Linear Algebra and Applications
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References:
-***Elementary Linear Algebra-Applications Version***,
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-***Elementary Linear Algebra***
Ron Larson and David Falvo, 6th Edition, HOUGHTON MIFFLIN HARCOURT PUBLISHING COMPANY, 2009

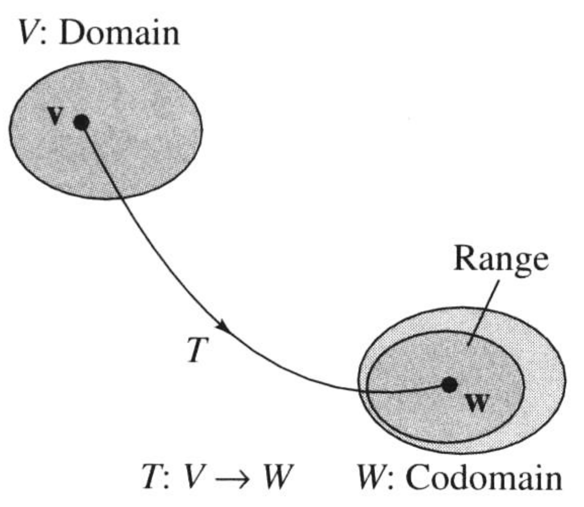
Introduction to Linear Transformations

- Function T that maps a vector space V into a vector space W:

$$T: V \xrightarrow{\text{mapping}} W, \quad V, W: \text{vector space}$$

V: the domain of T

W: the codomain of T



- Image of $\mathbf{v}$ under T:

  If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that

  $$T(\mathbf{v}) = \mathbf{w}$$

  Then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T .

- the range of T:

  The set of all images of vectors in V.

- the preimage of $\mathbf{w}$:

  The set of all $\mathbf{v}$ in V such that $T(\mathbf{v})=\mathbf{w}$.

- Ex: (A function from R^2 into R^2)

$$T : R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2). (b) Find the preimage of **w**=(-1,11)

Sol:

$$(a)\ \ \mathbf{v} = (-1, 2)$$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b)\ \ T(\mathbf{v}) = \mathbf{w} = (-1, 11)$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3,\ v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11).

- Linear Transformation (L.T.):

$V, W:$ vector space

$T: V \rightarrow W:$ V to W linear transformation

$$(1)\ T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

- Notes:

(1) A linear transformation is said to be operation preserving.

$$T(\mathbf{u}\underset{\text{Addition in } V}{+}\mathbf{v}) = T(\mathbf{u})\underset{\text{Addition in } W}{+}T(\mathbf{v}) \qquad T(\underset{\substack{\text{Scalar}\\ \text{multiplication}\\ \text{in } V}}{c\mathbf{u}}) = \underset{\substack{\text{Scalar}\\ \text{multiplication}\\ \text{in } W}}{cT(\mathbf{u})}$$

(2) A linear transformation $T:V \rightarrow V$ from a vector space into itself is called a **linear operator**.

- Ex: (Verifying a linear transformation T from R^2 into R^2)

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Proof:

$$\mathbf{u} = (u_1, u_2),\ \ \mathbf{v} = (v_1, v_2) : \text{vector in } R^2,\ \ c : \text{any real number}$$

$$\text{(1)Vector addition :}$$

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation.

- Ex: (Functions that are not linear transformations)

$(a)\ f(x) = \sin x$

$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$ $\Leftarrow f(x) = \sin x$ is not linear transformation

$\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{3})$

$(b)\ f(x) = x^2$

$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$ $\Leftarrow f(x) = x^2$ is not linear transformation

$(1+2)^2 \neq 1^2 + 2^2$

$(c)\ f(x) = x + 1$

$f(x_1 + x_2) = x_1 + x_2 + 1$

$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$

$f(x_1 + x_2) \neq f(x_1) + f(x_2) \Leftarrow f(x) = x + 1$ is not linear transformation

- Notes: Two uses of the term "linear".

  (1) $f(x)=x+1$ is called a linear function because its graph is a line.

  (2) $f(x)=x+1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication.

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = 0, \ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \forall \mathbf{v} \in V$$

- (Properties of linear transformations)

$$T: V \to W, \quad \mathbf{u}, \mathbf{v} \in V$$

$$(1)\ T(\mathbf{0}) = \mathbf{0}$$

$$(2)\ T(-\mathbf{v}) = -T(\mathbf{v})$$

$$(3)\ T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

(4) If $\mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

Then $T(\mathbf{v}) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

- Ex: (Linear transformations and bases)

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$

$$T(0,1,0) = (1,5,-2)$$

$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$.

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \qquad (T \text{ is a L.T.}) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1) \\ &= (7,7,0) \end{aligned}$$

- Ex: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2,-1)$

(b) Show that T is a linear transformation form R^2 into R^3

Sol: $(a)\mathbf{v} = (2,-1)$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{R^2 \text{ vector}} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{R^3 \text{ vector}}$$

$$\therefore T(2,-1) = (6,3,0)$$

$(b)\ T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

- Thm: (The linear transformation given by a matrix)

  Let A be an $m \times n$ matrix. The function T defined by

  $$T(\mathbf{v}) = A\mathbf{v}$$

  is a linear transformation from R^n into R^m.

- Note:

$$\begin{array}{ccccc} & & R^n \text{ vector} & & R^m \text{ vector} \\ A\mathbf{v} = & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} & = & \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix} \end{array}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$

The Kernel and Range of a Linear Transformation

- **Kernel of a linear transformation T:**

Let $T: V \to W$ be a linear transformation

Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = 0$ is called the kernel of T and is denoted by ker(T).

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = 0, \forall \mathbf{v} \in V\}$$

- **Ex: (Finding the kernel of a linear transformation)**

$$T(A) = A^T \quad (T: M_{3\times 2} \to M_{2\times 3})$$

Sol:

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

- Ex: (The kernel of the zero and identity transformations)

(a) $T(\mathbf{v})=\mathbf{0}$ (the zero transformation $T:V\rightarrow W$)

$$\ker(T)=V$$

(b) $T(\mathbf{v})=\mathbf{v}$ (the identity transformation $T:V\rightarrow V$)

$$\ker(T)=\{\mathbf{0}\}$$

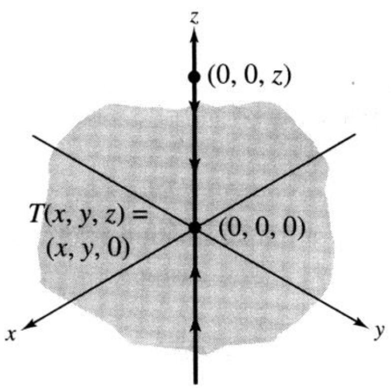
- Ex: (Finding the kernel of a linear transformation)

$$T(x,y,z)=(x,y,0) \qquad (T:R^3\rightarrow R^3)$$

$$\ker(T)=?$$

Sol:

$$\ker(T)=\{(0,0,z)\mid z \text{ is a real number}\}$$



The kernel of T is the set of all vectors on the z-axis.

- Ex: (Finding the kernel of a linear transformation)

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T : R^3 \rightarrow R^2)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0),\ x = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

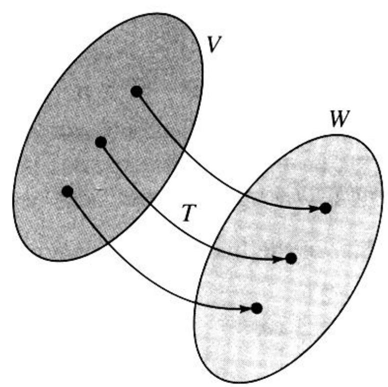
$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\}$$

$$= \text{span}\{(1,-1,1)\}$$

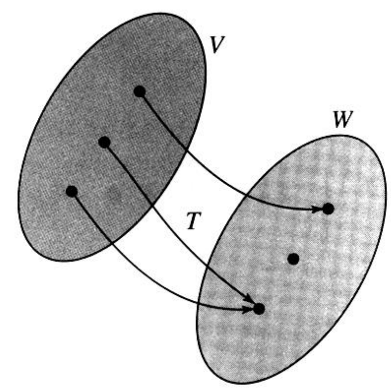
- One-to-one:

A function $T: V \to W$ is called one-to-one if the preimage of every $\mathbf{w}$ in the range consists of a single vector.

T is one-to-one iff for all $\mathbf{u}$ and $\mathbf{v}$ in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.



one-to-one



not one-to-one

- Onto:

A function $T: V \to W$ is said to be onto if every element in $\mathbf{w}$ has a preimage in V

(T is onto W when W is equal to the range of T.)

Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \rightarrow R^3$:

$$(1)\, T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\, T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write.
  - It is simpler to read.
  - It is more easily adapted for computer use.

- Thm 10: (Standard matrix for a linear transformation)

Let $T: R^n \rightarrow R^m$ be a linear trtansformation such that

$$T(e_1)=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(e_2)=\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad T(e_n)=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix whose n columns correspond to $T(e_i)$

$$A=\left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n)\right]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n.

A is called the standard matrix for T.

Proof:

$$\mathbf{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=v_1e_1+v_2e_2+\cdots+v_ne_n$$

$$T \text{ is a L.T.} \Rightarrow T(\mathbf{v})=T(v_1e_1+v_2e_2+\cdots+v_ne_n)$$
$$=T(v_1e_1)+T(v_2e_2)+\cdots+T(v_ne_n)$$
$$=v_1T(e_1)+v_2T(e_2)+\cdots+v_nT(e_n)$$

$$A\mathbf{v}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} a_{11}v_1+a_{12}v_2+\cdots+a_{1n}v_n \\ a_{21}v_1+a_{22}v_2+\cdots+a_{2n}v_n \\ \vdots \\ a_{m1}v_1+a_{m2}v_2+\cdots+a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex: (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $T: R^3 \rightarrow R^2$ define by

$$T(x, y, z) = (x - 2y,\ 2x + y)$$

Sol:

Vector Notation	Matrix Notation
$T(e_1) = T(1, 0, 0) = (1, 2)$	$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$T(e_2) = T(0, 1, 0) = (-2, 1)$	$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
$T(e_3) = T(0, 0, 1) = (0, 0)$	$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \left[T(e_1) \mid T(e_2) \mid T(e_3) \right]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ 2x + y \end{bmatrix}$$

i.e. $T(x, y, z) = (x - 2y, 2x + y)$

- Note:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} \leftarrow & 1x - 2y + 0z \\ \leftarrow & 2x + 1y + 0z \end{matrix}$$

- Ex: (Finding the standard matrix of a linear transformation)

The linear transformation $T: R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T.

Sol:

$$T(x, y) = (x, 0)$$

$$A = \left[T(e_1) \mid T(e_2)\right] = \left[T(1,0) \mid T(0,1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Notes:

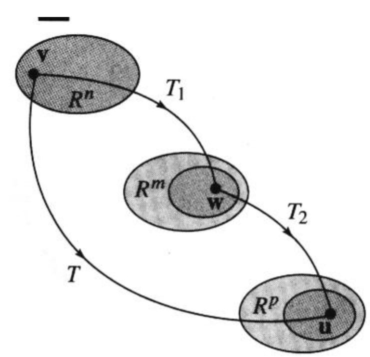
(1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix.

(2) The standard matrix for the zero transformation from R^n into R^n is the $n \times n$ identity matrix I_n

- Composition of $T_1:R^n \rightarrow R^m$ with $T_2:R^m \rightarrow R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$T = T_2 \circ T_1, \quad \text{domain of } T = \text{domain of } T_1$$



Composition of Transformations

- Thm : (Composition of linear transformations)

Let $T_1 : R^n \rightarrow R^m$ and $T_2 : R^m \rightarrow R^p$ be L.T.
with standard matrices A_1 and A_2 , then

(1) The composition $T : R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a L.T.

(2) The standard matrix A for T is given by the matrix product $A = A_2 A_1$

Proof:

(1)(T is a L.T.)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$

$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2)(A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- Note:

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

- Ex: (The standard matrix of a composition)

Let T_1 and T_2 be L.T. from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, \mathrm{z}, y)$$

Find the standard matrices for the compositions

$T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$,

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad (\text{standard matrix for } T_1)$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{standard matrix for } T_2)$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Inverse linear transformation:

$$\text{If } T_1 : R^n \to R^n \text{ and } T_2 : R^n \to R^n \text{ are L.T. s.t. for every } \mathbf{v} \text{ in } R^n$$

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

$$\text{Then } T_2 \text{ is called the inverse of } T_1 \text{ and } T_1 \text{ is said to be invertible}$$

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1}.

- Ex 4: (Finding the inverse of a linear transformation)

The L.T. $T: R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

- Ex: (Finding the inverse of a linear transformation)

The L.T. $T:\ R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{G.J.E}\left[\begin{array}{ccc|ccc}1&0&0&-1&1&0\\0&1&0&-1&0&1\\0&0&1&6&-2&-3\end{array}\right]=\left[I \mid A^{-1}\right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1}=\begin{bmatrix}-1&1&0\\-1&0&1\\6&-2&-3\end{bmatrix}$$

$$T^{-1}(\mathbf{v})=A^{-1}\mathbf{v}=\begin{bmatrix}-1&1&0\\-1&0&1\\6&-2&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_1+x_2\\-x_1+x_3\\6x_1-2x_2-3x_3\end{bmatrix}$$

In other words,

$$T^{-1}(x_1,x_2,x_3)=(-x_1+x_2,-x_1+x_3,6x_1-2x_2-3x_3)$$

- the matrix of T relative to the bases B *and* B':

$$T : V \rightarrow W \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis for } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_m\} \qquad \text{(a basis for } W\text{)}$$

Thus, the matrix of T relative to the bases B *and* B' is

$$A = \left[[T(v_1)]_{B'}, [T(v_2)]_{B'}, \cdots, [T(v_n)]_{B'} \right] \in M_{m \times n}$$

Transformation matrix for nonstandard bases:

Let V and W be finite-dimensional vector spaces with basis B and B', respectively, where $B = \{v_1, v_2, \cdots, v_n\}$

If $T: V \to W$ is a L.T. s.t.

$$[T(v_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad [T(v_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad [T(v_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $[T(v_i)]_{B'}$

$$A = \left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\left[T(\mathbf{v}) \right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

- Ex: (Finding a matrix relative to nonstandard bases)

Let $T: \ R^2 \to R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2,\ 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$[T(1, 2)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},\quad [T(-1, 1)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[[T(1, 2)]_{B'} \quad [T(1, 2)]_{B'} \right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex:

For the L.T. $T: R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v}=(2,1)$

Sol:

$$\mathbf{v}=(2,1)=1(1,2)-1(-1,1) \qquad B=\{(1,2),(-1,1)\}$$

$$\Rightarrow [\mathbf{v}]_B=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'}=A[\mathbf{v}]_B=\begin{bmatrix}3 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$

$$\Rightarrow T(\mathbf{v})=3(1,0)+3(0,1)=(3,3) \qquad B'=\{(1,0),(0,1)\}$$

- Check:

$$T(2,1)=(2+1,2(2)-1)=(3,3)$$

Transition Matrices and Similarity

$$T : V \rightarrow V \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis of } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_n\} \qquad \text{(a basis of } V\text{)}$$

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B \right] \qquad \text{(matrix of } T \text{ relative to } B\text{)}$$

$$A' = \left[[T(w_1)]_{B'}, [T(w_2)]_{B'}, \cdots, [T(w_n)]_{B'} \right] \qquad \text{(matrix of } T \text{ relative to } B'\text{)}$$

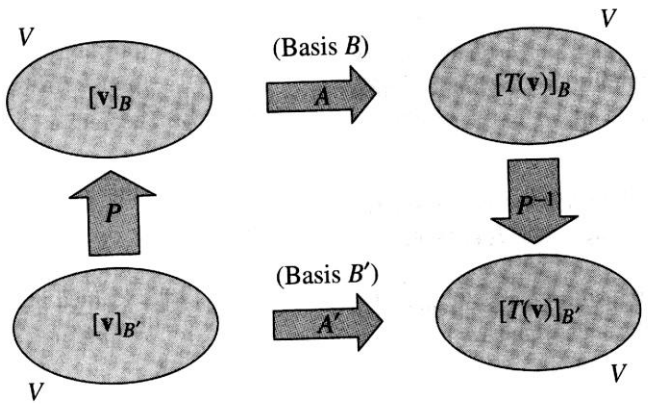
$$P = \left[[w_1]_B, [w_2]_B, \cdots, [w_n]_B \right] \qquad \text{(transition matrix from } B' \text{ to } B\text{)}$$

$$P^{-1} = \left[[v_1]_{B'}, [v_2]_{B'}, \cdots, [v_n]_{B'} \right] \qquad \text{(transition matrix from } B \text{ to } B'\text{)}$$

$$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \ [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$



- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:

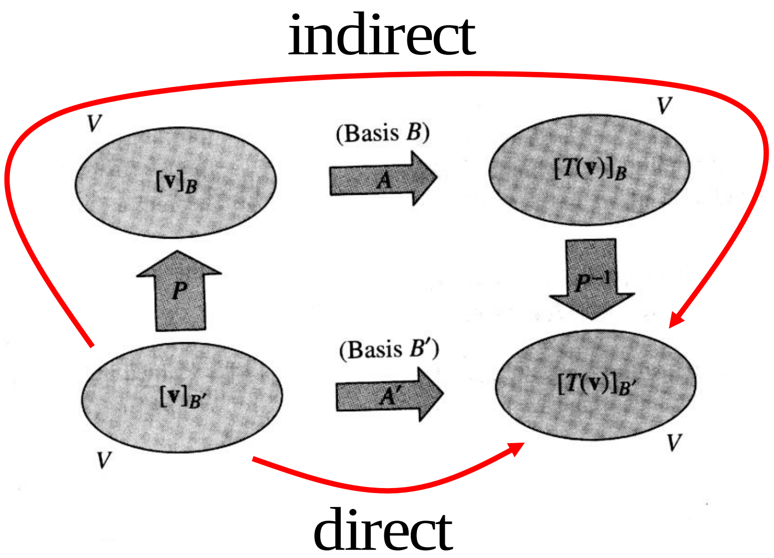
(1)(direct)

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

(2)(indirect)

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

$$\Rightarrow A' = P^{-1}AP$$



- Ex: (Finding a matrix for a linear transformation)

Find the matrix A' for $T: \; R^2 \rightarrow R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

reletive to the basis $B' = \{(1, 0), (1, 1)\}$

Sol:

$$\text{(I) } A' = \left[[T(1,0)]_{B'} \quad [T(1,1)]_{B'} \right]$$

$$T(1,0) = (2,-1) = 3(1,0) - 1(1,1) \;\Rightarrow\; [T(1,0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1,1) = (0,2) = -2(1,0) + 2(1,1) \;\Rightarrow\; [T(1,1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \left[[T(1,0)]_{B'} \quad [T(1,1)]_{B'} \right] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(II) standard matrix for T (matrix of T relative to $B = \{(1,0), (0,1)\}$)

$$A = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

transition matrix from B' to B

$$P = \begin{bmatrix} [(1,0)]_B & [(1,1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

transition matrix from B to B'

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

matrix of T relative B'

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

- Ex: (Finding a matrix for a linear transformation)

  Let $B=\{(-3, 2), (4, -2)\}$ and $B'=\{(-1, 2), (2, -2)\}$ be basis for R^2,

  and let $A=\begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$ be the matrix for $T: R^2 \to R^2$ relative to B.

  Find the matrix of T relative to B'.

Sol:

transition matrix from B' to B: $P=\begin{bmatrix} [(-1, 2)]_B & [(2, -2)]_B \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

transition matrix from B to B': $P^{-1}=\begin{bmatrix} [(-3, 2)]_{B'} & [(4, -2)]_{B'} \end{bmatrix}=\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

matrix of T relative to B':

$$A'=P^{-1}AP=\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

- Similar matrix:

  For square matrices A and A' of order n, A' is said to be similar to A if there exist an invertible matrix P $s.t.$ $A' = P^{-1}AP$

- Thm 6.13: (Properties of similar matrices)

  Let A, B, and C be square matrices of order n.

  Then the following properties are true.

  (1) A is similar to A.

  (2) If A is similar to B, then B is similar to A.

  (3) If A is similar to B and B is similar to C, then A is similar to C.

Proof:

$$(1) A = I_n A I_n$$

$$(2) A = P^{-1}BP \;\Rightarrow\; PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$PAP^{-1} = B \;\Rightarrow\; Q^{-1}AQ = B \;\; (Q = P^{-1})$$

- Ex: (Similar matrices)

$$(a)\ A=\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \text{ and } A'=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(b)\ A=\begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \text{ and } A'=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$

- Ex : (A comparison of two matrices for a linear transformation)

Suppose $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is the matrix for $T: R^3 \rightarrow R^3$ relative to the standard basis. Find the matrix for T relative to the basis

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

Sol:

The transition matrix from B' to the standard matrix

$$P = \left[[(1, 1, 0)]_B \quad [(1, -1, 0)]_B \quad [(0, 0, 1)]_B \right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

matrix of T relative to B':

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$