MTH201-PRACTICE PROBLEMS 2

Q 1. (i) Sketch the function $f(x) = x/|x|$. Check if the limit exists as $x \to 0$.

(ii) Sketch the function $f(x) = x^2/|x|$. Check if the limit exists as $x \to 0$.

(iii) Sketch the function $f(x) = x/|x^2|$. Check if the limit exists as $x \to 0$.

Q 2. Find the following limits.

(i) $\lim_{x\to a} \dfrac{x^2 - a^2}{x - a}$

(ii) $\lim_{x\to b} \dfrac{\sqrt{x} - \sqrt{b}}{x - b}, b > 0$

(iii) $\lim_{t\to 0} \dfrac{t}{\sqrt{4 - t} - 2}$

(iv) $\lim_{x\to 0} \dfrac{\sqrt{x + 25} - 5}{x}$

(v) $\lim_{x\to 0} \dfrac{2x \sin x}{1 - \cos x}$

(vi) $\lim_{x\to 0} \dfrac{\tan 3x - \sin 3x}{x^3}$

(vii) $\lim_{x\to\infty} \sqrt{x^2 + x} - x$, put $y = 1/x$. Then calculate using $y \to 0$.

(viii) Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$. Show that $\lim_{x\to a} f(x)$ does not exist for any real number a.

(ix) Let $f(x)$ be a function that satisfies the equation $f(x)^2 - 3f(x) + 2 = 0$. Calculate the possible limits $\lim_{x\to a} f(x)$, for any real number a.

Q 3. Suppose $y = f(x)$ defines a curve that always lies between the parabola $x^2 = y - 1$ and the hyperbola $yx + y - 1 = 0$. Then calculate $\lim_{x\to 0} f(x)$.

Q 4. Let $\lim_{x\to a} f(x) = A, \lim_{x\to a} g(x) = B$. If $f(x) < g(x)$, then is it true that $A < B$? Prove your claims.

Q 5. Use the sandwich theorem to calculate the given limits.

(i) Let $x^3 - 6x^2 + 12x - 3 \leq f(x) \leq x^2 - 4x + 9$ for $x \leq 3$. Calculate $\lim_{x\to 2} f(x)$.

(ii) $\lim_{x\to 0} x^2 \sin(\frac{\pi}{x})$.

(iii) $\lim_{x\to 0} x \cos(\frac{1}{x})$.

Q 6 (Challenging)**.** (i) Calculate $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$, without using L'Hospital's rule.

(ii) Show that $\lim_{u\to 0} \dfrac{\log(1+u)}{u} = 1$.

(iii) Use the previous one and calculate $\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$. [Take $\log_e$.]