Problem Set 1

Problem 1

Prove that $\log_4 6$ is irrational.
Proof:
I'm going to prove the proposition with contradiction. Suppose it can be expressed as a fraction m/n, where m and n are integers. Then,

$$log_4 6 = \frac{m}{n}$$

(use a definition of log)

$$4^{m/n} = 6$$

iff

$$4^m = 6^n$$

iff

$$2^{2m} = 2^n 3^n$$

iff

$$2^{2m-1} = 3^n$$

2^k $(k = 2m - 1)$ always produces an even number and 3^n always produces an odd number if m and n are integers. Because m and n are both integers, there is no such numbers satisfy the above equation. By the contradiction, $\log_4 6$ is irrational. QED

Problem 2

Use the Well Ordering Principle to prove that

$$n \leq 3^{n/3} \tag{1}$$

for every integer n.
Proof:
By contradiction. So I will prove an equation...

$$n > 3^{n/3} \tag{2}$$

By explicit calculation, the equation holds for $0 \leq n \leq 4$. By WOP, there is smallest value c in the set C, which is...

$$C = \{n \,|\, n > 3^{n/3}\}$$

The value $c - 3$ holds for (1) when $c > 4$ so,

$$(c - 3) \leq 3^{(c-3)/3}$$

iff

$$3(c - 3) \leq 3^{c/3}$$

$c < 3(c-3)$ holds for $c \geq 5$ because of its linearity so,

$$c < 3(c-3) \leq 3^{c/3}$$

iff

$$c < 3^{c/3}$$

which contradicts (2). QED

Problem 3

(a) Verify by truth table that

$$(\mathbf{P}\ IMPLIES\ \mathbf{Q})\ OR\ (\mathbf{Q}\ IMPLIES\ \mathbf{P}) \tag{3}$$

is valid. For the validity of (3), I made a truth table.

P	Q	(1)
T	T	T
T	F	T
F	T	T
F	F	T

Thus, the predicate (3) is valid.

(b) Let P and Q be propositional formulas. Describe a single formula, R, using only AND's, OR's, NOT's, and copies of P and Q, such that R is valid iff P and Q are equivalent.

For equivalence of P and Q, R would be

$$P \ \Leftrightarrow \ Q \tag{4}$$

If P and Q are equivalent, (4) is valid. However I should satisfy the constraint. So (4) iff,

$$NOT(P\ XOR\ Q) \tag{5}$$

iff

$$NOT(NOT(P\ AND\ Q)\ AND\ (P\ OR\ Q)) \tag{6}$$

by applying De Morgan's,

$$(A\ AND\ B)\ OR\ NOT(A\ OR\ B) \tag{7}$$

(7) is the R, which is valid iff P and Q are equivalent.

(c) Explain why
P is valid iff NOT(P) is not satisfiable.

If P is valid then NOT(P) is always false. So NOT(P) is not satisfiable.
If NOT(P) is not satisfiable, then NOT(P) is always false, so P is always true which means valid.

(d) Write a formula, S, so that the set $P = P_1, \cdots, P_k$ is not consistent iff S is valid.

P is consistent iff there's an environment s.t. all elements of P is true. So, P is not consistent iff $P_1 \wedge P_2 \wedge \cdots \wedge P_{k-1} \wedge P_k$ is always false. In order to make above formula always true, simply take NOT of it, which is S.

$$S = NOT(P_1 \wedge P_2 \wedge \cdots \wedge P_{k-1} \wedge P_k)$$

Problem 4

Parallel half adder

(a) A 1-bit add1 module.

$$p_0 = \overline{a_0} \tag{8}$$

$$c = a_0 \tag{9}$$

(b) Explain how to build an (n + 1)-bit parallel half-adder from an (n+1)-bit add1 module by writing a propositional formula for the half-adder output, o_i , using only the variables a_i , p_i , and c_{i-1}

$$o_i = c_{i-1} \; XOR \; p_i$$

(c) Write a formula for the carry, c, in terms of c(1) and c(2).
If all bits are 1 then c will be 1. Otherwise, c is 0.

$$c = c_{(1)} \;\; \wedge \;\; c_{(2)}$$

so as to satisfy it.

(d)
If $c_{(1)} = 0$ then p_i should be a_i, because no overflow occured. If $c_{(1)} = 1$ then p_i should be $r_{i-(n+1)}$ because of the overflow. So,

$$p_i = (\overline{c_{(1)}} \; AND \; a_i) \; OR \; (c_{(1)} \; AND \; r_{i-(n+1)})$$