Moments

4760

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

n'th moment of r.v. X.

$$\{m_0, m_1, \dots\} = f_{X}(x)$$

or

$$F_X(x)$$

$$\mathcal{N}_n = E\left[(\overbrace{X-\bar{X}}^{\text{zero mean}})^n\right] \longrightarrow E[\overbrace{X-\bar{X}}^{y}] = E[X] - E[\bar{X}] =$$

$$E[c] = \int_{-\infty}^{\infty} c \cdot f_X(x)\,dx = c \cdot \int_{-\infty}^{\infty} f_X(x)\,dx = c$$

$$\bar{X} = E[X]$$

$\downarrow$

n'th centralized moment.

$$\bar{X} - \bar{X} = 0 \Rightarrow E[y] = 0$$

$$\mathcal{N}_n = \int_{-\infty}^{\infty} (x-\bar{X})^n f_X(x)\,dx$$

we take interest in $n=1,2$ for m_n, (m_1, m_2)
and $n=2$ for $\mathcal{N}_n$ $(\mathcal{N}_2)$

$m_0 = E[\underbrace{X^0}_{1}] = 1$ does not give any information about r.v. X.

$m_1 = E[X^1] = E[X]$ is simply the mean value that we already know.

$m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$ refered to as second moment.

$$\left.\begin{array}{l}\mu_0 = E[(x-\bar{x})^0] = 1 \\ \\ \mu_1 = E[(x-\bar{x})^1] = 0\end{array}\right\}$$ does not give any info about X.

$$\mu_2 = E[(x-\bar{x})^2] = E(X^2 - 2X.\bar{X} + \bar{x}^2) = E[x^2] - 2.\bar{x}\underbrace{E[X]}_{\bar{X}} + \underbrace{E[\bar{x}^2]}_{\bar{x}^2}$$

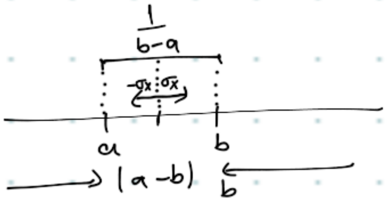
$$= E[X^2] - \bar{X}^2 = E[X^2] - E^2[X]$$ is called "the variance of the r.v. X" $= \sigma_X^2$

$$\sigma_X^2 = \mu_2 = \int_{-\infty}^{\infty} \underbrace{(X-\bar{X})^2}_{\geq 0} \underbrace{f_X(x)}_{\geq 0} dx \quad \geq 0$$

σ_X : standard deviation of r.v. X.
(has the same units as r.v. X)

Ex Consider uniform r.v. uniformly distributed between a and b.

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$



$$\sigma_X^2 = \underbrace{E[X^2]}_{m_2} - \underbrace{E^2[X]}_{m_1^2} \Rightarrow \sigma_X^2 = m_2 - m_1^2$$

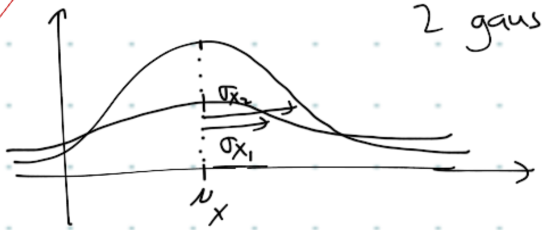
$$m_2 = E[X^2] = \int_a^b x^2 \frac{1}{b-a} dx = \frac{b^3 - a^3}{(b-a).3}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$m_1 = \frac{a+b}{2}$$

$$\Rightarrow \sigma_X^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12} \Rightarrow \boxed{\sigma_X = \frac{|a-b|}{\sqrt{12}}}$$

Ex/



2 gaussian r.v.'s

Systematic Determination of Higher Order Moments

Laplace transform

a) Moment generating function (like the s-transform of $f_X(x)$)

b) Characteristic function (like Fourier transform of $f_X(x)$)

$$F(s) = \mathcal{L}(f(t)) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

$s = jw$

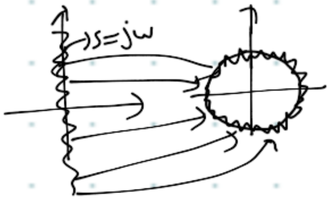
$$F(jw) = \int_{-\infty}^{\infty} e^{-jwt} f(t)\, dt$$

a) $M_X(s) \triangleq E[\underbrace{e^{sX}}_{g(X) = e^{sX}}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$

$\Rightarrow \underline{\underline{M_X(-s)}} = \underline{\underline{\mathcal{L}(f_X(x))}}$

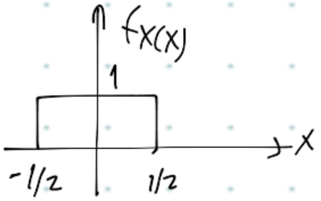
b) $\phi_X(\omega) = M_X(s)\Big|_{s=j\omega} = E[e^{j\omega X}]$

complex exponential



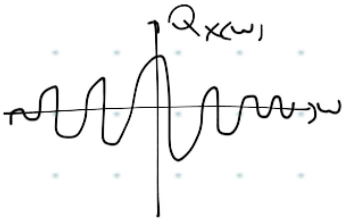
$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$\Rightarrow \underline{\underline{\phi_X(-\omega)}} = \underline{\underline{\text{F.T. of}}}\{f_X(x)\}$



$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x}\, f_X(x)\, dx = \int_{-1/2}^{1/2} e^{j\omega x}\, dx = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}$$

$$= \underbrace{\frac{(e^{j\omega/2} - e^{-j\omega/2})\cdot 2}{2j \cdot \omega}}_{\sin\omega/2} = \frac{\sin\omega/2}{\omega/2}$$



$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$\left.\frac{d\phi_X(\omega)}{j\,d\omega}\right|_{\omega=0} = \left.\int_{-\infty}^{\infty} x \cdot e^{j\omega x} \cdot f_X(x)\, dx\right|_{\omega=0} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = m_1$$

$$\left.\frac{d^2\phi_X(\omega)}{j^2\,d\omega^2}\right|_{\omega=0} = \left.\int_{-\infty}^{\infty} x^2 \cdot e^{j\omega x} \cdot f_X(x)\, dx\right|_{\omega=0} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = m_2$$

$$\vdots$$

$$\frac{d^n\phi_X(\omega)}{j^n\,d\omega^n} = \left.\int_{-\infty}^{\infty} x^n \cdot e^{j\omega x} \cdot f_X(x)\, dx\right|_{\omega=0} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx = \underline{\underline{m_n}}$$

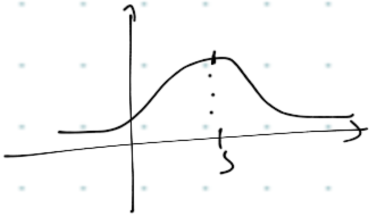
Ex/

$$X \sim N(\underset{\mu_X}{\underset{\downarrow}{0}},\underset{\sigma_X}{\underset{\downarrow}{1}}) \qquad \mu_X = E[X] = \bar{X}$$

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left(-\frac{1}{2}\cdot\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) dx$$

$$= \int_{-\infty}^{\infty} e^{sx}\cdot\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{1}{2}x^2\right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{1}{2}x^2+sx\right) dx$$

$$= e^{s^2/2}\underbrace{\int_{-\infty}^{\infty}\frac{e^{-(1/2)(x-s)^2}}{\sqrt{2\pi}}\,dx}_{\text{like a r.v. } \sim N(s,1)}$$



$$= e^{s^2/2}\cdot 1$$

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx = \frac{d}{ds}M_X(s)\Big|_{s=0} = \int_{-\infty}^{\infty} x\cdot \underbrace{e^{sx}\Big|_{s=0}}_{1} f_X(x)\,dx$$

$$m_1 = \frac{d}{ds}e^{s^2/2}\Big|_{s=0} = e^{s^2/2}\cdot s\Big|_{s=0} = 0$$

$$m_2 = \frac{d^2M_X(s)}{ds^2}\Big|_{s=0} \qquad m_2 = \left[\frac{d}{ds}\left(e^{s^2/2}\cdot s\right)\right]\Big|_{s=0} = \left(e^{s^2/2}\cdot s^2 + e^{s^2/2}\cdot 1\right)\Big|_{s=0}$$

$$= 1 \Rightarrow \sigma_X^2 = \mu_2 = m_2 - m_1^2 = E[X^2] - E^2[X] = \sigma_X^2 = 1 - 0^2 = \underline{\underline{1}}$$

Mean Value of Poisson r.v.

Ex/

$$f_X(x) = \sum_{n=0}^{\infty} \underbrace{\frac{e^{-a}.a^n}{n!}}_{Pr\{X=n\}} \delta(x-n)$$

$$p_n = Pr\{X=n\} = \frac{e^{-a}.a^n}{n!}$$

$$E[X] = \sum_{n=0}^{\infty} x_n p_n = \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!}.n$$

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} \underbrace{f_X(x)\,dx}_{\sum_{n=0}^{\infty} \frac{e^{-a}a^n}{n} \delta(x-n)} = \int_{-\infty}^{\infty} e^{sx} \sum_{n=0}^{\infty} \frac{e^{-a}a^n}{n!} \cdot \delta(x-n)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!} \cdot \int_{-\infty}^{\infty} \underbrace{e^{sx}\,\delta(x-n)}_{e^{sn}\delta(x-n)}\,dx = \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!} e^{sn} \cdot \underbrace{\int_{-\infty}^{\infty} \delta(x-n)\,dx}_{1}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!}.e^{sn}$$

$$m_1 = \frac{d}{ds} M_X(s) \Big|_{s=0} = \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!} \, n.e^{sn} \Big|_{s=0} = a$$

$$\Rightarrow M_X(s) = e^{-a} \sum_{n=0}^{\infty} \frac{a^n a^{\log_a e^{sn}}}{n!} \quad \ldots\ldots$$

$$m_1 = E[X] = \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!}.n = e^{-a}.\underbrace{\sum_{n=0}^{\infty} \frac{a^n.n}{n!}}$$

$$\Rightarrow 1 = \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!} \Rightarrow e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \Rightarrow e^a = \sum_{n=0}^{\infty} \frac{n.a^{n-1}}{n!} \overset{(a.e^{-a})}{\Rightarrow} a = \sum_{n=0}^{\infty} \frac{e^{-a}.n}{n!}$$

$$m_2 = E[X^2] = \sum_{n=0}^{\infty} \frac{e^{-a}.a^n}{n!}.n^2$$

$$e^a = \sum_{n=0}^{\infty} \frac{n.(n-1).a^{n-2}}{n!} \xrightarrow{(a^2.e^{-a})} a^2 = \sum_{n=0}^{\infty} \frac{n.(n-1).a^n.e^{-a}}{n!}$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{n^2 a^n e^{-a}}{n!}}_{m_2} - \underbrace{\sum_{n=0}^{\infty} \frac{n.a^n.e^{-a}}{n!}}_{-m_1 = -a} \Rightarrow m_2 - m_1 = a^2$$

$$\boxed{m_2 = a + a^2}$$

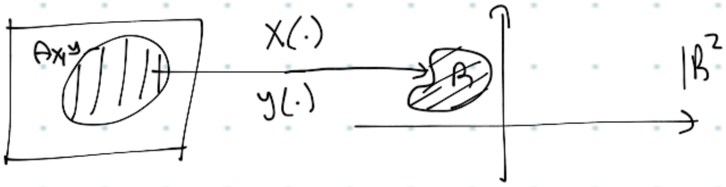
$$\sigma_X^2 = \mu_2 = E[X^2] - E^2[X] = m_2 - m_1^2 = a^2 + a - (a)^2 = \underline{\underline{a}}$$

$\boxed{m_1 = \mu_2 = a}$ for poisson.

$$a = n.p \begin{pmatrix} n \to \infty \\ p \to 0 \end{pmatrix}$$

Multiple Random Variables

Joint event $A_{x,y} = \{ \overset{\text{outcomes of } S}{\zeta} : (X(\zeta), Y(\zeta)) \in B \}$



Focus on special event: ! $A_x = \{ \zeta : X(\zeta) \leq x \}$

$$A_{xy} = \{ \zeta : X(\zeta) \leq x, Y(\zeta) \leq y \}$$

$Pr\{\zeta : X(\zeta) \leq x, Y(\zeta) \leq y\} = F_{X,Y}(x,y)$ → joint distribution function.

$= P_r\{X \leq x, Y \leq y\}$ (a region)

Properties

1.) $F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = \varnothing$

$$F_{X,Y}(-\infty,y) = Pr\{X \le -\infty, Y \le y\}$$

$$Pr\{\underbrace{\zeta : X(\zeta) \le -\infty, Y(\zeta) \le y}\}$$

$$\subseteq \{\zeta : X(\zeta) \le -\infty\} \longrightarrow Pr\{X \le -\infty\} = 0$$

$$Pr\{X \le -\infty\} \geq Pr\{X \le -\infty, Y \le y\}$$

$0 \longrightarrow$ has to be 0.

2) More like a definition.

$$F_{X,Y}(+\infty,y) = Pr\{\zeta : X(\zeta) \le \infty, Y(\zeta) \le y\} = Pr\{Y \le y\} = F_Y(y)$$

↳ marginal dist. function of Y.

$F_{X,Y}(x,+\infty) = F_X(x)$ ↳ marginal dist. func. of X.

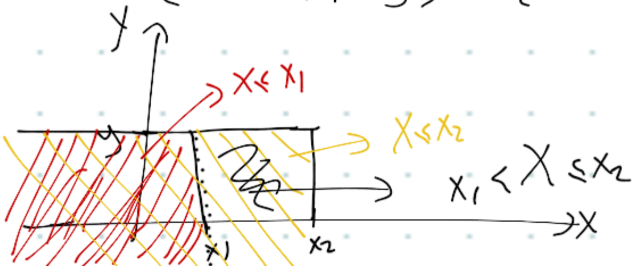
3) $F_{X,Y}(x,y)$ is non-decreasing function in either x or y when the other variable is held fixed.

$$x_1 < x_2$$

$$F_{X,Y}(x_1,y) \le F_{X,Y}(x_2,y)$$

$$\{\zeta : X(\zeta) \le x_2, Y(\zeta) \le y\} = \{\zeta : X(\zeta) \le x_1, Y(\zeta) \le y\} \cup \{\zeta : x_1 < X(\zeta) \le x_2, Y(\zeta) \le y\}$$

$$\{X \le x_2, Y \le y\} = \{X \le x_1, Y \le y\} \cup \{x_1 < X \le x_2, Y \le y\}$$

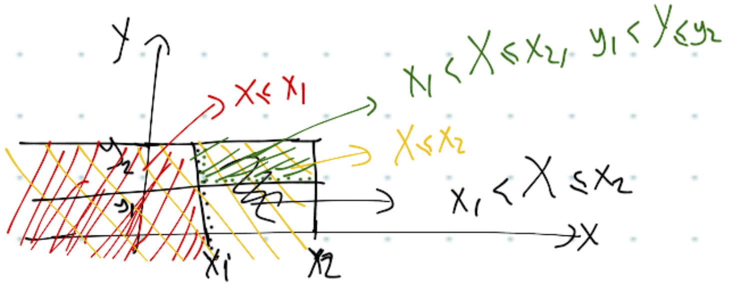


$$Pr\{X \le x_2, Y \le y\} = Pr\{X \le x_1, Y \le y\} + Pr\{x_1 < X \le x_2, Y \le y\}$$

$$F_{X,Y}(x_2,y) - F_{X,Y}(x_1,y) = \underbrace{Pr\{x_1 < X \le x_2, Y \le y\}}_{\rightarrow \geq 0}$$

$$Pr\{x_1 < X \le x_2, Y \le y_1\} = F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1)$$

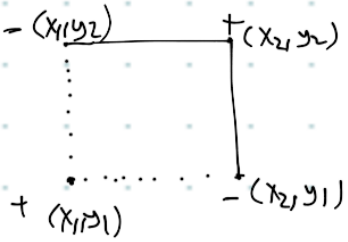
$$Pr\{x_1 < X \le x_2, Y \le y_2\} = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2)$$

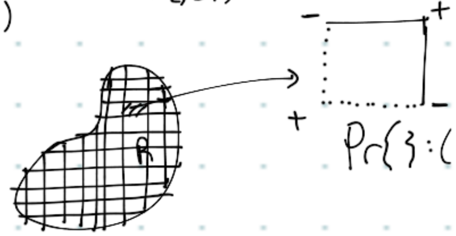


$$\{x_1 < X \le x_2, Y \le y_2\} = \{x_1 < X \le x_2, Y \le y_1\} \cup \{x_1 < X \le x_2, Y \le y_2\}$$

$$Pr\{x_1 < X \le x_2, Y \le y_2\} = Pr\{x_1 < X \le x_2, Y \le y_1\} + Pr\{x_1 < X \le x_2, Y \le y_2\}$$

$$Pr\{x_1 < X \le x_2, y_1 < Y \le y_2\} = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1)$$





$$Pr\{\xi : (X(\xi), Y(\xi)) \in \mathbb{R}\}$$

$$f_X(x) = \frac{d}{dx} F_X(x) \longrightarrow f_{X,Y}(x,y) = \frac{d^2}{dx\,dy} F_{X,Y}(x,y)$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \underbrace{\int_{-\infty}^{x} f_{X,Y}(\alpha,\beta)\, d\alpha}_{g(\beta)}\, d\beta$$