Most real-world problems are nonlinear

∴ Nonlinear programming forms an important part of mathematical optimization methods

An optimization problem in which objective function Z or some ~~/~~ all constraints are non-linear (which means x power is higher than 1) is called NLPP.

<u>3 types</u>

① No contraints
② Equality constraints
③ Inequality constraints

No constraints, we already saw some examples

Newton's method

<u>Equality constraints</u>

Several methods available to solve. Few of the important are

1) Direct substitution method
2) Constrained variation method
3) Lagrange multipliers method

Direct substitution method

→ Least complex method

Method is restricted to models that contain only equality constraints.

Method involves solving the constraint equation for one variable in term of another

This new expression is then substituted into the objective function, ~~effectivel~~ effctively eliminating the constraint.

A constrained optimization is converted / transformed into an unconstrained model.

Problem has n variables in m equality constrains ($m<n$).

Since $m = ns$ in m variables can be simultaneously solved, it is theoretically possible to reduce n variables to $(n-m)$ variables by suitable substitution

↓

These expression with $(n-m)$ variables can be substituted into original objective function to make it an unconstrained optimization problem

Algorithm

① Ensure the problem is of multivariable optimization with equality constraints such as x_i in $f(x)$ s.t. $g_j(x_i) = 0$; $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$.

② Express =ns in $(n-m)$ variables by suitable substitution of one equation in the other.

③ Substitute these expressions in $f(x)$ to make it unconstrained.

④ Solve it.

Ex Min. $f(x_1, x_2) = x_1^2 + (x_2 - 1)^2$

s.t. $-2x_1^2 + x_2 = 4$

Soln. $x_2 = 4 + 2x_1^2$

$$f(x_1, x_2) = x_1^2 + (2x_1^2 + 3)^2$$

$$= x_1^2 + 4x_1^4 + 12x_1^2 + 9$$

$$= 4x_1^4 + 13x_1^2 + 9 \longrightarrow$$

Convert to single variable unconstrained optimization apply necessary condition

$$\frac{df}{dx_1} = 16x_1^3 + 26x_1$$

f to be max/min. $f' = 0$

$$x_1(16x_1^2 + 26) = 0$$

$$\Rightarrow x_1^* = 0$$

then $x_2^* = 4$

Apply sufficient condition

$$\frac{d^2f}{dx_1^2} = 48x_1^2 + 26$$

at $x_1^* = 0$ $f'' \rightarrow$ +ve.

Hence f is relative minimum at $x_1^* = 0$.

$f(x_1^* = 0, x_2^* = 4) = 9$.

Q Minimize $f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

s.t. $g_1(x) = x_1 - x_2 = 0$ —①

$g_2(x) = x_1 + x_2 + x_3 - 1 = 0$ ②

$x_1 = x_2$ from ①

$g_2(x) = x_1 + x_1 + x_3 - 1 = 0$

$\Rightarrow +2x_1 + x_3 - 1 = 0$

$$x_3 = 1 - 2x_1$$

Scanned with OKEN Scanner

$$f(x) = \frac{1}{2}\left[2x_1^2 + (1-2x_1)^2\right]$$

$$= \frac{1}{2}\left[2x_1^2 + 1 + 4x_1^2 - 4x_1\right]$$

$$= \frac{1}{2}\left[6x_1^2 - 4x_1 + 1\right]$$

$$\frac{\partial f}{\partial x_1} = \frac{1}{2}(12x_1 - 4) = 0$$

$$x_1 = \frac{2}{6} = \frac{1}{3}$$

$$x_3 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$x_1 = x_2 = x_3 = \frac{1}{3}$$

$$\frac{\partial^2 f}{\partial x^2} = 6$$ is positive definite

$\therefore$ f^*_{min} at $x_1 = x_2 = x_3 = \frac{1}{3}$

$$f_{min} = \frac{1}{2}\left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = \frac{1}{6}.$$

Merit

Simple and straight forward method

Limitation

→ higher order → not easy to solve the constraint

→ Convenient to solve simple problem but fails for complex problem.

→ Method fail for $f(x)$ which is not continuous.

Constrained Variation Method

One find closed-form expression for first order differential at all point at which constraints
$g_j(x) = 0$; $j = 1, 2, \ldots m$ are satisfied

Then one find the desired optimum point by setting the differential of f equal to zero.

$$\min f(x_1, x_2)$$
$$\text{s.t. } g(x_1, x_2) = 0.$$

$f(x_1, x_2)$ → min. at (x_1^*, x_2^*)

Necessary condition is $f'(x_1^*, x_2^*) = 0$.

doing so

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

Since $g(x_1^*, x_2^*) = 0$ at minimum point,
any variation dx_1 and dx_2 about the point (x_1^*, x_2^*) are called admissible variation provided that new point lies on the constraint

i.e. $g(x_1^* + dx_1, x_2^* + dx_2) = 0$.

Taylor's series expansion at (x_1^*, x_2^*) gives

$$g(x_1^*+dx_1, x_2^*+dx_2) = g(x_1^*, x_2^*) + \frac{\partial g}{\partial x_1}(x_1^*, x_2^*)\,dx_1 + \frac{\partial g(x_1^*, x_2^*)}{\partial x_2}\,dx_2 = 0$$

where dx_1 & dx_2 are small

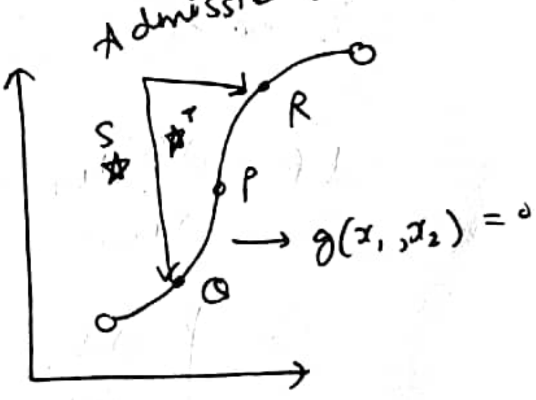
Since $g(x_1^*, x_2^*) = 0$.

we get

$$dg = \frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 = 0 \quad \text{at } (x_1^*, x_2^*)$$

has to be satisfied by all admissible variations

Set of variations (dx_1, dx_2) that satisfy the constraint curve is called admissible variation



S T are not admissible variation

Necessary condition for constrained variation

=n satisfied by admissible variation is

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0 \quad at\ (x_1^*, x_2^*)$$

It is rewritten as — (1)

$$\frac{\partial g}{\partial x_1} dx_1 = \frac{-\partial g}{\partial x_2} dx_2$$

$$dx_2 = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2} dx_1 \quad at\ (x_1^*, x_2^*)$$

assumption $\partial g/\partial x_2 \neq 0$.

Now ~~(1)~~ the necessary condition:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

becomes

$$df = \frac{\partial f}{\partial x_1} dx_1 - \frac{\partial f}{\partial x_2}\left[\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right] dx_1 = 0 \quad at\ (x_1^*, x_2^*)$$

$$= \left[\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\left(\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right)\right] = 0$$

$$\frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} = \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_1}$$

Thus in order to have $(\hat{x}_1, \hat{x}_2)$ as extreme point (maxima or minima) necessary condition is

$$\left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_1} \right) \Bigg|_{(\hat{x}_1, \hat{x}_2)} = 0$$

$\downarrow$

The above necessary condition can be applied if Jacobian is not equal to zero

i.e. $J = \left| \frac{f, g_1, g_2, \ldots, g_m}{x_1, x_2, \ldots, x_m} \right| = \begin{vmatrix} \frac{\partial f}{\partial x_k} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \\ \frac{\partial g_1}{\partial x_k} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \frac{\partial g_2}{\partial x_k} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_m} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_k} & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix} \neq 0$

Procedure

① Problem has to be multivariable constrained eq. optimization.

② $f(x_1, x_2)$ with $g(x_1, x_2) = 0$

③ Find partial derivative

$$\left(\frac{\partial f}{\partial x_1}\right), \left(\frac{\partial f}{\partial x_2}\right); \left(\frac{\partial g}{\partial x_1}\right) \text{ and } \left(\frac{\partial g}{\partial x_2}\right)$$

also $\left(\frac{\partial g}{\partial x_2}\right) \neq 0$

④ Apply necessary condition

$$\left(\frac{\partial f}{\partial x_1}\right)\cdot\left(\frac{\partial g}{\partial x_2}\right) - \left(\frac{\partial f}{\partial x_2}\right)\left(\frac{\partial g}{\partial x_1}\right) = 0$$

to get relation b/w x_1 + x_2

⑤ Find value of x_1^* + x_2^* using ③ condition + $g(x_1, x_2) = 0$

If problem has more than 2 variables then find Jacobian J.

If $J = 0$, one can't go to necessary condition ③

To find Jacobian, choose $(n-m)$ independent variables with remaining variables, find determinant of first partial derivatives

If Jacobian $\neq 0$, one apply necessary condition.

Use $\left(\frac{\partial f}{\partial x_k}\right)$ / $\left(\frac{\partial g_j}{\partial x_k}\right)$ $(j=1,2,..,m)$ as the first column where $k = m+1, m+2, .., n$ to find various possible relations.

(7) Solve relations and $g_j(x)$; $j = 1, 2, .., m$ to find values of x_i^*, $i = 1, 2, .., n$.

(8) Substitute x_i^* in f to find the extreme value

Example Max

$$f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

$$g_1(x) = x_1 + 2x_2 + 3x_3 + 5x_4 - 10 = 0$$

$$g_2(x) = x_1 + 2x_2 + 5x_3 + 6x_4 - 15 = 0$$

Problem has 4 variable with 2 constraint

$n = 4$, $m = 2$.

Scanned with OKEN Scanner

Must select 2 variables as independent to apply necessary condition.

i.e. Jacobian is not zero

Let say x_3, x_4 as independent, then Jacobian be

$$J = \left|\frac{g_1, g_2}{x_1, x_2}\right| = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

Since $J = 0$, necessary condition cannot be applied

Next x_2, x_4 as independent then Jacobian become

$$J = \left|\frac{g_1, g_2}{x_1, x_3}\right| = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_3} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2 \neq 0$$

apply necessary conditions $k = m+1$ i.e. x_2

$$\begin{vmatrix} \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} x_2 & x_1 & x_3 \\ 2 & 1 & 3 \\ 2 & 1 & 5 \end{vmatrix} = 0 \quad -(1)$$

and for x_4

$$\begin{vmatrix} \frac{\partial f}{\partial x_4} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g_1}{\partial x_4} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_4} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} x_4 & x_1 & x_3 \\ 5 & 1 & 3 \\ 6 & 1 & 5 \end{vmatrix} = 0 \quad -(2)$$

from (1) → $\cancel{x_2 (5-}$ $x_2 (5-3) - x_1 (10-8) + x_3 (2-2) = 0$

$2x_2 - 4x_1 = 0$

$x_2 = 2x_1$

(2) $x_4 (5-3) - x_1 (25-18) + x_3 (5-6) = 0$

$2x_4 - 7x_1 - x_3 = 0$

And we have $g_1(x) = 0$

$g_2(x) = 0$

One can know soln for x_1, x_2, x_3 & x_4

$x_3 = 2x_4 - 7x_1$ & $x_2 = 2x_1$ in g_1, g_2

$$x_1 + 4x_1 + 6x_4 - 21x_1 + 5x_4 - 10 = 0$$

$$-16x_1 + 11x_4 = 0 \quad - (a)$$

also $x_1 + 4x_1 + 10x_4 - 35x_1 + 6x_4 = 15$

$$-30x_1 + 16x_4 = 15 \quad - (b)$$

On get soln (a) & (b)

$$x_1^* = -5/74 \quad , \quad x_4^* = 30/37$$

Then we find $x_2^* = -5/37$, $x_3^* = 155/74$

The optimum soln is

$$x_1^* = \frac{-5}{74} \quad , \quad x_2^* = \frac{-5}{37} \quad ; \quad x_3^* = \frac{155}{74} \quad , \quad x_4^* = \frac{30}{37}$$

Next p

roblem with two variables & one equality constraint

minimize $f(x_1, x_2)$

s. t. $g(x_1, x_2) = 0$

For this problem, we know that necessary condition for the existence of the extreme point (max / min) at $x = x^*$ is

$$\left[\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_1}\left(\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right)\right]_{(x_1^*, x_2^*)} = 0$$

or

$$\left[\frac{\partial f}{\partial x_1} - \frac{(\partial f/\partial x_2)}{(\partial g/\partial x_2)}\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)} = 0$$

$\lambda \to$ Lagrange multiplier

$$\lambda = -\left(\frac{\partial f/\partial x_2}{\partial g/\partial x_2}\right)\Bigg|_{(x_1^*, x_2^*)}$$

so

$$\left[\frac{\partial f}{\partial x_1} + \lambda\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)} = 0$$

$$\left[\frac{\partial f}{\partial x_2} + \lambda\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)} = 0$$

Also have

$$g(x_1, x_2)\Big|_{(x_1^*, x_2^*)} = 0$$

Scanned with OKEN Scanner

Necessary condition for Lagrange multiplier meth

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

$L \rightarrow$ fn. of 3 variables $x_1, x_2 + \lambda$.

necessary condition for existence of the extreme point $(\overset{*}{x}_1, \overset{*}{x}_2)$ are given by

$$\frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) = \frac{\partial f}{\partial x_1}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_1}(x_1, x_2) = 0$$

$$\frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) = \frac{\partial f}{\partial x_2}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_2}(x_1, x_2) = 0$$

and $\frac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) = g(x_1, x_2) = 0$

Lagrange Multiplier method

Another powerful method is Lagrange multiple

Let say we want to minimize a function

$$\text{Min } f(x) \quad x = (x_1, \ldots x_n)^T \in \mathbb{R}^n$$

s.t. nonlinear equality constraint

$$g(x) = 0$$

then we combine the $f(x)$ with equality form a new function, one called Lagrangian.

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

Where λ is the Lagrange multiplier
λ is a unknown scalar to be determined.

This converts the constrained optimization into an ~~constrained~~ unconstrained problem:

$$L(x,\lambda) = f(x) + \lambda g(x)$$

M equalities

$$g_j(x) = 0 \quad (j = 1, \dots, M)$$

then we need M Lagrange multipliers λ_j $(j=1, \dots, M)$

$$L(x, \lambda_j) = f(x) + \sum_{j=1}^{M} \lambda_j g_j(x)$$

The requirements of stationary condition leads to

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{M} \lambda_j \frac{\partial g_j}{\partial x_i} \quad ; \quad (i = 1, \dots n)$$

and $\dfrac{\partial L}{\partial \lambda_j} = g_j = 0 \quad (j = 1, \dots, M)$

These $M+n$ = no will determine the n component of x and M Lagrange multipliers

Scanned with OKEN Scanner

One get the soln.

$$x^* = (x_1^*, x_2^*, \ldots x_n^*)^T \quad \& \quad \lambda^* = (\lambda_1^*, \lambda_2^*, \ldots \lambda_m^*)$$

Note

→ value of $\dfrac{\partial g}{\partial x_2} \neq 0$ is an essential condition.

If dx_2 is expressed in terms of dx_1 in necessary condition.

If dx_1 is expressed in terms of dx_2 then $\dfrac{\partial g}{\partial x_1} \neq 0$ is basic requirement. Thus, at least one of the partial derivative of $g(x_1, x_2)$ must be non-zero for existence of an extreme point on L.

→ Lagrange function can be taken as $L = f + \lambda g$

or $L = f - \lambda g$.

In second case we get values of λ with -ve sign. while they are +ve in first case.

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

$$\frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) = \frac{\partial f}{\partial x_1}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_1}(x_1, x_2) = 0$$

$$\frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) = \frac{\partial f}{\partial x_2}(x_1, x_2) + \lambda \frac{\partial g}{\partial x_2}(x_1, x_2) = 0$$

& $\dfrac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) = g(x_1, x_2) = 0$

$$\text{Min } f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$
$$g_1(x) = x_1 - x_2 = 0$$
$$g_2(x) = x_1 + x_2 + x_3 - 1 = 0$$

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) + \lambda_1 g_1(x_1, x_2) + \lambda_2 g_2(x_1, x_2, x_3)$$

$$L(x, \lambda) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda_1 (x_1 - x_2) + \lambda_2 (x_1 + x_2 + x_3 - 1)$$

Necessary condition for $L(x_i, \lambda_j)$ to have extreme point are

$$\frac{\partial L}{\partial x_1} = x_1 + \lambda_1 + \lambda_2 = 0 \quad - (1)$$

$$\frac{\partial L}{\partial x_2} = x_2 - \lambda_1 + \lambda_2 = 0 \quad - (2)$$

$$\frac{\partial L}{\partial x_3} = x_3 + \lambda_2 = 0 \quad - (3)$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 - x_2 = 0 \quad - (4)$$

$$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + x_3 - 1 = 0 \quad - (5)$$

$$(1) + (2) + (3) \quad (x_1 + x_2 + x_3) + 3\lambda_2 = 0 \quad - (6)$$

$$(5) \rightarrow x_1 + x_2 + x_3 = 1 \quad \Rightarrow \quad \lambda_2^* = -\frac{1}{3}$$

(3) $\Rightarrow x_3^* = \frac{1}{3}$

(4) $\Rightarrow x_1 = x_2$

Put $x_1 = x_2$ & $\lambda_2 = -\frac{1}{3}$.

(1) $x_1 + \lambda_1 + \lambda_2 \longrightarrow x_1 + \lambda_1 - \frac{1}{3} = 0$ — (7)

(2) $x_2 - \lambda_1 + \lambda_2 \longrightarrow x_1 - \lambda_1 - \frac{1}{3} = 0$ — (8)

(7) + (8) $x_1^* = \frac{1}{3}$

So $x_2^* = \frac{1}{3}$

$2\lambda_1 = 0 \Rightarrow \lambda_1^* = 0$

$$f(x)_{min} = \frac{1}{2}\left(\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right] = \frac{1}{2} \times \frac{3}{9} = \frac{1}{6}$$

Scanned with OKEN Scanner

What if we have inequality constraints

One can convert inequality constraints to equality constraints by introducing non-negative slacks variables. S_j^2 $j = 1, 2, \dots, m$

Problem now becomes

$$\min . f(x)$$

$$g_j(x, s) = g_j(x) + s_j^2 = 0 \quad ; j = 1, 2, \dots, m$$

where $s_j = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}$ is vector of slack variables

Now solve it using Lagrange multiplier method

$$L(x, s, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x, s)$$

$x = (x_1, x_2, \dots x_n)^T$ decision variables

$s = (s_1, s_2, \dots, s_m)^T$ slack variables

$\lambda = [\lambda_1, \lambda_2, \dots \lambda_m)^T$ Lagrange multiplier vector

Now write necessary conditions as

$$\frac{\partial L}{\partial x_i}(x, s, \lambda) = \frac{\partial f(x_i)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x, s)}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial \lambda_j}(x, s, \lambda) = g_j(x, s) = g_j + s_j^2 = 0$$

& $\frac{\partial L}{\partial s_j}(x, s, \lambda) = 2 \lambda_j s_j = 0$

where $i = 1, 2, \dots n$ & $j = 1, 2, \dots, m$

we have $(n + 2m)$ = no & $(\cancel{m} + 2m)$ variables & hence can solve system of = no for optimum soln. vector x^*

This concept was revised by Kuhn & Tucker

→ $g_j = 0$ at optimum point → active constraints

$g_j < 0$ → inactive constraints

Karush Kuhn-Tucker condition (KKT)

Necessary conditions to be satisfied at relative minimum of $f(x)$ with inequality constraint $g_j(x) \le 0$ expressed as

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n$$

& $\lambda_j \ge 0$

λ_j → Lagrange multipliers

If set of active constraint is not known, Karush Kuhn - Tucker conditions can be stated as follows. for case of minimize $f(x)$. subject to $g_j(x) \le 0$.

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \; ; \quad i = 1, 2, \ldots, n$$

$$\sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n$$

$$\lambda_j g_j = 0 \qquad j = 1, 2, \ldots, m$$

$$g_j \le 0 \qquad j = 1, 2, \ldots m$$

$$\lambda_j \ge 0 \qquad j = 1, 2, \ldots m$$

...traint qualification & the corresponding values of Lagrange multipliers will obey the following relations

① λ_j is non-negative ($\lambda_j \geq 0$) for minimization of f with $g_j \leq 0$.

② λ_j is non-negative ($\lambda \geq 0$) for maximization of f with $g_j \geq 0$

③ λ_j is negative ($\lambda_j \leq 0$) for maximization of f with $g_j \leq 0$

④ λ_j is negative ($\lambda_j \leq 0$) for minimization of f with $g_j \geq 0$

	Min $f(x)$	Max $f(x)$
$g_j(x) \leq 0$	$\lambda_j \geq 0$	$\lambda_j \leq 0$
$g_j(x) \geq 0$	$\lambda_j \leq 0$	$\lambda_j \geq 0$

Solve the

$$\max f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

s.t $\quad x_1 + x_2 \leq 3$

$3x_1 - x_3 \leq 6$

$x_1 + x_2 + x_3 \leq 6$

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n$$

$\lambda_j g_j = 0 \qquad j = 1, 2, \ldots, m$

$g_j \leq 0$

$\lambda_j \leq 0$

$$\max. f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

s.t. $g_1(x_1, x_2, x_3) = x_1 + x_2 \leq 3$

$g_2(x_1, x_2, x_3) = 3x_1 - x_3 \leq 6$

$g_3(x_1, x_2, x_3) = x_1 + x_2 + x_3 \leq 6$

Lagrang fn

$L(x, \lambda) = f(x) + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3$

$= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda_1(x_1 + x_2 - 3) + \lambda_2(3x_1 - x_3 - 6)$

$+ \lambda_3(x_1 + x_2 + x_3 - 6)$

where $\lambda_1 \leq 0$, $\lambda_2 \leq 0$ & $\lambda_3 \leq 0$

first set of KKT

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} + \lambda_3 \frac{\partial g_3}{\partial x_i} = 0$$

$i = 1, 2, 3.$

$$\frac{\partial L}{\partial x_1} = x_1 + \lambda_1 + 3\lambda_2 + \lambda_3 = 0 \quad - (1)$$

$$\frac{\partial L}{\partial x_2} = x_2 + \lambda_1 + \lambda_3 = 0 \quad - (2)$$

$$\frac{\partial L}{\partial x_3} = x_3 - \lambda_2 + \lambda_3 = 0 \quad - (3)$$

Second set of KKT

$$\lambda_j g_j = 0 \quad j = 1, 2, 3$$

$$\lambda_1 (x_1 + x_2 - 3) = 0 \quad - (4)$$

$$\lambda_2 (3x_1 - x_3 - 6) = 0 \quad - (5)$$

$$\lambda_3 (x_1 + x_2 + x_3 - 6) = 0. \quad - (6)$$

3^{rd} set of KT as $g_j \leq 0 \quad j = 1, 2, \ldots m$

$$g_1 (x_1, x_2, x_3) \equiv x_1 + x_2 - 3 \leq 0 \quad - (7)$$

$$g_2 \equiv 3x_1 - x_3 - 6 \leq 0 \quad - (8)$$

$$g_3 \equiv x_1 + x_2 + x_3 - 6 \leq 0 \quad - (9)$$

fourth set of KKT

$$\lambda_1 \leq 0 \quad - (10)$$

$$\lambda_2 \leq 0 \quad - (11)$$

$$\lambda_3 \leq 0 \quad - (12)$$

(1), (2), (3) rewritten as

$$x_1 = -\lambda_1 - 3\lambda_2 - \lambda_3 \quad - (13)$$

$$x_2 = -\lambda_1 - \lambda_3 \quad - (14)$$

$$x_3 = \lambda_2 - \lambda_3 \quad - (15)$$

by (13), (14), (15) rewrite (4), (5), (6)

$\lambda_1(-2\lambda_1 - 3\lambda_2 - 2\lambda_3 - 3) = 0$

$\lambda_1(2\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3) = 0$ (16)

$\lambda_2(3\lambda_1 + 10\lambda_2 + 2\lambda_3 + 6) = 0$ (17)

$\lambda_3(2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 6) = 0$ (18)

Eight cases of basic sol.

Case 1 $\lambda_1 = 0$, $\lambda_2 = 0$ + $\lambda_3 = 0$

$x_1 = x_2 = x_3 = 0$

all KKT condition, hence local max.

Case 2 $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$

(18). $\lambda_3(2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 6) = 0$

$2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 6 = 0$

$\Rightarrow 3\lambda_3 = -6$; $\lambda_3 = -2$

$x_1 = -\lambda_1 - 3\lambda_2 - \lambda_3 = 2$; $x_2 = -\lambda_1 - \lambda_3 = 2$; $x_3 = \lambda_2 - \lambda_3 = 2$

Violates (7). Discarded

Case 3. $\lambda_1 = 0$; $\lambda_2 \neq 0$ + $\lambda_3 = 0$.

(17) $\lambda_2(3\lambda_1 + 10\lambda_2 + 2\lambda_3 + 6) = 0$.

$\lambda_2 = -3/5$

$x_1 = \frac{9}{5}$, $x_2 = 0$, $x_3 = -3/5$

all KKT conditions satisfy. Local max.

Case 4 $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 = 0$

$\lambda_1(2\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3) = 0 \Rightarrow \lambda_1 = -3/2$

~~$x_1 = 9/5$~~ $x_1 = 3/2$, $x_2 = 3/2$, $x_3 = 0$

all KT condition satisfy Local max.

Case 5 $\lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0.$

(17), (18)

$$\lambda_2 \, (3\lambda_1 + 10\lambda_2 + 2\lambda_3 + 6) = 0$$

$$\lambda_3 \, (2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 6) = 0$$

$$10\lambda_2 + 2\lambda_3 + 6 = 0 \ \& \ 2\lambda_2 + 3\lambda_3 + 6 = 0$$

$$\lambda_2 = -3/13 \ \& \ \lambda_3 = \frac{24}{13}$$

$$x_1 = -\lambda_1 - 3\lambda_2 - \lambda_3 = \frac{33}{13}$$

$$x_2 = -\lambda_1 - \lambda_3 = \frac{24}{13}$$

$$x_3 = \lambda_2 - \lambda_3 = \frac{21}{13}$$

violates (7)

Case 6 $\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 \neq 0$

$$\lambda_1 (2\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3) = 0$$

$$\lambda_3 (2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 6) = 0$$

$$2\lambda_1 + 2\lambda_3 + 3 = 0 \ \& \ 2\lambda_1 + 3\lambda_3 + 6 = 0.$$

$$\lambda_1 = \frac{3}{2} \quad \lambda_3 = -3$$

violates (10) i.e. $\lambda_1 \leq 0$. discarded.

7 $\lambda_1 \neq 0, \lambda_2 \neq 0$ & $\lambda_3 = 0$

$$\lambda_1 (2\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3) = 0$$
$$\lambda_2 (3\lambda_1 + 10\lambda_2 + 2\lambda_3 + 6) = 0$$

$$2\lambda_1 + 3\lambda_2 + 3 = 0$$
$$3\lambda_1 + 10\lambda_2 + 6 = 0$$
$$\lambda_1 = \frac{-12}{11} \text{ \& } \lambda_2 = -3/11$$

$$x_1 = \frac{21}{11}, \quad x_2 = \frac{12}{11}, \quad x_3 = -3/11$$

violate (8) discarded

Case 8 $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$

$$2\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3 = 0$$
$$3\lambda_1 + 10\lambda_2 + 2\lambda_3 + 6 = 0$$
$$2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 6 = 0$$
$$\lambda_1 = 4, \lambda_2 = -1 \text{ \& } \lambda_3 = -4$$

violate ie. $\lambda_1 \leq 0$.

out of 8, local maximas in 3 cases (1, 3 & 4).

$$f_1(x) = 0$$
$$f_3(x) = 1.6$$
$$f_4(x) = 2.25$$

global maxima exist at

$$\lambda_1 = -3/2, \quad \lambda_2 = 0, \quad \lambda_3 = 0.$$
$$x_1^* = 3/2, \quad x_2^* = 3/2 \text{ \& } x_3^* = 0$$

counterpart of Lagrange multiplier for
near optimization with constraint inequality

$$\min f(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 \rightarrow f$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1 \rightarrow g$$

$$x_4 \leq A \longrightarrow \psi$$

$$\rightarrow x_4 - A \leq 0$$

$$L = f(x) + \lambda\, g(x) + \mu(\psi)$$

$$= x_1^2 + x_2^2 + x_3^2 + x_4^2 = \lambda(1 - x_1 - x_2 - x_3 - x_4) + \mu(x_4 - A)$$

KKT condition. (Karush-Kuhn-Tucker condition)

$$\frac{\partial L}{\partial x} = 0 \quad — (1)$$

$$x_1 + x_2 + x_3 + x_4 = 1 \quad — (2)$$

$$x_4 \leq A \quad — (3)$$

$$\mu \geq 0 \quad — (4)$$

$$\mu(x_4 - A) = 0 \quad — (5)$$

$$\frac{\partial L}{\partial x} : \begin{pmatrix} 2x_1 - \lambda \\ 2x_2 - \lambda \\ 2x_3 - \lambda \\ 2x_4 - \lambda + \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore x_1 = x_2 = x_3 = \frac{\lambda}{2}; \quad x_4 = \frac{\lambda - \mu}{2}$$

$x_1 + x_2 + x_3 + x_4 = 1$ ②

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} + \cancel{x_4 =} \ \frac{\lambda - \mu}{2} = 1$$

$$4\lambda - \mu = 2$$

$$\lambda = \frac{2+\mu}{4}$$

$$x_4 \leq A$$

$$\frac{\lambda - \mu}{2} \leq A$$

$$\frac{\frac{2+\mu}{4} - \mu}{2} \leq A \Rightarrow \frac{2 + \mu - 4\mu}{8} \leq A$$

$$\frac{1}{4} - \frac{3\mu}{8} \leq A$$

$$\frac{3\mu}{8} \geq \frac{1}{4} - A$$

① $A > \frac{1}{4}$

$$\frac{1}{4} - A \leq 0$$

then. as $\mu \geq 0$

then satisfy only when $\mu = 0$.

$$\therefore \lambda = \frac{2}{4} = \frac{1}{2}$$

$$x_1 = x_2 = x_3 = x_4 = \frac{1}{4}$$

II $A = 1/4$

Ignore inequality constraint

$$\frac{3\mu}{8} \geq 0$$

$$\frac{3\mu}{8} \geq 0$$

again $\mu = 0$

and $x_1 = x_2 = x_3 = x_4 = \frac{1}{4}$

III $A < 1/4$

x_4 is strictly less than A

$x_4 \leq A$

$$\mu(x_4 - A) = 0$$

$\downarrow$ as it is $-$ve

non μ is zero $\rightarrow$ $x_4 = 1/4$ which violate $x_4 \leq A$

But then

$x_4 = A$ &

$x_1 = x_2 = x_3 = \frac{1}{3}(1 - A)$

$$f = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$= \frac{1}{3}(1 - 2A + 4A^2)$$

$$f = \begin{cases} \frac{1}{4} & \text{if } A \geq \frac{1}{4} \\ \frac{1}{3}(1 - 2A + 4A^2) & \text{otherwise} \end{cases}$$

One can also simply write KKT conditions for

$$\text{Min } f(X) \qquad X = [x_1, x_2, \ldots, x_n]$$

$$\text{s.t.} \quad g_j(X) \leq 0 \quad \text{for } j = 1, 2, \ldots, p$$

$X^* = [x_1^*, x_2^*, \ldots x_n^*]$ to be local minima if following KKT conditions are satisfied

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \mu_j \frac{\partial g}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n$$

$$\mu_j g_j = 0 \quad j = 1, 2, \ldots, m$$

$$g_j \leq 0 \quad j = 1, 2, \ldots, m$$

$$\mu_j \geq 0 \quad j = 1, 2, \ldots, m$$

* In case of minimization problem, if constraints are of form $g_j(X) \geq 0$; then μ_j have to be negative

* If problem is one of maximization, with constraints in form $g_j(X) \geq 0$; then μ_j ~~to~~ is positive

$$\text{Min}: f = x_1^2 + x_2^2 + 60x_1$$

s.t. $x_1 - 80 \geq 0$

$x_1 + x_2 - 120 \geq 0$

KKT condition

$$\frac{\partial f}{\partial x_i} + \mu_1 \frac{\partial g_1}{\partial x_i} + \mu_2 \frac{\partial g_2}{\partial x_i} = 0 \rightarrow \begin{array}{ll} 2x_1 + 60 + \mu_1 + \mu_2 = 0 & -(1) \\ 2x_2 + \mu_2 = 0 & -(2) \end{array}$$

$$\mu_j g_j = 0 \Rightarrow \begin{array}{ll} \mu_1(x_1 - 80) = 0 & -(3) \\ \mu_2(x_1 + x_2 - 120) = 0 & -(4) \end{array}$$

$$g_j \leq 0 \Rightarrow \begin{array}{ll} x_1 - 80 \geq 0 & -(5) \\ x_1 + x_2 - 120 \geq 0 & (6) \end{array}$$

Here μ_1 & μ_2 will be ≤ 0.

(3) $\mu_1(x_1 - 80) = 0$

$\mu_1 = 0$ or $(x_1 - 80) = 0$

Case I

$\mu_1 = 0$

(1) + (2)

$$x_1 = -\frac{\mu_2}{2} - 30$$

$$x_2 = -\frac{\mu_2}{2}$$

Put in (4)

$$\mu_2(\mu_2 - 150) = 0$$

$\therefore \mu_2 = 0$ or -150

$\mu_2 = 0$; $X^* = [-30, 0] \rightarrow$ violate (5)

$\mu_2 = -150$; $X^* = [45, 75) \rightarrow$ violate (5)

$$2x_1 + 60 + \mu_1 + \mu_2 = 0$$

$$\mu_1 + \mu_2 = 220$$

$$2x_1 + \mu_2 = 0 \quad \Rightarrow \mu_2 = -2x_2$$

$$\mu_1 = 2x_2 - 220 \rightarrow$$

As $\mu_2 (x_1 + x_2 - 120) = 0$

$$-2x_2 (80 + x_2 - 120) = 0$$

$$-2x_2 (x_2 - 40) = 0$$

$x_2 = 0$ or $x_2 = 40$

$\rightarrow$ $x_2 = 0 \Rightarrow \mu_1 = -220$

$\downarrow$

$$x_1 + x_2 - 120 \geq 0$$

80. $80 + 0 - 120 \geq 0$ violate

$\rightarrow$ $x_2 = 40$; $\mu_1 = -140$ + $\mu_2 = -80$

satisfy (5) $x_1 - 80 \geq 0$

$$80 - 80 \geq 0$$

$$x_1 + x_2 - 120 \geq 0$$

$80 + 40 - 120 \geq 0$ OK

soln. set for this optimization problem is

$$X = [80, 40]$$

Scanned with OKEN Scanner