ROLL NO : MS _ _ _ _ _ NAME :

PHY303 MidSem I (Part A) Date : Sep 11, 2023 Inst: Abhishek Chaudhuri

- Time : 30 minutes, Max Marks : 10
- Attempt all questions. Please give your answers in the space provided.

1. Write a differential equation that a Green function $G(\mathbf{r}, \mathbf{r}')$ for Poisson's equation must satisfy, for Dirichlet boundary conditions. Include a statement of the boundary conditions. [2]

$$\nabla^2 G_D(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r} - \vec{r}') \quad \text{for } \vec{r}, \vec{r}' \text{ within } V \text{ — } 1$$

B.C. $G_D(\vec{r}, \vec{r}') = 0$ for $\vec{r}$ or $\vec{r}'$ on S. — 1

2. In an electrostatics problem with Neumann boundary conditions, what is the simplest allowable boundary condition on the the Green's function $G(\mathbf{r}, \mathbf{r}')$? Hint: The result must be consistent with the differential equation that G satisfies. [3]

$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r} - \vec{r}') \quad \text{for } \vec{r}, \vec{r}' \text{ within } V$$

Integrating over volume V,

$$\int_V \nabla'^2 G(\vec{r}, \vec{r}')\, d^3r' = -4\pi \int_V \delta^3(\vec{r} - \vec{r}')\, d^3r' = -4\pi \text{ — } 1$$

$$\int_V \vec{\nabla}' \cdot (\vec{\nabla}' G)\, d^3r' = \oint_S \nabla' G \cdot \hat{n}'\, da' \quad \text{Divergence theorem.}$$

$$\therefore \oint_S \nabla' G \cdot \hat{n}'\, da' = -4\pi$$

$$\Rightarrow \oint_S \frac{\partial G}{\partial n'}\, da' = -4\pi. \text{ — } 1$$

Choose, $\frac{\partial G}{\partial n'} = \text{const.} = A$ (say). Neumann B.C.

$$\therefore A \oint_S da' = -4\pi \Rightarrow A = -\frac{4\pi}{S}. \text{ — } 1$$

where S is total surface area of the system boundary.

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3. Use delta-function to express the charge density $\rho(\mathbf{r})$ for the following charge distribution, in the indicated coordinate system: In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R. [2]

By symmetry, $\rho(\vec{r}) = \rho(r) = Q\,\delta(r-R)\cdot A$ — 1

where A is some constant.

Now, $Q = \int \rho(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi \int r^2 \rho(r)\, dr$

$= 4\pi Q A \cdot \int r^2 \delta(r-R)\, dr = 4\pi Q A R^2.$

$\therefore A = \frac{1}{4\pi R^2}$ $\quad \therefore \rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta(r-R).$ — 1

4. The continuity equation follows from charge conservation and is given as $\partial\rho/\partial t + \nabla\cdot\mathbf{J} = 0$. If the charge density is expressed as $\rho(\mathbf{r},t) = \sum_k q_k \delta(\mathbf{r} - \mathbf{r}_k(t))$, find the expression for the current density $\mathbf{J}(\mathbf{r},t)$? [3]

$\rho = \sum_k q_k \delta(\vec{r} - \vec{r}_k(t)) \quad \therefore \frac{\partial\rho}{\partial t} = \sum_k q_k \frac{\partial}{\partial t}\delta(\vec{r} - \vec{r}_k(t))$ $\frac{1}{2}$

$\frac{\partial}{\partial t}\delta^3(\vec{r} - \vec{r}_k(t)) = \frac{\partial}{\partial t}\left[\delta(x - x_k(t))\,\delta(y-y_k(t))\,\delta(z - z_k(t))\right]$

$= \delta(y-y_k(t))\,\delta(z-z_k(t))\frac{\partial}{\partial t}\delta(x - x_k(t)).$

$+ \cdots\cdots$

$= -\delta(y-y_k(t))\,\delta(z-z_k(t))\frac{\partial}{\partial(x-x_k(t))}\delta(x-x_k(t))\cdot\frac{\partial x_k}{\partial t}$

$- \cdots\cdots$

$= -\frac{\partial}{\partial(x-x_k)}\left[\delta(x-x_k(t))\,\delta(y-y_k(t))\,\delta(z-z_k(t))\right]\cdot\frac{\partial x_k}{\partial t}$

$- \frac{\partial}{\partial(y-y_k)}\left[\quad\right]\frac{\partial y_k}{\partial t} - \frac{\partial}{\partial(z-z_k)}\left[\quad\right]\cdot\frac{\partial z_k}{\partial t}$

$= -\vec{v}_k\cdot\vec{\nabla}\,\delta^3(\vec{r} - \vec{r}_k(t)).$ — $1\frac{1}{2}$

$\therefore \frac{\partial\rho}{\partial t} = -\sum_k q_k \vec{v}_k\cdot\vec{\nabla}\delta^3(\vec{r} - \vec{r}_k(t))$

$= -\vec{\nabla}\cdot\left[\sum_k q_k \vec{v}_k\,\delta^3(\vec{r} - \vec{r}_k(t))\right]$

$\therefore \vec{J} = \sum_k q_k \vec{v}_k\,\delta^3(\vec{r} - \vec{r}_k(t)).$ — 1

1. $\Phi(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cosh y)$.

B.C. : (i) $\Phi = 0$ when $y = 0$ $\frac{1}{2}$

(ii) $\Phi = 0$ when $y = a$ $\frac{1}{2}$

(iii) $\Phi = \Phi_0$ when $x = b$ $\frac{1}{2}$

(iv) $\Phi = \Phi_0$ when $x = -b$. $\frac{1}{2}$

Since the region does not extend to $x \to \infty$, e^{kx} is fine. However Φ is symmetric w.r.t x.

$\therefore \Phi(-x,y) = \Phi(x,y) \Rightarrow Ae^{-kx} + Be^{kx} = Ae^{kx} + Be^{-kx}$

$\Rightarrow (A-B)(e^{-kx} - e^{kx}) = 0$

$\therefore A = B$. 1

$\therefore \Phi(x,y) = A(e^{kx} + e^{-kx})(C\sin ky + D\cos ky)$

$= \cosh kx\,(C'\sin ky + D'\cos ky)$.

where $C' = 2AC$, $D' = 2AD$.

B.C.(i) $\Rightarrow$ $0 = D'\cosh kx \Rightarrow D' = 0$.

B.C.(ii) $\Rightarrow$ $0 = C'\cosh kx \sin ka \Rightarrow \sin ka = 0$.

$\Rightarrow k = \frac{n\pi}{a}$, $n = 1, 2, \ldots$ 1

$\therefore \Phi(x,y) = c'\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$.

General sol^n: $\Phi(x,y) = \sum_{n=1}^{\infty} C_n' \cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$

Pick C_n such that, B.C.(iii) is satisfied

i.e. $\Phi(b,y) = \Phi_0 = \sum_{n=1}^{\infty} C_n' \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$.

Multiplying both sides by $\sin\left(\frac{n'\pi y}{a}\right)$ and integrating from 0 to a,

$$\sum_{n=1}^{\infty} C_n \left[\underbrace{\int_0^a dy \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)}_{\frac{a}{2}\delta_{n',n}}\right] \cosh\left(\frac{n\pi b}{a}\right) = \int_0^a dy\, \Phi_0 \sin\left(\frac{n'\pi y}{a}\right)$$

$$= \Phi_0 \int_0^a \sin\left(\frac{n'\pi y}{a}\right) dy$$

$$= -\Phi_0 \frac{a}{n'\pi} \cos\left(\frac{n'\pi y}{a}\right)\Big|_0^a$$

$$\Rightarrow \frac{a}{2}\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\delta_{n'n} = \frac{\Phi_0 a}{n'\pi}(1-\cos n\pi)$$

$$\Rightarrow C_{n'} \cosh\left(\frac{n'\pi b}{a}\right) = \frac{2\Phi_0}{n'\pi}(1-\cos n\pi) = \begin{cases} 0 & n' = \text{even} \\ \frac{4\Phi_0}{n'\pi} & n' = \text{odd.}\end{cases}$$

1

$$\therefore \Phi(x,y) = \frac{4\Phi_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \cdot \sin\left(\frac{n\pi y}{a}\right).$$

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2. Potential on the sphere is zero.

Then, $\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}')\, G_D(\vec{r},\vec{r}')\, d^3r'$

$\rho(\vec{r}') = Q\,\delta(\vec{r}' + a\hat{k}) - Q\,\delta(\vec{r}' - a\hat{k})$

$\therefore \Phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\int_V \left[\delta(\vec{r}' + a\hat{k}) - \delta(\vec{r}' - a\hat{k})\right] G_D(\vec{r},\vec{r}')\, d^3r'$

$= \frac{Q}{4\pi\epsilon_0}\left[G_D(\vec{r}, -a\hat{k}) - G_D(\vec{r}, a\hat{k})\right]$. 1

Now, $G_D(\vec{r},\vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R/r'}{\left|\vec{r} - \frac{R^2}{r'^2}\vec{r}'\right|}$

$= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{R^2} + R^2 - 2rr'\cos\gamma}}$

where γ is the angle between $\vec{r}$ & $\vec{r}'$.

Here, $\cos\gamma = \cos\theta$.

$\therefore G_D(\vec{r}, a\hat{k}) = \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{1}{\sqrt{\frac{r^2a^2}{R^2} + R^2 - 2ar\cos\theta}}$

$= \frac{1}{a}\frac{1}{\sqrt{1 + \frac{r^2}{a^2} - \frac{2r\cos\theta}{a}}} - \frac{R}{ra\sqrt{1 + \frac{R^4}{a^2r^2} - \frac{2R^2\cos\theta}{ar}}}$ 1

Similarly,

$G_D(\vec{r}, -a\hat{k}) = \frac{1}{a\sqrt{1 + \frac{r^2}{a^2} + \frac{2r\cos\theta}{a}}} - \frac{R}{ra\sqrt{1 + \frac{R^4}{a^2r^2} + \frac{2R^2\cos\theta}{ar}}}$ 1

When $a \gg R$, i.e. the sources are at infinity,

$G_D(\vec{r}, a\hat{k}) \simeq \frac{1}{a}\left(1 + \frac{r}{a}\cos\theta - \frac{r^2}{2a^2}\right) - \frac{R}{ar}\left(1 + \frac{R^2}{ar}\cos\theta - \frac{R^4}{2a^2r^2}\right)$

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& $G_D(\vec{r}, -R\hat{n}) \simeq \frac{1}{a}\left(1 - \frac{r}{a}\cos\theta + \frac{r^2}{2a^2}\right) - \frac{R}{ar}\left(1 \mp \frac{R^2}{ar}\cos\theta - \frac{R^4}{2a^2r^2}\right)$

$$\therefore \Phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}\left(1 - \frac{r}{a}\cos\theta + \frac{r^2}{2a^2}\right) - \frac{R}{ar}\left(1 \mp \frac{R^2}{ar}\cos\theta - \frac{R^4}{2a^2r^2}\right) - \frac{1}{a}\left(1 + \frac{r}{a}\cos\theta - \frac{r^2}{2a^2}\right) + \frac{R}{ar}\left(1 \pm \frac{R^2}{ar}\cos\theta - \frac{R^4}{2a^2r^2}\right)\right]$$

1

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{2}{a}\cdot\left(\frac{r^2}{2a^2} - \frac{r}{a}\cos\theta\right) + \frac{2R}{ar}\left(-\frac{R^4}{2a^2r^2} + \frac{R^2}{ar}\cos\theta\right)\right]$$

For large 'a', we can further approximate this:

$$\Phi(\vec{r}) \simeq \frac{Q}{4\pi\epsilon_0}\left[-\frac{2}{a}\cdot\frac{r}{a}\cos\theta + \frac{2R}{ar}\cdot\frac{R^2}{ar}\cos\theta\right]$$

1

$$= -\left(\frac{2Q}{4\pi\epsilon_0 a^2}\right) r\cos\theta + \left(\frac{2Q}{4\pi\epsilon_0 a^2}\right)\cdot\frac{R^3}{r^2}\cos\theta.$$

The uniform field produced by the external charges kept far apart is, $E_0 = \frac{2Q}{4\pi\epsilon_0 a^2}$.

$$\therefore \Phi(\vec{r}) = \underbrace{-E_0 r\cos\theta}_{\text{potential due to uniform field } E_0} + \underbrace{E_0\frac{a^3}{r^2}\cos\theta}_{\text{Potential due to induced surface charge}}.$$

Surface charge density,

$$\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = \epsilon_0 E_0\left(1 + \frac{2R^3}{r^3}\right)\Bigg|_{r=R}\cos\theta$$

$$= \epsilon_0 E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta$$

$$= 3\epsilon_0 E_0 \cos\theta.$$