1. (a) $\vec{E} = (\alpha, 0, 0)$; $\vec{B} = \left(\frac{\alpha}{c}, 0, \frac{2\alpha}{c}\right)$; $\vec{E}' = (E_x', \alpha, 0)$; $\vec{B}' = \left(\frac{\alpha}{c}, B_y', \frac{\alpha}{c}\right)$

For observer 'S', electromagnetic invariants are

$$\vec{E}\cdot\vec{B} = \alpha\cdot\frac{\alpha}{c} + 0\cdot 0 + 0\cdot\frac{2\alpha}{c} = \frac{\alpha^2}{c}$$

$$E^2 - c^2B^2 = \alpha^2 + 0 + 0 - c^2\left(\frac{\alpha^2}{c^2} + 0 + 4\frac{\alpha^2}{c^2}\right).$$

$$= -4\alpha^2$$

For observer 'S'', electromagnetic invariants are,

$$\vec{E}'\cdot\vec{B}' = E_x'\frac{\alpha}{c} + \alpha B_y' + 0\cdot\frac{\alpha}{c} = E_x'\frac{\alpha}{c} + \alpha B_y'$$

$$E'^2 - c^2B'^2 = E_x'^2 + \alpha^2 + 0 - c^2\left(\frac{\alpha^2}{c^2} + B_y'^2 + \frac{\alpha^2}{c^2}\right).$$

$$= E_x'^2 + \alpha^2 - 2\alpha^2 - c^2B_y'^2$$

$$= E_x'^2 - c^2B_y'^2 - \alpha^2$$

Now, $\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B}$

$$\Rightarrow \frac{\alpha^2}{c} = E_x'\frac{\alpha}{c} + \alpha B_y' \Rightarrow E_x' + cB_y' = \alpha \quad \text{—(1)}$$

$$E'^2 - c^2B'^2 = E^2 - c^2B^2 \Rightarrow E_x'^2 - c^2B_y'^2 - \alpha^2 = -4\alpha^2$$

$$\Rightarrow E_x'^2 - c^2B_y'^2 = -3\alpha^2 \quad \text{—(2)}$$

From (2), $(E_x' - cB_y')(E_x' + cB_y') = -3\alpha^2$

$$\Rightarrow \alpha\cdot(E_x' - cB_y') = -3\alpha^2 \Rightarrow E_x' - cB_y' = -3\alpha \quad \text{—(3)}$$

Adding (1) & (3) $\Rightarrow 2E_x' = -2\alpha \Rightarrow \boxed{E_x' = -\alpha}$

$$\therefore cB_y' = \alpha - E_x' = 2\alpha \Rightarrow \boxed{B_y' = \frac{2\alpha}{c}}.$$

$\therefore \vec{E}' = (-\alpha, \alpha, 0)$ & $\vec{B}' = \left(\frac{\alpha}{c}, \frac{2\alpha}{c}, \frac{\alpha}{c}\right)$

(b) Field transformation between S' & S'' moving with velocity $v\hat{x}$ relative to one another;

$$\vec{E}''_{\parallel} = \vec{E}'_{\parallel} \quad ; \quad \vec{E}''_{\perp} = \gamma(\vec{E}'_{\perp} + \vec{v}\times\vec{B}'_{\perp})$$

$$\vec{B}''_{\parallel} = \vec{B}'_{\parallel} \quad ; \quad \vec{B}''_{\perp} = \gamma(\vec{B}'_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}'_{\perp})$$

$$\therefore \vec{E}''_{\parallel} = -\alpha\hat{x} \,, \quad \vec{E}''_{\perp} = \gamma\left[(\alpha\hat{y} + 0.\hat{z}) + v\hat{x}\times\left(\frac{2\alpha}{c}\hat{y} + \frac{\alpha}{c}\hat{z}\right)\right]$$

$$= \gamma\left[\alpha\hat{y} + \frac{2\alpha v}{c}\hat{z} + \frac{\alpha v}{c}(-\hat{y})\right]$$

$$= \gamma\left[\alpha\left(1-\frac{v}{c}\right)\hat{y} + \frac{2\alpha v}{c}\hat{z}\right].$$

$$\therefore \vec{E}'' = \vec{E}''_{\parallel} + \vec{E}''_{\perp} = -\alpha\hat{x} + \alpha\gamma\left(1-\frac{v}{c}\right)\hat{y} + \frac{2\alpha\gamma v}{c}\hat{z}$$

$$\vec{B}''_{\parallel} = \frac{\alpha}{c}\hat{x} \quad ; \quad \vec{B}''_{\perp} = \gamma\left[\frac{2\alpha}{c}\hat{y} + \frac{\alpha}{c}\hat{z} - \frac{v}{c^2}\hat{x}\times(\alpha\hat{y} + 0.\hat{z})\right]$$

$$= \gamma\left[\frac{2\alpha}{c}\hat{y} + \frac{\alpha}{c}\hat{z} - \frac{v\alpha}{c^2}\hat{z}\right]$$

$$= \gamma\left[\frac{2\alpha}{c}\hat{y} + \frac{\alpha}{c}\left(1-\frac{v}{c}\right)\hat{z}\right]$$

$$\therefore \vec{B}'' = \vec{B}''_{\parallel} + \vec{B}''_{\perp} = \frac{\alpha}{c}\hat{x} + \frac{2\alpha\gamma}{c}\hat{y} + \frac{\alpha\gamma}{c}\left(1-\frac{v}{c}\right)\hat{z}$$

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2. $\dfrac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$; $\dfrac{\partial G^{\mu\nu}}{\partial x^\nu} = 0.$; $J^\mu = (c\rho, J_x, J_y, J_z)$

$\mu = 1$: $\dfrac{\partial F^{1\nu}}{\partial x^\nu} = \dfrac{\partial F^{10}}{\partial x^0} + \dfrac{\partial F^{11}}{\partial x^1} + \dfrac{\partial F^{12}}{\partial x^2} + \dfrac{\partial F^{13}}{\partial x^3}$

Now, $F^{10} = -\dfrac{E_x}{c}$; $F^{11} = 0$, $F^{12} = B_z$, $F^{13} = -B_y$

$x^0 = ct$; $x^1 = x$; $x^2 = y$; $x^3 = z$.

$$\therefore \frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial}{\partial (ct)}\left(-\frac{E_x}{c}\right) + \frac{\partial}{\partial x}(0) + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$$

$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}.$$

$$= \left(-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x$$

$\mu_0 J^1 = \mu_0 J_x.$

$$\therefore \left[-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right]_x = \mu_0 J_x$$

Similarly, putting $\mu = 2$ & $\mu = 3$. gives the other two components.

$$\therefore \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}. \quad \left(\because c^2 = \frac{1}{\mu_0\epsilon_0}\right)$$

$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$. For $\mu = 1$,

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3}$$

$G^{10} = -B_x$, $G^{11} = 0$, $G^{12} = -\frac{E_z}{c}$, $G^{13} = \frac{E_y}{c}$.

$x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\therefore \frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial}{\partial (ct)}(-B_x) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}\left(-\frac{E_z}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E_y}{c}\right)$$

$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z}$$

$$= -\frac{1}{c}\left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E}\right)_x$$

$$\therefore \frac{\partial G^{1\nu}}{\partial x^\nu} = 0 \Rightarrow \left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E}\right)_x = 0.$$

Similarly. $\mu = 2$ & $\mu = 3$ gives the other 2 components.

$$\therefore \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

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charges neutral. What is the magnitude of the force on the pion? What is the magnitude of the force on the proton? What about Newton's third law?

3. $\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,', t_r)}{|\vec{r}-\vec{r}\,'|} d^3r'$ $\quad t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$

$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,', t_r)}{|\vec{r}-\vec{r}\,'|} d^3r'$

Electric and magnetic fields are given by,

$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$; $\vec{B} = \vec{\nabla}\times\vec{A}$.

$\vec{\nabla}\Phi = \frac{1}{4\pi\epsilon_0}\int \vec{\nabla}\left(\frac{\rho}{\mathscr{r}}\right) d^3r'$ $\quad \mathscr{r} = |\vec{r}-\vec{r}\,'|$.

$= \frac{1}{4\pi\epsilon_0}\int \left[(\vec{\nabla}\rho)\frac{1}{\mathscr{r}} + \rho\,\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right] d^3r'$

$\vec{\nabla}\rho \equiv \vec{\nabla}\rho\left(\vec{r}\,', t - \frac{\mathscr{r}}{c}\right) = \dot{\rho}\,\vec{\nabla}t_r = -\frac{\dot{\rho}}{c}\vec{\nabla}\mathscr{r}$

$= -\frac{\dot{\rho}}{c}\hat{\mathscr{r}}$

Also, $\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$.

$\therefore \vec{\nabla}\Phi = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right] d^3r'$

$\therefore$ ~~$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$~~ $\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\dot{\vec{J}}}{\mathscr{r}} d^3r'$

$\therefore \vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}\,', t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \mu_0\epsilon_0\frac{\dot{\vec{J}}(\vec{r}\,', t_r)}{\mathscr{r}}\right] d^3r'$

$\therefore \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}}{c^2\mathscr{r}}\right] d^3r'}$

$$\vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left[\frac{\vec{J}(\vec{r}', t-\frac{\mathscr{r}}{c})}{\mathscr{r}}\right] d^3r'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}(\vec{\nabla}\times\vec{J}) - \vec{J}\times\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right] d^3r'$$

$$(\vec{\nabla}\times\vec{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z}.$$

$$\frac{\partial J_z}{\partial y} = \dot{J}_z \frac{\partial t_r}{\partial y} = \dot{J}_z \frac{\partial}{\partial y}\left(t-\frac{\mathscr{r}}{c}\right) = -\frac{\dot{J}_z}{c}\frac{\partial \mathscr{r}}{\partial y}$$

Similarly, $\frac{\partial J_y}{\partial z} = -\frac{\dot{J}_y}{c}\frac{\partial \mathscr{r}}{\partial z}$.

$$\therefore [\vec{\nabla}\times\vec{J}]_x = -\frac{1}{c}\left(\dot{J}_z\frac{\partial \mathscr{r}}{\partial y} - \dot{J}_y\frac{\partial \mathscr{r}}{\partial z}\right)$$

Note, $\dot{\vec{J}}\times\vec{\nabla}\mathscr{r} = (\dot{J}_x\hat{x} + \dot{J}_y\hat{y} + \dot{J}_z\hat{z})\times\left(\frac{\partial \mathscr{r}}{\partial x}\hat{x} + \frac{\partial \mathscr{r}}{\partial y}\hat{y} + \frac{\partial \mathscr{r}}{\partial z}\hat{z}\right).$

$$\therefore (\dot{\vec{J}}\times\vec{\nabla}\mathscr{r})_x = \left(\dot{J}_y\frac{\partial \mathscr{r}}{\partial z} - \dot{J}_z\frac{\partial \mathscr{r}}{\partial y}\right)$$

$$\therefore [\vec{\nabla}\times\vec{J}]_x = +\frac{1}{c}[\dot{\vec{J}}\times\vec{\nabla}\mathscr{r}]_x = +\frac{1}{c}[\dot{\vec{J}}\times\hat{\mathscr{r}}]_x$$

$$\therefore \vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}c}(\dot{\vec{J}}\times\hat{\mathscr{r}}) + \vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right] d^3r'$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}', t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\, d^3r'}$$

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proton. What about Newton's

4. $$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = R$$ ← 'new' equation of continuity.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} - \lambda\Phi$$ ← "new" Gauss' law.

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \lambda\vec{A}$$ ← "new" Ampere-Maxwell law.

Taking the divergence of the last equation,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J}) + \frac{1}{c^2}\vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right) - \vec{\nabla}\cdot(\lambda\vec{A})$$

$$\Rightarrow \quad 0 = \mu_0(\vec{\nabla}\cdot\vec{J}) + \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) - \lambda\vec{\nabla}\cdot\vec{A}.$$

$$= \mu_0\left[R - \frac{\partial \rho}{\partial t}\right] + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\rho}{\epsilon_0} - \lambda\Phi\right] - \lambda\vec{\nabla}\cdot\vec{A}$$

$$= \mu_0 R - \mu_0\frac{\partial \rho}{\partial t} + \frac{1}{\epsilon_0 c^2}\frac{\partial \rho}{\partial t} - \frac{\lambda}{c^2}\frac{\partial \Phi}{\partial t} - \lambda\vec{\nabla}\cdot\vec{A}$$

$$= \mu_0 R - \cancel{\mu_0\frac{\partial \rho}{\partial t}} + \cancel{\mu_0\frac{\partial \rho}{\partial t}} - \lambda\left[\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right]$$

$$= \mu_0 R - \lambda\left[\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right]$$

$$\therefore \quad \boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = \frac{\mu_0}{\lambda}R}$$

The theory is not gauge invariant because the equation chooses a gauge.

5. (a) Magnetic field for the solenoid,

$$B = \mu_0 n I_0 .$$

: uniform for $t > t_0$.

$\therefore$ Total field energy stored in the solenoid/time.

$$U_B = \frac{1}{2\mu_0} \int B^2 \, d\tau = \frac{1}{2\mu_0} (\mu_0 n I)^2 . (\pi R^2 L)$$

$$= \frac{\mu_0 \pi R^2 n^2 I_0^2 L}{2}$$

(b) During the time $0 < t < t_0$, current is changing & the electric field can be obtained as.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 n \frac{dI}{dt} \hat{z}$$

Using Stokes's theorem, $\oint \vec{E} . d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} . d\vec{a}$

$$\Rightarrow \quad E . 2\cancel{\pi} \cancel{r} = -\mu_0 n \frac{dI}{dt} (\cancel{\pi} r^{\cancel{2}})$$

$$\Rightarrow \quad \vec{E} = -\mu_0 n \frac{I_0}{2 t_0} r \hat{\phi} . \left(\because \frac{dI}{dt} = \frac{I_0}{t_0} \right).$$

(c) Poynting vector, (at $r = R$)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \left[-\mu_0 n \frac{I_0}{2t_0} R \hat{\phi} \right] \times \left[\mu_0 n \frac{I_0 t}{t_0} \hat{z} \right]$$

To see the last part, note that current increases linearly, i.e. $\frac{dI}{dt} =$ const. $= A$ (say)

$$\Rightarrow \quad I = At + \text{const}.$$

$$t = 0 , \ I = 0 \Rightarrow \text{const} = 0$$

$$t = t_0 , \ I = I_0 \Rightarrow A = \frac{I_0}{t_0} .$$

$$\therefore I(t) = \left(\frac{I_0}{t_0} \right) t .$$

$$\therefore \vec{S} = -\frac{\mu_0 n^2 I_0^2 t R}{2 t_0^2}\hat{r}$$

(d) Energy flux, $= \int \vec{S} \cdot d\vec{a}$

$$= \left(\frac{\mu_0 n^2 I_0^2}{\cancel{2} t_0^2} t R\right) \cdot (\cancel{2}\pi R L)$$

$$= \mu_0 \pi L n^2 R^2 \frac{I_0^2}{t_0^2} t.$$

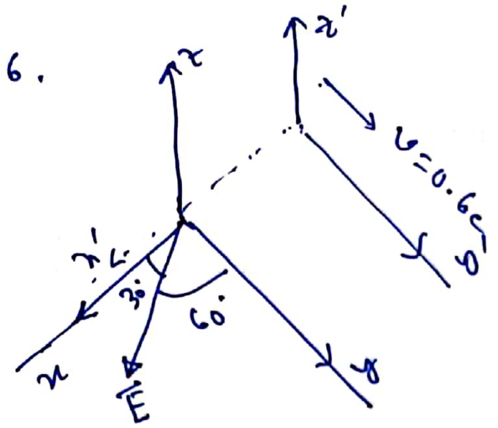
Integrated from $t=0$ to $t=t_0$,

$$\text{Energy/time} = \frac{\mu_0 \pi R^2 n^2 I_0^2 L}{\cancel{t_0^2}} \left.\frac{t^2}{2}\right|_0^{t_0}$$

$$= \frac{\mu_0 \pi R^2 n^2 I_0^2 L}{2}.$$

= same as in (c).

6.



Frame F : $\vec{E}$: $\begin{cases} E_x = 100\cos 30° \\ E_y = 100\sin 30° \\ E_z = 0. \end{cases}$

$\vec{B} = 0.$

Transformation equations:

$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$; $\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp})$

$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$; $\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp})$.

$\vec{v} = 0.6c\,\hat{y}$: The component $\parallel'$ to direction of relative motion is the $\hat{y}$ component.

$\therefore E_y' = E_y$ Also, $\because \vec{B} = 0$, $\therefore \vec{E}'_{\perp} = \gamma\vec{E}_{\perp}$

$\therefore E_x' = E_x\gamma$; $E_z' = E_z\gamma = 0$ ($\because E_z = 0$)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-(0.6)^2}} = 1.25.$$

$\therefore E_x' = 1.25\,E_x$, $E_y' = E_y$; $E_z' = 0$.

$$\therefore \tan\phi' = \frac{E_y'}{E_x'} = \frac{1}{1.25}\frac{E_y}{E_x} \Rightarrow \phi' = \tan^{-1}\left(\frac{E_y}{1.25\,E_x}\right)$$

$$|\vec{E}'| = \sqrt{E_x'^2 + E_y'^2 + E_z'^2} = \cancel{\sqrt{E_x'^2+E_y'^2}} = \sqrt{E_x'^2 + E_y'^2}$$

Now, $E_x = 100\cos 30° \simeq 86.6$ V/m ; $E_y = 100\sin 30° = 50$ V/m

$$\therefore \phi' = \tan^{-1}\left(\frac{50}{1.25\times 86.6}\right) \simeq 24.8°.$$

$$|\vec{E}'| = \sqrt{(1.25\times 86.6)^2 + (50)^2} \simeq 119.3 \text{ V/m}.$$

For the magnetic field, $\vec{B}'_{\parallel} = 0 \Rightarrow B_y' = 0$.

$$\vec{B}'_{\perp} = -\frac{\gamma}{c^2}\vec{v}\times\vec{E}_{\perp} \Rightarrow B_z' = -\frac{1.25}{c^{\cancel{2}}}\hat{y}(0.6\cancel{c})\times\hat{x}E_x.$$

$$= 0.75\frac{E_x}{c}\hat{z} \simeq (2.17\times 10^{-7}\text{ T})$$

$B_x' = 0$. $\therefore$ Magnetic field points in z-direction & has magnitude 2.17×10^{-7} T.

proton? What about Newton's third law?

7. $\vec{K}(\vec{r}) = \sigma \omega R \sin\theta \,\hat{\phi}$

Vector potential, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \oint \frac{\vec{K}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} da'$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 \sigma \omega R}{4\pi} \int \frac{\sin\theta' \hat{\phi}' R^2 \sin\theta' d\theta' d\phi'}{|\vec{r}-\vec{r}\,'|}.$$

$$= \frac{\mu_0 \sigma \omega R^3}{4\pi} \int \frac{\sin^2\theta' \hat{\phi}' d\theta' d\phi'}{|\vec{r}-\vec{r}\,'|}$$

$\hat{\phi}' = -\sin\phi' \hat{x}' + \cos\phi' \hat{y}'$

& $$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi).$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 \sigma \omega R^3}{4\pi} \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) \times$$

$$\int Y^*_{lm}(\theta',\phi') [-\sin\phi' \hat{x}' + \cos\phi' \hat{y}'] \sin\theta' d\Omega'$$

where, $d\Omega' = \sin\theta' d\theta' d\phi'$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 \sigma \omega R^3}{4\pi} \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi).$$

$$\times \int Y^*_{lm}(\theta',\phi') [-\sin\theta' \sin\phi' \hat{x}' + \sin\theta' \cos\phi' \hat{y}'] d\Omega'$$

Now, $$\sin\theta' \sin\phi' = i\sqrt{\frac{2\pi}{3}} [-Y_{11}(\theta',\phi') - Y_{1-1}(\theta',\phi')]$$

$$\sin\theta' \cos\phi' = \sqrt{\frac{2\pi}{3}} [-Y_{11}(\theta',\phi') + Y_{1-1}(\theta',\phi')]$$

$$\therefore \int_{\phi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1} Y^*_{lm}(\theta',\phi')\left[-\sin\theta'\sin\phi'\,\hat{x}+\sin\theta'\cos\phi'\,\hat{y}\right]d(\cos\theta')\,d\phi'$$

$$=\sqrt{\frac{2\pi}{3}}\int_{\phi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1} Y^*_{lm}(\theta',\phi')\Big[-i\left(Y_{11}(\theta',\phi')+Y_{1-1}(\theta',\phi')\right)\hat{x}$$
$$+\left(-Y_{11}(\theta',\phi')+Y_{1-1}(\theta',\phi')\right)\hat{y}\Big]d(\cos\theta')\,d\phi'$$

$$=\sqrt{\frac{2\pi}{3}}\int_{\phi'=0}^{2\pi}\int_{\cos\theta'=-1}^{1}\Big[-i\left\{Y^*_{lm}(\theta',\phi')Y_{11}(\theta',\phi')+Y^*_{lm}(\theta',\phi')Y_{1-1}(\theta',\phi')\right\}\hat{x}$$
$$+\left\{-Y^*_{lm}(\theta',\phi')Y_{11}(\theta',\phi')+Y^*_{lm}(\theta',\phi')Y_{1-1}(\theta',\phi')\right\}\hat{y}\Big]$$
$$d(\cos\theta')\,d\phi'$$

$$=\sqrt{\frac{2\pi}{3}}\left[-i\left(\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right)\hat{x}+\left(-\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right)\hat{y}\right]$$

$$\therefore \vec{A}(\vec{r})=\frac{\mu_0\sigma\omega R^3}{4\pi}\cdot\sqrt{\frac{2\pi}{3}}\cdot\sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y_{lm}(\theta,\phi)\Big[-i\left(\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right)\hat{x}$$
$$+\left(-\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right)\hat{y}\Big]$$

$$=\frac{\mu_0\sigma\omega R^3}{\cancel{4\pi}}\sqrt{\frac{2\pi}{3}}\cdot\frac{\cancel{4\pi}}{2.1+1}\frac{r_<}{r_>^2}\Big[-i\left\{Y_{11}(\theta,\phi)+Y_{1-1}(\theta,\phi)\right\}\hat{x}$$
$$+\left\{-Y_{11}(\theta,\phi)+Y_{1-1}(\theta,\phi)\right\}\hat{y}\Big]$$

$$=\frac{\mu_0\sigma\omega R^3}{3}\cdot\frac{r_<}{r_>^2}\left[-\sin\theta\sin\phi\,\hat{x}+\sin\theta\cos\phi\,\hat{y}\right].$$

$$=\frac{\mu_0\sigma\omega R^3}{3}\frac{r_<}{r_>^2}\sin\theta\left(-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}\right).$$

$$=\frac{\mu_0\sigma\omega R^3}{3}\frac{r_<}{r_>^2}\,\cancel{\phi}\,\sin\theta\,\hat{\phi}$$

proton? What about Newton's third law?

- Inside the sphere, $r_< = r$; $r_> = R$.

$$\therefore \vec{A}_{in} = \frac{1}{3}\mu_0 \sigma \omega R^3 \cdot \frac{r}{R^2} \sin\theta \,\hat{\phi}$$
$$= \frac{1}{3}\mu_0 \sigma \omega R r \sin\theta \,\hat{\phi}$$

- Outside the sphere, $r_< = R$, $r_> = r$

$$\therefore \vec{A}_{out} = \frac{1}{3}\mu_0 \sigma \omega R^3 \cdot \frac{R}{r^2} \sin\theta \,\hat{\phi}$$
$$= \frac{1}{3}\mu_0 \sigma \omega \frac{R^4}{r^2} \sin\theta \,\hat{\phi}.$$

(b) $\vec{B} = \vec{\nabla} \times \vec{A}$ Since $\vec{A}$ has only the $\hat{\phi}$ component,

$$\therefore \vec{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\hat{\theta}$$

$$\therefore \vec{B}_{in} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{1}{3}\mu_0\sigma\omega R r \sin^2\theta\right]\hat{r}$$
$$- \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{3}\mu_0\sigma\omega R r^2 \sin\theta\right]\hat{\theta}$$

$$= \frac{1}{\cancel{r}\cancel{\sin\theta}}\frac{1}{3}\mu_0\sigma\omega R\cancel{r}\cdot 2\cancel{\sin\theta}\cos\theta\,\hat{r}$$
$$- \frac{1}{\cancel{r}}\cdot\frac{1}{3}\mu_0\sigma\omega R\sin\theta\cdot 2\cancel{r}\,\hat{\theta}$$

$$= \frac{2}{3}\mu_0\sigma\omega R(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}).$$

$$= \frac{2}{3}\mu_0\sigma\omega R\,\hat{z} = \frac{2}{3}\mu_0\sigma R\,\vec{\omega}$$

$\therefore$ Field inside the spherical shell is uniform.

$$\vec{B}_{out} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{1}{3}\mu_0\sigma\omega\frac{R^4}{r^2}\sin^2\theta\right]\hat{r}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{3}\mu_0\sigma\omega\frac{R^4}{r}\sin\theta\right]\hat{\theta}$$

$$= \frac{1}{r\cancel{\sin\theta}}\,\frac{1}{3}\cdot\mu_0\sigma\omega\frac{R^4}{r^2}\,2\cancel{\sin\theta}\cos\theta\,\hat{r}$$

$$-\frac{1}{r}\cdot\frac{1}{3}\mu_0\sigma\omega\left(-\frac{R^4}{r^2}\right)\sin\theta\,\hat{\theta}.$$

$$= \frac{1}{3}\mu_0\sigma\omega\frac{R^4}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0}{4\pi}\cdot\left(\frac{4}{3}\pi R^3\right)(\sigma R\omega)\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right].$$

$$= \frac{\mu_0}{4\pi}\;\frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3}.$$

where $|\vec{m}| = \left(\frac{4}{3}\pi R^3\right)(\sigma R\omega)$, $\sigma R\omega$ = magnetization

$\Rightarrow$ magnetic dipole moment.

$\therefore$ Field outside is due to a magnetic dipole of magnetic moment $\vec{m}$.