PROBLEM SQUAD — #1*

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* Code and this write-up are available at `github.com/hrothgar/problemsquad`

> A particle starts at $(1, 0)$ and wanders around the (x, y) plane in standard Brownian motion.[1] Eventually (with probability 1), it will circle around the origin to cross the line $x > 0, y = 0$ at a moment when the winding number of its path so far with respect to the origin is nonzero. What is the probability that this happens at a time less than 1?

1. By *standard Brownian motion* we mean the 2D Wiener process, in which for any Δt, a particle's change in x coordinate in time Δt is given by the $N(0, \Delta t)$ normal distribution, and likewise for y. This can also be defined as the limit $h \to 0$ of a random walk on the square x–y h-lattice.

INTERPRETATION & PLAN OF ATTACK

The problem is of a multivariate probabilistic flavor. My instinct has been to place the planar Wiener process in the complex plane, where it is simpler to code and more intuitive to calculate the winding number. To me, the problem immediately presents two possible approaches: analytical solution by way of some ghastly multiple integral, or approximate solution by way of Monte Carlo simulation (which is of course just a different way to get at the same ghastly integral). I attempted both but had success only with the latter.

In perusing the literature, I discovered that many of the results pertaining to the winding number of planar random processes are asymptotic: most of the discussion I came across follows Spitzer [1958], who proved his eponymous theorem[2] that the asymptotic winding angle θ of a particle undergoing planar Brownian motion is independent of the particle's initial distance from the origin and obeys the Cauchy law

2. It seems that half a dozen other proofs have arisen since Spitzer first proved his theorem in 1958.

$$\lim_{t\to\infty} \mathbb{P}\left[x = \frac{2\theta(t)}{\log t}\right] = \frac{1}{\pi(1+x^2)}.$$

I came across a particularly inspiring monograph[3] about Monte Carlo simulation in the NA library. While the book does not contain anything directly pertinent to the present problem, it explains several methods to improve the efficiency of Monte Carlo simulations (and in fact numerically compares the efficiency of various methods against a test problem).

3. Hammersley et al. [1965]

MONTE CARLO SIMULATION

Crude Monte Carlo

My primary attempt to solve the problem was to conduct a crude Monte Carlo simulation. For a crude MC, one must realize some large number

(perhaps 10^6) of suitable Wiener processes before calculating the fraction of them that satisfy the winding criterion listed in the problem, which fraction is announced as the approximate solution. The time discretization is the only parameter in such a simulation; I partitioned the time interval $t \in [0, 1]$ into $N = 10^4$ uniform time steps. Then, after using Higham's `bpath2.m` code[4] to write a function `wiener.m` for a single complex planar trial, I expanded into a script `wiener_carlo.m` that runs a full simulation, periodically reporting results to a log file.

4. Higham [2001]

In the end, after 50160001 trials, the log reported the fraction of paths 'at least once wound' to be $\bar{x} = 0.0355925431$. That value is the midpoint of a 95% confidence interval whose endpoints are each out at a distance equal to the standard error estimate

$$\hat{s} = \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 0.0000261595.$$

Substituting data, we arrive at the interval

$$\begin{aligned} x &= 0.0355925431 \pm 0.0000261595, \\ \text{or} \quad & [0.0355663836, 0.0356187026], \end{aligned}$$

in which we can have confidence in three digits.

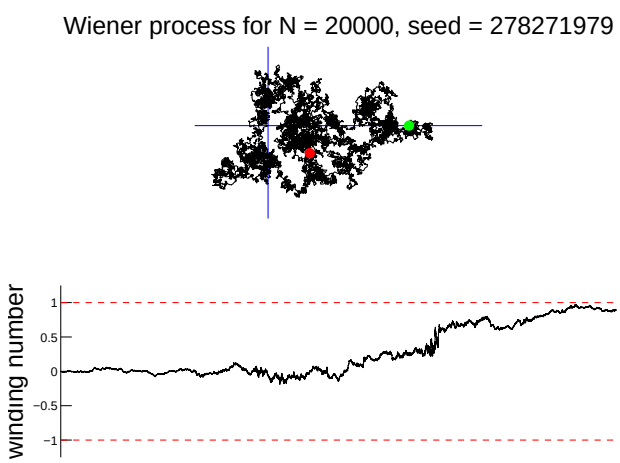

Figure 1: A sample planar Wiener process. The path appears to wind completely around the origin, but the graph of its winding number shows otherwise.

Adaptive Monte Carlo

One way to speed up the simulation would be to enlarge the time step. A large time step is stable and is no less accurate in determining the location of the particle in the future, but in lowering our resolution we may miss a transient winding. A solution to the problem is an adaptive Monte Carlo method: one in which the time step is dependent on the particle's distance from the origin.

If particle at distance R from the origin undergoes a perturbation of distance δ, its winding angle is perturbed by at most $\tan^{-1}\left({}^{\delta}\!/_{R}\right)$, indicating that outside the unit disc, small time steps have very small impact indeed on the winding angle. Instead of wasting computational power while the particle is far out, we can scale the time step in such a way that the expected $\Delta\theta$ is the same regardless of distance from the origin.[5] Such a scheme would greatly increase the efficiency of the MC method without sacrificing accuracy. Paths that steer away from the origin would be trivially inexpensive to compute, for instance.

Unfortunately, this idea came rather late and I did not have time to implement it. However, I might code it up soon to test its effectiveness.

5. In practice, the time step would scale everywhere except in a small neighborhood of the origin, where it would plateau at a minimum to prevent the path from freezing near the event horizon of the origin's black hole behavior.

CONCLUSION

My estimate to the solution of Problem 1 is $x = 0.035$. Although I was unable to find it this time around, I am convinced there is a feasible analytical solution and would like — if the problem remains unsolved — to work with others to find it. Refinements to the Monte Carlo method may allow an increase in efficiency sufficient to gain a few more digits as well.

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