MTH201-ASSIGNMENT 1

Q 1. If $a > 0, b > 0$ and $a > b$, then show that $a^{-1} < b^{-1}$.

Q 2. Give examples where the equality holds in the triangle inequality. In general, what are the conditions on a, b under which equality holds.

Q 3. In each of the following, check if an upper or lower bound exists. If so, then calculate their supremum and infimum.

(i) $\left\{\pm\frac{1}{n} : n \in \mathbb{N}\right\}$

(ii) $\left\{\left(\frac{-1}{n}\right)^n : n \in \mathbb{N}\right\}$

(iii) $\{\sin(\theta) : \theta \in \mathbb{R}\}$

(iv) $\{\tan(\theta) : \theta \in \mathbb{R}\}$

(v) $\{e^x : x \in \mathbb{R}\}$

Q 4. Check if the following defines a function.

(i) $f : \mathbb{Q} \longrightarrow \mathbb{Z}, f(m/n) = m$.

(ii) $f : \mathbb{Q} \longrightarrow \mathbb{Z}, f(m/n) = n$.

(iii) Let $n \in \mathbb{N}$, and define $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^n$.

(iv) $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ which is the sine function $\sin(x)$. Find the range.

(v) $f : \mathbb{R} \longrightarrow \mathbb{R}, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions. Show that the composite $g \circ f$ is a well-defined function.

(vi) $f : \mathbb{R} \longrightarrow \mathbb{R}, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions. If we define $f + g$ as $(f + g)(x) := f(x) + g(x)$, then show that $f + g$ is a well-defined function.

(vii) $f : \mathbb{R} \longrightarrow \mathbb{R}, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions. How will you naturally define $f - g, f.g$ and f/g if $g(x) \neq 0$ for each $x \in \mathbb{R}$? In each case, show that they are well-defined functions.

(viii) Take the circle defined by $x^2 + y^2 = 1$. Is the y-coordinate a well-defined function of the x-axis? For a single value of x, how many y-coordinates are there?

Q 5. Show that the following functions are injective.

(i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^3$.

(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^n$, for each odd $n \in \mathbb{N}$.

(iii) $f : \mathbb{R} \longrightarrow \mathbb{R}, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions. If both f, g are injective, then show that $g \circ f$ is also injective.

(iv) $f : \mathbb{R} \longrightarrow \mathbb{R}, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions. If both f, g are injective, is it necessary that $f + g, f - g, f.g, f/g$ are all injective? Give counterexamples if they are not injective.

Q 6. Same question as the previous one but with injective replaced by surjective.

Q 7. A function which is both injective and surjective is also called a bijective function. It is also called a one to one correspondence. Now, do the same questions in Q5, but with injective replaced by bijective.

Q 8. Let $a \geq 0$. Suppose $a < \dfrac{1}{n}$ for every $n \in \mathbb{N}$. Show that $a = 0$.