MTH201-3

1. Continuity

Definition 1. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to said to be continuous at a point $x_0 \in \mathbb{R}$, if

$$\text{for every real } \epsilon > 0, \text{ there is a real number } \delta > 0 \text{ such that}$$

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

This definition is equivalent to the definition in terms of left-hand and right hand limits, i.e., f is continuous at x_0 if and only if $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$.

In the above definition the domain can be replaced by a subset of $\mathbb{R}$. Let $X \subset \mathbb{R}$. A function $f : X \longrightarrow \mathbb{R}$ is said to said to be continuous at a point $x_0 \in X$, if

$$\text{for every real } \epsilon > 0, \text{ there is a real number } \delta > 0 \text{ such that}$$

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

If f is continuous at all the points of the domain, then we say that the function is continuous on the domain.

Example 2.
(i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x$. To show that this function is continuous at $a \in \mathbb{R}$. Let $\epsilon > 0$ be any positive real number. Then, we can see that $|f(x) - f(a)| = |x - a| < \epsilon$ will be achieved by $|x - a| < \delta$ if we choose $\delta = \epsilon$.
To check this, take $\delta = \epsilon$, then $|x - a| < \delta = |x - a| < \epsilon = |f(x) - f(a)| < \epsilon$. Hence f is continuous at a. As a is any point of $\mathbb{R}$, f is continuous on $\mathbb{R}$.
(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 2x$. To show that this function is continuous on $\mathbb{R}$. To prove this, let $\epsilon > 0$ be any positive real number. Then, we can see that $|f(x) - f(a)| = |2x - 2a| < \epsilon$ will be achieved by $|x - a| = |2x - 2a|/2 < \delta$ if we choose $\delta = \epsilon/2$.
To check this, take $\delta = \epsilon/2$, then $|x - a| < \delta = |x - a| < \epsilon/2 = |f(x) - f(a| < \epsilon$. Hence f is continuous at a.

It might be difficult to find a δ for a given ϵ for some functions. It might be helpful to have some other observations to determine the limit.

Proposition 3. *Let $f, g : X \longrightarrow \mathbb{R}$ be two functions. Let f and g be continuous on $X \subset \mathbb{R}$. Then*

(i) $f + g$ is continuous on X.
(ii) $f.g$ is continuous on X.
(iii) If g is continuous on the image set $f(X)$, then $g \circ f$ is continuous on X.
(iv) f/g is continuous on X if $g(x) \neq 0$ on X.

Proof. (i): Let $\epsilon > 0$. Since f is continuous on X, there exists $\delta_1 > 0$, such that $|x-a| < \delta_1 \implies |f(x) - f(a)| < \epsilon$.

Since g is continuous on X, there exists $\delta_2 > 0$, such that $|x-a| < \delta_2 \implies |g(x) - g(a)| < \epsilon$.

Both the statements above will be true simultaneously if $|x-a| < \min(\delta_1, \delta_2)$. Then, we have $|f(x) - f(a)| < \epsilon, |g(x) - g(a)| < \epsilon$. Then $|(f+g)(x) - f(a) - g(a)| = |f(x) - f(a) + g(x) - g(a)| < |f(x) - f(a)| + |g(x) - g(b)|\epsilon + \epsilon = 2\epsilon$, i.e., putting $\delta = \min(\delta_1, \delta_2)$, we have $|(f+g)(x) - (f+g)(a)| < 2\epsilon$. As ϵ varies over all the positive reals, 2ϵ also varies over all the positive reals. Hence, by definition, the limit of $f+g$ exists and $f+g$ is continuous on X.

(ii): First observe, for $A = f(a), B = g(a)$, the following:

$$\begin{aligned}
|(f.g)(x) - AB| = |f(x).g(x) - AB| &= |f(x)g(x) - Ag(x) + Ag(x) - AB| \\
&= |(f(x) - A)g(x) + A(g(x) - B)| \\
&\leqslant |(f(x) - A)||g(x)| + |A||(g(x) - B)|
\end{aligned}$$

by the triangle inequality. In these terms, we see that $|(f(x) - A)|$ small can be achieved, also $|(g(x) - B)|$ small can also be achieved. If the $|g(x)|$ term can be bounded, then the product $|(f(x) - A)||g(x)|$ will be small. This is what we do below.

Let $\epsilon > 0$, then there exists $\delta_1 > 0, \delta_2 > 0$, such that

$$|x-a| < \delta_1 \implies |(f(x) - A)| < \epsilon/(\epsilon + |B|)|,$$

and

$$|x-a| < \delta_2 \implies |(g(x) - B)| < \epsilon/(1 + |A|).$$

The second one means that $|g(x)| < \epsilon + |B|$, if $|x-a| < \delta_2$. So, now if we take $\delta = \min(\delta_1, \delta_2)$ and $|x-a| < \delta$, then all the above inequalities will hold simultaneously, and combining them together we get the following:

$$\begin{aligned}
|x-a| < \delta &\implies |(f(x) - A)| < \epsilon/(\epsilon + |B|)|, |(g(x) - B)| < \epsilon/(1 + |A|) \\
&\implies |(f(x) - A)||g(x)| + |A||(g(x) - B)| \leqslant \epsilon|g(x)|/(\epsilon + |B|) + |A|\epsilon/(1 + |A|) = 2\epsilon.
\end{aligned}$$

Therefore, $|x-a| < \delta \implies |(f.g)(x) - AB| < 2\epsilon$. As ϵ varies over all real numbers 2ϵ also varies over all real numbers. Hence, $f.g$ is continuous on X.

(iii), (iv): Try to prove on your own. □

Example 4. (i) $f(x) = x^3$. Then it a product of the three functions $g(x) = x$. By the results above, f is continuous.

(ii) $f(x) = 1/x$. Note that this a well-defined function from $\mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$. By (iv) of the previous proposition, f is continuous on the set $X = \mathbb{R}\backslash\{0\}$.

Theorem 5 (Fn is cont if and only if it takes conv seq to conv seq.)**.** *Let $X \subset \mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$. Then*
f is continuous at $a \in X$ if and only if for every sequence (a_n) in X that converges to a, the sequence $(f(a_n))$ converges to $f(a)$.

Proof. [$\implies$] Let f be continuous at a. Let (a_n) be a sequence in X such that $a_n \to a$.
Let $\epsilon > 0$ be any real mnumber.
Then $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.
As $a_n \to a$, for n sufficiently large, say $n > N$, $|a_n - a| < \delta$.
Hence $|f(a_n) - f(a)| < \epsilon$.
i.e., we began with $\epsilon > 0$ and obtained an $N \in \mathbb{N}$ such that $|f(a_n) - f(a)| < \epsilon$.
i.e., $f(a_n) \to f(a)$.

[$\impliedby$] Suppose f is not continuous at a. Then there is a positive $\epsilon_0 > 0$, such that no matter how small $|x - a|$ is, $|f(x) - f(a)| > \epsilon_0$.
So if $|x_1 - a| < 1$, then $|f(x_1) - f(a)| > \epsilon_0$
$|x_2 - a| < 1/2$, then $|f(x_2) - f(a)| > \epsilon_0$
$|x_3 - a| < 1/3$, then $|f(x_3) - f(a)| > \epsilon_0$
$\cdots$
$|x_n - a| < 1/n$, then $|f(x_n) - f(a)| > \epsilon_0$
Then $x_n \to a$, but $f(x_n) \nrightarrow f(a)$.
This proves the theorem. $\square$

Following are some properties of continuous functions on a closed interval.

Definition 6. Recall that a function $f : X \longrightarrow \mathbb{R}$ is

- bounded above if we can find a real number M, such that $f(x) \leqslant M$, for all $x \in X$,
- bounded below if we can find a real number m, such that $f(x) \geqslant m$, for all $x \in X$.
- bounded if f is bounded above as well as below.

Note that f is a bounded function if and only if the function defined by $|f(x)|$ is bounded.

Theorem 7 (Cont fn on a closed interval is bounded)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is a bounded function.*

Proof. Suppose, f is not bounded.
Then for every positive integer n, there will be an element x_n in the domain $[a, b]$, such that $|f(x_n)| > n$.
Then the elements $\{x_n : n \in \mathbb{N}\}$ form a sequence that lies in the closed interval $[a, b]$.
By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges, say to ℓ.
By continuity of f: $f(x_{n_k}) \to f(\ell)$.
By the reverse triangle inequality, we have $|f(x_{n_k})| \to |f(\ell)|$.
So for $\epsilon = 1$, we have $|f(x_{n_k})| - 1 < |f(\ell)|$ for k sufficiently large.
From this, we can see that $n_k - \epsilon < f(\ell)$, which is absurd as $n_k \to \infty$. $\square$

Above theorem is not necessarily true for continuous functions on a open interval. For example, $f(x) = 1/x, \forall x \in (0, 1]$.

Definition 8. Let $f : X \longrightarrow \mathbb{R}$ be a continuous function on X. Consider the set $\{f(x) : x \in X\}$. Then

- an element $x_0 \in X$ is called a maximum for f if $f(x) \leqslant f(x_0), \forall x \in X$, and
- an element $y_0 \in X$ is called a minimum for f if $f(y_0) \leqslant f(x), \forall x \in X$

Example 9. (i) $\sin : [0, 1] \longrightarrow \mathbb{R}$ is a continuous function. Here the minimum is $\sin 0 = 0$, and the maximum is $\sin 1$.

(ii) What if we change the domain to $(0, 1)$ and consider the function $\sin : (0, 1) \longrightarrow \mathbb{R}$? What is the minimum and the maximum?

Proposition 10 (Cont fn on a closed interval attains its bounds)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a closed interval $[0, 1]$. Then there exists α, β in $[a, b]$, such that the value of $f(\alpha)$ is the maximum value and $f(\beta)$ is the minimum value of f.*

Proof. Let $S = \{f(x) : x \in [a, b]\}$. We saw in the above theorem, that this is a bounded set. Let $M = \sup S, m = \inf S$. Then there is a sequence of elements in S, that converges to M. Let $\{s_n\}$ denote this sequence. Each $s_n = f(x_n)$ for some $x_n \in [a, b]$.
By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that is convergent, say $x_{n_k} \to \alpha$.
This limit L is in $[a, b]$, as it is a closed interval.
By continuity, we have $f(x_{n_k}) \to f(\alpha)$,
so $f(\alpha) = M$.

The next part $f(\beta) = m$ for some $\beta \in [a, b]$ is left as an exercise. Use infimum instead of supremum. □

Proposition 11 (Intermediate Value Property)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, and let $f(a) < t < f(b)$. Then, there exists $c \in (a, b)$, such that $f(c) = t$.*
i.e., Cont fn attains every value between the values at the end points.

Proof. Let $T := \{x \in [a, b] : f(x) \leqslant t\}$. $T \neq \varnothing$ (Why?)
T is bounded above, so T has a supremum.
Let $c = \sup T$.
Then we have a sequence (x_n), such that $f(x_n) \to t$.
Observe that (x_n) has a convergent subsequence (x_{n_k}), say $x_{n_k} \to c$. (Why $c \in [a, b]$?)
Then $f(x_{n_k}) \to f(c)$.
Therefore $f(c) \leqslant t$.
As c is the supremum of T, anything bigger than c is not in T.
So $y_n = c + \dfrac{b-a}{n} \in T$, (why $y_n \in [a, b]$?)
Therefore, $f(y_n) > t$.
Observe that $y_n \to c$, so $f(y_n) \to f(c)$.
Hence $t \geqslant f(c)$.
Finally, $f(c) = t$. □

Theorem 12. *Any polynomial $P(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$ of odd degree with real coefficients have a real root.*

Proof. Strategy is to show that as x is sufficiently large, $P(x)$ is positive, while if x is sufficiently small and negative, then $P(x)$ is negative. Then use the Intermediate value property as polynomials are continuous functions on $\mathbb{R}$.

Each coefficient $a_j \leqslant |a_j|$ and $x^j \leqslant |x|^j$. We will not change the highest power term x^{2n+1}. Then

$$\begin{aligned} P(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0 &\leqslant x^{2n+1} + |a_{2n}||x|^{2n} + \cdots + |a_1||x| + |a_0| \\ &\leqslant x^{2n+1} + |a_{2n}||x|^{2n} + \cdots + |a_1||x|^{2n} + |a_0||x|^{2n}, (\because |x|^{2n} \geqslant |x|^j, \forall j \\ &= x^{2n+1} + |x|^{2n}(|a_{2n}| + \cdots + |a_1| + |a_0|) \\ &= x^{2n+1} + x^{2n}(|a_{2n}| + \cdots + |a_1| + |a_0|) \\ &= x^{2n}(x + (|a_{2n}| + \cdots + |a_1| + |a_0|)) \end{aligned}$$

Now take x negative such that $x + (|a_{2n}| + \cdots + |a_1| + |a_0|)$ is negative. However, the term x^{2n} will be positive as the power is even. Eventually the product $x^{2n}(x + (|a_{2n}| + \cdots + |a_1| + |a_0|))$ is negative. Hence $P(x) < 0$ for x negative with $x + (|a_{2n}| + \cdots + |a_1| + |a_0|)$ negative.

To see that $P(x) > 0$ for x sufficiently large and positive, we use $a_j \geqslant -|a_j|$, and proceed similarly. Try this out.
Then the theorem follows by the IVP. □

Example 13. If $f : [0,1] \longrightarrow [0,1]$ is a continuous function, then show that there is an x_0 such that $f(x_0) = x_0$. Here consider the function $g(x) = f(x) - x$. If $g(0) = 0$ or $g(1) = 0$, then we are done. Now assume that $g(0) \neq 0 \neq g(1)$.
Then $g(0) \geqslant 0, g(1) \leqslant 0$.
Therefore the statement follows from the IVP.