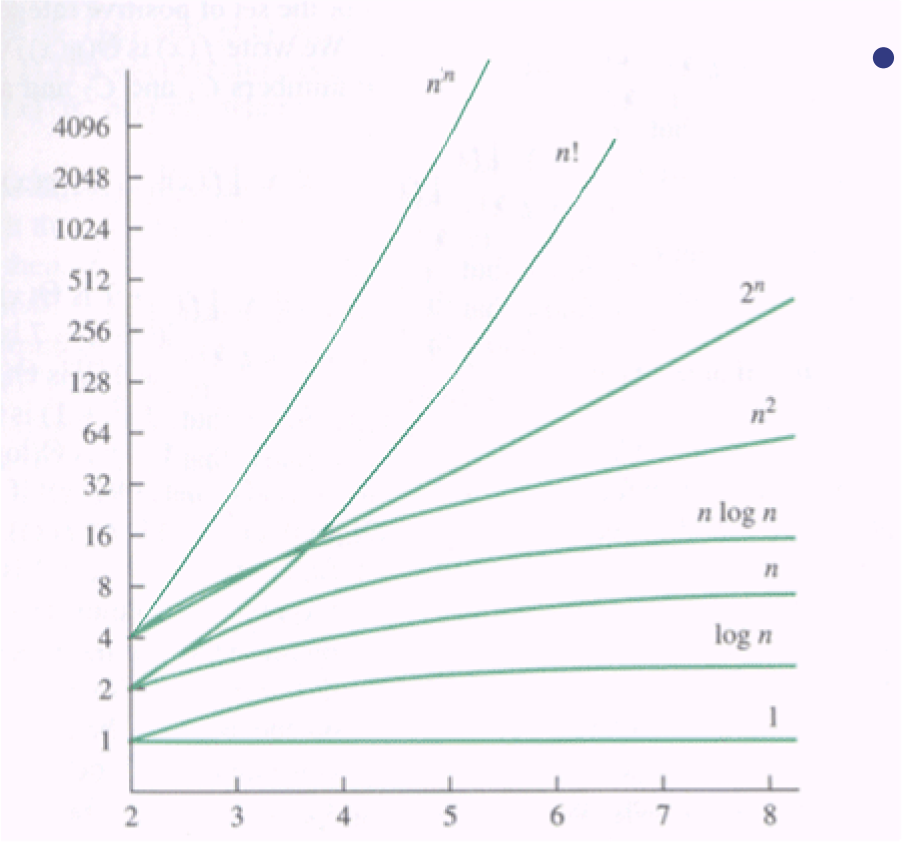
BLG 335E – Analysis of Algorithms I
Fall 2017, Recitation 2
03.10.2017

R.A. Doğan Altan
daltan@itu.edu.tr – 4316
R.A. Umut Sulubacak
sulubacak@itu.edu.tr – Res.Lab.1
R.A. Cumali Türkmen
turkmenc@itu.edu.tr – Res.Lab.2

Warm-up Problem

- Order the following functions by asymptotic growth rate:
  - $n^2 + 5n + 7$
  - $\log_2 n^3$
  - 95^{17}
  - $2^{\log_2 n}$
  - n^3
  - $nlog_2n + 9n$
  - $4 \log_2 n$
  - $\log_2 n + 3n$

Warm-up Problem



- Solution:
  - 95^{17}
  - $\log_2 n^3$
  - $4 \log_2 n$
  - $2^{\log_2 n}$
  - $\log_2 n + 3n$
  - $nlog_2n + 9n$
  - $n^2 + 5n + 7$
  - n^3

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

  a. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$

  b. $T(n) = T\left(\frac{9n}{10}\right) + n$

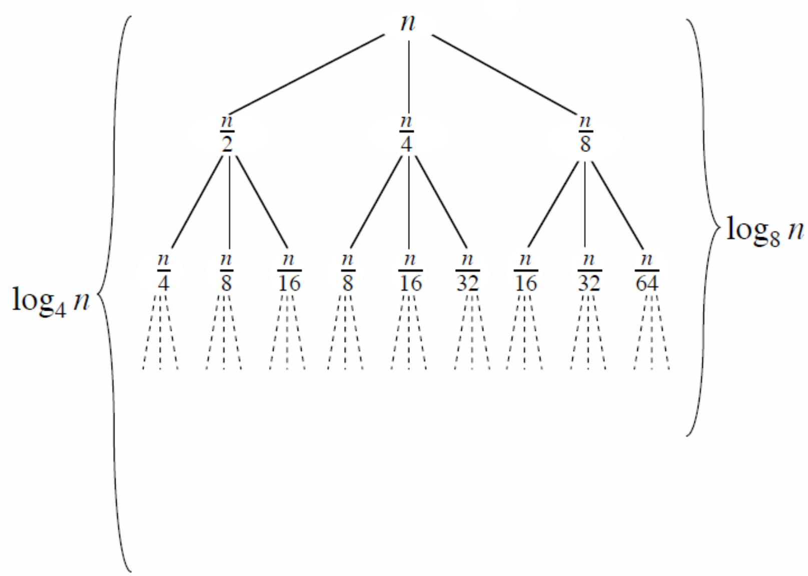
  c. $T(n) = 16T\left(\frac{n}{4}\right) + n^2$

  d. $T(n) = 7T\left(\frac{n}{2}\right) + n^2$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

  b. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$



$$n$$

$$n\left(\frac{4+2+1}{8}\right) = \frac{7}{8}n$$

$$n\left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{2}{32} + \frac{1}{64}\right)$$
$$= n\frac{16+16+12+4+1}{64}$$
$$= n\frac{49}{64} = \frac{7}{8}^{2} n$$

$$\vdots$$

$$\sum_{i=1}^{\log n} \left(\frac{7}{8}\right)^{i} n = \Theta(n)$$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

$b.\ \ T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$

Upper bound (*O*):

$$T(n) \leq \frac{cn}{2} + \frac{cn}{4} + \frac{cn}{8} + n = \frac{7cn}{8} + n \leq cn$$

$$true\ if\ c \geq 8$$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

$b.\ \ T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$

Lower bound (Ω):

$$T(n) \geq \frac{cn}{2} + \frac{cn}{4} + \frac{cn}{8} + n = \frac{7cn}{8} + n \geq cn$$

$$true\ if\ 0 < c \leq 8$$

Master Method

$$T(n) = aT(n/b) + f(n)$$

1 $f(n) = O\left(n^{\log_b a-\varepsilon}\right) \Rightarrow T(n) = \Theta\left(n^{\log_b a}\right)$

2 $f(n) = \Theta\left(n^{\log_b a}\right) \Rightarrow T(n) = \Theta\left(n^{\log_b a} \log_2 n\right)$

3 $f(n) = \Omega\left(n^{\log_b a+\varepsilon}\right) \quad and \quad af(n/b) \le cf(n),$

$for \quad \exists c \quad c < 1 \quad and \quad n > n_0$

$\Rightarrow T(n) = \Theta\left(f(n)\right)$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

c. $T(n) = T\left(\frac{9n}{10}\right) + n$

$$a = 1, b = \frac{10}{9}, f(n) = n = \Omega\left(\mathrm{n}^{\log_{\frac{10}{9}}1+1}\right)$$

$$possibly\ case\ 3, let's\ check\ c$$

$$1\frac{9n}{10} \leq cn\ holds\ for\ c = \frac{9}{10} \leq 1$$

$$certainly\ case\ 3:$$

$$T(n) = \Theta(\mathrm{n})$$

Problem 1

- Give tight asymptotic bounds for $T(n)$ in each of the following recurrences.

  d. $T(n) = 16T\left(\frac{n}{4}\right) + n^2$ $a = 16, b = 4, f(n) = n^2$

  $$n^2 = \Theta\left(n^{\log_4 16}\right), case\ 2:$$

  $$T(n) = \Theta(n^2 \log_2 n)$$

  e. $T(n) = 7T\left(\frac{n}{2}\right) + n^2$ $a = 7, b = 2, f(n) = n^2$

  $$n^2 = O\left(n^{\log_2 7 - \varepsilon}\right), case\ 1:$$

  $$T(n) = \Theta(n^{\log_2 7})$$

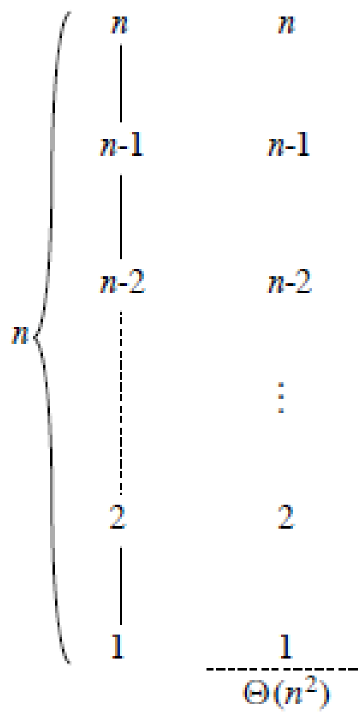
Problem 2

- Give asymptotic upper and lower bounds for $T(n)$ in for the following recurrence.

$$T(n) = T(n-1) + n$$

Solution 2

- Using the recursion tree shown below, we get a guess of $T\,(n) = \Theta(n^2)$.



Solution 2

- First, we prove the $T(n) = \Omega(n^2)$ part by induction. The inductive hypothesis is $T(n) \geq cn^2$ for some constant $c > 0$.

$$T(n) = T(n-1) + n$$
$$\geq c(n-1)^2 + n$$
$$= cn^2 - 2cn + c + n$$
$$\geq cn^2$$

- if $-2cn + n + c \geq 0$ or, equivalently, $n(1 - 2c) + c \geq 0$.
- This condition holds when $n \geq 0$ and $0 < c \leq 1/2$.

Solution 2

- For the upper bound, $T\,(n) = O(n^2)$, we use the inductive hypothesis that

$$T\,(n) \leq cn^2$$

for some constant $c > 0$.

- By a similar derivation, we get that $T(n) \leq cn^2$ if $-2cn + n + c \leq 0$ or, equivalently, $n(1 - 2c) + c \leq 0$.
- This condition holds for $c = 1$ and $n \geq 1$.
- Thus, $T\,(n) = \Omega(n^2)$ and $T\,(n) = O(n^2)$, so we conclude that $T\,(n) = \Theta(n^2)$.

Problem 3

- Determine a tight inclusion of the form **f(n) ∈ Δ(g(n))** for following functions using **limit method**.

$$\mathrm{f(n)} = \log(n^2)\ , g(n) = logn\ + 8$$

Solution 3

Using limit method we can set up a limit quotient between f and g functions as follows:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \text{then } f(n) \in \mathcal{O}(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ \infty & \text{then } f(n) \in \Omega(g(n)) \end{cases}$$

1. We look for algebraic simplifications first.
2. If **f** and **g** both diverge or converge on zero or infinity, then we need to apply **l'Hôpital's Rule**.

Solution 3

l'Hôpital's Rule:

Let **f** and **g**, if the limit between the quotient $\frac{f(n)}{g(n)}$ exists, it is equal to the limit of the derivative of the denominator and the numerator.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)'}{g(n)'}$$

Solution 3

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \text{then } f(n) \in \mathcal{O}(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ \infty & \text{then } f(n) \in \Omega(g(n)) \end{cases}$$

$$\lim_{n\to\infty} \frac{\log(n^2)}{\log(n\,)+8} = \lim_{n\to\infty} \frac{\frac{2}{nln10}}{\frac{1}{nln10}} =$$

$$\lim_{n\to\infty}(2) = 2$$

$$0 < \lim_{n\to\infty} \frac{\log(n^2)}{\log(n\,)+8} = 2 < \infty$$

We can say **f(n) = Θ(g(n))**

Problem 4

- Determine a tight inclusion of the form **f(n) ∈ Δ(g(n))** for following functions using **limit method**.

$$\mathrm{f(n)} = 2^{\mathrm{n}} \;, g(n) = 3^{\mathrm{n}}$$

Solution 4

- $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \lim\limits_{n \to \infty} \frac{2^{\mathrm{n}}}{3^{\mathrm{n}}}$
- **l'Hôpital's Rule:**

$$\frac{(2^n)'}{(3^n)'} = \frac{(\ln 2)2^n}{(\ln 3)3^n}$$

- Both numerator and denominator still diverge. We'll have to use an algebraic simplification.

Solution 4

- $\lim_{n\to\infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

$$\lim_{n\to\infty} \alpha^n = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

We can say $2^n \in O(3^n)$