MAT 281E Fall 2015-2016 HOMEWORK 2

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Q1:

a) $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$Ax = b$

A's dimensions are 3x3. b's dimensions are 3x1. x's dimensions are 3x1.

Applying Gauss-Jordan Elimination:

$$[A:b] = \begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 2 & 0 & 2 & : & 3 \\ -1 & 1 & 1 & : & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 2 & 0 & 2 & : & 3 \\ -1 & 1 & 1 & : & 2 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 0 & -2 & 8 & : & -5 \\ -1 & 1 & 1 & : & 2 \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 0 & -2 & 8 & : & -5 \\ 0 & 2 & -2 & : & 6 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 0 & -2 & 8 & : & -5 \\ 0 & 0 & 6 & : & 1 \end{bmatrix} \xrightarrow{(R_2/-2) \to R_2}$$

$$\begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 0 & 1 & -4 & : & 5/2 \\ 0 & 0 & 6 & : & 1 \end{bmatrix} \xrightarrow{(R_1/6) \to R_1} \begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 0 & 1 & -4 & : & 5/2 \\ 0 & 0 & 1 & : & 1/6 \end{bmatrix} \xrightarrow{R_1-R_2 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 1 & : & 3/2 \\ 0 & 1 & -4 & : & 5/2 \\ 0 & 0 & 1 & : & 1/6 \end{bmatrix} \xrightarrow{4R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 1 & : & 3/2 \\ 0 & 1 & 0 & : & 19/6 \\ 0 & 0 & 1 & : & 1/6 \end{bmatrix} \xrightarrow{R_1-R_3 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 8/6 \\ 0 & 1 & 0 & : & 19/6 \\ 0 & 0 & 1 & : & 1/6 \end{bmatrix}$$ Therefore, $x = \begin{bmatrix} 8/6 \\ 19/6 \\ 1/6 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & 1 \\ -1 & 2 & -4 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$Ax = b$

A's dimensions are 3x3. b's dimensions are 3x1. x's dimensions are 3x1.

Applying Gauss-Jordan Elimination:

$$[A:b] = \begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 1 & 0 & 1 & : & 1 \\ -1 & 2 & -4 & : & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 1 & 0 & 1 & : & 1 \\ -1 & 2 & -4 & : & 5 \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 1 & 0 & 1 & : & 1 \\ 0 & 4 & -7 & : & 6 \end{bmatrix} \xrightarrow{R_1-R_2 \to R_2}$$

$$\begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 0 & 2 & -4 & : & 0 \\ 0 & 4 & -7 & : & 6 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 0 & 2 & -4 & : & 0 \\ 0 & 0 & 1 & : & 6 \end{bmatrix} \xrightarrow{R_1-R_2 \to R_1}$$

$$\begin{bmatrix}1 & 0 & 1 & : & 1\\ 0 & 2 & -4 & : & 0\\ 0 & 0 & 1 & : & 6\end{bmatrix} \xrightarrow{4R_3+R_2\to R_2} \begin{bmatrix}1 & 0 & 1 & : & 1\\ 0 & 2 & 0 & : & 24\\ 0 & 0 & 1 & : & 6\end{bmatrix} \xrightarrow{-R_3+R_1\to R_1}$$

$$\begin{bmatrix}1 & 0 & 0 & : & -5\\ 0 & 2 & 0 & : & 24\\ 0 & 0 & 1 & : & 6\end{bmatrix} \xrightarrow{R_2/2\to R_2} \begin{bmatrix}1 & 0 & 0 & : & -5\\ 0 & 1 & 0 & : & 12\\ 0 & 0 & 1 & : & 6\end{bmatrix}$$ Therefore, $x = \begin{bmatrix}-5\\ 12\\ 6\end{bmatrix}$

c) $A = \begin{bmatrix}1 & 1 & -1 & 1\\ 2 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ $\quad b = \begin{bmatrix}5/2\\ -3/2\\ 0\\ 0\end{bmatrix}$ $\quad x = \begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}$

$Ax = b$

A's dimensions are 4x4. b's dimensions are 4x1. x's dimensions are 4x1.

Applying Gauss-Jordan Elimination:

$$[A{:}\,b] = \begin{bmatrix}1 & 1 & -1 & 1 & : & 1\\ 2 & 0 & 1 & 2 & : & 5\\ 0 & 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -1 & 1 & : & 1\\ 2 & 0 & 1 & 2 & : & 5\\ 0 & 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\end{bmatrix} \xrightarrow{-2R_1+R_2\to R_2} \begin{bmatrix}1 & 1 & -1 & 1 & : & 1\\ 0 & -2 & 3 & 0 & : & 3\\ 0 & 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\end{bmatrix} \xrightarrow{\frac{R_2}{-2}\to R_2}$$

$$\begin{bmatrix}1 & 1 & -1 & 1 & : & 1\\ 0 & 1 & -3/2 & 0 & : & -3/2\\ 0 & 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\end{bmatrix} \xrightarrow{R_1-R_2\to R_1} \begin{bmatrix}1 & 0 & 1/2 & 1 & : & 5/2\\ 0 & 1 & -3/2 & 0 & : & -3/2\\ 0 & 0 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 0 & : & 0\end{bmatrix}$$

Therefore, there is infinite solutions for x. If we assume that, $w = t$ and $z = 2k$ then $x = \begin{bmatrix}5/2 - k - t\\ 3k - 3/2\\ 2k\\ t\end{bmatrix}$

Q2: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix} \xrightarrow{2R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 3 & 10 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 3 & 10 \end{bmatrix} \xrightarrow{(-3/4)R_2+R_3 \to R_3}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ (Row Echelon Form)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 7 \end{bmatrix} \xrightarrow{R_3/7 \to R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2/4 \to R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-R_3 \to R_1}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2-R_3 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Reduced Row Echelon Form)

Q3: The matrix A is in row echelon form. Because this matrix satisfies those conditions:

- All rows with one or more nonzero elements are above any rows with only zeros
- The first nonzero element from left of a nonzero row is always at the right side of first nonzero element of the row above.

Q4: A is a matrix which is mxn so A^T must be nxm. In every case A's row size and A^T's column size must be same. Therefore multiplication of those matrices is exist and has the size of mxm.

Q5: C^T is 2x3, B^T is 4x3 so AB^T is 2x3. C^T and AB^T are had the same size so $(C^T - AB^T)$ process is possible and size of it is 2x3. Multiplying $(C^T - AB^T)$ with B gives the result matrix which is 2x3.

Q6: Let's assume A^{-1} exists,

$A\vec{x} = \vec{0}$ $\qquad A^{-1}A\vec{x} = A^{-1}\vec{0}$ $\qquad I\vec{x} = \vec{0}$ $\qquad \vec{x} = \vec{0}$

Since $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad \vec{x} \neq \vec{0}$ therefore A^{-1} $can't\ exist.$

Q7: $AB = [A_1 \quad | \quad A_2]\begin{bmatrix} B_1 \\ - \\ B_2 \end{bmatrix} = [A_1B_1 + A_2B_2]$

$$A_1B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad A_2B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$[A_1B_1 + A_2B_2] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} B_1 \\ - \\ B_2 \end{bmatrix}[A_1 \quad | \quad A_2] = \begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix}$$

$$B_1A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_1A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B_2A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \quad B_2A_2 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Q8: Only case satisfies $|i - j| \geq 1$ for natural numbers of i and j is $i \neq j$. Which causes matrix with the condition of ($a_{ij} = o$ if $|i - j| \geq 1$) to only have zero as elements except the elements on the main diagonal which makes that matrix let's say A, a diagonal matrix. $\det(A)$ must be other than zero to calculate inverse of the matrix. Determinant of the diagonal matrices are the multiplication of the elements on the main diagonal. So there can't be zero on the main diagonal.

$$A = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \quad x \neq 0, y \neq 0, z \neq 0, w \neq 0 \quad A^{-1} = \begin{bmatrix} 1/x & 0 & 0 & 0 \\ 0 & 1/y & 0 & 0 \\ 0 & 0 & 1/z & 0 \\ 0 & 0 & 0 & 1/w \end{bmatrix}$$

Q9: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad X = \begin{bmatrix} x & w \\ y & z \end{bmatrix}$ $\quad AX = I$ $\quad AA^{-1} = I$ $\quad X = A^{-1}$

$AX = \begin{bmatrix} ax + by & aw + bz \\ cx + dy & cw + dz \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ therefore,

$ax + by = 1 \qquad cx + dy = 0$

$x = \frac{-dy}{c} \qquad y\left(\left(\frac{-ad}{c}\right) + b\right) = 1 \qquad y\frac{bc-ad}{c} = 1 \qquad y = \frac{-c}{ad-bc}$

$ax + b\frac{-c}{ad-bc} = 1 \qquad ax + \frac{-bc}{ad-bc} = 1 \qquad x = (\frac{1}{a})(1 - \frac{-bc}{ad-bc}) \qquad x = \frac{d}{ad-bc}$

$aw + bz = 0 \qquad cw + dz = 1$

$w = -\frac{bz}{a} \qquad z\left(\left(-\frac{cb}{a}\right) + d\right) = 1 \qquad z\left(\frac{ad-bc}{a}\right) = 1 \qquad z = \frac{a}{ad-bc}$

$wa + \mathrm{b}\left(\frac{\mathrm{a}}{\mathrm{ad-bc}}\right) = 0 \qquad w = (\frac{-ba}{ad-bc})/a \qquad w = \frac{-b}{ad-bc}$

$X = A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Q10: $A^3 - 2A^2 + 3A = 0$

$A^{-1}AA^2 - 2A^{-1}AA + 3A^{-1}A = A^{-1}0$

$A^2 - 2A + 3I = 0$

$A^{-1}AA - 2A^{-1}A + 3A^{-1}I = A^{-1}0$

$A - 2I + 3A^{-1} = 0$

$A^{-1} = \frac{1}{3}(2I - A)$

Q11: If A has a row of zeros, the result from multiplication of A with any matrix must have at least one row of zeros too. Let's assume A^{-1} is exists therefore the result of multiple of A^{-1} with A must be identity matrix (I), but identity matrix (I) doesn't have a row of zeros, according to this and the statement before that there can't be A^{-1}.

Q12: $(A + B)^2 = A^2 + AB + BA + B^2$

If $AB = BA$ is true, we can write $(A + B)^2 = A^2 + 2AB + B^2$

Q13: $\begin{bmatrix}2 & 0 & -1\\0 & 1 & 0\\1 & 5 & 1\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -5 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & -1\\0 & 1 & 0\\1 & 5 & 1\end{bmatrix}=\begin{bmatrix}2 & 0 & -1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}$

$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & -1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}$ $\begin{bmatrix}1/3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}$

$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

We can apply many different row operations for that process and every one of them will be unique but it will give the same result.

Q14: If a matrix is invertible, determination of that matrix must be nonzero. Also if a matrix has two proportional rows, we can create a row of zeros by a single row operation. Determination of that matrix, which has a row of zeros, and determination of the matrix before applying row operations is zero. Therefore we cannot turn an invertible matrix into a matrix with two proportional rows.

Q15:

a) $[A\colon I]=\begin{bmatrix}1 & 0 & -1 & 0 & : & 1 & 0 & 0 & 0\\0 & 2 & 2 & 0 & : & 0 & 1 & 0 & 0\\1 & 0 & 0 & 1 & : & 0 & 0 & 1 & 0\\1 & 1 & 0 & 0 & : & 0 & 0 & 0 & 1\end{bmatrix}$

$$\begin{bmatrix}1 & 0 & -1 & 0 & : & 1 & 0 & 0 & 0\\0 & 2 & 2 & 0 & : & 0 & 1 & 0 & 0\\1 & 0 & 0 & 1 & : & 0 & 0 & 1 & 0\\1 & 1 & 0 & 0 & : & 0 & 0 & 0 & 1\end{bmatrix}\xrightarrow{-\frac{1}{2}R_2+R_4\to R_4}\begin{bmatrix}1 & 0 & -1 & 0 & : & 1 & 0 & 0 & 0\\0 & 2 & 2 & 0 & : & 0 & 1 & 0 & 0\\1 & 0 & 0 & 1 & : & 0 & 0 & 1 & 0\\1 & 0 & -1 & 0 & : & 0 & -1/2 & 0 & 1\end{bmatrix}$$

Since there is two proportional rows, this matrix' determinant must be zero, according to that there can't an inverse matrix.

b) $[A\colon I]=\begin{bmatrix}1 & 0 & -1 & : & 1 & 0 & 0\\0 & 2 & 2 & : & 0 & 1 & 0\\1 & 1 & 0 & : & 0 & 0 & 1\end{bmatrix}$

$$\begin{bmatrix}1 & 0 & -1 & : & 1 & 0 & 0\\0 & 2 & 2 & : & 0 & 1 & 0\\1 & 1 & 0 & : & 0 & 0 & 1\end{bmatrix}\xrightarrow{-R_1+R_3\to R_3}\begin{bmatrix}1 & 0 & -1 & : & 1 & 0 & 0\\0 & 2 & 2 & : & 0 & 1 & 0\\0 & 1 & 1 & : & -1 & 0 & 1\end{bmatrix}$$

Since there is two proportional rows, this matrix' determinant must be zero, according to that there can't an inverse matrix.

c) $[A{:}I] = \begin{bmatrix} 0 & 0 & a & : & 1 & 0 & 0 \\ 0 & b & 0 & : & 0 & 1 & 0 \\ c & 0 & 0 & : & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & a & : & 1 & 0 & 0 \\ 0 & b & 0 & : & 0 & 1 & 0 \\ c & 0 & 0 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{a}R_1 \to R_1} \begin{bmatrix} 0 & 0 & 1 & : & \frac{1}{a} & 0 & 0 \\ 0 & b & 0 & : & 0 & 1 & 0 \\ c & 0 & 0 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{b}R_2 \to R_2}$$

$$\begin{bmatrix} 0 & 0 & 1 & : & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & : & 0 & \frac{1}{b} & 0 \\ c & 0 & 0 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{c}R_3 \to R_3} \begin{bmatrix} 0 & 0 & 1 & : & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & : & 0 & \frac{1}{b} & 0 \\ 1 & 0 & 0 & : & 0 & 0 & \frac{1}{c} \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 0 & 0 & 1/c \\ 0 & 1 & 0 & : & 0 & 1/b & 0 \\ 0 & 0 & 1 & : & 1/a & 0 & 0 \end{bmatrix}$$

Q16: $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2 \to R_2} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_2 \to R_1} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Q17: If we compute the row echelon form of the given matrix by simply dividing every row to that row's first element (shown below); we get a matrix in row echelon form and this matrix has neither row of zeros or proportional rows therefore the given matrix is invertible.

$$\begin{bmatrix} 1 & 0 & 2 & -2 & -1/2 \\ 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A's row echelon form

Q18: $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & x & 3 & -1 \\ 0 & 0 & 1 & 0 \\ x+1 & 2 & 4 & 1 \end{bmatrix}$ $\quad \mathrm{Det}(A) = 1 \begin{bmatrix} 1 & 0 & 1 \\ 1 & x & -1 \\ x+1 & 2 & 4 \end{bmatrix}$

$$Det(A) = 1 \begin{bmatrix} x & -1 \\ 2 & 4 \end{bmatrix} + 1 \begin{bmatrix} 1 & x \\ x+1 & 2 \end{bmatrix} = (4x + 2) + (2 - 1) = 4x + 3$$

$4x + 3 = 0$ if $x = -\frac{4}{3}$ A is invertible.

Q19:

1. $\begin{bmatrix}1&0&0&*&*&*\\0&1&0&*&*&*\\0&0&1&*&*&*\end{bmatrix}$ 2. $\begin{bmatrix}1&*&0&0&*&*\\0&0&1&0&*&*\\0&0&0&1&*&*\end{bmatrix}$ 3. $\begin{bmatrix}1&*&*&0&0&*\\0&0&0&1&0&*\\0&0&0&0&1&*\end{bmatrix}$

4. $\begin{bmatrix}1&*&*&*&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$ 5. $\begin{bmatrix}1&0&*&*&*&*\\0&1&*&*&*&*\\0&0&0&0&0&0\end{bmatrix}$ 6. $\begin{bmatrix}1&*&0&*&*&*\\0&0&1&*&*&*\\0&0&0&0&0&0\end{bmatrix}$

7. $\begin{bmatrix}1&*&*&0&*&*\\0&0&0&1&*&*\\0&0&0&0&0&0\end{bmatrix}$ 8. $\begin{bmatrix}1&*&*&*&0&*\\0&0&0&0&1&*\\0&0&0&0&0&0\end{bmatrix}$ 9. $\begin{bmatrix}1&*&*&*&*&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$

10. $\begin{bmatrix}1&*&*&*&*&*\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ 11. $\begin{bmatrix}0&1&*&*&*&*\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ 12. $\begin{bmatrix}0&0&1&*&*&*\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$

13. $\begin{bmatrix}0&0&0&1&*&*\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ 14. $\begin{bmatrix}0&0&0&1&*&*\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ 15. $\begin{bmatrix}0&0&0&0&1&*\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$

16. $\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ 17. $\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ 18. $\begin{bmatrix}1&0&0&0&*&*\\0&1&0&0&*&*\\0&0&0&1&*&*\end{bmatrix}$

19. $\begin{bmatrix}1&0&*&*&0&*\\0&1&*&*&0&*\\0&0&0&0&1&*\end{bmatrix}$ 20. $\begin{bmatrix}1&0&*&*&*&0\\0&1&*&*&*&0\\0&0&0&0&0&1\end{bmatrix}$ 21. $\begin{bmatrix}1&*&0&*&0&*\\0&0&1&*&0&*\\0&0&0&0&1&*\end{bmatrix}$

22. $\begin{bmatrix}1&*&0&*&*&0\\0&0&1&*&*&0\\0&0&0&0&0&1\end{bmatrix}$ 23. $\begin{bmatrix}1&*&*&0&*&0\\0&0&0&1&*&0\\0&0&0&0&0&1\end{bmatrix}$ 24. $\begin{bmatrix}0&1&0&0&*&*\\0&0&1&0&*&*\\0&0&0&1&*&*\end{bmatrix}$

25. $\begin{bmatrix}0&1&0&*&0&*\\0&0&1&*&0&*\\0&0&0&0&1&*\end{bmatrix}$ 26. $\begin{bmatrix}0&1&0&*&*&0\\0&0&1&*&*&0\\0&0&0&0&0&1\end{bmatrix}$ 27. $\begin{bmatrix}0&1&0&*&*&*\\0&0&1&*&*&*\\0&0&0&0&0&0\end{bmatrix}$

28. $\begin{bmatrix}0&1&*&0&0&*\\0&0&0&1&0&*\\0&0&0&0&1&0\end{bmatrix}$ 29. $\begin{bmatrix}0&1&*&0&*&0\\0&0&0&1&*&0\\0&0&0&0&0&1\end{bmatrix}$ 30. $\begin{bmatrix}0&1&*&0&*&*\\0&0&0&1&*&*\\0&0&0&0&0&0\end{bmatrix}$

31. $\begin{bmatrix}0&1&*&*&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$ 32. $\begin{bmatrix}0&1&*&*&0&*\\0&0&0&0&1&*\\0&0&0&0&0&0\end{bmatrix}$ 33. $\begin{bmatrix}0&1&*&*&*&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$

34. $\begin{bmatrix}0&0&1&0&0&*\\0&0&0&1&0&*\\0&0&0&0&1&*\end{bmatrix}$ 35. $\begin{bmatrix}0&0&1&0&*&0\\0&0&0&1&*&0\\0&0&0&0&0&1\end{bmatrix}$ 36. $\begin{bmatrix}0&0&1&0&*&*\\0&0&0&1&*&*\\0&0&0&0&0&0\end{bmatrix}$

37. $\begin{bmatrix} 0 & 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ 38. $\begin{bmatrix} 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 39. $\begin{bmatrix} 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

40. $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ 41. $\begin{bmatrix} 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 42. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Q20: $A = \begin{bmatrix} 4 & 4 & 4 & 1 \\ 2 & 3 & 8 & 2 \\ 0 & 0 & -5 & 7 \\ 0 & 0 & -7 & -2 \end{bmatrix}$

$$\begin{vmatrix} 4 & 4 & 4 & 1 \\ 2 & 3 & 8 & 2 \\ 0 & 0 & -5 & 7 \\ 0 & 0 & -7 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 4 & 1 \\ 2 & 1 & 8 & 2 \\ 0 & 0 & -5 & 7 \\ 0 & 0 & -7 & -2 \end{vmatrix} \qquad \det(A) = 1\begin{vmatrix} 4 & 4 & 1 \\ 0 & -5 & 7 \\ 0 & -7 & -2 \end{vmatrix}$$

$$\det(A) = 4\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = 4\big((5x2) - (-7x7)\big) = 59x4 = 236$$

Q21: $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$ $A^3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$

$$A^{120} = ((A^2)^3)^{20} = A^{20} = ((A^2)^5)^2 = (A^5)^2 = (A^3XA^2)^2 = (A^2)^2 = A^2 = A$$