MTH201-6

1. Vectors in $\mathbb{R}^3$

Definition 1. An element of $\mathbb{R}^n$ is called a vector in $\mathbb{R}^n$.

Some prperties of vectors in $\mathbb{R}^3$:

(i) Let $\overline{v} = (x, y, z), \overline{w} = (a, b, c)$. Then $\overline{v} - \overline{w} = (x - a, y - b, z - c)$; $\lambda\overline{v} = (\lambda x, \lambda y, \lambda z)$, for $\lambda \in \mathbb{R}$.
(ii) Let $\overline{v} = (x, y, z) \in \mathbb{R}^3$. Then the norm of $\overline{v}$ is $||\overline{v}|| = \sqrt{x^2 + y^2 + z^2}$.
(iii) distance between $\overline{v}$ and $\overline{w}$ is $||\overline{v} - \overline{w}||$.
(iv) Dot product $\overline{v} \cdot \overline{w} = ax + by + cz$, also called scalar product. θ is the angle between $\overline{v}$ and $\overline{w}$, then $\overline{v} \cdot \overline{w} = ||\overline{v}||||\overline{w}|| \cos\theta$.
(v) Projection of a vector $\overline{v}$ along $\overline{w}$ is $\dfrac{\overline{v} \cdot \overline{w}}{\overline{w} \cdot \overline{w}}\overline{w}$.
(vi) cross product is defined by $\overline{v} \times \overline{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ x & y & z \\ a & b & c \end{pmatrix}$

Definition 2. A function f is called a vector valued function in $\mathbb{R}^2$. And similarly, a function $g : \mathbb{R} \longrightarrow \mathbb{R}^2$ is called a vector valued function in $\mathbb{R}^3$.

Example 3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^3$, $f(t) = t(a, b, c)$ for a point (a, b, c) is a vector valued function. It represents a line parring through $(0, 0, 0)$ along the point (a, b, c).

Definition 4. We say that $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is continuous at t_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|t - t_0| < \delta \implies ||f(t) - f(t_0)|| < \epsilon$. If f is continuous at every point of $\mathbb{R}$, then f is continuous on $\mathbb{R}$.

Example 5. Let $f(t) = (\sin t, \cos t)$. Then f is continuous on $\mathbb{R}$. In fact, if $t_0 \in \mathbb{R}$, then there exists $\delta_1 > 0, \delta_2 > 0$, such that $|t - t_0| < \delta_1 \implies |\sin t - \sin t_0| < \epsilon$ and $|t - t_0| < \delta_1 \implies |\cos t - \cos t_0| < \epsilon$. Therefore, for $\delta = \min\{\delta_1, \delta_2\}$, we have $|t - t_0| < \delta \implies ||(\sin t, \cos t) - (\sin t_0, \cos t_0)|| \leqslant |\sin t - \sin t_0| + |\cos t - \cos t_0| < 2\epsilon$. Hence, f is continuous at $t_0 \in \mathbb{R}$.

2. Limit and continuity in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let $X_n = (x_n, y_n, z_n) \in \mathbb{R}^3$.

Definition 6. The sequence X_n converges to $X \in \mathbb{R}^3$, if for any $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that $||X_n - X|| < \epsilon$ whenever $n \geqslant N$. We write $\lim_{n \to n} X_n = X$.

Example 7. (i) $(1/n + 1, 1/n - 1 \to (1, -1)$
(ii) $X_n \to X = (x, y, z)$ if and only if $x_n \to x, y_n \to y, z_n \to z$.

Definition 8. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function. Then f has a limit L as X tends to X_0 if for any $\epsilon > 0$ there exists $\delta > 0$ such that
$X \neq X_0, ||X - X_0|| < \delta \implies |f(X) - L| < \epsilon$.
We write $\lim_{X \to X_0} f(X) = L$.

Definition 9. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function. Then f is continuous at X_0 if
$\lim_{X \to X_0} f(X) = f(X_0)$.

Example 10. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, be defined by $f(x, y) = \begin{cases} \dfrac{\sin^2(x-y)}{|x| + |y|} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.
Then f is continuous at $(0, 0)$.
Consider the difference $|f(x, y) - f(0, 0)| \leqslant \dfrac{(x-y)^2}{|x| + |y|} \leqslant |x| + |y|$.
As $(x, y) \to (0, 0)$, $|x| + |y| \to 0$.
Therefore, $\lim_{(x,y) \to (0,0)} f(x, y) = f(0, 0)$, and f is continuous at $(0, 0)$.

Example 11. Let $f(x, y) = \dfrac{2xy}{x^2 + y^2}$, if $(x, y) \neq (0, 0)$.
Then $f(x, mx) = 2m/(1 + m^2)$ and as $x \to 0$ the $\lim_{(x,mx) \to (0,0)} f(x, mx) = m/(1 + m^2)$. Hence the limit does not exists as $(x, y) \to (0, 0)$

3. Differentiation in $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition 12. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Then the partial derivative of f at (x_0, y_0, z_0) with respect to x is defined to be limit
$\dfrac{\partial f}{\partial x}(X_0) = \lim_{h \to 0} \dfrac{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)}{h}$ provided the limit exists.
Similarly, we define
$\dfrac{\partial f}{\partial y}(X_0) = \lim_{h \to 0} \dfrac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$ provided the limit exists, and
$\dfrac{\partial f}{\partial z}(X_0) = \lim_{h \to 0} \dfrac{f(x_0, y_0, z_0 + h) - f(x_0, y_0, z_0)}{h}$ provided the limit exists.

Example 13. Consider the function $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$ Then as seen in the previous example this function is not continuous at $(0, 0)$, but check that the partial derivatives with respect to both x and y exists.

One situation where continuity happens is given below.

Theorem 14. *Let $S := (a, b) \times (c, d)$, and $f : S \longrightarrow \mathbb{R}$ such that the partial derivatives exists and are bounded in S. Then f is continuous in S.*

Definition 15. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $X = (a, b, c)$. Then f is differentiable at X, if there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that the error function

$$e(H) := \frac{f(X + H) - f(X) - \alpha \cdot H}{||H||} \to 0, \text{ as } ||H|| \to 0.$$

$\alpha \in \mathbb{R}^3$ is called the derivative of f at X.

Theorem 16. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be differentiable at X. then f is continuous at X.*

Proof. $|f(X + H) - f(X) - \alpha \cdot H| = ||H||e(H)$ and $e(H) \to 0$.
Therefore, $|f(X + H) - f(X)| \leqslant ||H||(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + ||H||e(H)$
As $||H|| \to 0$ and $e(H) \to 0$, $f(X + H) \to f(X)$.
Therefore, f is continuous at X. □

Theorem 17. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be differentiable at X. Then $\partial f/\partial x(X)\partial f/\partial y(X), \partial f/\partial z(X)$ exists and the derivative*
$f'(X) = (\alpha_1, \alpha_2, \alpha_3) = (\frac{\partial f}{\partial x}(X), \frac{\partial f}{\partial y}(X), \frac{\partial f}{\partial z}(X))$.

Proof. Take $H = (t, 0, 0)$, we have
$e(H) = \dfrac{f(X + H) - f(X) - \alpha_1 t}{|t|} \to 0$ as $t \to 0$,
i.e., $\dfrac{f(X + H) - f(X) - \alpha_1 t}{|t|} \to 0$
Therefore $\alpha_1 = \dfrac{\partial f}{\partial x}(X)$.
Similarly, $\alpha_2 = \dfrac{\partial f}{\partial y}(X)$, and $\alpha_2 = \dfrac{\partial f}{\partial z}(X)$. □

Example 18. $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$.
Then f is differentiable at $(0, 0)$.
It is easy to see that $(\frac{\partial f}{\partial x}(X), \frac{\partial f}{\partial y}(X))$ exists and equal to $(0, 0)$.
Therefore, it is enough to prove that $e(H) \to 0$ as $H \to 0$.
$|e(H)| = \dfrac{|f(0 + H) - f(0) - (0, 0) \cdot H|}{||H||} \leqslant \dfrac{|hk|}{\sqrt{h^2 + k^2}} \leqslant \sqrt{h^2 + k^2} \to 0$, as $H \to 0$.
Therefore f is differentiable at $(0, 0)$ and $f'(0, 0) = (0, 0)$.

The previous theorem requires f to be differentiable at X. The next theorem does not need differentiability at X, but the partial derivatives have to exist and they have to be continuous.

Theorem 19. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with all partial derivatives existing in a neighborhood of X_0 and continuous at X_0. Then f is differentiable at X_0.*

Theorem 20. *Let f be differentiable at (x_0, y_0). Then*
$f((x_0, y_0) + (h, k)) - f(x_0, y_0) = f_x(x_0, y_0)h + f_y(x_0, y_0)k + h\epsilon_1(h, k) + k\epsilon_2(h, k)$,
where $\epsilon_1(h, k) \to 0$ and $\epsilon_2(h, k) \to 0$ as $h \to 0$ and $k \to 0$.

Proof. Put $H = (h, k)$. Then, as f is differentiable at (x_0, y_0)
$f((x_0, y_0) + (h, k)) - f(x_0, y_0) = f_x(x_0, y_0)h + f_y(x_0, y_0)k + ||H||e(H)$ where $e(H) \to 0$ as $H \to 0$.

Claim: $||H||e(H) = h\epsilon_1 + k\epsilon_2$ for some ϵ_1, ϵ_2.

Note $||H||e(H) = \frac{e(H)}{||H||}(h^2+k^2) = (h\frac{e(H)}{||H||})h + (k\frac{e(H)}{||H||})k$

Take $\epsilon_1 = h\frac{e(H)}{||H||}, \epsilon_2 = k\frac{e(H)}{||H||}$

and $|\epsilon_1| = |h\frac{e(H)}{||H||}| \leqslant |e(H)| \to 0$ as $H \to 0$.

Similarly, $\epsilon_2(H) \to 0$ as $H \to 0$. □

Theorem 21 (Chain rule)**.** *Let $f(x,y)$ be differentiable and $x = u(t), y = v(t)$ be also differentiable with respect to t. Let $X_0 = (x_0, y_0) = (u(t_0), v(t_0))$. Then $f(u(t), v(t))$ is differentiable at t and*

$\frac{df}{dt}(X_0) = (\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t})(X_0).$

Proof. $f(X_0+H)-f(X_0) = f_x(X_0)h+f_y(X_0)k+h\epsilon_1+k\epsilon_2$, where $\epsilon_1, \epsilon_2 \to 0$ as $H = (h,k) \to 0$.

Therefore $\frac{f(X_0+H)-f(X_0)}{t-t_0} = (f_x\frac{h}{t-t_0} + f_y\frac{k}{t-t_0} + \frac{h}{t-t_0}\epsilon_1 + \frac{k}{t-t_0}\epsilon_2)(X_0)$.

As $t \to t_0$, we have $h \to 0, k \to 0$ and $\epsilon_1 \to 0, \epsilon_2 \to 0$.

Therefore, $\frac{df}{dt}(X_0) = (\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t})(X_0)$. □

Definition 22 (Gradient)**.** $(f_x(X_0), f_y(X_0), f_z(X_0))$ is called the gradient of f at X_0 and is denoted by $(\nabla f)(X_0)$.

If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, then $f_x : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and we can calculate $f_{x,x} := (f_x)_x$ and $f_{x,y} := (f_x)_y$, etc.

In general $f_{x,y} \neq f_{y,x}$.

Example 23 (Classic example)**.** $f(x,y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$

For $(x,y) \neq (0,0)$: $f_x(x,y) = \frac{-x^4y - 4x^2y^3 + y^5}{(x^2+y^2)2}$

For $(x,y) = (0,0)$: $f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$

For $(x,y) \neq (0,0)$: $f_y(x,y) = \frac{-x^5 + 4x^3y^2 + xy^4}{(x^2+y^2)^2}$

For $(x,y) = (0,0)$: $f_y(0,0) = 0$

Now, $f_{x,y}(0,0) = \lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{h^5/h^4}{h} = 1$, and

$f_{y,x}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{-h^5/h^4}{h} = -1$.

Therefore $f_{x,y}(0,0) \neq f_{y,x}(0,0)$.

Theorem 24. *If $f(x,y)$ and its partial derivatives $f_x, f_y, f_{x,y}, f_{y,x}$ exists in a neighbourhood of (x_0, y_0) and are all continuous, then $f_{x,y}(x_0,y_0) = f_{y,x}(x_0,y_0)$.*

Theorem 25. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ an d $X_0 = (x_0, y_0)$. Then there exists a point C in the line joining X_0 and X such that*

$f(X) - f(X_0) = f'(C) \cdot (X - X_0)$.
Equivalently, if $H = (h, k)$, then there exists $t_0 \in (0, 1)$ such that
$f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(C) + kf_y(C)$, where $C = (x_0 + t_0h, y_0 + t_0k)$.

Proof. Define $\phi : [0, 1] \longrightarrow \mathbb{R}$ by $\phi(t) = f(x_0 + th, y_0 + tk)$.
By Chain rule, ϕ is differentiable and
$\phi'(t) = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} = hf_x + kf_y$.
Therefore by Mean Value theorem for one variable, there exists $t_0 \in (0, 1)$ such that $\phi(1) - \phi(0) = \phi'(t_0)$.
The theorem follows from this. $\square$

Corollary 26. *$f(x, y)$ defined on $[a, b] \times [c, d]$ is a constant if and only if $f_x = 0 = f_y$ on $[a, b] \times [c, d]$.*

Proof. $f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(C) + kf_y(C) = 0$ for every (h, k).
Therefore f is a constant. $\square$

4. Extremum points in $\mathbb{R}^2$ and $\mathbb{R}^3$ and Lagrange Multiplier method

Theorem 27 (Existence of Maxima/Minima)**.** *Let D be a closed and bounded subset of $\mathbb{R}^2$ and $f : D \longrightarrow \mathbb{R}$ be continuous. Then f has a maximum and a minimum in D.*

Theorem 28 (Condition for local max/min)**.** *Let (x_0, y_0) be in the interior of the region D. Let f_x, f_y exists at (x_0, y_0). If f has a local maximum or minimum at (x_0, y_0), then $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$.*

Proof. $f(x, y_0)$ and $f(x_0, y)$ are one-variable functions with maximum/ minimum at x_0 and y_0 respectively.
Therefore $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$. $\square$

Remark 29. Condition above is not sufficient. For example, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, f(x, y) = xy$. $f_x(0, 0) = 0 = f_y(0, 0)$ but $(0, 0)$ is neither local maximum nor minimum for f.

Definition 30 (Lagrange Multiplier Method)**.** Let $f(x, y, z)$ be a continuous function with partial derivatives which are continuous. Consider the set $S : \{(x, y, z) : g(x, y, z) = 0\}$, where g is a function with continuous partial derivatives. This set S is the set of constraints. Assume that $(\nabla g)(x_0, y_0, z_0) \neq 0$ where (x_0, y_0, z_0) is a point in S.
If f has a local extremum point at (x_0, y_0, z_0), then there exists $\lambda \in \mathbb{R}$ such that $(\nabla f)(x_0, y_0, z_0) = \lambda(\nabla g)(x_0, y_0, z_0)$.

To implement this method, consider the following equations:

$$\begin{aligned}(\nabla f)(x, y, z) &= \lambda(\nabla g)(x, y, z)\\ g(x, y, z) &= 0\end{aligned}$$

Solve these set of equations for λ, x, y, z. The local extremum points are found among the solutions of these equations.

If we write $\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$,
then we solve the equations

$$\begin{aligned}(\nabla\mathcal{L})(x, y, z) &= 0\\ g(x, y, z) &= 0\end{aligned}$$

Example 31. Minimize $f(x, y) = x^2 + y^2$ subject to the constraints $g(x, y) = x + y - 1$.
Put $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y) = x^2 + y^2 - \lambda(x + y - 1)$.

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= 2x - \lambda = 0 \quad \Longrightarrow \quad 2x = \lambda\\ \frac{\partial\mathcal{L}}{\partial y} &= \lambda(\nabla g)(x, y, z)2y - \lambda = 0 \quad \Longrightarrow \quad 2y = \lambda\\ \frac{\partial\mathcal{L}}{\partial \lambda} &= -(x + y - 1) = 0 \quad \Longrightarrow \quad x + y = 1\end{aligned}$$

Solving we get $x = y = 1/2$ and $\lambda = 1$.
Therefore the minimum value is $f(1/2, 1/2) = 1/2$.

Example 32. Let $S : 2x = 3y - z = 5$. Find a point nearest to the origin that lies on S, i.e., minimize $f(x, y, z) = x^2 + y^2 + z^2$ with the constraint $g(x, y, z) = 2x + 3y - z - 5 = 0$. Here $\nabla g(X) \neq 0$ in S. Then the equations are as follows:
$2x = 2\lambda, 2y = 3\lambda, 2z = -\lambda, 2x + 3y - z - 5 = 0$.
Therefore $x = \lambda, y = 3\lambda/2, z = -\lambda/2, 2\lambda + 9\lambda/2 + \lambda/2 - 5 = 0$.
Therefore, $\lambda = 5/7$ and $(x, y, z) = (5/7, 15/14, -5/14)$.
Since f attains its minimum on the plane, therefore by the Lagrange multiplier method $(5/7, 15/14, -5/14)$ has to be the nearest point.

5. Principal Normal and curvature

Definition 33 (Parametric curves)**.** Consider the vector valued function $I \longrightarrow \mathbb{R}^3$. Then the set of points $\{(t, f(t)) : t \in I\}$ is called the graph of the function f. If f is a continuous function then we say that f is called a curve or a curve parametrized by f, or simply that f is a parametric curve with parameter t.

Let $r : (a, b) \longrightarrow \mathbb{R}^3$ be a parametric curve, which we write as $r(t) = (x(t), y(t), z(t))$. For $t \in (a, b)$, the directed distance measured along C from $r(t_0)$ to $r(t)$ is

$$s(t) := \int_{t_0}^{t} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}d\tau.$$

By the fundamental theorem of calculus, $ds/dt = ||dr/dt||$.
Recall that the unit tangent vector at t is $\overline{T} = r'(t)/||r'(t)||$ whenever $r'(t) \neq 0.$,
i.e., $\overline{T} = \dfrac{dr/dt}{ds/dt} = dr/ds$.

Theorem 34. *Let $u, v : \mathbb{R} \longrightarrow \mathbb{R}^3$ be vector valued functions, which are differentiable, then $(u \cdot v)' = u' \cdot v + u \cdot v'$.*

Theorem 35. *Let $r : (a, b) \longrightarrow \mathbb{R}^3$ such that $||r(t)|| = a$ constant $\forall t \in (a, b)$. Then $r \cdot r' = 0$ on (a, b), i.e, $r \perp r'$.*

Let $u(t) = r(t)/||r(t)||$, then $u(t) = 1, \forall t \in (a, b)$. Hence $u \perp u'$.

Definition 36. Principal normal to the curve at t is defined by $N(t) = u'/||u'||$, whenever $u'(t) \neq 0$.

Definition 37. Curvature of a curve at t is defined by $\kappa(t) = ||d\overline{T}/ds||$. As $d\overline{T}/ds = T'(t)/||dr/dt||$, so we have $\kappa(t) = ||\overline{T}'(t)/r'(t)||$.

Example 38. $C : r(t) = a \cos t\hat{i} + a \sin t\hat{j}$. Then $r'(t) = -a \sin t\hat{i} + a \cos t\hat{j}$. Hence, $\overline{T}(t) = -\sin t\hat{i} + \cos t\hat{j}$. Therefore $\kappa(t) = ||\overline{T}'(t)/r'(t)|| = 1/a$, i.e, the circle has constant curvature, and it is the reciprocal of the radius of the circle.

Note that the curvature decreases as the radius of the circle increases.

Theorem 39. *Let $v(t)$ and $a(t)$ denote the velocity and the acceleration vectors of the motion of a particle along a curve given by $r(t)$. Then*

$$\kappa(t) = \frac{||a(t) \times v(t)||}{||v(t)||^3}.$$

Example 40. C be defined by $y = f(x)$, where f is twice differentiable. Then curvature at $(x, f(x))$ is $\dfrac{|f''(x)|}{|1 + f'(x)|^{3/2}}$.

Proof. $v(t) = r'(t) = \hat{i} + f'(t)\hat{j}$, $a(t) = f''(t)\hat{j}$. Therefore $a(t) \times v(t) = f''(t)\hat{k}$, so $a(t) \times v(t)|| = |f''(t)|$, and $||v(t)|| = \sqrt{1 + f'(t)^2}$. Hence $\kappa(t) = \dfrac{|f''(x)|}{|1 + f'(x)|^{3/2}}$. □

6. Double and Triple integrals

6.1. **Double Integrals.** Let $Q = [a, b] \times [c, d]$ and $f : Q \longrightarrow \mathbb{R}$ be bounded.
Let P_1 be a partition of $[a, b]$ into m points,
and P_2 be a partition of $[c, d]$ into n points.
Then $P = P_1 \times P_2$ decomposes Q into mn rectangles.
Define $m_{i,j} = \min\{f(x, y) : (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$
and $M_{i,j} = \max\{f(x, y) : (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$.
Then the lower sum is denoted by $L(P, f) = \sum_{i=1}^{n} \sum_{j-1}^{m} m_{i,j}\Delta y_j \Delta x_i$.
Similarly, define the upper sum by $U(P, f) = \sum_{i=1}^{n} \sum_{j-1}^{m} M_{i,j}\Delta y_j \Delta x_i$.
The limit of these sums as the partition size increases are called the lower integrals and the upper integrals.

Definition 41. f is integrable if both the lower and upper integrals exists and are equal. If f is integrable on Q, then the double integral is denoted by $\int\int_Q f(x, y)dxdy$ or $\int\int_Q f(x, y)dA$, where $dA = dxdy$.

In general, calculation of this integral is difficult. It is simplified by the following theorem due to Fubini.

Theorem 42 (Fubini's theorem)**.** *Let $f : Q = [a, b] \times [c, d] \longrightarrow \mathbb{R}$ be a continuous function. Then*
$\iint_Q f(x, y)dxdy = \int_c^d (\int_a^b f(x, y)dx)dy = \int_a^b (\int_c^d f(x, y)dy)dx.$

Following is another generalization of Fubini's theorem.

Theorem 43 (Fubini's theorem)**.** *Let $f(x, y)$ be a function bounded on a domain D, which can be defined as follows:*

(i) $D := \{(x, y) : a \leqslant x \leqslant b, f_1(x) \leqslant y \leqslant f_2(x)\}$ *for some continuous functions $f_1, f_2 : [a, b] \longrightarrow \mathbb{R}$. Then*
$\iint_D f(x, y)dxdy = \int_a^b (\int_{f_1(x)}^{f_2(x)} f(x, y)dy)dx.$

(ii) $D := \{(x, y) : c \leqslant y \leqslant d, g_1(y) \leqslant x \leqslant g_2(y)\}$ *for some continuous functions $g_1, g_2 : [a, b] \longrightarrow \mathbb{R}$. Then*
$\iint_D f(x, y)dxdy = \int_c^d (\int_{g_1(x)}^{g_2(x)} f(x, y)dx)dy.$

Example 44. Let $D = [2, 4] \times [1, 2]$. Then $\iint_D 6xy^2 dxdy = \int_2^4 (\int_1^2 6xy^2 dy)dx = \int_2^4 (2xy^3)|_1^2 dx = \int_2^4 (16x - 2x)dx = 7x^2|_2^4 = 84$.

Example 45. Let D be the region bounded by the lies joining $(0, 0), (2, 0), (2, 2)$. Calculate $\iint_D (x + y)^2 dxdy$.
Take $a = 0, b = 2, f_1(x) = x, f_2(x) = 1$.
Then $\iint_D (x + y)^2 dxdy = \int_0^2 (\int_x^2 (x + y)^2 dy)dx = \int_0^2 \dfrac{(x + y)^3|_x^2}{3} dx.$

6.2. **Triple integrals.** Let $Q = [a, b] \times [c, d] \times [s, t]$ and $f : Q \longrightarrow \mathbb{R}$ be bounded.
Let P_1 be a partition of $[a, b]$ into m points,
and P_2 be a partition of $[c, d]$ into n points.
and P_3 be a partition of $[s, t]$ into p points.
Then $P = P_1 \times P_2 \times P_3$ decomposes Q into mnp rectangles.
Define $m_{i,j,k} = \min\{f(x, y, z) : (x, y, z) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]\}$
and $M_{i,j,k} = \max\{f(x, y, z) : (x, y, z) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]\}$.
Then the lower sum is denoted by $L(P, f) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k-1}^p m_{i,j,k} \Delta y_j \Delta x_i \Delta z_k$.
Similarly, define the upper sum by $U(P, f) = \sum_{i=1}^n \sum_{j=1}^m \sum_{k-1}^p M_{i,j,k} \Delta y_j \Delta x_i \Delta z_k$.
The limit of these sums as the partition size increases are called the lower integrals and the upper integrals.

Definition 46. f is integrable if both the lower and upper integrals exists and are equal.
If f is integrable on Q, then the double integral is denoted by $\iiint_Q f(x, y)dxdydz$ or $\iiint_Q f(x, y)dV$, where $dV = dxdydz$.

7. Change of variable

7.1. **Two variables.** Let $f : S \longrightarrow \mathbb{R}$ for a subset $S \subset \mathbb{R}^2$ be a continuous function. Let $x = X(u, v), y = Y(u, v)$.
Define $T := \{(u, v) \in \mathbb{R}^2 : x, y \in S\}$.
Assume the following:

(i) The map $T \longrightarrow S$ defined by $(u,v) \mapsto (X(u,v), Y(u,v))$ is an injective map.

(ii) X, Y are continuous and the partial derivatives $\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}, \frac{\partial Y}{\partial u}, \frac{\partial Y}{\partial v}$ are also continuous.

(iii) The Jacobian which is defined by the determinant $J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} \neq 0$.

Then $\iint_S f(x,y)dxdy = \iint_T f(X(u,v), Y(u,v))|J(u,v)|dudv$.

Example 47. Let S be the region bounded by $xy = 1, xy = 2, xy^2 = 3, xy^2 = 4$.
We can calculate the area of the region S, which is $\iint_S dxdy$.
Put $u = xy, v = xy^2$
Then $x = u^2/v, y = v/u$.
The region T is given by $1 \leqslant u \leqslant 2, 3 \leqslant v \leqslant 4$.
Here $J(u,v) = 1/v \neq 0$.
Therefore, by the change of variable formula:
Area of S=$\iint_S dxdy = \iint_T \frac{1}{v} dudv = \int_3^4 \int_1^2 \frac{1}{v} dudv = \int_3^4 \log v|_1^2 du = \int_3^4 \log 2 du = \log 2$.

7.2. **Three variables.** As in the case above for two variables, consider now $S \subset \mathbb{R}^3$, and a function $f : S \longrightarrow \mathbb{R}$, with $x = X(u,v,w), y = Y(u,v,w), z = Z(u,v,w)$.
Define $T := \{(u,v,w) \in \mathbb{R}^3 : x, y, z \in S\}$.
Assume the following:

(i) The map $T \longrightarrow S$ defined by $(u,v,w) \mapsto (X(u,v,w), Y(u,v,w), Z(u,v,w))$ is an injective map.

(ii) X, Y, Z are continuous and the partial derivatives $\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}, \frac{\partial X}{\partial w}, \frac{\partial Y}{\partial u}, \frac{\partial Y}{\partial v}, \frac{\partial Y}{\partial w}, \frac{\partial Z}{\partial u}, \frac{\partial Z}{\partial v}, \frac{\partial Z}{\partial w}$ are also continuous.

(iii) The Jacobian which is defined by the determinant $J(u,v,w) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \\ \frac{\partial X}{\partial w} & \frac{\partial Y}{\partial w} & \frac{\partial Z}{\partial w} \end{vmatrix} \neq 0$.

Then $\iiint_S f(x,y,z)dxdydz = \iiint_T f(X(u,v,w), Y(u,v,w), Z(u,v,w))|J(u,v,w)|dudvdw$.

8. Line Integrals

Consider a parametric curve $r : [a,b] \longrightarrow \mathbb{R}^2$, and let $C = \{r(t) : t \in [a,b]\}$ be the parametric curve parametrized by t. Let $f : C \longrightarrow \mathbb{R}^2$ be a bounded function.

Definition 48. Let $r(t)$ be a differentiable function of t. Then the integral of f along C is defined as

$$\int_C f \cdot dr = \int_a^b f(r(t)) \cdot r'(t)dt.$$

Note here that $f(r(t))$ and $r'(t)$ both are elements in $\mathbb{R}^2$ and the product between them is the dot product.

If we write $f = (f_1, f_2) = f_1\hat{i} + f_2\hat{j}$ and $r(t) = (x(t), y(t))$, then $\int_C f \cdot dr$ is also written as $\int_C (f_1 dx + f_2 dy)$.

Example 49. Let $f = x^2\hat{i} + y\hat{j}$ and C be the line joining $(0,0)$ to $(1,2)$. We can parametrize this line as $x = t, y = 2t$. Then
$\int_C f \cdot dr = \int_0^1 t^2 dt + 2td(2t) = \int_0^1 (t^2 + 4t)dt = (t^3/3 + 2t^2)|_0^1 = 7/3$.

Recall the second fundamental theorem for integrals. If $f : [a,b] \longrightarrow \mathbb{R}$ such that f is differentiable, then $\int_a^b f'(x)dx = f(b) - f(a)$. Here is the theorem for line integrals.

Theorem 50. *Let $S \subset \mathbb{R}^2$ and $f : S \longrightarrow \mathbb{R}$ be differentiable on S. Also assume that the gradient $\nabla f = (f_x, f_y)$ is continuous on S. Let A and B be two points on S, and let $C := \{r(t) : t \in [a,b]\}$ be the line or contour lying in S and joining the two points A and B, i.e., $A = r(a), B = r(b)$. Also assume that $r'(t)$ is continuous on $[a,b]$. Then*

$$\int_C (\nabla f) \cdot dr = f(B) - f(A).$$

Theorem 51 (Green's theorem)**.** *Let C be a curve defined by $r(t)$. Let $r(t)$ be a differentiable function, and D denote the region enclosed by C. Suppose $M, N, \partial N/\partial x, \partial M/\partial y$ be all continuous functions on an open set containing D. Then*

$$\iint_D (\partial N/\partial x - \partial M/\partial y) dxdy = \int_C (M\hat{i} + N\hat{j}) \cdot dr = \int_C M dx + N dy,$$

where the integral is taken along the counterclockwise direction.

Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a function given by $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$.

Definition 52. The curl is defined by

$$\text{curl} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times F,$$

where the product of $\dfrac{\partial}{\partial x}$ with, say Q, means $\dfrac{\partial Q}{\partial x}$.

Definition 53. The divergence of F is defined by

$$\text{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F.$$

Recall that maps $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are called vector valued functions or vector fields, and maps $\mathbb{R}^3 \longrightarrow \mathbb{R}$ are scalar fields. Finally, we have

(i) the gradient of a scalar field ∇f, that gives a function in $\mathbb{R}^3$
(ii) the curl of a vector field $\nabla \times F$, that gives a vector field in $\mathbb{R}^3$
(iii) the divergence of a vector field $\nabla \cdot F$, that gives a scalar field in $\mathbb{R}$.

Green's theorem can now be stated as

Theorem 54 (Green's theorem)**.** $\iint_D (\nabla \times F) \cdot \hat{k} dxdy = \int_C F \cdot dr.$

Theorem 55 (Stokes' theorem)**.** *Let $\hat{n}$ denote the unit normal to a smooth surface S whose boundary is given by a smooth curve C. Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field given by $F(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$, where P, Q, R are all continuous and their partial derivatives are all continuous in S. Then*

$$\iint\limits_S (\mathrm{curl} F) \cdot \hat{n} d\sigma = \int_C F \cdot dr,$$

where the line integral is taken around C in the direction of the orientation of C w.r.t $\hat{n}$.

Theorem 56 (Divergence theorem)**.** *Let D be a region in $\mathbb{R}^3$ whose boundary is a smooth orientable surface S. Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field given by $F(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$, where P, Q, R are all continuous and their partial derivatives are all continuous in D. Then*

$$\iiint\limits_S (\mathrm{div} F) \cdot \hat{n} dV = \iint\limits_S F \cdot \hat{n} d\sigma,$$

where the line integral is taken on S in the direction of the orientation of S w.r.t $\hat{n}$.

Example 57 (Area expressed by a line integral)**.** Consider a smooth simple closed curve, and D be the region enclosed by C. Take $N(x, y) = x/2, M(x, y) = -y/2$. Then, by Green's theorem, we have

$$\text{Area of } D = \iint\limits_D dxdy = \iint\limits_D (\partial N/\partial x - \partial M/\partial y) dxdy = \int_a^b M dx + N dy = \frac{1}{2} \int_C -ydx + xdy.$$

Example 58 (Area bounded by a circle of radius a)**.** $C : x^2 + y^2 = a^2$
Parametrize C by $(a \cos t, a \sin t), 0 \leqslant t \leqslant 2\pi$.
Then by the previous example, area bounded by the circle C
$=\frac{1}{2} \int_C -a \sin t(-a \sin t)dt + a \cos t(a \cos t)dt = \frac{1}{2} \int_0^{2\pi} -a \sin t(-a \sin t)dt + a \cos t(a \cos t)dt$
$= \pi a^2$

Example 59. The potential difference between two points (A and B) is the negative of the line integral of the electric field ($\vec{E}$) along a path connecting those points, expressed as $V_{AB} = V_A - V_B = -\int_A^B \vec{E} \cdot d\vec{r}$. This equation shows that potential difference is the work done per unit charge to move a charge from A to B against the electric field, or equivalently, the negative of the work done by the electric field.