# Problem Set 3

## Problem 1

Given the Fibonacci number $F_n$ is its n-th term, prove

$$F_n = \frac{p^n - q^n}{\sqrt{5}} \tag{1}$$

where,

$$p = \frac{1+\sqrt{5}}{2} \tag{2}$$

$$q = \frac{1-\sqrt{5}}{2} \tag{3}$$

Proof by strong induction.
First, the base case $F_0$ and $F_1$ must be verified.

$$\begin{aligned} F_0 &= \frac{p^0 - q^0}{\sqrt{5}} \\ &= \frac{1-1}{\sqrt{5}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} F_1 &= \frac{p^1 - q^1}{\sqrt{5}} \\ &= \frac{(\frac{1+\sqrt{5}}{2}) - (\frac{1-\sqrt{5}}{2})}{\sqrt{5}} \\ &= \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

Suppose $F_0, F_1, \ldots, F_n$ is true. Then $F_{n+1}$ must be true.

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \frac{p^n - q^n}{\sqrt{5}} + \frac{p^{n-1} - q^{n-1}}{\sqrt{5}} \\
&= \frac{p^n(1+p^{-1}) - (q^n(1+q^{-1}))}{\sqrt{5}} \\
&= \frac{p^n \cdot p - q^n \cdot q}{\sqrt{5}} \\
&= \frac{p^{n+1} - q^{n+1}}{\sqrt{5}}
\end{aligned}$$

So, (1) holds for all $n \in \mathbb{N}$. QED

## Problem 2

## Problem 3

$$\begin{aligned}
f &: A \to B \\
g &: B \to C \\
h &: A \to C
\end{aligned}$$

($f$ and $g$ are both functions.)
(a) If h is surjective, then f must be surjective.

If h is surjective,

$$(\#arrows\ of\ f) \geq |C| \tag{4}$$

is true. Because f is function,

$$|B| \geq (\#arrows\ of\ f) \tag{5}$$

is also true. Then f must be surjective function. □
(b) If h is surjective, then g must be surjective.
If g is injective function, still holds the lefthand side of the proposition. By the contradiction, it is false.
(c) If h is injective, then f must be injective.
If h is injective,

$$(\#arrows\ of\ f) \leq |C| \tag{6}$$

iff f is injective. (c) is true.
(d) If h is injective and f is total, then g must be injective.