

MTH201-2

1. LIMITS

Definition 1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be the limit of the function f at a point $a \in \mathbb{R}$ if there exists a real number L such that

for every real $\epsilon > 0$, there is a real number $\delta > 0$ such that

$$x \neq a, |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

If such a real number L exists then we write $L = \lim_{x \rightarrow a} f(x)$. Loosely, the definition expresses the possible value of the function f as the value of x gets close to a , i.e., when $|x - a|$ becomes small. Or, the value of f getting close to L is achieved by x getting close to a .

Proposition 2. *The limit of a function, if it exists, is unique.*

Example 3. (i) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x$. To find $\lim_{x \rightarrow a} f(x)$. We guess that the limit is a . To prove this, let $\epsilon > 0$ be any positive real number. Then, we can see that $|f(x) - a| = |x - a|, \epsilon$ will be achieved by $|x - a| < \delta$ if we choose $\delta = \epsilon$.

To check this, take $\delta = \epsilon$, then $|x - a| < \delta = |x - a| < \epsilon = |f(x) - a| < \epsilon$. Hence $\lim_{x \rightarrow a} f(x) = a$.

(ii) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x$. To find $\lim_{x \rightarrow a} f(x)$. We guess that the limit is $2a$. To prove this, let $\epsilon > 0$ be any positive real number. Then, we can see that $|f(x) - 2a| = |2x - 2a| < \epsilon$ will be achieved by $|x - a| = |2x - 2a|/2 < \delta$ if we choose $\delta = \epsilon/2$.

To check this, take $\delta = \epsilon/2$, then $|x - a| < \delta = |x - a| < \epsilon/2 = |f(x) - 2a| < \epsilon$. Hence $\lim_{x \rightarrow a} f(x) = 2a$.

It might be difficult to find a δ for a given ϵ for some functions. It might be helpful to have some other observations to determine the limit.

Proposition 4. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions. Let $\lim_{x \rightarrow a} f(x) = A$ and $\lim_{x \rightarrow a} g(x) = B$. Then

- (i) $\lim_{x \rightarrow a} (f + g)$ exists and equals $A + B$,
- (ii) $\lim_{x \rightarrow a} (f \cdot g)$ exists and equals $A \cdot B$,
- (iii) $\lim_{x \rightarrow a} (f \circ g)(x)$ may not exist as a real number, though it might exist in the extended reals,
- (iv) $\lim_{x \rightarrow a} (f/g)(x)$ exists and equals A/B if $B \neq 0$.

Proof. (i): Let $\epsilon > 0$. Since $\lim_{x \rightarrow a} f(x) = A$, there exists $\delta_1 > 0$, such that $|x - a| < \delta_1 \implies |f(x) - A| < \epsilon$.

Since $\lim_{x \rightarrow a} g(x) = B$, there exists $\delta_2 > 0$, such that
 $|x - a| < \delta_2 \implies |g(x) - B| < \epsilon$.

Both the statements above will be true simultaneously if $|x - a| < \min(\delta_1, \delta_2)$. Then, we have $|f(x) - A| < \epsilon, |g(x) - B| < \epsilon$. Then $|(f + g)(x) - A - B| = |f(x) - A + g(x) - B| < |f(x) - A| + |g(x) - B| + \epsilon = 2\epsilon$, i.e., putting $\delta = \min(\delta_1, \delta_2)$, we have $|(f + g)(x) - A - B| < 2\epsilon$. As ϵ varies over all the positive reals, 2ϵ also varies over all the positive reals. Hence, by definition, the limit of $f + g$ exists and $\lim_{x \rightarrow a} (f + g)(x) = A + B$.

(ii): First observe,

$$\begin{aligned} |(f \cdot g)(x) - AB| &= |f(x) \cdot g(x) - AB| = |f(x)g(x) - Ag(x) + Ag(x) - AB| \\ &= |(f(x) - A)g(x) + A(g(x) - B)| \\ &\leq |(f(x) - A)||g(x)| + |A||g(x) - B| \end{aligned}$$

by the triangle inequality. In these terms, we see that $|(f(x) - A)|$ small can be achieved, also $|(g(x) - B)|$ small can also be achieved. If the $|g(x)|$ term can be bounded, then the product $|(f(x) - A)||g(x)|$ will be small. This is what we do below.

Let $\epsilon > 0$, then there exists $\delta_1 > 0, \delta_2 > 0$, such that

$$|x - a| < \delta_1 \implies |(f(x) - A)| < \epsilon/(\epsilon + |B|),$$

and

$$|x - a| < \delta_2 \implies |(g(x) - B)| < \epsilon/(1 + |A|).$$

The second one means that $|g(x)| < \epsilon + |B|$, if $|x - a| < \delta_2$. So, now if we take $\delta = \min(\delta_1, \delta_2)$ and $|x - a| < \delta$, then all the above inequalities will hold simultaneously, and combining them together we get the following:

$$\begin{aligned} |x - a| < \delta &\implies |(f(x) - A)| < \epsilon/(\epsilon + |B|), |(g(x) - B)| < \epsilon/(1 + |A|) \\ &\implies |(f(x) - A)||g(x)| + |A||g(x) - B| \leq \epsilon|g(x)|/(\epsilon + |B|) + |A|\epsilon/(1 + |A|) = 2\epsilon. \end{aligned}$$

Therefore, $|x - a| < \delta \implies |(f \cdot g)(x) - AB| < 2\epsilon$. As ϵ varies over all real numbers 2ϵ also varies over all real numbers. Hence, $\lim_{x \rightarrow a} (f \cdot g)(x) = A \cdot B$. \square

(iii), (iv): Proofs are not necessary. \square

Example 5. (i) $f(x) = x^3$. Then it a product of the three functions $g(x) = x$. By the calculation that we have done above, we have $\lim_{x \rightarrow a} f(x) = a^3$.

(ii) $f(x) = 1/x$. Note that this a well-defined function from $\mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$. By (iv) of the previous proposition, $\lim_{x \rightarrow a} f(x) = 1/a$, if $a \neq 0$.

(iii) $f(x) = x^3 + c$, is a sum of two functions $g(x) = x^3$ and the constant function $h(x) = c$. We have seen that $\lim_{x \rightarrow a} g(x) = a^3, \lim_{x \rightarrow a} h(x) = c$. So, $\lim_{x \rightarrow a} f(x) = a^3 + c$.

(iv) Let $\theta : \mathbb{R} \setminus \{\text{roots of } x^3 + c\} \rightarrow \mathbb{R}$, given by $\theta(x) = 1/(x^3 + c)$. Then this is well-defined function and by the previous calculation, we have $\lim_{x \rightarrow a} \theta(x) = 1/(a^3 + c)$, by using (iv) of the previous proposition.

In this way, we can give several examples to calculate the limit of many functions.

Try to calculate, $\lim_{x \rightarrow a} |x|$.