## MTH201-4

### 1. Differentiability

**Definition 1.** Let $I$ be a set that contains an open interval around a point $c$. A function $f : I \longrightarrow \mathbb{R}$ is said to said to be differentiable at a point $c \in \mathbb{R}$, if

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists. The value of this limit is written as $f'(c)$, and is called the derivative of $f$ at $c$.

If $f$ is differentiable at all the points of the domain, then we say that the function is differentiable on the domain.

**Example 2.** (i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x$. To show that this function is continuous at $a \in \mathbb{R}$. Let $\epsilon > 0$ be any positive real number. Then, we can see that $\frac{f(x)-f(a)}{x-a} = 1$. So $\lim_{x\to c} \frac{f(x)-f(a)}{x-a} = 1$. Therefore $f'(c) = 1$.

**Proposition 3.** *Let $f, g : X \longrightarrow \mathbb{R}$ be two functions differentiable at $c$.*

*(i) $f + g$ is differentiable on $X$.*
*(ii) $f.g$ is differentiable on $X$, and $(f.g)'(c) = f'(c)g(c) + f(c)g'(c)$.*
*(iii) If $f$ is differentiable on the image set $g(X)$, then $f \circ g$ is differentiable on $X$, and $(f \circ g'(c) = f'(g(c))g'(c)$.*
*(iv) $f/g$ is differentiable on $X$ if $g(x) \neq 0$ on $X$.*

*Proof.* (i), (ii), (iv) are easy to prove.

(iii): Let $\phi(t) = \begin{cases} \frac{f(t)-f(g(c))}{t-g(c)}, & \text{if } t \neq g(c) \\ f'(g(c)), & \text{if } t = g(c). \end{cases}$

Then $\phi$ is continuous at $g(c)$ as $f$ is differentiable at $g(c)$.
$g$ is anyway continuous as it is differentiable at $c$.
Now $\frac{f(g(x))-f(g(c))}{x-c} = \phi(g(x))\frac{g(x)-g(c)}{x-c}$.
Therefore

$$\begin{aligned}\lim_{x\to c} \frac{f(g(x))-f(g(c))}{x-c} &= \lim_{x\to c} \phi(g(x)) \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \phi(g(c))g'(c) \\ &= f'(g(c))g'(c).\end{aligned}$$

Hence, $(f \circ g)'(c) = f'(g(c))g'(c)$. $\square$

**Example 4.** (i) $f(x) = x^3$. Then it a product of the three functions $g(x) = x$. By the results above, $f$ is differentiable.

(ii) $f(x) = 1/x$. Note that this a well-defined function from $\mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$. By (iv) of the previous proposition, $f$ is differentiable on the set $X = \mathbb{R}\backslash\{0\}$.

(iii) $f(x) = |x|$ is not differentiable at $x = 0$.

**Theorem 5.** *Let $f(x) = e^x$ (recall that $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ defines the exponential function.)*

*Proof.* $\dfrac{e^{x+h} - e^x}{h} = e^x \dfrac{e^h - 1}{h} = e^x(\sum_{n=1}^{\infty} \dfrac{h^{n-1}}{n!})$.
Here $\sum_{n=1}^{\infty} \dfrac{h^{n-1}}{n!} \leqslant \sum_{n=1}^{\infty} |h|^{n-1} = \dfrac{1}{1-|h|}$, as $|h|$ can be taken to be smaller than 1.
As $h \to 0$, we can see that $f'(c) = \lim_{h\to 0} \dfrac{e^{x+h} - e^x}{h} = e^c$. □

**Definition 6.** Let $f : X \longrightarrow \mathbb{R}$. Then

- $x_0 \in X$ is a point of local maximum if there is a $\delta > 0$ such that $f(x) \leqslant f(x_0)$, for all $x \in (x_0 - \delta, x_0 + \delta)$.
- $x_1 \in X$ is a point of local minimum if there is a $\delta' > 0$ such that $f(x_1) \leqslant f(x)$, for all $x \in (x_1 - \delta', x_1 + \delta')$.

These points are also referred to as the extremum points.

**Theorem 7** (Diff fn vanishes at extremum points)**.** *Let $f : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function, and $x_0$ is a point of local maximum. Then $f'(x_0) = 0$ .*

*Proof.* By the definition of local maximum, there exists $\delta > 0$ such that $f(x_0+h)-f(x_0) \leqslant 0$, for $h < \delta$.
Therefore, for $h > 0$, $\dfrac{f(x_0 + h) - f(x_0)}{h} \leqslant 0$, so the limit $f'(x_0) \leqslant 0$.
For $h < 0$, $\dfrac{f(x_0 + h) - f(x_0)}{h} \geqslant 0$, so the limit $f'(x_0) \geqslant 0$.
Together, we have $f'(x_0) = 0$. □

Try to prove the case when $x_0$ is a point of local minimum.