Linear Algebra and Applications
3 November 2014

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References:
-***Elementary Linear Algebra-Applications Version"***,
Howard Anton and Chris Rorres, 9^{th} Edition, Wiley, 2010.

Definition (Vector Space)

- Let V be an arbitrary nonempty set of objects on which two operations are defined:
  - Addition
  - Multiplication by scalars
- If the following *axioms* are satisfied by all objects **u**, **v**, **w** in V and all scalars k and l, then we call V a vector space and we call the objects in V vectors.
- (see Next Slide)

Definition (Vector Space)

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
9. $k(l\mathbf{u}) = (kl)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

Remarks

- Depending on the application, *scalars* may be real numbers or complex numbers.
  - Vector spaces in which the scalars are complex numbers are called complex vector spaces, and those in which the scalars must be real are called real vector spaces.
- The definition of a vector space specifies neither the nature of the vectors nor the operations.
  - *Any kind of object can be a vector*, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on R^n.
  - *The only requirement is that the ten vector space axioms be satisfied.*

Example (R^n Is a Vector Space)

- The set $V = R^n$ with the standard operations of addition and scalar multiplication is a vector space.

- Axioms 1 and 6 follow from the definitions of the standard operations on R^n; the remaining axioms follow from Theorem 4.1.1.

- The three most important special cases of R^n are R (the real numbers), R^2 (the vectors in the plane), and R^3 (the vectors in 3-space).

The Zero Vector Space

- Let V consist of *a signle object*, which we denote by $\mathbf{0}$, and define $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\,\mathbf{0} = \mathbf{0}$ for all scalars k.
- We called this the zero vector space.

Theorem 5.1.1

- Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:
  - $0\,\mathbf{u} = \mathbf{0}$
  - $k\,\mathbf{0} = \mathbf{0}$
  - $(-1)\,\mathbf{u} = -\mathbf{u}$
  - If $k\,\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Subspaces

- Definition
  - A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.
- Theorem 5.2.1
  - If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold:
    a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.
    b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.
- Remark
  - Theorem 5.2.1 states that W is a subspace of V if and only if W is a closed under addition (condition (a)) and closed under scalar multiplication (condition (b)).

Example

- Let W be any plane through the origin and let **u** and **v** be any vectors in W.
  - **u** + **v** must lie in W since it is the diagonal of the parallelogram determined by **u** and **v**, and k **u** must line in W for any scalar k since k **u** lies on a line through **u**.
- Thus, <u>W is closed under addition and scalar multiplication</u>, so it is a subspace of R^3.

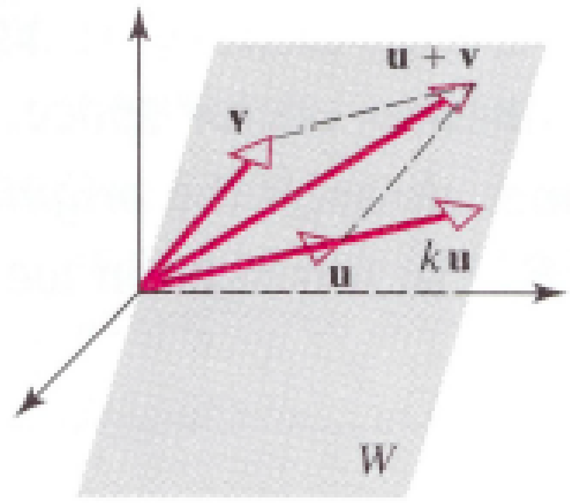
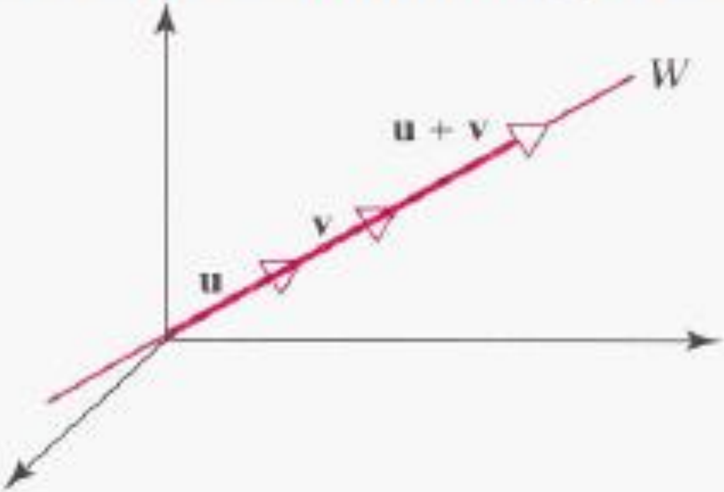


Figure 5.2.1
The vectors **u** + **v** and k**u** both lie in the same plane as **u** and **v**.

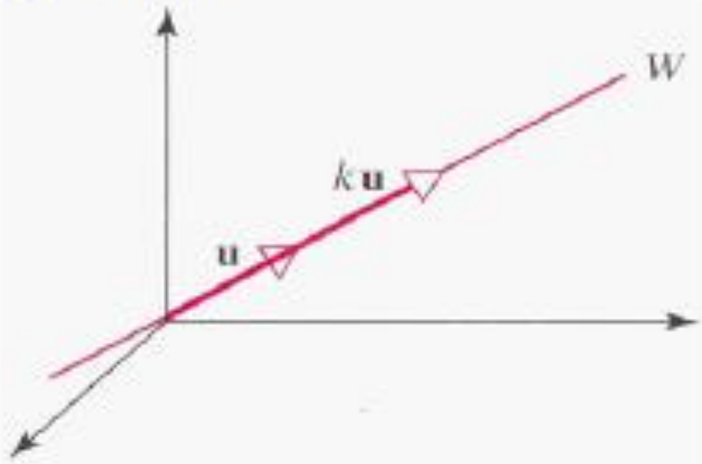
Example

- A line through the origin of R^3 is a subspace of R^3.

- Let W be a line through the origin of R^3.



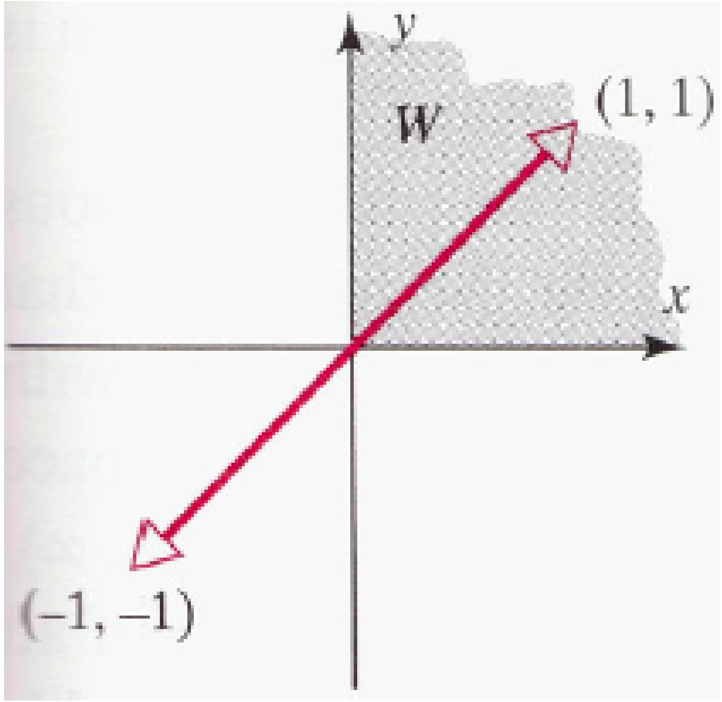
(*a*) W is closed under addition.



(*b*) W is closed under scalar multiplication.

Example (Not a Subspace)

- Let W be the set of all points (x, y) in R^2 such that $x \geq 0$ and $y \geq 0$. These are <u>the points in the first quadrant</u>.
- The set W is <u>not</u> a subspace of R^2 since it is <u>not closed under scalar multiplication</u>.
- For example, $\mathbf{v} = (1, 1)$ lines in W, but its negative $(-1)\mathbf{v} = -\mathbf{v} = (-1, -1)$ does not.



Remarks

Think about "set" and "empty set"!

- Every nonzero vector space V has at least two subspace: V itself is a subspace, and the set $\{\mathbf{0}\}$ consisting of just the zero vector in V is a subspace called the zero subspace.
- Examples of subspaces of R^2 and R^3:
  - Subspaces of R^2:
    - $\{\mathbf{0}\}$
    - Lines through the origin
    - R^2
  - Subspaces of R^3:
    - $\{\mathbf{0}\}$
    - Lines through the origin
    - Planes through origin
    - R^3
- They are actually the only subspaces of R^2 and R^3

Subspaces of M_{nn}

- Since the sum of two symmetric matrices is symmetric, and a scalar multiple of a symmetric matrix is symmetric. Thus, the set of $n \times n$ symmetric matrices is a subspace of the vector space M_{nn} of $n \times n$ matrices.
- Similarly, the set of $n \times n$ upper triangular matrices, the set of $n \times n$ lower triangular matrices, and the set of $n \times n$ diagonal matrices all form subspaces of M_{nn}, since each of these sets is closed under addition and scalar multiplication.

Solution Space

- Solution Space of Homogeneous Systems
  - If $A\mathbf{x} = \mathbf{b}$ is a system of the linear equations, then each vector $\mathbf{x}$ that satisfies this equation is called a solution vector of the system.
  - Theorem 5.2.2 shows that the solution vectors of a homogeneous linear system form a vector space, which we shall call the solution space of the system.

- Theorem 5.2.2
  - If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of R^n.

Example

- Find the solution spaces of the linear systems.

$$(a)\begin{bmatrix}1 & -2 & 3\\ 2 & -4 & 6\\ 3 & -6 & 9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (b)\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & 8\\ -2 & 4 & -6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$(c)\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ 4 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (d)\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

- Each of these systems has three unknowns, so the solutions form subspaces of R^3.
- Geometrically, each solution space must be a line through the origin, a plane through the origin, the origin only, or all of R^3.

Example

Solution.

(a) $x = 2s - 3t, \quad y = s, \quad z = t$

$x = 2y - 3z$ or $x - 2y + 3z = 0$

This is the equation of the plane through the origin with

$\mathbf{n} = (1, -2, 3)$ as a normal vector.

(b) $x = -5t, \quad y = -t, \quad z = t$

which are parametric equations for the line through the origin parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

(c) The solution is $x = 0, y = 0, z = 0$, so the solution space is the origin only, that is $\{\mathbf{0}\}$.

(d) The solution are $x = r, \quad y = s, z = t$, where r, s, and t have arbitrary values, so the solution space is all of R^3.

Linear Combination

- Definition
  - A vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$ where $k_1, k_2, \ldots, k_r$ are scalars.

- Vectors in R^3 are linear combinations of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$
  - Every vector $\mathbf{v} = (a, b, c)$ in R^3 is expressible as a linear combination of the standard basis vectors

$$\mathbf{i} = (1, 0, 0), \ \mathbf{j} = (0, 1, 0), \ \mathbf{k} = (0, 0, 1)$$

since

$$\mathbf{v} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

Example

- Consider the vectors **u** = (1, 2, -1) and **v** = (6, 4, 2) in R^3. Show that **w** = (9, 2, 7) is a linear combination of **u** and **v** and that **w'** = (4, -1, 8) is not a linear combination of **u** and **v**.

Solution.

In order for **w** to be a linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$;

$(9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$

Equating corresponding components gives

$$\begin{aligned} k_1 \ + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system yields $k_1 = -3$, $k_2 = 2$, so

$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$

Similarly, for **w'** to be a linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$;

$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2)$

or

$(4, -1, 8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$

Equating corresponding components gives

$$\begin{aligned} k_1 \ + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equation is inconsistent, so no such scalars k_1 and k_2 exist. Consequently, **w'** is not a linear combination of **u** and **v**.

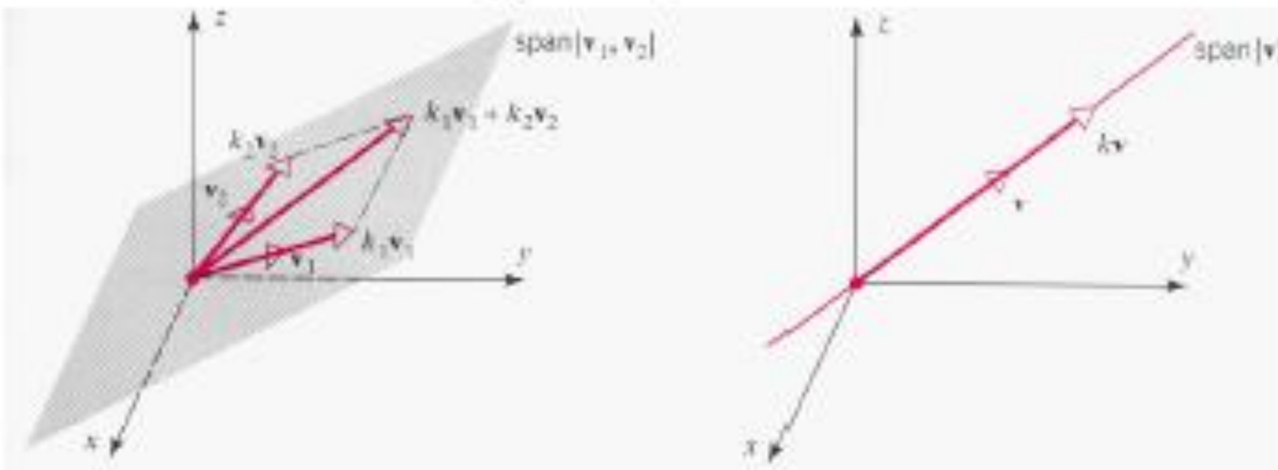
Linear Combination and Spanning

- Theorem 5.2.3
  - If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in a vector space V, then:
    - The set W of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is a subspace of V.
    - W is the smallest subspace of V that contain $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in the sense that every other subspace of V that contain $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ must contain W.

- Definition
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of vectors in a vector space V, then the subspace W of V containing of all linear combination of these vectors in S is called the space spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, and we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ span W.
  - To indicate that W is the space spanned by the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, we write $W = \text{span}(S)$ or $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$.

Example

- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-collinear vectors in R^3 with their initial points at the origin, then span$\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ is the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.
- Similarly, if $\mathbf{v}$ is a nonzero vector in R^2 and R^3, then span$\{\mathbf{v}\}$, which is the set of all scalar multiples $k\mathbf{v}$, is the linear determined by $\mathbf{v}$.



(a) Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Span$\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

Example

- Determine whether $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.
- Solution
  - Is it possible that an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$?
  - $\mathbf{b} = (b_1, b_2, b_3) = k_1(1, 1, 3) + k_2(1, 0, 1) + k_3(2, 1, 3) = (k_1+k_2+2k_3,\ k_1+k_3,\ 2k_1+k_2+3k_3)$ or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + \ k_3 &= b_2 \\ 2k_1 + k_2 + 3\,k_3 &= b_3 \end{aligned}$$

  - This system is consistent for *all* values of b_1, b_2, and b_3 if and only if the coefficient matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ has a nonzero determinant.
  - However, $\det(A) = 0$, so that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, do not span R^3.

Theorem 5.2.4

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ are two sets of vector in a vector space V, then

  $$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$$

  if and only if each vector in S is a linear combination of these in S' and each vector in S' is a linear combination of these in S.

Linearly Dependent & Independent

- Definition
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vector, then the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$ has at least one solution, namely $k_1 = 0,\ k_2 = 0, \ldots, k_r = 0$.
  - If this the only solution, then S is called a linearly independent set. If there are other solutions, then S is called a linearly dependent set.

- Examples
  - If $\mathbf{v}_1 = (2, -1, 0, 3)$, $\mathbf{v}_2 = (1, 2, 5, -1)$, and $\mathbf{v}_3 = (7, -1, 5, 8)$.
  - Then the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, since $3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0$.

Example

- Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ in R^3.
  - Consider the equation $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = 0$
    $\Rightarrow k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$
    $\Rightarrow (k_1, k_2, k_3) = (0, 0, 0)$
    $\Rightarrow$ The set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent.
- Similarly the vectors

  $$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

  form a linearly independent set in R^n.

- Remark:
  - To check whether a set of vectors is linear independent or not, write down the linear combination of the vectors and see if their coefficients all equal zero.

Example

- Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3),\ \mathbf{v}_2 = (5, 6, -1),\ \mathbf{v}_3 = (3, 2, 1)$$

  form a linearly dependent set or a linearly independent set.

- Solution
  - Let the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = 0$
    $\Rightarrow k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$
    $$\Rightarrow \quad \begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$
    $\Rightarrow \det(A) = 0$
    $\Rightarrow$ The system has nontrivial solutions
    $\Rightarrow \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ form a linearly dependent set

Theorems

- Theorem 5.3.1
  - A set with two or more vectors is:

    Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

    Linearly independent if and only if no vector in S *is expressible as a linear* combination of the other vectors in S.
- Theorem 5.3.2
  - A finite set of vectors that contains the zero vector is linearly dependent.
  - A set with exactly two vectors is linearly independently if and only if neither vector is a scalar multiple of the other.
- Theorem 5.3.3

  Let $S = \{v_1, v_2, \cdots, v_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

Examples

- In Example 1 we saw that the vectors

$v_1=(2, -1, 0, 3)$, $v_2=(1, 2, 5, -1)$, and $v_3=(7, -1, 5, 8)$

Form a linearly dependent set. In this example each vector is expressible as a linear combination of the other two since it follows from the equation $3v_1+v_2-v_3=0$ that

$v_1=-1/3v_2+1/3v_3$, $v_2=-3\ v_1+v_3$, and $v_3=3v_1+v_2$

- Consider the vectors $i=(1, 0, 0)$, $j=(0, 1, 0)$, and $k=(0, 0, 1)$ in R_3.

Suppose that k is expressible as

$k=k_1 i+k_2 j$

Then, in terms of components,

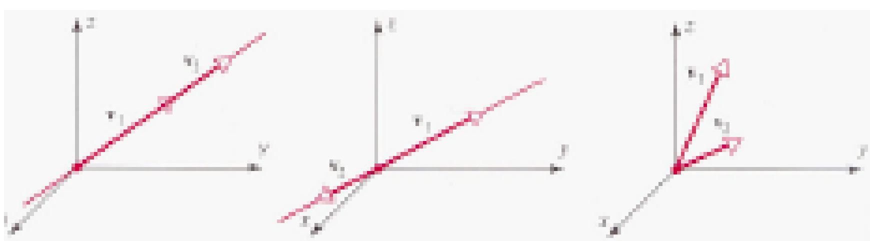
$(0, 0, 1)=k_1(1, 0, 0)+k_2(0, 1, 0)$ or $(0, 0, 1)=(k_1, k_2, 0)$

But the last equation is not satisfied by any values of k_1 and k_2, so k cannot be expressed as a linear combination of i and j. Similarly, i is not expressible as a linear combination of j and k, and j is not expressible as a linear combination of i and k.

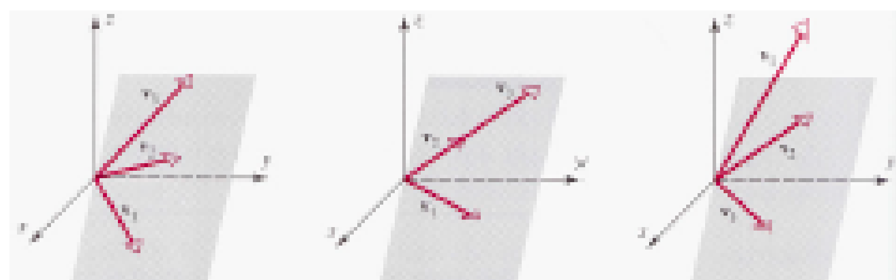
End of 1st Hour

Geometric Interpretation of Linear Independence

- In R^2 and R^3, a set of two vectors is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.
- In R^3, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.



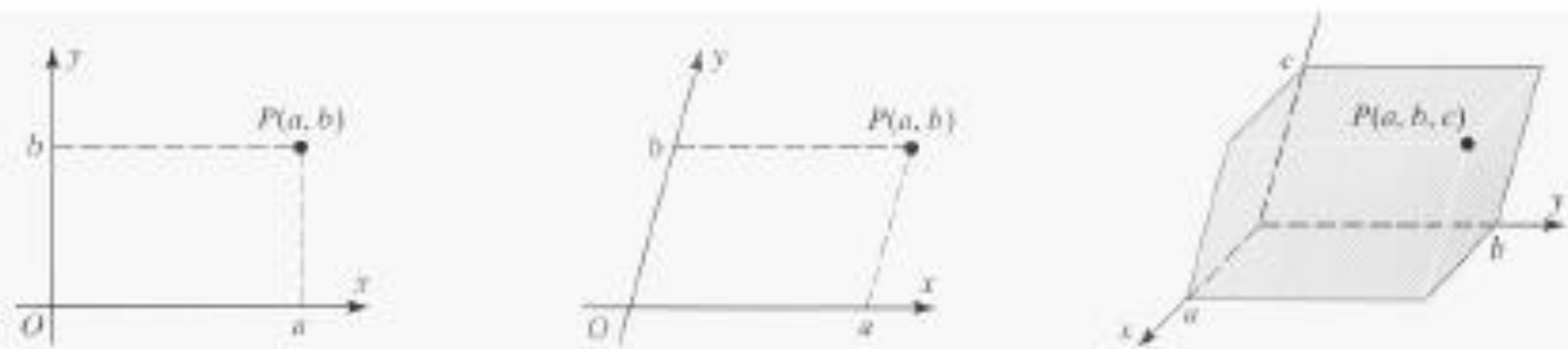
(a) Linearly dependent (b) Linearly dependent (c) Linearly independent



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Nonrectangular Coordinate System

- The coordinate system establishes a one-to-one correspondence between points in the plane and ordered pairs of real numbers.
- Although perpendicular coordinate axes are the most common, any two nonparallel lines can be used to define a coordinate system in the plane.



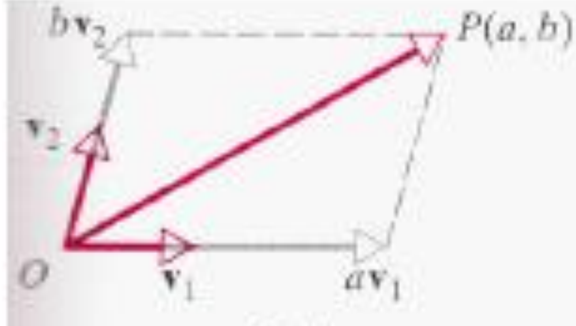
(a) Coordinates of P in a rectangular coordinate system in 2-space

(b) Coordinates of P in a nonrectangular coordinate system in 2-space

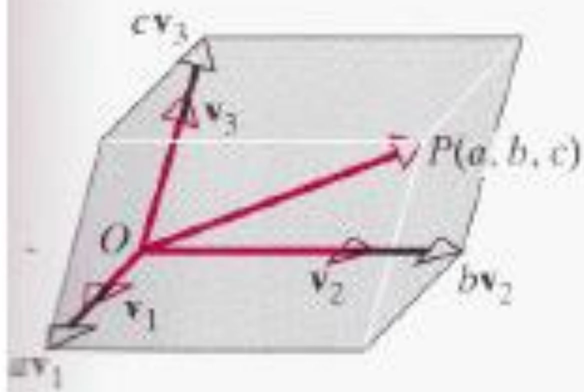
(c) Coordinates of P in a nonrectangular coordinate system in 3-space

Nonrectangular Coordinate System

- A coordinate system can be constructed by general vectors:
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors of length 1 that points in the positive direction of the axis: $\overline{OP} = a\mathbf{v}_1 + b\mathbf{v}_2$
  - Similarly, the coordinates (a, b, c) of the point P can be obtained by expressing $\overline{OP}$ as a linear combination of the vectors



(a)

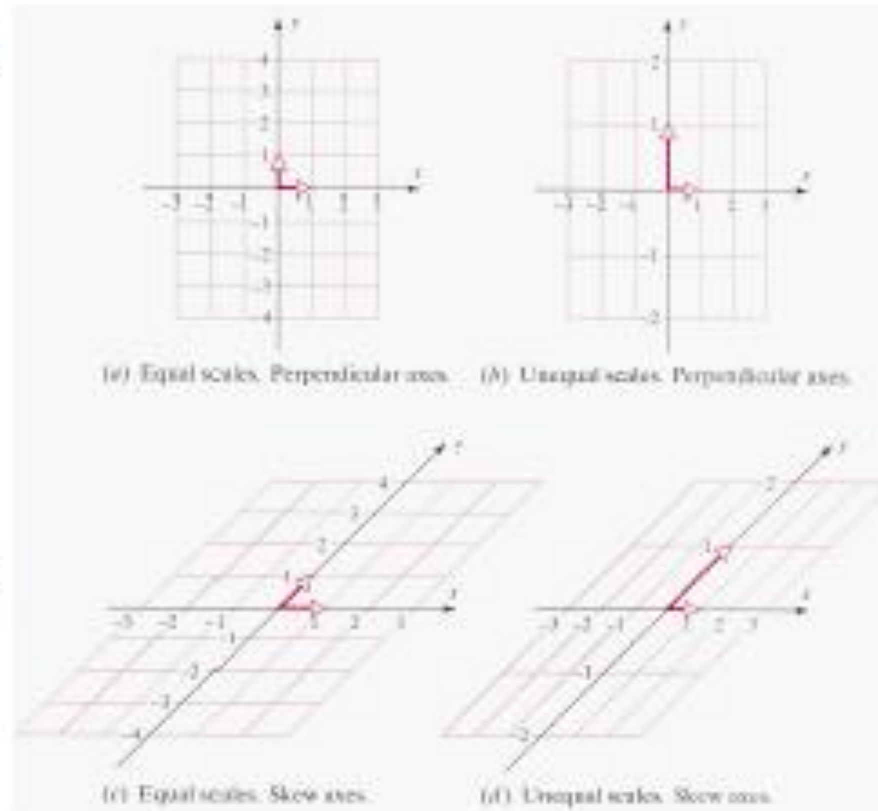


(b)

Nonrectangular Coordinate System

- Informally stated, vectors that specify a coordinate system are called “basis vectors” for that system.

- Although we used basis vectors of length 1 in the preceding discussion, this is not essential – nonzero vectors of any length will suffice.



Basis

- Definition
  - If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in V, then S is called a basis for V if the following two conditions hold:
    - S is linearly independent.
    - S spans V.
- Theorem 5.4.1 (Uniqueness of Basis Representation)
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

in exactly one way.

Coordinates Relative to a Basis

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$
is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars c_1, $c_2, \ldots, c_n$, are called the coordinates of $\mathbf{v}$ relative to the basis S.
- The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the coordinate vector of $\mathbf{v}$ relative to S; it is denoted by
$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

- Remark:
  - Coordinate vectors depend not only on the basis S but also on the order in which the basis vectors are written.
  - A change in the order of the basis vectors results in a corresponding change of order for the entries in the coordinate vector.

Example (Standard Basis for R^3)

- Suppose that $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, then $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent set in R^3.

- This set also spans R^3 since any vector $\mathbf{v} = (a, b, c)$ in R^3 can be written as

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

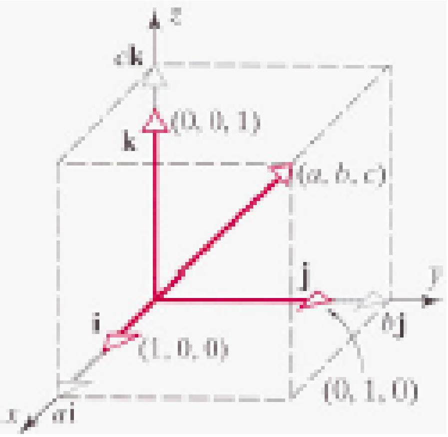
- Thus, S is a basis for R^3; it is called the standard basis for R^3.

- Looking at the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, it follows that the coordinates of $\mathbf{v}$ relative to the standard basis are a, b, and c, so

$$(\mathbf{v})_S = (a, b, c)$$

- Comparing this result to $\mathbf{v} = (a, b, c)$, we have

$$\mathbf{v} = (\mathbf{v})_S$$



Standard Basis for R^n

- If $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, 0, \ldots, 1)$, then
$$S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$
is a linearly independent set in R^n.

- This set also spans R^n since any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be written as
$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$
- Thus, S is a basis for R^n; it is called the standard basis for R^n.

- The coordinates of $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ relative to the standard basis are v_1, v_2, $\ldots$, v_n, thus
$$(\mathbf{v})_S = (v_1, v_2, \ldots, v_n)$$
- As the previous example, we have $\mathbf{v} = (\mathbf{v})_s$, so a vector $\mathbf{v}$ and its coordinate vector relative to the standard basis for R^n are the same.

Example

- Let $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$.
  Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for R^3.

- Solution:
  - To show that the set S spans R^3, we must show that an arbitrary vector

    $$\mathbf{b} = (b_1, b_2, b_3)$$

    can be expressed as a linear combination

    $$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

    of the vectors in S.
  - Let $(b_1, b_2, b_3) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$

    $$\begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad + 4c_3 &= b_3 \end{aligned}$$

    $\Rightarrow \det(A) \neq 0$

    $\Rightarrow S$ is a basis for R^3

Example (Representing a Vector Using Two Bases)

- Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis for R^3 in the preceding example.
  - Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ with respect to S.
  - Find the vector $\mathbf{v}$ in R^3 whose coordinate vector with respect to the basis S is $(\mathbf{v})_s = (-1, 3, 2)$.

- Solution (a)
  - We must find scalars c_1, c_2, c_3 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, or, in terms of components, $(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$
  - Solving this, we obtaining $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.
  - Therefore, $(\mathbf{v})_s = (-1, 3, 2)$.
- Solution (b)
  - Using the definition of the coordinate vector $(\mathbf{v})_s$, we obtain

$$\mathbf{v} = (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = (11, 31, 7).$$

Standard Basis for P_n

- $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials of the form $a_0 + a_1x + \ldots + a_nx^n$. The set S is called the standard basis for P_n.
  Find the coordinate vector of the polynomial $\mathbf{p} = a_0 + a_1x + a_2x^2$ relative to the basis $S = \{1, x, x^2\}$ for P_2 .

- Solution:
  - The coordinates of $\mathbf{p} = a_0 + a_1x + a_2x^2$ are the scalar coefficients of the basis vectors 1, x, and x^2, so

$$(\mathbf{p})_s = (a_0, a_1, a_2).$$

Standard Basis for M_{mn}

- Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- The set $S = \{M_1, M_2, M_3, M_4\}$ is a basis for the vector space M_{22} of 2×2 matrices.
- To see that S spans M_{22}, note that an arbitrary vector (matrix) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = aM_1 + bM_2 + cM_3 + dM_4$$

- To see that S is linearly independent, assume $aM_1 + bM_2 + cM_3 + dM_4 = 0$. It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, $a = b = c = d = 0$, so S is lin. indep.
- The basis S is called the standard basis for M_{22}.
- More generally, the standard basis for M_{mn} consists of the mn different matrices with a single 1 and zeros for the remaining entries.

Basis for the Subspace span(S)

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set in a vector space V, then S is a basis for the subspace span(S)

  since the set S span span(S) by definition of span(S).

Finite-Dimensional

- Definition
  - A nonzero vector V is called finite-dimensional if it contains a finite set of vector $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that forms a basis. If no such set exists, V is called infinite-dimensional. In addition, we shall regard the zero vector space to be finite-dimensional.

- Example
  - The vector spaces R^n, P_n, and M_{mn} are finite-dimensional.
  - The vector spaces $F(-\infty, \infty)$, $C(-\infty, \infty)$, $C^m(-\infty, \infty)$, and $C^\infty(-\infty, \infty)$ are infinite-dimensional.

Theorems

- Theorem 5.4.2
  - Let V be a finite-dimensional vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ any basis.
    - If a set has more than n vector, then it is linearly dependent.
    - If a set has fewer than n vector, then it does not span V.
- Theorem 5.4.3
  - All bases for a finite-dimensional vector space have the same number of vectors.

Dimension

- Definition
  - The dimension of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V.
  - We define the zero vector space to have dimension zero.

- Dimensions of Some Vector Spaces:
  - $\dim(R^n) = n$ [The standard basis has n vectors]
  - $\dim(P_n) = n + 1$ [The standard basis has $n + 1$ vectors]
  - $\dim(M_{mn}) = mn$ [The standard basis has mn vectors]

Theorems

- Theorem 5.4.4 (Plus/Minus Theorem)
  - Let S be a nonempty set of vectors in a vector space V.
    - If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.
    - If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is, span(S) = span($S - \{\mathbf{v}\}$)

- Theorem 5.4.5
  - If V is an n-dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if <u>either S spans V or S is linearly independent</u>.

Example

- Show that $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2 by inspection.
- Solution:
  - Neither vector is a scalar multiple of the other
    $\Rightarrow$ The two vectors form a linear independent set in the 2-D space R^2
    $\Rightarrow$ The two vectors form a basis by Theorem 5.4.5.

- Show that $\mathbf{v}_1 = (2, 0, 1)$, $\mathbf{v}_2 = (4, 0, 7)$, $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3 by inspection.
- Solution:
  - The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the *xy*-plane.
  - The vector $\mathbf{v}_3$ is outside of the *xy*-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.
  - Since R^3 is three-dimensional, Theorem 5.4.5 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for R^3.

Theorems

- Theorem 5.4.6
  - Let S be a finite set of vectors in a finite-dimensional vector space V.
    - If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
    - If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

- Theorem 5.4.7
  - If W is a subspace of a finite-dimensional vector space V, then $\dim(W) \leq \dim(V)$.
  - If $\dim(W) = \dim(V)$, then $W = V$.

Definition

- For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\cdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in R^n formed form the rows of A are called the row vectors of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the column vectors of A.

Example

- Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

- The row vectors of A are

$$\mathbf{r}_1 = [2\ 1\ 0] \text{ and } \mathbf{r}_2 = [3\ \text{-}1\ 4]$$

and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \text{and } \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

End

Row Space and Column Space

- Definition
  - If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the row space of A, and the subspace of R^m spanned by the column vectors is called the column space of A.
  - The solution space of the homogeneous system of equation $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the nullspace of A.

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- Theorem 5.5.1
  - A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Example

- Let $A\mathbf{x} = \mathbf{b}$ be the linear system $\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

  Show that $\mathbf{b}$ is in the column space of A, and express $\mathbf{b}$ as a linear combination of the column vectors of A.

- Solution:
  - Solving the system by Gaussian elimination yields

    $$x_1 = 2, x_2 = -1, x_3 = 3$$

  - Since the system is consistent, $\mathbf{b}$ is in the column space of A.
  - Moreover, it follows that

    $$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

General and Particular Solutions

- Theorem 5.5.2
  - If $\mathbf{x}_0$ denotes any single solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_k$ form a basis for the nullspace of A, (that is, the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$), then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

    Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$ the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

- Remark
  - The vector $\mathbf{x}_0$ is called a particular solution of $A\mathbf{x} = \mathbf{b}$.
  - The expression $\mathbf{x}_0 + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is called the general solution of $A\mathbf{x} = \mathbf{b}$, the expression $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is called the general solution of $A\mathbf{x} = \mathbf{0}$.
  - The general solution of $A\mathbf{x} = \mathbf{b}$ is the sum of any particular solution of $A\mathbf{x} = \mathbf{b}$ and the general solution of $A\mathbf{x} = \mathbf{0}$.

Example (General Solution of $A\mathbf{x} = \mathbf{b}$)

- The solution to the nonhomogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\
2x_1 + 5x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

  is

  $x_1 = -3r - 4s - 2t,\ x_2 = r,$
  $x_3 = -2s,\ x_4 = s,$
  $x_5 = t,\ x_6 = 1/3$

- The result can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 1/3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}}_{\mathbf{x}_0} + \underbrace{r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}}$$

  which is the general solution.

- The vector $\mathbf{x}_0$ is a <u>particular solution</u> of nonhomogeneous system, and the linear combination $\mathbf{x}$ is the <u>general solution</u> of the homogeneous system.

Example

- Find a basis for the nullspace of $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$
- Solution
  - The nullspace of A is the solution space of the homogeneous system

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \quad\quad + x_5 &= 0 \\ -x_1 - x_2 - 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \quad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

  - In Example 10 of Section 5.4 we showed that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the nullspace.

Theorems

- Theorem 5.5.3 & 5.5.4
  - Elementary row operations do not change both the nullspace and row space of a matrix.
- Theorem 5.5.5
  - If A and B are row equivalent matrices, then:
    - A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
    - A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.
- Theorem 5.5.6
  - If a matrix R is in row echelon form, then the row vectors with the leading 1's (i.e., the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

Bases for Row and Column Spaces

The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form. From Theorem 5.5.6 the vectors

$$\mathbf{r}_1 = [1\ {-2}\ 5\ 0\ 3]$$
$$\mathbf{r}_2 = [0\ 1\ 3\ 0\ 0]$$
$$\mathbf{r}_3 = [0\ 0\ 0\ 1\ 0]$$

form a basis for the row space of R, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

Example

- Find bases for the row and column spaces of

$$A=\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

- Solution:
  - Reducing A to row-echelon form we obtain

$$R=\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note about the correspondence!

  - By Theorem 5.5.6 and 5.5.5(b), the row and column spaces are

$\mathbf{r}_1 = [1\ -3\ 4\ -2\ 5\ 4]$

$\mathbf{r}_2 = [0\ 0\ 1\ 3\ -2\ -6]$

$\mathbf{r}_3 = [0\ 0\ 0\ 0\ 1\ 5]$

and

$$\mathbf{c}_1=\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{c}_3=\begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \mathbf{c}_5=\begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Example (Basis for a Vector Space Using Row Operations)

- Find a basis for the space spanned by the vectors

$$\mathbf{v}_1 = (1, -2, 0, 0, 3),\ \mathbf{v}_2 = (2, -5, -3, -2, 6),$$
$$\mathbf{v}_3 = (0, 5, 15, 10, 0),\ \mathbf{v}_4 = (2, 6, 18, 8, 6).$$

- Solution: (Write down the vectors as row vectors first!)

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - The nonzero row vectors in this matrix are

$$\mathbf{w}_1 = (1, -2, 0, 0, 3),\ \mathbf{w}_2 = (0, 1, 3, 2, 0),\ \mathbf{w}_3 = (0, 0, 1, 1, 0)$$

  - These vectors form a basis for the row space and consequently form a basis for the subspace of R^5 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

Example (Basis for the Row Space of a Matrix)

- Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

<u>consisting entirely of row vectors from A.</u>

- Solution:

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The column space of A^T are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \text{ and } \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

- Thus, the row space of A are

  $\mathbf{r}_1 = [1 \ -2 \ 0 \ 0 \ 3]$

  $\mathbf{r}_2 = [2 \ -5 \ -3 \ -2 \ 6]$

  $\mathbf{r}_3 = [2 \ -5 \ -3 \ -2 \ 6]$

Example (Basis and Linear Combinations)

- (a) Find a subset of the vectors $\mathbf{v}_1 = (1, -2, 0, 3)$, $\mathbf{v}_2 = (2, -5, -3, 6)$, $\mathbf{v}_3 = (0, 1, 3, 0)$, $\mathbf{v}_4 = (2, -1, 4, -7)$, $\mathbf{v}_5 = (5, -8, 1, 2)$ that forms a basis for the space spanned by these vectors.
- (b) Express each vector not in the basis as a linear combination of the basis vectors.

- Solution (a):

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{array} \end{array} \Rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{array} \end{array}$$

  - Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for the column space of the matrix.

Four Fundamental Matrix Spaces

- Consider a matrix A and its transpose A^T together, then there are six vector spaces of interest:
  - row space of A, row space of A^T
  - column space of A, column space of A^T
  - null space of A, null space of A^T
- However, the fundamental matrix spaces associated with A are
  - row space of A, column space of A
  - null space of A, null space of A^T
- If A is an $m \times n$ matrix, then <u>the row space of A and nullspace of A are subspaces of R^n</u> and <u>the column space of A and the nullspace of A^T are subspace of R^m</u>
- What is the relationship between the dimensions of these four vector spaces?

Dimension and Rank

- Theorem 5.6.1
  - If A is any matrix, then the row space and column space of A have the same dimension.

- Definition
  - The common dimension of the row and column space of a matrix A is called the rank of A and is denoted by $\text{rank}(A)$; the dimension of the nullspace of a is called the nullity of A and is denoted by $\text{nullity}(A)$.

Example (Rank and Nullity)

- Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- Solution:
  - The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Since there are two nonzero rows, the row space and column space are both two-dimensional, so $\operatorname{rank}(A) = 2$.

Continued,

Example (Rank and Nullity)

- The corresponding system of equations will be

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16\,x_5 + 5\,x_6 = 0$$

- It follows that the general solution of the system is

$$x_1 = 4r + 28s + 37t - 13u,\ x_2 = 2r + 12s + 16t - 5u,$$
$$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_6 = u$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Thus, nullity$(A) = 4$.

Theorems

- Theorem 5.6.2
  - If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.
- Theorem 5.6.3 (Dimension Theorem for Matrices)
  - If A is a matrix with n columns, then $\underline{\text{rank}(A) + \text{nullity}(A) = n}$.
- Theorem 5.6.4
  - If A is an $m \times n$ matrix, then:
    - $\text{rank}(A)$ = Number of leading variables in the solution of $A\mathbf{x} = \mathbf{0}$.
    - $\text{nullity}(A)$ = Number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Example (Sum of Rank and Nullity)

- The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

  has 6 columns, so

$$\text{rank}(A) + \text{nullity}(A) = 6$$

- This is consistent with the previous example, where we showed that

$$\text{rank}(A) = 2 \quad \text{and} \quad \text{nullity}(A) = 4$$

Example

- Find the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ if A is a 5×7 matrix of rank 3.
- Solution:
  - $\text{nullity}(A) = n - \text{rank}(A) = 7 - 3 = 4$
  - Thus, there are four parameters.

Dimensions of Fundamental Spaces

- Suppose that A is an $m \times n$ matrix of rank r, then
  - A^T is an $n \times m$ matrix of rank r by Theorem 5.6.2
  - $\text{nullity}(A) = n - r$, $\text{nullity}(A^T) = m - r$ by Theorem 5.6.3

Fundamental Space	Dimension
Row space of A	r
Column space of A	r
Nullspace of A	$n - r$
Nullspace of A^T	$m - r$

Maximum Value for Rank

- If A is an $m \times n$ matrix
  $\Rightarrow$ The row vectors lie in R^n and the column vectors lie in R^m.
  $\Rightarrow$ The row space of A is at most n-dimensional and the column space is at most m-dimensional.

- Since the row and column space have the same dimension (the rank A), we must conclude that if $m \neq n$, then the rank of A is at most the smaller of the values of m or n.
- That is,

$$\operatorname{rank}(A) \leq \min(m, n)$$

Example

- If A is a 7×4 matrix, then the rank of A is at most 4 and, consequently, the seven row vectors must be linearly dependent. If A is a 4×7 matrix, then again the rank of A is at most 4 and, consequently, the seven column vectors must be linearly dependent.

Theorems

- Theorem 5.6.5 (The Consistency Theorem)
  - If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.
    - $A\mathbf{x} = \mathbf{b}$ is consistent.
    - $\mathbf{b}$ is in the column space of A.
    - The coefficient matrix A and the augmented matrix $[A \mid \mathbf{b}]$ have the same rank.

- Theorem 5.6.6
  - If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.
    - $A\mathbf{x} = \mathbf{b}$ is consistent for every $m\times 1$ matrix $\mathbf{b}$.
    - The column vectors of A span R^m.
    - $\text{rank}(A) = m$.

Overdetermined System

- A linear system with more equations than unknowns is called an overdetermined linear system.
- If $A\mathbf{x} = \mathbf{b}$ is an overdetermined linear system of m equations in n unknowns (so that $m > n$), then the column vectors of A cannot span R^m.

- Thus, the overdetermined linear system $A\mathbf{x} = \mathbf{b}$ cannot be consistent for *every* possible $\mathbf{b}$.

Example

- The linear system
$$\begin{aligned} x_1 - 2x_2 &= b_1 \\ x_1 - \ \ x_2 &= b_2 \\ x_1 + \ \ x_2 &= b_3 \\ x_1 + 2x_2 &= b_4 \\ x_1 + 3x_2 &= b_5 \end{aligned}$$
is overdetermined, so it cannot be consistent for all possible values of b_1, b_2, b_3, b_4, and b_5. Exact conditions under which the system is consistent can be obtained by solving the linear system by Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 3b_2 + 2b_1 \\ 0 & 0 & b_4 - 4b_2 + 3b_1 \\ 0 & 0 & b_5 - 5b_2 + 4b_1 \end{bmatrix}$$

Example

- Thus, the system is consistent if and only if b_1, b_2, b_3, b_4, and b_5 satisfy the conditions

$$\begin{aligned} 2b_1 - 3b_2 + b_3 \qquad\qquad &= 0 \\ 2b_1 - 4b_2 \qquad + b_4 \qquad &= 0 \\ 4b_1 - 5b_2 \qquad\qquad + b_5 &= 0 \end{aligned}$$

or, on solving this homogeneous linear system, $b_1=5r-4s$, $b_2=4r-3s$, $b_3=2r-s$, $b_4=r$, $b_5=s$ where r and s are arbitrary.

Theorems

- Theorem 5.6.7
  - If $A\mathbf{x} = \mathbf{b}$ is consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

- Theorem 5.6.8
  - If A is an $m \times n$ matrix, then the following are equivalent.
    - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
    - The column vectors of A are linearly independent.
    - $A\mathbf{x} = \mathbf{b}$ has at most one solution (0 or 1) for every $m \times 1$ matrix $\mathbf{b}$.

Examples

- Number of Parameters in a General Solution:
  - If A is a 5×7 matrix with rank 4, and if $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then the general solution of the system contains $7 - 4 = 3$ parameters.

- An Undetermined System
  - If A is a 5×7 matrix, then for every 7×1 matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ is undetermined. Thus, $A\mathbf{x} = \mathbf{b}$ must be consistent for some $\mathbf{b}$, and for each such $\mathbf{b}$ the general solution must have $7 - r$ parameters, where r is the rank of A.

Theorem 5.6.9 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The range of T_A is R^n.
  - T_A is one-to-one.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.