MTH201-PRACTICE PROBLEMS 3

Q 1. Using the sage cell server type in the following commands and see for yourself

(i) Sketch the function $f(x) = \sin(x)/x$.

```
plot(sin(x)/x,(x,-10,10))
```

(ii) Sketch the function $f(x) = \sin(x)/|x|$.

```
plot(sin(x)/abs(x),(x,-10,10))
```

(iii) Sketch the function $f(x) = \cos(x)/|x|$.

```
plot(cos(x)/abs(x),(x,-10,10))
```

Q 2. Check for continuity.

(i) $f(x) = \dfrac{x^2 - a^2}{x - a}$, if $x \neq a, f(a) = 2a$ at $x = a$

(ii) $f(x) = \dfrac{\sqrt{x} - \sqrt{b}}{x - b}, b > 0,$ if $x \neq \sqrt{b}, f(b) = \frac{1}{2\sqrt{b}}$ at $x = b$

(iii) $f(x) = \lim_{t \to 0} \dfrac{t}{\sqrt{4 - t} - 2}$, if $t \neq 0, f(0) = 4$ at $t = 0$

(iv) $f(x) = \dfrac{\sqrt{x + 25} - 5}{x}$, if $x \neq 0, f(0) = 1/10$ at $x = 0$

(v) $f(x) = \dfrac{2x \sin x}{1 - \cos x}$, if $x \neq 0, f(0) = 0$ at $x = 0$

(vi) Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$. Is f continuous?

Q 3. Let $X \subset \mathbb{R}$.

(i) Let $f : X \longrightarrow \mathbb{R}$ be a continuous function. Show that $|f(x)|$ is also a continuous function on X.

(ii) Let $f : X \longrightarrow \mathbb{R}$ be a function, such that $|f(x) - f(y)| \leq |x - y|$, for all $x, y \in X$. Show that f is a continuous function.

(iii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function continuous at 0, such that $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$. Show that f is a continuous function on $\mathbb{R}$.

Q 4. From the definition of continuity, deduce the following.
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and $c \in (a, b)$ such that $f(c) > 0$. Then show that there is a $\delta > 0$ such that $|x - c| < \delta \implies f(x) > 0$.

Q 5. (i) Show that $f(x) = 1/x, x \in (0, 1]$ is not a bounded function.

(ii) Show that $f : X \longrightarrow \mathbb{R}$ is a bounded function if and only if the function defined by $|f(x)|$ is bounded.

Q 6. Using the intermediate value theorem, show that there is a solution in the specified interval.

(i) $3x^5 - 4x^2 = 3$ in $[0, 2]$

(ii) $e^x = 4 - x^3$ in $[-2, -1]$

Q 7. Using the intermediate value theorem, show that there is a solution.

(i) $x^3 + 2x - 5 = 0$

(ii) $e^x + x + 2 = 0$

(iii) $x^3 = 20 + \sqrt{x}$

(iv) $xe^x = x^2 - 1$

Q 8. If $f : [a, b] \longrightarrow [a, b]$ is a continuous function, then show that there is an x_0 such that $f(x_0) = x_0$.

Q 9. If $f : [0, 2] \longrightarrow \mathbb{R}$ is a continuous function such that $f(0) > 0$ and $f(2) < 4$, then show that there exists a $c \in (0, 2)$ such that $f(c) = c^2$.

Definition 1. A function $f : [a, b] \longrightarrow \mathbb{R}$ satisfies the intermediate value property (IVP) on $[a, b]$, if for every x_1, x_2, with $x_1 < x_2$, and for every t between $f(x_1)$ and $f(x_2)$ there is a c between x_1 and x_2 such that $f(c) = t$.

Q 10. Consider the following function defined on $[-2/\pi, 2\pi]$

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that it satisfies the IVP, but is not continuous at 0.

Q 11 (Challenging)**.**

(i) If the function f is one-one and satisfies the IVP on $[a, b]$, the show that f is continuous on $[a, b]$.

(ii) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function such that $f(a) = f(b)$, then show that there exists $x, y \in (a, b)$ such that $f(x) = f(y)$.

Q 12 (The puzzle)**.** A man starts walking up a winding hill at 8 a.m. and reaches the top at 5 p.m. He stays the night and begins his descent down the exact same path at 8 a.m. the next day, reaching the bottom at 5 p.m. He does not travel at a constant speed, and his watch is not working. Is there a point on the path where the man is at the same spot at the same time of day on both trips?