BLG456E
Robotics
2D spatial transforms

Lecture Contents:

Reference frames.

Representing rotations.

Composing transforms.

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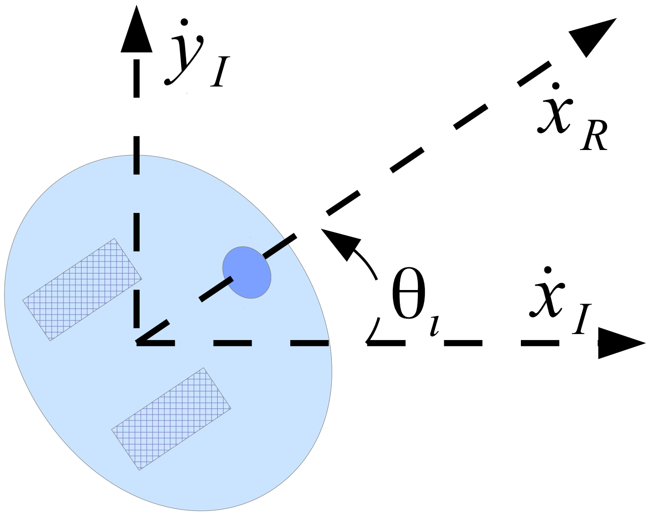
Reminder: differential drive robot velocity

Translational velocity depends on current orientation.

$$\dot{\chi}_I = \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = \begin{bmatrix} \dot{x}_R \cos\theta_I \\ \dot{x}_R \sin\theta_I \\ \dot{\theta}_R \end{bmatrix}$$

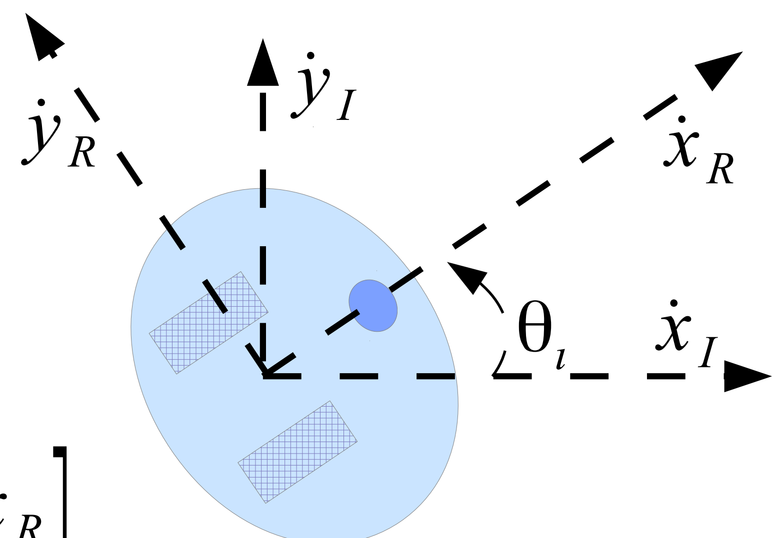


The general case: 2D velocity reference frames

If the robot could move sideways.

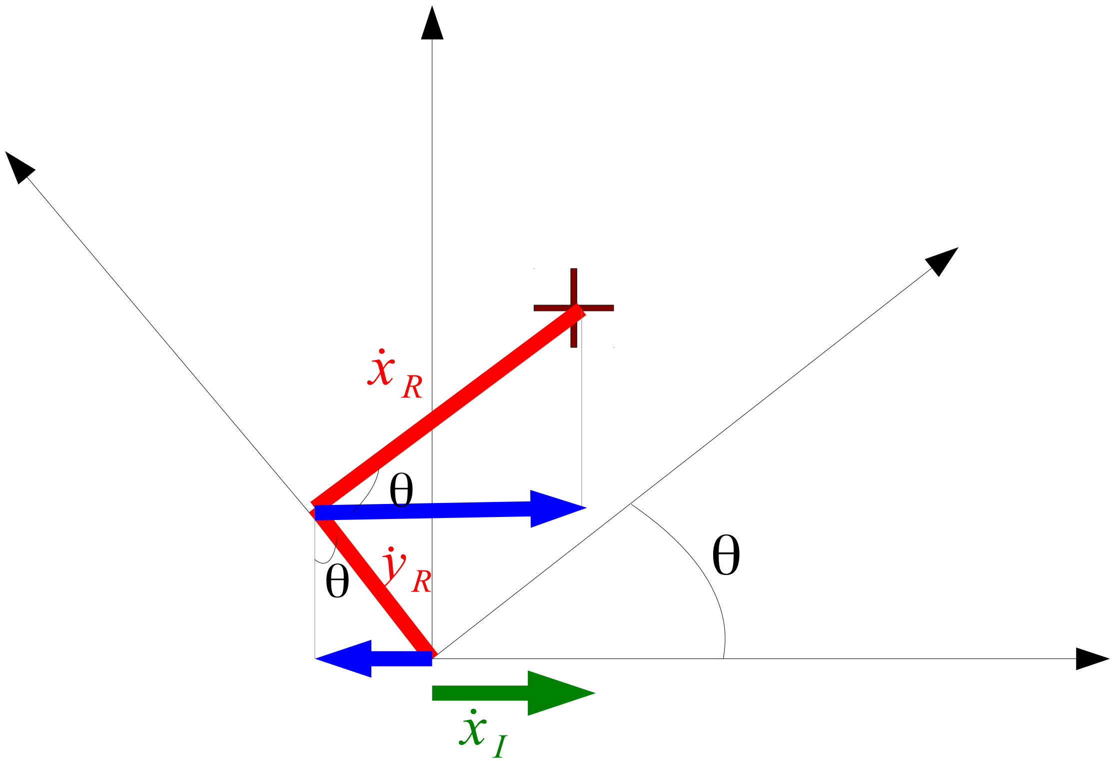
$$\dot{\chi}_I = \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = \begin{bmatrix} \dot{x}_R \cos\theta_I - \dot{y}_R \sin\theta_I \\ \dot{x}_R \sin\theta_I + \dot{x}_R \cos\theta_I \\ \dot{\theta}_R \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_I & -\sin\theta_I & 0 \\ \sin\theta_I & \cos\theta_I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$



Derivation from trig

$$\dot{x}_I = \cos(\theta)\,\dot{x}_R - \sin(\theta)\,\dot{y}_R$$



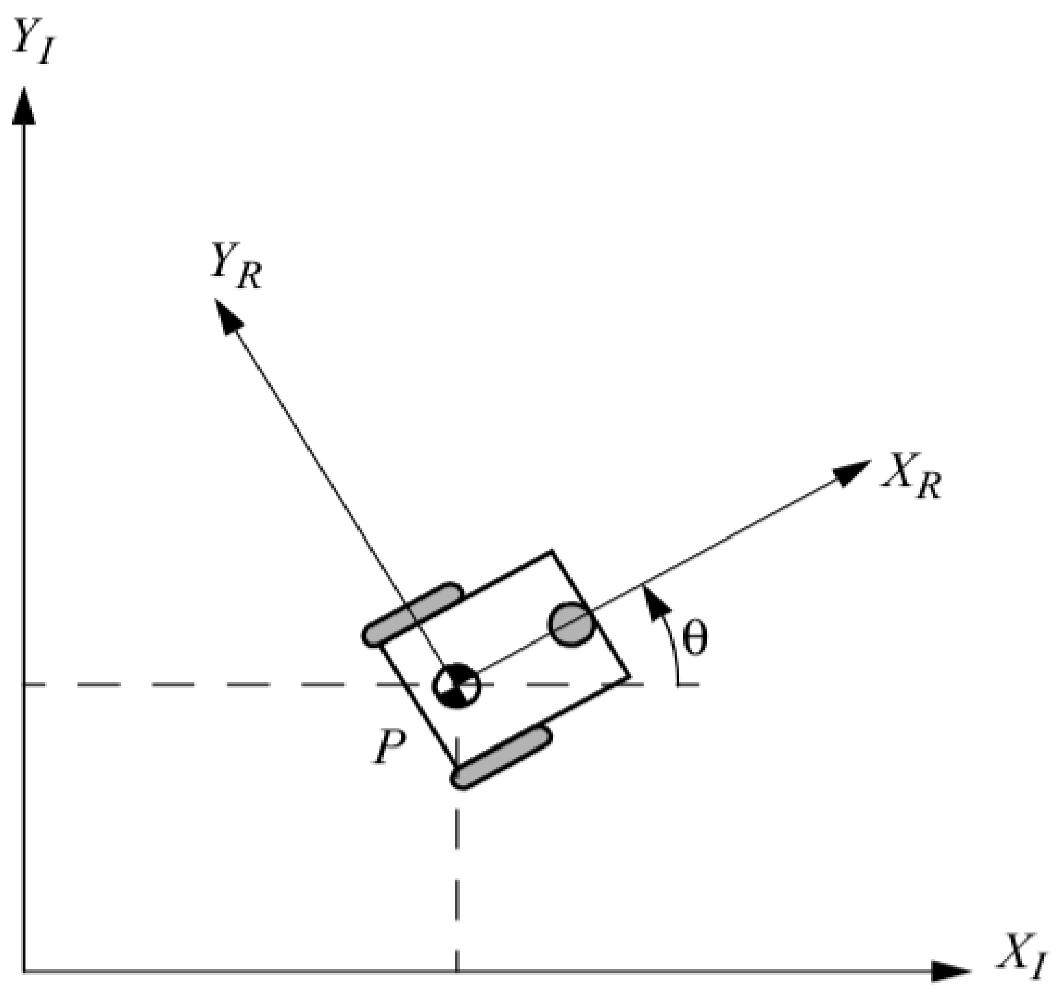
Take home exercises:

Derive $\dot{y}_I$ in terms of $\dot{x}_R, \dot{y}_R$

Derive $\dot{x}_R$ in terms of $\dot{x}_I, \dot{y}_I$

Derive $\dot{y}_R$ in terms of $\dot{x}_I, \dot{y}_I$

Robot and world reference frames



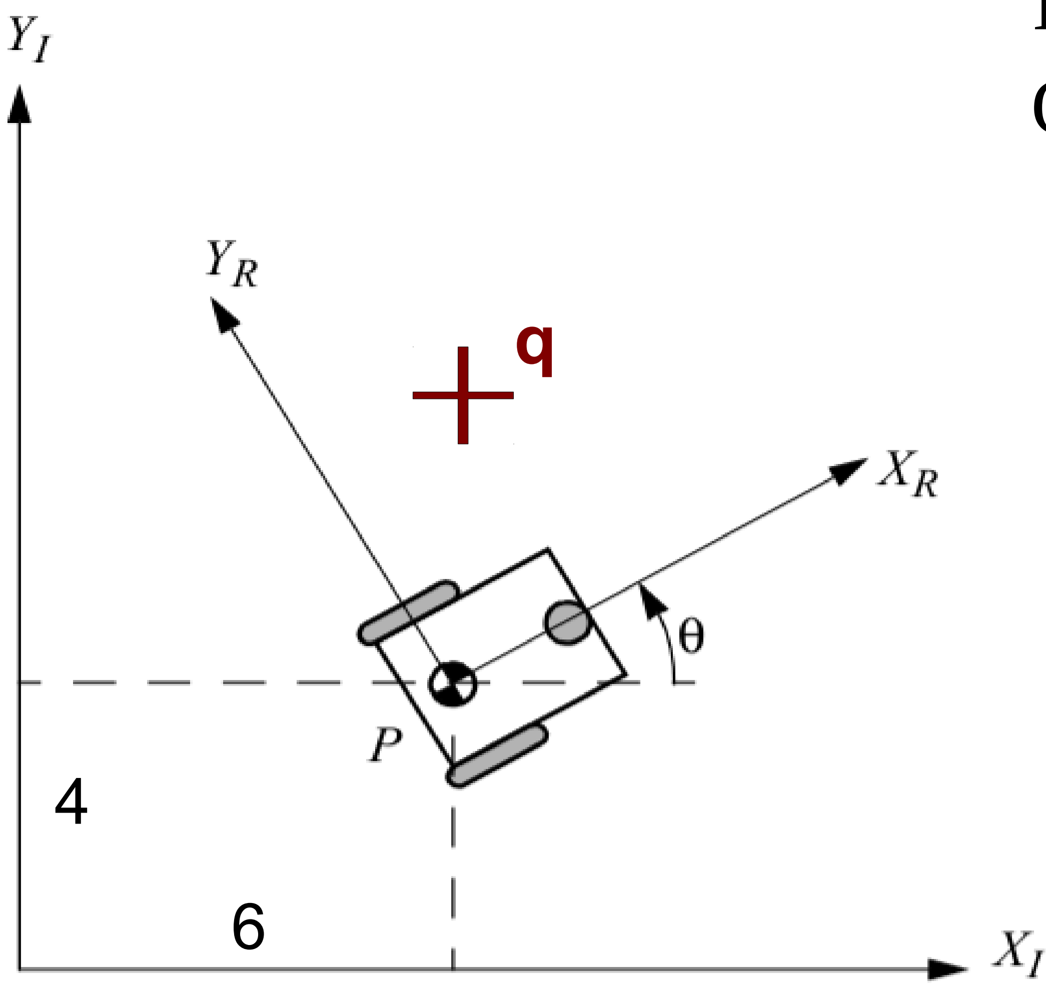
World reference frame:

$x_I \quad y_I$

Robot reference frame:

$x_R \quad y_R$

Points within reference frames



Local (to robot): x_R, y_R

Global (to the world): $x_I, y_I,$

Point q has two different addresses, q_I and q_R.

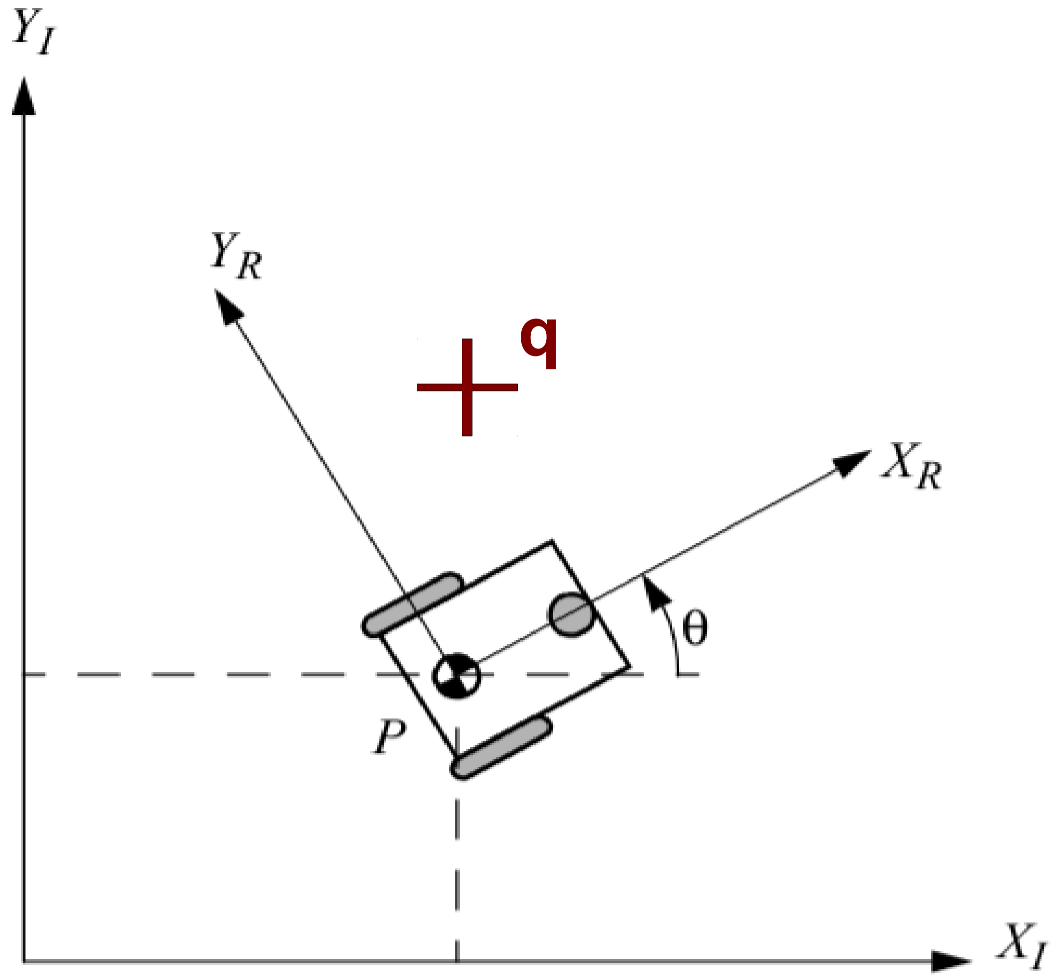
$$q_R = \begin{bmatrix} x_R \\ y_R \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$q_I = \begin{bmatrix} x_I \\ y_I \end{bmatrix} = \begin{bmatrix} 5.3 \\ 7.1 \end{bmatrix}$$

$$\theta = \frac{\pi}{6} \quad P = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

From Siegwart & Nourbaksh.

Transforming points between reference frames



$$\begin{bmatrix} X_I \\ Y_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_R \\ Y_R \end{bmatrix} + \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

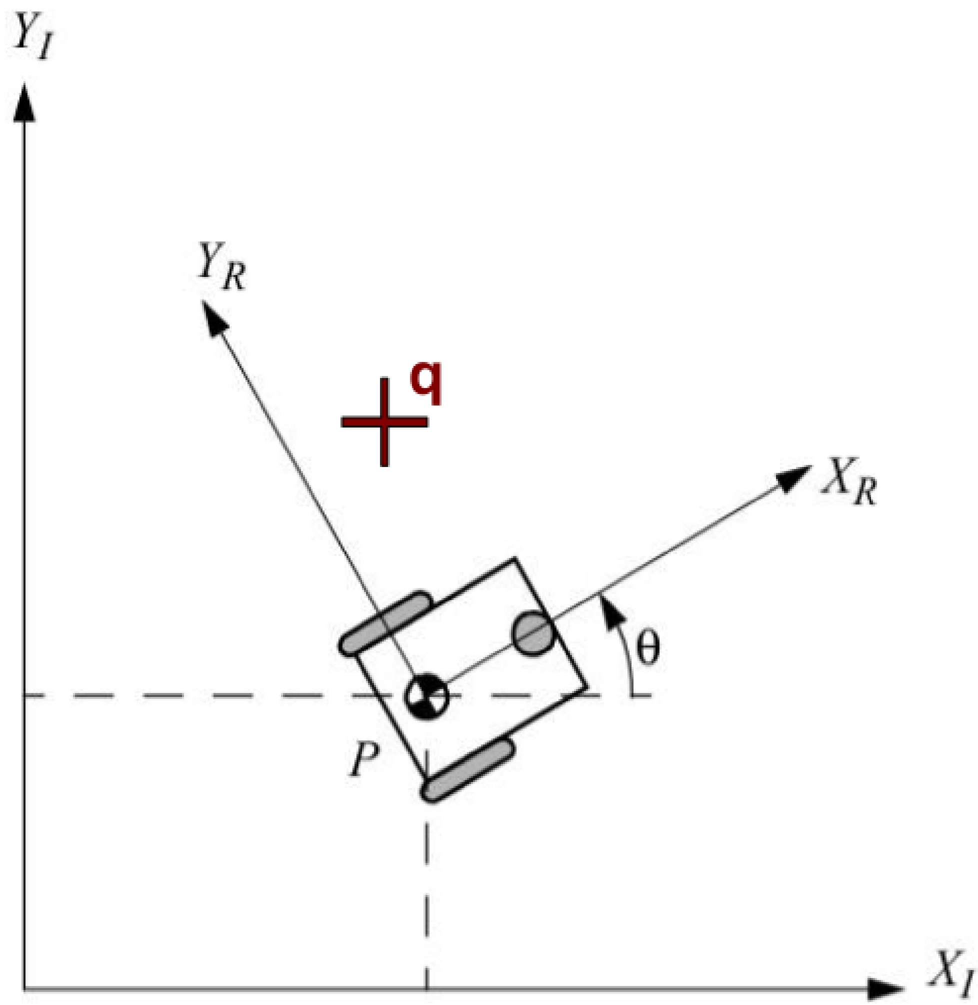
$$q_I = r^{IR}(\theta) q_R + P$$

$$r^{IR}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

From Siegwart & Nourbaksh.

$$\cos\frac{\pi}{2}=0$$
$$\sin\frac{\pi}{2}=1$$

Review: transform a point



$$\begin{bmatrix} X_I \\ Y_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_R \\ Y_R \end{bmatrix} + \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

$$q_I = r^{IR}(\theta) q_R + P$$

<u>Exercise 1:</u>
Rotate the point (3,3) by π/2.

<u>Exercise 2:</u>
Rotate the point (3,3) by π/2 and translate it by [2,2].

Transforming both ways between reference frames

$$q_I = r^{IR}(\theta) q_R + P$$

$$q_R = r^{RI}(\theta)(q_I - P)$$

$$r^{RI}(\theta)^{-1} = r^{IR}(\theta)$$

$$r^{IR}(\theta)^{-1} = r^{RI}(\theta),$$

Spatial vs Body Frames

- Static/Spatial/Fixed-frame transformations:
  - Transforms are applied with respect to a non-moving reference axis.
- Rotating/Body/Moving-frame transformations:
  - Transforms are applied with respect to the axes that moving as the transforms are applied.

Newtonian relativism in reference frames

If points in a moving rigid body are moving with respect to transform T

(in a spatial/fixed or body/moving frame)

Then from the perspective of the rigid body,
points outside the body are moving with respect to
T^{-1}

Combining rotation and translation into a matrix

$$\begin{bmatrix} X_I \\ Y_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_R \\ Y_R \end{bmatrix} + \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

$$\begin{bmatrix} X_I \\ Y_I \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & P_x \\ \sin\theta & \cos\theta & P_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_R \\ Y_R \\ 1 \end{bmatrix}$$

Homogeneous coordinates

Homogeneous coordinates

$$q_I = T(\theta, P) q_R$$
$$q_R = T(\theta, P)^{-1} q_I$$

Rigid body transforms are linear transforms

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Rotation is a linear transform

Rotation is a linear transform.

$$R_\theta(x+y)=R_\theta(x)+R_\theta(y)$$
$$aR_\theta(x)=R_\theta(ax)$$

Rotation is orthogonal and orthonormal.

$$\|R_\theta(x)-R_\theta(y)\|=\|x-y\|,\ R_\theta(x)\cdot R_\theta(y)=x\cdot y$$

→ Rotation is invertible.

$$R_\theta^{-1}(x)=y \quad s.t. \quad y=R_\theta(x)$$

→ Rotation can be expressed by a matrix.

$$R_\theta\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=R_\theta\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

General rigid body transforms are also linear.

2D rotation representation

- Rotation angle.

$$\theta$$

- Rotation matrix.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3D rotation representation (1)

- Angle-axis:
  - Angle of rotation + axis of rotation.

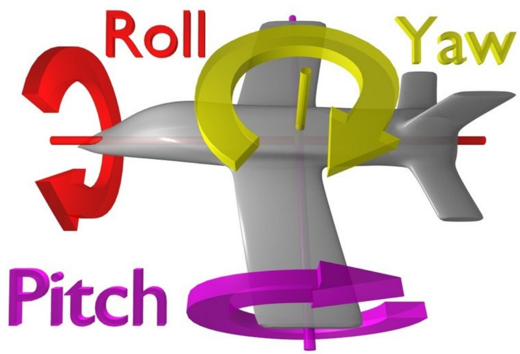
$$\left(\frac{\pi}{2}, 0, 1, 0\right)$$

- Rotation vector:
  - Length represents amount and direction represents axis (angle*axis).

$$\left(0, \frac{\pi}{2}, 0\right)$$

  - Analogous to angular velocity vector part in Twist.

3D rotation representation (II)



- Euler angles (*intrinsic rotations*):
  - Rotate around axes moving with body:
    - Normally roll, pitch, yaw (X,Y Z axes).

(Other directions possible.)

$$\left(0, \frac{\pi}{2}, 0\right)$$

- Fixed frame (*extrinsic rotations*).
  - Rotate around axes fixed in world frame.

$$\left(0, \frac{\pi}{2}, 0\right)$$

Exercise: Give a series of rotations in moving body frame that are different in fixed world frame

3D rotation representation (III)

- Rotation matrix.
  - The 3x3 matrix that, applied to a point, will rotate it.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- Unit quaternion.
  - 4 numbers constrained to length 1, with special properties.

$$\left(0, \sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}\right)$$

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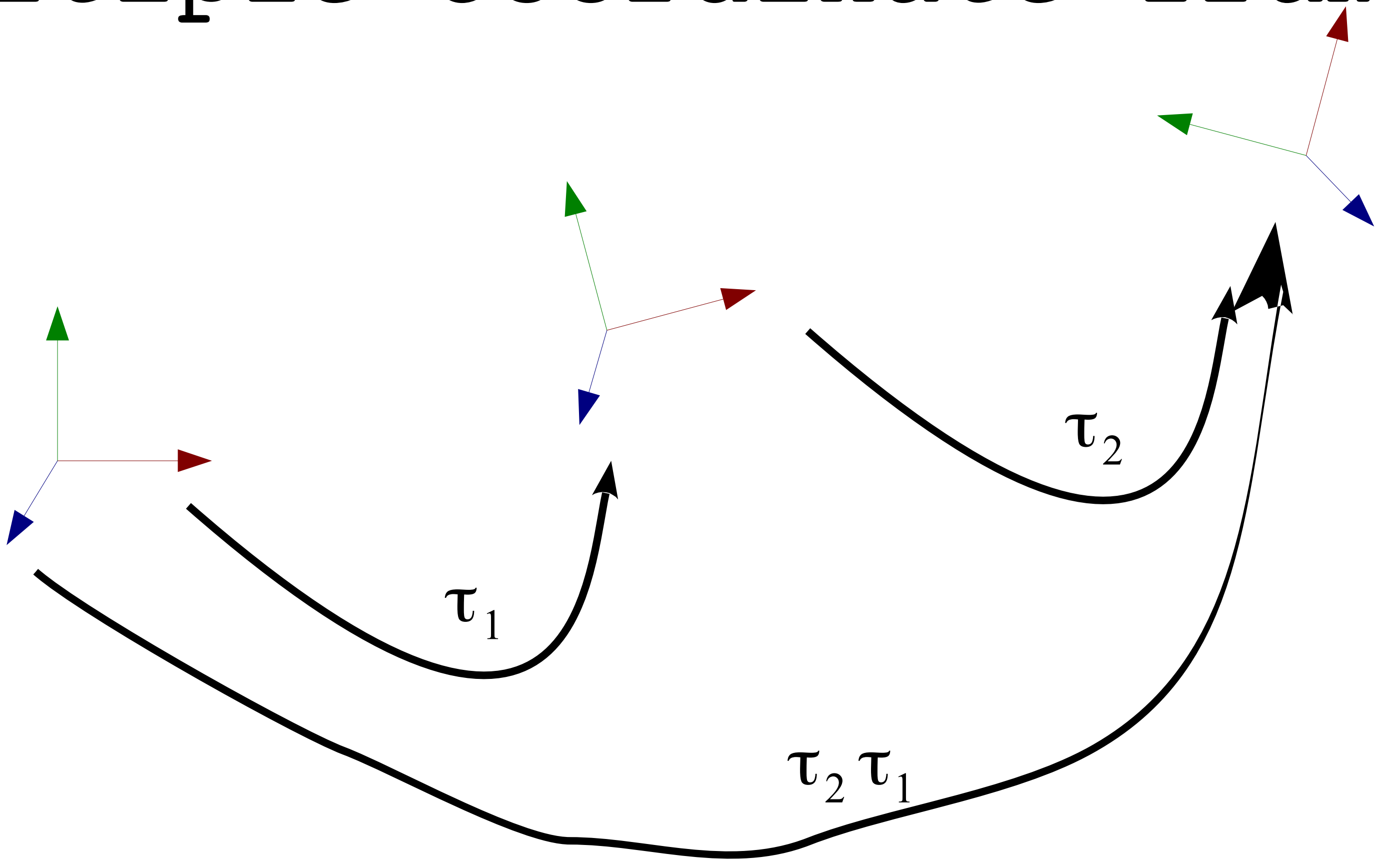
Transformations between multiple coordinate frames



Composition of transformations can be represented by composition of transformation matrices.

This is easiest using homogeneous coordinates.

Introduction to homogeneous coordinates

Non-homogenous

$$q_{inh} = \begin{bmatrix} x_{inh} \\ y_{inh} \end{bmatrix} \xrightarrow{} q_{hom} = \begin{bmatrix} x_{inh} z_{hom} \\ y_{inh} z_{hom} \\ z_{hom} \end{bmatrix}$$

Homogeneous

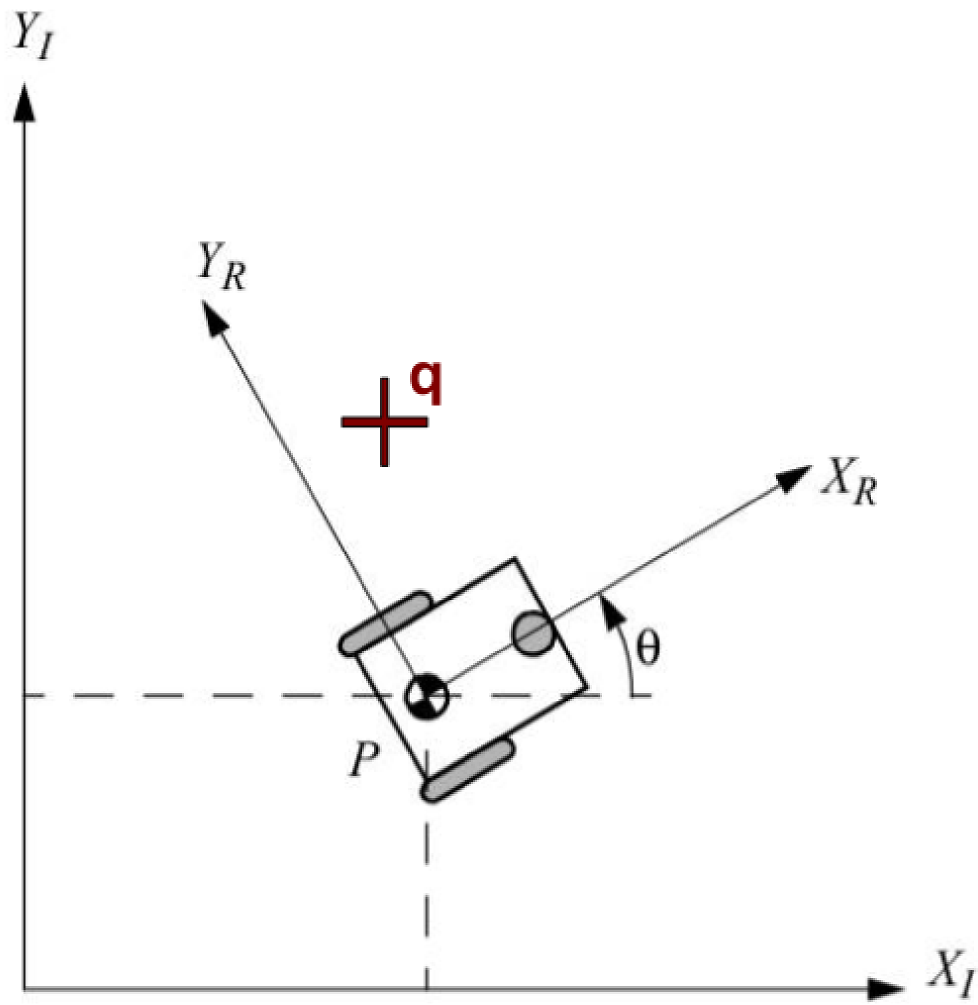
choose a z_{hom}
e.g. $z_{hom} = 1$

Represent 2D entities with 3 numbers!
Why?
Makes many calculations easier!
Can make the geometry easier too!

What to learn more? look up "projective geometry".

Exercise: (3 minutes) Convert the following vectors into homogeneous coordinates: $[5,4]^T$, $[0,0]^T$.

Transform a point with homogeneous coordinates



Homogeneous coordinates make writing transforms easier.

$$q_I = \begin{bmatrix} x_I \\ y_I \\ 1 \end{bmatrix} = T_{RI}(\theta, P)\, q_R$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & P_1 \\ \sin\theta & \cos\theta & P_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_R \\ y_R \\ 1 \end{bmatrix}$$

rotation matrix + translation vector

↓

translation matrix.

Converting from homogeneous coordinates

Homogenous

$$q_{hom} = \begin{bmatrix} x_{hom} \\ y_{hom} \\ z_{hom} \end{bmatrix}$$

$$\xrightarrow{\text{divide by } z_{hom}}$$

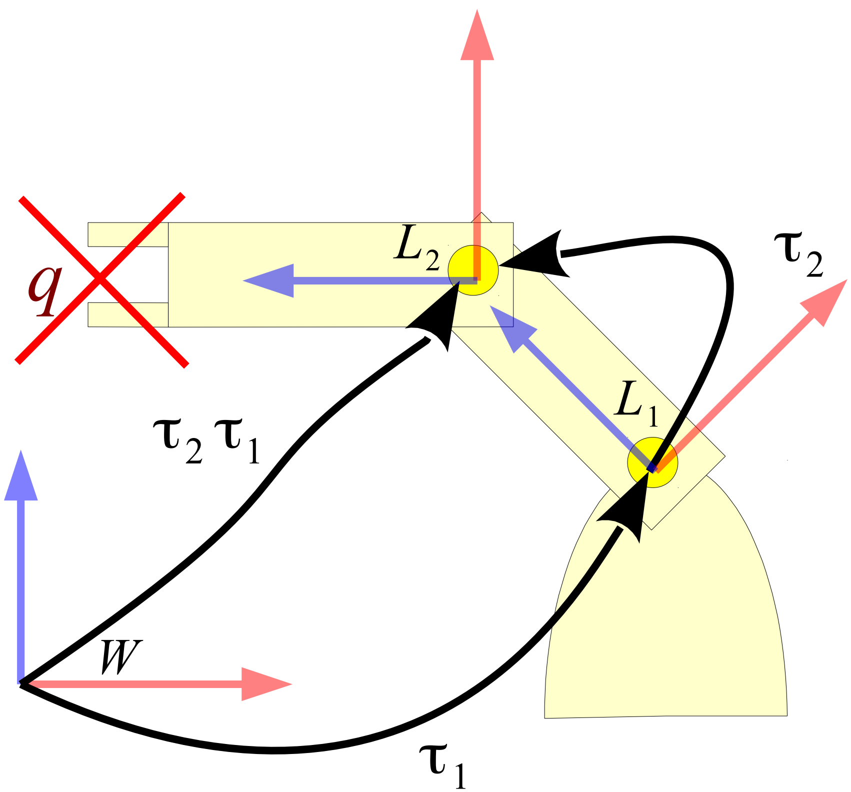
Non-homogeneous

$$q_{inh} = \begin{bmatrix} x_{hom} / z_{hom} \\ y_{hom} / z_{hom} \end{bmatrix}$$

Exercise: (3 minutes) Convert the following homogeneous vectors into non-homogeneous coordinates: $[0,3,6]^T$, $[1,2,0]^T$.

$[1,2,0]^T$ is called a "point at infinity".
can only be represented in homogeneous coordinates.

Forward kinematics in a multiple link arm



$$\tau_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & P_{1x} \\ \sin\theta_1 & \cos\theta_1 & P_{1y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tau_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & P_{2x} \\ \sin\theta_2 & \cos\theta_2 & P_{2y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$q_2 = \tau_2 q_1$$

$$q_1 = \tau_1 q_W$$

$$q_2 = \tau_2 \tau_1 q_W$$

Exercise: Compose the matrices for the two transformations. [note: cos(π/4)=1/sqrt(2) , sin(π/3)=1/sqrt(2)]

Exercise: If q_{L2} = (0,4) find q_W.

$$\cos\frac{\pi}{2}=0$$
$$\sin\frac{\pi}{2}=1$$

Class exercises

Exercise (3 minutes):

Rotate the point (2,2) by π/2 and translate it by [1,5].

$$q_1=\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \quad q_2=\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

$$q_2=\begin{bmatrix} \cos\theta & -\sin\theta & P_x \\ \sin\theta & \cos\theta & P_y \\ 0 & 0 & 1 \end{bmatrix} q_1$$