Yunus Güngör No:150150701 Date:17.04.2016

BLG202E CRN:21843 Homework 3

Q1)a)

	Regular Prices			Crash Prices		
Day	0	7	14	21	28	35
Stock Price	100	98	101	50	51	50

$$p(x) = a_1 + a_2 x$$

$$p(7) = a_1 + a_2 7\,,\, p(14) = a_1 + a_2 14$$

$$\begin{bmatrix} 1 & 7 \\ 1 & 14 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 98 \\ 101 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 95 \\ 0.428571428571429 \end{bmatrix}$$

$$p(x) = 95 + 0.428571428571429x$$

$$p(12) = 100.142857142857142857142857142857142857142857142857142857142857$$

Adding (0,100) data point:

$$p_2(x) = a_1 + a_2 x + a_3 x^2$$

$$p(7) = a_1 + a_2 7 + a_3 49\,,\; p(14) = a_1 + a_2 14 + a_3 196\,,\, p(0) = a_1$$

$$\begin{bmatrix} 1 & 7 & 49 \\ 1 & 14 & 196 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 98 \\ 101 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & 49 & 98 \\ 1 & 14 & 196 & 101 \\ 1 & 0 & 0 & 100 \end{bmatrix} \xrightarrow{-4R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 7 & 49 & 98 \\ -3 & -14 & 0 & -291 \\ 1 & 0 & 0 & 100 \end{bmatrix}$$

By using backward substitution:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 100 \\ -0.642857142857143 \\ 0.0510204081632653 \end{bmatrix}$$

$$p_2(x) = 100 - 0.642857142857143x + 0.0510204081632653x^2$$

$$p_2(12) = 99.6326530612244897959183673469387755102040816326530612244898$$

Adding (21,50) data point:

$$p_3(x) = a_1 + a_2 x + a_3 x^2$$

$$p(7) = a_1 + a_2 7 + a_3 49\,,\; p(14) = a_1 + a_2 14 + a_3 196\,,\, p(21) = a_1 + a_2 21 + a_3 441$$

$$\begin{bmatrix} 1 & 7 & 49 \\ 1 & 14 & 196 \\ 1 & 21 & 441 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 98 \\ 101 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 41 \\ 12 \\ -0.5510 \end{bmatrix}$$

$$p_3(x) = 41 + 12x - 0.5510x^2$$

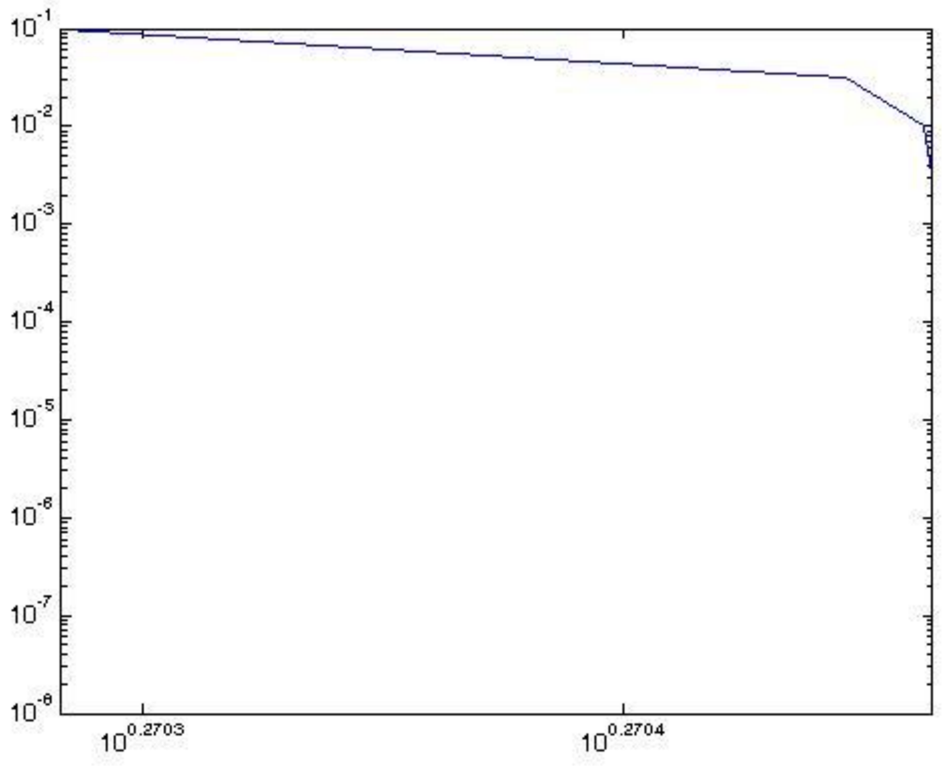
$$p_3(12) = 105.656$$

The most accurate interpolation is $p_2(12) = 99.6326530612244897959183673469 39$ because it does not involve extreme data points (stock prices after crash) and it includes all the other data points.

b)

Matlab Code:

```
>> w=[0 7 14 21 28 35];
>> p=[100 98 101 50 51 50];
>> p2=100-0.642857142857143*w+0.0510204081632653*power(w,2);
>> p3=41+12*w-0.5510*power(w,2);
>> figure
>> plot(p,w,p1,w,p2,w)
```



$p_2(x)$ and $p_3(x)$ is giving the right values before and after crash, in that order. But there is no function we can use for all data.

Q2)

a)

$x_0 = 0\ , x_1 = 0.5\ , x_2 = 1$

Using polynomial interpolation:

$p_2(x) = a_1 + a_2x + a_3x^2$

$p_2(0) = a_1\ , p_2(0.5) = a_1 + a_2 0.5 + a_3 0.25\ ,\ p_2(1) = a_1 + a_2 + a_3$

$p_2(0) = 1\ , p_2(0.5) = 1.64872127070013\ , p_2(1) = 2.71828182845905$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.64872127070013 \\ 2.71828182845905 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0.5 & 0.25 & 1.64872127070013 \\ 1 & 1 & 1 & 2.71828182845905 \end{bmatrix} \xrightarrow{-0.25R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0.75 & 0.25 & 0 & 0.969150813585367 \\ 1 & 1 & 1 & 2.71828182845905 \end{bmatrix}$$

By using backward substitution:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.12339674565853 \\ 1.84167857411758 \end{bmatrix}$$

$$p_2(x) = 1 - 0.12339674565853x + 1.84167857411758x^2$$

We need to find the maximum point of $|e^x - p_2(x)|$, by using derivation the maximum point is:

$$\frac{d}{dx} e^x - p_2(x) = 0$$

$$e^x + 0.12339674565853 - 3.68335714823516x = 0$$

$$e^x = 3.68335714823516x - 0.12339674565853$$

$$x = \ln(3.68335714823516x - 0.12339674565853)$$

$$x = 0.46626767363798821250847017753463$$

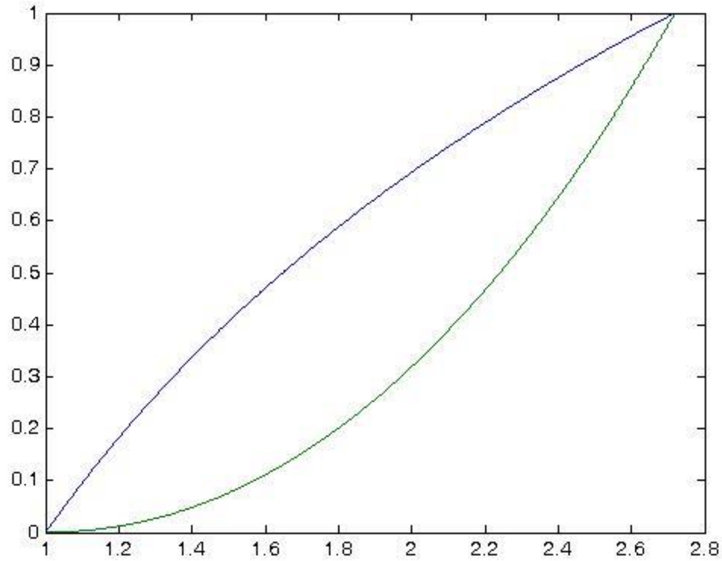
b)

$$p_2(x) = 1 - 0.12339674565853x + 1.84167857411758x^2$$

c)

Matlab code:

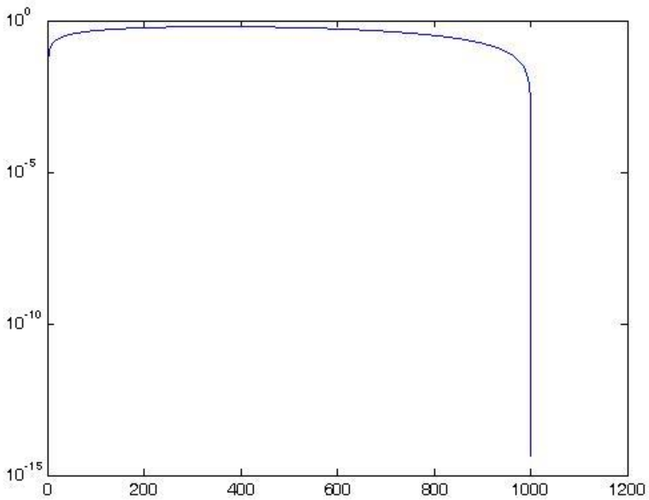
```
>> syms x;
>> p(x)=1-0.12339674565853*x+1.84167857411758*x^2;
>> x=0:0.001:1;
>> p=1-0.12339674565853*x+1.84167857411758*sqrt(x);
>> g=exp(x);
>> plot(g,x,p,x)
```



d)

Matlab code:

```
>> syms x;
>> p(x)=1-0.12339674565853*x+1.84167857411758*x^2;
>> x=0:0.001:1;
>> p=1-0.12339674565853*x+1.84167857411758*sqrt(x);
>> g=exp(x);
>> d=abs(g-p);
>> semilogy(d)
```



Q3)a)

By using taylor series:

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2}h^2+\frac{f'''(x)}{6}h^3+\frac{f^{iv}(x)}{24}h^4+\frac{f^{v}(x)}{120}h^5+\cdots$$

$$f(x-h)=f(x)-f'(x)h+\frac{f''(x)}{2}h^2-\frac{f'''(x)}{6}h^3+\frac{f^{iv}(x)}{24}h^4-\frac{f^{v}(x)}{120}h^5+\cdots$$

$$f(x+h)-f(x-h)=2f'(x)h+\frac{f'''(x)}{3}h^3+\frac{f^{v}(x)}{60}h^5+\cdots$$

$$f(x+h)-f(x-h)-2f'(x)h-\frac{f^{v}(x)}{60}h^5-\cdots=\frac{f'''(x)}{3}h^3$$

$$\frac{f(x+h)-f(x-h)-2f'(x)h}{h^3}-\frac{f^{v}(x)}{60}h^2-\cdots=\frac{f'''(x)}{3}$$

$$f'''(x)\cong 3(\frac{f(x+h)-f(x-h)}{h^3}-\frac{2f'(x)}{h^2})$$

Transaction error: $\frac{f^{v}(x)}{60}h^2$ and error's order: $O(h^2)$

b)

Matlab Code:

```
>> syms x;
>> f(x)=exp(x);
>> syms h;
>> d(x,h)=3*(((f(x+h)-f(x-h))/h^3)-(2*f(x)/h^2));%  f'(x)=f(x) for f(x)=e^x
>> for k=1:9
l=10^-k;
vpa(d(0,l))
l
end
ans =
1.0005001190641549423763011314301
l =
                        0.1
ans =
1.0000050000119047784391684704281
l =
                       0.01
```

```
ans =

1.0000000500000011904762070105818

l =

                   0.001

ans =

1.0000000005000000001190476193302

l =

                  0.0001

ans =

1.0000000000050000000000122870454

l =

                   1e-05

ans =

1.0000000000000499999999696164495

l =

                   1e-06

ans =

1.000000000000000498732999343332

l =

                   1e-07

ans =

1.0

l =

                   1e-08

ans =

1.0

l =

                   1e-09
```

For $h = 0.1$, $x = 0$,

$$(e^x)''' - f'''(x) = 1 - 1.0005001190641549423763011314301$$
$$= 0.0005001190641549423763011314301$$

Which is smaller than $h^2 = 10^{-2}$, this means that created formula is indeed second order accurate. For $h = 10^{-8}$ and $h = 10^{-9}$ formula gives the most accurate result, which is 1.

c) For very small h round-off error eliminates truncation error. Truncation error becomes so small and gets ignored because of the round-off error.

d) To get a fourth order formula for $f'''(x)$, $f(x+h) - f(x-h)$ can be used with Taylor series.

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{f'''(x)}{3}h^3 + \frac{f^{v}(x)}{60}h^5 + \frac{f^{vii}(x)}{2520}h^7 + \cdots$$

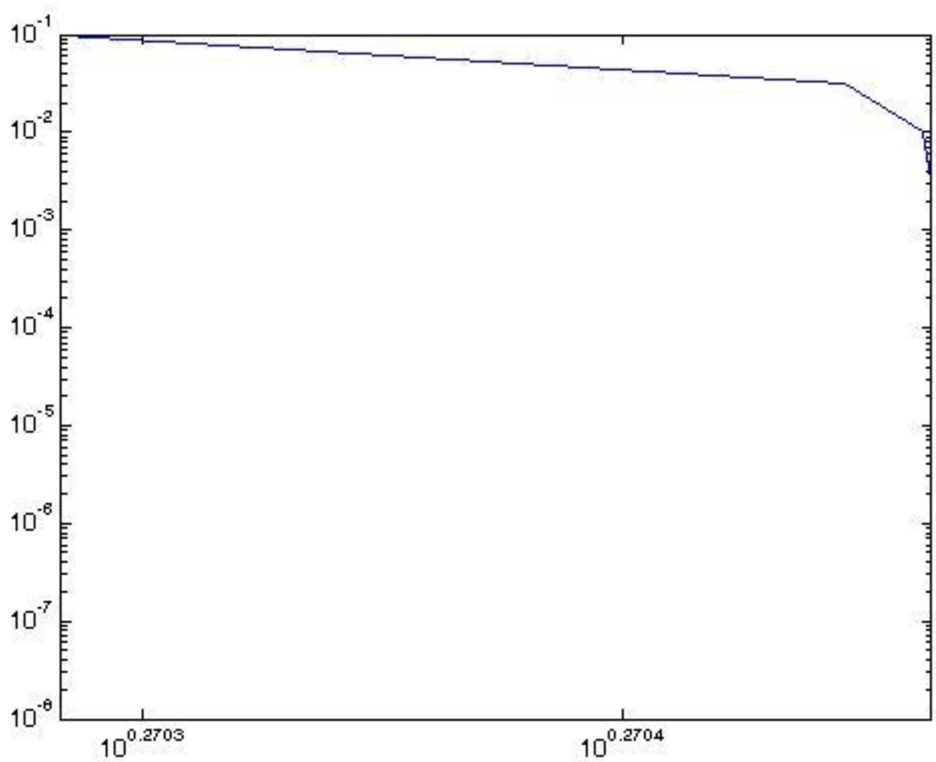
This method will require $(x+h)$, $f(x-h)$, $f'(x)$ and $f^{v}(x)$, which means 4 points required.

Q4)

$$f''(x) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

Matlab code:

```
>> syms x;

>> syms h;

>> d(x,h)=(sin(x+h)-2*sin(x)+sin(x-h))/h^2;

>> k=[1:0.5:8];

>> figure

>> h=power(10,-k);

>> g=sin(1.2)-d(1.2,h);

>> loglog(g,h)Th
```



Sudden change in the plot caused by round-off error. The most optimal h is $10^{-1.5}$.