~~L7-~~ 9

Lecture 9

Eigenvalues & Eigenvectors

Suppose $T \in L(V)$

A number $\lambda \in F$ is called an eigenvalue of T

if there exists $v \in F$ such that

→ $v \neq 0$ (non-zero vector)

→ $Tv = \lambda v$

In other words, linear transformation (which can be defined also as A) is applied on $\vec{v}$, then it simply scales v by an amount λ

$A \in \mathbb{R}^{n \times n}$

v → non zero vector $v \in \mathbb{R}^n$

↳ eigenvector of A corresponding to eigenvalue (λ)

if $\quad Av = \lambda v$

Scanned with OKEN Scanner

How to calculate

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Determining eigenvectors

A is 3×3 real mt

$$A v = \lambda v \;; \quad A v - \lambda v = 0$$

$$\Rightarrow (A - \lambda I) v = 0 \qquad \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

$$\det = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

So $\lambda = 1, 2, 3$ are 3 distinct eigenvalues of A

For λ

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\lambda = 1 \quad \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \begin{bmatrix} -x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + 2x_2 + 2x_3 \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$x_3 = 0 \qquad X_1 = C_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

a

$\lambda = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 + x_3 = 0,$ | $x_1 = -x_3$

$2x_1 + 2x_2 + x_3 = 0,$ | $x_2 = \frac{1}{2} x_3$

$$X_2 = c \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_1 = -$

$\lambda = 3$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 = -x_2$

$x_1 = -\frac{1}{2} x_3$

$$X_3 = c \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Note! a) If all the n eigenvalues of A are distinct.
→ There correspond n distinct linearly independent eigenvector

b) For an eigenvalue of A, repeated (twice or more) there may correspond one or several linearly independent eigenvectors.

Thus the set of eigenvectors "may not" form a set of ~~n~~ n linearly independent vectors

⇒ This depends upon Geometric multiplicity → no. of linearly independent eigenvectors associated with eigenvalues

Determinant of $A = \sigma_n = |A|$

$= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n$

A & A^T has same eigenvalues)

→ If all eigen values are non-zero, then $|A| \neq 0$.

i.e. A is non singular

→ If at least one eigenvalue is zero then A is singular

→ $A^{-1} A x = A^{-1} (\lambda x) \Rightarrow A^{-1} x = \frac{1}{\lambda} x$

→ $Ax = \lambda x \Rightarrow A^2 x = \lambda(\lambda x) \Rightarrow A^2 x = \lambda^2 x$

$A^n x = \lambda^n x$

→ Trace of A = sum of eigen values.

---

① Algebraic multiplicity of eigenvalue refers to no. of times the eigenvalue appears as root of characteristic polynomial of mtx.

Always +ve integer.

G.M. is always less or equal to A.M.

If G.M. = A.M. for all eigenvalues, mtx is diagonalizable

G.M. < A.M., mtx is defective → does not have enough l.i. eigenvectors & cannot be diagonalized

Quadratic form

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric m̄

The expression $q(x) = X^T A X$ is called Quadratic form

$$Q = X^T A X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + \dots + a_{1n} x_1 x_n +$$

$$a_{21} x_2 x_1 + a_{22} x_2^2 + \dots + a_{2n} x_2 x_n$$

$$+$$

$$+ a_{n1} x_n x_1 + a_{n2} x_n x_2 + \dots + a_{nn} x_n^2$$

A is known as coefficient m̄.

Rewriting

$$Q = X^T A X = a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + \dots + a_{nn} x_n^2$$

$$c_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) \text{ then } c_{ji} = c_{ij}$$

$$c_{ij} + c_{ji} = a_{ij} + a_{ji}$$

Thus Quadratic form can be rewritten as

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$a_{11} x_1^2 + a_{21} x_2 x_1 + a_{31} x_3 x_1 + a_{12} x_1 x_2 + a_{22} x_2^2 + a_{32} x_3 x_2 + a_{13} x_1 x_3 + a_{23} x_2 x_3 + a_{33} x_3^2$$

$$a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + (a_{21} + a_{12}) x_2 x_1 + (a_{31} + a_{13}) x_1 x_3 + (a_{32} + a_{23}) x_2 x_3$$

Rayleigh's Quotient is a scalar expression approximate eigenvalue of a mt for given vector.

For a $n \times n$ real mt;

$$R_A(X) = \frac{X^T A X}{X^T X} \quad \text{where } X \neq 0$$

$X^T A X \rightarrow$ quadratic form of mt A w.r.t. x. Produce scalar

$X^T X \rightarrow$ squared norm of x; ensure normalization

- If X is eigenvector of A with eigenvalue λ;
  then $R_A(X) = \lambda$

→ $\lambda_{min}(A) = \min\limits_{X \neq 0} R_A(X)$

→ For any X such that $\|X\|_2 = 1$

$$\lambda_{min}(A) \leq X^T A X \leq \lambda_{max}(A)$$

equality holds iff X is corresponding eigenvector

For any $X = 0$, one has

$$\lambda_{min}(A) \leq R_A(x) \leq \lambda_{max}(A)$$

Positive Semi Definite mtx

A symmetric mtx $A \in \mathbb{R}^{n \times n}$ is said to be P.S.D. if for all $X \in \mathbb{R}^n$ one has

$$q(X) = X^T A X \geq 0$$

always non-negative for all non-zero vectors X

$$A = \begin{pmatrix} 1 & 5 \\ 5 & 26 \end{pmatrix}$$

$$X^T A X = (x_1 \;\; x_2) \begin{pmatrix} 1 & 5 \\ 5 & 26 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 + 5x_2 \quad 5x_1 + 26x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1^2 + 5x_2 x_1 + 5x_1 x_2 + 26 x_2^2$$

$$= x_1^2 + 26x_2^2 + 10 x_2 x_1$$

$$= x_1^2 + 25x_2^2 + 10x_1 x_1 + x_2^2$$

$$= (x_1 + 5x_2)^2 + x_2^2 \geq 0$$

Positive Semidefinite mtx

P.S.D. of

Positive Definite mt . Symmetric mt A.

$A \in \mathbb{R}^{n \times n}$ is P.D. if for all non zero $X \in \mathbb{R}^n$

$$q(X) = X^T A X > 0$$

strictly +ve condition.

P.D. → all eigen value of A are +ve

P.S.D → All eigenvalues of A are ~~nonneg~~ non-negative (≥ 0) and at least one eigen value is zero.

Ex

$$A = \begin{bmatrix} 9 & -15 \\ -15 & 25 \end{bmatrix}$$

let x be 2×1 vector

$$X^T A X = [x_1 \quad x_2] \begin{bmatrix} 9 & -15 \\ -15 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \; x_2] \begin{bmatrix} 9x_1 - 15x_2 \\ -15x_1 + 25x_2 \end{bmatrix}$$

$$= 9x_1^2 - 15x_1x_2 - 15x_1x_2 + 25x_2^2$$

$$= (3x_1 - 5x_2)^2$$

$x^T A x \geq 0$ if $x \neq 0$. ∴ A is positive semidefinite

It is not positive definite ∵ there exists non-zero vector $x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ s.t. $x^T A x = 0$

symmetric mtx is ~~the (se~~ positive (semi-) definite
if its all pivots are +ve (non-negative)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

① $2 > 0$

② $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

③ $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$

$$\det A = 2(4-1) - (-1)(-2-0) + 0$$
$$= \cancel{2} \; 6 - 2 = 4 > 0$$

$2, 3, 4 > 0 \Rightarrow$ Positive definite

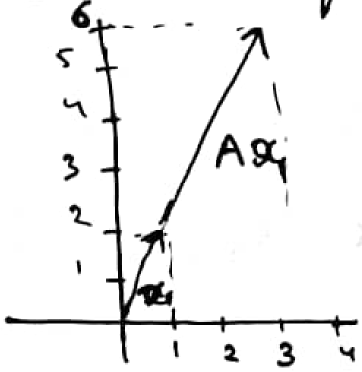
If x is an eigenvector of A then $x \neq 0$ and
$Ax = \lambda x$. In this case

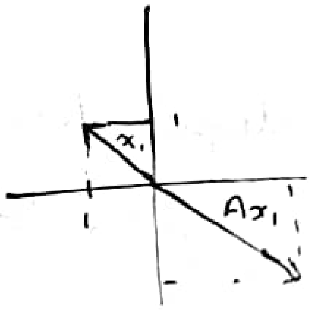
$x^T A x = \lambda x^T x$.

If $\lambda > 0$, then $x^T x > 0$ we must have $x^T A x > 0$

2a

Eigen vectors and eigen values are vectors + numbers associated to square mt.





Eigendecomposition

Let A be a square mt of order n. Also let $\{v_1, v_2, \ldots, v_n\}$ be the n linearly independent eigenvectors of A.

Then A can be factorized as

$$A = PDP^{-1}$$

The i^{th} column of P is the eigenvector v_i of A and D is the diagonal mt whose diagonal element are the corresponding eigenvalues of A

$\Rightarrow D_{ii} = \lambda_i$ ($\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

Please remember that not all square mt can be eigen decomposed.

A has non-degenerate eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ + corresponding eigenvectors (l.i) $x_1, x_2, x_3 \ldots x_n$

$$\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} \cdots \begin{bmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{bmatrix}$$

$$P = [x_1 \quad x_2 \quad \ldots \quad x_n] = \begin{bmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & O & \\ & O & \ddots & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix}$$

$$AP = A[x_1 \quad x_2 \ldots x_n] = \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \ldots & \lambda_n x_{n1} \\ \lambda_1 x_{12} & \lambda_2 x_{21} & \ldots & \lambda_n x_{n2} \\ \vdots & \vdots & & \\ \lambda_1 x_{1n} & \lambda_2 x_{2n} & & \lambda_n x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & x_{n1} \\ x_{12} & x_{22} & x_{n2} \\ \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} \ldots & x_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & O & \\ & O & \ddots & \\ 0 & & 0 & \lambda_n \end{bmatrix} = PD$$

$$APP^{-1} = PDP^{-1}$$

$$A = PDP^{-1}$$

Scanned with OKEN Scanner

a

$$\to \quad A^n = P D^n P^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PDIDP^{-1}$$

$$= PD^2P^{-1}$$

$$\to$$

$$A^{-1} = (PDP^{-1})^{-1} = (P[DP^{-1}])^{-1}$$

$$= \cancel{(D^{-1}P)} \qquad = [DP^{-1}]^{-1} P^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$$

$$= P D^{-1} P^{-1}$$

the inverse of diagonal mh is

$$D^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1/\lambda_n \end{bmatrix}$$

5a

Spectral decomposition

Let A be a real symmetric $n \times n$ mt with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ and corresponding orthonormal eigenvectors $v_1, v_2, \ldots v_n$. Then

$$A = \underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} \leftarrow & v_1 & \rightarrow \\ \leftarrow & v_2 & \rightarrow \\ & \vdots & \\ \leftarrow & v_n & \rightarrow \end{pmatrix}}_{Q^T}$$

If you see carefully Q is same as previous defined P.

→ Spectral decomposition typically give info. due to symmetric mt; while eigenvalue decomposition applies more generally to square mt.

As Q is orthogonal mt., we get $Q^{-1} = Q^T$

So, we get $A = Q D Q^T$ (or $P D P^T$)

$$A = \underbrace{\lambda_1 v_1 v_1^T}_{\substack{\text{mt of rank 1} \\ n \times n \text{ mt}}} + \underbrace{\lambda_2 v_2 v_2^T}_{\substack{\text{mt of rank 1} \\ n \times n \text{ mt}}} + \lambda_3 v_3 v_3^T + \ldots \underbrace{\lambda_n v_n v_n^T}_{\substack{\text{rank 1} \\ n \times n \text{ mt}}}$$

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

Spectral decomposition of A.

Note:→ Each mtx $v_i v_i^T$ for all $i = 1, \ldots n$ is the projection onto 1-D subspace spanned by v.

$P(x) = v_j v_j^T x$ projection map is an orthogonal projection onto the subspace spanned by the eigenvector v_j.

Consider 2×2 mtx. $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0 ; \quad (3-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda_1 = 1 + \lambda_2 = 5$$

$$\lambda_1 = \begin{pmatrix} 3-1 & 2 \\ 2 & 3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 2x_2 = 0 \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$2x_1 + 2x_2 = 0$$

$$\lambda_2 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{cases} -2x_1 + 2x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1a | $\lambda_1 = 1$; $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 5$; $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_1 \cdot v_2 = 0$. ortho gonal as eigenvalue .

$1^2 - 1^2 = 0$

normaliz. $\|v_1\|_2 = \sqrt{2}$

ortho normal

$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

$$A = Q D Q^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Spectral decomposition .

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T$$

$$v_1 v_1^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

$$v_2 v_2^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A = \cancel{3}, 1 \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 5 \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{5+1}{2} & \frac{5-1}{2} \\ \frac{5-1}{2} & \frac{5+1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

projection mt

Singular Value Decomposition (SVD)

We saw that if $A_{n\times n}$ is a symmetric mt then one can easily do a decomposition & get

$$A = PDP^T.$$

But if A is not a square mt

A is $m\times n$ mt Then ?

Singular Value Decomposition comes to the rescue!

Singular Value Decomposition is a central mt decomposition method in Linear Algebra.

Referred to as "Fundamental theorem of Linear Algebra"

Singular Values

Let $A \in \mathbb{R}^{m\times n}$. Consider the mt $A^T A$.

$A^T A$ will be symmetric $n\times n$ mt which is positive semi definite. The eigenvalues of $A^T A$ are

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n \geq 0.$$

Let $\sigma_i = \sqrt{\lambda_i}$

$$\Rightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_n \geq 0$$

The values $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n$ are called the singular values of A.

SVD

A singular value decomposition of a $m \times n$ mtx A is a factorization as

$$A = U \Sigma V^T$$

where

U is $m \times m$ orthogonal mtx

V is $n \times n$ orthogonal mtx

Σ is a $m \times n$ mtx where i^{th} diagonal entry equals to i^{th} singular value σ_i for $i = 1, 2, \ldots, r$

where
$$r = \text{rank}(A) \le \min(m, n)$$

All other non-diagonal entries Σ are zero.

$$\underset{m \times n}{\begin{bmatrix} & \end{bmatrix}} = \underset{\substack{m \times m \\ U}}{\begin{bmatrix} & \end{bmatrix}} \underset{m \times n}{\begin{bmatrix} \sigma_1 & 0 & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & \sigma_r & \\ & & & 0 \end{bmatrix}} \underset{\substack{n \times n \\ V}}{\begin{bmatrix} & \end{bmatrix}}$$

$U \rightarrow$ column of U are orthonormal eigenvectors $u_1, u_2, \ldots u_m$ of AA^T $\quad AA^T u_i = \sigma_i^2 u_i$

$V \rightarrow$ column of V are orthonormal eigenvectors $v_1, v_2, \ldots v_n$ of A^TA $\quad A^TA v_i = \sigma_i^2 v_i$

U

A is a $m \times n$ mt

$\Rightarrow AA^T$ is a $m \times m$ symmetric Positive Semi definite mt

$$AA^T u_i = \sigma_i^2 u_i \quad \text{for } i = 1, 2, \dots, m$$

$\sigma_i^2 = \lambda_i$

Take $u_1, u_2, \dots, u_m$ as orthonormal eigenvector of AA^T

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{m \times m}$$

V

A is a $m \times n$ mt

A^TA is $n \times n$ symmetric PSD mt

$$A^TA v_i = \sigma_i^2 v_i \qquad \sigma_i^2 = \lambda_i$$

for $i = 1, 2, \dots, n$

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{n \times n} \text{ mt}$$

Σ: ~~be A~~ A is $m \times n$ mt with rank$(A) = r$

then $r \leq \min(m, n)$

If $m \geq n$ then

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & & 0 \\ 0 & 0 & \sigma_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $m \geq n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \sigma_2 \quad 0 & \vdots \\ & 0 & \sigma_n \\ 0 & \cdots & 0 \end{bmatrix}$$

If $m < n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \cdots & & \cdots & 0 \\ \vdots & \sigma_2 & & 0 & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \cdots & \sigma_n & \cdots & 0 \end{bmatrix}$$

<u>Example</u>

$$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}$$

$\lambda_1 = 360$; $\lambda_2 = 90$ & $\lambda_3 = 0$ / $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $\sigma_3 = \sqrt{\lambda_3}$

$$v_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{3} & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix}$$

$\boxed{\text{L9 س1Σb.}}$

$$u_1 = \sigma_1^{-1} A v_1 = \frac{1}{6\sqrt{10}} \begin{pmatrix} 18 \\ 6 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$$

$$u_2 = \sigma_2^{-1} A v_2 = \frac{1}{3\sqrt{10}} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{pmatrix}$$

$$U = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix} \begin{pmatrix} 6/\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}^T$$