NUMERICAL METHODS

Week-5

11.03.2014

Interpolation

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Interpolation

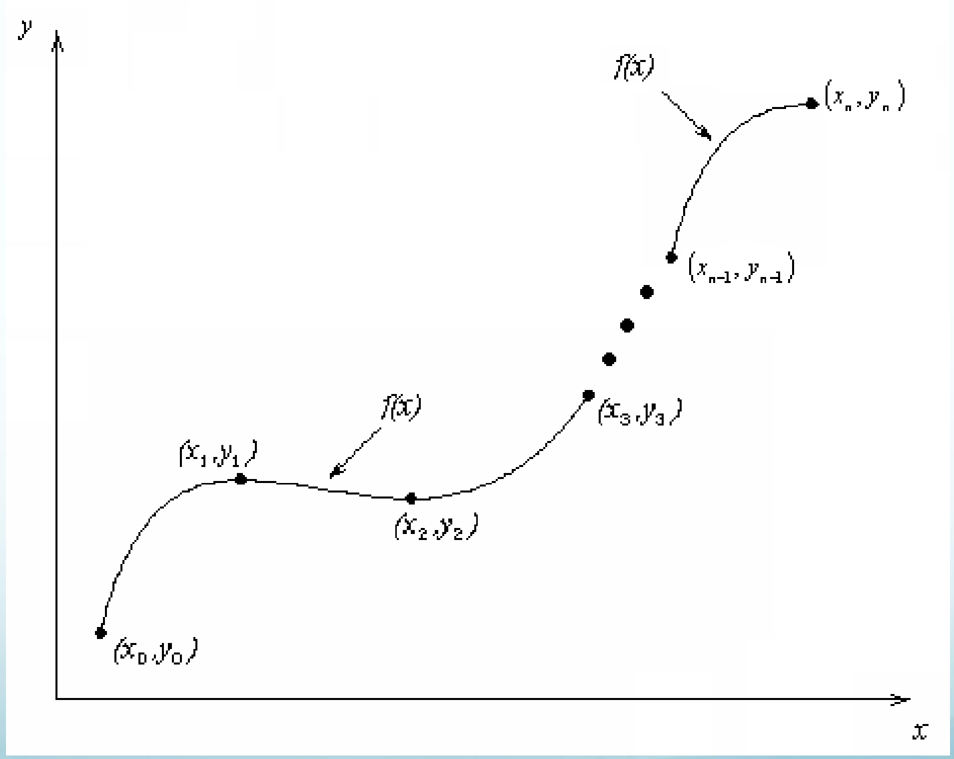
- **What** is an interpolation?
- **Why** do we use interpolation?
- **How** do we solve interpolation equations?

What is an Interpolation?

- Interpolation is the method of estimating unknown values with the help of given set of observations.
- The art of reading between the lines of the table.
- Having some number of points which are obtained by experiments or physical phenomenon forms the function of this experiment or phenomenon. The interpolation is to represent this phenomenon by a function and estimate (interpolate) the value of that function for an intermediate value within the range.
- (The process of computing the value of function outside the range of given values of the variable is called extrapolation.)

What is an interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



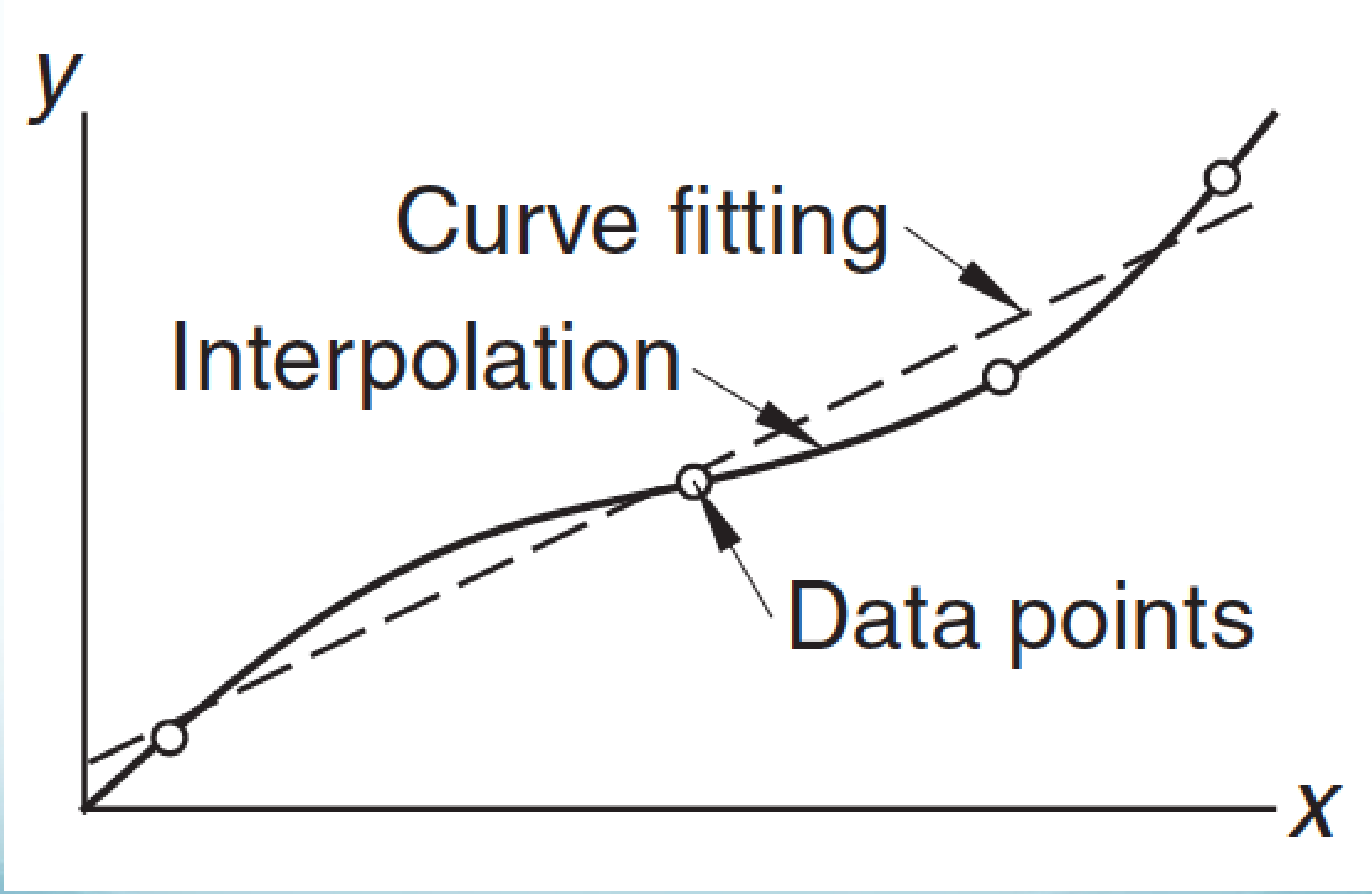
Why do we use Interpolation?

x_1	x_2	x_3	$\cdots$	x_n
y_1	y_2	y_3	$\cdots$	y_n

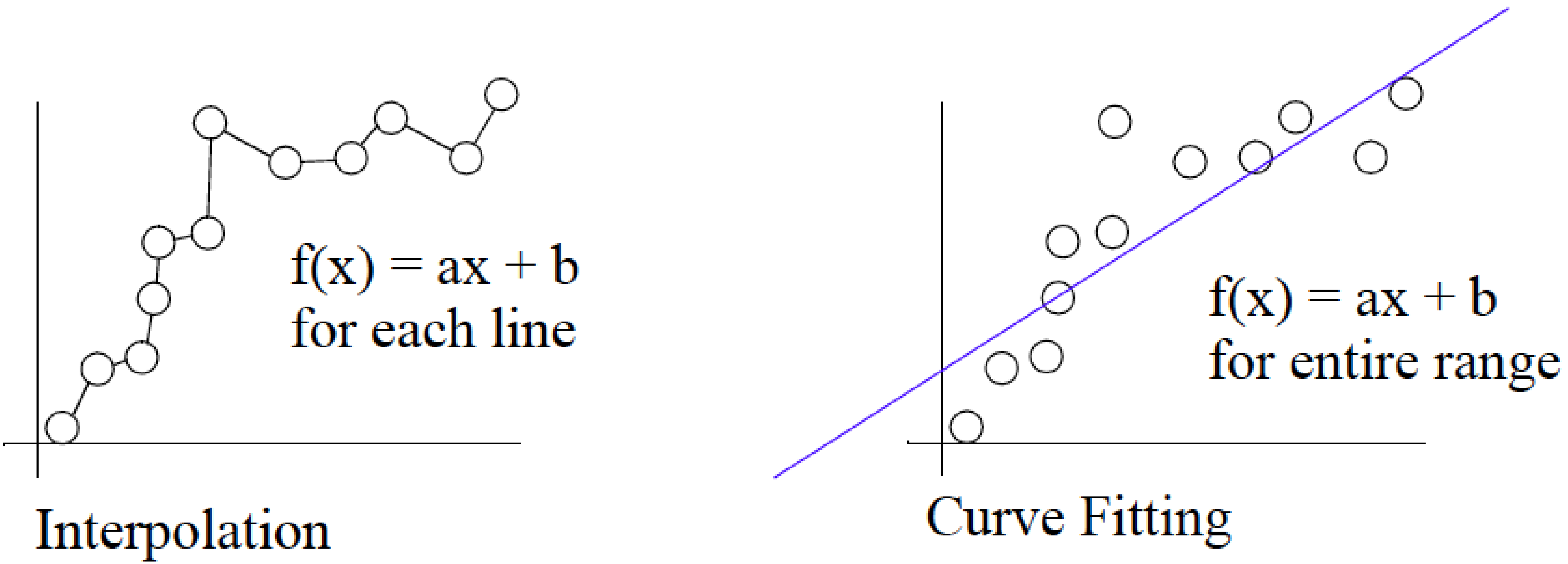
- In interpolation we **construct** a curve through **all** the data points. In doing so, we make the implicit assumption that the data points are accurate and distinct.

- Curve fitting is applied to data that contain scatter (noise), usually due to measurement errors. Here we want to find a smooth curve that **approximates the data in some sense**. Thus the curve does not have to hit the data points.

Difference between Interpolation and Curve fitting



Difference between Interpolation and Curve fitting



Why do we use Interpolation?

- Simply on every scientific problem
  - in which we have a set of experiments for a given range of data AND
  - IF we want to find out the function of this problem AND
  - IF we want to find a solution of this problem function for a specific value within the range.

How do we represent?

- Polynomial interpolation
  - Direct Method
  - Lagrange Method
  - Newton Method
  - …

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),………….. (x_n,y_n),

pass a polynomial of order 'n' through the data as given

below:

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n.$$

where a_0, a_1,………………. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example 1

The upward velocity of a rocket is given as a function of time in the table.

Find the velocity at t=16 seconds using the direct method for linear interpolation.

Table Velocity as a function of time.

t, (s)	$v(t)$, (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

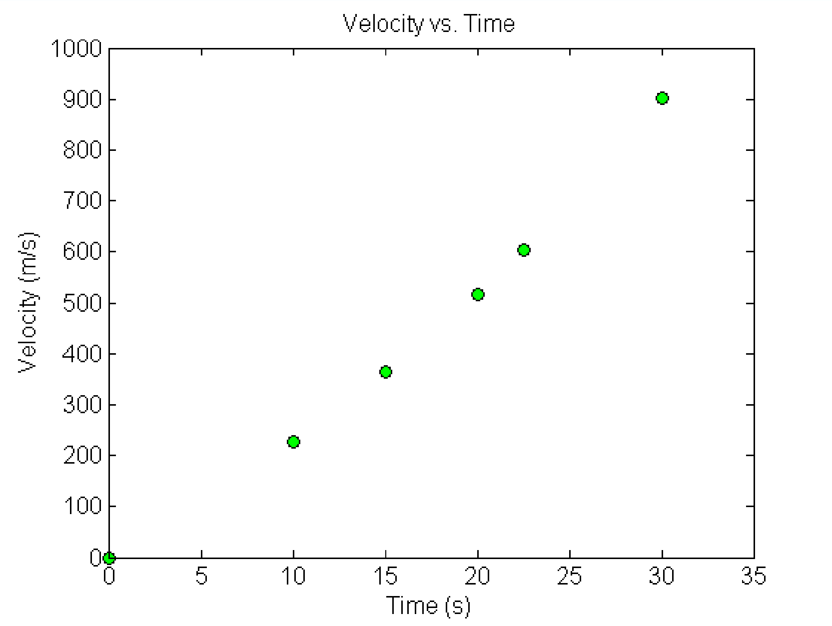


Figure: Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

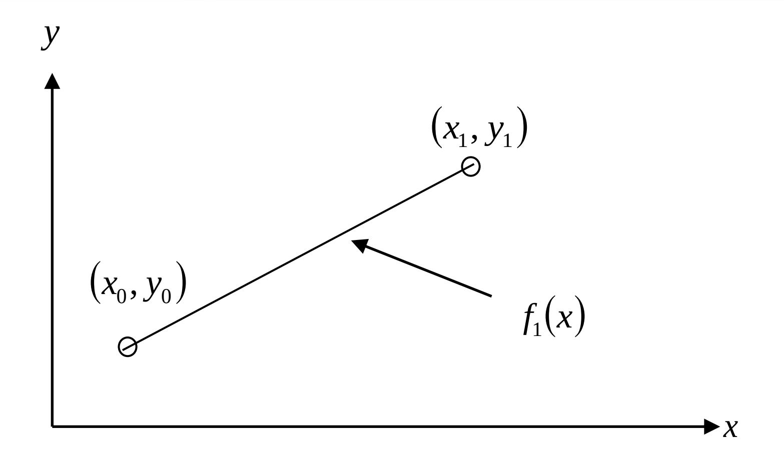


Figure Linear interpolation.

Solving the above two equations gives,

$$a_0 = -100.93 \quad a_1 = 30.914$$

Hence

$$v(t) = -100.93 + 30.914t, \ 15 \le t \le 20.$$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$

Example 2

The upward velocity of a rocket is given as a function of time in Table.

Find the velocity at t=16 seconds using the direct method for quadratic interpolation.

Table: Velocity as a function of time.

$t,(\mathrm{s})$	$v(t),(\mathrm{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

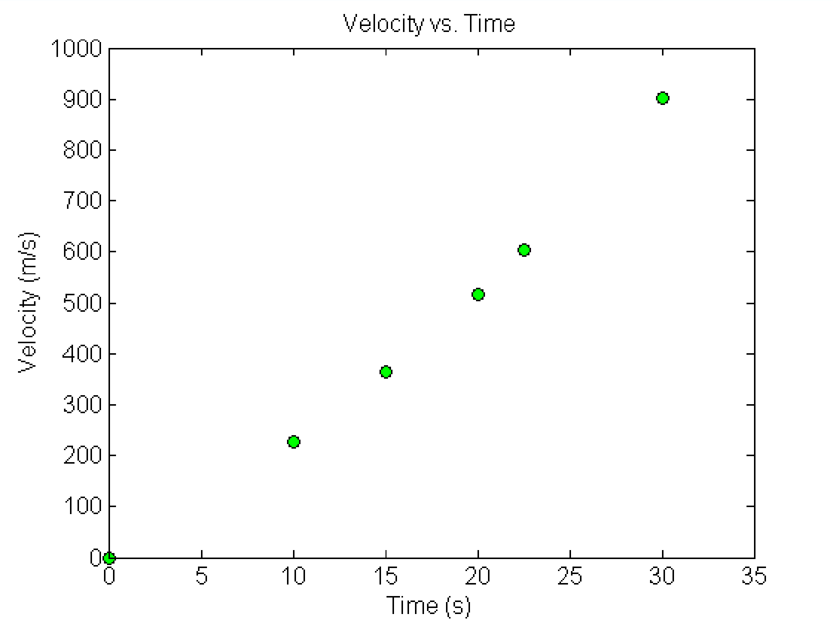


Figure Velocity vs. time data for the rocket example

Quadratic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

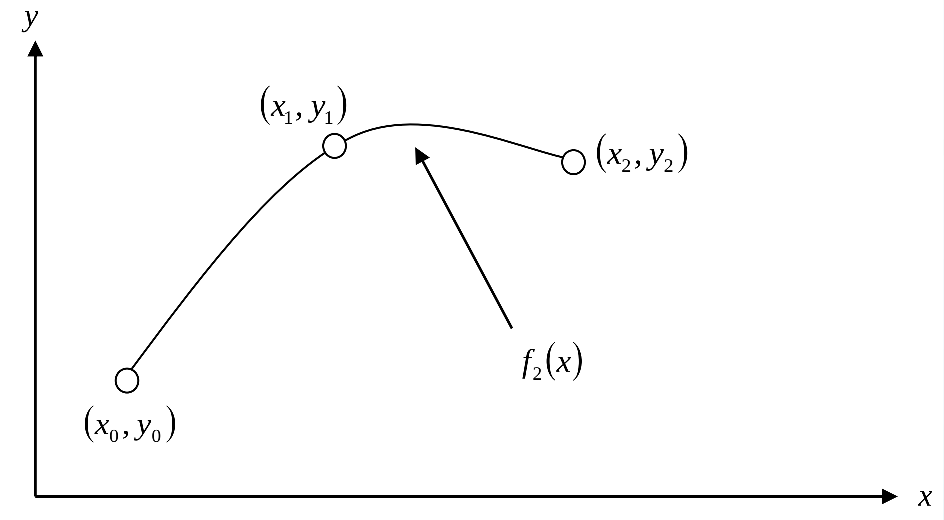


Figure Quadratic interpolation.

Solving the above three equations gives

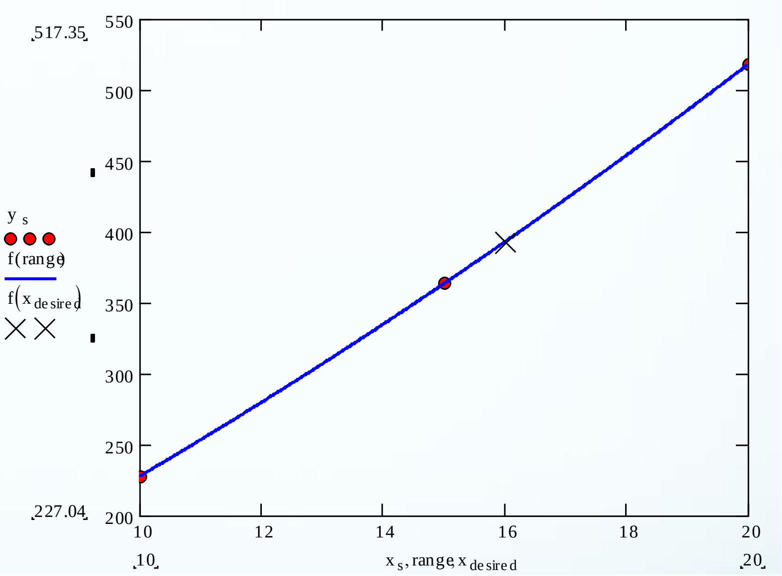
$$a_0 = 12.05 \quad a_1 = 17.733 \quad a_2 = 0.3766$$

Quadratic Interpolation (cont.)

$$v(t) = 12.05 + 17.733t + 0.3766t^2,\ 10 \le t \le 20$$

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2$$

$$= 392.19 \text{ m/s}$$



The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

$$= 0.38410\%$$

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

The upward velocity of a rocket is given as a function of time in the table. Find the velocity at t=16 seconds using the Lagrangian method for linear interpolation.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

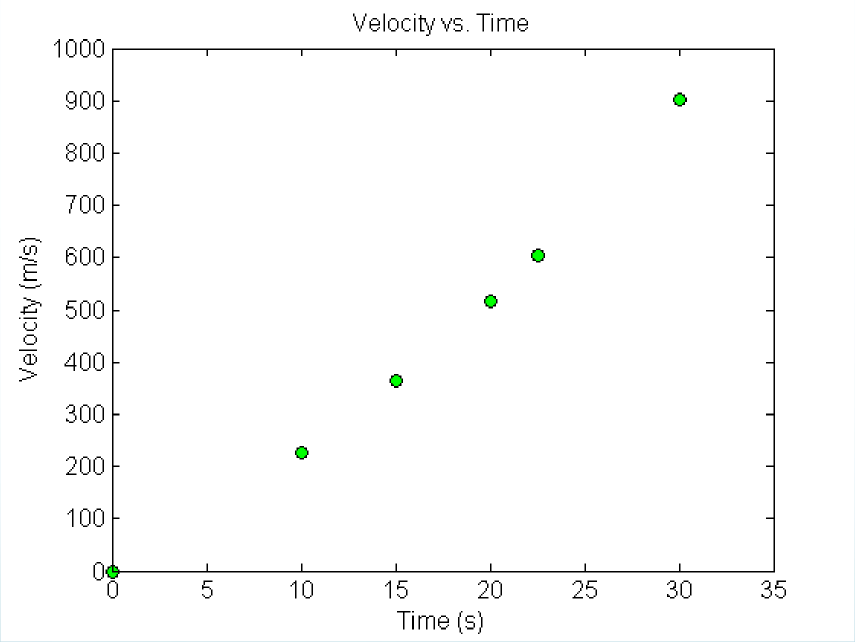


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = \sum_{i=0}^{1} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$

Linear Interpolation (contd)

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35)$$

$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$

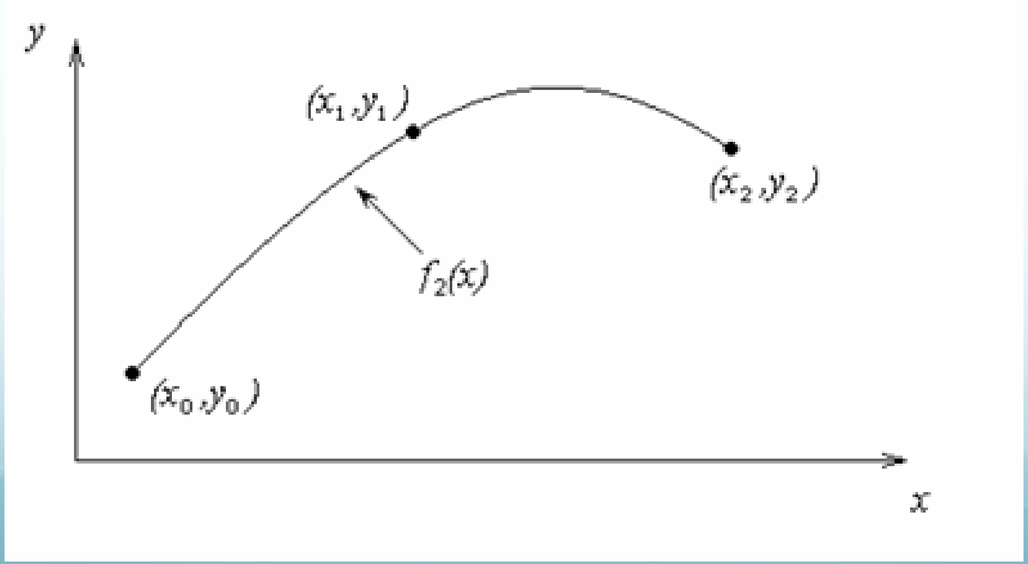
$$= 393.7 \text{ m/s.}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t) v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Quadratic Interpolation

$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j} = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j} = \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j} = \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

Quadratic Interpolation (contd)

$$v(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2)$$

$$v(16)=\left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)(227.04)+\left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)(362.78)+\left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)(517.35)$$

$$=(-0.08)(227.04)+(0.96)(362.78)+(0.12)(527.35)$$

$$=392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

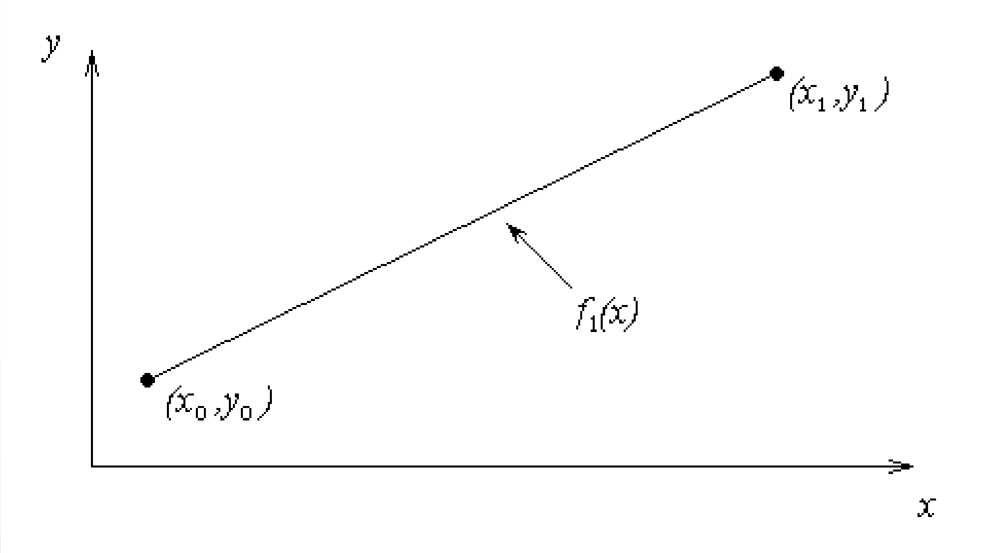
$$|\in_a|=\left|\frac{392.19-393.70}{392.19}\right|\times 100$$

$$=0.38410\%$$

Newton's Divided Difference Method

Linear interpolation: Given $(x_0, y_0), (x_1, y_1)$, pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where $b_0 = f(x_0)$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

The upward velocity of a rocket is given as a function of time in Table . Find the velocity at t=16 seconds using the Newton Divided Difference method for linear interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
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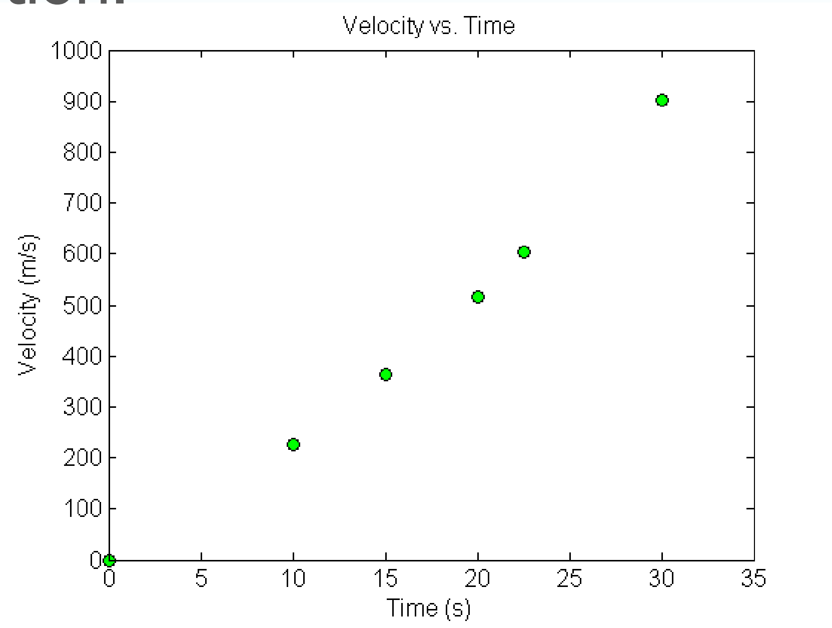


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = b_0 + b_1(t - t_0)$$

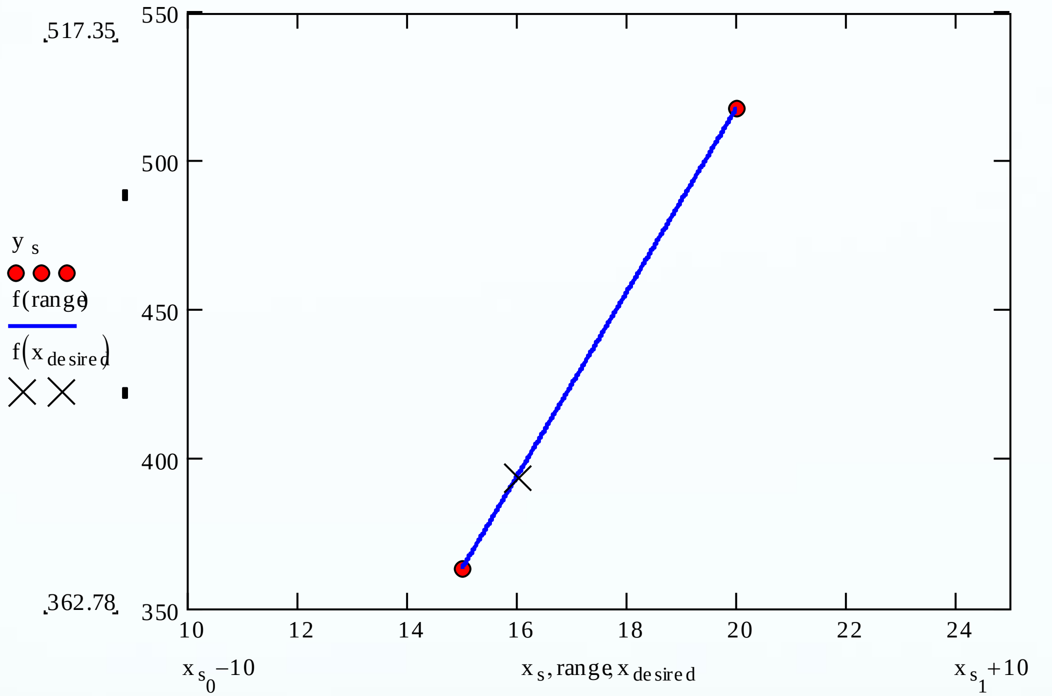
$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

$$b_0 = v(t_0) \quad = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} \quad = 30.914$$

Linear Interpolation (contd)



$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15),\ 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

Quadratic Interpolation

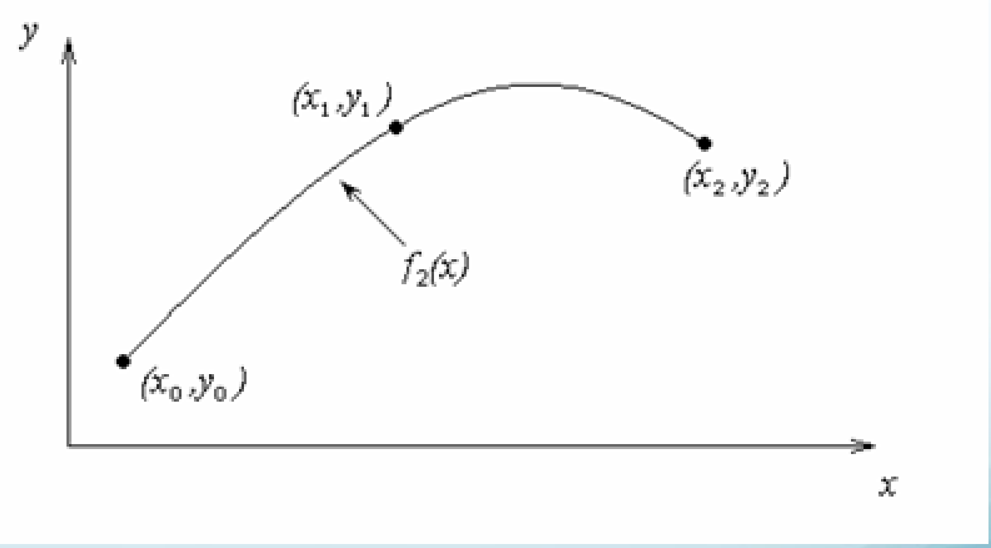
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

The upward velocity of a rocket is given as a function of time in the table . Find the velocity at t=16 seconds using the Newton Divided Difference method for quadratic interpolation.

Table. Velocity as a function of time

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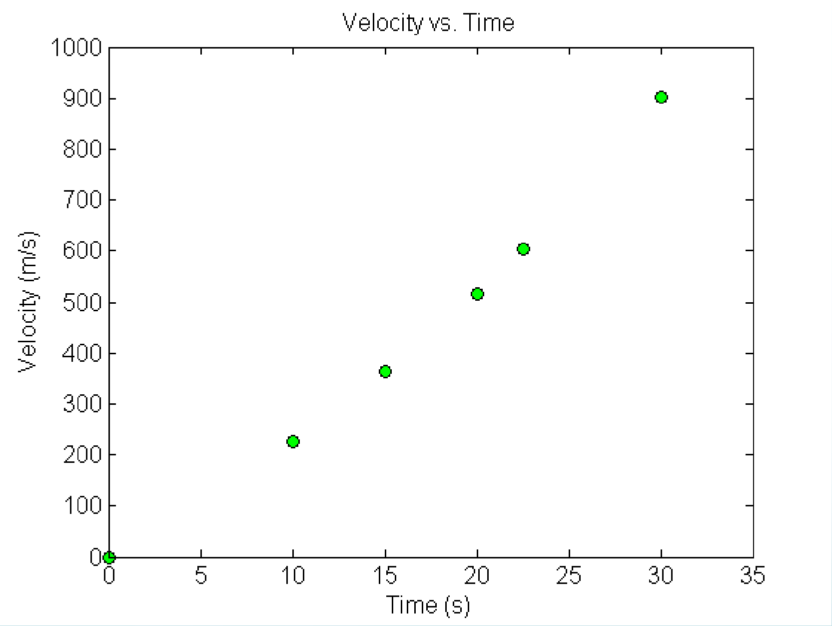
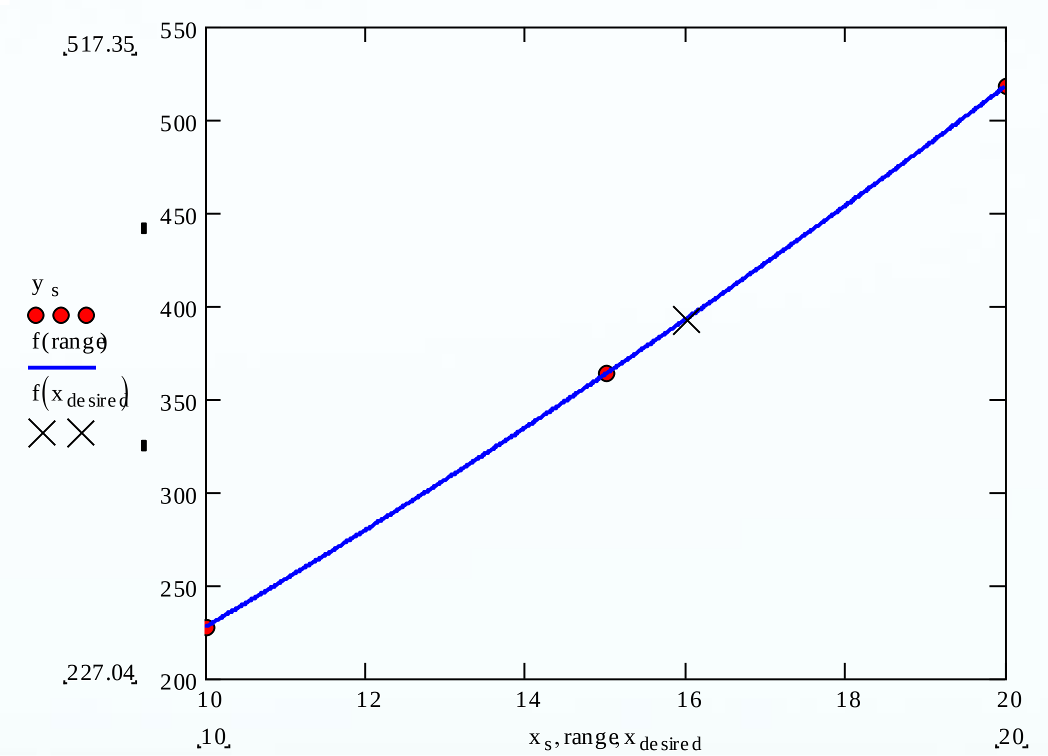


Figure. Velocity vs. time data for the rocket example

Quadratic Interpolation (contd)



$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

Quadratic Interpolation (contd)

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

Quadratic Interpolation (contd)

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) = 392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first order and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.69}{392.19}\right| \text{x}100$$

$$= 0.38502\ \%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

Divided Differences

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$
\begin{aligned}
b_0 &= f[x_0] \\
b_1 &= f[x_1, x_0] \\
b_2 &= f[x_2, x_1, x_0] \\
&\vdots \\
b_{n-1} &= f[x_{n-1}, x_{n-2}, \ldots., x_0] \\
b_n &= f[x_n, x_{n-1}, \ldots., x_0]
\end{aligned}
$$

General form

The third order polynomial, given $(x_0, y_0), (x_1, y_1), (x_2, y_2),$ and $(x_3, y_3),$ is

$$
\begin{aligned}
f_3(x) = {} & f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\
& + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)
\end{aligned}
$$

Divided Difference Table

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example to find divided differences and the Newton interpolation

- Construct a divided-difference diagram for the function f given in the following table, and write out the Newton form of the interpolating polynomial.

x	1	$\frac{3}{2}$	0	2
$f(x)$	3	$\frac{13}{4}$	3	$\frac{5}{3}$

The first entry for the table:

$$f[x_0, x_1] = \left(\tfrac{13}{4} - 3\right) / \left(\tfrac{3}{2} - 1\right) = \tfrac{1}{2}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{6} - \frac{1}{2}}{0 - 1} = \frac{1}{3}$$

The Complete Table:

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
1	3			
		$\frac{1}{2}$		
$\frac{3}{2}$	$\frac{13}{4}$		$\frac{1}{3}$	
		$\frac{1}{6}$		-2
0	3		$-\frac{5}{3}$	
		$-\frac{2}{3}$		
2	$\frac{5}{3}$			

We obtain:

$$p_3(x) = 3 + \tfrac{1}{2}(x-1) + \tfrac{1}{3}(x-1)\left(x-\tfrac{3}{2}\right) - 2(x-1)\left(x-\tfrac{3}{2}\right)x$$

Divided Differences Algorithm

integer i, j, n; **real array** $(a_{ij})_{0:n \times 0:n}$, $(x_i)_{0:n}$
for $i = 0$ **to** n **do**
    $a_{i0} \leftarrow f(x_i)$
end for
for $j = 1$ **to** n **do**
    for $i = 0$ **to** $n - j$ **do**
        $a_{ij} \leftarrow (a_{i+1,\, j-1} - a_{i,\, j-1})/(x_{i+j} - x_i)$
    end for
end for

Vandermonde Matrix

- **Theorem:** Every continuous function in the function space can be represented **as a linear combination of basis functions,** just as every vector in a vector space can be represented as a linear combination of basis vectors. Therefore,
- An Interpolating function f(x) can be represented by a set of basis functions ϕ_i for i=1,2,..,n.

$$f(x_i) = c_0\varphi_0(x_i) + c_1\varphi_1(x_i) + c_2\varphi_2(x_i) + \cdots + c_n\varphi_n(x_i) = y_i$$

Vandermonde Matrix

- For each i=1,2,…,n This is a system of linear equations→ we can represent in matrix form:

$$\mathbf{Ac} = \mathbf{y}$$

Here, A is the coefficient matrix with entries $a_{ij}=\phi_i(x_i)$.

Monomials are the simplest and most common basis functions. The monomials:

$$\varphi_0(x) = 1, \varphi_1(x) = x, \varphi_2(x) = x^2, \ldots, \varphi_n(x) = x^n$$

Vandermonde Matrix

Consequently, a given polynomial p can be the linear combination of monomials as:

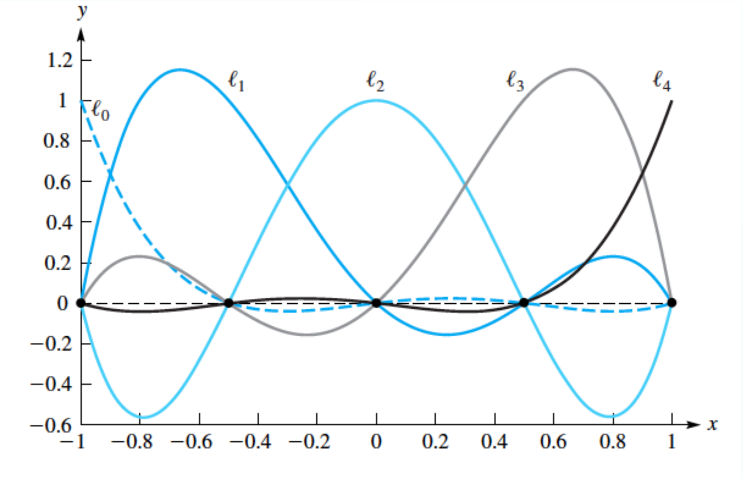
$$p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

The corresponding linear system Ac=y has the form:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

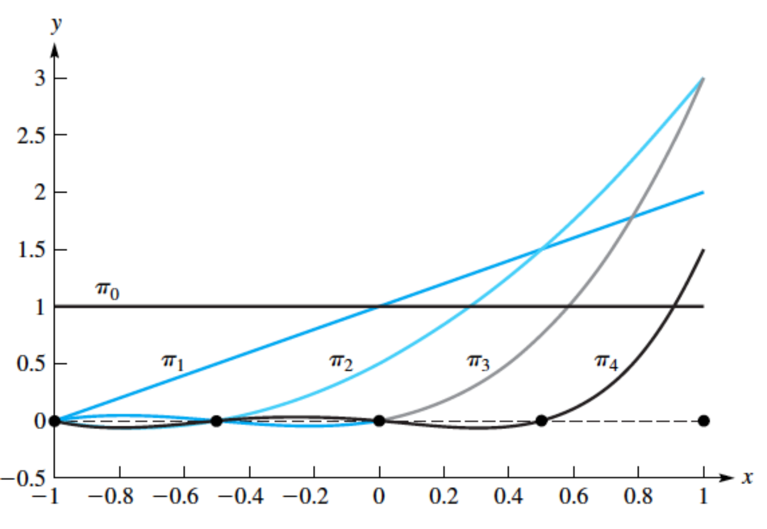
The coefficient matrix is called Vandermonde Matrix

Some basis functions we have just seen..



Lagrange Polynomials

Newton Polynomials



Monomials