Newton's laws
Inertial Frames and all that ...

* In a moving vehicle the trees appear to be moving backwards. To a kid it seems perplexing

- Because trees do not move (Our common sense)

* On a train on a platform, sometimes it becomes difficult to ascertain whether our train is moving or the adjacent one.

- Because both trains can move
(our common sense)

In the first case, the kid/we see that trees do not move backwards as long as train is not moving. If he/she/we see after a while that trees have started moving backwards then

→ Why do we correct the kid/us that trees are not moving backwards but it is we that are moving forward?

Lesson: (i) Motion is relative. If trees could move, we would not have been so sure to correct the kid. Just like for the train we don't!

(ii) So if trees do not move, then common sense suggests that non-moving objects can only move when a force is applied on them.

Newton's First law

★ There is a frame where "common sense" works !

Common Sense : If no force is applied then static things do not move and moving things do not stop / slow down.

Refinement : Common sense :

If no force is applied then things maintain their velocities.

⇒ Such a special frame is called Inertial Frame : where particles maintain inertia.

★ This is just a statement of Cause and effect

Acceleration is due to force
(Effect) (cause)

Corollary : In inertial frame force (only) causes acceleration.

But how much of acceleration due to an applied force is Newton's second law which quantifies.

★ Newton's second law

In the places of common sense, things move on application of force according to

$$m\vec{a} = \vec{F}$$

or

$$m\frac{d\vec{v}}{dt} = \vec{F}$$

clearly in this place if we do not apply any force $\frac{d\vec{v}}{dt} = 0$. This is inertial frame.

★ Second law is statement about how force commands velocity in an inertial frame.

Corollary : There can be frames in which

$m\vec{a} \neq \vec{F}$!! Those are non-inertial frames.

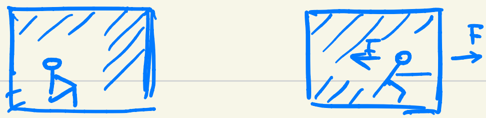
Easy Example : A boy sitting inside an accelerating vehicle. He finds everything around him accelerating for no reason !!

Third law : What common sense means

- How do we know the body exerting a force receives any opposite force by the body acted upon by it ?

- How do we know it is equal in magnitude? Why not twice or half ?

What is happening inside is not visible



A

B

The box does not start moving on its own.
That means if a force $\vec{F}$ is applied on the box wall, the person should get a backword force to keep the net system (Which here means) **the center of mass** not accelerating !

↳ Common Sense : Things unattended to (left alone) maintain their velocities.

For interacting particles,

If $m_1\ddot{x}_1 + m_2\ddot{x}_2 \neq 0$ $\left\{ \begin{array}{l} m_1\ddot{x}_1 = f_{12} \\ m\ddot{x}_2 = f_{21} \end{array} \right.$

$= (f_{12} + f_{21})$

$\Rightarrow$

$$\Rightarrow (m_1+m_2)\frac{d^2}{dt^2}\left[\frac{m_1 x_1 + m_2 x_2}{m_1+m_2}\right] \neq 0$$

$$\Rightarrow (m_1+m_2)\ddot{R}_{cm} \neq 0$$

That means purely due to internal forces the system's R_{cm} would have changed.

$\Rightarrow$ A cube put at somewhere would start moving on its own $\rightarrow$ i.e. no <u>external</u> force required to move.

★ If that does not happen, then

$$f_{12} = -f_{21}$$

$\rightarrow$ This is what we see and hence there is an empirical evidence for equal and opposite force.

★ Life in inertial Frames

Business as Usual

- Draw force diagram for each body
- Set up a co-ordinate system on all of the bodies (as per your convenience)
- Write down EOMs for all of them.
- Write constraint eqns.
- Solve the equations

Always remember: Newton's laws fail for non-inertial frames, massless particle, fast moving particles even in inertial frames and tiny particles

★ Life in non-inertial frames

- Newton's laws do not hold
- One should write applicable laws

Illustrative Example : A particle / Rocket is moving due to application of a force $\vec{f} = m\alpha\,\hat{i}$.

For an inertial observer sitting at $x=0, y=0, z=0$ its acceleration is

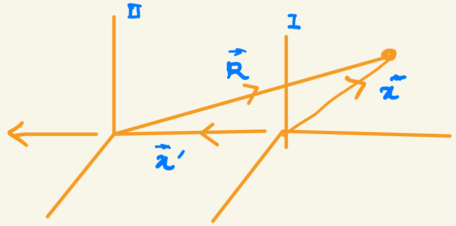
$$m\frac{d^2\vec{x}}{dt^2} = \vec{f} = m\alpha\,\hat{i}$$

$$\Rightarrow \vec{x} = \frac{\alpha t^2}{2}\hat{i} \Rightarrow \vec{V} = \alpha t\,\hat{i}$$

For another observer moving w.r.t. the static observer according to

$$\vec{x}\,' = -\frac{\beta t^3}{6}\hat{i}$$

$$\vec{V}\,' = -\frac{\beta t^2}{3}\hat{i}$$



In this new frame, the location of particle

$$\vec{R} = \vec{x} - \vec{x}\,' = \left(\frac{\alpha t^2}{2} + \frac{\beta t^3}{6}\right)\hat{i}$$

Hence acceleration of particle

$$\vec{A} = \frac{d^2\vec{R}}{dt^2} = (\alpha + \beta t)\,\hat{i}$$

$$m\vec{A} = m\alpha\,\hat{i} + m\beta t\,\hat{i}$$

In terms of observables on the particle

$$\boxed{m\vec{A} = \vec{f} + m(3\beta|V'|)^{1/2}\,\hat{i}}$$

OR

In terms of properties of frame

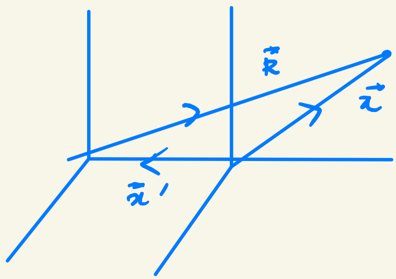
$$\boxed{m\vec{A} = \vec{f} + m\vec{a}\,'}$$

Force law in the non-inertial frame.

★ A fictious force has to be added which is different for different frames.

From a uniform accelerating frame

If $\vec{x} = -\frac{\beta t^2}{2}\hat{i}$



Then,

$$\vec{A} = \frac{d^2\vec{R}}{dt^2} = (\alpha + \beta)\hat{i}$$

$$\Rightarrow \boxed{\begin{aligned} m\vec{A} &= m\alpha\,\hat{i} + m\beta\,\hat{i} \\ m\vec{A} &= f\,\hat{i} + f_{\text{fictious}}\,\hat{i} \end{aligned}}$$

The Newton's law is corrected by a fictious / psuedo force.

▶ Be Cautious! – How do you know if you are in a not inertial frame or a force is acting? You can put f_{fiction} only if you know your own acceleration.

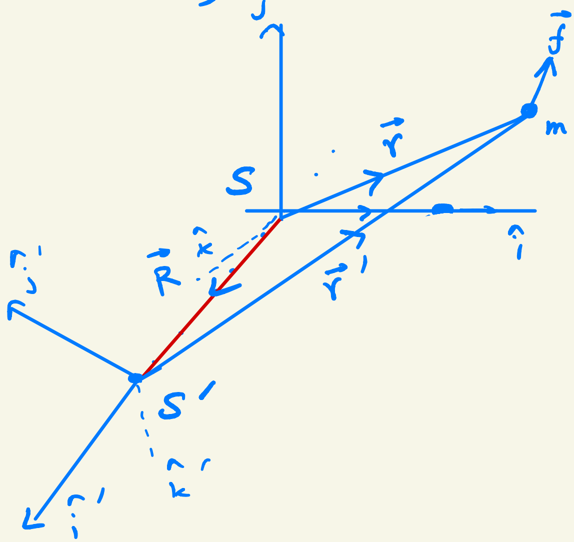
In such frames there is acceleration even when a body is not acted upon by any force ($\vec{f} = 0$).

▶ $\frac{d^2\vec{R}}{dt^2}$ will not always be $\vec{A}$ (as seen by non-inertial observer) if acceleration is not uniform

In 3-D (Uniform acceleration)

In the inertial frame

$$m \frac{d^2\vec{r}}{dt^2} = \vec{f}$$



The center of non-inertial frame S'

is a vector $\vec{R}(t)$

at time t

$$\vec{R}(t) + \vec{r}\,'(t) = \vec{r}(t)$$

Then $\vec{r}(t) - \vec{R}(t) = \vec{r}\,'(t)$

$$\therefore \; m \frac{d^2\vec{r}\,'}{dt^2} = m\left(\frac{d^2\vec{r}}{dt^2} - \frac{d^2\vec{R}}{dt^2}\right)$$

$$= m\left(\frac{\vec{f}}{m} - \vec{a}_{frame}\right)$$

If $\vec{a}_{frame} = 0$, Accelerations agree

$\vec{a}_{frame} = \frac{\vec{f}}{m}$, particle is uniformly moving.

* One can go to a particular frame where effect of force vanishes.

Equivalence principle :

For $\vec{F} = m\vec{g}$

you should be moving into downward falling frame with acceleration $\vec{g}$ where effect of gravity will vanish.

For electorns attracted by $\vec{E}$ you need to get to a frame with acceleration $\vec{a}^{\,e}_{frame} = -\frac{e\vec{E}}{m_e}$ in which effect of electric force does not bother you.

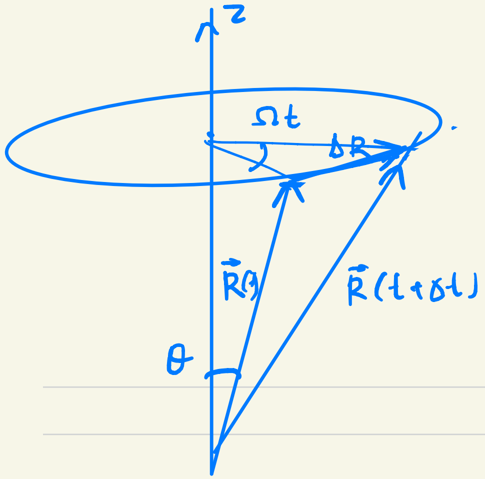
For protons $\vec{a}^{\,p}_{frame} = +\frac{eE}{m_p} \simeq -\frac{\vec{a}^{\,e}_{fram}}{2000}$

★ For gravitational force : Going to $\vec{a}_{frame} = \vec{g}$ eliminates gravity for all particles simultaneously.

Non uniform acceleration : Rotating frames

Coriolis force : Extra Force in a rotating frame

(A) A rotating vector



→ Think of vector $\vec{R}$ (pointing anywhere) which is rotating @ Ω about the z-axis

• At time t the vector is $\vec{R}(t)$

→ At time $t + \Delta t$ the vector is $\vec{R}(t+\Delta t)$

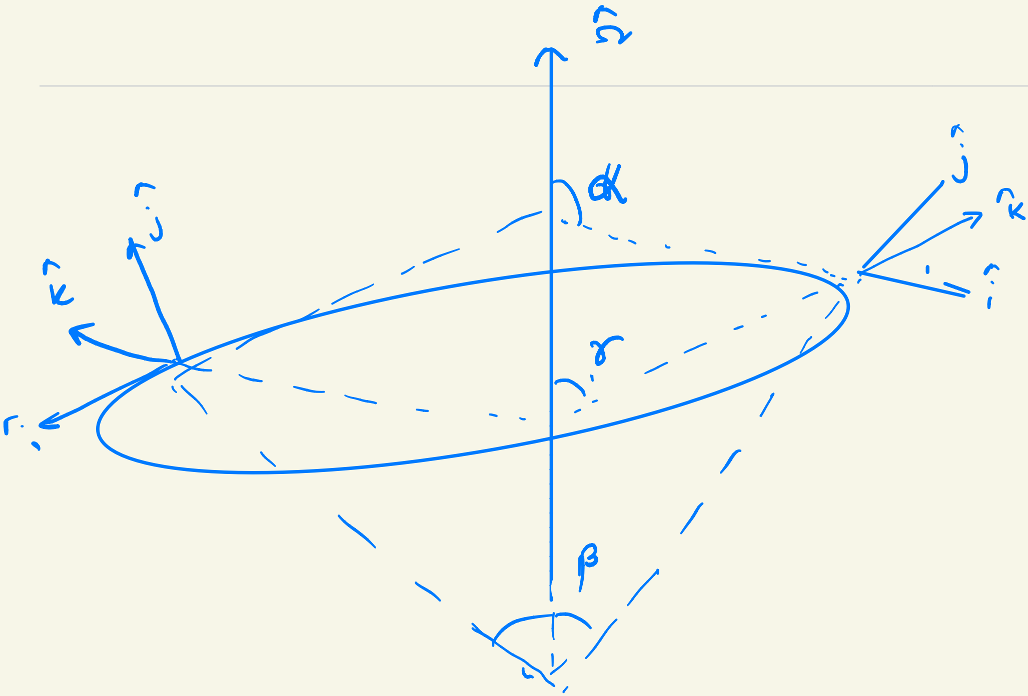
$$\Delta \vec{R} = \vec{R}(t+\Delta t) + \vec{R}(t)$$

For small Δt , $\Delta \vec{R} \sim R \sin\theta \ \hat{\phi}(t)$

$$\therefore \quad \frac{\Delta \vec{R}}{\Delta t} = |\vec{R} \sin\theta| \, \Omega \ \hat{\phi}(t)$$

$$\lim_{\Delta t \to 0} \frac{\Delta R}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t} = \lim_{\Delta t \to 0} R \sin\theta \, \Omega \, \hat{\phi}(t)$$

$$\frac{d\vec{R}}{dt} = \underset{\hat{\phi}}{R \sin\theta \ \Omega} = \vec{\Omega} \times \vec{R}$$



★ $\hat{i}$, $\hat{j}$, $\hat{k}$ can be seen to be rotating at angle α, β and γ respectively.

$$\therefore \left|\frac{d\hat{i}}{dt}\right| = |\Omega \sin\alpha| \quad , \quad \frac{d\hat{k}}{dt} = |\Omega \sin\gamma|$$

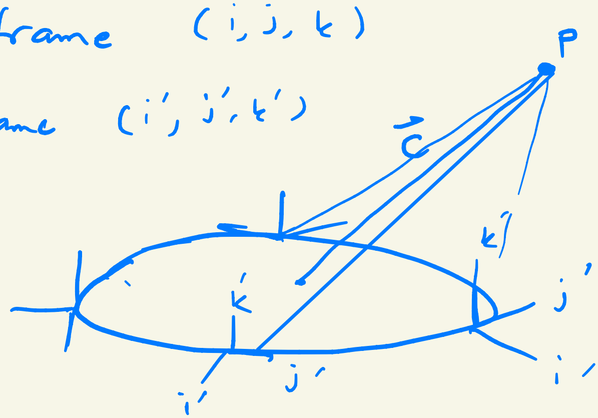
$$\left|\frac{d\hat{j}}{dt}\right| = |\Omega \sin\beta|$$

$$\frac{d(\hat{i}, \hat{j}, \hat{k})}{dt} = \vec{\Omega} \times (\hat{i}, \hat{j}, \hat{k})$$

(B) In a rotating co-ordinate system

S : Inertial frame (i, j, k)

S' : Rotating frame (i', j', k')



Let a particle P has position vector

$$\vec{c} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k} \quad \text{in } S$$

$$\left(\frac{d\vec{c}}{dt}\right) = \dot{c}_x \hat{i} + \dot{c}_y \hat{j} + \dot{c}_z \hat{k}$$

In S'

$$\vec{c} = c_x' \hat{i}' + c_y' \hat{j}' + c_z' \hat{k}'$$

$$\left(\frac{d\vec{c}}{dt}\right) = \underbrace{\dot{c}_x' \hat{i}' + \dot{c}_y' \hat{j}' + \dot{c}_z' \hat{k}'}_{} + c_x' \dot{\hat{i}}' + c_y' \dot{\hat{j}}' + c_z' \dot{\hat{k}}'$$

$$\left(\frac{d\vec{c}}{dt}\right)_{S'}$$

But $\frac{d\hat{i}'}{dt} = \vec{\Omega} \times \hat{i}'$, $\frac{d\hat{j}'}{dt} = \Omega \times \hat{j}'$

$$\frac{d\hat{k}'}{dt} = \vec{\Omega} \times \hat{k}'$$

$$\therefore \left(c_x' \dot{\hat{i}}' + c_y' \dot{\hat{j}}' + c_z' \dot{\hat{k}}' \right)$$

$$= \vec{\Omega} \times \left(c_x' \hat{i}' + c_y' \hat{j}' + c_z' \hat{k}' \right)$$

$$= \vec{\Omega} \times \vec{c}$$

$$\therefore \left(\frac{d\vec{c}}{dt} \right) = \left(\frac{d\vec{c}}{dt} \right)_{s'} + \vec{\Omega} \times \vec{c} \quad \text{— A}$$

This is change of particle w.r.t. fixed $\hat{i}', \hat{j}', \hat{k}'$ axes.

$$\Rightarrow \left(\frac{d\vec{r}}{dt} \right)_s = \left(\frac{d\vec{r}}{dt} \right)_{s'} + \vec{\Omega} \times \vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt} \left(\vec{V}_{s'} \right) + \vec{\Omega} \times \vec{r}$$

$$= \frac{d}{dt} \left(\vec{v}_{in} \right) + \vec{\Omega} \times v$$

$$\vec{v}_s = \vec{v}_{s'} + \vec{\Omega} \times \vec{r}$$

The relation (A) is true for any vector

$$\left(\frac{d\vec{\square}}{dt}\right)_s = \left(\frac{d\vec{\square}}{dt}\right)_{s'} + (\vec{\Omega} \times \vec{\square})$$

If $\vec{\square} = \vec{v}_s$ then,

$$\left(\frac{d\vec{v}_s}{dt}\right)_s = \left(\frac{d\vec{v}_s}{dt}\right)_{s'} + (\vec{\Omega} \times \vec{v}_s)$$

$$= \left(\frac{d}{dt}(\vec{v}_{s'} + \vec{\Omega} \times \vec{r})\right)_{s'} + \vec{\Omega} \times (\vec{v}_{s'} + \vec{\Omega} \times \vec{r})$$

$$= \left(\frac{d\vec{v}_s}{dt}\right)_{s'} + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_{s'}$$

$$+ \vec{\Omega} \times \vec{v}_{s'} + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

$$\vec{a}_s = \vec{a}_{s'} + 2(\vec{\Omega} \times \vec{v}_{s'}) + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

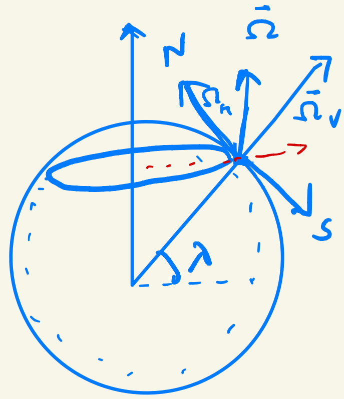
$$\therefore \vec{a}_{rot} = \vec{a}_{in} - 2(\vec{\Omega} \times \vec{v}_{s'}) - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

$$m\vec{a}_{rot} = m\vec{a}_{in} \underbrace{- 2m(\vec{\Omega} \times \vec{v}_{rot})}_{\text{Coriolis force}} - \underbrace{m(\vec{\Omega} \times \vec{\Omega} \times \vec{r})}_{\text{Centrifugal force}}$$

If particle is static in rotating frame, the in addition to $\vec{f}\,(= m\vec{a}_{in})$ it experiences a centrifugal force as well

Motion on Earth

On earth's surface $\vec{\Omega}$ can be broken along the surface $\vec{\Omega}_H$ and vertically up $\vec{\Omega}_V$



Thus, the coriolis force

$$\vec{F} = -2m(\vec{\Omega} \times \vec{v})$$
$$= -2m[(\vec{\Omega}_H + \vec{\Omega}_V) \times \vec{v}]$$
$$= \vec{F}_1 + \vec{F}_2 \qquad \text{where } F_{--} = -2m(\vec{\Omega}_{--} \times \vec{v})$$

Since for a particle moving on earth's surface both $\vec{\Omega}_H$ and $\vec{v}$ are along the tangents $\vec{F}_1$ (if non-zero) will point vertically upwards at any point.

$\vec{F}_2 \sim \vec{\Omega}_V \times \vec{V}$ will be on the surface and $\perp$ to $\vec{v}$

$$|\vec{\Omega}_V| = |\vec{\Omega}|\cos\left(\frac{\pi}{2} - \lambda\right) = |\Omega| \sin\lambda$$

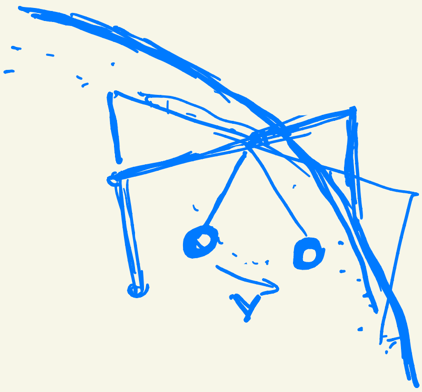
$$\therefore \; |\vec{F}_H| = 2mV\Omega\sin\lambda$$

Compare it with centrifugal

$$\frac{|F_H|}{|F_c|} \sim \frac{2V}{\Omega R}$$

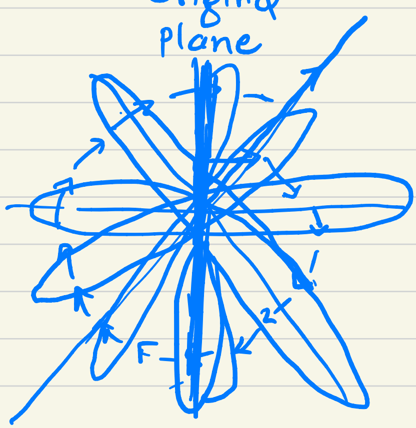
$$\Omega R \sim \frac{2\pi}{86400\text{ s}} \times 6400\text{ km}$$
$$\approx 500\text{ m s}^{-1}$$

Focault Pendulum



For a pendulum initially oscillating in a plane will start feeling a coriolis force perpendicular to its velocity

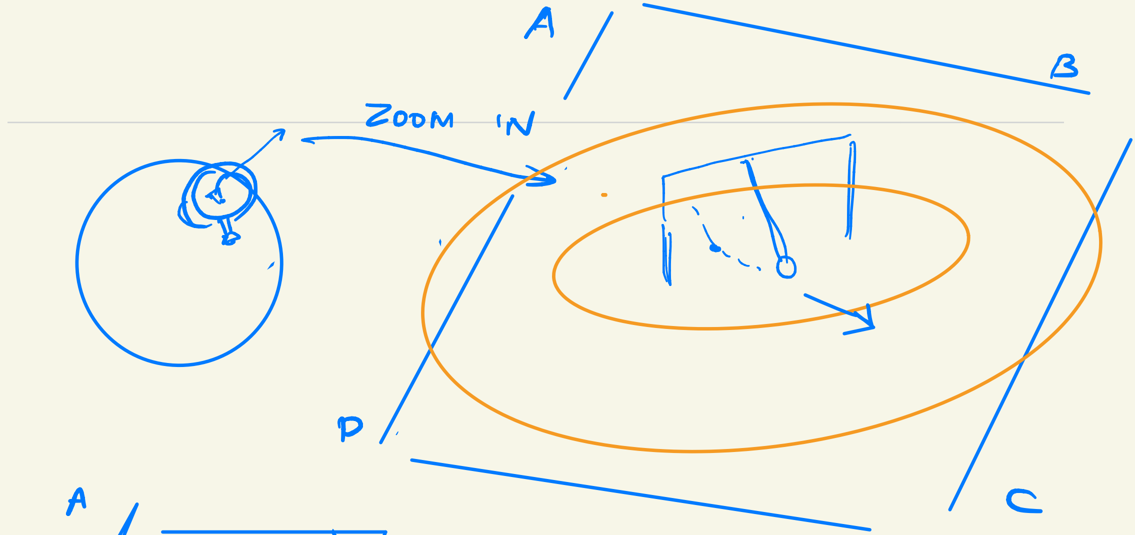
Newtonian Oscillator :
Remains oscillating in one plane

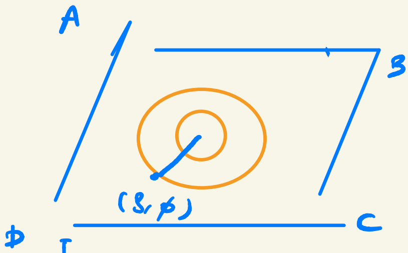


Top View

Coriolis force :
Tries to put a force perpendicular to vertical Ω_v and v , i.e. sideways

Thus the plane changes due to Coriolis force and oscillator precesses.





(Top view)

In the plane if we draw 2-D polar co-ordinates with center at the equilibrium position then the bob looks moving in the 2-D radial direction

★ Location in the plane

$$\vec{\rho} = \rho\hat{\rho}$$

$$\hookrightarrow \vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}$$ (Velocity as seen on earth's plane)

$$\vec{F}_H = -2m\left(\vec{\Omega}_v \times \left(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi}\right)\right)$$

$$= -2m\Omega\sin\lambda\,\dot{\rho}\hat{\rho} + \underbrace{2m\Omega\sin\lambda\,\rho\dot{\phi}\hat{\phi}}$$

Weak and in the $\perp$
direction to oscillation plane

Force in the angular direction

$$m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) = (F_H)_{\perp \text{ to plane}} = -2m\Omega\sin\lambda\,\dot{\rho}$$

$\therefore \quad \rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = -2\Omega \sin\lambda \, \dot{\rho}$

$\dot{\phi}$ is small, $\ddot{\phi}$ is way to smaller (otherwise $\dot{\phi}$ would have become larger) so that we can ignore, i.e. $\dot{\theta}$ is practically a constant

$$\therefore \quad \dot{\phi} \sim -\Omega \sin\lambda$$

If $\dot{\ } \sim$ const that $\dot{\phi} = \frac{2\pi}{T}$

where T is its period

$$T = \frac{2\pi}{|\dot{\phi}|} \Rightarrow T = \frac{2\pi}{\Omega \sin\lambda}$$

$$T = \frac{2\pi}{2\pi} \times \frac{24 \ h}{\sin\lambda} = \frac{24}{\sin\lambda} \ h$$

$$\boxed{\text{At Mohali} \qquad \lambda \sim 30^\circ}$$

$$T = \frac{24}{\sin 30^\circ} \ h \simeq 48 \ h$$

$$\boxed{\text{If a pendulum keeps oscillating for 12h its plane will turn orthogonal !!}}$$