CMSC 442/653 J. Hrudaya (1)
Instructor Dr. Lomonaco TA40935

(1) Construct the multiplication table of the group of symmetries of the equilateral triangle given by the presentation

$$(\rho, \sigma : \rho^3 = 1, \sigma^2 = 1, \rho\sigma = \sigma\rho^2)$$

Assume that the group distinct elements are

$$1, \rho, \rho^2, \sigma, \rho\sigma, \rho^2\sigma$$

For an equilateral triangle (D_3)

$$(\rho, \sigma : \rho^3 = 1, \sigma^2 = 1, \rho\sigma = \sigma\rho^2)$$

Set of elements of D_3 is

$$\{1, \rho, \rho^2, \sigma, \rho\sigma, \rho^2\sigma\} \qquad \rho^i\sigma^j;\ 0 \le i < 3,\ 0 \le j < 2$$

In order to construct multiplication table we have to perform certain calculations which is shown below

$\cdot$	1	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$
1	1	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$
ρ	ρ	ρ^2	$\rho^3 = 1$	$\rho\sigma = \cancel{\sigma\rho^2}$	$\rho(\rho\sigma) = \rho^2\sigma$	$\rho(\rho^2\sigma) = \sigma$
ρ^2	ρ^2	$\rho^3 = 1$	$\rho^4 = \rho$	$\rho^2\sigma$	$\rho^2(\rho\sigma) = \sigma$	$\rho^2(\rho^2\sigma) = \rho\sigma$
σ	σ	$\sigma\rho$ (given) $= \rho\sigma$	$\sigma\rho^2 = \rho\sigma$	1	$\sigma(\rho\sigma) = \sigma^2\rho^2 = \rho^2$	ρ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^2\sigma$	ρ	1	ρ^2
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^2	ρ	1

→ Identity so same

Identity so same

② 4^{th} row if we consider

① $\sigma \cdot 1 = \sigma$

② $\sigma \cdot \rho = \rho^2 \sigma$ (given from the above relations)

③ $\sigma \cdot \rho^2 = \rho\sigma$ (given) $(\sigma\rho)\rho = (\rho^2\sigma)\rho = \rho^2(\rho^2\sigma) = \rho\sigma$

④ $\sigma \cdot \sigma = \sigma^2 = 1$

⑤ $\sigma \cdot \rho\sigma = \sigma \cdot \sigma\rho^2 = \sigma^2\rho^2 = \rho^2$ (wkt $\rho\sigma = \sigma\rho^2$ & $\sigma^2 = 1$

⑥ $\sigma \cdot \rho^2\sigma = \sigma \cdot \sigma\rho = \sigma^2\rho = (1)\rho$ (w.r.t $\sigma\cdot\sigma = \sigma^2 = 1$ from above)

wkt $\rho^2\sigma = \sigma\rho$)

5^{th} row if we consider

① $\rho\sigma = \rho\sigma^2$ or $\rho\sigma$

② $\rho\sigma \cdot \rho = \cancel{\rho\sigma^2 = \rho(1)}$ ~~(WKT $\sigma^2 = 1$)~~

$= \rho \cdot \rho^2\sigma = \rho^3\sigma = \sigma$ (WKT $\sigma\rho = \rho^2\sigma$ & $\rho^3 = 1$)

③ $\rho\sigma \cdot \rho^2 = \sigma\rho^2 \cdot \rho^2 = \sigma\rho \cdot \rho^3 = \sigma\rho = \rho^2\sigma$ (WKT $\rho\sigma = \sigma\rho^2$

$\sigma\rho = \rho^2\sigma ; \rho^3 = 1$

④ $\rho\sigma \cdot \sigma = \rho\sigma^2 = \rho$ ($\sigma^2 = 1$)

⑤ $\rho\sigma \cdot \rho\sigma = \sigma\rho^2\rho\sigma = \sigma\rho^3\sigma = \sigma(1)\sigma = \sigma^2 = 1$

WKT $\rho\sigma = \sigma\rho^2$; $\rho^3 = 1$; $\sigma^2 = 1$

⑥ $\rho\sigma \cdot \rho^2\sigma = \sigma\rho^2 \cdot \rho^2 \rho\sigma = \sigma \cdot \rho\sigma = \sigma \cdot \sigma\rho^2 = \sigma^2\rho^2 = 1(\rho)^2 = \rho^2$

WKT $\rho\sigma = \sigma\rho^2$, $\rho^3 = 1$, $\sigma^2 = 1$

Scanned with CamScanner

For 6th row

① $\rho^2\sigma(1) = \rho^2\sigma$

② $\rho^2\sigma(\rho) = \rho^2(\sigma\rho) = \rho^2(\rho^2\sigma) = \rho^3(\rho\sigma)$
$= \rho\sigma$ (wrt $\sigma\rho = \rho^2\sigma$)

③ $\rho^2\sigma(\rho^2) = \rho^2(\sigma\rho)\rho = \rho^2(\rho^2\sigma)\rho = \rho^3(\rho\sigma)\rho = \rho(\sigma\rho) = \rho(\rho^2\sigma) = \sigma$
(wrt $\sigma\rho = \rho^2\sigma$, $\rho^3 = 1$, $\rho\sigma = \sigma\rho^2$)

④ $\rho^2\sigma(\sigma) = \rho^2\sigma^2 = \rho^2$

⑤ $\rho^2\sigma(\rho\sigma) = \rho^2(\rho^2\sigma)\sigma = \rho^4\sigma^2 = \rho.\underset{1}{\rho^3}\underset{1}{\sigma^2} = \rho$

⑥ $\rho^2\sigma(\rho^2\sigma) = \rho^2(\rho^2\sigma)\rho\sigma = (\rho\sigma)\rho\sigma = \rho(\rho^2\sigma)\sigma = \rho^3\sigma^2$
$= 1$

finally

$\cdot$	1	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$
1	1	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$
ρ	ρ	ρ^2	1	$\rho\sigma$	$\rho^2\sigma$	σ
ρ^2	ρ^2	1	ρ	$\rho^2\sigma$	σ	$\rho\sigma$
σ	σ	$\rho^2\sigma$	$\rho\sigma$	1	ρ^2	ρ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^2\sigma$	ρ	1	ρ^2
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^2	ρ	1

② Construct the multiplication table of the group of symmetries of the square given by the presentation ④

$$(\rho, \sigma : \rho^4 = 1, \sigma^2 = 1, \rho\sigma = \sigma\rho^3)$$

Assume that the distinct group elements are:

$$\{\rho^m \sigma^n : 0 \le m < 4, 0 \le n < 2\}$$

$$D_4 = (\rho, \sigma \mid \rho^4 = 1, \sigma^2 = 1, \rho\sigma = \sigma\rho^3)$$

$$= (\rho, \sigma \mid \rho^4 = 1, \sigma^2 = 1, \sigma\rho = \rho^3\sigma)$$

$\rho, \rho^2 = \rho^3 ; \rho^4 = 1$

$\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma$

$= \{1, \rho, \rho^2, \rho^3, \sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}$

Multiplication table

$\cdot$	1	ρ	ρ^2	ρ^3	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$
1	1	ρ	ρ^2	ρ^3	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$
ρ	ρ	ρ^2	ρ^3	1	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$	$\rho\sigma$
ρ^2	ρ^2	ρ^3	1	ρ	$\rho^2\sigma$	$\rho^3\sigma$	σ	$\rho\sigma$
ρ^3	ρ^3	1	ρ	ρ^2	$\rho^3\sigma$	σ	$\rho\sigma$	$\rho^2\sigma$
σ	σ	$\rho^3\sigma$	$\rho^2\sigma$	$\rho\sigma$	1	ρ^3	ρ^2	ρ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^3\sigma$	$\rho^2\sigma$	ρ	1	ρ^3	ρ^2
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	$\rho^3\sigma$	ρ^2	ρ	1	ρ^3
$\rho^3\sigma$	$\rho^3\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^3	ρ^2	ρ	1

2nd row

1. $\rho(1) = \rho$
2. $\rho(\rho) = \rho^2$
3. $\rho(\rho^2) = \rho^3$
4. $\rho(\rho^3) = \rho^4 = 1$ (w.k.t $\rho^4 = 1$)
5. $\rho(\sigma) = \rho\sigma$
6. $\rho(\rho\sigma) = \rho^2\sigma$
7. $\rho(\rho^3\sigma) = \rho^4\sigma = \sigma$ (WKT $\rho^4 = 1$)
8. $\rho(\rho^2\sigma) = \rho^3\sigma$

3rd row

1. $\rho^2(1) = \rho^2$
2. $\rho^2(\rho) = \rho^3$
3. $\rho^2(\rho^2) = \rho^4 = 1$ (WKT $\rho^4 = 1$)
4. $\rho^2(\rho^3) = \rho^4 \cdot \rho = \rho$ (WKT $\rho^4 = 1$)
5. $\rho^2(\sigma) = \rho^2\sigma$
6. $\rho^2(\rho\sigma) = \rho^3\sigma$
7. $\rho^2(\rho^2\sigma) = \rho^4\sigma = \sigma$ (WKT $\rho^4 = 1$)
8. $\rho^2(\rho^3\sigma) = \rho^5\sigma = \rho^4(\rho\sigma) = 1(\rho\sigma) = \rho\sigma$

4th row

1. $\rho^3(1) = \rho^3$
2. $\rho^3(\rho) = \rho^4 = 1$ (WKT $\rho^4 = 1$)
3. $\rho^3(\rho^2) = \rho^4 \cdot \rho = 1 \cdot \rho = e$ (WKT $\rho^4 = 1$)
4. $\rho^3(\rho^3) = \rho^4 \cdot \rho^2 = 1 \cdot \rho^2$ (WKT $\rho^4 = 1$)
5. $\rho^3(\sigma) = \rho^3\sigma$
6. $\rho^3(\rho\sigma) = \rho^4\sigma = \sigma(1)$ (WKT $\rho^4 = 1$)
7. $\rho^3(\rho^2\sigma) = \rho^4(\rho\sigma) = 1(\rho\sigma)$ (WKT ($\rho^4 = 1$)
8. $\rho^3(\rho^3\sigma) = \rho^4(\rho^2\sigma) = 1(\rho^2\sigma) = \rho^2\sigma$ (WKT $\rho^4 = 1$)

⑥

① $\sigma(1) = \sigma$

② $\sigma(\rho) = \sigma\rho = \rho^3\sigma$

③ $\sigma(\rho^2) = \sigma\rho^2$

④ $\sigma(\rho^3) = \rho\sigma$ (wkt $\rho\sigma = \sigma\rho^3$)

⑤ $\sigma(\sigma) = \sigma^2 = 1$

⑥ $\sigma(\rho\sigma) = \sigma(\sigma\rho^3) = \sigma^2\rho^3 = 1(\rho^3)$ wkt $\sigma^2 = 1$

⑦ $\sigma(\rho^2\sigma) = (\sigma\rho)(\rho\sigma) = (\rho^3\sigma)(\rho\sigma) = \rho^3(\rho^3\sigma)\sigma = \rho^4(\rho^2\sigma)\sigma = 1(\rho^2)1 = \rho^2$

wkt $\sigma\rho = \rho^3\sigma$, $\rho^4 = 1$, $\sigma^2 = 1$

~~⑧ $\sigma(\rho^3\sigma) = (\sigma\rho)\rho^2\sigma = (\rho^3\sigma)\rho^2\sigma = \rho^3(\sigma\rho)\rho^2\sigma = \rho^3(\rho^3\sigma)\rho^2\sigma$~~

~~$= \rho^4(\rho^2\sigma)\rho^2\sigma = 1(\rho^2\sigma 1)\rho\sigma = \rho^2(\rho^3\sigma)$~~

⑧ $\sigma(\rho^3\sigma) = (\sigma\rho)\rho^2\sigma = (\rho^3\sigma)\rho^2\sigma = \rho^3(\sigma\rho)\rho\sigma = \rho^3(\rho^3\sigma)\rho\sigma$

$= \rho^4(\rho^2\sigma)(\rho\sigma) = \rho^2(\sigma\rho)\sigma = \rho^2(\rho^3\sigma)\sigma = \rho^4(\rho\sigma)\sigma = 1\,\rho(\sigma^2) = \rho$

wkt ($\rho^4 = 1$, $\sigma^2 = 1$, $\rho\sigma = \sigma\rho^3$)

6th row

① $\rho\sigma(1) = \rho\sigma$

② $\rho\sigma(\rho) = \rho(\sigma\rho) = \rho(\rho^3\sigma) = \rho^4\sigma = 1(\sigma) = \sigma$

③ $\rho\sigma(\rho^2) = \rho(\sigma\rho)\rho = \rho(\rho^3\sigma)\rho = \rho^4(\sigma\rho) = 1(\rho^3\sigma) = \rho^3\sigma$

④ $\rho\sigma(\rho^3) = \rho(\sigma\rho)\rho^2 = \rho(\rho^3\sigma)\rho^2 = \rho^4\sigma\rho^2 = 1\sigma\rho^2 = (\sigma\rho)\rho = (\rho^3\sigma)\rho = \rho^3\rho\sigma$

$= \rho^3(\sigma\rho) = \rho^3(\rho^3\sigma) = \rho^4(\rho^2\sigma) = \rho^2\sigma$

⑤ $\rho\sigma(\sigma) = \rho\sigma^2 = \rho(1) = \rho$ wkt ($\sigma^2 = 1$)

⑥ $\rho\sigma(\rho\sigma) = \rho(\rho^3\sigma)\sigma = \rho^4\sigma^2 = 1$ ($\rho^4 = 1, \sigma^2 = 1$)

⑦ $\rho\sigma(\rho^2\sigma) = \rho(\sigma\rho)\rho\sigma = \rho(\rho^3\sigma)\rho\sigma = \sigma\rho\sigma = (\rho^3\sigma)\sigma = \rho^3\sigma^2 = \rho^3$

⑧ $\rho\sigma(\rho^3\sigma) = \rho(\sigma\rho)\rho^2\sigma = \rho(\rho^3\sigma)\rho^2\sigma = \sigma\rho^2\sigma = (\rho^3\sigma)\rho\sigma = $ ans $= \rho^2$

From 5th row ⑦

7th row

(7)

(1) $\rho^2\sigma(1) = \rho^2\sigma$

(2) $\rho^2\sigma(\rho) = \rho^2(\sigma\rho) = \rho^2(\rho^3\sigma) = \rho\sigma$

(3) $\rho^2\sigma(\rho^2) = \rho^2(\sigma\rho)\rho = \rho^2(\rho^3\sigma)\rho = \rho\sigma\rho = \sigma$ ↳ From 6th row (2)

(4) $\rho^2\sigma(\rho^3) = \rho^2(\sigma\rho)\rho^2 = \rho^2(\rho^3\sigma)\rho^2 = (\rho\sigma)\rho^2 = (\sigma\rho^3)\rho^2 = \sigma\rho = \rho^3\sigma$

(5) $\rho^2\sigma(\sigma) = \rho^2(\sigma^2) = \rho^2$

(6) $\rho^2\sigma(\rho\sigma) = \rho^2(\rho^3\sigma)\rho\sigma = \rho\sigma^2 = \rho$

(7) $\rho^2\sigma(\rho^2\sigma) = \rho^2(\rho^3\sigma\rho\sigma = \rho\sigma\rho\sigma =$ From 6th row (6) $= 1$

(8) $\rho^2\sigma(\rho^3\sigma) = \rho^2(\rho^3\sigma)\rho^2\sigma = (\rho\sigma)\rho^2\sigma = \rho(\rho^3\sigma)\rho\sigma = \sigma\rho\sigma = \rho^3$

From 5th (6)

8th row

(1) $\rho^3\sigma(1) = \rho^3\sigma$

(2) $\rho^3\sigma(\rho) = \rho^3(\sigma\rho) = \rho^3(\rho^3\sigma) = \rho^2\sigma$

(3) $\rho^3\sigma(\rho^2) = \rho^3(\sigma\rho)\rho = \rho^3(\rho^3\sigma)\rho = \rho^2\sigma\rho = \rho^2(\rho^3\sigma) = \rho\sigma$

(4) $\rho^3\sigma(\rho^3) = \rho^3(\sigma\rho)\rho^2 = \rho^3(\rho^3\sigma)\rho^2 = \rho^2\sigma\rho^2 \rightarrow$ From 7th row (3) $= \sigma$

(5) $\rho^3\sigma(\sigma) = \rho^3\sigma^2 = = \rho^2\cancel{(\rho\sigma)\sigma = \rho^2(\sigma\rho^3)\sigma} =$ $=\sigma$

$= \rho^3(1) = \rho^3$

(6) $\rho^3\sigma(\rho\sigma) = \rho^3(\rho^3\sigma)\sigma = \rho^2(1) = \rho^2$

(7) $\rho^3\sigma(\rho^2\sigma) = \rho^3(\rho^3\sigma)\rho\sigma = \rho^2\sigma\rho\sigma =$ From 7th row (6) $= \rho$

(8) $\rho^3\sigma(\rho^3\sigma) = \rho^3(\rho^3\sigma)\rho^2\sigma = \rho^2\sigma\rho^2\sigma =$ From 7th row (7) $= 1$

Scanned with CamScanner

③ let S be a set with an associate binary operation ⑤
$\cdot : S \times S \to S$. let e_L be a left identity of S ie ($e_L \cdot s = s \; \forall s \in S$),
and let e_R be a right identity of S ie $s \cdot e_R = s \; \forall s \in S$)

ⓐ prove that $e_L = e_R$

Given that

let S be a set with binary operation

$\cdot : S \times S \to S$

let e_L be a left identity and e_R be a right identity

$e_L \cdot s = s$ (As e_L is left identity)

$s \cdot e_R = s$ (As e_R is right identity)

Given S is a set with an associate binary operation

which means $a \times (b \times c) = (a \times b) \times c$

$e_L \cdot (s \cdot e_R) = (e_L \cdot s) \cdot e_R$

there e_L is left identity e_R is right identity

$\underline{1} \cdot (s \cdot e_R) = (e_L \cdot s) \cdot 1$

$(s \cdot e_R) = (e_L \cdot s)$ $\Rightarrow s = s$

Right identity left identity

As above S is a set

so

$e_R = e_L$

(or) $e_L \cdot e_R = e_R$ becz e_L is left id —①

$e_L \cdot e_R = e_L$ becz e_R is right identy —②

From ①&② $e_L = e_R$ Q.E.D

(b) Also prove that s can have atmost one 2-sided identity (1)

Given that to prove

s can have atmost 2sided identity

consider

let s has two sided identity say e_L and e_R

which means

$s \cdot e_L = e_L \cdot s = s$

$s \cdot e_R = e_R \cdot s = s$

So if we consider

$s \cdot e_L = s$

then $e_R \cdot e_L = e_R$ - (1)

and $s \cdot e_R = s = e_R \cdot s$

then ~~$e_R \cdot e_L =$~~

$e_R \cdot e_R = e_L$ - (2)

so $e_R = e_L$

therefore it has only one 2-sided identity which means it can have atmost 2-sided identity.

QED

④ S be a set with an associative binary operation ⑩
$\circ : S \times S \to S$ and a 2 sided identity e and let $s \in S$
let $\tilde{s}_L$ and $\tilde{s}_R$ be elements of S such that

$$\tilde{s}_L \cdot s = e = s \cdot \tilde{s}_R$$

Prove that $\tilde{s}_L = \tilde{s}_R$

Given that $\tilde{s}_L$ and $\tilde{s}_R$ be element of S

$$\tilde{s}_L \cdot s = e = s \cdot \tilde{s}_R$$

and given S is a set of associate binary operation

$$\odot : S \times S \to S$$

$$a \times (b \times c) = (a \times b) \times c$$

$$\tilde{s}_L \cdot \underbrace{(s \cdot \tilde{s}_R)}_{e} = \underbrace{(\tilde{s}_L \cdot s)}_{e} \cdot \tilde{s}_R \quad \text{given}$$

$$\tilde{s}_L \cdot e = e \cdot \tilde{s}_R$$

we know from identity element

$$\tilde{s}_L = \tilde{s}_R$$

Hence we proved

QED

Scanned with CamScanner