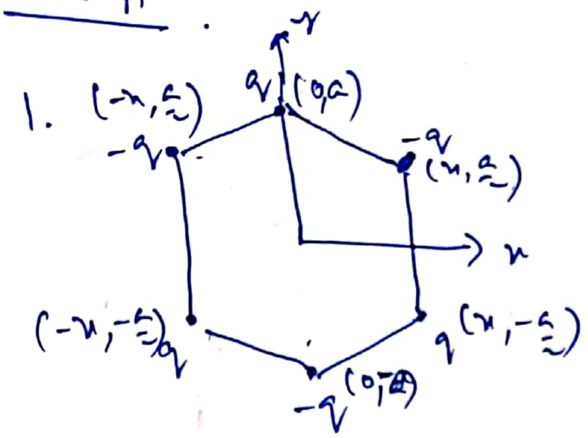
[...], motility induced phase separation [24-27] etc. In recent years, the

Part A

1.



$Q = \sum q_i = q - q + q - q + q - q$

$= 0.$

Dipole component,

$p_z = 0$ since $z = 0$ for all the charges.

$p_x = \sum_i q_i x_i = 0 - qx + qx + 0 - qx + qx = 0$

$p_y = \sum_i q_i y_i = qa - q\frac{a}{2} - q\frac{a}{2} + qa - q\frac{a}{2} - q\frac{a}{2} = 0$

2. $\vec{F} = (p \cdot \vec{\nabla})\vec{E}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$F_x = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0}\left[p_x\left\{\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right\} + p_y\left\{-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right\} + p_z\left\{-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right\}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{p}}{r^3} - \frac{3\vec{r}(\vec{p}\cdot\vec{r})}{r^5}\right]_x.$$

$$\therefore \vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\vec{p} - 3(\vec{p}\cdot\hat{r})\hat{r}\right].$$

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3. $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d^3r'$

$\vec{r}-\vec{r}' = \vec{\mathscr{r}}$, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{M}(\vec{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d^3r'$

$= \frac{\mu_0}{4\pi}\int \vec{M}(\vec{r}') \times \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d^3r'$

$\vec{\nabla}\times(f\vec{Q}) = f(\vec{\nabla}\times\vec{Q}) - \vec{Q}\times\vec{\nabla}f$

$\therefore \quad \vec{Q}\times\vec{\nabla}f = f(\vec{\nabla}\times\vec{Q}) - \vec{\nabla}\times(f\vec{Q})$.

$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}\left[\vec{\nabla}'\times\vec{M}(\vec{r}')\right] d^3r' - \int \vec{\nabla}'\times\left(\frac{\vec{M}(\vec{r}')}{\mathscr{r}}\right) d^3r'\right\}$

Now, $\int_V (\vec{\nabla}\times Q)\, d\tau = -\oint_S \vec{Q}\times d\vec{a}$.

$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{1}{\mathscr{r}}\left[\vec{\nabla}'\times\vec{M}(\vec{r}')\right] d^3r' + \frac{\mu_0}{4\pi}\oint_S \frac{1}{\mathscr{r}}\left[\vec{M}(\vec{r}')\times d\vec{a}'\right]$

$= \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_b(\vec{r}')}{\mathscr{r}} d^3r' + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{K}_b(\vec{r}')}{\mathscr{r}} da'$

where, $\vec{J}_b(\vec{r}) = \vec{\nabla}\times\vec{M}$ ≡ volume current

& $\vec{K}_b(\vec{r}) = \vec{M}\times\hat{n}$ ≡ surface current.

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Part B.

1. Multipole moments, $q_{lm} = \int Y^{*}_{lm}(\theta', \phi') \, r'^{l} \rho(\vec{r}') \, d^3 r'$,

For the uniformly charged disk,

$$\rho(\vec{r}) = \frac{\sigma}{r} \delta(\cos\theta) \, \Theta(R - r)$$

Check that, $\int \rho(\vec{r}) \, d^3 r' = 4\pi R^2 \sigma = Q$

$$\therefore q_{lm} = \sigma \int_{-1}^{1} d(\cos\theta') \, \delta(\cos\theta') \int_0^{2\pi} d\phi' \, Y^{*}_{lm}(\theta', \phi') \times \int_0^{\infty} dr \, r'^{l+1} \, \Theta(R - r)$$

Now, $$\frac{1}{2\pi} \int_0^{2\pi} d\phi \, Y^{*}_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta) \, \delta_{m,0}$$

$$\therefore q_{lm} = \sigma \frac{R^{l+2}}{l+2} \, 2\pi \sqrt{\frac{2l+1}{4\pi}} \, P_l(0) \, \delta_{m,0}$$

Now, $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q^{*}_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

$$= \frac{2\pi \sigma R^{l+2}}{2 \cdot 4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} \, \frac{P_l(0) \, Y_{lm}(\theta, \phi)}{(l+2) \, r^{l+1}} \delta_{m,0}$$

$$= \frac{(\sigma \cdot 4\pi R^2) \cdot R^l}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2l+1}} \, \frac{P_l(0) \, Y_{l0}(\theta, \phi)}{(l+2) \, r^{l+1}}$$

$$= \frac{Q}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^{l} \frac{2}{2(l+2)} \, P_l(0) \, P_l(\cos\theta)$$

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2. $\rho(r,\theta) = \rho_0 \frac{R}{r^2}(R-r)\sin^2\theta.$

$= \rho_0 \frac{R}{r^2}(R-r)(1-\cos^2\theta)$

$P_1(\cos\theta) = \cos\theta,\quad P_0(\cos\theta) = 1,\quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

$\cos^2\theta = \frac{2P_2+1}{3} = \frac{2P_2}{3} + \frac{P_0}{3}$

$\therefore \rho(r,\theta) = \rho_0 \frac{R}{r^2}(R-r)\left[P_0(\cos\theta) - \frac{2}{3}P_2(\cos\theta) - \frac{1}{3}P_0(\cos\theta)\right]$

$= \rho_0 \frac{R}{r^2}(R-r)\frac{2}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right]$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3r'$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$\therefore$ Potential inside the sphere, $r_< =$

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)$$

$$\times\int\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')\,\rho(\vec{r}\,')\,d^3r'$$

Now, $P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\,Y_{l0}(\theta,\phi).$

$$\therefore \rho(r,\theta) = \frac{2\rho_0 R}{3r^2}(R-r)\left[\sqrt{4\pi}\,Y_{00}(\theta,\phi) - \sqrt{\frac{4\pi}{5}}\,Y_{20}(\theta,\phi)\right]$$

$$\therefore \Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)$$

$$\times\frac{3\rho_0 R}{2}\int (R-r')\frac{r_<^l}{r_>^{l+1}}\frac{r'^2}{r'^2}\,dr'$$

$$\therefore \; \Phi(r,\theta) = \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \cdot \frac{2\rho_0 R}{3} \int (R-r') \frac{r_<^l}{r_>^{l+1}} \frac{r'^2}{r'^2} dr' \ldots$$

$$\times \int \left[\sqrt{4\pi}\, Y_{00} Y^*_{lm} - \sqrt{\frac{4\pi}{5}}\, Y_{20} Y^*_{lm} \right] d\Omega'$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta\phi) \frac{2\rho_0 R}{3} \int (R-r') \frac{r_<^l}{r_>^{l+1}} dr'$$

$$\times \left[\sqrt{4\pi}\, \delta_{l0}\delta_{m0} - \sqrt{\frac{4\pi}{5}}\, \delta_{l2}\delta_{m0} \right].$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \int_0^R dr' \frac{2\rho_0 R}{3} \sqrt{4\pi} \cdot 4\pi (R-r')$$

$$\times \left[\frac{1}{r_>} Y_{00}(\theta\phi) - \frac{1}{5} \frac{r_<^2}{r_>^3} \frac{1}{\sqrt{5}} Y_{20}(\theta\phi) \right]$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0} \int_0^R dr' \frac{2}{3}\rho_0 R \cdot \cancel{4\pi} (R-r')$$

$$\times \left[\frac{1}{r_>} \underbrace{\sqrt{4\pi}\, Y_{00}(\theta\phi)}_{P_0(\cos\theta)} - \frac{1}{5} \frac{r_<^2}{r_>^3} \underbrace{\sqrt{\frac{4\pi}{5}}\, Y_{20}}_{P_2(\cos\theta)} \right]$$

$$= \frac{1}{\epsilon_0} \frac{2}{3} \rho_0 R \int_0^R (R-r') \left[\frac{1}{r_>} P_0(\cos\theta) - \frac{1}{5} \frac{r_<^2}{r_>^3} P_2(\cos\theta) \right]$$

Integration to be split into 2 parts:

r' goes from 0 to r with $r_< = r'$, $r_> = r$

r' - - - - - - r to R ... $r_< = r$, $r_> = r'$

$$\int_0^R dr' \frac{R-r'}{r_>} = \int_0^r dr' \frac{R-r'}{r} + \int_r^R dr' \frac{R-r'}{r'}$$

$$= \frac{1}{r}\left(Rr - \frac{r^2}{2}\right) + R\ln\left(\frac{R}{r}\right) - (R-r)$$

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$$\int_0^R dr'(R-r')\frac{r_<^2}{r_>^3} = \int_0^r dr' \frac{(R-r')r'^2}{r^3} + \int_r^R dr' \frac{(R-r')r^2}{r'^3}$$

$$= \frac{1}{r^3}\left(\frac{Rr^3}{3}-\frac{r^4}{4}\right) + r^2\left[-\frac{2}{R}+\frac{2R}{r^2}+\frac{1}{R}-\frac{1}{r}\right]$$

$$\therefore \Phi(r,\theta) = \frac{\rho_0 R}{3\epsilon_0}\left[\left\{\frac{1}{2r}\left(Rr-\frac{r^2}{2}\right)+R\ln\frac{R}{r}-(R-r)\right\}P_0(\cos\theta)\right.$$

$$-\frac{1}{5}\left\{\frac{1}{r^3}\left(\frac{Rr^3}{3}-\frac{r^4}{4}\right)\right.$$

$$\left.\left.+\,r^2\left(-\frac{2}{R}+\frac{2R}{r^2}+\frac{1}{R}-\frac{1}{r}\right)\right\}P_2(\cos\theta)\right]$$