Indira Gandhi National
Open University
School of Sciences

MPH-001
MATHEMATICAL METHODS IN PHYSICS

Block

1

PARTIAL DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS

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MATHEMATICAL METHODS IN PHYSICS: COURSE INTRODUCTION

In your study of physics at the UG level, you must have realized that much of the beauty and elegance of physics stems from the language of mathematics that it uses. That is why we would like to repeatedly emphasize that you should acquire a very good grasp of the mathematics you need to understand physics at any level.

In this course we acquaint you with the areas of mathematics required for higher studies in physics. Specifically, you will learn about partial differential equations and special functions (Block 1), vector spaces, matrices and tensors (Block 2), complex analysis (Block 3), Laplace and Fourier transforms (Block 4) and group theory (Block 5).

Our focus is mainly on how these methods and techniques are used in various areas of physics ranging from classical mechanics to quantum mechanics, solid state physics and theories of relativity.

In order to study this course effectively, we advise you to study the physics electives PHE-04 entitled Mathematical Methods in Physics-I, PHE-05 entitled Mathematical Methods in Physics-II and PHE-14 entitled Mathematical Methods in Physics-III offered in the first B.Sc. programme of IGNOU. We are not taking up ordinary differential equations (ODEs) and vector analysis in this course as these are taught in detail at the UG level. These are listed for recapitulation in the syllabus. Therefore, we advise you to study Block 1 of the course PHE-05 and Block 1 of PHE-04 thoroughly to refresh your knowledge of ODEs. These courses are available in eGyankosh on the IGNOU website: www.ignou.ac.in. We will be referring to several units of these courses as we build various concepts.

Finally, you should always sit with a pen and paper at hand while studying the course material, work out all steps and all solved examples yourself. Avoid the temptation of looking up answers before working out the unsolved problems given in each unit.

We hope you find this course useful and enjoy studying it. We wish you good luck.

BLOCK 1: PARTIAL DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS

It is a truism that nothing is permanent except change. As you know, change in the physical world manifests itself in a variety of forms. Whether you consider variations in electromagnetic fields leading to electromagnetic waves, flow of heat in various bodies or current in electrical circuits, you come across parameters that change in time and/or space. As a student of physics, you know that the change in such functions is represented in terms of the 'rates of change' or the derivatives of these functions with respect to some variables. You have learnt in your UG physics courses that the behaviour of a variety of physical systems is modelled by ordinary or partial differential equations. Boltzmann once said, "Equations are more intelligent than the people who discover them". Probably he was arguing for their beauty and elegance in describing physical phenomena unfolded by nature!

Differential equations serve as useful tools in the study of change in the physical world. Most of the general laws of nature in physics, chemistry, biology, astronomy, engineering and many other areas find their most natural expression in the language of differential equations. Recall from UG physics that ordinary differential equations (ODEs) involve differential equations that have derivatives of the dependent variable with respect to only one independent variable. involving functions of more than one variable. As you know, PDEs are differential equations involving derivatives of functions of more than one variable. Partial differential equations arise in such diverse areas as wave motion, heat conduction, electrostatics, magnetism, hydrodynamics, aerodynamics, nuclear physics, to mention a few.

In this block, you will study partial differential equations (PDEs) and special functions, which are solutions of partial differential equations having very interesting properties, and find use in studying a wide variety of physical phenomena. The block is divided into **five units**. **Unit 1** entitled **Partial Differential Equations** is an introductory unit in which we quickly recapitulate certain basic concepts of PDEs and then discuss the solutions of Laplace's equations in the Cartesian, cylindrical and spherical polar coordinate systems. Laplace's equation has wide applications in physics. It can be applied to obtain the gravitational (electrostatic) potential in free space devoid of matter (charge), to study the steady (time independent) flow of heat across various bodies, to model surface waves on a fluid or to describe the irrotational motion of an incompressible fluid. In **Unit 2**, as its title **PDEs in Physics** reveals, we discuss how to solve other important PDEs in physics, namely, Poisson's equation, heat diffusion equation and the wave equation. We also discuss the Fredholm and Volterra integral equations. Then we turn our attention to special functions.

The study of special functions is beautiful even from a purely mathematical point of view and provides us very powerful tools to solve a wide variety of problems. For example, **Bessel Functions** (**Unit 3**) have many applications in physics, ranging from the study of planetary motion, cooling towers in power plants, diffusion of light at a circular aperture, e. m. waves in cavity resonators, waveguides, diffusion across variable cross section to scattering of neutrons by a nucleus, etc. The theoretical explanation of the description of an atom in terms of principal, orbital, azimuthal and spin quantum numbers on the basis of Schrodinger equation requires a knowledge of the properties of Legendre polynomials and spherical harmonics. We discuss these in **Unit 4** entitled **Special Functions – I**, which also deals with hypergeometric functions.

The problem of a one-dimensional quantum oscillator and hydrogen atom are of significant practical and theoretical interest. In **Unit 5** entitled **Special Functions – II**, you will study

Hermite and Laguerre polynomials that are required to understand these problems. In fact, these polynomials enable us to mathematically describe the microscopic world of molecules, atoms and sub-atomic particles. In the next block, we take up some very interesting areas of mathematical physics, namely, vector spaces, matrices and tensors.

One piece of advice before you start studying the block: You know that mathematical methods of physics are best learnt by working through the text, examples, exercises in the units called self-assessment questions (SAQs), and at the end of each unit called Terminal Questions (TQs). You must therefore study with a paper and pen in hand and solve as many problems as you can.

We hope that you will enjoy studying and working through the block. We wish you all the best!

UNIT 1

PARTIAL DIFFERENTIAL EQUATIONS

Structure

1.1 INTRODUCTION

You have studied ordinary and partial differential equations (ODEs and PDEs) and the Cartesian, cylindrical and spherical polar coordinate systems in your UG Mathematics and Physics courses. You would also be familiar with the Fourier series and their determination for various functions. We will assume that you have learnt about these topics in your UG courses. Otherwise, you may like to revise them from the references given in the margin and the Block Introduction before studying this unit.

You may like to refer to Unit 3 of PHE-04, Blocks 1 and 2 of PHE-05 and Block 4 of PHE-14 to study this unit. The complete reference of these courses is given in the Block Introduction.

As you are aware, partial differential equations are used to model a variety of physical phenomena such as the propagation of waves, heat conduction, diffusion of particles, electric potential distributions in various systems, etc. In this unit, you will recapitulate the method of separation of variables to reduce a given PDE into a set of ODEs. You will then learn how to solve Laplace's equation in Cartesian, cylindrical and spherical polar coordinate systems.

In the next unit, we will discuss how to solve other important PDEs in physics.

Expected Learning Outcomes

After studying this unit, you should be able to:

- apply the method of separation of variables to reduce a given PDE to a set of ODEs;
- obtain general solutions of simple PDEs; and
- solve Laplace's equation using Cartesian, cylindrical and spherical polar coordinates.

1.2 METHOD OF SEPARATION OF VARIABLES

You have learnt about PDEs in physics in your UG courses. You know that linear and non-linear second order PDEs form the backbone of theoretical physics. Apart from Laplace's equation and Poisson's equation, a few more examples of important PDEs in physics are the heat diffusion and wave equations; Integral equations; Fredholm and Volterra equations, Helmholtz equation, Telegraph equation, Klein-Gordon equation, Schrödinger equation and Dirac's equation, etc. We will assume that you know the basic concepts related to PDEs such as how to classify them on the basis of order, degree, linearity and homogeneity. You may like to revise these concepts by studying the relevant portions of the references given in the Block Introduction.

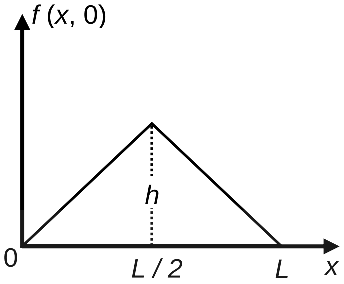

Fig. 1.1: **A plucked finite string fixed at both ends.**

In this section, we discuss the **method of separation of variables**, which is the simplest and most widely used method used to solve PDEs. We will apply this method to solve a few second order linear PDEs in physics in this unit and Unit 2 of this block. Let us now explain the method.

To illustrate the method of separation of variables, we consider a simple PDE that you are familiar with from your UG courses. You may recall that the vibrations of a plucked finite string (Fig. 1.1) fixed at both ends are described by the following partial differential equation known as the wave equation:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2} \tag{1.1}$$

where $f(x, t)$ gives the vertical displacement of the vibrating string from its equilibrium position at any instant t. The constant v is called the wave velocity because it is the velocity with which the disturbance at one point of the string would travel along the string. Note that we have taken the string to be along the x-axis.

Note that the PDE is linear as there is no term containing the powers of f or product of f and its derivatives.

We now assume that the solution of Eq. (1.1) can be written in the form of a product as:

$$f(x, t) = X(x)\,T(t) \tag{1.2}$$

Physically, it means that the dependence of the unknown function f on one variable is in no way affected by its dependence on the other variable. Does this imply that there is no connection at all between X and T? No, it only means that the function X does not depend on t and the function T does not depend upon x. For instance, the function

$$f(x, t) = x \sin \omega t \tag{1.3a}$$

is completely separable in x and t. On the other hand, the function

$$f(x, t) = x + t \tag{1.3b}$$

is inseparable because the function cannot be written as a product of two separate functions, each being a function of only x and only t, respectively. To illustrate the method, we differentiate Eq. (1.2) twice with respect to *x*. This gives:

$$\frac{\partial f}{\partial x} = X'T$$

and

$$\frac{\partial^2 f}{\partial x^2} = X''T \qquad (1.4)$$

where prime(s) denote ordinary differentiation with respect to *x*. This emphasizes the fact that the derivative is the total derivative and the function *X* has only one independent variable. Similarly, if we differentiate Eq. (1.2) with respect to *t*, we obtain

$$\frac{\partial f}{\partial t} = X\dot{T}$$

and

$$\frac{\partial^2 f}{\partial t^2} = X\ddot{T} \qquad (1.5)$$

where dot(s) denote ordinary differentiation with respect to *t*. We have used primes and dots just to distinguish the independent variables with respect to which differentiation has been carried out.

By inserting results contained in Eqs. (1.4 and 1.5) in Eq. (1.1), we obtain

$$X(x)\ddot{T}(t) = v^2 X''(x)T(t)$$

Dividing throughout by $v^2 X(x)T(t)$, we get

$$\frac{\ddot{T}(t)}{v^2 T(t)} = \frac{X''(x)}{X(x)} \qquad (1.6)$$

The left hand side of Eq. (1.6) involves functions which depend only on *t* whereas the expression on right-hand side is a function of *x* only. Note that *v* is constant. Thus, if we vary *t* and keep *x* fixed, the right-hand side cannot change. This means that $\ddot{T}(t)/v^2 T(t)$ must remain constant for all *t*.

Similarly, if we vary *x* holding *t* fixed, the left-hand side must not change. That is, the quantity $X''(x)/X(x)$ must be the same for all *x*. Mathematically, we express this fact by saying that both sides must be equal to a constant, say, *k*. Is this argument sound? To discover the answer to this question, let us write *k* to represent either side Eq. (1.6). i.e.,

$$\frac{\ddot{T}(t)}{v^2 T(t)} = k = \frac{X''(x)}{X(x)} \qquad (1.17)$$

Then from the right-hand side of the above equation, we have

$$\frac{\partial}{\partial t}(k) = \frac{\partial}{\partial t}\left[\frac{X''(x)}{X(x)}\right] = 0$$

and from the left-hand side, we have

$$\frac{\partial}{\partial x}(k) = \frac{\partial}{\partial x}\left[\frac{\ddot{T}(t)}{v^2 T(t)}\right] = 0$$

Since the first order partial derivative of k with respect to t or x is zero, k must be a constant. It is called the **separation constant**.

This step of **being able to equate the two sides of Eq. (1.7) to a constant is really the key to the process of separation of variables**.

Thus, you can now rewrite the given equation as the following two ordinary differential equations:

$$X''(x) - k\ X(x) = 0 \tag{1.8a}$$

and

$$\ddot{T}(t) - k\ v^2 T(t) = 0 \tag{1.8b}$$

That is, **by assuming a separable solution**, we have reduced a **partial differential equation in two variables into <u>two</u> equivalent ordinary differential equations**.

The ODEs so obtained can be solved using the methods you have learnt in the earlier UG courses. You may like to revise them from the reference given in the Block Introduction. You will learn how to solve a few important PDEs in physics in Sec. 1.3 of this unit and Unit 2. Now, before proceeding further let us revise the method of separation of variables.

METHOD OF SEPARATION OF VARIABLES
1. The unknown function of two (or more) variables is expressed as a product of two (or more) functions of each of these variables.
2. The assumed form of the solution is inserted in the given differential equation. A second-order PDE in two variables splits into two ODEs. When the number of independent variables is more than two, we get ODEs equal in number to the independent variables.
3. You can solve the ODEs so obtained using known methods.
4. The general solution of a given PDE is obtained by taking the product of the solutions of ODEs.

The same method can be extended to a PDE in three variables. Why don't you work out SAQ 1 to practice this method for three variables?

<u>SAQ 1</u>

Use the method of separation of variables to reduce the following PDE to a set of three ODEs:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

The equation in SAQ 1 describes the steady-state temperature distribution in a cylindrical body, such as control/fuel rods in the reactor core.

We hope that you can now use the method of separation of variables to reduce a PDE to a set of ODEs and solve it. **Remember**: **The number of ODEs equals the number of independent variables in the given PDE**.

We now solve **Laplace's equation** for different physical situations under specific initial and boundary conditions.

We expect you to know what the terms initial conditions and boundary conditions mean. You should revise these from the reference (PHE-05) given in the Block Introduction.

1.3 SOLVING LAPLACE'S EQUATION

Laplace's equation is one of the most important equations with many applications in physics. For example, it is used in gravitational and electrostatic potential theory, fluid mechanics, and for modelling steady state heat flow, etc. In general, Laplace's equation describes steady state situations (that do not depend explicitly on time), that is, situations of equilibrium. In this section, you will learn how to solve Laplace's equation in Cartesian, cylindrical and spherical polar coordinates. We will give examples for all three types of solutions.

Let us begin with the three-dimensional Laplace's equation:

$$\nabla^2 f = 0 \tag{1.9a}$$

You have to learn how to solve Eq. (1.9a) in Cartesian, cylindrical and spherical polar coordinates. Before studying Secs. 1.3.2 and 1.3.3, you may like to revise the cylindrical and spherical polar coordinates from Unit 3 of PHE-04 (refer to Block Introduction). We discuss these cases in the next three sections.

1.3.1 Cartesian Coordinates

Laplace's equation in Cartesian coordinates is given as:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1.9b}$$

Let us separate Eq. (1.9b) into three ODEs using the method of separation of variables. Following the method, we write:

$$f(x, y, z) = X(x)\,Y(y)\,Z(z) \tag{1.10}$$

Next, we differentiate the function *f* with respect to each one of the independent variables twice and write:

$$\frac{\partial f}{\partial x} = X'YZ$$

$$\frac{\partial^2 f}{\partial x^2} = X''YZ \tag{1.11a}$$

$$\frac{\partial f}{\partial y} = XY'Z$$

$$\frac{\partial^2 f}{\partial y^2} = XY''Z \tag{1.11b}$$

$$\frac{\partial f}{\partial z} = XYZ'$$

$$\frac{\partial^2 f}{\partial z^2} = XYZ'' \tag{1.11c}$$

Substituting Eqs. (1.11 a to c) in Eq. (1.9) and dividing the result by the product *XYZ*, we get:

$$\frac{1}{XYZ}\left(X''YZ + XY''Z + XYZ''\right) = 0$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \tag{1.12}$$

Notice that each of these terms on the LHS of Eq. (1.12) involves three functions which depend only on *single variables* *x*, *y* and *z*, respectively. Since *x*, *y* and *z* are independent variables, will the sum on LHS of Eq. (1.12) be equal to zero? It will be zero only if each of these terms is equal to a constant such that the sum of the three constants is zero. This means that we can equate each term to a constant and write the three terms as three ODEs:

$$\frac{X''(x)}{X(x)} = \frac{1}{X}\frac{d^2 X}{dx^2} = -k_1^2 \tag{1.13a}$$

$$\frac{Y''(y)}{Y(y)} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = -k_2^2 \tag{1.13b}$$

$$\frac{Z''(z)}{Z(z)} = \frac{1}{Z}\frac{d^2 Z}{dz^2} = k_3^2 \tag{1.13c}$$

Note that at least one of the three separation constants has to be negative for Eq. (1.12) to hold. We have taken two of these constants to be negative and one to be positive. Of course, the sum of the three constants has to be zero: $-k_1^2 - k_2^2 + k_3^2 = 0$ for Eq. (1.12) to hold.

So, we can now rewrite Laplace's equation as three separate ordinary differential equations:

$$X''(x) + k_1^2 \; X(x) = 0 \tag{1.14a}$$

$$Y''(y) + k_2^2 \, Y(y) = 0 \tag{1.14b}$$

and

$$Z''(z) - k_3^2 \, Z(z) = 0 \tag{1.14c}$$

You know very well the solutions of these three ODEs:

$$X(x) = A\cos k_1 x + B\sin k_1 x \tag{1.15a}$$

$$Y(y) = C\cos k_2 y + D\sin k_2 y \tag{1.15b}$$

and $$Z(z) = P\cosh k_3 z + Q\sinh k_3 z \tag{1.15c}$$

where $$k_1^2 + k_2^2 = k_3^2 \tag{1.15d}$$

The values of the constants *A*, *B*, *C*, *D*, *P* and *Q* are determined by the boundary conditions and initial conditions that apply in any given physical problem. The general solution of the Laplace's equation is obtained by taking a linear combination of all possible products of these solutions. We will now illustrate how to obtain the general solution of Laplace's equation in Cartesian coordinates for a given physical problem. Let us consider the example of determining the steady state temperature distribution in a rectangular metal plate.

Example 1.1

A thin rectangular metal plate of length L and width B is sandwiched between two sheets of insulation (Fig. 1.2a). The temperature of the plate is held at T_0 at its top edge, T_1 at its bottom edge and 0° C on its left edge. The plate is insulated on its right edge and so, no heat flows in that direction.

Since the plate is very thin, we can assume that the temperature does not vary in the *z*-direction. Therefore, the steady state temperature distribution of the plate obeys the two-dimensional Laplace's equation:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \qquad 0 < x < L, \; 0 < y < B \tag{1.16}$$

Fig. 1.2b shows the two-dimensional view of the plate in the *x*-*y* plane. Since the plate is insulated on its right edge, no heat flows in that direction and the partial derivative of T in the *x*-direction is zero (Fig. 1.2b).

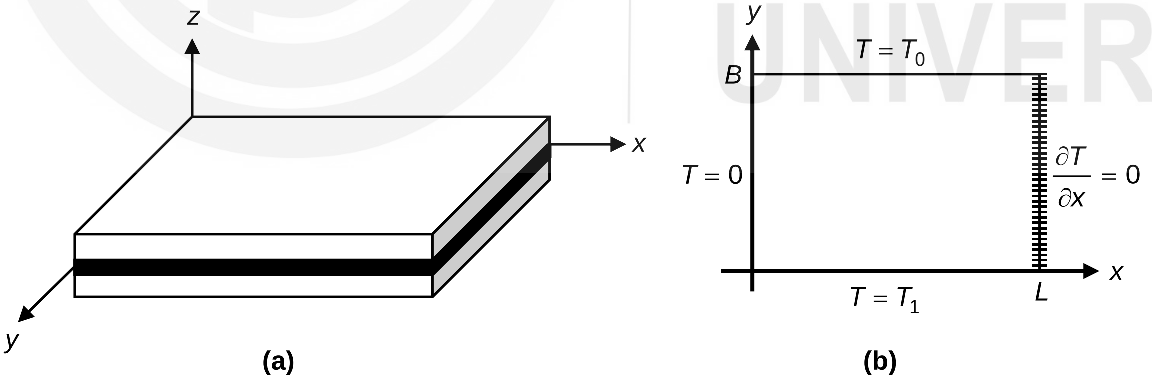

Fig. 1.2: a) A thin plate between sheets of insulation; b) transverse view of the plate showing boundary conditions for $T(x,y)$.

Let us write down the boundary conditions for this problem. You can see from Fig. 1.2b that

i) $T(0, y) = 0, \quad \dfrac{\partial T(L, y)}{\partial x} = 0 \qquad 0 < y < B$ (1.17a)

ii) $T(x, 0) = T_1, \quad T(x, B) = T_0 \qquad 0 < x < L$ (1.17b)

Solve Laplace's equation (1.16) for these boundary conditions.

Solution : We use the method of separation of variables to separate the two-dimensional Laplace's equation into two ODEs following the standard method:

$$T(x, y) = X(x)\,Y(y) \tag{1.18a}$$

so that

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \tag{1.18b}$$

You may like to obtain the solutions for $X(x)$ and $Y(y)$ as explained in this section. Solve SAQ 2.

SAQ 2

Obtain the general solutions $X(x)$ and $Y(y)$ of the ODEs given in Eq. (1.18b).

So, you have obtained the following solutions on solving SAQ 2:

$$X(x) = A\cos\lambda x + B\sin\lambda x \tag{1.18c}$$

and

$$Y(y) = C\cosh\lambda y + D\sinh\lambda y \tag{1.18d}$$

We now apply the boundary condition (i) given in Eq. (1.17a) and get

$$A = 0$$

So,

$$X(x) = B\sin\lambda x$$

Applying the second boundary condition in Eq. (1.17a), we get:

$$\frac{\partial T(L, y)}{\partial x} = B\lambda\cos\lambda x\big|_{x=L} = 0 \Rightarrow \cos\lambda L = 0$$

or

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2, ... \tag{1.18e}$$

This yields:

$$X_n(x) = B_n\sin\lambda_n x$$

Inserting the value of λ in the expression for $Y(y)$ given by Eq. (1.18d), we get:

$$Y_n(y) = C_n\cosh\lambda_n y + D_n\sinh\lambda_n y$$

The general solution of the Laplace's equation for $T(x, y)$ is given by:

$$T(x, y) = \sum_{n=1}^{\infty}(P_n\cosh\lambda_n y + R_n\sinh\lambda_n y)\sin\lambda_n x \tag{1.18f}$$

with λ_n given by Eq. (1.18c) and $P_n = C_n B_n$ and $R_n = D_n B_n$.

We next apply the boundary condition (ii) given by Eq. (1.17b) to determine the coefficients P_n and R_n. At $y = 0$,

$$T(x, 0) = \sum_{n=1}^{\infty} P_n \sin \lambda_n x = T_1, \qquad 0 < x < L \tag{1.19a}$$

From which we determine P_n to be:

$$P_n = \frac{2}{L}\int_0^L T_1 \sin\lambda_n x\, dx = \frac{4T_1}{(2n-1)\pi} \tag{1.19b}$$

At $y = B$,

$$T(x, B) = \sum_{n=1}^{\infty}(P_n \cosh \lambda_n B + R_n \sinh \lambda_n B)\sin \lambda_n x = T_0 \tag{1.20a}$$

or $$T(x, B) = \sum_{n=1}^{\infty} G_n \sin \lambda_n x = T_0 \tag{1.20b}$$

where $$G_n = P_n \cosh \lambda_n B + R_n \sinh \lambda_n B \tag{1.20c}$$

Then you can see that the coefficients G_n are given by:

$$G_n = \frac{2}{L}\int_0^L T_0 \sin\lambda_n x\, dx = \frac{4T_0}{(2n-1)\pi} \tag{1.21a}$$

Solve the integral in Eq. (1.21a) and the steps that follow yourself. From Eqs. (1.20c and 1.21a), we get the coefficients R_n in terms of the known coefficients G_n and P_n :

$$R_n = \frac{G_n - P_n \cosh \lambda_n B}{\sinh \lambda_n B} = \frac{4}{(2n-1)\pi}\frac{T_0 - T_1 \cosh \lambda_n B}{\sinh \lambda_n B} \tag{1.21b}$$

Thus, the unique solution of this problem is:

$$T(x, y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(T_1 \cosh \lambda_n y + \frac{T_0 - T_1 \cosh \lambda_n B}{\sinh \lambda_n B}\sinh \lambda_n y\right)\sin \lambda_n x \tag{1.22}$$

1.3.2 Cylindrical Coordinates

In cylindrical coordinates (ρ, ϕ, z), Laplace's equation $\nabla^2 f(\rho, \phi, z) = 0$ is given as:

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1.23}$$

Again we have to separate Eq. (1.23) into three ODEs using the method of separation of variables. We would like you to separate Eq. (1.23) into three ODES taking $f(\rho, \phi, z)$ as the product $R(\rho)\,\Phi(\phi)\,Z(z)$. Solve SAQ 3.

SAQ 3

Separate Eq. (1.23) into three ODEs.

So, you have obtained the following ODEs on solving SAQ 3.

$$\frac{1}{Z}\frac{d^2Z}{dz^2} - \alpha^2 = 0 \tag{1.24a}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\beta^2 \quad \text{or} \quad \frac{d^2\Phi}{d\phi^2} + \beta^2\Phi = 0 \tag{1.24b}$$

$$\frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \alpha^2\rho^2 - \beta^2 = 0 \tag{1.24c}$$

The general solutions of Eqs. (1.24a and b) are given by:

$$Z(z) = C_1 e^{\alpha z} + C_2 e^{-\alpha z}, \tag{1.25a}$$

$$\Phi(\phi) = C_3 e^{i\beta\phi} + C_4 e^{-i\beta\phi} \tag{1.25b}$$

> $x = \rho\alpha$
>
> $$\Rightarrow \frac{dR}{d\rho} = \frac{dR}{dx}\frac{dx}{d\rho} = \alpha\frac{dR}{dx}$$
>
> $$\frac{d^2R}{d\rho^2} = \frac{d}{d\rho}\left(\alpha\frac{dR}{dx}\right)$$
>
> $$= \alpha^2\frac{d^2R}{dx^2}$$

Let us now solve Eq. (1.24c). With the change of variable to $x = \rho\alpha$, we can recast it as the following equation (read the margin remark):

$$x^2\frac{d^2R}{\partial x^2} + x\frac{dR}{dx} + (x^2 - \beta^2)R = 0 \tag{1.25c}$$

This is the Bessel equation and you will solve it in Unit 3. Here we give the general solution, which is a linear combination of Bessel's functions of the first kind $J_\beta(x)$ and Bessel's functions of the second kind $Y_\beta(x)$:

$$R_\beta(\rho) = C_5 J_\beta(\alpha\rho) + C_6 Y_\beta(\alpha\rho) \tag{1.25d}$$

So, the general solution of Laplace's equation (1.23) in cylindrical coordinates is:

$$f(\rho, \phi, z) = \sum_\alpha\sum_\beta \begin{bmatrix} A_{\alpha\beta}e^{\alpha z}e^{i\beta\phi}J_\beta(\alpha\rho) + B_{\alpha\beta}e^{\alpha z}e^{i\beta\phi}Y_\beta(\alpha\rho) \\ + C_{\alpha\beta}e^{\alpha z}e^{-i\beta\phi}J_\beta(\alpha\rho) + D_{\alpha\beta}e^{\alpha z}e^{-i\beta\phi}Y_\beta(\alpha\rho) \\ + E_{\alpha\beta}e^{-\alpha z}e^{i\beta\phi}J_\beta(\alpha\rho) + F_{\alpha\beta}e^{-\alpha z}e^{i\beta\phi}Y_\beta(\alpha\rho) \\ + G_{\alpha\beta}e^{-\alpha z}e^{-i\beta\phi}J_\beta(\alpha\rho) + H_{\alpha\beta}e^{-\alpha z}e^{-i\beta\phi}Y_\beta(\alpha\rho) \end{bmatrix} \tag{1.26}$$

Let us now apply Laplace's equation to a system having cylindrical symmetry.

Example 1.2

Consider a solid cylindrical cooling tower of radius 2 units and height 4 units (see Fig. 1.3). Its base and the curved side are held at 0° and top at a constant temperature T_0. The steady-state temperature distribution is described by Laplace's equation $\nabla^2 T = 0$.

i) Specify the boundary conditions.

ii) Determine the steady-state temperature distribution in the cooling tower.

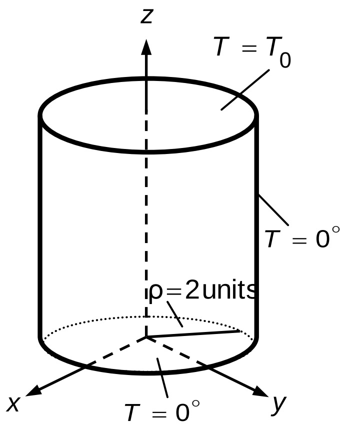

Fig. 1.3: A cylindrical cooling tower of radius 2 units and height 4 units.

Solution :

i) We can express the boundary conditions for the problem (Fig. 1.3) as:

$$T(2, z) = 0, \quad 0 < z < 4,$$

$$T(\rho, 0) = 0, \quad T(\rho, 4) = T_0, \; 0 < \rho < 2$$

ii) Since the tower possesses cylindrical symmetry, its steady state temperature distribution will be independent of ϕ. Hence, Laplace's equation in cylindrical polar coordinates for the steady state temperature $T(\rho, z)$ in the cooling tower is:

$$\nabla^2 T = \frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{\partial^2 T}{\partial z^2}, \quad 0 < \rho < 2; \quad 0 < z < 4 \qquad \text{(i)}$$

You can separate the PDE (i) into two ODEs in ρ and z. Substituting $T = R(\rho)\,Z(z)$ and separating variables, you will obtain the following ODEs:

$$\rho^2 R'' + \rho R' + \lambda^2 \rho^2 R = 0 \qquad \text{(ii)}$$

and

$$Z'' - \lambda^2 Z = 0 \qquad \text{(iii)}$$

You may like to work out the steps leading to Eqs. (ii and iii) before studying further.

The negative separation constant is used in Eq. (iii) since there is no reason to expect the solution to be periodic in z.

Eq. (ii) is the zeroth order Bessel's equation (see Sec. 3.2 of Unit 3 of this block). So, the solution of Eq. (ii) is:

$$R = c_1 J_0(\lambda\rho) + c_2 Y_0(\lambda\rho)$$

where J_0 and Y_0 are Bessel's functions of the first and the second kind of order zero.

Since the solution of Eq. (iii) is defined on the finite interval (0, 4), we write:

$$Z = c_3 \cosh \lambda z + c_4 \sin \lambda z$$

In order to have a bounded temperature $T(\rho, z)$ at $\rho = 0$, we must define $c_2 = 0$ since $Y_0(x) \to -\infty$ as $x \to 0$. The condition $T(2, z) = 0$ implies that $R(2) = 0$ or

$$J_0(2\lambda) = 0 \qquad \text{(iv)}$$

This equation will hold for $\lambda_1 = \alpha_1 / 2, \lambda_2 = \alpha_2 / 2, ...\lambda_n = \alpha_n / 2$ where $\alpha_1, \alpha_2, ...\alpha_n$ are zeros of the Bessel function. Lastly $Z(0) = 0$ implies $c_3 = 0$ since $\sinh 0 = 0$ and $\cosh 0 = 1$. Hence, we have $R = c_1 J_0(\lambda_n \rho)$, $Z = c_4 \sinh(\lambda_n z)$ and

$$u_n = A_n \sinh(\lambda_n z)\, J_0(\lambda_n \rho)$$

The general solution is, therefore, of the form:

$$u(r, z) = \sum_{n=1}^{\infty} A_n \sinh(\lambda_n z)\, J_0(\lambda_n \rho) \qquad \text{(v)}$$

Finally, we discuss Laplace's equation in spherical polar coordinates and solve it for a physical system.

1.3.3 Spherical Polar Coordinates

Laplace's equation in spherical coordinates (r, θ, ϕ) is given as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = 0 \qquad (1.27)$$

Once again, we leave it as a practice exercise for you to reduce it to three ODEs. You should write $f(r, \theta, \phi)$ as the product $R(r)\,\Theta(\theta)\,\Phi(\phi)$ and verify that the three ODEs are:

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \qquad (1.28a)$$

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n^2 R = 0 \qquad (1.28b)$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \left[n^2 - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \qquad (1.28c)$$

You should verify Eqs. (1.28a to c) before studying further.

The next step is to determine the solution of these ODEs. You know that the solution of Eq. (1.28a) is given by:

$$\Phi(\phi) = A_m e^{im\phi} = B_m \cos\phi + C_m \sin\phi \qquad (1.29a)$$

For solving Eq. (1.28b), we use the Frobenius method and obtain the solution for $n^2 = l(l+1)$ as:

$$R_l(r) = D_l r^l + F_l r^{-l-1} \tag{1.29b}$$

You may like to solve Eq. (1.28b) yourself and verify Eq. (1.29b). Solve SAQ 4.

SAQ 4

Solve Eq. (1.28b).

Eq. (1.28c) may be recast as the ODE for Associated Legendre polynomials with the change in variable $x = \cos\phi$ and with $m = 0, 1, ..., l$. We do not work this out as Associated Legendre polynomials are not a part of the syllabus (you can refer to the Appendix of Unit 14, PHE-14 for these polynomials). We just state the general solution here:

$$f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left([D_l r^l + F_l r^{-l-1}] P_l^m (\cos\theta) [B_m \cos\phi + C_m \sin\phi] \right) \tag{1.30}$$

The general solution is also written as:

$$f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left([D_l r^l + F_l r^{-l-1}] P_l^m (\cos\theta) e^{-im\phi} \right) \tag{1.31a}$$

$$f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left([D_l r^l + F_l r^{-l-1}] Y_l^m (\theta, \phi) \right) \tag{1.31b}$$

where $Y_l^m(\theta, \phi)$ are the spherical harmonics (Sec. 3.3 of Unit 3 in this block).

Let us now solve Laplace's equation in spherical coordinates for a physical problem.

Example 1.3

Consider a solid sphere of radius R and suppose that the surface of the upper half of the sphere is kept at a constant temperature T_0. Let the surface of its lower half be kept at zero temperature. Determine the steady state temperature at a point inside the sphere.

Solution : Since the problem has spherical geometry, we use Laplace's equation in spherical polar coordinates to solve it. The steady-state temperature distribution $T(r, \theta, \phi)$ satisfies the equation:

$$\nabla^2 T(r, \theta, \phi) = 0 \tag{i}$$

or $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}=0 \qquad \text{(ii)}$$

We will use the general solution given by Eq. (1.30).

Let us specify the boundary conditions for this problem. These are:

$$T(r=R)=\begin{cases} T_0, & 0<\theta<\frac{\pi}{2} \quad \text{or} \quad 0<\cos\theta<1 \\ 0, & \frac{\pi}{2}<\theta<\pi \quad \text{or} \quad -1<\cos\theta<0 \end{cases} \qquad \text{(iii)}$$

Note that the temperature of the sphere at its upper and lower surfaces is constant, i.e., independent of ϕ. So, in Eq. (1.30 or 1.31a), we must have $m=0$. Then $P_l^m(\cos\theta)=P_l(\cos\theta)$, the Legendre polynomials.

Note further that we have to determine the temperature distribution at a point **inside** the sphere. Therefore, the coefficient F_l of the term r^{-l-1} in the radial part of the solution, $[R_l(r)=D_l\, r^l+F_l\, r^{-l-1}]$, is to be taken as zero. Otherwise the solution will become infinite at the origin. So, we retain only the term containing r^l. The general solution then becomes:

$$T(r,\theta)=\sum_{l=0}^{\infty}C_l\, r^l P_l(\cos\theta) \qquad \text{(iv)}$$

We can determine the coefficients C_l by applying the boundary conditions specified above. Let us put $x=\cos\theta$ where x is not the coordinate x but just stands for cos θ. So, we have:

$$T\big|_{r=R}=\sum_{l=0}^{\infty}C_l\, R^l P_l(x)=T_0\, u(x) \qquad \text{(v)}$$

where $$u(x)=\begin{cases} 0, & -1<x<0 \\ 1, & 0<x<1 \end{cases} \qquad \text{(vi)}$$

As you have learnt in your UG course or will learn in Unit 4, $u(x)$ in Eq. (vi) can be expanded in a series of Legendre polynomials as:

$$u(x)=\sum_{l=0}^{\infty}A_l\, P_l(x)=\frac{1}{2}P_0(x)+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)+\ldots \qquad \text{(vii)}$$

From Eq. (v), the coefficients C_l are $\frac{T_0}{R^l} A_l$ and, therefore, substituting these values of C_l in Eq. (iv), we get the solution unique to this problem (TQ 5) as:

$$T = T_0 \left[\frac{1}{2} P_0(\cos\theta) + \frac{3}{4}\frac{r}{R} P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{11}{32}\left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \ldots \right] \qquad \text{(viii)}$$

With this discussion of Laplace's equation, we end this unit and recapitulate what you have learnt.

1.4 SUMMARY

In this unit, we have covered the following concepts:

- Method of separation of variables to reduce a given partial differential equation into a set of ordinary differential equations.
- Laplace's equation in Cartesian, cylindrical and spherical polar coordinates.
- General solutions of Laplace's equation in Cartesian, cylindrical and spherical polar coordinates.
- Applications of Laplace's equation in Cartesian, cylindrical and spherical polar coordinates to various physical systems.

1.5 TERMINAL QUESTIONS

1. Separate the PDE $\frac{\partial^2 \Psi}{\partial x^2} + \alpha \frac{\partial \Psi(x, t)}{\partial t} = 0$ into two ODEs.

2. The Helmholtz equation in Cartesian coordinates can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2 f(x, y, z) = 0$$

   Reduce it to three ODEs.

3. Obtain the steady state temperature $u(x, y)$ for the rectangular plate of Fig. 1.2 given the following boundary conditions:

$$u(0, y) = \frac{U_0 y}{B}, \qquad \frac{\partial u(L, y)}{\partial x} = -S \qquad 0 < y < B$$

$$u(x, 0) = 0, \qquad u(x, B) = 0 \qquad 0 < x < L$$

4. Verify Eqs (vii) and (viii) of Example 1.3.

1.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. To reduce the following PDE to a set of three ODEs,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

we take $T(r, z) = R(r)Z(z)$ (i)

Then $\frac{\partial T}{\partial r} = \frac{dR}{dr}Z,\ \frac{\partial^2 T}{\partial r^2} = \frac{d^2 R}{dr^2}Z$ and $\frac{\partial^2 T}{\partial z^2} = R\frac{d^2 Z}{dz^2}$ (ii)

Substituting these equations in the given PDE, we get

$$Z\frac{d^2 R}{dr^2} + \frac{Z}{r}\frac{dR}{dr} + R\frac{d^2 Z}{dz^2} = 0$$

On dividing throughout by ZR, we get

$$\frac{1}{R}\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] = -\frac{1}{Z}\frac{d^2 Z}{dz^2} \quad \text{(iii)}$$

The LHS of this equality involves functions which depend only on r, whereas the expression on RHS is a function of only z. So, both sides must be equal to a constant, say, k. Hence the given equation separates into the following two ODEs:

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - kR = 0 \quad \text{(iv)}$$

and $\frac{d^2 Z}{dz^2} + kZ = 0$ (v)

2. To obtain the general solutions of the separated ODEs,

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

we note from Eq. (1.17a) that $X(x)$ vanishes at the boundaries and so, the ratio cannot be positive as it will yield exponential function solutions that would be positive at $x = 0$. Hence, we write:

$$\frac{X''(x)}{X(x)} = -k^2 \quad \text{or} \quad X''(x) + k^2 X(x) = 0 \qquad 0 < x < L$$

and $Y''(y) - k^2 Y(y) = 0 \qquad 0 < y < B$

As you know, the general solutions of the two ODEs are:

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

and $Y(y) = C\cosh\lambda y + D\sinh\lambda y$

3. To separate the PDE $\dfrac{\partial^2 f}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$ into three ODEs, we substitute $f(\rho, \phi, z) = R(\rho)\,\Phi(\phi) Z(z)$ in it.

   So, we get:

$$\Phi Z \frac{d^2 R}{d\rho^2} + \Phi Z \frac{1}{\rho}\frac{dR}{d\rho} + RZ \frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} + \Phi R \frac{d^2 Z}{dz^2} = 0$$

   Dividing the above equation by $R\Phi Z$, we get

$$\frac{1}{R}\frac{d^2 R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 \Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

   We put $\dfrac{1}{Z}\dfrac{d^2 Z}{dz^2} = \alpha^2$ and

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\beta^2 \quad \Rightarrow \quad \frac{d^2\Phi}{d\phi^2} + \beta^2\Phi = 0$$

   Then we have: $\dfrac{\rho}{R}\dfrac{dR}{d\rho} + \dfrac{\rho^2}{R}\dfrac{d^2 R}{d\rho^2} + \alpha^2\rho^2 - \beta^2 = 0$

   These are Eqs. (1.24a to c).

4. We use the Frobenius method to solve the ODE. We expand R in the following series about $r = 0,$ and take its first and second order derivatives with respect to r :

$$R(r) = \sum_{m=0}^{\infty} a_m r^{m+k}$$

$$R'(r) = \sum_{m=0}^{\infty} a_m (m+k) r^{m+k-1}$$

$$R''(r) = \sum_{m=0}^{\infty} a_m (m+k)\,(m+k-1) r^{m+k-2}$$

   Substituting R and its derivatives in the given ODE, we get:

$$r^k \sum_{m=0}^{\infty} a_m \left[(m+k)(m+k-1) + 2(m+k) - n^2\right] = 0$$

   The indicial equation is the coefficient of r^k for $m = 0$:

$$a_0\left(k(k-1) + 2k - n^2\right) = 0 \quad \Rightarrow \quad \left(k^2 - k + 2k - n^2\right) = 0$$

   or $\quad k^2 + k - n^2 = 0 \quad \Rightarrow \quad k\,\dfrac{-1 \pm \sqrt{1+4n^2}}{2}$

   Now we substitute $n^2 = l(l+1),$ so that

$$k = \frac{-1 \pm \sqrt{1+4l^2+4l}}{2} = \frac{-1 \pm (2l+1)}{2}$$

This, we get 2 roots:

$$k_1 = l,\ k_2 = -l - 1$$

So the solution is of the form $R_l(r) = D_l r^l + F_l r^{-l-1},$ which is Eq. (1.29b).

Terminal Questions

1. The given ODE is: $\dfrac{\partial^2 \Psi}{\partial x^2} + \alpha \dfrac{\partial \Psi}{\partial t} = 0$

   We express $\Psi(x, t)$ as a product of two separable functions:

   $$\Psi(x, t) = X(x)T(t)$$

   Substituting it in the given PDE, we get:

   $$X''(x)T(t) + \alpha\ X(x)\dot{T}(t) = 0$$

   Dividing throughout by $X(x)\,T(t)$, we have:

   $$\frac{X''(x)}{X(x)} = -\alpha \frac{\dot{T}(t)}{T(t)} = -k^2$$

   so that the PDE is separated into the following ODEs:

   $$X''(x) + k^2 X(x) = 0$$

   and

   $$\dot{T}(t) - \lambda^2 T(t) = 0$$

   where $\lambda^2 = k^2 / \alpha.$

2. The Helmholtz equation in Cartesian coordinates is:

   $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2\ f(x, y, z) = 0 \qquad \text{(i)}$$

   Let us write

   $$f(x, y, z) = X(x)\,Y(y)\,Z(z) \qquad \text{(ii)}$$

   Substituting Eq. (ii) in Eq. (i), we get:

   $$Y\ Z \frac{d^2 X}{dx^2} + X\ Z \frac{d^2 y}{dy^2} + X\ Y \frac{d^2 Z}{dz^2} + k^2\ X\ Y\ Z = 0$$

   Dividing throughout by $X\ Y\ Z$ and rearranging terms, we get:

   $$\frac{1}{X}\frac{d^2 X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2 Y}{dy^2} - \frac{1}{Z}\frac{d^2 Z}{dz^2} \qquad \text{(iii)}$$

   The LHS is a function of x alone, whereas the RHS depends only on y and z. Let us choose

   $$\frac{1}{X}\frac{d^2 X}{dx^2} = -l^2 \qquad \text{(iv)}$$

Then we can write:

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + l^2 - \frac{1}{Z}\frac{d^2Z}{dz^2} \qquad \text{(v)}$$

Here we have a function of *y* equated to a function of *z*. We now set

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2 \qquad \text{(vi)}$$

so that

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 + l^2 + m^2 = -n^2 \qquad \text{(vii)}$$

where we have put $k^2 = l^2 + m^2 + n^2$. Eqs. (iv), (vi) and (vii) are the three ODEs into which Helmholtz equation separates.

3. The ODE for the problem is:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad 0 < x < L, \quad 0 < y < B \qquad \text{(i)}$$

We write $u(x,y) = X(x)Y(y)$ and separate the PDE into two ODEs as follows:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0, \quad 0 < x < L, \quad 0 < y < B$$

with the boundary conditions:

$$u(0, y) = \frac{U_0 y}{B}, \quad \frac{\partial u(L, y)}{\partial x} = -S \qquad 0 < y < B$$

$$X(x)Y(0) = 0, \;\; X(x)Y(B) = 0, \qquad 0 < x < L$$

or $$Y(0) = 0, \qquad Y(B) = 0 \qquad 0 < x < L$$

Since Y has to vanish at the boundaries $y = 0$ and $y = B$, the ratio $\frac{Y''}{Y}$ cannot be positive. Thus, we get the ODEs:

$$X'' - \lambda^2 X = 0, \;\; Y'' + \lambda^2 Y = 0$$

whence $\quad X(x) = M\cosh\lambda x + N\sinh\lambda x$ and $Y(y) = C\cos\lambda y + D\sin\lambda y$

The boundary conditions on Y yield the following values of C and λ:

$$C = 0, \;\; \lambda_n = \frac{n\pi}{B}, \qquad n = 1,2,3,...$$

Thus, $\quad Y_n(y) = D_n \sin\frac{n\pi y}{B}, \quad n = 1, 2,3,...$

Thus, the general solution is

$$u(x,y) = \sum_{n=1}^{\infty}(a_n\cosh\lambda_n x + b_n\sinh\lambda_n x)\sin\lambda_n y \qquad \text{(ii)}$$

where $\quad a_n = M_n D_n \quad$ and $\quad b_n = N_n D_n$.

Applying the remaining boundary conditions, we get

At $\qquad x = 0, \; \sum_{n=1}^{\infty} a_n \sin\lambda_n y = \frac{U_0 y}{B}, \qquad 0 < y < B$

Using the half-range expansion technique for Fourier series (see Unit 7 of PHE-05), we get the coefficients as follows:

$$a_n = \frac{2}{B}\int_0^B \frac{U_0 y}{B}\sin\lambda_n y\, dy$$

Integrating by parts, we get:

$$a_n = \frac{2U_0}{B^2}\left(\left[-\frac{y}{\lambda_n}\cos\lambda_n y\right]_0^B + \frac{1}{\lambda_n}\left[\frac{\sin\lambda_n y}{\lambda_n}\right]_0^B\right)$$

$$= \frac{2U_0}{B^2}\left(-\frac{B^2}{n\pi}\cos n\pi + 0\right)$$

or $\qquad a_n = -\dfrac{2U_0 \cos n\pi}{n\pi} \qquad$ (iii)

At $\qquad x = L, \quad \dfrac{\partial u}{\partial x}(L, y) = -S, \qquad 0 < y < B$

Differentiating the series (ii) for $u(x, y)$ term by term and applying the above boundary conditions, we get

$$\frac{\partial u}{\partial x}(L,y) = \sum_{n=1}^{\infty}\lambda_n (a_n \sinh\lambda_n L + b_n \cosh\lambda_n L)\sin\lambda_n y$$

$$= -S, \quad 0 < y < B$$

So we must choose b_n such that the coefficient of $\sin\lambda_n y$ will be

$$C_n = \lambda_n (a_n \sinh\lambda_n L + b_n \cosh\lambda_n L)$$

where $\qquad C_n = \dfrac{2}{B}\displaystyle\int_0^B (-S\sin\lambda_n y)\, dy$

$$= \frac{2S}{B}\left[\frac{\cos\lambda_n y}{\lambda_n}\right]_0^B = \frac{2S}{B\lambda_n}[\cos n\pi - 1] = \frac{2S}{n\pi}(\cos n\pi - 1)$$

Thus,

$$b_n = \frac{\dfrac{C_n}{\lambda_n} - a_n \sinh\lambda_n L}{\cosh\lambda_n L} \qquad \text{(iv)}$$

Thus, the unique solution of the problem is given by:

$$u(x,y) = \sum_{n=1}^{\infty}(a_n \cosh\lambda_n x + b_n \sinh\lambda_n x)\sin\lambda_n y$$

with $a_n = -\dfrac{2U_0 \cos n\pi}{n\pi}$, $b_n = \dfrac{\dfrac{C_n}{\lambda_n} - a_n \sinh\lambda_n L}{\cosh\lambda_n L}$ and $C_n = \dfrac{2S}{n\pi}(\cos n\pi - 1)$

4. We have to expand the function

$$u(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

into a series of Legendre polynomials. So, we write:

$$u(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

Comparing the above equation with Eq. (v) of Example 1.3, we can write:

$$\sum_{l=0}^{\infty} C_l R^l P_l(x) = T_0 \sum_{l=0}^{\infty} A_l P_l(x) \quad \Rightarrow \quad C_l = \frac{T_0}{R^l} A_l$$

To determine the coefficients A_l, we proceed as follows:

$$\int_{-1}^{+1} u(x)\, P_m(x)dx = \sum_{l=0}^{\infty} A_l \int_{-1}^{+1} P_m(x)P_l(x)dx$$

$$= A_m \cdot \frac{2}{2m+1} \; (\because \delta_{lm} = 0 \text{ for } l \neq m \text{ and } \delta_{lm} = 1 \text{ for } l = m)$$

$$A_m = \frac{2m+1}{2} \int_{-1}^{+1} u(x)\, P_m(x)dx \qquad \text{(v)}$$

From Eq. (v), for $m = 0$, we get:

$$A_0 = \frac{1}{2}\int_0^1 P_0(x)dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

For $m = 1$, we get:

$$A_1 = \frac{3}{2}\int_0^1 P_1(x)dx = \frac{3}{2}\int_0^1 x dx = \frac{3}{4}$$

For $m = 2$, we get

$$A_2 = \frac{5}{2}\int_0^1 P_2(x)dx = \frac{5}{2}\int_0^1 (3x^2 - 1)dx = 0$$

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)dx = \frac{7}{4}\int_0^1 (5x^3 - 3x)dx = -\frac{7}{16}$$

Here we have used the following results for Legendre polynomials:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

and so on.

$$\therefore \qquad u(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \ldots$$

So, Eq. (vii) is verified. To verify Eq. (viii), we substitute the values of C_l in the equation:

$$T(r, \theta) = \sum_{l=0}^{\infty} C_l\, r^l P_l(\cos\theta)$$

Since $C_l = \frac{T_0}{R^l} A_l$, we get:

$$C_0 = T_0 A_0 = \frac{T_0}{2}$$

$C_l = \frac{T_1}{R} A_1 \Rightarrow C_l = \frac{T_0}{R} . \frac{3}{4}$, and so on.

Therefore,

$$T = T_0 \left[\begin{array}{l} \frac{1}{2} P_0(\cos\theta) + \frac{3}{4} \frac{r}{R} P_1(\cos\theta) - \frac{7}{16} \left(\frac{r}{R}\right)^3 P_3(\cos\theta) \\ + \frac{11}{32} \left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \ldots \end{array} \right]$$

UNIT 2

PDEs IN PHYSICS

Structure

2.1 INTRODUCTION

In Unit 1, you have studied about the method of separation of variables to solve partial differential equations and used it to solve Laplace's equation in Cartesian, cylindrical and spherical polar coordinates. You have also learnt how to solve Laplace's equations for various physical systems in different geometries. In this unit, we discuss a few more important PDEs in Physics, most of which you will be solving in later courses. Specifically, we discuss Poisson equation, heat diffusion equation, wave equation, and the methods of solving these equations for given initial and boundary conditions. We also introduce integral equations and briefly discuss Fredholm and Volterra equations. You will also learn about many interesting applications of these PDEs in modelling a variety of physical phenomena.

In the next unit, we will discuss special functions like Legendre polynomials, spherical harmonics, hypergeometric functions that are solutions of various PDEs, some of which have been referred to in Unit 1.

Expected Learning Outcomes

After studying this unit, you should be able to:

- obtain the general solutions of Poisson's equation, heat diffusion equation and wave equation;
- solve Poisson's equation, heat diffusion equation and wave equation for various physical systems; and
- define an integral equation, write the Fredholm and Volterra equations of the first and second kind, and convert a given differential equation into an integral equation.

2.2 POISSON'S EQUATION

Poisson's equation is a very important equation in physics with major applications in potential field theory. The electrostatic potential field due to a given electric charge distribution or gravitational potential due to a given mass distribution satisfies Poisson's equation. You know from UG physics that if the respective potential fields are known, we can calculate electrostatic or gravitational force field. You will learn about many applications of Poisson's equations in the courses of this programme. Here we will discuss how to obtain the general solution of Poisson's equation and then apply it to a specific example.

The general form of Poisson's equation is:

$$\nabla^2 f(x,y,z) = u(x,y,z) \tag{2.1}$$

The term $u(x,y,z)$ on the right-hand side of Eq. (2.1) is known as the **source function**. Notice that Laplace's equation is a special case of Poisson's equation when $u(x,y,z)$ is zero. We also say that Poisson's equation is a generalization of Laplace's equation. Specifically, Poisson's equation for the electric potential ϕ due to a distribution of charges in some region having charge density $\rho(x,y,z)$ is given by:

$$\nabla^2 \phi = -\frac{\rho(x,y,z)}{\varepsilon_0} \tag{2.2a}$$

where ε_0 is the permittivity of free space. The corresponding equation for the gravitational potential due to some mass distribution having mass density $\rho(x,y,z)$ in a region is given by:

$$\nabla^2 \phi = 4\pi G \rho(x,y,z) \tag{2.2b}$$

where G is the universal gravitational constant. In this section, you will learn how to obtain the general solution of Eq. (2.1). Let us start with a particular case of gravitational potential and then generalise the result to any potential function. Recall from UG physics courses that the gravitational field is conservative and we can associate a gravitational potential function with it. Consider a point particle of mass m situated at the origin. You know that the gravitational potential $\phi(r)$ at a point P due to the particle at a distance r away from it is given by:

$$\phi(r) = -\frac{Gm}{r} \tag{2.3a}$$

Where as you know from UG physics, $\vec{F} = -\frac{Gm}{r^2}\hat{r}$ is the gravitational force field and $\hat{r}$ is the unit vector along r towards P. Now suppose there are many point particles of masses m_i at distances r_i from P. Then from UG physics, you also know that the total gravitational potential due to all these particles at P is the sum of the potentials due to individual particles:

$$\phi(r) = \sum_i \phi_i = -G\sum_i \frac{m_i}{r_i} \tag{2.3b}$$

If we consider a continuous distribution of these particles inside a volume V, with a volume mass density ρ, then you also know from UG physics that the total gravitational potential due to this mass distribution is given by:

$$\phi = -\iiint_V \frac{G\rho dV}{r} \tag{2.3c}$$

Note that, in general, ρ is not a constant. We now express the volume element dV in spherical polar coordinates and write Eq. (2.3c) as:

$$\phi = -\iiint_V \frac{G\rho}{r} r^2 \sin\theta \, dr d\theta d\phi$$

$$= -\iiint_V G\rho r \sin\theta \, dr d\theta d\phi \tag{2.3d}$$

Eq. (2.3c/2.3d) is the solution of the Poisson's equation (2.3a) for the gravitational potential ϕ. We can use this equation to write the general solution of Eq. (2.1). For this, we replace $4\pi G\rho$ by u and ϕ by f in Eqs. (2.3c and d). So, we can write:

$$f = -\frac{1}{4\pi}\iiint \frac{u}{r} dV \tag{2.4}$$

So, Eq. (2.4) is the general solution of Eq. (2.1). Now, if the coordinates of the point P are (x, y, z) and the (x', y', z') is some point in the mass distribution over which we integrate, such that r is the distance between these two points, then we can write Eq. (2.4) as:

$$f(x,y,z) = -\frac{1}{4\pi}\iiint \frac{u(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} dx' dy' dz' \tag{2.5}$$

Note that Eq. (2.5) is a solution of the Poisson's equation:

$$\nabla^2 f(x,y,z) = u(x,y,z)$$

Note also that Eq. (2.5) is a special solution of Poisson's equation because it tends to zero at infinity. Recall that generally we take the zero point for gravitational (and electrostatic) potential energy at infinity. However, we can obtain a solution of Eq. (2.1), which may be zero at any point rather than at infinity. To find such a solution, we note that if f is a solution of Poisson's equation, and g is any solution of Laplace's equation $(\nabla^2 g = 0)$, then

$$\nabla^2(f+g) = \nabla^2 f + \nabla^2 g = \nabla^2 f = u \tag{2.6}$$

Thus, $f+g$ is a solution of Poisson's equation. So, we can add any solution of Laplace's equation to the solution given by Eq. (2.5). We should however ensure that the solution satisfies any given boundary conditions just as we have done in Unit 1. Let us now apply this method to solve Eq. (2.2a) for the electrostatic potential of a uniformly charged sphere at some point outside it.

Example 2.1

A point charge Q is situated at a point (0, 0, R) outside a sphere that is grounded (see Fig. 2.1). Suppose that the centre of the sphere of radius a is at the origin. Determine the electrostatic potential ϕ due to the charge at any point P outside the sphere.

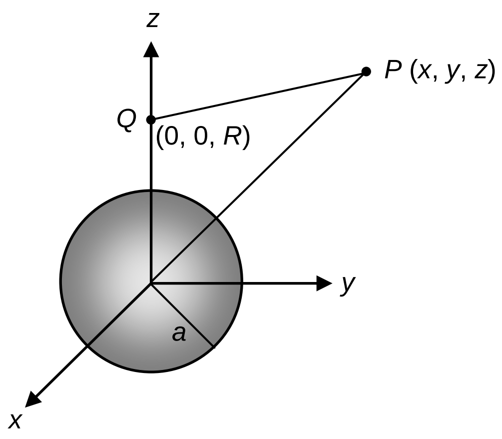

Fig. 2.1: **Electrostatic potential due to a point charge near a grounded sphere.**

Solution : We have to solve Poisson's equation for this problem. The potential Φ and the charge density ρ are related by Poisson's equation:

$$\nabla^2\Phi = -4\pi\rho \quad \text{(in units of } 1/4\pi\varepsilon_0\text{, i.e., Gaussian units)} \tag{2.6a}$$

We have used Gaussian units for ease of writing equations. From Eq. (2.5), the solution of Eq. (2.6) is:

$$\Phi(x,y,z) = -\frac{1}{4\pi}\iiint \frac{-4\pi\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,dx'\,dy'dz' \tag{2.6b}$$

In this problem, the sphere is grounded and hence, there is no charge on it. The only charge in the problem is the charge Q situated at (0, 0, z). So, we have:

$$Q = \iiint \rho(x',y',z')\,dx'\,dy'dz' \tag{2.6c}$$

and the potential Φ_Q due to charge Q is:

$$\Phi_Q(x,y,z) = \frac{Q}{\sqrt{x^2+y^2+(z-R)^2}} \tag{2.6d}$$

Of course, you know that the above equation could have been written down straight away! Now, we will add a solution of Laplace's equation such that the general solution, which is its linear combination with Eq. (2.6d), is zero on the grounded sphere. Since the problem has spherical symmetry, we will use spherical polar coordinates. Therefore, we write the expression for Φ_Q as:

$$\Phi_Q(r,\theta,\phi) = \frac{Q}{\sqrt{r^2-2Rr\cos\theta+R^2}} \tag{2.6e}$$

Note that the solution of Laplace's equation will be independent of the coordinate ϕ due to spherical symmetry. Also now the coordinate r denotes the distance from the origin to the point P (x, y, z). So, we use Eq. (1.31a) of Unit 1 with $m = 0$ to write the expression for Φ:

$$\Phi(x,y,z) = \Phi_Q(x,y,z) + \sum_{l=0}^{\infty}[D_l r^l + F_l r^{-l-1}]P_l(\cos\theta) \tag{2.7a}$$

Now, note that the point P is outside the sphere. Therefore, we retain only the term r^{-l-1} so that the solution does not become infinity as r $\to \infty$. Therefore, the general solution becomes:

$$\Phi(x,y,z) = \frac{Q}{\sqrt{r^2 - 2Rr\cos\theta + R^2}} + \sum_{l=0}^{\infty} C_l\, r^{-l-1} P_l(\cos\theta) \tag{2.7b}$$

The boundary condition for this problem is that the potential is zero at the surface of the sphere, which means that

$$\Phi(x,y,z) = 0 \text{ when } r = a \tag{2.7c}$$

Thus, we have:

$$\Phi(r,\theta)\big|_{r=a} = \frac{Q}{\sqrt{a^2 - 2Ra\cos\theta + R^2}} + \sum_{l=0}^{\infty} C_l\, a^{-l-1} P_l(\cos\theta) = 0 \tag{2.8a}$$

From Eq. (2.7d), we determine the constant C_l as follows:

Recall the expression for the generating function of Legendre polynomials from your UG courses (or see Sec. 3.2 of Unit 3 of this course):

$$g(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \tag{2.8b}$$

Substituting $x = \cos\theta$ and $t = \dfrac{a}{R}$ in Eq. (2.8b) and simplifying the expression, we can write:

$$\frac{Q}{\sqrt{a^2 - 2Ra\cos\theta + R^2}} = Q \sum_{l=0}^{\infty} \frac{a^l}{R^{l+1}} P_l(\cos\theta) \tag{2.8c}$$

We next substitute the RHS of Eq. (2.8c) in Eq. (2.8a) and obtain the value of the constant:

$$Q \sum_{l=0}^{\infty} \frac{a^l}{R^{l+1}} P_l(\cos\theta) + \sum_{l=0}^{\infty} C_l\, a^{-l-1} P_l(\cos\theta) = 0 \tag{2.8d}$$

or

$$C_l = -\frac{Q\,a^{2l+1}}{R^{l+1}} \tag{2.8e}$$

Therefore, the particular solution for this problem is given by:

$$\Phi(x,y,z) = \frac{Q}{\sqrt{r^2 - 2Rr\cos\theta + R^2}} - Q \sum_{l=0}^{\infty} \frac{a^{2l+1}}{R^{l+1}} r^{-l-1} P_l(\cos\theta) \tag{2.9a}$$

We can simplify this expression further by once again using Eq. (2.8b) with $x = \cos\theta$ and $t = \dfrac{a^2}{Rr}$. Then we get:

$$\Phi(x,y,z) = \frac{Q}{\sqrt{r^2 - 2Rr\cos\theta + R^2}} - \frac{(a/R)Q}{\sqrt{r^2 - 2r(a^2/R)\cos\theta + (a^2/R)^2}} \qquad (2.9b)$$

You may like to verify Eqs. (2.8c and 2.9b) before studying further. Solve SAQ 1.

SAQ 1

Verify Eqs. (2.8c and 2.9b).

We now consider the heat diffusion equation, which is also called the heat conduction equation or heat flow equation.

2.3 HEAT DIFFUSION EQUATION

The heat diffusion (alternately called the heat conduction or heat flow) equation is expressed as:

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \qquad (2.10)$$

Fig 2.2: **An insulated bar with its left end immersed in ice.**

The function $u(x, y, z, t)$ represents temperature of the object in which heat is flowing and k is a constant that is a characteristic of the material of which the object is made.

We now solve the heat diffusion equation and consider its application in Physics.

Let us first separate the variables in Eq. (2.10). So, we write:

$$u(x, y, z, t) = T(x, y, z)\,F(t) \qquad (2.11a)$$

Substituting Eq. (2.11a) in Eq. (2.10) and dividing by FT, we get:

$$F\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T(x, y, z) = \frac{1}{k} T \frac{dF}{dt}$$

$$\frac{1}{T}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T(x, y, z) = \frac{1}{k}\frac{1}{F}\frac{dF}{dt}$$

So, we get two separate ODEs:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T(x, y, z) = -c^2 T$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T(x, y, z) + c^2 T = 0 \qquad (2.11b)$$

and

$$\frac{dF}{dt} = -c^2 k F \qquad (2.11c)$$

Notice that we have chosen a negative separation constant. This is because temperature cannot increase indefinitely with time as it would if we chose c^2 to be positive.

Do you recognize Eq. (2.11b)? It is the Helmholtz equation, which you have reduced to three ODEs in Terminal Question 2 of Unit 1. We can solve it for a given problem in physics. We can obtain the solution of the first order ODE given by Eq. (2.11c) by simply integrating it. We get:

$$F(t) = \exp(-c^2 k\, t) \tag{2.11d}$$

Let us now solve the heat diffusion equation for a physical system.

Example 2.2

Consider the flow of heat in a uniform bar of length L, insulated along its length. The temperature of the bar is modelled by the one-dimensional heat diffusion equation

$$\frac{\partial T(x,t)}{\partial t} = k\frac{\partial^2 T(x,t)}{\partial x^2}, \quad (0 < x < L, \quad t > 0) \qquad \text{(i)}$$

Suppose one end of the block is immersed in a block of ice, maintained at 0°C, while the other end is insulated (Fig. 2.2). This gives rise to the boundary conditions:

$$T(0,t) = 0, \quad \text{and} \quad \frac{\partial T(L,t)}{\partial x} = 0, \quad t \geq 0 \qquad \text{(ii)}$$

If the initial temperature distribution is given by

$$T(x,\,0) = \frac{x}{2}(2L - x) \qquad (0 < x < L) \qquad \text{(iii)}$$

solve the heat equation (i). (Note that the initial condition is physically consistent with boundary conditions at $x = 0$ and $x = L$).

<u>Solution</u> : Taking $T(x,\,t) = X(x)\,Y(t)$ and $-\lambda^2$ as the separation constant, we separate Eq. (i) into the following two ODEs:

$$\frac{X''}{X} = \frac{Y'}{kY} = -\lambda^2 \qquad \text{(iv)}$$

or $$X'' + \lambda^2 X = 0 \qquad \text{(v)}$$

and $$Y' + k\lambda^2 Y = 0 \qquad \text{(vi)}$$

The solutions of (v) and (vi) for $X(x)$ and $Y(t)$ are well known:

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda\, x \qquad \text{(vii)}$$

and $$Y(t) = C_3 e^{-k\lambda^2 t} \qquad \text{(viii)}$$

From the boundary conditions for $T(x, t)$, we have

$$T(0,t) = X(0)Y(t) = 0$$

and $$\frac{\partial T}{\partial x}(L,t) = \left[\frac{dX(L)}{dx}\right]Y(t) = 0$$

Since $Y(t) \neq 0$, X must satisfy the conditions:

$$X(0) = X'(L) = 0$$

When we apply the first of these conditions to Eq. (vii), we get $C_1 = 0$. Thus,

$$X(x) = C_2 \sin\lambda x$$

The second boundary condition gives us:

$$X'(L) = C_2 \cos\lambda L = 0$$

For a non-trivial solution, for which $C_2 \neq 0$, we have

$$\cos\lambda L = 0$$

or $$\lambda L = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots$$

We call these values of λ as λ_n , then we can write the solution for the space part as:

$$X_n(x) = C_{2n} \sin\left[\frac{(2n+1)\pi}{2L}x\right], \quad n = 0, 1, 2, 3$$

Substituting λ_n in Eq. (viii), we get:

$$Y_n(t) = C_{3n} \exp\left[-\left(\frac{(2n+1)\pi}{2L}\right)^2 k\,t\right]$$

Thus,

$$T_n(x,t) = X_n(x)Y_n(t) = b_n \exp\left[-\left(\frac{(2n+1)\pi}{2L}\right)^2 k\,t\right]$$

$$\sin\left[\frac{(2n+1)\pi x}{2L}\right], \qquad n = 0, 1, 2, 3, \ldots$$

where we have put $b_n = C_{2n}\, C_{3n}$. The general solution of the heat diffusion equation for the given bar is:

$$T(x,t) = \sum_{n=0}^{\infty} b_n \exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right) \sin\left[\frac{(2n+1)\pi x}{2L}\right] \quad (2.12)$$

Applying the initial condition (iii), we have

$$T(x, 0) = \sum_{n=0}^{\infty} b_n \sin\left[\frac{(2n+1)\pi x}{2L}\right] = f(x), \quad 0 < x < L \qquad (2.13a)$$

where $$f(x) = \frac{x}{2}(2L - x) \tag{2.13b}$$

We now have to determine b_n in Eq. (2.12). For this, we use the half-range expansion of $f(x)$. **You may like to revise this method from Sec. 7.5.1, Unit 7 of PHE-05** before working out the steps that follow. Since $f(x)$ is defined on $0 < x < L$ and $T(x, 0)$ is the sum of a sine series, we take the odd extension of $f(x)$. Multiplying the LHS of Eq. (2.13a) by $\sin\left[\frac{(2m+1)\pi x}{2L}\right]$ and integrating from $-L$ to L, we have

$$\sum_{n=0}^{\infty} b_n \int_{-L}^{L} \left[\sin\frac{(2m+1)\pi x}{2L}\right] \sin\left[\frac{(2n+1)\pi x}{2L}\right] dx$$

$$= \int_{-L}^{L} g(x) \left[\sin\frac{(2m+1)\pi x}{2L}\right] dx$$

where

$$g(x) = \begin{cases} -f(-x) & \text{for } -L < x < 0 \\ 0 & \text{for } x = 0 \\ f(x) & \text{for } 0 < x < L \end{cases}$$

Thus, $$b_n = \frac{2}{L}\int_0^L g(x)\left[\sin\frac{(2n+1)\pi x}{2L}\right] dx$$

$$= \frac{2}{L}\int_0^L f(x)\left[\sin\frac{(2n+1)\pi x}{2L}\right] dx$$

$$= \frac{2}{L}\int_0^L \frac{x}{2}(2L - x)\left[\sin\frac{(2n+1)\pi x}{2L}\right] dx$$

You can integrate by parts twice and see that:

$$b_n = \frac{16L^2}{(2n+1)^3 \pi^3}$$

Hence, the solution is given by:

$$T(x,t) = \frac{16L^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right) \sin\left[\frac{(2n+1)\pi x}{2L}\right] \tag{2.14}$$

You may now like to work out an SAQ to solve the heat diffusion equation yourself.

Note that in this case the initial condition does not match the boundary conditions at $x = 0$ and $x = L$. In reality, when the ends of the bar are put into the ice, it would melt to match the temperature of the ends of the bar, which cool rapidly. Only later would we have $T(0, t) = 0$.

SAQ 2

In Example 2.2, solve the heat diffusion equation given that the initial temperature of the bar is constant (T_0) and

$$T(0, t) = \frac{\partial T}{\partial x}(L,t) = 0, \quad (t \geq 0),$$

$$T(x, 0) = T_0, \quad (0 < x < L)$$

Determine the expression for $T(x,t)$ and discuss its behaviour at large values of time.

Let us now discuss the wave equation.

2.4 WAVE EQUATION

For simplicity we shall restrict our discussion to the solution of the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L, \qquad t > 0 \tag{2.15}$$

with given initial and boundary conditions. You have learnt how to separate this PDE into ODEs in Unit 1. Let us now solve this equation, which has many applications in mechanical waves (vibrating strings, torsional vibrations), electromagnetic waves and other types of waves.

Let us consider the "plucked string" problem again [Fig. 1.1, Eqs. (1.8a and b].

A string is plucked at its mid-point and then released from rest from this position (Fig. 2.3). The resulting vibrations are modelled by the wave equation [Eq. (2.15)]. Suppose the string vibrates subject to the following boundary and initial conditions:

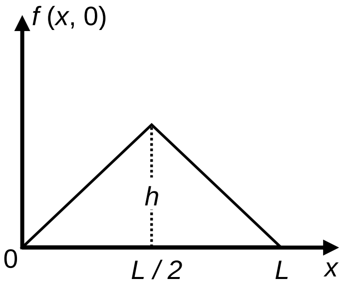

Fig. 2.3: **A plucked finite string fixed at both ends.**

$$u(0,t) = 0, \; u(L,t) = 0 \tag{2.16a}$$

$$u(x,0) = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\[2ex] 2h\left(1 - \dfrac{x}{L}\right), & \dfrac{L}{2} \leq x < L \end{cases} \tag{2.16b}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \tag{2.16c}$$

where h is a positive constant, which is small compared to L. These conditions correspond to an initial **triangular deflection** and zero initial velocity.

Let us first obtain the general solution of the wave equation for given boundary conditions [Eq. (2.16a)]. The separated ODEs may be written as:

$$X'' + \mu^2 X(x) = 0 \qquad (2.17a)$$

and $$\ddot{T}(t) + \mu^2 v^2 T(t) = 0. \qquad (2.17b)$$

The solutions of Eqs. (2.17a and b) are given as follows:

$$X(x) = A\cos\mu x + B\sin\mu x \qquad (2.18a)$$

and $$T(t) = C\cos\mu vt + D\sin\mu vt \qquad (2.18b)$$

Now since $u(0, t) = X(0)\,T(t) = 0$ and $u(L, t) = X(L)T(t) = 0,$ we must have $X(0) = 0$ and $X(L) = 0.$ Using the first of these conditions, we find that $A = 0.$ Therefore,

$$X(x) = B\sin\,\mu x \qquad (2.18c)$$

The second condition now implies that

$$X(L) = B\sin\mu L = 0 \qquad (2.18d)$$

This inequality will be satisfied if $B = 0$ or $\sin\mu L = 0.$ If $B = 0,$ then $X = 0$ so that $u = 0,$ which is a trivial solution. Hence, we must have $B \neq 0$ and the only option is $\sin\mu L = 0.$ This implies that $\mu L = n\pi$ or $\mu = n\pi / L$ for $n = 0, 1, 2, 3,.....$ The solution for $n = 0$ is a trivial solution. For any arbitrary value of *B*, we obtain infinite solutions of the form:

$$X(x) = X_n(x) = B_n \sin\left(\frac{n\pi}{L}\right)x \qquad n = 1, 2, 3,..... \qquad (2.19)$$

Hence, the general solution of Eq. (2.15) which satisfies the given boundary conditions for the plucked string is given by:

$$u_n(x, t) = \left[C\cos\left(\frac{n\pi vt}{L}\right) + D\sin\left(\frac{n\pi vt}{L}\right)\right]B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \left[a_n \cos\left(\frac{n\pi vt}{L}\right) + b_n \sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \qquad (2.20)$$

where we have put $CB_n = a_n$ and $DB_n = b_n$ since each value of *n* may require a different constant. You should note that the subscript *n* has been added to $u(x, t)$. Do you know why? This is just to allow for a different function for each value of *n*.

You would agree that $u_n(x, t)$ is not a solution of the given problem since initial conditions have not yet been imposed. Moreover, since the wave equation is linear and homogeneous, we expect that the most general solution, which satisfies the given boundary conditions, is given by the linear combination of all possible solutions:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi vt}{L}\right) + b_n \sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$(2.21)$$

Let us now apply the initial conditions [Eqs. 2.16b and c)] to Eq. (2.21):

$$u(x,\,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\ \\ 2h\left(1-\dfrac{x}{L}\right), & \dfrac{L}{2} \le x < L \end{cases} \qquad (2.22a)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}\left(-a_n\frac{n\pi v}{L}\sin\frac{n\pi vt}{L} + b_n\frac{n\pi v}{L}\cos\frac{n\pi vt}{L}\right)\sin\frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} b_n \frac{n\pi v}{L}\sin\frac{n\pi x}{L} = 0 \qquad (2.22b)$$

Note that Eq. (2.22b) will be satisfied only if $b_n = 0$ for all n. So now you have to determine a_n, i.e., you have to expand $u(x,0)$ and hence its half-range expansion in a Fourier sine series. Go through Sec. 7.5 of Unit 7, PHE-05 to revise the half-range expansion of Fourier series and obtain the unique solution for this problem.

SAQ 3

Show that the solution of the "plucked string" problem given above is:

$$u(x,t) = \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi vt}{L} + \ldots\right]$$

Another interesting application of the wave equation is in torsional vibrations. Such vibrations can result from unbalanced torques on shafts in a wide variety of machinery in cars, aircraft, turbines, railway engines, etc. Some common examples of shafts are axles connecting the wheels of a car, spindles on a spinning wheel, propeller shafts used for ship propulsion and shafts in belt and pulley arrangements. You will also use the wave equation for electromagnetic waves in the first and second semester courses on Electromagnetic Theory and Classical Electrodynamics.

So far, you have studied differential equations involving the unknown function and one or more of its ordinary or partial *derivatives*. You will now learn about **integral equations**. In particular, you will learn about **Fredholm** and **Volterra** equations. These equations have important applications in higher level physics courses and you will learn about some of them in the other courses of M. Sc. Moreover, some problems in physics cannot be modelled by differential equations. For example, transport phenomena such as neutron transport in reactors can be modelled using only integral equations or integro-differential equations. Similarly, the momentum representation of Schrödinger equation involves an integral equation. You may have learnt it in your UG course on Quantum Mechanics.

2.5 INTEGRAL EQUATIONS: FREDHOLM AND VOLTERRA EQUATIONS

First, let us give some definitions.

By definition, an **integral equation** is an equation that **involves a function and its integral (s)**. For example,

$$f(x) = u(x) + \int_a^b K(x, \eta) f(\eta) d\eta \quad \text{for all } x \in [a, b] \qquad (2.23)$$

is an integral equation. The function $K(x, \eta)$ is called the **kernel** of **the integral equation**. The limits of integration *a*, *b* may be constant or variable. Accordingly, we define Fredholm and Volterra equations as follows:

Fredholm integral equation is an integral equation in which the **limits of integration are constant**.

So, an example of Fredholm equation is:

$$f(x) = u(x) + \int_a^b K(x, \eta) f(\eta) d\eta \quad \text{for all } x \in [a, b] \qquad (2.24)$$

where the limits *a* and *b* are constant.

If either or both limits of integration in an integral equation are not constant, (i.e., these vary), it is called **Volterra integral equation**.

Examples of Volterra equation are:

$$f(x) = u(x) + \int_{a(x)}^b K(x, \eta) f(\eta) d\eta \quad \text{for all } x \in [a(x), b] \qquad (2.25a)$$

or $$f(x) = u(x) + \int_a^{b(x)} K(x, \eta) f(\eta) d\eta \quad \text{for all } x \in [a, b(x)] \qquad (2.25b)$$

or $$f(x) = u(x) + \int_{a(x)}^{b(x)} K(x, \eta) f(\eta) d\eta \quad \text{for all } x \in [a(x), b(x)] \qquad (2.25c)$$

Integral equations can further be classified as **integral equations of the first kind** and **second kind**. Let us define them.

If the unknown function appears **only** under the integral sign, it is called an **integral equation of the first kind**.

If the unknown function appears both inside and outside the integral sign, it is called an **integral equation of the second kind**.

An example of **Fredholm equation of the first kind** is:

$$\int_a^b K(x, \eta) f(\eta) d\eta = g(x) \quad \text{for all } x \in [a, b] \qquad (2.26a)$$

Examples of **Volterra equations of the first kind** are:

$$\int_{a(x)}^{b} K(x, \eta) f(\eta) d\eta = g(x) \quad \text{for all } x \in [a(x), b] \tag{2.26b}$$

$$\int_{a}^{b(x)} K(x, \eta) f(\eta) d\eta = g(x) \quad \text{for all } x \in [a, b(x)] \tag{2.26c}$$

and

$$\int_{a(x)}^{b(x)} K(x, \eta) f(\eta) d\eta = g(x) \quad \text{for all } x \in [a(x), b(x)] \tag{2.26d}$$

As you can see, Eqs. (2.24) is an example of **Fredholm equation of the second kind** for constant a and b. Eqs. (2.25a to c) are examples of **Volterra equation of the second kind**.

If the functions $u(x)$ and $g(x)$ in Eqs. (2.24 to 2.26d) are zero, the equations are called **homogeneous integral equations**.

You may like to revise the classification of integral equations by solving SAQ 4.

SAQ 4

Classify each of the following integral equations as Fredholm/Volterra equations of the first and second kind:

a) $f(x) = -\omega^2 \int_0^x (x-t) y(t) dt;$ b) $y(x) = f(x) + \frac{\omega^2}{b} \int_0^b (b-t) y(t) dt$

c) $y(x) = -\omega^2 \int_0^x (x-1)(t+1) y(t) dt;$ d) $f(x) = -\omega^2 \int_0^a (x-t) y(t) dt$

Sometimes, a problem in physics may be modelled using both differential and integral equations. And then we can choose which of the equations to solve. We can also convert a given ODE into an integral equation and vice versa. Let us explain how to convert an ODE into an integral equation. You will then appreciate the importance of integral equations.

Suppose, we wish to convert the following ODE into an integral equation:

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2.27a}$$

with initial conditions:

$$y(L) = y_0 \quad \text{and} \quad y'(L) = y_1 \tag{2.27b}$$

We integrate Eq. (2.27a) and get:

$$y' = -\int_L^x P(x)y'dx - \int_L^x Q(x)y dx + \int_L^x f(x)dx + y_1 \tag{2.27c}$$

Next we integrate the first integral on the RHS by parts and get:

$$y' = -Py - \int_L^x (Q - P')y\,dx + \int_L^x f(x)\,dx + P(L)y_0 + y_1 \qquad (2.27d)$$

You should note that the initial conditions are reflected in this integral equation through the constants of integration. Once again, we integrate Eq. (2.27d) and get:

$$y = -\int_L^x Py\,dx - \int_L^x\int_L^x [Q(t) - P'(t)]y(t)\,dt\,dx$$

$$+ \int_L^x\int_L^x f(t)\,dt\,dx + [P(L)y_0 + y_1](x - L) + y_0 \qquad (2.28)$$

You can verify that the following equation is correct by differentiating both its sides with respect to *x*:

$$\int_L^x\int_L^x f(t)\,dt\,dx = \int_L^x (x - t)f(t)\,dt \qquad (2.29)$$

We can use Eq. (2.29) to write Eq. (2.28) in a simpler form:

$$y(x) = -\int_L^x \{P(t) + (x - t)[Q(t) - P'(t)]\}y(t)\,dt$$

$$+ \int_L^x (x - t)f(t)\,dt + [P(L)y_0 + y_1](x - L) + y_0 \qquad (2.30a)$$

We can recast Eq. (2.30a) as an integral equation by substituting

$$K(x, t) = (t - x)[Q(t) - P'(t)] - P(t) \qquad (2.30b)$$

and

$$g(x) = \int_L^x (x - t)f(t)\,dt + [P(L)y_0 + y_1](x - L) + y_0 \qquad (2.30c)$$

Then Eq. (2.28) takes the form of an integral equation:

$$y(x) = g(x) + \int_L^x K(x,t)y(t)\,dt \qquad (2.31)$$

Can you identify which type of integral equation this is? It is a Volterra equation of the second kind.

You may now like to practice this kind of conversion of an ODE into an integral equation. Try SAQ 5.

SAQ 5

Convert the following ODE that models the oscillations of a simple harmonic oscillator into an integral equation, and classify it:

$$y''(x) + \omega_0^2 y(x) = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = 0$$

Integral equations are solved using numerical methods and you will learn how to solve them in the course on Computational Physics. We now end this unit and summarise the main points covered in it.

2.6 SUMMARY

In this unit, we have covered the following partial differential equations:

- Poisson's equation, with applications in determination of electrostatic and gravitational potentials.
- Heat diffusion equation, also known as heat conduction equation and heat flow equation with applications in heat flow in one-dimensional systems.
- Wave equation with application in plucked string problem.
- Integral equations such as Fredholm and Volterra equations of the first and second kind, and the conversion of an ODE into an integral equation.

2.7 TERMINAL QUESTIONS

1. Solve Poisson's equation for a uniformly charged sphere of radius R and surface charge q to obtain the electrostatic potential at a point inside the sphere. (Hint: Assume a solution of the form $Ar^2 + B$).

2. Consider a rod whose ends are kept at a constant temperature and its lateral surface is insulated. The heat flow is described by the one-dimensional heat diffusion equation subject to the conditions:

   $$f(0, t) = 0,\ f(L, t) = 0 \qquad \text{for} \qquad t > 0$$

   and $$f(x,0) = \sin(4\pi x / L) \qquad \text{for} \qquad 0 < x < L$$

   Obtain a unique solution.

3. The one-dimensional wave equation for e.m. wave propagation in free space is given by (for $\vec{E} \parallel \hat{y}$)

   $$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0$$

   Solve this equation if $E_y = 0$ at $x = 0$ and $x = L$.

4. Derive an integral equation corresponding to the ODE:

   $$y'' - 2y = 0$$

   subject to the conditions: $y(0) = 2;\ \ y'(0) = -2$

2.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We substitute $x = \cos\theta$ and $t = \dfrac{a}{R}$ in Eq. (2.8b) and get:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1-2\dfrac{a}{R}\cos\theta+\left(\dfrac{a}{R}\right)^2}} = \frac{R}{\sqrt{R^2-2aR\cos\theta+a^2}}$$

$$= \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{R}\right)^n$$

So, we get Eq. (2.8c):

$$\frac{Q}{\sqrt{a^2-2Ra\cos\theta+R^2}} = Q\sum_{l=0}^{\infty}\frac{a^l}{R^{l+1}}P_l(\cos\theta)$$

We substitute $x = \cos\theta$ and $t = \dfrac{a^2}{Rr}$ in Eq. (2.8b) and write it as:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1-2\dfrac{a^2}{Rr}\cos\theta+\left(\dfrac{a^2}{Rr}\right)^2}} = \frac{r}{\sqrt{r^2-2r(a^2/R)\cos\theta+\left(\dfrac{a^2}{R}\right)^2}}$$

$$= \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a^2}{Rr}\right)^n = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a^{2n}}{R^n r^n}\right)$$

or
$$\frac{1}{\sqrt{r^2-2r(a^2/R)\cos\theta+\left(\dfrac{a^2}{R}\right)^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a^{2n}}{R^n r^{n+1}}\right)$$

or
$$\frac{(a/R)Q}{\sqrt{r^2-2r(a^2/R)\cos\theta+\left(\dfrac{a^2}{R}\right)^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a^{2n+1}}{R^{n+1}r^{n+1}}\right)$$

which is Eq. (2.9b).

2. The general solution of this problem is given by Eq. (2.12):

$$T(x,t) = \sum_{n=0}^{\infty} b_n \exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right)\sin\left[\frac{(2n+1)\pi x}{2L}\right]$$

Using the half range expansion of $T(x, 0)$, we get:

$$b_n = \frac{2}{L}\int_0^L T_0\left[\sin\frac{(2n+1)\pi x}{2L}\right]dx = -\frac{4T_0}{(2n+1)\pi}\left[\cos\frac{(2n+1)\pi}{2} - 1\right]$$

$$= \frac{4T_0}{(2n+1)\pi} \quad \text{since } \cos\frac{(2n+1)\pi}{2} = 0 \quad \text{for all } n.$$

Hence, the particular solution is:

$$T(x,t) = \frac{4T_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right)\sin\left[\frac{(2n+1)\pi x}{2L}\right]$$

As t increases, all exponential terms tend to zero. Hence, $T(x,t)$ tends to zero.

3. Let us apply the given initial conditions to the general solution given by Eq. (2.21):

$$u(x, t) = \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi vt}{L}\right) + b_n \sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\ 2h\left(1-\dfrac{x}{L}\right), & \dfrac{L}{2} \le x < L \end{cases} \qquad \text{(i)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}\left(-a_n\frac{n\pi v}{L}\sin\frac{n\pi vt}{L} + b_n\frac{n\pi v}{L}\cos\frac{n\pi vt}{L}\right)\sin\frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} b_n\frac{n\pi v}{L}\sin\frac{n\pi x}{L} = 0 \qquad \text{(ii)}$$

Eq. (ii) will be satisfied only if $b_n = 0$ for all n. So now we have to determine a_n. So, we expand $u(x,0)$ in a Fourier sine series and get:

$$a_n = \frac{2}{L}\int_0^L u(x,0)\left[\sin\frac{n\pi x}{L}\right]dx$$

$$= \frac{2}{L}\int_0^{L/2}\frac{2hx}{L}\left[\sin\frac{n\pi x}{L}\right]dx + \frac{2}{L}\int_{L/2}^{L}2h\left(1-\frac{x}{L}\right)\left[\sin\frac{n\pi x}{L}\right]dx$$

$$= \frac{4h}{L^2}\left(-\left[\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\right]_0^{L/2} + \frac{L^2}{n^2\pi^2}\left[\sin\frac{n\pi x}{L}\right]_0^{L/2}\right)$$

$$-\frac{4h}{L}\frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_{L/2}^{L} - \frac{4h}{L^2}\left(-\left[\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\right]_{L/2}^{L}\right.$$

$$\left. + \frac{L^2}{n^2\pi^2}\left[\sin\frac{n\pi x}{L}\right]_{L/2}^{L}\right)$$

$$= \frac{4h}{L^2}\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{4h}{L^2}\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4h}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$+ \frac{4h}{L^2}\left(\frac{L^2}{n\pi}\cos n\pi - \frac{L^2}{2n\pi}\cos\frac{n\pi}{2}\right) - \frac{4h}{n^2\pi^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Thus, the unique solution of the given problem is:

$$u(x,t) = \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi vt}{L} + \ldots\right]$$

4. a) Volterra equation of the first kind.
   b) Fredholm equation of the second kind.
   c) Volterra equation of the second kind.
   d) Fredholm equation of the first kind.

5. Comparing

$$y''(x) + \omega_0^2\, y(x) = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1$$

with Eq. (2.27a) and the initial conditions, we can write:

$$P(x) = 0,\ Q(x) = \omega_0^2,\ f(x) = 0,\quad L = 0,\quad y_0 = 1,\quad y_1 = 1$$

Substituting these values in Eq. (2.30a), we get the equivalent integral equation, which is a Volterra equation of the second kind:

$$y(x) = -\int_0^x (x-t)\,\omega_0^2\, y(t)\,dt + x$$

or $$y(x) = x - \omega_0^2\int_0^x (x-t)\,y(t)\,dt$$

Terminal Questions

1. We will use spherical polar coordinates to solve Poisson's equation for the uniformly charged sphere. Poisson's equation $\nabla^2\phi = -\dfrac{\rho(x,y,z)}{\varepsilon_0}$ in spherical polar coordinates for the sphere is:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = -\frac{\rho}{\varepsilon_0}$$

because for the sphere, the potential is independent of both θ and ϕ.

Assuming a solution of the form $\phi = Ar^2 + B$, and substituting it in the above Poisson's equation, we get:

$$2A + 4A = -\frac{\rho}{\varepsilon_0} \Rightarrow 6A = -\frac{\rho}{\varepsilon_0} \Rightarrow A = -\frac{\rho}{6\varepsilon_0}$$

Now we apply the boundary condition that the charge at the sphere's surface $(r = R)$ is q, i.e., $q = \frac{4\pi}{3}R^3\rho$ So, we get:

$$\phi = -\frac{\rho}{6\varepsilon_0}R^2 + B = \frac{q}{4\pi\varepsilon_0 R} \Rightarrow B = \frac{\rho}{6\varepsilon_0}R^2 + \frac{q}{4\pi\varepsilon_0 R}$$

Therefore, the solution of Poisson's equation at a point inside a sphere is given by:

$$\phi = -\frac{\rho}{6\varepsilon_0}r^2 + \frac{\rho}{6\varepsilon_0}R^2 + \frac{q}{4\pi\varepsilon_0 R}$$

$$= \frac{\rho}{6\varepsilon_0}(R^2 - r^2) + \frac{4\pi}{3}R^3\rho\left(\frac{1}{4\pi\varepsilon_0 R}\right),$$

or $$\phi = \frac{\rho}{6\varepsilon_0}(R^2 - r^2) + \frac{\rho}{3\varepsilon_0}R^2 = \frac{\rho}{6\varepsilon_0}(3R^2 - r^2)$$

2. We have to solve the one-dimensional heat diffusion equation:

$$\frac{1}{v}\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \qquad \text{(i)}$$

subject to the conditions: $f(0, t) = 0$, $f(L, t) = 0$

and $$f(x, 0) = f(x) \qquad 0 < x < L \qquad \text{(ii)}$$

We write $f(x, t) = X(x)\,T(t)$ and separate the PDE into 2 ODEs as explained in Sec. 2.3 [you should work out all intermediate steps while solving such problems to get to Eq. (iii)]:

$$\frac{dT}{dt} + k^2 v\,T = 0 \qquad \text{(iii)}$$

and $$\frac{d^2X}{dx^2} + k^2 X = 0 \qquad \text{(iv)}$$

Eqs. (iii) and (iv) have solutions of the form:

$$T(t) = Ae^{-k^2 v t} \text{ and } X(x) = B\cos kx + C\sin kx$$

so that

$$f(x, t) = X(x)\,T(t) = (P\cos kx + Q\sin kx)e^{-k^2 v t}$$

The condition $f(0, t) = 0$ implies that $P = 0$ and for $f(L, t) = 0$ we get $k = n\pi L$. Hence, the general solution is given by:

$$f(x, t) = \sum_{n=0}^{\infty} Q_n \sin\left(\frac{n\pi x}{L}\right)\exp\left[-v\,t\left(\frac{n\pi}{L}\right)^2\right] \qquad \text{(v)}$$

We now apply the following condition to Eq. (v):

$$f(x,0) = \sin(4\pi x / L) \quad \text{for} \quad 0 < x < L \qquad \text{(vi)}$$

We notice that for the condition (vi), only the terms for which $n = 4$ is non-zero in Eq. (v). Hence, we get $Q_4 = 1$ and the unique solution is:

$$f(x, t) = \sin\left(\frac{4\pi x}{L}\right)\exp\left[-v\,t\left(\frac{4\pi}{L}\right)^2\right]$$

3. The given equation describes e.m. wave propagation in free space:

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0$$

Let us make the substitution $E_y(x,t) = X(x)T(t)$ and separate the PDE into ODEs:

$$X''\,T(t) - \frac{1}{c^2}X(x)\ddot{T} = 0$$

Dividing by $X(x), T(t)$, we get: $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{\ddot{T}}{T} = -k^2$

so that $X'' + k^2 X = 0$ and $\ddot{T} + \omega_0^2 T = 0$ where $\omega_0 = ck$.

The solutions of these equations are of the form:

$$X = A\cos k\,x + B\sin k\,x$$

and $$T = C\cos\omega_0 t + D\sin\omega_0 t$$

The condition

$$X(x)\,T(t) = E_y = 0 \text{ at } x = 0 \text{ and } x = L$$

implies that for all $t > 0$, $X(x) = 0$ at $x = L$. This leads to

$$X(L) = B\sin K\,L = 0$$

For a non-trivial solution, the eigenvalues are:

$$k_n L = n\pi \quad \text{or} \quad k_n = (n\pi / L)$$

with corresponding solutions: $X_n(x) = B_n \sin\left(\dfrac{n\pi x}{L}\right)$

The general solution is:

$$E_y(x,t) = \sum_n P_n \sin\left(\frac{n\pi x}{L}\right)\exp(i\omega_{0n} t)$$

4. Comparing

$$y'' - 2y = 0 \quad \text{with} \quad y(0) = 2; \;\; y'(0) = -2$$

with Eq. (2.27a) and the initial conditions, we can write:

$$P(x) = 0, \;\; Q(x) = -2, \quad f(x) = 0, \quad L = 0, \quad y_0 = 2, \quad y_1 = -2$$

Substituting these values in Eq. (2.30a), we get the equivalent integral equation, which is a Volterra equation of the second kind:

$$y(x) = \int_0^x 2(x - t)\,y(t)\,dt - 2x + 2$$

UNIT 3

BESSEL FUNCTIONS

Structure

3.1 INTRODUCTION

In Units 1 and 2, you have learnt how to solve various partial differential equations by separating them into ODEs. You have solved the ODEs using the standard methods you have learnt in UG physics and mathematics. In Units 3 to 5, we will introduce certain special functions that are solutions of given ODEs. This nomenclature has genesis in the fact that these functions are far more complex than elementary functions. Special functions find many useful applications in Physics as you have already learnt in Units 1 and 2, where we have referred to Bessel functions and Legendre polynomials. Special functions have numerous applications in physics. Solutions of many differential equations corresponding to physical problems can be expressed in terms of special functions being discussed in Units 3 to 5.

In this unit, we will discuss **Bessel functions**. We will first discuss **Bessel functions of the first kind** and solve the differential equation. You will learn about their generating functions, recurrence relations and orthogonality property. We will also discuss **Bessel functions of the second kind** and **spherical Bessel functions**. In the next unit, we discuss the special functions Legendre polynomials, spherical harmonics and hypergeometric functions.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ solve Bessel's differential equation for the Bessel functions of the first kind;
- ❖ obtain Bessel functions of the first kind from the generating function, write their recurrence relations and use them to solve physical problems;
- ❖ obtain Bessel functions of the second kind and use them to solve physical problems; and
- ❖ write the expressions for spherical Bessel functions and use them to solve physical problems.

3.2 BESSEL FUNCTIONS OF THE FIRST KIND

Bessel functions find wide uses in physics ranging from the study of planetary motion, cooling towers in power plants, diffraction of light at a circular aperture, e.m. waves in cavity resonators, waveguides, diffusion across variable cross section to scattering of neutrons by a nucleus. In this section, you will revisit how to solve Bessel's differential equation given as:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - m^2\right) y(x) = 0 \qquad (3.1)$$

The parameter m is real and non-negative. You may have solved this equation while studying UG physics. We will quickly revise the method of solving Eq. (3.1).

You can see that it is an ODE of order 2 and degree 1 with variable coefficients. You can also see that $x = 0$ is a regular singular point of this differential equation. Let us use the Frobenius' method (refer to Unit 3, Block 1, PHE-05) to solve
Eq. (3.1) about $x = 0$. So, we write the function $y(x)$ as:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

And obtain the indicial equation. You may like to obtain the indicial equation for practice. Solve SAQ 1.

SAQ 1

Obtain the indicial equation and its roots for Eq. (3.1).

You have obtained the roots of the indicial equation as $\pm m$. If m is real, positive and non-integral, one solution of Eq. (3.1) is

$$y_1(x) = a_0 x^m \left[1 - \frac{x^2}{2^2 1!(m+1)} + \frac{x^4}{2^4 2!(m+1)(m+2)} - \ldots \atop (-1)^k \frac{x^{2k}}{2^{2k} k!(m+1)(m+2)\ldots(m+k)} + \ldots \right]$$

$$= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+m}}{2^{2k} k!(m+1)(m+2)\ldots(m+k)} \qquad (3.2)$$

where a_0 is an arbitrary constant. For $k \neq 0$, the product $(m+1)(m+2)\ldots(m+k)$ occurring in the above expression is to be taken as unity. Practice these steps yourself.

SAQ 2

Verify Eq. (3.2).

We can write Eq. (3.2) in a more compact form using the gamma function, which is denoted by the symbol $\Gamma(m)$. An interesting property of the gamma function is

$$\Gamma(m+1) = m\Gamma(m) \tag{3.3}$$

with $\Gamma(1) = 1$ and $\Gamma(1/2) = \dfrac{\sqrt{\pi}}{2}$. You can read about the gamma function in some detail in Block 3. By choosing the constant a_0 to be:

$$a_0 = \frac{1}{2^m \Gamma(m+1)}$$

we can rewrite the solution of Bessel's differential equation as:

$$y_1(x) = J_m(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k+m} \tag{3.4}$$

This expression represents the **Bessel function of the first kind and of order** *m*. This series converges for all values of *x*, since there are no singular points other than $x = 0;$ this result in an infinite radius of convergence.

When the value of *m* is non-integral, the other linearly independent solution of Eq. (3.1) is obtained by replacing *m* by – *m*:

$$y_2(x) = J_{-m}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(-m+k+1)} \left(\frac{x}{2}\right)^{2k-m} \tag{3.5}$$

This expression defines the **Bessel function of the first kind and order** – *m*.

To make sure that you have understood these steps, you may like to solve SAQ 3.

SAQ 3

Show that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi}} x^{-1/2} \sin x$ and $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi}} x^{-1/2} \cos x$.

We have mentioned that for non-integral values of m, $J_m(x)$ and $J_{-m}(x)$ are two linearly independent solutions of Bessel's differential equation. This fact is reinforced by SAQ 3 since one solution involves the sine and the other solution involves the cosine function.

From Eq. (3.3), note that if *m* is zero or a positive integer, denoted by *n*, we can write

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \tag{3.6}$$

Note that when $n+k+1$ is a positive integer, $\Gamma(n+k+1) = (n+k)!$

For reference, we write explicit expressions of the first few Bessel functions:

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 \times (2!)^2} - \frac{x^6}{2^6 \times (3!)^2} + \ldots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \times (1!) \times (2!)} + \frac{x^5}{2^5 \times (2!) \times (3!)} - \frac{x^7}{2^7 \times (3!) \times (4!)} + \ldots$$

$$J_2(x) = \frac{x^2}{2^2 \times (2!)} - \frac{1}{3!}\frac{x^4}{2^4} + \frac{1}{(2!) \times (4!)}\frac{x^6}{2^6} - \frac{1}{(3!) \times (5!)}\frac{x^8}{2^8} + \ldots$$

Fig. 3.1 shows plots of the functions $J_0(x), J_1(x)$ and $J_2(x)$. Note that:

- these functions exhibit oscillatory behavior; and
- become zero for a number of values of *x*.

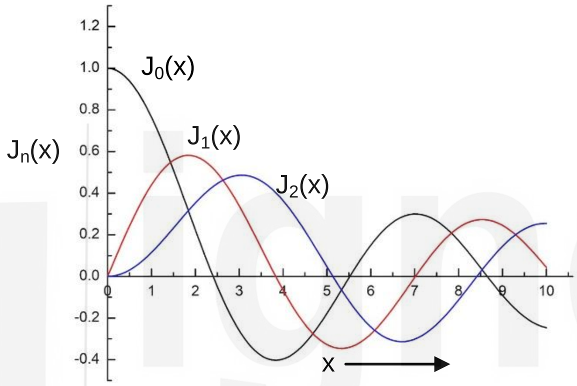

Fig. 3.1: **Plot of Bessel functions of the first kind and orders 0, 1 and 2.**

The values of *x* for which the Bessel functions become zero are said to be the zeros of the Bessel functions. Note also that at $x = 0$, $J_0(x)$ is 1, but $J_1(x)$ as well as $J_2(x)$ (and indeed higher order Bessel functions) are zero. This readily follows from the series given in Eq. (3.6). As $x \to 0$, we obtain $J_n(x)$, which corresponds to the first term with $k = 0$:

$$J_n(x) \xrightarrow{x \to 0} \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

From this it readily follows that for $n = 0$, $J_0(0) = 1$, whereas for $n = 1, 2, 3, \ldots,$ $J_n(x = 0) = 0.$.

It is important to note that it is not possible to use Eq. (3.5) to obtain the Bessel function of first kind for negative orders $-n$ $(n = 1,\ 2,\ 3, \ldots)$ since the gamma function occurring in the denominator will become infinite for $k \le (n-1)$. (The first *n* terms in the series will, therefore, be zero). If we omit these first *n* terms, we can write

$$J_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{2k-n}$$

To put this result in a more convenient form, we introduce a change of index by writing $p = k - n$. Then you may write

$$J_n(x) = \sum_{p=0}^{\infty} (-1)^{p+n} \frac{1}{(p+n)!\,\Gamma(p+1)}\left(\frac{x}{2}\right)^{2p+n}$$

$$= (-1)^n \sum_{p=0}^{\infty} (-1)^p \frac{1}{p!(n+p)!}\left(\frac{x}{2}\right)^{2p+n}$$

On comparing this expression with that given in Eq. (3.10) for $J_n(x)$, we get

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (3.7)$$

That is, for integral n, $J_{-n}(x)$ and $J_n(x)$ are related through the factor $(-1)^n$. This implies that when n is an even integer, $J_{-n}(x) = J_n(x)$. However, when n is odd, $J_{-n}(x)$ is just the negative of $J_n(x)$. We may, therefore, conclude that when n is integral, $J_n(x)$ and $J_{-n}(x)$ are solutions of Bessel's differential equation but these solutions are not linearly independent.

We now discuss the recurrence relations for Bessel functions of the first kind. We can use recurrence relations for Bessel functions to obtain the values of other Bessel functions and also to simplify expressions involving them.

3.2.1 Recurrence Relations

Let us first derive one of these recurrence relations. For this, we multiply $J_m(x)$ by x^m and then differentiate the product with respect to *x*. This gives

$$\frac{d}{dx}\left[x^m J_m(x)\right] = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2m)}{k!\Gamma(m+k+1)} \frac{x^{2k+2m-1}}{2^{2k+m}}$$

On simplification, we can write

$$m x^{m-1} J_m(x) + x^m \frac{dJ_m}{dx} = x^m \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(m+k)} \frac{x^{2k+m-1}}{2^{2k+m-1}}$$

$$= x^m J_{m-1}(x) \qquad (3.8)$$

Dividing throughout by x^m, we obtain

$$\boxed{\frac{m}{x} J_m(x) + \frac{dJ_m}{dx} = J_{m-1}(x)} \qquad (3.9)$$

If we multiply $J_m(x)$ by x^{-m} and differentiate the product with respect to *x*, we obtain a relation analogous to Eq. (3.9).

$$\boxed{\frac{m}{x} J_m(x) - \frac{dJ_m}{dx} = J_{m+1}(x)} \qquad (3.10)$$

You should solve SAQ 3 and obtain Eq. (3.10) yourself.

SAQ 4

Verify Eq. (3.10).

Note that Eqs. (3.9) and (3.10) connect $J_m(x)$ and its derivative with respect to *x* with Bessel functions of orders lower by one and higher by one. If you add

Eqs. (3.9) and (3.10) and simplify the resultant expression, the derivative term is eliminated.

$$\boxed{J_{m-1}(x) + J_{m+1}(x) = 2\frac{m}{x} J_m(x)} \tag{3.11}$$

On the other hand, by subtracting Eq. (3.10) from Eq. (3.9) we get a relation connecting the derivative of Bessel function of order m with Bessel functions of orders $m-1$ and $m+1$:

$$\boxed{J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x)} \tag{3.12}$$

where $J'_m(x)$ is the derivative of $J_m(x)$ with respect to the argument x. You may like to answer SAQ 5 to check whether you have understood the concepts.

SAQ 5

Show that $J_1(x) = -J'_0(x)$.

The derivation of recurrence relations given above holds for Bessel functions of the first kind whose orders may be integral or non-integral. However, it is more convenient to obtain the recurrence relations for the Bessel functions of integral order from the generating function. You will now learn about the generating function for Bessel functions of the first kind.

3.2.2 Generating Function

The generating function for Bessel functions of the first kind and integral order, is given by:

$$g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)t^n \tag{3.13}$$

Note that the right hand side is a series containing positive and negative integral (including zero) powers of t with coefficients which are Bessel functions of integral order. We differentiate both sides of Eq. (3.13) partially with respect to t, keeping x fixed. This gives:

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]\frac{x}{2}\left(1 + \frac{1}{t^2}\right) = \sum_{n=-\infty}^{+\infty} J_n(x)\,n\,t^{n-1}$$

or $$\frac{x}{2}(t^2+1)\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = t^2 \sum_{n=-\infty}^{+\infty} J_n(x)\,n\,t^{n-1}$$

Now we replace the factor $\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]$ occurring on the left hand side by the series given in Eq. (3.13) to write:

$$\frac{x}{2}(t^2+1)\sum_{n=-\infty}^{+\infty} J_n(x)t^n = \sum_{n=-\infty}^{+\infty} J_n(x)\,n\,t^{n+1}$$

We now equate the coefficients of like powers of t on the two sides of the above equation. Equating the coefficients of t^{n-1}, we get:

$$\frac{x}{2}J_{n-1}(x)+\frac{x}{2}J_{n+1}(x)=n\,J_n(x)$$

or $$J_{n-1}(x)+J_{n+1}(x)=2\frac{n}{x}J_n(x) \qquad (3.14)$$

This is the same recurrence relation as given in Eq. (3.11). To ensure that you have understood the procedure, you should obtain the recurrence relation given in Eq. (3.12) by starting from the generating function. Solve SAQ 6.

SAQ 6

Differentiate both sides of Eq. (3.13) partially with respect to x and obtain the recurrence relation given in Eq. (3.12).

In the following example, you will learn further uses of the generating function for the Bessel functions of the first kind.

Example 3.1

a) Starting from Eq. (3.13) show that

$$J_0(x)+2J_2(x)+2J_4(x)+2J_6(x)+\ldots+2J_{2k}(x)+\ldots=1$$

b) Starting from the generating function for the Bessel functions, show that $J_0(0)=1$ and $J_n(0)=0$ for $n=1, 2, 3,\ldots$

Solution : a) We put $t=1$ on both sides of Eq. (3.13) and note that the left hand side equals 1:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=e^0=1$$

The right hand side simplifies to:

$$\sum_{n=-\infty}^{+\infty}J_n(x)=J_0(x)+[J_1(x)+J_{-1}(x)]+[J_2+J_{-2}(x)]$$

$$+[J_3(x)+J_{-3}(x)]+[J_4(x)+J_{-4}(x)]+\ldots$$

You should note that here we have clubbed terms like $J_n(x)$ and $J_{-n}(x)$. Recall that when n is an odd integer, $J_{-n}(x)=-J_n(x)$ whereas when n is even, $J_{-n}(x)=J_n(x)$. Therefore, the second, fourth and other similar terms involving Bessel functions of odd integral order occurring on the right hand side of the above expression drop out. On using these results we get:

$$J_0(x)+2J_2(x)+2J_4(x)+\ldots=1$$

b) We put $x = 0$ on both sides of Eq. (3.13). This gives

$$1 = \sum_{n=-\infty}^{+\infty} J_n(0) t^n$$

Note that the left hand side is a constant. The only constant term on the right hand side will correspond to $n = 0$; other terms will involve t^n and should, therefore, be zero.

$$J_n(0) = 0, \text{ for } n \neq 0$$

For $n = 0$, it readily follows that $J_0(0) = 1$.

You have come across the applications of Bessel functions in Sec. 1.3.2 of Unit 1 in this course. Let us take up one more example of how Bessel functions are used in Physics.

Example 3.2

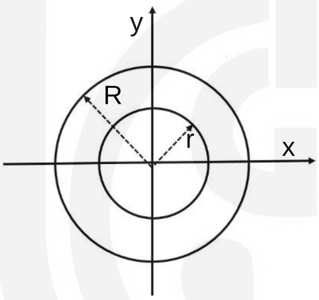

Fig 3.2: **A circular membrane fixed at the perimeter.**

We need to study the vibrations of a circular membrane fixed at the perimeter for understanding the characteristics of sound produced by beating of a drum or a table (Fig.3.2). You may have studied the nature of the transverse vibrations of a uniform flexible circular membrane fixed at its perimeter in your UG courses (refer to Unit 6 of PHE-05). For radially symmetrical case, under small oscillation approximation, transverse displacement $f(r, t)$ at a distance r from the centre O of the membrane satisfies the equation:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{(i)}$$

where $v = \sqrt{\frac{T}{m}}$, T, the force per unit length on the membrane edge and m, mass per unit area of the membrane. Solve Eq. (i) and obtain the modes of vibration of the membrane.

<u>Solution</u> : We separate Eq. (i) into ODEs, and assume the time variation to be harmonic. Then the radial function $R(r)$ satisfies the equation:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + r^2k^2R(r) = 0 \quad \text{(ii)}$$

where $k = \omega/v$. Comparing Eq. (ii) with Eq. (3.1), we can see that Eq. (ii) has solutions of the form:

$$R(r) = AJ_0(kr) + BY_0(kr) \quad \text{(iii)}$$

where $J_0(kr)$ and $Y_0(kr)$ are Bessel's functions of the first and second kind, respectively, of order 0. Since $N_0(kr) \to -\infty$ as $r \to 0$ but the displacement of the membrane at the centre $(r = 0)$ is finite, therefore, B has to be 0. Further, since the displacement is zero at $r = a$, the perimeter of the membrane is fixed, and we must have $A \neq 0$, which implies that

$$J_0(ka) = 0$$

so that $$ka = \beta_{0n} \qquad \text{(iv)}$$

where β_{0n}'s are the zeroes of the Bessel function $J_0(x)$. By denoting the permitted values of k by k_n, we can write:

$$k_n = \frac{\beta_{on}}{a} \qquad \text{(v)}$$

so that the corresponding angular frequencies of the normal modes of vibration are given by

$$\omega_n = k_n v = \frac{\beta_{0n} v}{a} \qquad \text{(vi)}$$

In particular, $\omega_1 = 2.405\frac{v}{a}$, $\omega_2 = 5.520\frac{v}{a}$, $\omega_3 = 8.654\frac{v}{a}$,...,etc.

You should note that though an infinite number of natural frequencies may occur, these are not integral multiples of the lowest frequency. The associated solutions in this case are given by:

$$R_n(r) = A_n J_0(k_n r) = A_n J_0\left(\beta_{0n}\frac{r}{a}\right) \qquad \text{(vii)}$$

where we have renamed the constant A as A_n. The general solution of Eq. (i) is obtained from the superposition principle for all n:

$$f(r,t) = \sum_{n=1}^{\infty} R_n(r)\cos(\omega t + \delta)$$

$$= \sum_{n=1}^{\infty} A_0 J_0\left(\beta_{0n}\frac{r}{a}\right)\cos\left(\beta_{0n}\frac{v}{a}t + \delta\right) \qquad \text{(viii)}$$

In Fig. 3.3, we show the nodes of three radially symmetrical modes of transverse vibration for a circular membrane. For the mode with angular frequency ω_t, the perimeter remains at rest throughout; no other node is present. For the second radially symmetric mode corresponding to angular frequency ω_2, the radial function $R_2(r) = A_2 J_0(\beta_{02}\, r/a)$ will be zero along the periphery as well as for

$$\beta_{02} = \frac{r}{a} = \beta_{01}$$

or $$\frac{r}{a} = \frac{\beta_{01}}{\beta_{02}} = \frac{2.405}{5.520} = 0.436$$

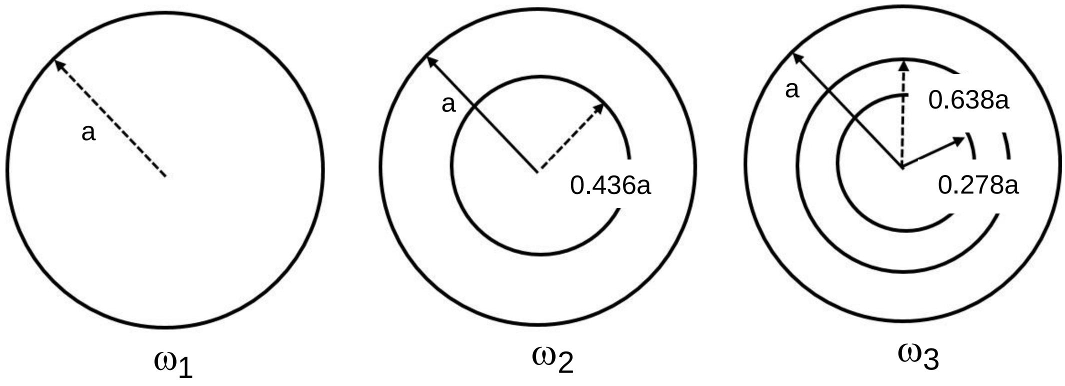

Fig. 3.3: **Nodes for three radially symmetric modes of transverse vibrations of circular membrane.**

So, there will be a node at $r = 0.436a$. All the points along the circumference of a circular membrane of radius a will always remain at rest for this mode of vibration. Similarly, for the third radially symmetrical mode with angular frequency ω_3, $R_3(r) = A_3(r) = A_3 J_0(\beta_{03}\, r / a)$, which is zero at the periphery and also when

$$\beta_{03}\frac{r}{a} = \beta_{01}$$

so that $$\frac{r}{a} = \frac{\beta_{01}}{\beta_{03}} = \frac{2.405}{8.654} = 0.278$$

or $$r = 0.278a$$

For $$\frac{r}{a} = \frac{\beta_{02}}{\beta_{03}} = \frac{5.520}{8.654} = 0.638$$

we will have $r = 0.638a$

These nodes are shown in Fig. 3.3.

We end this discussion on Bessel functions of the first kind by writing their integral representations. Such representations are very useful in dealing with Bessel functions occurring in different physical problems.

Integral Representation of Bessel functions of the First Kind

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\,d\theta \tag{3.15a}$$

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta \tag{3.15b}$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\,\theta - x\sin\theta)\,d\theta \text{ for } n = 0, 1, 2, 3, ... \tag{3.15c}$$

You will be proving these relations in TQ 3.

Let us take up an example to illustrate how integral representation of Bessel functions is used in physics.

Example 3.3

In your UG course on optics you must have studied Fraunhofer diffraction at a circular aperture. Use the integral representation of a Bessel function to arrive at Airy's formula.

Solution : Refer to Fig. 3.4, which shows a parallel beam of light incident on a circular aperture A of radius a. The diffraction pattern is formed on a suitably placed (effectively at infinite distance from A) screen S. Let α be the angle which the direction of the diffracted ray makes with central direction OZ.

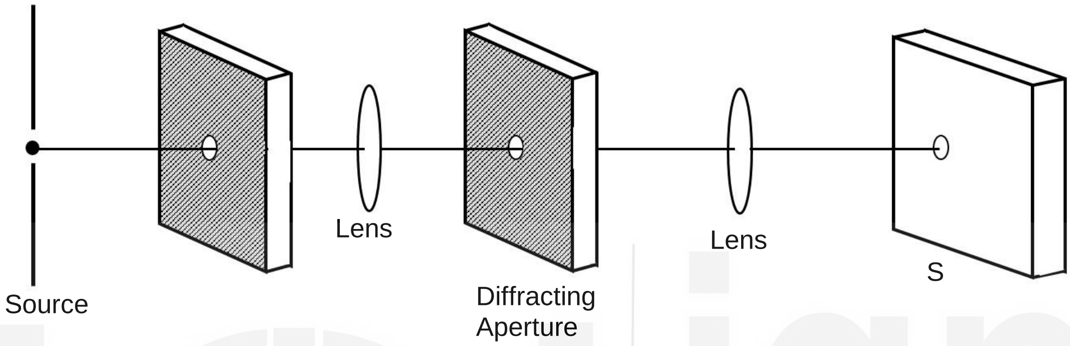

Fig. 3.4: **Fraunhofer diffraction of light at a circular aperture.**

According to the theory of diffraction of light, the amplitude of the light distribution at a point P on the screen is given by:

$$U(P) = C\int_0^n \int_0^{2\pi} e^{i\frac{2\pi}{\lambda} a \sin\alpha\cos\theta} \rho\, d\rho\, d\theta \qquad \text{(i)}$$

Here $\rho d\rho d\theta$ denotes an elementary area within the circular aperture in terms of polar coordinates ρ and θ, λ is the wavelength of light and C, a constant. From Eq. (3.15b) , we have the following integral representation of $J_0(x)$:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\, d\theta \qquad \text{(ii)}$$

If we choose $x = \frac{2\pi}{\lambda} a \sin\alpha$ and compare Eq. (i) with this result, we can express $U(P)$ in terms of $J_0(x)$:

$$U(P) = 2\pi C\int_0^a J_0\left(\frac{2\pi}{\lambda}\rho\sin\alpha\right)\rho\, d\rho \qquad \text{(iii)}$$

Now we have to evaluate this integral. From Eq. (3.9), we have for $m = 1$:

$$\frac{d}{dx}[x, J_1(x)] = x\, J_0(x)$$

so that $$xJ_1(x) = \int_0^{\pi} x' J_0(x')\, dx'$$

On inserting this result in Eq. (iii) with $x' = \frac{2\pi}{\lambda}\rho\sin\alpha$ and $x = \frac{2\pi}{\lambda}a\sin\alpha,$ we obtain

$$U(P) = C\,\pi a^2 \frac{2J_1\left(\frac{2\pi}{\lambda}a\sin\alpha\right)}{\frac{2\pi}{\lambda}a\sin\alpha} \qquad \text{(iv)}$$

The intensity of light at P is then given by

$$I(P) = |U(P)|^2 = I_0\left[\frac{2J_1\left(\frac{2\pi}{\lambda}\alpha\sin\alpha\right)}{\frac{2\pi}{2}a\sin\alpha}\right]^2 \qquad \text{(v)}$$

where $I_0 = C^2\pi^2a^4$. This is **Airy's formula.**

From Sec. 3.3, we have that as $x \to 0, J_n(x) \to x/2$.

$$J_1(x) \xrightarrow[x\to 0]{} \frac{x}{2}$$

In other words, as angle $\alpha \to 0,$ the expression within the square bracket in Eq. (v) will have the limiting value 1. This means that I_0 signifies the intensity at the centre of the diffraction pattern.

Let us now briefly discuss Bessel functions of the second kind.

3.3 BESSEL FUNCTIONS OF THE SECOND KIND

In physical problems where Bessel's differential equation occurs, you may need a general solution for integral order. However, from the discussion so far, you may be tempted to think that it is not possible to get a solution which is linearly independent of $J_n(x)$. But it is not so; for non-integer m, a function $Y_m(x)$ sometimes denoted by $N_m(x)$ defined as:

$$Y_m(x) = \frac{J_m(x)\cos(m\pi) - J_{-m}(x)}{\sin(m\pi)} \qquad \text{(3.16a)}$$

is a solution of the Bessel differential equation. It is called the **Bessel function** of the **second kind**. This function is also called **Weber function** [after Heinrich Martin Weber (1875-1913), with notation $Y_n(x)$, or **Neumann function** after Carl Neumann (1832-1925), with notation $N_n(x)$. We will use the notation $Y_n(x)$. When m is an integer, say n, we take a limiting value of the right side of the above equation as $m \to n$:

$$Y_n(x) = \lim_{m\to n} Y_m(x) = \frac{\cos n\pi J_n(x) - J_{-n}(x)}{\sin n\pi} \qquad \text{(3.16b)}$$

So, for non-integral $n, Y_n(x)$ is a solution of Bessel's differential equation (3.1), which is linearly independent of $J_n(x)$. In Fig. 3.5 we have shown the plots of the Neumann functions of orders 0, 1 and 2.

However, when n is an integer, Eq. (3.16b) is of the indeterminate form 0/0. We have to exclude these solutions sometimes in view of the physical boundary conditions of a problem. Therefore, we obtain $Y_n(x)$ for integer n, by using l'Hospital's rule:

$$Y_n(x) = \frac{\frac{d}{dm}[\cos m\pi J_m - J_{-m}(x)]}{\frac{d}{dm}(\sin m\pi)}$$

$$= \left. \frac{-\pi \sin m\pi J_m + \cos m\pi \frac{\partial J_m}{\partial m} - \frac{\partial J_{-m}}{\partial m}}{\pi \cos m\pi} \right|_{m=n}$$

or $$Y_n(x) = \frac{1}{\pi}\left[\frac{\partial J_m(x)}{\partial m} - \frac{\partial J_{-m}(x)}{\partial m}\right]_{m=n} \qquad (3.16c)$$

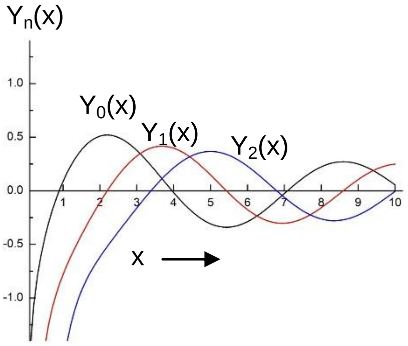

Fig. 3.5: **Plots of Bessel functions of the second kind of orders 0, 1 and 2.**

We can use Eq. (3.6) for Bessel function of the first kind:

$$J_m(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+m+1)}\left(\frac{x}{2}\right)^{2k+m}$$

and the digamma function defined as:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

to express Bessel function of the second kind as the following series. We write the final result here without going into the detailed steps:

$$Y_n(x) = \frac{1}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} \ln\left(\frac{x}{2}\right)\right.$$

$$\left. + \ (-1)^n \sum_{k\geq 0}\frac{(-1)^k}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n} \ln\left(\frac{x}{2}\right)\right]$$

$$= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)}{k!}\left(\frac{x}{2}\right)^{2k-n}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \qquad (3.16d)$$

where we have used the result: $\lim\limits_{z\to -n}\dfrac{\psi(z)}{\Gamma(z)} = (-1)^{n-1}n!$

For $n = 0$, the limiting value is:

$$Y_0(x) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{x}{2}\right)\right]J_0(x) - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)}{k!k!}\left(\frac{x}{2}\right)^{2k} \psi(k+1) \qquad (3.16e)$$

where $\gamma \approx 0.5772156649\ldots$ is the Euler constant.

Bessel functions of the second kind satisfy all recurrence relations satisfied by Bessel functions of the first order. These functions also have integral representations given by:

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt \tag{3.17a}$$

$$= -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{\sqrt{t^2-1}}\,dt, \qquad x > 0 \tag{3.17b}$$

The most general solution of Bessel's equation for any n is:

$$y(x) = CJ_n(x) + DY_n(x) \tag{3.18}$$

Since $Y_n(x)$ diverges logarithmically, any boundary condition that requires the solution to be finite at the origin must exclude it. If, however, there is no such boundary condition, then we may consider $Y_n(x)$ in the solution. Let us now discuss the spherical Bessel functions.

3.4 SPHERICAL BESSEL FUNCTIONS

When we separate Helmholtz equation in spherical polar coordinates, its radial part is given as:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - n(n+1)]R = 0 \tag{3.19}$$

Let us see how. In Unit 1, you have encountered Helmholtz equation in Cartesian coordinates and separated it into three ODEs:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2 f(x, y, z) = 0 \tag{3.20a}$$

Let us write this equation in spherical polar coordinates:

$$\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\right] = -k^2 f \tag{3.20b}$$

You can separate Eq. (3.20b) into three ODEs by putting $f(r, \theta, \phi)$ as the product $R(r)\,\Theta(\theta)\,\Phi(\phi)$. So, you will get:

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -k^2 \tag{3.21a}$$

Multiplying the above equation by $r^2\sin^2\theta$, we can write:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = r^2\sin^2\theta\left[-k^2 - \frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right. \tag{3.21b}$$

The LHS of Eq. (3.21b) is a function of ϕ alone and the RHS, a function of r and θ. Since r, θ and ϕ are independent variables, we put the LHS and RHS of Eq. (3.21b) equal to a constant:

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \quad \text{or} \quad \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{3.22a}$$

The normalized solution of Eq. (3.22a) is:

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \tag{3.22b}$$

The function $\Phi_m(\phi)$ is orthonormal with respect to the variable ϕ. Note that we have used a negative constant because in problems in physics, ϕ appears mostly as the azimuth angle, which suggests periodic solutions rather than exponential ones. Substituting Eq. (3.22a) in Eq. (3.21b), we can write:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2 = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta} \tag{3.23a}$$

You can see that in Eq. (3.23a), the variables are separated. Again, we equate each side of Eq. (3.23a) to a constant, say C. Then we have:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2\Theta}{\sin^2\theta} + C\Theta = 0, \tag{3.23b}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{CR}{r^2} = 0 \tag{3.23c}$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - n(n+1)]R = 0$$

which is Eq. (3.19) if we put $C = n(n+1)$. In Eq. (3.23c), if we substitute

$$R(kr) = \frac{Z(kr)}{(kr)^{1/2}}$$

then it becomes:

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - n(n+\frac{1}{2})^2\right]Z = 0 \tag{3.24}$$

This is Bessel's equation. Z is a Bessel function of order $n+\frac{1}{2}$ where n is an integer. The function $\frac{Z_{n+1/2}(kr)}{(kr)^{1/2}}$ occurs very often in physics. The functions Z are called spherical Bessel functions.

The spherical Bessel functions corresponding to Bessel functions of the first kind are denoted by $j_n(x)$ and defined as:

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\,J_{n+1/2}(x) \tag{3.25}$$

Fig. 3.6 shows the plots of spherical Bessel functions $j_n(x)$.

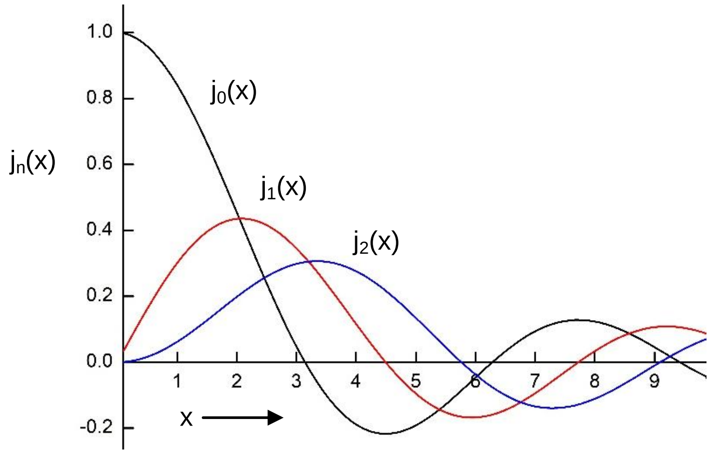

Fig 3.6: **Plots of spherical Bessel functions.**

There are some more functions like Hankel functions, modified Bessel functions and spherical Bessel functions corresponding to these Bessel functions, which will be discussed as and when you encounter them in other physics courses.

We now end this discussion and summarise the unit.

3.5 SUMMARY

In this unit, we have covered the following concepts:

- Bessel equation and its solutions by the Frobenius method.
- Bessel function of the first kind, its generating function, recurrence relations and integral representation.
- Bessel function of the second kind and its recurrence relations.
- Spherical Bessel functions.

3.6 TERMINAL QUESTIONS

1. If a function $Z_m(x)$ satisfies the recurrence relations

$$Z_{m-1}(x)+Z_{m+1}(x)=2\frac{m}{x}Z_m(x)$$

and $$Z_{m-1}(x)-Z_{m+1}(x)=2\frac{dZ_m(x)}{dx},$$

show that it satisfies Bessel's differential equation also.

2. Prove that $J_n(x)=(-1)^n\, x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$.

3. Show that $J_0(x+y)=J_0(x)J_0(y)+2\sum_{n=1}^{\infty}(-1)^n J_n(x)J_n(y)$.

4. Show that $J_0(x)=\frac{1}{2\pi}\int_0^{2\pi}e^{ix\cos\theta}d\theta$.

5. Recast the radial part of the Schrödinger equation for a spherically symmetric potential given below as an ODE for spherical Bessel function and solve it:

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[\frac{2mE}{\hbar^2}-\frac{l(l+1)}{r^2}\right]R(r)=0$$

where $R(r)$ is zero at $r=R$.

3.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We write Eq. (3.1) as follows and use Frobenius method to solve it. We assume a solution of the form:

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r} \qquad \text{(i)}$$

Differentiating Eq. (i) with respect to x, we get

$$\frac{dy}{dx}=\sum_n a_n(n+r)x^{n+r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_n a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting $y(x), \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in Bessel equation, we get

$$\sum_n (n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty}a_n x^{n+r+2}-m^2\sum_{n=0}^{\infty}a_n x^{n+r}=0 \qquad \text{(ii)}$$

On equating the coefficients of the lowest power of x to zero, we get the indicial equation:

$$a_0\left[r(r-1)+r-m^2\right]=0$$

For $a_0\neq 0$, the indicial equation $r^2-m^2=0$ has roots $r=\pm m$.

2. We continue from the solution of SAQ 1. Depending on the value of m, the solutions can differ vastly:

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+m} \qquad \text{(i)}$$

and $$y_2(x)=\sum_{n=0}^{\infty}b_n x^{n-m} \qquad \text{(ii)}$$

To find y_1, we put Eq. (ii) of SAQ 1 in expanded form of x equal to zero. Then all odd subscripted coefficients vanish. For even subscripted coefficients, it leads to the recurrence relation:

$$a_{2n}=\frac{(-1^n)}{2^{2n}n!(m+1)(m+2)\ldots(m+n)}a_0$$

so that

$$y_1(x) = a_0\left[x^m - \frac{x^{m+2}}{2^2(m+1)} + \frac{x^{m+1}}{2!2^n(m+1)(m+2)} + \ldots\right]$$

3. We put $m = 1/2$ in Eq. (3.4) to write:

$$J_{1/2}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!\,\Gamma\left(k+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2k+\frac{1}{2}}$$

Through repeated use of Eq. (3.3), we can write

$$\Gamma\left(k+\frac{3}{2}\right) = \left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right) = \left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)\ldots$$

$$= \left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\ldots\frac{5}{2}\times\Gamma\left(\frac{3}{2}\right)$$

Then

$$k!\,\Gamma\left(k+\frac{3}{2}\right)2^{2k+\frac{1}{2}} = 2\times4\times6\ldots(2k-2)2k\times3\times5\ldots$$

$$(2k-1)(2k+1)\Gamma\left(\frac{3}{2}\right)2^{1/2}$$

$$= (2k+1)!\,\Gamma\left(\frac{3}{2}\right)2^{1/2}$$

Since $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$, we have

$$J_{1/2}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)\times 2^{-1/2}\sqrt{\pi}}\, x^{2k+1}x^{-1/2}$$

$$= \sqrt{\frac{2}{\pi}}\, x^{-1/2}\sin x$$

because $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Similarly, putting $m = -1/2$ in Eq. (3.4), we get

$$J_{-1/2}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!\,\Gamma(k+1/2)}\left(\frac{x}{2}\right)^{2k-\frac{1}{2}}$$

In this case, the denominator of a general term is:

$$k!\,\Gamma\left(k+\frac{1}{2}\right)2^{2k-\frac{1}{2}} = 2\times4\times6\ldots(2k-2)\,2k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\ldots$$

$$\frac{3}{2}\times\frac{1}{2}\Gamma(1/2)2^k 2^{-1/2}$$

$$= 2\times4\times6\ldots1\times3(2k-3)(2k-1)\Gamma\left(\frac{1}{2}\right)2^{-1/2}$$

$$= (2k)!\sqrt{\pi}\,2^{-1/2}$$

Hence, $J_{-1/2}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} \frac{x^{2k-\frac{1}{2}}}{\sqrt{\pi}2^{-1/2}} = \sqrt{\frac{2}{\pi}}\, x^{-1/2} \cos x$

since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

4. Multiplying both sides of Eq. (3.4) by x^{-m}, we get

$$x^{-m} J_m(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(m+k+1)} \frac{x^{2k}}{2^{2k+m}}$$

$$\therefore \quad \frac{d}{dx}\left[x^{-m} J_m(x)\right] = \sum_{k=1}^{\infty} (-1)^k \frac{2k}{k!\,\Gamma(m+k+1)} \frac{x^{2k-1}}{2^{2k+m}}$$

$$= x^{-m} \sum_{k=1}^{\infty} (-1)^k \frac{k}{k!\,\Gamma(m+k+1)} \frac{x^{2k-1+m}}{2^{2k+m-1}}$$

$$= x^{-m} \sum_{k'=0}^{\infty} \frac{(k'+1)}{(k'+1)!\,\Gamma(m+k'+2)} \frac{x^{2k'+m+1}}{2^{2k'+m+1}}$$

where we have introduced $k' = k-1$. Then

$$\frac{d}{dx}\left[x^{-m} J_m(x)\right] = -x^{-m} \sum_{k'=0}^{\infty} (-1)^{k'} \frac{1}{k'!\,\Gamma(m+k'+2)} \left(\frac{x}{2}\right)^{2k'+m+1}$$

$$= -x^{-m} J_{m+1}(x)$$

On differentiating as indicated on the left hand side, we get:

$$-mx^{-m-1} J_m(x) + x^{-m} \frac{dJ_m}{dx} = -x^{-m} J_{m+1}(x)$$

Multiplying both sides by x^m and changing sign leads to:

$$\frac{m}{x} J_m(x) - \frac{dJ_m}{dx} = J_{m+1}(x)$$

This is the required Eq. (3.10).

5. In Eq. (3.10), we put $m = 0$. Then

$$-\frac{dJ_0}{dx} = J_1(x)$$

or $\quad J_1(x) = -J_0'(x)$

6. Differentiating both sides of Eq. (3.13) partially with respect to x, we get:

$$\frac{1}{2}\left(t - \frac{1}{t}\right) \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{+\infty} J_m'(x)\, t^m$$

or $\quad (t^2 - 1) \sum_{m=-\infty}^{\infty} J_m(x)\, t^m = 2t \sum_{m=-\infty}^{\infty} J_m'(x)\, t^m$

In the last step, $\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]$ has been replaced by the right side of Eq. (3.13). Equating coefficients of t^{m-1} on both sides of the above equation, we get

$$J_{m-1}(x) - J_{m+1}(x) = 2J_m'(x)$$

Terminal Questions

1. We are given the recurrence relations:

$$Z_{m-1}(x) + Z_{m+1}(x) = 2\frac{m}{x} Z_m(x) \qquad \text{(i)}$$

and
$$Z_{m-1}(x) - Z_{m+1}(x) = 2Z'_m(x) \qquad \text{(ii)}$$

On adding (i) and (ii) and multiplying the resultant expression by $x/2$, we get:

$$xZ_{m-1}(x) = mZ_m(x) + xZ'_m(x) \qquad \text{(iii)}$$

Differentiating both sides with respect to x, we get:

$$Z_{m-1}(x) + xZ'_{m-1}(x) = mZ'_m(x) + Z'_m(x) + xZ''_m(x)$$
$$= (m+1)Z'_m(x) + xZ''_m(x)$$

Multiplying all terms in the above equation by x, we get:

$$xZ_{m-1}(x) + x^2 Z'_{m-1}(x) = x(m+1)Z'_m(x) + x^2 Z''_m(x)$$

Subtracting from the two sides of the above equation, the corresponding sides of (iii) multiplied by m, we get:

$$(1-m)xZ_{m-1}(x) + x^2 Z'_{m-1}(x) = xZ'_m(x) + x^2 Z''_m(x) - m^2 Z_m(x) \qquad \text{(iv)}$$

Now we subtract the two sides of Eq. (ii) from the corresponding sides of Eq. (i) and multiply by x/2. We then get:

$$xZ_{m+1}(x) = mZ_m(x) - xZ'_m(x)$$

Next, we change m to $m-1$ in the above equation. Hence,

$$xZ_m(x) = (m-1)Z_{m-1}(x) - xZ'_{m-1}(x)$$

Multiplying the above equation by x and rearranging terms, we get:

$$(1-m)xZ_{m-1}(x) + x^2 Z'_{m-1}(x) = -x^2 Z_m(x) \qquad \text{(v)}$$

Comparing Eqs. (iv) and (v), we get:

$$xZ'_m(x) + x^2 Z''_m(x) - m^2 Z_m(x) = -x^2 Z_m(x)$$

or
$$x^2 Z''_m(x) + xZ'_m(x) + x^2 - m^2) Z_m(x) = 0$$

Thus $Z_m(x)$ satisfies Bessel's differential equation.

2. We will prove this result by the method of induction. So, we assume that the relation is true for $n = l$, so that

$$J_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l J_0(x) \qquad \text{(i)}$$

Now we have to show that the relation is also true for $n = l+1$. Let us start with

$$(-1)^{l+1} x^{l+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{l+1} J_0(x) = -(-1)^l\, x^{l+1} \frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)^l J_0(x)$$

$$= -(-1)^l x^l \frac{d}{dx}\left[(-1)^{-l} x^{-l} J_l(x)\right] \quad \text{[using (i)]}$$

$$= -x^l \frac{d}{dx}\left[x^{-l} J_l(x)\right] = -x^l \left[x^{-1}\frac{d}{dx} J_l(x) - lx^{-1-1} J_l(x)\right]$$

$$= \frac{1}{x} J_l(x) - \frac{dJ_l}{dx} = J_{l+1}(x) \quad \text{[Using Eq. (3.10) with } m = l.]$$

Thus if the relation to be proved is true for $n = l,$ it is also true for $n = l + 1.$ Let us check what happens when $n = 1.$ Then the right side of Eq.(i) becomes:

$$(-1)\,x\,\frac{1}{x}\frac{d}{dx} J_0(x) = -J_0'(x) = J_1(x)$$

Thus, the relation is true for $n = 1.$ Hence, from what has been proved above, the relation is true for $n = 2, 3, 4...$ The relation is, therefore, proved.

3. We start with the relation:

$$\exp\left[(x+y)\left(t - \frac{1}{t}\right)\right] = \exp\left[x\left(t - \frac{1}{t}\right)\right]\exp\left[y\left(t - \frac{1}{t}\right)\right]$$

In Eq. (3.14), the left hand side signifies the generating function for $J_l(x+y)$. Hence, we can write:

$$\sum_{l=-\infty}^{+\infty} J_l(x+y)t^l = \sum_{n=-\infty}^{+\infty} J_n(x)t^n \sum_{k=-\infty}^{+\infty} J_k(y)t^k$$

On the left hand side $J_0(x+y)$ occurs as the term independent of *t*. We will get terms independent of *t* on the right had side for $n + k = 0,$ i.e., for $k = -n.$ Therefore, we can write:

$$J_0(x+y) = \sum_{n=-\infty}^{+\infty} J_{-n}(x) J_{-n}(y)$$

Since $J_{-n}(y) = (-1)^n J_n(y)$ and $J_{-n}(x) = (-1)^n J_n(x),$ we get:

$$J_0(x+y) = J_0(x) J_0(y) + 2\sum_{n=1}^{\infty} (-1)^n J_n(x) J_n(y)$$

4. We take $n = 0$ in Eq. (3.15c). Then

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(-x\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\,d\theta$$

Since $\cos[x\sin(\pi+\theta)] = \cos(x\sin\theta)$ and $\sin[x\sin(\pi+\theta)] = -\sin(x\sin\theta)$ for any value of $\theta,$ we can write:

$$\int_{\pi}^{2\pi} \cos(x\sin\theta)\,d\theta = \int_0^{\pi} \cos(x\sin\theta)\,d\theta$$

and $$\int_{\pi}^{2\pi} \sin(x\sin\theta)\,d\theta = \int_0^{\pi} \sin(x\sin\theta)\,d\theta$$

$\therefore$ $$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta)\,d\theta$$

and $$0 = \frac{1}{2\pi}\int_0^{2\pi} \sin(x\sin\theta)\,d\theta$$

Hence, $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}[\cos(x\sin\theta) + i\sin(x\sin\theta)]\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\,d\theta$

Considering the range 0 to 2π, $\cos\theta$ has all the values that $\sin\theta$ assumes, and the above integral may be written as:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta$$

5. Multiplying the given equation:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{2mE}{\hbar^2} - \frac{l(l+1)}{r^2}\right]R(r) = 0$$

by r^2, we get the differential equation [Eq. (3.23c)]:

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + \left[k^2r^2 - l(l+1)\right]R(r) = 0 \qquad \text{(i)}$$

with $k^2 = \frac{2mE}{\hbar^2}$. In Eq. (i), if we substitute $R(kr) = \frac{Z(kr)}{(kr)^{1/2}}$

then it becomes:

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - n(n + \frac{1}{2})^2\right]Z = 0 \qquad \text{(ii)}$$

which is the differential equation for spherical Bessel function of order $n + \frac{1}{2}$ where n is an integer. The general solution of this equation is a linear combination of the spherical Bessel functions of the first and second kind. However, since the spherical Bessel functions of the second kind diverge at the r = 0, we usually do not consider them for physical problems. So, we consider only spherical Bessel functions of the first kind and the solution is:

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\,J_{n+1/2}(x)$$

UNIT 4

SPECIAL FUNCTIONS-I

Structure

4.1 INTRODUCTION

In this unit, we discuss **Legendre polynomials**, **spherical harmonics** and **hypergeometric functions**. We will solve respective differential equations, write their generating functions and obtain recurrence relations for each special function.

In the next unit, we discuss Hermite and Laguerre Polynomials, the Sturm-Liouville problem, and explain how to expand a given function in terms of orthogonal functions.

Expected Learning Outcomes

After studying this unit, you should be able to:

- obtain Legendre polynomials by solving Legendre's differential equation as well as from the generating function and Rodrigues's formula;
- derive the recurrence relations, orthogonality relation for Legendre polynomials and use them to solve problems in physics;
- obtain spherical harmonics by solving the ODE as well as from their generating function and derive their recurrence relations; and
- obtain Hypergeometric functions by solving the ODE as well as from their generating function and derive their recurrence relations.

4.2 LEGENDRE POLYNOMIALS

In Unit 1, you have learnt how to solve Laplace's equation in spherical polar co-ordinates (r, θ, ϕ) and seen that the solutions are expressed in terms of Legendre polynomials. Similarly, the solution of the (θ, ϕ) part of the Schrödinger equation for an electron is expressed in terms of Legendre polynomials. In your UG courses, you must have learnt how to solve Legendre's differential equation using the power series method. Its solutions are called Legendre polynomials, which are new functions with very interesting properties.

In this section, we revisit the solution of Legendre's differential equation and obtain the Legendre polynomials in two different ways: By solving the differential equation and from the generating function. (You should refresh your knowledge by studying Unit 3 of the course PHE-05 entitled Mathematical Methods in Physics-II.) Then we discuss the properties of Legendre polynomials; especially the orthogonality property. We have discussed many applications of Legendre polynomials. We expect you to study these examples carefully and link them up with the relevant topics. It is important to study Legendre's associated differential equation (refer to Appendix of Unit 14 of the course PHE-14) but we shall not go into the mathematical rigor of the properties of the associated Legendre polynomials.

You have learnt about **Legendre's differential equation** in your UG courses (see Examples 1 and 3 of Unit 3, Block 1, PHE-05):

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{4.1}$$

You can obtain the solution of this equation by solving SAQ 1. It is:

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{3!}x^5 - \ldots\right] \tag{4.2}$$

SAQ 1

Solve Eq. (4.1) and obtain its general solution given by Eq. (4.2).

Note from Eq. (4.2) that for an even integer $(n \geq 0)$, the first bracketed term in this series (with even powers of x) terminates leading to a polynomial solution. For an odd integer ($n > 0$), the latter term in the series (with odd powers of x) terminates and gives a polynomial solution. So, for any integer $(n \geq 0)$, Legendre's equation has a polynomial solution. For $n = 0, 1, 2, \ldots$, Eq. (4.2) leads to the following solutions of Eq. (4.1):

$$y = a_0, \qquad n = 0 \tag{4.3a}$$

$$y = a_1 x, \qquad n = 1 \tag{4.3b}$$

$$y = a_0(1 - 3x^2), \qquad n = 2 \tag{4.3c}$$

$$y = a_1\left(\frac{3x - 5x^3}{2}\right), \qquad n = 3 \tag{4.3d}$$

These expressions (of y) are, apart from multiplication constant, the **Legendre polynomials** $P_n(x)$. The multiplicative constant is chosen so that $P_n(1) = 1$.

We can write Eq. (4.2) as:

$$y = a_0 y_1(x) + a_1 y_2(x) \tag{4.4}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent. You should note that Eq. (4.4) does not give the general solution of Legendre's differential equation. To understand the general solution we have to reconsider the recursion formula arrived at while solving Legendre's equation (solution of SAQ 1):

$$a_{k+2} = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)} a_k; \quad n = 0,1,2,..., \tag{4.5}$$

Note that the coefficients a_{k+2} will vanish when (i) $n = k$ and/or (ii) $n = -(k+1)$. $a_{k+2} = 0$ implies that the series terminates with a_k as the last non-zero coefficient. That is how we get polynomial solutions. While obtaining the polynomial solutions $P_n(x)$, we considered the polynomial solutions with $n = k$ only.

When we take $n = -(k+1)$, the series obtained diverges for $x = \pm 1$. This solution is unbounded. It is called a **Legendre function of second kind** and denoted by $Q_n(x)$. Thus, the most general solution of Legendre's differential equation can be written as:

$$y = A_1 P_n(x) + A_2 Q_n(x) \tag{4.6}$$

Here we shall restrict ourselves to **Legendre polynomials of the first kind**, that is $P_n(x)$. Let us now derive the expressions for $P_n(x)$.

From Eq. (4.5), we note that if $k = n$, $a_{n+2} = 0$ and, by induction, $a_{n+4} = 0 = a_{n+6} ...$

The continue our discussion, we invert Eq. (4.5) to obtain

$$a_k = -\frac{(k+1)(k+2)}{(n-k)\times(n+k+1)} a_{k+2}$$

By taking $k = n-2, n-4, ...,$ we get:

$$a_{n-2} = -\frac{n(n-1)}{2(n-1)} a_n$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)} a_n$$

This yields the polynomial solution

$$y = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)} x^{n-4} - ... \right] \tag{4.7}$$

The Legendre polynomials $P_n(x)$ are defined by choosing

$$a_n = \frac{(2n-1)(2n-3)...3!}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

This choice of the coefficients a_n ensures that $P_n(1) = 1$ when $n = 1$. Thus,

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)}x^{n-4} - \ldots\right] \quad (4.8)$$

We now work out a simple example to enable you to fix these concepts

To show the equivalence of expressions in Eq. (4.8), we note that even terms are not occurring in the numerator defining the coefficient a_n. So, we multiply the numerator and denominator by $2n(2n-2)(2n-4)\ldots 4\times 2$.

This gives

$$a_n = \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots 4\times 3\times 2\times 1}{n!(2n)(2n-2)(2n-4)\ldots\times 4\times 2}$$

$$= \frac{2n!}{n!\,2n(2n-2)\ldots\times 4\times 2}$$

Now $(2n)(2n-2)\ldots\times 4\times 2$

$$= (2n)\times 2(n-1)\ldots\times 2(2)\times 2(1)$$

$$= 2^n n!$$

So that the expression for a_n reduces to

$$a_n = \frac{2n!}{2^n (n!)^2}$$

Example 4.1

Starting from Eq. (4.8), prove that

$$P_0(x) = 1, \quad P_1(x) = x \text{ and } P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Solution : From Eq. (4.8), we have

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

For n = 0, substituting 0! = 1 and $x^0 = 1$, we readily obtain $P_0(x) = 1$.

For $n = 1$,

$$P_1(x) = \frac{2!}{2^1\times(1!)^2}x = \frac{2x}{2} = x$$

$$P_2(x) = \frac{4!}{2^2\times(2!)^2}\left[x^2 - \frac{2\times(2-1)}{2\times(4-1)}x^0\right]$$

$$= \frac{24}{4\times 4}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}(3x^2 - 1)$$

In this way, you can obtain expressions for higher order Legendre polynomials. For practice, you may like to solve SAQ 2.

SAQ 2

Determine $P_3(x)$.

So far we have obtained expressions for Legendre polynomials by solving Legendre's differential equation. These expressions can also be generated from a function of two variables, say *x* and *t*. When expanded in powers of *t*, the *x*-dependent coefficients define the Legendre polynomials. This forms the subject matter of discussion for the following section.

4.2.1 Generating Function

Consider the function

$$g(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \quad (4.9)$$

We expand $(1-2xt+t^2)^{-1/2} = \{1-(2x-t)t\}^{-1/2}$ in powers of t for $|t|<1$ using binomial expansion:

$$[1-t(2x-t)]^{-1/2} = 1+\left(\frac{1}{2}\right)t(2x-t)+\frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)t^2(2x-t)^2$$

$$= 1+\frac{t(2x-t)}{2}+\left(\frac{1\times 3}{2\times 4}\right)t^2(2x-t)^2+\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)t^3(2x-t)^3+\ldots$$

$$+\frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)t^3(2x-t)+\ldots$$

$$= 1+\frac{1}{2}(2xt-t^2)+\frac{3}{8}t^2(4x^2+t^2-4xt)$$

$$+\frac{5}{16}t^3(8x^3-t^3-12x^2t+6xt^2)+\ldots$$

$$= 1+xt-\frac{1}{2}t^2+\frac{3}{2}x^2t^2+\frac{3}{8}t^4-\frac{3}{2}xt^3$$

$$+\frac{5}{2}x^3t^3-\frac{5}{16}t^6-\frac{15}{4}t^4x^2+\ldots$$

$$= 1+xt+\frac{1}{2}(3x^2-1)t^2+\frac{1}{2}(5x^3-3x)t^3+\ldots \qquad (4.10)$$

On equating the coefficients of t^n for $n=0,1,2$ and 3 in Eqs. (4.9) and (4.10), we find that the first few Legendre polynomials are given by

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

You should note that these expressions are the same as obtained in Example 4.1 and SAQ 2. The coefficient of t^n in the above expansion will be:

$$\frac{1\times 3\times 5\ldots\times(2n-1)}{2\times 4\times 6\ldots 2n}(2x)^n$$

$$-\frac{1\times 3\times 5\ldots\times(2n-3)}{2\times 4\times 6\ldots\times(2n-2)}\cdot\frac{(n-1)}{1!}(2x)^{n-2}$$

$$+\frac{1\times 3\times 5\ldots(2n-5)}{2\times 4\times 6\ldots(2n-4)}\cdot\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

This may be rewritten as:

$$\frac{1\times 3\times 5\ldots(2n-1)}{n!}\left\{x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)}x^{n-4}-\ldots\right\}$$

You can readily identify this with Eq. (4.8). So we may conclude that Eq. (4.9) with $t<1$ signifies the generating relation for Legendre polynomials and

$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}$ is called the **generating function** for Legendre polynomials.

For $x = 1$, Eq. (4.9) gives $(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\,t^n$

The left hand side now reduces to $(1-t)^{-1}$, which has series representation as $\sum_{n=0}^{\infty} t^n$. Then the above expression takes the form:

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)\,t^n$$

On comparing the coefficients of t^n, we get $P_n(1) = 1$ so that the expression of the generating function gets validated.

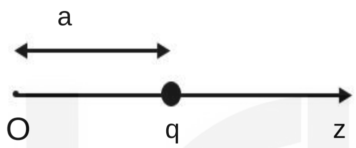

Fig. 4.1: **An electric charge *q* placed at *z* = *a*.**

The generating function is useful in solving physical problems involving the potential associated with any inverse square force. To illustrate this we consider an electric charge *q* placed on the *z*-axis at z = *a* (see Fig. 4.1). From UG courses, you will recall that the electrostatic potential at a non-axial point due to this charge at a distance r_1 from it is given by:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1} \tag{4.11}$$

From the properties of a triangle, we can write:

$$r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$$

On inserting this expression in Eq. (4.11), we get

$$V = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$$

$$= \frac{q}{4\pi\varepsilon_0 r}\frac{1}{\sqrt{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta}} \tag{4.12}$$

For $r > a$, the expression under the radical sign may be written as $(1-2xt+t^2)^{-1/2}$ where $x = \cos\theta$ and $t = \frac{a}{r}$; $|t| < 1$. From Eq. (4.10) we note that when $(1-2xt+t^2)^{-1/2}$ is expanded in powers of *t* for $|t| < 1$, the coefficient of t^n can be identified with $P_n(x)$.

On inserting this result in Eq. (4.12), we get

$$V = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n \tag{4.13}$$

Example 4.2

Calculate the values of $P_{2n}(0)$ and $P_{2n+1}(0)$.

Solution : Putting $x = 0$ in Eq. (4.9), we get

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n$$

Using binomial expansion, we can write:

$$(1+t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \ldots$$

$$+(-1)^n \frac{1\times 3\times 5\times\ldots\times(2n-1)}{2^n n!}t^{2n} + \ldots$$

$$= P_0(0) + P_1(0)t + P_2(0)t^2 + \ldots + P_{2n}(0)t^{2n}$$

On comparing the powers of t^{2n} in the above equation, we get:

$$P_{2n}(0) = (-1)^n \frac{1\times 3\times 5\ldots\times(2n-1)}{2^n n!}$$

Since in the expansion of $(1+t^2)^{-1/2}$ we only get even power of t, we have:

$$P_{2n+1}(0) = 0$$

You may now like to solve SAQ 3 on generating functions.

SAQ 3

Prove that

a) $P_n(-1) = (-1)^n$ and

b) $P_n(-x) = (-1)^n P_n(x)$

We shall now use the generating function to obtain the recurrence relations or recursion relations for Legendre polynomials. This nomenclature stems from the fact that expressions for higher order polynomial can be derived from knowledge of expressions for lower order polynomials.

4.2.2 Recurrence Relations

To obtain two primary recurrence relations, we differentiate $g(x,t)$ partially with respect to t and x, respectively.

You have learnt that the generating function [Eq. (4.9)] for Legendre polynomials is:

$$g(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Differentiating the above expression partially with respect to t, we get:

$$\frac{\partial g}{\partial t} = \left(-\frac{1}{2}\right)\frac{(-2x+2t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or $$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

We rewrite this result as:

$$\frac{(x-t)}{\sqrt{1-2xt+t^2)}} = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

and use the generating function to write:

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$\left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

On re-arrangement of terms, we obtain:

$$\sum_{n=0}^{\infty}\left[(n+1)P_n(x)t^{n+1} - (2n+1)xP_n(x)t^n + nP_n(x)t^{n-1}\right] = 0$$

To collect the coefficient of t^n, we replace n by $n - 1$ in the first term and by $n + 1$ in the last term. Then equating the resulting expression to zero, we get:

$$n\,P_{n-1}(x) - (2n+1)x\,P_n(x) + (n+1)\,P_{n+1}(x) = 0$$

$$\boxed{(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)} \qquad (4.14)$$

This is one of the two important recurrence relations and correlates any Legendre polynomial with its adjoining polynomials. For example, by putting $n=1$ in Eq. (4.14), we get:

$$3x\,P_1(x) = 2\,P_2(x) + P_0(x)$$

But we know that $P_0(x) = 1$ and $P_1(x) = x$. Hence, we find that:

$$3x^2 = 2P_2(x) + 1$$

or $$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Similarly, $P_3(x)$ can be obtained using expressions for $P_1(x)$ and $P_2(x)$; $P_4(x)$ can be obtained using expressions for $P_2(x)$ and $P_3(x)$, and so on.

To derive the second recurrence relation, we differentiate $g(x,t)$ partially with respect to x:

$$\frac{\partial g}{\partial x} = \frac{\left(-\frac{1}{2}\right)(-2t)}{\left(1-2xt+t^2\right)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n$$

so that

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

or

$$\frac{t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

On combining this result with Eq. (4.9), we get:

$$t\sum_{n=0}^{\infty} P_n(x)\, t^n = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

On rearranging terms, we obtain:

$$\sum_{n=0}^{\infty}[2x\,P_n'(x)+P_n(x)]\,t^{n+1} = \sum_{n=0}^{\infty}\left(1+t^2\right)P_n'(x)\,t^n$$

$$= \sum_{n=0}^{\infty} t^n P_n'(x) + \sum_{n=0}^{\infty} P_n'(x)\,t^{n+2}$$

As before, on collecting coefficients of t^{n+1} from both sides and equating them, we get the second recurrence relation:

$$2x\,P_n'(x)+P_n(x) = P_{n+1}'(x)+P_{n-1}'(x) \qquad (4.15)$$

Another very useful recurrence relation is obtained by combining Eqs. (4.14) and (4.15). To this end, we first differentiate Eq. (4.14) with respect to *x* and multiply the result by 2. This gives

$$2(2n+1)P_n(x)+2(2n+1)x\,P_n'(x) = 2(n+1)P_{n+1}'(x)+2n\,P_{n-1}'(x)$$

We rewrite it as:

$$2(2n+1)x\,P_n'(x) = 2(n+1)P_{n+1}'(x)+2nP_{n-1}'(x)-2(2n+1)P_n(x)$$

Next, we multiply Eq. (4.15) by (2*n*+1). This leads to:

$$2(2n+1)x\,P_n'(x)+(2n+1)P_n(x) = (2n+1)P_{n+1}'(x)+(2n+1)P_{n-1}'(x)$$

Substituting for $2(2n+1)x\,P_n'(x)$ in this relation, we obtain:

$$2(n+1)P_{n+1}'(x)+2n\,P_{n-1}'(x)-2(2n+1)P_n(x)$$

$$+\,2(2n+1)P_n(x) = (2n+1)P_{n+1}'(x)+(2n+1)P_{n-1}'(x)$$

On simplification, we obtain:

$$P_{n+1}'(x)-P_{n-1}'(x) = (2n+1)P_n(x) \qquad (4.16)$$

Eqs. (4.14) and (4.16) are the prime recurrence relations for the Legendre polynomials. Using these recurrence relations and Legendre's differential equation, you can obtain the following relations:

$$P'_{n+1}(x) = (n+1)P_n(x) + x\,P'_n(x) \quad (4.17a)$$

$$P'_{n-1}(x) = x\,P'_n(x) - n\,P_n(x) \quad (4.17b)$$

$$(1-x^2)P'_n(x) = n\,P_{n-1}(x) - nx\,P_n(x) \quad (4.17c)$$

and

$$(1-x^2)P'_n(x) = (n+1)\,x\,P_n(x) - (n+1)P_{n+1}(x) \quad (4.17d)$$

You should note that recurrence relations are identities in x and simplify proofs and derivations.

Two functions $A(x)$ and $B(x)$ are said to be orthogonal on the interval (a, b) if they satisfy the relation

$$\int_a^b A(x)B(x)\,dx = 0$$

From your UG courses (refer to Block 2 of PHE-05), you may recall that sine and cosine functions are orthogonal in the interval $(-1,1)$. A function whose norm is unity, i.e.,

$\int f^2\,dx = 1$ is said to be **normalised**. A system of normalised functions which are orthogonal is said to be **orthonormal**.

SAQ 4

Prove Eqs. (4.17a to d).

4.2.3 Orthogonality Relations

One of the important characteristics of Legendre polynomials is that they are orthogonal. This property enables us to express a given function defined on the interval (–1,1) in a series of Legendre polynomials. It is therefore important for you to master its applications. The mth and the nth order Legendre polynomials $P_m(x)$ and $P_n(x)$, respectively, satisfy the equations:

$$(1-x^2)P''_m - 2x\,P'_m + m(m+1)P_m(x) = 0 \quad (4.18a)$$

and

$$(1-x^2)P''_n - 2x\,P'_n + n(n+1)P_n(x) = 0 \quad (4.18b)$$

We multiply the first equation by $P_n(x)$ and the second equation by $P_m(x)$. Next we subtract the latter product from the former. This leads to the relation:

$$(1-x^2)(P_nP''_m - P_mP''_n) - 2x(P_nP'_m - P_mP'_n)$$
$$= [n(n+1) - m(m+1)]P_mP_n$$

You can verify quite easily that the left hand side is equal to $\frac{d}{dx}\{(1-x^2)(P_nP'_m - P_mP'_n)$ so that we can write

$$\frac{d}{dx}\left\{(1-x^2)(P_nP'_m - P_mP'_n)\right\} = [n(n+1) - m(m+1)]P_mP_n$$

On integrating both sides over x from x = –1 to x = +1, we obtain:

$$[n(n+1) - m(m+1)]\int_{-1}^{+1} P_m(x)P_n(x)\,dx$$

$$= (1-x^2)[P_nP'_m - P_mP'_n]\Big|_{-1}^{+1}$$

You should note that $(1-x^2)$ vanishes for both the limits $(x = \pm 1)$ implying that the right hand side will be zero always. Further, when $m \neq n$, for the above relation to hold we must have:

$$\int_1^{+1} P_m(x)P_n(x)\,dx = 0 \quad (4.19)$$

which means that the scalar product of Legendre polynomials of different orders (in the range $-1 \le x \le 1$) is zero. Eq. (4.19) constitutes the **orthogonality relation for Legendre polynomials.** Let us now evaluate the integral $\int_{-1}^{+1} P_m(x) P_n(x)\, dx$ for $m = n$. In other words, we have to evaluate the integral $\int_{-1}^{+1} [P_n(x)]^2\, dx$.

From the generating relation we recall that $\dfrac{1}{\sqrt{1-2t\,x+t^2}} = \sum_{m=0}^{\infty} P_n(x)\, t^n$

We can also write: $\dfrac{1}{\sqrt{1-2sx+s^2}} = \sum_{m=0}^{\infty} P_m(x)\, s^m$

Multiplying these equations and integrating with respect to x, between $x = -1$ and $x = +1$, we get

$$\int_{-1}^{+1} \frac{dx}{\left(\sqrt{1-2t\,x+t^2}\right) \times \left(\sqrt{1-2sx+s^2}\right)}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} P_m(x) P_n(x)\, dx \right\} s^m t^n$$

From Eq. (4.19) we understand that the RHS survives only for terms for which $m = n$. We also observe that when we consider terms for which $m = n$, the notations t and s become identical and we can write:

$$\int_{-1}^{+1} \frac{dx}{1-2t\,x+t^2} = \sum_{-1}^{+1} \left\{ \int_{-1}^{+1} [P_n(x)]^2\, dx \right\} t^{2n}$$

To evaluate the integral on the LHS, we put $1-2t\,x+t^2 = u$ so that $-2t\,dx = du$ or $dx = -\dfrac{du}{2t}$. When $x = -1$, the limit of integration changes to $u = 1+2t+t^2 = (1+t)^2$. Similarly, when $x = +1$, we have $u = 1-2t+t^2 = (1-t)^2$. Hence,

$$1 = \int_{(1+t)^2}^{(1-t)^2} -\frac{du}{2t\,u} = \frac{1}{2t} \int_{(1-t)^2}^{(1+t)^2} \frac{du}{u} = \frac{1}{2t} \ln|u| \Big|_{(1-t)^2}^{(1+t)^2}$$

$$= \frac{2}{2t} \ln\left(\frac{1+t}{1-t}\right) = \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)$$

We now recall the series expansion of a logarithmic function:

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots$$

and

$$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \ldots$$

so that

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \frac{1}{t}\{\ln(1+t) - \ln(1-t)\}$$

$$= \frac{1}{t}\left(2t + \frac{2t^3}{3} + \frac{2t^5}{5} + \ldots\right)$$

$$= 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \ldots\right) = 2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}$$

Thus, $$\sum_{n=0}^{\infty}\left\{\int_{-1}^{+1}[P_n(x)]^2\, dx\right\}t^{2n} = \sum_{n=0}^{\infty}\left(\frac{2}{2n+1}\right)t^{2n}$$

On equating the coefficient of t^{2n}, we get

$$\int_{-1}^{+1}[P_n(x)]^2\, dx = \frac{2}{2n+1} \tag{4.20}$$

We can combine Eqs. (4.19) and (4.20) to write

$$\int_{-1}^{+1}P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{mn} \tag{4.21}$$

where δ_{mn} is the *Kronechers's delta*, defined as

$$\left.\begin{aligned}\delta_{mn} &= 0, \text{when } m \neq n\\ &= 1, \text{when } m = n\end{aligned}\right\}$$

Eq. (4.21) tells us that the scalar product of a Legendre polynomial of a particular order with that of another order is zero, whereas it is non-zero for the product of a Legendre polynomial with itself (in the range $-1 \le x \le 1$). This suggests that Legendre polynomials of different orders are orthogonal.

Completeness of Legendre polynomials

In vector analysis, we define a set of orthogonal basis vectors as complete if there is no other vector orthogonal to them all in the number of dimensions under consideration. By analogy, we define a set of orthogonal functions as complete if there is no other function orthogonal to all of them. From UG courses, you know how to use an infinite series of sine and cosine terms to express a function, say temperature distribution, in a Fourier series on $(-\pi, \pi)$ (see Unit 7, Block 2 of PHE-05).

Now you will learn how to expand a function in a series of Legendre polynomials, which form a complete orthogonal set on $(-1, 1)$. The completeness means that any well-behaved function *f*(*x*) can be approximated to any desired accuracy by a series of $P_k(x)$ through the relation:

$$f(x) = \sum_{k=0}^{\infty}A_k P_k(x) \quad -1 \le x \le 1 \tag{4.22}$$

To obtain the coefficient A_K, we multiply both sides by $P_m(x)$ and integrate the resultant expression in the range -1 to $+1$:

$$\int_{-1}^{+1} P_m(x) f(x)\, dx = \sum_{k=0}^{\infty} A_k \int_{-1}^{+1} P_m(x) P_k(x)\, dx$$

Using orthogonality relation [Eq. (4.21)], we can write:

$$\int_{-1}^{+1} P_m(x) f(x)\, dx = \sum_{k=0}^{\infty} A_k \left(\frac{2}{2m+1} \delta_{km} \right)$$

$$= \frac{2}{2m+1} \sum_{k=0}^{\infty} A_k\, \delta_{km} = \frac{2A_m}{2m+1}$$

since $\delta_{km} = 0$ except for $k = m$ and $\delta_{mn} = 1$.

Hence, $$A_m = \frac{2m+1}{2} \int_{1}^{+1} P_m(x) f(x)\, dx$$

or $$A_k = \frac{2k+1}{2} \int_{1}^{+1} P_k(x) f(x)\, dx \qquad (4.23)$$

We shall now illustrate this with the help of an example.

Example 4.3

Expand the function $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases}$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)\, dx$.

Solution : From Eq. (4.23), we have

$$A_k = \frac{2k+1}{2} \int_{1}^{+1} P_k(x) f(x)\, dx$$

$$= \frac{2k+1}{2} \left[\int_{-1}^{0} P_k(x) f(x)\, dx + \int_{0}^{1} P_k(x) f(x)\, dx \right]$$

We are given that

$$f(x) = \begin{cases} 0 \;\; for \;\; -1 < x < 0 \\ 1 \;\; for \;\; 0 < x < 1 \end{cases}$$

On inserting these values in the expression for A_k, we get

$$A_k = \frac{2k+1}{2} \int_{0}^{1} P_k(x)\, dx$$

Now, refer to Example 4.1 and SAQ 2. We recall that $P_0(x) = 1, P_1(x) = x$,

$P_2(x) = \frac{(3x^2 - 1)}{2}$, $P_3(x) = \frac{5x^3 - 3x}{2}$, and so on. Therefore,

$$A_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

$$A_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{4}$$

$$A_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \frac{5}{4}\int_0^1 (3x^2 - 1)\,dx = 0$$

and

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\;dx = \frac{7}{4}\int_0^1 (5x^3 - 3x)\,dx = -\frac{7}{16}$$

Proceeding in this way, you will find that $A_4 = 0, A_5 = \frac{11}{32}$, and so on. Thus, we can write:

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + ...$$

You may now like to solve SAQ 5.

SAQ 5

a) In Example 4.3 you must have observed that $A_k = 0$ for even $k \neq 0$. Re-establish this result using the recurrence relation given by Eq. (4.16).

b) Expand $f(x) = x^2$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

You have learnt how to obtain Legendre polynomials from the generating function. There is another simple way of arriving at the Legendre polynomials. This is through Rodrigues' Formula, which we now discuss.

4.2.4 Rodrigues's Formula

Let us consider the function

$$v = (x^2 - 1)^n$$

and differentiate it with respect to *x*. This gives

$$\frac{dv}{dx} = n(x^2 - 1)^{n-1} \times 2x = 2nx(x^2 - 1)^{n-1}$$

so that

$$(x^2 - 1)\frac{dv}{dx} = 2nx(x^2 - 1)^n = 2nxv$$

That is, v satisfies the differential equation

$$(1-x^2)\frac{dv}{dx}+2nx\,v=0$$

Differentiating it again with respect to x, we get:

$$(1-x^2)\frac{d^2v}{dx^2}-2x\frac{dv}{dx}+2nx\frac{dv}{dx}+2nv=0$$

or

$$(1-x^2)\frac{d^2v}{dx^2}+2(n-1)x\frac{dv}{dx}+2nv=0 \qquad (4.24)$$

If you differentiate it again with respect to x, you can write the resultant expression as:

$$(1-x^2)\frac{d^{2+1}v}{dx^{2+1}}-2x\frac{d^{1+1}v}{dx^{1+1}}+2(n-1)\frac{dv}{dx}$$

$$+2(n-1)x\frac{d^{1+1}v}{dx^{1+1}}+2n\frac{dv}{dx}=0.$$

This can be re-arranged and written in a compact form as:

$$(1-x^2)\frac{d^{2+1}v}{dx^{2+1}}+2x(n-1-1)\frac{d^{1+1}v}{dx^{1+1}}+(1+1)(2n-1)\frac{dv}{dx}=0$$

After r differentiations, you will get

$$(1-x^2)\frac{d^{2+r}}{dx^{2+r}}+2x(n-r-1)\frac{d^{1+r}v}{dx^{1+r}}+(r+1)(2n-r)\frac{d^rv}{dx^r}=0$$

When $r=n,$ we get the nth derivative of Eq. (4.24):

$$(1-x^2)\frac{d^{n+2}v}{dx^{n+2}}-2x\frac{d^{n+1}}{dx^{n+1}}+n(n+1)\frac{d^nv}{dx^n}=0$$

This equation can be rewritten as:

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^nv}{dx^n}\right)-2x\frac{d}{dx}\left(\frac{d^nv}{dx^n}\right)+n(n+1)\frac{d^nv}{dx^n}=0$$

On comparing it with Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

you will note that d^nv/dx^n satisfies the Legendre's equation. So can write

$$P_n(x)=C\frac{d^nv}{dx^n}$$

That is, $P_n(x)$ is a constant multiplier times $\dfrac{d^nv}{dx^n}$. On substituting for v, we obtain

$$P_n(x)=C\frac{d^n}{dx^n}(x^2-1)^n$$

where C is a constant. To determine this constant, we have to consider terms with the highest power of x on both sides. From Eq. (4.8) we recall that the term with the highest power of x it the expression for $P_n(x)$ is $\dfrac{(2n)!}{2^n(n!)^2}x^n$.

Hence,

$$\frac{(2n)!}{2^n(n!)^2}x^n = C\frac{d^n}{dx^n}x^{2n} = C.2n(2n-1)(2n-2)\ldots[2n(n-1)]x^n$$

$$= C\frac{(2n)!}{n!}x^n$$

On comparing the coefficients of x^n on both sides, we get:

$$C = \frac{1}{(2^n)n!}$$

Hence, we can write:

$$\boxed{P_n(x) = \frac{1}{(2^n)n!}\frac{d^n}{dx^n}(x^2-1)^n} \qquad (4.25)$$

This is known as the **Rodrigues' formula** for $P_n(x)$. We now consider an application of this formula.

Example 4.5

Obtain the value of $P_3(x)$ using Rodrigues's formula.

Solution

From Eq. (4.25), we can write:

$$P_3(x) = \frac{1}{(2^3)3!}\frac{d^3}{dx^3}(x^3-1)^3 = \left(\frac{1}{48}\right)\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$$

Differentiating the expression in the small brackets with respect to x, you will get:

$$\frac{d}{dx}(x^6-3x^4+3x^2-1) = 6x^5-12x^3+6x$$

We differentiate again the resultant expression with respect to x and get:

$$\frac{d^2}{dx^2}(x^6-3x^4+3x^2-1) = \frac{d}{dx}(6x^5-12x^3+6x)$$

$$= 30x^4-36x^2+6$$

and

$$\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1) = \frac{d}{dx}(30x^4-36x^2+6)$$

$$= 120x^3-72x$$

$$\therefore \qquad P_3(x) = \left(\frac{1}{48}\right).24(5x^3-3x) = \frac{1}{2}(5x^3-3x)$$

You may now like to solve SAQ 6.

SAQ 6

Obtain $P_4(x)$ using Eq. (4.25).

We have discussed the basic operations involving Legendre polynomials. We now intend to discuss their applications in physics. The most instructive applications arise while solving Laplace's equation in spherical polar coordinates for potential and temperature related problems. You should go through the following examples carefully as you can learn a lot of good physics.

Example 4.6

A conducting sphere is placed in a uniform electric field of strength E_0, as shown in Fig. 4.2. Calculate the electrostatic potential at a point outside the sphere.

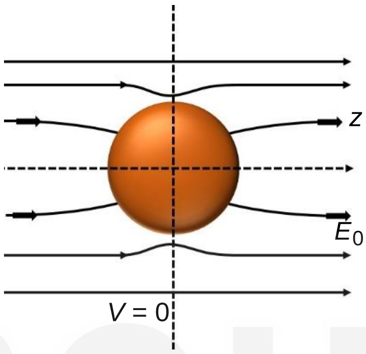

Fig. 4.2: **A conducting sphere placed in a uniform electric field.**

Solution : We note that due to spherical symmetry, the potential function will be independent of the azimuthal angle ϕ. From Unit 1, recall that Laplace's equation can be split into the following differential equations:

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0 \qquad \text{(i)}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \qquad \text{(ii)}$$

where $l(l+1)$ is the separation constant. You have seen that Eq. (i) of these equations is the radial part of Laplace's equation and has solutions of the form:

$$R_t(r) = A_l r^l + \frac{B_l}{r_{l+1}}$$

Note that Eq. (ii) is Legendre's equation, which has Legendre polynomials as solutions. Hence, the solution of Laplace's equation with azimuthal symmetry (no ϕ dependence) is given by:

$$V(r,\theta) = \sum_{t=0}^{\infty}\left(A_l r^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta) \qquad \text{(iii)}$$

To determine the constants A_n and B_n, we have to impose boundary conditions. Since the original uniform electric field (before the sphere is placed in electric field) is E_0, we must have

In this case, only the potential outside the sphere is relevant as potential inside a conducting sphere is zero.

$$V(r \to \infty) = -E_0 z = -E_0 r\cos\theta \quad (\because z = r\cos\theta)$$

or

$$V = -E_0 r\, P_1(\cos\theta) \quad (\because P_1(x) = x) \qquad \text{(iv)}$$

As $r \to \infty$, the second term in the bracket on the right side of Eq. (iii) will disappear. Thus, on comparing Eqs. (iii) and (iv), we get:

$$A_0 = 0, A_n = 0 \quad \text{for all } n > 1$$

and

$$A_1 = -E_0 \qquad \text{(v)}$$

For the second boundary condition, we choose the surface of the sphere to be at zero potential. Thus, from Eq. (iii) we have:

$$V(r = a) = \left(A_0 + \frac{B_0}{a}\right) + \left(\frac{B_1}{a^2} - E_0 a\right) P_1(\cos\theta) + \sum_{l=1}^{\infty} B_l \frac{P_l(\cos\theta)}{a^{l+1}} = 0$$

If this equality is to hold for all values of θ, each coefficient of $P_l(\cos\theta)$ must vanish. Hence, we have:

$$A_0 = B_0 = 0, \quad B_n = 0 \quad \text{for } n \geq 2$$

and $$B_1 = E_0\, a^3$$

Inserting these results in the above expression, we find that the electrostatic potential is given by:

$$V = -E_0\, r\, P_1(\cos\theta) + \frac{E_0 a^3}{r^2} P_1(\cos\theta)$$
$$= -E_0\, r \cos\theta \left(1 - \frac{a^3}{r^3}\right)$$

You should now solve SAQ 7 for practice.

SAQ 7

For a sphere of radius a such that $V(r,\theta)\big|_{r=a} = V_0 \cos^3\theta$, obtain the potential at a point inside the sphere. Assume that there are no charges at the origin.

We now discuss the spherical harmonics, which forms the basis of many important concepts in quantum mechanics, especially, those related to the structure of atom.

4.3 SPHERICAL HARMONICS

In Unit 1, you have encountered Helmholtz equation in Cartesian coordinates and separated it into three ODEs:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2\, f(x, y, z) = 0 \qquad \text{(4.26a)}$$

Let us write this equation in spherical polar coordinates:

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\right] = -k^2 f$$

(4.26b)

You can separate Eq. (4.26b) into three ODEs by putting $f(r, \theta, \phi)$ and then divide by $R(r)\,\Theta(\theta)\,\Phi(\phi)$. So, you will get:

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

$$+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=-k^2 \qquad (4.27a)$$

Multiplying the above equation by $r^2\sin^2\theta$, we can write:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=r^2\sin^2\theta\left[-k^2-\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right.$$

$$\left.+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] \qquad (4.27b)$$

The LHS of Eq. (4.27b) is a function of ϕ alone and the RHS, a function of r and θ. Since r, θ and ϕ are independent variables, we put the LHS and RHS of Eq. (4.27b) equal to a constant:

$$\frac{d^2\Phi}{d\phi^2}=-m^2\Phi \quad \text{or} \quad \frac{d^2\Phi}{d\phi^2}+m^2\Phi=0 \qquad (4.28a)$$

The normalized solution of Eq. (4.28a) is:

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \qquad (4.28b)$$

The function $\Phi_m(\phi)$ is orthonormal with respect to the variable ϕ. Note that we have used a negative constant because in problems in physics, ϕ appears mostly as the azimuth angle, which suggests periodic solutions rather than exponential ones. Substituting Eq. (4.28a) in Eq. (4.27b), we can write:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+r^2k^2=-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{m^2}{\sin^2\theta} \qquad (4.29a)$$

You can see that in Eq. (4.29a), the variables are separated. Again, we equate each side of Eq. (4.29a) to a constant, say C. Then we have:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2\Theta}{\sin^2\theta}+C\Theta=0, \qquad (4.29b)$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2R-\frac{CR}{r^2}=0 \qquad (4.29c)$$

Note that Eq. (4.29b) is the same as Eq. (1.28c) and now we will solve it. For simplicity of calculations, we put $C=l(l+1)$ as we did in Unit 1 while obtaining the solution given by Eq. (1.29b). Let us recast Eq. (4.29b)as follows:

$$\sin^2\theta\frac{d^2\Theta}{d\theta^2}+\sin\theta\cos\theta\frac{d\Theta}{d\theta}-\left(m^2-l(l+1)\sin^2\theta\right)\Theta=0 \qquad (4.29d)$$

This equation is called the **Legendre's associated differential equation**. If we set $m = 0$, i.e., there is no ϕ dependence, then Eq. (4.29d) reduces to Legendre's differential equation. We can transform Eq. (4.29d) by introducing a change of variables:

$$x = \cos\theta$$

so that $\sin^2\theta = 1 - x^2$ and by the chain rule:

$$\frac{d}{d\theta} = \frac{d}{dx}.\frac{dx}{d\theta} = -\sin\theta\frac{d}{dx}$$

$$\therefore \qquad \sin\theta\cos\theta\frac{d\Theta}{d\theta} = -\sin^2\theta\, x\frac{d\Theta}{d\theta} = -x\left(1 - x^2\right)\frac{d\Theta}{dx}$$

Similarly, we can write:

$$\frac{d^2\Theta}{d\theta^2} = \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) = -\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{dx}\right)$$

$$= -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d^2\Theta}{dx^2}\frac{dx}{d\theta}$$

so that $$\frac{d^2\Theta}{d\theta^2} = -\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2} = \left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - x\frac{d\Theta}{dx}$$

On inserting these results in Eq. (4.29d), we get

$$\left(1 - x^2\right)\left\{\left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - x\frac{d\Theta}{dx}\right\} - x\left(1 - x^2\right)\frac{d\Theta}{dx}$$
$$-\left[m^2 - l(l+1)\left(1 + x^2\right)\right]\Theta = 0$$

Dividing throughout by $(1 - x^2)$, we get Legendre's associated differential equation in *x*:

$$\left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left(l(l+1) - \frac{m^2}{1 - x^2}\right)\Theta = 0 \qquad (4.30)$$

We would like to point out here that solutions of Legendre's associated differential equation are labelled by both parameters *l* and *m* and written as $P_l^m(x)$.

For non-negative integral values of *m* and *l*, the general solution of Eq. (4.30) is given by:

$$\Theta = C_1 P_l^m(x) + C_2 Q_l^m(x)$$

$$= C_1 P_l^m(\cos\theta) + C_2 Q_l^m(\cos\theta) \qquad (4.31)$$

Note that $P_l^m(x)$ is finite for $-1 \le x \le 1$(which conforms with $x = \cos\theta$) and $Q_l^m(x)$ is unbounded for $x = \pm 1$. For this reason, we consider only $P_l^m(\cos\theta)$ as an acceptable solution and that too with integral values of *m* and *l*.

The associated Legendre polynomials are connected with Legendre polynomials through the relation:

$$P_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \qquad (4.32)$$

where $P_l(x)$ is the lth Legendre polynomial. You must note that if $m > l$, $P_l^m(x) = 0$.

Orthogonality

Just like Legendre polynomials, the associated Legendre polynomials $P_l^m(x)$ are also orthogonal in the range $-1 < x < 1$. Mathematically, we write

$$\int_{-1}^{1} P_l^m(x) P_k^m(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{kl} \qquad (4.33)$$

As before, we can expand $f(x)$ in a series of the form:

$$f(x) = \sum_{k=0}^{\infty} A_k P_k^m(z) \qquad (4.34a)$$

where

$$A_k = \frac{(2k+1)(k-m)!}{2(k+m)!} \int_{-1}^{1} f(x) P_k^m(x)\, dx \qquad (4.34b)$$

From Eq. (4.33), we can write the normalized associated Legendre polynomials as:

$$\wp_n^m(\cos\theta) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) \quad -n \le m \le n \qquad (4.34c)$$

We are now ready to define spherical harmonics. Note that the function $\Phi_m(\phi)$ is orthonormal with respect to the variable ϕ (the azimuthal angle) and the function $\wp_n^m(\cos\theta)$ is orthonormal with respect to the variable θ (the polar angle). We define **spherical harmonics** as the product of these two functions:

$$Y_n^m(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi} \qquad (4.35a)$$

Thus, spherical harmonics $Y_n^m(\theta, \phi)$ are functions of the two angles θ and ϕ, which are orthonormal over the spherical surface. Sometimes, a phase factor $(-1)^m$ known as the Condon Shortley phase is used in the expression for spherical harmonics, which then becomes:

$$Y_n^m(\theta, \phi) = (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi} \qquad (4.35b)$$

Eq. (4.35b) is useful in quantum theory of angular momentum. From Eq. (4.35b), the first few spherical harmonics are:

$$Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \qquad (4.36a)$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{4.36b}$$

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \tag{4.36c}$$

$$Y_1^{-1}(\theta,\phi) = +\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi} \tag{4.36d}$$

and so on.

The orthogonality relation for spherical harmonics is given as:

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{n_1}^{m_1 *}(\theta,\phi)\, Y_{n_2}^{m_2}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{n_1,n_2}\delta_{m_1,m_2} \tag{4.37}$$

You may like to know why this nomenclature is used for these functions. Firstly, these functions are defined over the surface of a sphere with θ, the polar angle, and ϕ, the azimuth. The term “harmonic” is included in the nomenclature because solutions of Laplace’s equations were called harmonic functions and the spherical harmonics represent the angular part of such solutions. You may like to solve an SAQ now.

SAQ 8

Write the expressions for the spherical harmonics $Y_2^0(\theta,\phi)$, $Y_2^1(\theta,\phi)$ and $Y_2^2(\theta,\phi)$.

In the last section of this unit, we discuss the hypergeometric functions.

4.4 HYPERGEOMETRIC FUNCTIONS

Hypergeometric function is a special function represented by the hypergeometric series, which is a generalization of a geometric series. The hypergeometric series has the form of a power series in which the coefficients are replaced by ratios of rational functions of constants. Hypergeometric function is the solution of the following differential equation:

$$x(1-x)\,y''(x) + [c-(a+b+1)x]\,y'(x) - ab\,y(x) = 0 \tag{4.38}$$

where a, b and c are constants. Eq. (4.38) is also called Gauss’s hypergeometric equation. You can verify that Eq. (4.38) has regular singularities at $x = 0$ and $x = 1$. For solving Eq. (4.38), we cast it in the standard form of a second order linear ODE:

$$y'' + p(x)y' + q(x)y = 0 \tag{4.39a}$$

where $$p(x) = \frac{[c-(a+b+1)x]}{x(1-x)} \quad\text{and}\quad q(x) = -\frac{ab}{x(1-x)} \tag{4.39b}$$

Let us determine the solutions of Eq. (4.39a) for the singularities using the Frobenius method (refer to Sec. 3.4, Unit 3 of PHE-05, IGNOU B. Sc. Course). To apply this method, we write Eq. (4.39a) as:

$$x^2y'' + x^2p(x)y' + x^2q(x)y = 0$$

or $$x^2 y'' + xb(x)y' + d(x)y = 0 \qquad (4.39c)$$

where

$$b(x) = xp(x) = \frac{[c - (a + b + 1)x]}{(1 - x)}$$

and $$d(x) = x^2 q(x) = -\frac{abx}{(1 - x)} \qquad (4.40)$$

We expand the functions $xp(x)$ and $x^2 q(x)$ in powers of x as follows:

$$b(x) = xp(x) = \frac{c - (a + b + 1)x}{(1 - x)}$$

$$= [c - (a + b + 1)x](1 + x + x^2 + ...) = \sum_{n=0}^{\infty} b_n x^n \qquad (4.41a)$$

and $$d(x) = x^2 q(x) = -\frac{abx}{(1 - x)}$$

$$= -abx(1 + x + x^2 + ...) = \sum_{n=0}^{\infty} d_n x^n \qquad (4.41b)$$

Let us now solve Eq. (4.39a) for both regular singular points.

Solution for the regular singularity at $x = 0$

We expand y in the following series about $x = 0$, and take its first and second order derivatives with respect to x:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+k}$$

$$y'(x) = \sum_{m=0}^{\infty} a_m (m + k) x^{m+k-1}$$

$$y''(x) = \sum_{m=0}^{\infty} a_m (m + k)(m + k - 1) x^{m+k-2}$$

We substitute $y(x)$ and its derivatives in Eq. (4.39c) and write:

$$y(x) = x^k \sum_{m=0}^{\infty} (m + k)(m + k - 1) a_m x^m$$

$$+ x^k \left[\sum_{m=0}^{\infty} (m + k) a_m x^m\right]\left[\sum_{n=0}^{\infty} b_n x^n\right]$$

$$+ x^k \left[\sum_{m=0}^{\infty} a_m x^m\right]\left[\sum_{n=0}^{\infty} d_n x^n\right] = 0 \qquad (4.42)$$

We now determine the indicial equation by equating the coefficient of x^k in Eq. (4.42) to zero. It is the lowest power of x in the equation obtained by putting $m = 0$ in it. Thus, Coefficient of x^k is: $k(k-1)+kb_0+d_0=0$

From Eq. (4.41a), $b_0=c$ and $d_0=0$

Hence, the indicial equation is:

$$k(k-1)+kc=0 \Rightarrow k=0 \quad \text{and} \quad k=1-c$$

$k=0$ corresponds to a Frobenius series solution if $1-c<0$ or the difference $(1-c)-0=1-c$ is not a positive integer. Note that the second condition implies the first one, so if $1-c$ is not a positive integer, then c is neither zero nor negative integer. Then $k=0$ corresponds to a Frobenius series solution of the form

$$y=x^0\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+... \tag{4.43}$$

where $a_0\neq 0.$ Substituting Eq. (4.43) into Eq. (4.38) and equating the coefficients of x^n to zero, we get:

$$a_1=\frac{ab}{c}a_0;\ a_2=\frac{(a+1)(b+1)}{2(c+1)}a_1=\frac{a(a+1)b(b+1)}{2c(c+1)}a_0$$

$$a_{n+1}=\frac{(a+n)(b+n)}{(n+1)(c+n)}a_n \tag{4.44}$$

With these coefficients and by letting $a_0=1$, the solution becomes:

$$\begin{aligned} y &= 1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{2c(c+1)}x^2+... \\ &= 1+\sum_{n=1}^{\infty}\frac{a(a+1)...(a+n-1)b(b-1)...(b+n-1)}{n!\,c(c+1)...(c+n-1)}x^n \end{aligned} \tag{4.45}$$

This is known as the **hypergeometric series**, and is denoted by the symbol $F(a,b,c,x)$. It is called by this name because it generalises the familiar geometric series as follows:

When $a=1$ and $c=b$ we obtain:

$$F(1,b,b,x)=1+x+x^2+...=\frac{1}{1-x} \tag{4.46}$$

If either a or b is either zero or negative integer, the series (4.46) breaks off and is a polynomial; otherwise the ratio tests shows that it converges for $|x|<1$, since Eq. (4.45) gives:

$$\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|=\left|\frac{(a+n)(b+n)}{(n+1)(c+n)}\right||x|\to|x| \quad \text{as } n\to\infty$$

When c is neither zero nor negative integer, $F(a,b,c,x)$ is an analytic function called **hypergeometric function** on the interval $|x|<1$.

It is the simplest particular solution of the hypergeometric equation and it remains unchanged when a and b are interchanged: $F(a,b,c,x)=F(b,a,c,x)$

Next we consider the solution corresponding to $k=1-c$.

If $1-c$ is neither zero nor negative integer, i.e., c is not a positive integer, then there is a second independent solution of Eq. (4.38) near $x=0$ for $k=1-c$. We can find this solution by substituting

$$y = x^{1-c}(a_0 + a_1 x + a_2 x^2 + ...)$$

in Eq. (4.38) and calculating the coefficients. The other way of finding the solution is to change the dependent variable in Eq. (4.38) from y to z by writing $y = x^{1-c} z$.

After mathematical manipulation, Eq. (4.38) becomes:

$$x(1-x)z'' + [(2-c) - ([a-c+1] + [b-c+1] + 1)x]z'$$
$$-(a-c+1)(b-c+1)z = 0 \qquad (4.46)$$

which is the hypergeometric equation with the constants a, b and c replaced with $a-c+1$, $b-c+1$ and $2-c$.

We already know that Eq. (4.46) has the solution:

$$z = F(a-c+1, b-c+1, 2-c, x)$$

near the origin, so our desired second solution is:

$$y = x^{1-c} z = x^{1-c} F(a-c+1, b-c+1, 2-c, x)$$

So, when c is not an integer, we have two independent Frobenius series solutions and hence,

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x) \qquad (4.47)$$

is the general solution of the hypergeometric equation (4.38) near the singular point $x = 0$. Note that the above solution is only valid near the origin. We now solve Eq. (4.38) near the other singular point $x = 1$.

Solution for the regular singularity at $x = 1$

The simplest way in which we can obtain this solution from the one we have already determined, is by introducing a new independent variable $t = 1-x$ Hence, $x = 1-t$, $dy/dx = -dy/dt$ and $d^2y/dx^2 = d^2y/dt^2$.

Then $x=1$ corresponds to $t=0$ and transforms Eq. (4.38) into:

$$t(1-t)y'' + [(a+b-c+1) - (a+b+1)t]y' - aby = 0$$

where the primes denotes the derivatives with respect to t. Since the above equation is a hypergeometric equation, its general solution near $t=0$ can be written down at once from Eq. (4.47) by replacing x by t and c by $a+b-c+1$ and then we replace t by $1-x$ to get the general solution of Eq. (4.38) near $x=1$.

$$y = c_1 F(a, b, a+b-c+1, 1-x)$$
$$+ c_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b-b+1, 1-x) \qquad (4.48)$$

In this case it is necessary to assume that $c - a - b$ is not an integer.

Eqs. (4.47 and 4.48) show that the adaptability of the constants in Eq. (4.38) makes it possible to express the general solution of this equation near each of its singular points in terms of the single function F.

Any differential equation in which the coefficients of y'', y' and y are polynomials of degree 2, 1 and 0, respectively, and also the first of these polynomials has distinct real roots, can be brought into the hypergeometric form by a linear change of the independent variable (TQ 5). Thus, such ODEs can be solved near their singular points in terms of the hypergeometric function.

Let us now summarise the unit.

4.5 SUMMARY

In this unit, we have covered the following concepts:

- Legendre equation and its solutions by the Frobenius method.
- Legendre polynomials, generating function, recurrence relations and Rodrigues's formula.
- Spherical harmonics.
- Hypergeometric equation, hypergeometric functions.

4.6 TERMINAL QUESTIONS

1. Calculate the values of $P_{2n}(0)$ and $P_{2n+1}(0)$.
2. Evaluate the integrals

   a) $\int_{-1}^{1} x\, P_{n-1}(x)\, P_n(x) dx$;

   b) $\int_{-1}^{1} xP_n(x)\, dx$

3. Determine the steady-state temperature inside a sphere of radius a given that its upper hemisphere is maintained at a temperature $T = T_0$ and the lower hemisphere is maintained at temperature $T = T_0$.
4. Prove Eq. (4.37).
5. With a suitable change of variable, obtain the solution of the following ODE near $x = 0$:

$$(1-e^x)\, y'' + \frac{y'}{2} + e^x y = 0$$

   Hint: Change the independent variable, $e^x = t$.

4.7 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We write Eq. (4.1) as follows and use power series method to solve it:

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{n(n+1)}{(1-x^2)} y = 0 \qquad (4.1)$$

Hence, $p(x) = -\dfrac{2x}{(1-x^2)}$ and $q(x) = \dfrac{n(n+1)}{(1-x^2)}$.

The functions $p(x)$ and $q(x)$ have regular singularities at the points $x = 1$ and $x = -1$. We use power series method to obtain the solution of this equation. Substituting

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$y'(x) = \sum_{m=1}^{\infty} a_m m x^{m-1}$$

$$y''(x) = \sum_{m=2}^{\infty} a_m m\,(m-1) x^{m-2}$$

in Eq. (4.1), we get:

$$(1-x^2)\sum_{m=2}^{\infty} m\,(m-1) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

Collecting the coefficient of the m^{th} power of x in the above equation, we can write:

$$(m+2)\,(m+1) a_{m+2} - m\,(m-1) a_m - 2m a_m + n(n+1) a_m = 0$$

or

$$a_{m+2} = \frac{m\,(m-1) + 2m - n(n+1)}{(m+2)\,(m+1)} a_m$$

$$= \frac{m\,(m+1) - n(n+1)}{(m+2)\,(m+1)} a_m$$

$$= -\frac{(n-m)(m+n+1)}{(m+2)\,(m+1)} a_m, \qquad m = 0, 1, 2, \ldots$$

From this recurrence relation for the coefficients a_n, we can write:

$$a_2 = -\frac{n(n+1)}{2!} a_0, \qquad a_3 = -\frac{(n-1)(n+2)}{3!} a_1,$$

$$a_4 = -\frac{(n-2)(n+3)}{12} a_2 = \frac{(n-2)\,n\,(n+1)(n+3)}{4!} a_0,$$

$$a_5 = -\frac{(n-3)(n+4)}{20} a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1,$$

and so on.

Substituting these values in the series for y, we get Eq. (4.2):

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{3!}x^5 - \ldots\right]$$

2. $$P_3(x) = \frac{6!}{2^3(3!)^2}\left(x^3 - \frac{3\times 2}{2\times 5}x\right)$$
$$= \frac{1\times 2\times 3\times 4\times 5\times}{8\times 6\times 6}\left(\frac{5x^3 - 3x}{5}\right) = \frac{1}{2}\left(5x^3 - 3x\right)$$

3. a) Putting $x = -1$ in Eq. (4.12), we get

$$\left(1 + 2t + t^2\right) = \sum_{n=0}^{\infty} P_n(-1)t^n$$

or $$(1+t)^{-1} = \sum_{n=0}^{\infty} P_n(-1)t^n$$

$$\therefore \quad 1 - t + t^2 - \ldots + (-1)^n t^n - \ldots = \sum_{n=0}^{\infty} P_n(-1)t^n$$

$$\therefore \quad P_n(-1) = (-1)^n$$

c) We observe that the generating function remains unchanged even if we replace x by $-x$ and t by $-t$. Thus,

$$g(t,x) = g(-t,-x) = \left[1 - 2(-t)(-x) + (-t)^2\right]^{-1/2}$$
$$= \sum_{n=0}^{\infty} P_n(-x)(-t)^n = \sum_{n=0}^{\infty} (-1)^n P_n(-x)t^n$$

so that $$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} (-1)^n P_n(-x)t^n$$

$$\therefore \quad P_n(x) = (-1)^n P_n(-x)$$

or $$P_n(x) = (-1)^n P_n(-x)$$

4. Upon adding Eqs. (4.15) and (4.16), we get

$$2P'_{n+1}(x) = 2(n+1)P_n(x) + 2x P'_n(x)$$

$$\therefore \quad P'_{n+1}(x) = (n+1)P_n(x) + x P'_n(x)$$

which is same as Eq. (4.17a).

On subtracting Eq. (4.16) from Eq. (4.15), we get

$$2P'_{n-1}(x) = -2n P_n(x) + 2x P'_n(x)$$

$$\therefore \quad P'_{n-1}(x) = xP'_n(x) - n P_n(x)$$

which is identical with Eq. (4.17b).

In Eq. (4.17a), we replace n by $(n-1)$. This gives

$$P'_n(x) = nPn - 1(x) + x P'_{n-1}(x)$$

Now, multiplying Eq. (4.17b) by x, we get

$$xP'_{n-1}(x) = x^2 P'_n(x) - nxP_x(x)$$

On comparing the above two equations, we get

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$

which is Eq. (4.17c).

Upon replacing n by $(n+1)$ in this expression, we get

$$(1-x^2)P'_{n+1}(x) = (n+1)P_n - (n+1)xP_{n+1}(x)$$

Using Eq. (4.17a), we get

$$(1-x^2)\,(n+1)P_n(x) + x(1-x^2)\,P'_n(x)$$

$$= (n+1)P_n(x) - (n+1)xP_{n+1}(x)$$

or $$-(n+1)x^2 P_n(x) + x(1-x^2)P'_n(x) = -(n+1)\,xP_{n+1}(x)$$

Since $x \neq 0$ (in general), we get

$$(1-x^2)P'_n(x) = (n+1)\,xP_n(x) - (n+1)P_{n+1}(x)$$

which is the result contained in Eq. (4.17d).

5. a) $$A_k = \frac{2k+1}{2}\int_0^1 P_k(x)\,dx$$

Using Eq. (4.16), we can write

$$A_k = \frac{1}{2}\int_0^1 [P'_{k+1}(x) - P'_{k-1}(x)]dx = \frac{1}{2}|P_{k+1}(x) - P_{k-1}(x)|_0^1$$

$$= \frac{1}{2}\{P_{k+1}(1) - P_{k-1}(1) + P_{k+1}(0) - P_{k-1}(0)\}$$

$$= \frac{1}{2}\{P_{k+1}(0) - P_{k-1}(0)\}$$

Hence, for even k (other than $k=0$), $A_k = 0$

b) $$f(x) = x^2 = \sum_{k=0}^{\infty} A_k P_k(x)$$

We have to find A_k, $k = 0, 1, 2, 3, \ldots$ such that

$$x^2 = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 P_3(x) + \ldots$$

$$= A_0(1) + A_1(x) + A_2\left(\frac{3x^2-1}{2}\right) + A_3\left(\frac{5x^3-3x}{2}\right) + \ldots$$

Since the left hand side of the above equation is a polynomial of degree 2, we must have $A_3 = 0$, $A_4 = 0$, $A_5 = 0$, and so on.

$$\therefore \quad x^2 = \left(A_0 - \frac{A_2}{2}\right) + A_1 x + \frac{3}{2}A_2 x^2$$

$$\Rightarrow \quad A_0 - \frac{A_2}{2} = 0,\ A_1 = 0,\ \text{and}\ \frac{3}{2}A_2 = 1$$

Thus, $A_0 = \frac{1}{3}, A_1 = 0$ and $A_2 = \frac{2}{3}$

$\therefore \quad x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$

Alternatively, you can calculate $A_k's$ using the relation

$$A_k = \frac{2k+1}{2}\int_{-1}^{+1} x^2 P_k(x)\,dx$$

6. $$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{16\times 24}\frac{d^4}{dx^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$$

$$= \frac{1}{16\times 24}(8\times 7\times 6\times 5x^4 - 4\times 6\times 5\times 4\times 3x^2 + 6\times 4\times 3\times 2\times 1)$$

$$= \frac{1}{8}(35x^2 - 30x^2 + 3)$$

7. We consider a sphere of radius a such that $V(r,\theta)\big|_{r=a} = V_0\cos^3\theta$ and assume that there are no charges at the origin. Since V must satisfy Laplace's equation and the boundary condition has no ϕ dependence, the solution will be obtained in terms of Legendre Polynomials. We write the general solution as:

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{t^{n+1}}\right)P_n(\cos\theta) \qquad \text{(i)}$$

From this form of the solution, we note that V can be finite at the origin only if $B_n = 0$ for all n. Then Eq. (i) reduces to:

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n\, r^n\, P_n(\cos\theta) \qquad \text{(ii)}$$

On applying the given boundary condition, we have

$$V(r,\theta) = V_0\cos^3\theta = \sum_{m=0}^{\infty} A_n\, a^n\, P_n(\cos\theta) \qquad \text{(iii)}$$

To solve for A_m we rewrite $\cos^3\theta$ in terms of Legendre polynomials. For this, we recall that $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$ and then we write

$$\cos^3\theta = \frac{2}{5}P_3\ (\cos\theta) + \frac{3}{5}\cos\theta$$

Since $\cos\theta = P_1(\cos\theta)$, we can write:

$$\cos^2\theta = \frac{2}{5}P_3\,(\cos\theta) + \frac{3}{5}P_1(\cos\theta)$$

Inserting this result in Eq. (iii) we obtain:

$$\frac{1}{5}[2V_0 P_3(\cos\theta) + 3V_0\, P_1(\cos\theta)] = \sum_{m=0}^{\infty} A_n\, a^n\, P_n(\cos\theta) \quad \text{(iv)}$$

Using the orthogonality property of Legendre polynomials, you can see that $A_1 = 3V_0/5a$ and $A_3 = 2V_0/5a^3$. Hence

$$V(r,\theta) = \frac{3}{5}V_0\,(r/a)P_1(\cos\theta) + \frac{2}{5}V_0(r/a)^3\,P_3(\cos\theta)$$

8. Using Eq. (4.35b), we get:

$$Y_n^m(\theta,\phi) = (-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\theta)e^{im\phi}$$

Hence, $$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta\, e^{-i\phi},$$

and $$Y_2^2(\theta,\phi) = \sqrt{\frac{5}{96\pi}}\,3\sin^2\theta\, e^{2i\phi}$$

Terminal Questions

1. Putting $x = 0$, in Eq. (4.12), we get

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n$$

Using binomial expansion, we can write

$$(1+t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \ldots$$

$$+(-1)^n\,\frac{1\times 3\times 5\ldots\times(2n-1)}{2^n\,n!}\,t^{2n} + \ldots$$

$$= P_0(0) + P_1(0)t + P_2(0)t^2 + \ldots + P_{2n}(0)t^{2n}$$

On comparing the powers of t^{2n}, we get

$$P_{2n}(0) = (-1)^n\,\frac{1\times 3\times 5\ldots\times(2n-1)}{2^n\,n!}$$

Since in the expansion of $(1+t^2)^{-1/2}$ we only get even powers of *t*, we have:

$$P_{2n+1}(0) = 0$$

2. a) Starting from the recurrence relation:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

we multiply both sides by $P_{n-1}(x)$ and integrate the terms with respect to *x* between the limits –1 and + 1. This yields:

$$(n+1)\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx - (2n+1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx$$

$$+ n\int_{-1}^{1} P_{n-1}^2(x)\,dx = 0$$

From Eq. (4.21), we have:

$$\int_{-1}^{1} P_{n+1}(x) P_{n-1}(x)\, dx = 0$$

and $\displaystyle\int_{-1}^{1} P_{n-1}^2(x)\, dx = \frac{2}{2(n-1)+1} = \frac{2}{2n-1}$

$\therefore \qquad 0 - (2n+1)\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\, dx + \frac{2n}{2n-1} = 0$

On rearranging terms, we get

$$(2n+1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\, dx = \frac{2n}{2n-1}$$

so that $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\, dx = \frac{2n}{(2n-1)(2n+1)}$

b) To evaluate the integral $\displaystyle\int_{-1}^{1} x\, P_n(x)\, dx,$ we note that $x = P_1(x)$, so that $\displaystyle\int_{-1}^{1} x\, P_n(x)\, dx = \int_{-1}^{1} P_1(x)\, P_n(x)\, dx.$ Therefore, the orthogonality relation for Legendre polynomials [Eq. (4.21)] implies that $\displaystyle\int_{-1}^{1} x\, P_n(x)\, dx = 0$ for $n \neq 1$ and $\dfrac{2}{2\times 1+1} = \dfrac{2}{3}$ for $n = 1$.

3. We need to solve Laplace's equation in spherical coordinates subject to the given boundary conditions. You know the solution of azimuthally symmetric Laplace's equation in spherical coordinates. Since temperature distribution is independent of ϕ, we can write

$$T(r,\theta) = \sum_{m=0}^{\infty} \left(A_m r^m + \frac{B_m}{r^{m+1}} \right) P_m(\cos\theta) \qquad \text{(i)}$$

Since temperature will be finite at the centre of the sphere ($r = 0$), we must have $B_m = 0$, for all m; otherwise the solution will diverge. Hence, Eq. (i) reduces to:

$$T(r,\theta) = \sum_{m=0}^{\infty} A_m\, r^m\, P_m(\cos\theta) \qquad \text{(ii)}$$

As per the given boundary conditions, we can write the temperature distribution on the surface of the sphere as:

$$f(x) = \begin{cases} -T_0 & -1 < x < 0 \\ T_0 & 0 < x < 1 \end{cases} \qquad \text{(iii)}$$

where $x = \cos\theta$. Applying the boundary conditions to Eq. (ii), we get

$$f(x) = \sum_{m=0}^{\infty} A_m\, a^m\, P_m(x) \qquad \text{(iv)}$$

To determine the constants A_m, we use the orthogonality relation. So, we multiply both sides of Eq. (iv) by $P_l(x)$ and integrate the resultant expression over x in the range –1 to +1. This yields:

$$\int_{-1}^{1} P_l(x) f(x)\, dx = \sum_{m=0}^{\infty} A_m\, a^m \int_{-1}^{1} P_l(x) P_m(x)\, dx \qquad \text{(v)}$$

Using the orthogonality relation for the Legendre Polynomials, we get

$$A_l = \frac{2l+1}{2a^l}\int_{-1}^{1} P_l(x)\, f(x) dx$$

$$= \frac{2l+1}{2a^l} T_0 \left[\int_{-1}^{0} P_l(x)\, dx + \int_{0}^{1} P_l(x)\, dx\right]$$

$$= \left(\frac{2l+1}{2}\right)\left(\frac{T_0}{a^l}\right)\left[\int_{0}^{1} P_l(x)\, dx - \int_{0}^{1} P_l(-x)\, dx\right] \qquad \text{(vi)}$$

Since $P_l(-x) = (-1)^l P_l(x)$, the above expression simplifies to

$$A_l = \begin{cases} (2l+1)(T_0 / a^l) \int_0^1 P_l\,(x)\, dx & l = \text{odd} \\ 0 & l = \text{even} \end{cases} \qquad \text{(vii)}$$

From this you can readily write the values of first few coefficients:

$$A_1 = \frac{3T_0}{a}\int_0^1 P_1(x)\, dx = \frac{3T_0}{a}\int_0^1 x\, dx = \frac{3T}{2a}$$

$$A_3 = \frac{7T_0}{2}\int_0^1 P_3\,(x)\, dx = \frac{7T_0}{a^3}\int_0^1 \left(\frac{5x^3 - 3x}{2}\right) dx = -\frac{7T_0}{8a^3}$$

Hence, the temperature distribution inside the given sphere is:

$$T(r,\theta) = T_0\left[\frac{3r}{2a}\cos\theta - \frac{7r^3}{16a^3}(5\cos^3\theta - 3\cos\theta) + \ldots\right] \qquad \text{(viii)}$$

4. Substituting $Y_n^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta) e^{im\phi}$ from Eq. (4.35a) in Eq. (4.37), we get:

$$\sqrt{\frac{2n_1+1}{4\pi}\frac{(n_1-m_1)!}{(n_1+m_1)!}}\sqrt{\frac{2n_2+1}{4\pi}\frac{(n_2-m_2)!}{(n_2+m_2)!}}$$

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} P_{n_1}^{m_1}(\cos\theta) e^{-im_1\phi}\, P_{n_2}^{m_2}(\cos\theta) e^{im_2\phi} \sin\theta\, d\theta\, d\phi$$

$$= \delta_{n_1,n_2}\delta_{m_1,m_2} \qquad \text{(i)}$$

Let us first consider the integral over ϕ, which is:

$$\int_{\phi=0}^{2\pi} e^{-im_1\phi} e^{im_2\phi} d\phi = 2\pi\delta_{m_1,m_2} \qquad \text{(ii)}$$

as you know from your UG integral calculus. Substituting Eq. (ii) in Eq. (i), its LHS becomes:

$$\sqrt{(2n_1+1)\frac{(n_1-m_1)!}{(n_1+m_1)!}}\sqrt{(2n_2+1)\frac{(n_2-m_2)!}{(n_2+m_2)!}}$$

$$\int_{\theta=0}^{\pi} P_{n_1}^{m_1}(\cos\theta)P_{n_2}^{m_2}(\cos\theta)\sin\theta\, d\theta \qquad \text{(iii)}$$

From Eq. (4.33), we have:

$$\int_{-1}^{1} P_l^m(x)P_k^m(x)\,dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{kl}$$

With $x = \cos\theta$, the above equation becomes:

$$\int_0^{\pi} P_l^m(\cos\theta)P_k^m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{kl} \qquad \text{(iv)}$$

Using Eq. (iv) in Eq. (iii), we get:

$$\sqrt{(2n_1+1)\frac{(n_1-m_1)!}{(n_1+m_1)!}}\sqrt{(2n_2+1)\frac{(n_2-m_2)!}{(n_2+m_2)!}}$$

$$\int_{\theta=0}^{\pi} P_{n_1}^{m_1}(\cos\theta)P_{n_2}^{m_2}(\cos\theta)\sin\theta\, d\theta = \delta_{n_1,n_2}$$

Combining the results of Eqs. (iv) and (ii), we get the orthonormality relation of the spherical harmonics [Eq. (4.37)].

5. We change the variable as $e^x = t$, and have:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^x\frac{dy}{dt} = t\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = t\frac{d}{dt}\left(t\frac{dy}{dt}\right) = t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}$$

The ODE then becomes:

$$(1-t)\left(t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right) + \frac{t}{2}\frac{dy}{dt} + ty = 0$$

or $$t(1-t)\frac{d^2y}{dt^2} + \left(\frac{3}{2} - t\right)\frac{dy}{dt} + y = 0$$

Comparing this equation with the hypergeometric equation [Eq. (4.38)], we find that:

$$c = \frac{3}{2}, \quad (a+b+1) = 1, \; ab = -1 \qquad \Rightarrow \qquad a = 1, \; b = -1$$

The general solution as per Eq. (4.48) at $t = 1$ or $x = 0$ is:

$$y = c_1F(a, b, a+b-c+1, 1-t)$$

$$+ c_2(1-t)^{c-a-b}F(c-b, c-a, c-a-b+1, 1-t)$$

or $$y = c_1F(1, -1, -\frac{1}{2}, 1-e^x)$$

$$+ c_2(1-e^x)^{3/2}F(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, 1-e^x)$$

UNIT 5

SPECIAL FUNCTIONS-II

Structure

5.1 INTRODUCTION

In Unit 4, you have learnt about Legendre polynomials, their generating function, recurrence relations and orthonormality. You have also learnt about spherical harmonics and hypergeometric functions. You have solved their respective differential equations and problems based on these special functions.

In this unit, we will discuss the special functions Hermite and Laguerre polynomials. You will learn about their generating functions, recurrence relations and orthogonality property. We will also discuss the Sturm-Liouville problem. Finally, we will explain how to expand a given function in terms of orthogonal functions.

Expected Learning Outcomes

After studying this unit, you should be able to:

- solve the differential equations for Hermite and Laguerre polynomials;
- write generating functions of Hermite and Laguerre polynomials and obtain recurrence relations for Hermite and Laguerre polynomials.
- state orthogonality properties of Hermite and Laguerre polynomials and use them to solve physical problems;
- define a Sturm-Liouville problem, and state and apply the properties of a Sturm-Liouville problem to solve ODEs; and
- expand any given function in terms of orthogonal functions.

5.2 HERMITE POLYNOMIALS

You may have studied Hermite's differential equation and its solutions in UG physics or mathematics courses (refer to SAQ 3 in Unit 3 of PHE-05). It is given as:

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny(x) = \theta \qquad (5.1)$$

Hermite's polynomials find applications in quantum theory of harmonic oscillators. You can obtain the general solution of Hermite's differential equation by solving SAQ 1. It is:

$$y = a_0\left[1 + \frac{(-2)n}{2!}x^2 + \frac{(-2)^2 n(n-2)}{4!}x^4 + \frac{(-2)^3 n(n-2)(n-4)}{6!}x^6 + \ldots\right]$$
$$+ a_1\left[x + \frac{(-2)(n-1)}{3!}x^3 + \frac{(-2)^2(n-1)(n-3)}{5!}x^5 + \ldots\right] \qquad (5.2)$$

The constants a_0 and a_1 may take arbitrary values.

SAQ 1

Solve Eq. (5.1) and obtain its general solution given by Eq. (5.2).

If n is a non-zero negative integer, the series given in Eq. (5.2) will be an infinite series. You may now like to know: What happens if n is zero or an even positive integer? To understand this, let us consider the case corresponding to $n = 6.$ You can see that the fifth and all subsequent terms of the series (not explicitly shown) in the first square bracket in Eq. (5.2) will be zero as their numerator will have a factor (n – 6). It means that for $n = 6,$ the first series will terminate at $\frac{(-2)^3\, 6\times4\times2}{6!}x^6.$

To extend this discussion for the general case, let us write $n = 2m,$ $m = 0, 1, 2\ldots$ Then the series in first square bracket of Eq. (5.2) will terminate at the term:

$$\frac{(-2)^m(2m)(2m-2)}{(2m)!}x^{2m} = \frac{(-1)^m\, m!}{(2m)!}(2x)^{2m}$$

and all subsequent terms will have their numerators equal to zero. The series in the second square bracket of Eq. (5.2) will however, remain an infinite series. But if we choose $a_1 = 0,$ we will obtain a particular solution of Eq. (5.1), which is a polynomial of degree 2m in x. However, you should note that such a polynomial will contain only even powers of x and the coefficient a_0 will still be arbitrary. For the series to appear more systematic, we choose the constant a_0 to be

$$a_0 = \frac{(-1)^m(2m)!}{m!}$$

The polynomial so obtained is called **Hermite polynomial of degree 2*m*** and is denoted by the symbol $H_{2m}(x)$:

$$H_{2m}(x) = \frac{(-1)^m(2m)!(-1)^{m-1}(2m)!}{m!} + \ldots$$
$$+ \frac{(-1)(2m)!}{(2m-2)!}(2x)^{2m-2} + (2x)^{2m} \qquad (5.3)$$

Note that the term containing the highest power of x is $(2x)^{2m}$.

Similarly, when n is a positive odd integer, say $2m+1$; $m = 0, 1, 2, ...,$ the second series in Eq. (5.2) will terminate at:

$$\frac{(-1)^m m!}{(2m+1)!} \frac{(2x)^{2m+1}}{2}$$

and the first series remains an infinite series. As before, we get a particular polynomial solution of Hermite's differential equation by putting $a_0 = 0$ and choosing a_1 as

$$a_1 = 2\frac{(-1)^m (2m+1)!}{m!}$$

This leads to **Hermite's polynomial of degree** $2m+1$**:**

$$H_{2m+1}(x) = \frac{(-1)^m (2m+1)!}{m!} 2x + \frac{(-1)^{m-1}(2m+1)!}{3!\,(m-1)!}(2x)^3$$

$$+ \frac{(-1)(2m+1)!}{(2m-1)!}(2x)^{2m-1} + (2x)^{2m+1} \qquad (5.4)$$

We would like to point out here that Eq. (5.4) contains only odd powers of x; the term containing the highest power of x is $(2x)^{2m+1}$.

We can now combine Eqs. (5.3) and (5.4) by taking n to be any positive integer including zero, and define the Hermite polynomial of degree n as:

$$H_n(x) = (2x)^n \; \frac{n(n-1)}{1!}(2x)^{n-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + ... + \frac{(-1)^{n/2} n!}{(n/2)!}$$

(if n is even) (5.5a)

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + ... + \frac{(-1)^{\frac{n-1}{2}} n!}{\left(\frac{n-1}{2}\right)!} 2x$$

(if n is odd) (5.5b)

> Any solution of Eq. (5.1) can be multiplied by an arbitrary constant to obtain another solution. Then the question arises: Why are we taking the constants a_0 and a_1 in this particular manner? This is because Hermite polynomial of degree n is defined in such a way that the term containing the highest power of x is $(2x)^n$.

where we have written the terms in decreasing powers of x.

From Eq. (5.5) we can write the first few Hermite polynomials as follows:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12 \qquad (5.6)$$

A plot of some of these polynomials is shown in Fig. 5.1. For ease in scaling, $H_n(x)(n \neq 0)'s$ are divided by n^3.

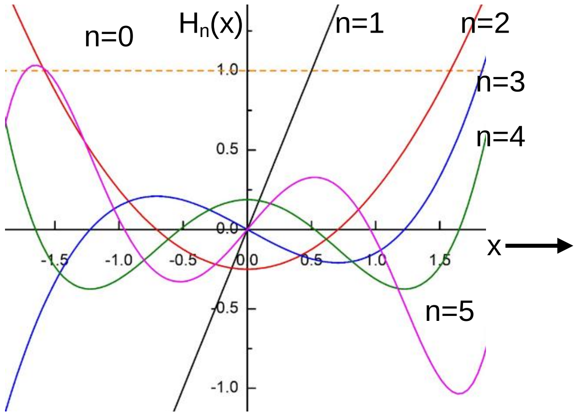

Fig. 5.1: **Plots of Hermite polynomials** $H_0(x), \dfrac{H_n(x)}{n^3} (n = 1 \text{ to } 5)$.

The series representation of Hermite polynomials becomes somewhat unwieldy, particularly when we have to evaluate integrals involving Hermite polynomials. In such situations and for the derivations of many other properties of Hermite polynomials, it is convenient to use the generating function for Hermite polynomials. So, we now state the generating function for the Hermite polynomials and derive the recurrence relations for Hermite polynomials of different orders and their derivatives.

5.2.1 Generating Function and Recurrence Relations for Hermite Polynomials

The generating function for Hermite polynomials is:

$$\boxed{g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}} \tag{5.7}$$

As in the case of Bessel functions and Legendre polynomials, the generating function for Hermite polynomials is expanded in powers of t and the x-dependent coefficients are related to the special function. Therefore, we rewrite $g(x, t)$ as

$$g(x,t) = e^{2xt} \times e^{-t^2}$$

and express the exponential functions in their respective power series to get

$$g(x,t) = \sum_{j=0}^{\infty} \frac{(2xt)^j}{j!} \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}}{m!}$$

$$= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(2x)^j}{j!\,m!} t^{2m+j}$$

To obtain the coefficient of t^n, we put $j + 2m = n$. You should note that this equality can be satisfied for various combinations of the values of j and m, namely $j = n,\ m = 1;\ j = n - 2,\ m = 1;\ j = n - 4,\ m = 2;$ and so on. If n is even, we will have $j = 0,\ m = \frac{n}{2}$ and if n is odd, we will have $j = 1,\ m = \frac{n-1}{2}$.

Thus, the coefficient of t^n is:

$$\frac{(2x)^n}{n!0!} - \frac{(2x)^{n-2}}{(n-2)!1!} + \frac{(2x)^{n-4}}{(n-4)!2!} - \ldots$$

The last term in this series is $\dfrac{(-1)^{\frac{n}{2}}}{0!\left(\frac{n}{2}\right)!}$ for even n and $\dfrac{(-1)^{\frac{n-1}{2}}}{1!\left(\frac{n-1}{2}\right)!}2x$ for odd n.

Thus the coefficient of $\dfrac{t^n}{n!}$, i.e. $H_n(x)$ will be obtained by multiplying the above expression by n!. You should recognize that the resultant expression is identical to that given in Eq. (5.5). Let us use the generating function to relate Hermite polynomials of different kinds.

Example 5.1

Establish the relation between $H_n(x)$ and $H_n(-x)$ using the generating function.

Solution : In Eq. (5.7) for the generating function, we change x to $-x$ and t to $-t$ .This will leave the exponential function unchanged so that

$$e^{-2xt-t^2} = \sum_n H_n(-x)\frac{(-t)^n}{n!} = \sum (-1)^n H_n(-x)\frac{t^n}{n!}$$

On comparing this expression with that given in Eq. (5.7), you will get

$$H_n(x) = (-1)^n H_n(-x)$$

or

$$H_n(-x) = (-1)^n H_n(x)$$

From this we note that if we change the sign of x, the Hermite polynomials for even positive integral values of n do not change whereas those with odd positive integral values of n just change sign. This result is, of course, obvious from the fact that Hermite polynomials contain only even (odd) powers of x when n is even (odd).

Now we will use the generating function to obtain the recurrence relations for Hermite polynomials.

Recurrence Relations

When we differentiate both sides of Eq. (5.7) partially with respect to t, we get:

$$(2x-2t)e^{2xt-t^2} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

Replacing e^{2xt-t^2} on the left hand side by the right hand side of Eq. (5.7), we get:

$$(2x-2t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

Equating coefficients of t^{n-1} from the two sides, we have:

$$2x\frac{H_{n+1}(x)}{(n+1)!} - 2\frac{H_n(x)}{n!} = \frac{H_{n+2}(x)}{(n+1)!}$$

On multiply throughout by $(n+1)!$, we can write

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x) \qquad (5.8)$$

You should note that this recurrence relation connects Hermite polynomials of three successive orders.

We will now illustrate how the generating function and this recurrence relation can be used to obtain expressions of some lower order Hermite polynomials. You should go through the following the example carefully.

Example 5.2

Starting from the generating function for the Hermite polynomials, obtain expressions for $H_0(x)$ and $H_1(x)$ and then use the recurrence relation given in Eq. (5.8) to obtain expressions for $H_2(x), H_3(x)$ and $H_4(x)$.

Solution : We first write the exponential in the generating function given in Eq. (5.7) as a power series in $2xt - t^2$:

$$e^{2xt-t^2} = 1 + (2xt - t^2) + \frac{(2xt - t^2)^2}{2!} + \ldots = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

You should note that $H_0(x)$ corresponds to a term independent of *t*. on the left side, the term independent of *t* is 1. Hence, we can write:

$$H_0(x) = 1$$

Further, $H_1(x)$ occurs as the coefficient of *t*. On the left hand side, the coefficient of *t* is $2x$. Therefore,

$$H_1(x) = 2x$$

From the recurrence relation in Eq. (5.8) with $n = 0$, we get:

$$H_2(x) = 2xH_1(x) - 2H_0(x)$$

On substituting for $H_1(x)$ and $H_0(x)$, we obtain:

$$H_2(x) = 4x^2 - 2$$

If we repeat this procedure for $n = 1$ and $n = 2$, we get:

$$H_3(x) = 2xH_2(x) - 4H_1(x) = 2x(4x^2 - 2) - 4(2x) = 8x^3 - 12x$$

and $$H_4(x) = 2xH_3(x) - 6H_2(x) = 16x^4 - 48x^2 + 12$$

Note that these expressions for Hermite polynomials are the same as given in Eq. (5.6) and obtained from the series given by Eq. (5.5). Proceeding in this way, you can calculate higher order Hermite polynomials. You are advised to obtain expressions of a couple of these.

You can obtain another recurrence relation by partially differentiating both sides of Eq. (5.7) with respect to *x*:

$$\boxed{H'_{n+1}(x) = 2(n+1)H_n(x)} \qquad (5.9)$$

where the prime denotes differentiation with respect to the argument *x*. We leave the derivation of this result as SAQ 2 for you.

SAQ 2

Differentiate the generating function for Hermite polynomials partially with respect to *x* and obtain the recurrence relation given in Eq. (5.9).

On combining the recurrence relations given in Eqs. (5.8) and (5.9), we get:

$$H_{n+2}(x) = 2_x H_{n+1}(x) - H'_{n+1}(x) \qquad (5.10)$$

By changing $(n + 1)$ to n in Eq. (5.10), we can write:

$$H_{n+1}(x) = 2xH_n(x) - H'_n(x) \qquad (5.11)$$

Again differentiating both sides of this equation with respect to x, we get:

$$H'_{n+1}(x) = 2H_n(x) + 2xH'_n(x) - H''_n(x)$$

On combining this with Eq. (5.9), we get:

$$2(n+1)H_n(x) = 2H_n(x) + 2xH'_n(x) - H''_n(x)$$

We can rearrange this equation and write:

$$\boxed{H''_n(x) - 2x\,H'_n(x) + 2nH_n(x) = 0} \qquad (5.12)$$

Do you recognise this equation? It is Hermite's differential equation (for positive integral or zero value of n) for $H_n(x)$. It means that if the polynomials $H_n(x)$ satisfy the recurrence relations in Eqs. (5.8) and (5.9), they must satisfy Hermite's differential equation.

The generating function has many other uses. You can utilize it to obtain Rodrigues' formula, which gives a compact expression for Hermite polynomials.

Rodrigues' Formula

We write Eq. (5.7) in expanded form:

$$e^{2xt-t^2} = 1 + t\,H_1(x) + \frac{t^2}{2!}H_2(x) + \ldots + \frac{t^n}{n!}H_n(x) + \ldots$$

Successive partial differentiation of both sides with respect to t yields:

$$\frac{\partial}{\partial t}\left(e^{2xt-t^2}\right) = H_1(x) + \frac{2t}{2!}H_2(x) + \ldots + \frac{nt^{n-1}}{n!}H_n(x) + \ldots$$

$$\frac{\partial^2}{\partial t^2}\left(e^{2xt-t^2}\right) = H_2(x) + \frac{3\times 2}{3!}H_3(x) + \ldots + \frac{n(n-1)}{n!}t^{n-2}H_n(x) + \ldots$$

$$\vdots$$

$$\frac{\partial^n}{\partial t^n}\left(e^{2xt-t^2}\right) = H_n(x) + \frac{(n+1)n(n-1)\ldots 2}{(n+1)!}t\,H_{n+1}(x) + \ldots$$

For $t = 0$, this expression simplifies to:

$$H_n(x) = \left[\frac{\partial^n}{\partial t^n}\left(e^{2xt-t^2}\right)\right]_{t=0}$$

Since $e^{2xt-t^2} = e^{x^2}e^{-(x-t)^2}$, we can rewrite this expression as:

$$H_n(x) = e^{x^2}\left[\frac{\partial^n}{\partial t^n}e^{-(x-t)^2}\right]_{t=0}$$

We note that $e^{-(x-t)^2}$ is a function of $x - t$, and for such a function the partial derivative with respect to t can be obtained from the partial derivative with respect to *x* by just changing the sign. So for *nth* order partial derivative, the sign will change *n* times. Hence, we can write:

$$H_n(x) = e^{x^2}(-1)^n \left[\frac{\partial^n}{\partial x^n} e^{-(x-t)^2}\right]_{t=0}$$

$$= (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \tag{5.13}$$

This is **Rodrigues's formula** for the Hermite polynomials.

You may like to apply Rodrigues's formula for a simple case. Solve SAQ 3.

SAQ 3

Use Rodrigues's formula to evaluate $H_4(x)$.

Yet another interesting application of the generating function for the Hermite polynomials is in evaluation of integrals involving their product with suitable polynomials. Of particular importance is the result that the integral over *x* from $-\infty$ to $+\infty$ of the product of two Hermite polynomials of different degrees with e^{-x^2} is zero. (The function e^{-x^2} is called the weight function.) These are termed the **orthogonality relations** of Hermite polynomials. You will now learn how to obtain as well as apply these relations. Along with the values of similar integrals for Hermite polynomials of the same degree, we can calculate the expansion coefficients when a function is expanded in terms of Hermite polynomials.

5.2.2 Orthogonality Relations for Hermite Polynomials

To obtain orthogonality relations for Hermite polynomial, we can start with Hermite differential equation as in the case of Legendre and Bessel polynomials, or the generating function for the Hermite polynomials. For the sake of variety here we consider the latter option:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

Next we change *t* to *u* and express the expansion as a power series in *u*:

$$e^{2xu-u^2} = \sum_{m=0}^{\infty} H_m(x)\frac{u^m}{m!}$$

Now we multiply these equations and the resultant expression by e^{-x^2}. This gives:

$$e^{-x^2+2xt-t^2+2xu-u^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-x^2} H_n(x)H_m(x)\frac{t^n u^m}{n!\,m!}$$

You can rewrite the left hand side as:

$$e^{2tu}e^{-(x^2+t^2+u^2-2xt-2xu+2tu)} = e^{2tu}e^{-(x-t-u)^2}$$

Proceeding further, we integrate both sides with respect to *x* from $-\infty$ to $+\infty$ and, on the right hand side, interchange the order of the summations and integration. The leads to

$$e^{2tu}\int_{-\infty}^{+\infty} e^{-(x-t-u)^2}dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)dx\,\frac{t^n u^m}{n!\,m!} \tag{5.14}$$

Do you recognise the integral on the left side? By changing the variable of integration from x to z through the substitution $x - t - u = z$, so that $dx = dz$, you can rewrite it as $\int_{-\infty}^{\infty} e^{-z^2}dz$ which is just $\Gamma(1/2)$ and is equal to $\sqrt{\pi}$ (read the margin remark). So, the left hand side of the above equation reduces to $\pi^{1/2}\,e^{2tu}$. On expanding the exponential in a power series, we can write

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)\,dx\,\frac{t^n u^m}{n!\,m!} = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2tu)^n}{n!}$$

$$= \sqrt{\pi}\left[1+\frac{(2tu)^1}{1!}+\frac{(2tu)^2}{2!}+\ldots\frac{(2tu)^n}{n!}+\ldots\right]$$

On equating the coefficients of $t^n u^m$, we get:

$$\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)dx = 0 \qquad \text{if } n \neq m$$

and

$$\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)\;dx = 2^n\,n!\,\pi^{1/2} \qquad \text{if } n = m$$

On combining these results, we can write:

$$\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)dx = 2^n\;n!\,\pi^{1/2}\delta_{nm} \tag{5.15}$$

where δ_{nm} is the Kronecker delta.

In words we can say that the Hermite polynomials of different degrees are orthogonal to each other on the interval $(-\infty, +\infty)$ with weight function e^{-x^2}. You may recall that for the Legendre polynomials, the weight function is unity and the range of integration varies from -1 to $+1$.

You may now like to solve SAQ 4 on evaluation of integral involving Hermite polynomials.

Consider the integral:

$$I = \int_{-\infty}^{\infty} e^{-z^2}dz$$

$$= 2\int_{0}^{\infty} e^{z}\;dz$$

We put $z^2 = p$ so that

$2z\,dz = dp$

or $\quad 2dz = \frac{dp}{z}$

$= p^{-1/2}dp$

Hence,

$$I = \int_{0}^{\infty} p^{-1/2}\,e^{-p}dp$$

$$= \Gamma(1/2)$$

SAQ 4

Use the generating function for Hermite polynomials to evaluate the integral

$$\int_{-\infty}^{+\infty} xe^{-x^2}\;H_n(x)H_m(x)\,dx$$

We hope that you have understood the properties of Hermite polynomials. We can apply this knowledge to solve the **one-dimensional harmonic oscillator** problem in quantum mechanics. This enables us to understand the nature of small vibrations of atoms in molecules and quantization of the electromagnetic field where energy of the field may be written as a sum of harmonic oscillator type energy terms. You will study this in the first semester course on Quantum Mechanics.

We will now discuss Laguerre polynomials.

5.3 LAGUERRE POLYNOMIALS

Laguerre's differential equation is a linear second order ODE:

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny(x)=0 \tag{5.16}$$

where n is a parameter. You can see that $x=0$ is a regular singular point of this equation. The solution around this point can be obtained using the power series method (refer to Unit 3, Block 1 of PHE-05):

$$y(x)=\sum_{j=0}^{\infty}a_j x^{a+j}, a_0\neq 0 \tag{5.17}$$

If n is not a positive integer or zero, this will remain an infinite series. However, for particular case when n is a positive integer or zero, the series will terminate at the $(n+1)th$ term and reduce to an nth degree polynomial in x. If we further choose $a_0=1$, the resultant expression defines the Laguerre polynomial, $L_n(x)$:

$$L_n(x)=1-\frac{n}{(1!)^2}x+\frac{n(n-1)}{(2!)^2}x^2-...+(-1)^n\frac{n(n-1)...1}{(n!)^2}x^n$$

$$=1-{}^nC_1\frac{x}{1!}+{}^nC_2\frac{x^2}{2!}-...+(-1)^n\,{}^nC_n\frac{x^n}{n!} \tag{5.18}$$

From this equation it readily follows that Laguerre polynomials for the first few values of n are:

$$L_0(x)=1$$

$$L_1(x)=1-x$$

$$L_2(x)=1-2x+\frac{x^2}{2}$$

$$L_3(x)=1-3x+\frac{3}{2}x^2-\frac{x^3}{6}$$

and $$L_4(x)=1-4x+3x^2-\frac{2}{3}x^3+\frac{x^4}{24} \tag{5.19}$$

Fig. 5.2 shows a plot of Laguerre polynomials versus x.

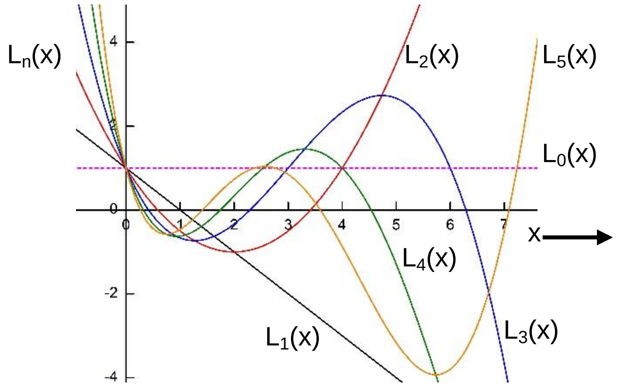

Fig. 5.2: **Laguerre polynomials** $L_n(x)$ **for** $n=1$ **to** $n=5$.

The **Rodrigues's formula** for $L_n(x)$ enables us to express it in a compact form:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \qquad (5.20)$$

You may now like to solve SAQ 5 for practice.

SAQ 5

Show that the Laguerre polynomial defined in Eq. (5.20) is identical with that given in Eq. (5.18).

5.3.1 Generating Function and Recurrence Relations for Laguerre Polynomials

The generating function for Laguerre polynomials is:

$$g(x,t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n \qquad |t| < 1 \qquad (5.21)$$

If we differentiate both sides of Eq. (5.21) partially with respect to *t*, we get:

$$(1-t-x)\frac{e^{-xt/(1-t)}}{(1-t)^3} = \sum_{n=1}^{\infty} L_n(x)nt^{n-1}$$

On combining this result with Eq. (5.21), we can write:

$$(1-t-x)\sum_{n=0}^{\infty} L_n(x)t^n = (1-t)^2\sum_{n=1}^{\infty} L_n(x)nt^{n-1}$$

On equating the coefficients of t^{n+1} from the two sides and rearranging terms, we obtain a recurrence relation which connects Laguerre polynomials of three successive degrees:

$$(n+2)L_{n+2}(x) = (2n+3-x)L_{n+1}(x) - (n+1)L_n(x) \qquad (5.22)$$

We now make a direct expansion of $g(x,t)$ given in Eq. (5.21) in powers of *t*:

$$g(x,t) = \left[1 - \frac{xt}{1-t} + \left(\frac{1}{2!}\right)\left(\frac{xt}{1-t}\right)^2 + \ldots\right]\left(\frac{1}{1-t}\right)$$

$$= \left(1 - \frac{xt}{1-t} + \ldots\right)(1-t)^{-1}$$

$$= 1 + 1 - xt - 2xt^2 - \ldots$$

From this we note that the coefficient of t^0 and t^1 are 1 and 1 – *x*, respectively. So we can say that $L_0(x) = 1$ and $L_1(x) = 1 - x$. If we now put $n = 0$ in Eq. (5.22) and substitute for $L_0(x)$ and $L_1(x)$, we obtain $L_2(x) = 1 - 2x + \dfrac{x^2}{2}$. Proceeding in the same way we can successively generate the Laguerre polynomials of all degrees.

In order to get another recurrence relation for the Laguerre polynomials, we differentiate both sides of Eq. (5.21) partially with respect to *x*. This gives:

The binomial series of e^x and $(1+x)^p$ (all *x* and *p*) are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

$$\frac{-t}{(1-t)^2}\exp\left[\frac{-xt}{(1-t)}\right] = \sum_{n=0}^{\infty}\frac{dL_n}{dx}t^n$$

This can be rewritten as:

$$-t\sum_{n=0}^{\infty}L_n(x)t^n = (1-t)\sum_{n=0}^{\infty}\frac{dL_n}{dx}t^n$$

On equating the coefficients of t^{n+1} on both sides, we get:

$$\frac{dL_{n+1}}{dx} = \frac{dL_n}{dx} - L_n(x) \tag{5.23}$$

Using binomial expansion we can write

$$\left(1-\frac{xt}{1-t}+\ldots\right)(1-t)^{-1}$$

$$= (1-xxt\,(1-t)^{-1}+\ldots)(1-t)^{-1}$$

$$= \left[1-xt\left(1+t+\frac{t^2}{2!}+\ldots\right)+\ldots\right]$$

$$\left(1+t+\frac{t^2}{2!}+\ldots\right)$$

$$= \left(1-xt-xt^2-x\frac{t^3}{2!}+\ldots\right)$$

$$(1+t+\ldots)$$

$$= 1-xt-xt^2-\frac{xt^3}{2!}+\ldots+$$

$$t-xt^2-xt^3-\ldots$$

The recurrence relations given in Eqs. (5.22 and 5.23) can be combined to obtain other relations. However, we will not go into these details.

You have earlier learnt the orthogonality relations for Legendre, Bessel and Hermite functions and their importance in understanding various physical problems. In the following section, we obtain orthogonality relations of Laguerre polynomials.

5.3.2 Orthogonality Relations for Laguerre Polynomials

There are different ways of obtaining orthogonality relations for Laguerre polynomials. We start with the differential equations satisfied by Laguerre polynomials of degrees *n* and *k*:

$$x\frac{d^2L_n}{dx^2} + (1-x)\frac{dL_n}{dx} + nL_n(x) = 0 \tag{5.24a}$$

$$x\frac{d^2L_k}{dx^2} + (1-x)\frac{dL_k}{dx} + kL_k(x) = 0 \tag{5.24b}$$

We multiply Eq. (5.24a) by $e^{-x}L_k(x)$ and Eq. Eq. (5.24b) by $e^{-x}L_n(x)$ and subtract the latter from the former. Then we can write the resultant expression as:

$$\frac{d}{dx}\left[x\,e^{-x}\left\{L_k(x)\frac{dL_n}{dx} - L_n(x)\frac{dL_k}{dx}\right\}\right] + (n-k)e^{-x}\,L_n(x)L_k(x) = 0$$

We integrate this expression over *x* from 0 to ∞:

$$x\,e^{-x}\left[L_k(x)\frac{dL_n}{dx} - L_n(x)\frac{dL_k}{dx}\right]_0^{\infty} + (n-k)\int_0^{\infty}e^{-x}\,L_n(x)L_k(x)\,dx = 0$$

Note that the expression within the square bracket is zero for both the limits (at ∞ because of the exponential factor and at zero because of the *x* factor). Hence, we obtain:

$$(n-k)\int_0^{\infty}e^{-x}\,L_n(x)L_k(x)\,dx = 0$$

Thus, if $n \neq k$

$$\int_0^{\infty}e^{-x}\,L_n(x)L_k\,dx = 0 \tag{5.25}$$

Thus, Laguerre polynomials of different degrees are orthogonal to each other on the interval $(0,\infty)$ with weight factor e^{-x}.

To obtain the orthogonality relation for $n = k$, we take the products of the two sides of Eq. (5.21) with themselves. This gives:

$$\frac{e^{-2xt/(1-t)}}{(1-t)^2} = \sum_{n=0}^{\infty} L_n(x) t^n \sum_{k=0}^{\infty} L_k(x) t^k$$

As before, we multiply both sides of this equation by e^{-x} and integrate from 0 to ∞. This gives:

$$\frac{1}{(1-t)^2}\int_0^{\infty} e^{-\frac{1+t}{1-t}x}dx = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^n t^k \int_0^{\infty} e^{-x} L_n(x) L_k(x)\, dx \qquad (5.26)$$

Note that we have changed the orders of summations and integration on the right hand side. For $n = k$, the right side of above equation reduces to

$$\sum_{n=0}^{\infty} t^{2n}\int_0^{\infty} e^{-x} L_n^2(x)\, dx.$$

To proceed further, we use the formula $\int e^{-ax}dx = -\frac{e^{-x}}{a}$. Thus, the left hand side of Eq. (5.26) can be written as:

$$\frac{1}{(1-t)^2}\times(-)\left(\frac{1-t}{1+t}\right)e^{-\frac{1+t}{1-t}}\Bigg|_0^{\infty} = \frac{1}{(1-t)(1+t)} = \frac{1}{1-t^2}$$

For $t << 1$, we have:

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + ... = \sum_{n=0}^{\infty} t^{2n}$$

so that $\sum_{n=0}^{\infty} t^{2n} = \sum_{n=0}^{\infty} t^{2n}\int_0^{\infty} e^{-x} L_n^2(x)\, dx$

On comparing the coefficients of t^{2n} for all n, we obtain:

$$\int_0^{\infty} e^{-x} L_n^2(X)\, dx = 1 \qquad (5.27)$$

We now combine Eqs. (5.25 and 5.27) to write the orthonormality relation for Laguerre polynomials as:

$$\int_0^{\infty} e^{-x} L_n(x) L_k(x)\, dx = \delta_{nk} \qquad (5.28)$$

Let us now discuss the Sturm-Liouville problem.

5.4 STURM-LIOUVILLE PROBLEM

You have learnt how to express a given function as a linear combination of an orthogonal set of functions, for example, Legendre polynomials in Unit… The question is: How do we determine or identify such orthogonal sets of functions? When is an orthogonal set of functions complete a complete set? Sturm Liouville theory helps us find the answers. So, in the final section of this unit and block, we discuss the **Sturm-Liouville problem**, as it is referred to in many texts.

We begin by stating: What is a Sturm-Liouville problem? The Sturm-Liouville problem comprises a second order linear differential equation along with certain boundary conditions. The Sturm-Liouville ODE on a finite interval $[a, b]$ is of the form:

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y + q(x) + \lambda r(x)y = 0, \quad x \in [a,b] \tag{5.29a}$$

In brief, we write it as:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \tag{5.29b}$$

Note that in the Sturm-Liouville equation (5.29a or 5.29b), p, q and r are specific functions of x, and λ is a parameter. Since λ is a parameter, it can be replaced by other variables or expressions. So, we can recast many ODEs that occur when we separate PDEs into ODEs by separation of variables in Sturm-Liouville form.

A Sturm-Liouville problem consists of:

- Sturm-Liouville equation (5.29a or 5.29b) on an interval $[a, b]$ and
- boundary conditions, which, as you know, specify the behaviour of y at $x = a$ and $x = b$.

It is also assumed that the functions p, p', q and r are continuous and $p > 0$ on (at least) the open interval $a < x < b$. This ensures that solutions of Eq. (5.29a or b) exist.

Thus, the **regular** Sturm-Liouville problem can be stated as follows:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \tag{5.29b}$$

$$c_1 y(a) + d_1 y'(a) = 0 \tag{5.29c}$$

$$c_2 y(b) + d_2 y'(b) = 0 \tag{5.29d}$$

So, in the regular Sturm-Liouville problem:

- $c_1, c_2 \neq 0$ and $d_1, d_2 \neq 0$;
- p, p', q and r are continuous on $[a, b]$;
- p and r are positive on $[a, b]$.

The boundary conditions (5.29b and c) are **homogeneous, mixed, separated** boundary conditions.

A Sturm-Liouville differential equation on an interval $[a, b]$ with periodic boundary conditions and $p(a) = p(b)$ is called a periodic Sturm-Liouville problem/system.

A Sturm-Liouville differential equation on an interval $[a, b]$ with any of the following conditions is called a singular Sturm-Liouville system.

1. $p(a) = 0$, boundary condition at $x = a$ is dropped, boundary condition at $x = b$ is homogenous mixed;

2. $p(a) = 0$, boundary condition at $x = b$ is dropped, boundary condition at $x = a$ is homogenous mixed;

3. $p(b) = p(a) = 0$, and no boundary condition; and

4. interval [*a*, *b*] is infinite.

You should note that:

1. If $p(a) = 0$ and there is no boundary condition at $x = a$, then *y* is considered a solution if $y(a) < \infty$. This is also true for the other cases listed at 2, 3, and 4 above.

2. If the interval is infinite, then *y* can be a solution only if it is square integrable.

You must remember that:

A non-zero solution *y* of the Sturm-Liouville problem [Eq. (5.29b)] along with the boundary conditions, is called an **eigenfunction**, and the corresponding value of λ is called its **eigenvalue**. The eigenvalues of a Sturm-Liouville problem are the values of λ for which non-zero solutions exist.

We can talk about eigenvalues and eigenfunctions for regular or singular problems. Our goal is to determine all values of λ for which a nontrivial solution *y* exists. Let us now consider an example of a Sturm-Liouville problem and convert a given equation into Sturm-Liouville equation.

Example 5.3

a) Determine whether the following ODEs are Sturm-Liouville equations and whether the given boundary conditions together with the Sturm-Liouville equations constitute the Sturm-Liouville problem:

   i) $y'' + my = 0, \quad 0 < x < a,$

   $y(0) = y(a) = 0,$

   ii) $x^2 y'' + xy' + (m^2 x^2 - n^2) y = 0, \qquad 0 < x < c,$

   $y(0) = 0$

b) Convert Legendre's differential equation (4.1) into a Sturm-Liouville problem.

Solution : a) i) Let us check if the ODE with the given boundary conditions is a Sturm-Liouville equation. Comparing with Eq. (5.29b), we note that in this ODE:

$$p(x) = r(x) = 1 \text{ and } q(x) = 0$$

So, *p*, p', *q* and *r* being constants are continuous and *p*, *r* > 0 on the open interval $0 < x < a$. Therefore, its solutions exist and it is a Sturm-Liouville problem.

ii) Dividing the ODE by x, we can write it as:

$$xy'' + y' + (\lambda x - \frac{n^2}{x})y = 0, \qquad 0 < x < c,$$

$$p(x) = r(x) = x \text{ and } q(x) = 0$$

Comparing it with Eq. (5.29b), we note that:

$$p(x) = r(x) = x \text{ and } q(x) = -\frac{n^2}{x}$$

Note that at $x = 0$, $p(x)$ and $r(x)$ are not positive. The function $q(x)$ diverges at $x = 0$, which means that it is not continuous at $x = 0$. Further, the boundary condition at $x = c$ is not given. So, this is not a regular Sturm-Liouville problem.

b) Legendre's differential equation is:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

Note that we can write:

$$(1 - x^2)y'' - 2xy' = [(1 - x^2)y']' :$$

Comparing the above equation with Eq. (5.29b), we note that:

$$p(x) = 1 - x^2, \quad q(x) = 0, \qquad r(x) = 1,$$

and the parameter λ is $n(n+1)$. So, the Sturm-Liouville form of Legendre's equation is:

$$[(1 - x^2)y']' + \lambda y = 0 \text{ with } \lambda = n(n+1).$$

We will now state a few properties of the Sturm-Liouville system of equations.

Properties of the Sturm Liouville System

We can get a lot of information about the eigen values and eigenfunctions without actually solving the Sturm-Liouville differential equation simply by virtue of its properties such as the following:

1. The eigen values are always real and bounded below but not above. So, the eigen values form an increasing sequence of real numbers $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \ldots$ with $\lim_{n\to\infty} \lambda_n \to \infty$.
2. If the interval [a, b] is finite, then eigen values are discrete.
3. The eigenfunction y_n corresponding to λ_n is unique (up to a scalar multiple), and has exactly $n - 1$ zeros in the interval $a < x < b$.
4. Eigen functions are oscillatory in nature.
5. Suppose that y_m and y_n are eigenfunctions corresponding to distinct eigenvalues λ_m and λ_n, then y_m and y_n are orthogonal on [a, b] with respect to the weight function $r(x)$:

$$\int_a^b y_m(x) y_n(x) r(x) dx = 0$$

This is the orthogonality property of the Sturm-Liouville system.

Let us take up an example to illustrate the above properties.

Example 5.4

Determine the eigenvalues of the regular Sturm-Liouville problem:

$$y'' + my = 0, \quad 0 < x < a,$$

$$y(0) = y(a) = 0,$$

Do you recognise this equation? This is the ODE separated for the space part of the one-dimensional wave equation and you know that non-zero solutions occur only for:

$$\lambda_n = \frac{n^2\pi^2}{a^2} \text{ (eigenvalues)}$$

The unique eigenfunctions corresponding to these eigenvalues are:

$$y_n = \sin\frac{n\pi x}{a}$$

for $n = 1, 2, 3, \ldots$ You also know that these eigenfunctions satisfy the orthogonality property:

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\,dx = 0 \qquad (5.30)$$

since $r(x) = 1$ for this problem.

Let us now discuss how to expand a given function in terms of orthogonal functions.

5.5 EXPANSION IN ORTHOGONAL FUNCTIONS

You have already learnt how to expand a given function in terms of Legendre polynomials in Unit 4. Let us now generalise this method for any given set of orthogonal functions.

Suppose, the functions $y_n(x)$ form a set of orthogonal functions on an interval [a, b] that satisfy Eq. (5.30). Then we can expand any given function $f(x)$ in the same interval in terms of the set of functions $y_n(x)$ as follows:

$$f(x) = \sum_n c_n y_n(x)$$

where the coefficients c_n are determined from the orthogonality property of the functions $y_n(x)$ [Eq. (5.30)] as:

$$c_n = \frac{1}{N_n}\int_a^b f(x)y_m(x)\,dx$$

where N_n is the normalization constant given by:

$$\int_a^b y_n(x)y_m(x)\,dx = N_n\,\delta_{mn}$$

This is how we can determine c_n and expand a given function in terms of orthogonal functions.

We will now summarise what you have learnt in this unit.

5.6 SUMMARY

In this unit, we have covered the following concepts:

- Hermite polynomials, their generating function, recurrence relations and orthogonality relations.
- Laguerre polynomials, their generating function, recurrence relations and orthogonality relations.
- Sturm-Liouville problem, properties of Sturm-Liouville problem and its solution.
- Expansion of any given function in terms of orthogonal functions.

5.7 TERMINAL QUESTIONS

1. Two operators

$$a = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right) \quad \text{and} \quad a^{\dagger} = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)$$

where $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$ operate on the harmonic oscillator wave function

$$\psi_n = N_n e^{-\frac{1}{2}\xi^2} H_n(\xi)$$

where $$N_n = \left[\left(\frac{m\omega}{\hbar}\right)^{1/2} \frac{1}{2^n n! \pi^{1/2}}\right]^{1/2}$$

Show that $a\psi_n = \sqrt{n}\,\psi_{n-1}$ and $a^{\dagger}\psi_n = \sqrt{n+1}\,\psi_{n+1}$

and $(aa^{\dagger} - a^{\dagger}a)\,\psi_n = \psi_n$

2. The 'zero' line in the fundamental band of the near infrared absorption spectrum of HCl^{35} gas occurs at 3.46×10^{-6}m. This corresponds to a transition from a state with vibrational quantum number zero to a state with quantum number one. Calculate the force constant for HCl bond assuming harmonic oscillator potential.

3. Expand the following function in terms of Legendre polynomials:

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1 & 0 < x < 1 \end{cases}$$

5.8 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. Solve and obtain its general solution given by Eq. (5.2).

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny(x) = \theta$$

Since $x = 0$ is an ordinary point of Hermite's differential equation [Eq. (5.1)], its solution in the form of a power series in x is given by

$$y = \sum_{j=0}^{\infty} a_j x^j \qquad \text{(i)}$$

with

$$a_{j+2} = -\frac{2(n-j)}{(j+1)(j+2)} a_j \qquad \text{(ii)}$$

You should verify Eq. (ii) on your own. Eq. (ii) tells us that for even positive integral value of j, the coefficients a_j can be expressed in terms of a_0 and for odd positive integral values of (j > 1) the coefficients can be expressed in terms of a_1:

$$a_2 = -\frac{2n}{1\times 2} a_0$$

$$a_4 = -\frac{2(n-2)}{3\times 4} \times a_2 = -\frac{(-2)^2\, n(n-2)}{1\times 2\times 3\times 4} a_0$$

and

$$a_3 = -\frac{2(n-1)}{2\times 3} a_1$$

$$a_5 = -\frac{(n-3)}{4\times 5} a_3 = \frac{(-2)^2 n(n-1)(n-3)}{2\times 3\times 4\times 5} a_1$$

Substituting these results in Eqs. (i) above, we obtain Eq. (5.2).

2. Differentiating both sides of the generating function for Hermite polynomials partially with respect to x, we get:

$$2t\, e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

We can rewrite it as:

$$2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

Equating coefficients of t^{n+1} from both sides, we get:

$$2\frac{H_n(x)}{n!} = \frac{H_{n+1}'(x)}{(n+1)!}$$

or

$$H_{n+1}'(x) = 2(n+1) H_n(x)$$

3. According to Rodrigues' formula:

$$H_4(x) = (-1)^4\, e^{x^2} \frac{d^4}{dx^4}\left(e^{-x^2}\right)$$

Since

$$\frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$$

$$\frac{d^2}{dx^2}\left(e^{-x^2}\right) = (2x)^2\, e^{-x^2} - 2e^{-x^2}$$

$$\frac{d^3}{dx^3}\left(e^{-x^2}\right) = -(2x)^3\, e^{-x^2} + 8x\, e^{-x^2} + 4x\, e^{-x^2}$$

$$= -(2x)^3\, e^{-x^2} + 12x\, e^{-x^2}$$

$$\frac{d^4}{dx^4}\left(e^{-x^2}\right) = (2x)^4\, e^{-x^2} - 24\, x^2 e^{-x^2} - 24x^2\, e^{-x^2} + 12e^{-x^2}$$

$$\therefore \qquad H_4(x) = 16x^4 - 48x^2 + 12$$

4. From the expression for the generating function of Hermite polynomials, we can write:

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} e^{2xt - t^2 + 2xu - u^2}\, dx$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-\infty}^{+\infty} x\, e^{-x^2}\, H_n(x) H_m(x)\, dx\, \frac{t^n u^m}{n!\, m!} \qquad \text{(i)}$$

Following the steps used in arriving at Eq. (5.14), we can rewrite the left hand side as:

$$I = e^{2tu} \int_{-\infty}^{+\infty} x e^{-(x-t-u)^2}\, dx$$

We now change the variable of integration and put $z = x - t - u$. Then the above integral takes the form:

$$I = e^{2tu} \int_{-\infty}^{+\infty} (z + t + u) e^{-z^2}\, dz$$

$$= e^{2tu}\left(0 + t\sqrt{\pi} + u\sqrt{\pi}\right) \;\; = \sqrt{\pi} \sum_{n=0}^{\infty} (t+u) \frac{(2tu)^n}{n!}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n}{n!} \left(t^{n+1} u^n + t^n u^{n+1}\right) \qquad \text{(ii)}$$

On equating the coefficients of $t^n\, u^m$ of the series in (ii) with the corresponding coefficients from the right hand side of (i), we obtain the required result:

$$\int_{-\infty}^{+\infty} x\, e^{-x^2} H_n(x) H_m(x)\, dx = 2^n \sqrt{\pi}(n+1)! \text{ if } m = n + 1$$

$$= 2^{n-1}\sqrt{\pi}\; n! \qquad \text{if } m = n - 1$$

$$= 0 \qquad \text{otherwise}$$

5. Show that the Laguerre polynomial defined in Eq. (5.20) is identical with that given in Eq. (5.18).

Recall Leibnitz rule for the *n*th derivative of the product of two polynomials $u(x)$ and $v(x)$:

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + {}^nC_1 \frac{d^{n-1}u}{dx^{n-1}} \frac{dv}{dx} + \ldots + {}^nC_n\, u \frac{d^n v}{dx^n}$$

Now we choose $u(x) = x^n$ and $v(x) = e^{-x}$. Then using Leibnitz rule, we can write:

$$\frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-n}\right) = \frac{e^x}{n!}\left[n!\,e^{-x} + {}^nC_1\frac{n!}{1!}x(-)e^{-x} + {}^nC_2\frac{n!}{2!}x^2(-1)^2 e^{-x}\right.$$

$$\left. + \ldots + {}^nC_n x^n (-1)^n e^{-x}\right]$$

$$= 1 - {}^nC_1\frac{x}{1!} + {}^nC_2\frac{x^2}{2!} - \ldots (-1)^n\, {}^nC_n\frac{x^n}{n!}$$

which is Eq. (5.18).

Terminal Questions

1. $$\frac{d\psi_n}{d\xi} = N_n e^{-(1/2)\xi^2}\left(-\xi H_n(\xi) + \frac{dH_n}{d\xi}\right)$$

$$= N_n e^{-(1/2)\xi^2}\left(-\xi H_n(\xi) + 2n H_{n-1}(\xi)\right)$$

$\therefore$ $$a\psi_n = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\psi_n$$

$$= \frac{N_n}{\sqrt{2}} e^{-(1/2)\xi^2}\, 2n H_{n-1}(\xi)$$

But $$\frac{N_n}{\sqrt{2}}2n = \left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{1}{2^n n!\,\pi^{1/2}}\right]^{1/2}\sqrt{2}\,n$$

$$= \left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{1}{2^{n-1}(n-1)!\,\pi^{1/2}}\right]^{1/2}\sqrt{n}$$

Hence, $$a\psi_n = \sqrt{n}\,N_{n-1}e^{-(1/2)\xi^2}H_{n-1}(\xi) = \sqrt{n}\,\psi_{n-1}$$

Further, $$a^{\dagger}\psi_n = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\psi_n$$

$$= \frac{N_n}{\sqrt{2}} e^{-(1/2)\xi^2}\left[2\xi H_n(\xi) - 2n H_{n-1}(\xi)\right]$$

$$= \frac{N_n}{\sqrt{2}} e^{-(1/2)\xi^2} H_{n+1}(\xi)$$

$$= \sqrt{n+1}\,\frac{1}{\sqrt{2}\sqrt{n+1}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{1}{2^n n!\,\pi^{1/2}}\right] e^{-(1/2)\xi^2} H_{n+1}(\xi)$$

$$= \sqrt{n+1}\,N_{n+1}e^{-(1/2)\xi^2}H_{n+1}(\xi) = \sqrt{n+1}\,\psi_{n+1}$$

The operators $a^{\dagger}$ and a are called raising and lowering operators (step up and step down operators) or (in the context of quantum field theory) creation and annihilation operators. Again,

$$\left(aa^{+} - a^{+}a\right)\psi_n = a\sqrt{n+1}\,\psi_{n+1} - a^{+}\sqrt{n}\,\psi_{n-1}$$

$$= \sqrt{n+1}\sqrt{n+1}\,\psi_n - \sqrt{n}\sqrt{n}\,\psi_n = (n+1-n)\,\psi_n$$

$$= \psi_n$$

2. Let ν be the frequency of the 'zero' line. Then

$$\nu = \frac{c}{\lambda} = \frac{3\times10^{8}}{3.46\times10^{-6}}\,\text{Hz}$$

Also

$$h\nu = E_{n=1} - E_{n=0} = \hbar\omega = \hbar\sqrt{\frac{k}{\mu}}$$

$$\therefore \qquad \frac{k}{\mu} = \left(\frac{2\pi\times3}{3.46}\times10^{14}\right)^2$$

where k is the force constant and μ is the reduced mass of H and Cl^{35}, i.e.

$$\mu = \frac{1\times35}{1+35}\times1.66\times10^{-27}\,\text{kg}$$

Then $$k = \frac{35}{36}\times1.66\times\left(\frac{6\pi}{3.46}\right)^2\times10\,\text{Nm}^{-1} = 479\,\text{Nm}^{-1}$$

3. We put:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \qquad \text{(i)}$$

We have to determine the coefficients a_n. For this, we multiply both sides of Eq. (i) by $P_m(x)$ and integrate from – 1 to + 1. You know that Legendre polynomials are orthogonal, all integrals on the RHS are zero except the one containing a_m. So, we use the orthogonality relation for Legendre polynomials and get:

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x)dx = c_m.\frac{2}{2m+1}$$

Using this result, we get:

$$\int_{-1}^{1} f(x)P_0(x)dx = c_0\int_{-1}^{1}[P_0(x)]^2 dx$$

or $$\int_{0}^{1} dx = c_0.2,\ c_0 = \frac{1}{2};$$

$$\int_{-1}^{1} f(x)P_1(x)dx = c_1\int_{-1}^{1}[P_1(x)]^2 dx$$

or $$\int_{0}^{1} xdx = c_1.\frac{2}{3},\ c_1 = \frac{3}{4};$$

$$\int_{-1}^{1} f(x)P_2(x)dx = c_2\int_{-1}^{1}[P_2(x)]^2 dx$$

or $$\int_{0}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx = c_2.\frac{2}{5},\ c_2 = 0.$$

$$\therefore \qquad f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \ldots$$