BLG 311E – FORMAL LANGUAGES AND AUTOMATA
SPRING 2017
HOMEWORK 2

1) Consider the inductive definition of the reverse operation on a string.

i. $|w| = 0 \Rightarrow w^R = w = \Lambda$

ii. $|w| = n + 1 \land n \in \mathbb{N} \Rightarrow |u| = n \land\ a \in \Sigma \land w = ua \Rightarrow w^R = au^R$

Using the definition above, show that $\left(w^i\right)^R = (w^R)^i$ where i is a natural number.

2) **a.** Prove that if a relation α is transitive, $r(\alpha)$ is transitive as well.

b. Consider the statement $ts(\alpha) \subseteq st(\alpha)$. Prove the statement if correct. Give a counter example over the set $A = \{a, b, c, d\}$ if wrong.

IMPORTANT: You must do this homework **by hand** and **submit it using the box in the department secretariat**.

SOLUTIONS:

1) This definition can be generalized for concatenation of two strings x and y:

$w = xy$
$|y| = m \Rightarrow y = y_1 y_2 \dots y_m, y_{1:m} \in \Sigma$
$w^R = (xy)^R = (xy_1y_2 \dots y_m)^R = y_m(xy_1y_2 \dots y_{m-1})^R = y_m y_{m-1}(xy_1y_2 \dots y_{m-2})^R = \cdots$
$= y_m y_{m-1} \dots y_1 x^R = y^R x^R$

Proof by induction

True for $i = 0$ as $(w^0)^R = (\Lambda)^R = \Lambda = (w^R)^0$

Assuming to be true for $i = n$ as $(w^n)^R = (w^R)^n$

For $i = n + 1$:

$(w^{n+1})^R = (w^n w)^R$
Using the generalization above: $(w^n w)^R = w^R (w^n)^R$
Using the assumption for $i = n$: $w^R (w^n)^R = w^R (w^R)^n = (w^R)^{n+1}$

2) **a.** We need to show that $\forall a, b, c,\ (a, b) \in r(\alpha) \wedge (b, c) \in r(\alpha) \Rightarrow (a, c) \in r(\alpha)$

$a = b$ case:

$(a, b) \in r(\alpha) \wedge (b, c) \in r(\alpha) \Rightarrow (a, c) = (b, c) \in r(\alpha)$

$b = c$ case:

$(a, b) \in r(\alpha) \wedge (b, c) \in r(\alpha) \Rightarrow (a, c) = (a, b) \in r(\alpha)$

$a \neq b \neq c$ case:

$(a, b) \in r(\alpha) \wedge (b, c) \in r(\alpha) \Rightarrow (a, c) \in \alpha$ as α is transitive

$(a, c) \in r(\alpha)$ as $r(\alpha) = \alpha \cup \alpha^0$

b. Counterexample:

$$\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad s(\alpha) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad ts(\alpha) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$t(\alpha) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad st(\alpha) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad ts(\alpha) \nsubseteq st(\alpha)$$