Boolean Algebra

George Boole (1815-1864) English mathematician and philosopher.

- Defined on the set B = {0,1}
- Binary operators: AND, OR { • , + }
- Unary operation: Complement (NOT) {'}
  Another symbol for the NOT operator: $\overline{a}$

a • b (AND)

a \ b	0	1
0	0	0
1	0	1

a + b (OR)

a \ b	0	1
0	0	1
1	1	1

Complement

a	a' ($\overline{a}$)
0	1
1	0

- **Axioms (Laws):**

Under assumption a , b ∈ B

1. Closure:	a + b ∈ B	a • b ∈ B
2. Commutative:	a + b = b + a	a • b = b • a
3. Associative:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
4. Identity :	a + 0 = a	a • 1 = a
5. Distributive:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
6. Inverse:	a + $\overline{a}$ = 1	a • $\overline{a}$ = 0

Precedence order of operators from higher to lower precedence:
1. Parenthesis, 2. NOT (Complement) (Negation), 3. AND, 4. OR

Properties and Theorems:

These properties and theorems are derived from the operations and axioms of Boolean algebra.
They can be proven using the axioms.

1. Annihilator (or Dominance):
   a + 1 = 1 a • 0 = 0
2. Involution: (a')' = a or $\overline{\overline{a}}$ = a
3. Idempotency:
   a+a+a+….+a = a a•a•a •... •a = a
4. Absorption: (Proof in 2.4)
   a + a·b = a a·(a+b) = a
5. De Morgan's Theorem: *Augustus De Morgan (1806 – 1871)*
   $\overline{(a + b)} = \overline{a} \cdot \overline{b}$ $\overline{(a \cdot b)} = \overline{a} + \overline{b}$
5. General form of De Morgan's Theorem:
   $\overline{f(X1, X2, ..., Xn, 0, 1, +, \cdot)} = f(\overline{X1}, \overline{X2}, ..., \overline{Xn}, 1, 0, \cdot, +)$
   - It establishes relations between binary operations (AND, OR): · and +

6. The Duality principle

A dual of a logical expression is obtained by replacing, • by +, + by •, 0 by 1, and 1 by 0, without changing the variables.

$$a + b + 0 \ldots \Leftrightarrow a \bullet b \bullet 1 \ldots$$

Principle: Duals of all proven theorems are also valid.

If we can prove a theorem, then we know that its dual is also true.

Example:

Absorption:

If we can prove the theorem $a + a \cdot b = a$, then its dual $a \cdot (a+b) = a$ is also true.

Note that in previous slides, axioms and theorems were presented with their duals.

General Duality Principle:

$$f(X1, X2, \ldots, Xn, 0, 1, +, \bullet) \Leftrightarrow f^D(X1, X2, \ldots, Xn, 1, 0, \bullet, +)$$

- It is different from the De Morgan's Theorem.
  - It establishes relation between proofs of theorems.
  - It is not used to convert one expression into another. Duals are not equal.

Proving the theorems:

a) With Axioms

Example:

Theorem: $X \bullet Y + X \bullet Y' = X$

Proof:

Distributive	$X \bullet Y + X \bullet Y'$	$= X \bullet (Y + Y')$
Complement	$X \bullet (Y + Y')$	$= X \bullet (1)$
Identity	$X \bullet (1)$	$= X$ ✓

Example:

Theorem: $X + X \bullet Y = X$ (*Absorption*)

Proof:

Identity	$X + X \bullet Y$	$= X \bullet 1 + X \bullet Y$
Distributive	$X \bullet 1 + X \bullet Y$	$= X \bullet (1 + Y)$
Annihilator	$X \bullet (1 + Y)$	$= X \bullet (1)$
Identity	$X \bullet (1)$	$= X$ ✓

Proving the theorems: **b) Using truth tables**

Note that to denote the negation (NOT), we can also use the notation $\overline{A}$.

De Morgan:

$$\overline{(X + Y)} = \overline{X} \bullet \overline{Y}$$

X	Y	$\overline{X}$	$\overline{Y}$	$\overline{(X + Y)}$	$\overline{X} \cdot \overline{Y}$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

$$\overline{(X \bullet Y)} = \overline{X} + \overline{Y}$$

X	Y	$\overline{X}$	$\overline{Y}$	$\overline{(X \cdot Y)}$	$\overline{X} + \overline{Y}$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

Although truth tables may contain many rows, computer programs can fill in and compare these tables very quickly.

Minimizing logical expressions using axioms and theorems:

Minimizing a logical expression means; finding the shortest expression with less operations and variables that generates same output values as the original expression.

Example:

Z = A' B C + A B' C + A B C' + A B C (Original expression)
= A' B C + A B' C + A B C' + A B C + A B C
= A' B C + A B C + A B' C + A B C' + A B C
= (A' + A) B C + A B' C + A B C' + A B C
= (1) B C + A B' C + A B C' + A B C
= B C + A B' C + A B C' + A B C + A B C
= B C + A B' C + A B C + A B C' + A B C
= B C + A (B' + B) C + A B C' + A B C
= B C + A (1) C + A B C' + A B C
= B C + A C + A B (C' + C)
= B C + A C + A B (1)
= B C + A C + A B (Minimized expression)

Logical (Boolean) Expressions

A logical expression, is a finite combination of variables, constants, and operators that are well-formed according to the rules.

It is represented as E(X), where X= (x_1, x_2, ... x_n) and each $x_i \in \{0,1\}$.

If E_1 and E_2 are logical expressions than, E_1', E_2', $E_1 + E_2$, $E_1 \bullet E_2$, and all possible combinations are also logical expressions.

Normal Forms of Logical expressions:

Each logical expression can be written in two special forms.

1. **Disjunctive normal form (DNF): ΣΠ**

   Logical sum of logical products (SOP). OR of ANDs.

   Example: bc'+ad+a'b

   The OR operation is also called *logical disjunction*.

2. **Conjunctive normal form (CNF): ΠΣ**

   Logical product of logical sums (POS). AND of ORs.

   Example: (a+b+c')(a+d)(a'+b)

   The AND operation is also called *logical conjunction*.

Any logical expression can be written in CNF (POS form) and DNF (SOP form).
Any expression in CNF can be converted to DNF and vice versa.

Value of a logical expression:

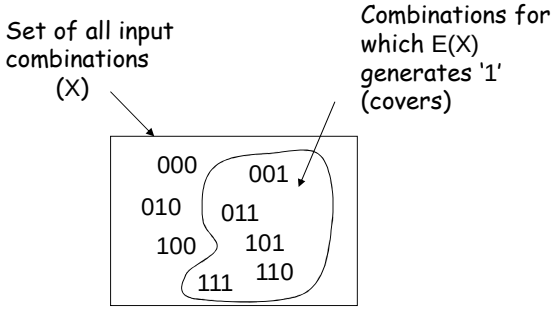
An expression, E(X), generates for each combination of the input vector X=(x_1, ... x_n), a value from the set of B={0,1}.

These values constitute the truth table for the expression.

Example: $E(X) = x_1x_2 + x_3$

Truth table for the expression

X= x_1	x_2	x_3	E(X)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



"Order Relation" between expressions:

To explain some properties of the logical expressions, we can use the following order relation:

An order relation between elements of set B={0,1}: $0 < 1$

Read as "0 precedes 1" or "0 is smaller than 1".

According to this, an order relation between X vectors can be defined as follows:

If all elements of X1 precede (are smaller than) elements in the same order of vector X2, then $X1 \leq X2$.

Example:

X1=1001 , X2 = 1101

$X1 \leq X2$.

The order relation may not exist between all vectors.

Example: X1=0011 , X2 = 1001

The order relation is not valid between X1 and X2.

Order relation between expressions:

$E_1(X) \leq E_2(X)$ denotes that, values of E_1 generated by combinations of input X are always smaller than (or equal to) all values of E_2 generated by the same input combinations.

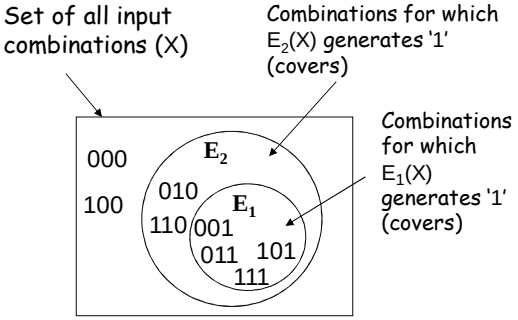
Example :

x_1	x_2	x_3	$E_1(X)$		$E_2(X)$
0	0	0	0	=	0
0	0	1	1	=	1
0	1	0	0	<	1
0	1	1	1	=	1
1	0	0	0	=	0
1	0	1	1	=	1
1	1	0	0	<	1
1	1	1	1	=	1

For each input combination where $E_1(X)$ generates 1, $E_2(X)$ also generates 1. (This is a special case.)

If $E_1(X) \leq E_2(X)$:

1. $E_1(X) + E_2(X) = E_2(X)$
2. $E_1(X) \bullet E_2(X) = E_1(X)$

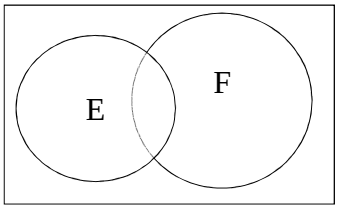


If $E_1(X) \leq E_2(X)$

$E_1(X)$ implies $E_2(X)$, $E_1(X) \Rightarrow E_2(X)$,

$E_2(X)$ covers $E_1(X)$.

The order relation (≤) may not be valid between all expressions.



IF E and F are two logical expressions, the following inequalities are always true:

$E \cdot F \le E \le E+F$

$E \cdot F \le F \le E+F$

Absorption Properties of expressions:

$\boxed{E+E \cdot F = E}$ dual $\boxed{E(E+F) = E}$

Proof: $E(E+F) = EE+EF = E+EF = E(1+F) = E$

$\boxed{E+E' \cdot F = E+F}$ dual $\boxed{E(E'+F) = E \cdot F}$

Proof: $E+E'F = (E+E')(E+F) = 1(E+F) = E+F$

These properties are used to minimize (simplify) logical expressions.

Example:

E(a,b,c,d) = abc' , F(a,b,c,d) = bd

E·F = abc'd E+F = abc' + bd

An order relation between E and F does not exist.

a b c d	E	F	E·F	E+F
0 0 0 0	0	0	0	0
0 0 0 1	0	0	0	0
0 0 1 0	0	0	0	0
0 0 1 1	0	0	0	0
0 1 0 0	0	0	0	0
0 1 0 1	0	①	0	①
0 1 1 0	0	0	0	0
0 1 1 1	0	①	0	①
1 0 0 0	0	0	0	0
1 0 0 1	0	0	0	0
1 0 1 0	0	0	0	0
1 0 1 1	0	0	0	0
1 1 0 0	①	0	0	①
1 1 0 1	①	①	①	①
1 1 1 0	0	0	0	0
1 1 1 1	0	①	0	①

E·F < E and E·F < F.

Therefore

E·F + E = E

abc'd + abc' = abc'

and

E·F + F = F

abc'd + bd = bd

E < E+F and F < E+F.

Therefore

E·(E + F) = E

abc' (abc' + bd) = abc'

and

F·(E + F) = F

bd (abc' + bd) = bd

The Consensus Theorem (SOP Form)

Assume that E_1 and E_2 are two expressions that do not include the literal x_1 :
$E_1(x_2, ... x_n)$ and $E_2(x_2, ... x_m)$

We can create a new expression by multiplying one term by x_1 and the other one by the complement of x_1 .

$$E = \mathbf{x_1}E_1 + \mathbf{x_1}'E_2$$

Here, x_1 is called a **biform** variable because it occurs both positively (as itself: $\mathbf{x_1}$) and negatively (as complement: $\overline{\mathbf{x_1}}$) in the expression.

Examples: $x_1(x_2+x_3') + x_1'(x_3+x_4)$,
$x_1x_2x_3 + x_1'x_4x_5$

•Given a pair of product terms that include a biform variable, **the consensus term** is formed by multiplying two original terms together, leaving out the selected variable and its complement.

• $\mathbf{E_1E_2}$ is **the consensus term** of $xE_1 + x'E_2$ with respect to the variable x.

Example: Consensus of **abc + a'cd** (respect to a) is bccd = **bcd**

Theorem: The consensus term is redundant and can be eliminated.

$$xE_1 + x'E_2 + E_1E_2 = xE_1 + x'E_2$$

The Consensus theorem (POS Form)

According to the duality principle, the consensus theorem is also valid for expressions written in product of sums (POS) form.

Assume that E_1 and E_2 are two expressions that do not include the literal x_1:
$E_1(x_2, ... x_n)$ and $E_2(x_2, ... x_m)$

We can create a new expression by adding x_1 to one term and the complement of x_1 to the other one.

$$E = (\mathbf{x_1}+E_1)(\mathbf{x_1}'+E_2)$$

Here, x_1 is a **biform** variable.

Examples: $(x_1+x_2+x_3')(x_1'+x_3+x_4)$,
$(x_1+x_2x_3')(x_1'+x_3x_4)$

•Given a pair of sums that include a biform variable, **the consensus term** is formed by adding two original terms together, leaving out the selected variable and its complement.

• $\mathbf{E_1 + E_2}$ is **the consensus term** of $(x+E_1)(x'+E_2)$ with respect to the variable x.

Example: Consensus of **(a+b+c) (a'+c+d)** is: b+c+c+d = **b+c+d**

Theorem: The consensus term is redundant and can be eliminated.

$$(x+E_1)(x'+E_2)(E_1+E_2) = (x+E_1)(x'+E_2)$$

Example: Minimization of a logical expression using the consensus theorem

F(A, B, C) = A'B'C + A'BC + AB'C + ABC + ABC'

Consensus (respect to C)

= A'B'C + A'BC + AB'C + ABC + ABC' + AB is added

= A'B'C + A'BC + AB'C + ~~ABC~~ + ~~ABC~~' + AB Absorption

= A'B'C + A'BC + AB'C + AB

Consensus (respect to B)

= A'B'C + A'BC + A'C + AB'C + AB is added

= ~~A'B'C~~ + ~~A'BC~~ + A'C + AB'C + AB Absorption

= A'C + AB'C + AB

Consensus (B) is added

= A'C + ~~AB'C~~ + AB + AC Absorption

= A'C + AB + AC

Consensus (A) is added

= ~~A'C~~ + AB + ~~AC~~ + C Absorption

= AB + C

Logical (Boolean) Functions

Logical functions are defined on the input set B^n (vectors with n binary variables). There are 3 types of logical functions:

1. Simple (basic) functions: Multiple inputs, single output

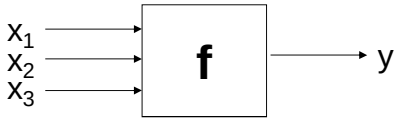
$y = f(X): B^n \rightarrow B$

$\forall X^0 \in B^n ; \exists! y^0 \in B ; y=f(X)$

"There exists a unique..."

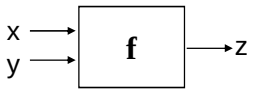
Example:

x_1 x_2 x_3	y
0 0 0	1
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1



There are $2^{(2^n)}$ possible basic logical functions for n binary variables (inputs).

For example,
there are 16 possible basic logical functions for 2 binary variables (inputs).



The truth table for 16 possible basic logical functions for 2 binary variables:

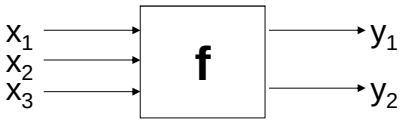
Inputs		Functions															
X	Y	F0	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		0	X AND Y	X	Y	X XOR Y	X OR Y			X NOR Y (X OR Y)'	X = Y	Y'	X'			X NAND Y (X AND Y)'	1

2. General functions: Multiple inputs, multiple outputs

$Y = f(X): B^n \rightarrow B^m$, $X=(x_1, ... x_n)$, $Y=(y_1, ... y_m)$,

Example:

x_1 x_2 x_3	y_1 y_2
0 0 0	1 1
0 0 1	1 0
0 1 0	0 0
0 1 1	0 0
1 0 0	1 1
1 0 1	0 1
1 1 0	0 1
1 1 1	1 0



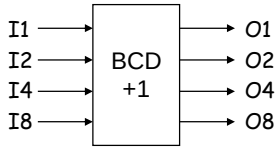
3. Incompletely specified functions:

They have unspecified outputs for some input combinations.

The input combinations of the function that the designer does not care about are called the **don't-care** terms (conditions).

A don't-care input combination never occurs as input to the function.

Example: A function that increments BCD numbers



I8	I4	I2	I1	O8	O4	O2	O1
0	0	0	0	0	0	0	1
0	0	0	1	0	0	1	0
0	0	1	0	0	0	1	1
0	0	1	1	0	1	0	0
0	1	0	0	0	1	0	1
0	1	0	1	0	1	1	0
0	1	1	0	0	1	1	1
0	1	1	1	1	0	0	0
1	0	0	0	1	0	0	1
1	0	0	1	0	0	0	0
1	0	1	0	X	X	X	X
1	0	1	1	X	X	X	X
1	1	0	0	X	X	X	X
1	1	0	1	X	X	X	X
1	1	1	0	X	X	X	X
1	1	1	1	X	X	X	X

For these input combinations the output values of the circuit (function) are not specified.
They cannot occur.
An X or Φ represents a don't care.

Representation of Logical (Boolean) Functions

There are different ways of representing (expressing) the same logical function.

When designing logical circuits, we choose the most suitable representation.

Truth Table Representation :

We write the output values for all possible input combinations (variables) in a table.

The input columns are usually constructed in the order of binary counting.

Input variables are encoded as binary numbers.

Numbered (Indexed) Representation:

Input variables are encoded as binary numbers.

We can assign a decimal number for each input combination according to its binary value.

To represent the function we can list the number of input combinations for which the function generates the logical value "1" (or logical "0" or "Φ").

Example: Representation of a completely specified basic logical function:

Num	Input x_1 x_2	Output y
0	0 0	1
1	0 1	0
2	1 0	1
3	1 1	0

$y = f(x_1,x_2) = \cup_1(0,2)$

$\cup$ denotes "union" or "set of".

$\cup_1$ denotes "set of 1-generating points".

The order of the variables (x_1,x_2) is important.
It must be the same as in the truth table.
Otherwise the numbers of the combinations will change.

The same function; only the order of the variables (x_2,x_1) is changed.

Num	Input x_2 x_1	Output y
0	0 0	1
1	0 1	1
2	1 0	0
3	1 1	0

$y = f(\mathbf{x_2,x_1}) = \cup_1(0,1)$

We can represent the same function with "0"-genereting combinations.

$y = f(x_1,x_2) = \cup_0(1,3)$

Example: Representation of a completely specified general logical function :
We apply the numbered representation to all outputs.

Num	x_1	x_2	y_1	y_2
0	0	0	1	1
1	0	1	0	1
2	1	0	1	0
3	1	1	0	0

$y_1 = f(x_1,x_2) = \cup_1(0,2)$

$y_2 = f(x_1,x_2) = \cup_1(0,1)$

Representation of the same function with "0"-genereting combinations:

$y_1 = f(x_1,x_2) = \cup_0(1,3)$

$y_2 = f(x_1,x_2) = \cup_0(2,3)$

Example: Representation of an incompletely general logical function:

In this case, writing only 1-generating or only 0-generating input combinations is not sufficient.

We must write at least two of the three different groups (1-generating, 0-generating, don't care).

No	x_1	x_2	y_1	y_2
0	0	0	1	1
1	0	1	0	Φ
2	1	0	Φ	0
3	1	1	0	Φ

$y_1 = f(x_1,x_2) = \cup_1(0) + \cup_0(1,3)$

or $y_1 = f(x_1,x_2) = \cup_1(0) + \cup_\Phi(2)$

or $y_1 = f(x_1,x_2) = \cup_0(1,3) + \cup_\Phi(2)$

$y_2 = f(x_1,x_2) = \cup_1(0) + \cup_0(2)$

or $y_2 = f(x_1,x_2) = \cup_1(0) + \cup_\Phi(1,3)$

or $y_2 = f(x_1,x_2) = \cup_0(2) + \cup_\Phi(1,3)$

Graphical Representation

The input variables (combinations) of a logical function are elements (vectors) of the set B^n. Example: B^3={000, 001, …, 111}

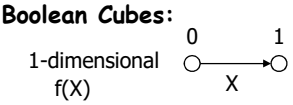
We can represent these variables as vertices of an n-dimensional hypercube.

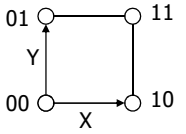
Vertices that correspond to 1-generating combinations are colored (marked).

The number of inputs of the function determines how many dimensions the hypercube has.

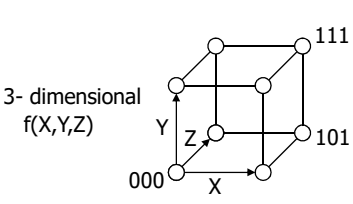
n-bit input → n-dimensional cube

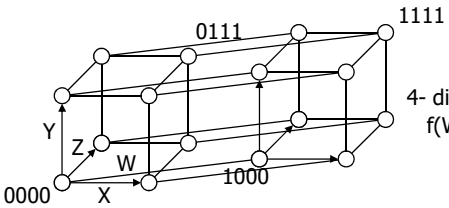
Boolean Cubes:





2- dimensional
f(X,Y)

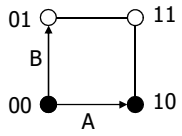




4- dimensional
f(W,X,Y,Z)

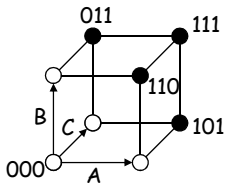
Example:

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0



Example:

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



As the number of variables (inputs) increases, drawing a Boolean cube becomes more difficult.

Therefore, Boolean cubes are not practical for representing Boolean functions with many inputs.

However, they make it easier to visualize some properties (especially, adjacent combinations) of the functions and to explain further topics.

Karnaugh Maps

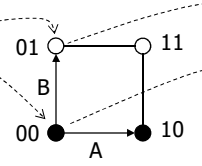
(*Maurice Karnaugh (1924-), American physicist and mathematician*)

The Karnaugh map is a tool for representing and simplifying Boolean functions.
The Boolean variables and related output values are transferred (generally from a truth table) into a table that is in a matrix form.

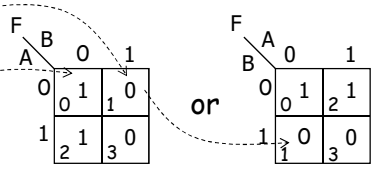
Truth table:

Num.	A	B	F
0	0	0	1
1	0	1	0
2	1	0	1
3	1	1	0

Boolean Cube:



Karnaugh maps:



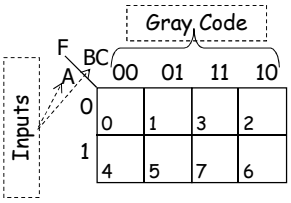
A: row, B: column A: column, B: row

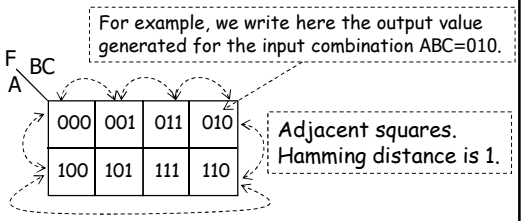
We can place different variables to columns or rows.

Karnaugh map (cont'd)

The rows and columns are labeled according to the **Gray** code property so that variables in adjacent squares (horizontal and vertical) of the map differ in only one variable.

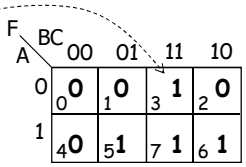
Format of the Karnaugh map for a function with 3 inputs:



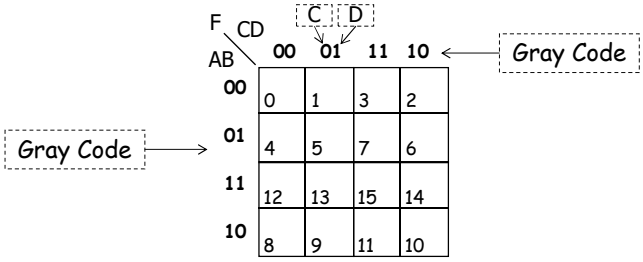


Example:

Num	A	B	C	F
0	0	0	0	0
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1



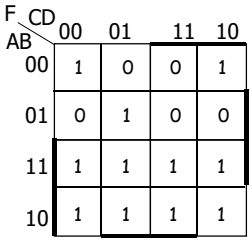
Format of the Karnaugh map of a function with 4 inputs:



Example:

Draw the Karnaugh map for the following function.

$F(A,B,C,D) = \cup_1 (0,2,5,8,9,10,11,12,13,14,15)$



We will use Karnaugh maps to simplify Boolean functions in the coming lectures.

Algebraic Representation (Expressions) and Canonical Forms

The word description of a real-world logic design problem can be translated into a truth table.

Example: Assume that input variable **A** represents the phrase "the car door is open", **B** represents "the key is inserted",
then the truth table can specify the action **Z** to be taken, where Z=1 means that the alarm sounds.

Num.	A	B	Z
0	0	0	0
1	0	1	0
2	1	0	0
3	1	1	1

Truth tables of real-world digital problems are more complicated.

To handle a logic design problem and implement the solution using logic gates, we need to obtain an algebraic **expression** for the output function. Z = f (A,B)

Logical expressions of the Boolean functions can be obtained in **canonical forms** from their truth tables.

There are two types of canonical forms:

- 1st canonical form: SOP (**ΣΠ**) form. Example: ab + cd
  Sum of products, each of which corresponds to a "1"-generating combination.
- 2nd canonical form: POS (**ΠΣ**) form. Example: (a+b) (c+d)
  Product of sums, each of which corresponds to a "0"-generating combination.

1st Canonical Form: Sum of Products

- The 1st canonical form is the sum of special products called minterm.
- In the 1st canonical form, each product in the sum corresponds to a row in the truth table with the output "1".
- **Minterm:** For a Boolean function of n variables, a product of n literals in which each variable appears exactly once (in either true or complemented form, but not both) is called a **minterm**.

  For example, a function with 3 variables (a, b, c) has 8 minterms:

  a'b'c', a'b'c, a'bc', a'bc, ab'c', ab'c, abc', abc

- Each minterm that appears in the 1st canonical form covers only one row of the truth table with the output "1".

  For example; the minterm a'b'c' can generate "1" only for the input value abc=000.

  For all other input combinations the minterm a'b'c' generates "0".
- The 1st canonical form of a function is the sum of minterms.

Finding minterms:

- To find minterms, we locate all rows in the truth table where the output is "1".
- To generate the individual minterms, we substitute variables (for example, A, B, or C) for ones (of the inputs) and complements of variables (A', B', or C') for zeros in the truth table.

Example:

"True" (1) combinations: 001 011 101 110 111

Sum of Minterms: F = A'B'C + A'BC + AB'C + ABC' + ABC

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

The 1st canonical form of the complement of a function:

We can similarly obtain the 1st canonical form of the complement of a function by considering the "false" (0) rows.

Example:

Obtain the 1st canonical form of the complement of a function F from the previous example.

1st canonical form of $\overline{F}$:

A	B	C	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

F' = A'B'C' + A'BC' + AB'C'

Simplification of the expressions in the 1st canonical form:

Canonical forms are usually not the simplest (optimal) algebraic expression of the function.

The can usually be simplified.

Minimization:

F(A, B, C) = A'B'C + A'BC + AB'C + ABC + ABC'
= (A'B' + A'B + AB' + AB)C + ABC'
= ((A' + A)(B' + B))C + ABC'
= C + ABC'
= ABC' + C
= AB + C

- A Boolean function may have many possible logical expressions. They produce the same result given the same inputs.
- Since the minterms present in the 1st canonical form are in one-to-one correspondence with the 1's of the truth table, the 1st canonical (standard) form expression for a function is unique.
- A function can have only one expression in the 1st canonical form.

Indexing minterms:

We assign each minterm an index (number) based on the binary encoding of the variables.

This is a decimal number that represents the row number (Row numbers start at 0).

For example, we assign the index 5 to the minterm ab'c (101) and designate it m5.

A	B	C	Minterms
0	0	0	A'B'C' = m0
0	0	1	A'B'C = m1
0	1	0	A'BC' = m2
0	1	1	A'BC = m3
1	0	0	AB'C' = m4
1	0	1	AB'C = m5
1	1	0	ABC' = m6
1	1	1	ABC = m7

Minterms for three variables

Example:

Expression of function F in (2.31) in 1st canonical form:

F(A, B, C) = Σm(1,3,5,6,7)
= m1 + m3 + m5 + m6 + m7
= A'B'C + A'BC + AB'C + ABC' + ABC
$F = \Sigma_{A, B, C}(1,3,5,6,7)$ (Sum of minterms)

2nd Canonical Form: Product of Sums

- The 2nd canonical form is the product of special sum terms called maxterm.
- In the 2nd canonical form, each of the sum terms (or factors) in the product corresponds to a row in the truth table with the output "0".
- **Maxterm:** For a Boolean function of n variables, a sum of n literals in which each variable appears exactly once (in either true or complemented form, but not both) is called a **maxterm**.
- For example, a function with 3 variables (a, b, c) has 8 maxterms:

  a+b+c, a+b+c', a+b'+c, a+b'+c', a'+b+c, a'+b+c', a'+b'+c, a'+b'+c'
- Each maxterm has a value of "0" for exactly one combination of values for the input variables.

  For example; the maxterm a+b+c can generate "0" only for the input value abc=000.

  For all other input combinations the minterm a+b+c generates "1".
- The 2nd canonical form of a function is the product of maxterms.

Finding maxterms:

- To find the maxterms, we locate all rows in the truth table where the output is "0".
- To generate the individual maxterms, we substitute variables (for example, A, B, or C) for zeros (of the inputs) and complements of variables (A', B', or C') for ones in the truth table.

Example:

"False" (0) generating combinations: *000* *010* *100*

Product of maxterms: F = (A + B + C) (A + B' + C) (A' + B + C)

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Note that this function F is the same as the function in slide 2.31.

Both expressions in 1st and 2nd canonical forms correspond to the same truth table.

The 2nd canonical form of the complement of a function:

We can similarly obtain the 2nd canonical form of the complement of a function by considering the "true" (1) rows:

Example:

Obtain the 2nd canonical form of the complement of a function F from the previous example.

2nd canonical form of $\overline{F}$:

F' = (A + B + C') (A + B' + C') (A' + B + C') (A' + B' + C) (A' + B' + C')

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Simplification of the expressions in the 2nd canonical form:

Canonical forms are usually not the simplest (optimal) algebraic expression of the function.

The can usually be simplified.

Minimization:

F(A, B, C) = (A+B+C) (A+B'+C) (A'+B+C)
= (A+C)(B+B') (A'+B+C)
= (A+C) (A'+B+C)
= (A+C) (A'+B+C) (B+C) (consensus)
= (A + C) (B + C)

- A Boolean function may have many possible logical expressions. They produce the same result given the same inputs.
- Since the maxterms present in the 2nd canonical form are in one-to-one correspondence with the 0's of the truth table, the 2nd canonical (standard) form expression for a function is unique.
- A function can have only one expression in the 2nd canonical form.

Indexing maxterms:

We assign each maxterm an index (number) based on the binary encoding of the variables. This is a decimal number that represents the row number (Row numbers start at 0).

For example, we assign the index 5 to the maxterm a'+b+c' (101) and designate it M5.

A	B	C	Maxterms	
0	0	0	A+B+C	M0
0	0	1	A+B+C'	M1
0	1	0	A+B'+C	M2
0	1	1	A+B'+C'	M3
1	0	0	A'+B+C	M4
1	0	1	A'+B+C'	M5
1	1	0	A'+B'+C	M6
1	1	1	A'+B'+C'	M7

Maxterms for 3 variables

Example:

2nd canonical form of F:

$F(A, B, C) = \Pi M(0,2,4)$

$= M0 \bullet M2 \bullet M4$

$= (A + B + C)(A + B' + C)(A' + B + C)$

$F = \Pi_{A,B,C}(0,2,4)$ product of maxterms.

Conversions Between Canonical Forms

- From 1st (sum of minterms) form to 2nd (product to maxterms) form
  - Replace the minterms with maxterms, and assign them numbers of minterms that don't appear in the 1st canonical form.
  - $F(A,B,C) = \Sigma m(1,3,5,6,7) = \Pi M(0,2,4)$
- From 2nd (product to maxterms) form to 1st (sum of minterms) form
  - Replace the maxterms with minterms, and assign them numbers of maxterms that don't appear in the 2nd canonical form.
  - $F(A,B,C) = \Pi M(0,2,4) = \Sigma m(1,3,5,6,7)$
- Finding the complement of the function in sum of minterms form
  - Select the minterms that don't appear in the 1st canonical form.
  - $F(A,B,C) = \Sigma m(1,3,5,6,7)$ $F'(A,B,C) = \Sigma m(0,2,4)$
- Finding the complement of the function in product of maxterms form
  - Select the maxterms that do not appear in the 2nd canonical form.
  - $F(A,B,C) = \Pi M(0,2,4)$ $F'(A,B,C) = \Pi M(1,3,5,6,7)$

Canonical Forms and the De Morgan's Theorem

Applying the De Morgan's theorem to the complement of the function in the 1st canonical form generates the expression in the 2nd canonical form.

- Complement of the function in SOP form

  F' = A'B'C' + A'BC' + AB'C'

  - De Morgan

  (F')' = (A'B'C' + A'BC' + AB'C')'

  F = (A + B + C) (A + B' + C) (A' + B + C) 2nd canonical form is obtained.

Applying the De Morgan's theorem to the complement of the function in the 2nd canonical form generates the expression in the 1st canonical form.

- Complement of the function in POS form

  F' = (A + B + C') (A + B' + C') (A' + B + C') (A' + B' + C) (A' + B' + C')

  - De Morgan

  (F')' = ((A + B + C')(A + B' + C')(A' + B + C')(A' + B' + C)(A' + B' + C'))'

  F = A'B'C + A'BC + AB'C + ABC' + ABC 1st canonical form is obtained.