PHY302: EndSem Supplementary Examination (50 points)

Instructor: Manabendra Nath Bera

1. Consider four Hermitian 2×2 matrices $\mathbb{I}$, σ_1, σ_2, and σ_3, where $\mathbb{I}$ is the unit matrix and others satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}$. Prove the following **without using a specific representation or form for the matrices.**

   (a) Prove that $\mathrm{tr}(\sigma_i) = 0$. [Point: 2]

   (b) Show that the eigenvalues of σ_i are ± 1 and that $\det(\sigma_i) = -1$. [Point: 2]

   (c) Show that the four matrices are linearly independent and therefore that any 2×2 matrix can be expanded in terms of them. [Points: 3]

   (d) From (c) we know that

   $$M = m_0\mathbb{I} + \sum_{i=1}^{3} m_i\sigma_i,$$

   where M is any 2×2 matrix. Determine the expression for m_i with $i = 0, 1, 2, 3$. [Points: 3]

2. (a) A particle of mass m moves in one dimension under the influence of a potential $V(x)$. Suppose it is in an energy eigenstate $\psi(x) = (\gamma^2/\pi)^{1/4} \exp\left(-\gamma^2 x^2/2\right)$ with energy $E = \hbar^2\gamma^2/2m$.

   (a) Find the mean position of the particle. [1 point]

   (b) Find the mean momentum of the particle. [1 point]

   (c) Find $V(x)$. [3 points]

   (b) Find out the energy uncertainty of a one-dimensional quantum Harmonic oscillator in the second excited state

   $$\psi_2(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(2\alpha x^2 - 1)e^{-\alpha x^2}/2,$$

   where $\alpha = \frac{m\omega}{\hbar}$. [5 points]

3. Consider a one-dimensional simple harmonic oscillator. Do the following algebraically, i.e., without using the explicit forms of the wavefunctions.

   (a) Construct a state as a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle \hat{x} \rangle$ is as large as possible. Here $|0\rangle$ and $|1\rangle$ are the ground and first excited states. [Points: 3]

   (b) Suppose the oscillator is in the state constructed in (a) at time $t = 0$. What is the state vector for $t > 0$? [Point: 1]

   (c) Evaluate the expectation value $\langle \hat{x} \rangle$ as a function of time for $t > 0$ using (i) the Schrödinger picture and (ii) the Heisenberg picture independently. [Point: 3]

   (d) Evaluate $\langle (\Delta x)^2 \rangle$ as a function of time using either picture. [Point: 3]

4. (a) Consider a two-state system with basis states $|1\rangle$ and $|2\rangle$ and a Hamiltonian

   $$H = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix} = -\Delta\sigma_1, \quad \text{with} \quad \Delta > 0.$$

   As you know, if the system is initially in the state $|1\rangle$, where $\sigma_3|1\rangle = |1\rangle$, then the probability of finding the system in $|1\rangle$ varies periodically between one and zero as a function of time.

   Now modify the Hamiltonian as $H' = H + \alpha\sigma_3$ (where α is some real number) and system is in the initial state $|1\rangle$. With this, the probability of finding the system in state $|1\rangle$ varies periodically between one and some minimum value $p_{\min} > 0$. Find out α in terms of Δ and $p_{\min}$. [Points: 5]

   Hint: It might be useful to write the Hamiltonian as the dot product of a vector with the spin operator and to think about time evolution as precession.

   (b) A state $|\psi\rangle$ is a simultaneous eigenstate of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$:

   $$\hat{\mathbf{L}}^2|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle, \quad \hat{L}_z|\psi\rangle = \hbar m|\psi\rangle. \tag{1}$$

   (a) Show that the square of the eigenvalue of $\hat{L}_z$ cannot exceed of the eigenvalue of $\hat{\mathbf{L}}^2$. [Points: 2]

   (b) Calculate $\langle \hat{L}_x \rangle$ and $\langle \hat{L}_x^2 \rangle$ for the state $|\psi\rangle$. [Points: 3]

   Hint: You may use the relations $L_x = (L_+ + L_-)/2$, $L_y = (L_+ - L_-)/2i$, and $L_\pm|lm\rangle \propto |l, m \pm 1\rangle$, where $|lm\rangle$ is a simultaneous eigenstate of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$.

5. (a) A particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$, with $b > a$. There is no other potential. Find the ground state energy and normalized wave function. [5 points]

   (b) Assume that the eigenstates of a hydrogen atom isolated in space are all known and designated as usual by

   $$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi).$$

   Suppose the nucleus of a hydrogen atom is located at a distance d from an infinite potential wall which, of course, tends to distort the hydrogen atom. Find the ground state wave function, in terms of $R_{nl}(r)Y_{lm}(\theta, \phi)$, of this hydrogen atom as d approaches zero. [5 points]

PHY302: 2nd Mid-Semester Examination (30 points)

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1. A particle is moving in a box divided into left and right compartments by a thin partition. The normalized eigenket $|l\rangle$ ($|r\rangle$) represents the state of the particle if it is known to be on the left (right) compartment and satisfy $\langle l|r\rangle = 0$. Consider a state of the particle at time $t = 0$,

$$|\psi(0)\rangle = \alpha|r\rangle + \beta|l\rangle,$$

where $|\alpha|^2 + |\beta|^2 = 1$. The particle can tunnel through the partition and that is characterized by the Hamiltonian

$$H = \delta\left(|r\rangle\langle l| + |l\rangle\langle r|\right),$$

where δ is a real number with the dimension of energy.

(a) Find the normalized energy eigenkets and the corresponding energy eigenvalue. (2 points)

(b) Find the state of the particle $|\psi(t)\rangle$ at time $t = t$ by applying the appropriate time-evolution operator to the initial state. (3 points)

(c) Suppose at $t = 0$ the particle is in the right compartment with certainty. What is the probability of finding the particle in the left compartment as a function of time? (3 points)

(d) Write down the coupled Schrödinger equations for α and β. (3 points)

(e) Consider a time-independent Schrödinger operator

$$A_S = |r\rangle\langle r|. \tag{1}$$

Find out the corresponding Heisenberg operator and its Heisenberg equation of motion. (2+2 points)

2. Consider a spin in a (unnormalized) state

$$(1 + i)\,|z;+\rangle + (1 + i\sqrt{3})\,|z;-\rangle,$$

where $|z;+\rangle$ and $|z;-\rangle$ are the orthonormal eigenvectors of S_z spin operator. Find out the direction of the spin. (5 points)

3. Consider a function, known as the correlation function, defined by

$$C(t) = \langle \hat{x}_H(t)\hat{x}_H(0)\rangle, \tag{2}$$

where $\hat{x}_H(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of one-dimensional simple harmonic oscillator. (10 points)