Assignment 2

1. (Jackson 2.3) A straight line charge with constant linear charge density λ is located perpendicular to the $x-y$ plane in the first quadrant (x_0, y_0). The intersecting planes $x = 0, y \geq 0$ and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

   (a) The well know potential for an isolated line charge at (x_0, y_0) is $\Phi(x, y) = (\lambda/4\pi\epsilon_0)\ln(R^2/r^2)$, where $r^2 = (x - x_0)^2 + (y - y^0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.

   (b) Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot σ/λ versus x for $(x_0 = 2, y_0 = 1)$, $(x_0 = 1, y_0 = 1)$ and $(x_0 = 1, y_0 = 2)$.

   (c) Show that the total charge per unit length in z on the plane $y = 0, x \geq 0$ is

   $$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right).$$

   What is the total charge on the plane $x = 0$?

   (d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{x^2 + y^2}$ and $\rho_0 = \sqrt{x_0^2 + y_0^2}$], the leading term in the potential is

   $$\Phi \to \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4}.$$

   Interpret.

2. (Jackson 2.7) Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

   (a) Write down the appropriate Green function $G(\mathbf{r}, \mathbf{r}')$.

   (b) If the potential on the plane $z = 0$ is specified to be V_0 inside a circle of radius R centered on the origin, and $V = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

   (c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

   $$V = V_0\left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right).$$

   (d) Show that at large distances ($\rho^2 + z^2 >> R^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$ and that the leading terms are

   $$V = \frac{V_0 R^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3R^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 R^2 + R^4)}{8(\rho^2 + z^2)^2} + \ldots\ldots\right]$$

   Verify that the results of parts (c) and (d) are consistent with each other in their common range of validity.

3. (Jackson 2.12 & 2.13)

   (a) Find the Green's function for the two dimensional potential problem with the potential specified on the surface of a cylinder of radius b, and show that the solution inside the cylinder is given by Poisson's integral :

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi' - \phi)} d\phi'$$

   (b) Two halves of a long conducting cylinder of radius b are separated by a small gap, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2b\rho}{b^2 - \rho^2} \cos\phi\right)$$

   where ϕ is measured from a plane perpendicular to the plabe through the gap.

   (c) Calculate the surface charge density on each half of the cylinder.

   (d) What modification is necessary in (a) if the potential is desired in the region of space bounded by the cylinder and infinity ?

4. (Jackson 2.23) A hollow cube has conducting walls defined by six planes $x = 0, y = 0, z = 0$, and $x = a, y = a, z = a$. The walls $z = 0$ and $z = a$ are held at constant potential V_0. The other four sides are at zero potential.

   (a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

   (b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls.

   (c) Find the surface-charge density on the surface $z = a$.