1) Consider the hermitian operators J_x, J_y, J_z, satisfying the relation

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k \; ; \; i,j,k = x,y,z.$$

Also, consider another hermitian operator $J^2 = J_x^2 + J_y^2 + J_z^2$.

Given that $[J^2, J_x] = 0$, there exists a set of common eigenstates $|j,m\rangle$, where

$$J^2 = \hbar^2 j(j+1) |j,m\rangle$$

& $J_x = \hbar m |j,m\rangle$.

Find out the constraints on the values of j and m.

2) a) Consider two independent quantum harmonic oscillators A & B with the annihilation operators $\hat{a}$ & $\hat{b}$ respectively.

Let us define the operators

$$J_z = \alpha\hbar\left(a^\dagger a - b^\dagger b\right),$$

$$J_+ = J_x + iJ_y = \beta\hbar\, a^\dagger b,$$

$$J_- = J_x - iJ_y = \beta\hbar\, a b^\dagger.$$

Now find out the values of α & β for which the following relation holds –

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k$$

where $i,j,k = x,y,z.$

b) With these values of α and β, find out

$$[J^2, J_i], \quad [J^2, J_\pm]$$

where $J^2 = J_x^2 + J_y^2 + J_z^2$.

3) Consider a quantum particle with total angular momentum $j = 1$.

Using the common eigen states of J^2 and J_3, ie, $|j, m\rangle$, find out the matrix representations of

$$J^2, J_\pm, J_1, J_2, J_3.$$

4) Consider a particle with spin $s = \frac{1}{2}$ and orbital angular momentum $l = 1$.

a) Label the eigenstates in the uncoupled basis (ie, common eigenstates of L^2, S^2, L_z, and S_z) by

$$|l\, s\, m_l\, m_s\rangle.$$

b) Label the states in coupled basis (ie, eigen states of J^2, J_z) by $|j, m\rangle$.

Here, $J = L + S$.

c) Find the state with maximum j and $m \left(= j_{max}\right)$ in terms of $|l\, s\, m_l\, m_s\rangle$ states.

d) Use $J_- = L_- + S_-$ to generate all states $|j_{max}, m\rangle$.

e) Use J_- to generate all states $|j_{max} - 1, m\rangle$.

f) What are the expectation values of L_z & S_z with the state $|j = \frac{1}{2}, m = \frac{1}{2}\rangle$?