MTH201-5

1. Integration

Definition 1. For any closed interval $[a,b]$, a partition P is a finite subset of $[a,b]$ that includes the points a,b. In other words, $P = \{t_0, t_1, \cdots, t_{n-1}, t_n\}$, where $t_0 = a, t_n = b$.

Example 2. For the closed interval $[0,1]$, the $P_n = \{0, 1/n, 2/n, 3/n, \cdots, n/n\}$ is a partition for each n.

Note that a union of two partitions is a partition. And, if we have partitions P, Q such that $P \subset Q$, then Q is a refinement of P.
For each partition P, we define

- the upper sums by $U(P,f) := \sum_{i=1}^{n} M_i \Delta x_i$, where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$;
- the lower sums by $L(P,f) := \sum_{i=1}^{n} m_i \Delta x_i$, where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$.

If $P \subset Q$, then $U(P,f) \geqslant U(Q,f)$ and $L(P,f) \leqslant L(Q,f)$.
Let $U(f) = \inf\{U(P,f) : P \text{ is a partition}\}$
and $L(f) = \sup\{L(P,f) : P \text{ is a partition}\}$.

Remark 3. The supremum and infimum may be taken over P_n for all n. If the integral exists, then the integral over $[a,b]$ is nothing but $\lim_{n\to\infty} U(P_n,f) = \lim_{n\to\infty} L(P_n,f)$.

Definition 4. A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to said to be integrable on $[a,b]$, if

$$U(f) = L(f).$$

The value of this limit is written as $\int_a^b f(x)dx$, and is called the integral of f from a to b.

Example 5. (i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x$.
Let $P : \{x_0, x_1, \cdots, x_n\}, x_0 = 1, x_n = 2$ be any partition of $[1,2]$.
$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$, where $M_i = \sup\{x : x \in x_{i-1} \leqslant x \leqslant x_i\} = x_i$.
$\therefore U(P,f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i x_{i-1}$
$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$, where $m_i = \inf\{x : x \in x_{i-1} \leqslant x \leqslant x_i\} = x_{i-1}$.
$\therefore L(P,f) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) = \sum_{i=1}^{n} x_{i-1}x_i - \sum_{i=1}^{n} x_{i-1}^2$
$U(P,f) - L(P,f) = \sum_{i=1}^{n}(x_i - x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1})^2$.
For any $n \in \mathbb{N}$, choose the partition P_n such that $\Delta x_i \leqslant 1/n$.
Then $U(P_n,f) - L(P_n,f) \leqslant \sum_{i=1}^{n} 1/n^2 = n/n^2 = 1/n$.
$\therefore$, $\inf U(P_n,f) - \sup L(P,f) < U(P_n,f) - L(P_n,f) < 1/n, \forall n \in \mathbb{N}$.
Hence, $\int_1^2 x dx$ exists.
To calculate the actual integral, we take the partitions
$P_n = \{1 + 1/n, 1 + 2/n, 1 + 3/n, \cdots, 1 + n/n\}$.

Then $U(P_n, f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) = \sum_{i=1}^{n}(1 + \dfrac{i+1}{n})\dfrac{1}{n} = \dfrac{(n-1)(3n+2)}{2n^2} = \dfrac{1}{2}(1 - \dfrac{1}{n})(3 + \dfrac{2}{n}))$.
$\therefore U(P_n, f) \to 3/2$ as $n \to \infty$.
Similarly, $L(P_n, f) \to 3/2$ as $n \to \infty$.
$\therefore \int_1^2 x dx = 3/2$, as expected.

Theorem 6. *Continous functions defined on a closed interval are integrable.*

Proof. Fact: Continuous functions defined on a closed interval $[a, b]$ are uniformly continuous.
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then it is uniformly continuous.
For every for $\varepsilon > 0, \exists \delta > 0$, such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon, \forall x, y \in [a, b]$. □

Proposition 7. *Let $f, g : X \longrightarrow \mathbb{R}$ be two functions integrable on $[a, b]$.*

(i) For any $\alpha \in [a, b]$, f is integrable on $[a, \alpha]$.
(ii) $f + g$ is integrable on X.
(iii) $f.g$ is integrable on X.
(iv) If f is integrable on the image set $g([a, b])$, and g is differentiable, then $f \circ g$ is integrable on $[a, b]$, and $\int_a^b (f \circ g)(x)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Proof. (i), (ii), (iii) are easy to prove.

(iv): Let $u = g(x)$. Then use the chain rule, and the mean value theorem. □

Example 8. (i) $f(x) = x^3$. Then it a product of the three functions $g(x) = x$. By the results above, f is integrable on $[a, b]$.
(ii) $f(x) = 1/x$. Note that this a continuous function $[a, b] \longrightarrow \mathbb{R}$ for $a \neq 0$. So, f is integrable on $[a, b]$.
(iii) $f(x) = |x|$ is integrable on any closed interval.
(iv) Recall that sin, cos are continuous functions on any closed interval. So, they are integrable on any closed interval.

Theorem 9 (Fundamental Theorem of calculus, first version)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and hence integrable on $[a, b]$. For any $x \in [a, b]$, $\int_a^x f(t)dt$ exists.*
Then the function $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$,
and $F'(x) = f(x), \forall x \in [a, b]$.

Theorem 10 (Fundamental Theorem of calculus, second version)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ such that f is differentiable on (a, b).*
Then $\int_a^b f'(t)dt = f(b) - f(a)$.
In particular, for any $x \in [a, b]$, $\int_a^x f'(t)dt = f(x) - f(a)$.

Remark 11. The second version says that the integral of the derivative f' differs from the integral by a constant. This is what allows us to think of the integral as coming from the anti-derivative but with a constant. This is the reason the definition of integral as anti-derivative makes sense, and by stripping off the limits of integration, we simply write the anti-derivative as $\int f'(x)dx = f(x) + C$, for a constant C, and $\int_a^b f'(t)dt = f(b) + C - f(a) - C = f(b) - f(a)$.

Theorem 12. *Let $f(x) = e^x$ (recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ defines the exponential function.) Then f is integrable on $[a, b]$ and $\int f(x) = e^x + C$.*

Proof. $f'(x) = e^x$. So, $\int f(x)dx = e^x + C$. $\square$

Example 13. (i) $\sin, \cos$ and all polynomial functions have anti-derivatives, so their indefinite integrals exists.
(ii) $1/x$ has an anti-derivative on $(0, \infty)$, which is $\log x$. However, we have to be careful not to evaluate the integral on closed intervals that include 0 and the negative reals.

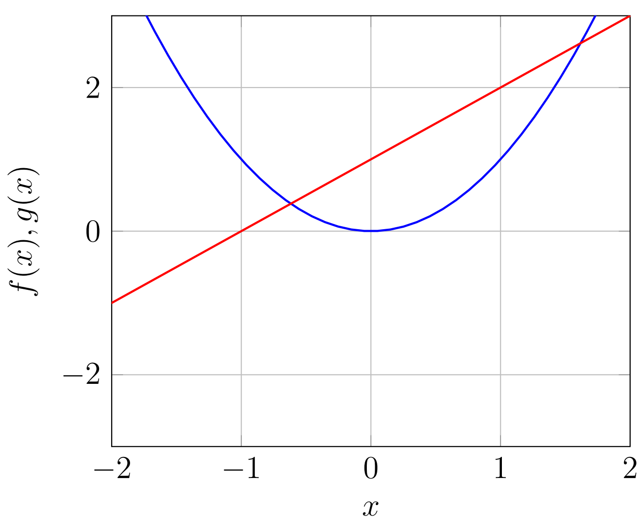
2. Area between two curves

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two continuous functions such that $f(x) \geqslant g(x), \forall x \in [a, b]$. Then the area between the curves defined by f and g is given by

$$\int_a^b (f(x) - g(x))dx.$$

Example 14. Let $f(x) = x^2, g(x) = x + 1$ be two curves on $\mathbb{R}$. Consider the interval $[-1, 1]$.

Plot of $f(x)$ in blue and $g(x)$ in red



The parabola and the line intersects at the points $(\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2})$ and $(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$. To find the area between the curves as x ranges from -1 to 1, we calculate:

$$\int_{-1}^{\frac{1-\sqrt{5}}{2}} (f(x) - g(x))dx + \int_{\frac{1-\sqrt{5}}{2}}^{1} (g(x) - f(x))dx = \int_{-1}^{\frac{1-\sqrt{5}}{2}} (x^2 - x - 1)dx + \int_{\frac{1-\sqrt{5}}{2}}^{1} (x + 1 - x^2)dx$$
$$= 2$$

Definition 15. Recall the conversion from cartesian to polar coordinates:
$x = r \sin\theta, y = r\cos\theta, x^2 + y^2 = r^2$.

The angle θ is a directed angle. It is considered positive if it is measured anticlockwise and negative if measured clockwise. We now calculate the area between two curves in polar coordinates.

Let $f : [\alpha, \beta] \longrightarrow \mathbb{R}$ be a continuous function. Let $r = f(\theta) \geqslant 0$ and $\beta \leqslant \alpha + 2\pi$.

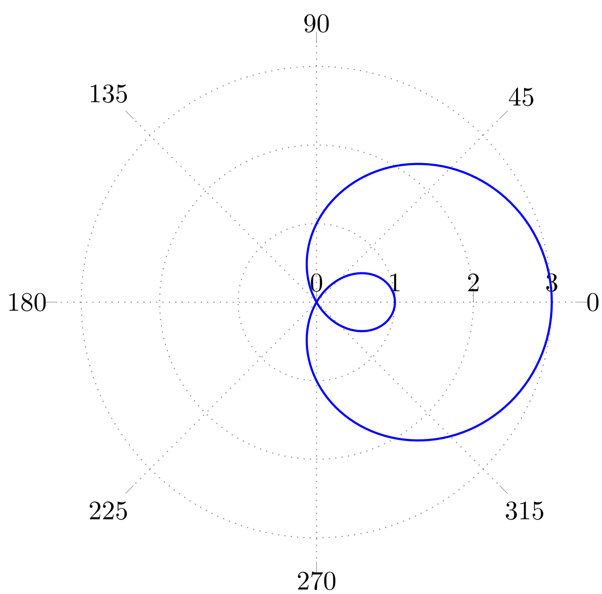
We now show how to calculate the area of the sector bounded by the angles α and β and the curve $r = f(\theta)$. Consider the partition $P = \{\theta_0, \theta_1, \cdots, \theta_n\}$, where $\theta_0 = \alpha, \theta_n = \beta$. Then the area of the each of the sector bounded by the angles $\theta_{i-1}\theta_i$ and the curve $f(\theta)$ is given by

$$\frac{\theta_i - \theta_{i-1}}{2}.f(\theta_i)^2 = \frac{1}{2}\Delta\theta_i f(\theta_i)^2.$$

$$\therefore \text{ sum of areas of all the sectors } = \sum_{i=1}^{n} \frac{1}{2}\Delta\theta_i f(\theta_i)^2.$$

$$\therefore \text{ area bounded by the curve and the angles } \alpha, \beta \text{ is given by } \frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta.$$

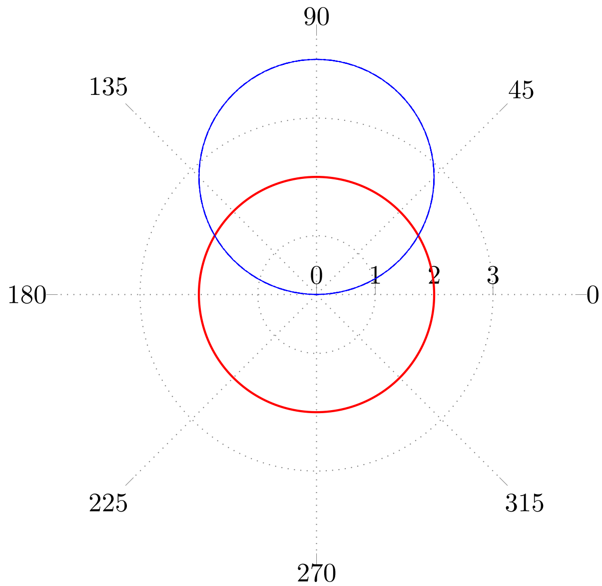
Example 16. Consider the function $r = 1 + \cos\theta$.



The area bounded by the angles 0 to 2π is

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta &= \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta \\ &= \frac{1}{2}\left(\frac{\theta}{2} + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_0^{2\pi} \\ &= \frac{3\pi}{2}.\end{aligned}$$

Example 17. To find the area between the circle $f(\theta) = 2$ and the curve $g(\theta) = 4\sin\theta$ as θ varies between their points of intersection.

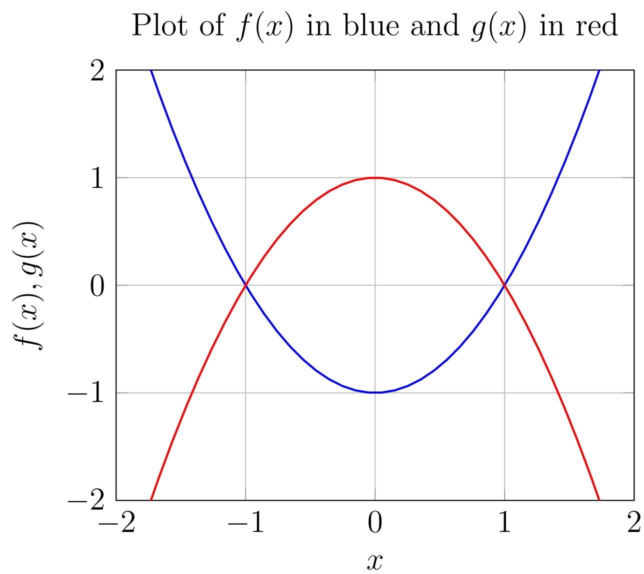


The points of intersection are given by $2 = 4\sin\theta$.
i.e., $\sin\theta = \dfrac{1}{2}$ so $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$.
$\therefore$ area $= \dfrac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (g(\theta)^2 - f(\theta)^2)d\theta = \dfrac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (16\sin^2\theta - 4)d\theta = \dfrac{4}{3}\pi + 2\sqrt{3}$.

3. Volume of a solid by slicing

Let $A(x)$ denote the area of the slice at x. Let $A(x)$ be a continuous function of x. Then the volume of the solid is given by $\int_a^b A(x)dx$.

Example 18. Consider the region between $f(x) = x^2 - 1$ and $g(x) = -x^2 + 1$, and cross sections which are perpendicular to the x-axis are equilateral triangles. Compute the volume of the solid.
The plot of the base of the solid is pictured below.



$A(x_i) = -(1 - x_i^2)\sqrt{3}(1 - x_i^2)$ is the area at x_i.
Then volume of the solid at x_i is $(1 - x_i^2)\sqrt{3}(x_i^2 - 1)\Delta x_i$
$\therefore$ volume $= \int_{-1}^{1} -\sqrt{3}(1 - x^2)^2 dx = \dfrac{16}{15}\sqrt{3}$.