Linear Algebra and Applications
8 October 2014

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References:
-***Elementary Linear Algebra-Applications Version"***,
Howard Anton and Chris Rorres, 9th Edition, Wiley, 2010.

Definitions

- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the sum $\mathbf{v} + \mathbf{w}$ is the vector determined as follows:
  - Position the vector $\mathbf{w}$ so that its initial point coincides with the terminal point of $\mathbf{v}$. The vector $\mathbf{v} + \mathbf{w}$ is represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$.
- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the difference of $\mathbf{w}$ from $\mathbf{v}$ is defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$.
- If $\mathbf{v}$ is a nonzero vector and k is nonzero real number (scalar), then the product $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k > 0$ and opposite to that of $\mathbf{v}$ if $k < 0$. We define $k\mathbf{v} = 0$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.
- A vector of the form $k\mathbf{v}$ is called a scalar multiple.

Norm of a Vector

- The length of a vector **u** is often called the norm of **u** and is denoted by ||**u**||.
- It follows from the Theorem of Pythagoras that the norm of a vector $\mathbf{u} = (u_1, u_2, u_3)$ in 3-space is

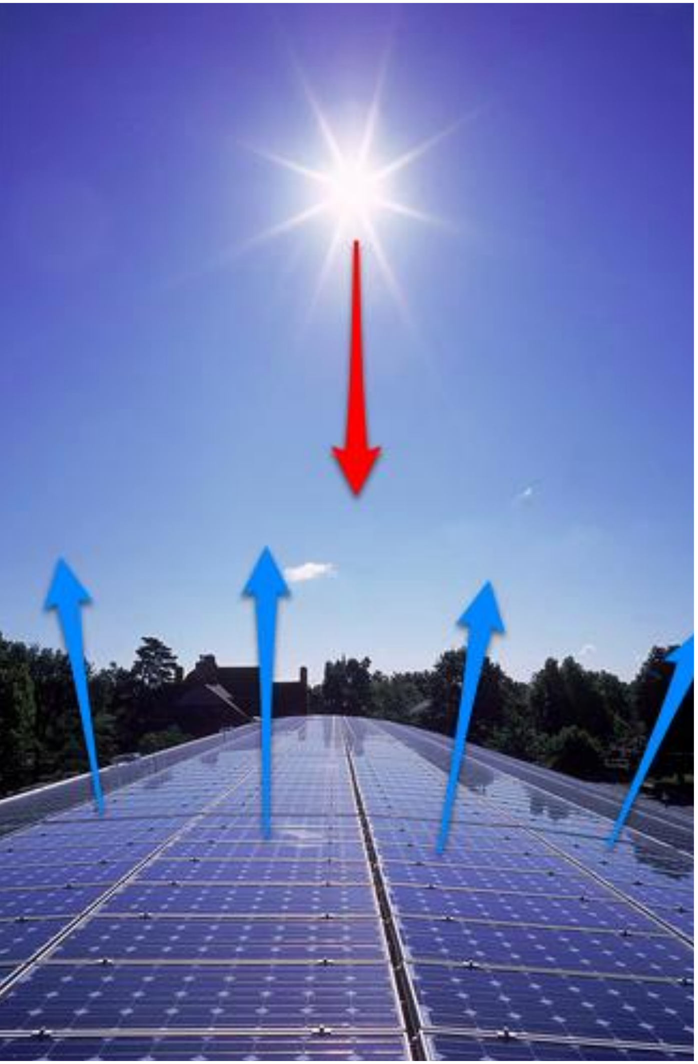
$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

- A vector of norm 1 is called a unit vector.
- The distance between two points is the norm of the vector.
- The length of the vector $k\mathbf{u} = \|k\mathbf{u}\| = |k|\ \|\mathbf{u}\|$.

Definitions

- Let **u** and **v** be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so their initial points coincided. By the angle between **u** and **v**, we shall mean the angle θ determined by **u** and **v** that satisfies $0 \leq \theta \leq \pi$.
- If **u** and **v** are vectors in 2-space or 3-space and θ is the angle between **u** and **v**, then the dot product or Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases}$$

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Example

- If the angle between the vectors $\mathbf{u} = (0,0,1)$ and $\mathbf{v} = (0,2,2)$ is 45°, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \sqrt{0+0+1}\sqrt{0+4+4} \cdot \left(\frac{1}{\sqrt{2}}\right) = 2$$

$$\mathbf{u} \cdot \mathbf{v} = (u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3 = 2$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{\sqrt{0+0+1}\sqrt{0+4+4}} = \frac{1}{\sqrt{2}}$$

Theorems

$$\mathbf{u}\cdot\mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u}\neq\mathbf{0} \text{ and } \mathbf{v}\neq\mathbf{0} \\ 0 & \text{if } \mathbf{u}=\mathbf{0} \text{ or } \mathbf{v}=\mathbf{0} \end{cases}$$

- Theorem 3.3.1
  - Let **u** and **v** be vectors in 2- or 3-space.
    - $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$; that is, $\|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2}$
    - If the vectors **u** and **v** are nonzero and θ is the angle between them, then
      - θ is acute if and only if $\mathbf{u}\cdot\mathbf{v} > 0$
      - θ is obtuse if and only if $\mathbf{u}\cdot\mathbf{v} < 0$
      - $\theta = \pi/2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$

- Theorem 3.3.2 (Properties of the Dot Product)
  - If **u**, **v** and **w** are vectors in 2- or 3-space, and k is a scalar, then
    - $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
    - $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
    - $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$
    - $\mathbf{v}\cdot\mathbf{v} > 0$ if $\mathbf{v}\neq 0$, and $\mathbf{v}\cdot\mathbf{v} = 0$ if $\mathbf{v} = 0$

Orthogonal Vectors

- Definition
  - Perpendicular vectors are also called orthogonal vectors.
  - Two *nonzero* vectors are orthogonal if and only if their dot product is zero.
  - To indicate that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors we write

$$\mathbf{u} \perp \mathbf{v}.$$

- Example
  - Show that in 2-space the nonzero vector $\mathbf{n} = (a,b)$ is perpendicular to the line $ax + by + c = 0$.

Solution

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points on the line, so that

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$

Since the vector $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$ runs along the line (Figure 3.3.5), we need only show that $\boldsymbol{n}$ and $\overrightarrow{P_1P_2}$ are perpendicular. But on subtracting the equations in (6), we obtain

$$a(x_2 - x_1) + b(y_2 - y_1) = 0$$

which can be expressed in the form

$$(a, b) \cdot (x_2 - x_1, y_2 - y_1) = 0 \qquad \text{or} \qquad \mathbf{n} \cdot \overrightarrow{P_1P_2} = 0$$

Thus $\boldsymbol{n}$ and $\overrightarrow{P_1P_2}$ are perpendicular.

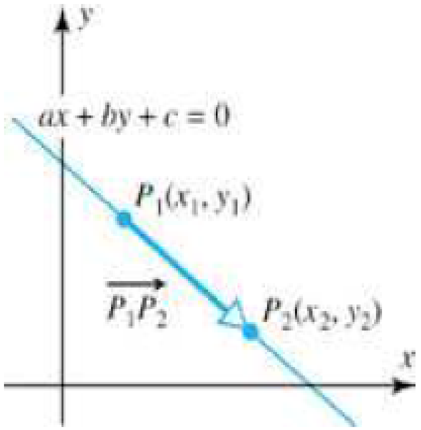
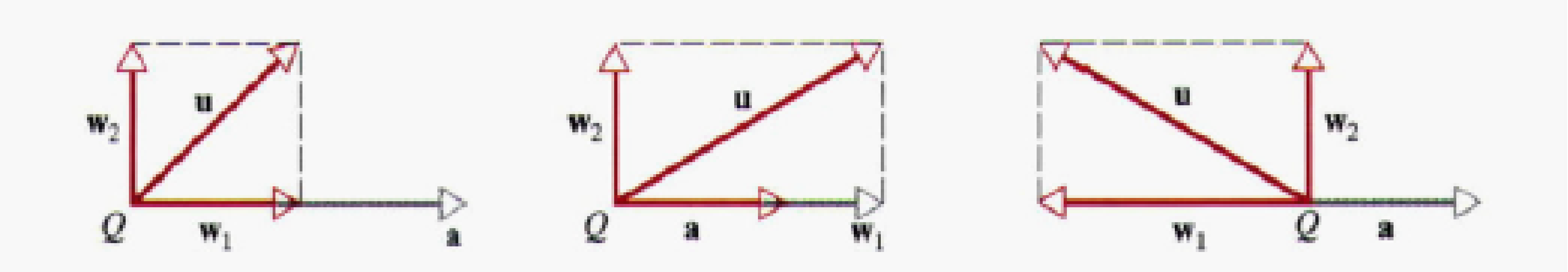


Figure 3.3.5

An Orthogonal Projection

- To "decompose" a vector **u** into a sum of two terms, one *parallel* to a specified nonzero vector **a** and the other *perpendicular* to **a**.
- We have $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ and $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$
- The vector $\mathbf{w}_1$ is called the orthogonal projection of **u** on **a** or sometimes the vector component of **u** along **a**, and denoted by $\mathrm{proj}_{\mathbf{a}}\mathbf{u}$
- The vector $\mathbf{w}_2$ is called the vector component of **u** orthogonal to **a**, and denoted by $\mathbf{w}_2 = \mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u}$



Theorem 3.3.3

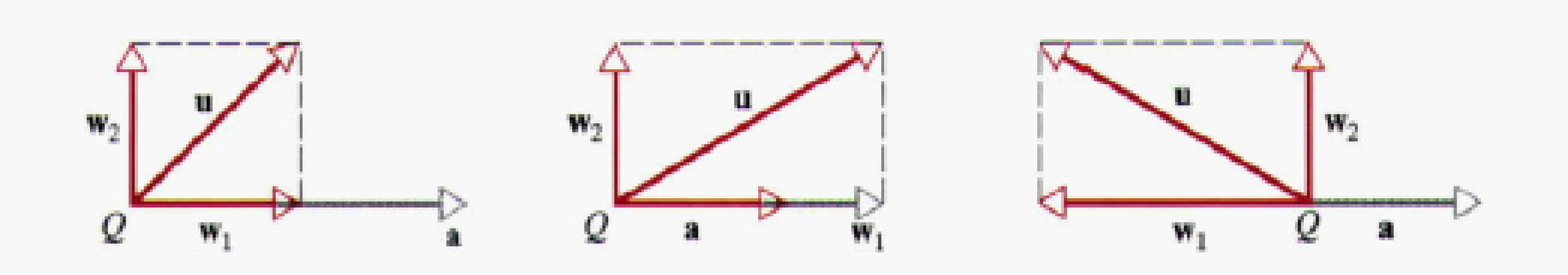
$$\mathbf{w}_1 = \operatorname{proj}_{\mathbf{a}} \mathbf{u}$$
$$\mathbf{w}_2 = \mathbf{u} - \operatorname{proj}_{\mathbf{a}} \mathbf{u}$$

- If $\mathbf{u}$ and $\mathbf{a}$ are vectors in 2-space or 3-space and if $\mathbf{a} \neq \mathbf{0}$, then

$$\operatorname{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \qquad \text{(vector component of } \mathbf{u} \text{ along } \mathbf{a}\text{)}$$

$$\mathbf{u} - \operatorname{proj}_{\mathbf{a}} \mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \qquad \text{(vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{a}\text{)}$$

$$\|\operatorname{proj}_{\mathbf{a}} \mathbf{u}\| = \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|} = \|\mathbf{u}\| |\cos\theta|$$



Example

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

Let $u = (2,-1,3)$ and $a = (4,-1,2)$. Find the vector component of u along a and the vector component of u orthogonal to a.

- Solution:

$$u \cdot a = (2)(4) + (-1)(-1) + (3)(2) = 15$$

$$\|a\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus, the vector component of u along a is

$$proj_a\, u = \frac{u \cdot a}{\|a\|^2} a = \tfrac{15}{21}(4,-1,2) = (\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7})$$

and the vector component of u orthogonal to a is

$$u - proj_a\, u = (2,-1,3) - (\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}) = (-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{11}{7})$$

Verify that the vector $u - proj_a\, u$ and a are perpendicular by showing that their dot product is zero.

Cross Product

- Definition
  - If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the cross product $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

or in determinant notation

$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

- Example
  - Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$.

$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$
$$= (2, -7, -6)$$

Theorems

- Theorem 3.4.1 (Relationships Involving Cross Product and Dot Product)
  - If **u**, **v** and **w** are vectors in 3-space, then
    - $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
    - $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
    - $\| \mathbf{u} \times \mathbf{v} \| = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (Lagrange's identity)
    - $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}$ (relationship between cross & dot product)
    - $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\, \mathbf{u}$ (relationship between cross & dot product)

- Theorem 3.4.2 (Properties of Cross Product)
  - If **u**, **v** and **w** are any vectors in 3-space and k is any scalar, then
    - $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
    - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
    - $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w}$
    - $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
    - $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
    - $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Standard Unit Vectors

- The vectors
$$\mathbf{i} = (1,0,0),\ \mathbf{j} = (0,1,0),\ \mathbf{k} = (0,0,1)$$
have length 1 and lie along the coordinate axes. They are called the standard unit vectors in 3-space.
- Every vector $\mathbf{v} = (v_1, v_2, v_3)$ in 3-space is expressible in terms of **i**, **j**, **k** since we can write
$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1,0,0) + v_2\,(0,1,0) + v_3\,(0,0,1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

- For example, $(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
- Note that
$$\mathbf{i}\times\mathbf{i} = \mathbf{0},\quad \mathbf{j}\times\mathbf{j} = \mathbf{0},\quad \mathbf{k}\times\mathbf{k} = \mathbf{0}$$
$$\mathbf{i}\times\mathbf{j} = \mathbf{k},\quad \mathbf{j}\times\mathbf{k} = \mathbf{i},\quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$
$$\mathbf{j}\times\mathbf{i} = -\mathbf{k},\quad \mathbf{k}\times\mathbf{j} = -\mathbf{i},\quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

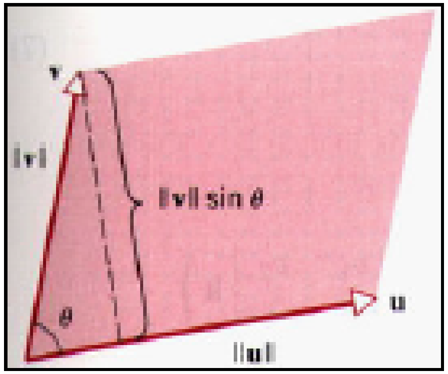
Cross Product

- A cross product can be represented symbolically in the form of 3×3 determinant:

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\mathbf{k}$$

- Geometric interpretation of cross product:
  - From Lagrange's identity, we have

$$\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$$



$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} - \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

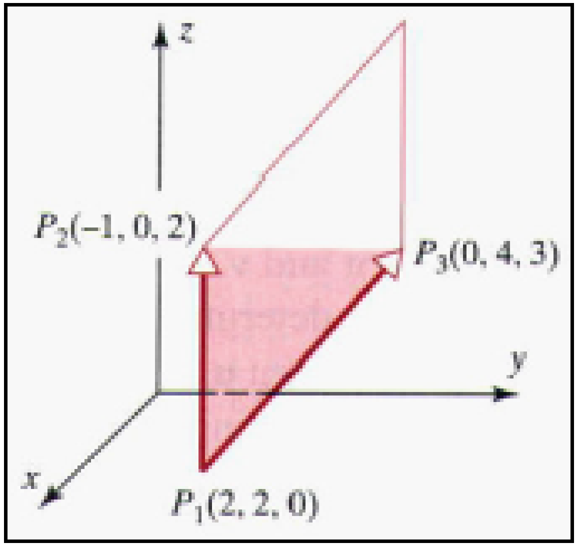
$$= i\,(2 \times 2 - 1 \times (-1)) - j\,(1 \times 2 - 1 \times 0) + k\,(1 \times (-1) - 2 \times 0)$$

$$= 5i - 2j - 1k = (5, -2, -1)^T$$

Area of a Parallelogram

- Theorem 3.4.3 (Area of a Parallelogram)
  - If **u** and **v** are vectors in 3-space, then $\|\mathbf{u} \times \mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**.

- Example
  - Find the area of the triangle determined by the point (2,2,0), (-1,0,2), and (0,4,3).



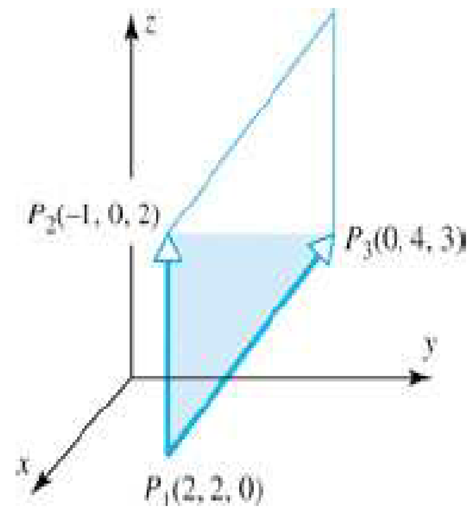
Solution

The area A of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ (Figure 3.4.5). Using the method discussed in Example 2 of Section 3.1, $\overrightarrow{P_1P_2} = (-3, -2, 2)$ and $\overrightarrow{P_1P_3} = (-2, 2, 3)$. It follows that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-10, 5, -10)$$

and consequently,

$$A = \frac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \frac{1}{2}(15) = \frac{15}{2}$$



Triple Product

- Definition
  - If **u**, **v** and **w** are vectors in 3-space, then $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called the scalar triple product of **u**, **v** and **w**.

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

- Remarks:
  - The symbol $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$ make no sense.
  - $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$

Theorem 3.4.4

- The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$, and $\mathbf{v} = (v_1, v_2)$,
- The absolute value of the determinant

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

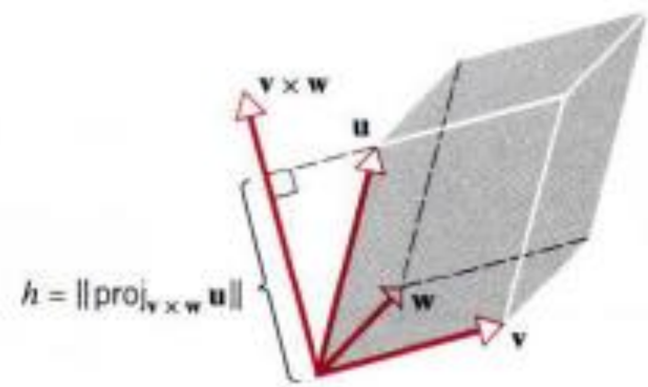
is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$,

Remark

$$V = \left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|$$

$$V = \begin{bmatrix} \text{volume of parallelepiped} \\ \text{determined by } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \end{bmatrix} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



(b)

Theorem 3.4.5

- If the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ have the same initial point, then they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

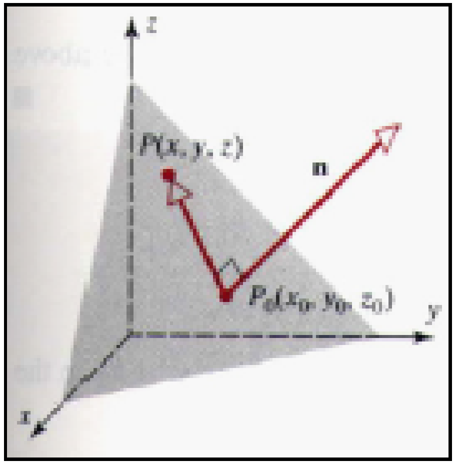
Planes in 3-Space

- One can specify a plane in 3-space by giving its inclination and specifying one of its points.
- A convenient method for a plane is to specify a nonzero vector, called a normal, that is perpendicular to the plane.
- The point-normal form of the equation of a plane:

$\mathbf{n} = (a, b, c)$

$$n \cdot \overrightarrow{P_0P} = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$



Example

Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = (4, 2, -5)$.

Solution.

From (2) a point-normal form is

$$4(x-3) + 2(y+1) - 5(z-7) = 0$$

By multiplying out and collecting terms, (2) can be rewritten in the form

$$ax + by + cz + d = 0$$

where a, b, c, and d are constants, and a, b, and c are not all zero. For example, the equation in Example 1 can be rewritten as

$$4x + 2y - 5z + 25 = 0$$

Theorem 3.5.1

- If a, b, c, and d are constants and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0$$

is a plane having the vector $\mathbf{n} = (a, b, c)$ as a normal.

- Remark:
  - The above equation is a linear equation in x, y, and z; it is called the general form of the equation of a plane.

- Theorem 3.5.2 (Distance between a Point and a Plane)
  - The distance D between a point $P_0(x_0,y_0,z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the equation of the plane passing through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

Solution.

Since the three points lie in the plane, their coordinates must satisfy the general equation $ax + by + cz + d = 0$ of the plane. Thus,

$$\begin{aligned} a + 2b - c + d &= 0 \\ 2a + 3b + c + d &= 0 \\ 3a - b + 2c + d &= 0 \end{aligned}$$

Example

Solving this system gives $a = -\frac{9}{16}t, b = -\frac{1}{16}t, c = \frac{5}{16}t, d = t$. Letting $t = -16$, for example, yields the desired equation

$$9x + y - 5z - 16 = 0$$

We note that any other choice of t gives a multiple of this equation, so that any value of $t \neq 0$ would also give a valid equation of the plane.

Alternative Solution.

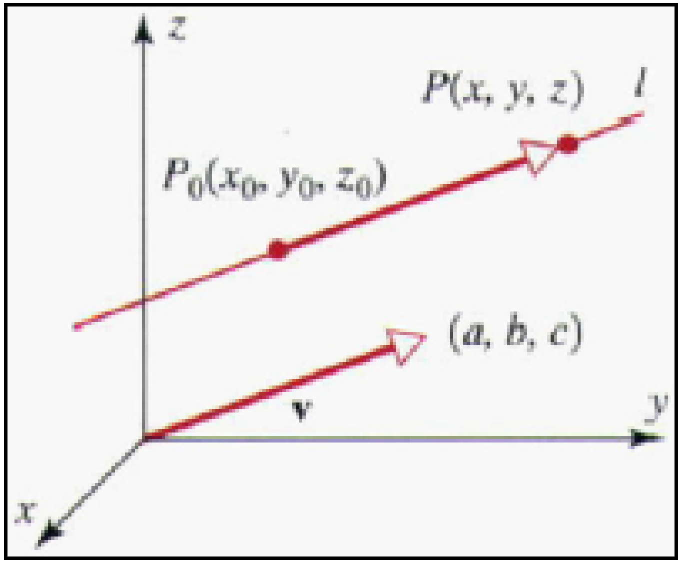
Since the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$ lie in the plane, the vectors $\overrightarrow{P_1P_2} = (1, 1, 2)$ and $\overrightarrow{P_1P_3} = (2, -3, 3)$ are parallel to the plane. Therefore, the equation $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (9, 1, -5)$ is normal to the plane, since it is perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. From this and the fact that P_1 lies in the plane, a point-normal form for the equation of the plane is

$$9(x - 1) + (y - 2) - 5(z + 1) = 0 \quad \text{or} \quad 9x + y - 5z - 16 = 0 \qquad \blacklozenge$$

Line in 3-Space

- Suppose that l is the line in 3-space through the point $P_0(x_0,y_0,z_0)$ and parallel to the nonzero vector $\mathbf{v} = (a, b, c)$.
- l consists precisely of those points $P_0(x_0,y_0,z_0)$ for which the vector $\overrightarrow{P_0P}$ is parallel to v, that is, for which there is a scalar t such that $\overrightarrow{P_0P} = t\mathbf{v}$
- Parametric equations for l:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$



Example

- Parametric equations of a line

The line through the point $(1, 2, -3)$ and parallel to the vector $\mathbf{v} = (4, 5, -7)$ has parametric equations

$$x = 1 + 4t, \quad y = 2 + 5t, \quad z = -3 - 7t \qquad (-\infty < t < +\infty)$$

♦

Example (Intersection of a Line and the xy-Plane)

(a) Find parametric equations for the line l passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

(b) Where does the line intersect the xy-plane?

Solution (a). Since the vector $\overrightarrow{P_1P_2} = (3, -4, 8)$ is parallel to l and $P_1(2, 4, -1)$ lies on l, the line l is given by

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \qquad (-\infty < t < +\infty)$$

Solution (b). The line intersects the xy-plane at the point where $z = -1 + 8t = 0$, that is, where $t = \frac{1}{8}$. Substituting this value of t in the parametric equations for l yields as the point of intersection

$$(x, y, z) = \left(\tfrac{19}{8}, \tfrac{7}{2}, 0\right)$$ ♦

Example (Line of Intersection of Two Planes)

Find parametric equations for the line of intersection of the planes

$$3x + 2y - 4z - 6 = 0 \quad \text{and} \quad x - 3y - 2z - 4 = 0$$

Solution.

The line of intersection consists of all points (x, y, z) that satisfy the two equations in the system

$$\begin{aligned} 3x + 2y - 4z &= 6 \\ x - 3y - 2z &= 4 \end{aligned}$$

Solving this system gives $x = \frac{26}{11} + \frac{16}{11}t$, $y = -\frac{6}{11} - \frac{2}{11}t$, $z = t$. Therefore, the line of intersection can be represented by the parametric equations

$$x = \tfrac{26}{11} + \tfrac{16}{11}t, \quad y = -\tfrac{6}{11} - \tfrac{2}{11}t, \quad z = t \qquad (-\infty < t < +\infty)$$ ♦

Example

- A line parallel to a given vector

The equation

$$(x, y, z) = (-2, 0, 3) + t(4, -7, 1) \quad (-\infty < t < +\infty)$$

is the vector equation of the line through the point $(-2, 0, 3)$ that is parallel to the vector $\mathbf{v} = (4, -7, 1)$. ♦

Theorem 3.5.2 (Distance Between a Point and a Plane)

- The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{\left|ax_0 + by_0 + cz_0 + d\right|}{\sqrt{a^2 + b^2 + c^2}}$$

Example (Distance Between a Pont and a Plane)

Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

Solution.

To apply (9), we first rewrite the equation of the plane in the form

$$2x - 3y + 6z + 1 = 0$$

Then

$$D = \frac{|2(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

Example (Distance Between Parallel Planes)

The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $(1, 2, -2)$ and $(2, 4, -4)$, are parallel vectors. Find the distance between these planes.

Solution.

To find the distance D between the planes, we may select an arbitrary point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (9), the distance between P_0 and the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|2(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6} \quad \blacklozenge$$

Euclidian Vector Spaces

Definitions

- If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuple is called n-space and is denoted by R^n.

- Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n are called equal if

  $$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$$

  The sum $\mathbf{u} + \mathbf{v}$ is defined by

  $$\mathbf{u} + \mathbf{v} = (u_1+v_1, u_1+v_1, \ldots, u_n+v_n)$$

  and if k is any scalar, the scalar multiple $k\mathbf{u}$ is defined by

  $$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

Remarks

- The operations of addition and scalar multiplication in this definition are called the *standard operations* on R^n.

- The zero vector in R^n is denoted by $\mathbf{0}$ and is defined to be the vector $\mathbf{0} = (0, 0, \ldots, 0)$.
- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is any vector in R^n, then the negative (or additive inverse) of $\mathbf{u}$ is denoted by $-\mathbf{u}$ and is defined by $-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$.
- The difference of vectors in R^n is defined by

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u}) = (v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n)$$

Theorem 4.1.1 (Properties of Vector in R^n)

- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n and k and l are scalars, then:
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  - $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - $\mathbf{u} + (-\mathbf{u}) = 0$; that is $\mathbf{u} - \mathbf{u} = 0$
  - $k(l\mathbf{u}) = (kl)\mathbf{u}$
  - $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
  - $(k+l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
  - $1\mathbf{u} = \mathbf{u}$

Euclidean Inner Product

- Definition
  - If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- Example
  - The Euclidean inner product of the vectors $\mathbf{u} = (-1,3,5,7)$ and $\mathbf{v} = (5,-4,7,0)$ in R^4 is

$$\mathbf{u} \cdot \mathbf{v} = (-1)(5) + (3)(-4) + (5)(7) + (7)(0) = 18$$

Properties of Euclidean Inner Product

- Theorem 4.1.2
  - If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n and k is any scalar, then
    - $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
    - $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
    - $(k\,\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
    - $\mathbf{v} \cdot \mathbf{v} \geq \mathbf{0}$; Further, $\mathbf{v} \cdot \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v} = 0$
- Example
  - $(3\mathbf{u} + 2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
    $= (3\mathbf{u}) \cdot (4\mathbf{u} + \mathbf{v}) + (2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
    $= (3\mathbf{u}) \cdot (4\mathbf{u}) + (3\mathbf{u}) \cdot \mathbf{v} + (2\mathbf{v}) \cdot (4\mathbf{u}) + (2\mathbf{v}) \cdot \mathbf{v}$
    $= 12(\mathbf{u} \cdot \mathbf{u}) + 11(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{v})$

Norm and Distance in Euclidean n-Space

- We define the Euclidean norm (or Euclidean length) of a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n by

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

- Similarly, the Euclidean distance between the points $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

- Example
  - If $\mathbf{u} = (1,3,-2,7)$ and $\mathbf{v} = (0,7,2,2)$, then in the Euclidean space R^4

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (7)^2} = \sqrt{63} = 3\sqrt{7}$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

Theorems

- Theorem 4.1.3 (Cauchy-Schwarz Inequality in R^n)
  - If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$$

- Theorem 4.1.4 (Properties of Length in R^n)
  - If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n and k is any scalar, then
    - $\|\mathbf{u}\| \ge 0$
    - $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
    - $\|k\mathbf{u}\| = |k| \|\mathbf{u}\|$
    - $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ (Triangle inequality)

Theorems

- Theorem 4.1.5 (Properties of Distance in R^n)
  - If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and k is any scalar, then
    - $d(\mathbf{u}, \mathbf{v}) \geq 0$
    - $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
    - $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
    - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (Triangle inequality)

- Theorem 4.1.6
  - If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n with the Euclidean inner product, then

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{4} \| \mathbf{u} + \mathbf{v} \|^2 - \tfrac{1}{4} \| \mathbf{u} - \mathbf{v} \|^2$$

Orthogonality

- Definition
  - Two vectors **u** and **v** in R^n are called orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$

- Example
  - In the Euclidean space R^4 the vectors

$$\mathbf{u} = (-2, 3, 1, 4) \text{ and } \mathbf{v} = (1, 2, 0, -1)$$

    are orthogonal, since $\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$

- Theorem 4.1.7 (Pythagorean Theorem in R^n)
  - If **u** and **v** are orthogonal vectors in R^n which the Euclidean inner product, then

$$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$$

Matrix Formulae for the Dot Product

- If we use column matrix notation for the vectors

  $$\mathbf{u} = [u_1\ u_2\ \ldots\ u_n]^T \text{ and } \mathbf{v} = [v_1\ v_2\ \ldots\ v_n]^T,$$

  or

  $$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

  then

  $$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{u}$$
  $$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$$
  $$\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v}$$

A Dot Product View of Matrix Multiplication

- If $A = [a_{ij}]$ is an $m\times r$ matrix and $B = [b_{ij}]$ is an $r\times n$ matrix, then the ij-the entry of AB is
$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{ir}b_{rj}$$
which is the dot product of the ith row vector of A and the jth column vector of B
- Thus, if the row vectors of A are $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ and the column vectors of B are $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1\cdot\mathbf{c}_1 & \mathbf{r}_1\cdot\mathbf{c}_2 & \cdots & \mathbf{r}_1\cdot\mathbf{c}_n \\ \mathbf{r}_2\cdot\mathbf{c}_1 & \mathbf{r}_2\cdot\mathbf{c}_2 & \cdots & \mathbf{r}_2\cdot\mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m\cdot\mathbf{c}_1 & \mathbf{r}_m\cdot\mathbf{c}_2 & \cdots & \mathbf{r}_m\cdot\mathbf{c}_n \end{bmatrix}$$

Functions from R^n to R

- A function is a rule f that associates with each element in a set A one and only one element in a set B.
- If f associates the element b with the element, then we write $b = f(a)$ and say that b is the image of a under f or that $f(a)$ is the value of f at a.
- The set A is called the domain of f and the set B is called the codomain of f.
- The subset of B consisting of all possible values for f as a varies over A is called the range of f.

Examples

Formula	Example	Classification	Description
$f(x)$	$f(x)=x^2$	Real-valued function of a real variable	Function from R to R
$f(x,y)$	$f(x,y)=x^2+y^2$	Real-valued function of two real variable	Function from R^2 to R
$f(x,y,z)$	$f(x,y,z)=x^2+y^2+z^2$	Real-valued function of three real variable	Function from R^3 to R
$f(x_1,x_2,\ldots,x_n)$	$f(x_1,x_2,\ldots,x_n)=x_1^2+x_2^2+\ldots+x_n^2$	Real-valued function of n real variable	Function from R^n to R

Function from R^n to R^m

- If the domain of a function f is R^n and the codomain is R^m, then f is called a map or transformation from R^n to R^m. We say that the function f maps R^n into R^m, and denoted by $f : R^n \to R^m$.

- If $m = n$ the transformation $f : R^n \to R^m$ is called an operator on R^n.

- Suppose $f_1, f_2, \ldots, f_m$ are real-valued functions of n real variables, say

$$
\begin{gathered}
w_1 = f_1(x_1, x_2, \ldots, x_n) \\
w_2 = f_2(x_1, x_2, \ldots, x_n) \\
\cdots \\
w_m = f_m(x_1, x_2, \ldots, x_n)
\end{gathered}
$$

  These m equations assign a unique point $(w_1, w_2, \ldots, w_m)$ in R^m to each point $(x_1, x_2, \ldots, x_n)$ in R^n and thus define a transformation from R^n to R^m. If we denote this transformation by $T: R^n \to R^m$ then

$$T(x_1, x_2, \ldots, x_n) = (w_1, w_2, \ldots, w_m)$$

Linear Transformations from R^n to R^m

- A linear transformation (or a linear operator if $m = n$) $T: R^n \rightarrow R^m$ is defined by equations of the form

$$\begin{array}{l} w_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ w_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ w_m = a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{array} \quad \text{or} \quad \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{13} \\ a_{21} & a_{22} & \cdots & a_{23} \\ \vdots & \vdots & & \vdots \\ a_{mn} & a_{mn} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

or

$$\mathbf{w} = A\mathbf{x}$$

- The matrix $A = [a_{ij}]$ is called the standard matrix for the linear transformation T, and T is called multiplication by A.

Example (Transformation and Linear Transformation)

- The equations

$$w_1 = x_1 + x_2$$
$$w_2 = 3x_1x_2$$
$$w_3 = x_1^2 - x_2^2$$

define a transformation $T: R^2 \rightarrow R^3$.

$T(x_1, x_2) = (x_1 + x_2, 3x_1x_2, x_1^2 - x_2^2)$

Thus, for example, $T(1,-2) = (-1,-6,-3)$.

- The linear transformation $T : R^4 \rightarrow R^3$ defined by the equations

$$w_1 = 2x_1 - 3x_2 + x_3 - 5x_4$$
$$w_2 = 4x_1 + x_2 - 2x_3 + x_4$$
$$w_3 = 5x_1 - x_2 + 4x_3$$

the standard matrix for T (i.e., $\mathbf{w} = A\mathbf{x}$) is $A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$

Remarks

- Notations:
  - If it is important to emphasize that A is the standard matrix for T. We denote the linear transformation $T: R^n \to R^m$ by $T_A: R^n \to R^m$. Thus,

$$T_A(\mathbf{x}) = A\mathbf{x}$$

  - We can also denote the standard matrix for T by the symbol $[T]$, or

$$T(\mathbf{x}) = [T]\mathbf{x}$$

- Remark:
  - We have establish a correspondence between $m \times n$ matrices and linear transformations from R^n to R^m:
    - To each matrix A there corresponds a linear transformation T_A (multiplication by A), and to each linear transformation $T: R^n \to R^m$, there corresponds an $m \times n$ matrix $[T]$ (the standard matrix for T).

Examples

- Zero Transformation from R^n to R^m
  - If 0 is the $m \times n$ zero matrix and **0** is the zero vector in R^n, then for every vector **x** in R^n

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

  - So multiplication by zero maps every vector in R^n into the zero vector in R^m. We call T_0 the zero transformation from R^n to R^m.

- Identity Operator on R^n
  - If I is the $n \times n$ identity, then for every vector in R^n

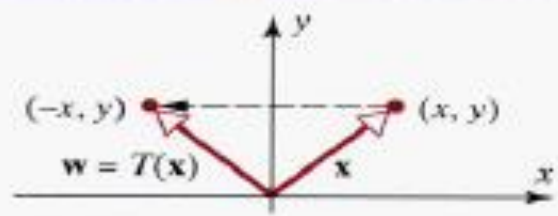
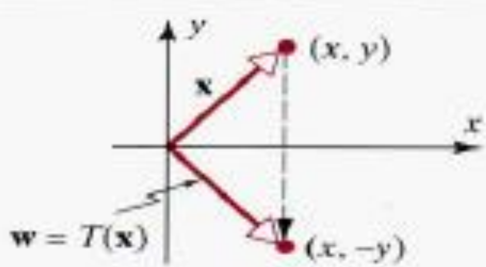
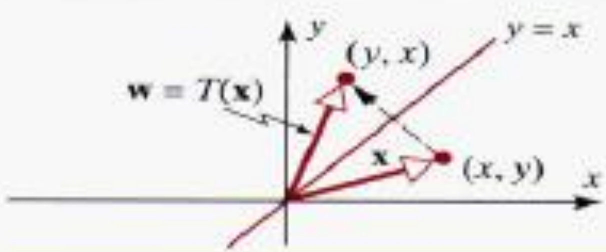
$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

  - So multiplication by I maps every vector in R^n into itself.
  - We call T_I the identity operator on R^n.

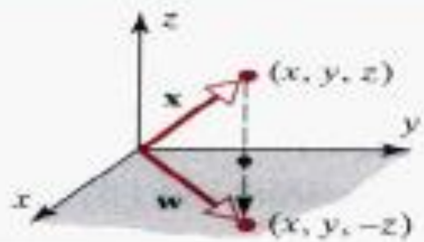
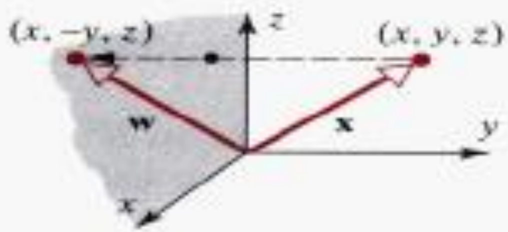
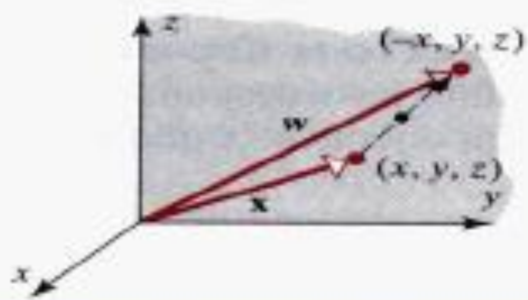
Reflection Operators

- In general, operators on R^2 and R^3 that map each vector into its symmetric image about some line or plane are called reflection operators.
- Such operators are linear.

Reflection Operators (2-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the y-axis		$w_1 = -x$ $w_2 = \ \ y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis		$w_1 = \ \ x$ $w_2 = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$		$w_1 = y$ $w_2 = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

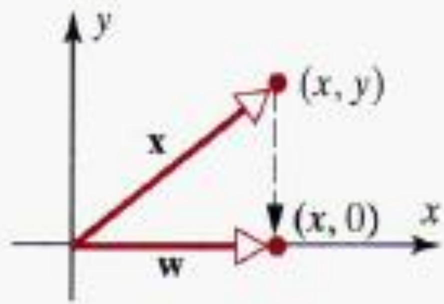
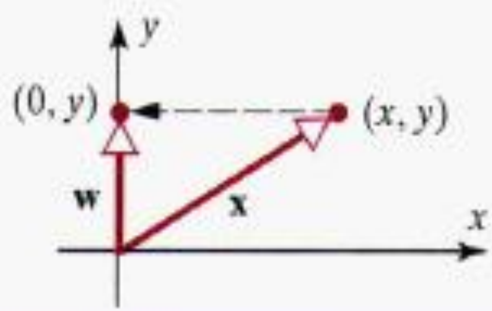
Reflection Operators (3-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = -z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane		$w_1 = x$ $w_2 = -y$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane		$w_1 = -x$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

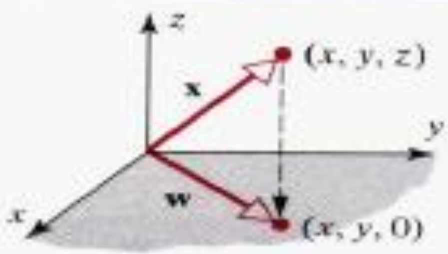
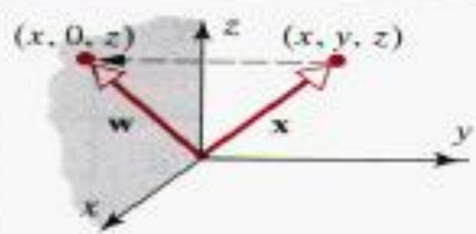
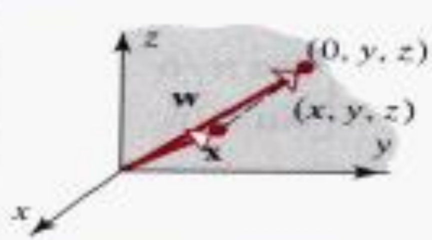
Projection Operators

- In general, a projection operator (or more precisely an orthogonal projection operator) on R^2 or R^3 is any operator that maps each vector into its orthogonal projection on a line or plane through the origin.
- The projection operators are linear.

Projection Operators

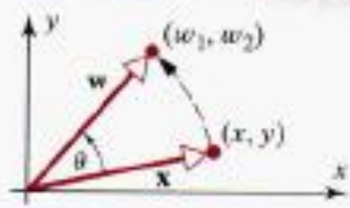
Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the x-axis		$w_1 = x$ $w_2 = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis		$w_1 = 0$ $w_2 = y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projection Operators

Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane		$w_1 = x$ $w_2 = 0$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane		$w_1 = 0$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation Operators

- An operator that rotate each vector in R^2 through a fixed angle θ is called a rotation operator on R^2.

Operator	Illustration	Equations	Standard Matrix
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example

- If each vector in R^2 is rotated through an angle of $\pi/6$ (30°) ,then the image **w** of a vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

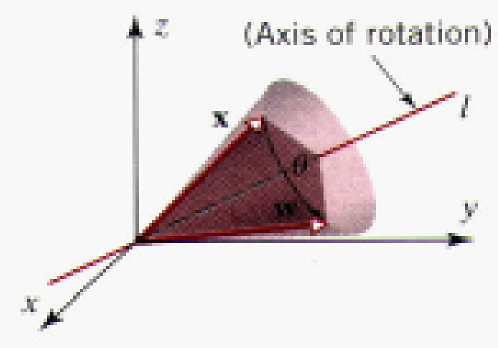
is $$\mathbf{w} = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2\, x - 1/2\, y \\ 1/2\, x + \sqrt{3}/2\, y \end{bmatrix}$$

- For example, the image of the vector

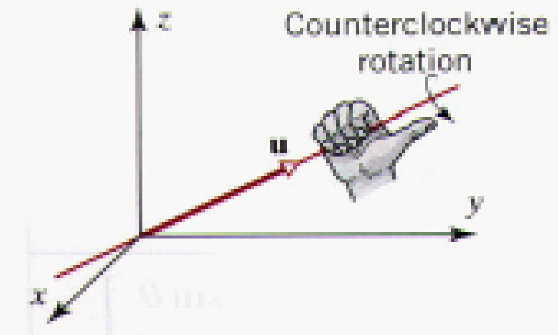
$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{is} \quad \mathbf{w} = \begin{bmatrix} \dfrac{\sqrt{3} - 1}{2} \\ \dfrac{1 + \sqrt{3}}{2} \end{bmatrix}$$

A Rotation of Vectors in R^3

- A rotation of vectors in R^3 is usually described in relation to a ray emanating from the origin, called the axis of rotation.
- As a vector revolves around the axis of rotation it sweeps out some portion of a cone.
- The angle of rotation is described as "clockwise" or "counterclockwise" in relation to a viewpoint that is along the axis of rotation *looking toward the origin*.
- The axis of rotation can be specified by a nonzero vector **u** that runs along the axis of rotation and has its initial point at the origin.
- The counterclockwise direction for a rotation about its axis can be determined by a "right-hand rule".

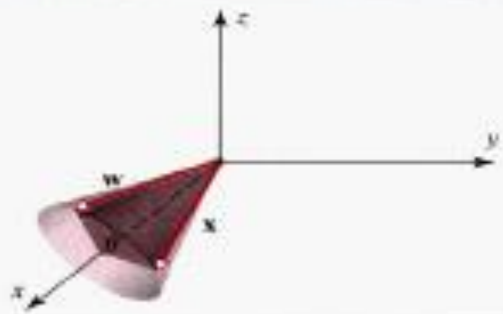
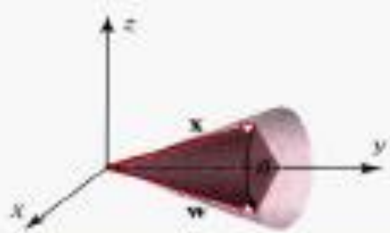
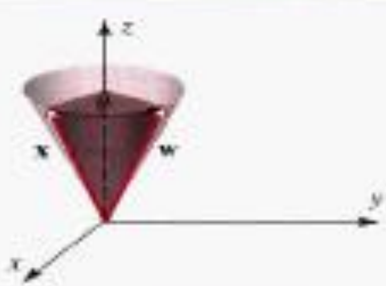


(*a*) Angle of rotation



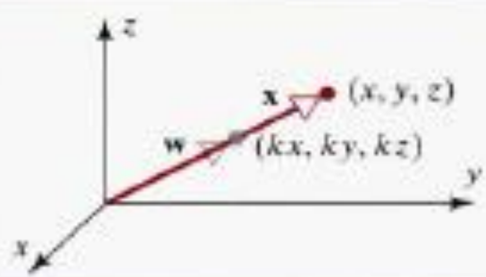
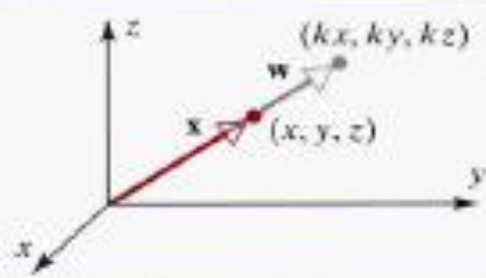
(*b*) Right-hand rule

A Rotation of Vectors in R^3

Operator	**Illustration**	**Equations**	**Standard Matrix**
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dilation and Contraction Operators

- If k is a nonnegative scalar, the operator on R^2 or R^3 is called a contraction with factor k if $0 \le k \le 1$ and a dilation with factor k if $k \ge 1$.

Operator	Illustration	Equations	Standard Matrix
Contraction with factor k on R^3 $(0 \le k \le 1)$		$w_1 = kx$ $w_2 = ky$ $w_3 = kz$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on R^3 $(k \ge 1)$		$w_1 = kx$ $w_2 = ky$ $w_3 = kz$	

Compositions of Linear Transformations

- If $T_A : R^n \rightarrow R^k$ and $T_B : R^k \rightarrow R^m$ are linear transformations, then for each $\mathbf{x}$ in R^n one can first compute $T_A(\mathbf{x})$, which is a vector in R^k, and then one can compute $T_B(T_A(\mathbf{x}))$, which is a vector in R^m.
- Thus, the application of T_A followed by T_B produces a transformation from R^n to R^m.
- This transformation is called the composition of T_B with T_A and is denoted by $T_B \circ T_A$. Thus
$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$
- The composition $T_B \circ T_A$ is linear since
$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = B(A\mathbf{x}) = (BA)\mathbf{x}$$
- The standard matrix for $T_B \circ T_A$ is BA. That is, $T_B \circ T_A = T_{BA}$
- *Multiplying matrices is equivalent to composing the corresponding linear transformations in the right-to-left order of the factors.*

Composition of Two Rotations

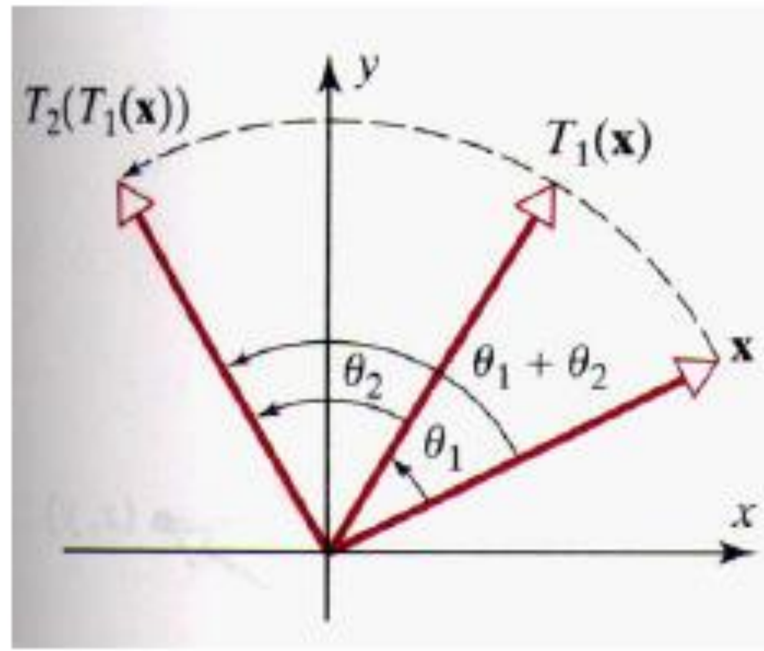
- Let $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ be linear operators that rotate vectors through the angle θ_1 and θ_2, respectively.
- The operation

  $$(T_2 \circ T_1)(\mathbf{x}) = (T_2(T_1(\mathbf{x})))$$

  first rotates $\mathbf{x}$ through the angle θ_1, then rotates $T_1(\mathbf{x})$ through the angle θ_2.
- It follows that the net effect of

  $$T_2 \circ T_1$$

  is to rotate each vector in R^2 through the angle $\theta_1 + \theta_2$

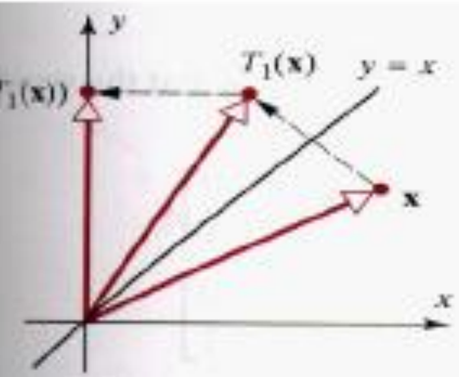


Composition Is Not Commutative

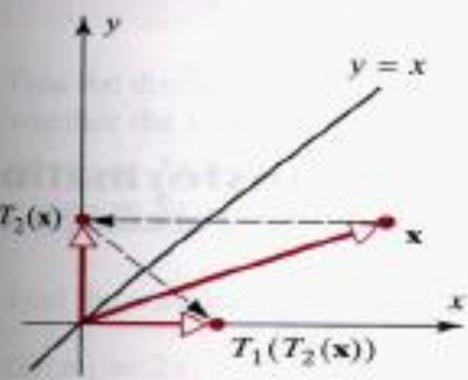
$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so $[T_1 \circ T_2] \neq [T_2 \circ T_1]$



(a) $T_2 \circ T_1$



(b) $T_1 \circ T_2$

Compositions of Three or More Linear Transformations

- Consider the linear transformations

$$T_1 : R^n \rightarrow R^k, \quad T_2 : R^k \rightarrow R^l, \quad T_3 : R^l \rightarrow R^m$$

- We can define the composition $(T_3 \circ T_2 \circ T_1) : R^n \rightarrow R^m$ by

$$(T_3 \circ T_2 \circ T_1)(\mathbf{x}) : T_3(T_2(T_1(\mathbf{x})))$$

- This composition is a linear transformation and the standard matrix for $T_3 \circ T_2 \circ T_1$ is related to the standard matrices for T_1, T_2, and T_3 by

$$[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$$

- If the standard matrices for T_1, T_2, and T_3 are denoted by A, B, and C, respectively, then we also have

$$T_C \circ T_B \circ T_A = T_{CBA}$$

Example

- Find the standard matrix for the linear operator $T: R^3 \to R^3$ that first rotates a vector counterclockwise about the z-axis through an angle θ, then reflects the resulting vector about the yz-plane, and then projects that vector orthogonally onto the xy-plane.

$$[T_1] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [T_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One-to-One Linear transformations

- Definition
  - A linear transformation $T : R^n \rightarrow R^m$ is said to be one-to-one if T maps distinct vectors (points) in R^n into distinct vectors (points) in R^m

- Remark:
  - That is, for each vector $\mathbf{w}$ in the range of a one-to-one linear transformation T, there is exactly one vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{w}$.

Theorem 4.3.1 (Equivalent Statements)

- If A is an $n \times n$ matrix and $T_A : R^n \rightarrow R^n$ *is* multiplication by A, then the following statements are equivalent.
  - A is invertible
  - The range of T_A is R^n
  - T_A is one-to-one

Examples

- The rotation operator $T : R^2 \to R^2$ is one-to-one
  - The standard matrix for T is

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  - $[T]$ is invertible since

$$\det\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

- The projection operator $T : R^3 \to R^3$ is not one-to-one
  - The standard matrix for T is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  - $[T]$ is invertible since $\det[T] = 0$

Inverse of a One-to-One Linear Operator

- Suppose $T_A : R^n \to R^n$ is a one-to-one linear operator
  $\Rightarrow$ The matrix A is invertible.
  $\Rightarrow T_{A^{-1}} : R^n \to R^n$ is itself a linear operator; it is called the inverse of T_A.
  $\Rightarrow T_A(T_{A^{-1}}(\mathbf{x})) = AA^{-1}\mathbf{x} = I\mathbf{x} = \mathbf{x}$ and $T_{A^{-1}}(T_A(\mathbf{x})) = A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x}$
  $\Rightarrow T_A \nwarrow T_{A^{-1}} = T_{AA^{-1}} = T_I$ and $T_{A^{-1}} \nwarrow T_A = T_{A^{-1}A} = T_I$

- If $\mathbf{w}$ is the image of $\mathbf{x}$ under T_A, then T_A^{-1} maps $\mathbf{w}$ back into $\mathbf{x}$, since

$$T_{A^{-1}}(\mathbf{w}) = T_{A^{-1}}(T_A(\mathbf{x})) = \mathbf{x}$$

- When a one-to-one linear operator on R^n is written as $T : R^n \to R^n$, then the inverse of the operator T is denoted by T^{-1}.
- Thus, by the standard matrix, we have $[T^{-1}]=[T]^{-1}$

Example

- Let $T : R^2 \rightarrow R^2$ be the operator that rotates each vector in R^2 through the angle θ:
$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
- Undo the effect of T means rotate each vector in R^2 through the angle -θ.

- This is exactly what the operator T^{-1} does: the standard matrix T^{-1} is
$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$
- The only difference is that the angle θ is replaced by -θ

Example

- Show that the linear operator $T: R^2 \rightarrow R^2$ defined by the equations

$$w_1 = 2x_1 + x_2$$
$$w_2 = 3x_1 + 4x_2$$

is one-to-one, and find $T^{-1}(w_1, w_2)$.

- Solution:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow [T] = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \Rightarrow [T^{-1}] = [T]^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$\Rightarrow [T^{-1}] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} w_1 - \frac{1}{5} w_2 \\ -\frac{3}{5} w_1 + \frac{2}{5} w_2 \end{bmatrix}$$

$$\Rightarrow T^{-1}(w_1, w_2) = (\frac{4}{5} w_1 - \frac{1}{5} w_2, -\frac{3}{5} w_1 + \frac{2}{5} w_2)$$

Linearity Properties

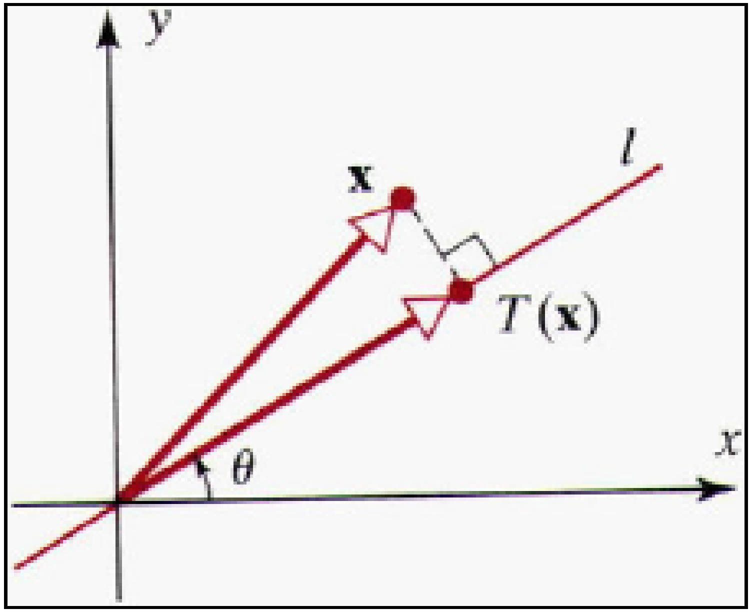
- Theorem 4.3.2 (Properties of Linear Transformations)
  - A transformation $T : R^n \to R^m$ is linear if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and every scalar c.
    - $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
    - $T(c\mathbf{u}) = cT(\mathbf{u})$

- Theorem 4.3.3
  - If $T : R^n \to R^m$ is a linear transformation, and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n, then the standard matrix for T is

$$A = [T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \ldots \mid T(\mathbf{e}_n)]$$

Example (Standard Matrix for a Projection Operator)

- Let l be the line in the xy-plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \le \theta \le \pi$. Let $T: R^2 \to R^2$ be a linear operator that maps each vector into orthogonal projection on l.
  - Find the standard matrix for T.
  - Find the orthogonal projection of the vector $\mathbf{x} = (1,5)$ onto the line through the origin that makes an angle of $\theta = \pi/6$ with the positive x-axis.



Example

- The standard matrix for T can be written as

$$[T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)]$$

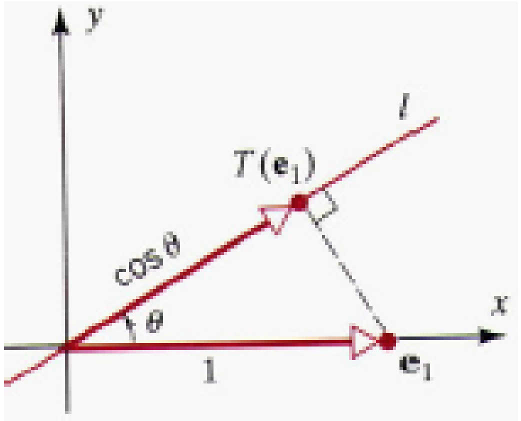
- Consider the case $0 \le \theta \le \pi/2$.
  - $\|T(\mathbf{e}_1)\| = \cos\theta$

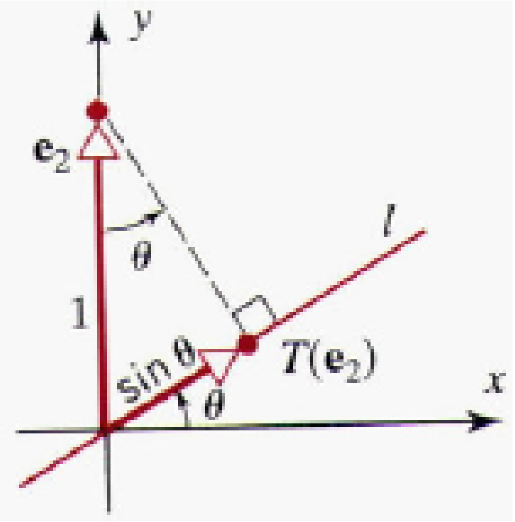
$$\Rightarrow \quad T(\mathbf{e}_1) = \begin{bmatrix} \|T(\mathbf{e}_1)\|\cos\theta \\ \|T(\mathbf{e}_1)\|\sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$$

  - $\|T(\mathbf{e}_2)\| = \sin\theta$

$$\Rightarrow \quad T(\mathbf{e}_2) = \begin{bmatrix} \|T(\mathbf{e}_2)\| = \cos\theta \\ \|T(\mathbf{e}_2)\| = \sin\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$





Example

$$[T]=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

- Since $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$, it follows from part (a) that the standard matrix for this projection operator is

$$[T]=\begin{bmatrix}3/4 & \sqrt{3}/4\\ \sqrt{3}/4 & 1/4\end{bmatrix}$$

Thus,

$$T\left(\begin{bmatrix}1\\5\end{bmatrix}\right)=\begin{bmatrix}3/4 & \sqrt{3}/4\\ \sqrt{3}/4 & 1/4\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}\dfrac{3+5\sqrt{3}}{4}\\ \dfrac{\sqrt{3}+5}{4}\end{bmatrix}$$

Eigenvalue and Eigenvector

- Definition
  - If $T: R^n \to R^n$ is a linear operator, then a scalar λ is called an eigenvalue of T if there is a nonzero $\mathbf{x}$ in R^n such that

    $$T(\mathbf{x}) = \lambda \mathbf{x}$$

    Those nonzero vectors $\mathbf{x}$ that satisfy this equation are called the eigenvectors of T corresponding to λ

- Remarks:
  - If A is the standard matrix for T, then the equation becomes

    $$A\mathbf{x} = \lambda \mathbf{x}$$

    - The eigenvalues of T are precisely the eigenvalues of its standard matrix A
    - $\mathbf{x}$ is an eigenvector of T corresponding to λ if and only if $\mathbf{x}$ is an eigenvector of A corresponding to λ
  - If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, then $A\mathbf{x} = \lambda \mathbf{x}$, so multiplication by A maps $\mathbf{x}$ into a scalar multiple of itself

Example

- Let $T : R^2 \to R^2$ be the linear operator that rotates each vector through an angle θ.
- If θ is a multiple of π, then every nonzero vector **x** is mapped onto the same line as **x**, so every nonzero vector is an eigenvector of T.
- The standard matrix for T is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- The eigenvalues of this matrix are the solutions of the characteristic equation

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = 0$$

- That is, $(\lambda - \cos\theta)^2 + \sin^2\theta = 0$.

Example

$$(\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

- If θ is not a multiple of π
  - $\Rightarrow \sin^2\theta > 0$
  - $\Rightarrow$ no real solution for λ
  - $\Rightarrow A$ has no real eigenvectors.
- If θ is a multiple of π
  - $\Rightarrow \sin\theta = 0$ and $\cos\theta = \pm 1$
- In the case that $\sin\theta = 0$ and $\cos\theta = 1$
  - $\Rightarrow \lambda = 1$ is the only eigenvalue
  - $\Rightarrow$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Thus, for all $\mathbf{x}$ in R^2, $T(\mathbf{x}) = A\mathbf{x} = I\mathbf{x} = \mathbf{x}$
- So T maps every vector to itself, and hence to the same line.
- In the case that $\sin\theta = 0$ and $\cos\theta = -1$,
  - $\Rightarrow A = -I$ and $T(\mathbf{x}) = -\mathbf{x}$
  - $\Rightarrow T$ maps every vector to its negative.

Example

- Let $T : R^3 \to R^3$ be the orthogonal projection on *xy*-plane.
- Vectors in the *xy*-plane are mapped into themselves under T, so each nonzero vector in the *xy*-plane is an eigenvector corresponding to the eigenvalue $\lambda = 1$.
- Every vector $\mathbf{x}$ along the *z*-axis is mapped into $\mathbf{0}$ under T, which is on the same line as $\mathbf{x}$, so every nonzero vector on the *z*-axis is an eigenvector corresponding to the eigenvalue.
- Vectors not in the *xy*-plane or along the *z*-axis are mapped into $\lambda = 0$ scalar multiples of themselves, so there are no other eigenvectors or eigenvalues.
- The standard matrix for T is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example

- The characteristic equation of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0 \quad \text{or} \quad (\lambda-1)^2\lambda = 0$$

- The eigenvectors of the matrix A corresponding to an eigenvalue λ are the nonzero solutions of

$$\begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- If $\lambda = 0$, this system is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

- The vectors are along the z-axis

Example

- If $\lambda = 1$, the system is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$$

- The vectors are along the *xy*-plane

Theorem 4.3.4 (Equivalent Statements)

- If A is an $n \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent.
  - A is invertible
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A is I_n
  - A is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
  - $\det(A) \neq 0$
  - The range of T_A is R^n
  - T_A is one-to-one