Example DTFS:

$$x[n] = \begin{cases} 1, & n=0,1,2 \\ 0, & n=3,4 \end{cases}$$ Find DFS coefficients

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k}{N} n}$$

$x[n]$ has freq. components at $\omega = 0, \pm\frac{2\pi}{N}, \pm\frac{2\pi}{N}(2), \dots$ and

the respective complex exponentials are

$$e^{j(\frac{2\pi}{N}(0))}, \quad e^{\pm j(\frac{2\pi}{N})}, \quad e^{\pm j(\frac{2\pi}{N}(2))}, \dots$$

There are only N distinct frequencies in $x[n]$

due to the periodicity of $e^{j\frac{2\pi k}{N}}$

So,

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k}{N} n}$$

multiply both side with $e^{-j\frac{2\pi}{N}mn}$ and sum

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn}$$

$$= \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

Because of orthogonality principle

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \begin{cases} N, & k=m \\ 0, & k\neq m \end{cases}$$

If $m=k$

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} a_k \overbrace{\cancel{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-k)n}}}^{N}$$

Very important point !! This expression is equal N for only one of the k value for other values it goes to zero.

Because of this

$$\sum_{k=0}^{N-1} a_k N = a_k \cdot N$$

So

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}} = a_k \cdot N$$

$$\Rightarrow \boxed{a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}}$$

$N = 5$

$$a_0 = \frac{1}{5}\sum_{n=0}^{4} x[n]\, e^{0} = \frac{1}{5}\left(\cancel{x[0]}_{1} + \cancel{x[1]}_{1} + \cancel{x[2]}_{1} + \cancel{x[3]}_{0} + \cancel{x[4]}_{0}\right)$$

$$= \frac{3}{5}$$

$$k \neq 0 \quad a_k = \frac{1}{5}\sum_{n=0}^{4} x[n]\, e^{-j\frac{2\pi}{5}nk} = \frac{1}{5}\sum_{n=0}^{2} \cancel{x[n]}^{\,1}\, e^{-j\frac{2\pi}{5}nk}$$

$$= \frac{1}{5}\sum_{n=0}^{2} e^{-j\frac{2\pi}{5}nk}$$

Using geometric sum formula

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \quad |\alpha| < 1$$

$$\sum_{n=0}^{2} e^{-j\frac{2\pi}{5}nk} = \frac{1-\left(e^{-j\frac{2\pi}{5}k}\right)^3}{1-e^{-j\frac{2\pi}{5}k}} = \frac{1-\left(e^{-j\frac{6\pi k}{5}}\right)}{1-e^{-j\frac{2\pi}{5}k}}$$

$$= \frac{e^{-j\frac{6\pi k}{10}}\left(e^{j\frac{6\pi k}{10}} - e^{-j\frac{6\pi k}{10}}\right)}{e^{-j\frac{2\pi}{10}k}\left(e^{j\frac{2\pi}{10}k} - e^{-j\frac{2\pi}{10}k}\right)}$$

side note

$$e^{-j\frac{\theta}{2}}\left(e^{j\theta/2} - e^{-j\theta/2}\right)$$

$$\Downarrow$$

$2j\sin(\theta/2)$ inverse euler

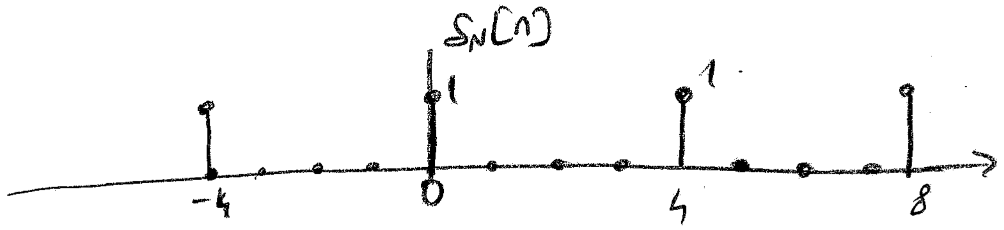
$$= e^{-j\frac{4\pi}{10}k} \cdot \frac{\cancel{2j}\,\sin\left(\frac{6\pi k}{10}\right)}{\cancel{2j}\,\sin\left(\frac{2\pi k}{10}\right)} = \frac{e^{-j\frac{4\pi}{10}k} \cdot \sin\left(\frac{6\pi k}{10}\right)}{\sin\left(\frac{2\pi k}{10}\right)}$$

$$k \neq 0 \qquad a_k = \frac{1}{5}\sum_{0}^{2} e^{-j\frac{2\pi}{5}nk} = \frac{1}{5}\,\frac{e^{-j\frac{4\pi k}{10}} \cdot \sin\left(\frac{6\pi k}{10}\right)}{\sin\left(\frac{2\pi k}{10}\right)}$$

$$a_k = \begin{cases} \dfrac{3}{5}, & k = 0, \pm N, \pm 2N, \dots \\[2ex] \dfrac{1}{5}\,\dfrac{e^{-j\frac{4\pi}{10}k} \cdot \sin\left(\frac{6\pi k}{10}\right)}{\sin\left(\frac{2\pi k}{10}\right)}, & \text{otherwise} \end{cases}$$

<u>Example</u>: Impulse train

$$\delta_4[n] = \begin{cases} 1, & n = 4m,\ m \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

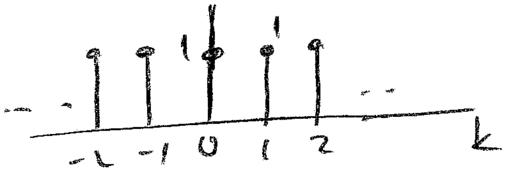


$$a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{0} = \frac{1}{4}$$

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}nk} = 1$$

$x[n] \rightarrow x[0] = 1$

$n \neq 0,\ x[n] = 0$



C.T. Fourier Transform

It is a special case of the Fourier Series when the period $T \to \infty$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$T_0 a_n = \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

As $T \to \infty$, $\omega_0 = \frac{2\pi}{T_0}$ becomes very small, the quantity $n\omega_0$ becomes cts. quantity.

So, $\omega = n\omega_0$, $T_0 \to \infty$

$T_0 a_n$ becomes function of ω

Forward FT.

$$T_0 a_n = X(\omega)$$

Analysis equation $\longrightarrow$ $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse FT

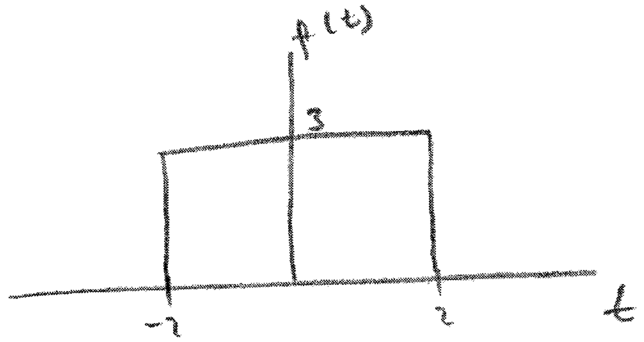
$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} T_0 a_n e^{jn\omega_0 t} \cdot \frac{1}{T_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} T a_n e^{jn\omega_0 t} \frac{1}{T} = \sum_{n=-\infty}^{\infty} X(\omega) e^{j\omega t} \frac{d\omega}{dt}$$

Inverse F.T. $\longrightarrow$ Synthesis Equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Example:



Calculate FT

$$\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-2}^{2} 3 e^{-j\omega t} dt$$

$$= 3 \int_{-2}^{2} e^{-j\omega t} dt = \frac{3}{-j\omega} \left[e^{-j\omega t} \Big|_{-2}^{2} \right]$$

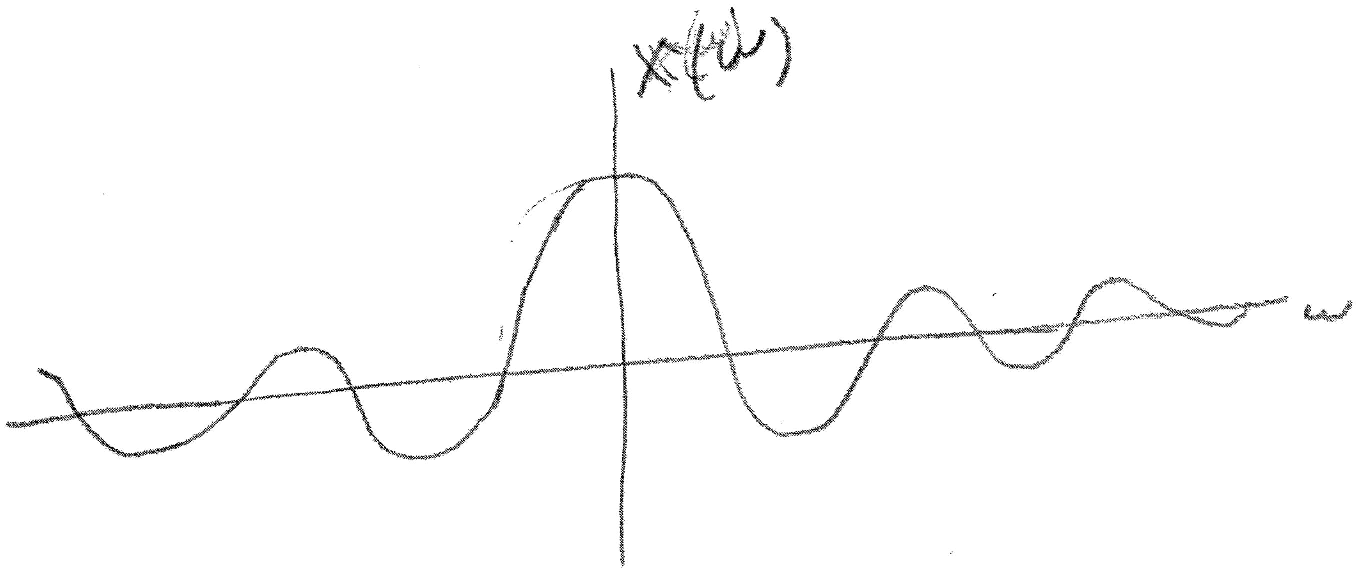
$$= \frac{3}{-j\omega} \left[e^{-j\omega 2} - e^{j\omega 2} \right] = \frac{3}{j\omega} \left(e^{2j\omega} - e^{-2j\omega} \right)$$

we know that $\sin\theta = \frac{1}{2j} \left(e^{j\theta} - e^{-j\theta} \right)$

$$\frac{3}{j\omega}\left(e^{2j\omega} - e^{-2j\omega}\right) = \frac{6}{\omega}\left(\frac{1}{2j}\left(e^{2j\omega} - e^{-2j\omega}\right)\right)$$

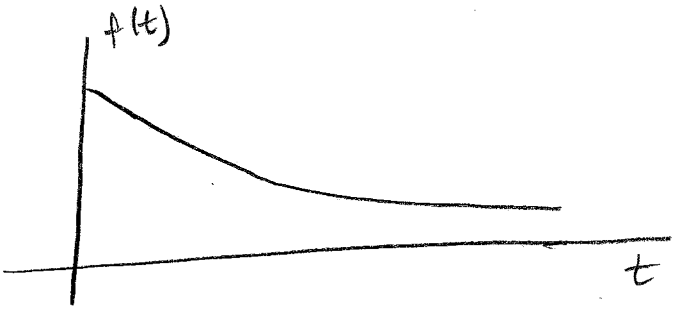
$$= \frac{6}{\omega}\sin 2\omega \longrightarrow \text{F.T.}$$

Graph of this function like this:



Example CT. F.T. (C.F.T.)

$$x(t) = e^{-at}\,u(t) \qquad \text{'a' is a real number } > 0.$$



Compute $X(j\omega)$

Def. of F.T.

$$X(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

so,

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} \underbrace{u(t)}_{\text{zero for } t<0} .\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

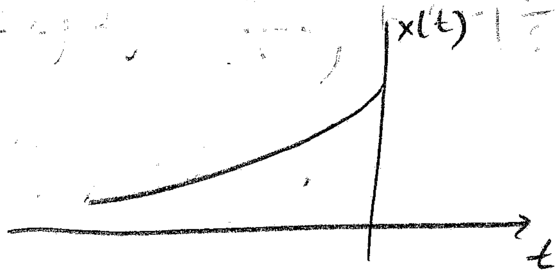
$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty} = \boxed{\frac{1}{a+j\omega}}$$

[redacted]

Between 7 and 8

Example C.F.T.

$$x(t) = e^{bt}\,u(-t) \qquad b>0$$



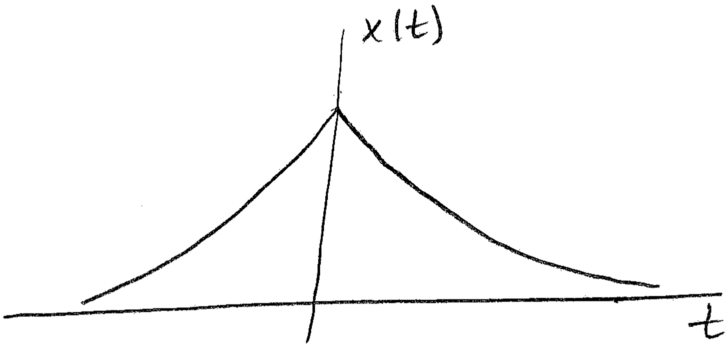
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{bt} u(-t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{bt}\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{(b-j\omega)t}\, dt$$

$$= \left. \frac{e^{(b-j\omega)t}}{b-j\omega} \right|_{-\infty}^{0} = \boxed{\frac{1}{b-j\omega}}$$

Example C.F.T.

$$x(t) = e^{-a|t|} \quad , \quad a > 0$$

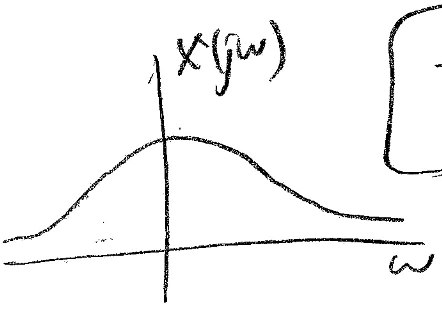


$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{-a|t|}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$= \underbrace{\int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt}_{\frac{1}{a-j\omega}} + \underbrace{\int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt}_{\frac{1}{a+j\omega}} = \frac{a+j\omega+a-j\omega}{a^2+\omega^2}$$

$$\boxed{= \frac{2a}{a^2+\omega^2}}$$



Example C.F.T.

$$x(t) = e^{j\omega_0 t}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{jt(\omega_0 - \omega)}\, dt = \left. \frac{e^{jt(\omega_0 - \omega)}}{j(\omega_0 - \omega)} \right|_{-\infty}^{\infty}$$

$$= \frac{e^{j\infty(\omega_0 - \omega)}}{j(\omega_0 - \omega)} - \frac{e^{-j\infty(\omega_0 - \omega_0)}}{j(\omega_0 - \omega_0)}$$

This means nothing

Instead we can use inverse F.T.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

So,

$$e^{j\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} ?\ e^{j\omega t}\, d\omega$$

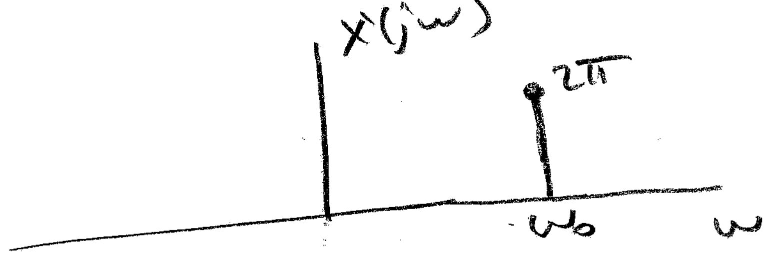
very similar. $e^{j\omega_0 t}$ is equal to $e^{j\omega t}$ at $\omega = \omega_0$

To sample a function at a point we can use impulse function

$$e^{j\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{2\pi\,\delta(\omega-\omega_0)}_{X(j\omega)}\, e^{j\omega t}\, d\omega$$

So.

$$F.T.\left\{ e^{j\omega_0 t} \right\} = 2\pi\,\delta(\omega-\omega_0)$$



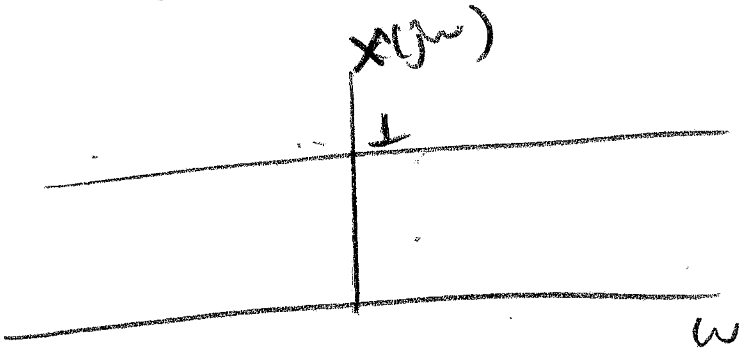
Example C.F.T.: $x(t) = \delta(t) \quad X(j\omega) = ?$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

use sifting property

(elemek, kalburdan geçirme
M. Türkçe)

$$= 1$$



Example:

$x(t) = a \qquad X(j\omega) = ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, dt \longrightarrow \text{Inverse F.T.}$$

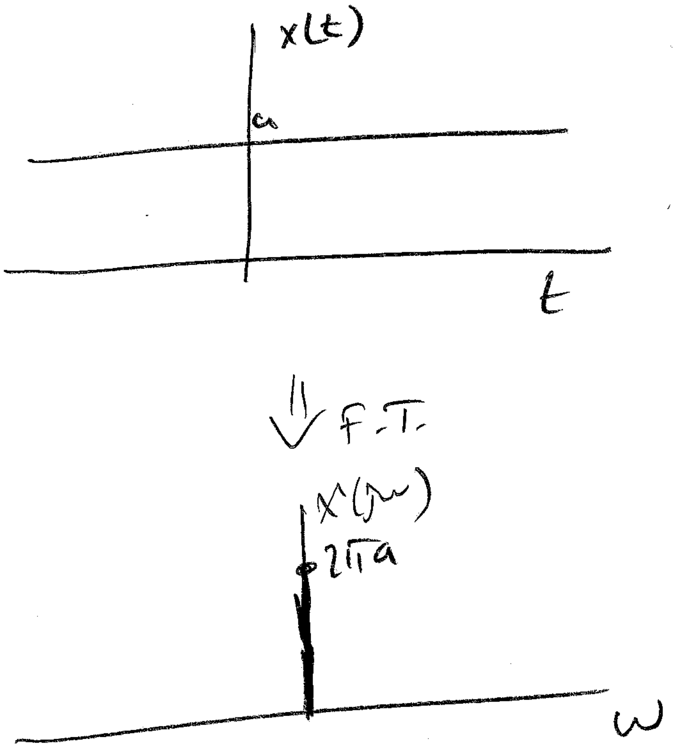
$$a = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\omega)}_{?}\, e^{j\omega t}\, dt$$

we know that $\displaystyle\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, dt = 1$

$$a = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{2\pi a\, \delta(\omega)}_{\longrightarrow X(j\omega)}\, e^{j\omega t}\, dt$$

So

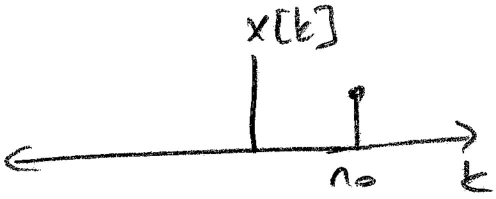
$$F.T.\{a\} = 2\pi a\,\delta(\omega)$$



$$F.T.\{\text{impulse}\} = \text{constant}$$

$$F.T.\{\text{constant}\} = \text{impulse}$$

Example D.T.F.T:



$$x[k] = \delta[k-n_0]$$

Remind C.F.T.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \, e^{j\omega t} \, dt$$

DTFT:

$\omega \longrightarrow \hat{\omega} = \omega T_s$ ← sampling period in s

↑ normalized angular freq.

So,

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\hat{\omega} n}$$

$X(e^{j\hat{\omega}})$ is periodic with 2π because

$$X(e^{j(\hat{\omega}+2\pi)}) = \sum_{n=-\infty}^{\infty} x(n) \, e^{-j(\hat{\omega}+2\pi) n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\hat{\omega} n} \, \underbrace{e^{-j2\pi n}}_{=1}$$

$$\boxed{= \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\hat{\omega} n}}$$

Inverse D.T.F.T

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}\, d\hat{\omega}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega} n} = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n-n_0]}\, e^{-j\hat{\omega} n}$$

If $n = n_0$, $\delta[n-n_0] = 1$
otherwise 0

$$= 0 + 0 + \dots + \delta[n_0 - n_0]\, e^{-j\hat{\omega} n_0} + 0 + 0 \dots$$

$$= 1 \cdot e^{-j\hat{\omega} n_0} = e^{-j\hat{\omega} n_0}$$

$$\text{D.T.F.T.} \left\{ \delta[k - n_0] \right\} = e^{-j\hat{\omega} n_0}$$

Example D.T.F.T.:

$$x[n] = a^{|n|} \quad -1 < a < 1$$

$$x[n] = a^n \;,\; n \geq 0$$

$$x[n] = a^{-n} \;,\; n < 0$$

$$X_1(e^{j\omega}) = \sum_{n=0}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} [a.e^{-j\omega}]^n$$

Using geometric series formula

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

$$X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{1}^{\infty} [a e^{j\omega}]^n$$

$$= \left(\sum_{0}^{\infty} [a.e^{j\omega}]^n\right) - 1 = \frac{1}{1-ae^{j\omega}} - 1 = \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

$$= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

Exemple D.T.F.T:

$$x[n] = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

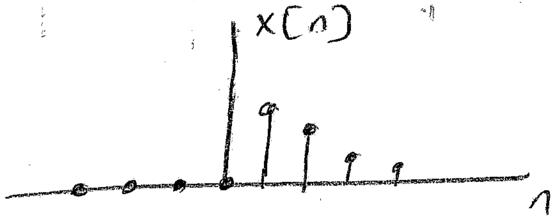
$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j\omega n}$$

using geometric sum formula you can solve

Example DTFT: Let $x[n] = \alpha^n u[n-1] \quad |\alpha| < 1$



$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\hat{\omega} n} = \sum_{n=-\infty}^{\infty} \alpha^n u[n-1]\, e^{-j\hat{\omega} n}$$

$$= \sum_{n=1}^{\infty} \alpha^n e^{-j\hat{\omega} n} = \sum_{k=1}^{\infty} \left(\alpha e^{-j\hat{\omega}}\right)^n$$

Sum of geometric series

$$\sum_{k=M}^{N} \beta^k = \frac{\beta^{N+1} - \beta^M}{\beta - 1}$$

$$\sum_{k=1}^{\infty} \left(\alpha e^{-j\hat{\omega}}\right)^n = \frac{\overset{0}{\cancel{\left(\alpha e^{-j\hat{\omega}}\right)^{\infty+1}}} - \left(\alpha e^{-j\hat{\omega}}\right)}{\alpha e^{-j\hat{\omega}} - 1}$$

$$\boxed{= \frac{\alpha e^{-j\hat{\omega}}}{1 - \alpha e^{-j\hat{\omega}}}}$$

$$\alpha^k u[n-1] \xleftrightarrow{D.T.F.T} \frac{\alpha e^{-j\hat{\omega}}}{1-\alpha e^{-j\hat{\omega}}}$$

Example D.T.F.T.:

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & o/w \end{cases}$$

$X(e^{j\hat{\omega}}) = ?$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega} n} = \sum_{n=0}^{4} e^{-j\hat{\omega} n}$$

use geometric series expansion formula

$$\sum_{n=0}^{N-1} c^n = \frac{1-c^N}{1-c}$$

$$\sum_{n=0}^{4} e^{-j\hat{\omega} n} = \frac{1-(e^{-j\hat{\omega}})^5}{1-e^{-j\hat{\omega}}} = \frac{\cancel{e^{-j\hat{\omega}5/2}}\, e^{-2j\hat{\omega}} \left[e^{j\hat{\omega}5/2} - e^{-j\hat{\omega}5/2}\right]}{\cancel{e^{-j\hat{\omega}/2}} \left[e^{j\frac{\hat{\omega}}{2}} - e^{-j\frac{\hat{\omega}}{2}}\right]}$$

inverse euler

$$= e^{-2j\hat{\omega}} \frac{\sin(5\hat{\omega}/2)}{\sin(\hat{\omega}/2)}$$

$$x[n] \xleftrightarrow{DTFT} e^{-2j\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

Example DFT:

$f(0) = 1, \ f(2) = 4, \ f(4) = 3, \ f(6) = 2$

$T_s = 2$

DFT of $f = ?$

DTFT : $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

If we assume $\hat{\omega} = \frac{2\pi}{N} k$, we can get rid of sampling time T_s

$$X[k] = \sum_{n=-\infty}^{\infty} x[n] e^{-j \frac{2\pi}{N} k n}$$

Many cases we have finite length

$$X[k] = \sum_{n=0}^{L-1} x[n] e^{-j \frac{2\pi}{N} k n}$$

Inverse DFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j 2\pi \frac{k}{N} n}$$

$$X[k] = \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$$

$$x[n] = \{1, 4, 3, 2\}$$

$$X[k] = x[0]\, e^{-j\frac{2\pi k}{4}\cdot 0} + x[1]\, e^{-j\frac{2\pi k}{4} 1} + x[2]\, e^{-j\frac{2\pi}{4}k2} + x[3]\, e^{-j\frac{2\pi}{4}k3}$$

$$= \underbrace{\cancel{e^{-j\frac{2\pi k}{4}\cdot 0}}}_{1} + 4e^{-j\frac{2\pi}{4}k} + 3e^{-j\frac{4\pi}{4}k} + 2e^{-j\frac{3\pi}{2}k}$$

$$= 1 + 4e^{-j\frac{\pi}{2}k} + 3e^{-j\pi k} + 2e^{-j\frac{3\pi}{2}k}$$

$$X[0] = 1+4+3+2 = 10$$

$$X[1] = -2-2j$$

$$X[2] = -2$$

$$X[3] = -2+2j$$

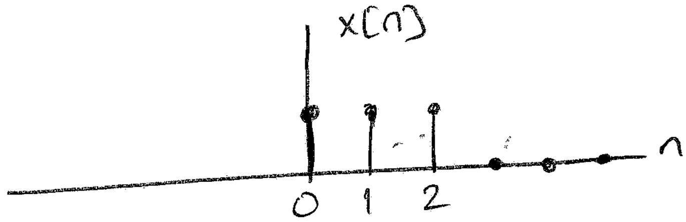
Example (Discrete Time Systems) :

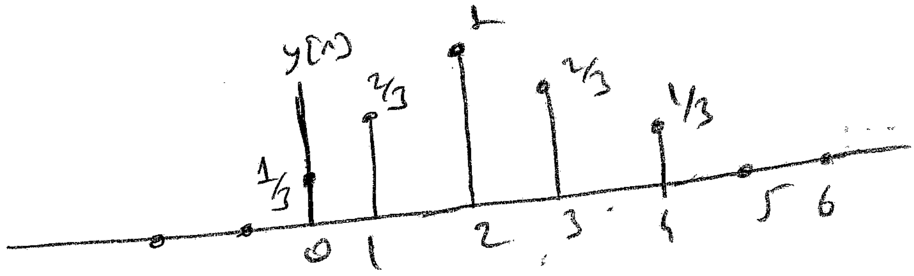
An averaging system

$$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$$

3 point averaging system

If we input following signal:





It smooths input signal. General Running Average Filter

$$y[n] = \sum_{k=0}^{M} \frac{1}{M+1} x[n-k]$$

Ex : square system $y[n] = (x[n])^2$

Example (FIR):

$$y[n] = 2x[n] - x[n-4]$$ Filter coefficients b_k?

General FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

So; $b_0 = 2$, $b_1 = b_2 = b_3 = 0$, $b_4 = -1$

Example (FIR): For FIR filter $b_k = \{3, -1, 2, 1\} \Rightarrow y[n] = ?$

$$y[n] = 3x[n] - 1x[n-1] + 2x[n-2] + x[n-3]$$

Example (LTI):

$$y(n) = (x(n))^2$$ Is Linear Time Invariant (LTI)?

Testing Linearity

$$\begin{tikzcd}
\text{I. } x_1[n] \arrow[r] & \boxed{Sys} \arrow[r] & y_1[n] = (x_1[n])^2 \arrow[r] & \otimes \arrow[dr] & & \\
 & & & a \arrow[u] & \oplus \arrow[r] & a x_1^2[n] + b x_2^2[n] \\
x_2[n] \arrow[r] & \boxed{Sys} \arrow[r] & y_2[n] = (x_2[n])^2 \arrow[r] & \otimes \arrow[ur] & & \\
 & & & b \arrow[u] & &
\end{tikzcd}$$

Not equal

$$\text{II} \quad (a x_1[n] + b x_2[n]) \longrightarrow \boxed{Sys} \longrightarrow (a x_1[n] + b x_2[n])^2$$

Not Linear

Time Invariance test

$$\text{I.} \quad x(n) \longrightarrow \boxed{System} \longrightarrow y(n) = x^2[n] \longrightarrow \boxed{Delay\ n_0} \longrightarrow x^2[n-n_0]$$

equal

$$\text{II.} \quad x(n) \longrightarrow \boxed{Delay\ n_0} \longrightarrow x[n-n_0] \longrightarrow \boxed{Sys} \longrightarrow y(n) = x^2[n-n_0]$$

Time Invariant

> So, system is not LTI

Ex: $y[n] = x[-n]$ Is it LTI?

$x_1[n] \xrightarrow{sys} y_1[n] = x_1[-n] \quad \times a \rightarrow a y_1[n] = a x_1[-n]$

$+$

$x_2[n] \xrightarrow{sys} y_2[n] = x_2[-n] \quad \times b \rightarrow b y_2[n] = b x_2[-n]$

$a y_1[n] + b y_2[n] = a x_1[-n] + b x_2[-n]$

$$\left(a x_1[n] + b x_2[n] \right) \rightarrow y[n] = \underbrace{a x_1[-n]}_{a y_1[n]} + \underbrace{b x_2[-n]}_{b y_2[n]}$$

equal

linear system

Test Time invariance

I. $x[n] \xrightarrow{sys} y[n] = x[-n] \xrightarrow{\text{delay } n_0} y[n-n_0] = x[-n+n_0]$

not equal

II. $x[n] \xrightarrow[n_0]{\text{delay}} x[n-n_0] \xrightarrow{sys} y[n] = x[-n-n_0]$

time variant

Ex: $y[n] = 5x[n-10]$

$x_1[n] \Rightarrow y_1[n] = 5x_1[n-10] \quad \times a = a y_1[n] = a5 x_1[n-10]$

$x_2[n] \rightarrow y_2[n] = 5x_2[n-10] \quad \times b = b y_2[n] \; b5 x_2[n-10]$

$+$

$a y_1[n] + b y_2[n] = 5a x_1[n-10] + 5b x_2[n-10]$

$$\left[a x_1[n] + b x_2[n] \right] \rightarrow y[n] = \underbrace{5a x_1[n-10]}_{a y_1[n]} + \underbrace{5b x_2[n-10]}_{b y_2[n-10]}$$

equal

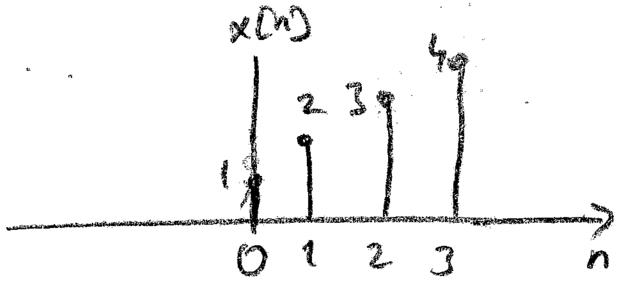
$x[n] \rightarrow \boxed{sys} \xrightarrow{5x[n-10]} \boxed{del} \rightarrow y[n] = 5x[n-10-n_0]$

$x[n] \rightarrow \boxed{del} \xrightarrow{x[n-n_0]} \boxed{sys} \rightarrow y[n] = 5x[n-10-n_0]$

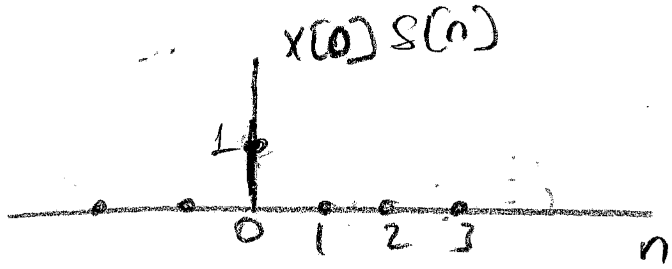
equal

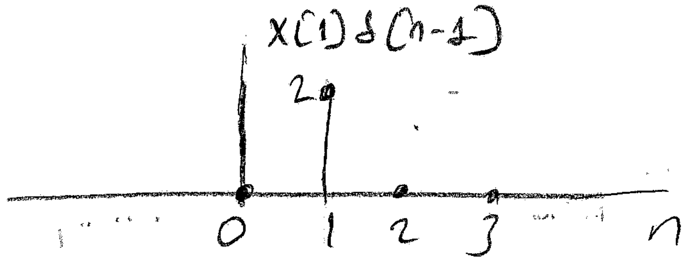
LTI system

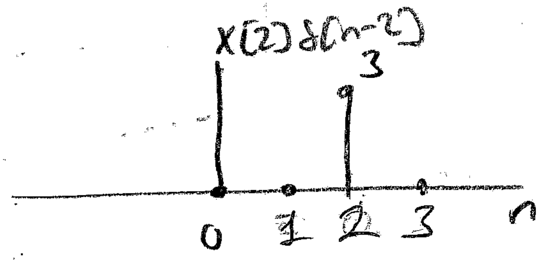
Example (Convolution)

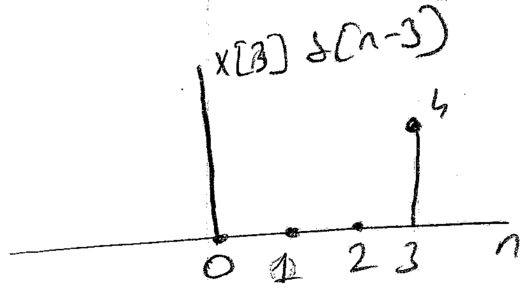


$\Downarrow$ write using impulse signal $\delta[n]$









+

$$x[n] = \sum_{k=\infty}^{\infty} x[k]\,\delta[n-k] = x[n] * \delta[n]$$

Example (Convolution)

In LTI system

$\delta[n] \longrightarrow \boxed{\text{Sys}} \longrightarrow h[n]$

Ask

$\delta[n-1] \longrightarrow \boxed{\text{Sys}} \longrightarrow h[n-1]$

$\delta[n-2] \longrightarrow \boxed{\text{Sys}} \longrightarrow h[n-2]$

$\vdots$

$\delta[n-k] \longrightarrow \boxed{\text{Sys}} \longrightarrow h[n-k]$

Time Invariance

$a_0\delta[n] \longrightarrow \boxed{\text{Sys}} \longrightarrow a_0h[n]$

$a_1\delta[n-1] \longrightarrow \boxed{\text{Sys}} \longrightarrow a_1h[n-1]$

$a_2\delta[n-2] \longrightarrow \boxed{\text{Sys}} \longrightarrow a_2h[n-2]$

$\vdots$

$a_k\delta[n-k] \longrightarrow \boxed{\text{Sys}} \longrightarrow a_kh[n-k]$

Linearity + Time invariance

$+$

$$a_0\delta[n] + \dots + a_k\delta[n-k] = a_0h[n] + \dots a_kh[n-k]$$

(2[illegible])

$$\left.\begin{array}{l} x[0] = a_0 \\ x[1] = a_1 \\ \vdots \\ x[k] = a_k \end{array}\right\} \Rightarrow \sum_{k=0}^{\infty} a_k \delta[n-k] = \sum_{k=0}^{\infty} x[k] \delta[n-k]$$

$$\Downarrow$$

$$\boxed{\text{Sys}}$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} a_k h[n-k] = \sum_{k=0}^{\infty} x[k] h[n-k]$$

We can extend lower limit to $k = -\infty$

because of LTI

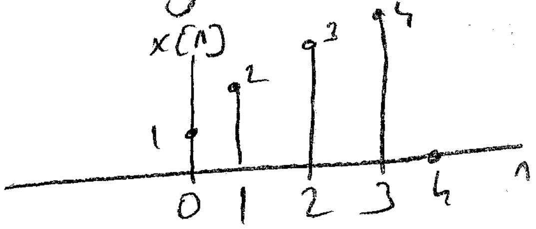
$$\sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \Rightarrow \boxed{\text{sys}} \Rightarrow \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

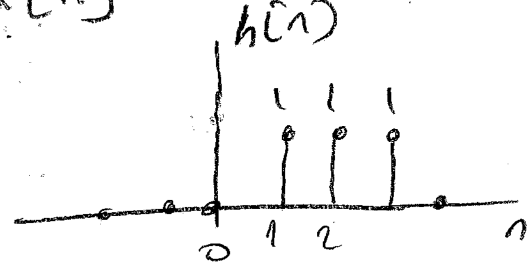
This is convolution

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

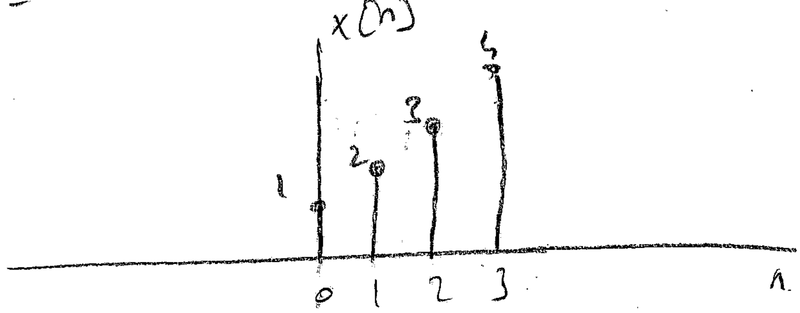
Ex (convolution):

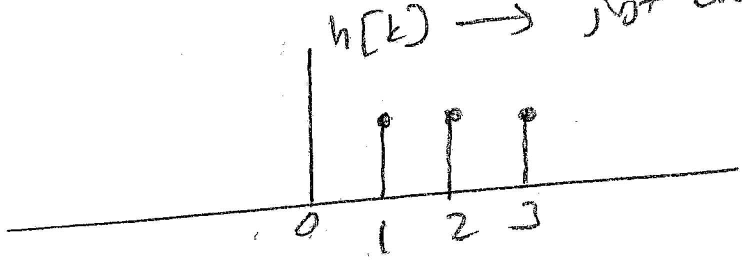
$$y[n] = h[n] * x[n]$$



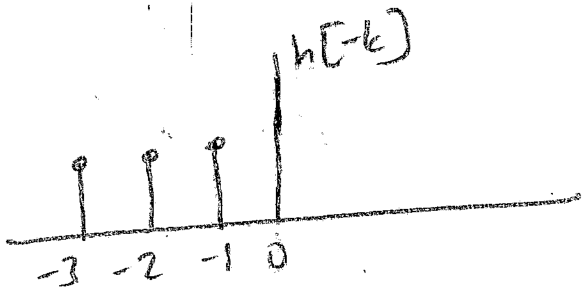


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

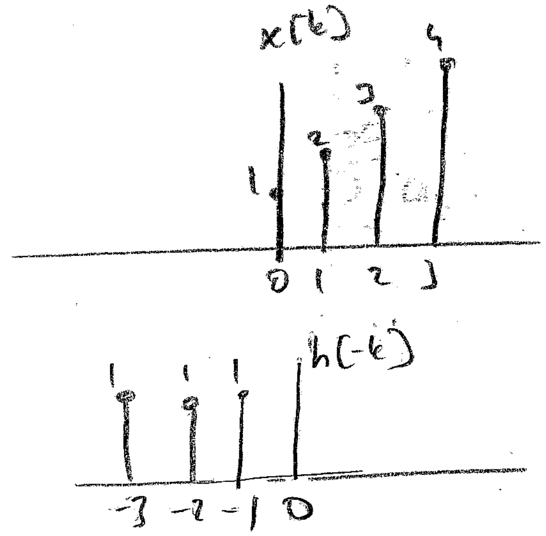




$\longrightarrow$ not changed parameter



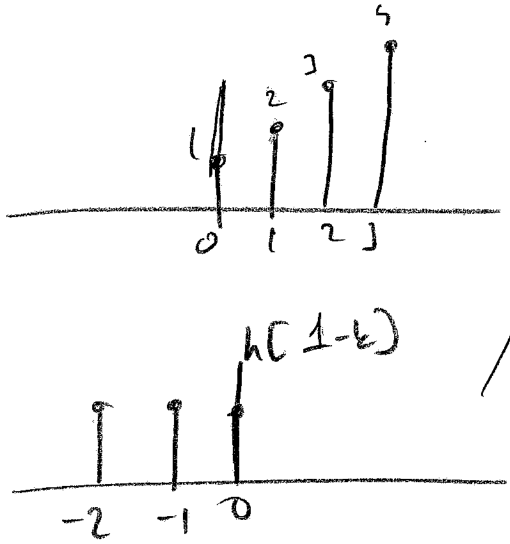
y[0]



multiply and sum

y[0] = 0

y[1]



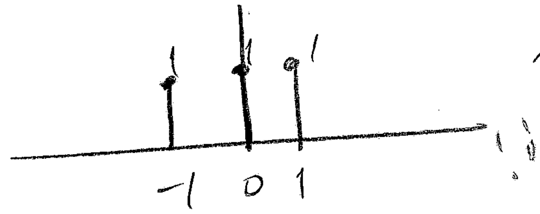
multiply and sum

y[-1] = 1

y[2]

x[k] → same

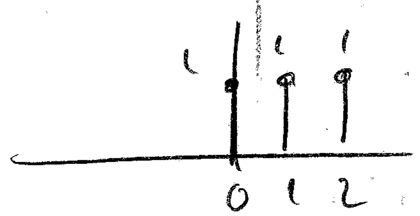
multiply and sum



y[-2] = 3

$y[+3]$

$x[k] \rightarrow$ same



mult. and sum

$y[+3] = 6$

$y[+4] \rightarrow$ calculate by the same way

$y[+4] = 9$

$y[+5] = 7$

$y[+6] = 4$

$y[+7] = 0$ $\quad y[-1] = 0 \quad \ldots \quad y[-\infty] = 0$