

## MTH201-2

### 1. LIMITS

**Definition 1.** A function  $f : \mathbb{R} \longrightarrow \mathbb{R}$  is said to be the limit of the function  $f$  at a point  $a \in \mathbb{R}$  if there exists a real number  $L$  such that

for every real  $\epsilon > 0$ , there is a real number  $\delta > 0$  such that

$$x \neq a, |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

If such a real number  $L$  exists then we write  $L = \lim_{x \rightarrow a} f(x)$ . Loosely, the definition expresses the possible value of the function  $f$  as the value of  $x$  gets close to  $a$ , i.e., when  $|x - a|$  becomes small. Or, the value of  $f$  getting close to  $L$  is achieved by  $x$  getting close to  $a$ .

**Proposition 2.** *The limit of a function, if it exists, is unique.*

**Example 3.** (i)  $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x$ . To find  $\lim_{x \rightarrow a} f(x)$ . We guess that the limit is  $a$ . To prove this, let  $\epsilon > 0$  be any positive real number. Then, we can see that  $|f(x) - a| = |x - a|, \epsilon$  will be achieved by  $|x - a| < \delta$  if we choose  $\delta = \epsilon$ .

To check this, take  $\delta = \epsilon$ , then  $|x - a| < \delta = |x - a| < \epsilon = |f(x) - a| < \epsilon$ . Hence  $\lim_{x \rightarrow a} f(x) = a$ .

(ii)  $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 2x$ . To find  $\lim_{x \rightarrow a} f(x)$ . We guess that the limit is  $2a$ . To prove this, let  $\epsilon > 0$  be any positive real number. Then, we can see that  $|f(x) - 2a| = |2x - 2a| < \epsilon$  will be achieved by  $|x - a| = |2x - 2a|/2 < \delta$  if we choose  $\delta = \epsilon/2$ .

To check this, take  $\delta = \epsilon/2$ , then  $|x - a| < \delta = |x - a| < \epsilon/2 = |f(x) - 2a| < \epsilon$ . Hence  $\lim_{x \rightarrow a} f(x) = 2a$ .

It might be difficult to find a  $\delta$  for a given  $\epsilon$  for some functions. It might be helpful to have some other observations to determine the limit.

**Proposition 4.** *Let  $f, g : \mathbb{R} \longrightarrow \mathbb{R}$  be two functions. Let  $\lim_{x \rightarrow a} f(x) = A$  and  $\lim_{x \rightarrow a} g(x) = B$ . Then*

- (i)  $\lim_{x \rightarrow a} (f + g)$  exists and equals  $A + B$ ,
- (ii)  $\lim_{x \rightarrow a} (f \cdot g)$  exists and equals  $A \cdot B$ ,
- (iii)  $\lim_{x \rightarrow a} (f \circ g)(x)$  may not exist as a real number, though it might exist in the extended reals,
- (iv)  $\lim_{x \rightarrow a} (f/g)(x)$  exists and equals  $A/B$  if  $B \neq 0$ .

*Proof.* (i): Let  $\epsilon > 0$ . Since  $\lim_{x \rightarrow a} f(x) = A$ , there exists  $\delta_1 > 0$ , such that  $|x - a| < \delta_1 \implies |f(x) - A| < \epsilon$ .

Since  $\lim_{x \rightarrow a} g(x) = B$ , there exists  $\delta_2 > 0$ , such that  
 $|x - a| < \delta_2 \implies |g(x) - B| < \epsilon$ .

Both the statements above will be true simultaneously if  $|x - a| < \min(\delta_1, \delta_2)$ . Then, we have  $|f(x) - A| < \epsilon$ ,  $|g(x) - B| < \epsilon$ . Then  $|(f + g)(x) - A - B| = |f(x) - A + g(x) - B| < |f(x) - A| + |g(x) - B| < \epsilon + \epsilon = 2\epsilon$ , i.e., putting  $\delta = \min(\delta_1, \delta_2)$ , we have  $|(f + g)(x) - A - B| < 2\epsilon$ . As  $\epsilon$  varies over all the positive reals,  $2\epsilon$  also varies over all the positive reals. Hence, by definition, the limit of  $f + g$  exists and  $\lim_{x \rightarrow a} (f + g)(x) = A + B$ .

(ii): First observe,

$$\begin{aligned} |(f \cdot g)(x) - AB| &= |f(x) \cdot g(x) - AB| = |f(x)g(x) - Ag(x) + Ag(x) - AB| \\ &= |(f(x) - A)g(x) + A(g(x) - B)| \\ &\leq |(f(x) - A)||g(x)| + |A|(g(x) - B)| \end{aligned}$$

by the triangle inequality. In these terms, we see that  $|(f(x) - A)|$  small can be achieved, also  $|(g(x) - B)|$  small can also be achieved. If the  $|g(x)|$  term can be bounded, then the product  $|(f(x) - A)||g(x)|$  will be small. This is what we do below.

Let  $\epsilon > 0$ , then there exists  $\delta_1 > 0, \delta_2 > 0$ , such that

$$|x - a| < \delta_1 \implies |(f(x) - A)| < \epsilon/(\epsilon + |B|),$$

and

$$|x - a| < \delta_2 \implies |(g(x) - B)| < \epsilon/(1 + |A|).$$

The second one means that  $|g(x)| < \epsilon + |B|$ , if  $|x - a| < \delta_2$ . So, now if we take  $\delta = \min(\delta_1, \delta_2)$  and  $|x - a| < \delta$ , then all the above inequalities will hold simultaneously, and combining them together we get the following:

$$\begin{aligned} |x - a| < \delta &\implies |(f(x) - A)| < \epsilon/(\epsilon + |B|), |(g(x) - B)| < \epsilon/(1 + |A|) \\ &\implies |(f(x) - A)||g(x)| + |A|(g(x) - B)| \leq \epsilon|g(x)|/(\epsilon + |B|) + |A|\epsilon/(1 + |A|) = 2\epsilon. \end{aligned}$$

Therefore,  $|x - a| < \delta \implies |(f \cdot g)(x) - AB| < 2\epsilon$ . As  $\epsilon$  varies over all real numbers  $2\epsilon$  also varies over all real numbers. Hence,  $\lim_{x \rightarrow a} (f \cdot g)(x) = A \cdot B$ .

(iii), (iv): Proofs are not necessary. □

**Example 5.** (i)  $f(x) = x^3$ . Then it a product of the three functions  $g(x) = x$ . By the calculation that we have done above, we have  $\lim_{x \rightarrow a} f(x) = a^3$ .

(ii)  $f(x) = 1/x$ . Note that this a well-defined function from  $\mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ . By (iv) of the previous proposition,  $\lim_{x \rightarrow a} f(x) = 1/a$ , if  $a \neq 0$ .

(iii)  $f(x) = x^3 + c$ , is a sum of two functions  $g(x) = x^3$  and the constant function  $h(x) = c$ . We have seen that  $\lim_{x \rightarrow a} g(x) = a^3$ ,  $\lim_{x \rightarrow a} h(x) = c$ . So,  $\lim_{x \rightarrow a} f(x) = a^3 + c$ .

(iv) Let  $\theta : \mathbb{R} \setminus \{\text{roots of } x^3 + c\} \longrightarrow \mathbb{R}$ , given by  $\theta(x) = 1/(x^3 + c)$ . Then this is well-defined function and by the previous calculation, we have  $\lim_{x \rightarrow a} \theta(x) = 1/(a^3 + c)$ , by using (iv) of the previous proposition.

In this way, we can give several examples to calculate the limit of many functions.

Try to calculate,  $\lim_{x \rightarrow a} |x|$ .