PHY 101 : Problem Sheet 9

1. Find out the moment of inertia of a uniform square of side a and mass m, about an axis passing through its center of mass, normal to its area. How does it change if a small square of size $a/2$ is cut out from it?

2. If the above mentioned frames are revolved about an axis through its center of mass but lying in the plane of the frame itself, how do the answers change?

3. If the cut out square frame is suspended from the center of one of its edges then find out the frequency of small oscillations. If the same frame is suspended from one of its corner find out the frequency of small oscillations.

4. Find out the moment of inertia of a conical slant surface of mass m having height h and angle θ, about its central axis. [Note : The surface area of a cone is $\pi rh/\cos\theta$ where r is the radius of the circle at its opening.]

5. If a uniformly dense cylinder of mass m, length ℓ and radius a is rotated about an axis on its edge parallel to its central axis with angular speed ω, find out the angular momentum of the cylinder.

6. For a rolling rigid coin the distance vector of any particle j of a particle within the body w.r.t. an inertial observer is $\mathbf{r_j}$. The location of the same particle w.r.t. the center of mass is $\mathbf{r'_j}$ such that $\mathbf{r_j} = \mathbf{R_{CM}} + \mathbf{r'_j}$ where $\mathbf{R_{CM}}$ marks the position of the center of mass. From this prove that total kinetic energy of the coin

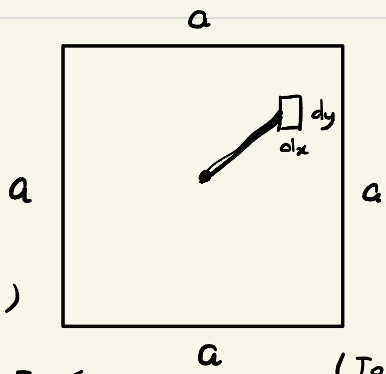
$$\sum_j \frac{1}{2} m_j \dot{\mathbf{r}}_\mathbf{j} \cdot \dot{\mathbf{r}}_\mathbf{j} = \frac{1}{2} M V_{CM}^2 + \frac{1}{2} I \omega^2$$

7. From the definition of the angular momentum of a rigid body, find out the expressions for I_{yx}, I_{yy}, I_{yz} and I_{zz}.

8. Evaluate the momentum of inertia tensor for a uniform sphere of radius R and mass m.

PS - 09

1. Let the frame has surface mass density σ : $M = \sigma a^2$



(Top view)

Consider an element $dx\,dy$ at (x, y)

Mass of small element $dm = \sigma\, dx\, dy$

$$I = \int dm\, \underbrace{(x^2+y^2)}$$

This is perpendicular distance from the axis of rotation

$$= \int_{-a/2}^{a/2} \int_{-a/2}^{+a/2} \sigma\, dx\, dy\, (x^2+y^2)$$

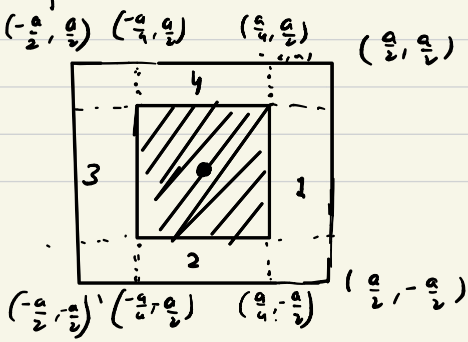
{ first do the y integration keeping x as constant

$$= \sigma \int_{-a/2}^{a/2} \left[x^2 y + \frac{y^3}{3} \right]_{-a/2}^{+a/2} dx = \sigma \int_{-a/2}^{a/2} \left[x^2 a + \frac{a^3}{12} \right] dx$$

$$= \sigma \left[\frac{x^3}{3} a + \frac{a^3}{12} x \right]_{-a/2}^{a/2} = \sigma \left[\frac{a^4}{12} + \frac{a^4}{12} \right]$$

$$= \frac{\sigma a^4}{6} = \frac{M a^2}{6}$$

If $a/2$ frame is cut out from the center then



$$I = I_1 + I_2 + I_3 + I_4$$

$$I_1 = \sigma \int_{a/4}^{a/2} \int_{-a/2}^{a/2} (x^2+y^2)\, dx\, dy = \sigma \left[\frac{x^3}{3} a + \frac{a^3}{12} x \right]_{a/4}^{a/2}$$

$$= 216 a^4 / 192$$

Verify : $I_3 = I_1$

Also, $I_2 = I_4$

$$I_4 = 6 \int_{-a/4}^{a/4} \int_{a/4}^{a/2} dx\, dy \,(x^2 + y^2) = \frac{6 a^4}{192}$$

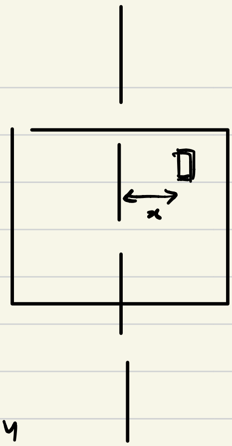
$$\therefore \quad I = 2 \times \left[\frac{21 a^4 6}{192} + \frac{6 a^4}{192} \right] = \frac{44 6 a^4}{192} = \frac{11 6 a^4}{48}$$

Mass of the frame : 6 (Area 1 + Area 2 + Area 3 + Area 4)

$$M_c = 6 \left(\frac{a^2}{4} + \frac{a^2}{8} + \frac{a^2}{8} + \frac{a^2}{4} \right) = \frac{3\cancel{6} a^2}{4}$$

$$\therefore \quad I = \frac{11}{36} M_c a^2$$

2. In this case

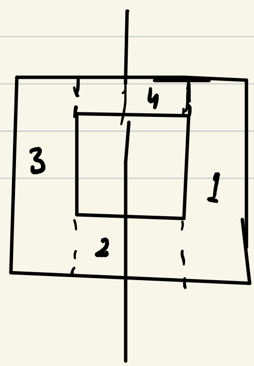


$\rho^2 = x^2$ (this will be the perpendicular distance from the axis of rotation

$$\therefore \quad I = \int_{-a/2}^{+a/2} \int_{-a/2}^{a/2} 6\, dx\, dy \; x^2$$

$$= 6\, a \left[\frac{x^3}{3} \right]_{-a/2}^{a/2} = \frac{6 a^4}{12} = \frac{Ma^2}{12}$$

For the cut out frame



$$I = I_1 + I_2 + I_3 + I_4$$

$$I_1 = 6 \int_{a/4}^{a/2} \int_{-a/2}^{a/2} 6\, dx\, dy \; x^2 = I_3$$

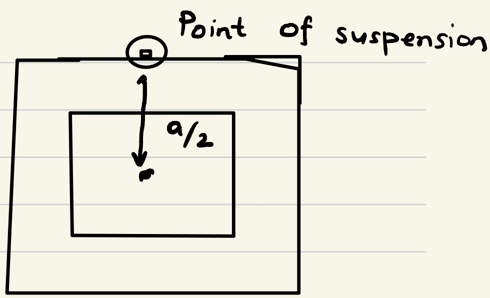
Evaluate this

$$I_2 = 6\int_{-a/4}^{a/4}\int_{-a/2}^{-a/4} dx\, dy \;\; x^2 = I_4$$

Evaluate this

$$I. = I_1 + I_2 + I_3 + I_4 = \frac{21}{192} 6\, a^4$$

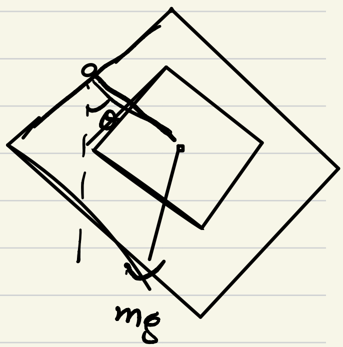
3. Since $I_{cM} = \frac{11}{36} M_c a^2$



$$I = I_{cM} + M_c\left(\frac{a}{2}\right)^2$$

$$= \frac{11}{36} M_c a^2 + \frac{M_c a^2}{4}$$

$$= \frac{20}{36} M_c a^2 = \frac{5}{9} M_c a^2$$



Thus, after small displacement

$$-M_c g \frac{a}{2} \sin\theta = I \frac{d^2\theta}{dt^2}$$

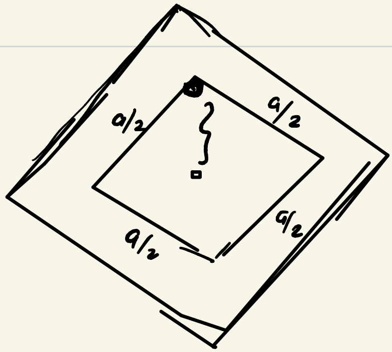
For small θ

$$\frac{d^2\theta}{dt^2} \approx -\frac{M_c g \frac{a}{2}\theta}{I} = -9\frac{g a}{10 a^2}\theta$$

$$= -\underbrace{\frac{9}{10}\frac{g}{a}}_{\omega^2}\theta$$

The point of suspension distance $\sim \sqrt{\left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^2}$

$= \frac{a}{2\sqrt{2}}$



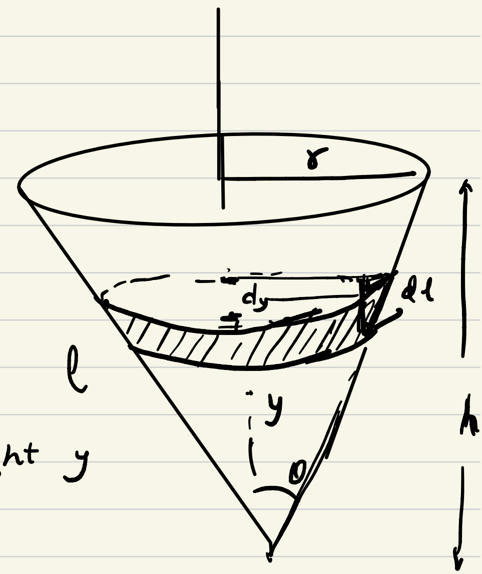
$$I = M_c \frac{a^2}{8} + \frac{11}{36} M_c a^2$$

$$= \frac{M_c a^2}{2}$$

$$\omega^2 = \frac{M_c g \, a/2}{\frac{M_c}{2} a^2} = \frac{g}{a}$$

4. Let us consider a strip at a height y from the tip of cone with thickness dl

$\frac{dy}{dl} = \cos\theta$



the radius of circle at height y

$x = y \tan\theta$

Area of the cone : $\pi r \frac{h}{\cos\theta} = \pi r l$

Mass $=$ $M = \sigma \pi r l$

$\uparrow$ surface mass density

$$I = \int \underbrace{(2\pi x)\, dl\, \sigma}_{dm} \underbrace{x^2}_{\text{perpendicular distance}}$$

$$= 2\pi\sigma \int_0^h (y \tan\theta)^3 \frac{dy}{\cos\theta}$$

$$= 2\pi\sigma \frac{\tan^3\theta}{\cos\theta} \int_0^h y^3\, dy = \frac{2\pi\sigma}{4} h^4 \frac{\tan^3\theta}{\cos\theta}$$

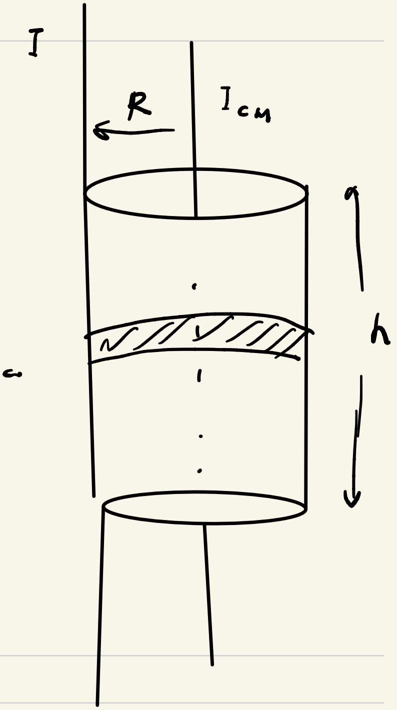
$$= \frac{\pi\sigma}{2} (h^2 \tan^2\theta)\, h \frac{\tan\theta}{\cos\theta}\, h$$

Since,

$$M = \sigma \pi r l = \sigma \pi h^2 \frac{\tan\theta}{\cos\theta}$$

$$I = \frac{M}{2} r^2$$

5. $I = I_{cm} + MR^2 \quad \text{— } 1$



$$I_{cm} = \underset{\text{density}}{\rho} \int_0^R \int_0^{2\pi} \int_0^h \underbrace{r\,dr\,d\theta\,dz}_{dm} \; \underbrace{r^2}_{\text{perpendicular distance}}$$

$$= 2\pi\rho \int_0^R \int_0^h dz\; r^3$$

$$= 2\pi\rho h \frac{R^4}{4} = \frac{\pi\rho h}{2} R^4$$

$$M = \rho \int_0^R \int_0^{2\pi} \int_0^h r\,dr\,d\theta\,dz = \rho\pi R^2 h$$

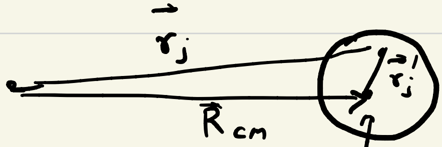
$$\therefore \quad I_{cm} = \frac{MR^2}{2}$$

Hence $I = \dfrac{3MR^2}{2}$ from 1

$$L_z = I\omega = \frac{3MR^2}{2}\omega$$

6.

For an observer



$$\vec{r}_j = \vec{R}_{cm} + \vec{r}_j'$$

$$\dot{\vec{r}}_j = \dot{\vec{R}}_{cm} + \dot{\vec{r}}_j'$$

$$\therefore \quad \sum_j \frac{1}{2} m_j \dot{\vec{r}}_j^{\,2} = \frac{1}{2} \sum m_j \left(\dot{\vec{R}}_{cm} + \dot{\vec{r}}_j'\right) \cdot \left(\dot{\vec{R}}_{cm} + \dot{\vec{r}}_j'\right)$$

$$= \frac{1}{2} \sum m_j \dot{\vec{R}}_{cm} \cdot \dot{\vec{R}}_{cm} + \frac{1}{2} \sum m_j \dot{\vec{r}}_j' \cdot \dot{\vec{r}}_j'$$

$$+ \sum m_j \dot{\vec{R}}_{cm} \cdot \dot{\vec{r}}_j'$$

since $\dot{\vec{r}}_j' = \vec{\omega} \times \vec{r}_j'$

$$= \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \sum m_j \left(r_j^2 \sin^2\theta\right) \omega^2$$

$$+ \frac{d}{dt}\left(\sum m_j \vec{r}_j'\right) \cdot \dot{\vec{R}}_{cm} \rightarrow 0$$

as $\left\{ \vec{R}_{cm}' = 0 \right.$ as viewed from cm itself

$$= \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega^2$$

7. We have seen that

$$\vec{L}_{cm} = \sum_j m_j \left[\vec{r}_j \times (\vec{\omega} \times \vec{r}_j) \right]$$

Thus,
$$\vec{L}_{cm} = \sum_j m_j \Big[(x_j\hat{i} + y_j\hat{j} + z_j\hat{k}) \times \left\{ (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}) \times (x_j\hat{i} + y_j\hat{j} + z_j\hat{k}) \right\} \Big]$$

$$= \sum_j m_j \Big[(x_j\hat{i} + y_j\hat{j} + z_j\hat{k}) \times \left\{ (\omega_y z_j - \omega_z y_j)\hat{i} + (\omega_z x_j - \omega_x z_j)\hat{j} + (\omega_x y_j - \omega_y x_j)\hat{k} \right\} \Big]$$

$$= \sum_j m_j \Big[\{ \underbrace{y_j(\omega_x y_j - \omega_y x_j) - z_j(\omega_z x_j - \omega_x z_j)}_{L_x} \}\hat{i}$$
$$+ \{ \underbrace{z_j(\omega_y z_j - \omega_z y_j) - x_j(\omega_x y_j - \omega_y x_j)}_{L_y} \}\hat{j}$$
$$+ \{ \underbrace{x_j(\omega_z x_j - \omega_x z_j) - y_j(\omega_y z_j - \omega_z y_j)}_{L_z} \}\hat{k} \Big]$$

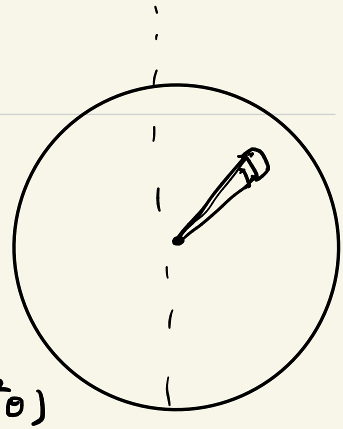
$\Rightarrow$

$$L_x = \sum_j m_j \{ y_j(\omega_x y_j - \omega_y x_j) - z_j(\omega_z x_j - \omega_x z_j) \}$$
$$= \sum_j \underbrace{m_j (y_j^2 + z_j^2)}_{I_{xx}} \omega_x - \sum_j \underbrace{m_j y_j x_j}_{I_{xy}} \omega_y - \sum_j \underbrace{m_j z_j x_j}_{I_{xz}} \omega_z$$

$$L_y = \underbrace{- \sum_j m_j y_j x_j}_{I_{yx}} \omega_x + \sum_j \underbrace{m_j (x_j^2 + z_j^2)}_{I_{yy}} \omega_y - \sum_j \underbrace{m_j y_j z_j}_{I_{yz}}$$

$$L_z = - \sum_j \underbrace{m_j z_j x_j}_{I_{zx}} \omega_x - \sum_j \underbrace{m_j z_j y_j}_{I_{zy}} \omega_y + \sum_j \underbrace{m_j (x_j^2 + y_j^2)}_{I_{zz}} \omega_z$$

8. The continuous representation



$$I_{xx} = \sum_j m_j \left(y_j^2 + z_j^2 \right)$$

$$\rightarrow \underbrace{\rho}_{\text{density}} \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi \left(r^2 \sin^2\theta \underbrace{\sin^2\phi}_{(1-\cos^2\phi)} + r^2 \cos^2\theta \right)$$

$$= \rho \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi \left(r^2 - r^2 \sin^2\theta \cos^2\phi \right)$$

$$= \rho \iiint r^2 \sin\theta \, dr \, d\theta \, d\phi \left(r^2 - r^2 \sin^2\theta \left(\frac{1 + \cos 2\phi}{2} \right) \right)$$

$$= \rho \iiint r^2 \sin\theta \, d\theta \, d\phi \, r^2 \left(1 - \frac{\sin^2\theta}{2} \right) = \frac{2}{5} M R^2 = I_{yy} = I_{zz}$$

Since $\cos 2\phi$ integral vanishes

where $M = \frac{4}{3} \pi R^3 \rho$

$$I_{xy} = - \sum_j m_j x_j y_j \rightarrow - \rho \int_0^R \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi \, (r \sin\theta \cos\phi)(r \sin\theta \sin$$

$$= 0 = I_{yz} = I_{zx}$$

$$\therefore \quad I = \begin{pmatrix} \frac{2}{5} MR^2 & 0 & 0 \\ 0 & \frac{2}{5} MR^2 & 0 \\ 0 & 0 & \frac{2}{5} MR^2 \end{pmatrix}$$