# Chapter 4: Linear Algebra Background

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Slides for the book
**A First Course in Numerical Methods** (published by SIAM, 2011)
`http://bookstore.siam.org/cs07/`

# Vector norms

A **vector norm** is a function "$\|\cdot\|$" from $\mathbb{R}^n$ to $\mathbb{R}$ that satisfies:

1. $\|\mathbf{x}\| \geq 0; \quad \|\mathbf{x}\| = 0 \text{ iff } \mathbf{x} = \mathbf{0}$,
2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\| \ \forall\ \alpha \in \mathbb{R}$,
3. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

This generalizes absolute value or magnitude of a scalar.

# Famous vector norms

- $\ell_2$-*norm*

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

- $\ell_\infty$-*norm*

$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|.$$

- $\ell_1$-*norm*

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

# Example

- Problem: Find the distance between

$$\mathbf{x} = \begin{pmatrix} 11 \\ 12 \\ 13 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 12 \\ 14 \\ 16 \end{pmatrix}.$$

- Solution: let

$$\mathbf{z} = \mathbf{y} - \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

  and find $\|\mathbf{z}\|$.
- Calculate

$$\begin{aligned} \|\mathbf{z}\|_1 &= 1 + 2 + 3 = 6, \\ \|\mathbf{z}\|_2 &= \sqrt{1 + 4 + 9} \approx 3.7417, \\ \|\mathbf{z}\|_\infty &= 3. \end{aligned}$$

# Matrix norms

**Induced matrix norm** of $m \times n$ matrix $A$ for a given vector norm:

$$\|A\| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \quad = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Then consistency properties hold,

$$\|AB\| \leq \|A\|\|B\|, \quad \|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|,$$

in addition to the previously stated three norm properties.

# Famous matrix norms

- $\ell_2$-*norm*

$$\|A\|_2 = \sqrt{\rho(A^T A)},$$

where $\rho$ is **spectral radius**

$$\rho(B) = \max\{|\lambda|;\ \ \lambda \text{ is an eigenvalue of } B\}.$$

- $\ell_\infty$-*norm*

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

- $\ell_1$-*norm*

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

# Symmetric positive definite matrices

Extend notion of positive scalar to matrices:

$$A = A^T, \quad \mathbf{x}^T A \mathbf{x} > 0, \quad \text{all } \mathbf{x} \neq \mathbf{0}.$$

A symmetric matrix is positive definite if and only if all its eigenvalues are positive:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0.$$

# Orthogonal matrices

**Orthogonal vectors**
Two vectors $\mathbf{u}$ and $\mathbf{v}$ of the same length are orthogonal if

$$\mathbf{u}^T\mathbf{v} = 0.$$

**Orthonormal vectors:** if *also* $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$.

Square matrix $Q$ is **orthogonal** if its columns are pairwise orthonormal, i.e.,

$$Q^TQ = I. \quad \text{Hence also } Q^{-1} = Q^T.$$

Important property: for any orthogonal matrix $Q$ and vector $\mathbf{x}$

$$\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2.$$

Hence

$$\|Q\|_2 = \|Q^{-1}\|_2 = 1.$$