the interval:

$$A_i[0,1] = \tfrac{1}{2}A_i[-1,1].$$

16.3.19 Using the matrix variational technique of Exercise 17.8.7, refine your calculation of the eigenvalue of Exercise 16.3.18(c) $[K(x,t) = |x-t|]$. Try a 40×40 matrix.
Note. Your matrix should be symmetric so that the (unknown) eigenvectors will be orthogonal.

ANS. (40-point Gauss-Legendre quadrature) 0.34727.

17.2.9 Find the root of $px_0 = \coth px_0$ (Eq. (17.39)) and determine the corresponding values of p and x_0 (Eqs. (17.41) and (17.42)). Calculate your values to five significant figures.

17.2.10 For the two-ring soap film problem of this section calculate and tabulate x_0, p, p^{-1}, and A, the soap film area for $px_0 = 0.00(0.02)1.30$.

17.2.11 Find the value of x_0 (to five significant figures) that leads to a soap film area, Eq. (17.43), equal to 2π, the Goldschmidt discontinuous solution.

ANS. $x_0 = 0.52770$.

17.6.2 Find the ratio of R (radius) to H(height) that will minimize the total surface area of a right-circular cylinder of fixed volume.

17.6.8 A **deformed** sphere has a radius given by $r = r_0\{\alpha_0 + \alpha_2 P_2(\cos\theta)\}$, where $\alpha_0 \approx 1$ and $|\alpha_2| \ll |\alpha_0|$. From Exercise 12.5.16 the area and volume are

$$A = 4\pi r_0^2 \alpha_0^2 \left\{1 + \frac{4}{5}\left(\frac{\alpha_2}{\alpha_0}\right)^2\right\}, \qquad V = \frac{4\pi r_0^3}{3} a_0^3 \left\{1 + \frac{3}{5}\left(\frac{\alpha_2}{\alpha_0}\right)^2\right\}.$$

Terms of order α_2^3 have been neglected.

(a) With the constraint that the enclosed volume be held constant, that is, $V = 4\pi r_0^3/3$, show that the bounding surface of minimum area is a sphere ($\alpha_0 = 1, \alpha_2 = 0$).
(b) With the constraint that the area of the bounding surface be held constant, that is, $A = 4\pi r_0^2$, show that the enclosed volume is a maximum when the surface is a sphere.

Note concerning the following exercises: In a quantum-mechanical system there are g_i distinct quantum states between energies E_i and $E_i + dE_i$. The problem is to describe how n_i particles are distributed among these states subject to two constraints:

(a) fixed number of particles,

$$\sum_i n_i = n.$$

(b) fixed total energy,

$$\sum_i n_i E_i = E.$$

17.6.10 For identical particles obeying the Pauli exclusion principle, the probability of a given arrangement is

$$W_{FD} = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}.$$

Show that maximizing W_{FD}, subject to a fixed number of particles and fixed total energy, leads to

$$n_i = \frac{g_i}{e^{\lambda_1 + \lambda_2 E_i} + 1}.$$

With $\lambda_1 = -E_0/kT$ and $\lambda_2 = 1/kT$, this yields Fermi-Dirac statistics. *Hint*. Try working with $\ln W$ and using Stirling's formula, Section 8.3. The justification for **differentiation** with respect to n_i is that we are dealing here with a large number of particles, $\Delta n_i/n_i \ll 1$.

17.6.11 For identical particles but no restriction on the number in a given state, the probability of a given arrangement is

$$W_{BE} = \prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}.$$

Show that maximizing W_{BE}, subject to a fixed number of particles and fixed total energy, leads to

$$n_i = \frac{g_i}{e^{\lambda_1 + \lambda_2 E_i} - 1}.$$

With $\lambda_1 = -E_0/kT$ and $\lambda_2 = 1/kT$, this yields Bose–Einstein statistics. *Note*. Assume that $g_i \gg 1$.

17.6.12 Photons satisfy W_{BE} and the constraint that total energy is constant. They clearly do **not** satisfy the fixed-number constraint. Show that eliminating the fixed-number constraint leads to the foregoing result but with $\lambda_1 = 0$.

17.7.6 Show that requiring J, given by

$$J = \int_a^b \left[p(x)y_x^2 - q(x)y^2\right] dx,$$

to have a stationary value subject to the normalizing condition

$$\int_a^b y^2 w(x)\, dx = 1$$

leads to the Sturm-Liouville equation of Chapter 10:

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy + \lambda wy = 0.$$

Note. The boundary condition

$$py_x y \,|_a^b = 0$$

is used in Section 10.1 in establishing the Hermitian property of the operator.

17.8.1 From Eq. (17.128) develop in detail the argument when $\lambda \geq 0$ or $\lambda < 0$. Explain the circumstances under which $\lambda = 0$, and illustrate with several examples.

17.8.7 In the matrix eigenvector, eigenvalue equation

$$\mathsf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i,$$

where λ is an $n \times n$ Hermitian matrix. For simplicity, assume that its n real eigenvalues (Section 3.5) are distinct, λ_1 being the largest. If $\mathbf{r}$ is an approximation to $\mathbf{r}_1$,

$$\mathbf{r} = \mathbf{r}_1 + \sum_{i=2}^{n} \delta_i \ \mathbf{r}_i,$$

show that

$$\frac{\mathbf{r}^\dagger \mathsf{A} \mathbf{r}}{\mathbf{r}^\dagger \mathbf{r}} \leq \lambda_1$$

and that the error in λ_1 is of the order $|\delta_i|^2$. Take $|\delta_i| \ll 1$.
Hint. The n $\mathbf{r}_i$ form a complete orthogonal set spanning the n-dimensional (complex) space.

17.8.8 The variational solution of Example 17.8.1 may be refined by taking $y = x(1-x)+a_2x^2(1-x)^2$. Using the numerical quadrature, calculate $\lambda_{\text{approx}} = F[y(x)]$, Eq. (17.128), for a fixed value of a_2. Vary a_2 to minimize λ. Calculate the value of a_2 that minimizes λ and calculate λ itself, both to five significant figures. Compare your eigenvalue λ with π^2.

18.2.8 Repeat Exercise 18.2.7 for Feigenbaum's α instead of δ.

18.2.11 Repeat Exercise 18.2.9 for Feigenbaum's α.

18.3.1 Use a programmable pocket calculator (or a personal computer with BASIC or FORTRAN or symbolic software such as Mathematica or Maple) to obtain the iterates x_i of an initial $0 < x_0 < 1$ and $f'_\mu(x_i)$ for the logistic map. Then calculate the Lyapunov exponent for cycles of period 2, 3,... of the logistic map for $2 < \mu < 3.7$. Show that for $\mu < \mu_\infty$ the Lyapunov exponent λ is 0 at bifurcation points and negative elsewhere, while for $\mu > \mu_\infty$ it is positive except in periodic windows.
Hint. See Fig. 9.3 of Hilborn (1994) in the Additional Readings.

18.4.4 Plot the intermittency region of the logistic map at $\mu = 3.8319$. What is the period of the cycles? What happens at $\mu = 1 + 2\sqrt{2}$?

ANS. There is a tangent bifurcation to period 3 cycles.

19.4.7 A piece of uranium is known to contain the isotopes $^{235}_{92}$U and $^{238}_{92}$U as well as from 0.80 g of $^{206}_{82}$Pb per gram of uranium. Estimate the age of the piece (and thus Earth) in years.
Hint. Assume the lead comes only from the $^{238}_{92}$U. Use the decay constant from Exercise 19.4.5.

19.6.4 If $x_1, x_2, \cdots, x_n$ are a sample of measurements with mean value given by the arithmetic mean $\bar{x}$ and the corresponding random variables X_j that take the values x_j with the same probability are independent and have mean value μ and variance σ^2, then show that $\langle \bar{x} \rangle = \mu$ and $\sigma^2(\bar{x}) = \sigma^2/n$. If $\bar{\sigma}^2 = \frac{1}{n}\sum_j (x_j - \bar{x})^2$ is the sample variance, show that $\langle \bar{\sigma}^2 \rangle = \frac{n-1}{n}\sigma^2$.