2016 – Bahar

Probability and Statistics HW #3 Solutions

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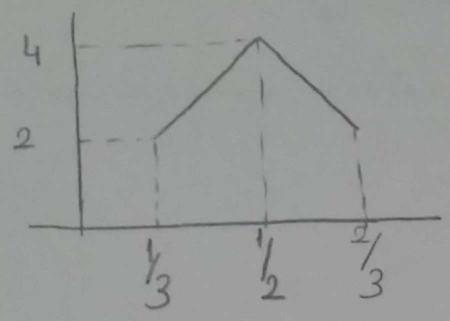
1)



$$f(x) = \begin{cases} 12x-2, & \frac{1}{3} < x < \frac{1}{2} \\ -12x+10, & \frac{1}{2} < x < \frac{2}{3} \end{cases}$$

i) $E[x] = \int_{1/3}^{1/2} 12x^2 - 2x \, dx + \int_{1/2}^{2/3} -12x^2 + 10x \, dx = \frac{1}{2}$

ii) $P(x > \frac{1}{2}) = \frac{1}{2}$

$$f_x(x \mid x > 0{,}5) = \frac{f_x(x)}{P(x > 1/2)} = \frac{-12x+10}{1/2} = \underline{-24x+20}, \quad \frac{1}{2} < x < \frac{2}{3}$$

iii) $E[x \mid x > 0{,}5] = \int_{1/2}^{2/3} -24x^2 + 20x \, dx = \frac{31}{54} \cong \underline{\underline{0{,}574}}$

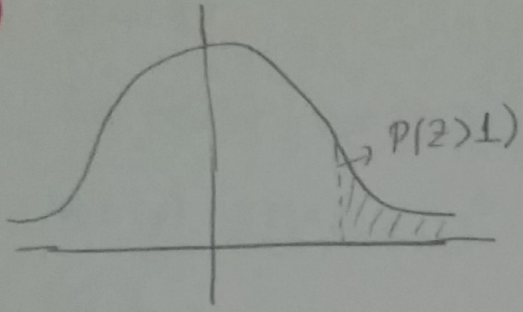
iv) $E[x^2] = \int_{1/3}^{1/2} 12x^3 - 2x^2 \, dx + \int_{1/2}^{2/3} -12x^3 + 10x^2 \, dx \cong \underline{\underline{2{,}6}}$

$Var(x) = E[x^2] - E[x]^2 = \sigma_x^2 = 0{,}26 - \frac{1}{4} = 0{,}01$

2)

X	0	1	2	3	4
f(x)	0,0625	0,25	0,375	0,25	0,625

3)



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad P(X>1) = 0{,}1587$$

$$f_{(X|X>1)} = \frac{f_X(x)}{P(X>1)} = \frac{1}{0{,}1587} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$E[X|X>1] = \int_1^{\infty} \frac{1}{0{,}1587 \cdot \sqrt{2\pi}} \cdot x \cdot e^{-\frac{x^2}{2}} dx$$

$$= e^{-1/2} \cdot \frac{1}{0{,}1587 \cdot \sqrt{2\pi}} \cong 1{,}5247 \;//$$

5) $P(X=i) = \frac{2}{3^i}$

X	1	2	3
P(x)	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$

a) $E[X] = \sum x \cdot P(x) = \sum_{i=1}^{\infty} i \cdot P(i) = 1 \cdot \frac{2}{3} + 2 \cdot \frac{2}{9} + 3 \cdot \frac{2}{27} + 4 \cdot \frac{2}{81} + \dots$

$$= 2\left(\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \dots\right)$$

$$= \frac{2}{3}\left(1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81} + \dots\right)$$

$$= \frac{2}{3} \sum_{n=1}^{\infty} \frac{n}{3^{n-1}} \cong \frac{2}{3} \cdot \frac{9}{4} = \frac{3}{2} \;//$$

b) $Y = X^2$

X	1	4	9	16	25
	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$	$\frac{2}{81}$	$\frac{2}{243}$

$E[Y] = \sum x^2 \cdot P(x) = \sum_{i=1}^{\infty} i^2 P(i)$

$$E[Y] = 1 \cdot \frac{2}{3} + 2^2 \cdot \frac{2}{9} + 3^2 \cdot \frac{2}{27} + 4^2 \cdot \frac{2}{81} \cdot 5^2 \cdot \frac{2}{243} + \dots$$

$$= \frac{2}{3} \sum_{n=1}^{\infty} \frac{n^2}{3^{n-1}} = \frac{2}{3} \cdot \frac{9}{2} = 3 \;//$$

$$E[Y^2] = 1^2 \cdot \frac{2}{3} + 4^2 \cdot \frac{2}{9} + 9^2 \cdot \frac{2}{27} + \dots = \frac{2}{3} \sum_{n=1}^{\infty} \frac{n^4}{3^{n-1}} = \frac{2}{3} \cdot 45 = 30$$

$$\sigma^2 = 30 - 3^2 = \underline{\underline{21}}$$

6) $g(x) = 3x$

$Var(g(x)) = Var(3x) = 9\,Var(x)$

$Var(x) = E[x^2] - E[x]^2$

$E[x] = \frac{2}{5}\int_0^1 x^2 + 2x\,dx = \frac{2}{5}\cdot\left(\frac{x^3}{3} + x^2\Big|_0^1\right) = \frac{2}{5}\left(\frac{1}{3}+1\right) = \frac{8}{15}$

$E[x^2] = \frac{2}{5}\int_0^1 x^3 + 2x^2\,dx = \frac{2}{5}\left(\frac{x^4}{4} + 2\cdot\frac{x^3}{3}\Big|_0^1\right) = \frac{2}{5}\left(\frac{1}{4}+\frac{2}{3}\right) = \frac{2}{5}\cdot\frac{11}{12} = \frac{11}{30}$

$Var(x) = \frac{11}{30} - \left(\frac{8}{15}\right)^2 \cong 0{,}082$

$Var(g(x)) = 9\cdot Var(x) = 9\cdot 0{,}082 = 0{,}74 \cong \underline{\underline{0{,}75}} = 3/4$

7)

$f(x,y)$		X: 1	2
y	1	1/30	4/30
	2	2/30	8/30
	3	3/30	12/30

X	1	2
f(x)	$\frac{6}{30}$	$\frac{24}{30}$

y	1	2	3
f(y)	$\frac{5}{30}$	$\frac{10}{30}$	$\frac{15}{30}$

a) $f(1,2) \overset{?}{=} f(1)\cdot f(2)$

$\frac{2}{30} = \frac{6}{30}\cdot\frac{10}{30}$ ✓ $\Rightarrow$ So, independent!

b) $f(x/y=1) = f(x)$

X	1	2
f(x)	$\frac{6}{30}$	$\frac{24}{30}$

c) $P(x\le 4,\ y=1) = f(1,1) + f(2,1)$

$= \frac{1}{30} + \frac{4}{30} = \frac{5}{30}$ ✓

d) $P(x\le 1,\ y>2) = P(1,3) = \frac{3}{30}$ //

e) $P(x < y+1) = 1 - f(2,1) = 1 - \frac{4}{30} = \underline{\underline{\frac{26}{30}}}$

f) $P(|x-y| = 1) = P(1,2) + P(2,1) + P(2,3) = \frac{18}{30}$ //

9) $A \int_{-1}^{0} \int_{0}^{1} (x-y) \cdot dx \cdot dy = 1$

a)

$$\frac{x^2}{2} - xy \Big|_0^1 = \frac{1}{2} - y \quad \Rightarrow \quad A \cdot \int_{-1}^{0} \frac{1}{2} - y \, dy = 1$$

$$A \left(\frac{y}{2} - \frac{y^2}{2} \Big|_{-1}^{0} \right) = 1$$

$$A \cdot \left(0 - \left(\frac{-1}{2} - \frac{+1}{2}\right)\right) = 1$$

$$\boxed{A = 1}$$

c) $f_y(y) = \int_0^1 x - y \, dx = \frac{x^2}{2} - xy \Big|_0^1 = \frac{1}{2} - y$

$f_x(x) = \int_{-1}^{0} x - y \, dy = xy - \frac{y^2}{2} \Big|_{-1}^{0} = 0 - \left(-x - \frac{1}{2}\right) = x + \frac{1}{2}$

if $f(x,y) = f(x) \cdot f(y) \Rightarrow$ independent

$x - y \neq \left(\frac{1}{2} - y\right)\left(x + \frac{1}{2}\right) \Rightarrow$ so, x and y are dependent!

d) $P(0 \leq x \leq 0{,}5, \quad 0{,}25 \leq y \leq 0{,}5) = \underline{\underline{0}}$

e) $P(\underbrace{x - y < 1}_{x < 1+y}) \Rightarrow \int_{-1}^{0} \int_{0}^{1+y} x - y \, dx \, dy \Rightarrow \frac{x^2}{2} - xy \Big|_0^{1+y} = \frac{(1+y)^2}{2} - \overbrace{(1+y) \cdot y}^{y^2 + y}$

$$= \frac{y^2 + 2y + 1 - 2y^2 - 2y}{2} = \frac{-y^2 + 1}{2}$$

$$\Rightarrow \frac{1}{2} \int_{-1}^{0} -y^2 + 1 \, dy = \frac{1}{2}\left(-\frac{y^3}{3} + y \Big|_{-1}^{0} \right) = \frac{1}{2}\left(0 - \left(\frac{+1}{3} - 1\right)\right) = \boxed{\frac{1}{3}}$$

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10) a)

		X	
		0	1
y	0	0,4	0
	1	0,2	0,4

X	0	1
F(x)	0,6	0,4

y	0	1
F(y)	0,4	0,6

b) $F(x|y=0) \Rightarrow$

y \ X	0	1
0	$\frac{0,4}{0,4}$	$\frac{0}{0,4}$

$\Rightarrow$

	0	1
0	1	0

$F(x|y=1) \Rightarrow$

	0	1
1	$\frac{0,2}{0,6}$	$\frac{0,4}{0,6}$

$\Rightarrow$

	0	1
1	$\frac{1}{3}$	$\frac{2}{3}$

c) $P(x-y<1) = F(0,0) + F(0,1) + F(1,1) = 1$ //

$0,4 + 0,2 + 0,4 = 1$ //

d) $F_{x,y}(0,0) \stackrel{?}{=} F_x(0) \cdot F_y(0)$

$0,4 \neq 0,6 \cdot 0,4 \Rightarrow$ so, dependent!

e) $E[x] = 0,4$ $\quad E[y] = 0,6$

$E[x,y] = \sum\sum x \cdot y \cdot F(x,y) = 0,4$

$\text{cov}(x,y) = E[xy] - E[x] \cdot E[y] = 0,4 - 0,24 = 0,16$

$$\rho = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{0,16}{\sqrt{0,24} \cdot \sqrt{0,24}} = \frac{2}{3}$$ //

$\text{cov}(x,y) \neq 0 \Rightarrow$ so, dependent!

12) i) $\int_0^1 \int_0^{1-y} 24xy \, dx \, dy = 24y \cdot \frac{x^2}{2} \Big|_0^{1-y} = 12y(y^2-2y+1) = 12(y^3-2y^2+y)$

$12\int_0^1 y^3-2y^2+y \, dy = 12\left(\frac{y^4}{4} - 2\cdot\frac{y^3}{3} + \frac{y^2}{2} \Big|_0^1\right) = 12\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) =$

$= 12\left(\frac{3}{12} - \frac{8}{12} + \frac{6}{12}\right) = 12 \cdot \frac{1}{12} = \underline{\underline{1}}$

VALID!

ii) $f(x) = \int_0^{1-x} 24xy \cdot dy = 24 \cdot x \cdot \frac{y^2}{2} \Big|_0^{1-x} = 12x(x^2-2x+1)$

$f_x(y|x) = \frac{24xy}{12x(x-1)^2} = \frac{2y}{(x-1)^2} \Rightarrow \frac{2y}{(0{,}5-1)^2} = \underline{\underline{8y}}$

$P\left(\frac{1}{4} < y < \frac{1}{2} \mid x=1\right) = \int_{1/4}^{1/2} \frac{2y}{(x-1)^2} dy$

iii) $\int_0^1 y \cdot \frac{2y}{(x-1)^2} \cdot dy = \frac{2}{(x-1)^2}\left(\frac{y^3}{3}\Big|_0^1\right) = \frac{2}{3} \cdot \frac{1}{(x-1)^2}$ ✓