Chapter 05.05
Spline Method of Interpolation – More Examples
Civil Engineering

Example 1

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 1.

Table 1 Temperature vs. depth for a lake.

Temperature, T $(°\text{C})$	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

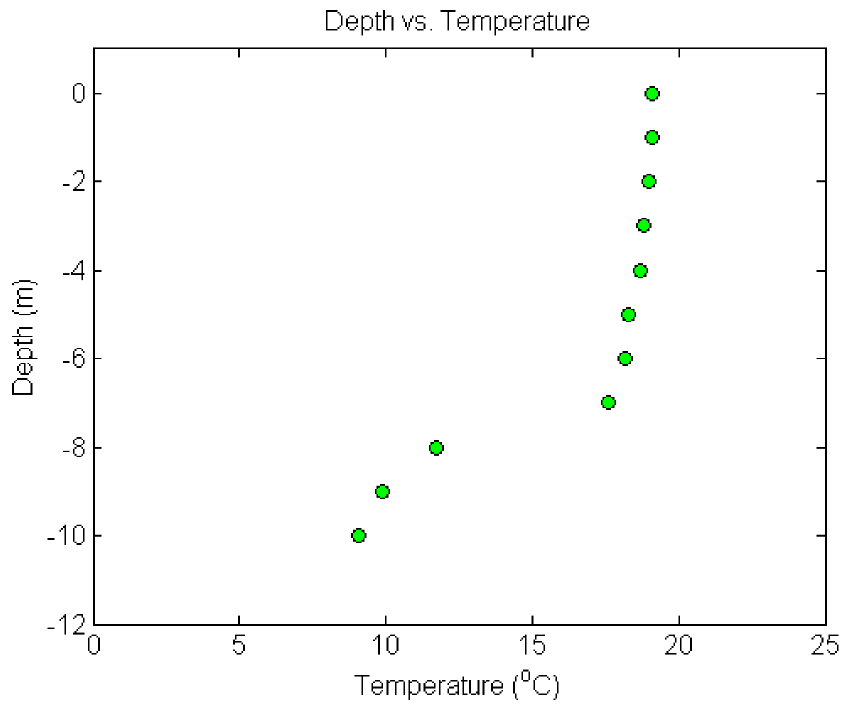

Figure 1 Temperature vs. depth of a lake.

Using the given data, we see the largest change in temperature is between $z = -8$ m and $z = -7$ m. Determine the value of the temperature at $z = -7.5$ m using linear splines.

Solution

Since we want to find the temperature at $z = -7.5$ and we are using linear splines, we need to choose the two data points that are closest to $z = -7.5$ that also bracket $z = -7.5$ to evaluate it. The two points are $z_0 = -8$ and $z_1 = -7$.

Then

$$z_0 = -8, \quad T(z_0) = 11.7$$
$$z_1 = -7, \quad T(z_1) = 17.6$$

gives

$$\begin{aligned} T(z) &= T(z_0) + \frac{T(z_1) - T(z_0)}{z_1 - z_0}(z - z_0) \\ &= 11.7 + \frac{17.6 - 11.7}{-7 + 8}(z + 8) \end{aligned}$$

Hence

$$T(z) = 11.7 + 5.9(z + 8), \; -8 \le z \le -7$$

At $z = -7.5$,

$$\begin{aligned} T(-7.5) &= 11.7 + 5.9(-7.5 + 8) \\ &= 14.65\,^{\circ}\mathrm{C} \end{aligned}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Example 2

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 2.

Table 2 Temperature vs. depth for a lake.

Temperature, T $(°C)$	Depth, z (m)
19.1	0
19.1	–1
19	–2
18.8	–3
18.7	–4
18.3	–5
18.2	–6
17.6	–7
11.7	–8
9.9	–9
9.1	–10

Using the given data, we see the largest change in temperature is between $z = -8$ m and $z = -7$ m. Determine the value of the temperature at $z = -7.5$ m using quadratic splines. Find the absolute relative approximate error for the second order approximation.

Solution

Since there are eleven data points, ten quadratic splines pass through them.

$$
\begin{aligned}
T(z) &= a_1 z^2 + b_1 z + c_1, && -10 \le z \le -9 \\
&= a_2 z^2 + b_2 z + c_2, && -9 \le z \le -8 \\
&= a_3 z^2 + b_3 z + c_3, && -8 \le z \le -7 \\
&= a_4 z^2 + b_4 z + c_4, && -7 \le z \le -6 \\
&= a_5 z^2 + b_5 z + c_5, && -6 \le z \le -5 \\
&= a_6 z^2 + b_6 z + c_6, && -5 \le z \le -4 \\
&= a_7 z^2 + b_7 z + c_7, && -4 \le z \le -3 \\
&= a_8 z^2 + b_8 z + c_8, && -3 \le z \le -2 \\
&= a_9 z^2 + b_9 z + c_9, && -2 \le z \le -1 \\
&= a_{10} z^2 + b_{10} z + c_{10}, && -1 \le z \le 0
\end{aligned}
$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1z^2+b_1z+c_1$ passes through $z=-10$ and $z=-9$.

$$a_1(-10)^2+b_1(-10)+c_1=9.1 \quad (1)$$

$$a_1(-9)^2+b_1(-9)+c_1=9.9 \quad (2)$$

$a_2z^2+b_2z+c_2$ passes through $z=-9$ and $z=-8$.

$$a_2(-9)^2+b_2(-9)+c_2=9.9 \quad (3)$$

$$a_2(-8)^2+b_2(-8)+c_2=11.7 \quad (4)$$

$a_3z^2+b_3z+c_3$ passes through $z=-8$ and $z=-7$.

$$a_3(-8)^2+b_3(-8)+c_3=11.7 \quad (5)$$

$$a_3(-7)^2+b_3(-7)+c_3=17.6 \quad (6)$$

$a_4z^2+b_4z+c_4$ passes through $z=-7$ and $z=-6$.

$$a_4(-7)^2+b_4(-7)+c_4=17.6 \quad (7)$$

$$a_4(-6)^2+b_4(-6)+c_4=18.2 \quad (8)$$

$a_5z^2+b_5z+c_5$ passes through $z=-6$ and $z=-5$.

$$a_5(-6)^2+b_5(-6)+c_5=18.2 \quad (9)$$

$$a_5(-5)^2+b_5(-5)+c_5=18.3 \quad (10)$$

$a_6z^2+b_6z+c_6$ passes through $z=-5$ and $z=-4$.

$$a_6(-5)^2+b_6(-5)+c_6=18.3 \quad (11)$$

$$a_6(-4)^2+b_6(-4)+c_6=18.7 \quad (12)$$

$a_7z^2+b_7z+c_7$ passes through $z=-4$ and $z=-3$.

$$a_7(-4)^2+b_7(-4)+c_7=18.7 \quad (13)$$

$$a_7(-3)^2+b_7(-3)+c_7=18.8 \quad (14)$$

$a_8z^2+b_8z+c_8$ passes through $z=-3$ and $z=-2$.

$$a_8(-3)^2+b_8(-3)+c_8=18.8 \quad (15)$$

$$a_8(-2)^2+b_8(-2)+c_8=19 \quad (16)$$

$a_9z^2+b_9z+c_9$ passes through $z=-2$ and $z=-1$.

$$a_9(-2)^2+b_9(-2)+c_9=19 \quad (17)$$

$$a_9(-1)^2+b_9(-1)+c_9=19.1 \quad (18)$$

$a_{10}z^2 + b_{10}z + c_{10}$ passes through $z = -1$ and $z = 0$.

$$a_{10}(-1)^2 + b_{10}(-1) + c_{10} = 19.1 \quad (19)$$

$$a_{10}(0)^2 + b_{10}(0) + c_{10} = 19.1 \quad (20)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $z = -9$

$$2a_1(-9) + b_1 - 2a_2(-9) - b_2 = 0 \quad (21)$$

At $z = -8$

$$2a_2(-8) + b_2 - 2a_3(-8) - b_3 = 0 \quad (22)$$

At $z = -7$

$$2a_3(-7) + b_3 - 2a_4(-7) - b_4 = 0 \quad (23)$$

At $z = -6$

$$2a_4(-6) + b_4 - 2a_5(-6) - b_5 = 0 \quad (24)$$

At $z = -5$

$$2a_5(-5) + b_5 - 2a_6(-5) - b_6 = 0 \quad (25)$$

At $z = -4$

$$2a_6(-4) + b_6 - 2a_7(-4) - b_7 = 0 \quad (26)$$

At $z = -3$

$$2a_7(-3) + b_7 - 2a_8(-3) - b_8 = 0 \quad (27)$$

At $z = -2$

$$2a_8(-2) + b_8 - 2a_9(-2) - b_9 = 0 \quad (28)$$

At $z = -1$

$$2a_9(-1) + b_9 - 2a_{10}(-1) - b_{10} = 0 \quad (29)$$

3. Assuming the first spline $a_1z^2 + b_1z + c_1$ is linear,

$$a_1 = 0 \quad (30)$$

$$
\begin{bmatrix}
100 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
81 & -9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 81 & -9 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 64 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 64 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 49 & -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 49 & -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-18 & 1 & 0 & 18 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -16 & 1 & 0 & 16 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -14 & 1 & 0 & 14 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 1 & 0 & 12 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 1 & 0 & 10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 1 & 0 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 1 & 0 & 6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \\ a_6 \\ b_6 \\ c_6 \\ a_7 \\ b_7 \\ c_7 \\ a_8 \\ b_8 \\ c_8 \\ a_9 \\ b_9 \\ c_9 \\ a_{10} \\ b_{10} \\ c_{10} \end{bmatrix}
=
\begin{bmatrix} 9.1 \\ 9.9 \\ 9.9 \\ 11.7 \\ 11.7 \\ 17.6 \\ 17.6 \\ 18.2 \\ 18.2 \\ 18.3 \\ 18.3 \\ 18.7 \\ 18.7 \\ 18.8 \\ 18.8 \\ 19 \\ 19 \\ 19.1 \\ 19.1 \\ 19.1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solving the above 30 equations gives the 30 unknowns as

i	a_i	b_i	c_i
1	0	0.8	17.1
2	1	18.8	98.1
3	3.1	52.4	232.5
4	–8.4	–108.6	–331
5	7.9	87	255.8
6	–7.6	–68	–131.7
7	7.3	51.2	106.7
8	–7.2	–35.8	–23.8
9	7.1	21.4	33.4
10	–7.2	–7.2	19.1

Therefore, the splines are given by

$$
\begin{aligned}
T(z) &= 0.8z + 17.1, && -10 \le z \le -9 \\
&= z^2 + 18.8z + 98.1, && -9 \le z \le -8 \\
&= 3.1z^2 + 52.4z + 232.5, && -8 \le z \le -7 \\
&= -8.4z^2 - 108.6z - 331, && -7 \le z \le -6 \\
&= 7.9z^2 + 87z + 255.8, && -6 \le z \le -5
\end{aligned}
$$

$$
\begin{aligned}
&= -7.6z^2 - 68z + 131.7, && -5 \le z \le -4 \\
&= 7.3z^2 + 51.2z + 106.7, && -4 \le z \le -3 \\
&= -7.2z^2 - 35.8z - 23.8, && -3 \le z \le -2 \\
&= 7.1z^2 + 21.4z + 33.4, && -2 \le z \le -1 \\
&= -7.2z^2 - 7.2z + 19.1, && -1 \le z \le 0
\end{aligned}
$$

At $z = -7.5$

$$
\begin{aligned}
T(-7.5) &= 3.1(-7.5)^2 + 52.4(-7.5) + 232.5 \\
&= 13.875°\text{C}
\end{aligned}
$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic splines is

$$
\begin{aligned}
|\in_a| &= \left|\frac{13.875 - 14.65}{13.875}\right| \times 100 \\
&= 5.5856\%
\end{aligned}
$$