# MTH201-1

## 1. The Real Number system

Some idea about the real number system and its properties.

Integers, denoted by $\mathbb{Z}$: set with operations $+, -, \times$. For $a, b \in \mathbb{Z}$ $a \geqslant b$ if and only if $a - b \geqslant 0$.

Positive integers are denoted by $\mathbb{N}$.

Rational Numbers, denoted by $\mathbb{Q}$: set with operations $+, -, \times, \div$. For $a, b \in \mathbb{Q}$, $a \geqslant b$ if and only if $a - b \geqslant 0$. New things emerge if we combine the operations with the rational numbers.

We denote the set of real numbers by $\mathbb{R}$. Recall a proof of the following theorem.

**Theorem 1.** *$\sqrt{2} \notin \mathbb{Q}$.*

The set of real numbers $\mathbb{R}$ has the following fundamental properties:

**Algebraic properties:** closed under addition and multiplication, commutative law, associative law and distributive property.

**Order properties:** for any two real numbers $a, b$, $a \geqslant b$ if and only if $a - b \geqslant 0$. Following are some of the properties:

(i) $a \geqslant b \implies a + c \geqslant b + c$
(ii) $a \geqslant b$ and $c \geqslant 0 \implies ac \geqslant bc$
(iii) $a \geqslant b, c \leqslant 0 \implies ac \leqslant bc$

Recall that, for all $x \in \mathbb{R}$:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$$

$|x|$ is called the absolute value of $x \in \mathbb{R}$. Following are some properties of the absolute value:

(i) $|a| = 0 \iff a = 0$
(ii) $a \leqslant |a|$
(iii) $|-a| = |a|$, for all $a \in \mathbb{R}$
(iv) $|ab| = |a||b|$, for all $a, b \in \mathbb{R}$
(v) if $r \geqslant 0$, then $|a| \leqslant r \iff -r \leqslant a \leqslant r$.
(vi) Triangle inequality: $|x + y| \leqslant |x| + |y|$.

*Proof.* $a \leqslant |a|$, so $-|a| \leqslant a \leqslant |a|$. Similarly, $-|b| \leqslant b \leqslant b$. Adding up, $-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$. Hence, $|a + b| = -(a + b)$ or $(a + b)$, both of which are $\leqslant |a| + |b|$. □

**Order completeness property of $\mathbb{R}$:** Let $A \subset \mathbb{R}$. Then $A$ is *bounded above* if there is a real number $x_0$ such that $a \leqslant x_0$, for every element $a$ of $A$. If there exists a real number $y_0$ such that $y_0 \leqslant a$ for all $a \in A$, then $A$ is said to be *bounded below.* Further, if $A$ has an upper bound, then the least upper bound or the smallest one among all the upper bounds is called the supremeum of $A$, and denoted $\sup(A)$. Similarly, if $A$ has a lower bound, then the greatest lower bound or the biggest one among all the lower bounds is called the infimum of $A$, and denoted $\inf(A)$.

The order completeness property of $\mathbb{R}$ says that any non-empty bounded set of real numbers bounded above has a supremum. Similarly, any non-empty bounded set of real numbers bounded below has an infimum.

**Example 2.** (i) $P = \{\pm\frac{1}{n} : n \in \mathbb{N}\}$ has upper bound 2, 3/2, 4, 5 and many more. It has lower bound -1, -2,-1.5,$\cdots$. $\sup P = 1, \inf P = -1$.

(ii) $\mathbb{Z}$ is a subset of $\mathbb{R}$ with no upper or lower bound.

(iii) $\mathbb{N}$ has lower bound 1, but no upper bound.

**Archimedean Property of $\mathbb{R}$:**

**Proposition 3.** *Let $a > 0, b > 0$. Then there exists $n \in \mathbb{N}$ such that $na \geqslant b$.*

*Proof.* If it fails then the set $S := \{na : n \in \mathbb{N}\}$ is bounded above by $b$. By the order completeness property, this implies that $S$ has a suprememum, say $s_0$. Then, there exists $s_0$ such that Let $s_0 = \sup S$. Therefore, $s_0 - a$ is not an upper bound of $S$. This means that there is an element in $S$ that is bigger than $s_0 - a$, i.e., there is a natural number $n_0$, such that $s_0 - a \leqslant n_0 a$. This means that $s_0 \leqslant (n_0 + 1)a$. This is a contradiction as $s_0$, being an upper bound is greater than or equal to every element of $S$. This contradiction proves the theorem. □

**Proposition 4.** *Let $a \in \mathbb{R}$ and $a \geqslant 0$. If $a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.*

*Proof.* Suppose $a \neq 0$. Then $a > 0$. Therefore $a/2 > 0$, and since $a < \varepsilon$ for every postive $\varepsilon$, in particular $a < a/2$. Hence $a/2 < 0$ and $a < 0$. This contradicts that $a \geqslant 0$. This contradiction shows that our assumption is false and hence $a = 0$. □

**Definition 5.** Let $A, B$ be two sets. Then to every element of $A$, we can associate an element of $B$. For each element $a$ of $A$, we denote the element associated to $B$ by $f(a)$. ( So far, more than two elements of $B$ may be associated to a single element of $A$.) Then this association is called a function if

(i) for every $a \in A$, we have $f(a) \in B$,
(ii) If $x = y$ in $A$, then $f(x) = f(y)$ in $B$.

We then write that $f$ is a function from $A$ to $B$. If any of these conditions fail, then we say that the function is not well-defined.

Recall for yourself injective, surjective, bijective functions.

Also recall for yourself relations between two sets, as well as reflexive, symmetric and transitive relations, and equivalence relations and equivalence classes.